PEAK FUNCTIONS ON FINITE TYPE DOMAINS IN $\mathbb{C}^2$

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DOMAINS IN $\mathbb{C}^2$

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TABLE OF CONTENTS

Chapter — Page

INTRODUCTION 1

The construction of Bedford and Fornaess 1

1. LEVEL CURVES OF HARMONIC POLYNOMIALS 6

1.1 Local behavior 6
1.2 Behavior at infinity 12
1.3 Global structure 18
1.4 Convergence of level curves 25

2. THE RIEMANN SURFACE $\mathcal{R}$ 31

2.1 Alternative description of $\mathcal{R}$ 31
2.2 The map $\xi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ and its critical points 42
2.3 The map $\chi : v \mapsto w$ 47
2.4 Properties of the map χ 65

CONCLUSION 82

BIBLIOGRAPHY 89

INTRODUCTION

The construction of Bedford and Fornaess

Given a domain $\Omega \subset \mathbb{C}^n$ and a point $z_0 \in \partial\Omega$, we say that z_0 is a peak point for Ω at z_0 if $\exists$ a function $f \in C(\overline{\Omega})$ satisfying that $f\big|_{\Omega} \in H(\Omega)$ and

$$f(z_0) = 1$$

$$|f(z)| < 1\,, \qquad z \in \overline{\Omega} \setminus \{z_0\}\,.$$

Such a function f is called a peak function for Ω at $z_0 \in \partial\Omega$.

The existence of peak functions at points of $\partial\Omega$ has many important implications in the theory of functions of several complex variables. For strongly pseudoconvex domains, sharp results have been established in this area of research. For weakly pseudoconvex domains, many questions remain open, however. We refer the reader to the introduction of [**Y**] for more information on the progress made in both the strongly and weakly pseudoconvex case.

For strongly pseudoconvex domains, see [**HS1**],[**P**] and [**G**]. For a bounded domain in $\mathbb{C}^2$, the existence of a peak function at a boundary point of finite type was proved by Bedford and Fornaess [**BF**] and, using a different method, by Fornaess and Sibony [**FS**]. Noell [**N**] extended the method of Bedford and Fornaess to a class of domains in higher dimensions. A method of proving similar results using estimates on the Bergman kernel was found by Fornaess and McNeal [**FM**]. More general results were proved by Herbort [**H**], Diedrich and Herbort [**DH**] and Yu [**Y**]. Also, see [**B**], [**HS2**] and [**R**] on the existence of peak function with certain smoothness up to the boundary.

In this paper we will use [**BF**] as the starting point of our discussion of peak functions on pseudoconvex domains of finite type in $\mathbb{C}^2$. We will show how the method of constructing such functions described there by the authors can be modified to prove more concrete and more precise results in this direction.

First, we will briefly describe the construction of Bedford and Fornaess. As in their paper, we only need to consider the following case. Let $m \in \mathbb{N} \setminus \{0\}$ and let

$$P_{2m}(z) = \sum_{j+k=2m} a_{j,k} z^j \bar{z}^k \,, \qquad z \in \mathbb{C}$$

be a real-valued subharmonic but not harmonic polynomial. Then $a_{j,k} = \bar{a}_{k,j}$ and we may assume also that $a_{2m,0} = 0$. Now consider the domain

$$\Omega = \left\{ (z,w) \in \mathbb{C}^2 : \mathrm{Re}(w^{2m}) + \delta P_{2m}(z) + \mathrm{Re}(z^{2m}) < 0 \right\}$$

where the number $\delta > 0$ is chosen so that the region

$$\left\{ u \in \mathbb{C} : \delta P_{2m}(u) + \mathrm{Re}(u^{2m}) < 0 \right\}$$

is the union of $2m$ disjoint open sectors.

Bedford and Fornaess show that we only need to study the geometry of Ω at $0 \in \partial\Omega$ to obtain a peak function on the domain

$$\Omega' = \left\{ (z',w') \in \mathbb{C}^2 : \mathrm{Re}(w') + \delta P_{2m}(z') + \mathrm{Re}\big[(z')^{2m}\big] < 0 \right\}$$

at $0 \in \Omega'$ of type $2m$, satisfying some useful properties described below. They also show how one can then easily obtain a peak function with similar properties at a finite type boundary point of a pseudoconvex domain in $\mathbb{C}^2$ in general. Therefore, we will focus our attention on a domain $\Omega \subset \mathbb{C}^2$ as given above.

Let

$$\mathcal{L} = \{(z, w, [\zeta : \eta]) \in \mathbb{C}^2 \times \mathbb{P} : \eta z = \zeta w\}$$

and $\forall [\zeta : \eta] \in \mathbb{P}$, put

$$L_{[\zeta:\eta]} = \{(z, w) \in \mathbb{C}^2 : \eta z = \zeta w\}.$$

Then $\mathcal{L}$ is a line bundle over $\mathbb{P}$ with projection $\pi : \mathcal{L} \to \mathbb{P}$.

Since $(0,0) \notin \Omega$, we can embed Ω into $\mathcal{L}$ using the map

$$(z, w) \mapsto (z, w, [z : w]) \in \mathcal{L}, \qquad (z, w) \in \Omega.$$

Using the local coordinates ζ on $U = \{[\zeta : 1] : \zeta \in \mathbb{C}\}$ and η on $V = \{[1 : \eta] : \eta \in \mathbb{C}\}$, we can describe $\Omega \subset \mathcal{L}$ as follows. We have

$$\mathcal{L}\big|_U \cong \mathbb{C}^2$$

$$(\zeta w, w, [\zeta : 1]) \leftarrow (w, \zeta) \in \mathbb{C}^2$$

and

$$\mathcal{L}\big|_V \cong \mathbb{C}^2$$

$$(z, \eta z, [1 : \eta]) \leftarrow (z, \eta) \in \mathbb{C}^2 .$$

Using the above local trivializations of $\mathcal{L}$, we obtain

$$\begin{aligned} \Omega \cap \pi^{-1}([\zeta : 1]) &\cong \{u \in \mathbb{C} : \mathrm{Re}(u^{2m}) + \delta P_{2m}(\zeta u) + \mathrm{Re}(\zeta^{2m} u^{2m}) < 0\} \\ &= \bigcup_{j=1}^{M(\zeta)} S_j(\zeta) \\ \Omega \cap \pi^{-1}([1 : \eta]) &\cong \{u \in \mathbb{C} : \mathrm{Re}(\eta^{2m} u^{2m}) + \delta P_{2m}(u) + \mathrm{Re}(u^{2m}) < 0\} \\ &= \bigcup_{j=1}^{N(\eta)} T_j(\eta) \end{aligned}$$

where $\forall \zeta \in \mathbb{C}$, we have $M(\zeta) \in \{1, \ldots, 2m\}$ and $S_1(\zeta), \ldots, S_{M(\zeta)}(\zeta) \subset \mathbb{C}$ are disjoint open sectors, and $\forall \eta \in \mathbb{C}$, we have $N(\eta) \in \{1, \ldots, 2m\}$ and $T_1(\eta), \ldots, T_{N(\eta)}(\eta) \subset \mathbb{C}$ are disjoint open sectors as well. Also, $\zeta\eta = 1$ implies $[\zeta : 1] = [1 : \eta]$, hence $M(\zeta) = N(\eta)$ and

$$S_j(\zeta) = T_{\sigma(j)}(\eta)\,, \qquad j \in \{1, \ldots, M(\zeta)\}$$

for some permutation σ.

In Lemma 1.1 and Lemma 1.2 of [**BF**], the authors describe some basic properties of the sectors S_j and T_k. They then use these properties to analyze the way these sectors vary with $[\zeta : \eta] \in \mathbb{P}$. This is applied to define a Riemann surface $\mathcal{R}$ with boundary $\partial\mathcal{R}$ the following way. $\mathcal{R}$ is a covering space of an open subset $W \subset \mathbb{P}$ with ∂W being a real-analytic curve in $U \cap V \subset \mathbb{P}$. Let $\varrho : \mathcal{R} \to W$ be the projection map. Then ϱ is is a locally biholomorphic map and $\forall[\zeta : \eta] \in W$, we have

$$\varrho^{-1}([\zeta : \eta]) = \begin{cases} \{S_1(\alpha), \ldots, S_{M(\alpha)}(\alpha)\} & \text{if} \quad [\zeta : \eta] = [\alpha : 1] \\ \\ \{T_1(\beta), \ldots, T_{N(\beta)}(\beta)\} & \text{if} \quad [\zeta : \eta] = [1 : \beta]\,. \end{cases}$$

Also, ϱ extends to a map $\varrho : \widetilde{\mathcal{R}} \to \widetilde{W}$ where $\widetilde{\mathcal{R}}$ is a Riemann surface with $\mathcal{R} \cup \partial\mathcal{R} \subset \widetilde{\mathcal{R}}$ and $\widetilde{W} \subset \mathbb{P}$ is an open set with $\overline{W} \subset \widetilde{W}$. Consider the pullback

$$\widetilde{\mathcal{L}} = \varrho^*(\mathcal{L})\,.$$

Then $\widetilde{\mathcal{L}}$ is a line bundle over $\widetilde{\mathcal{R}}$ and the sectors described above can be used to find a smooth nonvanishing section of $\widetilde{\mathcal{L}}$, after shrinking $\widetilde{\mathcal{R}}$ if necessary. Therefore, $\widetilde{\mathcal{L}}$ is topologically trivial. It follows that from the solution of the multiplicative Cousin problem, we can find a nonvanishing section of the dual bundle of $\widetilde{\mathcal{L}}$. This section can be used to define a function

$$G \in H(\Omega)$$

such that $\forall[\zeta : \eta] \in \mathbb{P}$, G is locally linear on $\Omega \cap L_{[\zeta:\eta]}$. Finally, the authors define a function $F \in H(\Omega')$ by

$$F(z', w') = \prod_{w^{2m} = w'} G(z', w)\,, \qquad (z', w') \in \Omega'\,, \quad w' \neq 0\,.$$

Using the symmetry built in the definition of F, we can take $H = F^{1/N} \in H(\Omega)$ where $N \in \mathbb{N}$ is chosen so large that we have

$$-\frac{\pi}{2} < \arg(H) < \frac{\pi}{2}.$$

Then $\exp(-H)$ is the desired peak function at $0 \in \Omega'$. Also, F is Hölder $1/(2m)$ near 0, hence H is Hölder $1/(2mN)$ near 0.

In this paper, we will make a convenient and concrete choice of the smooth section of $\mathcal{L}$ mentioned above, and for a wide variety of finite type pseudoconvex domains in $\mathbb{C}^2$, we will define a concrete $G \in H(\Omega)$ so that $\forall[\zeta : \eta] \in \mathbb{P}$, G is locally linear on $\Omega \cap L_{[\zeta:\eta]}$ with

$$-\frac{3\pi}{4m} < \arg(G) < \frac{3\pi}{4m}.$$

Then we define $F \in H(\Omega')$ as above, and obtain

$$-\frac{3\pi}{2} < \arg(F) < \frac{3\pi}{2}.$$

Therefore, we can take $N = 3$ and obtain a peak function $\exp(-H)$ where $H = F^{1/3}$ is Hölder $1/(6m) = 1/(3\text{type})$ near $0 \in \Omega'$.

CHAPTER 1

LEVEL CURVES OF HARMONIC POLYNOMIALS

Section 1.1 Local behavior

In this section, we describe the local geometry of the level curves of a nonconstant harmonic polynomial.

Let $r > 0$ and put $D = \{z \in \mathbb{C} : |z| < r\}$. Assume $\varphi \in H(D)$ and let

$$J = \left\{j \in \mathbb{N} : \varphi^{(j)}(0) \neq 0\right\}.$$

Then $\varphi \equiv 0 \Leftrightarrow J = \emptyset$. Assume $\varphi \not\equiv 0$ and let $m = \min J$. Let $\varphi(0) = 0$ and consider $w = \varphi(z)$, $z \in D$. Then $m \geq 1$ and

$$w = z^m \psi(z)$$

where $\psi \in H(D)$ satisfies $\psi(0) \neq 0$. After shrinking $r > 0$ if necessary, we may assume that $\Psi = \psi(D)$ is simply connected and that $0 \notin \Psi$. Then $\forall \psi \in \Psi\ \exists \varrho > 0$ and $\exists \theta \in [0, 2\pi)$ such that $\psi = \varrho e^{i\theta}$ and both $\varrho = \varrho(\psi)$ and $\theta = \theta(\psi)$ are continuous on Ψ. Then putting

$$\lambda(\psi) = \log(\varrho) + i\theta, \qquad \psi \in \Psi$$

we have that $\lambda \in H(\Psi)$. Therefore, the function $u = u(z)$ defined by

$$u = z e^{\frac{1}{m}\lambda[\psi(z)]}, \qquad z \in D$$

satisfies $u \in H(D)$ and $w = u^m$. We also have

$$u'(0) = e^{\frac{1}{m}\lambda[\psi(0)]} \neq 0\,.$$

Shrinking $r > 0$ further if necessary, we may then assume that u has a holomorphic inverse $v = v(u)$ defined in $\Omega = u(D)$.

Next, consider the equation

$$w = \varphi(z) = it\,, \qquad t \in \mathbb{R}. \tag{1.1.1}$$

Let $\alpha = \pi/2m$, let

$$\beta = \begin{cases} \pi/m & \text{if } m \text{ is even} \\ 2\pi/m & \text{if } m \text{ is odd} \end{cases}$$

and $\forall j \in \mathbb{Z}$, put

$$\ell_j = \left\{se^{i(\alpha+j\beta)} : s \in \mathbb{R}\right\}$$

and let $\gamma_j = v(\ell_j \cap \Omega)$.

Fix some $j \in \mathbb{Z}$. Then $u(\gamma_j) = u[v(\ell_j \cap \Omega)] = \ell_j \cap \Omega$ and so $\forall z \in \gamma_j$ $\exists s \in \mathbb{R}$ satisfying

$$u = u(z) = se^{i(\alpha+j\beta)}\,.$$

Therefore, we get $w = u^m = \pm is^m$, so $\forall z \in \gamma_j$ is a solution to equation (1.1.1) with $t = \pm s^m$.

Conversely, let $z \in D$ be a solution to equation (1.1.1) and consider $u = u(z)$. We have

$$u^m = w = it = \pm i|t| = |t|e^{\pm i\pi/2}$$

so clearly $\exists j \in \{1, \dots, m\}$ satisfying $u = |t|^{1/m}e^{i\omega}$ where

$$\omega = \pm\frac{\pi}{2m} + j\frac{2\pi}{m} = \begin{cases} \text{either} & \dfrac{\pi}{2m} + \dfrac{2j\pi}{m} \\ \text{or} & \dfrac{\pi}{2m} + \dfrac{(2j-1)\pi}{m}\,. \end{cases}$$

Therefore, if m is even, we get that

$$\omega = \begin{cases} \text{either} & \alpha + 2j\beta \\ \text{or} & \alpha + (2j-1)\beta \end{cases}$$

and for m odd, we get that

$$\omega = \begin{cases} \text{either} & \alpha + j\beta \\ \text{or} & \alpha + \pi + \left(j - \frac{m+1}{2}\right)\beta \,. \end{cases}$$

In each case $\exists k \in \mathbb{Z}$ such that

$$e^{i\omega} = \pm e^{i(\alpha + k\beta)}$$

and so

$$u = |t|^{1/m} e^{i\omega} = \pm |t|^{1/m} e^{i(\alpha+k\beta)} \in \ell_k \,.$$

Therefore, we have $z = v[u(z)] \in v(\ell_k \cap \Omega) = \gamma_k$.

We conclude that

$$\left\{ z \in D : \mathrm{Re}\left[\varphi(z)\right] = 0 \right\} = \bigcup_{j \in \mathbb{Z}} \gamma_j \,.$$

Assume that $j, k \in \mathbb{Z}$ satisfy $m | j - k$. Then $(j-k)\beta \in \pi\mathbb{Z}$, so we have $e^{i(\alpha + j\beta)} = \pm e^{i(\alpha+k\beta)}$. It follows that $\ell_j = \ell_k$.

Now assume that $\ell_j \cap \ell_k \neq \{0\}$. Then $\exists s, t \in \mathbb{R} \setminus \{0\}$ and $\exists j, k \in \mathbb{Z}$ such that

$$s e^{i(\alpha + j\beta)} = t e^{i(\alpha + k\beta)} \,.$$

Then $s = \pm t$ and so $(j-k)\beta \in \pi\mathbb{Z}$. If m is even, then $\beta = \pi/m$, therefore $m | j - k$. If m is odd, then $\beta = 2\pi/m$, therefore $m | 2(j-k)$. But m is odd, so $m | j - k$ in this case as well.

We obtain that

$$m | j - k \quad \Longleftrightarrow \quad \ell_j = \ell_k \quad \Longleftrightarrow \quad \gamma_j = \gamma_k \,.$$

We can sum up the above as follows.

Lemma 1.1.1 Let f be a nonconstant holomorphic function in a neighborhood of $z_0 \in \mathbb{C}$ and assume that $\mathrm{Re}[f(z_0)] = c$. Then $\exists$ an open neighborhood U of z_0, $\exists m \in \mathbb{N} \setminus \{0\}$, $\exists$ open intervals $I_j \subset \mathbb{R}$, $j = 1, \ldots, m$ and $\exists$ simple nonsingular analytic curves $\delta_j : I_j \to \mathbb{C}$, $j = 1, \ldots, m$ such that

$$\{z \in U : \mathrm{Re}[f(z)] = c\} = \bigcup_{j=1}^{m} \{\delta_j\}$$

where, as usual, $\forall \gamma : I \to \mathbb{C}$, we put $\{\gamma\} = \{\gamma(t) : t \in I\}$.

Proof. Pick an $r > 0$ small enough so that we may define

$$\varphi(z) = f(z_0 + z) - f(z_0), \qquad z \in D$$

where $D = \{z \in \mathbb{C} : |z| < r\}$. Then $\varphi(0) = 0$. Assume r is so small that the above discussion can be applied to φ. Using the same notation, we get that Ω is simply connected, so $\forall j \in \{1, \ldots, m\}$, the set $I_j = \{s \in \mathbb{R} : se^{i(\alpha + j\beta)} \in \Omega\}$ is an open interval. Then we may define

$$\delta_j(s) = z_0 + v\big(se^{i(\alpha+j\beta)}\big), \qquad s \in I_j .$$

We obtain the simple nonsingular analytic curves $\delta_j : I_j \to \mathbb{C}$, $j = 1, \ldots, m$, and the claim follows with $U = z_0 + D$.

Note 1.1.1 The m in Lemma 1.1.1 is unique and it can be given as follows. Let

$$K = \{k \in \mathbb{N} : k \geq 1 \quad \text{and} \quad f^{(k)}(z_0) \neq 0\}.$$

Here, the f is nonconstant, therefore $f' \not\equiv 0$ and so $K \neq \emptyset$. We have $m = \min K$.

Note 1.1.2 Consider Ω above. Clearly, $\exists$ disjoint open sets Ω^+ and Ω^- such that

$$\Omega \setminus \bigcup_{j=1}^{m} \ell_j = \Omega^+ \cup \Omega^-$$

$$\Omega^+ = \{z \in \Omega : \mathrm{Re}(z^m) > 0\}$$

$$\Omega^- = \{z \in \Omega : \mathrm{Re}(z^m) < 0\}.$$

Also, $\exists$ pairwise disjoint simply connected open sets Ω_j^+, Ω_j^-, $j = 1, \ldots, m$ such that

$$\Omega^\pm = \bigcup_{j=1}^{m} \Omega_j^\pm.$$

Correspondingly, we have the disjoint open sets $D^\pm = v(\Omega^\pm)$ and $U^\pm = z_0 + D^\pm$, as well as the pairwise disjoint simply connected open sets $D_j^\pm = v(\Omega_j^\pm)$, $j = 1, \ldots, m$ and $U_j^\pm = z_0 + D_j^\pm$, $j = 1, \ldots, m$, satisfying

$$D \setminus \bigcup_{j=1}^{m} \gamma_j = D^+ \cup D^-$$

$$U \setminus \bigcup_{j=1}^{m} \{\delta_j\} = U^+ \cup U^-$$

$$D^+ = \{z \in D : \mathrm{Re}[\varphi(z)] > 0\} = \bigcup_{j=1}^{m} D_j^+$$

$$D^- = \{z \in D : \mathrm{Re}[\varphi(z)] < 0\} = \bigcup_{j=1}^{m} D_j^-$$

$$U^+ = \{z \in U : \mathrm{Re}[f(z)] > c\} = \bigcup_{j=1}^{m} U_j^+$$

$$U^- = \{z \in U : \mathrm{Re}[f(z)] < c\} = \bigcup_{j=1}^{m} U_j^-.$$

Note 1.1.3 If f in Lemma 1.1.1 satisfies $f'(z_0) \neq 0$ or, equivalently, if $m = 1$, then $\exists!$ simple nonsingular analytic curve $\delta : I \to \mathbb{C}$ such that

$$\{z \in U : \mathrm{Re}[f(z)] = c\} = \{\delta\}.$$

Let $f(z_0) = c + it_0$ for some $t_0 \in \mathbb{R}$. After translating I if necessary, we may assume that $t_0 \in I$ and $\delta(t_0) = z_0$. Since $f'(z_0) \neq 0$, it follows that $\exists$ a neighborhood V of z_0

such that $f\big|_V$ is invertible with a holomorphic inverse. Therefore, $\exists \epsilon > 0$ such that

$$\gamma(s) = f^{-1}(c + is)\,, \qquad s \in (t_0 - \epsilon, t_0 + \epsilon)$$

is a nonsingular analytic curve satisfying $\{\gamma\} \subset \{\delta\}$. That is, $\exists$ a reparametrization $t : (t_0 - \epsilon, t_0 + \epsilon) \to I$ of δ such that $\gamma = \delta \circ t$ and γ satisfies

$$f\big[\gamma(s)\big] = c + is\,, \qquad s \in (t_0 - \epsilon, t_0 + \epsilon)\,.$$

It follows that we also have

$$i = \left[\frac{d}{ds}(f \circ \gamma)\right](s) = f'\big[\gamma(s)\big]\gamma'(s)\,, \qquad s \in (t_0 - \epsilon, t_0 + \epsilon)\,. \tag{1.1.2}$$

Assume $z_0 \neq 0$ and let $\epsilon > 0$ chosen above be so small that $\forall s \in (t_0 - \epsilon, t_0 + \epsilon)$, we have $z = \gamma(s) \neq 0$. Then (1.1.2) implies

$$\frac{dz/ds}{z} = \frac{i}{zf'(z)} = \frac{\mathrm{Im}\big[zf'(z)\big] + i\mathrm{Re}\big[zf'(z)\big]}{\big|zf'(z)\big|^2}\,.$$

Also, we may choose analytic functions $\varrho, \theta : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}$ such that $\varrho > 0$ and

$$z = \gamma(s) = \varrho(s)e^{i\theta(s)}\,, \qquad t \in (t_0 - \epsilon, t_0 + \epsilon)\,.$$

We obtain that $|z| = \varrho(s)$ and $\arg(z) = \theta(s)$ satisfy

$$\mathrm{sign}\big(d|z|/ds\big) = \mathrm{sign}\big(\mathrm{Im}\big[zf'(z)\big]\big)\,, \qquad s \in (t_0 - \epsilon, t_0 + \epsilon) \tag{1.1.3}$$

and

$$\mathrm{sign}\big(d\arg(z)/ds\big) = \mathrm{sign}\big(\mathrm{Re}\big[zf'(z)\big]\big)\,, \qquad s \in (t_0 - \epsilon, t_0 + \epsilon) \tag{1.1.4}$$

where $\forall x \in \mathbb{R}$ we put

$$\mathrm{sign}(x) = \begin{cases} 0 & \text{if } x = 0 \\ x/|x| & \text{otherwise}\,. \end{cases}$$

Section 1.2 Behavior at infinity

In this section, we study the asymptotic behavior at infinity of the level curves of a nonconstant harmonic polynomial.

Let $m \in \mathbb{N} \setminus \{0\}$ and consider the polynomial $q_m(z) = z^m + z^{m-1} + \cdots + 1$, $z \in \mathbb{C}$, and the closed sector $S_m = \left\{re^{i\theta} : r \geq 0, |\theta| \leq \frac{\pi}{m+1}\right\}$. Since the zeros of $q_m(z)$ are $e^{ij\omega}$, $j = 1, \ldots, m$, where $\omega = \frac{2\pi}{m+1}$, it follows that $\forall z \in S_m$, $q_m(z) \neq 0$. Therefore, $\exists c_m > 0$ such that $\forall z \in S_m$, $|q_m(z)| > c_m$.

Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $q(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{C}[z]$ with $\deg(q) = n$. Assume $a_{n-1} = 0$ and assume

$$q(z) = a_n z^n + a_k z^k + \cdots + a_0 \,, \qquad z \in \mathbb{C}$$

where $0 \leq k \leq n-2$ and $a_k \neq 0$. Let $A = 2|a_k/a_n|$. Then $\exists B > 0$ so large that

$$z \in \mathbb{C}, \quad |z| > B \quad \Longrightarrow \quad \left| \frac{a_k z^k + \cdots + a_0}{a_n z^n} \right| < \frac{A}{|z|^2} \,.$$

Let $z \in \mathbb{C}$ satisfy $|z| > B$. Then

$$\left| \frac{q(z)}{a_n z^n} - 1 \right| = \left| \frac{a_k z^k + \cdots + a_0}{a_n z^n} \right| < \frac{A}{|z|^2} \,.$$

Let $\theta \in [-\pi, \pi]$ and $r \geq 0$ satisfy

$$\frac{q(z)}{a_n z^n} = re^{i\theta} \,.$$

Clearly, $\exists C > 0$ so large that $\forall a \in \mathbb{C}$ with $|a| > C$ we have

$$q(z) = a \qquad \Longrightarrow \qquad |z| > B \,.$$

Let $a \in \mathbb{C}$ satisfy $|a| > C$, let $q(z) = a$ and put $b = zr^{1/n}e^{i\theta/n}$. Then we have

$$b^n = z^n re^{i\theta} = z^n \frac{q(z)}{a_n z^n} = \frac{a}{a_n}$$

and we have

$$|b/z - 1| = \frac{|(b/z)^n - 1|}{|(b/z)^{n-1} + (b/z)^{n-2} + \cdots + 1|} = \frac{\left|\dfrac{q(z)}{a_n z^n} - 1\right|}{|q_{n-1}(b/z)|} < \frac{A}{c_{n-1}|z|^2}$$

since $|\theta/n| \leq \pi/n$ and so $b/z \in S_{n-1}$. We obtain

$$|b - z| < \frac{A}{c_{n-1}|z|} < \frac{A}{c_{n-1}B}.$$

Lemma 1.2.1 Fix $n \in \mathbb{N}\backslash\{0\}$ and let $f(z) = \sum_{j=0}^{n} b_j z^j \in \mathbb{C}[z]$ with $\deg(f) = n$. Let $z_1, \ldots, z_n \in \mathbb{C}$ be the zeros of $f(z)$. Then $\forall \epsilon > 0$, $\exists C > 0$ such that $\forall a \in \mathbb{C}$ with $|a| > C$ and $\forall z \in \mathbb{C}$ with $f(z) = a$, $\exists b \in \mathbb{C}$ satisfying $b^n = a/a_n$ and

$$\left|\left(b + \frac{z_1 + \cdots + z_n}{n}\right) - z\right| < \epsilon.$$

Proof. Fix an arbitrary $\epsilon > 0$. For $n = 1$ the statement is trivial. Assume $n \geq 2$ and let

$$q(z) = f\left(z + \frac{z_1 + \cdots + z_n}{n}\right) = b_n \prod_{j=1}^{n} \left(z + \frac{z_1 + \cdots + z_n}{n} - z_j\right), \qquad z \in \mathbb{C}.$$

Then $q(z) = \sum_{j=0}^{n} a_j z^j$ where $a_n = b_n$ and

$$a_{n-1} = b_n \sum_{j=1}^{n} \left(\frac{z_1 + \cdots + z_n}{n} - z_j\right) = 0.$$

We claim that $\exists C > 0$ such that $\forall a \in \mathbb{C}$, $|a| > C$ and $q(w) = a$ imply the existence of $b \in \mathbb{C}$ with $b^n = a/a_n$ and $|b - w| < \epsilon$. This is clear if $q(z) = a_n z^n$, otherwise it follows from the above discussion by choosing $B > 0$ so large that

$$\frac{A}{c_{n-1}B} < \epsilon.$$

Now let $a \in \mathbb{C}$ satisfy $|a| > C$ and let $f(z) = a$. Put

$$w = z - \frac{z_1 + \cdots + z_n}{n}.$$

Then $q(w) = f(z) = a$, so $\exists b \in \mathbb{C}$ with $b^n = a/a_n = a/b_n$ satisfying

$$\left|\left(b + \frac{z_1 + \cdots + z_n}{n}\right) - z\right| = |b - w| < \epsilon$$

and Lemma 1.2.1 follows.

Lemma 1.2.2 Using the same notation as in Lemma 1.2.1, we have the following converse of the statement there: $\forall \epsilon > 0$, $\exists C > 0$ such that $\forall t \in \mathbb{R}$ with $|t| > C$ and $\forall b \in \mathbb{C}$ with $b^n = it/a_n$, $\exists z \in \mathbb{C}$ satisfying $f(z) = it$ and

$$\left|\left(b + \frac{z_1 + \cdots + z_n}{n}\right) - z\right| < \epsilon .$$

Proof. Fix an arbitrary $\epsilon > 0$. Choose $C > 0$ with the property stated in Lemma 1.2.1. Let $t \in \mathbb{R}$ satisfy $|t| > C$ and let $u_1, \ldots, u_n$ be the solutions to the equation

$$f(u) = it, \qquad u \in \mathbb{C}$$

and let $b_1, \ldots, b_n$ be the solutions to the equation

$$b^n = it/a_n, \qquad b \in \mathbb{C}.$$

By the choice of $C > 0$, we have a map $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\forall j \in \{1, \ldots, n\}$ we have

$$\left|\left(b_{\sigma(j)} + \frac{z_1 + \cdots + z_n}{n}\right) - u_j\right| < \epsilon .$$

We need to show that we may choose $C > 0$ so large that the above map σ is onto. Since $\{1, \ldots, n\}$ is a finite set, it suffices to show that σ is one-to-one if we choose $C > 0$ large enough.

Let $r > 0$ satisfy that $\forall j \in \{1, \ldots, n\}$, $|z_j| < 1/r$. Let $D = \{z \in \mathbb{C} : |z| < r\}$. Then $\forall z \in D \setminus \{0\}$, we have $1/r < |1/z|$, hence $1/z \notin \{z_1, \ldots, z_n\}$. Therefore, $\forall z \in D$

we may define

$$g(z) = \begin{cases} \dfrac{1}{f(1/z)} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0\,. \end{cases}$$

It clearly follows that $\lim_{z\to 0} g(z) = 0$, so we obtain $g \in H(D)$. Fix $\varrho \in \mathbb{R}$ with $0 < \varrho < r$. Then $\exists R > 0$ such that $\forall a \in \mathbb{C}$ with $|a| > R$, the equation

$$f(z) = a\,, \qquad z \in \mathbb{C}$$

has exactly n solutions in the region $\{z \in \mathbb{C} : |z| > 1/\varrho\}$. Therefore, $\forall a \in \mathbb{C}$ with $|a| > R$, the equation

$$g(z) = 1/a\,, \qquad z \in \mathbb{C}$$

has exactly n solutions in the region $\{z \in \mathbb{C} : |z| < \varrho\}$ and no solution in its boundary, hence

$$\frac{1}{i2\pi} \int\limits_{|\zeta|=\varrho} \frac{g'(\zeta)d\zeta}{g(\zeta) - 1/a} = n\,.$$

Taking the limit of the above integral as $a \to \infty$, we get

$$\frac{1}{i2\pi} \int\limits_{|\zeta|=\varrho} \frac{g'(\zeta)d\zeta}{g(\zeta)} = n\,.$$

Letting $\varrho \to 0$ in the above integral, we get that $g(z)$ has a zero of multiplicity n at $z = 0$. Thus, $g(z) = z^n \psi(z)$ where $\psi \in H(D)$ with $\psi(0) \neq 0$. We obtain $g(0) = \cdots = g^{(n-1)}(0) = 0$ and $g^{(n)}(0) \neq 0$. Applying Lemma 1.1.1 to g at $z_0 = 0$, we obtain an open neighborhood U of 0, open intervals $I_j \subset \mathbb{R}$, $j = 1, \ldots, n$, and simple nonsingular analytic curves $\delta_j : I_j \to \mathbb{C}$, $j = 1, \ldots, n$, satisfying

$$\{z \in U : \mathrm{Re}[g(z)] = 0\} = \bigcup_{j=1}^{n} \{\delta_j\}\,.$$

$\forall j \in \{1, \ldots, n\}$, let L_j be the tangent line to $\{\delta_j\}$ at 0. The proof of Lemma 1.1.1 implies that $\exists \omega \in \mathbb{R}$ such that $\forall j \in \{1, \ldots, n\}$, we have

$$L_j = \left\{ t e^{i(\omega + j2\pi/n)} : t \in \mathbb{R} \right\}.$$

Fix $j \in \{1,\dots,n\}$. We may assume that $0 \in I_j$ and $\delta_j(0) = 0$. $\forall t \in I_j$, let $z_t = \delta_j(t)$ and let $w_t \in L_j$ be the orthogonal projection of z_t onto L_j. Since L_j is tangent to $\{\delta_j\}$ at 0, we may choose $\varrho_j > 0$ with $(-\varrho_j, \varrho_j) \subset I_j$ and $K_j > 0$ such that $\forall t \in (-\varrho_j, \varrho_j)$, we have

$$|z_t| \leq 2|w_t|$$

$$|z_t - w_t| \leq K_j |z_t|^2 .$$

Therefore, $\forall t \in (-\varrho_j, \varrho_j)$, we have

$$\left|\frac{1}{z_t} - \frac{1}{w_t}\right| = \frac{|z_t - w_t|}{|z_t||w_t|} \leq K_j \frac{|z_t|}{|w_t|} \leq 2K_j . \tag{1.2.1}$$

Let $I^- = (-\varrho_j, 0)$, $I^+ = (0, \varrho_j)$ and consider the curves

$$\gamma^{\pm}(t) = \frac{1}{\delta_j(t)}, \qquad t \in I^{\pm} .$$

Then $\forall t \in I^{\pm}$, we have

$$\mathrm{Re}\big(f\left[\gamma^{\pm}(t)\right]\big) = \mathrm{Re}\big[f(1/z_t)\big] = \mathrm{Re}\big[1/g(z_t)\big] = 0$$

because $\mathrm{Re}\big[g(z_t)\big] = 0$. We may choose $d_j \in \mathbb{R}$ with $d_j > 0$ satisfying

$$|z_t| < d_j \quad \Longrightarrow \quad |t| < \varrho_j .$$

$\forall j \in \{1,\dots,n\}$, we obtain K_j and d_j as described above. Put

$$K = 2\max\{K_1,\dots,K_n\}$$

$$d = \max\left\{\frac{2(K+\epsilon)}{|e^{i2\pi/n} - 1|}, 1/d_1, \dots, 1/d_n\right\} .$$

Let $C > 0$ chosen above be so large that $\forall a \in \mathbb{C}$ with $|a| > C$, we have

$$f(z) = a \quad \Longrightarrow \quad |z| > d .$$

Fix $t \in \mathbb{R}$ with $|t| > C$. Let $u_1, \dots, u_n \in \mathbb{C}$ be the solutions to the equation

$$f(u) = it\,, \qquad u \in \mathbb{C}$$

and let $b_1, \dots, b_n \in \mathbb{C}$ be the solutions to the equation

$$b^n = it/a_n\,, \qquad b \in \mathbb{C}.$$

$\forall j \in \{1, \dots, n\}$, we have $|u_j| > d \geq 1/d_j$, hence $|1/u_j| < d_j$. Since $\mathrm{Re}\big[g(1/u_j)\big] = 0$, $j = 1, \dots, n$, we may assume that $1/u_j = \delta_j(t_j)$ for some $t_j \in (-\varrho_j, \varrho_j)$, $j = 1, \dots, n$. $\forall j \in \{1, \dots, n\}$, let $w_j \in L_j$ be the orthogonal projection of $1/u_j$ onto L_j and let $v_j = 1/w_j$. Then $|w_j| \leq |1/u_j|$, hence $|v_j| \geq |u_j| > d$. Since $v_j = |v_j|e^{-i(\omega + j2\pi/n)}$, $j = 1, \dots, n$, it follows that $\forall j, k \in \{1, \dots, n\}$ with $j \neq k$, we clearly have

$$|v_j - v_k| > d|e^{i2\pi/n} - 1| \geq 2(K + \epsilon)\,.$$

By (1.2.1) above, $\forall j \in \{1, \dots, n\}$, we also have

$$|u_j - v_j| \leq 2K_j \leq K\,.$$

We obtain that $\forall j, k \in \{1, \dots, n\}$ with $j \neq k$, we have

$$2(K + \epsilon) < |v_j - v_k| \leq |v_j - u_j| + |u_j - u_k| + |u_k - v_k| \leq K + |u_j - u_k| + K$$

therefore

$$2\epsilon < |u_j - u_k| \leq |u_j - b_{\sigma(j)}| + |b_{\sigma(j)} - b_{\sigma(k)}| + |b_{\sigma(k)} - u_k| < \epsilon + |b_{\sigma(j)} - b_{\sigma(k)}| + \epsilon\,.$$

We get $|b_{\sigma(j)} - b_{\sigma(k)}| > 0$, therefore $\sigma(j) \neq \sigma(k)$, and Lemma 1.2.2 follows.

Section 1.3 Global structure

Based on Sections 1.1 and 1.2, we will now analyze the global geometry of level sets of nonconstant harmonic polynomials.

First, we show in Lemma 1.3.2 below that each smooth level curve in such a level set is a 1-dimensional analytic real submanifold of $\mathbb{C}$, extending to infinity in both directions.

As usual, $\forall z \in \mathbb{C}$ and $\forall r > 0$, we put $D(z,r) = \{u \in \mathbb{C} : |u - z| < r\}$. Fix $z \in \mathbb{C}$ and $r > 0$, put $U = D(z,r)$ and let $f \in H(U)$ with $f' \not\equiv 0$. Let $c = \mathrm{Re}[f(z)]$. Then by Lemma 1.1.1, $\exists$ an open interval $I \subset \mathbb{R}$ and $\exists$ a nonsingular analytic curve $\gamma : I \to U$ with $z \in \{\gamma\}$ and $\forall t \in I$ we have

$$\mathrm{Re}\big(f[\gamma(t)]\big) = c.$$

Since γ is nonsingular, we may assume that γ is parametrized by arclength. Let $w \in \overline{\{\gamma\}} \cap U$, let $\varrho > 0$ and put $V = D(w, \varrho)$. Assume that $g \in H(V)$ satisfies $g\big|_{U \cap V} = f\big|_{U \cap V}$. Then $g' \not\equiv 0$. Since f is continuous at w and $w \in \overline{\{\gamma\}}$, we have

$$\mathrm{Re}\big[g(w)\big] = \mathrm{Re}\big[f(w)\big] = \lim_{\{\gamma\} \ni u \to w} \mathrm{Re}\big[f(u)\big] = c.$$

It follows from Lemma 1.1.1 that $\exists$ an open neighborhood W of w, $\exists m \in \mathbb{N} \setminus \{0\}$, $\exists$ open intervals $I_j \subset \mathbb{R}$, $j = 1, \ldots, m$, and $\exists$ nonsingular analytic curves $\delta_j : I_j \to \mathbb{C}$, $j = 1, \ldots, m$, such that

$$\{u \in W : \mathrm{Re}\big[g(u)\big] = c\} = \bigcup_{j=1}^{m} \{\delta_j\}.$$

Since $w \in \overline{\{\gamma\}}$, we have that $\exists j \in \{1, \ldots, m\}$ with $\{\gamma\} \cap \{\delta_j\} \neq \emptyset$, Therefore, we may continue γ across w in arclength parametrization.

Note 1.3.1 Let $n \in \mathbb{N}$ and let $f(z) \in \mathbb{C}[z]$ with $\deg(f) = n$. Let $a, b, c \in \mathbb{R}$ with $a < b$ and assume that we have some nonsingular analytic curve $\gamma : (a,b) \to \mathbb{C}$ satisfying

$$f\big[\gamma(t)\big] = c + it\,, \qquad t \in (a,b)\,.$$

We claim that we cannot have

$$\lim_{t \to a^+} \big|\gamma(t)\big| = \infty\,.$$

Otherwise, $\forall z \in \{\gamma\}$, we would have

$$\mathrm{Re}\big[f(z)\big] = c \qquad \text{and} \qquad a < \mathrm{Im}\big[f(z)\big] < b\,.$$

That is, the nonconstant polynomial $f(z)$ would stay bounded on the unbounded set $\{\gamma\}$, a contradiction.

Therefore, $\exists w = \lim_{t \to a^+} \gamma(t)$, and, using the above process at $w \in \overline{\{\gamma\}}$, we can continue γ across $a \in \mathbb{R}$ to a slightly larger interval $(a', b) \supset [a, b)$ as a nonsingular curve such that we still have

$$\mathrm{Re}\big(f\big[\gamma(t)\big]\big) = c\,, \qquad t \in (a', b)\,.$$

Similarly, we can continue γ across $b \in \mathbb{R}$ in the same manner.

Lemma 1.3.1 Let $f \in H(\mathbb{C})$ with $f' \not\equiv 0$. Fix $z \in \mathbb{C}$ and put $c = \mathrm{Re}\big[f(z)\big]$. Then $\exists$ a nonsingular analytic curve $\gamma : \mathbb{R} \to \mathbb{C}$, parametrized by arclength, such that $\{\gamma\} \ni z$ and $\forall t \in \mathbb{R}$ we have

$$\mathrm{Re}\big(f\big[\gamma(t)\big]\big) = c\,.$$

Also, let Γ be the set of all nonsingular analytic curves $\gamma : \mathbb{R} \to \mathbb{C}$ parametrized by arclength and let

$$S_c = S_c(f) = \big\{u \in \mathbb{C} : \mathrm{Re}\big[f(u)\big] = c\big\}$$

$$\Gamma_c = \Gamma_c(f) = \big\{\gamma \in \Gamma : \forall t \in \mathbb{R}, \quad \mathrm{Re}\big(f\big[\gamma(t)\big]\big) = c\big\}\,.$$

Then we have

$$S_c(f) = \bigcup_{\gamma \in \Gamma_c} \{\gamma\}$$

and $\forall \gamma, \delta \in \Gamma_c$ with $\{\gamma\} \neq \{\delta\}$, we have $\{\gamma\} \cap \{\delta\} \subset \{u \in \mathbb{C} : f'(u) = 0\}$.

Proof Except for the very last statement, we can prove Lemma 1.3.1 by repeated applications of the above process. For this last claim, let $\gamma, \delta \in \Gamma_c$. If $u \in \{\gamma\} \cap \{\delta\}$ and $f'(u) \neq 0$, then $\exists$ a neighborhood U of u such that $U \cap \{\gamma\} = U \cap \{\delta\}$. Since $\gamma, \delta : \mathbb{R} \to \mathbb{C}$ are both analytic curves, we get $\{\gamma\} = \{\delta\}$ and Lemma 1.3.1 follows.

Notation 1.3.1 Fix $n \in \mathbb{N} \setminus \{0\}$ and let $f(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{C}[z]$ with $\deg(f) = n$. Let $Z = \{z_1, \ldots, z_n\}$ be the zeros of $f(z)$, put $a = (z_1 + \cdots + z_n)/n$, and let $Z' = \{z'_1, \ldots, z'_{n-1}\}$ be the zeros of $f'(z)$. $\forall c \in \mathbb{R}$, let $S_c = S_c(f)$ and $\Gamma_c = \Gamma_c(f)$ as defined in Lemma 1.3.1. Let $\mathcal{M}$ be the set of all 1-dimensional analytic real submanifolds of $\mathbb{C}$ and $\forall c \in \mathbb{R}$, put

$$\mathcal{M}_c = \{\{\gamma\} : \gamma \in \Gamma_c\}.$$

By an argument similar to the one used in Section 1.1, we may choose $\omega \in \mathbb{R}$ with the following property. $\forall j \in \{1, \ldots, 2n\}$, let

$$N_j = \begin{cases} \{re^{i(\omega + j2\pi/n)} : r \leq 0\} & \text{if } 1 \leq j \leq n \\ \{re^{i(\omega + j2\pi/n)} : r \geq 0\} & \text{if } n+1 \leq j \leq 2n\,. \end{cases}$$

Then we have

$$\left\{u \in \mathbb{C} : u^n \in \frac{i}{a_n}\mathbb{R}\right\} = \bigcup_{j=1}^{2n} N_j\,.$$

Lemma 1.3.2 $\forall c \in \mathbb{R}$, we have $\mathcal{M}_c \subset \mathcal{M}$.

Proof Fix $c \in \{\mathrm{Re}[f(z)] : z \in \mathbb{C}\}$ and let $\gamma \in \Gamma_c$. Assume $\exists a, b \in \mathbb{R}$ with $a \neq b$ such that $\gamma(a) = \gamma(b)$. Then $\exists$ a bounded open connected set $\Omega \subset \mathbb{C}$ with

$\partial\Omega \subset \{\gamma\}$. $\forall j \in \{1, \ldots, n-1\}$, put $c_j = \mathrm{Re}\big[f(z'_j)\big]$. Let $w \in \Omega \setminus Z'$. Then $f'(w) \neq 0$, so $\exists$ an open neighborhood U of w such that $f(U)$ is open. Thus, the projection $\big\{\mathrm{Re}(v) : v \in f(U)\big\}$ of $f(U)$ on the real axis is also open, hence $\exists v \in f(U)$ with $d = \mathrm{Re}(v) \notin \{c, c_1, \ldots, c_{n-1}\}$. Let $u \in U$ satisfy $v = f(u)$. Consider a curve $\delta \in \Gamma_d$ with $u \in \{\delta\}$. Since $d \neq c$, we have $\{\delta\} \cap \partial\Omega \subset \{\delta\} \cap \{\gamma\} = \emptyset$, therefore $\{\delta\} \subset \Omega$. Since $d \notin \{c_1, \ldots, c_{n-1}\}$, it follows that $\forall j \in \{1, \ldots, n-1\}$ and $\forall y \in \{\delta\}$, we have $\mathrm{Re}\big[f(y)\big] = d \neq \mathrm{Re}\big[f(z'_j)\big]$. That is, $\forall t \in \mathbb{R}$, we have $f'\big[\delta(t)\big] \neq 0$. Then Note 1.1.3 implies that $\forall t_0 \in \mathbb{R}$, $\exists \epsilon > 0$ and a reparametrization $t : (t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}$ of δ such that $\eta = \delta \circ t$ satisfies

$$f\big[\eta(s)\big] = d + is\,, \qquad s \in (t_0 - \epsilon, t_0 + \epsilon)\,.$$

Piecing such local parametrizations together, we obtain $\eta : \mathbb{R} \to \mathbb{C}$ with $\{\eta\} = \{\delta\}$ satisfying

$$f\big[\eta(s)\big] = d + is\,, \qquad s \in \mathbb{R}.$$

But then $\eta(s)$ cannot remain bounded as $s \to \pm\infty$, contradicting $\{\eta\} = \{\delta\} \subset \Omega$. Therefore, $\gamma : \mathbb{R} \to \mathbb{C}$ must be a simple curve and Lemma 1.3.2 follows.

Next, we describe the global structure of the entire set $\mathcal{M}_c$ for any fixed $c \in \mathbb{R}$.

Consider an arbitrary $M \in \mathcal{M}_c$ and choose $\gamma \in \Gamma_c$ with $\{\gamma\} = M$. $\forall j \in \{1, \ldots, n-1\}$, let $r_j = \mathrm{Im}\big[f(z'_j)\big]$ and let

$$r_{\min} = \min\{r_1, \ldots, r_{n-1}\} \qquad \text{and} \qquad r_{\max} = \max\{r_1, \ldots, r_{n-1}\}\,.$$

Put $r = \max\{|r_{\min}|, |r_{\max}|\}$ and fix $R > r$.

Case 1: $\{\gamma\} \cap Z' = \emptyset$. Then by the same argument as for the δ in the proof of Lemma 1.3.2, we may choose a global reparametrization $t : \mathbb{R} \to \mathbb{R}$ of γ such that $\eta = \gamma \circ t$ satisfies

$$f\big[\eta(s)\big] = c + is\,, \qquad s \in \mathbb{R}.$$

Put $u_M = \eta(-R)$ and $v_M = \eta(R)$. Then $u_M \neq v_M$ since γ is simple, and $u_M, v_M \notin Z'$.

Case 2: $\{\gamma\} \cap Z' \neq \emptyset$. Then, using the fact that γ is simple and Z' is finite, we may define $t_{\min} = \min\{t \in \mathbb{R} : \gamma(t) \in Z'\}$ and $t_{\max} = \max\{t \in \mathbb{R} : \gamma(t) \in Z'\}$. Let $I_{\text{left}} = (-\infty, t_{\min})$ and $I_{\text{right}} = (t_{\max}, \infty)$. Since $\{\gamma|_{I_{\text{left}}}\} \cap Z' = \{\gamma|_{I_{\text{right}}}\} \cap Z' = \emptyset$, it follows that $\forall t \in I_{\text{left}} \cup I_{\text{right}}$, we have $f'[\gamma(t)] \neq 0$. Then, as before, $\exists$ reparametrizations $t_p : I_p \to I_{\text{left}}$ and $t_q : I_q \to I_{\text{right}}$, where I_p is either $(-\infty, r_{\min})$ or $(r_{\max}, \infty)$ and I_q is either $(-\infty, r_{\min})$ or $(r_{\max}, \infty)$, such that $\eta_p = \gamma \circ t_p$ and $\eta_q = \gamma \circ t_q$ satisfy

$$f[\eta_p(s)] = c + is\,, \qquad s \in I_p$$

$$f[\eta_q(s)] = c + is\,, \qquad s \in I_q\,.$$

By the choice of η_p and η_q, we have $\{\eta_p\} \subset \{\gamma|_{I_{\text{left}}}\}$ and $\{\eta_q\} \subset \{\gamma|_{I_{\text{right}}}\}$. Then $\exists s \in I_p$ and $\exists t \in I_q$ with $|s| = |t| = R$. Put $u_M = \eta_p(s)$, $v_M = \eta_q(t)$. Then $u_M \in \{\eta_p\} \subset \{\gamma|_{I_{\text{left}}}\}$ and $v_M \in \{\eta_q\} \subset \{\gamma|_{I_{\text{right}}}\}$, so $u_M \neq v_M$ since γ is simple, and $u_M, v_M \notin Z'$.

In both cases we find $u_M, v_M \in M$ with $u_M \neq v_M$, satisfying

$$f(u_M) = c \pm iR$$

$$f(v_M) = c \pm iR\,.$$

Also, $\forall N \in \mathcal{M}_c$ with $N \neq M$, we have $u_M, v_M \notin N$ since $u_M, v_M \notin Z'$.

Let $X, Y \subset \mathbb{C}$ be the solution sets to the equations

$$f(x) = c - iR\,, \qquad x \in \mathbb{C}$$

$$f(y) = c + iR\,, \qquad y \in \mathbb{C}$$

respectively. By the above, we have the one-to-one maps $\sigma, \tau : \mathcal{M}_c \to X \cup Y$, $\sigma(M) = u_M$ and $\tau(M) = v_M$, $M \in \mathcal{M}_c$, satisfying $\sigma(\mathcal{M}_c) \cap \tau(\mathcal{M}_c) = \emptyset$. Therefore, $\mathcal{M}_c$ must be a finite set, satisfying

$$2|\mathcal{M}_c| = \big|\sigma(\mathcal{M}_c)\big| + \big|\tau(\mathcal{M}_c)\big| = \big|\sigma(\mathcal{M}_c) \cup \tau(\mathcal{M}_c)\big| \leq |X \cup Y| \leq 2n\,.$$

That is, $\exists \ell = \ell(c) \in \mathbb{N} \setminus \{0\}$ with $\ell \leq n$ and $\exists M_1, \ldots, M_\ell \in \mathcal{M}$ such that $\mathcal{M}_c = \{M_1, \ldots, M_\ell\}$.

Let $\ell = \ell(0)$ and consider $\mathcal{M}_0 = \{M_1, \ldots, M_\ell\}$.

Fix $j \in \{1, \ldots, \ell\}$ and choose some $\gamma \in \Gamma_0$ with $\{\gamma\} = M_j$. $\forall \epsilon > 0$, put

$$d = d(\epsilon) = \left(\frac{2\epsilon}{|e^{i2\pi/n} - 1|}\right)^n |a_n|\,.$$

Using the same notations as in Case 1 and Case 2 above, we have the following.

Case 1: $\{\gamma\} \cap Z' = \emptyset$. We obtain $\eta : \mathbb{R} \to \mathbb{C}$ with $\{\eta\} = \{\gamma\}$ satisfying

$$f\big[\eta(s)\big] = is\,, \qquad s \in \mathbb{R}.$$

It follows from Lemma 1.2.1 that $\forall \epsilon > 0$, $\exists C > d$ such that $\forall s \in \mathbb{R}$ with $|s| > C$, $\exists b, c \in \mathbb{C}$ with $b^n = -is/a_n$ and $c^n = is/a_n$ satisfying

$$\big|(b + a) - \eta(-s)\big| < \epsilon \qquad \text{and} \qquad \big|(c + a) - \eta(s)\big| < \epsilon\,.$$

Case 2: $\{\gamma\} \cap Z' \neq \emptyset$. We obtain $\eta_p : I_p \to \mathbb{C}$ with $\{\eta_p\} \subset \{\gamma|_{I_{\text{left}}}\}$ and $\eta_q : I_q \to \mathbb{C}$ with $\{\eta_q\} \subset \{\gamma|_{I_{\text{right}}}\}$ satisfying

$$f\big[\eta_p(s)\big] = is\,, \qquad s \in I_p$$

$$f\big[\eta_q(s)\big] = is\,, \qquad s \in I_q\,.$$

Again Lemma 1.2.1 implies that $\forall \epsilon > 0$, $\exists C > d$ such that $\forall s \in I_p$ and $\forall t \in I_q$ with $|s|, |t| > C$, $\exists b, c \in \mathbb{C}$ with $b^n = is/a_n$ and $c^n = it/a_n$ satisfying

$$\big|(b + a) - \eta_p(s)\big| \qquad \text{and} \qquad \big|(c + a) - \eta(q)\big| < \epsilon\,.$$

Because of the choice $C > d$, in both Case 1 and Case 2 we have

$$|b|, |c| > \frac{2\epsilon}{|e^{i2\pi/n} - 1|}$$

therefore $\exists! \mu \in \{1, \ldots, 2n\}$ and $\exists! \nu \in \{1, \ldots, 2n\}$ such that $b \in N_\mu$ and $c \in N_\nu$. It follows that M_j is asymptotic to both $a + N_\mu$ and $a + N_\nu$ since in Case 1 we have

$$\lim_{s \to \pm\infty} \big|\eta(s)\big| = \infty$$

$$\lim_{s \to -\infty} \operatorname{dist}\big[\eta(s), a + N_\mu\big] = \lim_{s \to \infty} \operatorname{dist}\big[\eta(s), a + N_\nu\big] = 0$$

and in Case 2 we have

$$\lim_{\substack{s \in I_p \\ |s| \to \infty}} \big|\eta_p(s)\big| = \lim_{\substack{t \in I_q \\ |t| \to \infty}} \big|\eta_q(t)\big| = \infty$$

$$\lim_{\substack{s \in I_p \\ |s| \to \infty}} \operatorname{dist}\big[\eta_p(s), a + N_\mu\big] = \lim_{\substack{t \in I_q \\ |t| \to \infty}} \operatorname{dist}\big[\eta_q(t), a + N_\nu\big] = 0\,.$$

$\forall j \in \{1, \ldots, \ell\}$, define $\mu = \mu(j), \nu = \nu(j) \in \{1, \ldots, 2n\}$ as above. We obtain the maps $\mu, \nu : \{1, \ldots, \ell\} \to \{1, \ldots, 2n\}$. It follows from Lemma 1.2.2 that $\forall \epsilon > 0$, $\exists C > 0$ such that $\forall k \in \{1, \ldots, 2n\}$ and $\forall b \in N_k$ with $|b| > C$, $\exists z \in \mathbb{C}$ satisfying $f(z) = a_n b^n \in i\mathbb{R}$, hence $z \in \Gamma_0$, and

$$\big|(b + a) - z\big| < \epsilon\,.$$

Therefore, $\forall k \in \{1, \ldots, 2n\}$, $\exists j \in \{1, \ldots, \ell\}$ such that either $k = \mu(j)$ or $k = \nu(j)$. Let $A = \mu\big(\{1, \ldots, \ell\}\big)$ and $B = \nu\big(\{1, \ldots, \ell\}\big)$. Then $A \cup B = \{1, \ldots, 2n\}$. Since $|A| \le \ell$ and $|B| \le \ell$, we have

$$2n \le |A \cup B| \le |A| + |B| \le 2\ell\,.$$

Using Notation 1.3.1, we obtain the following.

Proposition 1.3.1 $\forall c \in \mathbb{R}$, $\exists M_1, \ldots, M_n \in \mathcal{M}$ such that

$$S_c = \bigcup_{j=1}^{n} M_j$$

and $\exists$ maps $\mu, \nu : \{1, \ldots, n\} \to \{1, \ldots, 2n\}$ with $\mu(\{1, \ldots, n\}) \cup \nu(\{1, \ldots, n\}) = \{1, \ldots, 2n\}$ such that $\forall j \in \{1, \ldots, n\}$, M_j is asymptotic to both $a + N_{\mu(j)}$ and $a + N_{\nu(j)}$.

Proof Fix $c \in \mathbb{R}$ and $\forall z \in \mathbb{C}$, let $g(z) = f(z) - c$. Then $S_c(f) = S_0(g)$, and Proposition 1.3.1 follows from the above argument applied to $S_0(g)$.

Section 1.4 Convergence of level curves

Before leaving the topic of level curves, we prove in Proposition 1.4.1 below a simple statement about level curves belonging to the same constant $c \in \mathbb{R}$ of a locally uniformly convergent sequence of nonconstant harmonic functions. This will be used in Chapter 3.

Lemma 1.4.1 Let $U \subset \mathbb{C}$ be a convex domain and let $f \in H(U)$. Assume that $\exists$ an open half plane $H \subset \mathbb{C}$ bounded by a line through $0 \in \mathbb{C}$ such that $f'(U) \subset H$. Then $f\big|_U$ is an invertible map.

Proof Let $f = u + iv$ where $u, v : U \to \mathbb{R}$. Assume on the contrary that $\exists a, b \in U$ with $a \neq b$ such that $f(a) = f(b)$. Since U is convex, the line segment $[a, b] = \big\{a + t(b-a) : t \in [0,1]\big\} \subset U$. Choose some arbitrary $\alpha, \beta \in \mathbb{R}$ with $(\alpha, \beta) \neq (0,0)$, and consider the function

$$g(t) = \alpha u\big[a + t(b-a)\big] + \beta v\big[a + t(b-a)\big], \qquad t \in [0,1].$$

Then $g(0) = \alpha u(a) + \beta v(a) = \alpha u(b) + \beta v(b) = g(1)$, so $\exists t_0 \in (0,1)$ such that $g'(t_0) = 0$. Let $z_0 = a + t_0(b-a)$. We obtain

$$0 = g'(t_0) = \left(\alpha \frac{\partial u}{\partial x}(z_0) + \beta \frac{\partial v}{\partial x}(z_0), \ \alpha \frac{\partial u}{\partial y}(z_0) + \beta \frac{\partial v}{\partial y}(z_0)\right) \begin{pmatrix} \mathrm{Re}(b-a) \\ \mathrm{Im}(b-a) \end{pmatrix}$$

$$= \left(\alpha\frac{\partial u}{\partial x}(z_0) + \beta\frac{\partial v}{\partial x}(z_0)\,,\, -\alpha\frac{\partial v}{\partial x}(z_0) + \beta\frac{\partial u}{\partial x}(z_0)\right)\begin{pmatrix}\mathrm{Re}(b-a)\\ \mathrm{Im}(b-a)\end{pmatrix}$$

$$= \left(\frac{\partial u}{\partial x}(z_0)\,,\, \frac{\partial v}{\partial x}(z_0)\right)\begin{pmatrix}\mathrm{Re}(b-a) & \mathrm{Im}(b-a)\\ \\ -\mathrm{Im}(b-a) & \mathrm{Re}(b-a)\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

where we put $z = x + iy$. By letting $\alpha + i\beta$ vary on the unit circle, we get that the argument of the non-zero number

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$$

must change by an angle $\theta \geq \pi$ as z varies in $[a,b]$. But this contradicts the fact that $f'\big([a,b]\big) \subset f'(U) \subset H$, and Lemma 1.4.1 follows.

Lemma 1.4.2 Let $U \subset \mathbb{C}$ be a domain, let $z_0 \in U$ and let $f \in H(U)$ with $f'(z_0) \neq 0$. Given a sequence $(f_n)_{n=0}^{\infty} \subset H(U)$ with $f_n \to f$ as $n \to \infty$, uniformly on compact sets in U, we have the following. $\exists N \in \mathbb{N}$, $\exists r > 0$ and $\exists$ an open set $W \subset \mathbb{C}$ with $f(z_0) \in W$ satisfying the properties below. Put $D = D(z_0, r)$.

(1) $\overline{D} \subset U$, and $f\big|_D$ and $\forall n \geq N$, $f_n\big|_D$ are all invertible in U.

(2) $W \subset f(D)$ and $\forall n \geq N$, $W \subset f_n(D)$.

Proof (1) Let $L \subset \mathbb{C}$ be the line through $0 \in \mathbb{C}$ for which $f'(z_0) \perp L$, and let H be the open half plane bounded by L with $f'(z_0) \in H$. Choose $r > 0$ such that the disc $D = D(z_0, r)$ satisfies $\overline{D} \subset U$ and $\forall z \in D$, we have

$$\big|f'(z) - f'(z_0)\big| < \frac{1}{4}\big|f'(z_0)\big|\,.$$

Then $f'(D) \subset H$.

Since $f_n \to f$ as $n \to \infty$, uniformly on compact sets in U, we have $f_n' \to f'$ as $n \to \infty$, uniformly on compact sets in U. Since $\overline{D} \subset U$ is compact, it follows that $\exists N \in \mathbb{N}$ such that $\forall z \in D$ and $\forall n \geq N$, we have

$$\big|f_n'(z) - f'(z)\big| < \frac{1}{4}\big|f'(z_0)\big|\,.$$

Then $\forall z \in D$ and $\forall n \geq N$, we have

$$\left|f_n'(z) - f'(z_0)\right| \leq \left|f_n'(z) - f'(z)\right| + \left|f'(z) - f'(z_0)\right| < \frac{1}{4}\left|f'(z_0)\right| + \frac{1}{4}\left|f'(z_0)\right| = \frac{1}{2}\left|f'(z_0)\right|.$$

Therefore, $\forall n \geq N$, we have $f_n'(D) \subset H$. Now (1) follows by Lemma 1.4.1.

(2) Consider the compact set $C = \{z \in \mathbb{C} : |z - z_0| = r\} \subset U$. Let $\gamma = f(C)$. Then $f(z_0) \in \operatorname{int}(\gamma)$. Let $d = \operatorname{dist}[f(z_0), \gamma]$. Then $d > 0$, so we may assume that the above $N \in \mathbb{N}$ is so large that $\forall n \geq N$, we have

$$\left|f_n(z) - f(z)\right| < d/2, \qquad z \in C.$$

Let $W = D[f(z_0), d/2]$. Then $W \subset \operatorname{int}(\gamma)$ and $\forall n \geq N$, we have $W \subset \operatorname{int}(\gamma_n)$ where $\gamma_n = f_n(C)$. By the argument principle, we have $W \subset f(D)$ and $W \subset f_n(D)$, $n \geq N$, and (2) follows.

Proposition 1.4.1 Let $U \subset \mathbb{C}$ be a domain, and let $f \in H(U)$ and $(f_n)_{n=0}^{\infty} \subset H(U)$ satisfy $f_n \to f$ as $n \to \infty$, uniformly on compact sets in U. Given an open interval $I \subset \mathbb{R}$ and a smooth curve $\zeta : I \to U$ such that

$$f[\zeta(t)] = it, \qquad t \in I$$

we have the following. $\forall$ compact subinterval $K \subset I$, $\exists$ an open neighborhood V of $\{\zeta|_K\}$, $\exists$ a subsequence $(m_\ell)_{\ell=0}^{\infty} \subset (n)_{n=0}^{\infty}$, $\exists$ an open interval $J \subset \mathbb{R}$ with $K \subset J$ and $\exists$ smooth curves $\eta : J \to U$ and $\eta_{m_\ell} : J \to U$, $\ell \in \mathbb{N}$, satisfying the properties below.

(1) $f[\eta(t)] = it, \quad t \in J.$

(2) $\forall \ell \in \mathbb{N}$, $f_{m_\ell}[\eta_{m_\ell}(t)] = it$, $t \in J$.

(3) $\eta_{m_\ell} \to \eta$ as $\ell \to \infty$, uniformly on compact sets in J.

(4) $\eta_{m_\ell}' \to \eta'$ as $\ell \to \infty$, uniformly on compact sets in J.

(5) $\{\eta\} \cap V = \{z \in V : \mathrm{Re}[f(z)] = 0\}$.

(6) $\forall \ell \in \mathbb{N}$, $\{\eta_{m_\ell}\} \cap V = \{z \in V : \mathrm{Re}[f_{m_\ell}(z)] = 0\}$.

Proof Fix some compact interval $K \subset I$.

Fix $t \in K$. Then $f'[\zeta(t)] \neq 0$ since $f'[\zeta(t)]\zeta'(t) = i$. By Lemma 1.4.2, it follows that $\exists N_t \in \mathbb{N}$, $\exists r_t > 0$, $\exists$ an open set $W_t \ni f[\zeta(t)] = it$ such that $D_t = D[\zeta(t), r_t)]$ satisfies that $f|D_t$ and $\forall n \geq N_t$, $f_n|D_t$ are all invertible and $W_t \subset f(D_t)$ and $\forall n \geq N_t$, $W_t \subset f_n(D_t)$.

We obtain

$$iK \subset \bigcup_{t \in K} W_t .$$

But iK is compact, therefore $\exists m \in \mathbb{N} \setminus \{0\}$ and $\exists t_1, \ldots, t_m \in K$ with $t_1 < \cdots < t_m$ such that $iK \subset \bigcup_{j=1}^m Wt_j$. Put

$$D_j = D_{t_j}, \qquad j \in \{1, \ldots, m\}, \qquad \text{and} \qquad D = \bigcup_{j=1}^m D_j$$

$$W_j = W_{t_j}, \qquad j \in \{1, \ldots, m\}, \qquad \text{and} \qquad W = \bigcup_{j=1}^m W_j .$$

Shrinking the W_j if necessary, we may assume that $W_j \cap W_{j+1} \neq \emptyset$, $j = 1, \ldots, m-1$, but $W_j \cap W_{j+2} = \emptyset$, $j = 1, \ldots, m-2$ in case $m \geq 3$. Let $N = \max\{N_{t_1}, \ldots, N_{t_m}\}$.

$\forall j \in \{1, \ldots, m\}$, put $g_j = (f|D_j)^{-1}$. Then the function elements $(g_j|_{W_j}, W_j)$, $j = 1, \ldots, m$, define some $g \in H(D)$. It follows that g is invertible and $g^{-1} = f|_{g(W)}$.

$\forall n \geq N$ and $\forall j \in \{1, \ldots, m\}$, put $g_{n,j} = (f_n|_{D_j})^{-1}$. Then $\forall n \geq N$, the function elements $(g_{n,j}|_{W_j}, W_j)$, $j = 1, \ldots, m$, define some $g_n \in H(W)$. It follows that $\forall n \geq N$, g_n is invertible and $g_n^{-1} = f_n|g_n(W)$.

Now $\forall n \geq N$, we have $g_n(W) \subset D$. Therefore, the sequence $(g_n)_{n=N}^\infty \subset H(W)$ is uniformly bounded. By the Vitali-Montel theorem, $\exists$ a subsequence $(n_k)_{k=0}^\infty \subset (n)_{n=N}^\infty$ and $\exists h \in H(W)$ such that $g_{n_k} \to h$ as $k \to \infty$, uniformly on compact sets in W. We claim that $g = h$. Fix an arbitrary $w \in W$ and let $z = h(w) = \lim_{k\to\infty} g_{n_k}(w)$. Now $\forall y \in D$ and $\forall (y_k)_{k=0}^{\infty} \subset D$ with $y_k \to y$ as $k \to \infty$, we have

$$f_k(y_k) \to f(y) \qquad \text{as} \qquad k \to \infty\,.$$

Indeed, $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ such that $\forall k \geq M$, we have

$$\bigl|f(y_k) - f(y)\bigr| < \epsilon/2 \qquad \text{and} \qquad \bigl|f_k(x) - f(x)\bigr| < \epsilon/2\,, \qquad x \in D$$

since $\overline{D}$ is compact. It follows that $\forall k \geq M$, we have

$$\bigl|f_k(y_k) - f(y)\bigr| \leq \bigl|f_k(y_k) - f(y_k)\bigr| + \bigl|f(y_k) - f(y)\bigr| < \epsilon/2 + \epsilon/2 = \epsilon\,.$$

Therefore,

$$w = \lim_{k\to\infty} f_{n_k}\bigl[g_{n_k}(w)\bigr] = f\bigl[\lim_{k\to\infty} g_{n_k}(w)\bigr] = f(z)\,.$$

We obtain $g(w) = g\bigl[f(z)\bigr] = z = h(w)$.

We use a similar argument for $g \in H(W)$ and $(g_{n_k})_{k=0}^{\infty} \subset H(W)$ as the one we used for $f \in H(U)$ and $(f_n)_{n=0}^{\infty} \subset H(U)$ above.

$\forall t \in K$, $\exists M_t \in \mathbb{N}$ and $\exists \varrho_t > 0$ such that $E_t = D(it, \varrho_t)$ satisfies $E_t \subset W$, and $\exists V_t \ni g(it) = \zeta(t)$ satisfying $V_t \subset g(E_t)$ and $\forall k \geq M_t$, $V_t \subset g_{n_k}(E_t)$. It follows that $iK \subset \bigcup_{t\in K} E_t$. Therefore, $\exists \ell \in \mathbb{N}\setminus\{0\}$ and $\exists \tau_1, \ldots, \tau_\ell \in K$ such that $iK \subset \bigcup_{j=1}^{\ell} E_{\tau_\ell}$. Put $E = \bigcup_{j=1}^{\ell} E_{\tau_j}$, $V = \bigcup_{j=1}^{\ell} V_{\tau_j}$ and $M = \max\{M_{\tau_1}, \ldots, M_{\tau_\ell}\}$. Let $J = \{t \in \mathbb{R} : it \in E\}$. Then $K \subset J$. Also, let

$$\eta(t) = g(it)\,, \qquad t \in J$$

and $\forall k \geq M$, let

$$\eta_{n_k}(t) = g_{n_k}(it)\,, \qquad t \in J\,.$$

It follows that $\eta_{n_k} \to \eta$ as $k \to \infty$, uniformly on compact sets in J, since $g_{n_k} \to g$ as $k \to \infty$, uniformly on compact sets in W. Also, $\eta'_{n_k} \to \eta'$ as $k \to \infty$, uniformly on compact sets in J, since $\eta'(t) = ig'(it)$, $t \in J$ and $\forall k \geq M$, $\eta'_{n_k}(t) = ig'_{n_k}(it)$, $t \in J$, and $g'_{n_k} \to g'$ as $k \to \infty$, uniformly on compact sets in W.

This proves properties (1),(2),(3) and (4). But (5) and (6) are also clear since we have $V \subset g(E)$ and $\forall k \geq M$, $V \subset g_{n_k}(E)$, and $iJ = i\mathbb{R} \cap E$. Therefore, Proposition 1.4.1 follows.

CHAPTER 2

THE RIEMANN SURFACE $\mathcal{R}$

Section 2.1 Alternative description of $\mathcal{R}$

Let $m \in \mathbb{N} \setminus \{0\}$ and consider a polynomial

$$P_{2m}(z) = \sum_{j+k=2m} a_{j,k} z^j \bar{z}^k \,, \qquad z \in \mathbb{C}$$

so that P_{2m} is real-valued and subharmonic but not harmonic at each $z \in \mathbb{C}$. Then $a_{j,k} = \bar{a}_{k,j}$ and $a_{m,m} > 0$. Put

$$\Omega = \big\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}(w^{2m}) + \delta P_{2m}(z) + \mathrm{Re}(z^{2m}) < 0\big\}$$

where $\delta > 0$ is chosen as in the Introduction. Then we have the Riemann surface $\mathcal{R}$ associated with Ω, and we can use $\mathcal{R}$ to define a peak function on the domain

$$\Omega' = \big\{(z',w') \in \mathbb{C}^2 : \mathrm{Re}(w') + \delta P_{2m}(z') + \mathrm{Re}\big[(z')^{2m}\big] < 0\big\}$$

as also described in the Introduction.

Fix some $\eta \in \mathbb{C}$. Then we have

$$\big\{z \in \mathbb{C} : \mathrm{Re}(\eta^{2m} z^{2m}) + \delta P_{2m}(z) + \mathrm{Re}(z^{2m}) < 0\big\} = \bigcup_{j=1}^{N(\eta)} T_j(\eta)$$

where $T_1(\eta), \ldots, T_{N(\eta)}(\eta) \subset \mathbb{C}$ are disjoint open sectors, and the definition of $\mathcal{R}$ is based on the way these sectors vary with $\eta \in \mathbb{C}$. Put $\xi = \eta^{2m} + 1$ and let

$$P(\xi, z) = \mathrm{Re}(\xi z^{2m}) + \delta P_{2m}(z) \,, \qquad z \in \mathbb{C}.$$

Put $a_0 = a_{m,m}$ and $\forall j \in \{1, \dots, m-1\}$, put $a_j = a_{m+j,m-j}$. Define

$$p(\xi, u) = \xi u^{2m} + \delta \sum_{j=0}^{m-1} a_j u^{2j}\,, \qquad z \in \mathbb{C}. \tag{2.1.1}$$

Then $\forall z = |z|e^{i\theta} \in \mathbb{C} \setminus \{0\}$, we have

$$P(\xi, z) = |z|^{2m} \mathrm{Re}\big[p(\xi, e^{i\theta})\big]\,.$$

The fact that the function $z \mapsto P(\xi, z)$, $z \in \mathbb{C}$, is subharmonic but not harmonic for any fixed $\xi \in \mathbb{C}$ is equivalent to

$$\mathrm{Re}\big[4m^2 f_\xi(u) - u f_\xi'(u) - u^2 f_\xi''(u)\big] \geq 0\,, \qquad u \in \mathbb{C}, \quad |u| = 1 \tag{2.1.2}$$

whenever $\xi \in \mathbb{C}$ is fixed and we put

$$f_\xi(u) = p(\xi, u)\,, \qquad z \in \mathbb{C}.$$

This is further equivalent to the statement that $\forall$ such f_ξ, we have

$$\mathrm{Re}\big[4m^2 f_\xi(u) - u f_\xi'(u) - u^2 f_\xi''(u)\big] > 0\,, \qquad u \in \mathbb{C}, \quad |u| < 1$$

since $\forall \xi \in \mathbb{C}$, the function

$$r_\xi(u) = \mathrm{Re}\big[4m^2 f_\xi(u) - u f_\xi'(u) - u^2 f_\xi''(u)\big]\,, \qquad u \in \mathbb{C}$$

is a real-valued harmonic polynomial with constant term $4m^2 a_0 = 4m^2 a_{m,m} > 0$.

Fix $\eta \in \mathbb{C}$, put $\xi = \eta^{2m} + 1$ and let

$$g_\xi(\theta) = \mathrm{Re}\big[p(\xi, e^{i\theta})\big]\,, \qquad \theta \in \mathbb{R}.$$

Then the intersections of the open sectors $T_1(\eta), \dots, T_{N(\eta)}(\eta) \subset \mathbb{C}$ with the unit circle are the open arcs $A_1(\eta), \dots, A_{N(\eta)}(\eta)$ given by

$$\bigcup_{j=1}^{N(\eta)} A_j(\eta) = \{e^{i\theta} : g_\xi(\theta) < 0\}\,.$$

Therefore, the authors in [**BF**] study properties of these open arcs, using the fact that $\forall \xi \in \mathbb{C}$, we have

$$4m^2 g_\xi(\theta) + g_\xi''(\theta) = r_\xi(e^{i\theta}) \geq 0\,, \qquad \theta \in \mathbb{R}$$

where equality holds for at most a finite number of values of $\theta \in [0, 2\pi)$.

Put in this light, the Riemann surface $\mathcal{R}$ is really associated with the polynomial $p(\xi, z)$, $\xi, z \in \mathbb{C}$, as in (2.1.1), satisfying (2.1.2). Such a Riemann surface can be defined in the slightly more general setting described below.

Let $n \in \mathbb{N} \setminus \{0\}$, let $a_0, \ldots, a_{n-1} \in \mathbb{C}$ and put

$$p(\xi, z) = \xi z^n + \sum_{j=0}^{n-1} a_j z^j\,, \qquad z \in \mathbb{C}. \tag{2.1.3}$$

As above, $\forall \xi \in \mathbb{C}$, define

$$f_\xi(z) = p(\xi, z)\,, \qquad z \in \mathbb{C}$$

and assume that $\forall \xi \in \mathbb{C}$, we have

$$\mathrm{Re}\big[n^2 f_\xi(z) - z f_\xi'(z) - z^2 f_\xi''(z)\big] > 0\,, \qquad z \in \mathbb{C}, \quad |z| < 1\,.$$

Now $\forall \xi \in \mathbb{C}$, we will study the properties of the set

$$\big\{z \in \mathbb{C} : |z| < 1, \mathrm{Re}\big[f_\xi(z)\big] < 0\big\}$$

instead of studying the set

$$\{e^{i\theta} : g_\xi(\theta) < 0\} = \big\{e^{i\theta} : \mathrm{Re}\big[f_\xi(e^{i\theta})\big] < 0\big\}\,.$$

This way, we will be able to give a concrete definition of a Riemann surface $\mathcal{R}$, associated with the polynomial p given in (2.1.3) in analogy with the Bedford-Fornaess Riemann surface $\mathcal{R}$ associated with the domain Ω above.

The next 2 lemmas correspond to Lemma 1.1 and Lemma 1.2 in [**BF**].

Lemma 2.1.1 Let $c > 0$ and let $g : \mathbb{R} \to \mathbb{R}$ be a real-analytic function such that

$$(2.1.4) \qquad c^2 g(\theta) + g''(\theta) \geq 0\,, \qquad \theta \in \mathbb{R}.$$

If $g \not\equiv 0$ and $g(\alpha) = g'(\alpha) = 0$ for some $\alpha \in \mathbb{R}$, then $\exists m \in \mathbb{N} \setminus \{0\}$ such that $\forall j \in \{1, \ldots, 2m-1\}$, we have $g^{(j)}(\alpha) = 0$ and $g^{(2m)}(\alpha) > 0$.

Proof Since $g \not\equiv 0$ is real-analytic, $\exists n \geq 2$ such that $g^{(n)}(\alpha) \neq 0$ and

$$g(\theta) = \frac{g^{(n)}(\alpha)}{n!}(\theta - \alpha)^n + O\big[(\theta - \alpha)^{n+1}\big]\,.$$

By (2.1.4), it follows that

$$\begin{aligned} 0 &\leq c^2 g(\theta) + g''(\theta \\ &= \big[c^2 + n(n-1)\big]\frac{g^{(n)}(\alpha)}{n!}(\theta - \alpha)^{n-2} + O\big[(\theta - \alpha)^{n-1}\big]\,, \qquad \theta \in \mathbb{R}. \end{aligned}$$

Therefore, $g^{(n)}(\alpha) > 0$ and n is even. This proves Lemma 2.1.1.

Lemma 2.1.2 Let $g : \mathbb{R} \to \mathbb{R}$ be as in Lemma 2.1.1 and let $\alpha < \beta$. Then we have the following.

(1) If $\forall \theta \in (\alpha, \beta)$, we have $g(\theta) < 0$, then $\beta - \alpha \leq \pi/c$.

(2) If $\forall \theta \in (\alpha, \beta)$, we have $g(\theta) > 0$ and $g(\alpha) = g(\beta) = 0$, then $\beta - \alpha \geq \pi/c$.

In either case, strict inequality holds unless $\exists c_1, c_2 \in \mathbb{R}$ such that

$$(2.1.5) \qquad g(\theta) = c_1 \sin(\theta) + c_2 \cos(\theta)\,, \qquad \theta \in \mathbb{R}.$$

Proof Since $\forall \theta \in (\alpha, \beta)$, we have $g(\theta) \neq 0$, we may define

$$h(\theta) = \arctan\left[\frac{g'(\theta)}{c g(\theta)}\right], \qquad \theta \in \mathbb{R}$$

so that $h : (\alpha, \beta) \to (-\pi/2, \pi/2)$ is a real-analytic function.

(1) In this case (2.1.4) implies

$$c^2\left[g(\theta)\right]^2 + g''(\theta)g(\theta) \le 0\,, \qquad \theta \in (\alpha,\beta)$$

and equality holds everywhere if and only if we have (2.1.5). Therefore, we have

$$h'(\theta) = \frac{g''(\theta)g(\theta) - \left[g'(\theta)\right]^2}{\left(1 + \left[\dfrac{g'(\theta)}{cg(\theta)}\right]^2\right)c\left[g(\theta)\right]^2} \le -c\,, \qquad \theta \in (\alpha,\beta)$$

hence

$$-\pi \le \int_\alpha^\beta h'(\theta)d\theta < -c(\beta-\alpha)$$

unless (2.1.5) holds. This proves (1).

(2) Since $\forall\theta \in (\alpha,\beta)$, we have $g(\theta) > 0$ and $g(\alpha) = g(\beta) = 0$, we have $g'(\alpha) \ge 0 \ge g'(\beta)$. We claim that

$$\lim_{\theta\to\alpha+} \frac{g'(\theta)}{g(\theta)} = \infty \qquad \text{and} \qquad \lim_{\theta\to\beta-} \frac{g'(\theta)}{g(\theta)} = -\infty\,.$$

The first equality is obvious if $g'(\alpha) > 0$, and it follows from Lemma 2.1.1 if $g'(\alpha) = 0$. The other limit is similar.

Now (2.1.4) implies $c^2g(\theta) + g''(\theta) \ge 0$, $\theta \in (\alpha,\beta)$, with equality everywhere if and only if (2.1.5) is true. Computing as before, we get $h'(\theta) \ge -c$, $\theta \in (\alpha,\beta)$, hence

$$-\pi = \int_\alpha^\beta h'(\theta)d\theta > -c(\beta-\alpha)$$

unless we have (2.1.5). This completes the proof of Lemma 2.1.2.

Lemma 2.1.3 Let $f(z) = \sum_{j=0}^n a_j z^j \not\equiv 0$, $z \in \mathbb{C}$, and let $r > 0$ satisfy

(2.1.6) $$\mathrm{Re}\left[n^2 f(z) - zf'(z) - z^2 f''(z)\right] \ge 0\,, \qquad z \in \mathbb{C}, \quad |z| = r\,.$$

Then we have the following.

(1) If $\alpha < \beta$ so that $\forall\theta \in (\alpha, \beta)$, we have $\mathrm{Re}\big[f(re^{i\theta})\big] < 0$, then $\beta - \alpha < \pi/n$.

(2) If $\alpha < \beta$ so that $\mathrm{Re}\big[f(re^{i\alpha})\big] = \mathrm{Re}\big[f(re^{i\beta})\big] = 0$ and $\forall\theta \in (\alpha, \beta)$, we have $\mathrm{Re}\big[f(re^{i\theta})\big] > 0$, then $\beta - \alpha > \pi/n$.

(3) If $\mathrm{Re}\big[f(re^{i\alpha})\big] = \mathrm{Im}\big[re^{i\alpha} f'(re^{i\alpha})\big] = 0$, then $\forall\theta \in [\alpha - \pi/n, \alpha) \cap (\alpha, \alpha + \pi/n]$, we have $\mathrm{Re}\big[f(e^{i\theta})\big] > 0$.

Proof Let $g(\theta) = \mathrm{Re}\big[f(re^{i\theta})\big]$, $\theta \in \mathbb{R}$. Then $g \not\equiv 0$ since $f \not\equiv 0$. Also, (2.1.6) implies

$$\mathrm{Re}\big[n^2 f(z) - zf'(z) - z^2 f''(z)\big] > 0\,, \qquad z \in \mathbb{C}, \quad |z| < r$$

hence $a_0 = \mathrm{Re}\big[f(0)\big] > 0$. It follows that g is not of the form (2.1.5). Now $g'(\theta) = -\mathrm{Im}\big[re^{i\theta} f'(re^{i\theta})\big]$, hence (1), (2) and (3) follow from Lemma 2.1.1 and Lemma 2.1.2.

Lemma 2.1.4 Let $f(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$, $z \in \mathbb{C}$, satisfy

$$\mathrm{Re}\big[n^2 f(z) - zf'(z) - z^2 f''(z)\big] \geq 0\,, \qquad z \in \mathbb{C}, \quad |z| = 1\,. \tag{2.1.7}$$

Then we have the following.

If $a = |a|e^{i\alpha}$ with $0 < |a| < 1$ so that $\mathrm{Re}\big[f(a)\big] = \mathrm{Im}\big[af'(a)\big] = 0$, then we have

$$\mathrm{Re}\big[f(re^{i\alpha})\big] < 0\,, \qquad r \in (|a|, 1]\,.$$

Proof Let $a = |a|e^{i\alpha}$ as above and fix $r \in (|a|, 1]$.

Let

$$q(z) = f(z^2) = \sum_{j=0}^{n} a_j z^{2j}\,, \qquad z \in \mathbb{C}.$$

Now

$$n^2 f(z) - zf'(z) - z^2 f''(z) = \sum_{j=0}^{n} (n^2 - j^2) a_j z^j\,, \qquad z \in \mathbb{C}$$

and

$$4n^2 q(z) - zq'(z) - z^2 q''(z) = 4\sum_{j=0}^{n} (n^2 - j^2) a_j z^{2j}\,, \qquad z \in \mathbb{C}$$

hence we have

$$\operatorname{Re}\big[4n^2q(z) - zq'(z) - z^2q''(z)\big] \geq 0\,, \qquad z \in \mathbb{C}, \quad |z| = 1\,. \tag{2.1.8}$$

Let $b = |b|e^{i\beta}$ with $|b| = |a|^{1/2}$ and $\beta = \alpha/2$, let $s = r^{1/2}$ and $\forall z \in \mathbb{C}$, put

$$Q(z) = \operatorname{Re}\left[2n\sum_{j=0}^{n-1} a_j s^{2j} z^{n+j}\bar{z}^{n-j} - \sum_{j=0}^{n-1}(n+j)a_j s^{2j} z^{2j} - \sum_{j=0}^{n-1}(n-j)\bar{a}_j s^{2j}\bar{z}^{2j}\right].$$

Then $\forall z = |z|e^{i\theta} \in \mathbb{C} \setminus \{0\}$, we have

$$4\partial\bar{\partial}Q(z) = 2n|z|^{2n-2}\operatorname{Re}\big[4n^2q(se^{i\theta}) - se^{i\theta}q'(se^{i\theta}) - s^2e^{i2\theta}q''(se^{i\theta})\big] \geq 0\,. \tag{2.1.9}$$

Now $q \not\equiv 0$ and (2.1.8) together imply that

$$\operatorname{Re}\big[4n^2q(z) - zq'(z) - z^2q''(z)\big] > 0\,, \qquad z \in \mathbb{C}, \quad |z| < 1$$

hence $Q(0) = -2n\operatorname{Re}(a_0) < 0$. Therefore, $Q \not\equiv 0$. But $\forall\theta \in \mathbb{R}$, we have $Q(e^{i\theta}) = 0$ and Q is subharmonic by (2.1.9), hence $Q(z) < 0$ whenever $|z| < 1$. In particular, we have

$$\begin{aligned} 0 &> Q(bs^{-1}) \\ &= \operatorname{Re}\left[2ns^{-2n}\sum_{j=0}^{n-1} a_j s^{2j} b^{n+j}\bar{b}^{n-j} - \sum_{j=0}^{n-1}(n+j)a_j b^{2j} + \sum j = 0^{n-1}(n-j)\bar{a}_j\bar{b}^{2j}\right]. \end{aligned}$$

But by assumption, we have

$$\begin{aligned} 0 = 4n\operatorname{Re}\big[f(a)\big] + i4\operatorname{Im}\big[af'(a)\big] &= 2n\operatorname{Re}\big[q(b)\big] + i2\operatorname{Im}\big[bq'(b)\big] \\ &= 2na_nb^{2n} + \sum_{j=0}^{n-1}(n+j)a_jb^{2j} + \sum_{j=0}^{n-1}(n-j)\bar{a}_j\bar{b}^{2j}\,. \end{aligned}$$

Therefore, we have

$$0 > Q(bs^{-1}) = 2n|b|^{2n}s^{-2n}\operatorname{Re}\big[q(se^{i\beta})\big] = 2n|a|^n r^{-n}\operatorname{Re}\big[f(re^{i\alpha})\big]$$

and Lemma 2.1.4 follows.

Next, we will apply the results of Chapter 1 to give a convenient description of the Riemann surface $\mathcal{R}$ mentioned above.

Let $p(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$, $z \in \mathbb{C}$, and let

$$\Gamma = \{z \in \mathbb{C} : \mathrm{Re}[f(z)]\}.$$

Then we have

$$\Gamma = \bigcup_{j=1}^{n} C_j$$

where $C_1, \ldots, C_n$ are distinct smooth curves.

We have $a \in C_j \cap C_k$ with $j \neq k$ if and only if $\mathrm{Re}[f(a)] = f'(a) = 0$. Let $a \in \mathbb{C}$ be such a point and assume that $f'(a) = \cdots = f^{(m-1)}(a) = 0$ but $f^{(m)}(a) \neq 0$. Then exactly m of the C_j intersect at a and their tangent lines at a divide a circle around a into $2m$ equal parts.

Let $a \neq 0$ be such that $f(a) = ir$ for some $r \in \mathbb{R}$ and $f'(a) \neq 0$. Then $\exists \epsilon > 0$ such that the unique curve $C \subset \Gamma$ through a has a real-analytic parametrization $t \mapsto \gamma(t)$, $t \in (r - \epsilon, r + \epsilon)$ satisfying $\gamma(t_0) = a$ and

$$f[\gamma(t)] = it, \qquad t \in (r - \epsilon, r + \epsilon).$$

Write

$$\gamma(t) = \varrho(t) e^{i\theta(t)}, \qquad t \in (r - \epsilon, r + \epsilon).$$

Then $\varrho'(r) = 0$ is equivalent to $\mathrm{Re}[\gamma'(r)/\gamma(r)] = 0$ which is further equivalent to

$$\mathrm{Im}[af'(a)] = 0.$$

That is, the tangent line to C at a is perpendicular to a if and only if $\mathrm{Im}[af'(a)] = 0$.

Now we also assume that

$$\mathrm{Re}[n^2 f(z) - zf'(z) - z^2 f''(z)] \geq 0, \qquad z \in \mathbb{C}, \quad |z| = 1$$

as before. Then $\mathrm{Re}[f(0)] = \mathrm{Re}(a_0) > 0$, hence $0 \notin \Gamma$.

Let $w_1 = r_1 e^{i\omega_1}$ with $0 < r < 1$ satisfy

$$r_1 = \min\{|z| : z \in \Gamma\}\,. \tag{2.1.10}$$

If there were 2 distinct $C_j \subset \Gamma$ through w_1, then they would meet at a positive angle, so one of them would intersect the circle $T_1 = \{z \in \mathbb{C} : |z| = r_1\}$ transversally, contradicting (2.1.10). Therefore, $\exists$ a unique curve $C_1 \subset \Gamma$ through w_1.

Now C_1 has to be tangential to T_1 by (2.1.10), so we have

$$\mathrm{Im}[w_1 f'(w_1)] = 0$$

by the above discussion. It follows that $\forall r \in (r_1, 1]$, we have

$$\mathrm{Re}[f(re^{i\omega_1})] < 0$$

by Lemma 2.1.3.

For each $r \in (r_1, 1]$, let $\alpha_1(r) < \omega_1 < \beta_1(r)$ satisfy

$$\mathrm{Re}[f(re^{i\alpha_1(r)})] = \mathrm{Re}[f(re^{i\beta_1(r)})] = 0$$

$$\mathrm{Re}[f(re^{i\theta})] < 0\,, \qquad \alpha_1(r) < \theta < \beta_1(r)\,.$$

Then by Lemma 2.1.3 (1), we have

$$\beta_1(r) - \alpha_1(r) < \pi/n$$

and by Lemma 2.1.3 (3), we have

$$\mathrm{Re}[f(r_1 e^{i\theta})] > 0\,, \qquad \theta \in [\omega_1 - \pi/n, \omega_1) \cap (\omega_1, \omega_1 + \pi/n]$$

hence we must have

$$\alpha_1(r) \to \omega_1 \qquad \text{and} \qquad \beta_1(r) \to \omega_1 \qquad \text{as} \qquad r \to r_1^+\,.$$

Put $\alpha_1(r_1) = \beta_1(r_1) = \omega_1$ and define

$$B_1 = \left\{re^{i\alpha_1(r)} : r \in [r_1, 1]\right\} \cup \left\{re^{i\beta_1} : r \in [r_1, 1]\right\}.$$

Now Lemma 2.1.3 (3) also implies that $\forall z \in B_1 \setminus \{w_1\}$, we must have

$$\mathrm{Im}\big[zf'(z)\big] \neq 0.$$

That is, $\forall z \in B_1$, $\exists$ a unique smooth curve in Γ through z. Bycompactness, B_1 must be contained in the same curve $C_1 \subset \Gamma$ and no other smooth curve in Γ intersects B_1. Define also

$$W_1 = \{re^{i\theta} : r \in (r_1, 1) \quad \text{and} \quad \alpha_1(r) < \theta < \beta_1(r)\}$$

and let

$$A_1 = \{e^{i\theta} : \alpha_1(1) < \theta < \beta_1(1)\}.$$

Now let $w_2 = r_2 e^{i\omega_2}$ with $0 < r_2 < 1$ satify

$$r_2 = \min\{|z| : z \in \Gamma \setminus B_1\}.$$

Repeat the above argument to obtain the set

$$B_2 = \left\{re^{i\alpha_2(r)} : r \in [r_2, 1]\right\} \cup \left\{re^{i\beta_2} : r \in [r_2, 1]\right\}$$

contained in some smooth curve $C_2 \subset \Gamma$ and intersecting no other such curve, satisfying

$$\mathrm{Re}\big[f(re^{i\alpha_2(r)})\big] = \mathrm{Re}\big[f(re^{i\beta_2(r)})\big] = 0$$

$$\mathrm{Re}\big[f(re^{i\theta})\big] < 0, \qquad r \in (r_2, 1], \quad \alpha_2(r) < \theta < \beta_2(r).$$

Also, define the open set

$$W_2; \{re^{i\theta} : r \in (r_2, 1) \quad \text{and} \quad \alpha_2(r) < \theta < \beta_2(r)\}$$

and the open arc

$$A_2 = \{e^{i\theta} : \alpha_2(1) < \theta < \beta_2(1)\}.$$

Keep repeating the same until we exhaust $\Gamma \cap \{z \in \mathbb{C} : |z| < 1\}$, that is, until we get

$$\Gamma \cap \{z \in \mathbb{C} : |z| \leq 1\} = \bigcup_{j=1}^{m} B_j$$

for some $m \in \{1, \dots, n\}$, where $B_1, \dots, b_m$ are disjoint smooth curves. Then we will also have the disjoint unions

$$\{z \in \mathbb{C} : |z| < 1 \quad \text{and} \quad \mathrm{Re}[f(z)] < 0\} = \bigcup_{j=1}^{m} W_j$$

$$\{z \in \mathbb{C} : |z| = 1 \quad \text{and} \quad \mathrm{Re}[f(z)] < 0\} = \bigcup_{j=1}^{m} A_j$$

satisfying $\partial W_j = B_j \cup A_j$, $j \in \{1, \dots, m\}$.

Since $\forall j \in \{1, \dots, m\}$, $w_j = r_j e^{i\omega_j}$ and $\omega_j \in A_j$, Lemma 2.1.3 (2) implies that the w_j are separated by an angle of at least π/n.

Let

$$p(\xi, z) = \xi z^n + \sum_{j=0}^{n-1}, \qquad \xi, z \in \mathbb{C}$$

and for each fixed $\xi \in \mathbb{C}$, consider the polynomial

$$f_\xi(z) = p(\xi, z), \qquad z \in \mathbb{C}.$$

By the above arguments, we have that $\forall \xi \in \mathbb{C}$, $\exists m = m(\xi) \in \mathbb{N}$ and $\exists w_1(\xi), \dots, w_m(\xi)$, $B_1(\xi), \dots, B_m(\xi)$, $W_1(\xi), \dots, W_m(\xi)$ and $A_1(\xi), \dots, A_m(\xi)$ as described above.

Clearly, $w_1(\xi), \dots, w_m(\xi)$ are precisely the solutions to the system of equations

$$\mathrm{Re}[f_\xi(w)] = \mathrm{Im}[w f'_\xi(w)] = 0, \qquad w \in \mathbb{C}, \quad |w| < 1$$

which is equivalent to the single equation

$$2n\xi w^n + \sum_{j=0}^{n-1}(n+j)a_j w^j + \sum_{j=0}^{n-1}(n-j)\bar{a}_j \bar{w}^j = 0\,, \qquad w \in \mathbb{C}, \quad |w| < 1\,. \tag{2.1.11}$$

The way the $w_j(\xi)$ are separated for each $\xi \in \mathbb{C}$ for which $m(\xi) \neq 0$ implies that each w_j can be viewed as a local real-analytic diffeomorphism. Therefore, (2.1.11) can be used to define a Riemann surface with boundary $\partial\mathcal{R}$ given by

$$\mathcal{R} = \big\{(\xi, w) \in \mathbb{C}^2 : |w| < 1 \quad \text{and} \quad (\xi, w) \quad \text{satisfies (2.1.11)}\big\}$$

where the projection $\pi : \mathcal{R} \to U$ onto $U \subset \mathbb{C}$, $\pi(\xi, w) = \xi$, is the local coordinate, and

$$\partial\mathcal{R} = \big\{(\xi, w) \in \mathbb{C}^2 : |w| = 1 \quad \text{and} \quad (\xi, w) \quad \text{satisfies (2.1.11)}\big\}\,.$$

Since clearly $\exists K > 0$ such that $|\xi| > K$ implies $m(\xi) = n$, and also $w_1(\xi), \ldots, w_n(\xi) \to 0$ as $|\xi| \to \infty$, $\mathcal{R}$ extends to a Riemann surface over $U \cup \{\infty\} \subset \mathbb{P}$.

Section 2.2 The map $\xi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ and its critical points

Fix $n \in \mathbb{N} \setminus \{0\}$ and $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Let

$$p(\xi, z) = \xi z^n + \sum_{j=0}^{n-1} ajz^j\,, \qquad \xi, z \in \mathbb{C}.$$

We will use the notation $\partial_2 p = \partial p/\partial z$ and $\partial_2^2 p = \partial^2 p/\partial z^2$. Consider the equation

$$2n\mathrm{Re}\big[p(\xi, z)\big] + i2\mathrm{Im}\big[z\partial_2 p(\xi, z)\big] = 0\,, \qquad \xi, z \in \mathbb{C}.$$

The above equation can be put in the equivalent form

$$2n\xi z^n + \sum_{j=0}^{n-1}(n+j)a_j z^j + \sum_{j=0}^{n-1}(n-j)\bar{a}_j \bar{z}^j = 0\,, \qquad \xi, z \in \mathbb{C} \tag{2.2.1}$$

which defines the real-analytic function $\xi = \xi(z)$, $z \in \mathbb{C} \setminus \{0\}$.

Let

$$g(z)=\sum_{j=0}^{n-1}(n-j)a_jz^j \qquad \text{and} \qquad h(z)=\sum_{j=0}^{n-1}(n^2-j^2)a_jz^j\,, \qquad z\in\mathbb{C}.$$

Lemma 2.2.1 The map $\xi:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ has a real-analytic inverse $z=z(\xi)$ defined in a neighborhood of $\xi_0=\xi(z_0)$, satisfying $z(\xi_0)=z_0$, if and only if we have both $\mathrm{Re}\big[g(z_0)\big]\neq 0$ and $\mathrm{Re}\big[h(z_0)\big]\neq 0$.

Proof Let $I\subset\mathbb{R}$ be an open interval and let $z=w(t)$, $t\in I$, be a smooth curve. Then, taking d/dt of both sides of equation (2.2.1) and assuming that $w(t)\neq 0$, $t\in I$, we get

$$\begin{aligned}2n\frac{d\xi}{dt}w^n+2n^2\xi w^n\frac{dw/dt}{w}+\frac{dw/dt}{w}\sum_{j=0}^{n-1}j(n+j)a_jw^j&\\ +\left(\overline{\frac{dw/dt}{w}}\right)\sum_{j=0}^{n-1}j(n-j)\bar{a}_j\bar{w}^j=0\,,&\qquad t\in I.\end{aligned}\tag{2.2.2}$$

Here, we used the fact that $d(\bar{w})/dt=\overline{dw/dt}$.

Fix some $t_0\in I$, some $z_0\in\mathbb{C}$ with $z_0\neq 0$ and put $\xi_0=\xi(z_0)$.

First, pick a smooth curve $u:I\to\mathbb{C}$ with $u(t_0)=z_0$, satisfying

$$\frac{du/dt}{u}=1 \qquad \text{at} \qquad t=t_0\,.$$

Then equation (2.2.2) with $w=u$ implies that, at $t=t_0$, we have

$$\begin{aligned}-2n\frac{d\xi}{dt}z_0^n&=2n^2\xi_0z_0^n+\sum_{j=0}^{n-1}j(n+j)a_jz_0^j+\sum_{j=0}^{n-1}j(n-j)\bar{a}_j\bar{z}_0^j\\ &=2n\mathrm{Re}\left(n\xi_0z_0^n+\sum_{j=0}^{n-1}ja_jz_0^j\right)+i2\mathrm{Im}\left(n^2\xi_0z_0^n+\sum_{j=0}^{n-1}j^2a_jz_0^j\right)\\ &=2n\mathrm{Re}\big[z_0\partial_2p(\xi_0,z_0)\big]+i2\mathrm{Im}\big[z_0\partial_2p(\xi_0,z_0)\big]+i2\mathrm{Im}\big[z_0^2\partial_2^2p(\xi_0,z_0)\big]\\ &=2n\mathrm{Re}\big[z_0\partial_2p(\xi_0,z_0)\big]+i2\mathrm{Im}\big[z_0^2\partial_2^2p(\xi_0,z_0)\big]\,.\end{aligned}$$

That is, the directional derivative of $\xi:\mathbb{C}\to\mathbb{C}$ at $z_0=u(t_0)$ in the direction of z_0 is given by

$$\left[\frac{d}{dt}(\xi\circ u)\right](t_0)=-\frac{1}{nz_0^n}\Big(n\mathrm{Re}\big[z_0\partial_2p(\xi_0,z_0)\big]+i\mathrm{Im}\big[z_0^2\partial_2^2p(\xi_0,z_0)\big]\Big)\,.\tag{2.2.3}$$

Next, pick a smooth curve $v : I \to \mathbb{C}$ with $v(t_0) = z_0$, satisfying

$$\frac{dv/dt}{v} = i \qquad \text{at} \qquad t = t_0 \,.$$

Then, as above, equation (2.2.2) with $w = v$ and $t = t_0$ implies

$$\begin{aligned}
-2n\frac{d\xi}{dt}z_0^n &= i2n^2\xi_0 z_0^n + i\sum_{j=0}^{n-1} j(n+j)a_j z_0^j - i\sum_{j=0}^{n-1} j(n-j)\bar{a}_j \bar{z}_0^j \\
&= i\left[i2n\mathrm{Im}\left(n\xi_0 z_0^n + \sum_{j=0}^{n-1} j a_j z_0^j\right) + 2\mathrm{Re}\left(n^2\xi_0 z_0^n + \sum_{j=0}^{n-1} j^2 a_j z_0^j\right)\right] \\
&= i\Big(i2n\mathrm{Im}\big[z_0\partial_2 p(\xi_0, z_0)\big] + 2\mathrm{Re}\big[z_0\partial_2 p(\xi_0, z_0)\big] + 2\mathrm{Re}\big[z_0^2\partial_2^2 p(\xi_0, z_0)\big]\Big) \\
&= i2\mathrm{Re}\big[z_0\partial_2 p(\xi_0, z_0) + z_0^2\partial_2^2 p(\xi_0, z_0)\big] \,.
\end{aligned}$$

It follows that the directional derivative of ξ at z_0 in the direction of iz_0 is given by

$$\left[\frac{d}{dt}(\xi \circ v)\right](t_0) = -\frac{1}{nz_0^n} i\mathrm{Re}\big[z_0\partial_2 p(\xi_0, z_0) + z_0^2\partial_2^2 p(\xi_0, z_0)\big] \,. \tag{2.2.4}$$

Now $\mathrm{Re}\big[p(\xi_0, z_0)\big] = 0$ by (2.2.1), therefore we have

$$\begin{aligned}
\mathrm{Re}\big[z_0\partial_2 p(\xi_0, z_0)\big] &= -\mathrm{Re}\big[np(\xi_0, z_0) - z_0\partial_2 p(\xi_o, z_0)\big] \\
&= -\mathrm{Re}\left(\sum_{j=0}^{n-1}(n-j)a_j z_0^j\right) .
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mathrm{Re}\big[z_0\partial_2 p(\xi_0, z_0) + z_0^2\partial_2^2 p(\xi_0, z_0)\big] &= -\mathrm{Re}\big[n^2 p(\xi_0, z_0) - z_0\partial_2 p(\xi_0, z_0) - z_0^2\partial_2^2 p(\xi_0, z_0)\big] \\
&= -\mathrm{Re}\left(\sum_{j=0}^{n-1}(n^2 - j^2)a_j z_0^j\right) .
\end{aligned}$$

Using the usual identification $\mathbb{R}^2 = \mathbb{R} + i\mathbb{R}$, let $A : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^{2\times 2}$ be the derivative map of the map $\xi : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$.

Fix $z_0 \in \mathbb{R}^2 \setminus \{(0,0)\}$ and let $\xi_0 = \xi(z_0)$. It follows from (2.2.3) and (2.2.4) that the real 2×2 matrix $A(z_0)$, written in the basis $\{z_0, iz_0\}$, has the form

$$A(z_0) = \frac{1}{nz_0}\begin{pmatrix} n\mathrm{Re}\big[g(z_0)\big] & 0 \\ & \\ \mathrm{Im}\big[z_0^2\partial_2^2 p(\xi_0, z_0)\big] & \mathrm{Re}\big[h(z_0)\big] \end{pmatrix} . \tag{2.2.5}$$

Now ξ has the reqired inverse at $z = z_0$ if and only if $\det[A(z_0)] \neq 0$, and Lemma 2.2.1 follows.

Before studying the critical points of the map $\xi : \mathbb{C} \setminus \{0\}$, we make a few observations.

Note 2.2.1 Fix $m \in \mathbb{N}$ and $b_1, \ldots, b_n \in \mathbb{C}$. Let $f(z) = \sum_{j=0}^{m} b_j z^j$, $z \in \mathbb{C}$, let $z = re^{i\theta}$, $r \in \mathbb{R} \setminus \{0\}$, $\theta \in \mathbb{R}$, and consider the real-analytic map

$$F(r,\theta) = \mathrm{Re}\big[f(z)\big]\,, \qquad r \in \mathbb{R} \setminus \{0\}, \quad \theta \in \mathbb{R}.$$

Then a simple computation shows that

$$r\frac{\partial F}{\partial r}(r,\theta) = \mathrm{Re}\big[zf'(z)\big]$$

$$\frac{\partial F}{\partial \theta}(r,\theta) = -\mathrm{Im}\big[zf'(z)\big]\,.$$

Let $r_0 > 0$ and $\theta_0 \in \mathbb{R}$ and assume that $z_0 = r_0 e^{i\theta_0}$ satisfies

$$\mathrm{Re}\big[z_0 f'(z_0)\big] \neq 0\,.$$

It follows from the implicit function theorem that $\exists$ an open interval $I \subset \mathbb{R}$ with $\theta_0 \in I$ and $\exists$ a real-analytic function $r = r(\theta)$, $\theta \in I$, with $r(\theta_0) = r_0$ such that

$$F(r,\theta) = 0\,, \qquad \theta \in I\,. \tag{2.2.6}$$

Let $I \subset \mathbb{R}$ be an open interval, consider some smooth real-valued function $r = r(\theta)$, $\theta \in I$, with $0 \notin r(I)$, and let $z = re^{i\theta}$. Then we have

$$\begin{aligned}
\frac{d}{d\theta}\mathrm{Re}\big[f(z)\big] &= \frac{dr/d\theta}{r}\mathrm{Re}\big[zf'(z)\big] - \mathrm{Im}\big[zf'(z)\big]\,, \qquad \theta \in I \\
\frac{d}{d\theta}\mathrm{Im}\big[f(z)\big] &= \frac{dr/d\theta}{r}\mathrm{Im}\big[zf'(z)\big] + \mathrm{Re}\big[zf'(z)\big]\,, \qquad \theta \in I\,.
\end{aligned} \tag{2.2.7}$$

Again, consider $r = r(\theta)$, $\theta \in I$ given by (2.2.6). Assume further that

$$\mathrm{Im}\big[z_0 f'(z_0\big] = 0\,.$$

Then $dr/d\theta = 0$ at $\theta = \theta_0$. Applying (2.2.7) once again, we get

$$\frac{d^2r}{d\theta^2}\Big|_{\theta=\theta_0} = r\frac{\mathrm{Re}\big[z_0 f'(z_0) + z_0^2 f''(z_0)\big]}{\mathrm{Re}\big[z_0 f'(z_0)\big]} = r\frac{\mathrm{Re}\big[h(z_0)\big]}{\mathrm{Re}\big[g(z_0)\big]}$$

since $\mathrm{Re}\big[f(z_0)\big] = 0$, $g(z) = nf(z) - zf'(z)$ and $h(z) = n^2 f(z) - zf'(z) - z^2 f''(z)$.

Note 2.2.2 Let $f(z)$, $z \in \mathbb{C}$, and $F(r,\theta)$, $r \in \mathbb{R}\setminus\{0\}$, $\theta \in \mathbb{R}$, be as in Note 2.2.1. Now let $r_0 > 0$, $\theta_0 \in \mathbb{R}$ and let $z_0 = r_0 e^{i\theta_0}$ satisfy

$$\mathrm{Im}\big[z_0 f'(z_0)\big] \neq 0\,.$$

Then $\exists$ an open interval $I \subset \mathbb{R}\backslash\{0\}$ with $r_0 \in I$ and $\exists$ a real-analytic function $\theta = \theta(r)$, $r \in I$, with $\theta(r_0) = \theta_0$ such that

$$F(r,\theta) = 0\,, \qquad r \in I\,. \tag{2.2.9}$$

Assume that $I \subset \mathbb{R}\setminus\{0\}$ is an open interval, $\theta = \theta(r)$, $r \in I$, is a smooth real-valued function and $z = re^{i\theta}$. Then we have

$$\begin{aligned}
\frac{d}{dr}\mathrm{Re}\big[f(z)\big] &= \frac{1}{r}\mathrm{Re}\big[zf'(z)\big] - \frac{d\theta}{dr}\mathrm{Im}\big[zf'(z)\big]\,, \qquad r \in I \\
\frac{d}{dr}\mathrm{Im}\big[f(z)\big] &= \frac{1}{r}\mathrm{Im}\big[zf'(z)\big] + \frac{d\theta}{dr}\mathrm{Re}\big[zf'(z)\big]\,, \qquad r \in I\,.
\end{aligned} \tag{2.2.10}$$

Again, consider $\theta = \theta(r)$, $r \in I$, given by (2.2.9). Assume also that

$$\mathrm{Re}\big[z_0 f'(z_0)\big] = 0\,.$$

Then $d\theta/dr = 0$ at $r = r_0$. Applying (2.2.10) one more time yields

$$\frac{d^2\theta}{dr^2}\Big|_{r=r_0} = -\frac{1}{r^2}\frac{\mathrm{Re}\big[z_0^2 f''(z_0)\big]}{\mathrm{Im}\big[z_0 f'(z_0)\big]} = -\frac{1}{r^2}\frac{\mathrm{Re}\big[h(z_0)\big]}{\mathrm{Im}\big[z_0 f'(z_0)\big]} \tag{2.2.11}$$

since $\mathrm{Re}\big[f(z_0)\big] = \mathrm{Re}\big[z_0 f'(z_0)\big] = 0$ and $h(z) = n^2 f(z) - zf'(z) - z^2 f''(z)$.

Next, we study critical points of $\xi : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ where $\mathrm{Re}\big[h(z)\big] = 0$.

Note 2.2.3 Let $z_0 \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}\big[h(z)\big] = 0$, let $\xi_0 = \xi(z_0)$ and put

$$f(z) = p(\xi_0, z)\,, \qquad z \in \mathbb{C}.$$

Assume that $f'(z_0) \neq 0$. Since we have

$$\mathrm{Im}\big[z_0 f'(z_0)\big] = \mathrm{Im}\big[z_0 \partial_2 p(\xi_0, z_0)\big] = 0$$

by definition of the map ξ, it follows that

$$\mathrm{Re}\big[z_0 f'(z_0)\big] \neq 0\,.$$

Put $z_0 = r_0 e^{i\theta_0}$. Then Note 2.2.1 implies that $\exists$ an open interval $I \subset \mathbb{R}$ with $\theta_0 \in I$ and $\exists$ a real-analytic function $r = r(\theta)$, $\theta \in I$, satisfying $r(\theta_0) = r_0$ and

$$F(r, \theta) = \mathrm{Re}\big[f(z)\big] = 0\,, \qquad \theta \in I$$

where $\forall \theta \in I$, we put $z = re^{i\theta}$. By repeated applications of (2.2.6), we obtain that

$$\frac{dr}{d\theta}\Big|_{\theta=\theta_0} = 0 \qquad \text{and} \qquad \frac{d^2 r}{d\theta^2}\Big|_{\theta=\theta_0} = 0$$

$$\frac{1}{r_0}\left(\frac{d^3 r}{d\theta^3}\Big|_{\theta=\theta_0}\right) = -\frac{\mathrm{Im}\big[3z_0^2 f''(z_0) + z_0^3 f'''(z_0)\big]}{\mathrm{Re}\big[z_0 f'(z_0)\big]} = \frac{\mathrm{Im}\big[z_0 h'(z_0)\big]}{\mathrm{Re}\big[z_0 f'(z_0)\big]} = -\frac{\mathrm{Im}\big[z_0 h'(z_0)\big]}{\mathrm{Re}\big[g(z_0)\big]}$$

since $\mathrm{Im}\big[z_0 f'(z_0)\big] = 0$, $h(z) = n^2 f(z) - zf'(z) - z^2 f''(z)$ and $g(z) = nf(z) - zf'(z)$.

Section 2.3 The map $\chi : v \mapsto w$

Let $k, n \in \mathbb{N}$ with $0 < k < n$ and consider the polynomial

$$p(x, y) = xy^n + n^2 y^k + n^2 - k^2\,, \qquad x, y \in \mathbb{C}.$$

Definition 2.3.1 The map $\varphi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

$\forall v \in \mathbb{C} \setminus \{0\}$, define $\varphi = \varphi(v)$ by the equation

$$p(\varphi, v) = \varphi v^n + n^2 v^k + n^2 - k^2 = 0\,.$$

$\forall v_0 \in \mathbb{C} \setminus \{0\}$, let $A(v_0)$ be the derivative matrix at v_0 of the map φ. Then we have

$$A(v_0) = \frac{n(n-k)}{v_0^n} \begin{pmatrix} \mathrm{Re}(nv_0^k + n + k) & -\mathrm{Im}(nv_0^k) \\ \\ \mathrm{Im}(nv_0^k) & \mathrm{Re}(nv_0^k + n + k) \end{pmatrix}$$

written in the basis $\{v_0, iv_0\}$. $\forall v \in \mathbb{C} \setminus \{0\}$, we have that v is a critical point of φ if and only if

$$\mathrm{Re}(nv^k + n + k) = \mathrm{Im}(v^k) = 0\,.$$

Definition 2.3.2 The map $\psi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

$\forall w \in \mathbb{C} \setminus \{0\}$, define $\psi = \psi(w)$ by

$$\mathrm{Re}\big[p(\psi, w)\big] = \mathrm{Im}\big[w \partial_2 p(\psi, w)\big] = 0\,.$$

This ψ is the same map as the map ξ that we introduced in Section 2.2 for a more general polynomial. Therefore, all the statements there apply to ψ.

$\forall w_0 \in \mathbb{C} \setminus \{0\}$, let $B(w_0)$ be the derivative matrix at w_0 of the map ψ. Then we have

$$B(w_0) = \frac{n(n-k)}{w_0^n} \begin{pmatrix} \mathrm{Re}(nw_0^k + n + k) & 0 \\ \\ \mathrm{Im}(kw_0^k) & \mathrm{Re}\big[(n+k)w_0^k + n + k\big] \end{pmatrix}$$

written in the basis $\{w_0, iw_0\}$. The critical points of the map ψ form the curves given by the equation

$$\mathrm{Re}(nw^k + n + k) = 0\,, \qquad w \in \mathbb{C}$$

and the curves given by the equation

$$\operatorname{Re}(w^k + 1) = 0\,, \qquad w \in \mathbb{C}.$$

We state a few simple facts about these curves without proofs.

No two of the curves given by the above equations intersect. Consider the half rays starting at 0 that form the set $\{v \in \mathbb{C} : \operatorname{Im}(v^k) = 0\}$. Then each half ray intersects none or two of the above curves, and every point of intersection of a half ray with a curve is the unique point on that curve which is closest to 0.

Let

$$V = \bigl\{v \in \mathbb{C} : \operatorname{Re}(nv^k + n + k) > 0\bigr\}.$$

Then all curves given by $\operatorname{Re}(w^k + 1) = 0$ are contained in V. Let

$$W = \bigl\{w \in \mathbb{C} : \operatorname{Re}(w^k + 1) > 0\bigr\}.$$

Then $\overline{W} \subset V$. Finally, let

$$V_0 = \bigl\{v \in V \setminus \{0\} : 0 < \arg(v) < \pi/k\bigr\}$$

$$W_0 = \bigl\{w \in W \setminus \{0\} : 0 < \arg(w) < \pi/k\bigr\}.$$

We will use the following construction to define a map $\chi : V_0 \to W_0$.

Construction 2.3.1 Fix $v_0 \in V_0$. Let $\varphi_0 = \varphi(v_0)$ and put

$$f(z) = p(\varphi_0, z)\,, \qquad z \in \mathbb{C}.$$

Then $f(v_0) = 0$. Also, $f'(v_0) \neq 0$ since $v_0 \in V_0$ implies $\operatorname{Im}(v_0^k) > 0$ and so we have

$$v_0 f'(v_0) = v_0 f'(v_0) - nf(v_0) = -\bigl[(n-k)n^2 v_0^k + n(n^2 - k^2)\bigr] \neq 0\,.$$

Therefore, $\exists \alpha > 0$ and a simple nonsingular analytic curve $\gamma : (-\alpha, \alpha) \to \mathbb{C}$ with $\gamma(0) = v_0$ satisfying

$$f\bigl[\gamma(t)\bigr] = it\,, \qquad t \in (-\alpha, \alpha)\,.$$

Since $v_0 \neq 0$, (1.1.3) and (1.1.4) imply that $z = \gamma(t)$, $t \in (-\alpha, \alpha)$, satisfies

$$\begin{aligned}\operatorname{sign}\left(\frac{d|z|}{dt}\bigg|_{t=0}\right) &= \operatorname{sign}\left(\operatorname{Im}\left[v_0 f'(v_0)\right]\right)\\ &= -\operatorname{sign}\left(\operatorname{Im}\left[nf(v_0) - v_0 f'(v_0)\right]\right)\\ &= -\operatorname{sign}\left[(n-k)n^2 \operatorname{Im}(v_0^k)\right] = -1\end{aligned}$$

and

$$\begin{aligned}\operatorname{sign}\left(\frac{d\arg(z)}{dt}\bigg|_{t=0}\right) &= \operatorname{sign}\left(\operatorname{Re}\left[v_0 f'(v_0)\right]\right)\\ &= -\operatorname{sign}\left(\operatorname{Re}\left[nf(v_0) - v_0 f'(v_0)\right]\right)\\ &= -\operatorname{sign}\left(\operatorname{Re}\left[(n-k)n^2 v_0^k + n(n^2-k^2)\right]\right)\\ &= -\operatorname{sign}\left[\operatorname{Re}(nv_0^k + n + k)\right] = -1\,.\end{aligned}$$

This last equality follows from the fact that $v_0 \in V_0 \subset V$.

Continue the given parametrization of γ analytically on $(-\alpha, \infty)$ as far as possible. Assume $b > 0$ is such that γ can be extended to $(-\alpha, b)$, and that $z = \delta(t)$, $t \in (-\alpha, b)$, satisfies $d|z|/dt < 0$ and $d\arg(z)/dt < 0$. Then $\forall t \in (-\alpha, b)$, we have

$$\begin{aligned}0 < -\operatorname{Im}\left[zf'(z)\right] &< nt - \operatorname{Im}\left[zf'(z)\right]\\ &= n\operatorname{Im}\left[f(z)\right] - \operatorname{Im}\left[zf'(z)\right] \qquad (2.3.1)\\ &= \operatorname{Im}\left[(n-k)n^2 z^k\right]\end{aligned}$$

and

$$\operatorname{Re}\left[zf'(z)\right] = -\operatorname{Re}\left[nf(z) - zf'(z)\right] = -(n-k)n\operatorname{Re}(nz^k + n + k)$$

as above. Therefore, $\forall t \in (-\alpha, b)$, $z = \gamma(t)$ satisfies $\operatorname{Im}(z^k) > 0$ and $\operatorname{Re}(nz^k + n + k) > 0$, hence $z \in V_0$.

Let $z_0 = \lim_{t \to b^-} z$, which exists by Note 1.3.1. Then we have $f(z_0) = ib$. Also, (2.3.1) implies

$$0 \leq -\operatorname{Im}\left[z_0 f'(z_0)\right] \leq (n-k)n^2 \operatorname{Im}(z_0^k)\,.$$

If it were the case that $\mathrm{Im}(z_0^k)=0$, then we would have

$$0=\mathrm{Im}\big[z_0 f'(z_0)\big]=\mathrm{Im}\big[z_0 f'(z_0)\big]+(n-k)n^2\mathrm{Im}(z_0^k)=n\mathrm{Im}\big[f(z_0\big]\,.$$

But it follows from the definitions of z_0 and $z=\gamma(t)$, $t\in(-\alpha,b)$, that $\mathrm{Im}\big[f(z_0)\big]=b\neq 0$, a contradiction. Therefore, it must be that $\mathrm{Im}(z_0^k)>0$, hence $0<\arg(z_0)<\pi/k$.

Next, we claim that $\mathrm{Re}(nz_0^k+n+k)\neq 0$. We can prove this as follows. Consider $g(z)=nz^k+n+k$, $z\in\mathbb{C}$. $\forall t>0$, write

$$-\frac{n+k}{n}+it=re^{i\theta}$$

with $r>0$ and $\pi/2<\theta<\pi$. Define the curve $\delta:(0,\infty)\to\mathbb{C}$ by

$$\delta(t)=r^{1/k}e^{i\theta/k}\,,\qquad t\in(0,\infty)\,.$$

Then we have

$$g\big[\delta(t)\big]=it\,,\qquad t\in(0,\infty)$$

and

$$\{\delta\}=\big\{z\in\mathbb{C}:0<\arg(z)<\pi/k\quad\text{and}\quad\mathrm{Re}\big[g(z)\big]=0\big\}\,.$$

Assume that $\mathrm{Re}\big[g(z_0)\big]=0$. Since the curve $\gamma:(-\alpha,b)\to\mathbb{C}$ satisfies

$$\mathrm{Im}(z^k)>0\,,\qquad z\in\overline{\{\gamma\}}$$

we have that $\overline{\{\gamma\}}\cap\big\{z\in\mathbb{C}:\mathrm{Im}(z^k)=0\big\}=\emptyset$. Now $v_0\in\overline{\{\gamma\}}$ satisfies $0<\arg(v_0)<\pi/k$ by assumption, so we have that

$$\overline{\{\gamma\}}\subset\{z\in\mathbb{C}:0<\arg(z)<\pi/k\}$$

since $\overline{\{\gamma\}}$ does not intersect any of the half rays that bound this last set. We obtain that $z_0\in\{\delta\}$. Since $\forall t\in(0,\infty)$, $z=\delta(t)$ satisfies

$$zg'(z)=nkz^k=nk\Big(-\frac{n+k}{n}+it\Big)$$

we obtain that $\mathrm{Re}\big[zg'(z)\big] < 0$ and $\mathrm{Im}\big[zg'(z)\big] > 0$. It follows that

$$\frac{d|\delta|}{dt} > 0 \qquad \text{and} \qquad \frac{d\arg(\delta)}{dt} < 0 \qquad \text{on} \qquad (0, \infty)\,.$$

Put

$$A = \big\{z \in \mathbb{C} : |z| \leq |z_0|\big\}$$

$$B = \big\{z \in \mathbb{C} : 0 < \arg(z) \leq \arg(z_0)\big\}\,.$$

Then we have $\{\delta\} \subset A \cup B$. Since $z = \gamma(t)$, $t \in (-\alpha, b)$, satisfies $d|z|/dt < 0$ and $d\arg(z)/dt < 0$, it follows that $\{\gamma\} \cap \big(A \cup B\big) = \emptyset$.

Also, $\forall z \in \{\gamma\}$, we have $\mathrm{Re}\big[g(z)\big] > 0$. But the above arguments show that $\forall z \in \{\gamma\}$, we have that the closed line segment $[0, z]$ satisfies

$$[0, z] \cap \{\delta\} \neq \emptyset\,.$$

In fact, $[0, z] \cap \{\delta\} = \{w\}$ for some $w \in \mathbb{C}$ and so

$$\mathrm{Re}\big[ug'(u)\big] > 0\,, \qquad u \in [0, w)$$

$$\mathrm{Re}\big[vg'(v)\big] < 0\,, \qquad v \in (w, z]\,.$$

We get $\mathrm{Re}\big[zg'(z)\big] < 0$, a contradiction.

Therefore, it must be that $\mathrm{Re}(nz_0^k + n + k) > 0$, which together with $0 < \arg(z_0) < \pi/k$ implies $z_0 \in V_0$.

Now

$$\mathrm{Re}\big[z_0 f'(z_0)\big] = \mathrm{Re}\big[z_0 f'(z_0)\big] - n\mathrm{Re}\big[f(z_0)\big] = -(n-k)n\mathrm{Re}(nz_0^k + n + k) \neq 0$$

so $f'(z_0) \neq 0$. Therefore, $\gamma : (-\alpha, b) \to \mathbb{C}$ extends analytically across b. Since $\forall t \in (-\alpha, b)$, $z = \gamma(t)$ satisfies $\mathrm{Im}\big[zf'(z)\big] < 0$, we have $\mathrm{Im}\big[z_0 f'(z_0)\big] \leq 0$. If $\mathrm{Im}\big[z_0 f'(z_0)\big] < 0$, then $\exists c > b$ such that γ extends to $(-\alpha, c)$ analytically, and $z = \gamma(t)$, $t \in (-\alpha, c)$, satisfies $d|z|/dt < 0$ and $d\arg(z)/dt < 0$. This means that if $d > 0$ is the smallest number for which γ extends analytically to $(-\alpha, d)$ as above but $w_0 = \lim_{t\to d^-}\gamma(t)$ satisfies $\mathrm{Im}\big[w_0 f'(w_0)\big] = 0$, then $w_0 \in V_0$.

Now we will show that $w_0 \in W_0$ as well, that is, $\mathrm{Re}(w_0^k + 1) > 0$. Assume on the contrary that $\mathrm{Re}(w_0^k + 1) = 0$. The same way as for z_0 above, we can prove that $\mathrm{Re}\big[w_0 f'(w_0)\big] \neq 0$. We obtain $\mathrm{Re}\big[w_0 f'(w_0)\big] < 0$ because $d\arg(z)/dt < 0$ on $(-\alpha, d)$. But $h(z) = (n^2 - k^2)n^2(w^k + 1)$ satisfies $\mathrm{Re}\big[h(w_0)\big] = 0$ and $\mathrm{Im}\big[w_0 h'(w_0)\big] = (n^2 - k^2)n^2 k \mathrm{Im}(w_0^k) > 0$ since $w_0 \in V_0$. Let $w_0 = r_0 e^{i\theta_0 i}$. Then Note 2.2.2 implies that $\exists$ an open interval $I \subset \mathbb{R}$ with $\theta_0 \in I$ and $\exists$ a real-analytic function $r = r(\theta)$, $\theta \in I$, satisfying $r(\theta_0) = r_0$ and

$$\mathrm{Re}\big[f(z)\big] = 0\,, \qquad \theta \in I$$

where $\forall\theta \in I$, we put $z = |z| re^{i\theta}$. We obtain that

$$\frac{dr}{d\theta}\Big|_{\theta=\theta_0} = 0\,, \qquad \frac{d^2 r}{d\theta^2}\Big|_{\theta=\theta_0} = 0 \qquad \text{and} \qquad \frac{d^3 r}{d\theta^3}\Big|_{\theta=\theta_0} < 0\,.$$

Therefore, $r = r_0 - C\theta^3 + O(\theta^4)$ with some $C > 0$, and so r is a strictly decreasing function near $\theta = \theta_0$. But $z = \gamma(t)$, $t \in (-\alpha, d)$, satisfies

$$\frac{d\arg(z)}{dt} < 0\,, \qquad t \in (-\alpha, d)$$

hence $\theta = \theta(t) = \arg(z)$ is a strictly decreasing function on $(-\alpha, d)$. We obtain that the composition $r = r \circ \theta(t)$, t near d, is strictly increasing. But $r = r(t) = |z|$, t near d, contradicting the fact that $d|z|/dt < 0$ on $(-\alpha, d)$.

Definition 2.3.3 The map $\chi : V_0 \to W_0$.

Let $v_0 \in V_0$, let $\varphi_0 = \varphi(v_0)$ and put $f(z) = p(\varphi_0, z)$, $z \in \mathbb{C}$. Let $d > 0$ and $\gamma : [0, d] \to \mathbb{C}$ be such that

$$f\big[\gamma(t)\big] = it\,, \qquad t \in [0, d]$$

as above. That is, putting $z = \gamma(t)$, $t \in [0, d]$, we have $d|z|/dt < 0$ and $d\arg(z)/dt < 0$ on $(0, d)$, and putting $w_0 = \gamma(d)$, we have $\mathrm{Im}\big[w_0 f'(w_0)\big] = 0$. Then we define

$$\chi(v_0) = w_0 \,.$$

Proposition 2.3.1 (1) The map $\chi : v_0 \to \chi(V_0)$ is a global real-analytic diffeomorphism.

(2) χ extends continuously to $\overline{V_0}$ with $\chi(\overline{V_0}) \subset W_0$.

(3) $\lim_{|v|\to\infty} |\chi(v)| = \infty$.

Proof Since $\forall v \in V_0$, we have $w = \chi(v) \in W_0$, that is, $\mathrm{Re}(w^k + 1) > 0$, it follows that w is not a critical point of the map $\psi : \mathbb{C} \setminus \{0\} \to \mathbb{C}$. Therefore, ψ is invertible on a neighborhood of w. But $\varphi = \varphi(v)$ satisfies

$$\mathrm{Re}\big[p(\varphi, w)\big] = \mathrm{Im}\big[w\partial_2(\varphi, w)\big] = 0$$

by the definition of $w = \chi(v)$, so $\varphi = \psi(w)$ by the definition of ψ. Therefore, if χ is continuous, then we have

$$\chi = \psi^{-1} \circ \varphi$$

on a neighborhood of v. Since $v \in V_0$ and $\varphi = \varphi(v)$ imply $p(\varphi, v) = 0$ and

$$\mathrm{Re}\big[v\partial_2 p(\varphi, v)\big] = \mathrm{Re}\big[v\partial_2(\varphi, v)\big] - n\mathrm{Re}\big[p(\varphi, v)\big] = -(n-k)n\mathrm{Re}(nv^k + n + k) < 0$$

we have that $\partial_2 p(\varphi, v) \neq 0$. Therefore, φ is invertible on a neighborhood of v. We obtain that χ is also invertible on a neighborhood of v and

$$\chi^{-1} = \varphi^{-1} \circ \psi$$

on this neighborhood. It follows that χ is a local real-analytic diffeomorphism if it is continuous.

It follows that to prove (1), it is enough to show that χ is continuous and $1-1$.

Fix $v_0 \in V_0$ and consider some sequence $(v_j)_{j=0}^{\infty} \subset V_0$ such that $v_j \to v_0$ as $j \to \infty$. Let $w_j = \chi(v_j)$, $j \in \mathbb{N}$.

Assume that

$$\lim_{j\to\infty} w_j = w_0$$

is false. Then $\exists\ \epsilon > 0$ and a subsequence $(w_{j_\ell})_{\ell=0}^{\infty} \subset (w_j)_{j=1}^{\infty}$ such that

$$|w_{j_\ell} - w_0| \geq \epsilon\,, \qquad \ell \in \mathbb{N}. \tag{2.3.2}$$

Let $\varphi_j = \varphi(v_j)$, $j \in \mathbb{N}$, and $\forall \ell \in \mathbb{N}$, let

$$f_{j_\ell}(z) = p(\varphi_{j_\ell}, z)\,, \qquad z \in \mathbb{C}.$$

Then $f_{j_\ell} \to f_0$ as $\ell \to \infty$, uniformly on compact sets in $\mathbb{C}$, where

$$f_0(z) = p(\varphi_0, z)\,, \qquad z \in \mathbb{C}.$$

By construction, $\exists \alpha, d > 0$ and $\exists$ a simple nonsingular analytic curve $\gamma : (-\alpha, d+\alpha) \to \mathbb{C}$ satisfying

$$f_0\big[\gamma(t)\big] = it\,, \qquad t \in (-\alpha, d+\alpha)$$

$$f_0\big[\gamma(0)\big] = v_0 \qquad \text{and} \qquad f_0\big[\gamma(d)\big] = w_0\,.$$

Consider the compact interval $K = [0, d] \subset (-\alpha, d+\alpha)$. Then Proposition 1.4.1 implies that $\exists$ an open neighborhood U of $\{\gamma\big|_K\}$, $\exists$ a subsequence $(m_\lambda)_{\lambda=0}^{\infty} \subset (j_\ell)_{\ell=0}^{\infty}$, $\exists$ an open interval $J \subset \mathbb{R}$ with $K \subset J$ and $\exists$ smooth curves $\eta : J \to U$ and $\eta_{m_\lambda} : J \to U$, $\lambda \in \mathbb{N}$, satisfying properties (1)–(6) there.

Since $v_j \to v_0 \in \{\eta\}$ as $j \to \infty$, property (6) of Proposition 1.4.1 implies that $v_{m_\lambda} \in \{\eta_{m_\lambda}\}$, $\lambda \in \mathbb{N}$.

Now $w_0 \in W_0$ implies that ψ is a diffeomorphism on an open neighborhood D of w_0 with $D \subset W_0$. But φ is continuous, hence $\varphi_j \to \varphi_0$ as $j \to \infty$. It follows that $\exists N \in \mathbb{N}$ such that $\forall j \geq N$, we have

$$\varphi_j \in \psi(D)\,.$$

Therefore, we have that $\forall j \geq N$, $\exists! z_j \in D$ with $\psi(z_j) = \varphi_j$ and

$$\lim_{N \leq j \to \infty} z_j = w_0\,.$$

We may assume that the above D is chosen so small that $D \subset U$.

Let $\lambda \in \mathbb{N}$. Then by property (6) in Proposition 1.4.1, $\exists t_{m_\lambda} \in J$ such that

$$z_{m_\lambda} = \eta_{m_\lambda}(t_{m_\lambda})\,.$$

Since $z_{m_\lambda} \in W_0$, it follows that $|\eta_{m_\lambda}|$ has a local minimum at t_{m_λ}. By construction, we have

$$w_{m_\lambda} \in \{\eta_{m_\lambda}\big|_{[0,t_{m_\lambda}]}\}\,.$$

Also by construction, we have

$$|w_j| < |v_j|\,, \qquad j \in \mathbb{N}$$

hence the sequence $(w_{m_\lambda})_{\lambda=0}^{\infty}$ is bounded. Therefore, $\exists y \in \mathbb{C}$ and $\exists$ a subsequence $(w_{\mu_\nu})_{\nu=0}^{\infty} \subset (w_{m_\lambda})_{\lambda=0}^{\infty}$ such that

$$\lim_{\nu \to \infty} w_{\mu_\nu} = y\,.$$

It follows that

$$\mathrm{Re}\big[f(y)\big] = \mathrm{Im}\big[yf'(y)\big] = 0\,.$$

Therefore, $y \in \{\eta\big|_{[0,d]}\}$ since $z_{\mu_\nu} \to w_0 = \eta(d)$ and $w_{\mu_\nu} \in \{\eta_{\mu_\nu}\big|_{[0,t_{\mu_\nu}]}\}$ where $\eta_{\mu_\nu}(t_{\mu_\nu}) = z_{\mu_\nu}$. But (2.3.2) implies that we must have $y \in \{\eta\big|_{[0,d)}\}$, contradicting the choice of d.

Therefore, we must have

$$\lim_{j\to\infty} w_j = w_0$$

and continuity of χ is proved.

Now let $v_1, v_2 \in V_0$ and assume that $\chi(v_1) = \chi(v_2) = w_0$. Then $\varphi(v_1) = \psi(w_0) = \varphi(v_2)$. Let $\varphi_0 = \varphi(v_1) = \varphi(v_2)$ and put

$$f(z) = p(\varphi_0, z)\,, \qquad z \in \mathbb{C}.$$

Then $\exists \alpha, d_1, d_2 > 0$ and $\exists$ smooth curves $\gamma_j : (-\alpha, d_j + \alpha) \to \mathbb{C}$, $j = 1, 2$, such that

$$f\big[\gamma_j(t)\big] = it\,, \qquad t \in (-\alpha, d_j + \alpha)\,, \quad j = 1, 2$$

$$f\big[\gamma_j(0)\big] = v_j \qquad \text{and} \qquad f\big[\gamma_j(d_j)\big] = w_0\,, \qquad j = 1, 2\,.$$

But then we get

$$id_1 = f\big[\gamma_1(d_1)\big] = w_0 = f\big[\gamma_2(d_2)\big] = id_2$$

hence $d_1 = d_2$. Also, we have $w_0 \in \{\gamma_1\} \cap \{\gamma_2\}$ and $f'(w_0) \neq 0$, so we must have

$$\{\gamma_1\} = \{\gamma_2\}\,.$$

It follows that $v_1 = v_2$, and (1) follows.

Proof of (2) Define $\chi(t) = t$, $t \geq 0$, and $\chi(te^{i\pi/k}) = te^{i\pi/k}$, $0 \leq t \leq 1$.

We can use Construction 2.3.1 to define $\chi(v)$ for

$$v \in \left\{te^{i\pi/k} : 1 < t < \left(\tfrac{n+k}{n}\right)^{1/k}\right\} \cup \{\delta\}$$

where δ is the same as in Construction 2.3.1.

Then we can use the same argument as in (1) above to prove that the extended χ is continuous on

$$\overline{V_0} \setminus \left\{\left(\tfrac{n+k}{n}\right)^{1/k} e^{i\pi/k}\right\}.$$

We need to handle the case $v_0 = \left(\frac{n+k}{n}\right)^{1/k} e^{i\pi/k}$ differently, since then it is no longer true that $\exists \gamma : (-\alpha, d+\alpha) \to \mathbb{C}$ such that $\gamma(0) = v_0$ and

$$f_0\big[\gamma(t)\big] = it\,, \qquad t \in (-\alpha, d+\alpha)\,.$$

Let $v_0 = \left(\frac{n+k}{n}\right)^{1/k} e^{i\pi/k}$, let $(v_j)_{j=1}^{\infty} \subset \overline{V_0} \setminus \{v_0\}$ satisfy $v_j \to v_0$ as $j \to \infty$, and $\forall j \in \mathbb{N}$, put $\varphi_j = \varphi(v_j)$ and

$$f_j(z) = p(\varphi_j, z)\,, \qquad z \in \mathbb{C}.$$

Assume that $\exists w_0 \in \mathbb{C}$ such that $w_j = \chi(v_j)$, $j \in \mathbb{N} \setminus \{0\}$, satisfy

$$\lim_{j \to \infty} w_j = w_0\,.$$

Then we will show that $\exists \alpha, d > 0$ and $\exists$ a smooth curve $\gamma : (0, d+\alpha) \to V_0$ such that $|\gamma|$ and $\arg(\gamma)$ both decrease on $(0, d+\alpha)$ and we have

$$\lim_{t \to 0^+} \gamma(t) = v_0 \qquad \text{and} \qquad \gamma(d) = w_0\,.$$

First of all, we must have $w_0 \in \overline{W_0}$ since $(w_j)_{j=1}^{\infty} \subset W_0$ by construction of χ on $\overline{V_0} \setminus \{v_0\}$.

Next, observe that $\forall j \in \mathbb{N}$, $\exists \alpha_j, d_j > 0$ and $\exists$ a smooth curve $\gamma_j : (-\alpha_j, d_j + \alpha_j) \to \overline{V_0} \setminus \{v_0\}$ such that

$$f_j\big[\gamma_j(t)\big] = it\,, \qquad t \in (-\alpha_j, d_j + \alpha_j)$$

$$\gamma_j(0) = v_j \qquad \text{and} \qquad \gamma_j(d_j) = w_j\,.$$

Now $\{\delta\} \cap \overline{W_0} = \emptyset$, so $f'(w_0) \neq 0$. Let $f_0(w_0) = id$ for some $d \in \mathbb{R}$. Then $\exists \beta > 0$ and $\exists$ a smooth curve $\gamma : (d-\beta, d+\beta) \to \mathbb{C}$ such that $\gamma(d) = w_0$ and

$$f_0\big[\gamma(t)\big] = it\,, \qquad t \in (d-\beta, d+\beta)\,.$$

But $lim_{j\to\infty} w_j = w_0$ implies that $f_j \to f_0$ as $j \to \infty$, uniformly on compact sets in $\mathbb{C}$. Consider the compact interval $K = [d - \beta/2, d + \beta/2] \subset (d - \beta, d + \beta)$. Then by Proposition 1.4.1, $\exists$ an open neighborhood U of $\{\gamma\big|_K\}$, $\exists$ an open interval $J \subset \mathbb{R}$ with $K \subset J$, $\exists$ a subsequence $(j_\ell)_{\ell=0}^{\infty} \subset (j)_{j=1}^{\infty}$, $\exists$ a smooth curve $\eta : J \to U$ and $\forall \ell \in \mathbb{N}$, $\exists$ a smooth curve $\gamma_{j_\ell} : J \to U$ satisfying properties (1)–(6) there.

Now $\exists N \in \mathbb{N}$ such that $\forall \ell \geq N$, we have $w_{j_\ell} \in U$. It follows from Proposition 1.4.1 (6) that

$$w_{j_\ell} \in \{\gamma_{j_\ell}\} \cap \{\eta_{j_\ell}\}, \qquad \ell \in \mathbb{N}.$$

Therefore, $\forall \ell \in \mathbb{N}$, we have $d_{j_\ell} \in J$ and γ_{j_ℓ} is a continuation of η_{j_ℓ} to $(-\alpha_{j_\ell}, d_{j_\ell} + \alpha_{j_\ell})$. Let $I_{j_\ell} = (-\alpha_{j_\ell}, d_{j_\ell} + \alpha_{j_\ell}) \cup J$, $\ell \geq N$. Then $\forall \ell \geq N$, $\exists$ a smmooth curve $\zeta_{j_\ell} : I_{j_\ell} \to \mathbb{C}$ such that

$$\zeta_{j_\ell}(t) = \begin{cases} \gamma_{j_\ell}(t) & \text{if} \quad t \in (-\alpha_{j_\ell}, d_{j_\ell} + \alpha_{j_\ell}) \\ \eta_{j_\ell}(t) & \text{if} \quad t \in I_{j_\ell}\,. \end{cases}$$

It is clear that we have $d_{j_\ell} \to d$ as $\ell \to \infty$ since

$$id_{j_\ell} = f_{j_\ell}\left[\eta_{j_\ell}(d_{j_\ell})\right] = f_{j_\ell}(w_{j_\ell}) \to f_0(w_0) = f\left[\eta(d)\right] = id$$

as $\ell \to \infty$. Then $d \geq 0$. We claim that $d > 0$, that is, $f_0(w_0) \neq 0$. Indeed, assuming $f_0(w_0) = 0$, we get a contradiction as follows. Since $f_{j_\ell} \to f_0$ as $\ell \to \infty$, uniformly on compact sets in $\mathbb{C}$, Hurwitz' theorem implies that $\exists$ a subsequence $(m_\lambda)_{\lambda=0}^{\infty} \subset (j_\ell)_{\ell=0}^{\infty}$ such that $\forall \lambda \in \mathbb{N}$, $\exists z_{m_\lambda} \in U$ satisfying $z_{m_\lambda} \to w_0$ as $\lambda \to \infty$ and

$$f_{m_\lambda}(z_{m_\lambda}) = 0\,.$$

Again, Proposition 1.4.1 (6) implies that $\forall \lambda \in \mathbb{N}$, $z_{m_\lambda} \in \{\eta_{m_\lambda}\} \subset \{\zeta_{m_\lambda}\}$. But $\forall \ell \geq N$, we have

$$f_{j_\ell}\left[\zeta_{j_\ell}(t)\right] = it\,, \qquad t \in I_{j_\ell}$$

hence $\{\zeta_{j_\ell}\}$ can contain at most 1 zero of f_{j_ℓ}. Therefore, $\forall\lambda \in \mathbb{N}$, we have $v_{m_\lambda} = z_{m_\lambda}$ and

$$v_0 = \lim_{\lambda\to\infty} v_{m_\lambda} = \lim_{\lambda\to\infty} z_{m_\lambda} = w_0$$

contradicting $v_0 \notin \overline{W_0}$.

Next, we claim that $\mathrm{Re}(w_0^k + 1) \neq 0$. Indeed, assuming $\mathrm{Re}(w_0^k + 1) = 0$, we get a contradiction in the following way. We have $\mathrm{Im}\big[w_0 f_0'(w_0)\big] = 0$. As in Construction 2.3.1, we may then assume that $\beta > 0$ is chosen so small that $z = \eta(t)$, $t \in (d - \beta/2, d)$, satisfies

$$\frac{d|z|}{dt} > 0 \qquad \text{on} \qquad (d - \beta/2, d)\,.$$

but we have that $\forall\ell \in \mathbb{N}$, $z = \eta_{j_\ell}(t)$, $t \in (0, d_{j_\ell})$, satisfies

$$\frac{d|z|}{dt} < 0 \qquad \text{on} \qquad (0, d_{j_\ell})$$

contradicting $\eta_{j_\ell}' \to \eta'$ as $\ell \to \infty$, uniformly on K, since we have $d_{j_\ell} \to d > 0$.

It follows that $w_0 \in W_0$ since we have $\mathrm{Re}\big[f_0(w_0)\big] = \mathrm{Im}\big[w_0 f_0'(w_0)\big] = 0$ and we have just seen that $f_0(w_0) \neq 0$ and $\mathrm{Re}(w_0^k + 1) \neq 0$, hence $w_0 \notin \partial W_0$.

Now we have that $|\eta(t)|$ has a local minimum at $t = d$. Let $a \in (0, d)$ satisfy that η extends to $(a, d + \beta/2)$ with $|\eta|$ and $\arg(\eta)$ both decreasing on (a, d) and

$$f_0\big[\eta(t)\big] = it\,, \qquad t \in (a, d + \beta/2)\,.$$

Observe that this implies $\{\eta\} \subset \overline{V_0}$ since $\forall$ compact subinterval $K' \subset (a, d + \beta/2)$, a subsequence of $(\zeta_{j_\ell}\big|_{K'})_{\ell=0}^{\infty}$ must converge to $\eta\big|_{k'}$ uniformly on K' by Proposition 1.4.1, and we have $\{\zeta_{j_\ell}\} \subset V_0$, $\ell \in \mathbb{N}$.

Let $z_0 = \lim_{t\to b^-} \eta(t)$ and assume that $\mathrm{Im}\big[z_0 f'(z_0)\big] = 0$.

We claim that z_0 satisfies $\mathrm{Re}(z_0^k + 1) \neq 0$ and $\mathrm{Re}(n z_0^k + n + k) \neq 0$.

Assume on the contrary that $\mathrm{Re}(z_0^k + 1) = 0$. Then $z_0 \in \overline{W_0}$. Since $\{\delta\} \cap \overline{W_0} = \emptyset$, we have $f_0'(z_0) \neq 0$. Therefore, we can extend η to $(a-\epsilon, d+\beta/2)$ for some $\epsilon > 0$. Now we get a contradiction the same way as for w_0 above.

Next assume that $\mathrm{Re}(nz_0^k + n + k) = 0$. Then $z_0 \in \{\delta\}$. Since $f_0(z_0) = ia \neq 0$, we have $z_0 \neq v_0$. Therefore, $|z_0| > |v_0|$. Then $\exists \epsilon > 0$ such that

$$\left|\eta(a+\epsilon)\right| > |v_0| .$$

Consider the compact interval $K_1 = [a+\epsilon, d] \subset (a, d+\beta/2)$. Then by Proposition 1.4.1, $\exists$ an open neighborhood U_1 of $\{\eta\big|_{K_1}\}$, $\exists$ an open interval $J_1 \subset \mathbb{R}$ with $K_1 \subset J_1$, $\exists$ a subsequence $(m_\lambda)_{\lambda=0}^{\infty} \subset (j_\ell)_{\ell=0}^{\infty}$, $\exists$ a smooth curve $\delta_0 : J_1 \to U_1$ and $\forall \lambda \in \mathbb{N}$, $\exists$ a smooth curve $\delta_{m_\lambda} : J_1 \to U_1$ such that we have properties (1)–(6) there. But then Proposition 1.4.1 (6) implies that we must have $\{\delta_{m_\lambda}\big|_{I_{m_\lambda}}\} \subset \{\zeta_{m_\lambda}\}$. By Construction 2.3.1, we have

$$\left|\zeta_{m_\lambda}(t)\right| < |v_0| , \qquad t \in (0, d_{m_\lambda})$$

contradicting $\zeta_{m_\lambda}(a+\epsilon) = \delta_{m_\lambda}(a+\epsilon) \to \delta_0(a+\epsilon) = \eta(a+\epsilon)$.

It follows that $\exists b \in (0, a)$ such that η extends to $(b, d+\beta/2)$ with

$$f_0\big[\eta(t)\big] = it , \qquad t \in (b, d+\beta/2) .$$

But as we saw above, $z_0 = \eta(a)$ is not a critical point of the map ψ, so ψ is invertible in an open neighborhood D of z_0. We have $\varphi_0 = \varphi(v_0) = \psi(z_0)$. Therefore, $\exists M \in \mathbb{N}$ such that $\forall \ell \geq M$, we have $\varphi_{j_\ell} = \varphi(v_{j_\ell}) \in \psi(D)$. Put $u_{j_\ell} = \psi^{-1}(\varphi_{j_\ell})$, $\ell \geq M$. Then

$$\mathrm{Re}\big[f_{j_\ell}(u_{j_\ell})\big] = \mathrm{Im}\big[u_{j_\ell} f_{j_\ell}'(u_{j_\ell})\big] = 0 , \qquad \ell \geq M .$$

Consider the compact interval $K_2 = [a, d] \subset (b, d+\beta/2)$. Using Proposition 1.4.1 as above, we obtain a subsequence $(u_{m_\lambda})_{\lambda=0}^{\infty} \subset (u_{j_\ell})_{\ell=0}^{\infty}$ such that $\forall \lambda \in \mathbb{N}$, we have

$$u_{m_\lambda} \in \{\zeta_{m_\lambda}\}$$

contradicting that $\forall \ell \in \mathbb{N}$, $|\zeta_{j_\ell}|$ decreases on $(-\alpha_{j_\ell}, d_{j_\ell} + \alpha_{j_\ell})$.

It follows that we must have $\mathrm{Im}\big[z_0 f_0'(z_0)\big] \neq 0$.

We obtain that η must extend to $(0, d + \beta/2)$ with $|\eta|$ and $\arg(\eta)$ both decreasing on $(0, d+\beta/2)$ and

$$f_0\big[\eta(t)\big] = it\,, \qquad t \in (0, d+\beta/2)\,.$$

Let $u_0 = \lim_{t\to 0^+} \eta(t)$. Then $f_0(u_0) = 0$ and $u_0 \in \overline{V_0}$.

If $u_0 \neq v_0$, then we claim that $\mathrm{Re}(u_0^k + n + k) \neq 0$. This can be proved the same way as for z_0 above. Then $\exists \epsilon > 0$ such that η extends to $(-\epsilon, d+\beta/2)$ satisfying

$$f_0\big[\eta(t)\big] = it\,, \qquad t \in (-\epsilon, d+\beta/2)\,.$$

Consider the compact interval $K_2 = [0, d] \subset (-\epsilon, d+\beta/2)$. Applying Proposition 1.4.1 and Hurwitz' theorem as for w_0 above, we obtain a subsequence $(m_\lambda)_{\lambda=0}^\infty \subset (j_\ell)_{\ell=0}^\infty$ and a sequence $(u_{m_\lambda})_{\lambda=0}^\infty$ with $u_{m_\lambda} \to u_0$ as $\lambda \to \infty$ such that $\forall \lambda \in \mathbb{N}$, we have

$$f_{m_\lambda}(u_{m_\lambda}) = 0\,.$$

But then we must have $u_{m_\lambda} = v_{m_\lambda}$, $\lambda \in \mathbb{N}$, implying

$$u_0 = \lim_{\lambda\to\infty} u_{m_\lambda} = \lim_{\lambda\to\infty} v_{m_\lambda} = v_0$$

a contradiction.

Now we prove that χ has a limit at $v_0 = \left(\frac{n+k}{n}\right)^{1/k} e^{i/k}$.

Let $\varphi_0 = \varphi(v_0)$ and put

$$f_0(z) = p(\varphi_0, z)\,, \qquad z \in \mathbb{C}\,.$$

Let $(v_j)_{j=1}^\infty \subset \overline{V_0} \setminus \{v_0\}$ satisfy

$$v_j \to v_0 \qquad \text{as} \qquad j \to \infty$$

and $\forall j \in \mathbb{N} \setminus \{0\}$, put $w_j = \chi(v_j)$. As before, we conclude that $(w_j)_{j=1}^{\infty}$ is a bounded sequence. Therefore, $\exists w_0 \in \mathbb{C}$ and $\exists$ a subsequence $(w_{j_\ell})_{\ell=0}^{\infty} \subset (w_j)_{j=1}^{\infty}$ such that

$$w_{j_\ell} \to w_0 \qquad \text{as} \qquad \ell \to \infty\,.$$

It follows from the above discussion that $w_0 \in W_0$, $\exists \alpha, d > 0$ and $\exists$ a smooth curve $\gamma : (0, d+\alpha) \to V_0$ such that $|\gamma|$ and $\arg(\gamma)$ both decrease on $(0, d+\alpha)$ and we have

$$f_0\big[\gamma(t)\big] = it\,, \qquad t \in (0, d+\alpha)$$

$$\lim_{t\to 0^+} \gamma(t) = v_0 \qquad \text{and} \qquad \gamma(d) = w_0\,.$$

Assume that

$$w_j \not\to w_0 \qquad \text{as} \qquad j \to \infty\,.$$

Then $\exists \epsilon > 0$ and $\exists$ a subsequence $(w_{m_\lambda})_{\lambda=0}^{\infty} \subset (w_j)_{j=1}^{\infty}$ such that

$$|w_{m_\lambda} - w_0| \geq \epsilon\,, \qquad \lambda \in \mathbb{N}.$$

Again, $(w_{m_\lambda})_{\lambda=0}^{\infty}$ is a bounded sequence, therefore $\exists z_0 \in \mathbb{C}$ and $\exists$ a subsequence $(w_{\mu_\nu})_{\nu=0}^{\infty} \subset (w_{m_\lambda})_{\lambda=0}^{\infty}$ satisfying

$$\lim_{\nu\to\infty} w_{\mu_\nu} = z_0\,.$$

The same way as above, we obtain that $z_0 \in W_0$, $\exists \beta, c > 0$ and $\exists$ a smooth curve $\zeta : (0, c+\beta) \to V_0$ such that $|\zeta|$ and $\arg(\zeta)$ both decrease on $(0, c+\beta)$ and we have

$$f_0\big[\zeta(t)\big] = it\,, \qquad t \in (0, c+\beta)$$

$$\lim_{t\to 0^+} \zeta(t) = v_0 \qquad \text{and} \qquad \zeta(c) = z_0\,.$$

Clearly, $z_0 \neq w_0$, hence $\gamma \neq \zeta$. This leads to a contradiction as follows. Since $f_0'(v_0) = 0$ but $f_0''(v_0) \neq 0$, we have that $\exists$ 2 distinct smooth curves in $\Gamma_0(f_0)$ intersct-ing at v_0 at a right angle. Since $\varphi_0 = \varphi(v_0)$ implies

$$\varphi_0 v_0^n = -n^2 v_0^k - n^2 + k^2 = k(n+k)$$

we have that

$$f_0(v_0 u) = \varphi_0 v_0^n u^n + n^2 v_0^k u^k + n^2 - k^2 = k(n+k)u^n - n(n+k)u^k + n^2 - k^2 = \overline{f_0(v_0 \bar{u})}\,.$$

That is, $\Gamma_0(f_0)$ is symmetric about the line L through 0 and v_0.

It follows that the tangent lines at v_0 of the 2 curves in $\Gamma_0(f_0)$ intersecting at v_0 are either $L_1 = L$ and $L_2 = \{v_0 + tiv_0 : t \in \mathbb{R}\}$, or $L_3 = \{v_0 + te^{-i\frac{\pi}{2k}} : t \in \mathbb{R}\}$ and $L_4 = \{v_0 + te^{i\frac{\pi}{2k}} : t \in \mathbb{R}\}$. But L_1 cannot be a tangent line because $L_1 \not\subset \Gamma_0(f_0)$, hence the mentioned symmetry about $L_1 = L$ would force 2 distinct curves in $\Gamma_0(f_0)$ to have L_1 as tangent line at v_0, an impossibility.

Thus, L_3 and L_4 must be the tangent lines at v_0. But a smooth curve through v_0, tangent to L_3, must have points arbitrarily close to v_0 on either side of v_0 that are not in V_0, contradicting the existence of the 2 distinct curves γ and ζ with $\{\gamma\}, \{\zeta\} \subset \Gamma_0(f_0)$, satisfying $\{\gamma\}, \{\zeta\} \subset V_0$ and $v_0 \in \overline{\{\gamma\}}, \overline{\{\zeta\}}$.

Therefore, we have that

$$w_0 = \lim_{W_0 \ni v \to v_0} \chi(v)$$

exists. If we define

$$\chi(v_0) = w_0$$

then we have continuity of χ at v_0 and (2) is proved.

(3) This easily follows from the fact that $\forall v_0 \in V_0$, $w_0 = \chi(v_0)$ and $\varphi_0 = \varphi(v_0)$ imply that $\varphi_0 = \psi(w_0)$.

This concludes the proof of Proposition 2.3.1.

Section 2.4 Properties of the map χ

In this section, we make the assumption that $n, k \in \mathbb{N}$ are chosen so that $n/2 \leq k < n$.

Lemma 2.4.1 Let $v_0 = \left(\frac{n+k}{n}\right)^{1/k} e^{i\pi/k}$, let $\varphi_0 = \varphi(v_0)$ and put

$$f_0(z) = p(\varphi_0, z), \qquad z \in \mathbb{C}.$$

Let $v = r^{1/k} e^{i\pi/k}$, $r \in \left[1, \frac{n+k}{n}\right]$. Then $v^k = -r$. Let $w = \chi(v)$. Then

$$0 < \arg(v) - \arg(w) \leq \frac{\pi}{2n}, \qquad r \in \left[1, \tfrac{n+k}{n}\right].$$

Proof By definition of χ, we have

$$\chi(e^{i\pi/k}) = e^{i\pi/k}.$$

Therefore, $\exists \epsilon > 0$ such that $\forall r \in (1, 1+\epsilon)$, we have

$$0 < \arg(v) - \arg(w) < \frac{\pi}{2n}, \qquad v = r^{1/k} e^{i\pi/k}, \quad w = \chi(v).$$

Assume $\exists r_1 \in \left(1, \frac{n+k}{n}\right]$ such that $v_1 = r_1^{1/k} e^{i\pi/k}$ and $w_1 = \chi(v_1)$ satisfy

$$\arg(v_1) - \arg(w_1) > \frac{\pi}{2n}.$$

It follows that $\exists r_2 \in (1, r_1)$ such that $v_2 = r_2^{1/k} e^{i\pi/k}$ and $w_2 = \chi(v_2)$ satisfy

$$\arg(v_2) - \arg(w_2) = \frac{\pi}{2n}.$$

Then $\exists t_2 \in (0, 1)$ such that

$$w_2 = v_2 t_2 e^{-i\frac{\pi}{2n}}.$$

That is, the system of equations in $z \in \mathbb{C}$

$$\text{(2.4.1)} \qquad \begin{aligned} \operatorname{Re}\big[f_0(z)\big] &= 0 \\ \operatorname{Im}\big[z f_0'(z)\big] &= 0 \end{aligned}$$

has a solution with $z = z(r,t) = r^{1/k} e^{i\pi/k} t e^{-i\frac{\pi}{2n}}$, $r \in \left(1, \frac{n+k}{n}\right)$, $t \in (0,1)$.

Equivalently, the system of equations in $(r,t) \in \mathbb{R}^2$

$$\begin{aligned} x(r,t) &= -n^2 r t^k \cos\left(\frac{k\pi}{2n}\right) + n^2 - k^2 = 0 \\ y(r,t) &= (n^2 - k^2 - n^2 r)t^n + nkrt^k \sin\left(\frac{k\pi}{2n}\right) = 0 \end{aligned}$$

has a solution with $r \in \left(1, \frac{n+k}{n}\right)$, $t \in (0,1)$.

Now $x(r,t) = 0$ implies $rt^k = C$ where

$$C = \frac{n^2 - k^2}{n^2 \cos\left(\frac{k\pi}{2n}\right)} > 0\,.$$

Therefore, $x(r,t) = 0$, $t > 0$, defines $t = t(r)$, $r \in \left[1, \frac{n+k}{n}\right]$, as a decreasing function. To obtain a contradiction with (2.4.1), it suffices to show that we have

$$\text{(2.4.2)} \qquad y\big[r, t(r)\big] > 0\,, \qquad r \in \left(1, \tfrac{n+k}{n}\right).$$

We compute

$$\begin{aligned} y(r,t) &= t^{n-k}\left[(n^2 - k^2)t^k - n^2 r t^k\right] + nkrt^k \sin\left(\frac{k\pi}{2n}\right) \\ &= t^{n-k}\left[(n^2 - k^2)t^k - n^2 C\right] + nk \sin\left(\frac{k\pi}{2n}\right) \end{aligned}$$

if $rt^k = C$. Put

$$u(t) = t^{n-k}\left[(n^2 - k^2)t^k - n^2 C\right] + nkC \sin\left(\frac{k\pi}{2n}\right), \qquad t \in [a,b]$$

where $a, b > 0$ satisfy

$$a^k = \frac{n}{n+k} C \qquad \text{and} \qquad b^k = C\,.$$

Then u is an increasing function. To see this, we compute

$$u'(t) = n(n-k)t^{n-k-1}\big[(n+k)t^k - nC\big] > 0\,, \qquad t \in (a,b)\,.$$

The equation $rt^k = C$, $r,t > 0$ defines a decreasing function $t = t(r)$, $r \in \big[1, \frac{n+k}{n}\big]$. It follows that the function

$$q(r) = y\big[r, t(r)\big]$$

decreases on $\big[1, \frac{n+k}{n}\big]$ since $u(t)$ increases on $[a,b]$, $t = t(r)$ decreases on $\big[1, \frac{n+k}{n}\big]$ and $t : \big[1, \frac{n+k}{n}\big] \to [a,b]$.

Therefore, to prove (2.4.2), it is enough to show that

$$y\big[r, t(r)\big] \geq 0 \qquad \text{at} \qquad r = \frac{n+k}{n}\,.$$

But $r = \frac{n+k}{n}$ implies

$$t^k = \frac{n-k}{n\cos\left(\frac{k\pi}{2n}\right)}$$

and $t = t(r)$ implies

$$\operatorname{Im}\big[zf'(z)\big] = -k(n+k)t^k\left[t^{n-k} - \sin\left(\frac{k\pi}{2n}\right)\right].$$

Now we need to show

$$\left[\frac{n-k}{n\cos\left(\frac{k\pi}{2n}\right)}\right]^{\frac{n-k}{k}} \leq \sin\left(\frac{k\pi}{2n}\right). \tag{2.4.3}$$

Let $x = k/n$. Then $x \in [\frac{1}{2}, 1)$ and (2.4.3) is equivalent to

$$\left[\frac{1-x}{\cos\left(\frac{\pi}{2}x\right)}\right]^{\frac{1}{x}-1} \leq \sin\left(\frac{\pi}{2}x\right), \qquad x \in (\tfrac{1}{2}, 1)\,.$$

We have $\cos(\pi/4) = \sqrt{2}/2$ and $\cos(\pi/2) = 0$, and $\cos(\frac{\pi}{2}x)$ is concave down $\forall x \in (\frac{1}{2}, 1)$. It follows that

$$\sqrt{2}(1-x) < \cos\left(\frac{\pi}{2}x\right), \qquad x \in (\tfrac{1}{2}, 1)$$

hence we need to show

$$\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{1/x} \leq \sin\left(\frac{\pi}{2}x\right), \qquad x \in [\tfrac{1}{2}, 1].$$

Therefore, Lemma 2.4.1 follows from Lemma 2.4.2 below.

Lemma 2.4.2 Let

$$u(x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^{1/x} \qquad \text{and} \qquad v(x) = \sin\left(\frac{\pi}{2}x\right), \qquad x \in [\tfrac{1}{2}, 1].$$

Then $u(x) \leq v(x)$, $x \in [\frac{1}{2}, 1]$.

Proof We have $u(1/2) = \sqrt{2}/2 = v(1/2)$ and $u(1) = 1 = v(1)$.

Assume $\exists x_0 \in (\frac{1}{2}, 1)$ such that $u(x_0) > v(x_0)$.

Since both $u(x)$ and $v(x)$ are positive in $[\frac{1}{2}, 1]$, we may define

$$y(x) = \log\big[u(x)\big] \qquad \text{and} \qquad z(x) = \log\big[v(x)\big], \qquad x \in [\tfrac{1}{2}, 1].$$

Then $y(1/2) = z(1/2)$ and $y(1) = z(1)$. Since log is an increasing function, we have $y(x_0) > z(x_0)$. Also, we have

$$y'(x) = \frac{\log 2}{2x^2} \qquad \text{and} \qquad z'(x) = \frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right), \qquad x \in [\tfrac{1}{2}, 1].$$

It follows that $y'(1/2) = 2\log 2 < \pi/2 = z'(1/2)$. To see this, note that $16 < (27/10)^3 < e^3$, therefore $\log 16 < 3 < \pi$. Also, we have $y'(1) = \frac{1}{2}\log 2 > 0 = z'(1)$. These together with $y(1/2) = z(1/2)$ and $y(1) = z(1)$ imply that $\exists \epsilon > 0$ such that

$$y(x) < z(x), \qquad x \in (\tfrac{1}{2}, \tfrac{1}{2} + \epsilon) \qquad \text{and} \qquad x \in (1 - \epsilon, 1).$$

Then the assumption $y(x_0) > z(x_0)$ for some $x_0 \in (\frac{1}{2}, 1)$ implies that $\exists x_1, x_2 \in (\frac{1}{2}, 1)$ with $x_1 \neq x_2$ such that $y(x_j) = z(x_j)$, $j = 1, 2$. This together with $y(1/2) = z(1/2)$ and $y(1) = z(1)$ implies that $\exists t_1, t_2, t_3 \in (\frac{1}{2}, 1)$ with $t_1 < t_2 < t_3$ such that

$$y'(t_j) = z'(t_j), \qquad j = 1, 2, 3.$$

Let $y_1(x) = 2y'(x)/\pi$ and $z_1(x) = 2z'(x)/\pi$. Then $y_1(t_j) = z_1(t_j)$, $j = 1, 2, 3$. It follows that

$$\arctan\left[y_1(t_j)\right] = \arctan\left[z_1(t_j)\right], \qquad j = 1, 2, 3.$$

Therefore, $\exists s_1, s_2 \in (\frac{1}{2}, 1)$ with $s_1 \neq s_2$ such that

$$\frac{d}{dx}\arctan\left[y_1(x)\right] = \frac{d}{dx}\arctan\left[z_1(x)\right] \qquad \text{at} \qquad x = s_1, s_2$$

hence $\exists r \in (\frac{1}{2}, 1)$ satisfying

$$\frac{d^2}{dx^2}\arctan\left[y_1(x)\right] = \frac{d^2}{dx^2}\arctan\left[z_1(x)\right] \qquad \text{at} \qquad x = r. \tag{2.4.4}$$

Now an easy computation shows that

$$\begin{aligned} &\frac{d^2}{dx^2}\arctan\left[y_1(x)\right] = 0 \qquad \iff \qquad x = \pm\left(\frac{\sqrt{2}\log 2}{\pi\sqrt{3}}\right)^{1/2} \\ &\qquad\qquad\qquad \frac{d^2}{dx^2}\arctan\left[z_1(x)\right] \equiv 0. \end{aligned} \tag{2.4.5}$$

Using $\log 16 < 3 < \pi\sqrt{3/2}$ once again, we obtain

$$\left(\frac{\sqrt{2}\log 2}{\pi\sqrt{3}}\right)^{1/2} < \frac{1}{2}$$

leading to a contradiction between (2.4.4) and (2.4.5). It follows that the assumption $u(x_0) > v(x_0)$ for some $x_0 \in (\frac{1}{2}, 1)$ cannot be true, and we have Lemma 2.4.2.

Lemma 2.4.3 Let $v = v(r)$, $r > 0$ be given by

$$v^k = -\frac{n+k}{n} + ir \qquad \text{with} \qquad 0 < \arg(v) < \pi/k.$$

Then

$$0 < \arg(v) - \arg(w) \leq \frac{\pi}{2n}, \qquad r > 0$$

where $w = \chi(v)$.

Proof As before, put $\varphi = \varphi(v)$ and

$$f(u) = p(\varphi, u)\,, \qquad u \in \mathbb{C}.$$

Let $z = z(r,t) = vte^{-i\frac{\pi}{2n}}$, $r, t > 0$, $v = v(r)$. Then we have

$$\begin{aligned}\frac{1}{n}\mathrm{Im}\big[zf'(z)\big] &= -k(n+k)t^n + t^k\left[k(n+k)\sin\left(\frac{k\pi}{2n}\right) + nkr\cos\left(\frac{k\pi}{2n}\right)\right] \\ &= t^k\left[-k(n+k)t^{n-k} + k(n+k)\sin\left(\frac{k\pi}{2n}\right) + nkr\cos\left(\frac{k\pi}{2n}\right)\right].\end{aligned}$$

Fix $r > 0$ and consider the equation

$$\mathrm{Im}\big[zf'(z)\big] = 0\,, \qquad z = z(r,t)\,, \quad t > 0\,.$$

Clearly, $\exists! t = t(r)$, $r > 0$, satisfying this equation.

$\forall r > 0$, r near 0, we have that $\varphi = \varphi(v)$ satisfies

$$\frac{d\varphi}{dr} = \frac{kn(n-k)r}{|v|^{2k}v^n}(n+k+inr)$$

since

$$A(v) = \frac{n(n-k)}{v^n}\begin{pmatrix} 0 & -n\mathrm{Im}(v^k) \\ n\mathrm{Im}(v^k) & 0 \end{pmatrix}$$

written in the basis $\{v, iv\}$ as in Definition 2.3.1, and

$$\frac{dv}{dr} = \frac{kr}{|v|^{2k}}v - \frac{k(n+k)}{n|v|^{2k}}iv$$

Here, we used the identification $\mathbb{R}^2 = \mathbb{R} + i\mathbb{R}$ as usual. We obtain

$$\begin{aligned}\frac{kn(n-k)r}{|v|^{2k}v^n}(n+k+inr) = \frac{d\varphi}{dr} &= \frac{n(n-k)}{w^n}B(w)\frac{dw}{dr} \\ &= \frac{n(n-k)}{w^n}\begin{pmatrix} \mathrm{Re}(nw^k+n+k) & 0 \\ k\mathrm{Im}(w^k) & (n+k)\mathrm{Re}(w^k+1) \end{pmatrix}\frac{dw}{dr}\end{aligned}$$

where $B(w)$ is written in the basis $\{w, iw\}$ as in Definition 2.333.

It follows from Proposition 2.3.1 and Lemma 2.4.1 that $\exists \epsilon = \epsilon(r) > 0$, $r > 0$, such that $\epsilon \to 0$ as $r \to 0$ and

$$0 < \arg(v^n) - \arg(w^n) < \frac{\pi}{2} + \epsilon .$$

Therefore, in the coordinate system $\{1/w^n, i/w^n\}$, we have that $1/v^n$ approaches the vector $1/v_0^n$

$= 1/\left[v(0)\right]^n \in \overline{QIV}$ but

$$B(w)w = \frac{1}{w^n}\mathrm{Re}(nw^k + n + k) + \frac{i}{w^n} n\mathrm{Im}(w^k) \in S$$

where S is an open sector with $\overline{S} \subset QI$. That is, $\arg[B(w)w]$ stays bounded away from both 0 and $\pi/2$, $\forall w$ near $w_0 = \chi(v_0)$.

We obtain that

$$\frac{d\varphi}{dr} = \frac{kn(n-k)r}{|v|^{2k}v^n}(n + k + inr) \in L^-_{B(w)w}, \qquad \forall r \quad \text{near} \quad 0 .$$

It follows that $dw/dr \in L^-_w$, $\forall r$ near 0, hence $\arg(w)$ is decreasing $\forall r$ near 0.

Therefore, we have

$$0 < \arg(v) - \arg(w) < \frac{\pi}{2n}, \qquad r > 0, \quad r \quad \text{near} \quad 0$$

since $0 < \arg(v_0) - \arg(w_0) \leq \frac{\pi}{2n}$ where $v_0 = v(0)$ and $w_0 = \chi(v_0)$.

We claim that if $|v|$ is large enough, then

$$\arg(v) - \arg(w) < \frac{\pi}{2n} .$$

$\forall v, w \in \mathbb{C} \setminus \{0\}$, put

$$\alpha = \arg\left(\frac{v^n}{v^k + \frac{n^2-k^2}{n^2}}\right) \qquad \text{and} \qquad \beta = \arg\left(\frac{w^n}{\frac{n+k}{2n}w^k + \frac{n^2-k^2}{n^2} + \frac{n-k}{2n}\bar{w}^k}\right)$$

$$\theta = \arg(v^{n-k}) \qquad \text{and} \qquad \omega = \arg\left(\frac{w^n}{\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k}\right).$$

Clearly, $\forall \epsilon > 0$, $\exists C > 0$ such that $\forall v, w \in \mathbb{C}$ with $|v|, |w| > C$, we have

$$|\alpha - \theta| < \epsilon/2$$

$$|\beta - \omega| < \epsilon/2 .$$

therefore

$$|\theta - \omega| \leq |\theta - \alpha| + |\alpha - \beta| + |\beta - \omega| < |\alpha - \beta| + \epsilon .$$

Now $x, y \in \mathbb{C} \setminus \{0\}$ implies

$$|\arg(x) - \arg(y)| = |\arg(1/x) - \arg(1/y)|$$

and $w = \chi(v)$ implies

$$\frac{v^n}{v^k + \frac{n^2-k^2}{n^2}} = \frac{w^n}{\frac{n+k}{2n}w^k + \frac{n^2-k^2}{n^2} + \frac{n-k}{2n}\bar{w}^k}$$

hence $\alpha = \beta$.

Fix $\epsilon > 0$ and let $C > 0$ be as above. Since

$$\lim_{|v| \to \infty} |\chi(v)| = \infty$$

it follows that $\exists D > C$ such that $\forall v \in V_0$ with $|v| > D$, we have $|w| > C$ where $w = \chi(v)$. Therefore, we have

$$\left|\arg\left(\frac{w^{n-k}}{v^{n-k}}\right) - \arg\left(\frac{\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k}{w^k}\right)\right| = \left|\arg\left(\frac{1}{v^{n-k}}\right) - \arg\left(\frac{\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k}{w^n}\right)\right|$$
$$= |\theta - \omega| < |\alpha - \beta| + \epsilon = \epsilon$$

whenever $v \in V_0$ with $|v| > D$ and $w = \chi(v)$. Here, we used the fact that

$$\arg\left(\frac{1}{v^{n-k}}\right) = \arg\left(\frac{w^{n-k}}{v^{n-k}}\right) - \arg(w^{n-k})$$

$$\arg\left(\frac{\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k}{w^n}\right) = \arg\left(\frac{\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k}{w^k}\right) - \arg(w^{n-k})$$

since $n - k \le k$ and we have

$$0 < \arg\left(\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k\right) < \arg(w^k) < \arg(v^k) < \pi\,.$$

Therefore, it is enough to show that

$$\left|\arg\left(\frac{\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k}{w^n}\right)\right| < \frac{(n-k)\pi}{2n}\,, \qquad w \in \mathbb{C}\setminus\{0\}\,. \tag{2.4.6}$$

Clearly, it is enough to check the above inequality when w varies on the unit circle, so $w^k = e^{i\sigma}$, $\sigma \in \mathbb{R}$. Then

$$\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k = \cos(\sigma) + i\frac{k}{n}\sin(\sigma)\,.$$

If the left hand side of (2.4.6) reaches its maximum with $|w| = 1$, then $w^k = e^{i\sigma}$ for some $\sigma \in (0, \pi/2)$ and we have

$$\frac{d\arg\left[\cos(\sigma) + i\frac{k}{n}\sin(\sigma)\right]}{d\sigma} = 1\,.$$

We obtain

$$1 = \operatorname{Im}\left[\frac{-\sin(\sigma) + i\frac{k}{n}\cos(\sigma)}{\cos(\sigma) + i\frac{k}{n}\sin(\sigma)}\right] = \frac{k/n}{(k/n)^2 + \left[1 - (k/n)^2\right]\cos^2(\sigma)}$$

hence

$$\cos^2(\sigma) = \frac{k}{n+k}\,, \qquad \text{so} \qquad \sin^2(\sigma) = \frac{n}{n+k} \qquad \text{and} \qquad \tan(\sigma) = \sqrt{n/k}\,.$$

Note that $\sigma \in (0, \pi/2)$ implies $\tan(\sigma) > 0$. Let

$$\tau = \arg\left(\frac{n+k}{2n}w^k + \frac{n-k}{2n}\bar{w}^k\right) = \arg\left[\operatorname{Re}(e^{i\sigma}) + i\frac{k}{n}\operatorname{Im}(e^{i\sigma})\right].$$

It follows that $\tan(\tau) = \frac{k}{n}\tan(\sigma) = \sqrt{k/n}$. Therefore, we have

$$\tan(\sigma - \tau) = \frac{\tan(\sigma) - \tan(\tau)}{1 + \tan(\sigma)\tan(\tau)} = \frac{\tan(\sigma) - \tan(\tau)}{2} = \frac{1}{2}\left(\sqrt{n/k} - \sqrt{k/n}\right).$$

We obtain

$$\tan^2(\sigma-\tau) = \frac{(n-k)^2}{4nk} \qquad \text{and} \qquad \sin^2(\sigma-\tau) = \frac{\tan^2(\sigma-\tau)}{1+\tan^2(\sigma-\tau)} = \left(\frac{n-k}{n+k}\right)^2$$

hence $\sin(\sigma-\tau) = \frac{n-k}{n+k}$. But clearly, $\sigma - \tau < \pi/2$, therefore

$$\frac{2}{\pi}(\sigma-\tau) < \sin(\sigma-\tau) \qquad \text{and} \qquad \sigma-\tau < \frac{(n-k)\pi}{2(n+k)} < \frac{(n-k)\pi}{2n}$$

as was required. This proves (2.4.6).

Assume that $\exists r_0 > 0$ such that $v_0 = v(r_0)$ and $w_0 = \chi(v_0)$ satisfy

$$\arg(v_0) - \arg(w_0) > \frac{\pi}{2n}.$$

We obtain a contradiction as follows.

Since $\forall r > 0$ with r either near 0 or near ∞, we have

$$\arg(v) - \arg(w) < \frac{\pi}{2n}, \qquad w = \chi(v), \quad v = v(r)$$

it follows that $\exists r_1, r_2 > 0$ with $r_1 \neq r_2$ such that $v_j = v(r_j)$ and $w_j = \chi(v_j)$, $j = 1, 2$, satisfy

$$\arg(v_j) - \arg(w_j) = \frac{\pi}{2n}, \qquad j = 1, 2.$$

Putting $z = z(r,t) = vte^{i\frac{\pi}{2n}}$, $r, t > 0$, $v = v(r)$, we obtain that the system of equations

$$\mathrm{Re}\big[f(z)\big] = 0$$

$$\mathrm{Im}\big[zf'(z)\big] = 0$$

have at least 2 distinct solutions, $(r_1, t_1) \neq (r_2, t_2)$. Here, as before, we put $\varphi = \varphi(v)$ and

$$f(u) = p(\varphi, u), \qquad u \in \mathbb{C}.$$

Now we compute

$$0 = \mathrm{Re}\big[f(z)\big] = -n^2 r t^n + t^k \left[n^2 r \sin\!\left(\frac{k\pi}{2n}\right) - n(n+k)\cos\!\left(\frac{k\pi}{2n}\right)\right] + n^2 - k^2$$

$$0 = \frac{1}{n}\mathrm{Im}\big[zf'(z)\big] = t^k\left[-k(n+k)t^{n-k} + k(n+k)\sin\!\left(\frac{k\pi}{2n}\right) + nkr\cos\!\left(\frac{k\pi}{2n}\right)\right].$$

Since the second equation clearly defines $t = t(r)$ as an increasing function of r, $r_1 \neq r_2$ implies $t_1 \neq t_2$.

After eliminating r from these equations, we obtain that

$$\begin{aligned} nt^k \cos^2\!\left(\frac{k\pi}{2n}\right) &- (n-k)\cos\!\left(\frac{k\pi}{2n}\right) \\ &= -nt^{2n-k}\left[1 - \sin\!\left(\frac{k\pi}{2n}\right)\right] + nt^n\left[1 - \sin\!\left(\frac{k\pi}{2n}\right)\right]\sin\!\left(\frac{k\pi}{2n}\right) \end{aligned}$$

has at least 2 distinct solutions, $t_1 \neq t_2$. Note that $t_1, t_2 < 1$ since by construction, $|w| = |\chi(v)| < |v|\ \forall v \in V_0$. We may assume that $t_1 < t_2$.

Taking derivatives of both sides, multiplying by t and collecting like terms, we get that

(2.4.7)

$$n(2n-k)t^{2(n-k)}\left[1 - \sin\!\left(\frac{k\pi}{2n}\right)\right] - n^2 t^{n-k}\left[1 - \sin\!\left(\frac{k\pi}{2n}\right)\right]\sin\!\left(\frac{k\pi}{2n}\right) + kn\cos\!\left(\frac{k\pi}{2n}\right) = 0$$

has at least 1 solution $t = s \in \mathbb{R}$ with $t_1 < s < t_2 < 1$. But this is a quadratic equation in t^{n-k} with discriminant

$$(1-y)^2\left[n^2y^2 - 4k(2n-k)(1+y)\right]$$

where we put $y = \sin(\frac{k\pi}{2n})$. Then $1/\sqrt{2} \leq y < 1$ since $n/2 \leq k < n$. But $\forall$ such values of y and k, we have $y^2 < y$ and $4k(2n-k)(1+y) > 2n^2$, hence

$$(1-y)^2\left[n^2y^2 - 4k(2n-k)(1+y)\right] < -n^2(1-y)^2 < 0$$

contradicting the existence of a solution $t = s$ of (2.4.7). This proves Lemma 2.4.3.

Lemma 2.4.4 Consider $v = r^{1/k}e^{i\pi/k}$ and $w = \chi(v)$, $r \in \left[1, \frac{n+k}{n}\right]$. Then $|w|$ is an increasing function of r.

Proof $\forall r \in \left[1, \frac{n+k}{n}\right]$, $v = r^{1/k}e^{i\pi/k}$ satisfies $\mathrm{Im}(v^k) = 0$. Therefore, using the basis $\{v, iv\}$, we have

$$A(v) = \frac{n(n-k)}{v^n}\begin{pmatrix} -nr+n+k & 0 \\ 0 & -nr+n+k \end{pmatrix}$$

where $-nr + n + k > 0$. Now

$$\frac{dv}{dr} = \frac{1}{k}r^{1/k-1}e^{i\pi/k} = \frac{1}{kr}v\,.$$

It follows that $d\arg(v)/dr \equiv 0$ and

$$\frac{d\varphi}{dr} = A(v)\frac{dv}{dr} = \frac{n(n-k)i(-nr+n+k)}{v^n}\frac{1}{kr}\,.$$

Also, we have

$$\frac{d\varphi}{dr} = B(w)\frac{dw}{dr} = \frac{n(n-k)}{w^n}\begin{pmatrix} \mathrm{Re}(nw^k+n+k) & 0 \\ k\mathrm{Im}(w^k) & (n+k)\mathrm{Re}(w^k+1) \end{pmatrix}\frac{dw}{dr}$$

where the entries of $B(w)$ are all positive. Since Lemma 2.4.1 implies $0 < \arg(v) - \arg(w) \leq \frac{\pi}{2n}$, it follows that $v \in QIV$ in the coordinate system $\{1/w^n, i/w^n\}$. Therefore, we have $d\varphi/dr \in QIV$ as well, since $d\varphi/dr \parallel v$. Then we must have $d\varphi/dr \in L^-_{i/w} = L^-_{B(w)iw}$. But

$$\frac{n(n-k)}{w^n}\begin{pmatrix} \mathrm{Re}(nw^k+n+k) \\ k\mathrm{Im}(w^k) \end{pmatrix} \in L^-_{i/w^n}$$

hence $|w|$ increases as claimed. This proves Lemma 2.4.4.

Lemma 2.4.5 Consider $v = v(r)$, $r > 0$ given by $v^k = -\frac{n+k}{n} + ir$ with $0 < \arg(v) < \pi/2$. Let $w = \chi(v)$. Then $|w|$ is an increasing function of r.

Proof By a similar argument as above, we get that

$$\frac{d\varphi}{dr} \in L^-_{i/w^n} = L^-_{b(w)iw}$$

since $0 < \arg(v) - \arg(w) \leq \frac{\pi}{2n}$. Therefore, $|w|$ increases $\forall r > 0$ in the same way. This proves Lemma 2.4.5.

Lemma 2.4.6 $\forall v \in V_0$, $w = \chi(v)$ satisfies

$$0 < \arg(v) - \arg(w) < \frac{\pi}{2n}.$$

Proof Assume $v_0 \in V_0$ and $w_0 = \chi(v_0)$ satisfy

$$\arg(v_0) - \arg(w_0) \geq \frac{\pi}{2n}.$$

If the above inequality is strict, then $\exists t > 1$ such that $v = tv_0 \in V_0$ satisfies $\arg(v) - \arg(w) = \frac{\pi}{2n}$ since the half ray $L = \{tv : t > 0\}$ intersects $\{\delta\}$ and $v_1 \in L \cap \{\delta\}$ satisfies $\arg(v_1) - \arg(w_1) \leq \frac{\pi}{2n}$ where $w_1 = \chi(v_1)$.

Therefore, it is enough to consider the case $\arg(v_0) - \arg(w_0) = \frac{\pi}{2n}$. Then we have $\arg(1/w^n) - \arg(1/v^n) = \pi/2$, hence $1/v^n$ lines up with $-i/w^n$. It follows that $1/v^n$ lines up with $B(w)(-iw)$.

Let $\zeta(t) = tv$, $t > 0$. Then $\zeta'(t) \equiv v$ and $\arg[\zeta'(t)] \equiv \arg(v)$. Therefore, $\zeta'(1) \in QIV$ in the coordinate system $\{1/w^n, i/w^n\}$, hence $w = w(t) = \chi[\zeta(t)]$ satisfies that $\arg(w)$ is decreasing at $t = 1$. It follows that

$$\arg[\zeta(t)] - \arg(w)$$

increases at $t = 1$. As above, we conclude that $\exists t_1 > 1$ such that $v_1 = \zeta(t_1) \in V_0$ satisfies

$$\arg[\zeta(t_1)] - \arg(w_1) = \frac{\pi}{2n}$$

where $w_1 = \chi(v_1)$.

Repeating the same argument over and over, we obtain a sequence $(t_j)_{j=1}^{\infty} \subset \mathbb{R}$ such that $1 < t_1 < \cdots < t_j < \ldots$ and

$$\lim_{j\to\infty} t_j v_0 = v_\infty \in \{\delta\} \qquad \text{or} \qquad \lim_{j\to\infty} t_j = \infty$$

$$\arg(t_j v_0) - \arg(w_j) = \frac{\pi}{2n}, \qquad j \in \mathbb{N}\setminus\{0\} \tag{2.4.8}$$

where $w_j = \chi(t_j v_0)$, $j \in \mathbb{N}\setminus\{0\}$.

Assume that we have the first possibility above.

Then the same considerations at v_∞ show that $\arg(tv_0) - \arg(w)$ increases at $t = t_\infty$ where $t_\infty v_0 = v_\infty$ and $w = \chi(tv_0)$. But we have a sequence $(t_j v_0)_{j=1}^{\infty}$ converging to v_∞ satisfying (2.4.8). Therefore, $\arg(tv_0) - \arg(w)$, $w = \chi(tv_0)$, cannot be increasing at $t = t_\infty$, a contradiction.

In the other case, we get a contradiction because

$$\arg(v) - \arg(w) < \frac{\pi}{2n}$$

if $|v| \to \infty$ and $w = \chi(v)$ as we have seen, yet we obtained a sequence $(v_j)_{j=1}^{\infty}$ such that $|v_j| = t_j|v_0| \to \infty$ and $w_j = \chi(v_j)$, $j \in \mathbb{N}\setminus\{0\}$, satisfies (2.4.8).

This proves Lemma 2.4.6.

Lemma 2.4.7 Let

$$V = \{z \in \mathbb{C} : \mathrm{Re}[g(z)] > 0\} \setminus \{re^{ij\pi/k} : 1 \le r \le \left(\tfrac{n+k}{n}\right)^{1/k}, j = 1, \ldots, 2k\}$$

where $g(z) = nz^k + n + k$, $z \in \mathbb{C}$, as before. Then we have the following.

(1) χ extends continuously to V as a global real-analytic diffeomorphism with $\chi(0) = 0$.

(2) Let $v_0 \in V \setminus \{0\}$, let $\varphi_0 = \varphi(v_0)$ and put

$$f_0(z) = p(\varphi_0, z)\,, \qquad z \in \mathbb{C}.$$

If $w_0 = \chi(v_0)$ satisfies $|w_0| < 1$, then $\exists \alpha, \beta \in \mathbb{R}$ such that the open arc

$$A_0 = \{e^{i\omega} : \alpha < \omega < \beta\}$$

satisfies $w_0/|w_0| \in A_0$ and

$$\mathrm{Re}\big[f_0(e^{i\alpha})\big] = \mathrm{Re}\big[f_0(e^{i\beta})\big] = 0$$

$$\mathrm{Re}\big[f_0(z)\big] < 0\,, \qquad z \in A_0\,.$$

Also, we have

$$-\frac{3\pi}{2n} < \arg(v_0) - \arg(z) < \frac{3\pi}{2n}\,, \qquad z \in A_0\,.$$

(3) $\Delta \subset \chi(V)$ where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Proof (1) and (2) Let $v_0 \in V_0$, let $w_0 = \chi(v_0$ and put

$$f_0(z) = p(\varphi_0, z)\,, \qquad z \in \mathbb{C}$$

where $\varphi_0 = \varphi(v_0)$. Assume $|w_0| < 1$. Then by Section 2.1, $\exists \alpha, \beta \in \mathbb{R}$ with

$$0 < \beta - \alpha < \frac{\pi}{n}$$

such that the open arc

$$A_0 = \{e^{i\omega} : \alpha < \omega < \beta\}$$

satisfies $w_0/|w_0| \in A_0$ and

$$\mathrm{Re}\big[f_0(e^{i\alpha})\big] = \mathrm{Re}\big[f_0(e^{i\beta})\big] = 0$$

$$\mathrm{Re}\big[f_0(z)\big] < 0\,, \qquad z \in A_0\,.$$

Since

$$0 < \arg(v_0) - \arg(w_0) < \frac{\pi}{2n}$$

it follows that we have

$$-\frac{3\pi}{2n} < \arg(v_0) - \arg(z) < \frac{3\pi}{2n}, \qquad z \in A_0 .$$

Let V_1 be the reflection of V_0 about the line

$$L_1 = \{te^{i\pi/k} : t \in \mathbb{R}\} .$$

Fix $v_0 = r^{1/k}e^{i\pi/k}e^{i\theta} \in V_0$ and let $\varphi_0 = \varphi(v_0)$. Then

$$\varphi_0 v_0^n = -n^2 v_0^k - n^2 + k^2 = n^2 re^{ik\theta} - n^2 + k^2 .$$

Let $v_1 = r^{1/k}e^{i\pi/k}e^{-i\theta} \in V_1$ and let $\varphi_1 = \varphi(v_1)$. Then

$$\varphi_1 v_1^n = n^2 re^{-ik\theta} - n^2 + k^2 .$$

We obtain

$$\begin{aligned} p(\varphi_0, v_0 u) &= \varphi_0 v_0^n u^n + n^2 v_0^k u^k + n^2 - k^2 \\ &= (n^2 re^{ik\theta} - n^2 + k^2)u^n - n^2 re^{ik\theta} u^k + n^2 - k^2 , \qquad u \in \mathbb{C} \end{aligned}$$

$$\begin{aligned} p(\varphi_1, v_1 \bar{u}) &= \varphi_1 v_1^n \bar{u}^n + n^2 v_1^k \bar{u}^k + n^2 - k^2 \\ &= (n^2 re^{-ik\theta} - n^2 + k^2)\bar{u}^n - n^2 re^{-ik\theta} \bar{u}^k + n^2 - k^2 \\ &= \overline{p(\varphi_0, v_0 u)} , \qquad u \in \mathbb{C}. \end{aligned}$$

It follows that whenever $v_0 \in V_0$ and $v_1 \in V_1$ are reflections of each other about L_1 and $\varphi_j = \varphi(v_j)$, $j = 0, 1$, then

$$f_j(z) = p(\varphi_j, z) , \qquad z \in \mathbb{C}, \quad j = 0, 1$$

satisfy

$$f_1(u_1) = \overline{f_0(u_0)}$$

whenever u_0 and u_1 are reflections of each other about L_1. Therefore, for such corresponding $u_j \in V_j$, $j = 0, 1$, we have

$$\mathrm{Re}\big[f_0(u_0)\big] = \mathrm{Re}\big[f_1(u_1)\big] .$$

Let $w_0 = \chi(v_0)$, let w_1 be the reflection of w_0 about L_1 and define

$$\chi(v_1) = w_1 .$$

Assume $|w_0| = |w_1| < 1$. Then the corresponding arcs A_0 and its reflection A_1 about L_1 satisfy $w_j/|w_j| \in A_j$, $j = 0, 1$ and

$$-\frac{3\pi}{2n} < \arg(v_j) - \arg(z) < \frac{3\pi}{2n}, \qquad z \in A_j, \quad j = 0, 1 .$$

Let $\ell \in \{1, \dots, 2k\}$ and assume that we have defined $V_0, \dots, V_{\ell-1}$ and extended χ to $V_0 \cup \dots \cup A_{\ell-1}$ as above. Put

$$L_\ell = \{te^{i\ell\pi/k} : t \in \mathbb{R}\}$$

and let V_ℓ be the reflection of $V_{\ell-1}$ about L_ℓ. Let $v_j \in V_j$, $j = \ell - 1, \ell$, and assume that $v_{\ell-1}$ and v_ℓ are reflections of each other about L_ℓ. Let $w_{\ell-1} = \chi(v_{\ell-1})$, let w_ℓ be the reflection of $w_{\ell-1}$ about L_ℓ and define

$$\chi(v_\ell) = w_\ell .$$

Assume $|w_{\ell-1}| = |w_\ell| < 1$. Then the corresponding arcs $A_{\ell-1}$ and its reflection A_ℓ about L_ℓ satisfy $w_j/|w_j| \in A_j$, $j = \ell - 1, \ell$ and

$$= \frac{3\pi}{2n} < \arg(v_j) - \arg(z) < \frac{3\pi}{2n}, \qquad z \in A_j, \quad j = \ell - 1, \ell .$$

Continuing inductively in this fashion, we obtain (1) and (2) since clearly, V_{2k} and V_0 are again reflections of each other about the real line $L_0 = \mathbb{R}$, so $\chi\big|_{V_{2k}}$ and $\chi\big|_{V_0}$ will match up in the same way. Note that we need to define

$$\chi(z) = z\,, \qquad z \in \{re^{ij\pi/k} : 0 \le r < 1\,,\, j = 1, \ldots, 2k\}$$

to get the desired extension of χ to all of V.

(3) This clearly follows from Lemma 2.4.4 and Lemma 2.4.5.

This concludes the proof of Lemma 2.4.7.

CONCLUSION

Fix $\ell, m \in \mathbb{N}$ with $m/2 \leq \ell < m$ and let

$$P_{2m}(z) = 2m^2 z^{m+\ell}\bar{z}^{m-\ell} + 4(m^2 - \ell^2)|z|^{2m} + 2m^2 z^{m-\ell}\bar{z}^{m+\ell}$$

$$= |z|^{2m}\mathrm{Re}\big[4m^2 e^{i2\ell\theta} + 4(m^2 - \ell^2)\big]$$

where $z = |z|e^{i\theta} \in \mathbb{C}\backslash\{0\}$. Then $P_{2m}(z)$ is a subharmonic but not harmonic real-valued homogeneous polynomial in z and $\bar{z}$ of degree $2m$.

Consider the domain

$$\Omega = \big\{(z,w) \in \mathbb{C}^2 : \mathrm{Re}(w^{2m}) + \delta P_{2m}(z) + \mathrm{Re}(z^{2m})\big\}$$

where $\delta > 0$ is chosen so that the region

$$\big\{z \in \mathbb{C} : \delta P_{2m}(z) + \mathrm{Re}(z^{2m})\big\}$$

is the union of $2m$ disjoint open sectors, as in [**BF**].

Based on the preceding analysis of the Riemann surface $\mathcal{R}$ associated with the above domain $\Omega \subset \mathbb{C}^2$, we now prove the following main result of this paper.

Theorem Let

$$\Omega' = \big\{(z',w') \in \mathbb{C}^2 : \mathrm{Re}(w') + \delta\mathbb{P}_{2m}(z') + \mathrm{Re}\big[(z')^{2m}\big]\big\}\,.$$

Then $0 \in \partial\Omega'$ is of finite type $2m$ and $\exists$ a function H continuous on $\overline{\Omega'}$ and holomorphic on a neighborhood U' of $\overline{\Omega'} \setminus \{0\}$ with $0 \notin U'$ such that H is a peak function for U' at $0 \in \partial U'$. Also, $H = \exp(-G)$ where G is Hölder $1/(6m)$=$1/(3\text{type})$ near $0 \in \mathbb{C}^2$.

Proof Let $(z, w) \in \Omega$. Then $\exists [\zeta : \eta] \in \mathbb{P}$ such that

$$(z, w) \in \Omega \cap L_{[\zeta:\eta]}$$

where

$$L_{[\zeta:\eta]} = \left\{ (z, w) \in \mathbb{C}^2 : \eta z = \zeta w \right\}.$$

Assume $z \neq 0$. Then we may assume that $\zeta = 1$, that is, we have

$$w = \eta z .$$

Put

$$\left\{ u \in \mathbb{C} : (u, \eta u) \in \Omega \right\} = \bigcup_{j=1}^{N(\eta)} S_j$$

where $S_1, \ldots, S_{N(\eta)} \subset \mathbb{C}$ are disjoint open sectors. Then $\exists ! k \in \{1, \ldots, N(\eta)\}$ such that

$$z \in S_k .$$

Let $w_k \in \Delta = \{u \in \mathbb{C} : |u| < 1\}$ be the corresponding point of $\mathcal{R}$ as given in Section 2.1. Then $\exists ! v_k \in \mathbb{C} \setminus \{0\}$ satisfying

$$\chi(v_k) = w_k$$

where χ is the map associated with the polynomial

$$p(\xi, z) = \xi z^{2m} + \delta\left[4(m^2 - \ell^2) z^{2\ell} + 4m^2\right], \qquad \xi, z \in \mathbb{C}$$

with $\xi = \eta^{2m} + 1$, as defined in Definition 2.4.3.

Define

$$g(z, w) = \frac{z}{v_k} .$$

Then by construction, we have

$$-\frac{3\pi}{4m} < \arg\left[g(z, w)\right] < \frac{3\pi}{4m} .$$

Next, assume that $(0, w_0) \in \Omega$ and let

$$\big((z_j, w_j)\big)_{j=1}^{\infty} \subset \Omega$$

satisfy $z_j \neq 0$, $j \in \mathbb{N} \setminus \{0\}$, and $(z_j, w_j) \to (0, w_0)$ as $j \to \infty$. Put

$$\eta_j = \frac{w_j}{z_j}, \qquad j \in \mathbb{N} \setminus \{0\}$$

$$\{u \in \mathbb{C} : (u, \eta_j u) \in \Omega\} = \bigcup_{\nu=1}^{N(\nu)} S_\nu(\eta_j), \qquad j \in \mathbb{N} \setminus \{0\}$$

as above, and $\forall j \in \mathbb{N} \setminus \{0\}$, let $k_j \in \{1, \ldots, N(\eta_j)\}$ satisfy

$$z_j \in S_{k_j}(\eta_j).$$

Then we have

$$g(z_j, w_j) = \frac{z_j}{v_{k_j}} = \frac{w_j}{\eta_j^{2m} v_{k_j}}.$$

But we have

$$(\eta_j^{2m} + 1) v_{k_j}^{2m} + \delta\big[4(m^2 - \ell^2) v_{k_j}^{2\ell} + 4m^2\big] = 0$$

hence $\eta_j^{2m} v_{k_j} \to -4m^2\delta$ as $j \to \infty$, because $|\eta_j| \to \infty$ and $v_{k_j} \to 0$ as $j \to \infty$. We obtain that

$$g(z_j, w_j) \to -\frac{w_0}{4m^2\delta} \qquad \text{as} \qquad j \to \infty.$$

Define

$$g(0, w_0) = -\frac{w_0}{4m^2\delta}.$$

Now $(0, w_0) \in \Omega$ implies that

$$\operatorname{Re}(w_0^{2m}) < 0$$

hence we have

$$-\frac{\pi}{4m} < \arg\big[g(0, w_0)\big] < \frac{\pi}{4m}.$$

We obtain the function $g(z,w)$, $(z,w) \in \Omega$, such that $\forall[\zeta : \eta] \in \mathbb{P}$, g is locally linear on $\Omega \cap L_{[\zeta:\eta]}$. Therefore, $g \in H(\Omega)$ and g clearly extends continuously to $\overline{\Omega}$ with

$$g(0,0) = 0$$

and g extends to a function holomorphic on an open neighborhood U of $\overline{\Omega} \setminus \{0\}$ with $0 \notin U$ such that $\forall[\zeta : \eta] \in \mathbb{P}$, the extended g is locally linear on $U \cap L_{[\zeta:\eta]}$, satisfying

$$-\frac{3\pi}{4m} < \arg\big[g(z,w)\big] < \frac{3\pi}{4m}, \qquad (z,w) \in U.$$

We define a peak function on Ω' as follows. Let $(z',w') \in \Omega'$ satisfy $w' \neq 0$. Then put

$$F(z',w') = \prod_{w^{2m}=w'} g(z',w)$$

as in [**BF**]. Let $w = w_1, \ldots, w_{2m}$ be the distinct solutions to the equation

$$w^{2m} = w', \qquad w \in \mathbb{C}.$$

We obtain

$$g(z',w_1) = \cdots = g(z',w_{2m})$$

hence

$$F(z',w') = \big[g(z',w_1)\big]^{2m}.$$

Also, put

$$F(z',0) = \big[g(z',0)\big]^{2m}, \qquad (z',0) \in \Omega'.$$

Then $\forall(z',w') \in \Omega'$, we clearly have

$$-\frac{3\pi}{2} < \arg\big[F(z',w')\big] < \frac{3\pi}{2}.$$

Clearly, F extends to a continuous function on $\overline{\Omega'}$ and extends to a holomorphic function on an open neighborhood U' of $\overline{\Omega'} \setminus \{0\}$ with $0 \notin U'$, so that we have

$$-\frac{3\pi}{2} < \arg\bigl[F(z',w')\bigr] < \frac{3\pi}{2}\,, \qquad (z',w') \in U'\,.$$

Finally, define

$$G(z',w') = \bigl[F(z',w')\bigr]^{1/3}\,, \qquad (z',w') \in U'$$

$$H(z',w') = \exp\bigl[-G(z',w')\bigr]\,, \qquad (z',w') \in \Omega'$$

with $G(0,0) = 0$, hence $H(0,0) = 1$.

Then H is the required peak function since $G = F^{1/3}$ and F is easily seen to be Hölder $1/(2m)$=1/type near $0 \in \mathbb{C}^2$ as in [**BF**].

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