A GENERAL LATTICE APPROACH TO PRICING AMERICAN OPTIONS WITH NONLOGNORMAL DISTRIBUTIONS

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CHAPTER ONE

INTRODUCTION

Accurate option valuation formulas are highly demanded by practitioners in financial and commodity markets. Companies use option valuation formulas to predict prices, make trading decisions, and manage the related risk (Lapan, Moschini, and Hanson; Wilson, and Fung). Among the option pricing formulas developed so far, the Black-Scholes option valuation formula has been the most popular since its publication in 1973. While ease of use is its advantage, the inaccuracy of the formula is widely recognized. At least for short maturity stock options, the biases in the Black-Scholes are serious (Backus et al.). Due to the inaccuracy of Black-Scholes, traders often adjust the volatility parameter in the formula by personal experience, use alternatives such as binomial tree or trinomial tree models, or use a different volatility for every strike price and every maturity. However, Dumas, Fleming and Whaley argue that there is no evidence showing superior performance of those models over the Black-Scholes formula.

The inaccuracy problem of the Black-Scholes formula mainly arises from the assumptions on which the model is based being incorrect. The lognormal distribution is the most likely weakness. While such a distribution can describe some asset price processes well, most asset prices can not be well modeled by a lognormal distribution. For example, futures and stock price distributions tend to be more leptokurtic than a lognormal distribution and are sometimes skewed (Hall, Brorsen, and Irwin; Akgiray and Boothe).

Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of volatility. The volatility level that makes the function equal to the actual option price is called implied volatility. If the model were perfect, the implied volatility would be the same for all option market prices at a given time. However, many empirical studies have revealed that the implied volatility strongly depends on the strike price and the maturity of European options (Myers, and Hanson). If we plot implied volatilities of exchange-traded options against their strike prices for fixed maturity, the curve is typically convex in shape, rather than a straight line as suggested by the Black-Scholes model. The phenomenon that the volatility depends on the strike price is usually called the "volatility smile", since such a volatility strike structure ends up looking like a smile. In reality, the volatility strike structure does not have to be a smile. It can take on various shapes, depending on what the actual price distribution is, compared to the lognormal price distribution. When the actual price distribution has fatter tails than the lognormal model distribution does, the out-of-the-money calls and puts tend to show volatilities that increase as the option strikes make the options go further and further out-of-the-money. If the option pricing model is built to incorporate the exact price distribution, the volatility strike structure would be flat, i.e., the same volatility would reflect all the options of the same expiration time but of varying strike prices. Although such a perfect option pricing model, that incorporates all the price distribution characteristics, is not likely to exist, a relatively accurate model should be expected to capture the main, if not all, characteristics of the price distribution. Since the Black-Scholes formula is not satisfactory, traders are often forced to incorporate the strike structure volatility effects in the implementation of the model: the traders maintain the volatilities for various out-of-the-money and in-the-money options.

Another drawback of the Black-Scholes formula is that it is applicable only to European options. A European option only allows exercise on a single date. Most options traded allow for more than one date as the possible exercise date. These options are called American options. A holder of an American option will compare the value he or she would obtain by exercising the option with the market value of the option if the right were held and not exercised. Since the additional early exercise privilege should not be worthless, an American option is worth more than its European counterpart. Thus, if we use the Black-Scholes formula to value an American option, under-valuation usually occurs.

Many option valuation models have been developed to overcome the drawbacks of the Black-Scholes. However, all these models can handle only one of the two inaccuracy sources in the Black-Scholes, not both. For example, using the idea analogous to Taylor series as an approximation for an arbitrary analytical function, Jarrow and Rudd (1982) developed an approximate option valuation formula, in which the biases of an option formula are captured by a function of additional parameters measuring skewness and kurtosis. This model has potential to increase the accuracy of the Black-Scholes, but is limited to European options. Another example is the Merton jump-diffusion model (1976), in which the underlying price process is assumed to consist of a Poisson process generating the jumps. Similar to the Jarrow-Rudd approximate formula, the Merton's jump-diffusion model only applies to European options and, more limitedly, is only suitable for options with jump-diffusion asset price processes. Moreover, both the

Jarrow-Rudd model and the Merton jump-diffusion model require additional input parameters that are not directly observable in the markets. They must thus be estimated using various statistical techniques. The additional accuracy that might be offered by these models is outweighed by the complexity of estimating additional parameters required. Due to these limitations, these two models assuming underlying distributions other than lognormal have received attention mainly by academics. In practice, various binomial tree and trinomial tree models such as the models developed by Cox, Ross and Rubinstein in 1979, and by Boyle in 1986 are widely used. The tree models can handle European options as well as American options. However, like the Black-Scholes formula, all these tree models assume that the underlying asset prices follow a lognormal distribution.

The objective of this study is to develop a general option valuation model that is suitable for an arbitrary underlying asset price distribution yet can handle both European and American options. Such a model is valuable for practitioners because in practice the information about the distribution of the underlying asset price is often very limited. It is risky to rely on a model based on the assumption of a lognormal distribution when the true underlying distribution is unclear. It is also risky to simply modify a model based on the lognormal distribution with personal experience when it is known that the true distribution is significantly different from a lognormal.

In general, a model suitable for an arbitrary distribution must capture the effects of higher moments. Previous empirical work has shown that the underlying distribution does have skewness and excess kurtosis (Sherrick, Garcia, and Tirupattur; O'Brien, Hayenga, and Babcock). On the other hand, a model suitable for American options can not use an analytic formula because analytic valuation formulas are generally not suitable for American options. Therefore, the general option valuation model developed in this study will be a numerical scheme that incorporates higher moment parameters.

The common numerical methods employed in option valuation include the binomial and trinomial tree schemes, finite difference algorithms and Monte Carlo simulation. This study will develop a general binomial tree model and a general trinomial tree model because the tree schemes are the most widely used in the finance community for valuation of a wide variety of option models and because of their ease of implementation and pedagogical appeal.

This study is presented as follows. First, previous analytic option valuation models based on non-lognormal distributions and the tree schemes based on lognormal distributions are reviewed. The theoretical framework then follows on how a general binomial tree is developed and extended to a general trinomial tree, how the general binomial scheme converges to a higher order differential equation, for which the well known Black-Scholes differential equation is a special case, and converges to a general option valuation formula, for which the Black-Scholes formula is a special case, and how these general tree schemes can be implemented. Then, through numerical simulations and empirical analysis on the real data from the futures options on three commodities, the accuracy of the general binomial and trinomial tree models developed in this study is examined by comparison with the previous models reviewed. Finally, the conclusions and practical applications are discussed.

CHAPTER TWO

REVIEW OF PREVIOUS OPTION PRICING MODELS

Past option pricing models can be categorized into two groups: the models assuming the lognormal underlying distribution and the ones assuming underlying distributions other than lognormal. Most option pricing models are based on the lognormal distribution assumption. Only a few option pricing models are applicable to non-lognormal distributions. This chapter first introduces the lognormal distribution and the option pricing models based on it. Then the Edgeworth series and Merton's jump-diffusion process as well as the option valuation models based on them are reviewd.

Models Based on the Lognormal Distribution

Lognormal Distribution and Black-Scholes Model

A commonly used assumption in financial modeling is that the price process of an asset is governed by a Brownian motion (Hull, 2000):

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where S is asset price, μ is the expected rate of return per unit of time from the asset, σ is the volatility of the asset price, z denotes a standard Brownian motion with no drift, that is, $dz = \varepsilon\sqrt{dt}$ with $\varepsilon \sim N(0, 1)$, and the values of dz for any two different short intervals of time dt are independent. Both the expected drift rate and variance rate of the process can change over time. But, during a very short period, they are considered constant.

To show S is lognormally distributed, the process followed by the logarithm of the asset price is derived by Ito's lemma. From Ito's lemma (Ito), the process followed by a function of S, denoted as G, is

$$dG = \left(\frac{\partial G}{\partial t} + \mu \frac{\partial G}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 G}{\partial S^2} \right) dt + \sigma \frac{\partial G}{\partial S} dz .$$

Define $G = \ln S$, then $\frac{\partial G}{\partial t} = 0$, $\frac{\partial G}{\partial S} = \frac{1}{S}$, and $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$. It follows from Ito's lemma that the process followed by $\ln S$ is

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz .$$

Because μ and σ are constant, this equation indicates that $\ln S$ follows a Brownian process with drift $\left(\mu - \sigma^2 / 2\right)$ and variance σ^2. Let S and S_T denote the asset price at the current time and at any later time T periods from now, according to the definition of a Brownian process (Ross), $\ln S_T - \ln S = \ln \frac{S_T}{S}$ is normally distributed with mean $\left(\mu - \sigma^2 / 2\right)$ and variance $\sigma^2 T$. Given the current asset price S, this is to say that S_T is lognormally distributed.

Assuming a lognormal distribution and no riskless arbitrage, Black and Scholes (1973) derived a parabolic partial differential equation called the Black-Scholes equation:

$$\frac{\partial c}{\partial t} + rS \frac{\partial c}{\partial S} + S^2 \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial S^2} - rc = 0 \qquad (2)$$

where c denotes the premium of a European vanilla, i.e. without special features, call option. By using some initial and boundary conditions, the solution for the Black-Scholes equation is:

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$

$$d_1 = \frac{\ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{3}$$

$$d_2 = \frac{\ln(S/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where X denotes the strike price corresponding to a maturity T, other notations are as defined before. By the put-call parity relation, the price of a European vanilla put option, denoted by p is given by

$$p = Xe^{-rT}N(-d_2) - SN(-d_1).$$

Binomial Tree Models

The Black-Scholes is an analytic formula for European options. Generally, an analytic formula can not handle the dynamics of value comparison between holding and exercising the right required with an American option. For American option valuation, binomial tree or trinomial tree option pricing models are widely used. Unlike the continuous Black-Scholes framework, the tree methods bypass the derivation of partial differential equations and so the comprehension of the method is accessible to a much wider audience in the finance community. There are various versions of the binomial and trinomial models. The different versions of the tree models differ in the method used to determine the parameters in the model. All versions of the binomial tree models have the same lattice structure. So do the various trinomial models. In a standard tree model, the asset price movement is assumed to be a discrete random walk, which converges to a continuous lognormal diffusion as the time interval between successive steps tends to zero. The tree models are consistent with the risk neutrality argument where the option price obtained from the model depends only on the growth rate of a riskless bond but is independent of the expected rate of return of the asset.

Let S and $S_{\Delta t}$ denote, respectively, the asset prices at the current time and at one period later. Suppose $S_{\Delta t}$ be either Su with probability q or Sd with probability $1 - q$, where $u > d > 0$. Also assume that it is possible to borrow or lend at a risk-free interest rate r. Then to avoid risk-less arbitrage opportunities, it must be true that $u > e^{r\Delta t} > d$. To see this, suppose $e^{r\Delta t} \geq u > d$ and $0 < q < 1$. Then one could short the asset and loan the proceeds, thereby obtaining a profit of either $e^{r\Delta t} - u$ or $e^{r\Delta t} - d$, depending on the outcome state. The initial cost is zero, but in either case the profit is positive, which is not possible if there are no arbitrage opportunities. A similar argument rules out $u > \mathrm{d} \geq e^{r\Delta t}$. To find the value of the call option, we use a no-arbitrage argument by referring to Figure 1. The c in the figure represents the call option price and the X denotes the strike price. This figure shows the two period binomial lattices for the asset price and the value of the option.

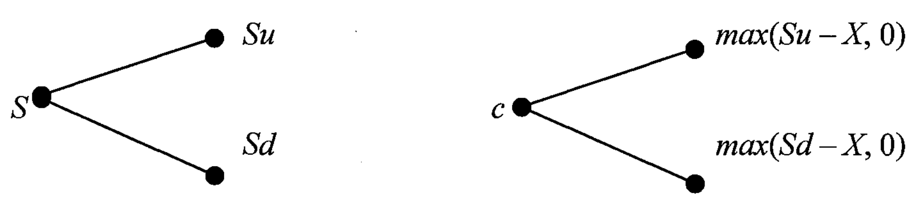


Figure 1. Two-period binomial lattices

To illustrate the pricing mechanism of the binomial tree model, consider a portfolio consisting of a long position in Δ units of asset valued at S and a short position in one call option with a value c. The current value of the portfolio is $S\Delta$ - c. Suppose the asset price moves to Su or Sd, then the corresponding call option value a period later is

$$c_u = \max(Su - K,\ \ 0)$$

or

$$c_d = \max(Sd - K,\ \ 0)$$

where c_u is the call option price corresponding to the upward movement of the asset price and c_u to the downward movement. The number Δ can be chosen so that the portfolio is risk-less no matter which of the two uncertain possibilities realizes. For the portfolio to be risk-less, the portfolio should have the same value in either of the possible situations. This requires that

$$Su\Delta - c_u = Sd\Delta - c_d$$

or

$$\Delta = (c_u - c_d)/(Su - Sd).$$

In the absence of arbitrage opportunities, a risk-less portfolio must earn the risk free interest rate r. It follows that

$$S\Delta - c = (Su\Delta - c_u)e^{-r\Delta t}$$

or

$$c = S\Delta - (Su\Delta - c_u)e^{-r\Delta t}.$$

Substituting Δ into the equation, we have

$$c = \left(qc_u + (1-q)c_d\right)e^{-r\Delta t} \tag{4}$$

where

$$q = \frac{e^{r\Delta t} - d}{u - d}.$$

q can be interpreted as the probability that the asset price moves upward, so the probability that the asset price moves downward is $1 - q$. By a similar way, the binomial tree valuation of a put option is

$$p = (qp_u + (1-q)p_d)e^{-r\Delta t}$$

where p_u and p_d are the put options prices in situations of upward and downward asset prices respectively.

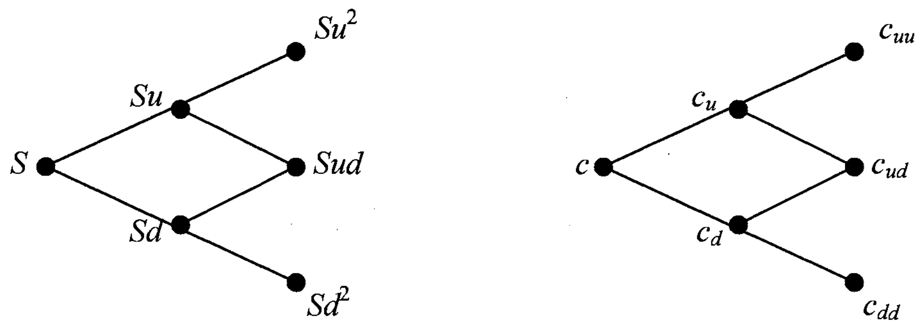


Figure 2. Two-step binomial lattices

The extension of the binomial model with two periods from expiry is quite straightforward. By assuming that u and d stay the same for all binomial steps, the corresponding dynamics of the binomial process for the asset price and the call price are shown in Figure 2, in which c_{uu} denotes the call value at two periods beyond the current time with two consecutive upward moves of the asset price and the similar notations are for c_{ud} and c_{dd}.

Based on a similar argument as in formula (4), the call values c_u and c_d are related to c_{uu}, c_{ud} and c_{uu} as follows:

$$c_u = \left(qc_{uu} + (1-q)c_{ud}\right)e^{-r\Delta t}$$

$$c_d = \left(qc_{ud} + (1-q)c_{dd}\right)e^{-r\Delta t}$$

where c_{uu} = max($Su^2 - K$, 0), c_{ud} = max($Sud - K$, 0), c_{dd} = max($Sd^2 - K$, 0). Next, by substituting the above results, the call value at the current time that is two periods from expiry is

$$c = \left(q^2 c_{uu} + 2q(1-q)c_{ud} + (1-q)^2 c_{dd}\right)e^{-2r\Delta t}$$

Deductively, for an n-period binomial process, the call value is

$$c = \left[\sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} \max(Su^j d^{n-j} - X, 0)\right] e^{-nr\Delta t} \tag{5}$$

where $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$ is the binomial coefficient.

At this point we have introduced the general lattice structure that is followed by various versions of the binomial model. There are three parameters in the binomial model. The probability parameter $q = \dfrac{e^{r\Delta t} - d}{u - d}$ is already determined by the procedure of forming the replicating portfolio. However, u and d have not yet been determined. In the Blakc-Scholes continuous model, the asset price dynamics are assumed to follow the geometric Brownian motion where $\dfrac{S_{\Delta t}}{S}$ is lognormally distributed. In the risk neutral world, $\ln\dfrac{S_{\Delta t}}{S}$ becomes normally distributed with mean $\left(r - \dfrac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2 \Delta t$, where r is the riskless interest rate and σ^2 is the variance rate of the lognormal process. Correspondingly, the first and second moments of $\dfrac{S_{\Delta t}}{S}$ are $e^{r\Delta t}$ and $e^{(2r+\sigma^2)\Delta t}$

respectively. On the other hand, for the one-period binomial option model under the risk neutral environment, the first and second moments of the asset price ratio $\frac{S_{\Delta t}}{S}$ at one period after the current time are given by $qu + (1-q)d$ and $qu^2 + (1-q)d^2$. By equating the first and second moments of the asset price ratio $\frac{S_{\Delta t}}{S}$ in both continuous and discrete models, the following equations are obtained:

$$qu + (1-q)d = e^{r\Delta t} \tag{6}$$

$$qu^2 + (1-q)d^2 = e^{(2r+\sigma^2)\Delta t} \tag{7}$$

Equations (6) and (7) provide only two equations for the three unknowns u, d and q. The remaining condition is conventionally, and somewhat arbitrarily, chosen as $u = \frac{1}{d}$, so that the lattice nodes associated with the binomial tree are symmetrical. Writing $\tilde{\sigma}^2 = e^{(2r+\sigma^2)\Delta t}$, the solution to the system of equations is (Hull and White, 1988)

$$u = \frac{1}{d} = \frac{\tilde{\sigma}^2 + 1 + \sqrt{(\tilde{\sigma}^2+1)^2 - 4e^{2r\Delta t}}}{2e^{r\Delta t}}, \quad q = \frac{e^{r\Delta t} - d}{u - d} \tag{8}$$

The expression for u in the above formula appears to be quite cumbersome. A simpler formula for u, but without sacrificing the degree of accuracy, is based on expanding the above u in Taylor series in powers of $\sqrt{\Delta t}$:

$$u = 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t + \frac{4r^2 + 4\sigma^2 r + 3\sigma^4}{8\sigma}\Delta t^{3/2} + O(\Delta t^2)$$

Observe that the first three terms in the above Taylor series agree with that of $e^{\sigma\sqrt{\Delta t}}$ up to $O(\Delta t)$ term. This suggests the judicious choice of the following set of parameter values (Cox, Ross and Rubinstein, 1979):

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{e^{r\Delta t} - d}{u - d} \tag{9}$$

which appear to be of simpler forms compared to those in formula (8). With the parameters in formula (9), the second moments of the price ratio $\frac{S_{\Delta t}}{S}$ in the continuous and discrete models agree only up to $O(\Delta t^2)$. More precisely, equations in (9) are satisfied up to $O(\Delta t^2)$ since the second moment in the discrete model is

$$\begin{aligned} qu^2 + (1-q)d^2 &= q(u-d)(u+d) + d^2 \\ &= (e^{r\Delta t} - d)(u+d) + d^2 \\ &= e^{r\Delta t}(u+d) - du \\ &= \left(1 + r\Delta t + \frac{r^2}{2}\Delta t^2 + O(\Delta t^3)\right)\left(2 + \sigma^2\Delta t + O(\Delta t^2)\right) - 1 \\ &= 1 + (2r + \sigma^2)\Delta t + r(r+\sigma^2)\Delta t^2 + O(\Delta t^3) \end{aligned}$$

and the second moment in the continuous model is

$$e^{(2r+\sigma^2)\Delta t} = 1 + (2r+\sigma^2)\Delta t + \frac{(2r+\sigma^2)^2}{2}\Delta t^2 + O(\Delta t^3).$$

Besides the two sets of parameter values given in equations (8) and (9), two other versions of the set of binomial parameters have been proposed. Jarrow and Rudd (1983) relaxed the reconnecting condition $u = \frac{1}{d}$ and chose $q = \frac{1}{2}$ as the third condition.

Solving together with the first and second moment conditions, the values for u, d and q are

$$u = e^{r\Delta t}\left(1+\sqrt{e^{\sigma^2\Delta t}-1}\right),\ d = e^{r\Delta t}\left(1-\sqrt{e^{\sigma^2\Delta t}-1}\right),\ q = \frac{1}{2}.$$

Following the same line of thought of seeking simplified approximate formulas, Jarrow and Rudd proposed the following parameter values in their binomial model:

$$u = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}},\ d = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t-\sigma\sqrt{\Delta t}},\ q = \frac{1}{2} \tag{10}$$

It can be checked easily that the Taylor expansions of

$$e^{r\Delta t}\left(1+\sqrt{e^{\sigma^2\Delta t}-1}\right) \text{ and } e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}}$$

agree up to $O(\Delta t)$, since

$$e^{r\Delta t}\left(1+\sqrt{e^{\sigma^2\Delta t}-1}\right) = 1+O(\Delta t) = e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}}$$

Similarly, by using Taylor expansions, it can be shown that the first and second moment conditions are now not satisfied exactly, but only up to $O(\Delta t^2)$, for the simplified set of parameter values in formula (10).

A more recent version of the binomial model was proposed by Tian (1993), who chose the third condition to be

$$qu^3 + (1-q)d^3 = e^{3(r+\sigma^2)\Delta t}.$$

The above relation is derived from matching the third moment of the discrete-time process and the continuous-time process for the asset price ratio. Solving the moment conditions from the first to the third together, the parameter values in the binomial model are

$$u = \frac{MV}{2}\left((V+1) + \sqrt{V^2 + 2V - 3}\right)$$

$$d = \frac{MV}{2}\left((V+1) - \sqrt{V^2 + 2V - 3}\right)$$

$$q = \frac{M - d}{u - d} \qquad (11)$$

where $M = e^{r\Delta t}$ and $V = e^{\sigma^2 \Delta t}$. Note that $ud = (MV)^2$ instead of $ud = 1$. The binomial tree loses symmetry about S whenever $ud \neq 1$.

Trinomial Tree Models

A further extension of the binomial tree is the trinomial tree model introduced by Boyle(1986). Like the binomial tree, the trinomial tree can be used to price both European and American options on a single underlying asset. Because the asset price can move in three directions from a given node, compared with only two in a binomial tree, the number of time steps can be reduced to attain the same accuracy as in the binomial tree. This makes trinomial trees more efficient than binomial trees. In a trinomial pricing model, there are three possible asset price jumps. The asset price S will become Su, Sm or Sd with probabilities q_u, q_m and q_d respectively after one time period Δt, where $u > m > d$. The middle jump ratio m is chosen to be 1. There are five parameters in Boyle's trinomial model. The moment conditions used in the model are the following equations

$$q_u + q_m + q_d = 1$$

$$q_u u + q_m + q_d d = e^{r\Delta t}$$

$$q_u u^2 + q_m + q_d d^2 = e^{(2r + \sigma^2)\Delta t}$$

The remaining two conditions are chosen freely by Boyle to be

$$ud = 1 \quad \text{and} \quad u = e^{\lambda\sigma\sqrt{\Delta t}}$$

where λ is a free parameter. Observe that Boyle's trinomial model reduces to the Cox-Ross-Rubinstein binomial scheme when $\lambda = 1$. The five parameters can then be determined by the five equations. For example, with $\lambda = \sqrt{2}$, the parameters are given by

$$u = e^{\sigma\sqrt{2\Delta t}}, \ m = 1, \ d = e^{-\sigma\sqrt{2\Delta t}},$$

$$q_u = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$q_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2$$

$$q_m = 1 - q_u - q_d \tag{12}$$

One advantage of Boyle's trinomial scheme is that $\ln u$ and $\ln d$ can be chosen to be any multiple of $\sigma\sqrt{\Delta t}$. By choosing the parameter λ appropriately, negative probability values in the calculations can be avoided. In his numerical experiments, Boyle claimed that best results were obtained when the probabilities are roughly equal and the accuracy of the trinomial scheme with five time steps is comparable to that of the Cox-Ross-Rubinstein binomial scheme with 20 time steps.

By dropping the restriction $m = 1$ in Boyle's trinomial model, Tian proposed two modified trinomial models. In both models, the moment conditions in Boyle's model are remained:

$$q_u + q_m + q_d = 1$$

$$q_u u + q_m m + q_d d = M$$

$$q_u u^2 + q_m m^2 + q_d d^2 = M^2 V$$

where, as defined in Tian's binomial model, $M = e^{r\Delta t}$ and $V = e^{\sigma^2 \Delta t}$. In addition, to guarantee the trinomial lattice recombining properly, the following restriction is used:

$$ud = m^2 .$$

For the other two restrictions needed to determine the parameters, Tian proposed two methods. The first method follows Boyle's argument that best results were obtained when the probabilities were roughly equal. Thus, the additional restriction in Tian's first modified trinomial model is

$$q_u = q_m = q_d .$$

Solving the system of the restriction equations in his first modified model, Tian expressed the parameter values as

$$p_u = p_m = p_d = \frac{1}{3}$$

$$u = K + \sqrt{K^2 - m^2}$$

$$d = K - \sqrt{K^2 - m^2}$$

$$m = \frac{M(3-V)}{2} \tag{13}$$

where $K = M(V+3)/4$.

The additional restrictions in Tian's second modified trinomial model are based on the argument that the third and fourth moments of the trinomial distribution should match their counterparts of the continuous distribution. According to the third and fourth moments in the discrete and continuous models, the following two restrictions are imposed:

$$q_u u^3 + q_m m^3 + q_d d^3 = M^3 V^3$$

$$q_u u^4 + q_m m^4 + q_d d^4 = M^4 V^6$$

By solving the restriction system in his second modified trinomial model, Tian presented the following solution:

$$p_u = \frac{md - M(m+d) + M^2 V}{(u-d)(u-m)},$$

$$p_m = \frac{M(u+d) - ud - M^2 V}{(u-m)(m-d)},$$

$$p_d = \frac{um - M(u+m) + M^2 V}{(u-d)(m-d)},$$

$$u = K + \sqrt{K^2 - m^2},$$

$$d = K - \sqrt{K^2 - m^2},$$

$$m = MV^2, \quad K = \frac{M}{2}\left(V^4 + V^3\right). \tag{14}$$

Models Based on Non-lognormal Distributions

Since the bias of the Black-Scholes model is mainly due to the strong assumption that the underlying asset price follows a lognormal distribution, the valuation of options on assets that are assumed to follow stochastic processes other than Brownian motion has received attention by academics. One representative study in this line is the Jarrow-Rudd (1982) approximate option valuation model. Another example of such work is the Merton (1976) jump-diffusion model.

Jarrow-Rudd Approximate Model

Jarrow-Rudd approximate model is based on the method of generalized Edgeworth series expansion. The Edgeworth series expansion is quite similar to the Taylor series expansion for analytic functions in function theory. The main idea of the Edgeworth series expansion is that a given distribution, though generally unknown, can be approximated by another distribution to any desired level of accuracy. Applying this method to the asset prices means that the true probability density function of the asset price can always be approximated as an Edgeworth series containing a lognormal density function (Johnson et al.):

$$f(s) = g(s) + \frac{(k_2 - k_2(G))}{2!}\frac{d^2 g(s)}{ds^2} - \frac{(k_3 - k_3(G))}{3!}\frac{d^3 g(s)}{ds^3} + \frac{((k_4 - k_4(G)) + 3(k_2 - k_2(G))^2)}{4!}\frac{d^4 g(s)}{ds^4} + \varepsilon(s) \tag{15}$$

where *f(s)* is the true probability density function of the asset price, and

$$g(s) = \frac{1}{s\sigma\sqrt{2\pi t}} \exp\left(-\frac{(\ln s - \mu t)^2}{2\sigma^2 t} \right)$$

is a lognormal density function, $\varepsilon(s)$ is the residual error, and the cumulants k_js are defined by the following equations:

$$\alpha_j = \int_{-\infty}^{\infty} s^j f(s) ds$$

$$\mu_j = \int_{-\infty}^{\infty} (s - \alpha_1)^j f(s) ds, \quad j = 1, 2, 3, 4$$

$$k_2 = \mu_2 \quad k_3 = \mu_3 \quad k_4 = \mu_4 - 3\mu_2$$

The cumulant $k_j(G)$ is defined similarly to k_j by changing the density function from *f(s)* to *g(s)*.

Thus if an Edgeworth series containing cumulants up to the fourth is used to approximate the true density function, the error is $\varepsilon(s)$. If the lognormal density function is used as an approximation for the true density function, the error will be a series of higher cumulants plus $\varepsilon(s)$. As long as the true distribution is not lognormal, the Edgeworth series is a more accurate approximation for the true probability density function than the lognormal density function.

Based on the assumption that the underlying unknown probability density function can be approximated by (15), Jarrow and Rudd derived an approximate formula for European call options:

$$c = c(G) + e^{-rT} \frac{(k_2 - k_2(G))}{2!} g(X) - e^{-rT} \frac{(k_3 - k_3(G))}{3!} \frac{dg(X)}{dS}$$

$$+ e^{-rT} \frac{((k_4 - k_4(G)) + 3(k_2 - k_2(G))^2)}{4!} \frac{d^2 g(X)}{dS^2} + \varepsilon(X) \qquad (16)$$

where $c(G)$, with G representing that the underlying probability density function is $g(s)$, is the Black-Scholes formula for the call option, and other symbols are defined as before. The Jarrow-Rudd formula for European put options can be obtained by the put-call parity, that is

$$p = c - S + Xe^{-rT}.$$

Merton Jump-Diffusion Model

The critical assumption required for both the Black-Scholes model and the Jarrow-Rudd approximate formula is that the underlying asset dynamics can be described by a stochastic process with a continuous sample path. In the Merton jump-diffusion model, an option pricing formula is derived for the more-general case when the underlying asset price returns are generated by a mixture of both continuous and jump processes. To highlight the impact of non-continuous asset price dynamics on option pricing, all the other assumptions made by Black and Scholes are maintained in the Merton model.

This model assumes the underlying asset price distribution satisfies the following jump-diffusion process:

$$\frac{dS}{S} = (\mu - \lambda k)dt + \sigma dz + dq$$

where μ is the instantaneous expected return on the asset, λ is the mean number of important new information arrivals per unit of time, σ is the instantaneous standard deviation of the return conditional on no arrivals of important new information, dz is a standard Gauss-Wiener process, q is the independent Poisson process describing the arrivals of the important new information, dz and dq are assumed to be independent. Let E_1 denote the event that no important new information arrives in the time interval $(t, t+\Delta t)$, E_2 denote the event that important new information arrives once in the interval, and E_3 the event that important new information arrives more than once. Then the Poisson process is described as a set of probabilities:

$$\text{Prob}(E_1) = 1 - \lambda\Delta t + O(\Delta t),$$

$$\text{Prob}(E_2) = \lambda\Delta t + O(\Delta t), \text{ and}$$

$$\text{Prob}(E_3) = O(\Delta t).$$

Assuming the jump component to be diversifiable, Merton obtained the following option valuation formulas:

$$c = \sum_{i=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^i}{i!} c_i(S, X, T, r, \sigma_i)$$

$$p = \sum_{i=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^i}{i!} p_i(S, X, T, r, \sigma_i),$$

$$\sigma_i = \sqrt{z^2 + \delta^2 (i/T)}$$

$$\delta = \sqrt{\frac{\gamma\sigma^2}{\lambda}},\ z = \sqrt{\sigma^2 - \lambda(e^{\delta^2} - 1)} \tag{17}$$

where c_i and p_i are the Black-Scholes formulas for call and put options respectively. In addition to the volatility parameter σ, there are two parameters to be estimated: the expected number of jumps per year λ and the percentage of the total volatility explained by the jumps γ.

Summary

The Black-Scholes option pricing formula could be seriously inaccurate because it is a European option pricing formula based on the assumption that the underlying asset price follows a lognormal distribution. Since most option contracts in practice are American, the Black-Scholes model usually undervalues put options. Since the lognormal distribution is a strong assumption, when the true distribution differs significantly from a lognormal, the bias from the Black-Scholes could be serious.

Various versions of the binomial tree and the trinomial tree are powerful yet easy to use when handling American options. However, present tree models are mostly based on the lognormal distribution assumption. Since all versions of the tree models converge well when the number of the time steps is large enough, the accuracy among the different tree models that assume lognormality should differ little.

The Jarrow-Rudd approximate option valuation formula is attractive because it is based on an Edgeworth series expansion that can approximate an arbitrary distribution at an acceptable level of error. However, this model is still a European option pricing formula. This drawback of the Jarrow-Rudd model limits its use in practice.

The Merton jump-diffusion model is more general than the Black-Scholes because it incorporates the impact of the jumps of important new information. However, relative to the Jarrow-Rudd model, it is still a special case that is only suitable for the jump-diffusion underlying price process, not applicable to arbitrary processes. Moreover, similar to the Jarrow-Rudd model, the Merton jump-diffusion model can only be used to value European options.

In summary, in practice, the tree models are more powerful than the Black-Scholes since they can handle American options. Even for European options, since the tree models can converge well by using a large number of time steps, the accuracy can be controlled to a satisfactory level as long as the true underlying distribution is not far from lognormal. In the situation that the true underlying distribution can not be approximated well by a lognormal distribution, all the tree models reviewed in this chapter would be inaccurate with about the same error level. Thus, the inaccuracy of the tree models does not arise from the lattice structures of the trees, but from the common assumption underlying the different trees — the lognormal distribution. This suggests that the accuracy of a tree model is more likely to be improved by modifying the distributional assumption than the lattice structure. Of course, the more general the underlying distribution assumed, the more useful the tree model. Directed by such an idea, a general tree option pricing model suitable for arbitrary underlying distributions will follow next.

CHAPTER THREE

THEORETICAL MODEL

A general binomial tree model and a general trinomial tree model will be developed in this chapter. The properties such as the asymptotic limits of the general tree models are derived. Like the binomial and trinomial models introduced in Chapter Two, the general tree models have recombining lattices. The difference in the general tree models is the method of determining the move magnitudes and the probabilities in the trees. All the previous binomial and trinomial models assume that the underlying asset prices are lognormally distributed. The move magnitudes and the probabilities in the previous tree models are uniquely determined by the volatility parameter according to some formulas. In the general binomial and trinomial tree models developed in this study, any underlying distribution is allowable. The move magnitudes and the probabilities in the general tree models are determined by Gaussian quadrature. Gaussian quadrature sets the moments of the approximating discrete distribution equal to the moments of the continuous distribution. Since the move magnitudes and the probabilities in our general tree models are determined by moments that can be from any distributions, we first present a general expression of the moments of an arbitrary distribution. Under this general form, moments from any familiar distribution such as lognormal can be expressed as a special case. Then, based on the Guassian quadrature with the general form of moments, we will establish our general binomial tree model. The standard algorithm solving a Guassian quadrature equation system is already available (Miller and Rice; Preckel and DeVuyst). However, adopting this standard method to develop an algorithm for option pricing is an innovation. In the case of the trinomial tree, a restriction is added to form a recombining tree.

Since the binomial tree is a discrete model, it is natural to explore its asymptotic properties. This can be done in two ways. One way is to examine the limit of the general binomial tree model for a very short time. Another way is to find the limit of the general binomial model with a range of time to expiration. It will be shown that, by using Taylor series expansion, a one period general binomial tree model converges to a third order partial differential equation with the well-known Black-Scholes differential equation as a special case. On the other hand, according to the central limit theorem, the limit of a general binomial tree model with multiple periods is a formula similar to, but different from, the Black-Scholes. Again, the Black-Scholes formula can be considered as a special case of the limit of the general binomial tree model.

General Form of Moments

Let S and $S_{\Delta t}$ denote, respectively, the asset prices at the current time and at one period later. Suppose S follows a specific dynamic process. Though this process is usually unknown, the kth moment of the price ratio, $Y = S_{\Delta t}/S$, at a point in time can be generally written as a mathematical expectation of Y at the current time:

$$E(Y^k) = \int_{-\infty}^{\infty} y^k f(y)dy = \exp\left\{\left[A(\theta,k) + rk + \frac{\sigma^2 k(k-1)}{2}\right]\Delta t\right\}$$

where $f(y)$ is the probability density function of Y, E is an expectation operator, $A(\theta, k)$ is a function with a parameter vector θ, σ is a volatility parameter, r is risk-free interest rate. Since $E(Y^0) = 1$ is always true, we know that $A(\theta, 0) = 0$. Also, in a risk neutral world, the expected rate of return on the asset is equal to the risk-free interest rate, so under this assumption we have $E(Y) = e^{r\Delta t}$ and, thus, $A(\theta,\ 1) = 0$.

Under this general expression, the moments of any specific distributions can be considered as special cases. For example, the case that $A(\theta, k) = 0$ for any k represents a lognormal distribution, i.e. $\ln Y$ is normally distributed with mean $(r - \sigma^2/2)\Delta t$ and variance $\sigma^2\Delta t$. In case of $A(\theta,k) = \lambda\left(e^{\delta^2 k(k-1)/2} - 1\right)$, the asset price distribution is generated by the Merton (1976) jump-diffusion process.

General Binomial Tree Model

Now we begin to establish our general binomial tree model as well as the algorithm for the move magnitudes and the probabilities in the model. In a discrete random walk model the ratio of the asset price over a period of time is assumed to have finite possible outcomes: y_i with probability q_i, where $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} q_i = 1$. In this study, the method to determine the outcomes and probabilities is based on an approximation called Gaussian quadrature. Simply speaking, Gaussian quadrature is a discrete approximation of an integral:

$$\int_{-\infty}^{\infty} g(y) f(y) dy \approx \sum_{i=1}^{n} g(y_i) q_i ,$$

where $g(y)$ is a function of the random variable Y and $f(y)$ is the continuous density function. With Gaussian quadrature the evaluation points (y_i) and the probabilities (q_i) are selected so that the first k moments of the discrete distribution match those of the continuous distribution.

In our general binomial model, $n = 2$. The lattice of our binomial model is the same as the one illustrated in Figure 1. We use u and d to denote the upward and downward moves, and use q to represent the upward move probability. These move magnitudes and probabilities are determined by the moments of the asset price ratio as follows.

The asset price ratio is a continuous random variable. Its first moment is $E(Y)$. On the other hand, in a binomial tree framework, Y is assumed to take value u with probability q and d with probability $1-q$. So its first moment is assumed to be $qu+(1-q)d$. Thus, for the first moment based on the binomial framework to be a correct approximation of the one based on the real continuous distribution, there should be an equation between the two first moments from the discrete and the continuous versions. Similarly, we can approximate the kth moment of Y based on the continuous dynamic process by the kth moment based on the binomial framework:

$$qu^k + (1-q)d^k \approx E(Y^k) = \int_{-\infty}^{\infty} y^k f(y)dy \qquad k = 0, 1, 2, \ldots$$

To determine the parameters u, d and q in the binomial model, we write out and solve the equations corresponding to the moments up to the third:

$$qu^0 + (1-q)d^0 = E(Y^0) = 1$$

$$qu + (1-q)d = E(Y)$$

$$qu^2 + (1-q)d^2 = E(Y^2)$$

$$qu^3 + (1-q)d^3 = E(Y^3) \tag{18}$$

The first equation above is simply a repetition of $q+(1-q)=1$. Under the risk-neutral probability structure, i.e. $E(Y) = e^{r\Delta t}$, the second equation is actually $q = \dfrac{e^{r\Delta t}-d}{u-d}$, the one obtained before.

Now define a polynomial

$$P(y) = (y-u)(y-d) = y^2 + C_1 y + C_0$$

It is obvious that $P(u) = P(d) = 0$. By comparing the coefficients of the polynomial $P(y)$, it is easy to find that

$$C_0 = ud \quad \text{and} \quad C_1 = -(u+d) \tag{19}$$

Solving u and d in (18), we have

$$u = \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{2}$$

$$d = \frac{-C_1 - \sqrt{C_1^2 - 4C_0}}{2}$$

Next we try to express C_0 and C_1 as functions of the moments so that u and d can be determined given the moments known. Take the first three equations. Multiplying the first equation in (18) by C_0, the second by C_1 and the last by 1, we have

$$C_0 q + C_0(1-q) = C_0,$$

$$C_1 qu + C_1(1-q)d = C_1 E(Y),$$

$$\text{and} \quad qu^2 + (1-q)d^2 = E(Y^2).$$

Adding them together, we have

$$C_0 + C_1 E(Y) + E(Y^2) = qP(u) + (1-q)P(d) = 0 \tag{20}$$

Similarly, take the second through the fourth equations. Multiplying the three equations by C_0, C_1 and 1 respectively, and adding them together, we have

$$C_0 E(Y) + C_1 E(Y^2) + E(Y^3) = quQ(u) + (1-q)dQ(d) = 0 \tag{21}$$

The solution for the linear equations (20) and (21) about C_0 and C_1 is

$$C_0 = \frac{E(Y)E(Y^3) - (E(Y^2))^2}{E(Y^2) - (E(Y))^2}$$

$$C_1 = \frac{E(Y)E(Y^2) - E(Y^3)}{E(Y^2) - (E(Y))^2} \tag{22}$$

Thus, given C_0 and C_1 expressed by (22), the three parameters of our general binomial tree are determined by the following formulas:

$$u = \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{2},$$

$$d = \frac{-C_1 - \sqrt{C_1^2 - 4C_0}}{2} \quad \text{and}$$

$$q = \frac{e^{r\Delta t} - d}{u - d} \tag{23}$$

Now we see that the parameters in our binomial option pricing model are determined by C_0 and C_1, which are, in turn, functions of the general form of moments up to the third. The algorithm for the parameters in the binomial tree is general in the sense that it is applicable to any distribution with finite second and third moments.

Special Cases of the General Binomial Model

Since the binomial tree model developed above is general, all previous binomial models based on the lognormal assumption can be considered as the special cases of it.

Also, it can handle distributions other than the lognormal as well. The following two examples illustrate this advantage.

Case 1: Lognormal

If Y follows a lognormal distribution, then, by denoting $e^{r\Delta t}$ as M and $e^{\sigma^2\Delta t}$ as V, so that $E(Y) = M$, $E(Y^2) = M^2V$, and $E(Y^3) = M^3V^3$, we have

$$C_0 = \frac{E(Y)E(Y^3) - (E(Y^2))^2}{E(Y^2) - (E(Y))^2} = \frac{M(M^3V^3) - (M^2V)^2}{M^2V - M^2} = M^2V^2$$

$$C_1 = \frac{E(Y)E(Y^2) - E(Y^3)}{E(Y^2) - (E(Y))^2} = \frac{M(M^2V) - M^3V^3}{M^2V - M^2} = -MV(V+1)$$

Thus

$$u = \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{2} = \frac{MV}{2}\left[(V+1) + \sqrt{V^2 + 2V - 3}\right] \tag{24}$$

$$d = \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{2} = \frac{MV}{2}\left[(V+1) - \sqrt{V^2 + 2V - 3}\right]$$

$$q = \frac{M - d}{u - d}$$

This is exactly Tian's solution for the parameters in the binomial tree model as shown in (11). If a further constraint of $ud = 1$ is added, instead of using the third moment condition based on Guassian quadrature, the other versions of the previous binomial models are obtained. It is now clear that all previous binomial tree models other than Tian's match only the first two moments of the continuous distribution. While Tian's binomial model is based on Gaussian quadrature, it is restricted by the assumption that the underlying distribution is lognormal.

Case 2: Jump-Diffusion

In case of $A(\theta,k)=\lambda\left(e^{\gamma^2 k(k-1)/2}-1\right)$, the asset price distribution is generated by the jump-diffusion process (Merton 1976). While all previous binomial tree models are invalid in this case, options can still be valued with the general binomial tree model because the moments are available. The moments in this case are

$$E(Y)=\exp(r\Delta t)$$

$$E(Y^2)=\exp\left[\left((2r+\sigma^2)+\lambda(e^{\gamma^2}-1)\right)\Delta t\right]$$

$$E(Y^3)=\exp\left[\left(3(r+\sigma^2)+\lambda(e^{3\gamma^2}-1)\right)\Delta t\right]$$

Given σ, λ and γ, the values of u, d and q can be found by the general formulas (21). Here the powerfulness of the general binomial tree model over the previous binomial models is that the impact of parameters λ and γ is reflected in the general model but not in any version of the previous tree models.

Asymptotic Limit of One-step General Binomial Tree Model

Given the general binomial tree model developed so far, one is tempted to ask the question: what is the continuous counterpart of the general binomial tree? The query can be answered by examining the asymptotic limit of the general binomial formula with one step and with multiple steps. We will see in this section that the limit of the one-step binomial process is a third order partial differential equation. In the next section, we will show that the limit of multi-period binomial model is an option pricing formula similar to the Black-Scholes.

We use the notation of $O(\Delta t^k)$, where k is a positive integer, to express the concept of convergence of an approximation. We say that a function $f(\Delta t)$ is $O(\Delta t^k)$ if

$$\lim_{\Delta t \to 0} \frac{f(\Delta t)}{\Delta t^k} < \infty$$

Note that the statement that $f(\Delta t)$ is $O(\Delta t^k)$ implies that $f(\Delta t)/\Delta t$ is $O(\Delta t^{k-1})$ since

$$\lim_{\Delta t \to 0} \frac{f(\Delta t)/\Delta t}{\Delta t^{k-1}} = \lim_{\Delta t \to 0} \frac{f(\Delta t)}{\Delta t^k} < \infty$$

We now consider the limit $\Delta t \to 0$ in the binomial formula (4)

$$c = \left(qc_u + (1-q)c_d\right)e^{-r\Delta t}.$$

We want to perform the Taylor expansion of the binomial scheme at (S, t). To avoid a two-variable Taylor series, we rewrite the formula as

$$c(S, t-\Delta t)e^{r\Delta t} = \left(qc(Su, t) + (1-q)c(Sd, t)\right)$$

By this way, the only variable at the left side of the equation is t and the right side is S. Then the Taylor expansion is

$$\begin{aligned} e^{r\Delta t}\left[c(S,t) - \frac{\partial c}{\partial t}(S,t)\Delta t + O(\Delta t^2)\right] &= c(S,t) \\ &+ \left[q(u-1) + (1-q)(d-1)\right]S\frac{\partial c}{\partial S}(S,t) \\ &+ \frac{1}{2}\left[q(u-1)^2 + (1-q)(d-1)^2\right]S^2\frac{\partial^2 c}{\partial S^2}(S,t) \\ &+ \frac{1}{6}\left[q(u-1)^3 + (1-q)(d-1)^3\right]S^3\frac{\partial^3 c}{\partial S^3}(S,t) + \cdots \end{aligned} \tag{25}$$

According to the Gaussian quadrature equations that determine the parameters in the general binomial tree, in (23),

$$q(u-1) + (1-q)(d-1) = qu - q + (1-q)d - 1 + q = E(Y) - 1$$

$$q(u-1)^2 + (1-q)(d-1)^2 = qu^2 + (1-q)d^2 - 2[qu + (1-q)d] + q + (1-q)$$

$$= E(Y^2) - 2E(Y) + 1$$

$$q(u-1)^2 + (1-q)(d-1)^2 = qu^3 + (1-q)d^3 - 3[qu^2 + (1-q)d^2] + 3[qu + (1-q)d] - [q + (1-q)]$$

$$= E(Y^3) - 3E(Y^2) + 3E(Y) - 1$$

Also, recall that, by Taylor expansion,

$$E(Y) = \exp(r\Delta t) = 1 + r\Delta t + O(\Delta t^2)$$

$$E(Y^2) = \exp\left[\left(A(\theta,2) + 2r + \sigma^2\right)\Delta t\right] = 1 + \left(A(\theta,2) + 2r + \sigma^2\right)\Delta t + O(\Delta t^2),$$

$$E(Y^3) = \exp\left[\left(A(\theta,3) + 3(r + \sigma^2)\right)\Delta t\right] = 1 + \left(A(\theta,3) + 3(r + \sigma^2)\right)\Delta t + O(\Delta t^2)$$

Substituting the results above into the Taylor expansion of the binomial formula (4) and rearranging the equation, we have

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\left(\sigma^2 + A_2\right)S^2\frac{\partial^2 c}{\partial S^2} + \frac{1}{6}\left(A_3 - 3A_2\right)S^3\frac{\partial^3 c}{\partial S^3} - rc + O(\Delta t) = 0 \qquad (26)$$

where $A_k = A(\theta, k)$ is a parameter related to the kth moment (k = 2, 3). This equation indicates that $c(S, t)$ almost satisfies a third order partial differential equation, with the truncation error being $O(\Delta t)$. We call this partial differential equation *the general option pricing equation* because the result is valid for all versions of the binomial tree model.

If the true distribution is lognormal, then $A_2 = A_3 = 0$, so that the general option pricing equation reduces to the familiar Black-Scholes equation. Alternatively, if the true asset price distribution is generated by Merton's jump-diffusion process, $A_2 = \lambda\left(e^{\gamma^2} - 1\right)$ and $A_3 = \lambda\left(e^{3\gamma^2} - 1\right)$, the value of the call option is still determined by a third order partial differential equation with three parameters λ, γ, and σ.

Now we go further to a multiplicative n-period binomial process. The derivation method we use is from the classical article of Cox, Ross and Rubinstein (1979), in which the limit of the binomial model is the Black-Scholes formula. However, since the underlying distribution in our general binomial model is arbitrary rather than the lognormal, the result derived will be a formula other than the Black-Scholes.

Recall Eg. (5). The call value corresponding to an n-step binomial tree model can be expressed as

$$c = \left[\sum_{j=0}^{n} \binom{n}{j} q^j (1-q)^{n-j} \max(Su^j d^{n-j} - X,\ \ 0) \right] e^{-nr\Delta t}$$

We define k to be the smallest non-negative integer such that $Su^k d^{n-k} \geq X$, that is, $k \geq \ln(X / Sd^n) / \ln(u / d)$. It is seen that

$$\max(Su^j d^{n-j} - X, 0) = \begin{cases} 0 & \text{when} \quad j < k \\ Su^j d^{n-j} - X & \text{when} \quad j \geq k \end{cases}.$$

The integer k gives the minimum number of upward moves required for the asset price in the multiplicative binomial process such that the call expires in-the-money. The call formula for the above n-step binomial tree model can then be simplified as

$$c = S \sum_{j=k}^{n} \binom{n}{j} q^j (1-q)^{n-j} u^j d^{n-j} e^{-nr\Delta t} - X e^{-nr\Delta t} \sum_{j=k}^{n} \binom{n}{j} q^j (1-q)^{n-j}$$

Let $\Phi(n,k,q)$ denote the probability in the risk neutral world that the call will expire in-the-money, i.e. the probability that at least k successes in n trials of a binomial experiment, where q is the probability of success in each trial, we know

$$\Phi(n,k,q) = \sum_{j=k}^{n} \binom{n}{j} q^j (1-q)^{n-j}$$

Further, by writing $q' = uqe^{-r\Delta t}$ so that $1 - q' = d(1-q)e^{-r\Delta t}$, the call price formula for the n-step binomial tree model can be expressed as

$$c = S\Phi(n,k,q') - Xe^{-rn\Delta t}\Phi(n,k,q) \tag{27}$$

The first term of formula (27) gives the discounted expectation of the asset price at expiration given that the call expires in-the-money and the second term gives the present value of the expected cost incurred by exercising the call. The two terms together give the discounted expectation taken under the adjusted risk neutral discrete binomial probability distribution. The formula above is very similar to the Black-Scholes formula for a European call option. However, we will show that, except for a special case, the asymptotic limit of this general binomial tree model is generally different from the Black-Scholes.

We want to find the limit of the call option formula for an n-step general binomial tree model as $n \to \infty$, or equivalently $t \to 0$ (since $n\Delta t = T - t$ is finite). The analysis relies on the Central Limit Theorem stated below:

The central limit theorem (Ross, p399)

Let X_1, X_2, ... be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

$$P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx \qquad \text{as} \quad n \to \infty.$$

When X_1, X_2, … are a sequence of independent and identically distributed Bernoulli random variables each having mean q and variance $(1 - q)$, $Y = X_1 + \ldots + X_n$ is a binomial random variable with parameters n and q. According to the central limit theorem above, we have the following corollary:

Corollary (Normal approximation to the binomial distribution)

Let Y be a binomial random variable with parameters n and q, where n is the number of binomial trials and q is the probability of success. For large n, Y is approximately normal with mean nq and variance $nq(1-q)$.

Recall that $\Phi(n,k,q)$ is the probability that the number of upward moves in the asset price is greater than or equal to k in the n-step binomial model, where q is the probability of an upward move. Let j denote the random integer variable that gives the number of upward moves during the n periods. Consider

$$1-\Phi(n,k,q)=P_r(j \le k-1)=P_r\left(\frac{j-nq}{\sqrt{nq(1-q)}} \le \frac{k-1-nq}{\sqrt{nq(1-q)}}\right) \tag{28}$$

where $\dfrac{j-nq}{\sqrt{nq(1-q)}}$ is the normalized binomial variable. Let S and S' denote the asset price at the current time and at n periods later. Since S and S' are related by $S' = Su^j d^{n-j}$, we then have

$$\ln\frac{S'}{S} = j\ln\frac{u}{d} + n\ln d$$

For the binomial random variable j, its mean and variance are known to be $E(j) = nq$ and $Var(j) = nq(1-q)$ respectively. Since $\ln(S'/S)$ and j are linearly related, the mean and variance of $\ln(S'/S)$ are given by

$$E\left(\ln\frac{S'}{S}\right)=E(j)\ln\frac{u}{d}+n\ln d=n\left(q\ln\frac{u}{d}+\ln d\right)$$

$$Var\left(\ln\frac{S'}{S}\right)=Var(j)\left(\ln\frac{u}{d}\right)^2=nq(1-q)\left(\ln\frac{u}{d}\right)^2$$

In the limit $n\to\infty$, the mean and variance of the logarithm of the price ratio of the discrete binomial model and the continuous counterpart should agree with each other, that is

$$\lim_{n\to\infty}n\left(q\ln\frac{u}{d}+\ln d\right)=M(\psi,T-t)$$

$$\lim_{n\to\infty}nq(1-q)\left(\ln\frac{u}{d}\right)^2=V(\psi,T-t),\quad T=t+n\Delta t$$

where $M(\psi,T-t)$ and $V(\psi,T-t)$ are the mean and the variance of the continuous logarithm price ratio random variable, both are functions of a parameter vector ψ and the period $T-t$. Since k is the smallest non-negative integer greater than or equal to $\ln(X/Sd^n)/\ln(u/d)$, we have

$$k-1=\frac{\ln\frac{X}{Sd^n}}{\ln\frac{u}{d}}-\alpha,\quad \text{where } 0<\alpha\leq 1,$$

so that (28) can be rewritten as

$$1-\Phi(n,k,q)=P_r(j\leq k-1)=P_r\left(\frac{j-nq}{\sqrt{nq(1-q)}}\leq\frac{\ln\frac{X}{S}-n\left(q\ln\frac{u}{d}+\ln d\right)-\alpha\ln\frac{u}{d}}{\sqrt{nq(1-q)}\ln\frac{u}{d}}\right)$$

In the limit $n \to \infty$, $n\left(q \ln \frac{u}{d} + \ln d\right)$ converges to $M(\psi, T-t)$ and $\sqrt{nq(1-q)} \ln \frac{u}{d}$ to $\sqrt{V(\psi, T-t)}$. Taking the limit $n \to \infty$, or equivalently $\Delta t \to 0$, and by the virtue of normal approximation to a binomial distribution, we obtain

$$\lim_{n\to\infty}(1-\Phi(n,k,q)) = N\left(\frac{\ln \frac{X}{S} - M(\psi,T-t) - \lim_{\Delta t \to 0} \alpha \ln \frac{u}{d}}{\sqrt{V(\psi,T-t)}}\right) \tag{29}$$

The limit $\lim_{\Delta t\to 0} \alpha \ln \frac{u}{d}$ in the above equation is finite. This can be shown as follows. Recall that

$$u = \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{2} \text{ and } d = \frac{-C_1 - \sqrt{C_1^2 - 4C_0}}{2}$$

$$\text{where } C_0 = \frac{E(X)E(X^3) - (E(X^2))^2}{E(X^2) - (E(X))^2} \text{ and } C_1 = \frac{E(X)E(X^2) - E(X^3)}{E(X^2) - (E(X))^2},$$

So that $\lim_{\Delta t\to 0} \alpha \ln \frac{u}{d} = \alpha \lim_{\Delta t \to 0} \ln \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{-C_1 - \sqrt{C_1^2 - 4C_0}} = \alpha \ln \frac{-\lim_{\Delta t\to 0} C_1 + \sqrt{\lim_{\Delta t\to 0} C_1^2 - 4\lim_{\Delta t\to 0} C_0}}{-\lim_{\Delta t\to 0} C_1 - \sqrt{\lim_{\Delta t\to 0} C_1^2 - 4\lim_{\Delta t\to 0} C_0}}$

By using *L'Hôpital*'s Rule and the fact that, for any integer k,

$$\frac{dE(Y^k)}{d(\Delta t)} = \left[A(\theta,k) + rk + \frac{\sigma^2 k(k-1)}{2}\right] E(Y^k) \text{ and } \lim_{\Delta t\to 0} E(Y^k) = 1,$$

we have

$$\lim_{\Delta t\to 0} C_0 = \lim_{\Delta t\to 0} \frac{rE(Y)E(Y^3) + [A(\theta,3) + 3r + 3\sigma^2]E(Y)E(Y^3) - 2[A(\theta,2) + 2r + \sigma^2]E(Y^2)^2}{[A(\theta,2) + 2r + \sigma^2]E(Y^2) - 2rE(X)^2}$$

$$= \frac{A(\theta,3) - 2A(\theta,2) + \sigma^2}{A(\theta,2) + \sigma^2}$$

$$\lim_{\Delta t \to 0} C_1 = \lim_{\Delta t \to 0} \frac{rE(Y)E(Y^2) + \left[A(\theta,2) + 2r + \sigma^2\right]E(Y)E(Y^2) - \left[A(\theta,3) + 3r + 3\sigma^2\right]E(Y^3)}{\left[A(\theta,2) + 2r + \sigma^2\right]E(Y^2) - 2rE(X)^2}$$

$$= \frac{A(\theta,2) - A(\theta,3) - 2\sigma^2}{A(\theta,2) + \sigma^2}$$

Since both $\lim_{\Delta t \to 0} C_0$ and $\lim_{\Delta t \to 0} C_0$ are finite, $\lim_{\Delta t \to 0} \ln \frac{u}{d}$ is finite too. A special case is that, when Y is lognormally distributed, $A(\theta,2) = A(\theta,3) = 0$, thus $\lim_{\Delta t \to 0} C_0 = 1$ and $\lim_{\Delta t \to 0} C_1 = -2$, so that $\lim_{\Delta t \to 0} \ln \frac{u}{d} = \ln 1 = 0$.

Now, rearranging the terms in equation (26), we have

$$\lim_{n \to \infty} \Phi(n,k,q) = N\left(\frac{\ln \frac{S}{X} + M(\psi, T-t) + \lim_{\Delta t \to 0} \alpha \ln \frac{u}{d}}{\sqrt{V(\psi, T-t)}}\right) \tag{30}$$

By a similar procedure, we can also obtain

$$\lim_{n \to \infty} \Phi(n,k,q') = N\left(\frac{\ln \frac{S}{X} + M'(\psi, T-t) + \lim_{\Delta t \to 0} \alpha \ln \frac{u}{d}}{\sqrt{V'(\psi, T-t)}}\right) \tag{31}$$

where $M'(\psi, T-t)$ and $V'(\psi, T-t)$, the mean and variance of the continuous random variable corresponding to a binomial distribution with the parameters n and q', are defined as

$$M'(\psi, T-t) = \lim_{n \to \infty} n\left(q' \ln \frac{u}{d} + \ln d\right)$$

$$V'(\psi, T-t) = \lim_{n \to \infty} nq'(1-q')\left(\ln \frac{u}{d}\right)^2$$

At this point, we can find the asymptotic limit of the general binomial tree valuation of a European call option:

$$\lim_{n\to\infty} c = S\lim_{n\to\infty}\Phi(n,k,q') - Xe^{-r(T-t)}\lim_{n\to\infty}\Phi(n,k,q)$$

$$= SN\left(\frac{\ln\dfrac{S}{X} + M'(\psi,T-t) + \lim\limits_{\Delta t\to 0}\alpha\ln\dfrac{u}{d}}{\sqrt{V'(\psi,T-t)}}\right)$$

$$- Xe^{-r(T-t)}N\left(\frac{\ln\dfrac{S}{X} + M(\psi,T-t) + \lim\limits_{\Delta t\to 0}\alpha\ln\dfrac{u}{d}}{\sqrt{V(\psi,T-t)}}\right) \tag{32}$$

This formula is similar to the Black-Scholes except that the arguments in the standard normal distribution functions are different. When the asset price ratio is lognormally distributed, we have

$$M(\psi,T-t) = \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

$$V(\psi,T-t) = \sigma^2(T-t)$$

$$M'(\psi,T-t) = \lim_{n\to\infty} n\left(q'\ln\frac{u}{d} + \ln d\right) = \lim_{n\to\infty} n\left(uqe^{-r\Delta t}\ln\frac{u}{d} + \ln d\right) = \left(r + \frac{\sigma^2}{2}\right)(T-t)$$

$$V'(\psi,T-t) = \lim_{n\to\infty} nq'(1-q')\left(\ln\frac{u}{d}\right)^2 = \sigma^2(t-t)$$

Substituting the above results as well as $\lim\limits_{\Delta t\to 0}\ln u/d = 0$ into (27) and taking a limit, we have the familiar Black-Scholes formula

$$\lim_{n\to\infty} c = S\lim_{n\to\infty}\Phi(n,k,q') - Xe^{-r(T-t)}\lim_{n\to\infty}\Phi(n,k,q)$$

$$= SN\left(\frac{\ln\frac{S}{X}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma^2(T-t)}}\right)-Xe^{-r(T-t)}N\left(\frac{\ln\frac{S}{X}+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sqrt{\sigma^2(T-t)}}\right)$$

Now we have completed the derivation and concluded that the multiple-period binomial model generally converges to a formula similar to the Black-Scholes. When the underlying distribution is lognormal, this limit is exactly the Black-Scholes.

American Options

So far we have developed and examined the general binomial tree model for European options. An additional dynamic programming procedure is required in the binomial scheme in order to price an American option. When early exercise is considered, the option value calculated by the binomial model should be compared with the option's intrinsic value, which is the payoff function upon exercise at each binomial node.

As an example, we consider the valuation of an American put option. First, as usual, we build the binomial tree that gives a discrete representation of the stochastic movement of the asset price. We use (n, j) to represent the node on the tree that corresponds to *j* upward moves and *n-j* downward moves and it is *n* time steps from the current time. Let S_j^n and p_j^n denote the asset price and put value at the (n, j) node respectively. The procedure for computing the option price proceeds backward from the expiration time to the current time. At each binomial node we compare the calculated value without immediate exercise and intrinsic value and take the maximum among them as the option value. The intrinsic value of a put option is $X - S_j^n$ at the (n, j) node. Hence, the dynamic programming procedure applied at each node is given by

$$p_j^n = \max\left[\left(qp_{j+1}^{n+1} + (1-q)p_j^{n+1}\right)e^{-r\Delta t},\ X - S_j^n\right] \tag{33}$$

where j = 0, 1, …, n. This dynamic procedure is continued until the node (n, j), i.e. the current time. Then the American put option premium is obtained. Since the discussion here is limited to non-dividend paying stock options, American call options have the same values as their European counterparts and thus do not need to be discussed again.

The Sources of Pricing Biases

From the analysis above, we see that the pricing biases in an option valuation model can arise from three sources: distribution bias, truncation error, and using the European formula to price American options.

Distribution biases can exist in both analytic and numerical option pricing models. Since the true probability distribution of the underlying asset price is generally unknown, as long as a specific distribution is adopted to model the price, there will always be biases due to the inconsistency of the true distribution and the distribution assumed by the model. For example, if the true distribution is not lognormal but a lognormal distribution is used to model the ratio of the asset prices, the effect of higher moments would be ignored in the resulting differential equation. Similarly, if Merton's jump-diffusion process is assumed but the true price process is significantly different from the jump-diffusion, distribution bias still exists. In this case multiple parameters are included in the equation, but they are generally not the true parameters that should be included.

Truncation errors exist in a numerical option-pricing model. In analysis of accuracy for a numerical scheme, a conventional method is to use the continuous solution as a criterion. Then the truncation error of a given numerical scheme is obtained by substituting the exact solution of the continuous problem into the numerical scheme, and the order of accuracy of a scheme is defined to be the order in powers of Δt. A numerical scheme is said to be kth order time accurate if the local truncation error of the numerical scheme tends to zero for vanishing time step. For example, assuming the underlying asset price ratio follows a lognormal distribution, the Black-Scholes is usually used as a criterion and the order of accuracy of a binomial model is measured in terms of $O(\Delta t)$. However, in practical use, the Δt in the binomial model cannot be small enough. Therefore, the pricing solution can be biased from the continuous solution due to this approximation.

As mentioned before, an analytic valuation formula is only suitable for European options and analytic formulas are generally not available for American options. Therefore, if an analytic option formula is used to price American options, pricing biases will arise. For example, if the Black-Scholes formula is used to price American put options, there will be some biases.

General Trinomial Tree Model

A natural extension of the general binomial tree model is the general trinomial tree model. In a trinomial tree model, we assume three possible asset price movements. The current asset price S will become either Su, Sm or Sd after one time period Δt, where u >

$m > d$. As an extension to the binomial formula given in equation (4), the two-period trinomial formula for call option value is

$$c = \left(q_u c_u + q_m c_m + q_d c_d\right)e^{-r\Delta t} \tag{34}$$

Here, c_u and q_u denote the option price and the probability respectively for the situation when the asset price takes the value Su one period later, and similar meaning for c_m and q_m as well as c_d and q_d. It seems that the parameters u, m, and d as well as q_u, q_m, and q_d can be determined in a similar way as in the binomial model except that the Gaussian quadrature equations corresponding to the moments up to the fifth are needed. However, since the trinomial lattice cannot automatically recombine as the binomial counterpart, an additional recombining condition is needed. If we use the fifth moment equation, altogether there will be seven equations for six unknowns. The equation system will be over-determined and the solution will not be unique. Thus, to assure a unique solution, we only need the Gaussian quadrature equations corresponding to the moments up to the fourth and an equation representing the recombining condition:

$$q_u + q_m + q_d = E(Y^0) = 1$$

$$q_u u + q_m m + q_d d = E(Y)$$

$$q_u u^2 + q_m m^2 + q_d d^2 = E(Y^2)$$

$$q_u u^3 + q_m m^3 + q_d d^3 = E(Y^3)$$

$$q_u u^4 + q_m m^4 + q_d d^4 = E(Y^4)$$

$$ud = m^2$$

The last equation $ud = m^2$ is the recombining condition.

Such a system cannot be solved analytically as in the general binomial tree model because the recombining condition is nonlinear. Therefore, we use a numerical solution. The set of nonlinear simultaneous equations are solved for the parameters of the general trinomial tree model by minimizing the sum of the squared differences between the two sides of the equations in the system. Specifically, we can determine the parameters by solving the following nonlinear programming problem:

$$\min_{u,m,d,q_u,q_m,q_d} \sum_{i=0}^{4} \left(q_u u^i + q_m m^i + q_d d^i - E(Y^i)\right)^2$$

$$\text{subject to } q_u + q_m + q_d = 1$$

$$ud - m^2 = 0$$

$$q_u \geq 0,\ q_m \geq 0,\ q_d \geq 0$$

By this way, the solution for the parameters will be unique yet the recombining condition holds.

Due to the complexity of the trinomial model and the fact that the analytical solution for the parameters in the general trinomial tree model is not available, it is difficult to examine the asymptotic limit of the model as in the general binomial tree model. Though clear analysis is not available, the same as the general binomial tree model, the trinomial model developed here is also a general model in the sense that all previous trinomial models can be considered as special cases of it. This argument is based on the fact that the only information required to determine the parameters in this trinomial model are the moments and the underlying distribution can be anything. The general binomial model considers the third moment that is related to skewness, while the general trinomial model considers the skewness as well as kurtosis. Empirical work has generally not looked at moments higher than the fourth. The Gaussian quadrature approach could easily allow considering higher moments, but there is no evidence yet that there is a need to go beyond the fourth moment.

Summary

The general binomial and trinomial tree models developed in this study use parameters determined based on Gaussian quadrature. Like past binomial and trinomial models, American options are handled using dynamic programming. The new general tree models are the first option pricing models that can handle American options and can easily handle any i.i.d. price distribution that has finite moments. The binomial tree can handle skewness and the trinomial tree can handle skewness and kurtosis. Gaussian quadrature offers a general approach that could also capture even higher moments if more branches were added at each node. Since previous binomial and trinomial tree models have ad hoc restrictions on the parameters, we, from the theory, would expect them to perform worse than the general binomial and trinomial models.

When compared with analytic option pricing models such as the Black-Scholes formula, the Merton jump-diffusion model, and the Jarrow-Rudd approximate model, we expect that the relative accuracy of the general binomial and trinomial models depends on whether the underlying distribution is known or not, and whether the options concerned are European or American. For example, for European options with lognormally distributed underlying asset prices, the Black-Scholes is of course the most accurate. However, when the distribution underlying the European options is unknown, the Jarrow-Rudd model should be considered most accurate. Furthermore, for American options with unknown underlying distributions, it seems reasonable to expect that the general binomial and trinomial tree models can perform better than all other models. Based on such hypotheses, the accuracy of the general binomial and trinomial tree models relative to the other option pricing models will be examined under various conditions in the next chapter.

CHAPTER FOUR

NUMERICAL ANALYSIS AND EMPIRICAL EVIDENCE

The general binomial tree and general trinomial tree models developed in Chapter Three sound good in theory. However, their performances in practice need to be investigated. In this chapter, the accuracy of the general binomial tree and general trinomial tree models will be examined by comparing them to other option pricing models. The accuracy will be examined by two methods. The first method is numerical analysis while the second one is an empirical study. With the study of numerical accuracy, the underlying asset price process and the parameters required for the various tree models are known. Given the known price process and parameters, option premiums based on different tree models can be calculated. For European options, premiums calculated by a closed form option pricing formula are used as the comparison criterion. The accuracy of the different tree models is measured by the difference between the premiums calculated by the tree models and the analytical solution. With the empirical study, the underlying asset price processes and the parameters in the option pricing models are not available. Instead, historical daily data, such as option premiums, strike prices, asset price and interest rate, are available. The parameters selected are those that minimize the sum of the squared errors between actual and predicted premiums. The parameters obtained are called implied parameters. An out-of-sample evaluation is then performed by using yesterday's parameters and today's futures prices to predict today's option premiums. The difference between the actual option premium and the one predicted by the implied parameters can be considered as the forecasting error of the model. Thus, the accuracy of various option pricing models can be measured by such forecasting errors.

Numerical Analysis

Procedure

The accuracy of the general binomial and trinomial tree models is compared with the accuracy of the other tree models under two cases of the underlying asset price process: a lognormal distribution and a jump-diffusion process. The lognormal represents the standard case while the jump-diffusion represents the non-standard case. In the lognormal distribution case, the option values calculated by the Black-Scholes formula are used as the comparison criterion. The jump-diffusion process is a particular challenge because four moments may not be sufficient to well approximate it. It is chosen as a representative non-lognormal case not only because of its practical use (Hilliard and Reis), but also because the analytic Merton jump-diffusion formula can be used as the criterion and there is no other analytic formula based on a distribution other than the lognormal available. The underlying asset is supposed to be a non-dividend paying stock with the input values given in Table 1.

All versions of binomial tree models use the same dynamic programming procedure to obtain the option values. So do the various trinomial tree models. The difference among the tree models is how to determine the move magnitudes and the probabilities in the models. To use the general binomial and trinomial tree models, the move magnitudes and the probabilities are obtained using the Gaussian quadrature formulas and the analytical moments. The moments are obtained in turn by using some given parameter or parameters, depending on the underlying asset price process assumed. Specifically, if a lognormal underlying distribution is assumed, the only parameter needed is the volatility. On the other hand, with the jump-diffusion price process, besides volatility, the expected number of jumps per year λ and the percentage of the total volatility explained by the jumps γ are needed to calculate the moments. Since all tree models other than the general tree models developed in this study are based on the lognormal distribution assumption, the only parameter needed for those models is the volatility.

Table 1. Input Values for Numerical Analysis

Parameter Values	
Stock Price	S = 100
Exercise Price	X = 90, 100, 110
Time to maturity	T = 6 months
Interest rate	r = 5%
Volatility	σ = 30%
Number of jumps	λ = 5
Volatility explained by jumps	γ = 50%

The common dynamic programming procedure for various binomial tree models is illustrated as follows. The calculation is implemented in a backward way. That is, the option values at the nodes of the final step of the binomial tree are calculated first. Then, the values at the step before the final one are calculated. This procedure is continued until the initial step, i.e. the current time. An n-step binomial tree has n + 1 nodes at the final, or the nth, step. By using c_n^i to denote the call value and p_n^i the put value at the ith node of the nth step, the option values at the $n+1$ nodes of the final step are calculated by

$$c_n^i = \max\left(0, Su^i d^{n-i} - X\right) \quad \text{and}$$

$$p_n^i = \max\left(0, X - Su^i d^{n-i}\right)$$

where $\quad i = 0, 1, \ldots, n$

Then, for European options, the n call and n put values at the nodes of the $n-1$th step are given by

$$c_{n-1}^i = \left(qc_n^{i+1} + (1-q)c_n^i\right)e^{-r\Delta t} \quad \text{and}$$

$$p_{n-1}^i = \left(qp_n^{i+1} + (1-q)p_n^i\right)e^{-r\Delta t}$$

where $i = 0, 1, \ldots, n-1$.

For American options, the option values at the $n-1$th step are given by

$$c_{n-1}^i = \max\left(Su^i d^{|i-n+1|} - X,\ \left(qc_n^{i+1} + (1-q)c_n^i\right)e^{-r\Delta t} \right)$$

$$p_{n-1}^i = \max\left(X - Su^i d^{|i-n+1|},\ \left(qp_n^{i+1} + (1-q)p_n^i\right)e^{-r\Delta t} \right)$$

$$i = 0, 1, \ldots, n-1$$

By continuing the backward computation until the current time, the call and put option values can finally be obtained.

For a trinomial, there are $2n+1$ nodes at the nth step. The $2n+1$ option values at the final step of an n-step trinomial tree are calculated as

$$c_n^i = \max\left(Su^{\max(i-n,0)} m^{(n-|i-n|)} d^{\max(n-i,0)} - X, 0\right)$$

$$p_n^i = \max\left(X - Su^{\max(i-n,0)} m^{(n-|i-n|)} d^{\max(n-i,0)}, 0\right)$$

$$i = 0, 1, \ldots, 2n$$

Then the European options at the $n-1$th step are valued as

$$c_{n-1}^{i} = \left(q_u c_n^{i+2} + q_m c_n^{i+1} + q_d c_n^{i}\right)e^{-r\Delta t} \quad \text{and}$$

$$p_{n-1}^{i} = \left(q_u p_n^{i+2} + q_m p_n^{i+1} + q_d p_n^{i}\right)e^{-r\Delta t}$$

where $i = 0, 1, \ldots, n-1$.

If the options are American, the options at the $(n-1)$th step are valued as

$$c_{n-1}^{i} = \max\left(Su^{\max(i-n+1,0)} m^{(n-1-|i-n+1|)} d^{\max(n-1-i,0)} - X, (q_u c_n^{i+2} + q_m c_n^{i+1} + q_d c_n^{i})e^{-r\Delta t}\right)$$

$$p_{n-1}^{i} = \max\left(Su^{\max(i-n+1,0)} m^{(n-1-|i-n+1|)} d^{\max(n-1-i,0)} - X, (q_u p_n^{i+2} + q_m p_n^{i+1} + q_d p_n^{i})e^{-r\Delta t}\right)$$

$$i = 0, 1, \ldots, n-1$$

Similar to the procedure for the binomial tree models, the backward computation is continued until the current time to obtain the call and put option values.

The accuracy of the various tree models for European options is measured by a percentage average pricing error that is defined as

$$\text{Average pricing error } = \frac{1}{M}\sum_{M}\frac{|V_s - V_c|}{V_c}\times 100\%$$

where M is the number of strike prices considered (in our case, $M = 3$), V_s is the option value calculated by a tree model and V_c is the criterion option value obtained by a closed form option pricing formula. For American put options with an underlying lognormal distribution, no exact closed form pricing formula is available, option values simulated by different tree models are listed to see the difference among them. In fact, since American call option values are the same as the European counterparts for non-dividend paying stocks and thus the relative accuracy among the tree models can be measured as mentioned before, it is reasonable to expect about the same relative accuracy when American put options are valued by the different tree models.

The methods to determine the move magnitudes and the probabilities in various binomial and trinomial tree models have been outlined in the previous chapters. For clearness, the different methods are restated here. We denote the general binomial tree model by GBIN and the general trinomial tree model by GTRIN. The parameters in the general tree models are determined by the moments up to the third in GBIN and the fourth in GTRIN. When the underlying asset price follows a lognormal distribution, the kth moment of the stock price ratio Y, as introduced in Chapter Three, is

$$E(Y^k) = \exp\left\{\left[rk + \frac{\sigma^2 k(k-1)}{2}\right]\Delta t\right\}.$$

On the other hand, when the stock price process is jump-diffusion, the kth moment of the price ratio, also as introduced in Chapter Three, is

$$E(Y^k) = \exp\left\{\left[\lambda\left(e^{\delta^2 k(k-1)/2} - 1\right) + rk + \frac{\sigma^2 k(k-1)}{2}\right]\Delta t\right\}$$

$$\text{with } \delta = \sqrt{\frac{\gamma\sigma^2}{\lambda}}.$$

For GBIN, the parameters are calculated by the following formulas:

$$u = \frac{-C_1 + \sqrt{C_1^2 - 4C_0}}{2},$$

$$d = \frac{-C_1 - \sqrt{C_1^2 - 4C_0}}{2}, \quad \text{and}$$

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

where $C_0 = \dfrac{E(Y)E(Y^3) - (E(Y^2))^2}{E(Y^2) - (E(Y))^2}$ and $C_1 = \dfrac{E(Y)E(Y^2) - E(Y^3)}{E(Y^2) - (E(Y))^2}$.

For GTRIN, the parameters are obtained by solving the nonlinear programming problem

$$\min_{u,m,d,q_u,q_m,q_d} \sum_{i=0}^{4}\left(q_u u^i + q_m m^i + q_d d^i - E(Y^i)\right)^2$$

$$\text{subject to } q_u + q_m + q_d = 1$$

$$ud - m^2 = 0$$

$$q_u \geq 0,\ q_m \geq 0,\ q_d \geq 0.$$

The parameters in the other tree models used for comparison are determined as follows: In the Cox-Ross-Rubinstein binomial tree model, denoted as CRR,

$$u = e^{\sigma\sqrt{\Delta t}},\quad d = e^{-\sigma\sqrt{\Delta t}},\quad q = \frac{e^{r\Delta t} - d}{u - d}.$$

In the Boyle trinomial tree model, denoted as BOYLE,

$$u = e^{\sigma\sqrt{2\Delta t}},\ m = 1,\ d = e^{-\sigma\sqrt{2\Delta t}},$$

$$q_u = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2,$$

$$q_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2, \text{ and}$$

$$q_m = 1 - q_u - q_d.$$

Tian's two trinomial tree models are denoted as TTRIN1 and TTRIN2. In TTRIN1,

$$q_u = q_m = q_d = \frac{1}{3},$$

$$u = K + \sqrt{K^2 - m^2},$$

$$d = K - \sqrt{K^2 - m^2}, \text{ and}$$

$$m = \frac{M(3 - V)}{2}.$$

In TTRIN2,

$$q_u = \frac{md - M(m+d) + M^2V}{(u-d)(u-m)},$$

$$q_m = \frac{M(u+d) - ud - M^2V}{(u-m)(m-d)},$$

$$q_d = \frac{um - M(u+m) + M^2V}{(u-d)(m-d)},$$

$$u = K + \sqrt{K^2 - m^2},$$

$$d = K - \sqrt{K^2 - m^2},$$

$$m = MV^2, \quad \text{and } K = \frac{M}{2}\left(V^4 + V^3\right).$$

In both TTRIN1 and TTRIN2, M and V are defined as illustrated in Chapter Two.

Results

Altogether six tree models are investigated and compared. The numerical calculations are performed with Visual Basic for Application (VBA). Table 2 reports the values of the European call options obtained from all six lattice procedures using time steps ranging from 5, 10, 20, …, 100, 200, …, 500. Since it is assumed that the stock does not pay dividends, American call options will have the same value as the European call options. Similar results, obtained from a study of European put options using the same option parameter values, are summarized in Table 3. It is clear that option values obtained from all models converge closely enough to the correct values, the Black-Scholes prices. The average pricing errors of the six models for the three strike prices, in terms of percentage, are reported in Table 4. It is observed that, by using 500 steps, the

Table 2. European Calls Valued by Various Models under Lognormal Distribution
($S = 100$, $r = 5\%$, $\sigma = 30\%$, $T = 6$ months)

	Numerical Model					
Time Steps	CRR	GBIN	BOYLE	TTRIN1	TTRIN2	GTRIN
		Exercise price X = 90, Black-Scholes = 15.4860				
5	15.2607	15.6792	15.5495	15.4928	15.5870	15.2722
10	15.5495	15.5307	15.4593	15.5526	15.4255	15.5690
20	15.4593	15.4646	15.5235	15.4999	15.5147	15.4806
30	15.5267	15.4938	15.4732	15.5058	15.4963	15.5100
40	15.5235	15.5094	15.4917	15.5061	15.4870	15.4593
50	15.5010	15.4613	15.5012	15.4851	15.4882	15.5012
60	15.4732	15.4830	15.4938	15.4935	15.4967	15.4839
70	15.4741	15.5005	15.4792	15.4983	15.4649	15.4795
80	15.4917	15.4935	15.4848	15.4942	15.4936	15.4675
90	15.4996	15.4723	15.4923	15.4853	15.4915	15.4850
100	15.5012	15.4784	15.4936	15.4886	15.4700	15.4906
200	15.4936	15.4836	15.4827	15.4871	15.4900	15.4820
300	15.4893	15.4843	15.4847	15.4874	15.4882	15.4843
400	15.4827	15.4868	15.4842	15.4878	15.4820	15.4847
500	15.4890	15.4863	15.4859	15.4875	15.4874	15.4863
		Exercise price X = 100, Black-Scholes = 9.63487				
5	10.0474	9.5655	9.4278	9.5928	9.7596	9.8295
10	9.4278	9.8244	9.5306	9.6166	9.4394	9.7258
20	9.5306	9.6832	9.5826	9.6290	9.6803	9.6018
30	9.5652	9.6081	9.6000	9.6327	9.6510	9.5841
40	9.5826	9.5953	9.6087	9.6343	9.5895	9.6412
50	9.5930	9.6352	9.6139	9.6352	9.6348	9.6534
60	9.6000	9.6522	9.6174	9.6357	9.6509	9.6503
70	9.6049	9.6581	9.6199	9.6360	9.6473	9.6390
80	9.6087	9.6581	9.6218	9.6363	9.6340	9.6474
90	9.6116	9.6550	9.6232	9.6364	9.6161	9.6463
100	9.6139	9.6500	9.6244	9.6365	9.6281	9.6347
200	9.6244	9.6419	9.6296	9.6367	9.6397	9.6390
300	9.6279	9.6341	9.6314	9.6366	9.6381	9.6372
400	9.6296	9.6379	9.6323	9.6365	9.6357	9.6364
500	9.6307	9.6374	9.6328	9.6364	9.6308	9.6361
		Exercise price X = 110, Black-Scholes = 5.5871				
5	5.1367	5.9665	5.7062	5.6916	5.4322	5.4600
10	5.7062	5.6228	5.4754	5.6030	5.5587	5.5847
20	5.4754	5.6419	5.6295	5.6361	5.5988	5.6174
30	5.6056	5.6378	5.6022	5.5787	5.6050	5.6139
40	5.6295	5.6318	5.5601	5.6034	5.6055	5.6085
50	5.6214	5.6260	5.5943	5.6076	5.6045	5.6061
60	5.6022	5.6210	5.6020	5.5970	5.6030	5.6025
70	5.5795	5.6166	5.5973	5.5807	5.6015	5.6006
80	5.5601	5.6128	5.5866	5.5926	5.6000	5.5792
90	5.5820	5.6094	5.5755	5.5976	5.5985	5.5670
100	5.5943	5.6064	5.5874	5.5971	5.5972	5.5916
200	5.5874	5.5878	5.5916	5.5860	5.5881	5.5909
300	5.5848	5.5820	5.5901	5.5885	5.5829	5.5902
400	5.5916	5.5883	5.5889	5.5878	5.5840	5.5893
500	5.5832	5.5903	5.5863	5.5863	5.5869	5.5865

Table 3. European Puts Valued by Various Models under Lognormal Distribution
($S = 100$, $r = 5\%$, $\sigma = 30\%$, $T = 6$ months)

	Numerical Model					
Time Steps	CRR	GBIN	BOYLE	TTRIN1	TTRIN2	GTRIN
		Exercise price X = 90, Black-Scholes = 3.2639				
5	3.0386	3.4571	3.3274	3.2707	3.3649	3.0501
10	3.3274	3.3086	3.2372	3.3305	3.2034	3.3469
20	3.2372	3.2425	3.3014	3.2778	3.2926	3.2585
30	3.3046	3.2717	3.2511	3.2836	3.2742	3.2879
40	3.3014	3.2873	3.2696	3.2839	3.2649	3.2372
50	3.2789	3.2392	3.2791	3.2629	3.2661	3.2791
60	3.2511	3.2608	3.2717	3.2714	3.2746	3.2618
70	3.2520	3.2784	3.2571	3.2762	3.2428	3.2574
80	3.2696	3.2713	3.2627	3.2721	3.2715	3.2454
90	3.2775	3.2502	3.2702	3.2632	3.2694	3.2629
100	3.2791	3.2563	3.2715	3.2664	3.2479	3.2682
200	3.2715	3.2615	3.2606	3.2649	3.2679	3.2599
300	3.2672	3.2622	3.2626	3.2652	3.2661	3.2619
400	3.2606	3.2646	3.2621	3.2656	3.2599	3.2625
500	3.2669	3.2642	3.2638	3.2652	3.2653	3.2642
		Exercise price X = 100, Black-Scholes = 7.1659				
5	7.5784	7.0965	6.9588	7.1238	7.2906	7.3605
10	6.9588	7.3554	7.0616	7.1476	6.9704	7.2568
20	7.0616	7.2142	7.1135	7.1599	7.2113	7.1328
30	7.0962	7.1391	7.1309	7.1637	7.1820	7.1151
40	7.1135	7.1263	7.1397	7.1653	7.1205	7.1722
50	7.1240	7.1662	7.1449	7.1662	7.1658	7.1844
60	7.1309	7.1832	7.1484	7.1667	7.1819	7.1813
70	7.1359	7.1891	7.1509	7.1670	7.1783	7.1700
80	7.1397	7.1891	7.1528	7.1673	7.1650	7.1784
90	7.1426	7.1859	7.1542	7.1674	7.1471	7.1773
100	7.1449	7.1810	7.1554	7.1675	7.1591	7.1653
200	7.1554	7.1729	7.1606	7.1677	7.1707	7.1700
300	7.1589	7.1651	7.1624	7.1675	7.1690	7.1680
400	7.1606	7.1689	7.1632	7.1674	7.1666	7.1672
500	7.1617	7.1684	7.1638	7.1673	7.1618	7.1671
		Exercise price X = 110, Black-Scholes = 12.8712				
5	12.4208	13.2506	12.9903	12.9757	12.7162	12.7441
10	12.9903	12.9069	12.7595	12.8871	12.8428	12.8688
20	12.7595	12.9260	12.9136	12.9202	12.8829	12.9015
30	12.8897	12.9219	12.8863	12.8628	12.8891	12.8980
40	12.9136	12.9159	12.8442	12.8875	12.8896	12.8926
50	12.9055	12.9101	12.8784	12.8917	12.8886	12.8902
60	12.8863	12.9051	12.8861	12.8811	12.8871	12.8866
70	12.8636	12.9007	12.8814	12.8648	12.8856	12.8847
80	12.8442	12.8969	12.8707	12.8767	12.8841	12.8633
90	12.8660	12.8935	12.8596	12.8817	12.8826	12.8511
100	12.8784	12.8905	12.8715	12.8812	12.8813	12.8753
200	12.8715	12.8719	12.8757	12.8702	12.8721	12.8750
300	12.8689	12.8661	12.8742	12.8727	12.8670	12.8741
400	12.8757	12.8724	12.8730	12.8719	12.8681	12.8732
500	12.8673	12.8744	12.8704	12.8705	12.8710	12.8706

Table 4. Average European Options Pricing Errors of Various Models under Lognormal Distribution

(S = 100, r = 5%, σ = 30%, T = 6 months)

	Numerical Model					
Time Steps	CRR	GBIN	BOYLE	TTRIN1	TTRIN2	GTRIN
			Call Prices			
5	4.599%	2.920%	1.564%	0.784%	1.573%	1.892%
10	1.564%	0.965%	1.085%	0.301%	0.976%	0.507%
20	1.085%	0.540%	0.515%	0.343%	0.289%	0.307%
30	0.439%	0.412%	0.239%	0.100%	0.185%	0.387%
40	0.515%	0.454%	0.264%	0.143%	0.269%	0.207%
50	0.382%	0.287%	0.148%	0.125%	0.109%	0.210%
60	0.239%	0.269%	0.166%	0.079%	0.174%	0.150%
70	0.174%	0.288%	0.127%	0.069%	0.174%	0.109%
80	0.264%	0.250%	0.051%	0.055%	0.096%	0.130%
90	0.141%	0.232%	0.123%	0.069%	0.145%	0.161%
100	0.148%	0.184%	0.055%	0.071%	0.118%	0.037%
200	0.055%	0.034%	0.052%	0.015%	0.031%	0.046%
300	0.045%	0.037%	0.033%	0.017%	0.041%	0.030%
400	0.052%	0.019%	0.024%	0.013%	0.030%	0.021%
500	0.044%	0.029%	0.012%	0.013%	0.018%	0.009%
			Put Prices			
5	5.386%	3.279%	1.920%	0.536%	2.013%	3.418%
10	1.920%	1.431%	1.047%	0.807%	1.601%	1.277%
20	1.047%	0.585%	0.737%	0.297%	0.535%	0.287%
30	0.788%	0.336%	0.332%	0.234%	0.227%	0.551%
40	0.737%	0.538%	0.251%	0.250%	0.269%	0.357%
50	0.437%	0.355%	0.272%	0.064%	0.068%	0.291%
60	0.332%	0.199%	0.200%	0.107%	0.226%	0.133%
70	0.280%	0.333%	0.166%	0.148%	0.310%	0.120%
80	0.251%	0.251%	0.074%	0.105%	0.115%	0.267%
90	0.261%	0.291%	0.149%	0.042%	0.174%	0.115%
100	0.272%	0.197%	0.128%	0.060%	0.221%	0.057%
200	0.128%	0.059%	0.069%	0.021%	0.066%	0.070%
300	0.072%	0.034%	0.037%	0.025%	0.049%	0.037%
400	0.069%	0.025%	0.035%	0.027%	0.052%	0.026%
500	0.060%	0.023%	0.013%	0.022%	0.034%	0.011%

average pricing error from each of the models, except the CRR for puts, is less than 0.05%. If we rank the accuracy of the various models with 500 steps, though thedifference is small, the most accurate one is the GTRIN. The GBIN is usually not as accurate as the trinomial tree models, but definitely more accurate than the CRR. With a 266 MHz Pentium II computer, for 500 steps, the CRR and GBIN models spend the minimum time, less than1 second, the BOYLE, TTRIN1 and TTRIN2 spend medium, 2 seconds, and the GTRIN model spends the most, 3 seconds. Thus, actually, computing efficiency is not a problem, though convergence speed is somewhat different from model to model.

For American put options, since there is no analytic pricing formula available as the comparison criterion, the values simulated by all six tree models are reported in Table 5. The values from all models with 500 steps are very close to each other. In summary, if the true underlying distribution is lognormal, the general tree models, especially the general trinomial tree model, can be at least as accurate as any other tree models.

For the case of a jump-diffusion process, the European option values simulated by the general binomial and trinomial trees are summarized in Table 6. Since all tree models other than the general binomial and trinomial trees are based on the lognormal distribution and thus do not capture the impact of the additional parameters λ and γ, the option values simulated by these tree models are the same as shown in Tables 2 and 3. The average pricing errors by all six models under the jump-diffusion price process are reported in Table 7. We see that if the number of the steps is large enough, the average pricing errors of the two general tree models developed by this study are consistently below one percent while the errors of the other tree models are consistently larger than one percent. Though the accuracy seems not as satisfactory as in the case of the lognormal distribution, both the general binomial and trinomial tree models are significantly more accurate than the CRR, BOYLE, TTRIN1 and TTRIN2.

Table 5. American Puts Valued by Various Models under Lognormal Distribution
(S = 100, r = 5%, σ = 30%, T = 6 months)

	Numerical Model					
Time Steps	CRR	GBIN	BOYLE	TTRIN1	TTRIN2	GTRIN
			Exercise price X = 90			
5	3.1183	3.5038	3.3967	3.3686	3.3891	3.1168
10	3.4343	3.3675	3.3050	3.4029	3.2768	3.4089
20	3.3189	3.3353	3.3780	3.3638	3.3653	3.3278
30	3.3838	3.3477	3.3353	3.3634	3.3473	3.3616
40	3.3846	3.3687	3.3483	3.3657	3.3435	3.3151
50	3.3655	3.3274	3.3584	3.3476	3.3421	3.3550
60	3.3401	3.3406	3.3528	3.3527	3.3524	3.3418
70	3.3358	3.3586	3.3400	3.3574	3.3246	3.3360
80	3.3511	3.3539	3.3435	3.3543	3.3493	3.3268
90	3.3588	3.3355	3.3504	3.3466	3.3488	3.3419
100	3.3609	3.3377	3.3521	3.3482	3.3300	3.3475
200	3.3533	3.3443	3.3428	3.3471	3.3480	3.3417
300	3.3488	3.3438	3.3442	3.3469	3.3467	3.3432
400	3.3435	3.3460	3.3439	3.3471	3.3416	3.3440
500	3.3485	3.3456	3.3454	3.3467	3.3462	3.3456
			Exercise price X = 100			
5	7.7896	7.3958	7.2135	7.3960	7.4632	7.4946
10	7.2768	7.5649	7.3063	7.3938	7.1837	7.4392
20	7.3358	7.4495	7.3522	7.3948	7.4090	7.3524
30	7.3558	7.3829	7.3664	7.3957	7.3954	7.3403
40	7.3654	7.3612	7.3732	7.3966	7.3520	7.3884
50	7.3710	7.3918	7.3776	7.3970	7.3845	7.4010
60	7.3748	7.4073	7.3805	7.3971	7.4014	7.4023
70	7.3777	7.4137	7.3824	7.3970	7.4001	7.3933
80	7.3798	7.4147	7.3840	7.3970	7.3907	7.4005
90	7.3814	7.4127	7.3851	7.3969	7.3753	7.3990
100	7.3827	7.4090	7.3861	7.3968	7.3848	7.3906
200	7.3884	7.4001	7.3901	7.3964	7.3959	7.3974
300	7.3903	7.3945	7.3915	7.3960	7.3955	7.3953
400	7.3912	7.3967	7.3921	7.3958	7.3937	7.3947
500	7.3918	7.3967	7.3925	7.3956	7.3902	7.3951
			Exercise price X = 110			
5	13.1908	13.6652	13.4161	13.4303	13.1815	13.2582
10	13.4744	13.4547	13.3010	13.4280	13.2886	13.3278
20	13.3386	13.4483	13.4124	13.4304	13.3477	13.3826
30	13.4224	13.4374	13.3922	13.3870	13.3889	13.3939
40	13.4311	13.4280	13.3680	13.4054	13.3797	13.3904
50	13.4199	13.4208	13.3930	13.4066	13.3930	13.3913
60	13.4029	13.4167	13.3975	13.3972	13.3901	13.3933
70	13.3860	13.4121	13.3938	13.3873	13.3893	13.3892
80	13.3769	13.4088	13.3857	13.3954	13.3933	13.3761
90	13.3917	13.4059	13.3799	13.3983	13.3910	13.3697
100	13.3995	13.4033	13.3885	13.3973	13.3887	13.3875
200	13.3917	13.3892	13.3912	13.3890	13.3873	13.3910
300	13.3878	13.3859	13.3902	13.3903	13.3837	13.3898
400	13.3927	13.3901	13.3895	13.3895	13.3850	13.3891
500	13.3868	13.3912	13.3878	13.3883	13.3867	13.3879

Table 6. European Options Valued by Various Models under Jump-Diffusion Process

($S = 100$, $r = 5\%$, $\sigma = 30\%$, $T = 6$ months, $\lambda = 5$, $\gamma = 50\%$)

	Call		Put	
Time Steps	GBIN	GTRIN	GBIN	GTRIN
	Exercise price X = 90, Merton = 15.4279		Exercise price X = 90, Merton = 3.2065	
5	15.6452	15.1594	3.4230	2.9373
10	15.4244	15.4707	3.2023	3.2486
20	15.4215	15.3584	3.1994	3.1363
30	15.4182	15.3976	3.1961	3.1755
40	15.4487	15.4251	3.2266	3.2030
50	15.4348	15.4254	3.2127	3.2033
60	15.4281	15.4287	3.2060	3.2066
70	15.4318	15.4317	3.2097	3.2096
80	15.4351	15.4233	3.2130	3.2011
90	15.4226	15.4107	3.2005	3.1886
100	15.4097	15.4140	3.1876	3.1918
200	15.4263	15.4179	3.2042	3.1958
300	15.4217	15.4243	3.1996	3.2022
400	15.4243	15.3648	3.2022	3.1267
500	15.4234	15.4198	3.2013	3.1977
	Exercise price X = 100, Merton = 9.5229		Exercise price X = 100, Merton = 7.0541	
5	9.2772	9.7138	6.8082	7.2448
10	9.3956	9.6687	6.9266	7.1997
20	9.5805	9.6463	7.1115	7.1773
30	9.6604	9.5851	7.1914	7.1161
40	9.5476	9.6105	7.0786	7.1414
50	9.6296	9.5928	7.1606	7.1238
60	9.6278	9.5769	7.1588	7.1079
70	9.6190	9.5758	7.1500	7.1068
80	9.6183	9.6086	7.1493	7.1396
90	9.6188	9.5906	7.1498	7.1216
100	9.6049	9.6052	7.1359	7.1362
200	9.5994	9.5950	7.1304	7.1260
300	9.5860	9.6028	7.1170	7.1338
400	9.6001	9.5591	7.1311	7.0741
500	9.5939	9.5986	7.1249	7.1296
	Exercise price X = 110, Merton = 5.4859		Exercise price X = 110, Merton = 12.7695	
5	5.9923	5.7325	13.2764	13.0165
10	5.7500	5.6114	13.0341	12.8955
20	5.5932	5.6092	12.8773	12.8933
30	5.6282	5.6247	12.9123	12.9088
40	5.6374	5.5784	12.9215	12.8625
50	5.5730	5.5987	12.8571	12.8827
60	5.6054	5.5638	12.8895	12.8479
70	5.6185	5.5891	12.9026	12.8732
80	5.6151	5.5912	12.8992	12.8753
90	5.6125	5.6016	12.8966	12.8857
100	5.6073	5.5965	12.8914	12.8806
200	5.5973	5.5936	12.8814	12.8777
300	5.5956	5.5871	12.8797	12.8711
400	5.5947	5.5501	12.8788	12.8182
500	5.5828	5.5864	12.8669	12.8705

Table 7. Average European Options Pricing Errors of Various Models under Jump-Diffusion Process

($S = 100$, $r = 5\%$, $\sigma = 30\%$, $T = 6$ months, $\lambda = 5$, $\gamma = 0.5$)

	Numerical Model					
Time Steps	CRR	GBIN	BOYLE	TTRIN1	TTRIN2	GTRIN
			Call Prices			
5	4.319%	4.406%	1.934%	1.635%	1.499%	2.746%
10	1.934%	2.058%	0.159%	1.309%	0.740%	1.365%
20	0.159%	0.867%	1.288%	1.440%	1.424%	1.331%
30	1.089%	1.367%	1.074%	1.116%	1.320%	1.127%
40	1.288%	1.052%	0.889%	1.273%	1.088%	0.874%
50	1.226%	0.918%	1.136%	1.256%	1.243%	0.935%
60	1.074%	1.094%	1.179%	1.212%	1.308%	0.664%
70	0.956%	1.151%	1.127%	1.124%	1.218%	0.820%
80	0.889%	1.135%	1.081%	1.189%	1.224%	0.950%
90	1.049%	1.117%	1.035%	1.200%	1.148%	0.977%
100	1.136%	1.064%	1.114%	1.204%	1.136%	0.990%
200	1.114%	0.948%	1.135%	1.135%	1.164%	0.928%
300	1.101%	0.901%	1.136%	1.150%	1.123%	0.902%
400	1.135%	0.939%	1.130%	1.146%	1.108%	0.653%
500	1.101%	0.847%	1.120%	1.136%	1.120%	0.893%
			Put Prices			
5	5.134%	4.736%	2.283%	1.535%	2.903%	4.345%
10	2.283%	1.337%	0.381%	2.038%	0.620%	1.455%
20	0.381%	0.627%	1.644%	1.635%	1.934%	1.635%
30	1.532%	1.130%	1.131%	1.563%	1.620%	0.979%
40	1.644%	0.721%	1.255%	1.639%	1.234%	0.692%
50	1.437%	0.796%	1.468%	1.435%	1.458%	0.659%
60	1.131%	0.814%	1.428%	1.498%	1.619%	0.460%
70	1.105%	0.833%	1.275%	1.506%	1.267%	0.552%
80	1.255%	0.857%	1.314%	1.496%	1.499%	0.736%
90	1.408%	0.847%	1.370%	1.417%	1.389%	0.809%
100	1.468%	0.902%	1.421%	1.450%	1.218%	0.831%
200	1.421%	0.677%	1.343%	1.406%	1.457%	0.734%
300	1.385%	0.657%	1.368%	1.415%	1.418%	0.687%
400	1.343%	0.694%	1.364%	1.417%	1.344%	1.051%
500	1.391%	0.643%	1.377%	1.408%	1.385%	0.713%

From the numerical simulations we see that, when the information about the underlying distribution is given, the general binomial and trinomial models are at least as accurate as the other tree models when the distribution is lognormal. When the underlying distribution is different from the lognormal, the general models are more accurate than the other models. Next, we will examine the relative accuracy of the general models with unknown underlying distributions by using real data on futures options.

Empirical Study

Procedure

The accuracy of the general binomial and trinomial models relative to the other models is examined by measuring the forecasting errors of the alternative models. The empirical study consists of three steps. First, the necessary parameters in various option pricing models are daily implied by using the historical data on three futures options. Second, the implied parameters for every business day and the real data on the next day are substituted into the pricing models to calculate the option values. These are the predicted tomorrow's option premiums based on today's information. The daily forecasting errors of the various models are then measured. Finally, the daily forecasting errors from different pricing models are presented by graphs and summarized by descriptive statistics.

The parameters in all six option pricing models will be implied. Given a model and the data of a business day, the implied parameters are found by solving a non-linear programming problem

$$\min_{\theta}\sum_{i=1}^{N}\left(Call_i - c(F,r,T,X_i,\theta)\right)^2 + \sum_{j=1}^{M}\left(Put_j - p(F,r,T,X_j,\theta)\right)^2$$

where $Call_i$ is the observed option premium for the i-th call contract, Put_j is the observed option premium for the j-th put contract, $c(\cdot)$ and $p(\cdot)$ are option formulas for calls and puts respectively, F is the observed futures price, r is the observed risk-free interest rate, T is the time of expiration of the option contract, X_i and X_j are the strike prices for the i-th call and the j-th put respectively, θ is the parameter vector to be implied. The six option pricing models considered are the Black-Scholes formula, the Jarrow-Rudd approximate formula, the Cox-Ross-Rubinstein binomial tree model, the Boyle trinomial tree model, and the general binomial and trinomial tree models developed in this study. All these models have been discussed before. However, since the underlying assets are futures rather than stocks, some modifications for the models should be noted. Specifically, the Black-Scholes formulas for calls and puts are

$$c = e^{-rT}\left[FN(d_1) - XN(d_2)\right] \quad \text{and}$$

$$p = e^{-rT}\left[XN(-d_2) - FN(-d_1)\right]$$

with

$$d_1 = \frac{\ln(F/X) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad \text{and}$$

$$d_2 = \frac{\ln(F/X) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where F is the futures price, and the other notations are the same as defined before. The Jarrow-Rudd approximate option pricing formula for a call is still defined as Eq. (16) except that the Black-Scholes formula in it is the futures version as presented above. The formula for put options can be obtained by put-call parity. Since the futures price itself is an expectation of a spot price, the expected futures price in the future is the current futures price. Thus the expected ratio of the futures price in a future time over the current futures price is equal to one, so that the first moment condition in the Cox-Ross-Rubinstein binomial tree model and in the general binomial tree model is

$$uq + d(1-q) = 1 \qquad \text{or} \qquad q = \frac{1-d}{u-d}$$

instead of

$$uq + d(1-q) = e^{r\Delta t} \qquad \text{or} \qquad q = \frac{e^{r\Delta t} - d}{u-d}$$

For the same reason, the first moment condition in the Boyle trinomial tree model and in the general trinomial tree model is

$$uq_u + mq_m + dq_d = 1$$

rather than

$$uq_u + mq_m + dq_d = e^{r\Delta t}.$$

Only one parameter, the volatility σ, is implied in the Black-Scholes, the Cox-Ross-Rubinstein binomial, and the Boyle trinomial models. In the Jarrow-Rudd model, by using an assumption that the second cumulant of the true underlying distribution is equal to the one of the lognormal distribution, i.e. $k_2 = k_2(G)$ (refer to Eq. (16)), there are three parameters, i.e. σ, k_2 and k_3, to be implied. For the general binomial tree model, two parameters u and d are implied while the parameter q is related to u and d by the first moment condition. For the general trinomial tree model, move magnitudes u, m, d, and any two of the three probabilities corresponding to the move magnitudes need to be implied.

For the tree models, the number of time periods needs to be determined. Intuitively, it seems that the larger the number of the steps, the smaller the fit error. However, experiments showed that increasing the step number did not reduce the fit errors. Generally, five steps can guarantee the fit error, i.e. the objective value in the nonlinear programming for implying the parameters, very small.

Once the implied parameters are obtained, the forecasting performance for each of the six models is measured by out-of-sample fit errors. This is a procedure of using yesterday's implied parameters to predict today's option prices. The forecasting error is the measure of the performance of the implied parameters. This study adopts the following measurement for the absolute forecasting error at time t:

$$error_t = \sum_{i=1}^{N} | Call_{it} - c(F_t, r_t, T_t, X_{it}, \theta_{t-1}) | + \sum_{j=1}^{M} | Put_{jt} - p(F_t, r_t, T_t, X_{jt}, \theta_{t-1}) |$$

To compare the performances of the models, we measure the forecasting errors in terms of per dollar value of the total option premium:

$$\frac{error_t}{\sum_{i=1}^{N} Call_{it} + \sum_{j=1}^{M} Put_{jt}} \tag{35}$$

For comparison convenience, the forecasting errors obtained by (35) are further normalized in terms of the errors from the Black-Scholes model. That is, the forecasting errors from each of the six models are divided by the errors from the Black-Scholes model. By this way, the forecasting errors of the Black-Scholes always equal one. The accuracy of the other models relative to the Black-Scholes can be easily detected by watching whether the normalized forecasting errors are larger than one or not.

Finally, all forecasting error series are graphed together to present the relative accuracy intuitively. Since the forecasting errors are normalized, only the series other than the one from the Black-Scholes need to be graphed. The sample mean and standard deviation for all six series are also calculated for comparison.

Data

The data that will be used in the empirical study consist of daily futures prices and premiums of futures options for each of three commodities — corn, soybeans and wheat. Daily 3-month U.S. treasury bill yields will be used as the risk-free interest rates. All the data are from the Chicago Board of Trade. The time periods covered vary by the futures options but run from 200 days before the expirations of the options. The contracts for the futures options are July Corn, November Soybean and July Wheat. All the contracts are in 1998.

For a given futures in a business day, option quotes corresponding to different strike prices can be as many as 40. Not all these observations reflect the true relationships between the option premiums and the parameters in the option pricing models. The quotes that do not reflect the true price relationship are called "stale" prices. Stale prices can be easily identified by trading volume, since there would be no trade for incorrectly priced options. Thus an ideal method to filter the data is to assign a level of trading volume and only use the quotes with trading volume above the level. Since the trading volume data are not available, this study adopts a proxy method. Only quotes with strike prices higher or lower than the futures price by no more than 15% are considered valid. This method is based on the fact that there would be no transactions for options that are too deep in-the-money or too deep out-of-the-money.

Results

By using the forecasting error of the Black-Scholes formula as the basis, the relative forecasting errors of other models with the three commodity futures are shown in the three figures. For corn (see Figure 3), by using yesterday's implied parameters, all the models other than the Black-Scholes, except for the Boyle trinomial model, forecast the option prices more accurate than the Black-Scholes formula. Among the models, the most accurate ones are the general binomial tree and general trinomial tree models, whose forecasting errors are less than 80% of Black-Scholes. The Jarrow-Rudd model, though more accurate than the Black-Scholes, is not as accurate as the general binomial and trinomial models. The Cox-Ross-Rubinstein binomial model is about as accurate as the Black-Scholes. The Boyle trinomial model is obviously less accurate than Black-Scholes.

Figure 4 shows that the relative accuracy among the models for soybean futures is about the same as for corn futures. The two general tree models are still the most accurate ones. However, different from the corn futures, the Jarrow-Rudd model performs about as well as the two general tree models. Also, both the Cox-Ross- Rubinstein and the Boyle models are less accurate than the Black-Scholes. Between these two models, the Cox-Ross-Rubinstein binomial model is a little better. Generally, the American option pricing models did not perform as well for pricing November 1998 soybeans as their European counterparts.

As shown in Figure 5, there are only small differences among the models for the wheat futures options. The general binomial and trinomial trees as well as the Jarrow-Rudd model are more accurate than the Black-Scholes, but not very much. The Cox-Ross-Rubinstein binomial model is less accurate than the Black-Scholes, but not very much either. The least accurate model is still the Boyle trinomial tree model. The descriptive statistics for the forecasting errors of the six option pricing models on the three commodity futures are summarized in Table 8.

Figure 3. Forecasting Error Comparison for 1998 July Corn Contract (Black-Scholes = 1)

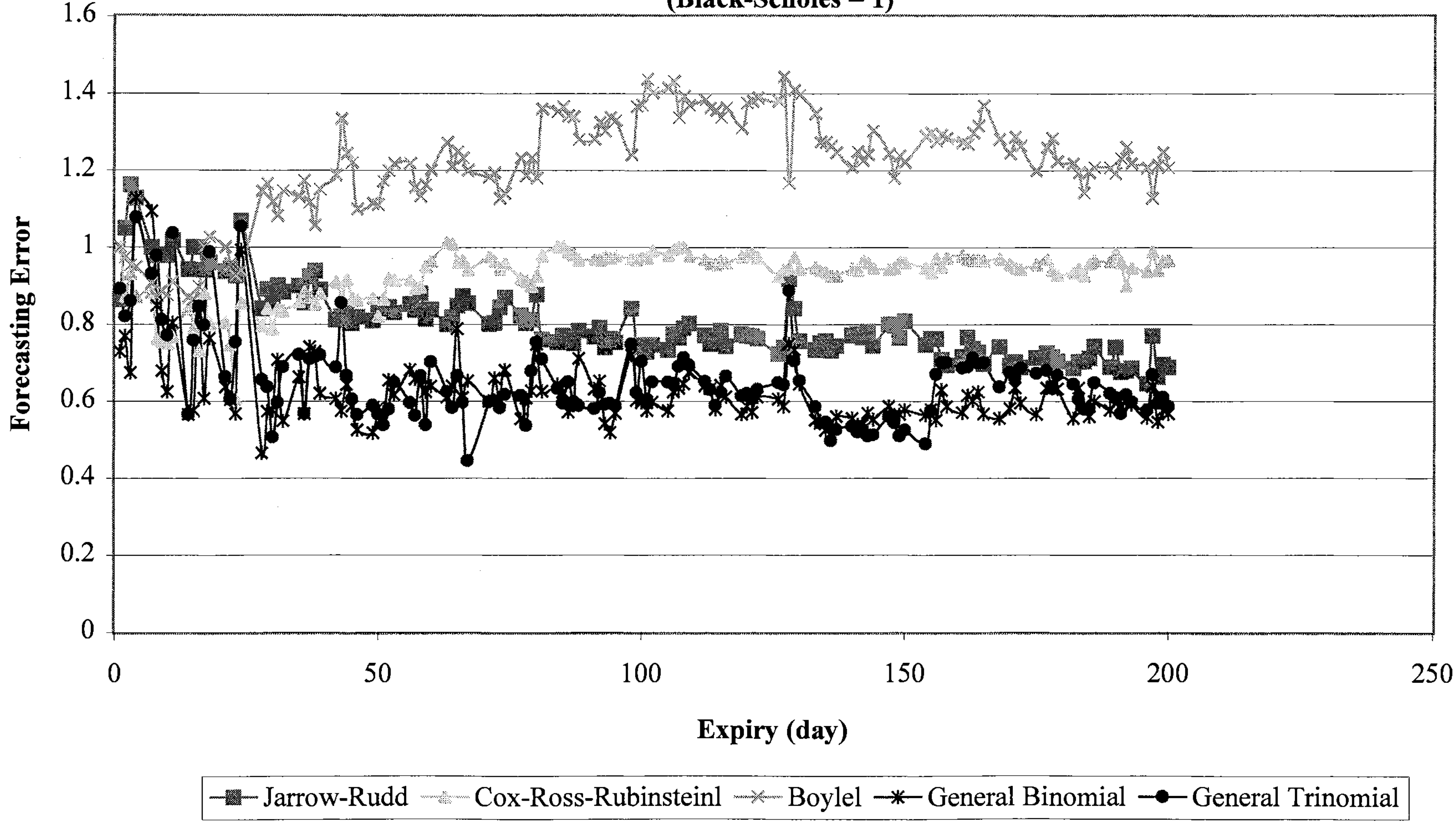


Figure 4. Forecasting Error Comparison for 1998 September Soybean Contract
(Black-Scholes = 1)

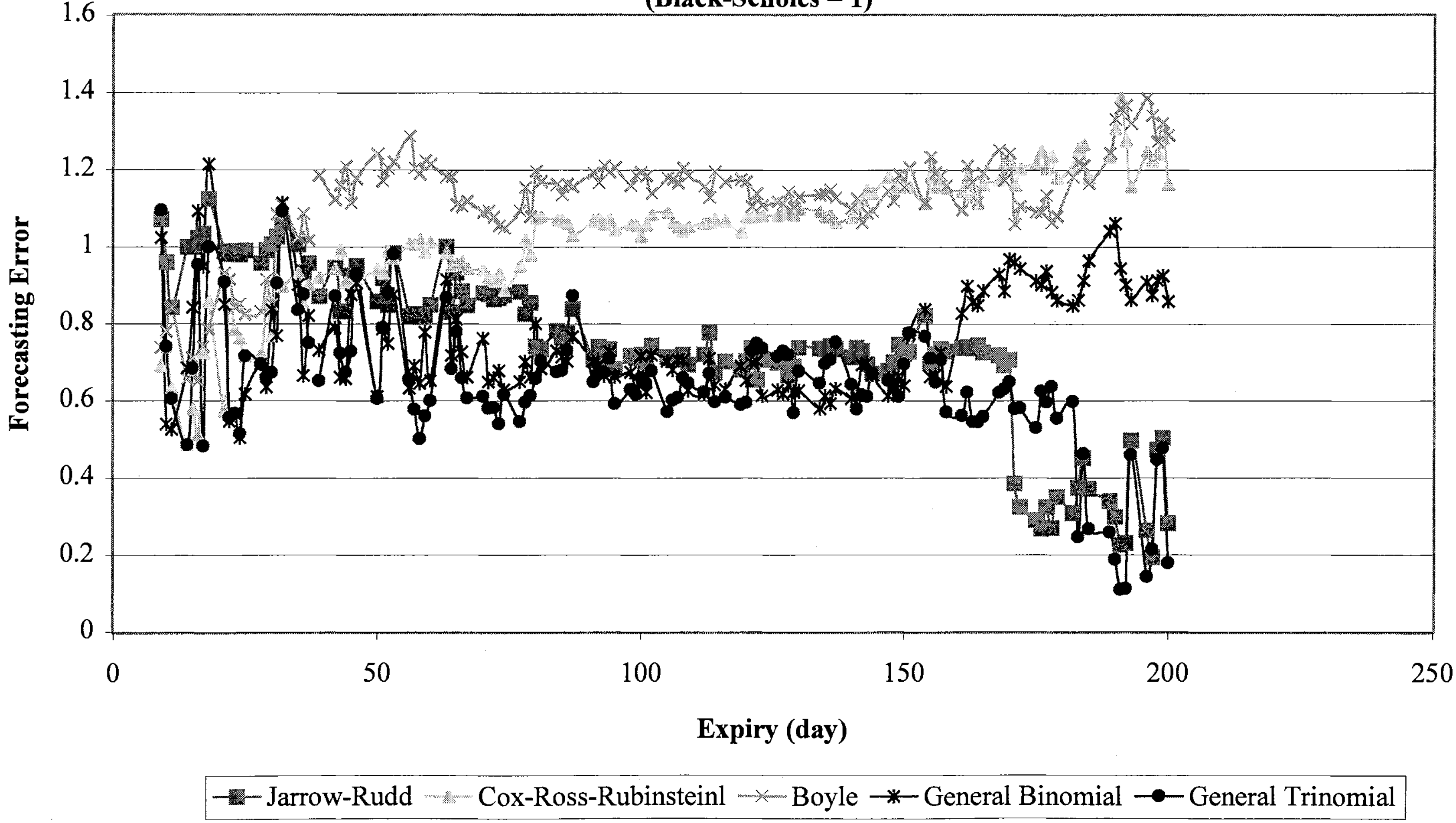


Figure 5. Forecasting Error Comparison for 1998 July Wheat Contract
(Black-Scholes = 1)

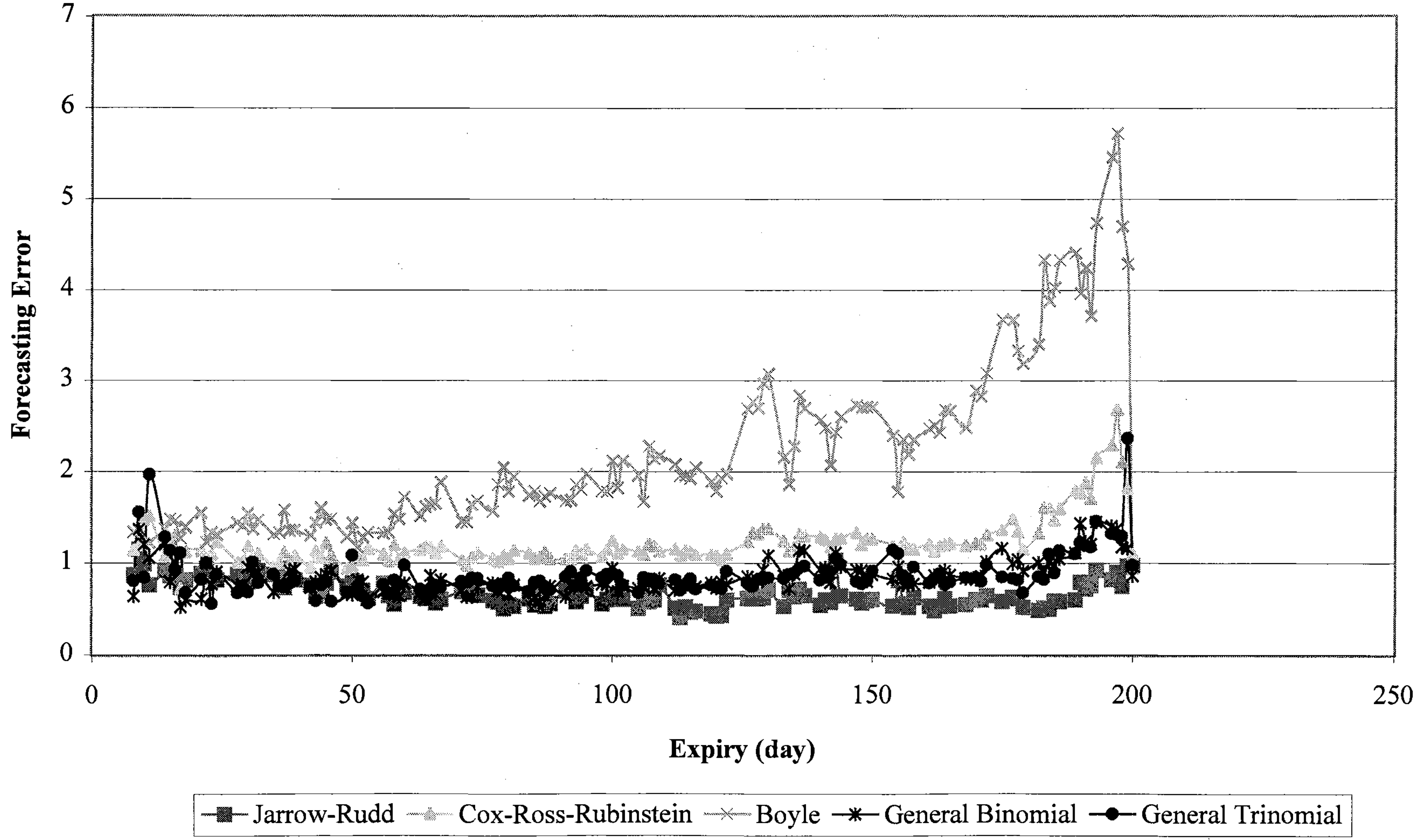


Table 8. Descriptive Statistics of Forecasting Errors of Various Models (%)

Commodity		Model					
		Black-Scholes	Jarrow-Rudd	Cox-Ross-Rubinstein	Boyle	General Binomial	General Trinomial
Corn	Mean	4.55	3.49	4.32	5.74	2.75	2.85
	Std	2.17	1.53	2.14	2.84	1.29	1.37
Soybean	Mean	8.46	5.60	9.33	9.83	6.36	5.64
	Std	4.26	2.74	5.10	5.00	3.48	4.00
Wheat	Mean	3.07	2.03	3.61	6.28	2.54	2.60
	Std	1.02	0.70	1.02	2.14	0.68	0.77

Note: The forecasting error is defined as $\dfrac{\sum_{i=1}^{N}\left|Call_{ti}-c(\theta_{t-1})\right|+\sum_{j=1}^{N}\left|Put_{tj}-p(\theta_{t-1})\right|}{\sum_{i=1}^{N}Call_{it}+\sum_{j=1}^{M}Put_{jt}}$, where $Call_{ti}$ is the premium for the i-th call option contract at the t-th day, Put_{tj} is the premium for the j-th put option contract at the t-th day, $c(\theta)$ and $p(\theta)$ are the pricing formulas for call and put options respectively with θ as the parameter vector in the formulas.

In summary, the general binomial and trinomial trees are more accurate than the Black-Scholes formula. Between the general binomial and trinomial trees, the general binomial tree performs as accurate as the general trinomial tree. This suggests that for these grain futures, capturing skewness was important, but kurtosis was not. So see this, the skewness and kurtosis of the three commodity futures price ratios, calculated by using the implied parameters and Gaussian quadrature equations, are graphed in Figure 6 to Figure 11. In view of the fact that the binomial tree is simpler than the trinomial tree, the former should be more favorable than the latter. Since the Cox-Ross-Rubinstein binomial and the Boyle trinomial trees are always the two poorest among the six models examined, they should never be used to imply the volatility parameter. Such results are consistent with intuition. Since some constraints in the Cox-Ross-Rubinstein and the Boyle models are artificial, there must be some correct relationship among the parameters being violated. Since more artificial constraints are added in the Boyle trinomial, its forecasting

Figure 6. Skewness of 1998 July Corn Futures

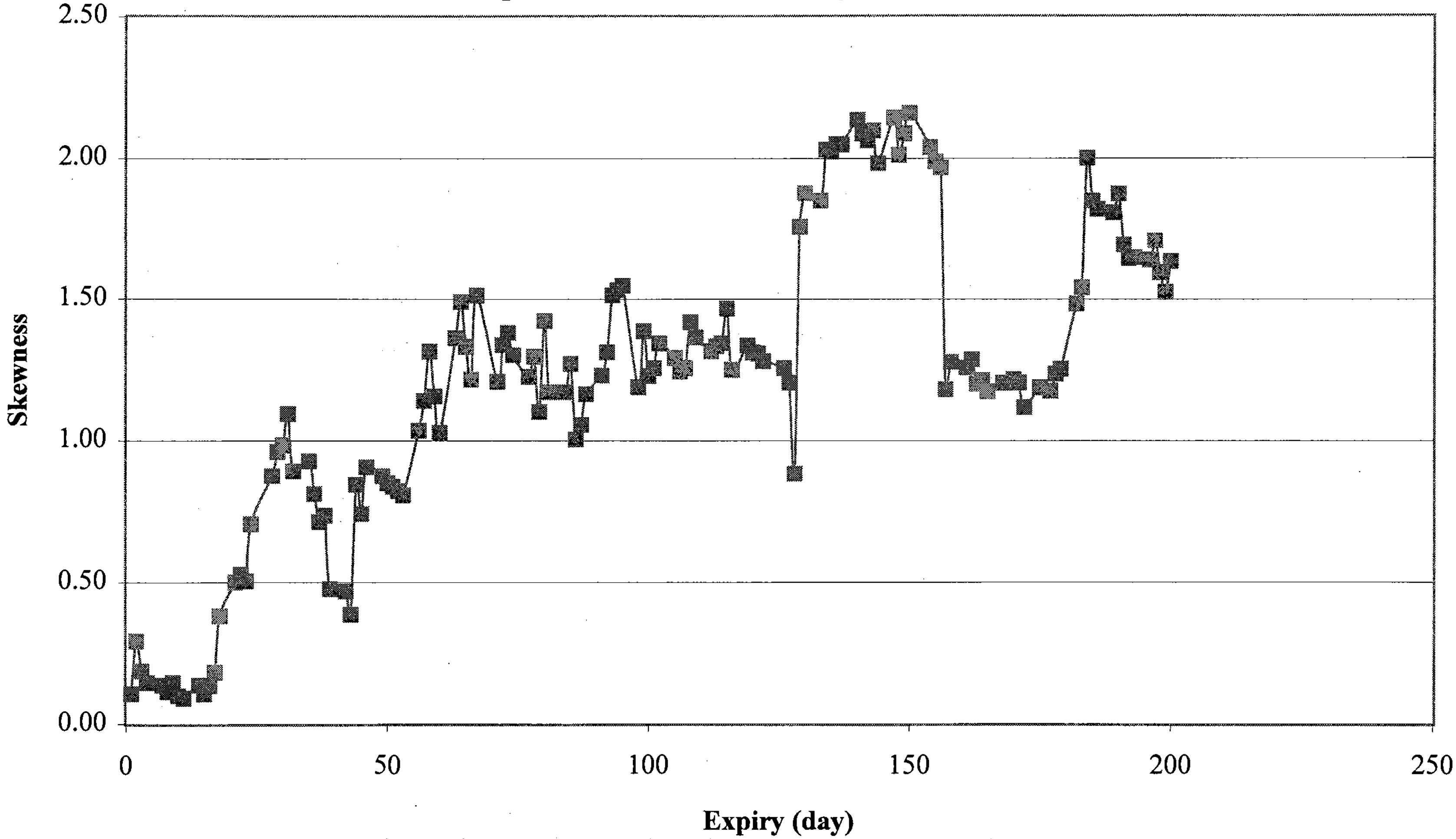


Figure 7. Kurtosis of 1998 July Corn Futures

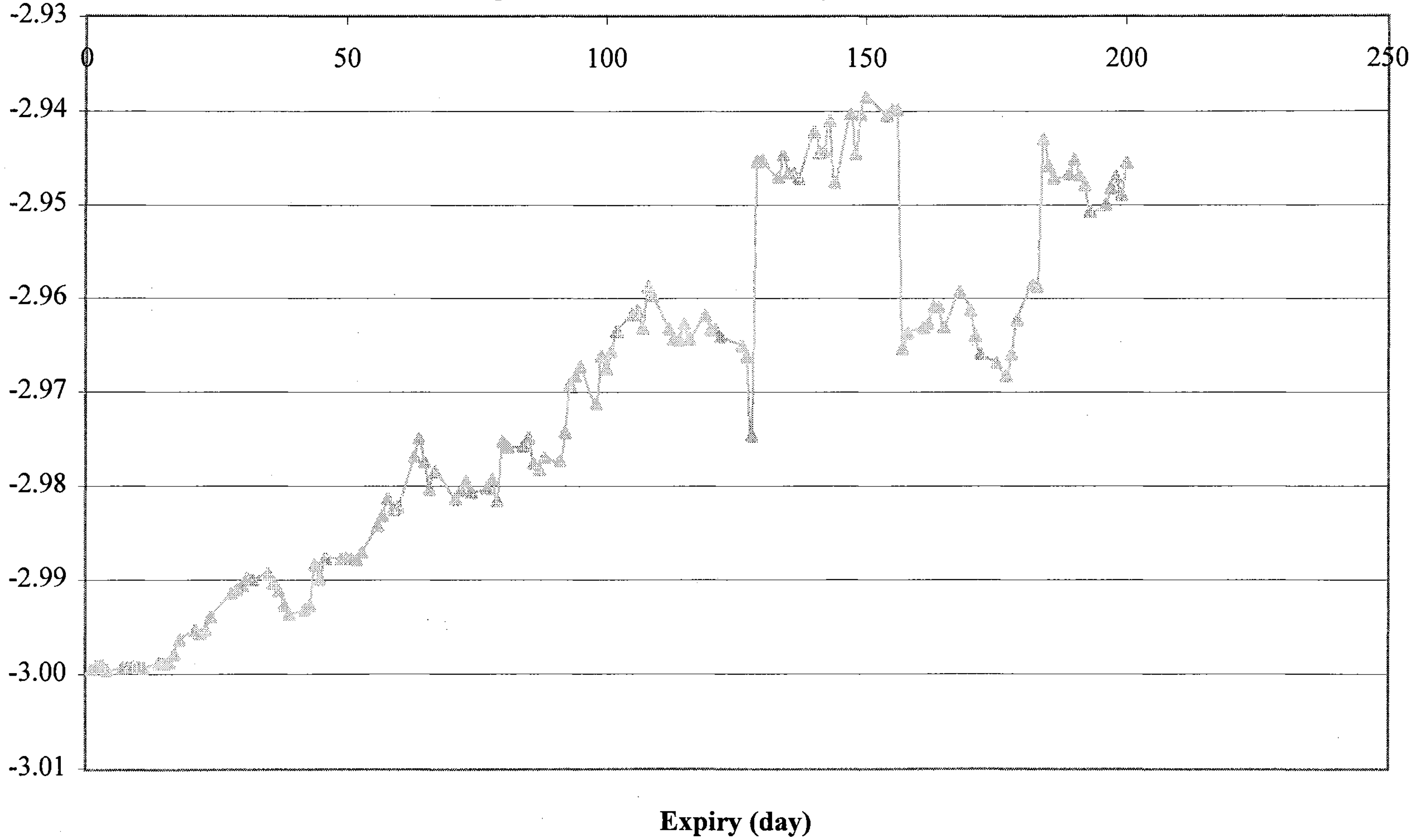


Figure 8. Skewness of 1998 September Soybean Futures

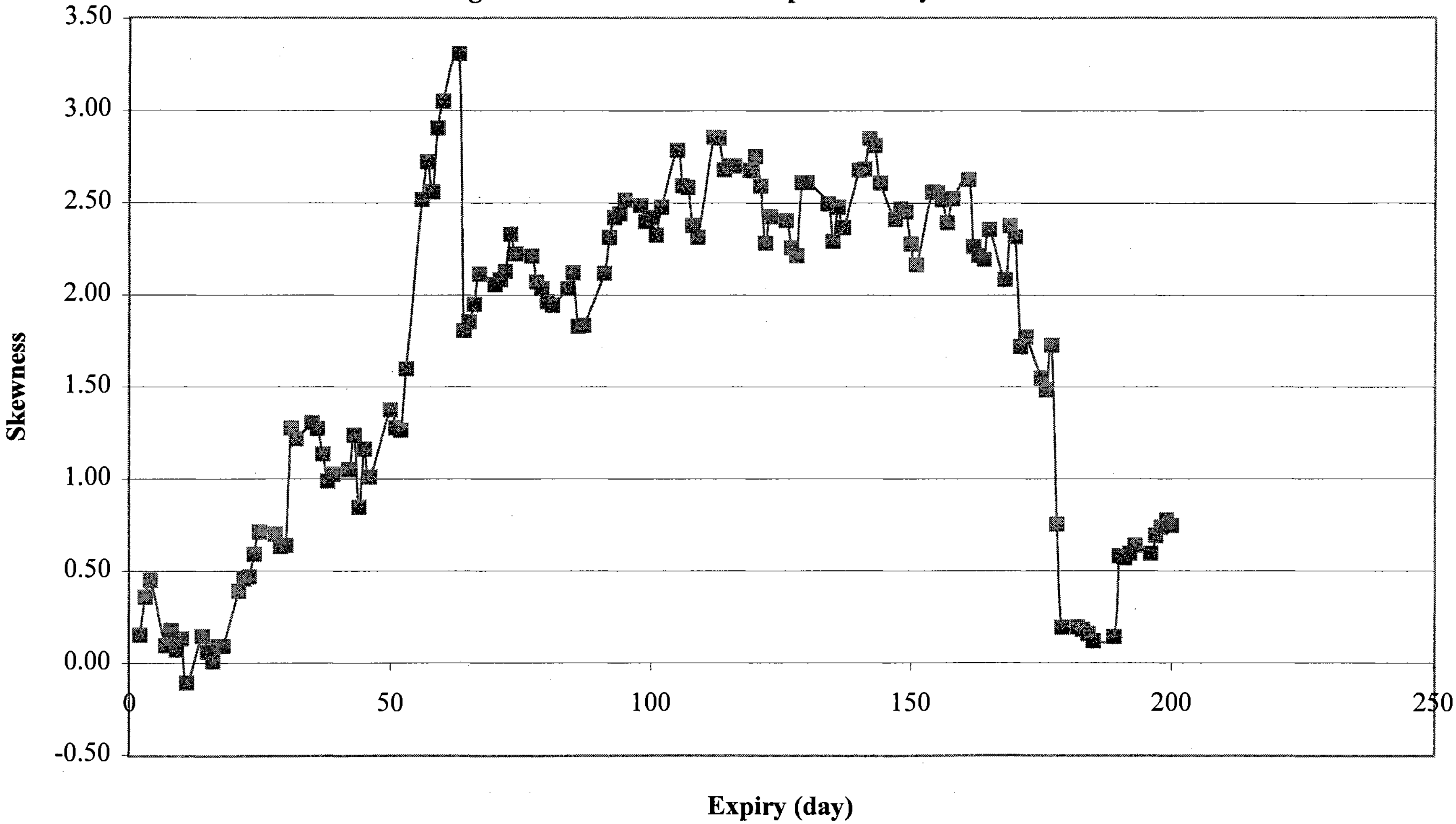


Figure 9. Kurtosis of 1998 September Soybean Futures

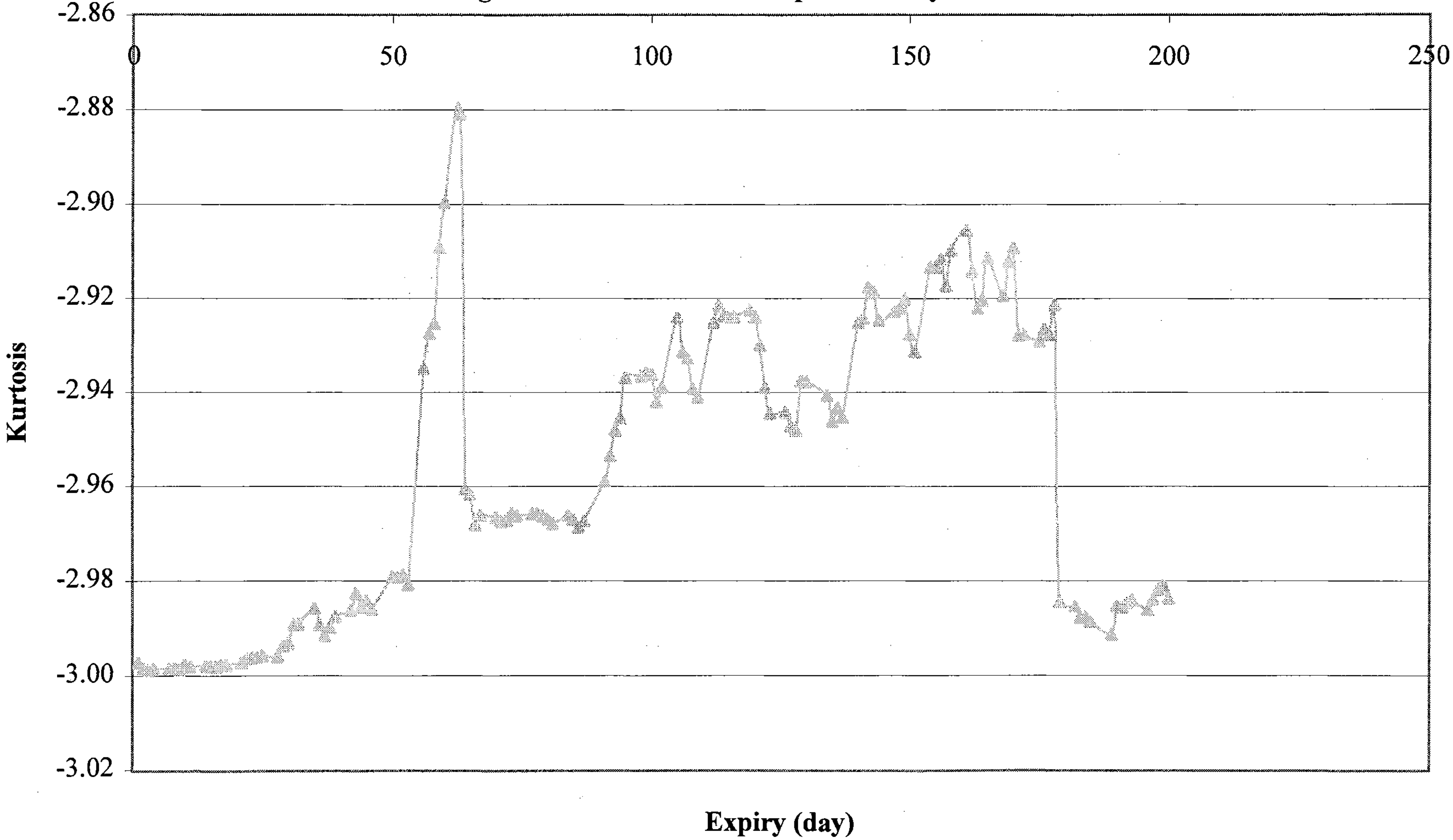


Figure 10. Skewness of 1998 July Wheat Futures

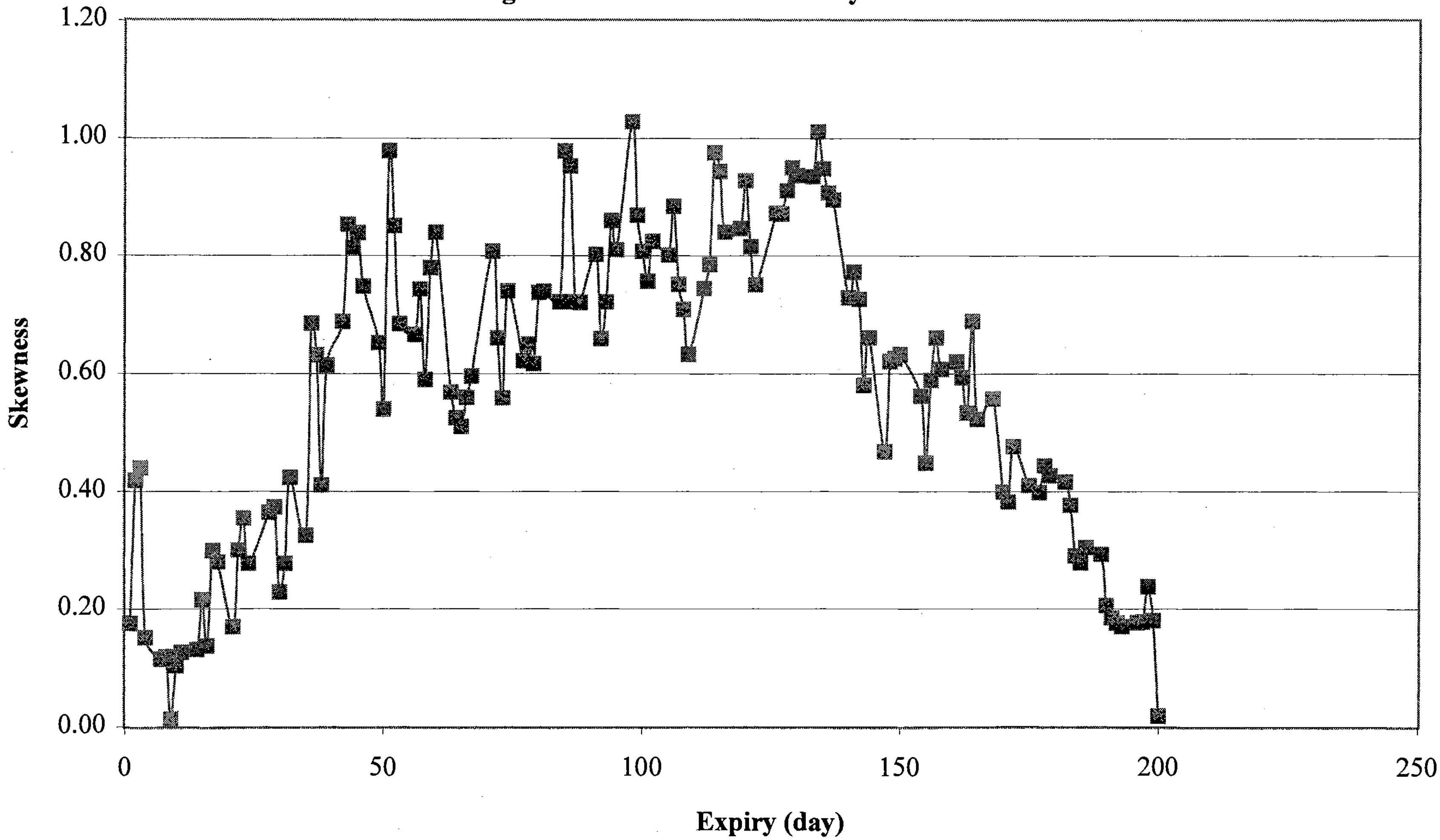
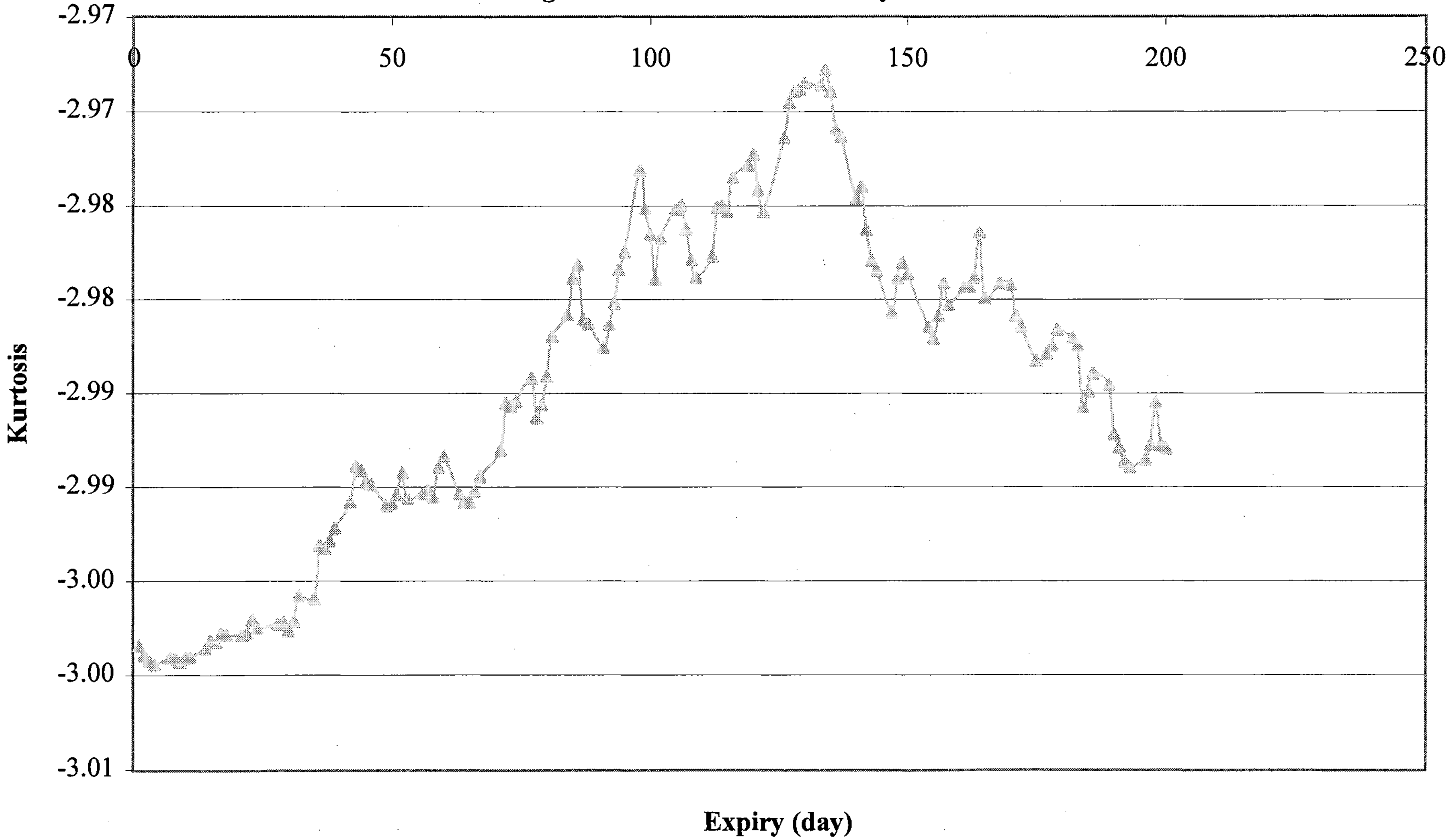


Figure 11. Kurtosis of 1998 July Wheat Futures



performance is worse than the less restricted one, the Cox-Ross-Rubinstein binomial tree model. It is clear that, to imply the parameters in the option pricing models and use the implied parameters to predict the option prices in the future, the Jarrow-Rudd model is always more accurate than the Black-Scholes. The fact that Jarrow-Rudd model has accuracy similar to the general binomial and trinomial trees offers further evidence that the bias in Black-Scholes for these data series is in ignoring skewness rather than ignoring kurtosis or any American option premium.

CHAPTER FIVE

CONCLUSIONS

The inaccuracy problem of the Black-Scholes formula is due in part to assuming a lognormal distribution and the fact that the formula is only applicable to European options while most options in practice are American. Previously developed option pricing models can handle one of the inaccuracy sources, but not both. Specifically, the Jarrow-Rudd model allows arbitrary underlying distributions, but is not suitable for American options. The Cox-Ross-Rubinstein binomial tree model and the Boyle trinomial tree model can handle American options, but are not suitable when the underlying distribution is non-lognormal. The advantage of the general binomial tree and the general trinomial tree option pricing models developed in this study is that they can value American options with arbitrary underlying asset price distributions.

The general binomial tree and the general trinomial tree models developed in this study have the same lattices as the previous binomial and trinomial trees. The unnecessary ad hoc restrictions imposed in previous studies are dropped. This study derives the parameters based on Gaussian quadrature. This allows calculating the parameters in the binomial tree and the trinomial tree based on the parameters of the underlying distribution. Implied moments can also be calculated directly from implied parameters. The solution of a Gaussian quadrature equation system is a standard method of numerical quadrature, but past binomial and trinomial models have been developed without using the power and generality of Gaussian quadrature. Tian's binomial and trinomial trees are also based on Gaussian quadrature. However, Tian's tree models only apply to lognormally distributed underlying asset prices and thus are only special cases of the general tree models. Moreover, Tian presented an analytical solution for his trinomial tree model based on the moments up to the fourth and a recombining condition, but did not explain how the analytical solution is derived. It is not practical to find an analytical solution for such an equation system in the general trinomial case. For the general trinomial tree model, this study uses a numerical solution based on nonlinear programming, but other ways of solving a system of nonlinear equations could also work.

Using a Taylor series expansion, this study shows that the general binomial tree model converges to a third order partial differential equation. The well-known Black-Scholes equation can be considered as a special case of this general differential equation. Furthermore, using the derivation similar to the method in Cox-Ross-Rubinstein (1979), this study expresses the general binomial tree formula for a call option with a fixed period of time before expiration as a linear combination of two binomial distribution functions. Then, by the central limit theorem, it is shown that the asymptotic limit of the general binomial tree model is a linear combination of two standard normal distribution functions with a form similar to the Black-Scholes formula. Generally, the arguments in the two standard normal distribution functions are unknown so that the limit of the general binomial tree is generally different from the Black-Scholes formula. When the underlying asset prices follow a lognormal distribution, the limit of the general binomial tree reduces to the Black-Scholes.

The numerical accuracy and efficiency of the general tree models was measured for examples where the true premium was known since it could be obtained analytically. Under the lognormal distribution, where the Black-Scholes formula gives the exact premium, the general tree models are at least as accurate as, or more accurate than, the other tree models. Under Merton's jump-diffusion process, with Merton's jump-diffusion model as the criterion, the general binomial and trinomial tree models are much more accurate than the other tree models. The comparison results under these two underlying asset price processes demonstrate that the superiority of the general tree models over the previous tree models is the "insensitivity to distribution". Though, strictly speaking, the calculation for the general trinomial tree is little less efficient compared to the previous trinomial models, this one-second more time spending for a 500-step tree can be ignored in practice.

Next, the parameters in the option pricing models were implied by using real data. Then the parameters implied today were used to forecast tomorrow's option premiums. The empirical results from this examination show that, for the futures options on corn, soybean and wheat, the general tree models always forecast option premiums more accurately than the Cox-Ross-Rubinstein binomial model and the Boyle trinomial model. The general tree models are more accurate than the previous tree models because dropping the unnecessary constraints allows capturing skewness with the general binomial and skewness as well as kurtosis with the general trinomial models. While these constraints make the algorithm more convenient, they may violate some correct relationship between discrete moments and continuous moments. The impact of the incorrect constraints on the forecasting accuracy can also be detected by watching the accuracy difference between the Cox-Ross-Boyle binomial model and the Boyle trinomial model. The Boyle trinomial model predicts the option premiums less accurately than the Cox-Ross-Rubinstein binomial model because there are more additional constraints in the Boyle than in the Cox-Ross-Rubinstein. The forecasting accuracy of the Jarrow-Rudd model verses the general tree models can be explained by the different sources of pricing biases. The bias of the Jarrow-Rudd model is mainly from valuing American options by the European option pricing model, while the truncation errors in numerical option pricing models is the main bias source of the general binomial and trinomial trees. Thus, relative accuracy of the Jarrow-Rudd to the general trees depends on the magnitude of the "American-European" error compared to the truncation error in the general trees. The empirical results on the three commodity futures options show that there is no consistent pattern about the accuracy comparison between the Jarrow-Rudd and the general tree models. Sometimes, the Jarrow-Rudd may forecast premiums more accurately than the general trees. Other times, the general trees may be more accurate. The Jarrow-Rudd model having accuracy close to that of the general tree models suggests that for this set of data, the lognormal distribution assumption is a more important source of error in the Black-Scholes model than using an European option pricing model to price American options.

This study contributes to theory about the binomial tree and trinomial tree models in several aspects. It provides a general way to derive binomial and trinomial tree models for any arbitrary distribution with finite moments. By approaching the pricing problem as a dynamic programming problem using Gaussian quadrature, the model can much more easily handle varied situations than present restricted models that are not as closely linked with the vast numerical analysis literature. A general differential equation is shown to be the limit of the binomial tree model. The limit of the general binomial tree has a general closed formula, which is similar to the Black-Scholes. Within this theoretical framework, all previous binomial and trinomial option pricing models can be considered as special cases.

The results of the numerical analysis and the empirical study show two practical ways to use the general binomial tree and general trinomial tree models. In practice, the choice between the two applications depends on the information available. When the standard option contracts on an asset are not available, but some information about the underlying distribution is available, so that the moments are available, we can use the moments to determine the parameters in a general tree model with Gaussian quadrature. Then the parameters determined by Gaussian quadrature can be substituted into the general tree to value the non-standard options. More attractively, even if we do not know the distribution form of the underlying prices, we can still determine the parameters in a general tree by using Gaussian quadrature equations as long as the moments of the underlying prices can be estimated. On the other hand, if some standard option contracts on an asset are available, we can use a general tree model with the observed option premiums to imply the parameters. Then, with the implied parameters, a non-standard option on the same underlying asset can be valued by the general tree. The empirical evidence in research about other futures already showed the superiority of the parameter implied by option data (Jorion). Also, the parameters implied by a general binomial or trinomial model provide an alternative to other methods (for instance, O'Brien, Hayenga, and Babcock) to forecast probability distributions of commodity futures prices.

One of the great potential places to use the general tree models is energy market. Energy markets are being transformed by derivatives and risk management. What makes energies so different from the traditional financial markets is the excessive number of fundamental price drivers, which cause extremely complex price behavior. This complexity frustrates people's ability to capture the essence of the market by specific distributions. In addition, due largely to the needs of end users, energy contracts often exhibit a complexity of price averaging and customized characteristics of commodity delivery. All these pricing difficulties present a terrific challenge to quantitative analysts and risk managers in the energy markets. A likely use of the general tree models is to derive moments from an actively traded market and use these moments in pricing over-the-counter derivatives. Without requiring information about the underlying distributions, the general binomial and trinomial trees developed in this study provide an alternative method that captures the price behaviors, such as volatility, skewness and kurtosis, into a quantitative models that is also simple enough for quick and efficient everyday use on trading desks.

We already know that previous binomial option pricing models are based on the lognormal distribution. When the number of the steps in those binomial models is large enough, the binomial models converge to the Black-Scholes formula, the closed form model based on the lognormal distribution. By analogy, a possible limit of the general binomial tree model is the Jarrow-Rudd approximate formula, because both the general binomial tree model and the Jarrow-Rudd model are based on arbitrary underlying distributions. This study derived a general closed form as the limit of the general binomial tree model. Subsequent theoretical research may want to investigate the relationship between the Jarrow-Rudd formula and the limit of the general tree models.

In summary, the tree option pricing models developed in this study are general not only in theory but also, more importantly, in practice. Previous work about futures options of agricultural commodities have shown that options can be part of an optimal marketing strategy (Catlett, and Boehlje; Frank, Irwin, Pfeiffer, and Curtis; Hauser, and Liu; Sakong, Hayes, and Hallam). The empirical results based on the general tree models in this study contribute to the knowledge about agricultural futures. The general tree models are superior over the previous option pricing models because they are appropriate for various availability of the information about the underlying distributions and, at the same time, reserve all the advantages of previous tree models. It can be expected that these versatile general models will become a choice of modeling tool for the practitioners in investment and risk management, especially in young derivatives markets such as weather markets.

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