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LOCALLY FINITE TREE ACTIONS FOR GROUPS OF DIMENSION TWO

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LOCALLY FINITE TREE ACTIONS FOR GROUPS OF DIMENSION TWO

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DEDICATION

to

My Parents,

Michael and Phyllis Martino

who trusted me to pursue my own interests even when I didn't

Acknowledgments

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Abstract

We investigate finitely generated groups of cohomological dimension 2 and certain actions on locally finite trees. Our setting includes examples of Daniel Wise, which possess two canonical splittings. We show that under suitable hypotheses such a group cannot admit more than two finite-index splittings, up to deformation.

Chapter 1

Introduction

We are interested in groups acting on locally finite trees. By Bass-Serre theory this corresponds to groups that split as a finite index graph of groups. When working with splittings there is already a notion of similarity given by elementary deformations. We say two graphs of groups lie in the same deformation space if they are related by a sequence of moves that transform edges of the form $A *_C C$ to a vertex A or the reverse.

In order to have access to results from homological algebra and make the problem tractable we restrict ourselves to cocompact actions where the stabilizers are groups of type FP. Property FP along with FP_n, FP_∞, and F are all related finiteness properties. We will only need Property FP; for examples take groups G with cocompact finite dimensional $K(G, 1)$ spaces. Our setting then is finite index graphs of groups where the vertex and edge groups are FP and the quotient graph is finite. (Note, finite index implies that if any stabilizer is FP then they all are)

One basic question to ask of a group is its cohomological dimension. Trivial groups have dimension zero, non-abelian free groups have dimension one, and so on. Even at dimension two there are interesting examples that fit within our setup. Daniel Wise invented a kind of 2-complex called a VH-complex and used them to construct groups with no finite quotients [Wis96]. Burger and Mozes produced infinite, finitely presented, torsion-free simple groups in the same class [BM00].

Our goal is to fix a group G of dimension two (as this is the first interesting case after free groups) and give a uniqueness result on the number of splittings. Already, Wise has shown that G comes with two splittings but there is no comment on uniqueness and indeed the torus shows that $\mathbb{Z} \times \mathbb{Z}$ has an infinite number of splittings all lying in different deformation spaces. We will

show that for certain actions there are exactly two splittings up to deformation.

The proof proceeds by encoding two tree actions into a single square complex called Guirardel's core. Then, using a supposed third action we are able to construct a three dimensional complex that can be built up from lower dimensional complexes in such a way that results of Bieri on cohomological dimension apply. The result is an extended core with dimension three that G acts on; this contradicts the original assumption that G only had dimension two.

By analogy with VH-complexes that come with a vertical and horizontal splitting thanks to the work of Wise, our extended core will satisfy the VHD property (see 6.6.1, here "D" stands for "Depth") that we introduce below. Not all VHD complexes will have a third splitting but we show our extended core does. Lastly, it's possible to have a VHD-complex where the splittings are in a sense degenerate for our purposes; essentially the splitting process can be iterated until only graphs remain and we require at least one graph of positive rank. We call these primitive 6.6.2 VHD-complexes and account for them.

The main result is the following.

Theorem 1.0.1 (Main Theorem). *If G is finitely generated and has dimension 2 then either G is the fundamental group of a primitive VHD complex or there are at most two pairwise transverse deformation spaces of finite type G-trees.*

We conjecture that primitive VHD complexes all have a fundamental group of $\mathbb{Z} \times \mathbb{Z}$ so the main conjecture is the following.

Conjecture 1.0.2 (Main Conjecture). *If G is finitely generated, dimension 2, and not $\mathbb{Z} \times \mathbb{Z}$, then there are at most two pairwise transverse deformation spaces of finite type G-trees.*

Chapter 2

Preliminaries

This section collects general results and fixes terminology.

2.1 Actions on Trees

In this section we introduce G-trees, their equivalence with graphs of groups, and related results.

A *graph* Γ is a set of vertices $V(\Gamma)$ and edges $E(\Gamma)$ together with a fixed point free involution on $E(\Gamma)$ denoted by $e \mapsto \bar{e}$ and a map $\partial_0 : E(\Gamma) \to V(\Gamma)$. Define ∂_1 via $\partial_1(e) := \partial_0(\bar{e})$. An edge e for which $\partial_0 e = \partial_1 e$ is called a *loop*. A finite sequence of edges $e_1, \ldots, e_n$ such that $\partial_1(e_k) = \partial_0(e_{k+1})$ for $1 \leqslant k \leqslant n-1$ is called a *path*. If additionally, $e_{i+1} \neq \bar{e}_i$ then it is a *path without reversals*. A *circuit* is a path without reversals where the starting and ending vertex are the same. A *tree* is a connected graph with no circuits.

A *G-tree* is an action of a group G on a simplicial tree without inversions. An element $g \in G$ is *elliptic* if g fixes a vertex and *hyperbolic* otherwise. If $g \in G$ is hyperbolic then there is a line called the *axis* that it acts on by translation. A subgroup $H \leqslant G$ is *elliptic* if there is a vertex that is fixed by all elements of H. We denote the set of all elliptic subgroups by $\mathcal{E}(G)$.

A *graph of groups* consists of a connected graph Γ, groups G_v indexed by $V(\Gamma)$, groups indexed by $E(\Gamma)$ satisfying $G_e = G_{\bar{e}}$, and injective homomorphisms $\phi_e : G_e \to G_{\partial_0(e)}$ indexed by edges. Given a vertex $v \in V(\Gamma)$ there is a notion of a *fundamental group* $\pi_1(\Gamma, v)$ as defined in [Ser80].

A *graph of spaces* consists of a connected graph Γ, vertex spaces X_v indexed by $V(\Gamma)$, edge spaces X_e indexed by $E(\Gamma)$ that satisfy $X_e = X_{\bar{e}}$, and π_1-injective maps $f_e : X_e \to X_{\partial_0(e)}$ indexed

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by edges. The *total space* of a graph of spaces is the quotient

$$\left(\bigsqcup_{v \in V(\Gamma)} X_v \sqcup \bigsqcup_{e \in E(\Gamma)} X_e \times [0,1] \right) \Big/ \sim$$

where $\sim$ is given by identifying the following:

$$X_e \times [0,1] \to X_{\bar{e}} \times [0,1] \text{ by } (x,t) \mapsto (x, 1-t)$$

$$X_e \times 0 \to X_{\partial_0(e)} \text{ by } (x,0) \mapsto f_e(x).$$

These definitions work well together in the following sense. Given a graph of groups with fundamental group G it is possible to construct a graph of spaces where the fundamental group of the total space is also G. We also recover two special cases of interest. A graph of groups with exactly two vertices and one edge corresponds to an amalgamated free product $A *_C B$; should the single edge be a loop then it corresponds to an HNN extension $A*_C$. These special cases can be seen as applications of Van Kampen's theorem.

Definition 2.1.1 (Minimal Action). We say a G-tree is *minimal* if there is no proper invariant subtree.

Under the correspondence between G-trees and graphs of groups, minimality is equivalent to requiring that every proper subgraph corresponds to a proper subgroup of G.

From [BF91] we have the following classification of G-trees. Note, any tree action can be made to act without inversions by subdividing once.

Theorem 2.1.2 (*G*-tree Classification). *Let G be a finitely generated group and T a minimal G-tree acting without inversions. Then T satisfies exactly one of the following:*

1. *Elliptic: T is a point or equivalently every element of G is elliptic.*

2. *Dihedral: T is a line and there exists an epimorphism from G to the infinite dihedral group.*

3. *Parabolic: T has one end fixed by G. Additionally, there is an epimorphism from G to $\mathbb{Z}$.*

4. *Hyperbolic (or irreducible): There exists two axes in T with compact intersection. When this happens, G contains a non-abelian free group.*

Given a G-tree, or equivalently a graph of groups, a *collapse move* produces a new graph where an edge corresponding to $A *_C C$ is collapsed to a vertex A. The reverse is called an *expansion move*. An *elementary deformation* is a finite sequence of such moves. We say a G-tree is *reduced* when one can no longer perform any collapse moves. For details and examples see [For02].

Definition 2.1.3 (Deformation Space). We say two G-trees T_1 and T_2 are in the same *deformation space* if there is an elementary deformation from T_1 to T_2.

Lemma 2.1.4. *If X is a G-tree that is reduced then it is minimal*

Proof. One can show that if e is an edge outside of a G-invariant subtree, then it admits an elementary collapse move. For details see [For02]. $\qquad\square$

The following is a corollary of the main theorem from [For02] that relates elliptic subgroups and deformations.

Theorem 2.1.5 (Forester). *Let G be a group and let X and Y be cocompact G−trees. The following conditions are equivalent.*

1. *X and Y are related by an elementary deformation.*

2. *X and Y have the same elliptic subgroups.*

If all vertex stabilizers are finitely generated then we may also include:

3. *X and Y have the same elliptic elements.*

Example 2.1.6. An important example: The Torus has infinitely many graph of spaces decompositions each lying in a different deformation space. To see this, note that simple closed curves on the torus correspond to graph of spaces decompositions with the curve as the edge space. Up

to homotopy, such curves correspond to extended rationals e.g. r/s, 1/0, and 0/1 each telling you how many times a curve wraps around both directions of the torus. The wrapping numbers correspond to coordinates of $\mathbb{Z} \times \mathbb{Z}$, so r/s corresponds to the cyclic subgroup $\langle (r, s) \rangle \leqslant \mathbb{Z} \times \mathbb{Z}$. With this perspective, $\mathbb{Z} \times \mathbb{Z}$ can be viewed as an HNN extension (i.e. a graph of groups with exactly one edge and one vertex) where the vertex and edge groups are $\langle (r, s) \rangle$. The elliptic subgroups are exactly the vertex group and its subgroups therefore by 2.1.5 each splitting is in a different deformation space.

If one G-tree contains the elliptic subgroups of another it is possible to produce a G-map between the two trees. This map in turn can be analyzed using Stallings folds as in 2.2.7.

Proposition 2.1.7 (Equivariant Map equivalent to elliptic subgroup containment). *Suppose X and Y are simplicial G-trees. Then the following are equivalent:*

1. *$\mathcal{E}(X) \subseteq \mathcal{E}(Y)$*

2. *There exists a simplicial G-map from a subdivision of X to Y.*

Proof. For (1)$\Rightarrow$(2) we will construct a G-map. As in 6.3.1 we start by defining a map on representatives of vertex orbits and extending equivariantly. Pick a vertex $x \in X$, by (1) G_x is also elliptic in Y and so fixes at least one vertex, say y. Define $f(x) = y$ and $f(gx) = gy$ for all $g \in G$. Next, we check that the resulting map is well-defined.

Indeed, if $gv = hv$ then $g^{-1}h = v$ which means $g^{-1}h \in G_v$. Then by the definition of f the element $g^{-1}h$ fixes $f(v) \in Y$. Hence,

$$f(gv) = gf(v) = g(g^{-1}h)f(v) = hf(v) = f(hv).$$

Once defined on vertices, define on edges by sending the edge vw to the unique path in Y from $f(v)$ to $f(w)$. To make this definition work on edges it may be necessary to subdivide the edge $vw \in X$. If the endpoints of an edge go to the same vertex, send the entire edge to that vertex.

For (2)⇒(1), suppose $f : X \to Y$ was a simplicial G-map. Consider a vertex group of X, say G_x. Let $g \in G_x$, then $g \cdot f(x) = f(g \cdot x) = f(x)$ a vertex in Y since the map is simplicial. Hence, G_x also fixes the vertex $f(x) \in Y$. Hence, every vertex group for X fixes a vertex of Y. Therefore, every elliptic subgroup for X is also elliptic for Y. □

Bass-Serre Theory is responsible for the correspondence between G-trees and graph of groups decompositions of G. The original construction creates a tree from cosets but the following definition is inspired by the topological reformulation given by Scott and Wall in [SW79].

Definition 2.1.8 (Bass-Serre Map). The universal cover of a graph of spaces is a tree of spaces. Define a map to the underlying tree taking X_v to the vertex v and $X_e \times I$ to the edge e by collapsing onto the second factor. This map is called a Bass-Serre map.

Here we list some technical definitions that are not part of the standard background for G-trees but are needed here or appear in results we cite.

To start, we are most interested in G-trees of the following type.

Definition 2.1.9 (Finite Type). We say a G-tree is of *finite type* if the tree is locally finite, the vertex stabilizers have property FP, and the quotient is a finite graph.

Next, we fix terminology for discussing Guirardel's work from [Gui05].

Definition 2.1.10 (Directions and Halfspaces). A *direction* based at a point $p \in T$ where T is a tree is a connected component of $T \setminus p$. If p is a midpoint of an edge then we call the resulting directions *halfspaces*.

A *halfspace in a product of trees* is a preimage of a direction of a tree under projection. A *quadrant* in $T_1 \times T_2$ based at (x, y) is the product of two directions at x and y respectively or equivalently the intersection of two halfspaces, one from each factor. An *orthant* in $T_1 \times T_2 \times T_3$ based at (x, y, z) is the product of three directions at x and y and z respectively or equivalently the intersection of three halfspaces, one from each factor.

Finally, there is a notion of a geometric action on an $\mathbb{R}$-tree. The full details do not concern us but [LP97] gives a condition 2.1.11 that applies when specialized to our setup.

Lemma 2.1.11 (Geometric Condition). *(Theorem 0.6 in [LP97]) A minimal simplicial action of a finitely generated group is geometric if and only if all edge groups are finitely generated.*

2.2 Folding

Here we define a special kind of map between G-trees called a fold, first introduced by Stallings in [Sta83]. Later we will factor G-maps into a sequence of folds and analyze each fold separately.

Definition 2.2.1 (Folding). Folding is an operation on G-trees without inversions. Essentially, two edges sharing a vertex are folded together followed by folding their translates together equivariantly. After folding a new G-tree is produced, denote it by $T/\sim$. It may happen that $T/\sim$ inverts some edge but this can be remedied by first subdividing the original G-tree T.

Let e_1 and e_2 be edges with a common vertex v and denote the vertices opposite to v by u_1 and u_2 respectively. Suppose that in the quotient, e_1 and e_2 are not loops, that is $v \notin G u_1 \cup G u_2$. Then folding e_1 and e_2 together will be called a *type A* fold. The other types of folds will not concern us as they are all compositions of type A folds and subdivisions. For details see [BF91] and 2.2.2.

Example 2.2.2 (Type IA). Suppose the orbit map embeds $e_1 \cup e_2$. Then the fold will force a new identification. See figure 2.2.2.

Example 2.2.3 (Type IIIC). Suppose there exists some $g \in G$ that identifies e_1 with the reverse of e_2, then the new edge $e_1/\sim = e_2/\sim$ is inverted by G. This can be avoided by first subdividing.
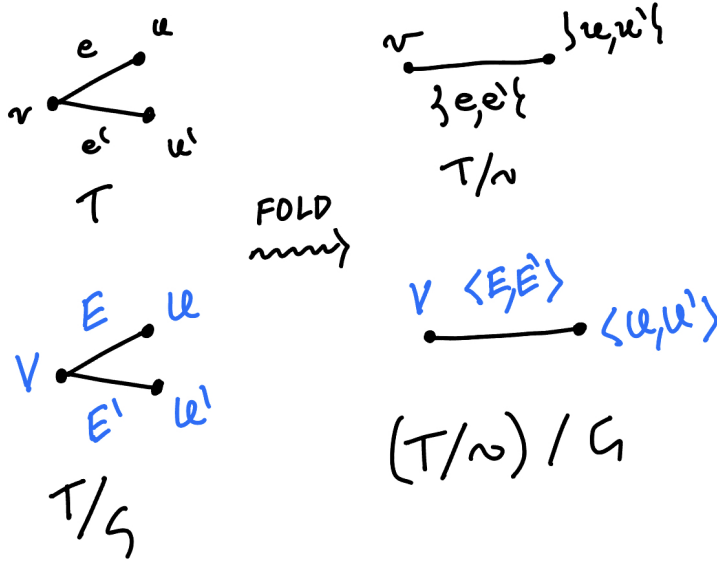
Figure 2.1: The left side shows T and T/G. The right side shows the new G-tree $T/\sim$ and the resulting quotient $(T/\sim)/G$ after a type IA fold. Stabilizers are written in blue.

Definition 2.2.4 (Morphism). A *morphism* is a map between graphs where vertices go to vertices and edges go to edges. (Alternatively, this is a simplicial map where no edges are collapsed)

Definition 2.2.5 (Collapse Map). A *collapse map* is a map between two trees obtained by quotienting each connected component of a union of subtrees to a point. (Alternatively, it's a simplicial map where the preimage of every vertex is connected)

We need a result from Bestvina and Feighn on factoring morphisms.

Theorem 2.2.6 (Bestvina-Feighn [BF91]). *Let G be a finitely generated group. Suppose that $\alpha : T' \to T$ is a simplicial equivariant map from a G-tree T' to a minimal G-tree T such that no edge in T' is mapped to a point by α. If all edge stabilizers of T are finitely generated and if T'/G is finite, then α can be represented as a finite composition of folds.*

The following lemma lets us fully factor our G-maps.

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Lemma 2.2.7. *Every surjective G-map between trees factors as a collapse map, followed by a morphism.*

Proof. Consider all edges that map to vertices. The connected components of this set of edges constitutes a forest which gives the collapse map. □

2.3 Cohomological Dimension and Groups of Type FP

We write A to mean an arbitrary ring with a unit. Take R to be a non-zero commutative ring with a unit. A resolution of a module M is a sequence of modules $(P_k)_{0 \leqslant k \leqslant \infty}$ and homomorphisms $(\phi_k)_{0 \leqslant k \leqslant \infty}$ where

$$\cdots \to P_0 \xrightarrow{\phi_0} M$$

is exact. We say a resolution

$$\cdots 0 \to P_n \to \cdots \to P_0 \to M$$

has length n if n is the greatest integer such that P_n is a non-zero module.

Definition 2.3.1 (Projective Module). A module P is projective if for every surjective module homomorphism $f : N \to M$ and for every module homomorphism $g : P \to M$ there exists a module homomorphism $h : P \to N$ such that $fh = g$.

Definition 2.3.2 (Projective Resolution). A *projective resolution* $(P_k)_{0 \leqslant k \leqslant \infty}$ of a module M is a resolution of M where all the P_k are projective modules.

Definition 2.3.3 (Cohomological Dimension). The cohomological dimension of a group G is given by:

$$\mathrm{cd}(G) = \inf\{n \mid \text{The trivial } \mathbb{Z}G\text{-module } \mathbb{Z} \text{ admits a projective resolution of length } n \text{ over } \mathbb{Z}G\}$$

Definition 2.3.4 (Property FP_n for a Module). A module M over A has property FP_n with $0 \leqslant n \leqslant \infty$ if there is a projective resolution $(P_k)_{0 \leqslant k \leqslant \infty}$ of M where the P_k are finitely generated for all $0 \leqslant k \leqslant n$.

Remark 2.3.5. The property FP_n generalizes familiar properties. For a module M, being finitely generated is equivalent to having property FP_0, being finitely presented is equivalent to having property FP_1.

Definition 2.3.6 (Property FP_n for a Group). A group G has property FP_n over a ring R with $0 \leqslant n \leqslant \infty$ if the trivial RG-module R is of type FP_n as an RG-module.

For our purposes we will only need the case where R is the integers $\mathbb{Z}$ and will therefore omit the ring. Our resolutions will then be taken over the group ring $\mathbb{Z}G$.

Proposition 2.3.7. *A group G is finitely generated if and only if G is of type FP_1 over R.*

Definition 2.3.8 (Property FP). A module is of type FP if there exists a finitely generated projective resolution of finite length. This is equivalent to being of type FP_∞ and having finite cohomological dimension.

2.4 VH-complexes

A *cube complex* is a CW-complex consisting of cubes glued together via isometries between their faces. Here an *n-cube* is a copy of $[-1, 1]^n$ with the Euclidean metric, a *k-face* is a subset of an n-cube where all but k coordinates are restricted to -1 or 1, a *midcube* of a cube $[-1, 1]^n$ is the result of restricting exactly one coordinate to zero. A *hyperplane* of a cube complex comes from taking a single midcube and extending it along glued faces of the cube complex to include other midcubes. More specifically, a hyperplane is a minimal closed subset whose intersection with any closed cube is a union of midcubes. For details see [?], [HW08].

There are several notions of what it means for a space to have non-positive curvature. For a cube complex there is a local combinatorial condition.

Definition 2.4.1 (Link of a vertex). Let X be a cube complex. The *link of a vertex v* is the complex formed by taking a small spherical neighborhood of the vertex v.

Definition 2.4.2 (Flag simplicial complex). A simplicial complex K satisfies the *flag simplicial* property if every complete subgraph of the 1-skeleton of K bounds a simplex

Definition 2.4.3 (Non-positively curved (NPC)). A cube complex X is *non-positively curved (NPC)* if the link of every vertex is simplicial and satisfies the flag condition. If X is also simply connected then we say it is *CAT(0)*.

Definition 2.4.4 (Square Complex). A *square complex* is a 2-complex whose 2-cells are attached by combinatorial paths of length 4.

Definition 2.4.5 (VH-complex). A square complex X is a *VH-complex* if the following hold,

1. the link at each vertex is simplicial;

2. each edge is labelled vertical or horizontal;

3. each hyperplane is two-sided;

4. attaching maps alternate between horizontal and vertical edges.

Let V and H denote the set of vertical and horizontal edges. Take $V_X = V \cup X^{(0)}$ and $H_X = H \cup X^{(0)}$ to be the *vertical* and *horizontal* skeletons.

Remark 2.4.6. Note that the link of every VH-complex is a bipartite graph; this follows from conditions (2) and (3), see also remark 1.4 from Wise [Wis96]. In dimension 2 the NPC property for a cube complex is equivalent to requiring that all links be simplicial graphs i.e. no double edges or loops. Wise does not require his VH-complexes to be NPC. However, for convenience, we assume property (1) and so our VH-complexes will be NPC. The link condition also generalizes to an object we need later called a VHD-complex (see 6.6.1).

Definition 2.4.7 (Two-sided Hyperplanes). A hyperplane is *two-sided* if there is a choice of orientation on each edge dual to the hyperplane that is consistent across squares.

Remark 2.4.8. An embedded hyperplane H is *two-sided* if and only if the open cubical neigh-borhood is homeomorphic to $H \times (-1, 1)$. This orientation condition can always be achieved by subdividing a cube complex twice which allows us to avoid any twisting (e.g. Mobius bands) and ensures our cube complex will decompose easily into a graph of spaces.

The following is necessary to state an important result of Wise about the existence of splittings for VH-complexes. The notion of a foliation will appear in Guirardel's work as well.

Definition 2.4.9 (Decomposition Graph). Given a VH-complex X we define a map $\rho : X \to \Gamma_X$ from X to a graph. The vertices of Γ_X are defined to be the connected components of V_X which we call vertex spaces and the edges are given by the connected components of $X \smallsetminus V_X$. After foliating each square with vertical segments, given $x \in X$ define V_x to be the smallest subset of X that contains x and any vertical segment that it intersects. Because X has two-sided hyperplanes there are no singular leaves and V_a and V_b are parallel to each other if a and b lie in the interior of the same horizontal edge. The set of all V_x then gives a foliation of X. Collapsing each leaf of the foliation is enough to define a map to Γ_X viewed as a 1-dimensional cell complex.

The following is a restatement of theorem 2.16 [Wis96] from Wise:

Theorem 2.4.10 (Wise Graph Decomposition). *Suppose X is a VH-complex. Then the vertical and horizontal decomposition graphs 2.4.9 each determine a splitting of $\pi_1(X)$ as a graph of free groups.*

Chapter 3

Locally finite trees and elliptic subgroups

In this section we prove various utility lemmas related to locally finite trees. We also give a key sufficient condition for when elliptic subgroup containment can be upgraded to equality.

Proposition 3.0.1 (Hyperbolic gives unique minimal tree). *If G is acting on a tree X and contains a hyperbolic element then there is a unique minimal subtree equal to the union of all hyperbolic axes. In particular it is non-empty.*

Proof. See Proposition 3.1 from [Bas93]. □

Proposition 3.0.2 (Commensurable groups have the same minimal tree). *Suppose G acts on a tree X and H and K are commensurable subgroups. If H contains a hyperbolic element, then so does K and the minimal subtrees for H and K are equal.*

Proof. This follows from 3.0.1, for details see Corollary 7.7 from [Bas93]. □

The reader should think about a sequence of folds from 2.2.6 while reading the statements of 3.0.3, 3.0.5, and 3.0.6. Combined these lemmas allow us to inductively push properties across folds.

Proposition 3.0.3 (Folding preserves properties). *Suppose X and Y are G-trees. In addition suppose Y has a locally finite minimal subtree that is not a point. Let $\phi : X \to Y$ be a type A fold. Then if an element of G is hyperbolic for X it is also hyperbolic for Y.*

Proof. For the fold ϕ pick two edges e and e' that are folded and adjacent to some v. The element g is hyperbolic and so comes with an axis that has a positive translation length, hence Gv, the orbit of v is infinite. Because ϕ is a type A fold, Gv is taken injectively to Y. That is, $G(\phi(v))$ is infinite. If $G(\phi(v))$ intersects $Y_{\min}$ then $G(\phi v)$ is entirely contained in $Y_{\min}$ because minimal trees are G-invariant. However, g is elliptic in Y and we are acting by isometries so the infinte set $G(\phi(v))$ lies within a bounded distance of $\phi(v)$ in $Y_{\min}$; but this is impossible becauase $Y_{\min}$ is locally finite. That is, $G(\phi(v)) \subset Y \smallsetminus Y_{\min}$.

Consider the images of the folded edges e and e' along with their orbits. Folds preserve adjacency so every edge in $G(\phi(e))$ (which is equal to $G(\phi(e'))$) is adjacent to a vertex in $G(\phi(v))$. In particular, this means at least one vertex of each edge in $G(\phi(e))$ and $G(\phi(e'))$ is outside of $Y_{\min}$. Therefore, $G(\phi(e))$ and $G(\phi(e'))$ are not contained in $Y_{\min}$. Said differently, $\phi(Ge \cup Ge')$ is disjoint from $Y_{\min}$.

Finally, since $Y_{\min}$ is not a point, it contains an edge, call it f. Consider an edge $\hat{f}$ that maps to f. Since f is contained in $Y_{\min}$ it is disjoint from $\phi(Ge \cup Ge')$ and therefore $\hat{f}$ is not part of a fold and therefore $G\hat{f}$ goes injectively. But this is impossible because then Gf is an infinite set within a bounded distance of a single point in a locally finite tree. $\qquad\square$

Lemma 3.0.4. *Let X and Y be G–trees in the same deformation space and suppose that X contains a locally finite subtree that is G–invariant. Then Y also contains a G–invariant locally finite subtree.*

Proof. It suffices to consider a single elementary collapse move $q\colon X \to Y$ along the edge $e \in E(X)$ with $G_{\partial_0(e)} = G_e$. First suppose that X has a G–invariant subtree X'. Let $Y' = q(X')$, a G–invariant subtree of Y. If $e \notin X'$ then X' maps isomorphically to Y', and so Y' is locally finite. Otherwise, the restriction $q\colon X' \to Y'$ is an elementary collapse. In the proof of [For02, Theorem 7.3] it was observed that q is a $(3, 2/3)$–quasi-isometry, and that $\frac{1}{3}(d(x, x') - 2) \leqslant d(q(x), q(x'))$ for all $x, x' \in X'$. It follows that the pre-image of a ball of radius 1 is contained in a ball of radius

5 in X'. The latter is finite, and so every ball of radius 1 in Y' is finite.

Next consider the same collapse move $q\colon X \to Y$ but suppose that Y contains a locally finite G–invariant subtree Y'. We wish to find the same in X. Let $Z \subset X$ be the G–invariant subgraph whose edges are $\{e' \in E(X) - (Ge \cup G\bar{e}) \mid q(e') \in E(Y')\}$. If Z is connected then let $X' = Z$; it maps isomorphically to Y' by q and hence is locally finite.

Otherwise, $\partial_0(e)$ is in Z. We define X' to be $Z \cup (Ge \cup G\bar{e}) = q^{-1}(Y')$. For any vertex $v \in V(X')$ consider the ball $B_v(1)$ of radius 1 at v. Edges of Z in $B_v(1)$ map injectively to Y', so there are finitely many. It remains to bound the number of edges of $Ge \cup G\bar{e}$ in $B_v(1)$. Note that each component of $Ge \cup G\bar{e}$ is a cone on some subset $S \subset G\partial_0(e)$ (with cone point in $G\partial_1(e)$). Each vertex of S is incident to an edge of Z and these edges are all distinct. Hence S is finite, because collapsing e results in a locally finite tree. Hence $Ge \cup G\bar{e}$ is locally finite, and therefore $B_v(1)$ is finite. □

From the above lemma we get the following corollary.

Corollary 3.0.5. *If X and Y are in the same deformation space and Y_{min} is is locally finite then X_{min} is also locally finite.*

Lemma 3.0.6 (Collapse map preserves hyperbolic elements). *Suppose $X \to Y$ is a collapse map with Y locally finite. Suppose Y is not a single point. Then, if an element is hyperbolic for X it is also hyperbolic for Y.*

Proof. Suppose for sake of a contradiction that $g \in G$ were hyperbolic for X and elliptic for Y. Let $y \in Y$ be some vertex fixed by the elliptic element g and G_y it's stabilizer. Since Y is not a single point, there is another vertex $y \neq z \in Y$. Because Y is locally finite, G_y and G_z are commensurable. For G-maps, pre-images are invariant. By the construction of a collapse map, the preimage of vertices are connected and non-empty. Putting these together we have that the preimages of vertices are invariant trees. This means that the minimal subtrees of G_y and G_z acting on X are contained in the disjoint preimages of y and z respectively. However, since they

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are commensurable and G_y contains the hyperbolic element g, these minimal trees are non-empty and equal by 3.0.2. This is a contradiction. □

We already saw that being in the same deformation space is equivalent to having the same elliptic subgroups. This further reduces the problem to having the same elliptic elements.

Theorem 3.0.7 (Elliptic elements determine elliptic subgroups). *Let X and Y be cocompact G-trees with finitely generated vertex groups. Then the following are equivalent:*

1. *X and Y define the same partition of G into elliptic and hyperbolic elements.*

2. *X and Y have the same elliptic subgroups.*

Proof. By Proposition 2.6, Theorem 4.2, and Corollary 4.3 of [For02]. □

Remark 3.0.8. For an interesting example of how the conclusion of 3.0.7 may not hold when a group fails to be finitely generated consider a particular subgroup of the Baumslag-Solitar group $BS(1,2)$ acting on a trivalent tree with a fixed end. One can produce a group isomorphic to the dyadic rationals along with two tree actions where every element is elliptic but the elliptic subgroups differ. For details see [For02].

The following is used whenever we need to apply Bieri's results on dimension iteratively. This happens once during the transverse construction 4.0.8 as well as part of the final argument 6.7.4 in the proof of the main theorem. The proof works by factoring a tree map into a sequence of folds and showing that the decomposition of the group into elliptic and hyperbolic elements is preserved across folds. In our setting, this decomposition is enough to force the elliptic subgroups to be preserved as well.

Theorem 3.0.9 (Elliptic containment implies equality). *If X and Y are locally finite cocompact G-trees with finitely generated vertex and edge stabilizers and Y has no global fixed point then*
$$\mathcal{E}(X) \subseteq \mathcal{E}(Y) \implies \mathcal{E}(X) = \mathcal{E}(Y).$$

Proof. Let X and Y as in the hypotheses. Without loss of generality we may assume that X and Y are reduced and therefore minimal by 2.1.4. By 2.1.7 there exists a G-map from X to Y. This G-map is onto because Y is minimal and the image of a G-map is an invariant set. Using 2.2.7 we can factor the G-map into a collapse map followed by a morphism. Apply 2.2.6 to factor the morphism into a finite sequence of folds. From a remark in [BF91] after possibly subdividing we can take all of the folds to be of type A. Collapse maps and folds preserve the finite generation of vertex and edge stabilizers. In the next step we repeatedly apply 3.0.5. Using 3.0.6 and 3.0.3 we see each stage of the composition starting from the right preserves the property of hyperbolicity for an element $g \in G$. Hence, X and Y partition G into the same elliptic and hyperbolic elements. Under our setup 3.0.7 applies, therefore X and Y have the same elliptic subgroups as needed. $\square$

Chapter 4

Transverse deformation spaces

In this section, we introduce the transverse property and use it to construct a VH-complex with two predetermined splittings. This will be the first step in bootstrapping the construction of the extended core.

Definition 4.0.1 (Transverse). We say that two locally finite G-trees X and Y are *transverse* if they are not in the same deformation space and there exist two vertex stabilizers, one for each tree, such that their intersection has Property FP.

Remark 4.0.2. The definition of transverse does not depend on the vertices chosen and remains unchanged up to deformation. From the definition of transverse one gets that $G_x \cap G_y$ is FP for a specific x and y. Consider $G_{x'}$. Local finiteness of X implies that $G_{x'}$ and G_x are commensurable so they share a finite index subgroup. Taking the subgroup diagram and intersecting everything with G_y and checking the inclusions are still of finite index gives that $G_{x'} \cap G_y$ is commensurable with $G_x \cap G_y$ and so also must be FP. A similar argument works for vertices y' in the second tree. An elementary deformation either pulls a subgroup out into a new vertex group or pushes one back in. In either case, the tree remains transverse to the same trees.

Given two tree actions one can take their product and use the diagonal action to obtain a new object that can fail to be cocompact. Guirardel's theorem below gives a way to find a cocompact subset in this situation. It's worth noting that the problem of finding a compact subset that carries the fundamental group of a space isn't always solvable. For a graph with a finitely generated fundamental group one can find a compact core but Wise was able to find a 2-complex in [Wis02] where no compact subcomplex carried the fundamental group.

Theorem 4.0.3 (Guirardel Core Theorem). *Let T_1, T_2 be two minimal actions of G on $\mathbb{R}$-trees having non-homothetic length functions, or being irreducible (see 2.1.2). Assume that T_1 and T_2 are not the refinement of a common simplicial non-trivial action. Then there exists a subset $C \subseteq T_1 \times T_2$ which is the smallest non-empty closed invariant connected subset of $T_1 \times T_2$ having convex fibers. Moreover, C is CAT(0) for the induced path-metric, and $T_1 \times T_2$ equivariantly deformation retracts to C. We call C the core of $T_1 \times T_2$.*

Remark 4.0.4. As Guirardel explains after Theorem 8.1 in [Gui05], if T_1 and T_2 are simplicial trees then the core from 4.0.3 is cocompact.

Definition 4.0.5 (Refinement). For G-trees T_1 and T_2 we say that T_1 *is a refinement of* T_2 if there is an equivariant collapse map from T_1 to T_2.

Lemma 4.0.6 (Not refinements of a common tree). *Let X and Y be two locally finite G-trees that lie in different deformation spaces. Then X and Y are not refinements of a common non-trivial simplicial G-tree.*

Proof. Suppose the two trees were refinements of a common non-trivial simplicial G-tree. This would mean there is an edge in the common tree that has an edge above it in both trees. Let K be the stabilizer of this edge. It appears in all three trees. Since X and Y are locally finite, the vertex groups of X are commensurable to each other, similarly for Y. But the property of fixing a point is invariant under commensurability. Therefore, all vertex groups of the first tree are elliptic in the second tree and vice versa. Hence, both actions have the same elliptic subgroups which means they are in the same deformation space which contradicts our initial assumptions. □

Lemma 4.0.7 (Bass finitely generated conditions). *Consider a group G acting on a tree X*

1. *If X/G is finite and G_x is finitely generated for all $x \in X$ then G is finitely generated*

2. *If G is finitely generated and acts minimally on X then X/G is finite*

Proof. For details see Proposition 7.9 in [Bas93]. □

The following should be thought of as a prequel to the main theorem 1.0.1. In one of the implications below we take two trees, impose a condition on its vertex stabilizers, apply the Guirardel Core machinery 4.0.3, and finally produce a compact 2-dimensional complex with a prescribed fundamental group. In the proof of the sequel, we will take three trees, impose a similar condition on vertex stabilizers, carefully arrange that a key lemma of Guirardel 5.0.6 applies, and finally produce a compact 3-dimensional complex.

Theorem 4.0.8 (Transverse Construction). *Let G be a group of cohomological dimension 2. If X and Y are non-trivial minimal G-trees of finite type that are in different deformation spaces then the following are equivalent:*

1. *X and Y are transverse;*

2. *$G_x \cap G_y = \{1\}$ for all vertices $x \in V(X), y \in V(Y)$;*

3. *There exists a compact VH-complex K with $\pi_1(K) \cong G$ whose horizontal and vertical splittings are X and Y.*

Proof. $1 \Rightarrow 2$: Fix $x_0 \in V(X)$. Let $y \in V(Y)$. Then $G_{x_0} \cap G_y = (G_{x_0})_y$. By (1) X is transverse to Y hence $G_{x_0} \cap G_y$ is FP. Since the choice of $y \in V(Y)$ was arbitrary, the vertex groups of the G_{x_0} action on Y are FP. Note, Y locally finite implies its edge groups are finite index subgroups of its vertex groups. Hence the edge groups are also FP.

We claim that the action of G_{x_0} on Y is non-trivial. We will apply Bieri 6.7.3 twice. The third equality below comes from applying Bieri to the action of G_{x_0} on Y restricted to the minimal subtree. To know this action is cocompact we use 4.0.7.

$$2 = dG$$
$$= dG_{x_0} + 1$$
$$= d(G_{x_0})_y + 1 + 1$$
$$= d(G_{x_0} \cap G_y) + 2$$

The equation shows that $d(G_{x_0} \cap G_y) = 0$ so $G_{x_0} \cap G_y$ is trivial.

Claim. The action of G_{x_0} on Y is non-trivial.

Proof. Suppose the action were trivial. That is, there exists some $y \in V(Y)$ such that $(G_{x_0})_y = G_{x_0}$. Hence, G_{x_0} is elliptic for the action of G on Y. By the local finiteness of Y, for all $x \in V(X)$, G_x acts elliptically on Y. Hence, $\mathcal{E}(X) \subset \mathcal{E}(Y)$. Again by local finiteness we can promote this using 3.0.9 to $\mathcal{E}(X) = \mathcal{E}(Y)$ which by Theorem 2.1.5 gives $X \sim Y$ contradicting the fact that X and Y were assumed to be in different deformation spaces. $\square$

$2 \Rightarrow 1$: Trivial groups are FP.

$2 \Rightarrow 3$: Take $X \times Y$ and give it the VH-structure where X and Y correspond to horizontal and vertical edges respectively. We first check a few conditions.

First note that X and Y are minimal G-trees by assumption.

If our trees had homothetic length functions (i.e. the length functions were a constant multiple of each other) then they would vanish on the same elements, which would imply they had the same elliptic elements. Our trees are cocompact with FP vertex groups, and FP implies finitely generated so we can apply 2.1.5 of [For02] which says in this case having the same elliptic elements is enough to conclude that the elliptic subgroups are also the same. Hence, both trees lie in the same deformation space, a contradiction. Therefore, X and Y have non-homothetic length functions. By 4.0.6 our trees are not refinements of a common non-trivial simplicial G-tree.

Apply the Guirardel Core Theorem 4.0.3 to obtain C a certain subset of $X \times Y$ that we call the core. The core C has convex fibers. It's also CAT(0). Our G-trees are simplicial so C is a subcomplex. Condition (2) says that G acts freely on the vertices of the product $X \times Y$ and therefore also on the the vertices of the core, a subset of $X \times Y$. Since the G-trees are simplicial and the product action is free on vertices, and the VH property rules out rotating a square by 90 degrees, and the product action prevents rotations by 180 degrees (a rotation by 180 degrees

would invert an edge in the projection) we get that the product action on the cell complex C will be a covering space action.

We also need that C/G is VH. Is it enough to observe that the product action respects the tree factors. The edge partition on the cover C descends to a well-defined edge partition on the quotient and attaching maps constructed in the standard way for the quotient alternate between vertical and horizontal edges as needed.

From Guirardel, C is CAT(0) and therefore NPC. We also get compactness from 4.0.4. However, NPC is a local condition and under a covering map it descends to C/G. Following Wise in [Wis96] the VH-complex C/G has a decomposition into vertex and edge spaces. The NPC condition ensures the attaching maps are π_1-injective, hence C/G is a graph of spaces with horizontal and vertical splittings.

Because $C \subseteq X \times Y$ the leaves of C coming from the vertical foliation as a square complex are equal to the connected components of the fibers from projecting C to the X coordinate, however by Guirardel the latter are connected. Collapsing leaves then is the same as collapsing connected fibers which gives projection to the X factor. On the other hand, as C is a cover of C/G we get that C is also a graph of spaces. In this case, the edge and vertex spaces of C correspond to certain fibers from the foliation. Therefore, the Bass-Serre map from C given by collapsing vertex spaces to a point and mapping edge spaces to edges is the restriction of the projection map. With the product action projection is G-invariant and X is minimal so the image of the Bass-Serre map is all of X. Hence, the G-tree X matches the Bass-Serre tree for the horizontal splitting of C/G where we collapse the vertical fibers. In a similar way, Y is the Bass-Serre tree of the vertical splitting of C/G.

$3 \Rightarrow 2$:

We have $\widetilde{K}$ and its vertical and horizontal foliations. Collapsing leaves to points gives G-maps $\widetilde{K}$ to X and $\widetilde{K}$ to Y. These are the Bass-Serre maps. The product map gives a map from $\widetilde{K}$ to $X \times Y$.

Claim. This map is 1-1.

Proof. It's enough to be 1-1 on vertices. Suppose v, v' go to the same vertex of $X \times Y$. This means v, v' are on the same leaf of the vertical foliation, and are also on the same leaf of the horizontal foliation. Call these leaves V_v and H_v. By Lemma 3.7 of Wise in [Wis07], the intersection of V_v and H_v is at most one point. (Remark: the lemma says exactly one point, for $\widetilde{K}$ a Complete Square Complex (CSC). But for "at most one point", it only needs that $\widetilde{K}$ is VH and CAT(0), not complete.) Hence $v = v'$. □

Both $\widetilde{K}$ and $X \times Y$ are CAT(0) square complexes, and we regard $\widetilde{K}$ as a subcomplex that is invariant under the product G-action on $X \times Y$. The action on $\widetilde{K}$ is free because it is a covering space action.

By Cor 5.2 of Haglund in [Hag07] if G acts on a CAT(0) cube complex X without inversions then every element either has a fixed vertex or a combinatorial axis, and not both.

Now take any non-trivial $g \in G$. Acting on $\widetilde{K}$, it has a combinatorial axis L, because the action is free. But L is also a combinatorial axis in $X \times Y$. By the lemma, g acting on $X \times Y$ cannot fix a vertex. So, no non-trivial element fixes a vertex of $X \times Y$. Hence $G_x \cap G_y = 1$ for all vertices $x \in X$, $y \in Y$.

□

Chapter 5

Notions of convexity in products of trees

This section develops the definitions for fibers, filling, and quadrant convexity along with the tools needed to apply them during the construction of the extended core.

Definition 5.0.1 (Quadrant Convex). A subset $S \subset T_1 \times T_2$ is *quadrant convex* if the complement is the union of quadrants. The *quadrant hull* of a set S is the smallest quadrant convex subset containing S i.e. it's the intersection of all quadrant convex subsets containing S. Equivalently, it's the complement of the union of all quadrants disjoint from S.

The following proposition is the first important application of quadrant convexity. Guirardel uses it to prove that his core is simply connected and later we will use it to show that the hyperplanes in our extended core are simply connected as a step towards showing the entire space is also simply connected.

Proposition 5.0.2 (Quadrant Convex implies Deformation Retraction). *Let T_1 and T_2 be simplicial trees, and S a connected quadrant convex subset of $T_1 \times T_2$, then there is a deformation retraction from $p_1(S) \times p_2(S)$ onto S.*

Proof. This follows from the proof of proposition 4.17 from Guirardel [Gui05]. □

Remark 5.0.3. Later we will find subsets S where $p_i(S)$ is a tree for $i \in \{1, 2\}$ and we will conclude S is simply connected.

Definition 5.0.4 (Fibers in trees). A *fiber* of a product of trees is the inverse image of a point under a projection map; e.g. $\pi_1^{-1}(x) = \{x\} \times Y$ is a fiber of $X \times Y$. A *one-dimensional fiber* of a product of three trees is the inverse image of a point under a map $\pi_{jk} : T_1 \times T_2 \times T_3 \to T_j \times T_k$ given by $p = (p_1, p_2, p_3) \mapsto (p_j, p_k)$. We use similar terminology for subsets of tree products, that is a one-dimensional fiber of a subset of a product of trees is the intersection of that subset with a one-dimensional fiber of the product.

Next, we give an equivalence between connected fibers and quadrant convexity. The forward direction gives the second important application of quadrant convexity. During the construction of the extended core we will show a set has connected fibers, is quadrant convex, and simply connected with each proving the next in turn.

Lemma 5.0.5 (Guirardel Lemma 5.4, Corollary 5.5). *Let T_1, T_2 be two $\mathbb{R}$-trees and let F be a nonempty connected subset of $T_1 \times T_2$ with convex fibers. Then the complement of $\overline{F}$ is a union of quadrants. That is, $\overline{F}$ is also nonempty, connected, and has convex fibers.*

Theorem 5.0.6 (Equivalent Quadrant Convex Condition). *Let X and Y be two simplicial trees and $F \subset X \times Y$ a closed subset. Then F is quadrant convex if and only if all of its fibers are connected.*

Proof. Follows immediately from 5.0.5 and the fact that quadrant convex implies connected fibers. $\square$

Keeping the above theorem in mind, we generalize the notion of quadrant convexity to three dimensions.

Definition 5.0.7 (Orthant Convexity). A subset $S \subseteq T_1 \times T_2 \times T_3$ is *orthant convex* if S has convex one-dimensional fibers.

5.1 Building the orthant hull

In this section we work by analogy with Guirardel's results on connected fibers and quadrant hulls. To create a quadrant hull one can either work subtractively by taking away quadrants or additively by connecting fibers with new material. We will work additively one dimension up and build an object by filling in fibers.

Proposition 5.1.1 (Slices are connected). *Put $f = f_1 \times f_2 \times f_3 : X \to T_1 \times T_2 \times T_3$ and $J = Im(f)$. Then $J \cap T_1 \times T_2 \times \{z\} = Im_f(f_3^{-1}(z))$.*

Proof. Let $p = (p_1, p_2, p_3) \in T_1 \times T_2 \times T_3$. Then we have the following,

$$
\begin{aligned}
p \in \text{ LHS} &\iff p \in Im(f) \wedge p_3 = z \\
&\iff \exists x \in X(f(x) = p \wedge f_3(x) = z) \\
&\iff \exists x \in X(f(x) = p \wedge x \in f_3^{-1}(z)) \\
&\iff p \in Im_f(f_3^{-1}(z)).
\end{aligned}
$$

$\square$

The following definition serves only to give a name to an object created by Guirardel during the proof of the extension lemma 6.3.2.

Definition 5.1.2 (Subgraph Coning). Let X be a 2-complex with a G-action. Let T be a simplicial G-tree. Let $f : X \to T$ be a map. Let K be a subgraph of the 1-skeleton of X. Then the *subgraph cone of f with K* denoted $\Lambda(f, K)$ is the set

$$
X \sqcup (G \times C_K)
$$

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modulo the relation sending $(g, (x, 0))$ to $g \cdot x$ where C_K is the set

$$K \times [0, 1]$$

modulo the relation that glues $(x, 1)$ to $(x', 1)$ if and only if $f(x) = x'$.

Definition 5.1.3 (Filling). Given $S \subseteq X := X_1 \times X_2 \times X_3$ define S_k for $k \in \{1, 2, 3\}$ via:

$$p \in S_k \iff \exists\, q, r \in S\, \forall j \in \{1, 2, 3\} : j \neq k \implies (p_j = q_j = r_j \text{ and } p_k \in \mathrm{cvxhull}_k(\{q_k, r_k\}))$$

where $\mathrm{cvxhull}_k$ is the convex hull operation on subsets of X_k.

For us, when we apply the above definition, the factor spaces will be trees and therefore the filling operation that produces, say S_3 from $S \subset X_1 \times X_2 \times X_3$ is ensuring the one-dimensional fibers of $\pi_{12} : S_3 \to X_1 \times X_2$ are connected. In the case of a tree, fibers are connected by drawing unique geodesics between points and connectedness and convexity are equivalent properties.

Definition 5.1.4 (Connected in coordinate planes (CCP)). Let $S \subseteq X_1 \times X_2 \times X_3$. Then S is *connected in all coordinate planes (CCP)* if $S \cap \pi_k^{-1}(p)$ path is connected for all $p \in X_k$ for all $k \in \{1, 2, 3\}$.

Lemma 5.1.5 (Coning Connected Fibers). *Let $f : K \to T$ be a map from a 2-complex K to a simplicial tree T with connected fibers. Let Γ be a connected subgraph of $K^{(1)}$, the 1-skeleton of K. Pick $t_0 \in f(\Gamma)$. Define $F : \Gamma \times I \to T$ by,*

$$F(x, s) = \gamma_{f(x), t_0}(s)$$

where $\gamma_{x,y}$ is the embedded path between $x, y \in T$ with domain $[0, 1]$ of constant speed or a constant map if $x = y$. Then the map F is continuous and points have connected preimages.

Proof. Let (x, s) be an arbitrary point in $\Gamma \times I$. Put $t := F(x, s)$. We will show there is a path

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in $\Gamma \times I$ from (x, s) to some $(x', 0) \in \Gamma \times I$ that stays inside of the point preimage $F^{-1}(t)$. This is enough because f already has connected preimages in K and $F(x, 0) = f(x)$.

If $s = 0$ then the constant path at (x, s) suffices.

If $s = 1$ then $t = F(x, s) = F(x, 1) = \gamma_{f(x), t_0}(1) = t_0$. We can take a path that runs along the top of $\Gamma \times I$ and then goes down using a vertical fiber. Since $t_0 \in f(\Gamma)$ there exists some $z \in \Gamma$ such that $F(z, 0) = t_0 = t$. Take g to be the concatenation of a path from $(x, 1)$ to $(z, 1)$ contained in $\Gamma \times \{1\}$ and the path $(z, 1)$ to $(z, 0)$ given by $t \to (z, 1 - t)$. When restricted to a vertical fiber, F is either injective or a constant map. Since $F(z, 0) = F(z, 1) = t_0 = t$ we have the image of our path lies in $F^{-1}(t)$ as desired.

Suppose $0 < s < 1$. Consider the image of $x \times I$, a fiber of the product $\Gamma \times I$, under F. Denote the map $\gamma_{f(x), t_0}$ by γ. From the definition of γ, the fiber $x \times I$ is either sent to a single point or goes injectively to T. In the former, we take g to be the straight path from (x, s) to $(x, 0)$.

Finally, suppose the fiber $x \times I$ goes injectively to T. In this case, $F(x \times I) = (\gamma)$, recall γ is a reparameterized geodesic between two points in (f) which by the continuity of f is both connected and convex in T. Hence, $t \in (f)$ by convexity. This means $f^{-1}(t)$ is a non-empty closed subset of K. Let $x' \in f^{-1}(t)$ be a point that minimizes $d_K(x, x')$. The distance cannot be zero because our fiber goes injectively. Hence, the distance is positive, so $x \neq x'$.

Take η to be an geodesic path in K from x to x'. The image of η is disjoint from $f^{-1}(t)$.

Hence for $r \in [0, 1]$, the geodesic from $f\eta(r)$ to t_0 contains t. In fact, $d_T(f(k), t) + d_T(t, t_0) = d_T(f(k), t_0)$ for $k \in (\eta)$. Define a function $g : (\eta) \to I$ by $g(k) = d_T(f(k), t)/d_T(f(k), t_0)$. The function g is continuous and so the graph of g is connected, contained in $F^{-1}(t)$, and contains (x, s) and $(x', 0)$. That is, the path $\eta(r) \times g(\eta(r))$ connects our preimage point to a connected fiber below. $\square$

Lemma 5.1.6 (Reduction to vertical subpath). *Suppose $S \subseteq T_1 \times T_2 \times T_3$ is a subcomplex with CCP. Let $p, q, r \in S$ satisfy*

1. $r \notin S$

2. $p, q \in S$

3. $p_2 = q_2 = r_2$ and $p_3 = q_3 = r_3$

4. $r_1 \in cvx_{T_1}(\{p_1, q_1\})$

then there is a path $\sigma : [0, 1] \to S$ between p and q such that $\sigma(t)$ is contained in $S \cap (T_1 \times \{r_2\} \times \delta)$ where δ is an open direction in T_3 at r_3 provided $t \neq 0, 1$.

Proof. Property CCP implies that $S \cap (T_1 \times \{r_2\} \times T_3)$ is path connected. Let σ be a path in that set from p to q. Consider the pre-image of $T_1 \times r_2 \times r_3$ by σ, call it K. Note that the complement of K is a countable disjoint union of open intervals in $[0, 1]$ – we will choose one later. Each
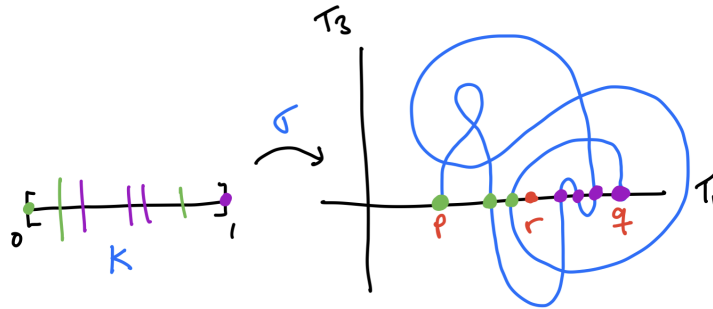

Figure 5.1: Possible configuration when T_1 and T_3 are both $\mathbb{R}$. Note, interval length on the LHS may not correspond to path length on the RHS. The curve may not even be rectifiable. In general, the set on the LHS may not be discrete as drawn; it may include limit points or closed intervals.

open interval is connected so considering projection and the fact that r_3 is separating in T_3 we have that under σ each open interval is mapped so the third coordinate lies in a single direction of T_3 at r_3. After identifying, K maps into T_1. Color the points of K by which direction at r_1 in T_1 they map into. (See figure 5.1 and 5.3.) Here we use the fact that σ is a path that is disjoint from r. In fact, because S is a subcomplex it is closed and so there is an open neighborhood of r that is disjoint from S and therefore also σ. (As in figure 5.2.) Intersecting this neighborhood with $T_1 \times r_2 \times r_3$ gives an open neighborhood in T_1 that is disjoint from σ. The upside is that
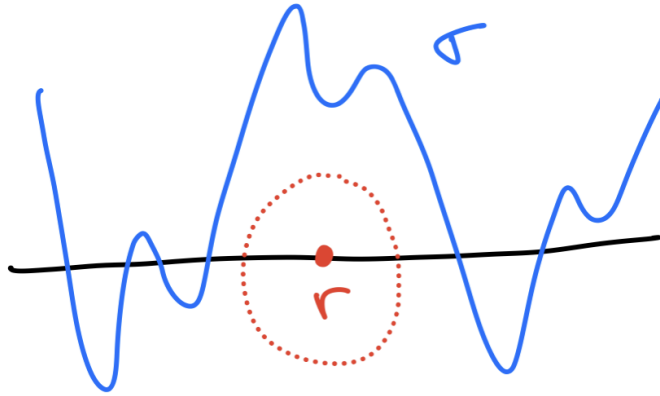
Figure 5.2: The path σ avoids a neighborhood of r.
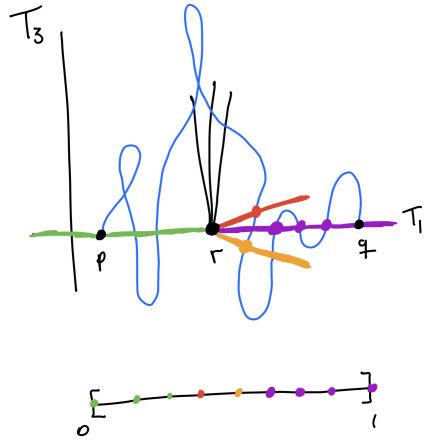
Figure 5.3: Sketch of a possible configuration. This generalizes the diagram from figure 5.1.

each monocolored subset of K is closed by looking at the image of σ in the slice and taking intersections with a closed halfspace pointing away from r_1. Take the smallest pairwise distance between the finite number of colored closed sets. This distance is non-zero because σ is disjoint from a neighborhood of r_1 in T_1. Consider two points in K that achieve that distance. There cannot be any points of K between them because we chose the smallest distance. This picks out an interval with endpoints that map to different directions as needed. □

Lemma 5.1.7 (Switching). *Let $S \subseteq T_1 \times T_2 \times T_3$ be a subcomplex that is connected in all*

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coordinate planes 5.1.4. Then $S_x, S_y,$ and S_z are as well.

Figure 5.4: The goal is to show the vertical green interval contained in the xy-plane between r and r' lying on σ is in S_x. This connects points in yz-planes to S.

Proof. Without loss of generality, consider S_x, note that S_x will be connected in all xy and xz planes because S was. Consider the $yz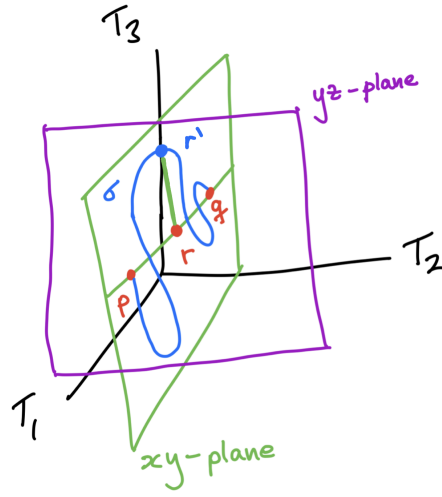$-planes in S_x, if there were no new points added then the planes are connected and we are done. Suppose that $r \in (S_x \smallsetminus S)$, we need to connect r to a point in S. We will show that there is a path in $S_x \cap \pi_2^{-1}(r_2)$ between $r \notin S$ and some point $r' \in S$ as in 5.4.

Since r is in $S_x \smallsetminus S$ there exist distinct points p and q in S that agree in all coordinates except the first where we have that $r_1 \in \text{cvxhull}_{T_1}(\{p_1, q_1\})$. Now, because S is connected in all coordinate planes there is a path σ from p to q that lies in $S \cap \pi_2^{-1}(r_2)$. In fact, we can take σ to be a path that begins at p and ends at q with T_3 coordinates lying in exactly one closed direction of T_3 at r_3. We have factored out this situation into claim 5.1.6.

Take σ as in the claim 5.1.6. Consider $D = r_1 \times r_2 \times \bar{\delta}$ a closed set. Let A be the set of points where σ crosses D. Note, A does not include the root of δ. This is a closed set so we can consider the preimage under σ and look at the complement in $[0, 1]$. Color each interval by the direction in T_1 at r_1 that $\pi_1\sigma$ takes it to. Now, identifying D with a closed direction at r_3 in T_3

we say that the colors at $x \in D$ are the set of colors of intervals that the preimage of x under $\pi_3\sigma$ hits. (See figure 5.5.)
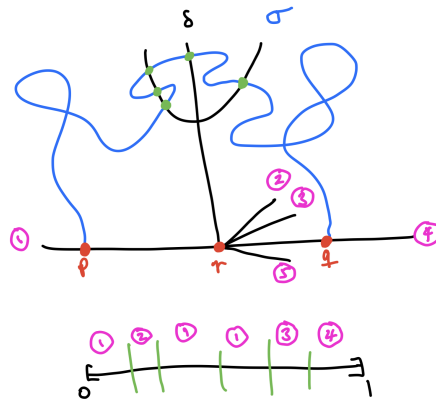
Figure 5.5: A possible configuration
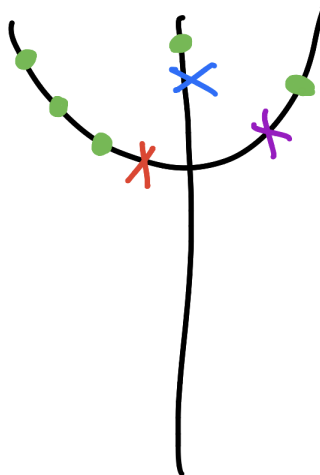
Figure 5.6: A finite number of monocolored points separating the root from A.

Our goal is to find a geodesic from the root of D to a point in A that is multicolored. (See figures 5.6 and 5.7.) Note, because points in trees are separating, if the image of a continuous map contains two points it also contains the geodesic between those points. Suppose the claim were false. Then between every point of A and the root there is a point that is monocolored. A finite number of these points suffices to separate all of A from the root. Consider the first interval, it must cross one of these points. There must be another interval that eventually leaves that

Figure 5.7: Continuing from 5.6. Note, the left branch is not multicolored. A particular multicolored path is highlighted.

half space and enters a halfspace not entered yet. Because the geodesics have one color, and we always cross a monocolored point upon leaving, each segment has the same color. Including the last segment that goes back to the root. But this is impossible because we assumed the path begins and ends in different directions in T_1 at r_1.

$\square$

Let T_1 and T_2 be trees with $x_2 \in T_2$. Then

$$T_1 \times x_2 = \bigcup_{\delta \in D(x_1)} \bar{\delta} \times \{x_2\}.$$

Proposition 5.1.8.

$$(T_1 \times T_2) \smallsetminus (T_1 \times x_2) = \bigcup_{\delta \in D(x_2)} T_1 \times \delta$$

Lemma 5.1.9 (Slice Switching). *If $R \subseteq T_1 \times T_2$ is connected then $(R_x)_y = (R_y)_x$ where x corresponds to T_1 and y corresponds to T_2 in the definition of filling 5.1.3.*

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Proof. We first show that $(R_x)_y$ has connected 1-dimensional fibers. The set $(R_x)_y$ has connected x-fibers, this follows from the definition of filling in the y-direction. That is, $(R_x)_y \cap \{x_0\} \times T_2$ is connected for all $\{x_0\}$ in T_1. It remains to show that $(R_x)_y$ has connected y-fibers.

For sake of a contradiction, suppose $(R_x)_y$ had a disconnected y-fiber at y_0, denoted F. That is, there exists y_0 such that $F = (R_x)_y \cap T_1 \times \{y_0\}$ is disconnected. Because we are working in a tree we can find a point $x_0 \in T_1$ such that (x_0, y_0) separates F. Using 5.1 we proceed by cases on how many directions meet R_x. Specifically, subtract (x_0, y_0) from both sides of 5.1 to obtain a disjoint union of sets and ask how many intersect $F \cap R_x$.

1. zero: Suppose $F \cap R_x \cap (\delta \times y_0) = \varnothing$ for all $\delta \in D(x_0)$ then by 5.1, $R_x \cap F = \varnothing$. However, the fiber is still disconnected so must intersect some of $(R_x)_y$. This means there exists $p \in (F \cap (R_x)_y) \setminus R_x$. That is, p was obtained by filling in the vertical (y, T_2) direction. There exist $\eta_1, \eta_2 \in D(y_0)$ such that $F \cap R_x \cap (p_1 \times \eta_k) \neq \varnothing$ for $k = 1, 2$. By 5.1.8 $F \cap R_x \cap (p_1 \times \eta_1) \subset T_1 \times \eta_1$ and $F \cap R_x \cap (p_1 \times \eta_2) \subset T_1 \times \eta_2$ are disjoint. Pick out points in each, since they are in R_x they are either already in R or were obtained by filling in the x-direction. Hence, $F \cap R_x \cap (T_1 \times \eta_1)$ and $F \cap R_x \cap (T_1 \times \eta_2)$ both intersect R. However, these are disjoint and R is connected a contradiction.

2. at least two: Suppose $F \cap R_x \cap (\delta_k \times y_0) \neq \varnothing$ for $k = 1, 2$ with $\delta_k \in D(x_0)$. Then because R_x has connected y-fibers we have that $(x_0, y_0) \in R_x \subset (R_x)_y$ which is a contradiction since (x_0, y_0) separates F.

3. exactly one: Suppose $F \cap R_x \cap (\delta_0 \times y_0) \neq \varnothing$ but is $\varnothing$ for all $\delta_k \in D(x_0)$ where $k \neq 0$. (See figure 5.8 for a diagram of this case) Since $\delta_0 \times y_0$ intersects R_x it also intersects R. Consider the sets $(x_0 \times \eta)$ for $\eta \in D(y_0)$, at most one intersects R_x. If more than one did then because $(R_x)_y$ has connected x-fibers we would have $(x_0, y_0) \in (R_x)_y$ a contradiction since (x_0, y_0) separates F. Because F is disconnected, there exists $\delta_1 \in D(x_0)$ with $\delta_1 \neq d_0$ and $(\delta_1 \times y_0) \cap F \neq \varnothing$. Combined with our assumption, pick a point $p \in (\delta_1 \times y_0) \cap F \setminus R_x$. Now, $p \in (R_x)_y \setminus R_x$ so there exist directions $\eta_1, \eta_2 \in D(y_0)$ such that $(p_1 \times \eta_i) \cap R_x \neq \varnothing$

for $i = 1, 2$. From above, take η_1 to be one of the directions such that $(x_0 \times \eta_1) \cap R_x = \varnothing$. Suppose $q \in (p_1 \times \eta_1) \cap R_x$. If $q \in R$ take $x' = q_1$. If $q \notin R$ then we use the fact that $q \in R_x \setminus R$ to find a point in R. We can take $x' \in \omega$ where ω denotes a direction in $D(p_1) = D(q_1)$ that does not contain δ_0 with the property that $(x', q_2) \in R$. Then $(x', q_2) \in \delta_1 \times \eta_1$ and $(x', q_2) \in R$, however the boundary of the quadrant $(\delta_1 \times y_0) \cup (x_0 \times \eta_1)$ does not intersect R_x and so separates $(x', q_2) \in R$ from a point in $R \cap (\delta_0 \times y_0)$. This is a contradiction since R is connected.

Hence, the set $(R_x)_y$ has connected 1-dimensional fibers. By Guirardel 5.0.5 then the complement of $(R_x)_y$ is a union of quadrants and so $QH(R) \subset (R_x)_y \subset QH((R_x)_y)$. Next note that $QH((R_y)_x) = QH(R_y) = QH(R) = QH(R_x) = QH((R_x)_y)$. Therefore, $(R_x)_y = QH(R) = (R_y)_x$.
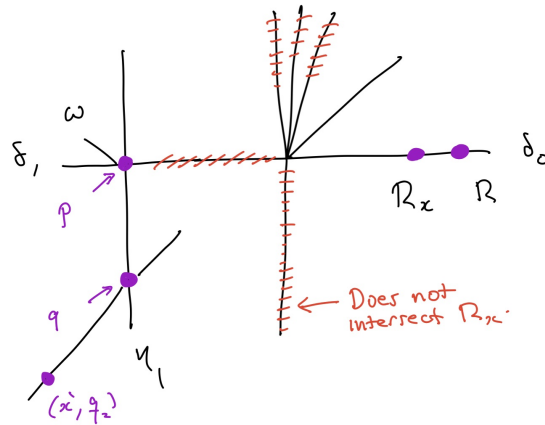

Figure 5.8: Illustrating case 3 of Lemma 5.1.9

□

5.2 Cocompactness of orthant hulls

Lemma 5.2.1 (Filling preserves cocompactness). *Let T_1, T_2, T_3 be simplicial G-trees that are pairwise transverse and $S \subset T_1 \times T_2 \times T_3$ an invariant cocompact subcomplex. Then S_x is also*

cocompact.

Proof. Observe that $S \subseteq T_1 \times B$ where $B = \pi_{23}(S)$. Under the product action B is an invariant set. Projection is a continuous map so B is cocompact. Next, consider a one-dimensional fiber of S above a point, that is $S' = S \cap (T_1 \times \{b\})$ where $b \in B$ is a vertex. Because T_2 and T_3 are transverse the stabilizer of b is trivial. Hence, the quotient map on the one-dimensional fiber S' is an embedding into a compact set S/G. Hence, S' is also compact. For each vertex $b \in B$ consider the one-dimensional fiber $S \cap (T_1 \times \{b\})$, the action in T_1 is by simplicial automorphisms so distances are preserved. Hence, every one-dimensional fiber in an orbit has the same finite diameter. Because B is cocompact there are a finite number of vertex orbits and therefore a universal bound on the diameters of one-dimensional fibers above vertices. After identifying a one-dimensional fiber with T_1 we see that each vertex of S_x is obtained by filling in the convex hull of some one-dimensional fiber; more precisely, if $v \in S_x$ then for some $b \in B$ a vertex, $v_1 \in \text{cvx}(\pi_1(S \cap (T_1 \times \{b\})))$ where v_1 is the first coordinate of v. In particular, S_x is contained within a bounded neighborhood of S with the product metric. Since S was a subcomplex, so is S_x. Hence, S_x is cocompact. $\square$

Chapter 6

Proof of main theorem

Starting with the hypotheses from the main theorem, this section breaks the proof into pieces.

6.1 Setup

Suppose for sake of a contradiction that there were three non-trivial G-trees T_1, T_2, and T_3 of finite type 2.1.9 that are pairwise transverse 4.0.1 and no two are in the same deformation space. Transversality and finite quotients are preserved by deformations. Therefore, by 2.1.4, without loss of generality we may assume that these are minimal G-trees after performing a sequence of elementary collapses.

6.2 Construct square complex with two splittings and given fundamental group

Applying the transverse construction lemma 4.0.8 we obtain X_{12}. For now think of X_{12} as a square complex that encodes the two G-trees T_1 and T_2 with fundamental group G.

6.3 Construct CCP set

So far we have only used T_1 and T_2 to define X_{12}. We need to include our third action.

Lemma 6.3.1 (Affine Equivariant Map). *Suppose that G acts freely on a simplicial complex K and acts on a simplicial tree T. Then there exists an equivariant map $f : K \to T$ where*

the connected components of the fibers of f are the leaves of a measured foliation and f is an isometry on edges transverse to F.

Proof. Construct an equivariant map.

We start by defining f on $K^{(0)}$ the 0-skeleton. It is enough to define the map on a single vertex in each vertex orbit, and extend equivariantly. These choices can be arbitrary. Next we check that the resulting map is well-defined.

Indeed, if $gv = hv$ then $g^{-1}h = 1$ by freeness and so

$$f(gv) = gf(v) = g(g^{-1}h)f(v) = hf(v) = f(hv).$$

Next we define the map on the 1-skeleton by mapping each edge. If vw is an edge, map it to the unique geodesic $[f(v), f(w)]$ in T.

Lastly, for 2-cells we use the standard fibration coming from mapping triangles to tripods. There are four cases based on where the 3 vertices land in T. See figure 6.1.
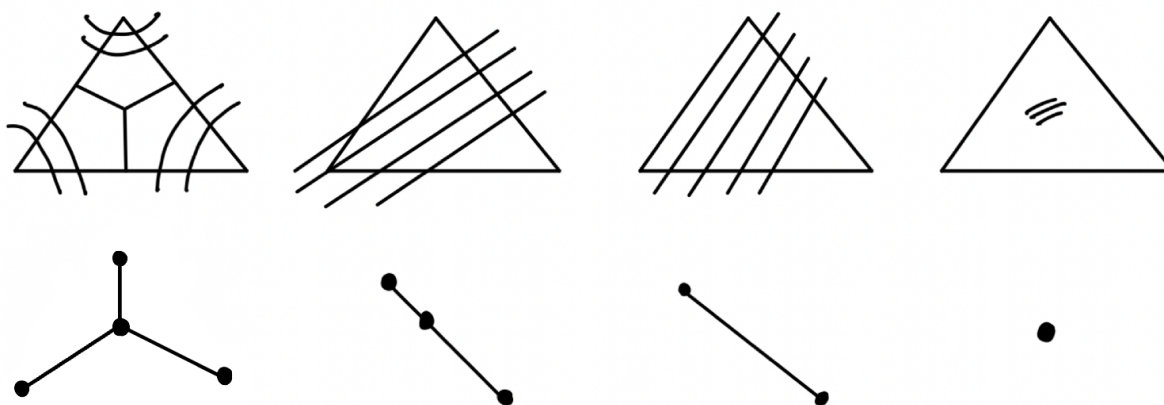

Figure 6.1: Possible fibrations for each simplex

To obtain a transverse measure, take a small arc α that is transverse to the leaves of the foliation. The length of α is defined to be the length of $f \circ \alpha$ in the tree. Hence, by construction f is an isometry on edges transverse to F. □

Let $\widetilde{X_{12}}$ denote the universal cover of X_{12}. Applying the affine equivariant map construction 6.3.1 to the actions on $\widetilde{X_{12}}$ and T_3 gives an equivariant map and a foliation that we denote by f_{123} and F_{123} respectively.

The content of the next lemma is that a map from a 2-complex that is constant on leaves can be extended to a map with connected leaves by enlarging the 2-complex.

Lemma 6.3.2. *(Guirardel's Extension Lemma) Consider a geometric action of a finitely generated group G on an $\mathbb{R}$-tree T, and let X be a 2-complex endowed with a free properly discontinuous cocompact action of G. Let F be a G-invariant measured foliation on X. Consider a map $f : X \rightarrow T$ which is constant on leaves of F, and isometric in restriction to transverse edges of X. Then there exists a 2-complex X' containing X, endowed with a free properly discontinuous cocompact action of G, a measured foliation F' extending F, and which induces an isometry between X'/F' and T. Moreover, the inclusion $X \subseteq X'$ induces an epimorphism of fundamental groups.*

Next, we check that the extension lemma can be used within our setting.

Lemma 6.3.3 (Technical assumptions for Guirardel). *The following properties hold for $\widetilde{X_{12}}$ and the map $f_{123} : \widetilde{X_{12}} \rightarrow T_3$ constructed via 6.3.1. The action of G on T_3 is geometric. The action of G on X_{12} is free, properly discontinuous, and cocompact. (Compare to 6.3.2)*

Proof. Our trees are simplicial and the actions are minimal, hence by 2.1.11 they are also geometric. The action on X_{12} is a covering space action from 4.0.8 and is therefore free, properly discontinuous, and cocompact. $\qquad\square$

With 6.3.3 we now have everything needed to apply the Guirardel Extension Lemma 6.3.2 to $\widetilde{X_{12}}$, f_{123} and F_{123} in order to obtain $\widetilde{X_{12}}^+$ and f_{123}^+ and F_{123}^+ respectively. Having connected the fibers for f_{123} we continue with work on the other functions.

Next, let f_{121} and f_{122} denote the Bass-Serre 2.1.8 maps from $\widetilde{X_{12}}$ to T_1 and T_2 respectively.

Remark 6.3.4. The following is an informal sketch of what the Bass-Serre maps f_{122} and f_{121} look like. In our case $\widetilde{X_{12}}$ is a VH-complex so edge and vertex spaces are graphs. Each square has a vertical and horizontal foliation. When x is a vertex or a point on a horizontal edge, call the subset we get from extending the vertical foliation V_x (recall 2.4.9). Then collapsing all the V_x spaces to a point gives the map to T_1 called f_{121} defined above as the Bass-Serre map.

Our goal is to produce a map from the extended complex $\widetilde{X_{12}}^+$ to $T_1 \times T_2 \times T_3$. The map f_{123}^+ defined on $\widetilde{X_{12}}^+$ is already determined by the extension lemma, so it remains to extend the Bass-Serre maps f_{121} and f_{122} from $\widetilde{X_{12}}$ to $\widetilde{X_{12}}^+$.

By construction, $\widetilde{X_{12}}^+$ has the form $\Lambda(f_{123}, K)$ from 5.1.2 for some choice of graph K.

Remark 6.3.5. The following is an explanation of how the graph K from applying 6.3.2 in Guirardel's paper is used. In short, for G finitely presented Guirardel's proof picks a specific compact subgraph of the 1-skeleton K that is large enough to make $\widetilde{X_{12}}^+$ simply connected. Strictly speaking, we do not need to know the nature of K, only that the extension has some graph coning structure so we can extend the Bass-Serre maps.

Once we know that the extension has this graph coning structure we can extend the Bass-Serre maps to $\widetilde{X_{12}}^+$ by applying 5.1.5. Denote the resulting extensions by $f_{121}^\wedge$ and $f_{122}^\wedge$ and foliations by $F_{121}^\wedge$ and $F_{122}^\wedge$.

Form the product map $f := f_{121}^\wedge \times f_{122}^\wedge \times f_{123}^+$. Finally, define $J := \text{Im}(f) \subset T_1 \times T_2 \times T_3$. Each map is equivariant so G acts on J. Since J is the image of a cocompact set under a continuous G-map it is cocompact.

Finally, by 5.1.1 J has property CCP.

6.4 Take cellular neighborhood

Lemma 6.4.1 (Cell Respects Slices). *If $A \subset T_1 \times T_2 \times T_3$ is CCP and A is path connected then cell(A) is also CCP.*

Proof. Let $z \in T_3$. Pick $p, q \in \text{cell}(A) \cap (T_1 \times T_2 \times \{z\})$. For a product of trees, the closure of a cell is a subcomplex. For all p in $\text{cell}(A)$ there exists a cell P and a point $p' \in A$ such that $p' \in P$ and p is in the topological closure of P. Similarly for q, q', and Q. Since p and p' are contained in a closed n-cube but p in P, we have that $d_{T_3}(p_3, p'_3) < 1$. Throughout, remember that if p'_3 is a vertex, then $p'_3 = z$. Similarly for q'_3. This ensures that later when points are pushed along an edge the result remains within $\text{cell}(A)$. Note, closed cubes project to closed cubes in T_3. Since $p3 = q3 = z$ the projections of $\text{cl}(P)$ and $\text{cl}(Q)$ are closed cubes in T_3 both containing z.

Let η' be a path in A from p' to q'. The goal is to create a modified path η such that $d_{T_3}(\pi_3 \eta(t), z) < 1$. Points in $\pi_3 \eta'$ that are not vertices of T_3 but lie in A can be pushed to a new path η in $\text{cell}(A) \cap (T_1 \times T_2 \times \{z\})$.

Using the CCP property of A, for a given point $r \in T_3$ we can note the first and last time a path enters the slice $T_1 \times T_2 \times \{r\}$ and replace that (possibly degenerate) segment by a path contained in $A \cap T_1 \times T_2 \times \{r\}$. Call this a path snip. (For the degenerate case, concatenate paths instead)

Consider the following cases.

1. $p'_3 = q'_3$: Use the CCP property of A to draw a path η contained in the slice $A \cap (T_1 \times T_2 \times \{z\})$.

2. $p'_3 \neq q'_3$ and $z = p'_3$ or $z = q'_3$: Snip at p'_3 and at q'_3. This limits the path to one edge and prevents it from wandering when z is a vertex of T_3.

3. $p'_3 \neq q'_3$ and $z \in [p'_3, q'_3] \setminus \{p'_3, q'_3\}$: As before, begin by snipping at p'_3 and q'_3. This case includes the situation where z is a vertex of T_3 lying between p'_3 and q'_3. This allows a path to wander arbitrarily far from z in T_3. To prevent this, we also snip at z.

If necessary, use the open cells containing p' and q', to draw paths to p and q. Require the interior of their domains to map to P and Q respectively so their T_3 coordinates remain close to z. Concatenate with η' to create a new path from p to q. Denote this path by τ and form $t \to (\tau_1(t), \tau_2(t), z)$ a path from p to q contained in $\text{cell}(A)$ as τ was contained in A and

remained close to z.

$\square$

Next, define $K := \mathrm{cell}(J)$. As J is G invariant so is K. Because our trees are locally finite and a cellular neighborhood is contained in a bounded neighborhood we have that K is cocompact. Taking a cellular neighborhood respects slices by 6.4.1 so K has property CCP as well.

6.5 Filling

Filling is a way to ensure an object has connected 1-dimensional fibers; at least in the direction of the filling. Each time you fill, material is added that could in principle create disconnected fibers in the other two directions. The worry is that attempting to refill will simply create more disconnected fibers so no amount of repeated fillings will be enough. We will apply the switching lemma 5.1.7 in the context of the extended core construction so this does not happen and we will obtain an object with connected 1-dimensional fibers.

We are now ready to complete the construction of the extended core. Let $E := ((K_x)_y)_z$ be the extended core. Repeatedly apply lemma 5.2.1 and use the fact that filling preserves CCP by construction to see that E is cocompact and has CCP. By the switching lemma 5.1.7 E has connected 1-dimensional fibers.

6.6 Topological properties of the Extended Core

Here we verify that E is a universal cover and that E/G has three honest graph of spaces decompositions.

Definition 6.6.1 (VHD-Complex). We say X is a VHD-complex if it is a 3 dimensional cube complex with simplicial links and two-sided hyperplanes where edges are labelled one of V, H, or D, parallel edges have the same label, and incident edges of square have different labels.

Definition 6.6.2 (Primitive). A VHD complex is *primitive* if its 1-skeleton contains no non-trivial circuit with edges of the same label. That is, for each label, the subgraph spanned by edges with that label is a forest.

To start, $T_1 \times T_2 \times T_3$ has a natural VHD-complex structure given by labelling T_1 edges with V, T_2 edges with H and T_3 edges with D. The product $T_1 \times T_2 \times T_3$ contains E as a G-invariant subset. The action is a product action so orbits of edges remain in the same parallelism class, therefore the quotient inherits the VHD-complex labeling. The tree actions do not have any inversions and therefore E and the quotient E/G have two-sided hyperplanes. Hence, E and E/G are VHD-complexes.

Remark 6.6.3. Our definition of VHD-Complex does not require that hyperplanes are injective on fundamental groups, only that hyperplanes are two-sided. The injectivity property is however needed to obtain an honest graph of groups decomposition. For the VHD-Complex E/G we are considering the injectivity property follows from fact that there is a universal cover E with simply-connected hyperplanes. For comparison, in dimension two, in a VH-Complex the π_1-injectivty of hyperplanes follows automatically from the two-sided requirement and the vertex link requirement.

This lemma and the next one ensures E/G will have fundamental group G.

Lemma 6.6.4. *The action of G on E is a covering space action.*

Proof. Consider the product action of G on $T_1 \times T_2 \times T_3$. The trees T_1, T_2, and T_3 are mutually transverse which tells us that the action on the vertices of a product of any two trees is free; hence free on the vertices of the product of all three trees. Acting by simplicial automorphisms with finite vertex stabilizers (in our case trivial) gives a PD action.

For a cube of any dimension, if you fix a point on it's interior then that cube is taken to itself.

Being a product action rules out rotations of cubes or squares and disallowing edge inversions rules out reflections.

Hence, no non-trivial element takes a cube to itself. Thus, the action is free. This implies that the product action is a covering space action.

Lastly, $E \subseteq T_1 \times T_2 \times T_3$ is a G-invariant subset and therefore the action of G on E is also a covering space action. □

Lemma 6.6.5 (Extended Core is simply connected). *The Extended core E is simply connected.*

Proof. We will show that E is the total space of a graph of spaces where the edge and vertex spaces are simply connected and the underlying graph is a tree. The edge spaces are hyperplanes and the vertex spaces are the connected components of the complement of the corridors of the hyperplanes. Now, E is a subset of a product so hyperplanes are embedded and two-sided and come with maps to each side of their corridor.

The hyperplanes of E are the connected components of the intersections of fibers (e.g. $\pi_1^{-1}(p)$) with E. By Property CCP then, the hyperplanes are path connected. Since the fibers are separating, the hyperplanes are also separating. This means the underlying graph is a tree. Because E has connected 1-dimensional fibers the hyperplanes also have connected 1-dimensional fibers and so are quadrant convex by Guirardel 5.0.6 and so are also simply connected. Hence, E is a tree of simply connected spaces and is therefore simply connected.

□

Proposition 6.6.6 (Injective on fundamental groups). *Let $A \subseteq X$ and $\rho : \tilde{X} \to X$ the universal cover of X. Then the inclusion map of A into X is π_1-injective if and only if the connected components of $\rho^{-1}(A)$ are simply connected.*

The final topological property we need is for the hyperplanes of E/G to be π_1-injective. Fortunately, during the proof of 6.6.5 we show that the hyperplanes of E are simply connected. This combined with 6.6.6 gives the injectivity property. As a corollary, E/G has three splittings with Bass-Serre trees T_1, T_2, and T_3.

6.7 Bieri dimension argument

At this stage the construction of E/G is complete. It's a three dimensional complex that splits in three ways over 2-dimensional VH-complexes. By Wise, VH-complexes themselves come with two splittings over 1-dimensional graphs. Lastly, graphs are themselves splittings over trivial groups. Using a result of Bieri, we show this iterated splitting pushes the cohomological dimension of G higher which gives the final contradiction needed for the proof of the main theorem.

In order to use our assumptions we need to show the first iteration of splittings coming from the VHD complex E correspond exactly to the original trees.

Earlier we proved that E is a graph of spaces, essentially because it sits within a product of trees. With this setup, after identification, the Bass-Serre maps that collapse vertex spaces to points and fibers of edge space products to points are projection maps. We're acting via a product action so projection maps are equivariant maps which means the image of E under projection is invariant. Lastly, our G-trees are minimal so the Bass-Serre maps from E map onto the original trees.

Next, we restate corollaries of Bieri's results.

Lemma 6.7.1. *(See Corollary 6.5, Bieri, p87) Let $G = G_1 *_H G_2$ be an amalgamated product of groups of type FP_∞ over R, with H a proper subgroup of finite index in both factors. Then for $k = 1, 2$ we have:*

$$cd_R(G) = cd_R(G_k) + 1.$$

Lemma 6.7.2. *(See Corollary 6.7, Bieri, p92) Let $G = G_{1*H,\phi}$ be an HNN extension where G_1 is of type FP_∞ over R with subgroups H and $\phi(H)$ of finite index in G_1. Then*

$$cd_R(G) = cd_R(G_1) + 1.$$

The following result is not proved as a direct consequence of the previous lemmas, instead Bieri

is able to run the argument again to obtain a similar result for general graphs of groups. This is the version we will be applying.

Theorem 6.7.3 (Bieri dimension plus one). *(Exercise p.93 from Bieri [Bie77]) For a non-trivial cocompact locally finite simplicial G-tree with FP_∞ vertex and edge groups we have that the dimension of G is exactly one more than the dimension of any vertex or edge group.*

The following lemma shows that the second step in the iterative splitting is non-trivial.

Lemma 6.7.4 (Iterated Splitting). *Let T_1 and T_2 be two non-trivial locally finite G-trees in different deformation spaces. Then the vertex groups of each tree act non-trivially on the other tree.*

Proof. Suppose x were a vertex of T_1, Let K be it's stabilizer. Now K is a subgroup of G and so also acts on T_2. If K had a global fixed point in T_2 then by local finiteness of T_1 every vertex group of T_1 would as well. Then by 3.0.9 T_1 and T_2 are in the same deformation space; a contradiction. Therefore, K acts non-trivially on T_2 as needed. □

We will now iterate the splitting process and then apply Bieri. To begin, by assumption each of the original splittings T_1, T_2, and T_3 are non-trivial. Now, without loss of generality, consider a vertex group G_v from the T_3 splitting corresponding to cutting along the D edges. By lemma 6.7.4, G_v acts non-trivially on T_1 and T_2. These non-trivial actions are the second iteration of the splitting process. The vertex group G_v is the fundamental group of the vertex space X_v, a VH-complex with two splittings from 2.4.10; in fact X_v is a graph of spaces where the vertex and edge spaces are also graphs. Cut along the horizontal edges of X_v to obtain a splitting where each vertex space is a graph composed entirely of edges with a vertical label. Let w be a vertex in the graph of groups splitting and Γ_w the corresponding vertex space. The graph Γ_w can be regarded as a splitting over the trivial group – but is it a trivial splitting? It could happen that Γ_w has trivial fundamental group. Here we need to assume that at least one sequence of iterated splittings ends in a graph with positive rank. Suppose this were the case for G_v, that is when

X_v splits there is a vertex space Γ which is a graph of positive rank. The fundamental group of Γ is a non-abelian free group and therefore has dimension one. This fact can also be seen as an application of Bieri to Γ where Γ is viewed as a non-trivial splitting over the trivial group. In this case the trivial group has dimension zero, so Bieri gives that $\pi_1(\Gamma)$ has dimension one. After a second application of Bieri we get that G_v must have dimension two. Finally, after a third application of Bieri's dimension theorem 6.7.3 we know G has dimension 3, a contradiction. This completes the proof of the main theorem.

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