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SHAPE GROUPS, ANR-SYSTEMS AND RELATED TOPICS

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CHAPTER I

INTRODUCTION

The theory of shapes was first introduced by K. Borsuk in [2]. His approach was through the notion of fundamental classes as mappings from one compactum into another. These fundamental classes are a generalization of the usual notion of the homotopy classes of continuous functions. His purpose in introducing this concept was to study the global homotopy properties of compacta and to alleviate the local difficulties that arise in the application of homotopy theory to compacta.

In [10] and [11], S. Mardešić and J. Segal gave an alternate approach to the theory of shapes. They used ANR-sequences, which are special types of inverse systems, and maps of systems, which are generalizations of the usual notion of maps of inverse systems. Their approach generalized the theory to include compact Hausdorff spaces.

In this paper both approaches are used. Chapter II is devoted to the notions used by Borsuk. In many cases these notions are given in a more general setting. In Chaper III the fundamental groups defined by Borsuk in [2] are obtained through the ANR-system approach. These groups may be useful in the study of pointed compact Hausdorff spaces. They extend the usual homotopy groups, but may give more information in cases where local difficulties arise. For example, it is known that the usual homotopy groups of the "Warsaw circle" are trivial. However, the first shape group is infinite cyclic.

In Chapter IV the notion of extensions of maps of systems is given and their relation to the shape groups and shape retractions given by Mardešić in [9], are discussed. Chapter V deals with products in shape theory and some of the results of Borsuk are extended to arbitrary products.

For convenience all spaces are assumed to be Hausdorff. Unless otherwise stated, X, X_0, Y, Y_0, Z, Z_0 will denote compact spaces. The reader is referred to [13] for pertinent definitions; e.g., ANR, homotopic maps.

CHAPTER II

SOME RESULTS RELATED TO SHAPE THEORY

1. Weak Homotopy Category. Let X, X_0, Y, Y_0, Z, Z_0 denote compact spaces such that $X_0 \subset X \subset M$, $Y_0 \subset Y \subset N$ and $Z_0 \subset Z \subset P$. If two maps $f, g\colon (X,X_0) \to (Y,Y_0)$ are homotopic in each neighborhood (V,V_0) of (Y,Y_0) in N, then they will be said to be weakly homotopic in N, denoted by $f \underset{N}{\simeq} g$. (compare [2] §2). It is clear that the relation of weak homotopy in N is an equivalence relation. The equivalence class to which a map $f\colon (X,X_0) \to (Y,Y_0)$ belongs is called the weak homotopy class of f in N and is denoted by $[f]_N$.

Theorem 1.1: If N is normal, P is an ANR and $f, f'\colon (X,X_0) \to (Y,Y_0)$ and $g, g'\colon (Y,Y_0) \to (Z,Z_0)$ are maps such that $f \underset{N}{\simeq} f'$ and $g \underset{P}{\simeq} g'$ then $gf \underset{P}{\simeq} g'f'$.

Proof: Let (W,W_0) be an open neighborhood of (Z,Z_0) in P. There is a homotopy $G\colon (Y,Y_0) \times I \to (W,W_0)$ such that $G(y,0) = g(y)$, $G(y,1) = g'(y)$ for all $y \in Y$. Since $W \subset P$ is an ANR there is a neighborhood V of Y in N and a map $G'\colon V \times I \to W$ such that $G'(y,t) = G(y,t)$ if $y \in Y$. Since $G'^{-1}(W_0)$ is a neighborhood of $Y_0 \times I$ in $V \times I$ there is a neighborhood V_0 of Y_0 in V such that $Y_0 \times I \subset V_0 \times I \subset G'^{-1}(W_0)$. Then $G'\colon (V,V_0) \times I \to (W,W_0)$ is an extension of G. Since $f \underset{N}{\simeq} f'$ there is a homotopy $F\colon (X,X_0) \times I \to (V,V_0)$ such that $F(x,0) = f(x)$, $F(x,1) = f'(x)$ for all $x \in X$. Let $E\colon (X,X_0) \times I \to (W,W_0)$ be given by $E(x,t) = G'(F(x,t),t)$. Then $E(x,0) = G'(F(x,0),0) = G'(f(x),0) = gf(x)$ since $f(x) \in Y$.

Similarly, $E(x,1) = g'f'(x)$. Thus $gf \underset{P}{\simeq} g'f'$.

If $M = N = P \in$ ANR, Theorem 1.1 allows one to define a composition of weak homotopy classes in M as follows:

$$[g]_M \, [f]_M = [gf]_M$$

whenever the composition of the maps is defined. It is clear that the weak homotopy class $[1_{X,X_0}]_M$ with representative $1_{X,X_0}: (X,X_0) \to (X,X_0)$, the identity map, is an identity for this composition.

Theorem 1.2: If $M \in$ ANR there is a category, called the weak homotopy category of M, whose objects are pairs (X,X_0) of compact subsets of M and whose morphisms are weak homotopy classes in M.

A map $f: (X,X_0) \to (Y,Y_0)$ of pairs of compact subsets of M is said to be a weak homotopy equivalence in M if there is a map $g: (Y,Y_0) \to (X,X_0)$ such that $gf \underset{M}{\simeq} 1_{X,X_0}$ and $fg \underset{M}{\simeq} 1_{Y,Y_0}$. Two pairs of compact subsets of M are said to be of the same weak homotopy type in M, $(X,X_0) \underset{M}{\simeq} (Y,Y_0)$, provided there is a weak homotopy equivalence in M, $f: (X,X_0) \to (Y,Y_0)$. It is clear that the relation $\underset{M}{\simeq}$ on pairs of compact subsets of M is an equivalence relation. The equivalence class with representative (X,X_0) is denoted by $[X,X_0]_M$ and is referred to as the weak homotopy type of (X,X_0) in M.

If $X_0 = \{x_0\}$ or $X_0 = \emptyset$ one has the concepts of weak homotopy type for pointed compact subsets of M and compact subsets of M, respectively. A neighborhood of (X,x_0) in M is a pair (V,x_0) such that V is a neighborhood of X in M.

Theorem 1.3: Let (X,X_0) be a compact pair of subsets of M and (Y,Y_0) a compact pair of subsets of N. If $f, g: (X,X_0) \to (Y,Y_0)$ are maps such that $f \simeq g$ then $f \underset{N}{\simeq} g$.

<u>Proof</u>: Since $f \simeq g$ in (Y,Y_0), $f \simeq g$ in every neighborhood (V,V_0) of (Y,Y_0) in N.

<u>Corollary 1.4</u>: If (X,X_0) and (Y,Y_0) are compact pairs of subsets of $M \in \mathrm{ANR}$ then $[X,X_0] = [Y,Y_0]$ implies $[X,X_0]_M = [Y,Y_0]_M$ where $[X,X_0]$ denotes the usual homotopy class of (X,X_0), see [13].

<u>Proof</u>: Let $f: (X,X_0) \to (Y,Y_0)$ and $g: (Y,Y_0) \to (X,X_0)$ be maps such that $gf \simeq 1_{X,X_0}$ and $fg \simeq 1_{Y,Y_0}$. By Theorem 1.3, $gf \underset{M}{\simeq} 1_{X,X_0}$ and $fg \underset{M}{\simeq} 1_{Y,Y_0}$ so that f is a weak homotopy equivalence in M.

<u>Corollary 1.5</u>: Let (X,X_0), (Y,Y_0) be compact pairs in $M \in \mathrm{ANR}$ and (X',X_0'), (Y',Y_0') compact pairs in $N \in \mathrm{ANR}$ such that $(X,X_0) \cong (X',X_0')$ and $(Y,Y_0) \cong (Y',Y_0')$. Then $[X,X_0]_M = [Y,Y_0]_M$ implies that $[X',X_0']_N = [Y',Y_0']_N$.

<u>Proof</u>: Let $h_1: (X,X_0) \to (X',X_0')$ and $h_2: (Y,Y_0) \to (Y',Y_0')$ be homeomorphisms. Let $f: (X,X_0) \to (Y,Y_0)$ and $g: (Y,Y_0) \to (X,X_0)$ be maps such that $gf \underset{M}{\simeq} 1_{X,X_0}$ and $fg \underset{M}{\simeq} 1_{Y,Y_0}$. Then $h_2fh_1^{-1}: (X',X_0') \to (Y',Y_0')$ and $h_1gh_2^{-1}: (Y',Y_0') \to (X',X_0')$ are maps such that

$$(h_2fh_1^{-1})(h_1gh_2^{-1}) = h_2fgh_2^{-1} \underset{N}{\simeq} h_2 1_{Y,Y_0} h_2^{-1} = 1_{Y',Y_0'}$$

and similarly

$$(h_1gh_2^{-1})(h_2fh_1^{-1}) = h_1gfh_1^{-1} \underset{N}{\simeq} 1_{X',X_0'}.$$

This completes the proof.

2. <u>Fundamental Sequences and Fundamental Classes</u>.

A <u>fundamental sequence</u> (compare [4] p. 127) $\underline{f} = \{f_k, (X,X_0),(Y,Y_0)\}_{M,N}$ consists of a sequence of maps $f_k: M \to N$ such that for every neighborhood (V,V_0) of (Y,Y_0) in N there is a neighborhood (U,U_0) of (X,X_0) in M and an index k_0 such that if $k \geq k_0$ then

$$f_k|(U,U_0) \simeq f_{k+1}|(U,U_0) \text{ in } (V,V_0).$$

If $\underline{f} = \{f_k, (X,X_0), (Y,Y_0)\}_{M,N}$ and $\underline{g} = \{g_k, (Y,Y_0), (Z,Z_0)\}_{N,P}$ are fundamental sequences then $\underline{g}\ \underline{f} = \{g_k f_k, (X,X_0), (Y,Y_0)\}_{M,P}$ is a fundamental sequence, called the composition of the fundamental sequences $\underline{f}$ and $\underline{g}$, [4] p. 128. A fundamental sequence $\underline{f}$ is said to be generated by a map $f: (X,X_0) \to (Y,Y_0)$ if $f_k(x) = f(x)$ for all $x \in X$. The identity fundamental sequence is generated by the identity $1_{X,X_0}$ and is denoted by $\underline{1}_{X,X_0} = \{1_k, (X,X_0), (X,X_0)\}_{M,M}$. There is a category whose objects are triplets $(X,X_0;M)$ where X and X_0 are compact and $X_0 \subset X \subset M$ and whose morphisms are fundamental sequences.

Two fundamental sequences $\underline{f} = \{f_k, (X,X_0), (Y,Y_0)\}_{M,N}$ and $\underline{g} = \{g_k, (X,X_0), (Y,Y_0)\}_{M,N}$ are said to be homotopic, $\underline{f} \simeq \underline{g}$, if for every neighborhood (V,V_0) of (Y,Y_0) in N there is a neighborhood (U,U_0) of (X,X_0) in M and an index k_0 such that if $k \geq k_0$ then

$$f_k\big|(U,U_0) \simeq g_k\big|(U,U_0) \text{ in } (V,V_0).$$

It is clear that this is an equivalence relation. The equivalence class to which a fundamental sequence $\underline{f}$ belongs is called the fundamental class of $\underline{f}$ and is denoted by $[\underline{f}]$.

By inspecting the proof of [2] Theorem 6.4, one has that if $\underline{f} = \{f_k, (X,X_0), (Y,Y_0)\}_{M,N}$ and $\underline{f}' = \{f'_k, (X,X_0), (Y,Y_0)\}_{M,N}$ are homotopic fundamental sequences and if $\underline{g} = \{g_k, (Y,Y_0), (Z,Z_0)\}_{N,P}$ and $\underline{g}' = \{g'_k, (Y,Y_0), (Z,Z_0)\}_{N,P}$ are homotopic fundamental sequences then $\underline{g}\ \underline{f}$ and $\underline{g}'\underline{f}'$ are homotopic fundamental sequences. Thus there is a category, denoted by $\mathfrak{F}$, whose objects are triplets $(X,X_0;M)$ such that X and X_0 are compact, $X_0 \subset X \subset M$ and whose morhpisms are fundamental classes of fundamental sequences.

A fundamental sequence $\underline{f} = \{f_k, (X,X_0), (Y,Y_0)\}_{M,N}$ is a fundamental

<u>equivalence</u> if there is a fundamental sequence $\underline{g} = \{g_k, (Y,Y_0), (X,X_0)\}_{N,M}$ such that $\underline{g}\ \underline{f} \simeq \underline{1}_{X,X_0}$ and $\underline{f}\ \underline{g} \simeq \underline{1}_{Y,Y_0}$. In this case, (X,X_0) and (Y,Y_0) are said to be <u>fundamentally equivalent</u> in M,N; denoted by $(X,X_0) \underset{\mathfrak{F}}{\simeq} (Y,Y_0)$ in M,N. Two pairs of compact spaces (X,X_0) and (Y,Y_0) are of the <u>same shape</u>, $Sh(X,X_0) = Sh(Y,Y_0)$, if there exist AR-sets M and N and spaces $X_0' \subset X' \subset M$ and $Y_0' \subset Y' \subset N$ such that $(X,X_0) \cong (X',X_0')$, $(Y,Y_0) \cong (Y',Y_0')$ and $(X',X_0') \underset{\mathfrak{F}}{\simeq} (Y',Y_0')$ in M,N. The relation of having the same shape is an equivalence relation and the equivalence class to which (X,X_0) belongs is called the <u>shape</u> of (X,X_0), denoted $Sh(X,X_0)$. This notion of shape is due to Borsuk [4].

<u>Theorem 2.1</u>: If $N \in ANR$, M is normal and $\underline{f} = \{f_k, (X,X_0), (Y,Y_0)\}_{M,N}$, $\underline{g} = \{g_k, (X,X_0), (Y,Y_0)\}_{M,N}$ are fundamental sequences satisfying for each neighborhood (V,V_0) of (Y,Y_0) in N there is an index k_0 such that if $k \geq k_0$ then $f_k|(X,X_0) \simeq g_k|(X,X_0)$ in (V,V_0) then $\underline{f} \simeq \underline{g}$.

<u>Proof</u>: Let (V,V_0) be an open neighborhood of (Y,Y_0) in N. Then there is a neighborhood (U,U_0) of (X,X_0) in M and an index k_1 such that if $k \geq k_1$ then

$$f_k|(U,U_0) \simeq f_{k_1}|(U,U_0) \text{ in } (V,V_0),$$
$$g_k|(U,U_0) \simeq g_{k_1}|(U,U_0) \text{ in } (V,V_0)$$

and

$$f_k|(X,X_0) \simeq g_k|(X,X_0) \quad \text{in } (V,V_0).$$

Let $F: (X,X_0) \times I \to (V,V_0)$ be a homotopy such that $F(x,0) = f_{k_1}(x)$ and $F(x,1) = g_{k_1}(x)$ for all $x \in X$. Define $F': (X \times I) \cup (U \times \dot{I}) \to V$ by

$$F'(x,t) = \begin{cases} F(x,t) \text{ if } (x,t) \in X \times I \\ f_{k_1}(x) \text{ if } (x,t) \in U \times 0. \\ g_{k_1}(x) \text{ if } (x,t) \in U \times 1 \end{cases}$$

By standard arguments, F' is a map. Since V is an ANR and X is compact, there is a neighborhood W of X in M such that $W \times I \subset U \times I$ and an extension $\hat{F}: W \times I \to V$ of F'. Since $X_0 \times I \subset \hat{F}^{-1}(V_0)$ and X_0 is compact there is a neighborhood W_0 of X_0 in M such that $W_0 \times I \subset \hat{F}^{-1}(V_0)$. Then $\hat{F}: (W,W_0) \times I \to (V,V_0)$ is such that $\hat{F}(x,0) = f_{k_1}(x)$, $\hat{F}(x,1) = g_{k_1}(x)$ for all $x \in W$; that is,

$$\hat{F}: f_{k_1}\big|(W,W_0) \simeq g_{k_1}\big|(W,W_0) \text{ in } (V,V_0).$$

Hence if $k \geq k_1$ then

$$f_k\big|(W,W_0) \simeq f_{k_1}\big|(W,W_0) \simeq g_{k_1}\big|(W,W_0) \simeq g_k\big|(W,W_0) \text{ in } (V,V_0).$$

Therefore, $\underline{f} \simeq \underline{g}$.

This theorem also holds for fundamental sequences of compact spaces and of pointed compact spaces. As a corollary, one has a result similar to [3] 1.1.

<u>Corollary 2.2</u>: If $N \in$ ANR, M is normal and two fundamental sequences $\underline{f}$ and $\underline{g}$ satisfy the condition that $f_k(x) = g_k(x)$ for every point $x \in X$ and for $k = 1, 2, \ldots$, then $\underline{f} \simeq \underline{g}$.

<u>Proof</u>: If (V,V_0) is any neighborhood of (Y,Y_0) in N then there is an index k_0 such that if $k \geq k_0$ then $f_k(X,X_0) \subset (V,V_0)$ and $g_k(X,X_0) \subset (V,V_0)$. Since $f_k(x) = g_k(x)$ for all $x \in X$, the hypothesis of Theorem 2.1 is satisfied so that $\underline{f} \simeq \underline{g}$.

<u>3. Extensions and Retractions</u>. In this section, attention is restricted to the absolute case. Let $X \subset X' \subset M$ and $Y \subset N$ be compact. A fundamental sequence $\underline{f}' = \{f'_k, X', Y\}_{M,N}$ is said to be an <u>extension</u> of the fundamental sequence $\underline{f} = \{f_k, X, Y\}_{M,N}$ if $f'_k(x) = f_k(x)$ for all k and $x \in X$.

<u>Theorem 3.1</u>: If $X \subset X' \subset M$ and $Y \subset N$ where M is normal and N is an AR-set then a fundamental sequence $\underline{f} = \{f_k, X, Y\}_{M,N}$ has an extension iff there is a fundamental sequence $\underline{f}' = \{f'_k, X', Y\}_{M,N}$ such that $\underline{f} \simeq \underline{f}'\underline{i}$ where $\underline{i} = \{1_M, X, X'\}_{M,M}$.

<u>Proof</u>: If $\underline{f}' = \{f'_k, X', Y\}_{M,N}$ is an extension of $\underline{f}$ then for all $x \in X$, $f'_k(x) = f_k(x)$. By Theorem 2.1, $\underline{f} \simeq \underline{f}'\underline{i}$.

Conversely, assume $\underline{f}'$ is a fundamental sequence such that $\underline{f} \simeq \underline{f}'\underline{i}$. Then $\underline{f}'$ is an extension of $\underline{f}'\underline{i}$. By inspecting the proof given in [12], $\underline{f} \simeq \underline{f}'\underline{i}$ and $\underline{f}'\underline{i}$ having an extension implies that $\underline{f}$ has an extension.

A fundamental sequence $\underline{r} = \{r_k, X', X\}_{M,M}$ is called an <u>M-fundamental retraction</u> if $r_k(x) = x$ for all $x \in X$. That is, $\underline{r}$ is an extension of the identity fundamental sequence $\underline{1}_X = \{1_M, X, X\}_{M,M}$. If there is an M-fundamental retraction from X' to X then X is called an <u>M-fundamental retract</u> of X'.

<u>Theorem 3.2</u>: If $X \subset Y \subset M$, $Y' \subset N$ with $M, N \in AR$ and if $h: Y \to Y'$ is a homeomorphism then X is an M-fundamental retract of Y iff $X' = h(X)$ is an N-fundamental retract of Y'.

<u>Proof</u>: Let $\underline{r} = \{r_k, Y, X\}_{M,M}$ be such that if $x \in X$ then $r_k(x) = x$. Let $f: N \to M$ be an extension of h^{-1} and $g: M \to N$ an extension of h. Since $f(Y') = Y$ and $g(X) = X'$, both $\underline{f} = \{f, Y', Y\}_{N,M}$ and $\underline{g} = \{g, X, X'\}_{M,N}$ are fundamental sequences. Consider the fundamental sequence $\underline{g}\,\underline{r}\,\underline{f} = \{gr_kf, Y', X'\}_{N,N}$. If $x \in X'$ then $gr_kf(x) = hr_kh^{-1}(x) = x$ since $h^{-1}(X') = X$. Thus $\underline{g}\,\underline{r}\,\underline{f}$ is an N-fundamental retraction from Y' to X'.

<u>Corollary 3.3</u>: If $X \subset Y \subset Q$ then X is a fundamental retract of Y (in the sense of Borsuk [3]) iff X is a Q-fundamental retraction of Y.

In Section 6 of this chapter, attention is focused on the compact metric case (i.e., when X can be embedded in Hilbert cube Q).

4. Fundamental Groups. Let X be a compact subset of M and $x_0 \in X$. Let (S,a) denote the pointed n-dimensional sphere. An M-approximative map of (S,a) toward (X,x_0), $\underline{\xi} = \{\xi_k,(S,a)\to(X,x_0)\}_M$ is a fundamental sequence $\underline{\xi} = \{\xi_k,(S,a),(X,x_0)\}_{S,M}$.

If $\xi,\eta : (S,a)\to(M,x_0)$ are maps, their join, $\xi * \eta : (S,a)\to(M,x_0)$ is defined as follows. Let P, P' be n-dimensional balls on S such that $a \in S-\mathring{P}$, $a \in S-\mathring{P}'$ and $\mathring{P}' \subset S-\mathring{P}$. Let $\alpha, \beta : (S,a)\times I \to (S,a)$ be homotopies such that $\alpha(x,0) = \beta(x,0) = x$ for every point $x \in S$ and $\alpha(S-\mathring{P},1) = a = \beta(S-\mathring{P}',1)$. Define

$$(\xi * \eta)(x) = \begin{cases} \xi\alpha(x,1) & \text{if } x \in S-\mathring{P}' \\ \eta\beta(x,1) & \text{if } x \in S-\mathring{P}. \end{cases}$$

Note, if $[\xi], [\eta] \in \pi_n(X,x_0)$ where $(X,x_0) \subset (M,x_0)$ then $[\xi]*[\eta] = [\xi*\eta]$ is the group operation $*$ in $\pi_n(X,x_0)$. Let $\underline{\pi}_n^M(X,x_0)$ denote the set of fundamental classes of M-approximative maps of (S,a) toward (X,x_0). By inspecting the proof given in [2] §14, one can define a group operation $*$ in $\underline{\pi}_n^M(X,x_0)$ by setting

$$[\underline{\xi}]*[\underline{\eta}] = [\{\xi_k * \eta_k,(S,a)\to(X,x_0)\}_M].$$

If H denotes Hilbert space then $\underline{\pi}_n^H(X,x_0)$ is the $n^{\underline{th}}$ fundamental group as defined by Borsuk in [2].

By inspecting the proof in [2] 15.1, one has the following theorem.

Theorem 4.1: If $\underline{f} = \{f_k,(X,x_0),(Y,y_0)\}_{M,N}$ is a fundamental sequence then $[\underline{f}]$ induces a homomorphism $[\underline{f}]_*: \underline{\pi}_n^M(X,x_0)\to\underline{\pi}_n^N(Y,y_0)$ given by $[\underline{f}]_*([\underline{\xi}]) = [\{f_k\,\xi_k,(S,a)\to(Y,y_0)\}_N]$.

The following theorem is evident; see [2] 15.2.

Theorem 4.2: The homomorphism of the fundamental group $\underline{\pi}_n^M(X,x_0)$ induced by the identity fundamental class $[\underline{1}_{X,x_0}]$ is the identity.

The proof of the following theorem is clear, see [2] 15.3.

Theorem 4.3: If $[\underline{f}]$ is a fundamental class from (X,x_0) to (Y,y_0) in M,N and $[\underline{g}]$ is a fundamental class from (Y,y_0) to (Z,z_0) in N,P then the homomorphism of $\underline{\pi}_n^M(X,x_0)$ into $\underline{\pi}_n^P(Z,z_0)$ induced by the composition $[\underline{g}][\underline{f}]$ is the composition of the homomorphisms induced by these fundamental classes; i.e., $([\underline{g}][\underline{f}])_* = [\underline{g}]_*[\underline{f}]_*$.

The following theorem is an analogue of [2] 15.4.

Theorem 4.4: If one assigns to each fundamental class $[\underline{f}]$ from (X,x_0) to (Y,y_0) in M,N the induced homomorphism $[\underline{f}]_*$ from $\underline{\pi}_n^M(X,x_0)$ to $\underline{\pi}_n^N(Y,y_0)$ then one gets a covariant functor $\underline{\pi}_n$ from the category $\mathfrak{F}$ to the category of groups (abelian if $n > 1$).

As in [2] 15.5, one has the following corollary.

Corollary 4.5: If the fundamental class $[\underline{g}]$ from (Y,y_0) to (X,x_0) in N,M is a right inverse of the fundamental class $[\underline{f}]$ from (X,x_0) to (Y,y_0) in M,N then the homomorphism $[\underline{g}]_*: \underline{\pi}_n^N(Y,y_0) \to \underline{\pi}_n^M(X,x_0)$ induced by $[\underline{g}]$ is a right inverse of the homomorphism $[\underline{f}]_*: \underline{\pi}_n^M(X,x_0) \to \underline{\pi}_n^N(Y,y_0)$ induced by $[\underline{f}]$.

Corollary 4.6: If $f: M \to N$ is a map, $f(X) \subset Y$ and $y_0 = f(x_0)$ then there is an induced homomorphism $f_*: \underline{\pi}_n^M(X,x_0) \to \underline{\pi}_n^N(Y,y_0)$ given by $f_*[\underline{\xi}] = [\{f\xi_k, (S,a) \to (Y,y_0)\}_N]$.

Proof: The map $f: M \to N$ induces a fundamental sequence $\underline{f} = \{f, (X,x_0), (Y,y_0)\}_{M,N}$.

Corollary 4.7: If N is a deformation retract of M such that $x_0 \in X \subset N$ then $\underline{\pi}_n^M(X,x_0) \approx \underline{\pi}_n^N(X,x_0)$.

Proof: By hypothesis, $i: N \subset M$, the inclusion map, is a homotopy equivalence. By Corollary 4.5 and Corollary 4.6, $i_*: \underline{\pi}_n^N(X,x_0) \to \underline{\pi}_n^M(X,x_0)$ is an isomorphism.

In particular, Corollary 4.7 implies that if H denotes Hilbert space, Q denotes the Hilbert cube and $x_0 \in X \subset Q$ then $\underline{\pi}_n^H(X,x_0)$ is isomorphic to $\underline{\pi}_n^Q(X,x_0)$.

Using Borsuk's method of proof of [2] 10.1, one has the following theorem.

<u>Theorem 4.8</u>: If $X \subset M$ and X_0 is the component of X containing x_0 then $\underline{\pi}_n^M(X,x_0)$ is isomorphic to $\underline{\pi}_n^M(X_0,x_0)$.

<u>Proof</u>: Since $X_0 \subset X$ the identity map 1_m: $M \to M$ induces a homomorphism 1_*: $\underline{\pi}_n^M(X_0,x_0) \to \underline{\pi}_n^M(X,x_0)$. Let $[\underline{\xi}]$ be an M-approximative class with representative $\underline{\xi} = \{\xi_k, (S,a) \to (X,x_0)\}_M$. For every neighborhood (V,x_0) of (X_0,x_0) in M there is a neighborhood (V,x_0) of (X,x_0) in M such that the component of the set V containing X_0 lies in V_0. Since $\underline{\xi}$ is an approximative map there is an index k_0 such that if $k \geq k_0$ then there is a homotopy F: $(S,a) \times I \to (V,x_0)$ such that $F(x,0) = \xi_k(x)$ and $F(x,1) = \xi_{k+1}(x)$ for all $x \in S$. Now $S \times I$ is connected and $F(a,0) = x_0$ so that $F(S \times I) \subset V_0$. Thus $\xi_k \simeq \xi_{k+1}$ in (V_0,x_0). Hence $\underline{\xi}' = \{\xi_k, (S,a) \to (X_0,x_0)\}_M$ is an M-approximative map. By a similar argument, it is easily seen that if $\underline{\xi}$ and $\underline{\eta}$ are homotopic M-approximative maps of (S,a) toward (X,x_0) then $\underline{\xi}'$ and $\underline{\eta}'$ are homotopic M-approximative maps of (S,a) toward (X_0,x_0). Thus by defining $\tau([\underline{\xi}]) = [\underline{\xi}']$ one has a function τ: $\underline{\pi}_n^M(X,x_0) \to \underline{\pi}_n^M(X_0,x_0)$. It is clear that τ is a homomorphism since

$$\tau([\underline{\xi}]*[\underline{\eta}]) = \tau[\underline{\zeta}] = [\underline{\zeta}'] = [\underline{\xi}']*[\underline{\eta}'].$$

But $\tau 1_* = 1_{\underline{\pi}_n^M(X_0,x_0)}$ and $1_* \tau = 1_{\underline{\pi}_n^M(X,x_0)}$ are immediate consequences of the definitions of τ and 1_*. This completes the proof.

<u>Theorem 4.9</u>: If M, $N \in AR$ and (X,x_0) is homeomorphic to (Y,y_0) then $\underline{\pi}_n^M(X,x_0)$ is isomorphic to $\underline{\pi}_n^N(Y,y_0)$.

Proof: Let $h: (X,x_0) \to (Y,y_0)$ be a homeomorphism. Let $h_1: M \to N$ and $h_2: N \to M$ be extensions of h and h^{-1}, respectively. If $\underline{h}_1 = \{h_1,(X,x_0),(Y,y_0)\}_{M,N}$ and $\underline{h}_2 = \{h_2,(Y,y_0),(X,x_0)\}_{N,M}$ then by Corollary 2.2,

$$[\underline{h}_2][\underline{h}_1] = [\underline{1}_{X,x_0}]$$

and

$$[\underline{h}_1][\underline{h}_2] = [\underline{1}_{Y,y_0}].$$

By Corollary 4.5, $[h_1]_*: \underline{\pi}_n^M(X,x_0) \to \underline{\pi}_n^N(Y,y_0)$ is an isomorphism.

A pointed compactum $(Y,y_0) \subset (H,y_0)$ is said to be approximately n-connected [6] if for every neighborhood V of Y there is a neighborhood V_0 of Y such that every map of (S^n,a) into (V_0,y_0) is null-homotopic in (V,y_0). That is, the homomorphism $j_*: \pi_n(V_0,y_0) \to \pi_n(V,y_0)$ induced by the inclusion map $j: (V_0,y_0) \to (V,y_0)$ is trivial. It is clear that an approximately n-connected pointed compactum (Y,y_0) has a trivial $n^{\underline{th}}$ fundamental group (in H).

Theorem 4.10: Let X be a compactum in H which is the union of compacta X_1 and X_2 such that X_1, X_2 and $X_0 = X_1 \cap X_2$ are connected and non-void. Let $x_0 \in X_0$; then if (X_1,x_0), (X_2,x_0) and (X_0,x_0) are approximately 1-connected so is (X,x_0).

Proof: Let V be a neighborhood of X. There are neighborhoods $V_i \subset V$ of X_i $(i = 0, 1, 2)$ such that the homomorphisms j_{i*} from $\pi_1(V_i,x_0)$ into $\pi_1(V,x_0)$ induced by the inclusion maps $j_i: (V_i,x_0) \to (V,x_0)$ are trivial. Let $W_i \subset V_i$ $(i = 1, 2)$ be open path-connected neighborhoods of X_i $(i = 1, 2)$ such that $W_0 = W_1 \cap W_2 \subset V_0$ is path-connected. Let $\omega_i: W_i \to W = W_1 \cup W_2$ $(i = 0, 1, 2)$ be the inclusion maps. By van Kampen's Theorem, $\pi_1(W,x_0)$ is generated by

$(\omega_0)_*(\pi_1(W_0,x_0))$, $(\omega_1)_*(\pi_1(W_1,x_0))$ and $(\omega_2)_*(\pi_1(W_2,x_0))$. Let $i\colon (W,x_0)\subset(V,x_0)$ be the inclusion map. Then $i_*\colon \pi_1(W,x_0)\to\pi_1(V,x_0)$ is trivial since each $(j_i\omega_i)_*\colon \pi_1(W_i,x_0)\to\pi_1(V,x_0)$ is trivial. Thus (X,x_0) is approximately 1-connected.

5. Some Relations Between Weak Homotopy Type and Shape. The following theorem gives a converse to [2] 4.3.

Theorem 5.1: If $N\in$ ANR, M is normal and $f, g\colon (X,X_0)\to(Y,Y_0)$ generate fundamental sequences $\underline{f} = \{f_k,(X,X_0),(Y,Y_0)\}_{M,N}$ and $\underline{g} = \{g_k,(X,X_0),(Y,Y_0)\}_{M,N}$, respectively, then $f \underset{N}{\simeq} g$ iff $\underline{f}\simeq\underline{g}$.

Proof: Assume that $\underline{f}\simeq\underline{g}$. Then for every neighborhood (V,V_0) of (Y,Y_0) in N there is a neighborhood (U,U_0) of (X,X_0) in M and an index k_0 such that if $k\geq k_0$ then

$$f_k|(U,U_0)\simeq g_k|(U,U_0) \text{ in } (V,V_0).$$

Since $(X,X_0)\subset(U,U_0)$ and $f = f_k|(X,X_0)$, $g = g_k|(X,X_0)$, one has that $f\simeq g$ in (V,V_0). That is, $f\underset{N}{\simeq} g$.

Conversely, assume that $f\underset{N}{\simeq} g$. Then if (V,V_0) is any neighborhood of (Y,Y_0) in N,

$$f = f_k|(X,X_0)\simeq g = g_k|(X,X_0) \text{ in } (V,V_0).$$

Applying Theorem 2.1, $\underline{f}\simeq\underline{g}$.

A subset X of E^n is cellular in E^n if there is a sequence $\{Q_i\}$ of n-cells such that $Q_1\supset\mathring{Q}_1\supset Q_2\supset\mathring{Q}_2\supset\ldots$ and $X = \bigcap_1^\infty Q_i$. If there is an embedding $j\colon X\to E^n$ such that $j(X)$ is cellular in E^n then X is said to be cell-like [7].

Theorem 5.2: If $X\subset H$ is a finite dimensional compactum then X is cell-like iff X has the weak homotopy type in H of a point.

Proof: If $X \subset H$ has the weak homotopy type of a point, then by Theorem 5.1, Sh(X) is trivial (i.e., X has the fundamental homotopy type of a point). Lacher [8] has shown this implies X is cell-like.

Conversely, if X is cell-like, Sh(X) is trivial [8]. By Corollary 1.5, we may assume that $X \subset E^n$ is cellular; i.e., there is a sequence $\{Q_i\}$ of n-cells such that $Q_1 \supset \mathring{Q}_1 \supset Q_2 \supset \mathring{Q}_2 \supset \dots$, and $X = \bigcap_1^\infty Q_i$. Then $X = X \times 0 \subset E^n \times 0 \subset H$ is such that $X = \bigcap_1^\infty (Q_i \times B_i)$ where B_i is the open ball in H centered at 0 with radius $\frac{1}{i}$, and thus each $Q_i \times B_i$ is an AR. Let $[\underline{f}]$ be a fundamental class from X to x_0 and $[\underline{g}]$ a fundamental class from x_0 to X such that $[\underline{g}][\underline{f}] = [\underline{1}_X]$ and $[\underline{f}][\underline{g}] = [\underline{1}_{x_0}]$. Since x_0 is an ANR, we may choose a representative $\underline{f} = \{f_k, X, x_0\}_{H,H}$ of $[\underline{f}]$ that is generated by a map $f\colon X \to x_0$ [2], 5.1. Let $\underline{g} = \{g_k, x_0, X\}_{H,H}$ be a representative of $[\underline{g}]$. Since each $Q_i \times B_i$ is a neighborhood of X in H, there is an increasing sequence of indices $\{k_i\}$ such that $g_{k_i}(x_0) \in Q_i \times B_i$. Then $\underline{g}' = \{g_{k_i}, x_0, X\}_{H,H}$ is a fundamental sequence homotopic to $\underline{g}$ [2], 3.4. Let $x_1 \in X$ and $g\colon x_0 \to X$ be given by $g(x_0) = x_1$. If V is any neighborhood of X, there is an index i such that $Q_i \times B_i \subset V$. Then if $j \geq i$, $g_{k_j}(x_0) \in Q_i \times B_i$. This implies that $g \simeq g_{k_j}|x_0$ in V. Let $\hat{\underline{g}}$ be a fundamental sequence generated by g. By Theorem 2.1, $\hat{\underline{g}} \simeq \underline{g}'$; and hence, $\hat{\underline{g}} \simeq \underline{g}$. Thus $[\hat{\underline{g}}][\underline{f}] = [\underline{1}_X]$ and $[\underline{f}][\hat{\underline{g}}] = [\underline{1}_{x_0}]$. But fg generates $\underline{f}\,\hat{\underline{g}}$ and gf generates $\hat{\underline{g}}\,\underline{f}$ so by Theorem 5.1, $fg \underset{H}{\cong} 1_X$ and $gf \underset{H}{\cong} 1_{x_0}$. Thus X has the weak homotopy type in H of a point.

6. Fundamental Retracts. In the remainder of this chapter, only compacta are considered. Thus, all spaces are assumed to be embedded in Hilbert space H and all fundamental sequences are in H. The subscripts H are therefore omitted.

The following theorem may be found in [3], p. 210; however, the following proof is somewhat shorter.

Theorem 6.1: Let X, X' and Y' be compacta in H and let $h\colon X' \to Y'$ be a homeomorphism. Then the set $Y = h(X)$ is a fundamental retract of Y' if X is a fundamental retract of X'.

Proof: Let $h_1\colon H \to H$ be a map such that $h_1(x) = h(x)$ for every point $x \in X'$, and let $h_2\colon H \to H$ be a map such that $h_2(y) = h^{-1}(y)$ for every point $y \in Y'$. Then $\underline{h} = \{h_1, X, Y\}$ and $\underline{h}^{-1} = \{h_2, Y', X'\}$ are fundamental sequences. Assume that there is a fundamental retraction $\underline{r} = \{r_k, X', X\}$. Then $\underline{h}\ \underline{r}\ \underline{h}^{-1} = \{h_1 r_k h_2, Y', Y\}$ is a fundamental sequence. If $y \in Y$ then for $k = 1, 2, \ldots,$ $h_1 r_k h_2(y) = h r_k h^{-1}(y) = y$ since $h(X) = Y$. Thus $\underline{h}\ \underline{r}\ \underline{h}^{-1}$ is a fundamental retraction of Y' to Y.

Theorem 6.2: Let X, X' be two compacta in E^n such that $X \subset X'$. If X is a fundamental retract of X' then no component of the set $E^n - X$ is contained in X'.

Proof: Since no unbounded component of $E^n - X$ can be contained in the compactum X', we need consider only the bounded ones. Let $p\colon H \to E^n$ be a retraction and $j\colon E^n \to H$ the inclusion map. Assume that $\underline{r} = \{r_k, X', X\}$ is a fundamental retraction and that G is a bounded component of $E^n - X$ such that $G \subset X'$. For each k, $s_k = p r_k j\colon E^n \to E^n$ is a map such that $s_k(x) = x$ if $x \in X$. By [7] p. 190, for each k, $\bar{G} \subset s_k(G)$. Let $q \in G$; then $p^{-1}(q)$ is a closed subset of H so that $V = H - p^{-1}(q)$ is a neighborhood of X in H. Since $\underline{r}$ is a fundamental retraction, there exists an index k_0 such that if $k \geq k_0$ then $r_k(X') \subset V$. Then

$$s_k(X') = p(r_k(X')) \subset p(V).$$

But $\bar{G} \subset X'$ implies that

$$s_k(\bar{G}) \subset s_k(X') \subset p(V)$$

which is impossible since $q \notin p(V)$.

Corollary 6.3: If X is a fundamental retract of X', $X \subset X' \subset E^n$, then $E^n - X$ cannot have more components than $E^n - X'$.

7. FAR and FANR-sets. A compactum $X \subset H$ is said to be a fundamental absolute retract ($X \in FAR$) if it is a fundamental retract of every compactum $X' \subset H$ containing X. Borsuk [3] 6.3, has shown that FAR-sets are the same as fundamental retracts of AR-sets lying in H.

Theorem 7.1: If $X \subset E^n$ is a FAR-set then $E^n - X$ has no bounded components.

Proof: Since $X \subset E^n$ is compact, it is bounded. Let B^n be any n-cell such that $X \subset B^n$. Since $X \in FAR$, X is a fundamental retract of B^n. This together with Theorem 6.2 gives the desired result.

Corollary 7.2: If $X \subset E^n$ $(n > 1)$ is a FAR-set then X cannot decompose E^n.

A closed subset X_0 of a compactum $X \subset H$ is said to be a fundamental neighborhood retract of X if there is a closed neighborhood W of X_0 such that X_0 is a fundamental retract of the set $W \cap X$. If for every compactum X' such that $X \subset X' \subset H$ the compactum X is a fundamental neighborhood retract of X', then X is said to be a fundamental absolute neighborhood retract (FANR). Borsuk [3] 6.8, has shown that FANR-sets are the same as fundamental retracts of ANR-sets lying in H.

The following theorem may be found in [3] 6.14. This proof is somewhat shorter.

Theorem 7.3: If Y is an ANR-set lying in H, then every fundamental neighborhood retract of Y is an FANR-set.

Proof: Since $Y \in$ ANR, there is a neighborhood V of Y and a map $s: H \to H$ such that $s(V) = Y$ and $s(y) = y$ for every point $y \in Y$. Let Y_0 be a fundamental neighborhood retract of Y, then there is a closed neighborhood V_0 of Y_0 and a fundamental retraction $\underline{r} = \{r_k, V_0 \cap Y, Y_0\}$. Select a closed neighborhood $V_1 \subset V$ of Y_0 such that $s(V_1) \subset \text{int}\, V_0 \cap Y$. Let Y' be a compactum such that $Y_0 \subset Y' \subset H$. Since $s(V_1 \cap Y') \subset V_0 \cap Y$, $\underline{s} = \{s, V_1 \cap Y', V_0 \cap Y\}$ is a fundamental sequence. Then $\underline{r}\,\underline{s} = \{r_k s, V_1 \cap Y', Y_0\}$ is a fundamental retraction, since if $y \in Y_0 \subset Y$,

$$r_k s(y) = r_k(y) = y.$$

The following theorem may be found in [3], 8.1. This proof is somewhat shorter.

Theorem 7.4: If X is a FAR-set, then every set $Y \subset H$ homeomorphic to X is also a FAR-set.

Proof: Let $h: X \to Y$ be a homeomorphism of X onto Y. Let $f: H \to H$ and $g: H \to H$ be extensions of h and h^{-1}, respectively. Consider a compactum $Y' \subset H$ such that $Y \subset Y'$. Let $X' = g(Y')$. Then $X \subset X' \subset H$ and X' is compact. Since $X \in$ FAR there is a fundamental retraction $\underline{r} = \{r_k, X', X\}$. Since $g(Y') = X'$, $\underline{g} = \{g, Y', X'\}$ is a fundamental sequence. Also, $\underline{f} = \{f, X, Y\}$ is a fundamental sequence. Then $\underline{f}\,\underline{r}\,\underline{g} = \{f r_k g, Y', Y\}$ is a fundamental retraction since if $y \in Y$,

$$f r_k g(y) = h r_k h^{-1}(y) = y.$$

The following theorem may be found in [3], 8.2. This proof is somewhat shorter.

Theorem 7.5: If X is a FANR-set, then every set $Y \subset H$ homeomorphic to X is also a FANR-set.

Proof: Let $h: X \to Y$ be a homeomorphism of X onto Y. Let $f: H \to H$

and $g: H \to H$ be extensions of h and h^{-1}, respectively. Let Y' be a compactum such that $Y \subset Y' \subset H$. Let $X' = g(Y')$; then X' is compact and $X \subset X' \subset H$. Since $X \in$ FANR, there is a closed neighborhood M of X and a fundamental retraction $\underline{r} = \{r_k, M \cap X', X\}$. Set $N = g^{-1}(M)$; then N is a closed neighborhood of Y and $\underline{g} = \{g, N \cap Y', M \cap X'\}$ is a fundamental sequence since $g(N \cap Y') \subset M \cap X'$. Also, $\underline{f} = \{f, X, Y\}$ is a fundamental sequence. Thus $\underline{f}\ \underline{r}\ \underline{g} = \{fr_kg, N \cap Y', Y\}$ is a fundamental retraction since if $y \in Y$,

$$fr_kg(y) = hr_kh^{-1}(y) = y.$$

Using Borsuk's method of proof of [2], 10.1, one has the following.

Lemma 7.6: If $X \subset X'$ and $\underline{r} = \{r_k, X', X\}$ is a fundamental retraction then if C is any component of X and C' is the component of X' containing C then $\{r_k, C', C\}$ is a fundamental retraction.

Proof: For every neighborhood V of C there is a neighborhood V_0 of X such that the component of the set V_0 containing C lies in V. Since $\underline{r}$ is a fundamental sequence, there is a neighborhood U_0 of X' and an index k_0 such that if $k \geq k_0$ then

$$r_k|U_0 \simeq r_{k+1}|U_0 \text{ in } V_0.$$

If U denotes the component of U_0 containing C' then

$$r_k|U \simeq r_{k+1}|U \text{ in } V.$$

Since $r_k(x) = x$ for $x \in X$ we have shown that $\{r_k, C', C\}$ is a fundamental retraction.

Corollary 7.7: Every component of a FANR-set is a FANR-set.

Proof: If $X \in$ FANR then there is a $X' \in$ ANR such that X is a fundamental retract of X'. If C is a component of X and C' is the component of X' containing C then by Lemma 7.6, C is a fundamental retract of C'. But $C' \in$ ANR [1], 2.11, and hence, [3], 6.8, $C \in$ FANR.

Lemma 7.8: If X is a fundamental retract of X' then X cannot have more components than X'.

Proof: Suppose C_1, C_2 are components of X and $\underline{r} = \{r_k, X', X\}$ is a fundamental retraction. It suffices to show that C_1 and C_2 are not contained in the same component of X'. Suppose $C_1 \cup C_2 \subset C$ where C is a component of X'. By Lemma 7.6, $\{r_k, C, C_1\}$ and $\{r_k, C, C_2\}$ are fundamental retractions. Let U_i $(i = 1, 2)$ be disjoint neighborhoods of C_i $(i = 1, 2)$. Then there is an index k_0 such that if $k \geq k_0$ then $r_k(C) \subset U_1$ and $r_k(C) \subset U_2$, a contradiction.

Corollary 7.9: If $X \in$ FAR then X is connected.

Proof: If $X \in$ FAR then X is a fundamental retract of an AR-set. But an AR-set is connected, so that X can have at most one component.

Corollary 7.10: Every FANR-set has only a finite number of components.

Proof: If $X \in$ FANR then X is a fundamental retract of an ANR-set. But an ANR-set has only a finite number of components [1], 2.7. Thus X has only a finite number of components.

Lemma 7.11: If X_1, X_2, Y are compacta such that $X_1 \cap X_2 = \emptyset$ and if $\underline{f} = \{f_k, X_1, Y\}$ and $\underline{g} = \{g_k, X_2, Y\}$ are fundamental sequences then there is a fundamental sequence $\underline{h} = \{h_k, X_1 \cup X_2, Y\}$ extending both $\underline{f}$ and $\underline{g}$.

Proof: Since X_1 and X_2 are disjoint compacta there are disjoint closed neighborhoods U_i $(i = 1, 2)$ of X_i $(i = 1, 2)$. For each k, let h_k': $U_1 \cup U_2 \rightarrow H$ be given by

$$h_k'(x) = \begin{cases} f_k(x) & \text{if } x \in U_1 \\ g_k(x) & \text{if } x \in U_2. \end{cases}$$

Let $h_k\colon H \to H$ be any extension of h_k', $k = 1, 2, \ldots$. If V is any neighborhood of Y in H, there are an index k_0 and neighborhoods W_i $(i = 1, 2)$ of X_i $(i = 1, 2)$ such that if $k \geq k_0$ there exist homotopies $F_i\colon W_i \times I \to V$ $(i = 1, 2)$ such that $F_1(x,0) = f_k(x)$, $F_1(x,1) = f_{k+1}(x)$ if $x \in W_1$ and $F_2(x,0) = g_k(x)$, $F_2(x,1) = g_{k+1}(x)$ if $x \in W_2$. Let $W = (W_1 \cap U_1) \cup (W_2 \cap U_2)$ and $F\colon W \times I \to V$ be given by $F(x,t) = F_i(x,t)$ if $(x,t) \in (W_i \cap U_i) \times I$. Then $F\colon h_k|_W \simeq h_{k+1}|_W$ in V and $\underline{h} = \{h_k, X_1 \cup X_2, Y\}$ is a fundamental sequence extending both $\underline{f}$ and $\underline{g}$.

Lemma 7.12: If X_1, $X_2 \in$ FANR and $X_1 \cap X_2 = \emptyset$ then $X = X_1 \cup X_2 \in$ FANR.

Proof: Suppose X' is a compactum such that $X \subset X'$. Let U_i be disjoint closed neighborhoods of X_i $(i = 1, 2)$. Since $X_i \subset X'$ and $X_i \in$ FANR, there are closed neighborhoods $W_i \subset U_i$ and fundamental retractions $\underline{r}_i' = \{r_k^i, X' \cap W_i, X_i\}$ $(i = 1, 2)$. Since $X_i \subset X$, $\underline{r}_i = \{r_k^i, X' \cap W_i, X\}$ are also fundamental sequences. Since $(X' \cap W_1) \cap (X' \cap W_2) = \emptyset$ there is a fundamental sequence $\underline{r} = \{r_k, X' \cap (W_1 \cup W_2), X\}$ that is an extension of $\underline{r}_i$ $(i = 1, 2)$. If $x \in X$ then for some i, $x \in X_i$ so that $r_k(x) = r_k^i(x) = x$ for all k. That is, $\underline{r}$ is a fundamental retraction. Thus $X \in$ FANR.

Theorem 7.13: A compactum X is a FANR-set iff X has a finite number of components, each of which is a FANR-set.

Example: The Cantor set can be written as an intersection of compact sets each of which is the union of a finite number of disjoint closed intervals. However, it has an infinite number of components and thus is an example of a compact set which is not a FANR-set but is the intersection of a decreasing sequence of FANR-sets.

Theorem 7.13: If $X \subset E^n$ is a FANR-set then $E^n - X$ has only a finite number of components.

Proof: Since $X \in$ FANR there is a compact ANR-set X' lying in the closure of the convex hull $\hat{X}$ of X, hence also in E^n, such that X is a fundamental retract of X' [3], pp. 66-67. But $E^n - X'$ has only a finite number of components [7], 5.1, so by Corollary 6.3, $E^n - X$ has only a finite number of components.

CHAPTER III

SHAPE GROUPS

1. Preliminary Definitions. Let $\mathcal{A}$ be a category. An object X of $\mathcal{A}$ is said to be a terminal object (universally attracting) if whenever Y is an object of $\mathcal{A}$ there is a unique $\mathcal{A}$-morphism $f\colon Y\to X$. An inverse system in $\mathcal{A}$, $\underline{X} = \{X_a, p_{aa'}, A\}$, consists of a family $\{X_a\colon a\in A\}$ indexed by a directed set A and a family $\{p_{aa'}\colon a\leq a'\}$ of $\mathcal{A}$-morphisms such that if $a\leq a'$ then $p_{aa'}\colon X_{a'}\to X_a$ satisfies

(1) $p_{aa} = 1_{X_a}$,

(2) if $a\leq a'\leq a''$ then $p_{aa''} = p_{aa'}p_{a'a''}$.

An inverse limit $X_\infty = \varprojlim \underline{X}$ of the inverse system $\underline{X} = \{X_a, p_{aa'}, A\}$ is a terminal object in the following category $\mathrm{inv}(\underline{X})$. The objects of $\mathrm{inv}(\underline{X})$ consist of pairs $(X, \{p_a\colon a\in A\})$ where X is an object in $\mathcal{A}$ and $p_a\colon X\to X_a$ are $\mathcal{A}$-morphisms such that if $a\leq a'$ then $p_a = p_{aa'}p_{a'}$. An $\mathrm{inv}(\underline{X})$-morphism $f\colon (X,\{p_a\})\to(Y,\{q_a\})$ consists of an $\mathcal{A}$-morphism $f\colon X\to Y$ such that if $a\in A$ then $p_a = q_a f$. Inverse limits exist in the category of topological spaces and in the category of groups.

If $\underline{X} = \{X_a, p_{aa'}, A\}$ and $\underline{Y} = \{Y_b, q_{bb'}, B\}$ are inverse systems in $\mathcal{A}$, a morphism of inverse systems $\underline{f}\colon \underline{X}\to\underline{Y}$ consists of an increasing function $f\colon B\to A$ and a collection of $\mathcal{A}$-morphisms $f_b\colon X_{f(b)}\to Y_b$ such that if $b\leq b'$ then $f_b p_{f(b)f(b')} = q_{bb'}f_{b'}$. If $X_\infty = \varprojlim \underline{X}$ and $Y_\infty = \varprojlim \underline{Y}$ exist and $\underline{f}\colon \underline{X}\to\underline{Y}$ is a morphism of inverse systems then for each $b \in B$ the composition $f_b p_{f(b)}: X_\infty \to Y_b$ satisfies if $b \leq b'$ then

$$f_b p_{f(b)} = f_b p_{f(b)f(b')} p_{f(b')} = q_{bb'} f_{b'} p_{f(b')}.$$

By the universal mapping property of Y_∞ there is a unique $\mathcal{A}$-morphism $f_\infty: X_\infty \to Y_\infty$ such that if $b \in B$ then $q_b f_\infty = f_b p_{f(b)}$. The $\mathcal{A}$-morphism f_∞ is said to be induced by $\underline{f}$.

There is a category whose objects are inverse systems $\underline{X} = \{X_a, p_{aa'}, A\}$ in $\mathcal{A}$ and whose morphisms $\underline{f}: \underline{X} \to \underline{Y}$ are morphisms of inverse systems. If $\underline{f}: \underline{X} \to \underline{Y}$ and $\underline{g}: \underline{Y} \to \underline{Z} = \{Z_c, r_{cc'}, C\}$ are morphisms of inverse systems then the composition $\underline{h} = \underline{g}\,\underline{f}: \underline{X} \to \underline{Z}$ is given by: $h: C \to A$ is the composition of $f: B \to A$ and $g: C \to B$; for each $c \in C$, $h_c: X_{h(c)} \to Z_c$ is the composition of $f_{g(c)}: X_{fg(c)} = X_{h(c)} \to Y_{g(c)}$ and $g_c: Y_{g(c)} \to Z_c$. The identity $\underline{1}_{\underline{X}}: \underline{X} \to \underline{X}$ is given by $1: A \to A$ and $1_a: X_a \to X_a$ (identity morphisms). It is easily seen by the uniqueness that if X_∞, Y_∞ and $Z_\infty = \varprojlim \underline{Z}$ exist then $(gf)_\infty = g_\infty f_\infty$.

A (pointed) <u>ANR-system</u> (compare [9]) is an inverse system $\underline{X} = \{(X_a, x_a), p_{aa'}, A\}$ in the category of pointed topological spaces where A is closure-finite (each $a \in A$ has only a finite number of predecessors) and each X_a is a compact ANR (for normal spaces). A <u>map of ANR-systems</u> (compare [10] p. 42) $\underline{f}: \underline{X} \to \underline{Y} = \{(Y_b, y_b), q_{bb'}, B\}$ consists of an increasing function $f: B \to A$ and a collection of maps $f_b: (X_{f(b)}, x_{f(b)}) \to (Y_b, y_b)$ such that if $b \leq b'$ then $f_b p_{f(b)f(b')} \simeq q_{bb'} f_{b'}$. Two maps of systems $\underline{f}, \underline{g}: \underline{X} \to \underline{Y}$ are said to be homotopic (compare [10] p. 43), written $\underline{f} \simeq \underline{g}$, provided for every $b \in B$ there is an $a \in A$, $a \geq f(b), g(b)$ such that $f_b p_{f(b)a} \simeq g_b p_{g(b)a}$. A map of systems $\underline{f}: \underline{X} \to \underline{Y}$ is said to be a homotopy equivalence provided there is a map of systems $\underline{g}\colon \underline{Y}\to\underline{X}$ such that $\underline{g}\,\underline{f}\simeq \underline{1}_{\underline{X}}$ and $\underline{f}\,\underline{g}\simeq \underline{1}_{\underline{Y}}$. Two systems $\underline{X}$ and $\underline{Y}$ are of the same homotopy type, $\underline{X}\simeq\underline{Y}$, if there is a homotopy equivalence $\underline{f}\colon \underline{X}\to\underline{Y}$. If X is a compact Hausdorff space, $x_0\in X$ and $(X,x_0) = \varprojlim \underline{X}$ then the ANR-system $\underline{X}$ is said to be associated with (X,x_0). Mardešić and Segal [10] have shown that every pair (X,x_0), where X is a compact Hausdorff space and $x_0\in X$, has an associated homotopy class of ANR-systems, called the shape of (X,x_0) and denoted $Sh(X,x_0)$, such that if $\underline{X}$ is associated with (X,x_0) then $\underline{X}\in Sh(X,x_0)$. They show in [11] that this definition of shape is equivalent to that given by Borsuk. Similar definitions can be made for relative ANR-systems, $\underline{X} = \{(X_a,X_{0a}),p_{aa'},A\}$, and absolute ANR-systems, $\underline{X} = \{X_a,p_{aa'},A\}$ (see [10]).

If $x_0\in X\subset M$, an inclusion ANR-system in M associated with (X,x_0) is an ANR-system $\underline{X} = \{(X_a,x_0),i_{aa'},A\}$ associated with (X,x_0) such that

(1) each X_a is a neighborhood of X in M,

(2) $X = \bigcap_{a\in A} X_a$,

(3) if $a \leq a'$ then $i_{aa'}\colon (X_{a'},x_0)\to(X_a,x_0)$ is an inclusion map.

If $A = N$ (the set of natural numbers) then $\underline{X}$ is said to be an inclusion ANR-sequence and is denoted $\underline{X} = \{(X_k,x_0),i_{kk'}\}$. Mardešić has shown [9] that by embedding a compact Hausdorff space X in a parallelotope I^{Ω} one can construct an inclusion ANR-system (sequence if Ω is countable, see [10]) in I^{Ω} associated with (X,x_0).

2. The Shape Groups. Let $\mathcal{A}$ be a category. Two morphisms $\underline{f}$, $\underline{g}\colon \underline{X}\to\underline{Y}$ of inverse systems in $\mathcal{A}$ are $\sim$-related ($\underline{f}\sim\underline{g}$) if

for each $b \in B$ there is an index $a \in A$, $a \geq f(b)$, $g(b)$ such that $f_b p_{f(b)a} = g_b p_{g(b)a}$.

Theorem 2.1: The relation $\sim$ is an equivalence relation.

Proof: The proof is as in Theorem 2 of [10].

Theorem 2.2: Let $\underline{f}$, $\underline{f}'$: $\underline{X} \to \underline{Y}$ and $\underline{g}$, $\underline{g}'$: $\underline{Y} \to \underline{Z}$ be morphisms of inverse systems. If $\underline{f} \sim \underline{f}'$ and $\underline{g} \sim \underline{g}'$ then $\underline{g}\,\underline{f} \sim \underline{g}'\underline{f}'$.

Proof: See Theorem 3 of [10].

A morphism $\underline{f}$: $\underline{X} \to \underline{Y}$ is a $\sim$-equivalence provided there is a morphism $\underline{g}$: $\underline{Y} \to \underline{X}$ (called the $\sim$-inverse of $\underline{f}$) such that $\underline{g}\,\underline{f} \sim \underline{1}_{\underline{X}}$ and $\underline{f}\,\underline{g} \sim \underline{1}_{\underline{Y}}$. In this case, $\underline{X}$ and $\underline{Y}$ are said to be $\sim$-equivalent ($\underline{X} \sim \underline{Y}$).

Theorem 2.3: The relation $\sim$ is an equivalence relation on inverse systems in $\mathcal{A}$.

Proof: See Theorem 4 of [10].

Theorem 2.4: If $\underline{f}$, $\underline{g}$: $\underline{X} \to \underline{Y}$ are $\sim$-related morphisms and X_∞ and Y_∞ both exist then $f_\infty = g_\infty$.

Proof: By definition, f_∞: $X_\infty \to Y_\infty$ is the unique $\mathcal{A}$-morphism satisfying for all $b \in B$, $q_b f_\infty = f_b p_{f(b)}$. Similarily, g_∞: $X_\infty \to Y_\infty$ is the unique $\mathcal{A}$-morphism satisfying for all $b \in B$, $q_b g_\infty = g_b p_{g(b)}$. Choose $a \geq f(b)$, $g(b)$ such that $f_b p_{f(b)a} = g_b p_{g(b)a}$. Now $p_{f(b)} = p_{f(b)a} p_a$ and $p_{g(b)} = p_{g(b)a} p_a$ so that

$$q_b g_\infty = g_b p_{g(b)} = g_b p_{g(b)a} p_a = f_b p_{f(b)a} p_a = f_b p_{f(b)}.$$

By the uniqueness, $f_\infty = g_\infty$.

Corollary 2.5: If $\underline{X} \sim \underline{Y}$ and X_∞, Y_∞ both exist then X_∞ and Y_∞ are $\mathcal{A}$-equivalent objects.

Proof: If $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{X}$ are such that $\underline{g}\,\underline{f} \sim \underline{1}_{\underline{X}}$ and

$\underline{f}\ \underline{g} \sim \underline{1}_{\underline{Y}}$ then $g_\infty f_\infty = (gf)_\infty = 1_{X_\infty}$ and $f_\infty g_\infty = (fg)_\infty = 1_{Y_\infty}$.

If $\underline{X} = \{(X_a, x_a), p_{aa'}, A\}$ is an ANR-system, let $\pi_n(\underline{X}) = \{\pi_n(X_a, x_a), \rho_{aa'}, A\}$ denote the inverse system of groups where $\pi_n(X_a, x_a)$ is the n-th homotopy group of (X_a, x_a) and if $a \leq a'$ then $\rho_{aa'}: \pi_n(X_{a'}, x_{a'}) \to \pi_n(X_a, x_a)$ is the homomorphism induced by $p_{aa'}$; i.e., if $[\xi] \in \pi_n(X_{a'}, x_{a'})$ then $\rho_{aa'}[\xi] = [p_{aa'}\xi]$.

If $\underline{f}: \underline{X} \to \underline{Y}$ is a map of systems, $\underline{f}$ induces a morphism of inverse systems $\underline{f}_*: \pi_n(\underline{X}) \to \pi_n(\underline{Y})$ where $f_* = f: B \to A$ and $(f_b)_*: \pi_n(X_{f(b)}, x_{f(b)}) \to \pi_n(Y_b, y_b)$ is the homomorphism induced by f_b. This gives a covariant functor π_n between the category of ANR-systems and the category of inverse systems of groups.

<u>Theorem 2.6</u>: If $\underline{f}$, $\underline{g}$: $\underline{X} \to \underline{Y}$ are homotopic maps of systems ($\underline{f} \simeq \underline{g}$) then the induced morphisms $\underline{f}_*$, $\underline{g}_*$: $\pi_n(\underline{X}) \to \pi_n(\underline{Y})$ are $\sim$-related ($\underline{f}_* \sim \underline{g}_*$).

<u>Proof</u>: For each $b \in B$, choose $a \in A$ such that $a \geq f(b)$, $g(b)$ and $f_b p_{f(b)a} = g_b p_{g(b)a}$. Then if $[\xi] \in \pi_n(X_a, x_a)$,

$$(f_b)_* \rho_{f(b)a}[\xi] = [f_b p_{f(b)a}\xi] = [g_b p_{g(b)a}\xi] = (g_b)_* \rho_{g(b)a}[\xi].$$

<u>Corollary 2.7</u>: If $\underline{X} \simeq \underline{Y}$ then $\pi_n(\underline{X}) \sim \pi_n(\underline{Y})$.

Mardešić and Segal have shown in [9] that if $\underline{X}$ and $\underline{Y}$ are ANR-systems associated with (X, x_0) and (Y, y_0), respectively, then a map $f: (X, x_0) \to (Y, y_0)$ has an associated map of systems $\underline{f}: \underline{X} \to \underline{Y}$. If $\underline{X}$ and $\underline{X}'$ are ANR-systems associated with (X, x_0) then the map of systems $\underline{i}: \underline{X} \to \underline{X}'$ associated with the identity $1_{X,x_0}: (X, x_0) \to (X, x_0)$ is a homotopy equivalence. By Theorem 2.6 and Theorem 2.4 one has that $i_\infty: \varprojlim \pi_n(\underline{X}) \to \varprojlim \pi_n(\underline{X}')$ is an isomorphism. Suppose $f: (X, x_0) \to (Y, y_0)$ is a map, $\underline{X}$, $\underline{X}'$ are ANR-systems associated with

(X,x_0) and $\underline{Y}$, $\underline{Y}'$ are ANR-systems associated with (Y,y_0). Let $\underline{i}$: $\underline{X}\to\underline{X}'$ and $\underline{j}$: $\underline{Y}\to\underline{Y}'$ be the homotopy equivalences associated with $1_{X,x_0}$ and $1_{Y,y_0}$, respectively. Let $\underline{f}$: $\underline{X}\to\underline{Y}$ and $\underline{f}'$: $\underline{X}'\to\underline{Y}'$ be the maps of systems associated with f. It follows [9] that $\underline{j}\ \underline{f}\simeq\underline{f}'\underline{i}$: $\underline{X}\to\underline{Y}'$. By Theorem 2.6 and Theorem 2.4 one has that

$j_\infty f_\infty = f'_\infty i_\infty$: $\varprojlim \pi_n(\underline{X})\to\varprojlim \pi_n(\underline{Y}')$.

If (X,x_0) is a pointed compact Hausdorff space and $\underline{X}$ is any ANR-system associated with (X,x_0) then the <u>n-th shape group</u> of (X,x_0) is given by $\underline{\pi}_n(X,x_0) = \varprojlim \pi_n(\underline{X})$. If f: $(X,x_0)\to(Y,y_0)$ then the homomorphism f_∞: $\underline{\pi}_n(X,x_0)\to\underline{\pi}_n(Y,y_0)$ is said to be induced by f. It is easy to show that $(1_{X,x_0})_\infty = 1_{\underline{\pi}_n(X,x_0)}$ and $(fg)_\infty = f_\infty g_\infty$. Corollary 2.7 also shows that the n-th shape group is a shape invariant. It is shown in Section 3 that this definition of $\underline{\pi}_n$ extends that given by Borsuk in [2].

<u>Theorem 2.8</u>: There is a homomorphism p: $\pi_n(X,x_0)\to\underline{\pi}_n(X,x_0)$ such that for all $a\in A$, $(p_a)_* = \rho_a p$ where $(p_a)_*$: $\pi_n(X,x_0)\to\pi_n(X_a,x_a)$ is the homomorphism induced by p_a: $(X,x_0)\to(X_a,x_a)$.

<u>Proof</u>: The collection of maps p_a: $(X,x_0)\to(X_a,x_a)$ induces homomorphisms $(p_a)_*$: $\pi_n(X,x_0)\to\pi_n(X_a,x_a)$ such that if $a\le a'$ then $(p_a)_* = \rho_{aa'}(p_{a'})_*$. By the universal mapping property of $\underline{\pi}_n(X,x_0)$ there is a unique homomorphism p: $\pi_n(X,x_0)\to\underline{\pi}_n(X,x_0)$ such that for all $a\in A$, $(p_a)_* = \rho_a p$.

<u>Theorem 2.9</u>: If $X\in$ ANR then $\pi_n(X,x_0)\approx\underline{\pi}_n(X,x_0)$.

<u>Proof</u>: Since $X\in$ ANR there is a special ANR-system $\underline{X} = \{(X,x_0),1_{X,x_0}\}$ associated with (X,x_0). Then $\pi_n(\underline{X}) = \{\pi_n(X,x_0),1_{\pi_n(X,x_0)}\}$ has as inverse limit the group $\pi_n(X,x_0)$.

<u>Theorem 2.10</u>: If X is a compact Hausdorff space, $x_0\in X$ and X_0

is the component of X containing x_0 then $\underline{\pi}_n(X,x_0) = \underline{\pi}_n(X_0,x_0)$.

Proof: Assume $X \subset I^{\Omega}$ and $\underline{X} = \{(X_a,x_0),i_{aa'},A\}$ is an inclusion ANR-system associated with (X,x_0). For each $a \in A$ let X_{a0} denote the component of X_a containing x_0. Since a compact ANR is locally contractible, it is locally path connected. It follows that each X_{a0} is a compact path connected ANR.

Claim: $\underline{X}_0 = \{(X_{a0},x_0),i_{aa'}|X_{a'0},A\}$ is an inclusion ANR-system associated with (X_0,x_0). It suffices to show that $X_0 = \bigcap_{a \in A} X_{a0}$. Certainly $X_0 \subset \bigcap_{a \in A} X_{a0}$ since X_0 a compact connected subset of I^{Ω} implies that if N is any neighborhood of X_0 there is a path connected neighborhood U of X_0 such that $U \subset N$. Let $x \in \bigcap_{a \in A} X_{a0} - X_0$. Then $x \in X - X_0$ so let X_1 denote the component of X to which x belongs. There are disjoint open sets U_0, U_1 such that $U_i \cap X = X_i$ $(i = 0,1)$. Since I^{Ω} is normal there are open sets V_0, V_1 such that $X_i \subset V_i \subset \overline{V}_i \subset U_i$ $(i = 0, 1)$. Since $V = V_0 \cup V_1 \cup [I^{\Omega} - (\overline{V}_0 \cup \overline{V}_1)]$ is a neighborhood of X in I^{Ω} there is an $a \in A$ such that $X_a \subset V$. Then $X_{a0} \subset V_0$ and $x \in V_1$ a contradiction since $V_0 \cap V_1 = \emptyset$. Thus $X_0 = \bigcap_{a \in A} X_{a0}$ and the claim is proven.

By a well-known theorem, $\pi_n(X_a,x_0) = \pi_n(X_{a0},x_0)$ so that $\pi_n(\underline{X}) = \pi_n(\underline{X}_0)$. It follows then that $\underline{\pi}_n(X,x_0) = \underline{\pi}_n(X_0,x_0)$.

If x_0, $x_1 \in X$ and $\omega: I \to X$ is a path in X connecting x_0 and x_1 then for each $a \in A$, ω induces an isomorphism $\omega_a: \pi_n(X_a,x_0) \to \pi_n(X_a,x_1)$. If $a \le a'$ then $i_{aa'}\omega_{a'} = \omega_a$ and it is not hard to show that $\underline{\pi}_n(X,x_0) \approx \underline{\pi}_n(X,x_1)$. Thus we have the following theorem.

Theorem 2.11: If x_0 and x_1 are in the same path component of X then $\underline{\pi}_n(X,x_0) \approx \underline{\pi}_n(X,x_1)$.

Question: Is Theorem 11 valid if one replaces path component with component? Using Theorem 4.1 of [5], one can easily show the following is true.

Theorem 2.12: If X is a movable compact metric space and if x_0 and x_1 are in the same component of X then $\underline{\pi}_n(X,x_0) \approx \underline{\pi}_n(X,x_1)$.

3. Equivalence of the inverse limit and Borsuk's definition of $\underline{\pi}_n$. Let X be a compact metric space and $x_0 \in X$. Assume that X is embedded in Q (Hilbert cube).

Theorem 3.1: If $\underline{X} = \{(X_k,x_0),i_{kk'}\}$ is an inclusion ANR-sequence in Q associated with (X,x_0) then $\underline{\pi}_n^Q(X,x_0) \approx \varprojlim \pi_n(\underline{X}) = \underline{\pi}_n(X,x_0)$.

Proof: Let $\lambda_k\colon \underline{\pi}_n^Q(X,x_0) \to \pi_n(X_k,x_0)$ be given as follows. If $[\underline{\xi}] \in \underline{\pi}_n^Q(X,x_0)$ then since (X_k,x_0) is a neighborhood of (X,x_0) in Q there is an index m_k such that if $m \geq m_k$ then $\xi_m \simeq \xi_{m_k}$ in (X_k,x_0). Define $\lambda_k[\underline{\xi}] = [\xi_{m_k}] \in \pi_n(X_k,x_0)$. If $[\underline{\xi}] = [\underline{\eta}]$ then there is an m_0 such that if $m \geq m_0$ then $\xi_m \simeq \xi_m$ in (X_k,x_0), so that λ_k is a well-defined function. If $[\underline{\xi}]$, $[\underline{\eta}] \in \underline{\pi}_n^Q(X,x_0)$ and m_0 is "large enough" then

$$\begin{aligned}\lambda_k([\underline{\xi}]*[\underline{\eta}]) &= \lambda_k[\{\xi_m*\eta_m,(S,a)\to(X,x_0)\}]\\ &= [\xi_{m_0}*\eta_{m_0}]\\ &= [\xi_{m_0}]*[\eta_{m_0}]\\ &= \lambda_k[\underline{\xi}]*\lambda_k[\underline{\eta}].\end{aligned}$$

Thus each λ_k is a group homomorphism.

Note: If $\lambda_k[\underline{\xi}] = \lambda_k[\underline{\eta}]$ for all k then $[\underline{\xi}] = [\underline{\eta}]$. Let (V,x_0) be a neighborhood of (X,x_0) in Q. Choose k so that $(X_k,x_0) \subset (V,x_0)$. Then $\lambda_k[\underline{\xi}] = \lambda_k[\underline{\eta}]$ implies there is an m_0 such that if $m \geq m_0$ then $\xi_m \simeq \eta_m$ in $(X_k,x_0) \subset (V,x_0)$.

We will now show that $(\underline{\pi}_n^Q(X,x_0), \{\lambda_k\})$ is a terminal object in the category $\mathrm{inv}(\pi_n(\underline{X}))$, from which it will follow by uniqueness of inverse limit that $\underline{\pi}_n^Q(X,x_0) \approx \underline{\pi}_n(X,x_0)$. To show $(\underline{\pi}_n^Q(X,x_0), \{\lambda_k\})$ is in the category $\mathrm{inv}(\pi_n(\underline{X}))$, one must show that if $k \leq k'$ then $\lambda_k = \rho_{kk'}\lambda_{k'}$ where $\rho_{kk'}: \pi_n(X_{k'},x_0) \to \pi_n(X_k,x_0)$ is the homomorphism induced by $i_{kk'}: (X_{k'},x_0) \to (X_k,x_0)$. Choose $m_0 \geq m_k, m_{k'}$. Then $\lambda_k[\underline{\xi}] = [\xi_{m_0}] = \rho_{kk'}\lambda_{k'}[\underline{\xi}]$.

It remains to show that $(\underline{\pi}_n^Q(X,x_0), \{\lambda_k\})$ is a terminal object; i.e., if G is any group and $\sigma_k: G \to \pi_n(X_k,x_0)$ are group homomorphisms such that if $k \leq k'$ then $\sigma_k = \rho_{kk'}\sigma_{k'}$, then there is a unique group homomorphism $\sigma: G \to \underline{\pi}_n^Q(X,x\ )$ such that $\sigma_k = \lambda_k\sigma$ for all k. The uniqueness follows immediately from the above note.

Existence: Let $g \in G$. Define $\sigma(g) = [\{\xi_k, (S,a) \to (X,x_0)\}]$ where $\xi_k: (S,a) \to (Q,x_0)$ satisfies $\xi_k \in \sigma_k(g) \in \pi_n(X_k,x_0)$. First, $\{\xi_k, (S,a) \to (X,x_0)\}$ is an approximative map of (S,a) toward (X,x_0). If (U,x_0) is any neighborhood of (X,x_0) in Q choose k_0 such that $k \geq k_0$ implies that $(X_k,x_0) \subset (U,x_0)$. Then $\sigma_k(g) = \rho_{k\ k+1}\sigma_{k+1}(g)$ so that $\xi_k \simeq \xi_{k+1}$ in $(X_k,x_0) \subset (U,x_0)$. Next, σ is a well-defined function for if $\underline{\xi} = \{\xi_k, (S,a) \to (X,x_0)\}$ and $\underline{\xi}' = \{\xi_k', (S,a) \to (X,x_0)\}$ are such that $\xi_k, \xi_k' \in \sigma_k(g)$ for each k, then if (U,x_0) is any neighborhood of (X,x_0) in Q choose k_0 such that if $k \geq k_0$ then $(X_k,x_0) \subset (U,x_0)$. Then $\xi_k \simeq \xi_k'$ in (U,x_0) and hence $[\underline{\xi}] = [\underline{\xi}']$. Also, σ is a group homomorphism. Each σ_k is a homomorphism so that $\sigma_k(g_1g_2) = \sigma_k(g_1) * \sigma_k(g_2)$. Thus if $\xi_k \in \sigma_k(g_1)$, $\eta_k \in \sigma_k(g_2)$ then $\xi_k * \eta_k \in \sigma_k(g_1) * \sigma_k(g_2) = \sigma_k(g_1g_2)$. That is, $\sigma(g_1g_2) = [\{\xi_k * \eta_k, (S,a) \to (X,x_0)\}]$. But $\sigma(g_1) * \sigma(g_2) = [\{\xi_k * \eta_k, (S,a) \to (X,x_0)\}]$ so that σ is a group homomorphism.

Finally, $\sigma_k = \lambda_k\sigma$ for each k. Since $\lambda_k\sigma(g) = [\xi_{m_k}] \in \pi_n(X_k,x_0)$, it suffices to show $\xi_{m_k} \simeq \xi_k$ in (X_k,x_0). If $k \geq m_k$ then by the definition of m_k, $\xi_k \simeq \xi_{m_k}$ in (X_k,x_0). If $m_k \geq k$ then $\sigma_k(g) = \rho_{km_k}\sigma_{m_k}(g)$ so that $\xi_{m_k} \simeq \xi_k$ in (X_k,x_0). This completes the proof of the theorem.

CHAPTER IV

EXTENSIONS IN SHAPE THEORY

1. Shape Retracts. In [9], Mardešić has defined the notion of retraction in the ANR-system approach to shape, called a shape retraction. He then uses this notion to define absolute shape retracts (ASR) and absolute neighborhood shape retracts (ANSR). In the compact metric case, these correspond to Borsuk's FAR and FANR-sets, respectively.

If X and Y are compact Hausdorff spaces and $j: X \to Y$ is an embedding then X is a **shape retract** of Y (compare [9], Definition 3) if whenever $\underline{X}$ and $\underline{Y}$ are ANR-systems associated with X and Y, respectively, there is a map of systems $\underline{r}: \underline{Y} \to \underline{X}$ that has as right homotopy inverse the map of systems $\underline{j}: \underline{X} \to \underline{Y}$ associated with the embedding j; i.e., $\underline{r}\,\underline{j} \simeq \underline{1}_{\underline{X}}: \underline{X} \to \underline{X}$. The map of systems $\underline{r}: \underline{Y} \to \underline{X}$ is called a **shape retraction.** Using an analogous definition in the case of pointed compact Hausdorff spaces, one has the following theorem.

Theorem [illegible].1: If (X,x_0) is a shape retract of (Y,y_0) then $\underline{\pi}_n(X,x_0)$ is isomorphic to a factor of $\underline{\pi}_n(Y,y_0)$, $n > 1$.

Proof: Since $\underline{r}\,\underline{j} \simeq \underline{1}_{\underline{X}}$, the homomorphism $j_*: \underline{\pi}_n(X,x_0) \to \underline{\pi}_n(Y,y_0)$ has as left inverse the homomorphism $r_*: \underline{\pi}_n(Y,y_0) \to \underline{\pi}_n(X,x_0)$. The result then follows from standard arguments in group theory.

A compact Hausdorff space X is an **absolute shape retract** (ASR) (compare [9], Definition 4) if for every compact Hausdorff space Y and embedding $j\colon X \to Y$, X is a shape retract of Y.

Theorem 1.2: A compact Hausdorff space X is an ASR iff there is a compact AR, Y, and an embedding $j\colon X \to Y$ such that X is a shape retract of Y.

Proof: Let $j\colon X \to I^{\Omega}$ be an embedding. Since $X \in$ ASR, X is a shape retract of $I^{\Omega} \in$ AR.

Conversely, suppose $Y \in$ AR, $j\colon X \to Y$ is an embedding and $\underline{r}\colon \underline{Y} \to \underline{X}$ is a shape retraction where $\underline{Y} = \{Y, 1_Y\}$ is the special ANR-system associated with Y. Suppose $i\colon X \to Z$ is an embedding. Consider $ji^{-1}\colon i(X) \to Y$ where $i^{-1}\colon i(X) \to X$ is the homeomorphism determined by $i\colon X \to Z$. Since $Y \in$ AR, there is an extension $f\colon Z \to Y$ of ji^{-1} such that $fi = j\colon X \to Y$. Let $\underline{f}\colon \underline{Z} \to \underline{Y}$ be a map of systems associated with f. Then

$$\underline{r}\,\underline{f}\,\underline{i} \simeq \underline{r}\,\underline{j} \simeq \underline{1}_X$$

so that $\underline{r}\,\underline{f}\colon \underline{Z} \to \underline{X}$ is a shape retraction and $X \in$ ASR.

A compact Hausdorff space X is an <u>absolute neighborhood shape retract</u> (ANSR), compare [9] Definition 5, if for every compact Hausdorff space Z and embedding $j\colon X \to Z$, there is a closed neighborhood W of $j(X)$ in Z such that X is a shape retract of W. Mardešić, [9] Theorem 6, proves the following characterization of ANSR's.

Theorem 1.3: A compact Hausdorff space X is an ANSR iff there is a compact ANR, Y, and an embedding $j\colon X \to Y$ such that X is a shape retract of Y.

An embedding $j\colon X \to M$ is said to have <u>property UV^{∞}</u> if for every neighborhood U of $j(X)$ in M there is a neighborhood V of $j(X)$ in M,

$V \subset U$, such that V is contractible in U.

The following theorem is similiar to one proven by R. C. Lacher [8] and indicates that ASR-sets may be thought of as a generalization of cell-like spaces.

Theorem 1.4: Let X be a compact Hausdorff space. Then the following are equivalent:

(a) X is an ASR-set,

(b) $Sh(X)$ is trivial,

(c) some embedding $X \to I^{\Omega}$ has property UV^{∞},

(d) for any neighborhood retract of a parallelotope, N, any embedding $f: X \to N$ has property UV^{∞}.

Proof: Mardesic, [9] Theorem 3, proves (a)$\Leftrightarrow$(b), while (d)$\Rightarrow$(c) is immediate.

(b)$\Rightarrow$(c): Assume $Sh(X)$ is trivial and let $j: X \to I^{\Omega}$ be an embedding. Let $\underline{X} = \{X_a, i_{aa'}, A\}$ be an inclusion ANR-system associated with X. If U is a neighborhood of X in I^{Ω}, let $A' = \{a \in A: X_a \subset U\}$. It is not hard to show that A' is cofinal in A so that $\underline{X}' = \{X_a, i_{aa'}, A'\}$ is an inclusion ANR-system associated with X. Since $Sh(X)$ is trivial, there are maps of systems $\underline{f}: \underline{X}' \to \underline{p}$, $\underline{g}: \underline{p} \to \underline{X}'$ such that $\underline{f}\,\underline{g} \simeq \underline{1}_{\underline{p}}$ and $\underline{g}\,\underline{f} \simeq \underline{1}_{\underline{X}'}$ where $\underline{p} = \{p, 1_p\}$ is the special ANR-system associated with the singleton p. The map $\underline{f}: \underline{X}' \to \underline{p}$ consists of a map $f: X_a \to p$ for some fixed $a \in A'$. Since $\underline{g}\,\underline{f} \simeq \underline{1}_{\underline{X}'}$, there is an $a' \in A'$, $a' \geq a$ such that $i_{aa'} \simeq g_a f i_{aa'}$. That is, the inclusion map $i_{aa'}: X_{a'} \to X_a$ is null homotopic. Let $V = X_{a'}$, then $V \subset U$ and V is contractible in X_a, hence in U. Therefore, $j: X \to I^{\Omega}$ has property UV^{∞}.

(c)$\Rightarrow$(b): Assume $X \subset I^{\Omega}$ has property UV^{∞}. Let $\underline{X} = \{X_a, i_{aa'}, A\}$ be an inclusion ANR-system associated with X. For each $a \in A$, there is a neighborhood U_a of X such that the inclusion map $i_a: U_a \to X_a$ is homotopic to the constant map $c_a: U_a \to x_0 \in X$. Let $\underline{f}: \underline{X} \to \underline{p}$ be given by $f: X_\alpha \to p$, where $\alpha \in A$ is fixed. Let $\underline{g}: \underline{p} \to \underline{X}$ be given by $g_a: p \to X_a$ is the constant map $g_a(p) = x_0 \in X \subset X_a$. Then $\underline{f}\ \underline{g} = \underline{1}_{\underline{p}}$ and $\underline{g}\ \underline{f} \simeq \underline{1}_{\underline{X}}$ since for all $a' \in A$ one can choose $a'' \geq \alpha, a'$ such that $X_{a''} \subset U_{a'}$ and

$$i_{a'a''} \simeq c_{a'}\big|X_{a''} = g_\alpha f i_{\alpha a''}.$$

Thus Sh(X) is trivial.

(c) $\Rightarrow$ (d): Assume $X \subset N \subset I^\Omega$. Let W be an open neighborhood of N in I^Ω and $r: W \to N$ a retraction. If U is any neighborhood of X in N then $r^{-1}(U)$ is a neighborhood of X in I^Ω. Since some embedding of X into an AR has property UV^∞, it is not hard to show that all embeddings of X into I^Ω have property UV^∞. Thus there is a neighborhood V of X in I^Ω, $X \subset V \subset r^{-1}(U)$, such that V is contractible in $r^{-1}(U)$. Let $F: V \times I \to r^{-1}(U)$ be a homotopy such that $F(x,0) = x$, $F(x,1) = x_0 \in X$ for all $x \in V$. Let $V' = V \cap N$, a neighborhood of X in N. Then $rF\big|_{V' \times I}: V' \times I \to U$ is a homotopy such that $rF(x,0) = r(x) = x$ and $rF(x,1) = r(x_0) = x_0$ for $x \in V'$. That is, V' is contractible in U and the embedding $X \subset N$ has property UV^∞.

<u>2. Extensions of Maps of Systems</u>. If X and X' are compact Hausdorff spaces and $j: X \to X'$ is an embedding then there is a map of systems $\underline{j}: \underline{X} \to \underline{X}'$ associated with j, where $\underline{X}$ and $\underline{X}'$ are any ANR-systems associated with X and X' respectively. If $\underline{f}: \underline{X} \to \underline{Y}$ is a map of systems then $\underline{f}': \underline{X}' \to \underline{Y}$ is said to be an <u>extension</u> of $\underline{f}$ if $\underline{f} \simeq \underline{f}'\underline{j}$. The following theorem follows immediately from the definitions.

<u>Theorem 2.1</u>: If $j: X \to X'$ is an embedding then a shape retraction

$\underline{r}$: $\underline{X}' \to \underline{X}$ is an extension of the identity map $\underline{1}_{\underline{X}}$: $\underline{X} \to \underline{X}$, and conversely.

The following theorem relates extensions of maps of systems and the shape groups, compare [3] Theorem 1.6.

Theorem 2.2: If $\underline{f}'$: $\underline{X}' \to \underline{Y}$ is an extension of $\underline{f}$: $\underline{X} \to \underline{Y}$ then the induced homomorphisms f_∞: $\underline{\pi}_n(X,x_0) \to \underline{\pi}_n(Y,y_0)$ and f'_∞: $\underline{\pi}_n(X',x_0') \to \underline{\pi}_n(Y,y_0)$ satisfy the condition $f_\infty = f'_\infty j_\infty$ where j_∞: $\underline{\pi}_n(X,x_0) \to \underline{\pi}_n(X',x_0')$ is the homomorphism induced by the embedding j: $(X,x_0) \to (X',x_0')$.

Proof: Since $\underline{f} \simeq \underline{f}'\underline{j}$ one has that $f_\infty = f'_\infty j_\infty$.

As an immediate consequence, one has the following corollary, compare [3] Theorem 1.7.

Corollary 2.3: If $\underline{f}$: $\underline{X} \to \underline{Y}$ has an extension $\underline{f}'$: $\underline{X}' \to \underline{Y}$ then the kernel of the homomorphism j_∞: $\underline{\pi}_n(X,x_0) \to \underline{\pi}_n(X',x_0')$ induced by the embedding j: $(X,x_0) \to (X',x_0')$ is contained in the kernel of the homomorphism f_∞: $\underline{\pi}_n(X,x_0) \to \underline{\pi}_n(Y,y_0)$ induced by $\underline{f}$.

The above results show that the extensions of maps of systems enjoy many of the same properties as extensions of fundamental classes. Indeed, it is now shown (Theorem 2.6) that these concepts are equivalent in the compact metric case.

Lemma 2.4: If $\underline{f}$, $\underline{g}$: $\underline{X} \to \underline{Y}$ are maps of systems such that $\underline{f} \simeq \underline{g}$ then $\underline{f}$ has an extension to $\underline{X}'$ iff $\underline{g}$ has an extension to $\underline{X}'$.

Proof: Suppose j: $X \to X'$ is an embedding and $\underline{f}'$: $\underline{X}' \to \underline{Y}$ is an extension of $\underline{f}$. Then $\underline{f}'$ is also an extension of $\underline{g}$ since

$$\underline{g} \simeq \underline{f} \simeq \underline{f}'\underline{j}.$$

Theorem 2.5: Suppose $X \subset X' \subset I^{\Omega}$, $Y \subset I^{\Omega}$ and $\underline{X} = \{X_a, i_{aa'}, A\}$, $\underline{X}' = \{X'_\alpha, i'_{\alpha\alpha'}, A'\}$, $\underline{Y} = \{Y_b, j_{bb'}, B\}$ are inclusion ANR-systems associated with X, X' and Y, respectively. If $\underline{f}$: $\underline{X}\to\underline{Y}$ has an extension $\underline{f}'$: $\underline{X}'\to\underline{Y}$ then there exists a $\underline{g}$: $\underline{X}'\to\underline{Y}$ such that

(1) $\underline{g} \simeq \underline{f}'$ (and is therefore an extension of $\underline{f}$),

(2) $g_b(x) = f_b(x)$ for all $b \in B$ and $x \in X$.

Conversely, if $\underline{g}$: $\underline{X}'\to\underline{Y}$ is a map of systems such that for all $b \in B$ and $x \in X$, $g_b(x) = f_b(x)$ then $\underline{g}$ is an extension of $\underline{f}$.

Proof: Since $\underline{f}'$ is an extension of $\underline{f}$ one has that $\underline{f} \simeq \underline{f}'\underline{j}$, where $\underline{j}$ is chosen such that each $j_\alpha\colon X_{j(\alpha)} \to X_\alpha$ is an inclusion map. That is, for each $b \in B$ there is an $a \in A$, $a \geq f(b)$, $jf'(b)$ such that

$$f_b i_{f(b)a} \simeq f'_b j_{f'(b)} i_{jf'(b)a}.$$

In other words,

$$f'_b\big|_{X_a} \simeq f_b\big|_{X_a} : X_a \to Y_b.$$

By Borsuk's Homotopy Extension Theorem, there is a $g_b\colon X'_{f'(b)} \to Y_b$ such that $f'_b \simeq g_b$ and $g_b(x) = f_b(x)$ for all $x \in X_a$. The map $\underline{g}$: $\underline{X}'\to\underline{Y}$ thus defined satisfies the required conditions.

Conversely, if $g_b(x) = f_b(x)$ for all $x \in X$, then since $Y_b \in$ ANR there is a neighborhood U of X such that $U \subset X_{f(b)} \cap X'_{g(b)}$ and $f_b\big|_U \simeq g_b\big|_U$. Choose $a \in A$ such that $X_a \subset U$ and $a \geq f(b)$, $jg(b)$. Then

$$f_b i_{f(b)a} \simeq g_b j_{g(b)} i_{jg(b)a};$$

that is, $\underline{f} \simeq \underline{g}\,\underline{j}$.

Theorem 2.6: Suppose $X \subset X' \subset Q$, $Y \subset Q$ and $\underline{X}$, $\underline{X}'$, $\underline{Y}$ are inclusion ANR-sequences associated with X, X', Y, respectively. If $\underline{f}$: $\underline{X}\to\underline{Y}$ is related to the fundamental sequence $\underline{\xi} = \{\xi_k, X, Y\}$ then there is an extension $\underline{f}'$: $\underline{X}'\to\underline{Y}$ of $\underline{f}$ iff there is an extension $\underline{\xi}'$ of $\underline{\xi}$.

Proof: If $\underline{\xi}'$ is an extension of $\underline{\xi}$ then $\underline{\xi} \simeq \underline{\xi}'\underline{i}$ where $\underline{i} = \{i_k, X, X'\}$ is the fundamental sequence generated by the inclusion map of X into X'.

Let $\underline{f}'$: $\underline{X}' \to \underline{Y}$ be the map of systems related to $\underline{\xi}'$. Then [11], Lemma 6, $\underline{\xi} \simeq \underline{\xi}'\underline{i}$ implies $\underline{f} \simeq \underline{f}'\underline{j}$ since $\underline{j}$ is related to $\underline{i}$.

Conversely, assume without loss that $\underline{f}'$ is a regular map. Let $\underline{\xi}'$ be a fundamental sequence related to $\underline{f}'$. Then [11], Lemma 6, $\underline{f} \simeq \underline{f}'\underline{j}$ implies $\underline{\xi} \simeq \underline{\xi}'\underline{i}$. By Therorem I.2.3, this implies that $\underline{\xi}$ has an extension.

Let us now return to a study of some of the relationships that exist between shape retractions and extensions.

<u>Theorem 2.7</u>: If X is a shape retract of X' and $\underline{f}$: $\underline{X} \to \underline{Y}$ is a map then $\underline{f}$ has an extension to $\underline{X}'$.

<u>Proof</u>: Let $\underline{r}$: $\underline{X}' \to \underline{X}$ be a shape retraction. Then the composition $\underline{f}\ \underline{r}$: $\underline{X}' \to \underline{Y}$ is easily seen to be an extension of $\underline{f}$.

Mardešić, [9] Corollary 1, proves the following theorem.

<u>Theorem 2.8</u>: If $Y \in$ ASR and j: $X \to X'$ is an embedding then any map of systems, $\underline{f}$: $\underline{X} \to \underline{Y}$, has an extension to $\underline{X}'$.

An analogous theorem holds for ANSR-sets.

<u>Theorem 2.9</u>: If Y is a compact ANR and j: $X \to X'$ is an embedding then every map of systems $\underline{f}$: $\underline{X} \to \underline{Y}$ has an extension $\underline{f}'$: $\underline{W} \to \underline{Y}$ where W is a closed neighborhood of j(X) in X'.

<u>Proof</u>: Assume that $\underline{Y} = \{Y, 1_Y\}$ is the special ANR-system associated with Y and j: $X \to X'$ is the inclusion map. The map of systems $\underline{f}$: $\underline{X} \to \underline{Y}$ consists of a map f: $X_a \to Y$ for some fixed $a \in A$. Since $Y \in$ ANR, there is a closed neighborhood W of X in X' and an extension f': $W \to Y$ of fi_a: $X \to Y$. Let $\underline{W}$ be an ANR-system associated with W, and $\underline{f}'$: $\underline{W} \to \underline{Y}$ a map of systems associated with f'. That is, $\underline{f}'$ consists of a map f": $W_\alpha \to Y$ for some fixed $\alpha \in A'$ such that

$f''p'_\alpha \simeq f' \colon W \to Y$. Choose $a' \geq a,\ j(\alpha)$, then

$$\begin{aligned} fp_{aa'}p_{a'} &= fp_a \\ &= f'j \\ &\simeq f''p'_\alpha j \\ &\simeq f''j_\alpha p_{j(\alpha)} \\ &= f''j_\alpha p_{j(\alpha)a'}p_{a'}. \end{aligned}$$

By Lemma 4 of [10] there is an $a'' \geq a'$ such that

$$\begin{aligned} fp_{aa''} &= fp_{aa'}p_{a'a''} \\ &\simeq f''j_\alpha p_{j(\alpha)a'}p_{a'a''} \\ &= f''j_\alpha p_{j(\alpha)a''}. \end{aligned}$$

Thus $\underline{f} \simeq \underline{f}'\underline{j}$.

Corollary 2.10: If $Y \in$ ANSR and $j \colon X \to X'$ is an embedding then every map $\underline{f} \colon \underline{X} \to \underline{Y}$ has an extension $\underline{f}' \colon \underline{W} \to \underline{Y}$ where W is a closed neighborhood of $j(X)$ in X'.

Proof: Since $Y \in$ ANSR, there is an embedding $i \colon Y \to Y'$ where $Y' \in$ ANR and a shape retraction $\underline{r} \colon \underline{Y}' \to \underline{Y}$. Consider $\underline{i}\ \underline{f} \colon \underline{X} \to \underline{Y}'$. By Theorem 2.9, there is a closed neighborhood W of $j(X)$ in X' and an extension $\underline{f}' \colon \underline{W} \to \underline{Y}'$ of $\underline{i}\ \underline{f}$; i.e., $\underline{i}\ \underline{f} \simeq \underline{f}'\underline{j}$. Then $\underline{r}\ \underline{f}' \colon \underline{W} \to \underline{Y}$ is an extension of $\underline{f}$ since

$$\underline{f} \simeq \underline{r}\ \underline{i}\ \underline{f} \simeq \underline{r}\ \underline{f}'\underline{j}.$$

CHAPTER V

PRODUCTS IN SHAPE THEORY

1. Introduction. In recent papers; e.g., [3], Borsuk has stated some results in shape theory concerning products. In particular he has proven the following. First, the shape of the product of two (and hence finitely many) compacta depends only on the shape of the factors. Secondly, he has proven that if $X = \prod_{k=1}^{\infty} X_k$ is the product of a countable number of compacta then X is a FAR-set iff each X_k is a FAR-set. Finally, he has shown that if $X = \prod_{k=1}^{\infty} X_k$ is the product of a countable number of compacta then X is a FANR-set iff each X_k is a FANR-set and all but finitely many are FAR-sets. In this chapter these results are extended to arbitrary products using the ANR-system approach. A result relating (direct) products and shape groups is also obtained.

2. The Product of a Family of Inverse Systems. Let Ω be an index set. For each $\omega \in \Omega$, let $\underline{X}^{\omega} = \{X_a^{\omega}, p_{aa'}^{\omega}, A^{\omega}\}$ be an inverse system of topological spaces (a similar construction can be made for groups, R-modules, etc.). Let $\Gamma = \{(F,\sigma)$: F is a finite non-empty subset of Ω and $\sigma: F \to \bigcup_{\omega\in\Omega} A^{\omega}$ is a function such that $\sigma(\omega) \in A^{\omega}$ for all $\omega \in F\}$ Order Γ by $(F,\sigma) \leq (F',\sigma')$ iff $F \subset F'$ and $\sigma(\omega) \leq \sigma'(\omega)$ for all $\omega \in F$. For $(F,\sigma) \in \Gamma$ let $X_{(F,\sigma)} = \prod_{\omega\in F} X^{\omega}_{\sigma(\omega)}$. If $(F,\sigma) \leq (F',\sigma')$ then let $p_{(F,\sigma)(F',\sigma')}: X_{(F',\sigma')} \to X_{(F,\sigma)}$ be the

composition of the natural projection $\eta: \prod_{\omega\in F'} X^{\omega}_{\sigma'(\omega)} \to \prod_{\omega\in F} X^{\omega}_{\sigma'(\omega)}$ and the product map $\prod p^{\omega}_{\sigma(\omega)\,\sigma'(\omega)}: \prod_{\omega\in F} X^{\omega}_{\sigma'(\omega)} \to \prod_{\omega\in F} X^{\omega}_{\sigma(\omega)}$. It is not difficult to show that $\underline{X} = \{X_{(F,\sigma)}, p_{(F,\sigma)(F',\sigma')}, \Gamma\}$ is an inverse system. The inverse system $\underline{X}$ is called the <u>product</u> of the family $\{\underline{X}^{\omega}: \omega\in\Omega\}$.

<u>Example</u>: If each A^{ω} is a singleton, each $X^{\omega}_a = I_{\omega} = I$ is the unit interval and $\underline{X}^{\omega} = \{I_{\omega}, 1_{\omega}\}$ where $1_{\omega}: I_{\omega} \to I_{\omega}$ is the identity map then the above construction gives the usual representation of $I^{\Omega} = \prod_{\omega\in\Omega} I_{\omega}$ as the inverse limit of $\{I^{\alpha}, p_{\alpha\alpha'}, F(\Omega)\}$ where $F(\Omega)$ is the set of all non-empty finite subsets of Ω ordered by inclusion and $p_{\alpha\alpha'}: I^{\alpha'} = \prod_{\omega\in\alpha'} I_{\omega} \to I^{\alpha}$, is the natural projection (see [10]).

<u>Theorem 2.1</u>: $\varprojlim \prod_{\omega\in\Omega} \underline{X}^{\omega} = \prod_{\omega\in\Omega} \varprojlim \underline{X}^{\omega}$.

<u>Proof</u>: Let $X^{\omega} = \varprojlim \underline{X}^{\omega}$. We show $\prod_{\omega\in\Omega} X^{\omega}$ is a terminal object in the category $\mathrm{inv}(\prod \underline{X}^{\omega})$. For $(F,\sigma)\in\Gamma$, let $p_{(F,\sigma)}: \prod_{\omega\in\Omega} X^{\omega} \to X_{(F,\sigma)}$ be the composition of the natural projection $\eta: \prod_{\omega\in\Omega} X^{\omega} \to \prod_{\omega\in F} X^{\omega}$, and the product map $\prod p^{\omega}_{\sigma(\omega)}: \prod_{\omega\in F} X^{\omega} \to \prod_{\omega\in F} X^{\omega}_{\sigma(\omega)}$. It is not hard to show that if $(F,\sigma) \le (F',\sigma')$ then $p_{(F,\sigma)(F',\sigma')} p_{(F',\sigma')} = p_{(F,\sigma)}$. Thus $(\prod_{\omega\in\Omega} X^{\omega}, \{p_{(F,\sigma)}\})$ is in the category $\mathrm{inv}(\prod \underline{X}^{\omega})$.

It remains to show that $\prod_{\omega\in\Omega} X^{\omega}$ is a terminal object. That is, if Y is any space and $f_{(F,\sigma)}: Y \to X_{(F,\sigma)}$ is a family of maps such that if $(F,\sigma) \le (F',\sigma')$ then $p_{(F,\sigma)(F',\sigma')} f_{(F',\sigma')} = f_{(F,\sigma)}$, then there is a unique map $f: Y \to \prod_{\omega\in\Omega} X^{\omega}$ such that for all $(F,\sigma)\in\Gamma$, $p_{(F,\sigma)} f = f_{(F,\sigma)}$. If $\omega\in\Omega$ and $a\in A^{\omega}$ let $\sigma_a: \{\omega\} \to A^{\omega}$ be the function $\sigma_a(\omega) = a$. Then $(\{\omega\}, \sigma_a)\in\Gamma$ and $f^{\omega}_a = f_{(\{\omega\}, \sigma_a)}: Y \to X^{\omega}_a$ is a family of maps such that if $a \le a'$ then $(\{\omega\}, \sigma_a) \le (\{\omega\}, \sigma_{a'})$, so that $p^{\omega}_{aa'} f^{\omega}_{a'} = f^{\omega}_a$. By the universal mapping property of X^{ω}, there is a unique $f^{\omega}: Y \to X^{\omega}$

such that $p_a^{\omega} f^{\omega} = f_a^{\omega}$ for all $a \in A^{\omega}$. Let $f: Y \to \prod_{\omega \in \Omega} X^{\omega}$ be the unique map thus defined. Then f satisfies $p_{(F,\sigma)} f = f_{(F,\sigma)}$. Furthermore, if $g: Y \to \prod_{\omega \in \Omega} X^{\omega}$ is any map that satisfies $p_{(F,\sigma)} g = f_{(F,\sigma)}$ then $p_{(\{\omega\},\sigma_a)} g = f_{(\{\omega\},\sigma_a)} = f_a^{\omega}$. It follows then that $f = g$.

<u>Corollary 2.2</u>: If $\underline{X}^{\omega} = \{X_a^{\omega}, p_{aa'}^{\omega}, A^{\omega}\}$, $\omega \in \Omega$, is a family of ANR-systems where $\underline{X}^{\omega}$ is associated with X^{ω}, then $\prod_{\omega \in \Omega} \underline{X}^{\omega}$ is associated with $\prod_{\omega \in \Omega} X^{\omega}$.

<u>Proof</u>: It suffices to note that if each A^{ω} is closure-finite then so is Γ and that the product of a finite number of ANR's is an ANR.

Suppose $\underline{X}^{\omega} = \{X_a^{\omega}, p_{aa'}^{\omega}, A^{\omega}\}$, $\omega \in \Omega$; $\underline{Y}^{\lambda} = \{Y_b^{\lambda}, q_{bb'}^{\lambda}, B^{\lambda}\}$, $\lambda \in \Lambda$, are inverse systems (or ANR-systems) and $\theta: \Lambda \to \Omega$ is a one-to-one function such that for each $\lambda \in \Lambda$ there is a map $\underline{f}^{\lambda}: \underline{X}^{\theta(\lambda)} \to \underline{Y}^{\lambda}$. Recall, a map $\underline{f}^{\lambda}: \underline{X}^{\theta(\lambda)} \to \underline{Y}^{\lambda}$ consists of an increasing function $f^{\lambda}: B^{\lambda} \to A^{\theta(\lambda)}$ together with a family of maps $f_b^{\lambda}: X_{f^{\lambda}(b)}^{\theta(\lambda)} \to Y_b^{\lambda}$, $b \in B^{\lambda}$, such that if $b \leq b'$ then $q_{bb'}^{\lambda} f_{b'}^{\lambda} = f_b^{\lambda}$ (in the ANR-system case, $q_{bb'}^{\lambda} f_{b'}^{\lambda} \simeq f_b^{\lambda}$). Define $f: \Gamma_Y \to \Gamma_X$ by $f(F,\sigma) = (\theta(F), f_{\sigma})$ where $f_{\sigma}: \theta(F) \to \bigcup_{\omega \in \Omega} A^{\omega}$ is given by $f_{\sigma}(\theta(\lambda)) = f^{\lambda}(\sigma(\lambda)) \in A^{\theta(\lambda)}$. Then $X_{f(F,\sigma)} = \prod_{\omega \in \theta(F)} X_{f_{\sigma}(\omega)}^{\omega} = \prod_{\lambda \in F} X_{f^{\lambda}(\sigma(\lambda))}^{\theta(\lambda)}$ so define $f_{(F,\sigma)}: X_{f(F,\sigma)} \to Y_{(F,\sigma)}$ as the product map

$\prod f_{\sigma(\lambda)}^{\lambda}: \prod_{\lambda \in F} X_{f^{\lambda}(\sigma(\lambda))}^{\theta(\lambda)} \to \prod_{\lambda \in F} Y_{\sigma(\lambda)}^{\lambda}$. One then checks that if $(F,\sigma) \leq (F',\sigma')$ then $f(F,\sigma) \leq f(F',\sigma')$ and $q_{(F,\sigma)(F',\sigma')} f_{(F',\sigma')} = f_{(F,\sigma)}$ (in the ANR-system case,

$q_{(F,\sigma)(F',\sigma')}f_{(F',\sigma')} \simeq f_{(F,\sigma)}$). Thus there is a map $\underline{f}: \prod_{\omega\in\Omega} \underline{X}^{\omega} \to \prod_{\lambda\in\Lambda} \underline{Y}^{\lambda}$.

If $\underline{Z}^{\tau} = \{Z_c^{\tau}, r_{cc'}^{\tau}, C^{\tau}\}$, $\tau\in T$, is another family of inverse systems and $\phi: \Omega \to T$ is a one-to-one function such that for all $\omega\in\Omega$ there is a $\underline{g}^{\omega}: \underline{Z}^{\phi(\omega)} \to \underline{X}^{\omega}$ then there is a "natural composition" given by $\phi\theta: \Lambda \to T$ and $\underline{f}^{\lambda}\underline{g}^{\theta(\lambda)}: \underline{Z}^{\phi\theta(\lambda)} \to \underline{Y}^{\lambda}$. It is left to the reader to verify that the map determined by the composition is the same as the composition of the respective determined maps.

There is a "natural identity", $\theta: \Omega \to \Omega$ the identity function and each $\underline{1}^{\omega}: \underline{X}^{\omega} \to \underline{X}^{\omega}$ the identity map. It is left to the reader to verify that the identity $1: \prod_{\omega\in\Omega} \underline{X}^{\omega} \to \prod_{\omega\in\Omega} \underline{X}^{\omega}$ is determined by the natural identity.

We now restrict our attention to the ANR-system case when $\Omega = \Lambda$ and θ is the identity.

Theorem 2.3: If $\underline{f}^{\omega}, \underline{g}^{\omega}: \underline{X}^{\omega} \to \underline{Y}^{\omega}$ are families of maps of systems such that $\underline{f}^{\omega} \simeq \underline{g}^{\omega}$ for all $\omega\in\Omega$ then $\underline{f} \simeq \underline{g}: \prod_{\omega\in\Omega} \underline{X}^{\omega} \to \prod_{\omega\in\Omega} \underline{Y}^{\omega}$.

Proof: For each $b\in B^{\omega}$ there is an $a_b \in A^{\omega}$, $a_b \geq f^{\omega}(b), g^{\omega}(b)$ such that $f_b^{\omega} p_{f^{\omega}(b) a_b}^{\omega} \simeq g_b^{\omega} p_{g^{\omega}(b) a_b}^{\omega}$. Let $\tau_{\omega}: B^{\omega} \to A^{\omega}$ be an increasing function such that $\tau_{\omega}(b) \geq a_b$ for all $b\in B^{\omega}$. If $(F,\sigma)\in \Gamma_Y$, consider $(F,\tau)\in \Gamma_X$ where $\tau: F \to \bigcup_{\omega\in\Omega} A^{\omega}$ is given by $\tau(\omega) = \tau_{\omega}(\sigma(\omega))$. First, $(F,\tau) \geq f(F,\sigma), g(F,\sigma)$. Since θ is the identity, $f(F,\sigma) = (F, f_{\sigma})$ where $f_{\sigma}(\omega) = f^{\omega}(\sigma(\omega))$. Then $(F,\tau) \geq (F, f_{\sigma})$ since

$$\tau(\omega) = \tau_{\omega}(\sigma(\omega)) \geq a_{\sigma(\omega)} \geq f^{\omega}(\sigma(\omega)) = f_{\sigma}(\omega).$$

Similarly, $(F,\sigma) \geq g(F,\sigma)$. Furthermore,

$f_{\sigma(\omega)}^{\omega} p_{f^{\omega}(\sigma(\omega))\, \tau_{\omega}(\sigma(\omega))}^{\omega} \simeq g_{\sigma(\omega)}^{\omega} p_{g^{\omega}(\sigma(\omega))\, \tau_{\omega}(\sigma(\omega))}^{\omega}$ implies

$$f_{(F,\sigma)} p_{f(F,\sigma)(F,\tau)} \simeq g_{(F,\sigma)} p_{g(F,\sigma)(F,\tau)}.$$

Thus $\underline{f} \simeq \underline{g}$.

Corollary 2.4: If $Sh(X^\omega) = Sh(Y^\omega)$ for all $\omega\in\Omega$ then $Sh(\prod_{\omega\in\Omega} X^\omega) = Sh(\prod_{\omega\in\Omega} Y^\omega)$.

Corollary 2.4 allows one to define the product of shapes as follows: $\prod_{\omega\in\Omega} Sh(X^\omega) = Sh(\prod_{\omega\in\Omega} X^\omega)$.

Corollary 2.5: If $\underline{r}^\omega: \underline{Y}^\omega \to \underline{X}^\omega$ is a shape retraction for all ω then $\underline{r}: \prod_{\omega\in\Omega} \underline{Y}^\omega \to \prod_{\omega\in\Omega} \underline{X}^\omega$ is also a shape retraction

Proof: Let $\underline{Y}^\omega$, $\underline{X}^\omega$ be associated with Y^ω, X^ω, respectively, and $j^\omega: X^\omega \to Y^\omega$ the required embeddings. Let $j: \prod_{\omega\in\Omega} X^\omega \to \prod_{\omega\in\Omega} Y^\omega$ be the embedding determined by the family $\{j^\omega : \omega\in\Omega\}$. It is routine to verify that the map determined by the family $\{\underline{j}^\omega : \underline{X}^\omega \to \underline{Y}^\omega\}$ is associated with j. We have that $\underline{r}^\omega \underline{j}^\omega \simeq \underline{1}_\omega$ where $\underline{1}_\omega : \underline{X}^\omega \to \underline{X}^\omega$ is the map associated with the identity. By the above theorem, $\underline{r}\,\underline{j} \simeq \underline{1}_{\prod_{\omega\in\Omega} \underline{X}^\omega}$.

3. Products of ASR and ANSR-sets.

Theorem 3.1: If $X = \prod_{\omega\in\Omega} X^\omega$ then $X \in$ ASR iff $X^\omega \in$ ASR for all $\omega\in\Omega$.

Proof: If $X \in$ ASR there is a $Y \in$ AR, an embedding $j: X \to Y$ and a shape retraction $\underline{r}: \underline{Y} \to \underline{X}$. Since each natural projection $p_\omega : X \to X^\omega$ is a retraction, the associated maps of systems $\underline{p}_\omega : \underline{X} \to \underline{X}^\omega$ are shape retractions. It follows [9] that $\underline{p}_\omega \underline{r}: \underline{Y} \to \underline{X}^\omega$ is a shape retraction. By Theorem IV.1.2, each X^ω is an ASR.

Conversely, if $X^\omega \in$ ASR for all $\omega\in\Omega$, then for each $\omega\in\Omega$ there is an AR-set Y^ω such that $\underline{X}^\omega$ is a shape retract of $\underline{Y}^\omega$. Since the product of any family of AR-sets is an AR-set, we have by Corollary 2.5 that $X \in$ ASR.

Theorem 3.2: If $X = \prod_{\omega\in\Omega} X^{\omega}$ then $X \in$ ANSR iff $X^{\omega} \in$ ANSR for all ω and $X^{\omega} \in$ ASR for all but a finite number of ω.

Proof: If $X^{\omega} \in$ ANSR for all ω and $X^{\omega} \in$ ASR for all but finitely many ω, say $\omega_1, \omega_2, \ldots, \omega_n$, then for all ω there is an ANR-set Y^{ω} and a shape retraction $\underline{r}^{\omega}: \underline{Y}^{\omega} \to \underline{X}^{\omega}$ such that $Y^{\omega} \in$ AR if $\omega \neq \omega_k$ $(k = 1, 2, \ldots, n)$. Then $\prod_{\omega\in\Omega} Y^{\omega} \in$ ANR and there is a shape retraction $\underline{r}: \prod_{\omega\in\Omega} \underline{Y}^{\omega} \to \prod_{\omega\in\Omega} \underline{X}^{\omega}$ so that $\prod_{\omega\in\Omega} X^{\omega} \in$ ANSR.

Conversely, if $X \in$ ANSR then as in the proof of Theorem 3.1, each $X^{\omega} \in$ ANSR. We may assume without loss that $X^{\omega} \subset I^{\Lambda_\omega} = \prod_{\lambda\in\Lambda_\omega} I^{\lambda}$ and $X \subset I^{\Lambda} = \prod_{\omega\in\Omega} I^{\Lambda_\omega}$. By Theorem IV.2.9, there is a closed neighborhood W of X in I^{Λ} and a shape retraction $\underline{r}: \underline{W} \to \underline{X}$. There is a finite subset of Ω, $\{\omega_1, \omega_2, \ldots, \omega_n\}$ and neighborhoods U_i of X^{ω_i} in $I^{\Lambda_{\omega_i}}$ $(i = 1, 2, \ldots, n)$ such that

$$X = \prod_{\omega\in\Omega} X^{\omega} \subset \prod_{i=1}^{n} U_i \times \prod_{\omega \neq \omega_i} I^{\Lambda_\omega} \subset W.$$

Let $i: X \to W$, $j_\omega: X^{\omega} \to I^{\Lambda_\omega}$ denote the inclusion maps and let $p_\omega: X \to X^{\omega}$ be the natural projections. Choose inclusion maps $j'_\omega: I^{\Lambda_\omega} \to W$ for $\omega \neq \omega_i$ $(i = 1, 2, \ldots, n)$ and $i_\omega: X^{\omega} \to X$ such that $j'_\omega j_\omega = i\, i_\omega$ and $p_\omega i_\omega = 1_{X_\omega}$. Then $\underline{r}\,\underline{i} \simeq \underline{1}_X$ so that for $\omega \neq \omega_i$ $(i = 1, 2, \ldots, n)$,

$$\underline{p}_\omega \underline{r}\, \underline{j}'_\omega \underline{j}_\omega \simeq \underline{p}_\omega \underline{r}\, \underline{i}\, \underline{i}_\omega \simeq \underline{p}_\omega \underline{i}_\omega \simeq \underline{1}_{X^\omega}.$$

Hence $\underline{p}_\omega \underline{r}\, \underline{j}'_\omega: \underline{I}^{\Lambda_\omega} \to \underline{X}^{\omega}$ is a shape retraction for $\omega \neq \omega_i$ $(i = 1, 2, \ldots, n)$. By Theorem IV.2.1, X^{ω}, $\omega \neq \omega_i$ $(i = 1, 2, \ldots, n)$, is an ASR-set.

4. Products and Shape Groups. An inspection of Theorem 2.1 shows that the proof does not involve the fact that each X^{ω} is a topological space. It remains valid, for example, whenever the objects are groups. This fact together with the fact that the (usual) homotopy group of a product is the direct product of the (usual) homotopy groups of its factors, [13] Exercise B.5 p. 419, gives the following theorem.

Theorem 4.1: If $(X,x_0) = \prod_{\omega\in\Omega}(X^{\omega},x_0^{\omega})$ then $\underline{\pi}_n(X,x_0) = \prod_{\omega\in\Omega}\underline{\pi}_n(X^{\omega},x_0^{\omega})$.

Proof: For each ω let $\underline{X}^{\omega} = \{(X_a^{\omega},x_a^{\omega}),p_{aa'}^{\omega},A^{\omega}\}$ be an ANR-system associated with $(X^{\omega},x_0^{\omega})$. Then

$$\begin{aligned}
\underline{\pi}_n(X,x_0) &= \underline{\pi}_n\Big(\prod_{\omega\in\Omega}(X^{\omega},x_0^{\omega})\Big)\\
&= \varprojlim \pi_n\Big(\prod_{\omega\in\Omega}\underline{X}^{\omega}\Big)\\
&= \varprojlim \pi_n\{X_{(F,\sigma)},p_{(F,\sigma)(F',\sigma')},\Gamma\}\\
&= \varprojlim \Big\{\pi_n\Big(\prod_{\omega\in F}(X^{\omega}_{\sigma(\omega)},x^{\omega}_{\sigma(\omega)})\Big),\rho_{(F,\sigma)(F',\sigma')},\Gamma\Big\}\\
&= \varprojlim\Big\{\prod_{\omega\in F}\pi_n(X^{\omega}_{\sigma(\omega)},x^{\omega}_{\sigma(\omega)}),\prod_{\omega\in F}\rho_{\sigma(\omega)\sigma'(\omega)}^{\omega},\Gamma\Big\}\\
&= \varprojlim \prod_{\omega\in\Omega}\{\pi_n(X_a^{\omega},x_a^{\omega}),\rho_{aa'}^{\omega},A\}\\
&= \prod_{\omega\in\Omega}\varprojlim\{\pi_n(X_a^{\omega},x_a^{\omega}),\rho_{aa'}^{\omega},A^{\omega}\}\\
&= \prod_{\omega\in\Omega}\underline{\pi}_n(X^{\omega},x_0^{\omega}).
\end{aligned}$$

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