AN APPROACH TO MODELLING THE COEFFICIENT OF VARIATION IN FACTORIAL EXPERIMENTS

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CHAPTER ONE

INTRODUCTION

As the ratio of the sample standard deviation to the sample mean, the sample coefficient of variation (CV) provides a useful and unitless measure of relative variability. As Ahmed (1994) notes, the CV can sometimes be more relevant than the standard deviation alone, such as when the precision of measuring instruments or the volatility of stocks is considered. Hurlimann (1995) points out that the CV is useful in insurance risk assessment as a measure of the heterogeneity of insurance portfolios. Williams (1991) cites the importance of the CV in the determination of detection limits in instrumental analysis. Feltz and Miller (1996) notes that in medical studies, the CV often determines the feasibility of combining results from separate clinical trials.

Payton (1997) suggests that the types of populations for which the CV has relevance are those which are of the ratio type. In such populations, an observation equal to zero represents the absence of the measured characteristic, such as with populations of volumes, yields, or weights, since only in this context does the CV ratio itself have meaning. Negative observations are not possible.

Although theoretically not of the ratio type, normal populations have long been considered in connection with the behavior of sample CVs. In such cases, negative sample means are assumed to be highly improbable. However, in contrast with the mean of the normal distribution, comparatively little work has been done in connection with hypothesis tests and confidence intervals for unknown population CVs based on observed data. Papers which have addressed these subjects for a single population CV include Koopmans, et al. (1964), Vangel (1996), and Payton (1997), which utilize exact and approximate distributions of the sample CV from a normal population. Tests for the equality of k normal population CVs that employ approximate distributions and the normal density include Bennett (1976), Miller and Karson (1977), Doornbos and Dijkstra (1983), and Shafer and Sullivan (1986). Gupta and Ma (1996) extends a Wald test developed by Rao and Vidya (1992) for two populations based on the normal density to k populations and introduces a score test which also utilizes the actual density of the observations. A test based on the asymptotic moments of the CV is provided by Feltz and Miller (1996).

Less work has addressed the analysis of population CVs in the context of designed factorial experiments. Taguchi (1992) discusses a well-known approach to the analysis of product quality using fractional factorial designs that often models a log-transformed CV. However, his approach has yielded recent criticisms (see, for example, Box, 1988) and corrections because of biased tests of factor effects. More recent work by McCullagh and Nelder (1989) and Nelder and Lee (1991) has utilized models of the CVs of gamma populations within a larger theory of joint modelling of mean and dispersion in designed industrial experiments. An alternative approach to modelling the CVs of gamma distributions from a sociological standpoint is provided by Eliason (1993).

Absent from the current literature, however, is a technique for constructing factorial models of the CVs of normal populations that makes use of known approximate distributions and asymptotic moments of the sample CV. The current work addresses this situation by first establishing a proper structure for a model of population CVs in a general setting. Next, the theory of generalized linear modelling is applied in the context of maximum- and quasi-likelihood estimation to achieve a simplified iterative algorithm for estimation of model parameters that parallels methods currently used to fit models in categorical data analysis. An application of model diagnostics like those used in categorical analysis is proposed, and simulations to investigate the power of these diagnostics in the context of the approximate distributions and asymptotic moments are discussed. The effects of departures from the normal assumption also are determined.

CHAPTER TWO

REVIEW OF LITERATURE

In this chapter, several approximations to the exact distribution of the sample CV when data are drawn from a normal population are discussed, and comparisons to the exact distribution are made. Several one-factor tests for the equality of k normal population CVs currently in the literature are reviewed, and variations of the Taguchi approach, which often implicitly models a log-transformed CV in a (fractional) factorial design, are summarized.

Terminology and Definitions

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population with $E(X_i) = \mu > 0$ and $Var(X_i) = \sigma^2$, $i = 1, 2, \ldots, n$, and let $R = \sigma / \mu$ be the population CV. Define $\overline{X} = \sum_{i=1}^{n} X_i / n$ to be the sample mean and assume that $P(\overline{X} < 0)$ is negligible. Let $S^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2 / (n-1)$ and $S_n^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2 / n$ be the unbiased and maximum-likelihood estimates of σ^2, respectively, and let $r = S / \overline{X}$ and $r_n = S_n / \overline{X}$ be the corresponding point estimates of R. Note that r_n is the maximum-likelihood estimate of R and that $r_n = ((n-1)/n)^{1/2} r$. Although neither r nor r_n is an unbiased estimate of R, both are strongly consistent; that is, $P\left(\lim_{n\to\infty} r = R\right) = P\left(\lim_{n\to\infty} r_n = R\right) = 1$ (Serfling, 1980, pp. 24-26, 136-137). Hence, both are reasonable estimators of R, particularly when computed from large samples.

For later convenience, define the h-function $h(x) = x^2 / (1 + x^2)$ for $x > 0$. Then h has an inverse, and $h^{-1}(x) = \left(x/(1-x)\right)^{1/2}$ for $0 < x < 1$. Additionally, define a random variable Y to have the gamma distribution with parameters λ and ν if and only if its density is given by

$$f(y) \;=\; \frac{1}{y\Gamma(\nu)}\left(\frac{\nu y}{\lambda}\right)^{\nu} \exp\left(-\frac{\nu y}{\lambda}\right), \quad y \geq 0$$

$$\;=\; 0, \quad y < 0,$$

where $\lambda > 0$, $\nu > 0$, and $\Gamma(\bullet)$ is the gamma function. It follows that $E(Y) = \lambda$ and $Var(Y) = \lambda^2 / \nu$. The parameter ν is sometimes called the index (McCullagh and Nelder, 1989, p. 287).

Approximate Distributions of the Sample CV

Under normal theory, the exact distribution of r is a multiple ($\sqrt{n}$) of the inverse of a non-central t distribution having $(n-1)$ degrees of freedom and non-centrality parameter $\sqrt{n}/R$. The density of the non-central t for degrees of freedom p and non-centrality parameter q is given by Lehmann (1959, p. 200) as

$$f(t) = \left(2^{(p+1)/2}\Gamma(p/2)(\pi p)^{1/2}\right)^{-1}\int_0^{\infty} y^{(p-1)/2} \exp\left[-\frac{y}{2} - \frac{1}{2}\left(t\sqrt{\frac{y}{p}} - q\right)^2\right] dy, \quad (2.1)$$

for $-\infty < t < \infty$. Given the density of r, the density of r_n can be obtained, in turn, by transforming r according to $r_n = \left((n-1)/n\right)^{1/2} r$. Difficulties associated with direct application of the non-central t distribution itself have prompted the study of several approximations to the exact distributions of r and r_n.

McKay's and David's Approximations

McKay (1932) gives the earliest approximation to the distribution of r_n when samples are drawn from a normal population. By utilizing a contour-integral expression of the density of r_n, he is able to show that $nh(r_n) / h(R)$ has an approximate χ^2 distribution with (n - 1) degrees of freedom, provided that $R \in (0, 1/3)$. This requirement on R is consistent with the added assumption that negative observations also are highly improbable, in addition to a negative sample mean. Equivalently, $(n / (n-1))h(r_n)$ has an approximate gamma distribution with expectation h(R) and index (n - 1) / 2. Vangel (1996) observes that McKay utilizes an asymptotic approximation in his derivation, so that his approximation is, in fact, most accurate for large n, although its small sample properties also are very good.

David (1949) obtains an approximation to the distribution of r by reexpressing McKay's approximation in terms of r and deleting a negligible term. Beginning with $nh(r_n) / h(R)$, she writes

$$\frac{nh(r_n)}{h(R)} = \frac{n}{h(R)}\frac{r_n^2}{1+r_n^2} = \frac{n}{h(R)}\frac{\left(\frac{n-1}{n}\right)r^2}{1+\left(\frac{n-1}{n}\right)r^2}$$

$$= \frac{n-1}{h(R)} \frac{r^2}{1+r^2-\frac{r^2}{n}} \approx \frac{n-1}{h(R)} \frac{r^2}{1+r^2} = \frac{(n-1)h(r)}{h(R)},$$

since r^2 / n is typically close to zero for large n. She thus obtains that $(n - 1)h(r) / h(R)$ also has an approximate χ^2 distribution with (n - 1) degrees of freedom, or, equivalently, that h(r) is distributed approximately gamma with expectation h(R) and index (n - 1) / 2.

Iglewicz and Myers' Approximation

A third approximation for consideration is discussed by Iglewicz and Myers (1970). They derive asymptotic expansions for the moments of the exact distribution of r under normal theory and conclude that an adequate approximation for even relatively small n can be obtained by assuming that r itself is normally distributed with mean R and variance $\left(\frac{R^2}{n}\right)\left(R^2 + \frac{1}{2}\right)$. This variance was apparently given originally by Pearson (David, 1949). Both Serfling (1980, pp. 136-137) and Feltz and Miller (1996) note that r is, in fact, asymptotically normal with these same moments. Hence, an application of Slutsky's Theorem gives that r_n likewise possesses these asymptotic properties (Serfling, p. 19). Simulation results reported by Iglewicz and Myers suggest that this approximation is superior to other normal approximations with higher-order expansions for the mean and variance.

Comparisons to Exact Quantiles

Owen (1968) outlines a process to determine cumulative probabilities of the exact distribution of r based on the non-central t distribution. Making use of (2.1), he notes that, for c > 0,

$$P(r > c) = P(\frac{S}{\overline{X}} > c) = P(0 < \frac{\overline{X}}{S} < \frac{1}{c}) = P(0 < \frac{\sqrt{n}\,\overline{X}}{S} < \frac{\sqrt{n}}{c})$$

$$= P(0 < t < \frac{\sqrt{n}}{c}),$$

where t has the non-central t distribution with (n - 1) degrees of freedom and non-centrality parameter $\sqrt{n}$ / R. Hence,

$$P(r < c) = P(t < 0) + P(t > \frac{\sqrt{n}}{c}). \qquad (2.2)$$

Using (2.2), exact quantiles can be computed for r, from which quantiles for r_n can be obtained using $r_n = \left((n-1)/n\right)^{1/2} r$. Tables I through IX give selected exact quantiles for r and r_n, as well as corresponding quantiles for each of the approximate distributions discussed above. The SAS program used to calculate these quantiles is included in Appendix B.

The tables suggest that both McKay's and David's approximations perform very well, especially for smaller values of R and for large n. Iglewicz and Myers' approximation generally performs worse than David's approximation but improves for large n. All three approximations are less accurate as R increases. There is a clear disparity between Iglewicz and Myers' approximation and the exact distribution of r near the first and third quartiles, particularly for small n. David (1949) comments that the normal approximation to the distribution of r works best for values of n > 40.

TABLE I

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR R = 0.1 AND n = 10

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.04556	0.04551	0.04802	0.04798	0.04746
0.05	0.05753	0.05747	0.06064	0.06059	0.06285
0.10	0.06444	0.06437	0.06792	0.06787	0.07106
0.20	0.07325	0.07318	0.07722	0.07716	0.08099
0.30	0.07989	0.07981	0.08422	0.08416	0.08816
0.40	0.08575	0.08566	0.09038	0.09033	0.09428
0.50	0.09135	0.09126	0.09630	0.09624	0.10000
0.60	0.09709	0.09700	0.10234	0.10230	0.10572
0.70	0.10337	0.10326	0.10896	0.10891	0.11184
0.80	0.11088	0.11077	0.11688	0.11684	0.11901
0.90	0.12158	0.12146	0.12816	0.12814	0.12894
0.95	0.13065	0.13053	0.13772	0.13772	0.13715
0.99	0.14821	0.14806	0.15622	0.15626	0.15254

IM = Iglewicz and Myers' Approximation

TABLE II

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION
OF r AND r_n FOR R = 0.1 AND n = 50

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.04556	0.04551	0.04802	0.04798	0.04746
0.05	0.08226	0.08225	0.08310	0.08309	0.08339
0.10	0.08572	0.08570	0.08659	0.08658	0.08706
0.20	0.08997	0.08995	0.09088	0.09087	0.09150
0.30	0.09309	0.09307	0.09403	0.09402	0.09470
0.40	0.09578	0.09576	0.09675	0.09674	0.09744
0.50	0.09832	0.09830	0.09932	0.09931	0.10000
0.60	0.10089	0.10087	0.10192	0.10191	0.10256
0.70	0.10366	0.10364	0.10472	0.10471	0.10530
0.80	0.10694	0.10692	0.10803	0.10802	0.10850
0.90	0.11154	0.11152	0.11268	0.11267	0.11294
0.95	0.11540	0.11537	0.11657	0.11656	0.11661
0.99	0.12274	0.12271	0.12398	0.12398	0.12349

IM = Iglewicz and Myers' Approximation

TABLE III

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR R = 0.1 AND n = 100

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.08309	0.08308	0.08350	0.08350	0.08339
0.05	0.08768	0.08768	0.08813	0.08812	0.08825
0.10	0.09017	0.09017	0.09063	0.09062	0.09085
0.20	0.09323	0.09322	0.09370	0.09369	0.09399
0.30	0.09545	0.09544	0.09593	0.09592	0.09626
0.40	0.09736	0.09735	0.09785	0.09785	0.09819
0.50	0.09917	0.09916	0.09966	0.09966	0.10000
0.60	0.10098	0.10097	0.10149	0.10148	0.10181
0.70	0.10293	0.10292	0.10345	0.10345	0.10374
0.80	0.10524	0.10523	0.10577	0.10576	0.10601
0.90	0.10846	0.10845	0.10901	0.10900	0.10915
0.95	0.11115	0.11114	0.11171	0.11170	0.11175
0.99	0.11625	0.11624	0.11683	0.11683	0.11661

IM = Iglewicz and Myers' Approximation

TABLE IV

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR R = 0.2 AND n = 10

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.09034	0.08997	0.09523	0.09488	0.09188
0.05	0.11428	0.11382	0.12046	0.12006	0.12355
0.10	0.12816	0.12764	0.13509	0.13467	0.14044
0.20	0.14594	0.14536	0.15384	0.15340	0.16088
0.30	0.15941	0.15878	0.16803	0.16760	0.17563
0.40	0.17132	0.17065	0.18059	0.18017	0.18823
0.50	0.18280	0.18208	0.19268	0.19228	0.20000
0.60	0.19458	0.19382	0.20511	0.20473	0.21177
0.70	0.20754	0.20673	0.21877	0.21843	0.22437
0.80	0.22315	0.22229	0.23522	0.23496	0.23912
0.90	0.24560	0.24465	0.25888	0.25875	0.25956
0.95	0.26483	0.26382	0.27915	0.27917	0.27645
0.99	0.30263	0.30151	0.31900	0.31943	0.30812

IM = Iglewicz and Myers' Approximation

TABLE V

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR R = 0.2 AND n = 50

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.15103	0.15089	0.15257	0.15246	0.15165
0.05	0.16385	0.16371	0.16551	0.16541	0.16581
0.10	0.17087	0.17073	0.17261	0.17251	0.17336
0.20	0.17955	0.17940	0.18137	0.18128	0.18251
0.30	0.18592	0.18577	0.18781	0.18772	0.18910
0.40	0.19144	0.19129	0.19339	0.19330	0.19473
0.50	0.19667	0.19651	0.19866	0.19858	0.20000
0.60	0.20195	0.20179	0.20400	0.20393	0.20527
0.70	0.20767	0.20751	0.20978	0.20971	0.21090
0.80	0.21445	0.21430	0.21663	0.21657	0.21749
0.90	0.22401	0.22386	0.22629	0.22625	0.22664
0.95	0.23205	0.23189	0.23440	0.23437	0.23419
0.99	0.24744	0.24729	0.24996	0.24996	0.24835

IM = Iglewicz and Myers' Approximation

TABLE VI

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR R = 0.2 AND n = 100

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.16548	0.16539	0.16631	0.16625	0.16581
0.05	0.17483	0.17475	0.17571	0.17566	0.17583
0.10	0.17991	0.17983	0.18082	0.18077	0.18117
0.20	0.18615	0.18607	0.18709	0.18704	0.18763
0.30	0.19071	0.19063	0.19167	0.19162	0.19229
0.40	0.19464	0.19456	0.19562	0.19557	0.19628
0.50	0.19834	0.19826	0.19934	0.19930	0.20000
0.60	0.20207	0.20200	0.20309	0.20306	0.20372
0.70	0.20610	0.20603	0.20714	0.20711	0.20771
0.80	0.21086	0.21079	0.21192	0.21190	0.21237
0.90	0.21754	0.21747	0.21863	0.21862	0.21883
0.95	0.22312	0.22305	0.22424	0.22423	0.22417
0.99	0.23376	0.23370	0.23493	0.23494	0.23419

IM = Iglewicz and Myers' Approximation

TABLE VII

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR $R = 0.3\bar{3}$ AND n = 10

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.14770	0.14603	0.15568	0.15411	0.14164
0.05	0.18756	0.18546	0.19770	0.19586	0.19779
0.10	0.21090	0.20855	0.22230	0.22037	0.22773
0.20	0.24110	0.23845	0.25415	0.25215	0.26398
0.30	0.26422	0.26134	0.27851	0.27653	0.29012
0.40	0.28488	0.28180	0.30029	0.29837	0.31246
0.50	0.30496	0.30170	0.32145	0.31964	0.33333
0.60	0.32581	0.32236	0.34343	0.34178	0.35421
0.70	0.34899	0.34536	0.36787	0.36648	0.37655
0.80	0.37735	0.37350	0.39776	0.39679	0.40268
0.90	0.41899	0.41486	0.44165	0.44154	0.43894
0.95	0.45560	0.45127	0.48024	0.48115	0.46887
0.99	0.53048	0.52591	0.55918	0.56308	0.52503

IM = Iglewicz and Myers' Approximation

TABLE VIII

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR $R = 0.3\bar{3}$ AND $n = 50$

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.24862	0.24787	0.25115	0.25054	0.24760
0.05	0.27058	0.26982	0.27333	0.27276	0.27272
0.10	0.28270	0.28194	0.28557	0.28503	0.28611
0.20	0.29777	0.29702	0.30079	0.30031	0.30232
0.30	0.30891	0.30817	0.31205	0.31160	0.31401
0.40	0.31861	0.31789	0.32185	0.32145	0.32400
0.50	0.32784	0.32713	0.33117	0.33081	0.33333
0.60	0.33721	0.33653	0.34064	0.34034	0.34267
0.70	0.34742	0.34677	0.35095	0.35072	0.35266
0.80	0.35960	0.35898	0.36325	0.36311	0.36435
0.90	0.37691	0.37636	0.38074	0.38074	0.38056
0.95	0.39159	0.39112	0.39557	0.39571	0.39395
0.99	0.42012	0.41982	0.42438	0.42485	0.41906

IM = Iglewicz and Myers' Approximation

TABLE IX

EXACT AND APPROXIMATE QUANTILES OF THE DISTRIBUTION OF r AND r_n FOR R = $0.3\bar{3}$ AND n = 100

Quantile	Exact r_n	McKay	Exact r	David	IM
0.01	0.27322	0.27273	0.27459	0.27420	0.27271
0.05	0.28939	0.28893	0.29085	0.29050	0.29047
0.10	0.29823	0.29778	0.29973	0.29941	0.29994
0.20	0.30912	0.30870	0.31068	0.31041	0.31140
0.30	0.31712	0.31672	0.31872	0.31848	0.31967
0.40	0.32404	0.32367	0.32568	0.32547	0.32673
0.50	0.33060	0.33024	0.33226	0.33209	0.33333
0.60	0.33722	0.33689	0.33892	0.33878	0.33994
0.70	0.34440	0.34410	0.34614	0.34604	0.34700
0.80	0.35291	0.35265	0.35469	0.35465	0.35526
0.90	0.36493	0.36473	0.36677	0.36681	0.36673
0.95	0.37504	0.37489	0.37693	0.37705	0.37619
0.99	0.39447	0.39445	0.39646	0.39675	0.39395

IM = Iglewicz and Myers' Approximation

One-Factor Tests for Population CVs

Suppose $X_{i1}, X_{i2}, \ldots, X_{in_i}$, i = 1, 2, ..., k are independent random samples from k normal populations having $E(X_{ij}) = \mu_i > 0$, $Var(X_{ij}) = \sigma_i^2$, and CVs $R_i = \sigma_i / \mu_i$. Assume $P(\overline{X}_i < 0)$ is negligible for all i. Let S_i^2 and $S_{n,i}^2$ be the unbiased and maximum-likelihood estimates of σ_i^2, respectively, and let $r_i = S_i / \overline{X}_i$ and $r_{n,i} = S_{n,i} / \overline{X}_i$ be the corresponding point estimates of R_i.

Bennett's and Shafer and Sullivan's Tests

Bennett (1976) proposes a procedure for testing $H_o : R_1 = R_2 = \ldots = R_k$ that makes apparent use of McKay's approximation for r_n. He notes that since $h(R_i)$ is a monotone function of R_i, then the null hypothesis $H_o' : h(R_1) = h(R_2) = \ldots = h(R_k)$ is equivalent to H_o and corresponds to a test of the equality of means of k gamma distributions, since $(n_i / (n_i - 1))h(r_{n,i})$ is distributed approximately gamma with expectation $h(R_i)$ and index $(n_i - 1) / 2$ according to McKay (1932).

Under this distributional assumption and hypothesis, Bennett applies a likelihood-ratio statistic suggested by Pitman (1939) and obtains

$$(N-k)\log\sum_{i-1}^{k}\left(n_i h(r_i)/(N-k)\right) - \sum_{i=1}^{k}\left(n_i - 1\right)\log\left(n_i h(r_i)/\left(n_i - 1\right)\right), \qquad (2.3)$$

where $N = \sum_i n_i$, which, he argues, is approximately distributed χ^2 with (k - 1) degrees of freedom under H_o'.

However, Bennett makes the erroneous assumption that McKay's approximation applies to r_i, not $r_{n,i}$, as McKay intended. That is, Bennett assumes that $(n_i / (n_i - 1))h(r_i)$ is distributed approximately gamma with mean $h(R_i)$ and index $(n_i - 1) / 2$, or, equivalently, that $n_i h(r_i) / h(R_i)$ is approximately distributed χ^2 with $(n_i - 1)$ degrees of freedom, which is a slightly less accurate approximation to the distribution of r_i than McKay's approximation is of $r_{n,i}$ (Umphrey, 1983). Curiously, Warren (1982) also makes this mistake in a paper documenting apparent discrepancies between McKay's approximation and the exact distribution of r.

This fact led Shafer and Sullivan (1986) to investigate the effect of replacing $h(r_i)$ by the more appropriate $h(r_{n,i})$ in (2.3). They find a slight increase in power, but recommend Bennett's test since it employs a more familiar form of the sample CV.

Doornbos and Dijkstra's Likelihood-Ratio Test

Doornbos and Dijkstra (1983) proposes a likelihood-ratio test for the equality of k normal population CVs that utilizes a reparameterized normal density and extends an earlier procedure by Miller and Karson (1977), which deals only with two populations and equal sample sizes. Doornbos and Dijkstra substitute $\frac{\sigma_i}{R_i}$ for μ_i in the density (since $R_i = \sigma_i / \mu_i$) and solve their chosen likelihood equations in R^{-1} and σ_i, i = 1, 2, ..., k under the null hypothesis H_o: $R_1 = R_2 = \ldots = R_k = R$ (unknown). However, Doornbos and Dijkstra offer an iterative algorithm for solving these equations which Gupta and Ma (1996) calls "questionable". In response, Gupta and Ma provide an alternative reparameterization, substituting $R_i\mu_i$ for σ_i, and suggest an improved algorithm for solving the resulting likelihood equations.

Under H_o: $R_1 = R_2 = \ldots = R_k = R$ (unknown), using Gupta and Ma's parameterization, the log-likelihood is given by $L_o = -\sum_{i=1}^{k} n_i \log((2\pi)^{1/2}\mu_i R) - \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(X_{ij}-\mu_i)^2}{2\mu_i^2 R^2}$. Differentiating with respect to R and μ_i gives the likelihood equations

$$\frac{\partial L_o}{\partial R} = -\sum_{i=1}^{k}\frac{n_i}{R} + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(X_{ij}-\mu_i)^2}{2\mu_i^2 R^3} = 0\text{, and}$$

$$\frac{\partial L_o}{\partial \mu_i} = -\frac{n_i}{\mu_i} + \sum_{j=1}^{n_i}\frac{X_{ij}(X_{ij}-\mu_i)}{\mu_i^3 R^2} = 0,\ i = 1, 2, \ldots, k.$$

Simplifying the equations gives

$$\sum_{i=1}^{k}\frac{n_i(1+\sqrt{1+4(1+r_i^2)R^2})}{2(1+r_i^2)} - \sum_{i=1}^{k} n_i = 0\text{, and} \qquad (2.4)$$

$$\mu_i = \frac{2(1+r_i^2)\overline{X}_i}{1+\sqrt{1+4(1+r_i^2)R^2}},\ i = 1, 2, \ldots, k. \qquad (2.5)$$

Equation (2.4) has no closed form solution in R for $k > 2$ and requires an iterative solution.

Using Gupta and Ma's algorithm, let $m = \min\{r_1, r_2, \ldots, r_k\}$ and $M = \max\{r_1, r_2, \ldots, r_k\}$. Let G(R) equal the left-hand side of (2.4). Then $G(m) \le 0 \le G(M)$, so that the solution is in the interval [m, M]. Bisecting [m, M], the solution falls into the half having left endpoint m_1 satisfying $G(m_1) \le 0$ and right endpoint M_1 satisfying $G(M_1) \ge 0$. Bisecting $[m_1, M_1]$ in turn, the solution now falls into the half having left endpoint m_2

satisfying $G(m_2) \leq 0$ and right endpoint M_2 satisfying $G(M_2) \geq 0$. The process is continued until, at the t^{th} iteration, the bisection point $(m_t + M_t) / 2$ gives $G((m_t + M_t) / 2)$ sufficiently close to zero.

Denoting the resulting approximate solution by $\widetilde{R}$, the value may be substituted into (2.5) to obtain the restricted estimators $\widetilde{\mu}_i$. Under H_o, the restricted maximum of L_o is then given by

$$L_o^* \;=\; -\frac{N}{2}\log(2\pi) \;-\; \sum_{i=1}^{k} n_i \log(\widetilde{\mu}_i \widetilde{R}) \;-\; \frac{N}{2},$$

where $N = \sum_i n_i$. The unrestricted maximum is given by

$$L^* \;=\; -\frac{N}{2}\log(2\pi) \;-\; \sum_{i=1}^{k} n_i \log(S_{n,i}) \;-\; \frac{N}{2}.$$

Hence, the traditional likelihood-ratio statistic is given by

$$\begin{aligned}
-2\left(L_o^* - L^*\right) &= -2\left(-\sum_{i=1}^{k} n_i \log\left(\widetilde{\mu}_i \widetilde{R}\right) + \sum_{i=1}^{k} n_i \log\left(S_{n,i}\right)\right) \\
&= \sum_{i=1}^{k} n_i \log\left(\widetilde{\mu}_i^2 \widetilde{R}^2\right) - \sum_{i=1}^{k} n_i \log\left(S_{n,i}^2\right) \\
&= \sum_{i=1}^{k} n_i \log\left(\frac{\widetilde{\mu}_i^2 \widetilde{R}^2}{S_{n,i}^2}\right).
\end{aligned}$$

Under H_o, Doornbos and Dijkstra suggest that this statistic is asymptotically distributed as χ^2 with (k-1) degrees of freedom. However, an apparent requirement that all $n_i \to \infty$ is not stressed (Silvey, 1975, pp. 112-114). This approach illustrates how methods which utilize the normal density are sometimes complicated by the fact that restrictions (and models) on the CV often cannot be made without addressing additional nuisance parameters.

Doornbos and Dijkstra's Non-Central t Test

Doornbos and Dijkstra (1983) also suggests an alternative test for the equality of k normal population CVs based on the non-central t distribution (2.1). Let $b_i = \frac{1}{r_i}$, i = 1, 2, ..., k, and define $\bar{b} = \frac{1}{N}\sum_{i=1}^{k} n_i b_i$, where $N = \sum_i n_i$. Under H_o: $R_1 = R_2 = \ldots = R_k = R$ (unknown), $\sqrt{n_i}\, b_i$ has a non-central t distribution with (n_i - 1) degrees of freedom and non-centrality parameter $\frac{\sqrt{n_i}}{R}$. It follows that

$$E(b_i) \;=\; \left(\frac{n_i - 1}{2}\right)^{1/2} \frac{\Gamma\left[\frac{1}{2}(n_i - 2)\right]}{\Gamma\left[\frac{1}{2}(n_i - 1)\right] R} \;=\; \frac{\xi_i}{R}, \text{ and} \tag{2.6}$$

$$E(b_i^2) \;=\; \frac{n_i - 1}{n_i - 3}\left(\frac{1}{n_i} + \frac{1}{R^2}\right), \; i = 1, 2, \ldots, k. \tag{2.7}$$

See, for example, Owen (1968). Hence,

$$E\left(\sum_{i=1}^{k} n_i b_i^2\right) \;=\; \sum_{i=1}^{k} \frac{n_i - 1}{n_i - 3} \;+\; \frac{1}{R^2}\sum_{i=1}^{k} \frac{n_i(n_i - 1)}{n_i - 3}, \tag{2.8}$$

so that an unbiased estimate of $\frac{1}{R^2}$ is

$$\widetilde{R}^{-2} = \frac{\sum_{i=1}^{k} n_i b_i^2 - \sum_{i=1}^{k} \frac{n_i - 1}{n_i - 3}}{\sum_{i=1}^{k} \frac{n_i(n_i - 1)}{n_i - 3}}. \qquad (2.9)$$

Define $T = \sum_{i=1}^{k} n_i(b_i - \bar{b})^2$. For large n_i, it follows from the moments of the non-central t given above that T is distributed approximately as $\left(1 + \frac{1}{2R^2}\right)\chi_{k-1}^2$ under H_o. Further,

$$E(T) = E\left[\sum_{i=1}^{k} n_i b_i^2 - N\bar{b}^2\right] = E\left[\sum_{i=1}^{k} n_i b_i^2\right] - \frac{1}{N}\left[E\left(\sum_{i=1}^{k} n_i b_i\right)^2\right]$$

$$= E\left[\sum_{i=1}^{k} n_i b_i^2\right] - \frac{1}{N}\left[\mathrm{Var}\left(\sum_{i=1}^{k} n_i b_i\right) + \left(E\left(\sum_{i=1}^{k} n_i b_i\right)\right)^2\right]$$

$$= E\left[\sum_{i=1}^{k} n_i b_i^2\right] - \frac{1}{N}\left[\sum_{i=1}^{k} n_i^2 \mathrm{Var}(b_i) + \left(E\left(\sum_{i=1}^{k} n_i b_i\right)\right)^2\right]$$

$$= E\left[\sum_{i=1}^{k} n_i b_i^2\right] - \frac{1}{N}\left[\sum_{i=1}^{k} n_i^2\left[E(b_i^2) - (E(b_i))^2\right] + \left(\sum_{i=1}^{k} n_i E(b_i)\right)^2\right].$$

Substituting (2.6), (2.7), and (2.8) into this last equation gives

$$E(T) = \sum_{i=1}^{k} \frac{(N - n_i)(n_i - 1)}{N(n_i - 3)}$$

$$+ \frac{1}{R^2}\left\{\sum_{i=1}^{k} \frac{n_i(N - n_i)(n_i - 1)}{N(n_i - 3)} + \frac{1}{N}\left[\sum_{i=1}^{k} n_i^2 \xi_i^2 - \left(\sum_{i=1}^{k} n_i \xi_i\right)^2\right]\right\}.$$

Finally, substituting (2.9) for $\frac{1}{R^2}$ and denoting the result by $\overline{E(T)}$, it follows that for large samples $\overline{E(T)} \approx E(T) \approx \left(1+\frac{1}{2R^2}\right)(k-1)$, so that under H_o, $(k-1)\frac{T}{\overline{E(T)}}$ has approximately a χ^2 distribution with (k - 1) degrees of freedom.

Gupta and Ma's Wald Test

Gupta and Ma (1996) proposes a Wald procedure for testing H_o: $R_1 = R_2 = \ldots = R_k$ based on an earlier form by Rao and Vidya (1992), which deals only with two populations and equal sample sizes. Gupta and Ma suggest the following general theory: Under regularity conditions satisfied by the normal log-likelihood, suppose that a random sample of size n is taken from a distribution with parameter vector $\theta = \left(\theta_1, \theta_2, \ldots, \theta_p\right)'$.

Let $\hat{\theta} = \left(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_p\right)'$ be the unrestricted maximum-likelihood estimate of θ obtained via the log-likelihood $L(\theta)$. Then a test for H_o: $\mathbf{k}(\theta) = ((k_1(\theta), k_2(\theta), \ldots, k_m(\theta))' = 0$, where the k_i are differentiable with respect to θ, is given via the statistic

$$\mathbf{k}'(\hat{\theta})\left[\hat{\mathbf{K}}'(\mathbf{I\hat{n}f})^{-1}\hat{\mathbf{K}}\right]^{-1}\mathbf{k}(\hat{\theta}) \qquad (2.10)$$

where $\hat{\mathbf{K}}$ is a p x m matrix having entries $k_{ij} = \partial k_j(\theta)/\partial\theta_i$, i = 1, 2, ..., p, j = 1, 2, ..., m estimated at $\hat{\theta}$ and where $\mathbf{I\hat{n}f}$ is the p x p Fisher's information matrix having entries $i_{jk} = -E\left(\frac{\partial^2 L(\theta)}{\partial\theta_j \partial\theta_k}\right)$, j, k = 1, 2, ..., p, also evaluated at $\hat{\theta}$. Under H_o, (2.10) has a χ^2 distribution with m degrees of freedom for large n (Silvey, 1975, pp. 115-116).

Gupta and Ma take $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \ldots, \mu_k, \sigma_k)'$ and $k_i(\theta) = \frac{\sigma_i}{\mu_i} - \frac{\sigma_{i+1}}{\mu_{i+1}}$, i = 1, 2, ..., k -1 to test $H_o : R_1 = R_2 = \ldots = R_k$, where $L(\theta) = -\sum_{i=1}^{k} n_i \log((2\pi)^{1/2}\sigma_i) - \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{(X_{ij} - \mu_i)^2}{2\sigma_i^2}$. From (2.10), for k = 2, they obtain

$$\frac{(r_{n,1} - r_{n,2})^2}{\frac{r_{n,1}^2}{2n_1} + \frac{r_{n,1}^4}{n_1} + \frac{r_{n,2}^2}{2n_2} + \frac{r_{n,2}^4}{n_2}}, \qquad (2.11)$$

while for k = 3, (2.10) gives

$$\begin{pmatrix} r_{n,1} - r_{n,2} \\ r_{n,2} - r_{n,3} \end{pmatrix}' \begin{pmatrix} \frac{r_{n,1}^2}{2n_1} + \frac{r_{n,1}^4}{n_1} + \frac{r_{n,2}^2}{2n_2} + \frac{r_{n,2}^4}{n_2} & -\frac{r_{n,2}^2}{2n_2} - \frac{r_{n,2}^4}{n_2} \\ -\frac{r_{n,2}^2}{2n_2} - \frac{r_{n,2}^4}{n_2} & \frac{r_{n,2}^2}{2n_2} + \frac{r_{n,2}^4}{n_2} + \frac{r_{n,3}^2}{2n_3} + \frac{r_{n,3}^4}{n_3} \end{pmatrix}^{-1} \begin{pmatrix} r_{n,1} - r_{n,2} \\ r_{n,2} - r_{n,3} \end{pmatrix}. \qquad (2.12)$$

They state that for k > 3, the general formula is omitted "because of its complexity". In order for this application of Wald theory to apply, however, it is apparently necessary that all $n_i \to \infty$ (Silvey, 1975, pp. 115-118), a requirement which Gupta and Ma do not address.

Gupta and Ma also do not simplify their statistics. For example, (2.11) may be reexpressed as

$$\frac{(r_{n,1} - r_{n,2})^2}{\frac{r_{n,1}^2}{n_1}\left(r_{n,1}^2 + \frac{1}{2}\right) + \frac{r_{n,2}^2}{n_2}\left(r_{n,2}^2 + \frac{1}{2}\right)} = (\mathbf{Cr})'\left[\mathbf{C}\hat{\mathbf{V}}\mathbf{C}'\right]^{-1}(\mathbf{Cr}),$$

where $\mathbf{C} = (1, -1)$, $\mathbf{r} = (r_{n,1}, r_{n,2})'$, and $\hat{\mathbf{V}} = \text{diag}\left\{\frac{r_{n,i}^2}{n_i}\left(r_{n,i}^2 + \frac{1}{2}\right)\right\}$, i = 1, 2. Similarly,

(2.12) may be simplified in this way by taking $\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and expanding $\mathbf{r}$ and $\hat{\mathbf{V}}$ to contain a third element. These simplifications suggest that Gupta and Ma's test may also be obtained from the discussion surrounding the Iglewicz and Myers' approximation by noting that since the samples are independent, $\mathbf{r} = \left(r_{n,1}, r_{n,2}, \ldots r_{n,k}\right)'$ is asymptotically normal with mean $\mathbf{R} = \left(R_1, R_2, \ldots R_k\right)'$ and covariance matrix $\mathbf{V} = \text{diag}\left\{\frac{R_i^2}{n_i}\left(R_i^2 + \frac{1}{2}\right)\right\}$.

Hence, under H_o: $\mathbf{CR} = 0$, where $\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}$ is a (k-1) x k matrix of restrictions on $\mathbf{R}$ corresponding to $R_1 = R_2 = \ldots = R_k$, $\mathbf{Cr}$ is asymptotically normal with mean 0 and covariance matrix $\mathbf{CVC}'$. Evaluating $\mathbf{V}$ at $\mathbf{r}$ to obtain $\hat{\mathbf{V}}$, it follows that $(\mathbf{Cr})'\left[\mathbf{C}\hat{\mathbf{V}}\mathbf{C}'\right]^{-1}(\mathbf{Cr})$ is asymptotically distributed as χ^2 with (k-1) degrees of freedom under H_o for large n_i (Serfling, 1980, pp. 128-130, 155; Judge, et al., 1988, pp. 52, 109-110; Eliason, 1993, pp. 34-35).

<u>Gupta and Ma's Score Test</u>

Gupta and Ma (1996) also develops a likelihood-based test that utilizes a reparameterized normal density and the following general theory: Under regularity conditions satisfied by the normal log-likelihood, suppose that a random sample of size n is taken from a distribution with parameter vector $\theta = \left(\theta_1, \theta_2, \ldots, \theta_p\right)'$. Assume

$\widetilde{\theta} = \left(\widetilde{\theta}_1, \widetilde{\theta}_2, \ldots, \widetilde{\theta}_p\right)'$ is the restricted maximum-likelihood estimate of θ obtained via the log-likelihood $L(\theta)$ under H_o: $\mathbf{k}(\theta) = ((k_1(\theta),\ k_2(\theta), \ldots, k_m(\theta))' = 0$. Let $\mathbf{U}(\theta)$ be a p x 1 vector having elements $u_i = \partial L(\theta) / \partial \theta_i$, i = 1, 2, ..., p. Then a test of H_o is given via the statistic

$$\left(\mathbf{U}(\widetilde{\theta})\right)' (\mathbf{I\widetilde{n}f})^{-1} \left(\mathbf{U}(\widetilde{\theta})\right), \qquad (2.13)$$

where $\mathbf{I\widetilde{n}f}$ is the p x p Fisher's information matrix having entries $i_{jk} = -E\left(\dfrac{\partial^2 L(\theta)}{\partial \theta_j \partial \theta_k}\right)$, j, k = 1, 2, ..., p, evaluated at $\widetilde{\theta}$. Under H_o, (2.13) has a χ^2 distribution with m degrees of freedom for large n (Rao, 1973, pp. 418-420; Silvey, 1975, pp. 118-120).

Gupta and Ma take $\theta = \left(R_1, R_2, \ldots, R_k, \mu_1, \mu_2, \ldots, \mu_k\right)'$ and $k_i(\theta) = R_i - R_{i+1}$, i = 1, 2, ..., k-1 to test H_o: $R_1 = R_2 = \ldots = R_k = R$ (unknown), where the reparameterized normal likelihood $L(\theta) = -\sum_{i=1}^{k} n_i \log((2\pi)^{1/2} \mu_i R_i) - \sum_{i=1}^{k} \sum_{j=1}^{n_i} \dfrac{\left(X_{ij} - \mu_i\right)^2}{2\mu_i^2 R_i^2}$. From (2.13), they obtain

$$\widetilde{R}^2 \left(\widetilde{R}^2 + \frac{1}{2}\right) \sum_{i=1}^{k} \frac{a_i^2}{n_i},$$

where $a_i = \dfrac{\sum_{j=1}^{n_i} \left(X_{ij} - \widetilde{\mu}_i\right)^2}{\widetilde{\mu}_i^2 \widetilde{R}^3} - \dfrac{n_i}{\widetilde{R}}$, i = 1, 2, ..., k, and where $\widetilde{R}$ and $\widetilde{\mu}_i$ are the restricted parameter estimates under H_o, obtained via the iterative algorithm outlined above for

Doornbos and Dijkstra's likelihood-ratio test. Once again, however, an apparent requirement that all $n_i \to \infty$ is not addressed.

Feltz and Miller's Test

Feltz and Miller (1996) suggests a test for the equality of k normal population CVs which is developed solely from the standpoint of asymptotic moments, much like the simplification of the Wald test offered above. Feltz and Miller note that since the samples are independent, $\mathbf{r} = (r_1, r_2, \ldots r_k)'$ is asymptotically normal with mean $\mathbf{R} = (R_1, R_2, \ldots R_k)'$ and covariance matrix $\mathbf{V} = \text{diag}\left\{\frac{R_i^2}{n_i}\left(R_i^2 + \frac{1}{2}\right)\right\}$, i = 1, 2, ..., k.

Under H_o: $R_1 = R_2 = \ldots = R_k = R$ (unknown), $\mathbf{V}$ simplifies to $R^2\left(R^2 + \frac{1}{2}\right)\text{diag}\{n_i^{-1}\}$.

Constructing the quadratic form $\mathbf{r}'\mathbf{Ar}$ under H_o, with $\mathbf{A} = \mathbf{V}^{-1} - \left(\mathbf{V}^{-1}\mathbf{J}\mathbf{V}^{-1}\right)/\left(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}\right)$, where $\mathbf{J}$ is a k x k matrix of ones and $\mathbf{1}$ is a k x 1 vector of ones, Feltz and Miller note that since $\mathbf{AV}$ is idempotent, then $\mathbf{r}'\mathbf{Ar}$ is asymptotically distributed as χ^2 with (k-1) degrees of freedom (for large n_i) (Serfling, 1980, pp. 128-129). Simplified and evaluated at $\mathbf{r}$,

$$\mathbf{r}'\mathbf{Ar} \quad = \quad \left[\tilde{R}^2\left(\tilde{R}^2 + \frac{1}{2}\right)\right]^{-1}\left(\sum_{i=1}^{k} n_i\left(r_i - \tilde{R}\right)^2\right),$$

where, for $N = \sum_i n_i$, $\tilde{R} = \left(\sum_{i=1}^{k} n_i r_i\right) / N$ is offered as a reasonable estimate of R under H_o. Note that $\tilde{R}$ is simply the weighted average of the r_i.

Factorial Designs for Population CVs

The Taguchi Approach

The methods of statistical experimental design have taken an increasingly important role in industry worldwide, as businesses seek to improve product quality and consistency, while minimizing cost. At the core of this movement have been the Taguchi methods, which utilize some of the more basic concepts of experimental design to great effect.

Among the responses of interest are the Taguchi signal-to-noise ratios, which are calculated within treatment combinations and are designed to reflect the effect of the individual treatments on the ability of a process to attain a designated target value. In particular, if it is desired to identify treatment factors that are important for maintaining closeness to a finite, positive average with minimum variation, Taguchi (1992, pp. 120-124) suggests that the response statistic $10\log_{10}\left(\frac{1}{r_i^2}-\frac{1}{n_i}\right)$ be analyzed in the context of a fractional factorial design. Noting that for values of $r_i \leq 0.3$ and $n_i \geq 2$, the term $1 / n_i$ is proportionately small, an alternative ratio is often given as $10\log_{10}\left(\frac{1}{r_i^2}\right) = -20\log_{10} r_i$ (Maghsoodloo, 1990; Schmidt and Launsby, 1994, Ch. 5, p. 18), which is simply the log-transformed sample CV.

Under a null hypothesis of no factor effects, these statistics have constant variance. A typical approach, then, is to conduct a standard normal-theory analysis of variance, treating the signal-to-noise ratios as the responses. However, there is only a single statistic per treatment combination, so that no estimate of the experimental error is available unless at least one mean square (corresponding to the highest-order interaction in a full factorial, presumably) is used for this purpose. Unfortunately, in the context of fractional factorials, there is usually no clearly defined hierarchy of effects, so that the one or several very small effects are pooled to create an estimate of the error. This process tends to produce tests of factor effects that have inflated Type I error rates because of the post-test selection of small effects (Box, 1988; Bissell, 1989; Zacks, 1991). In addition, at least one factor must always be declared negligible, even though experimental results may suggest that all factors are potentially important.

Bissell's Approach

Bissell (1989) proposes two procedures that simultaneously solve the bias problem and the lack of a test for all factor effects while maintaining a normal-theory analysis of variance of the signal-to-noise ratios.

According to his first solution, suppose there are a total of k factors, arranged in his example according to a fractional factorial. Calculate the mean squares $M_1, M_2, \ldots, M_k$ of the response signal-to-noise ratios and compute the overall average mean square $\overline{M} = \sum_{i=1}^{k} M_i / k$. If each factor has, say, κ degrees of freedom, then under the assumption of homogeneity of mean squares (that is, no factor effects), the common variance of the M_i may be estimated as $2\overline{M}^2 / \kappa$. A statistic for testing the deviation of at least one M_i from this hypothesis is then

$$\frac{(k-1)\mathrm{Var}(M_i)}{2\overline{M}^2/\kappa} = \frac{(k-1)\kappa\mathrm{Var}(M_i)}{2\overline{M}^2},$$

where $\mathrm{Var}(M_i) = \sum_{i=1}^{k}\left(M_i - \overline{M}\right)^2 / (k-1)$ is the observed variance of the M_i. Under the null hypothesis of homogeneity, this statistic has an approximate χ^2 distribution with (k - 1) degrees of freedom. If the hypothesis of homogeneity is rejected, Bissell suggests identifying the largest mean square as corresponding to a significant effect, removing it from consideration, and repeating the entire process until the hypothesis of homogeneity is not rejected.

Bissell's application of this procedure is to fractional factorials, which typically do not have hierarchy restrictions on factors since interactions are often not considered. However, it also could be used in more traditional full factorial settings by examining specific terms in order.

Bissell's second solution also addresses the problem from a homogeneity standpoint using the well-known Bartlett's test for equality of variances. Assuming that each of the k factors has κ degrees of freedom, Bissell's variant, applied to the mean squares, is

$$B = \log\left(\frac{1}{k}\sum_{i=1}^{k} M_i\right) - \frac{1}{k}\sum_{i=1}^{k}\log(M_i),$$

where Box's small-sample correction gives that $(k\kappa f_2 B)/(f_1(b - k\kappa B))$ has approximately an F distribution with f_1 and f_2 degrees of freedom under the null hypothesis of homogeneity of mean squares, with $f_1 = k - 1$ and $f_2 = (k - 1) / A^2$, where $A = 1 + (k + 1) / (2k\kappa)$, and where $b = f_2 / (1 - A + 2 / f_2)$. Bissell utilizes a top-down approach for the elimination of factors as in his first solution, wherein the effects are removed in descending order according to size until the null hypothesis of homogeneity is not rejected. He notes general agreement between his two procedures, although a power analysis was not conducted.

Zacks' Approach

An alternative correction for the bias induced by selecting the several smallest effects post-test is discussed by Zacks (1991), who considers modified F critical values. However, this approach does not address the lack of a test of all factor effects. An additional apparent shortcoming of Zacks' approach and of the related approaches discussed above is that subsequent analysis of the CV itself is somewhat compromised by the log transformation and the corresponding assumption of equal variance, which is incorrect outside the null hypothesis of no factor effects.

CHAPTER THREE

REVIEW OF THEORY

In this chapter, the theories of maximum- and quasi-likelihood estimation and their application in the context of the generalized linear model are discussed. In Chapter Four, these techniques will be applied to the approximate distributions of the sample CV discussed in Chapter Two in order to estimate the parameters of a factorial model of the population CV, once a proper form for such a model is proposed.

The Exponential Family

Let Y be a random variable whose probability function may be expressed in the form

$$f(y;\theta,\phi) = \exp\{(y\theta - b(\theta))/a(\phi) + c(y, a(\phi))\} \qquad (3.1)$$

with parameters θ and ϕ for suitably chosen functions $a(\bullet), b(\bullet),$ and $c(\bullet)$. The parameter θ is called the natural parameter and ϕ is called the dispersion parameter. If ϕ is known, such a function is said to belong to the exponential family. Examples include the binomial and Poisson. For unknown ϕ, (3.1) encompasses the two-parameter exponential family, which includes the gamma and the normal.

McCullagh and Nelder (1989, pp. 28-29), and Agresti (1990, p. 446-447) demonstrate how the first two moments of Y can be expressed in terms of θ and ϕ. Let

$\ell(\theta,\phi;y) = \log f(y;\theta,\phi)$ be the log-likelihood of Y. Then

$$\ell(\theta,\phi;y) = (y\theta - b(\theta)) / a(\phi) + c(y, a(\phi))$$

and so

$$\frac{\partial \ell}{\partial \theta} = \frac{(y - b'(\theta))}{a(\phi)}, \qquad \frac{\partial^2 \ell}{\partial \theta^2} = -\frac{b''(\theta)}{a(\phi)}.$$

Under regularity conditions satisfied by (3.1), it follows that $E\left(\frac{\partial \ell}{\partial \theta}\right) = 0$ and $E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) +$

$E\left(\frac{\partial \ell}{\partial \theta}\right)^2 = 0$. Hence,

$$E\left(\frac{Y - b'(\theta)}{a(\phi)}\right) = \frac{E(Y) - b'(\theta)}{a(\phi)} = 0,$$

which implies that $E(Y) = \psi = b'(\theta)$. Similarly,

$$E\left(-\frac{b''(\theta)}{a(\phi)}\right) + E\left(\frac{Y - b'(\theta)}{a(\phi)}\right)^2 = -\frac{b''(\theta)}{a(\phi)} + \frac{\mathrm{Var}(Y)}{[a(\phi)]^2} = 0,$$

which implies that $\mathrm{Var}(Y) = a(\phi)\, b''(\theta)$. The function $b''(\theta)$ depends only on the mean ψ via the natural parameter θ and is called the variance function. The notation $V(\psi)$ is typically used. The function $a(\phi)$ typically has the form $a(\phi) = \phi / w$, where w is a known weight. In the future, the weight w will be absorbed into $V(\psi)$, so that the notation $\mathrm{Var}(Y) = \phi V(\psi)$ will be employed. Additionally, noting that $\frac{\partial \psi}{\partial \theta} = b''(\theta) = \frac{\mathrm{Var}(Y)}{a(\phi)}$, it follows that

$$\frac{\partial \ell}{\partial \psi} = \frac{\partial \ell}{\partial \theta}\frac{\partial \theta}{\partial \psi} = \frac{y-\psi}{a(\phi)}\frac{a(\phi)}{\mathrm{Var}(Y)} = \frac{y-\psi}{\mathrm{Var}(Y)}. \tag{3.2}$$

Wedderburn (1974) shows that (3.2) is, in fact, a property possessed solely by probability functions of the form (3.1).

Maximum-Likelihood Estimation

Let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)'$ be a vector of independent observations with expectation $\psi = (\psi_1, \psi_2, \ldots, \psi_N)'$ and covariance matrix $\phi\mathbf{V}(\psi) = \phi\mathrm{diag}\{V_1(\psi_1), V_2(\psi_2), \ldots, V_N(\psi_N)\}$, and let the probability function of the Y_i, $i = 1, 2, \ldots, N$, have the form (3.1). Using conditions established by McCullagh (1983, 1986), suppose that ψ is related to a p-dimensional parameter vector $\beta = (\beta_1, \beta_2, \ldots, \beta_p)'$ through an arbitrary (possibly nonlinear) regression model $\psi = \psi(\beta)$ such that $\partial^3 \psi_i(\beta)/\partial\beta_j^3$ are bounded for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, p$, and such that if $\beta \neq \beta'$ then $\psi(\beta) \neq \psi(\beta')$ (that is, assume that the model is identifiable). Let ℓ_i denote the log-likelihood of the i^{th} observation. Then the log-likelihood of the N observations as a function of β via $\psi(\beta)$ is $L(\beta) = \sum_{i=1}^{N} \ell_i$. A commonly used iterative method for determining the vector $\hat{\beta}$ that maximizes $L(\beta)$, that is, determines the solution of $\partial L(\beta)/\partial\beta = \mathbf{0}$, is a variation of the Newton-Raphson algorithm known as Fisher scoring, discussed in Judge, et al. (1988, pp. 524-527), Agresti (1990, pp. 447-451) and Eliason (1993, pp. 41-45).

The Newton-Raphson Algorithm

Given a suitable initial estimate $\beta^{(0)}$ of β, let $\beta^{(t)}$ denote the approximation of β at the t^{th} iteration. Then the $(t + 1)^{th}$ estimate of β is given via the Newton-Raphson algorithm as

$$\beta^{(t+1)} = \beta^{(t)} - \left(\mathbf{H}^{(t)}\right)^{-1} \mathbf{q}^{(t)}, \tag{3.3}$$

where $\mathbf{q}^{(t)}$ is the vector of estimating equations $\partial L(\beta) / \partial \beta$ having elements

$$\frac{\partial L(\beta)}{\partial \beta_j} = \sum_{i=1}^{N} \frac{\partial \ell_i}{\partial \beta_j},$$

with

$$\frac{\partial \ell_i}{\partial \beta_j} = \frac{\partial \ell_i}{\partial \psi_i} \frac{\partial \psi_i}{\partial \beta_j} = \frac{y_i - \psi_i}{\phi V_i(\psi_i)} \frac{\partial \psi_i}{\partial \beta_j},$$

evaluated at $\beta^{(t)}$, and where $\mathbf{H}^{(t)}$ is the Hessian matrix (assumed nonsingular) having elements $h_{kj} = \partial^2 L(\beta) / \partial \beta_k \partial \beta_j$, with

$$\frac{\partial^2 L(\beta)}{\partial \beta_k \partial \beta_j} = \frac{\partial}{\partial \beta_k} \left(\sum_{i=1}^{N} \frac{y_i - \psi_i}{\phi V_i(\psi_i)} \frac{\partial \psi_i}{\partial \beta_j} \right)$$

$$= \frac{1}{\phi} \sum_{i=1}^{N} \left(\left(y_i - \psi_i\right) \frac{\partial}{\partial \beta_k} \left[\frac{1}{V_i\left(\psi_i\right)} \frac{\partial \psi_i}{\partial \beta_j} \right] - \left[\frac{1}{V_i\left(\psi_i\right)} \frac{\partial \psi_i}{\partial \beta_j} \frac{\partial \psi_i}{\partial \beta_k} \right] \right), \tag{3.4}$$

also evaluated at $\beta^{(t)}$. In vector form, the estimating equations may be written as

$$U(\beta) = \frac{\partial L(\beta)}{\partial \beta} = \mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \psi(\beta)\right) / \phi,$$

where $\mathbf{D}$ has elements $d_{ij} = \partial\psi_i / \partial\beta_j$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)'$, and where, under the identifiability condition on the model, $\mathbf{D}$ has full rank. The vector $U(\beta)$ is commonly called the score vector. Note that the (possibly) unknown dispersion parameter ϕ cancels in the iterative equation (3.3) and does not affect the estimation of β.

The Fisher Scoring Algorithm

Fisher scoring replaces $-\mathbf{H}$ with its expectation, also known as the Fisher's information matrix. In this case $\mathbf{Inf} = -E(\mathbf{H})$ has elements

$$i_{kj} = \frac{1}{\phi}\sum_{i=1}^{N} \frac{1}{V_i(\psi_i)} \frac{\partial\psi_i}{\partial\beta_j} \frac{\partial\psi_i}{\partial\beta_k},$$

a fact obtained from (3.4) by noting that the first term of the summand has expectation zero. Hence, at the t^{th} iteration, $\mathbf{Inf}^{(t)}$, not $\mathbf{H}^{(t)}$, is evaluated at $\beta^{(t)}$. In matrix form, $\mathbf{Inf} = \mathbf{D}'\mathbf{V}^{-1}\mathbf{D} / \phi$. Substitution into the Newton-Raphson algorithm gives two forms of the iterative equations:

$$\beta^{(t+1)} = \beta^{(t)} + \left(\mathbf{Inf}^{(t)}\right)^{-1} \mathbf{q}^{(t)} \qquad (3.5)$$

$$= \beta^{(t)} + \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1} \mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \psi\left(\beta^{(t)}\right)\right), \qquad (3.6)$$

where, as before, both $\mathbf{D}$ and $\mathbf{V}$ are evaluated at $\beta^{(t)}$. For both the Newton-Raphson and Fisher scoring methods, iteration continues until changes in $\beta^{(t)}$ are acceptably small.

Alternate Step Lengths

In order to reduce the possibility that the iterative equations (3.3) or (3.5) will overstep the maximum of the likelihood surface and fail to converge, the Newton-Raphson and Fisher scoring algorithms are often modified to include a step length (Judge, et al., 1988, pp. 517, 524; Eliason, 1993, p. 45). In particular, the Newton-Raphson algorithm may be rewritten as

$$\beta^{(t+1)} \quad = \quad \beta^{(t)} - s_t\left(\mathbf{H}^{(t)}\right)^{-1}\mathbf{q}^{(t)},$$

where s_t is a constant which may be adjusted at each iteration. The Fisher scoring algorithm is modified according to

$$\beta^{(t+1)} \quad = \quad \beta^{(t)} \quad + \quad s_t\left(\mathbf{Inf}^{(t)}\right)^{-1}\mathbf{q}^{(t)}.$$

If $s_t = 1$ for all t, these algorithms reduce to the forms (3.3) and (3.5) given above.

Some techniques call for s_t to be adjusted at each iteration in order to achieve the optimum movement toward the maximum. However, for simplicity, a fixed step other than one can also be used. Often, a fixed step length of 0.5 can greatly improve the odds that the iterative equations will converge. On occasion, a step of 0.1 or 0.2 may be required. In general, the smaller the step, the greater the chance of convergence, although an increasing number of iterations may become necessary.

Quasi-Likelihood Estimation

Wedderburn (1974) establishes a method of estimation for nonlinear models that makes assumptions only about the first two moments of the observed data. Proceeding much like before, let $\mathbf{Y} = \left(Y_1, Y_2, \ldots, Y_N\right)'$ be a vector of independent observations with expectation $\psi = \left(\psi_1, \psi_2, \ldots, \psi_N\right)'$ and covariance matrix $\phi\mathbf{V}(\psi) = \phi\mathrm{diag}\{V_1(\psi_1), V_2(\psi_2), \ldots, V_N(\psi_N)\}$. Suppose that ψ is related to a p-dimensional parameter vector $\beta =$

$(\beta_1, \beta_2, \ldots, \beta_p)'$ through an arbitrary (possibly nonlinear) regression model $\psi = \psi(\beta)$ such that $\partial^3 \psi_i(\beta)/\partial\beta_j^3$ are bounded for i = 1, 2, ..., N , j = 1, 2, ..., p, and such that if $\beta \neq \beta'$ then $\psi(\beta) \neq \psi(\beta')$ (that is, assume identifiability of the model). Note, however, that no distributional assumptions about **Y** have been made.

Under these conditions, Wedderburn defines the log-quasi-likelihood, or simply the quasi-likelihood, of the i$^{\text{th}}$ observation $Q_i(\psi_i ; y_i)$ by the relation

$$\frac{\partial Q_i}{\partial \psi_i} = \frac{y_i - \psi_i}{\phi V_i(\psi_i)}, \qquad (3.7)$$

so that for the given variance function $V_i(\psi_i)$, the function Q_i possesses the same property (3.2) uniquely associated with log-likelihoods of probability functions having the form (3.1). Any function Q_i satisfying (3.7) may serve as a quasi-likelihood, including functions which are not actual likelihoods; hence, the term quasi-likelihood. In particular, Q_i cannot correspond to an actual likelihood unless $V_i(\psi_i)$ is a variance function of a distribution with probability function satisfying (3.1). McCullagh and Nelder (1989, p. 325) defines Q_i as

$$Q_i(\psi_i ; y_i) = \int_{y_i}^{\psi_i} \frac{y_i - t}{\phi V_i(t)}\, dt, \qquad (3.8)$$

provided that the integral exists. Note that by the Fundamental Theorem of Calculus, this definition satisfies (3.7).

Under the assumption of independence, and provided that each of the Q_i exist, the log-quasi-likelihood of the N observations, as a function of β, is given by $Q(\beta)$ =

$\sum_{i=1}^{N} Q_i$ (McCullagh, 1983; McCullagh and Nelder, 1989, p. 325). The vector $\hat{\beta}$ that maximizes Q(β), that is, provides the solution of $\partial Q(\beta)/\partial\beta = \mathbf{0}$, can be determined using Fisher scoring. The estimating equations are given in matrix form by

$$U(\beta) = \frac{\partial Q(\beta)}{\partial \beta} = \mathbf{D}'\mathbf{V}^{-1}(\mathbf{y} - \psi(\beta))/\phi, \qquad (3.9)$$

where **D** has full rank with elements $d_{ij} = \partial\psi_i/\partial\beta_j$ and the $(t+1)^{th}$ estimate of β is given as

$$\beta^{(t+1)} = \beta^{(t)} + \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}\mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \psi\left(\beta^{(t)}\right)\right), \qquad (3.10)$$

where current estimates of **D** and **V** are obtained from $\beta^{(t)}$ as before. The vector (3.9) is commonly called the quasi-score vector. It is of interest to note that the estimating equations (3.9) do not explicitly require that Q(β) exist as a function (McCullagh, 1986).

For the particular case where $V_i(\bullet)$ is constant for each i = 1, 2, ..., N, (3.10) reduces to the Gauss-Newton method for obtaining the solution $\hat{\beta}$ that minimizes the nonlinear weighted least squares criterion $(\mathbf{y} - \psi(\beta))'\mathbf{V}^{-1}(\mathbf{y} - \psi(\beta))$ (Wedderburn, 1974; McCullagh, 1983, 1986). For the general case where the $V_i(\bullet)$ are functions of the ψ_i, (3.10) offers a computationally attractive alternative to the generalized least squares technique discussed by Carroll and Ruppert (1988, pp. 13-15), since the latter approach usually requires several successive applications of an iterative nonlinear weighted least squares algorithm. When the observation vector **Y** represents a sample from a distribution with probability function satisfying (3.1), quasi- and maximum-likelihood estimation coincide.

Asymptotic Properties of the Maximum- and Quasi-Likelihood Estimators

A convenient by-product of the Newton-Raphson and Fisher scoring algorithms for maximum-likelihood estimation is that estimated covariance matrices of $\hat{\beta}$ are available upon convergence. For Newton-Raphson, this matrix is the iterated solution for $-\mathbf{H}^{-1}$, once a suitable estimate of ϕ is obtained (if necessary) (Judge, et al., 1988, pp. 519-527; Agresti, 1990, p. 116). For Fisher scoring, $\mathbf{Inf}^{-1} = -(\mathrm{E}(\mathbf{H}))^{-1} = \phi(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D})^{-1}$ is the asymptotic covariance matrix of $\hat{\beta}$ (Judge, et al., pp. 521-523; Agresti, p. 451; Eliason, 1993, p. 40). Wedderburn (1974) shows that the asymptotic covariance matrix of the quasi-likelihood estimator $\hat{\beta}$ can similarly be expressed as $\phi(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D})^{-1}$.

McCullagh (1983) establishes that the desirable asymptotic properties of $\hat{\beta}$ in the context of maximum-likelihood, namely consistency of $\hat{\beta}$ and asymptotic normality of both $\hat{\beta}$ and $\mathrm{U}(\beta)$, can be applied to quasi-likelihood under the model and moment assumptions of the previous section with the additional requirement that $\frac{1}{\phi \mathrm{N}}(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D})$ has a positive definite limit as $\mathrm{N} \to \infty$.

For the case where N remains fixed, and the responses Y_i, i = 1, 2, ..., N, correspond, for example, to proportions or counts, these results are contingent on the assumption that the elements of $\frac{1}{\phi}(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D})$ increase without limit (McCullagh and

Nelder, 1983, p. 133; 1989, p. 328). This requirement generally holds, at least for $N > p$, provided that the number of sample elements n_i contributing to each of the Y_i increases without bound (McCullagh and Nelder, 1983, pp. 82-83, 133). In particular, assuming that the fitted model is correct, for large n_i, it follows that $\hat{\beta} \dot{\sim} N\left(\beta, \phi\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}\right)$ and

$U(\beta) \dot{\sim} N\left(\mathbf{0}, \frac{1}{\phi}\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)\right)$.

The Generalized Linear Model

The algorithms (3.6) and (3.10) may be simplified in terms of iteratively reweighted least squares equations, provided that the nonlinear regression equation $\psi = \psi(\beta)$ can be linearized in β via a properly chosen transformation. Let $\mathbf{Y} = \left(Y_1, Y_2, \ldots, Y_N\right)'$ be a vector of independent observations with expectation $\psi = \left(\psi_1, \psi_2, \ldots, \psi_N\right)'$ and covariance matrix $\phi\mathbf{V}(\psi) = \phi\mathrm{diag}\{V_1(\psi_1), V_2(\psi_2), \ldots, V_N(\psi_N)\}$, and suppose that there exists a monotone, differentiable function $g(\bullet)$ relating ψ_i to a p-dimensional parameter vector $\beta = \left(\beta_1, \beta_2, \ldots, \beta_p\right)'$ of the form

$$g(\psi_i) = \mathbf{x}_i'\beta, \qquad i = 1, 2, \ldots, N,$$

where $\mathbf{x}_i = \left(x_{i1}, x_{i2}, \ldots, x_{ip}\right)'$ is the i^{th} set of covariates. A model of ψ in β which may be expressed in this form is called a generalized linear model, and $g(\bullet)$ is called a link function. The form of the iterative equations when such a function exists is summarized in the following important theorem, given in Nelder and Wedderburn (1972), Wedderburn

(1974), McCullagh and Nelder (1989, p. 40-43), and Agresti (1990, p. 449-451), and examined by Hillis and Davis (1994).

Theorem 3.1 Let $\mathbf{Y}$ be defined as above with $E(\mathbf{Y}) = \psi$ and $\text{cov}(\mathbf{Y}) = \phi\mathbf{V}(\psi)$, and suppose that $g(\bullet)$ exists as defined above with

$$g(\psi_i) = \eta_i = \mathbf{x}_i'\beta,\ i = 1, 2, \ldots, N. \qquad (3.11)$$

Then a method equivalent to the iterative equations (3.6) and (3.10) is to calculate repeatedly a weighted linear regression of

$$z_i \quad = \quad g(\psi_i) \quad + \quad g'(\psi_i)(y_i - \psi_i)$$

on $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ using weight

$$w_i \quad = \quad [g'(\psi_i)]^{-2}[V_i(\psi_i)]^{-1}$$

for $i = 1, 2, \ldots, N$, where the current estimates of ψ_i are computed from the current estimates of $\beta_1, \beta_2, \ldots, \beta_p$.

Proof From (3.6) and (3.10), the $(t + 1)^{\text{th}}$ estimate of β is given by

$$\beta^{(t+1)} \quad = \quad \beta^{(t)} \quad + \quad \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}\mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \psi\left(\beta^{(t)}\right)\right),$$

where $\mathbf{D}$ has elements $d_{ij} = \partial\psi_i / \partial\beta_j$. Multiplying through by $\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}$ gives

$$\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)\beta^{(t+1)} \quad = \quad \left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)\beta^{(t)} \quad + \quad \mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \psi\left(\beta^{(t)}\right)\right). \qquad (3.12)$$

However, from (3.11), it follows that

$$\frac{\partial\psi_i}{\partial\beta_j} \quad = \quad \frac{\partial\psi_i}{\partial\eta_i}\frac{\partial\eta_i}{\partial\beta_j} \quad = \quad \frac{1}{g'(\psi_i)}x_{ij}.$$

Hence, $\mathbf{D}'\mathbf{V}^{-1}\mathbf{D} = \mathbf{X}'\mathbf{W}\mathbf{X}$, where $\mathbf{X}$ is an N x p matrix of full rank having elements x_{ij} and where

$$\mathbf{W} = \operatorname{diag}\left\{\left[g'(\psi_1)\right]^{-2}\left[V_1(\psi_1)\right]^{-1},\ldots,\left[g'(\psi_N)\right]^{-2}\left[V_N(\psi_N)\right]^{-1}\right\}$$

$$= \operatorname{diag}\left\{w_1, w_2, \ldots w_N\right\},$$

evaluated at $\beta^{(t)}$. Further, $\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)\beta^{(t)}$ is a vector, say **u**, having elements

$$u_k = \sum_{j=1}^{p}\sum_{i=1}^{N}\frac{x_{ik}x_{ij}}{V_i\left(\psi_i^{(t)}\right)}\frac{1}{\left[g'\left(\psi_i^{(t)}\right)\right]^2}\beta_j^{(t)}, \quad k = 1, 2, \ldots, p,$$

and $\mathbf{D}'\mathbf{V}^{-1}\left(\mathbf{y} - \psi\left(\beta^{(t)}\right)\right)$ is a vector, say **v**, of estimating equations having elements

$$v_k = \sum_{i=1}^{N}\frac{\left(y_i - \psi_i^{(t)}\right)x_{ik}}{g'\left(\psi_i^{(t)}\right)V_i\left(\psi_i^{(t)}\right)}, \quad k = 1, 2, \ldots, p,$$

where $\psi_i^{(t)} = g^{-1}\left(\mathbf{x}_i'\beta^{(t)}\right)$. Adding the vectors on the right-hand side of (3.12), that is, taking **u** + **v**, gives a vector having elements

$$u_k + v_k = \sum_{i=1}^{N}\left(\sum_{j=1}^{p}\frac{x_{ik}x_{ij}}{V_i\left(\psi_i^{(t)}\right)}\frac{1}{\left[g'\left(\psi_i^{(t)}\right)\right]^2}\beta_j^{(t)}\right) + \frac{\left(y_i - \psi_i^{(t)}\right)x_{ik}}{g'\left(\psi_i^{(t)}\right)V_i\left(\psi_i^{(t)}\right)}$$

$$= \sum_{i=1}^{N}\left[g'\left(\psi_i^{(t)}\right)\right]^{-2}\left[V_i\left(\psi_i^{(t)}\right)\right]^{-1}x_{ik}\left[\sum_{j=1}^{p}x_{ij}\beta_j^{(t)} + g'\left(\psi_i^{(t)}\right)\left(y_i - \psi_i^{(t)}\right)\right].$$

Hence, $\mathbf{u} + \mathbf{v} = \mathbf{X}'\mathbf{W}\mathbf{z}$, where $\mathbf{z} = \left(z_1, z_2, \ldots, z_N\right)'$, and (3.12) may be written as

$$\left(\mathbf{X}'\mathbf{W}\mathbf{X}\right)\beta^{(t+1)} = \mathbf{X}'\mathbf{W}\mathbf{z}$$

or

$$\beta^{(t+1)} = \left(\mathbf{X}'\mathbf{W}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{W}\mathbf{z} \qquad \text{Q.E.D.}$$

An immediate corollary of the theorem is that the asymptotic covariance matrix of $\hat{\beta}$ may be reexpressed as $\phi(\mathbf{X'WX})^{-1}$, which may be estimated using the iterated solution of β. Further, the estimating equations $\mathbf{D'V}^{-1}(\mathbf{y}-\psi(\beta))$ may be rewritten as

$$\mathbf{D'V}^{-1}(\mathbf{y}-\psi(\beta)) \quad = \quad \mathbf{X'G}^{-1}\mathbf{V}^{-1}(\mathbf{y}-\psi(\beta)) \quad = \quad \mathbf{X'GW}(\mathbf{y}-\psi(\beta)),$$

where $\mathbf{G} = \text{diag}\{g'(\psi_1), g'(\psi_2), \ldots, g'(\psi_N)\}$. Adequate starting values for the z_i and w_i may be obtained by substituting y_i for ψ_i (Wedderburn, 1974; McCullagh and Nelder, 1989, p. 41; Agresti, 1990, p. 450). If necessary, a step length s_t can be introduced to aid with convergence, in which case the response z_i is given by

$$z_i \quad = \quad g(\psi_i) \quad + \quad s_t g'(\psi_i)(y_i - \psi_i).$$

If the fitted model is saturated, that is, has as many parameters as observations, then the iteratively reweighted least squares estimates may be computed directly via ordinary least squares. This result holds because weighted and ordinary least squares are equivalent in the saturated case and because observed and predicted responses coincide, so that the starting substitution in the z_i is unchanged. Hence, the estimate $\hat{\beta}$ is given in closed form by

$$\hat{\beta} \quad = \quad (\mathbf{X'X})^{-1}\mathbf{X'z},$$

where $\mathbf{z} = (g(y_1), g(y_2), \ldots, g(y_N))'$. The estimated asymptotic covariance matrix is given in closed form by $\phi(\mathbf{X'\hat{W}X})^{-1}$, where $\hat{\mathbf{W}} = \text{diag}\{[g'(y_i)]^{-2}[V_i(y_i)]^{-1}\}$, i = 1, 2, ..., N.

Model Diagnostics

Several techniques for examining the adequacy of a model fit made utilizing quasi- and, in particular, maximum-likelihood estimation are available. These include the Wald test, the likelihood-ratio test, and the score test. The behavior of these tests has, in large part, been determined by the general asymptotic results of McCullagh (1983).

The Wald Test

Let $\hat{\beta} = \left(\hat{\beta}_1', \hat{\beta}_2'\right)'$ be the unrestricted quasi-likelihood estimate of a p x 1 vector of model parameters with subvector dimensions β_1: (p-q) x 1 and β_2: q x 1, $0 < q < p < N$. Then under conditions where $\hat{\beta}$ has an approximate p-variate normal distribution with mean β and covariance matrix $\phi\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)^{-1}$, or, in the context of a generalized linear model, with covariance matrix $\phi\left(\mathbf{X}'\mathbf{W}\mathbf{X}\right)^{-1}$, a Wald-type test can be used (Serfling, 1980, pp. 128-130; Carroll and Ruppert, 1988, pp. 213-214; Judge, et al., 1988, pp. 52, 109-110; Eliason, 1993, pp. 34-35).

In particular, a test of H_o: $\beta_2 = \mathbf{0}$, assuming that $\phi = 1$, is given by

$$\hat{\beta}_2'\left(\left(\mathbf{X}'\mathbf{W}\mathbf{X}\right)^{-1}_{qxq}\right)^{-1}\hat{\beta}_2, \qquad (3.13)$$

where $\mathbf{W}$ is estimated at $\hat{\beta}$, and the q x q subscript denotes the q x q submatrix of $\left(\mathbf{X}'\mathbf{W}\mathbf{X}\right)^{-1}$ corresponding to $\hat{\beta}_2$. Under H_o, (3.13) has an approximate χ^2 distribution with q degrees of freedom.

The Likelihood-Ratio Test

McCullagh (1983) discusses the asymptotic behavior of a test of model fit based on a difference of log-quasi-likelihoods which extends a traditional test based on the log-likelihoods of probability distributions having the form (3.1) detailed in Agresti (1990, p. 452).

Let $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)'$ be a vector of independent observations with expectation $\psi = (\psi_1, \psi_2, \ldots, \psi_N)'$ such that the probability function of the Y_i, $i = 1, 2, \ldots, N$ is of the form (3.1), and suppose initially that $\beta = (\beta_1', \beta_2', \beta_3')'$ is an associated N x 1 parameter vector corresponding to a saturated model, with subvector dimensions β_1: (p-q) x 1, β_2: q x 1, and β_3: (N-p) x 1, $0 < q < p < N$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_N)'$ be the vector of natural parameters of the observations. Let $\hat{\theta} = \theta(\hat{\beta})$ denote its estimate in the saturated case, and let $L(\hat{\beta})$ denote the unrestricted maximum of the log-likelihood.

Suppose, however, without loss of generality, that a model containing only the parameters in $(\beta_1', \beta_2')'$ is also under consideration. Take $\hat{\theta}_{1,2} = \theta(\hat{\beta}_{1,2})$ as the estimate of θ for this model and $L(\hat{\beta}_{1,2})$ as the restricted maximum of the log-likelihood. Letting $a_i(\phi) = \phi / w_i$ in (3.1), it follows that

$$
\begin{aligned}
-2\left(L(\hat{\beta}_{1,2}) - L(\hat{\beta})\right) &= 2\sum_{i=1}^{N} w_i\left[y_i(\hat{\theta}_i - \hat{\theta}_{1,2,i}) - b(\hat{\theta}_i) + b(\hat{\theta}_{1,2,i})\right] / \phi \qquad (3.14) \\
&= D(y; \hat{\psi}_{1,2}), \text{ say,}
\end{aligned}
$$

where $\hat{\psi}_{1,2} = \psi(\hat{\beta}_{1,2})$ and $\mathbf{y} = (y_1, y_2, \ldots, y_N)'$. The function $D(\mathbf{y}; \hat{\psi}_{1,2})$ is called the scaled deviance and is often employed as a relative measure of the discrepancy of fit of the reduced model (McCullagh and Nelder, 1989, pp. 33-34; Agresti, 1990, p. 452). For the more general quasi-likelihood, using McCullagh and Nelder's definition (3.8), the scaled deviance (3.14) becomes

$$D(\mathbf{y}; \hat{\psi}_{1,2}) \quad = \quad -2\left[Q(\hat{\beta}_{1,2}) - Q(\hat{\beta})\right] \quad = \quad -2Q(\hat{\beta}_{1,2}).$$

In certain cases to be discussed shortly, the distribution of the scaled deviance under the null hypothesis that the reduced model is correct may be approximated by a χ^2 distribution with (N - p) degrees of freedom.

Suppose for the moment, however, without loss of generality, that two reduced models, one containing the parameters $(\beta_1', \beta_2')'$ and the other containing only the parameters β_1, are to be compared. Take $\hat{\theta}_1 = \theta(\hat{\beta}_1)$ as the estimate of θ in the latter case and $L(\hat{\beta}_1)$ as the corresponding restricted maximum. Then, letting $D(\mathbf{y}; \hat{\psi}_1) = -2\left(L(\hat{\beta}_1) - L(\hat{\beta})\right)$ be the corresponding scaled deviance for the latter model, it follows that the difference

$$D(\mathbf{y}; \hat{\psi}_1) \quad - \quad D(\mathbf{y}; \hat{\psi}_{1,2}) \quad = \quad 2\sum_{i=1}^{N} w_i\left[y_i\left(\hat{\theta}_{1,2,i} - \hat{\theta}_{1,i}\right) - b\left(\hat{\theta}_{1,2,i}\right) + b\left(\hat{\theta}_{1,i}\right)\right] / \phi \qquad (3.15)$$

also has the form of the scaled deviance as in (3.14). McCullagh shows that under the general model and moment assumptions for quasi- and, hence, maximum-likelihood estimation, if $\frac{1}{\phi N}\left(\mathbf{D}'\mathbf{V}^{-1}\mathbf{D}\right)$ has a positive definite limit as $N \to \infty$ (or all $n_i \to \infty$ for fixed

N), then the difference (3.15) has a limiting χ^2 distribution with q degrees of freedom, the difference in the number of parameters between the reduced models, under H_o: $\beta_2 = 0$ (see also Silvey, 1975, pp. 112-114, and McCullagh and Nelder, 1989, pp. 118-119).

The asymptotic behavior of the scaled deviance itself, however, may be determined only under certain restrictions. Results reported by McCullagh and Nelder (1983, pp. 82-83, 133; 1989, pp. 118-119) suggest that, for fixed N, with all $n_i \to \infty$, the scaled deviance can generally be approximated by a χ^2 distribution, although a detailed theory is apparently unavailable. Conversely, the approximation appears to be generally invalid as $N \to \infty$ except when the observations are drawn from normal distributions (McCullagh and Nelder, 1989, p. 36). Evidently, the requirement that all $n_i \to \infty$ is not considered by either Bennett (1976) or Shafer and Sullivan (1986) in the development of their (scaled deviance) tests, which utilize a fixed N.

<u>The Score Test</u>

Let $\tilde{\beta} = (\tilde{\beta}_1', 0')'$ be the restricted quasi-likelihood estimate of a p x 1 vector of model parameters $\beta = (\beta_1', \beta_2')'$ with subvector dimensions β_1: (p-q) x 1 and β_2: q x 1, $0 < q < p < N$ under the hypothesis H_o: $\beta_2 = 0$. Then under conditions where $U(\beta)$ has an approximate p-variate normal distribution with mean 0 and covariance matrix $\frac{1}{\phi}(\mathbf{D'V^{-1}D})$, or, in the context of a generalized linear model, with covariance matrix $\frac{1}{\phi}(\mathbf{X'WX})$, a score test can be used (Rao, 1973, pp. 418-420; Serfling, 1980, pp. 156-158; McCullagh, 1986; Fahrmeir, 1987; Carroll and Ruppert, 1988, pp. 215-216).

In particular, a test of H_o: $\beta_2 = 0$, assuming that $\phi = 1$, is given by

$$\left[U(\tilde{\beta})\right]'(\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\left[U(\tilde{\beta})\right], \qquad (3.16)$$

where $\mathbf{W}$ and $U(\tilde{\beta})$, the quasi-score vector of estimating equations, are both evaluated at $\tilde{\beta}$. Under H_o, (3.16) has an approximate χ^2 distribution with q degrees of freedom.

Tests for the Saturated Model

Although McCullagh and Nelder (1983, 1989) effectively argue that the scaled deviance itself can, for fixed N and $n_i \to \infty$, be used as a test of fit of a reduced model versus a saturated model, there is little discussion in the literature of the asymptotic behavior of the Wald and score tests in this same scenario. However, given the form of the direct estimate of $\hat{\beta}$ in the saturated generalized linear model, for the responses considered in the next chapter, the results of Serfling (1980, pp. 24-25, 118, 128-130) provide for the asymptotic normality of $\hat{\beta}$ and the large-sample distribution of the Wald test.

CHAPTER FOUR

THE MODELLING APPROACH

In this chapter, a proper form for a factorial model of the population CV is proposed. In the context of each of the approximate distributions discussed in Chapter Two, the proposed multiplicative model is shown to satisfy the form of the generalized linear model reviewed in Chapter Three, and the corresponding iteratively reweighted least squares equations for estimating its parameters are established. In addition, a form for the iterative equations for an additive model is also suggested, and the equivalence of some associated one-factor model diagnostics to several of the one-factor tests currently in the literature is shown.

Choice of Model

In contrast with the one-factor tests discussed in the review of literature, a model of the population CV in a factorial experiment must accommodate the fact that additive restrictions may not adequately describe interactions and main effects. Models of the μ_i in classical analysis of variance are typically linear, for example, under the often implicit assumption that the μ_i may take any value on the real number line. However, the population CV is, by assumption, strictly positive, suggesting that multiplicative models are more appropriate.

Some justification for this argument is provided by McCullagh and Nelder (1989) and Eliason (1993). In the notation of Chapter Three, for models with a gamma-distributed response $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_N)'$, McCullagh and Nelder (p. 286) argues that an appropriate model for the mean vector $\psi = (\psi_1, \psi_2, \ldots, \psi_N)'$ based on a p-dimensional parameter vector $\beta = (\beta_1, \beta_2, \ldots, \beta_p)'$ is the multiplicative model

$$\psi_i \quad = \quad \exp(\mathbf{x}_i'\beta), \; i = 1, 2, \ldots, N,$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ is the i^{th} set of covariate values. Similarly, Eliason (pp. 22-23, 47-48) argues for such a model for gamma-distributed responses because of the restriction of the range of the ψ_i to positive values.

If it is assumed that the dispersion parameter ϕ may vary from observation to observation, then the dispersion vector $\phi = (\phi_1, \phi_2, \ldots, \phi_N)'$ can likewise be modelled. Eliason (pp. 22-23) notes that since the parameters of ϕ cannot be negative, the corresponding model structure should reflect this fact and not allow for unrealistic, that is, negative, values.

Hence, a model of the population CV in a factorial experiment may be argued in the following way. Take a collection of CVs $R_1, R_2, \ldots, R_k$ of normal populations, where, for convenience, a single subscript is used, but where any number of associated fixed factors may be supposed. Assume that the i^{th} population has mean $\mu_i > 0$ and variance σ_i^2, so that $R_i = \sigma_i / \mu_i$, $i = 1, 2, \ldots, k$. An appropriate model structure for the μ_i is then

$$\mu_i \quad = \quad \exp(\mathbf{x}_i'\alpha), \; i = 1, 2, \ldots, k,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)'$ is a parameter vector of fixed factor effects ($p \le k$) and $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ip})'$ is the i^{th} set of covariate values. For a factorial model, these covariates are properly assigned values of zero or positive or negative one under some identifiability constraint; for example, that the associated parameters summed across any single subscript must equal zero. Similarly, a model for the σ_i^2 might be

$$\sigma_i^2 \quad = \quad \exp(\mathbf{x}_i'\gamma), \; i = 1, 2, \ldots, k,$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)'$ is the corresponding parameter vector for the variances, so that a model for the σ_i may be written as

$$\sigma_i \quad = \quad \exp(\mathbf{x}_i'\gamma^*), \; i = 1, 2, \ldots, k,$$

where $\gamma^* = 0.5\gamma$.

Combining these models gives a multiplicative model for the R_i:

$$\begin{aligned} R_i &= \frac{\sigma_i}{\mu_i} = \frac{\exp(\mathbf{x}_i'\gamma^*)}{\exp(\mathbf{x}_i'\alpha)} = \exp(\mathbf{x}_i'(\gamma^* - \alpha)) \\ &= \exp(\mathbf{x}_i'\delta), \quad i = 1, 2, \ldots, k, \qquad (4.1) \end{aligned}$$

where $\delta = \gamma^* - \alpha$. This approach is corroborated for the case of gamma-distributed responses (as opposed to normal) by Eliason (1993, pp. 48-51).

As an example of such a model, suppose that two fixed factors, A and B, with a total of a and b levels, respectively, are arranged in a factorial experiment. Then model (4.1) may initially be expressed in the form

$$R_{ij} \quad = \quad \exp\left(R^* + \alpha_i + \beta_j + (\alpha\beta)_{ij}\right), \; i = 1, 2, \ldots, a, \; j = 1, 2, \ldots, b,$$

where $\exp(R^*)$ is the overall population CV, $\exp(\alpha_i)$ is the multiplicative effect caused by the i^{th} level of A, $\exp(\beta_j)$ is the multiplicative effect caused by the j^{th} level of B, and the terms $\exp((\alpha\beta)_{ij})$ describe the multiplicative effect caused by an interaction between A and B. In order to estimate the model, an identifiability constraint that, say, $\sum_{i=1}^{a}\alpha_i = \sum_{j=1}^{b}\beta_j = \sum_{i=1}^{a}(\alpha\beta)_{ij} = \sum_{j=1}^{b}(\alpha\beta)_{ij} = 0$ is also imposed.

The relationship between the parameters of (4.1) and those of models of the μ_i and the σ_i demonstrates one of the traditional criticisms of the CV, namely, that simultaneous factor effects on both the mean and standard deviation which are of equal magnitude leave the CV unchanged. Hence, a factor declared not to be significant in (4.1) might have no effect on either the mean or the standard deviation, or the same effect on both.

The Model-Fitting Algorithm

Suppose, now, that independent random samples of size n_i are drawn from each of the k normal populations, and that the sample CVs $r_i = S_i / \overline{X}_i$ and $r_{n,i} = S_{n,i} / \overline{X}_i$ are computed, where, as before, S_i^2 and $S_{n,i}^2$ are the unbiased and maximum-likelihood estimates of σ_i^2, respectively. Further, suppose that $R_i \in (0, 1/3)$, $i = 1, 2, \ldots, k$, that is, that each of the k populations essentially consists of positive values. Although this restriction is not made in the literature except in the context of McKay's approximation, it is largely consistent with the suggestion by Payton (1997) that the populations be of the ratio type and will be assumed throughout the remainder of this thesis.

McKay's and David's Approximations

According to David's approximation, $h(r_i)$ is distributed approximately gamma with expectation $h(R_i)$ and index $(n_i - 1) / 2$, so that $\text{Var}(h(r_i)) \approx \dfrac{2[h(R_i)]^2}{n_i - 1} = V_i(h(R_i))$ (taking $\phi = 1$). Supposing the model (4.1) for the R_i gives, as a model for the $h(R_i)$,

$$h(R_i) \quad = \quad h(\exp(x_i'\delta)), \; i = 1, 2, \ldots, k,$$

for which a linearizing transformation is

$$\log h^{-1}(h(R_i)) \quad = \quad x_i'\delta. \qquad (4.2)$$

Model (4.2) is a generalized linear model of the $h(R_i)$ with link function $\log h^{-1}(\bullet)$, but in the parameters of the original model of the R_i, so that estimating (4.2) simultaneously estimates (4.1). Additionally, for $0 < x < 1$,

$$\log h^{-1}(x) \quad = \quad \log\left(\frac{x}{1-x}\right)^{1/2} \quad = \quad \frac{1}{2}\log\left(\frac{x}{1-x}\right) \quad = \quad \frac{1}{2}\operatorname{logit}(x),$$

so that (4.2) also has the form of a logit model but with a gamma-distributed response.

As provided by Theorem 3.1, iteratively reweighted least squares may be employed to fit (4.2). Letting $R_i^* = h(R_i)$ and $r_i^* = h(r_i)$, it follows that

$$\begin{aligned} z_i \quad &= \quad \log h^{-1}(R_i^*) \quad + \quad \frac{d(\log h^{-1}(R_i^*))}{dR_i^*}(r_i^* - R_i^*) \\ &= \quad \log h^{-1}(R_i^*) \quad + \quad \frac{r_i^* - R_i^*}{2R_i^*(1 - R_i^*)} \quad , \qquad (4.3) \end{aligned}$$

and

$$w_i = \left[\frac{d\left(\log h^{-1}\left(R_i^*\right)\right)}{dR_i^*}\right]^{-2} [V_i\left(R_i^*\right)]^{-1} = \left[\frac{1}{2R_i^*\left(1-R_i^*\right)}\right]^{-2}\left[\frac{2\left(R_i^*\right)^2}{n_i-1}\right]^{-1}$$

$$= 2\left(n_i-1\right)\left(1-R_i^*\right)^2. \qquad (4.4)$$

Appropriate starting values for z_i and w_i may be obtained by substituting r_i^* for R_i^* in (4.3) and (4.4). Given that the t^{th} iteration has been made and that the t^{th} estimate $\delta^{(t)}$ has been obtained, the $(t + 1)^{th}$ estimate of δ can be computed after the substitution of $\left(R_i^*\right)^{(t)} = h\left(\exp\left(\mathbf{x}_i'\delta^{(t)}\right)\right)$ into (4.3) and (4.4).

If the alternative approximation of McKay is used, then $(n_i / (n_i - 1))h(r_{n,i})$ is supposed to be distributed approximately gamma with expectation $h(R_i)$ and index $(n_i - 1) / 2$. Hence, $r_{n,i}^* = (n_i / (n_i - 1))h(r_{n,i})$ may be substituted in z_i and w_i in place of r_i^*.

Upon convergence of the iteratively reweighted least squares algorithm to the maximum-likelihood estimate $\hat{\delta}$, any of the Wald test, the likelihood-ratio test, or the score test may be used to determine the significance of interactions and main effects.

In order to construct the likelihood-ratio test, it is necessary to know the form of the scaled deviance. By the parameterization given in (3.1), the approximate distributions of McKay and David give

$$\theta_i = -\frac{1}{h(R_i)} = -\frac{1}{R_i^*}, \qquad b(\theta_i) = \log h\left(R_i\right) = \log R_i^*,$$

with $\phi = 1$ and $w_i = (n_i - 1) / 2$, $i = 1, 2, \ldots, k$. Hence, by (3.14) the scaled deviance associated with a fitted model giving $\hat{R}_i^* = h\left(\exp\left(\mathbf{x}_i'\hat{\delta}\right)\right)$ in terms of David's approximation, with $\mathbf{r}^* = \left(r_1^*, r_2^*, \ldots, r_k^*\right)'$ and $\hat{\mathbf{R}}^* = \left(\hat{R}_1^*, \hat{R}_2^*, \ldots, \hat{R}_k^*\right)'$, is

$$
\begin{aligned}
D\left(\mathbf{r}^*; \hat{\mathbf{R}}^*\right) &= 2\sum_{i=1}^{k}\left(\frac{n_i - 1}{2}\right)\left[r_i^*\left(-\frac{1}{r_i^*} + \frac{1}{\hat{R}_i^*}\right) - \log r_i^* + \log \hat{R}_i^*\right] \\
&= -\sum_{i=1}^{k}\left(n_i - 1\right)\left[\log\left(\frac{r_i^*}{\hat{R}_i^*}\right) - \left(\frac{r_i^* - \hat{R}_i^*}{\hat{R}_i^*}\right)\right]
\end{aligned}
$$

(McCullagh and Nelder, 1989, p. 290). For McKay's approximation, $r_{n,i}^*$ should be substituted for r_i^* in the scaled deviance.

Iglewicz and Myers' Approximation

According to Iglewicz and Myers' approximation, r_i is distributed approximately normal with mean R_i and variance $\left(\frac{R_i^2}{n_i}\right)\left(R_i^2 + \frac{1}{2}\right) = V_i(R_i)$ (taking $\phi = 1$). However, the small-sample behavior of this approximation is inferior to that of McKay's and David's approximations, suggesting that the incorporation of the normal likelihood into the model estimation process here is less desirable than was the previous use of the gamma likelihood. Further, when the variance of a normal distribution is a function of its mean, the probability function no longer has the form (3.1), so that maximum-likelihood estimation via iteratively reweighted least squares is not possible. Carroll and Ruppert (1988, pp. 21-23) suggests that generalized least squares estimation, of which quasi-likelihood estimation is a special case, is generally preferred in these settings. Since quasi-likelihood estimation may be achieved through the same least squares process used to fit models for the McKay's and David's approximations, an opportunity to construct a single algorithm which incorporates all three approximations is available, provided that a model of the R_i in the context of the Iglewicz and Myers' approximation may be expressed as a generalized linear model.

This is easily achieved since, under the Iglewicz and Myers' approximation, the r_i have expectation R_i, and a log transformation of (4.1) gives the desired form. Specifically,

$$\log R_i \quad = \quad \mathbf{x}_i'\delta , \qquad (4.5)$$

which has the structure of a log-linear model. Theorem 3.1 may be applied to obtain the quasi-likelihood estimates of the model parameters in (4.5) with

$$z_i \quad = \quad \log R_i \quad + \quad \frac{d(\log R_i)}{dR_i}(r_i - R_i)$$

$$= \quad \log R_i \quad + \quad \frac{r_i - R_i}{R_i} \qquad (4.6)$$

and

$$w_i \quad = \quad \left[\frac{d(\log R_i)}{dR_i}\right]^{-2}\left[\left(\frac{R_i^2}{n_i}\right)\left(R_i^2 + \frac{1}{2}\right)\right]^{-1} \quad = \quad \left[\frac{1}{R_i}\right]^{-2}\left[\frac{n_i}{R_i^2\left(R_i^2 + \frac{1}{2}\right)}\right]$$

$$= \quad \frac{n_i}{R_i^2 + \frac{1}{2}}. \qquad (4.7)$$

In this case, appropriate starting values for z_i and w_i are obtained by substituting r_i for R_i in (4.6) and (4.7). Subsequently, once the t^{th} iterated estimate of δ is obtained, the $(t + 1)^{th}$ is computed by substitution of $R_i^{(t)} = \exp\left(\mathbf{x}_i'\delta^{(t)}\right)$ into (4.6) and (4.7), with the process being repeated until convergence to the quasi-likelihood estimate $\hat{\delta}$. Upon convergence, tests of factor effects may be conducted as before.

Using the form of the quasi-likelihood function (3.8), the scaled deviance associated with a fitted model giving $\hat{R}_i = \exp\left(\mathbf{x}_i'\hat{\delta}\right)$, with $\mathbf{r} = \left(r_1, r_2, \ldots, r_k\right)'$ and $\hat{\mathbf{R}} = \left(\hat{R}_1, \hat{R}_2, \ldots, \hat{R}_k\right)'$, is

$$D\left(\mathbf{r};\hat{\mathbf{R}}\right) \quad = \quad -2\sum_{i=1}^{k} Q_i(\hat{R}_i; r_i),$$

where

$$\begin{aligned} Q_i\left(\hat{R}_i; r_i\right) &= \int_{r_i}^{\hat{R}_i} \frac{r_i - t}{V_i(t)}\,dt = \int_{r_i}^{\hat{R}_i} \frac{r_i - t}{\frac{t^2}{n_i}\left(t^2 + \frac{1}{2}\right)}\,dt \\ &= 2n_i\left[\sqrt{2}r_i\left(\tan^{-1}\left(\sqrt{2}r_i\right) - \tan^{-1}\left(\sqrt{2}\hat{R}_i\right)\right) - \left(\frac{r_i - \hat{R}_i}{\hat{R}_i}\right)\right] \\ &\quad + \; n_i \log\left(\frac{r_i^2\left(\hat{R}_i^2 + \frac{1}{2}\right)}{\hat{R}_i^2\left(r_i^2 + \frac{1}{2}\right)}\right) \end{aligned}$$

(Burington, 1947, p. 64).

Existence and Uniqueness of Estimators

The existence of maximum- and quasi-likelihood estimators in the context of logit and log links is often a function of whether any of the responses, y_i, equal zero (McCullagh and Nelder, 1989, p. 117; Agresti, 1990, pp. 245, 249). Because for the current models, such an occurrence is possible only if all of the observations at a given treatment combination are identical (making the sample CV zero), existence of $\hat{\delta}$ is apparently not an issue. Unfortunately, in the context of these same link functions, the log- and quasi-likelihood surfaces considered above are not strictly concave in δ. In general, this implies that unique maximum- and quasi-likelihood estimates cannot be guaranteed for small n_i (Wedderburn, 1976; McCullagh, 1983; Fahrmeir and Kaufmann, 1985). However, empirical examination of these likelihood surfaces in simulation suggests that unique maxima do, in fact, exist even for relatively small n_i. In any event, McCullagh shows that for sufficiently large n_i, the iterative equations will converge to the correct maximum with high probability.

Model Selection

Although the algorithm for fitting generalized linear models has been established, and diagnostics for determining the adequacy of these models have been summarized, a broader technique for selecting the best model from a collection of potential models is necessary. In a regression setting, several techniques such as forward selection, backward elimination, and stepwise regression are available for determining the best subset of potential covariates. However, for factorial models, the number and type of terms available are limited. Despite this fact, Agresti (1990, pp. 218-222) considers both forward selection and backward elimination in his discussion of log-linear modelling. Nevertheless, he apparently prefers backward elimination, stating, "It is usually safer to delete terms from an overspecified model than to add terms to an underspecified one" (p. 218). For this reason, and also in an attempt to retain much of the spirit of an analysis of variance for normally-distributed data (which is a special case of the generalized linear model approach discussed here), the backward elimination approach is advocated.

According to this approach, the highest-order interaction of the full or saturated model is tested first, followed in turn, if necessary, by lower-order interactions and main effects according to the standard hierarchy. However, estimated factor effects are generally not independent, so that a reduced model must be iteratively refitted following the deletion of any factor judged not to be significant (McCullagh and Nelder, 1989, pp. 35-36). Further, when testing the significance of several factors with the same hierarchy -- for example, the three two-way interactions in a 2^3 or 3^3 factorial -- it is necessary when using the likelihood-ratio or score test to temporarily delete each individual factor in turn in order to build statistics to indicate which, if any, of these are not significant. Examples of model selection are given in Chapter Six.

Additive Models for Population CVs

Although not consistent with the multiplicative argument given earlier in this chapter, Gupta and Ma's Wald test, for example, expresses relationships among CVs in additive terms. It should be noted that the Wald tests of the significance of the single factor are not the same for additive and multiplicative models (the likelihood-ratio and score tests are unaffected). Because an additive model of the population CV could, conceivably, be desired even in the factorial case, the iterative algorithms for the additive model are given below. In particular, for the one-factor case, a model of the R_i may be written as $R_i = R + \alpha_i$, $i = 1, 2, \ldots, k$, where R is the overall population CV and α_i is the additive effect caused by the i^{th} factor level. While it is unnecessary to distinguish between additive and multiplicative models when testing for the presence of the single factor, conceptually, a multiplicative model may be preferred if certain contrasts are desired based on the saturated model (see Chapter Six, Applied Example #1).

McKay's and David's Approximations

For McKay's and David's approximations, since a proposed model of the R_i is now $R_i = \mathbf{x}_i'\delta$, the associated model of the $h(R_i)$ is given by

$$h(R_i) = h(\mathbf{x}_i'\delta), i = 1, 2, \ldots, k$$

and by the generalized linear model

$$h^{-1}(h(R_i)) = \mathbf{x}_i'\delta.$$

Using Theorem 3.1 and letting $R_i^* = h(R_i)$ and $r_i^* = h(r_i)$ as before, David's approximation gives

$$\begin{aligned} z_i &= h^{-1}(R_i^*) + \frac{d(h^{-1}(R_i^*))}{dR_i^*}(r_i^* - R_i^*) \\ &= h^{-1}(R_i^*) + \frac{r_i^* - R_i^*}{2\sqrt{R_i^*(1-R_i^*)^3}} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} w_i &= \left[\frac{d(h^{-1}(R_i^*))}{dR_i^*}\right]^{-2} [V_i(R_i^*)]^{-1} = \left[\frac{1}{2\sqrt{R_i^*(1-R_i^*)^3}}\right]^{-2}\left[\frac{2(R_i^*)^2}{n_i - 1}\right]^{-1} \\ &= \frac{2(n_i-1)(1-R_i^*)^3}{R_i^*}. \end{aligned} \tag{4.9}$$

As before, appropriate starting values for z_i and w_i may be obtained by substituting r_i^* for

R_i^* in (4.8) and (4.9). Further, the $(t + 1)^{th}$ iterated estimate of δ can be computed after the substitution of $\left(R_i^*\right)^{(t)} = h\left(\mathbf{x}_i'\delta^{(t)}\right)$ into (4.8) and (4.9). For McKay's approximation, $r_{n,i}^* = (n_i / (n_i - 1))h(r_{n,i})$ may be substituted for r_i^*.

Iglewicz and Myers' Approximation

For Iglewicz and Myers' approximation, the model $R_i = \mathbf{x}_i'\delta$ may be estimated via Theorem 3.1 using simply $z_i = r_i$ and $w_i = \left[\left(\frac{R_i^2}{n_i}\right)\left(R_i^2 + \frac{1}{2}\right)\right]^{-1}$, since the derivative of the link function with respect to R_i is one. In this case, only w_i is updated after each iteration. Values for the w_i are obtained initially by substituting r_i for R_i and, after the t^{th} iteration, by substituting $R_i^{(t)} = \mathbf{x}_i'\delta^{(t)}$.

Existent One-Factor Tests as Special Cases

In certain one-factor cases, with the appropriate approximation, tests discussed in the review of literature are special cases of model diagnostics discussed in Chapter Three. In particular, Shafer and Sullivan's test, Gupta and Ma's Wald test, and Feltz and Miller's test have a more general form applicable to factorial experiments.

Shafer and Sullivan's Test

It is easily shown that Shafer and Sullivan's test is equivalent to the likelihood-ratio test using McKay's approximation in the one-factor case. For simplicity, let $R_i^* = h(R_i)$ and $r_{n,i}^* = \left(n_i / \left(n_i - 1\right)\right)h(r_{n,i})$ as before. Under the null hypothesis H_o: $R_1 = R_2 = \ldots = R_k$, or, equivalently, supposing that the model $R_i = R$, $i = 1, 2, \ldots, k$ holds, where

R is the common population CV, the maximum-likelihood estimate of the single parameter R may be obtained by solving the estimating equation

$$\sum_{i=1}^{k} \frac{r_{n,i}^{*} - R_i^{*}}{\left[\frac{2\left(R_i^{*}\right)^2}{n_i - 1}\right]}\left(2\sqrt{R_i^{*}\left(1 - R_i^{*}\right)^3}\right) = 0$$

for R. The solution gives, as an estimate of the common predicted mean $R^{*} = h(R)$ under McKay's approximation, $\hat{R}^{*} = \left(\sum_{i=1}^{k}\left(n_i - 1\right) r_{n,i}^{*}\right) / (N - k)$, where $N = \sum_i n_i$. Substituting into the scaled deviance gives

$$-\sum_{i=1}^{k}\left(n_i - 1\right)\left(\log\left(\frac{r_{n,i}^{*}}{\hat{R}^{*}}\right) - \left(\frac{r_{n,i}^{*} - \hat{R}^{*}}{\hat{R}^{*}}\right)\right)$$

$$= -\sum_{i=1}^{k}\left(n_i - 1\right)\log r_{n,i}^{*} + \sum_{i=1}^{k}\left(n_i - 1\right)\log\left(\sum_{i=1}^{k}\frac{\left(n_i - 1\right) r_{n,i}^{*}}{N - k}\right)$$

$$= -\sum_{i=1}^{k}\left(n_i - 1\right)\log\left(\frac{n_i h\left(r_{n,i}\right)}{n_i - 1}\right) + \sum_{i=1}^{k}\left(n_i - 1\right)\log\left(\sum_{i=1}^{k}\frac{n_i h\left(r_{n,i}\right)}{N - k}\right)$$

$$= -\sum_{i=1}^{k}\left(n_i - 1\right)\log\left(\frac{n_i h\left(r_{n,i}\right)}{n_i - 1}\right) + (N - k)\log\left(\sum_{i=1}^{k}\frac{n_i h\left(r_{n,i}\right)}{N - k}\right),$$

which is distributed as χ^2 with (k - 1) degrees of freedom under H_o for large n_i. However, this is the Shafer and Sullivan statistic for testing the same hypothesis.

<u>Gupta and Ma's Wald Test</u>

For the case k = 2, it is easily shown that the Wald test using Iglewicz and Myers' approximation is equivalent to Gupta and Ma's Wald test if the approximation is applied to the $r_{n,i}$ as opposed to the r_i. Since the decision to use r_i in place of $r_{n,i}$ is largely unimportant, this shows that the Gupta and Ma test is also a special case of the current results.

For the model $R_i = R + \alpha_i$, i = 1, 2, where R is the overall population CV and α_i is the deviation due to the i^{th} factor level, a corresponding model in matrix form, subject to the identifiability constraint $\sum_{i=1}^{2} \alpha_i = 0$, is

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} R \\ \alpha_1 \end{bmatrix}.$$

Because the model is saturated, quasi-likelihood estimates of R and α_1 may be obtained directly via ordinary least squares as

$$\begin{bmatrix} \hat{R} \\ \hat{\alpha}_1 \end{bmatrix} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{r},$$

where $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $\mathbf{r} = \left(r_{n,1}, r_{n,2}\right)'$. Hence,

$$\hat{R} = \frac{1}{2}r_{n,1} + \frac{1}{2}r_{n,2}, \text{ and } \qquad \hat{\alpha}_1 = \frac{1}{2}r_{n,1} - \frac{1}{2}r_{n,2}.$$

The estimated asymptotic covariance matrix of $\begin{bmatrix} \hat{R} \\ \hat{\alpha}_1 \end{bmatrix}$ is given by

$$\left(\mathbf{X}'\hat{\mathbf{W}}\mathbf{X}\right)^{-1} = \frac{1}{4}\begin{bmatrix} V_1\left(r_{n,1}\right) + V_2\left(r_{n,2}\right) & V_1\left(r_{n,1}\right) - V_2\left(r_{n,2}\right) \\ V_1\left(r_{n,1}\right) - V_2\left(r_{n,2}\right) & V_1\left(r_{n,1}\right) + V_2\left(r_{n,2}\right) \end{bmatrix},$$

where $\hat{\mathbf{W}} = \operatorname{diag}\left\{\left[\frac{r_{n,i}^2}{n_i}\left(r_{n,i}^2 + \frac{1}{2}\right)\right]^{-1}\right\} = \operatorname{diag}\left\{\left[V_i\left(r_{n,i}\right)\right]^{-1}\right\}$, i = 1, 2. Testing the null hypothesis that $R_1 = R_2$ is equivalent to testing $\alpha_1 = 0$, and the Wald test for the latter equality is

$$\hat{\alpha}_1\left[\frac{1}{4}\left(V_1\left(r_{n,1}\right)+V_2\left(r_{n,2}\right)\right)\right]^{-1}\hat{\alpha}_1 \quad = \quad \frac{4\hat{\alpha}_1^2}{V_1\left(r_{n,1}\right)+V_2\left(r_{n,2}\right)}$$

$$= \quad \frac{\left(r_{n,1}-r_{n,2}\right)^2}{\frac{r_{n,1}^2}{n_1}\left(r_{n,1}^2+\frac{1}{2}\right)+\frac{r_{n,2}^2}{n_2}\left(r_{n,2}^2+\frac{1}{2}\right)},$$

which is distributed as χ^2 with one degree of freedom under H_o for large n_i. This is also the Gupta and Ma Wald test for k = 2. A similar equivalence holds for k > 2 if the hypothesis H_o: $\alpha_i = 0$, i = 1, 2, ..., k - 1 is tested.

Feltz and Miller's Test

For the case k = 2, it is easily shown that the quasi-score test using Iglewicz and Myers' approximation is equivalent to Feltz and Miller's test. Under the null hypothesis that $R_1 = R_2 = \ldots = R_k$, the model for the R_i may be written as $R_i = R$, i = 1, 2, ...,k, where R is the common population CV. The restricted quasi-likelihood estimate of the single model parameter R is obtained by solving the estimating equation

$$\sum_{i=1}^{k}\frac{r_i - R_i}{\frac{R_i^2}{n_i}\left(R_i^2+\frac{1}{2}\right)} \quad = \quad 0,$$

which gives $\widetilde{R} = \left(\sum_{i=1}^{k} n_i r_i\right)/N$, where $N = \sum_i n_i$. This estimate of R is the weighted average of r_i given by Feltz and Miller as a reasonable estimate of the common CV under H_o. Substituting $\widetilde{R}$ into the formula for the quasi-score test for the specific case k = 2 initially gives

$$\left[U(\tilde{R})\right]'\left(X'\tilde{W}X\right)^{-1}\left[U(\tilde{R})\right] = \left[\left(r-\tilde{R}\right)'\tilde{W}X\right]\left(X'\tilde{W}X\right)^{-1}\left[X'\tilde{W}\left(r-\tilde{R}\right)\right], \quad (4.10)$$

where $\mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the design matrix of the saturated model $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} R \\ \alpha_1 \end{bmatrix}$,

$\tilde{\mathbf{W}} = \mathrm{diag}\left\{\left[\frac{\tilde{R}^2}{n_i}\left(\tilde{R}^2 + \frac{1}{2}\right)\right]^{-1}\right\} = \mathrm{diag}\left\{\left[V_i(\tilde{R})\right]^{-1}\right\}$, i = 1, 2, and where $\mathbf{r} = \left(r_1, r_2\right)'$ and

$\tilde{\mathbf{R}} = \left(\tilde{R}_1, \tilde{R}_2\right)' = \left(\tilde{R}, \tilde{R}\right)'$. However, (4.10) may be reexpressed as

$$\begin{bmatrix} \dfrac{r_1-\tilde{R}}{V_1(\tilde{R})} + \dfrac{r_2-\tilde{R}}{V_2(\tilde{R})} \\ \dfrac{r_1-\tilde{R}}{V_1(\tilde{R})} - \dfrac{r_2-\tilde{R}}{V_2(\tilde{R})} \end{bmatrix}' \left(\frac{1}{4}\begin{bmatrix} V_1(\tilde{R})+V_2(\tilde{R}) & V_1(\tilde{R})-V_2(\tilde{R}) \\ V_1(\tilde{R})-V_2(\tilde{R}) & V_1(\tilde{R})+V_2(\tilde{R}) \end{bmatrix}\right) \begin{bmatrix} \dfrac{r_1-\tilde{R}}{V_1(\tilde{R})} + \dfrac{r_2-\tilde{R}}{V_2(\tilde{R})} \\ \dfrac{r_1-\tilde{R}}{V_1(\tilde{R})} - \dfrac{r_2-\tilde{R}}{V_2(\tilde{R})} \end{bmatrix}$$

$$= \frac{\left(r_1-\tilde{R}\right)^2}{V_1(\tilde{R})} + \frac{\left(r_2-\tilde{R}\right)^2}{V_2(\tilde{R})}$$

$$= \frac{1}{\tilde{R}^2\left(\tilde{R}^2+\frac{1}{2}\right)}\left[n_1\left(r_1-\tilde{R}\right)^2 + n_2\left(r_2-\tilde{R}\right)^2\right],$$

which is the Feltz and Miller statistic. A similar result holds for k > 2 but becomes difficult to demonstrate because of the complexity of $(\mathbf{X'WX})^{-1}$ in closed form.

Confidence Intervals for Fitted Models

Once the significant interactions and main effects in a fitted factorial model have been determined, confidence intervals for estimated contrasts may be desired. For the multiplicative model (4.1), such contrasts estimate ratios of unknown population CVs rather than differences, as in normal-theory analysis of variance. If the additive model of the CVs is used, these contrasts will estimate differences.

For simplicity, suppose that two population CVs, R_1 and R_2, are to be contrasted, and assume that the multiplicative model (4.1) has been fitted. Note that although a single subscript is used, these CVs may be associated with either main or simple effects of factors. In this context, the unknown ratio of R_1 and R_2 may be expressed as

$$\log\left(\frac{R_1}{R_2}\right) \quad = \quad \log R_1 - \log R_2 \quad = \quad \mathbf{x}_1'\delta - \mathbf{x}_2'\delta \quad = \quad \left(\mathbf{x}_1' - \mathbf{x}_2'\right)\delta \quad = \quad \mathbf{x}_{12}'\delta .$$

Once the maximum- or quasi-likelihood estimates of δ are obtained via one of the three approximations under consideration, an asymptotic 100(1 - α)% confidence interval for the log-ratio is then

$$\widehat{\log\left(\frac{R_1}{R_2}\right)} \quad = \quad \mathbf{x}_{12}'\hat{\delta} \quad \pm \quad z_{\alpha/2}\sqrt{\mathbf{x}_{12}'\left(\mathbf{X}'\hat{\mathbf{W}}\mathbf{X}\right)^{-1}\mathbf{x}_{12}} \, ,$$

where $z_{\alpha/2}$ is a value from a standard normal distribution having right-tail probability α/2, and where $\left(\mathbf{X}'\hat{\mathbf{W}}\mathbf{X}\right)^{-1}$ is the appropriate estimated asymptotic covariance matrix of $\hat{\delta}$. Denoting the lower and upper endpoints of this interval by $\hat{L}$ and $\hat{U}$, respectively, a corresponding 100(1 - α)% confidence interval for R_1 / R_2 is then given by $\left(\exp(\hat{L}), \exp(\hat{U})\right)$. Examples of interval estimation are provided in Chapter Six.

For the additive model,

$$R_1 - R_2 \quad = \quad \mathbf{x}_1'\delta - \mathbf{x}_2'\delta \quad = \quad \left(\mathbf{x}_1' - \mathbf{x}_2'\right)\delta \quad = \quad \mathbf{x}_{12}'\delta ,$$

so that a 100(1 - α)% confidence interval for the difference between population CVs may be constructed as

$$\widehat{\mathbf{R}_1 - \mathbf{R}_2} = \mathbf{x}_{12}'\hat{\delta} \pm z_{\alpha/2}\sqrt{\mathbf{x}_{12}'\left(\mathbf{X}'\hat{\mathbf{W}}\mathbf{X}\right)^{-1}\mathbf{x}_{12}}\,,$$

where $z_{\alpha/2}$ and $\left(\mathbf{X}'\hat{\mathbf{W}}\mathbf{X}\right)^{-1}$ are defined as before.

CHAPTER FIVE

SIMULATION METHODS AND RESULTS

In this chapter, the objectives and methods of the adopted simulation strategy are discussed. Simulation results, as well as their potential ramifications on the overall modelling approach, are summarized. For reference throughout this chapter, tables of simulation results are included in Appendix A. The simulation programs for SAS are included in Appendixes C through F.

Simulation Objectives

Asymptotic Behavior of the Scaled Deviance

Since there are apparently no conclusive results pertaining to the asymptotic behavior of the scaled deviance, the sufficiency of large samples in the current modelling context is investigated. In particular, the capability of the scaled deviance as a test of interaction in a 2 x 2 factorial experiment is simulated. The corresponding Wald and score tests also are considered.

Asymptotic Behavior of a Difference of Scaled Deviances

Agresti (1990, p. 250) notes that for log-linear models for count data, where the scaled deviance is known to have a limiting χ^2 distribution, a difference of scaled deviances for comparing a reduced model to an intermediate but unsaturated model converges to its limiting distribution more quickly than the scaled deviance, provided that the reduced model holds. This suggests that the use of a difference as a test of significance of, say, a main effect in a 2 x 2 factorial model of population CVs may be superior to the required use of the scaled deviance as a test of interaction in the same experiment. The relative behavior of these diagnostics is determined for this case. The corresponding Wald and score tests also are considered.

Relative Capabilities of Model Diagnostics

The combination of the three approximations under consideration (McKay's, David's, and Iglewicz and Myers') with the three potential diagnostic tests (Wald, likelihood-ratio, and score) results in nine ways of testing the significance of an effect(s) in a fitted factorial model. The relative powers and Type I error rates of these nine tests in the context of the 2 x 2 factorial experiment discussed above are investigated.

The One-Factor Experiment

For the one-factor experiment, the likelihood-ratio test using McKay's approximation and the Wald and score tests using Iglewicz and Myers' approximation correspond to established tests (see Chapter Four). However, the six remaining tests are new in this context. For this reason, the capabilities of all nine tests in the one-factor experiment are compared. In addition, three other existent tests discussed in the review of literature are simulated.

Non-Normal Data

If Payton's (1997) suggestion that the CV be associated primarily with ratio-level data is followed, the possibility exists that data will actually be taken from extremely right-skewed populations. To investigate the impact of skewed data, simulated observations are drawn from gamma distributions having CVs marginally within and clearly outside the range (0, 1/3) of values consistent with the assumption of "ratio-normality".

Simulation of a 2 x 2 Factorial Experiment

In order to assess and compare the capabilities of the approximation-diagnostic combinations under consideration, tests of interaction and a main effect were conducted on normal data that were generated using a 2 x 2 multiplicative factorial model of the form

$$R_{ij} = \exp\left(R^* + \alpha_i + \beta_j + (\alpha\beta)_{ij}\right), \quad i, j = 1, 2, \qquad (5.1)$$

where, for identifiability, $\sum_{i=1}^{2} \alpha_i = \sum_{j=1}^{2} \beta_j = \sum_{i=1}^{2} (\alpha\beta)_{ij} = \sum_{j=1}^{2} (\alpha\beta)_{ij} = 0$. For convenience, the data were drawn from normal populations having means $\mu_{ij} = 1$ and standard deviations $\sigma_{ij} = R_{ij}$.

For the simulations, the overall CV, $\exp(R^*)$, was set at both 0.1 and 0.2. For an overall CV of 0.1, tests of interaction were conducted with $\exp((\alpha\beta)_{11})$ set to 1, 1.1, 1.2, ..., 1.6, and, for simplicity, with $\alpha_1 = \beta_1 = 0$. The main-effect tests were conducted only for a single factor. In particular, effect sizes of $\exp(\beta_1)$ = 1, 1.1, 1.2, ..., 1.6 were considered. In this case, the interaction terms $(\alpha\beta)_{ij}$ were removed from the generating model so that the main-effect tests could be conducted in the proper context. For simplicity, α_1 was set to zero. For an overall CV of 0.2, tests were conducted on interaction and main effect sizes of 1, 1.05, 1.1, 1.15, ..., 1.3. Smaller sizes were chosen to preserve the range (0, 1/3) of population CVs of "ratio-normal" distributions.

For an overall CV of 0.1, equal sample sizes of 10, 20, 30, and 50 were taken from each of the 2 x 2 = 4 factor level combinations. Sample size combinations of 10 with 20, 10 with 30, 20 with 30, and 30 with 50 also were considered. For an overall CV of 0.2, samples with equal sizes of 30, 50, and 100 were drawn, the larger samples reflecting the smaller effect sizes considered. Combinations of 30 with 50 and 50 with 100 also were examined.

For each combination of overall CV, effect type (main or interaction), effect size, and sample size configuration, 10,000 data sets were simulated, and the appropriate effect was tested at the 0.05 level using each of the nine approximation-diagnostic combinations under consideration. On occasion, this strategy produced duplicate simulations when effect sizes were set to one. These simulations were combined to give improved estimated Type I error rates based on 20,000 data sets and are marked in the tables by a single asterisk (*).

Simulation of a One-Factor Experiment

In order to compare the new approximation-diagnostic combinations in the one-factor case, normal data also were generated from a one-factor multiplicative model having three levels. A multiplicative model was chosen to provide greater flexibility when varying the magnitude of the single factor effect. However, an additive model was actually fitted in the simulation so that Gupta and Ma's Wald test would have the proper form. In particular, the model

$$R_i = \exp(R^* + \alpha_i), \; i = 1, 2, 3,$$

was used to generate the data. Like the 2 x 2 factorial model above, data were generated from normal distributions having means $\mu_i = 1$ and standard deviations $\sigma_i = R_i$. The identifiability constraint $\alpha_2 = 0$ was used. For simplicity, α_1 was set equal to $-\alpha_3$, with $\alpha_3 \geq 0$. Numerically, this produced $R_1 \leq R_2 \leq R_3$, with R_2 equal to the overall population CV, $\exp(R^*)$.

The overall CV was set at both 0.1 and 0.2. For an overall CV of 0.1, tests were conducted with $\exp(\alpha_3) = 1, 1.1, 1.2, ..., 1.6$. In this case, equal sample sizes of 10, 20, 30, and 50 were taken, while combinations of 10 with 20, 10 with 30, 20 with 30, and 30 with 50 also were considered. For an overall CV of 0.2, $\exp(\alpha_3)$ was set at 1, 1.05, 1.1, 1.15, ..., 1.3, the smaller effect sizes preserving the "ratio-normal" range of population CVs. Equal sample sizes of 30, 50, and 100 were simulated, as well as combinations of 30 with 50 and 50 with 100.

For each combination of overall CV, effect size, and sample size configuration, 10,000 data sets were simulated. The effect was tested at the 0.05 level using each of the nine approximation-diagnostic combinations under consideration and also using Doornbos and Dijkstra's likelihood-ratio and non-central t tests, and Gupta and Ma's score test. Duplicate simulations for effect sizes of one were combined to give 30,000 data sets to use in assessing the Type I error rate. These are marked by a double asterisk (**) in the tables.

Simulation of Non-Normal Data

In order to determine the impact of non-normal data, in particular, right-skewed data, on the approximation-diagnostic combinations, observations also were generated from gamma distributions having CVs determined by model (5.1). Since the mean of a gamma distribution is λ and the standard deviation is $\frac{\lambda}{\sqrt{\nu}}$, the corresponding population CV is given by $\frac{\lambda/\sqrt{\nu}}{\lambda} = \frac{1}{\sqrt{\nu}}$. For convenience, the data satisfying (5.1) were drawn from gamma distributions having means $\lambda_{ij} = 1$ and index parameters $\nu_{ij} = \frac{1}{R_{ij}^2}$.

Skewed data were simulated using overall population CVs of 0.3 and 0.6. Larger overall values were chosen so that the generated data would possess a noticeable level of skewness. Tests of interaction only were conducted with $\exp((\alpha\beta)_{11})$ set to 1, 1.1, 1.2, ..., 1.6 for sample configurations involving sizes of 10, 20, and 30 at both overall CV values. Sample configurations involving sizes of 50 and 100 also were investigated for interaction effect sizes of 1, 1.05, 1.1, 1.15, ..., 1.3 at both overall CV values.

Simulation Results

The Interaction Test

Tables XV through XXXIX summarize the simulations of the scaled deviance as a test of interaction in a 2 x 2 factorial experiment, as well as the corresponding Wald and score tests. For equal sample sizes, Type I error rates tended to be high for all tests except the score tests using McKay's (M) and David's (D) approximations, but tended to improve as the sample size increased. Type I error rates for Iglewicz and Myers' (IM) approximation were consistently higher than M and D.

Except for the score tests using M and D, Type I error rates were adversely affected by unequal sample sizes, but more so when overall sample sizes were small. In particular, effects of unequal sample sizes were most pronounced for combinations of 10 with 20 and 10 with 30.

For unequal sample sizes, the Wald, likelihood-ratio, and score tests performed comparably when larger samples were drawn from populations with smaller CVs for all approximations, while the Wald test was consistently more powerful than the likelihood-ratio and score tests when larger samples were associated with larger CVs. This last pattern was present though less pronounced when sample sizes were evenly split between the "low" and "high" levels of interaction. Overall, the effects of unequal sample sizes were strongest when sample sizes were small (in particular, 10 with 20 and 10 with 30).

In general, sample size configurations common to simulations involving overall CVs of 0.1 and 0.2 (all 30, all 50, 30 with 50) showed that powers tended to be slightly lower when the overall CV was larger.

The Main-Effect Test

Tables XL through LXIV summarize the use of a difference of scaled deviances as a test of a main effect in a 2 x 2 factorial experiment, as well as the corresponding Wald and score tests. For both overall CV values and equal sample sizes, the performance of these main-effect tests, based on a single degree of freedom, was virtually identical to the interaction tests above, which also were based on one degree of freedom. However, Type I error rates were slightly improved.

For unequal sample size configurations, for both overall CV values, the main-effect tests performed better than the interaction tests when large samples were coupled with small population CVs and when sample sizes were split evenly between "low" and "high" levels for all approximations. On the other hand, the interaction tests performed better when large samples were combined with large CVs. As in other previous cases involving unequal samples, these results were most pronounced when overall sizes were small (in particular, 10 with 20 and 10 with 30). In addition, the main-effect tests were typically more successful than the interaction tests at maintaining the stated Type I error rate when small and disparate sample sizes were used.

Relative Capabilities of Model Diagnostics

Relative performance among the approximation-diagnostic combinations under consideration was virtually the same for the interaction and main-effect tests. In general, diagnostics using IM tended to have higher power than M or D, while M tended to have slightly higher power than D. However, IM also tended to exceed the Type I error rate more often than M or D.

Among diagnostics associated with a given approximation, when larger samples were associated with smaller population CVs, the likelihood-ratio test tended to have the highest power, followed closely by both the Wald and score tests. When larger samples were associated with larger CVs, the Wald test typically has the highest power, followed closely by the likelihood-ratio test. However, the score test had much lower power when

M or D was used and moderately so when IM was used. This last pattern was also present when sample sizes were unequal but split evenly between "low" and "high" levels of the relevant effect, although the score tests associated with M and D were more powerful. When sample sizes were equal, all tests and approximations were comparable.

<u>The One-Factor Experiment</u>

Tables LXV through CVII summarize the use of the approximation-diagnostic combinations under consideration as tests of the single effect in a one-factor experiment. For equal sample sizes, at both overall CV values, the Type I error rates for IM and Doornbos and Dijkstra's likelihood-ratio test (DDL) were extremely poor but improved as the sample sizes increased. Among all tests, the likelihood-ratio test using IM was the most powerful but had the most difficulty achieving the Type I error rate. Doornbos and Dijkstra's t test (DDT) and Gupta and Ma's score test (GM) consistently had the lowest power but improved as the sample sizes increased.

For one small and two large samples, when the largest samples corresponded to the largest population CVs, the Wald and likelihood-ratio tests using IM were the most powerful. For other configurations involving one small and two large samples, the likelihood-ratio test using IM and DDL were consistently the most powerful. Overall, the score tests using M and D, DDT, and GM preserved the Type I error rate, while the Wald and likelihood-ratio tests using IM and DDL had the worst Type I rates. As overall sample sizes increased, the score test using IM also tended to have a good Type I error rate.

For two small and one large sample, when the single large sample corresponded to the largest population CV, the Wald test using IM was the single most powerful test but had the worst Type I error rate. For other configurations involving two small and one large sample, the likelihood-ratio test using IM and DDL were the most powerful but were among the worst at preserving the Type I rate. Overall, as before, the score tests using M and D, DDT, and GM preserved the Type I error rate and were joined by the score test using IM as overall sample sizes increased.

Among all simulations involving unequal sample size configurations, as sample sizes became more disparate, Type I error rates tended to worsen, but improved as overall sample sizes increased. However, the Wald and likelihood-ratio tests using IM and DDL continued to have difficulty preserving the Type I rate. Among sample size configurations common to both overall CV values, powers tended to be less for an overall CV value of 0.2 as opposed to 0.1.

The Interaction Test for Non-Normal Data

Tables CVIII through CXXV summarize the capabilities of the approximation-diagnostic combinations under consideration as tests of interaction in a 2 x 2 factorial experiment when the observations belong to right-skewed populations. For an overall CV of 0.3, which resulted in some population CV values falling outside the range expected for "ratio-normal" distributions, powers of all tests were somewhat lower for every sample size configuration than for "ratio-normal" data. However, except for IM in some small sample cases, the simulated Type I error rates were below the stated 0.05 level.

For an overall CV of 0.6, powers were substantially lower among all sample size configurations compared to the corresponding tests when the overall CV was at 0.3. The powers for the score tests summarized in Table CXXIII were particularly poor. However, all Type I error rates attained the stated level. Overall, the likelihood-ratio test using IM retained the most power.

In some cases where the sample sizes were moderately small, the score tests actually decreased in power when an extremely large interaction effect was present! In order to determine if these results were due to sampling error alone, all affected rejection rates were tested for a significant decrease at the 0.05 level, and those found to be significant are bolded in the tables (in particular, tables CXIV, CXV, CXIX, and CXXIII are affected). These results can apparently be attributed to the inability of the approximate likelihood surfaces under consideration to completely incorporate extreme effect sizes associated with right-skewed populations having CVs outside the range expected for "ratio-normal" data.

Recommendations

For factorial experiments, if sample sizes are equal, the score tests using M and D are preferred. These tests preserve the Type I error rate but have powers comparable to the other approximation-diagnostic combinations for hypotheses involving both saturated and reduced models as indicated by the interaction and main-effect tests simulated here.

On the other hand, if sample sizes are small and unequal, and large samples are associated with the largest population CVs, these score tests can perform poorly. In this case, the likelihood-ratio tests using M and D are preferred. Practically, however, this situation can be avoided by insuring that all sample sizes exceed 20.

For one-factor experiments, if the sample sizes are equal, the score tests using M and D are preferred since they preserve the Type I error rate and have higher power than DDT and GM for very small samples. These tests also are generally better when sample sizes are unequal, though DDT and GM are best in some cases.

If data are suspected of belonging to right-skewed distributions in a factorial experiment, but the population CVs are within the range (0, 1/3) for "ratio-normal" data, the same recommendations given above apply. If the population CVs are outside this range, however, the likelihood-ratio test using IM is generally preferred. In this case, based on simulation results, the score tests are not recommended.

CHAPTER SIX

APPLIED EXAMPLES

In this chapter, two applied examples of factorial experiments are introduced and the appropriate models are fitted. The first example is a component of a data set originally discussed by Gerig and Sen (1980) involving relative variability of duck kills in Canadian provinces for the years 1969 and 1970. Gupta and Ma (1996) utilizes one-factor tests for each province to test for a difference in years. With the new method, a global test for interaction between province and year is available. The second example is based on two data sets given in Ott (1993) where the pH level of drug vials stored at two temperatures in two different labs is the variable of interest.

Applied Example #1

During each of the years 1969 and 1970, as part of the Canada migratory game bird surveys, random samples of hunters were drawn from each Canadian province using lists of the previous year's permit holders. Among hunters reporting at least one duck kill (excluding sea ducks), the log (base 10) of the number of kills per hunter was recorded. Although the log transformation compromises the ratio-level nature of the data, it was claimed by Gerig and Sen (1980) to be necessary in order to induce normality. The observed sample means, standard deviations, and CVs for the four westernmost provinces are given in Table X. Although the data are no longer of the ratio type, they are strictly non-negative. As a result, judging from the observed CV values, the assumption of normality is questionable.

Gupta and Ma (1996) uses a variety of one-factor tests to assess the hypothesis claimed by Gerig and Sen (1980) that the relative variability of log-duck-kills per hunter for 1969 and 1970 were equal for each province. However, no global test of interaction between province and year was available.

For simplicity, only the four westernmost provinces were reanalyzed using a 2 x 4 factorial model. The saturated model has the form

$$R_{ij} = \exp(R^* + \alpha_i + \beta_j + (\alpha\beta)_{ij}), \ i = 1, 2, \ j = 1, 2, 3, 4,$$

where $\exp(R^*)$ is the overall population CV, $\exp(\alpha_i)$ is the effect of the i^{th} year, $\exp(\beta_j)$ is the effect of the j^{th} province, and the terms $(\alpha\beta)_{ij}$ describe the interaction between year and province. The identifiability constraint $\sum_{i=1}^{2}\alpha_i = \sum_{j=1}^{4}\beta_j = \sum_{i=1}^{2}(\alpha\beta)_{ij} = \sum_{j=1}^{4}(\alpha\beta)_{ij} = 0$ was used. In matrix form, the resulting generalized linear model is given by

$$\begin{bmatrix} \log R_{11} \\ \log R_{12} \\ \log R_{13} \\ \log R_{14} \\ \log R_{21} \\ \log R_{22} \\ \log R_{23} \\ \log R_{24} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} R^* \\ \alpha_1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ (\alpha\beta)_{11} \\ (\alpha\beta)_{12} \\ (\alpha\beta)_{13} \end{bmatrix}.$$

For brevity, only Iglewicz and Myers' approximation was applied to fit the saturated model. The resulting tests of significance of the interaction, based on three degrees of freedom, are appended to Table XI. All tests were clearly significant at the 0.05 level,

TABLE X

OBSERVED MEANS, STANDARD DEVIATIONS, AND CVs OF LOG-DUCK-KILLS (BASE 10) PER HUNTER BY PROVINCE AND YEAR

Province	Year	n	$\overline{x}$	s	r
British Columbia	1969	503	0.9299	0.4680	0.5034
	1970	743	0.9539	0.4906	0.5143
Alberta	1969	654	1.0817	0.4350	0.4021
	1970	882	1.0474	0.4645	0.4435
Saskatchewan	1969	863	1.0085	0.4080	0.4046
	1970	977	1.1084	0.4214	0.3802
Manitoba	1969	1,102	0.9653	0.4301	0.4455
	1970	1,031	1.0080	0.4261	0.4228

TABLE XI

ESTIMATED PARAMETERS FOR SATURATED MODEL OF CVs OF LOG-DUCK-KILLS (BASE 10) PER HUNTER

Parameter	Estimate	Standard Error	Effect
R^*	-0.8271	0.01049	log overall CV
α_1	-0.000615	0.01049	1969
β_1	0.1514	0.02064	British Columbia
β_2	-0.03496	0.01831	Alberta
β_3	-0.1089	0.01699	Saskatchewan
$(\alpha\beta)_{11}$	-0.01010	0.02064	1969 / B.C.
$(\alpha\beta)_{12}$	-0.04838	0.01831	1969 / Alberta
$(\alpha\beta)_{13}$	0.03172	0.01699	1969 / Sask.

Tests for Interaction: Wald χ^2: 10.152, $p = 0.0173$; LR χ^2: 10.089, $p = 0.0178$; Score χ^2: 9.626, $p = 0.0220$, each based on 3 df.

suggesting that the ratio of the relative variabilities of log-duck-kills per hunter (1969 to 1970) was not consistent across provinces. Estimated parameters for the saturated model are given in Table XI. Asymptotic 95% confidence intervals for the log-ratio and ratio of population CVs (1969 to 1970) for each province are given in Table XII. Apparently, the relative variability of log-duck kills per hunter for Alberta was only between 0.83 and 0.99 times as large in 1969 as in 1970. No significant difference was found for the other provinces.

To demonstrate the versatility of the current modelling technique, the relative variability of the western provinces (British Columbia and Alberta) was contrasted with that of the central provinces (Saskatchewan and Manitoba) for each year. For the multiplicative model, these contrasts estimate the ratio of the geometric average of the "western" population CVs to the geometric average of the "central" population CVs.

Because a normal population CV is a ratio of distinct parameters, a geometric average of two or more CVs preserves information about the contributing means and standard deviations that is typically lost by taking an arithmetic average. This suggests that the multiplicative model should generally be used even in the one-factor case.

For 1969, the contrast has the form

$$\log\left(\frac{\left(R_{11}R_{12}\right)^{1/2}}{\left(R_{13}R_{14}\right)^{1/2}}\right) = \frac{1}{2}\log R_{11} + \frac{1}{2}\log R_{12} - \frac{1}{2}\log R_{13} - \frac{1}{2}\log R_{14}$$

$$= \beta_1 + \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{12}\,.$$

The estimated log-ratio and ratio are 0.05799 and 1.0597, respectively, and the corresponding asymptotic 95% confidence intervals are (-0.002843, 0.1188) and (0.9972,

TABLE XII

ASYMPTOTIC 95% CONFIDENCE INTERVALS FOR LOG-RATIO AND RATIO OF CVs OF LOG-DUCK-KILLS (BASE 10) PER HUNTER (1969 TO 1970) BY PROVINCE

	Log-Ratio		Ratio	
Province	Estimate	CI	Estimate	CI
B. C.	-0.02142	(-0.1199, 0.07710)	0.9788	(0.8870, 1.0802)
Alberta	-0.09800	(-0.1812, -0.01480)	0.9067	(0.8343, 0.9853)
Sask.	0.06220	(-0.01189, 0.1363)	1.0642	(0.9882, 1.1460)
Manitoba	0.05230	(-0.01816, 0.1228)	1.0537	(0.9820, 1.1306)

1.1262). For 1970, the estimated log-ratio and ratio are 0.1750 and 1.1912, with asymptotic 95% confidence intervals (0.1204, 0.2295) and (1.1279, 1.2580).

It would appear that the relative variability of the western provinces was significantly higher than that of the central provinces in 1970 but not in 1969, which helps to explain the significant interaction.

Applied Example #2

Ott (1993, pp. 916, 919) lists the observed pH levels of 2-mL vials of a drug product stored at each of two temperatures (30°C and 40°C) in two labs (#1 and #2). Twelve vials were examined from each temperature-lab combination. The data, along with the sample means, standard deviations, CVs, and Shapiro-Wilk statistics for testing normality (SAS Institute, Inc., 1990, p. 627) are given in Table XIII. The objective in this applied example is to estimate a factorial model that describes how each factor influences the relative variability of the pH.

Technically, pH is not a ratio-level variable, partly because a negative pH is possible, and partly because the pH is computed as a (negative) log (base 10) of hydrogen ion concentration in a solution (Holtzclaw and Robinson, 1988, pp. 479-480). However, it represents a variable whose relative consistency is potentially of interest and so is considered here.

The saturated model has the form

$$R_{ij} = \exp(R^* + \alpha_i + \beta_j + (\alpha\beta)_{ij}),\ i = 1, 2,\ j = 1, 2,$$

where $\exp(R^*)$ is the overall population CV, $\exp(\alpha_i)$ is the effect of the i^{th} temperature, $\exp(\beta_j)$ is the effect of the j^{th} lab, and the terms $\exp((\alpha\beta)_{ij})$ describe the interaction

TABLE XIII

OBSERVED MEANS, STANDARD DEVIATIONS, AND CVs OF pH LEVELS BY TEMPERATURE AND LAB

Temperature	Lab	pH Data	$\overline{x}$	s_n	r_n
30°C	#1	3.45, 3.48, 3.50, 3.55	3.5883	0.09754	0.02718
(W = 0.905, p = 0.173)		3.56, 3.57, 3.59, 3.60			
		3.60, 3.61, 3.74, 3.81			
30°C	#2	3.70, 3.74, 3.75, 3.76	3.8108	0.06689	0.01755
(W = 0.921, p = 0.277)		3.77, 3.80, 3.80, 3.84			
		3.87, 3.90, 3.90, 3.90			
40°C	#1	3.29, 3.32, 3.38, 3.39	3.5108	0.1348	0.03838
(W = 0.931, p = 0.367)		3.45, 3.51, 3.59, 3.60			
		3.61, 3.63, 3.65, 3.71			
40°C	#2	3.60, 3.64, 3.68, 3.70	3.7233	0.06587	0.01769
(W = 0.906, p = 0.179)		3.70, 3.70, 3.70, 3.75			
		3.80, 3.80, 3.80, 3.81			

Values given in parentheses are the Shapiro-Wilk statistics and p-values for testing the null hypotheses that the samples were drawn from normal distributions.

between temperature and lab. The identifiability constraint $\sum_{i=1}^{2} \alpha_i = \sum_{j=1}^{2} \beta_j = \sum_{i=1}^{2} (\alpha\beta)_{ij} =$

$\sum_{j=1}^{2} (\alpha\beta)_{ij} = 0$ was used. In matrix form, the resulting generalized linear model is given by

$$\begin{bmatrix} \log R_{11} \\ \log R_{12} \\ \log R_{21} \\ \log R_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} R^* \\ \alpha_1 \\ \beta_1 \\ (\alpha\beta)_{11} \end{bmatrix}.$$

McKay's approximation was applied to fit the model. The corresponding tests for interaction, based on one degree of freedom, are appended to Table XIV. Note that there is clearly no evidence of interaction, so that a reduced model with only main effects was considered.

In order to determine if both main effects are significant, Wald tests were conducted on the estimated parameters of the main-effects model, while each main effect was individually removed in turn so that associated likelihood-ratio and score tests also could be constructed. These conditional χ^2 statistics for assessing the significance of temperature and lab, each based on one degree of freedom, also are appended to Table XIV.

Apparently, temperature can be removed from the model. The estimated parameters of the resulting "lab" model are given in Table XIV. Updated tests for the significance of the lab effect also are included. To assess the overall adequacy of this model, a global test for interaction and temperature effects also was conducted which corroborated these findings (Wald χ^2: 1.307, $p = 0.520$; LR χ^2: 1.282, $p = 0.527$; Score χ^2: 1.210, $p = 0.546$, each based on 2 df).

TABLE XIV

ESTIMATED PARAMETERS FOR LAB
MODEL OF CVs OF pH LEVELS

Parameter	Estimate	Standard Error	Effect
R*	-3.6775	0.1067	log overall CV
β_1	0.3175	0.1067	Lab #1

Tests for Interaction: Wald χ^2: 0.624, p = 0.429; LR χ^2: 0.621, p = 0.431; Score χ^2: 0.613, p = 0.434, each based on 1 df.

Tests for Temperature | Lab: Wald χ^2: 0.683, p = 0.409; LR χ^2: 0.661, p = 0.416; Score χ^2: 0.633, p = 0.426, each based on 1 df.

Tests for Lab | Temperature: Wald χ^2: 8.065, p = 0.005; LR χ^2: 7.437, p = 0.006; Score χ^2: 6.185, p = 0.013, each based on 1 df.

Tests for Lab Only: Wald χ^2: 8.859, p = 0.003; LR χ^2: 8.323, p = 0.004; Score χ^2: 6.930, p = 0.008, each based on 1 df.

Based on the fitted "lab" model, the estimated log-ratio and ratio for lab (#1 to #2), irrespective of storage temperature, are 0.6350 and 1.8871, respectively, while the asymptotic 95% confidence intervals are (0.2168, 1.0532) and (1.2421, 2.8668). It appears that vials stored in lab #1 have a significantly higher relative variability than those stored in lab #2.

CHAPTER SEVEN

CONCLUSION

The modelling approach developed in this thesis is significant because it expands the settings in which the normal population CV may be analyzed to include designed factorial experiments. In particular, the use of approximations of the distribution of the sample CV provides a context well suited to the application of the generalized linear model and its iterative algorithms for model estimation. When the CV is the population characteristic of interest, the approach is apparently superior to the modelling efforts associated with Taguchi because it incorporates estimable model and covariance structures for the observed sample CVs rather than use transformed CVs that are assumed to have constant variance. As a result, estimated model parameters are easily interpreted, tests of all effects in a fitted factorial model are available, and asymptotic confidence intervals for ratios of contrasted population CVs are readily obtained. Further, the approach incorporates several tests for the equality of population CVs in a one-factor experiment which have previously been discussed in the literature.

Several related topics are available for future research. Of principal importance is a detailed investigation of the behavior of the exact and approximate likelihood surfaces under consideration in the context of the score test. Evidently, in both factorial and one-factor experiments, when larger samples are associated with populations having larger CV values, and the overall sample sizes are small, the likelihood surfaces are poorly behaved, since the powers of the associated score tests are very low. The existence of a unique maximum is apparently not an issue, but rather the behavior of the surfaces at points other than the maximum.

The current modelling approach is based on approximate distributions because the structure of these distributions is simple and easily incorporated into the theory of generalized linear models. However, an exact model of the population CV also could be obtained using the normal likelihood of the observed data reparameterized in terms of the R_i and μ_i as in Gupta and Ma (1996). In this case, model (4.1) and a corresponding multiplicative model of the μ_i could be estimated via maximum-likelihood and compared to the current models, although a direct application of the Fisher scoring algorithm without the benefit of generalized linear models would be necessary.

Rather than estimate the variances of the observed sample CVs in the context of a generalized linear model, the observed data might also be resampled via a bootstrap or jackknife technique to obtain estimated variances which could be incorporated into a weighted least squares model. For example, if a multiplicative model of the R_i is desired, then (4.5) might be estimated as an additive model using the log r_i as the responses, with weights obtained by estimating the variances of the log r_i using a resampling scheme. Estimated model parameters could be tested using a Wald procedure and reduced models could be fitted using the resampled variance estimates as weights. Presumably, resulting parameter estimates would at least be comparable to those obtained via generalized linear models and better than those obtained using the Taguchi approach.

Lastly, the likelihood-ratio test using Iglewicz and Myers' approximation often has the best power as a test of an effect in a factorial or one-factor experiment. However, its Type I error rate is extremely poor, especially for small samples. As a result, Bartlett's correction factor could likely be used to improve the χ^2 approximation for this test when sample sizes are small (Shafer and Sullivan, 1986), although its effect on power would need to be investigated.

BIBLIOGRAPHY

Agresti, A. (1990), *Categorical Data Analysis*, New York: John Wiley and Sons, Inc.

Ahmed, S. E. (1994), "Improved Estimation of the Coefficient of Variation," *Journal of Applied Statistics*, 21(6), pp. 565-573.

Bennett, B. M. (1976), "On an Approximate Test for Homogeneity of Coefficents of Variation," *Contributions to Applied Statistics*, ed. W. J. Ziegler, pp. 169-171.

Bissell, A. F. (1989), "Interpreting Mean Squares in Saturated Fractional Designs," *Journal of Applied Statistics*, 16(1), pp. 7-18.

Box, G. (1988), "Signal-to-Noise Ratios, Performance Criteria, and Transformations," *Technometrics*, 30(1), pp. 1-17.

Burington, R. S. (1947), *Handbook of Mathematical Tables and Formulas*, Sandusky: Handbook Publishers, Inc.

Carroll, R. J. and Ruppert D. (1988), *Transformations and Weighting in Regression*, New York: Chapman and Hall.

David, F. N. (1949), "Note on the Application of Fisher's k-statistics," *Biometrika*, 36, pp. 383-393.

Doornbos, R. and Dijkstra, J. B. (1983), "A Multi Sample Test for the Equality of Coefficients of Variation in Normal Populations," *Communications in Statistics - Simulation and Computation*, 12(2), pp. 147-158.

Eliason, S. R. (1993), *Maximum Likelihood Estimation: Logic and Practice*, Newbury Park: Sage Publicatons.

Fahrmeir, L. (1987), "Asymptotic Testing Theory for Generalized Linear Models," *Statistics*, 18(1), pp. 65-76.

Fahrmeir, L. and Kaufmann, H. (1985), "Consistency and Asymptotic Normality of the Maximum Likelihood Estimator in Generalized Linear Models," *The Annals of Statistics*, 13(1), pp. 342-368.

Feltz, J. F. and Miller, G. E. (1996), "An Asymptotic Test for the Equality of Coefficients of Variation from k Populations," *Statistics in Medicine*, 15, pp. 647-658.

Gerig, T. M. and Sen, A. R. (1980), "MLE in Two Normal Samples with Equal but Unknown Population Coefficients of Variation," *Journal of the American Statistical Association*, 75, pp. 704-708.

Gupta, R. C. and Ma, S. (1996), "Testing the Equality of Coefficients of Variation in k Normal Populations," *Communications in Statistics - Theory and Methods*, 25(1), pp. 115-132.

Hillis, S. L. and Davis, C. S. (1994), "A Simple Justification of the Iterative Fitting Procedure for Generalized Linear Models," *The American Statistician*, 48(4), pp. 288-289.

Holtzclaw, Jr., H. F. and Robinson, W. R. (1988), *General Chemistry, 8th Ed.*, Lexington, MA: D. C. Heath and Co.

Hurlimann, W. (1995), "A Uniform Approximation to the Sampling Distribution of the Coefficient of Variation," *Statistics and Probability Letters*, 24, pp. 263-268.

Iglewicz, B. and Myers, R. H. (1970), "Comparisons of Approximations to the Percentage Points of the Sample Coefficient of Variation," *Technometrics*, 12(1), pp. 166-169.

Judge, G. G., Hill, R. C., Griffiths, W. E., Lutkepohl, H. and Lee, T. (1988), *Introduction to the Theory and Practice of Econometrics*, New York: John Wiley and Sons, Inc.

Koopmans, L. H., Owen, D. B., and Rosenblatt, J. I. (1964), "Confidence Intervals for the Coefficient of Variation for the Normal and Log-Normal Distributions," *Biometrika*, 51, pp. 25-32.

Lehmann, E. L. (1959), *Testing Statistical Hypotheses*, New York: John Wiley and Sons, Inc.

McCullagh, P. (1983), "Quasi-Likelihood Functions," *The Annals of Statistics*, 11(1), pp. 59-67.

McCullagh, P. (1986), "Quasi-Likelihood Functions," *Encyclopedia of Statistical Sciences*, eds. S. Kotz and N. L. Johnson, 7, pp. 464-467.

McCullagh, P. and Nelder, J. A. (1983), *Generalized Linear Models*, New York: Chapman and Hall.

McCullagh, P. and Nelder, J. A. (1989), *Generalized Linear Models, 2nd Ed.*, New York: Chapman and Hall.

McKay, A. T. (1932), "Distribution of the Coefficient of Variation and the Extended 't' Distribution," *Journal of the Royal Statistical Society*, 95, pp. 695-698.

Maghsoodloo, S. (1990), "The Exact Relation of Taguchi's Signal-to-Noise Ratio to His Quality Loss Function," *Journal of Quality Technology*, 22(1), pp. 57-67.

Miller, E. C. and Karson, M. J. (1977), "Testing Equality of Two Coefficients of Variation," *American Statistical Association: Proceedings of the Business and Economics Section,Part I*, pp. 278-283.

Nelder, J. A. and Wedderburn, R. W. M. (1972), "Generalized Linear Models," *Journal of the Royal Statistical Society*, 135, pp. 370-384.

Nelder, J. A. and Lee, Y. (1991), "Generalized Linear Models for the Analysis of Taguchi-Type Experiments," *Applied Stochastic Models and Data Analysis*, 7, pp. 107-120.

Owen, D. B. (1968), "A Survey of Properties and Applications of the Noncentral t-Distribution," *Technometrics*, 10(3), pp. 445-477.

Ott, R. L. (1993), *An Introduction to Statistical Methods and Data Analysis, 4th Ed.*, Belmont, CA: Duxbury Press.

Payton, M. E. (1997), "Confidence Intervals for the Coefficient of Variation," *Applied Statistics in Agriculture*, pp. 82-89.

Pitman, E. J. G. (1939), "Tests of Hypotheses Concerning Location and Scale Parameters," *Biometrika*, 31, pp. 200-215.

Rao, C. R. (1973), *Linear Statistical Inference and Its Application*, New York: John Wiley and Sons, Inc.

Rao, K. A. and Bhatt, A. R. S. (1995), "Tests for Coefficients of Variation," *Journal of the Indian Society of Agricultural Statistics*, 47(3), pp. 225-229.

Rao, K. A. and Vidya, R. (1992), "On the Performance of a Test for Coefficients of Variation," *Calcutta Statistical Association Bulletin*, 42, March & June, pp. 87-95.

Reh, W. and Scheffler, B. (1996), "Significance Tests and Confidence Intervals for Coefficients of Variation," *The Statistical Software Newsletter*, 22, pp. 449-452.

SAS Institute, Inc. (1989), *SAS / IML Software: Usage and Reference, Version 6*, Cary, NC: SAS Institute, Inc.

SAS Institute, Inc. (1990), *SAS Procedures Guide, Version 6, 3rd Ed.*, Cary, NC: SAS Institute, Inc.

Schmidt, S. R. and Launsby, R. G. (1994), *Understanding Industrial Designed Experiments*, Colorado Springs: Air Academy Press.

Serfling, R. J. (1980), *Approximation Theorems of Mathematical Statistics*, New York: John Wiley and Sons, Inc.

Shafer, N. J. and Sullivan, J. A. (1986), "A Simulation Study of a Test for the Equality of the Coefficients of Variaton," *Communications in Statistics - Simulation*, 15(3), pp. 681-695.

Silvey, S. D. (1975), *Statistical Inference, 2nd Ed.*, London: Chapman and Hall.

Taguchi, G. (1992), *Taguchi Methods: Research and Development*, Dearborn, MI: American Supplier Institute.

Umphrey, G. J. (1983), "A Comment on McKay's Approximation for the Coefficient of Variation," *Communications in Statistics - Simulation and Computation*, 12(5), pp. 629-635.

Vangel, M. G. (1996), "Confidence Intervals for a Normal Coefficient of Variation," *The American Statistician*, 15(1), pp. 21-26.

Warren, W. G. (1982), "On the Adequacy of the Chi-Squared Approximation for the Coefficient of Variation," *Communications in Statistics - Simulation and Computation*, 11(6), pp. 659-666.

Wedderburn, R. W. M. (1974), "Quasi-Likelihood Functions, Generalized Linear Models, and the Gauss-Newton Method," *Biometrika*, 61(3), pp. 439-447.

Wedderburn, R. W. M. (1976), "On the Uniqueness of the Maximum Likelihood Estimates for Certain Generalized Linear Models," *Biometrika*, 63(1), pp. 27-32.

Williams, R. R. (1991), "Fundamental Limitations on the Use and Comparison of Signal-to-Noise Ratios," *Analytical Chemistry*, 63(15), pp. 1638-1643.

Zacks, S. (1991), "Inference Based on Taguchi's Saturated Designs," *Communications in Statistics - Theory and Methods*, 20(2), pp. 497-510.

APPENDIXES

APPENDIX A

TABLES OF SIMULATION RESULTS

TABLE XV

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0613	0.0574	0.0452
	D	0.0610	0.0569	0.0452
	IM	0.0755	0.0744	0.0708
1.1	M	0.1422	0.1347	0.1095
	D	0.1419	0.1345	0.1085
	IM	0.1631	0.1618	0.1551
1.2	M	0.3414	0.3291	0.2882
	D	0.3409	0.3285	0.2876
	IM	0.3745	0.3723	0.3631
1.3	M	0.5976	0.5869	0.5389
	D	0.5964	0.5863	0.5380
	IM	0.6324	0.6301	0.6218
1.4	M	0.7963	0.7887	0.7519
	D	0.7957	0.7883	0.7505
	IM	0.8233	0.8215	0.8144
1.5	M	0.9154	0.9111	0.8892
	D	0.9150	0.9108	0.8888
	IM	0.9287	0.9277	0.9239
1.6	M	0.9689	0.9669	0.9574
	D	0.9687	0.9669	0.9571
	IM	0.9763	0.9758	0.9734

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XVI

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 20$

Effect Size		Wald	LR	Score
1	M	0.0539	0.0521	0.0457
	D	0.0538	0.0519	0.0456
	IM	0.0590	0.0588	0.0572
1.1	M	0.2165	0.2117	0.1969
	D	0.2163	0.2117	0.1969
	IM	0.2315	0.2300	0.2254
1.2	M	0.5953	0.5879	0.5675
	D	0.5946	0.5876	0.5671
	IM	0.6149	0.6138	0.6088
1.3	M	0.8904	0.8880	0.8769
	D	0.8902	0.8879	0.8769
	IM	0.8995	0.8991	0.8970
1.4	M	0.9823	0.9816	0.9790
	D	0.9823	0.9816	0.9790
	IM	0.9840	0.9838	0.9837
1.5	M	0.9977	0.9976	0.9973
	D	0.9977	0.9976	0.9973
	IM	0.9980	0.9980	0.9980
1.6	M	0.9999	0.9999	0.9999
	D	0.9999	0.9999	0.9999
	IM	0.9999	0.9999	0.9999

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XVII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0519	0.0506	0.0471
	D	0.0516	0.0506	0.0470
	IM	0.0562	0.0559	0.0543
1.1	M	0.3088	0.3061	0.2925
	D	0.3088	0.3059	0.2924
	IM	0.3217	0.3207	0.3175
1.2	M	0.7894	0.7859	0.7770
	D	0.7892	0.7857	0.7767
	IM	0.7988	0.7982	0.7960
1.3	M	0.9759	0.9748	0.9731
	D	0.9759	0.9748	0.9731
	IM	0.9783	0.9781	0.9772
1.4	M	0.9984	0.9983	0.9983
	D	0.9984	0.9983	0.9983
	IM	0.9986	0.9986	0.9985
1.5	M	0.9999	0.9999	0.9999
	D	0.9999	0.9999	0.9999
	IM	0.9999	0.9999	0.9999
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XVIII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0534	0.0524	0.0511
	D	0.0534	0.0523	0.0511
	IM	0.0557	0.0556	0.0551
1.1	M	0.4627	0.4598	0.4539
	D	0.4627	0.4595	0.4536
	IM	0.4694	0.4690	0.4678
1.2	M	0.9432	0.9428	0.9404
	D	0.9432	0.9428	0.9404
	IM	0.9464	0.9462	0.9451
1.3	M	0.9993	0.9993	0.9993
	D	0.9993	0.9993	0.9993
	IM	0.9993	0.9993	0.9993
1.4	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XIX

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 10$, $n_{12} = n_{21} = 20$

Effect Size		Wald	LR	Score
1*	M	0.0646	0.0601	0.0307
	D	0.0646	0.0600	0.0307
	IM	0.0772	0.0752	0.0551
1.1	M	0.1330	0.1468	0.1390
	D	0.1325	0.1462	0.1381
	IM	0.1522	0.1594	0.1478
1.2	M	0.3908	0.4205	0.4106
	D	0.3899	0.4196	0.4103
	IM	0.4257	0.4389	0.4224
1.3	M	0.6811	0.7022	0.6964
	D	0.6804	0.7013	0.6957
	IM	0.7059	0.7190	0.7040
1.4	M	0.8706	0.8834	0.8804
	D	0.8702	0.8831	0.8794
	IM	0.8857	0.8922	0.8847
1.5	M	0.9572	0.9627	0.9608
	D	0.9568	0.9625	0.9606
	IM	0.9631	0.9657	0.9627
1.6	M	0.9872	0.9889	0.9884
	D	0.9870	0.9888	0.9883
	IM	0.9891	0.9907	0.9890

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XX

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 20$, $n_{12} = n_{21} = 10$

Effect Size		Wald	LR	Score
1*	M	0.0646	0.0601	0.0307
	D	0.0646	0.0600	0.0307
	IM	0.0772	0.0752	0.0551
1.1	M	0.2063	0.1782	0.0665
	D	0.2063	0.1783	0.0665
	IM	0.2275	0.2159	0.1561
1.2	M	0.4782	0.4296	0.2229
	D	0.4782	0.4295	0.2226
	IM	0.5101	0.4917	0.3932
1.3	M	0.7579	0.7179	0.4844
	D	0.7578	0.7179	0.4842
	IM	0.7846	0.7692	0.6881
1.4	M	0.9288	0.9085	0.7573
	D	0.9288	0.9085	0.7572
	IM	0.9379	0.9328	0.8936
1.5	M	0.9844	0.9787	0.9167
	D	0.9844	0.9787	0.9167
	IM	0.9878	0.9863	0.9734
1.6	M	0.9981	0.9972	0.9809
	D	0.9981	0.9972	0.9809
	IM	0.9986	0.9982	0.9959

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXI

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$ FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 20$, $n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0645	0.0602	0.0301
	D	0.0640	0.0601	0.0301
	IM	0.0747	0.0743	0.0563
1.1	M	0.1630	0.1549	0.0926
	D	0.1626	0.1542	0.0925
	IM	0.1841	0.1825	0.1479
1.2	M	0.4457	0.4337	0.3143
	D	0.4453	0.4328	0.3136
	IM	0.4764	0.4733	0.4206
1.3	M	0.7216	0.7091	0.5893
	D	0.7211	0.7085	0.5887
	IM	0.7486	0.7461	0.6980
1.4	M	0.8968	0.8908	0.8191
	D	0.8967	0.8906	0.8188
	IM	0.9091	0.9078	0.8837
1.5	M	0.9730	0.9712	0.9443
	D	0.9730	0.9712	0.9442
	IM	0.9781	0.9776	0.9679
1.6	M	0.9930	0.9923	0.9837
	D	0.9830	0.9923	0.9837
	IM	0.9943	0.9941	0.9914

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 20$, $n_{12} = n_{21} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0546	0.0522	0.0402
	D	0.0546	0.0522	0.0402
	IM	0.0602	0.0597	0.0522
1.1	M	0.2345	0.2436	0.2414
	D	0.2343	0.2435	0.2406
	IM	0.2491	0.2546	0.2503
1.2	M	0.6710	0.6797	0.6780
	D	0.6706	0.6793	0.6774
	IM	0.6840	0.6882	0.6857
1.3	M	0.9251	0.9282	0.9274
	D	0.9247	0.9281	0.9274
	IM	0.9303	0.9313	0.9306
1.4	M	0.9905	0.9909	0.9909
	D	0.9904	0.9909	0.9908
	IM	0.9912	0.9915	0.9913
1.5	M	0.9991	0.9992	0.9992
	D	0.9991	0.9992	0.9992
	IM	0.9993	0.9993	0.9993
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXIII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 30$, $n_{12} = n_{21} = 20$

Effect Size		Wald	LR	Score
1*	M	0.0546	0.0522	0.0402
	D	0.0546	0.0522	0.0402
	IM	0.0602	0.0597	0.0522
1.1	M	0.2782	0.2620	0.2004
	D	0.2782	0.2618	0.2004
	IM	0.2919	0.2847	0.2555
1.2	M	0.6989	0.6793	0.6120
	D	0.6989	0.6791	0.6119
	IM	0.7135	0.7059	0.6724
1.3	M	0.9477	0.9432	0.9192
	D	0.9477	0.9431	0.9192
	IM	0.9512	0.9497	0.9403
1.4	M	0.9963	0.9955	0.9925
	D	0.9963	0.9955	0.9925
	IM	0.9970	0.9967	0.9952
1.5	M	0.9994	0.9993	0.9991
	D	0.9994	0.9993	0.9991
	IM	0.9994	0.9994	0.9993
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXIV

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 30$, $n_{21} = n_{22} = 20$

Effect Size		Wald	LR	Score
1	M	0.0547	0.0529	0.0422
	D	0.0545	0.0527	0.0422
	IM	0.0605	0.0601	0.0532
1.1	M	0.2525	0.2482	0.2209
	D	0.2524	0.2479	0.2204
	IM	0.2658	0.2649	0.2492
1.2	M	0.6851	0.6801	0.6412
	D	0.6851	0.6798	0.6405
	IM	0.7010	0.6995	0.6805
1.3	M	0.9321	0.9305	0.9154
	D	0.9320	0.9304	0.9152
	IM	0.9370	0.9365	0.9311
1.4	M	0.9935	0.9935	0.9902
	D	0.9935	0.9935	0.9902
	IM	0.9941	0.9941	0.9935
1.5	M	0.9996	0.9996	0.9996
	D	0.9996	0.9996	0.9996
	IM	0.9996	0.9996	0.9996
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXV

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 10$, $n_{12} = n_{21} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0657	0.0577	0.0160
	D	0.0655	0.0575	0.0159
	IM	0.0771	0.0729	0.0325
1.1	M	0.1307	0.1530	0.1262
	D	0.1305	0.1523	0.1256
	IM	0.1503	0.1635	0.1235
1.2	M	0.4017	0.4418	0.4036
	D	0.4006	0.4406	0.4024
	IM	0.4320	0.4532	0.3947
1.3	M	0.6971	0.7322	0.6997
	D	0.6962	0.7315	0.6987
	IM	0.7219	0.7422	0.6887
1.4	M	0.8988	0.9155	0.8991
	D	0.8983	0.9151	0.8989
	IM	0.9103	0.9191	0.8959
1.5	M	0.9664	0.9718	0.9667
	D	0.9663	0.9717	0.9665
	IM	0.9709	0.9732	0.9659
1.6	M	0.9903	0.9923	0.9905
	D	0.9903	0.9923	0.9904
	IM	0.9920	0.9925	0.9900

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXVI

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$ FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 30$, $n_{12} = n_{21} = 10$

Effect Size		Wald	LR	Score
1*	M	0.0657	0.0577	0.0160
	D	0.0655	0.0575	0.0159
	IM	0.0771	0.0729	0.0325
1.1	M	0.2302	0.1877	0.0215
	D	0.2302	0.1877	0.0215
	IM	0.2549	0.2352	0.1123
1.2	M	0.5363	0.4729	0.1079
	D	0.5364	0.4730	0.1078
	IM	0.5705	0.5431	0.3449
1.3	M	0.8206	0.7753	0.3133
	D	0.8208	0.7755	0.3132
	IM	0.8420	0.8275	0.6523
1.4	M	0.9612	0.9426	0.5883
	D	0.9613	0.9426	0.5884
	IM	0.9684	0.9624	0.8836
1.5	M	0.9928	0.9883	0.8268
	D	0.9928	0.9883	0.8267
	IM	0.9943	0.9930	0.9742
1.6	M	0.9996	0.9992	0.9468
	D	0.9996	0.9992	0.9467
	IM	0.9997	0.9996	0.9966

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXVII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 30$, $n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0651	0.0617	0.0138
	D	0.0651	0.0615	0.0138
	IM	0.0754	0.0739	0.0395
1.1	M	0.1828	0.1722	0.0543
	D	0.1825	0.1717	0.0541
	IM	0.2041	0.2014	0.1224
1.2	M	0.4715	0.4538	0.2170
	D	0.4705	0.4533	0.2165
	IM	0.5026	0.5001	0.3713
1.3	M	0.7720	0.7577	0.5119
	D	0.7717	0.7573	0.5110
	IM	0.7929	0.7910	0.6855
1.4	M	0.9324	0.9258	0.7791
	D	0.9323	0.9257	0.7787
	IM	0.9427	0.9419	0.8887
1.5	M	0.9831	0.9812	0.9270
	D	0.9830	0.9812	0.9267
	IM	0.9865	0.9862	0.9709
1.6	M	0.9968	0.9958	0.9801
	D	0.9968	0.9958	0.9801
	IM	0.9976	0.9975	0.9935

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXVIII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 30$, $n_{12} = n_{21} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0545	0.0530	0.0425
	D	0.0545	0.0530	0.0425
	IM	0.0583	0.0579	0.0493
1.1	M	0.3453	0.3558	0.3498
	D	0.3448	0.3553	0.3495
	IM	0.3542	0.3608	0.3494
1.2	M	0.8522	0.8593	0.8559
	D	0.8521	0.8593	0.8556
	IM	0.8586	0.8623	0.8552
1.3	M	0.9898	0.9904	0.9901
	D	0.9898	0.9904	0.9901
	IM	0.9902	0.9907	0.9901
1.4	M	0.9998	0.9998	0.9998
	D	0.9998	0.9998	0.9998
	IM	0.9998	0.9998	0.9998
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXIX

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{22} = 50$, $n_{12} = n_{21} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0545	0.0530	0.0425
	D	0.0545	0.0530	0.0425
	IM	0.0583	0.0579	0.0493
1.1	M	0.3878	0.3684	0.2978
	D	0.3879	0.3684	0.2978
	IM	0.3981	0.3908	0.3469
1.2	M	0.8781	0.8694	0.8206
	D	0.8781	0.8694	0.8206
	IM	0.8851	0.8796	0.8564
1.3	M	0.9955	0.9950	0.9903
	D	0.9955	0.9950	0.9903
	IM	0.9961	0.9957	0.9945
1.4	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXX

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 50$, $n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0533	0.0519	0.0422
	D	0.0532	0.0517	0.0421
	IM	0.0570	0.0568	0.0488
1.1	M	0.3625	0.3573	0.3190
	D	0.3624	0.3572	0.3188
	IM	0.3722	0.3719	0.3426
1.2	M	0.8639	0.8621	0.8385
	D	0.8637	0.8617	0.8383
	IM	0.8708	0.8705	0.8545
1.3	M	0.9932	0.9925	0.9899
	D	0.9931	0.9924	0.9899
	IM	0.9936	0.9936	0.9916
1.4	M	1.0000	1.0000	0.9999
	D	1.0000	1.0000	0.9999
	IM	1.0000	1.0000	0.9999
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXI

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0552	0.0542	0.0493
	D	0.0547	0.0540	0.0488
	IM	0.0606	0.0604	0.0585
1.05	M	0.1193	0.1179	0.1106
	D	0.1188	0.1173	0.1101
	IM	0.1269	0.1263	0.1245
1.1	M	0.2912	0.2885	0.2766
	D	0.2904	0.2877	0.2756
	IM	0.3014	0.3008	0.2978
1.15	M	0.5256	0.5231	0.5110
	D	0.5246	0.5221	0.5099
	IM	0.5387	0.5384	0.5350
1.2	M	0.7493	0.7462	0.7367
	D	0.7488	0.7453	0.7358
	IM	0.7587	0.7584	0.7553
1.25	M	0.9016	0.8996	0.8945
	D	0.9009	0.8991	0.8941
	IM	0.9064	0.9064	0.9043
1.3	M	0.9714	0.9711	0.9689
	D	0.9712	0.9708	0.9688
	IM	0.9737	0.9736	0.9732

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL R = 0.2, $n_{11} = n_{12} = n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0494	0.0488	0.0470
	D	0.0492	0.0485	0.0469
	IM	0.0512	0.0512	0.0507
1.05	M	0.1585	0.1574	0.1532
	D	0.1581	0.1572	0.1528
	IM	0.1639	0.1639	0.1624
1.1	M	0.4442	0.4421	0.4334
	D	0.4439	0.4411	0.4327
	IM	0.4514	0.4510	0.4487
1.15	M	0.7527	0.7515	0.7443
	D	0.7525	0.7509	0.7431
	IM	0.7581	0.7580	0.7567
1.2	M	0.9360	0.9354	0.9336
	D	0.9358	0.9353	0.9335
	IM	0.9392	0.9390	0.9378
1.25	M	0.9864	0.9863	0.9853
	D	0.9863	0.9862	0.9853
	IM	0.9869	0.9869	0.9869
1.3	M	0.9984	0.9984	0.9983
	D	0.9984	0.9984	0.9983
	IM	0.9986	0.9986	0.9984

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXIII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL R = 0.2, $n_{11} = n_{12} = n_{21} = n_{22} = 100$

Effect Size		Wald	LR	Score
1	M	0.0539	0.0537	0.0527
	D	0.0539	0.0536	0.0526
	IM	0.0554	0.0554	0.0550
1.05	M	0.2656	0.2643	0.2619
	D	0.2654	0.2641	0.2617
	IM	0.2688	0.2687	0.2679
1.1	M	0.7301	0.7289	0.7261
	D	0.7297	0.7287	0.7256
	IM	0.7337	0.7337	0.7332
1.15	M	0.9634	0.9631	0.9626
	D	0.9632	0.9631	0.9625
	IM	0.9641	0.9641	0.9640
1.2	M	0.9984	0.9984	0.9983
	D	0.9984	0.9984	0.9983
	IM	0.9984	0.9984	0.9984
1.25	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.3	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXIV

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{22} = 30$, $n_{12} = n_{21} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0560	0.0536	0.0422
	D	0.0559	0.0533	0.0421
	IM	0.0593	0.0586	0.0485
1.05	M	0.1164	0.1235	0.1181
	D	0.1159	0.1227	0.1173
	IM	0.1226	0.1273	0.1194
1.1	M	0.3290	0.3436	0.3382
	D	0.3276	0.3420	0.3365
	IM	0.3401	0.3476	0.3380
1.15	M	0.6179	0.6313	0.6254
	D	0.6161	0.6298	0.6237
	IM	0.6273	0.6351	0.6252
1.2	M	0.8322	0.8414	0.8378
	D	0.8310	0.8401	0.8368
	IM	0.8394	0.8441	0.8376
1.25	M	0.9467	0.9505	0.9488
	D	0.9462	0.9501	0.9481
	IM	0.9494	0.9509	0.9489
1.3	M	0.9858	0.9869	0.9865
	D	0.9857	0.9867	0.9863
	IM	0.9866	0.9875	0.9864

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXV

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{22} = 50$, $n_{12} = n_{21} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0560	0.0536	0.0422
	D	0.0559	0.0533	0.0421
	IM	0.0593	0.0586	0.0485
1.05	M	0.1436	0.1332	0.0964
	D	0.1436	0.1331	0.0964
	IM	0.1515	0.1452	0.1210
1.1	M	0.3637	0.3453	0.2752
	D	0.3641	0.3453	0.2752
	IM	0.3730	0.3654	0.3198
1.15	M	0.6458	0.6280	0.5516
	D	0.6460	0.6280	0.5517
	IM	0.6548	0.6474	0.6060
1.2	M	0.8571	0.8475	0.7973
	D	0.8571	0.8475	0.7973
	IM	0.8642	0.8580	0.8323
1.25	M	0.9621	0.9586	0.9355
	D	0.9622	0.9586	0.9355
	IM	0.9648	0.9624	0.9512
1.3	M	0.9922	0.9906	0.9839
	D	0.9922	0.9906	0.9839
	IM	0.9925	0.9923	0.9885

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXVI

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = 50$, $n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0522	0.0514	0.0404
	D	0.0519	0.0509	0.0402
	IM	0.0561	0.0560	0.0478
1.05	M	0.1298	0.1275	0.1049
	D	0.1296	0.1268	0.1045
	IM	0.1353	0.1351	0.1186
1.1	M	0.3471	0.3442	0.3026
	D	0.3463	0.3436	0.3021
	IM	0.3577	0.3568	0.3282
1.15	M	0.6277	0.6245	0.5872
	D	0.6269	0.6234	0.5861
	IM	0.6389	0.6384	0.6109
1.2	M	0.8441	0.8424	0.8135
	D	0.8436	0.8419	0.8129
	IM	0.8502	0.8494	0.8328
1.25	M	0.9572	0.9564	0.9454
	D	0.9567	0.9562	0.9453
	IM	0.9594	0.9591	0.9523
1.3	M	0.9899	0.9898	0.9871
	D	0.9899	0.9898	0.9870
	IM	0.9910	0.9910	0.9890

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXVII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{22} = 50$, $n_{12} = n_{21} = 100$

Effect Size		Wald	LR	Score
1*	M	0.0525	0.0512	0.0362
	D	0.0524	0.0511	0.0363
	IM	0.0544	0.0544	0.0404
1.05	M	0.1815	0.1919	0.1746
	D	0.1802	0.1912	0.1737
	IM	0.1857	0.1927	0.1689
1.1	M	0.5278	0.5437	0.5224
	D	0.5265	0.5422	0.5209
	IM	0.5346	0.5441	0.5139
1.15	M	0.8530	0.8616	0.8488
	D	0.8520	0.8604	0.8482
	IM	0.8562	0.8619	0.8435
1.2	M	0.9725	0.9743	0.9712
	D	0.9720	0.9742	0.9710
	IM	0.9733	0.9744	0.9703
1.25	M	0.9965	0.9969	0.9959
	D	0.9964	0.9968	0.9959
	IM	0.9965	0.9969	0.9957
1.3	M	0.9999	0.9999	0.9999
	D	0.9999	0.9999	0.9999
	IM	0.9999	0.9999	0.9999

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXVIII

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{22} = 100$, $n_{12} = n_{21} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0525	0.0512	0.0362
	D	0.0524	0.0511	0.0363
	IM	0.0544	0.0544	0.0404
1.05	M	0.2032	0.1925	0.1338
	D	0.2036	0.1929	0.1339
	IM	0.2076	0.2019	0.1561
1.1	M	0.5756	0.5555	0.4587
	D	0.5758	0.5558	0.4589
	IM	0.5839	0.5733	0.5019
1.15	M	0.8756	0.8664	0.8079
	D	0.8760	0.8665	0.8085
	IM	0.8794	0.8744	0.8350
1.2	M	0.9812	0.9794	0.9660
	D	0.9812	0.9794	0.9662
	IM	0.9816	0.9807	0.9726
1.25	M	0.9988	0.9986	0.9968
	D	0.9988	0.9986	0.9968
	IM	0.9989	0.9988	0.9977
1.3	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XXXIX

REJECTION RATES FOR INTERACTION TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = 100$, $n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0519	0.0508	0.0360
	D	0.0516	0.0507	0.0357
	IM	0.0542	0.0541	0.0402
1.05	M	0.1923	0.1905	0.1555
	D	0.1918	0.1902	0.1553
	IM	0.1961	0.1960	0.1658
1.1	M	0.5536	0.5510	0.4933
	D	0.5530	0.5505	0.4928
	IM	0.5610	0.5603	0.5115
1.15	M	0.8691	0.8679	0.8328
	D	0.8690	0.8675	0.8324
	IM	0.8730	0.8726	0.8447
1.2	M	0.9784	0.9780	0.9692
	D	0.9783	0.9779	0.9691
	IM	0.9790	0.9789	0.9716
1.25	M	0.9978	0.9977	0.9963
	D	0.9978	0.9977	0.9963
	IM	0.9979	0.9979	0.9969
1.3	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XL

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0665	0.0544	0.0362
	D	0.0660	0.0542	0.0361
	IM	0.0815	0.0782	0.0707
1.1	M	0.1437	0.1259	0.0929
	D	0.1431	0.1253	0.0925
	IM	0.1669	0.1620	0.1521
1.2	M	0.3498	0.3212	0.2607
	D	0.3494	0.3207	0.2601
	IM	0.3832	0.3766	0.3623
1.3	M	0.5978	0.5621	0.4947
	D	0.5962	0.5612	0.4938
	IM	0.6365	0.6273	0.6102
1.4	M	0.7993	0.7723	0.7157
	D	0.7989	0.7718	0.7150
	IM	0.8271	0.8218	0.8092
1.5	M	0.9148	0.9006	0.8656
	D	0.9145	0.9001	0.8649
	IM	0.9294	0.9267	0.9194
1.6	M	0.9694	0.9615	0.9433
	D	0.9690	0.9614	0.9430
	IM	0.9746	0.9730	0.9711

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLI

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 20$

Effect Size		Wald	LR	Score
1	M	0.0558	0.0510	0.0427
	D	0.0558	0.0510	0.0427
	IM	0.0618	0.0599	0.0573
1.1	M	0.2154	0.2048	0.1830
	D	0.2150	0.2046	0.1825
	IM	0.2288	0.2246	0.2202
1.2	M	0.5964	0.5798	0.5488
	D	0.5960	0.5793	0.5482
	IM	0.6154	0.6099	0.6036
1.3	M	0.8897	0.8825	0.8623
	D	0.8896	0.8822	0.8623
	IM	0.8989	0.8969	0.8938
1.4	M	0.9831	0.9813	0.9761
	D	0.9829	0.9813	0.9760
	IM	0.9850	0.9844	0.9836
1.5	M	0.9982	0.9977	0.9971
	D	0.9982	0.9977	0.9971
	IM	0.9984	0.9984	0.9982
1.6	M	0.9999	0.9999	0.9999
	D	0.9999	0.9999	0.9999
	IM	0.9999	0.9999	0.9999

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0560	0.0521	0.0460
	D	0.0560	0.0518	0.0459
	IM	0.0592	0.0584	0.0570
1.1	M	0.2965	0.2866	0.2688
	D	0.2962	0.2862	0.2686
	IM	0.3079	0.3057	0.3007
1.2	M	0.7797	0.7705	0.7545
	D	0.7791	0.7703	0.7541
	IM	0.7890	0.7875	0.7828
1.3	M	0.9756	0.9738	0.9705
	D	0.9756	0.9736	0.9705
	IM	0.9777	0.9773	0.9761
1.4	M	0.9987	0.9987	0.9984
	D	0.9987	0.9987	0.9984
	IM	0.9988	0.9988	0.9988
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLIII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$ FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0511	0.0490	0.0460
	D	0.0510	0.0489	0.0460
	IM	0.0535	0.0532	0.0523
1.1	M	0.4615	0.4552	0.4434
	D	0.4612	0.4549	0.4433
	IM	0.4691	0.4673	0.4634
1.2	M	0.9462	0.9447	0.9409
	D	0.9460	0.9447	0.9407
	IM	0.9480	0.9477	0.9471
1.3	M	0.9990	0.9990	0.9989
	D	0.9990	0.9990	0.9989
	IM	0.9990	0.9990	0.9990
1.4	M	0.9999	0.9999	0.9999
	D	0.9999	0.9999	0.9999
	IM	0.9999	0.9999	0.9999
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLIV

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 10$, $n_{12} = n_{22} = 20$

Effect Size		Wald	LR	Score
1*	M	0.0608	0.0519	0.0267
	D	0.0606	0.0517	0.0266
	IM	0.0733	0.0707	0.0492
1.1	M	0.1509	0.1570	0.1412
	D	0.1500	0.1564	0.1411
	IM	0.1631	0.1695	0.1533
1.2	M	0.4074	0.4176	0.3955
	D	0.4065	0.4161	0.3946
	IM	0.4212	0.4331	0.4111
1.3	M	0.7122	0.7211	0.6998
	D	0.7115	0.7197	0.6995
	IM	0.7248	0.7335	0.7155
1.4	M	0.8848	0.8880	0.8761
	D	0.8843	0.8876	0.8755
	IM	0.8893	0.8933	0.8846
1.5	M	0.9673	0.9686	0.9632
	D	0.9672	0.9686	0.9631
	IM	0.9691	0.9707	0.9669
1.6	M	0.9895	0.9905	0.9890
	D	0.9895	0.9905	0.9889
	IM	0.9905	0.9911	0.9901

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLV

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 20$, $n_{12} = n_{22} = 10$

Effect Size		Wald	LR	Score
1*	M	0.0608	0.0519	0.0267
	D	0.0606	0.0517	0.0266
	IM	0.0733	0.0707	0.0492
1.1	M	0.1759	0.1416	0.0480
	D	0.1758	0.1415	0.0480
	IM	0.2084	0.1921	0.1357
1.2	M	0.4534	0.3922	0.1789
	D	0.4534	0.3921	0.1788
	IM	0.5001	0.4800	0.3791
1.3	M	0.7471	0.6936	0.4372
	D	0.7471	0.6936	0.4370
	IM	0.7864	0.7697	0.6772
1.4	M	0.9164	0.8912	0.7058
	D	0.9164	0.8912	0.7056
	IM	0.9346	0.9251	0.8817
1.5	M	0.9810	0.9722	0.8878
	D	0.9810	0.9722	0.8878
	IM	0.9865	0.9844	0.9697
1.6	M	0.9963	0.9934	0.9655
	D	0.9963	0.9934	0.9655
	IM	0.9975	0.9970	0.9931

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLVI

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 20$, $n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0613	0.0524	0.0418
	D	0.0611	0.0523	0.0415
	IM	0.0692	0.0675	0.0632
1.1	M	0.1738	0.1613	0.1367
	D	0.1737	0.1609	0.1367
	IM	0.1894	0.1857	0.1816
1.2	M	0.4719	0.4529	0.4055
	D	0.4714	0.4526	0.4051
	IM	0.4983	0.4927	0.4828
1.3	M	0.7778	0.7596	0.7221
	D	0.7775	0.7587	0.7219
	IM	0.7965	0.7917	0.7848
1.4	M	0.9316	0.9234	0.9055
	D	0.9313	0.9231	0.9050
	IM	0.9388	0.9371	0.9340
1.5	M	0.9861	0.9845	0.9789
	D	0.9861	0.9844	0.9787
	IM	0.9887	0.9883	0.9870
1.6	M	0.9976	0.9972	0.9956
	D	0.9976	0.9972	0.9955
	IM	0.9979	0.9979	0.9977

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLVII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 20$, $n_{12} = n_{22} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0572	0.0525	0.0401
	D	0.0571	0.0525	0.0400
	IM	0.0618	0.0601	0.0534
1.1	M	0.2426	0.2448	0.2372
	D	0.2420	0.2442	0.2368
	IM	0.2516	0.2555	0.2505
1.2	M	0.6681	0.6692	0.6589
	D	0.6674	0.6683	0.6581
	IM	0.6769	0.6815	0.6744
1.3	M	0.9335	0.9338	0.9303
	D	0.9333	0.9337	0.9302
	IM	0.9364	0.9380	0.9352
1.4	M	0.9945	0.9940	0.9933
	D	0.9945	0.9940	0.9933
	IM	0.9945	0.9945	0.9943
1.5	M	0.9992	0.9993	0.9993
	D	0.9992	0.9993	0.9993
	IM	0.9993	0.9993	0.9993
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLVIII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 30$, $n_{12} = n_{22} = 20$

Effect Size		Wald	LR	Score
1*	M	0.0572	0.0525	0.0401
	D	0.0571	0.0525	0.0400
	IM	0.0618	0.0601	0.0534
1.1	M	0.2574	0.2360	0.1752
	D	0.2573	0.2360	0.1752
	IM	0.2770	0.2681	0.2378
1.2	M	0.6903	0.6623	0.5797
	D	0.6903	0.6620	0.5795
	IM	0.7113	0.7015	0.6639
1.3	M	0.9438	0.9337	0.9000
	D	0.9438	0.9336	0.9000
	IM	0.9482	0.9462	0.9357
1.4	M	0.9941	0.9930	0.9868
	D	0.9941	0.9930	0.9868
	IM	0.9953	0.9948	0.9934
1.5	M	0.9997	0.9995	0.9988
	D	0.9997	0.9995	0.9988
	IM	0.9998	0.9997	0.9995
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE XLIX

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 30$, $n_{21} = n_{22} = 20$

Effect Size		Wald	LR	Score
1	M	0.0572	0.0515	0.0409
	D	0.0572	0.0513	0.0409
	IM	0.0624	0.0611	0.0527
1.1	M	0.2518	0.2427	0.2097
	D	0.2516	0.2424	0.2093
	IM	0.2653	0.2632	0.2463
1.2	M	0.6885	0.6751	0.6329
	D	0.6880	0.6744	0.6326
	IM	0.7043	0.7009	0.6803
1.3	M	0.9382	0.9326	0.9155
	D	0.9382	0.9326	0.9151
	IM	0.9427	0.9410	0.9342
1.4	M	0.9934	0.9923	0.9890
	D	0.9934	0.9922	0.9890
	IM	0.9944	0.9940	0.9927
1.5	M	1.0000	0.9999	0.9995
	D	1.0000	0.9999	0.9995
	IM	1.0000	1.0000	0.9999
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE L

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 10$, $n_{12} = n_{22} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0605	0.0529	0.0187
	D	0.0604	0.0528	0.0185
	IM	0.0747	0.0712	0.0319
1.1	M	0.1543	0.1730	0.1364
	D	0.1531	0.1724	0.1354
	IM	0.1629	0.1733	0.1312
1.2	M	0.4480	0.4776	0.4259
	D	0.4472	0.4762	0.4246
	IM	0.4570	0.4751	0.4144
1.3	M	0.7503	0.7739	0.7314
	D	0.7492	0.7729	0.7303
	IM	0.7573	0.7715	0.7190
1.4	M	0.9124	0.9223	0.9013
	D	0.9119	0.9220	0.9007
	IM	0.9147	0.9207	0.8959
1.5	M	0.9738	0.9774	0.9702
	D	0.9738	0.9773	0.9700
	IM	0.9747	0.9769	0.9687
1.6	M	0.9950	0.9955	0.9933
	D	0.9949	0.9955	0.9933
	IM	0.9948	0.9953	0.9928

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LI

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 30$, $n_{12} = n_{22} = 10$

Effect Size		Wald	LR	Score
1*	M	0.0605	0.0529	0.0187
	D	0.0604	0.0528	0.0185
	IM	0.0747	0.0712	0.0319
1.1	M	0.1948	0.1548	0.0146
	D	0.1951	0.1549	0.0146
	IM	0.2320	0.2120	0.1014
1.2	M	0.4993	0.4266	0.0793
	D	0.4996	0.4267	0.0794
	IM	0.5503	0.5248	0.3188
1.3	M	0.7995	0.7404	0.2605
	D	0.7997	0.7407	0.2605
	IM	0.8360	0.8176	0.6413
1.4	M	0.9450	0.9224	0.5307
	D	0.9451	0.9224	0.5304
	IM	0.9602	0.9515	0.8690
1.5	M	0.9923	0.9866	0.7908
	D	0.9923	0.9866	0.7908
	IM	0.9950	0.9937	0.9715
1.6	M	0.9991	0.9984	0.9348
	D	0.9991	0.9984	0.9349
	IM	0.9996	0.9994	0.9952

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 30$, $n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0535	0.0484	0.0410
	D	0.0534	0.0482	0.0409
	IM	0.0607	0.0591	0.0566
1.1	M	0.2239	0.2105	0.1877
	D	0.2235	0.2101	0.1872
	IM	0.2386	0.2339	0.2273
1.2	M	0.5976	0.5818	0.5522
	D	0.5972	0.5818	0.5521
	IM	0.6132	0.6107	0.6035
1.3	M	0.8904	0.8838	0.8652
	D	0.8903	0.8834	0.8652
	IM	0.8986	0.8976	0.8946
1.4	M	0.9827	0.9805	0.9758
	D	0.9827	0.9803	0.9758
	IM	0.9844	0.9842	0.9835
1.5	M	0.9979	0.9977	0.9971
	D	0.9979	0.9976	0.9971
	IM	0.9981	0.9980	0.9979
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LIII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 30$, $n_{12} = n_{22} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0544	0.0518	0.0402
	D	0.0541	0.0518	0.0401
	IM	0.0587	0.0580	0.0480
1.1	M	0.3635	0.3709	0.3596
	D	0.3631	0.3703	0.3587
	IM	0.3687	0.3739	0.3599
1.2	M	0.8548	0.8577	0.8505
	D	0.8545	0.8576	0.8503
	IM	0.8564	0.8595	0.8509
1.3	M	0.9927	0.9929	0.9923
	D	0.9927	0.9929	0.9923
	IM	0.9929	0.9930	0.9925
1.4	M	0.9995	0.9995	0.9995
	D	0.9995	0.9995	0.9995
	IM	0.9995	0.9995	0.9995
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LIV

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{21} = 50$, $n_{12} = n_{22} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0544	0.0518	0.0402
	D	0.0541	0.0518	0.0401
	IM	0.0587	0.0580	0.0480
1.1	M	0.3648	0.3447	0.2713
	D	0.3650	0.3447	0.2713
	IM	0.3881	0.3709	0.3297
1.2	M	0.8825	0.8681	0.8141
	D	0.8825	0.8681	0.8141
	IM	0.8915	0.8866	0.8587
1.3	M	0.9934	0.9924	0.9864
	D	0.9934	0.9924	0.9864
	IM	0.9947	0.9941	0.9917
1.4	M	1.0000	1.0000	0.9997
	D	1.0000	1.0000	0.9997
	IM	1.0000	1.0000	0.9999
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LV

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_{11} = n_{12} = 50$, $n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0535	0.0515	0.0449
	D	0.0535	0.0515	0.0449
	IM	0.0563	0.0554	0.0541
1.1	M	0.3825	0.3746	0.3598
	D	0.3825	0.3742	0.3594
	IM	0.3926	0.3902	0.3864
1.2	M	0.8827	0.8788	0.8713
	D	0.8827	0.8788	0.8712
	IM	0.8869	0.8861	0.8843
1.3	M	0.9959	0.9956	0.9949
	D	0.9959	0.9956	0.9949
	IM	0.9960	0.9960	0.9959
1.4	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LVI

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0544	0.0506	0.0454
	D	0.0536	0.0503	0.0451
	IM	0.0584	0.0576	0.0559
1.05	M	0.1188	0.1129	0.1044
	D	0.1181	0.1127	0.1038
	IM	0.1245	0.1228	0.1193
1.1	M	0.2855	0.2770	0.2605
	D	0.2847	0.2763	0.2599
	IM	0.2964	0.2941	0.2893
1.15	M	0.5473	0.5352	0.5165
	D	0.5458	0.5347	0.5148
	IM	0.5569	0.5549	0.5506
1.2	M	0.7623	0.7543	0.7377
	D	0.7616	0.7534	0.7369
	IM	0.7720	0.7701	0.7665
1.25	M	0.9023	0.8984	0.8889
	D	0.9020	0.8979	0.8887
	IM	0.9076	0.9061	0.9038
1.3	M	0.9679	0.9657	0.9612
	D	0.9677	0.9654	0.9607
	IM	0.9696	0.9695	0.9687

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LVII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0543	0.0522	0.0485
	D	0.0540	0.0517	0.0484
	IM	0.0569	0.0563	0.0552
1.05	M	0.1574	0.1530	0.1440
	D	0.1572	0.1525	0.1438
	IM	0.1610	0.1603	0.1585
1.1	M	0.4410	0.4345	0.4225
	D	0.4401	0.4341	0.4218
	IM	0.4479	0.4465	0.4441
1.15	M	0.7582	0.7525	0.7418
	D	0.7577	0.7520	0.7416
	IM	0.7641	0.7637	0.7598
1.2	M	0.9343	0.9316	0.9273
	D	0.9343	0.9314	0.9271
	IM	0.9370	0.9364	0.9352
1.25	M	0.9869	0.9865	0.9849
	D	0.9868	0.9865	0.9849
	IM	0.9878	0.9876	0.9875
1.3	M	0.9984	0.9984	0.9984
	D	0.9984	0.9984	0.9984
	IM	0.9986	0.9986	0.9985

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LVIII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = n_{21} = n_{22} = 100$

Effect Size		Wald	LR	Score
1	M	0.0496	0.0491	0.0474
	D	0.0494	0.0489	0.0472
	IM	0.0513	0.0512	0.0501
1.05	M	0.2610	0.2586	0.2545
	D	0.2607	0.2585	0.2541
	IM	0.2650	0.2642	0.2627
1.1	M	0.7395	0.7366	0.7297
	D	0.7395	0.7364	0.7292
	IM	0.7429	0.7422	0.7404
1.15	M	0.9639	0.9631	0.9615
	D	0.9638	0.9630	0.9615
	IM	0.9645	0.9644	0.9643
1.2	M	0.9992	0.9992	0.9991
	D	0.9992	0.9992	0.9991
	IM	0.9992	0.9992	0.9992
1.25	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.3	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LIX

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{21} = 30$, $n_{12} = n_{22} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0521	0.0493	0.0364
	D	0.0520	0.0491	0.0363
	IM	0.0556	0.0550	0.0449
1.05	M	0.1295	0.1353	0.1268
	D	0.1288	0.1342	0.1256
	IM	0.1338	0.1371	0.1295
1.1	M	0.3484	0.3575	0.3464
	D	0.3463	0.3564	0.3443
	IM	0.3546	0.3616	0.3494
1.15	M	0.6197	0.6293	0.6181
	D	0.6179	0.6272	0.6162
	IM	0.6245	0.6326	0.6199
1.2	M	0.8382	0.8444	0.8365
	D	0.8367	0.8430	0.8356
	IM	0.8411	0.8470	0.8375
1.25	M	0.9502	0.9528	0.9496
	D	0.9497	0.9524	0.9490
	IM	0.9515	0.9536	0.9505
1.3	M	0.9881	0.9884	0.9877
	D	0.9879	0.9884	0.9877
	IM	0.9883	0.9886	0.9879

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LX

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{21} = 50$, $n_{12} = n_{22} = 30$

Effect Size		Wald	LR	Score
1*	M	0.0521	0.0493	0.0364
	D	0.0520	0.0491	0.0363
	IM	0.0556	0.0550	0.0449
1.05	M	0.1323	0.1203	0.0856
	D	0.1321	0.1203	0.0855
	IM	0.1421	0.1357	0.1098
1.1	M	0.3545	0.3313	0.2583
	D	0.3545	0.3315	0.2583
	IM	0.3707	0.3606	0.3155
1.15	M	0.6385	0.6155	0.5297
	D	0.6387	0.6157	0.5298
	IM	0.6545	0.6430	0.5998
1.2	M	0.8578	0.8419	0.7837
	D	0.8580	0.8419	0.7837
	IM	0.8672	0.8605	0.8324
1.25	M	0.9582	0.9499	0.9222
	D	0.9582	0.9499	0.9222
	IM	0.9624	0.9599	0.9448
1.3	M	0.9913	0.9898	0.9820
	D	0.9913	0.9898	0.9820
	IM	0.9926	0.9917	0.9885

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LXI

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = 50$, $n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0511	0.0494	0.0442
	D	0.0510	0.0493	0.0440
	IM	0.0535	0.0527	0.0517
1.05	M	0.1282	0.1238	0.1159
	D	0.1274	0.1231	0.1155
	IM	0.1336	0.1324	0.1299
1.1	M	0.3713	0.3660	0.3523
	D	0.3703	0.3652	0.3520
	IM	0.3799	0.3790	0.3751
1.15	M	0.6612	0.6521	0.6353
	D	0.6608	0.6515	0.6339
	IM	0.6711	0.6690	0.6643
1.2	M	0.8738	0.8687	0.8582
	D	0.8734	0.8684	0.8579
	IM	0.8792	0.8779	0.8751
1.25	M	0.9676	0.9657	0.9633
	D	0.9674	0.9656	0.9631
	IM	0.9693	0.9692	0.9680
1.3	M	0.9932	0.9927	0.9916
	D	0.9931	0.9926	0.9915
	IM	0.9937	0.9935	0.9933

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LXII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{21} = 50$, $n_{12} = n_{22} = 100$

Effect Size		Wald	LR	Score
1*	M	0.0499	0.0475	0.0323
	D	0.0498	0.0472	0.0321
	IM	0.0523	0.0512	0.0371
1.05	M	0.1861	0.1961	0.1771
	D	0.1854	0.1948	0.1764
	IM	0.1862	0.1923	0.1706
1.1	M	0.5517	0.5638	0.5383
	D	0.5503	0.5625	0.5366
	IM	0.5525	0.5614	0.5286
1.15	M	0.8591	0.8675	0.8511
	D	0.8589	0.8667	0.8501
	IM	0.8598	0.8651	0.8460
1.2	M	0.9769	0.9780	0.9753
	D	0.9767	0.9778	0.9753
	IM	0.9768	0.9776	0.9738
1.25	M	0.9978	0.9980	0.9976
	D	0.9978	0.9979	0.9976
	IM	0.9978	0.9979	0.9974
1.3	M	0.9998	0.9998	0.9995
	D	0.9998	0.9998	0.9995
	IM	0.9998	0.9998	0.9994

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LXIII

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL R = 0.2, $n_{11} = n_{21} = 100$, $n_{12} = n_{22} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0499	0.0475	0.0323
	D	0.0498	0.0472	0.0321
	IM	0.0523	0.0512	0.0371
1.05	M	0.2016	0.1861	0.1288
	D	0.2016	0.1884	0.1291
	IM	0.2128	0.2026	0.1560
1.1	M	0.5642	0.5435	0.4426
	D	0.5645	0.5438	0.4429
	IM	0.5801	0.5668	0.4957
1.15	M	0.8684	0.8573	0.7993
	D	0.8688	0.8576	0.7996
	IM	0.8747	0.8694	0.8344
1.2	M	0.9811	0.9787	0.9637
	D	0.9811	0.9788	0.9637
	IM	0.9823	0.9813	0.9723
1.25	M	0.9985	0.9985	0.9970
	D	0.9986	0.9985	0.9970
	IM	0.9987	0.9985	0.9978
1.3	M	0.9998	0.9998	0.9996
	D	0.9998	0.9998	0.9996
	IM	0.9998	0.9998	0.9998

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LXIV

REJECTION RATES FOR MAIN-EFFECT TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_{11} = n_{12} = 100$, $n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0525	0.0513	0.0494
	D	0.0524	0.0512	0.0491
	IM	0.0546	0.0544	0.0532
1.05	M	0.2115	0.2087	0.2044
	D	0.2112	0.2081	0.2042
	IM	0.2158	0.2154	0.2134
1.1	M	0.6055	0.6022	0.5940
	D	0.6053	0.6017	0.5936
	IM	0.6095	0.6081	0.6063
1.15	M	0.8996	0.8983	0.8949
	D	0.8996	0.8982	0.8947
	IM	0.9021	0.9015	0.9002
1.2	M	0.9879	0.9872	0.9862
	D	0.9878	0.9871	0.9862
	IM	0.9883	0.9883	0.9881
1.25	M	0.9992	0.9991	0.9991
	D	0.9992	0.9991	0.9991
	IM	0.9992	0.9992	0.9992
1.3	M	0.9999	0.9999	0.9999
	D	0.9999	0.9999	0.9999
	IM	0.9999	0.9999	0.9999

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE LXV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0530	0.0560	0.0312	DDL	0.0730
	D	0.0528	0.0559	0.0310	DDT	0.0297
	IM	0.0717	0.0837	0.0713	GM	0.0227
1.1	M	0.0722	0.0793	0.0472	DDL	0.1056
	D	0.0714	0.0790	0.0471	DDT	0.0392
	IM	0.0945	0.1178	0.1045	GM	0.0342
1.2	M	0.1171	0.1452	0.0973	DDL	0.1793
	D	0.1162	0.1448	0.0970	DDT	0.0687
	IM	0.1479	0.1954	0.1802	GM	0.0768
1.3	M	0.2037	0.2593	0.1880	DDL	0.3074
	D	0.2023	0.2583	0.1872	DDT	0.1252
	IM	0.2514	0.3272	0.3072	GM	0.1563
1.4	M	0.3199	0.3985	0.2999	DDL	0.4558
	D	0.3183	0.3968	0.2985	DDT	0.2157
	IM	0.3809	0.4753	0.4537	GM	0.2594
1.5	M	0.4560	0.5445	0.4192	DDL	0.5950
	D	0.4527	0.5433	0.4184	DDT	0.3337
	IM	0.5210	0.6170	0.5899	GM	0.3761
1.6	M	0.5964	0.6809	0.5339	DDL	0.7281
	D	0.5945	0.6797	0.5332	DDT	0.4624
	IM	0.6634	0.7497	0.7220	GM	0.4897

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXVI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0540	0.0547	0.0419	DDL	0.0622
	D	0.0537	0.0546	0.0418	DDT	0.0415
	IM	0.0616	0.0661	0.0616	GM	0.0361
1.1	M	0.0981	0.1084	0.0881	DDL	0.1220
	D	0.0979	0.1084	0.0881	DDT	0.0770
	IM	0.1107	0.1276	0.1213	GM	0.0782
1.2	M	0.2333	0.2625	0.2269	DDL	0.2859
	D	0.2331	0.2618	0.2264	DDT	0.1981
	IM	0.2564	0.2942	0.2889	GM	0.2087
1.3	M	0.4703	0.5088	0.4556	DDL	0.5367
	D	0.4693	0.5082	0.4552	DDT	0.4210
	IM	0.4984	0.5474	0.5348	GM	0.4326
1.4	M	0.7004	0.7275	0.6649	DDL	0.7484
	D	0.6998	0.7269	0.6644	DDT	0.6534
	IM	0.7245	0.7589	0.7448	GM	0.6417
1.5	M	0.8619	0.8781	0.8271	DDL	0.8920
	D	0.8616	0.8779	0.8268	DDT	0.8315
	IM	0.8761	0.8979	0.8887	GM	0.8123
1.6	M	0.9458	0.9514	0.9233	DDL	0.9580
	D	0.9457	0.9514	0.9229	DDT	0.9307
	IM	0.9531	0.9605	0.9562	GM	0.9145

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXVII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL R = 0.1, $n_1 = n_2 = n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0551	0.0580	0.0482	DDL	0.0633
	D	0.0550	0.0578	0.0482	DDT	0.0457
	IM	0.0605	0.0655	0.0628	GM	0.0437
1.1	M	0.1304	0.1394	0.1225	DDL	0.1505
	D	0.1303	0.1392	0.1220	DDT	0.1122
	IM	0.1399	0.1561	0.1501	GM	0.1146
1.2	M	0.3704	0.3931	0.3642	DDL	0.4116
	D	0.3699	0.3923	0.3637	DDT	0.3430
	IM	0.3890	0.4193	0.4100	GM	0.3516
1.3	M	0.6797	0.6992	0.6593	DDL	0.7163
	D	0.6792	0.6988	0.6591	DDT	0.6521
	IM	0.6952	0.7219	0.7121	GM	0.6462
1.4	M	0.8870	0.8940	0.8704	DDL	0.9015
	D	0.8867	0.8939	0.8703	DDT	0.8736
	IM	0.8953	0.9045	0.9008	GM	0.8611
1.5	M	0.9719	0.9739	0.9630	DDL	0.9768
	D	0.9718	0.9738	0.9630	DDT	0.9667
	IM	0.9745	0.9779	0.9758	GM	0.9604
1.6	M	0.9953	0.9959	0.9928	DDL	0.9963
	D	0.9953	0.9959	0.9928	DDT	0.9943
	IM	0.9962	0.9966	0.9961	GM	0.9917

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXVIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0523	0.0528	0.0488	DDL	0.0554
	D	0.0522	0.0528	0.0487	DDT	0.0472
	IM	0.0553	0.0574	0.0569	GM	0.0465
1.1	M	0.1918	0.1991	0.1876	DDL	0.2059
	D	0.1914	0.1989	0.1874	DDT	0.1782
	IM	0.1988	0.2091	0.2062	GM	0.1822
1.2	M	0.5974	0.6113	0.5913	DDL	0.6215
	D	0.5972	0.6109	0.5911	DDT	0.5801
	IM	0.6074	0.6243	0.6216	GM	0.5828
1.3	M	0.9049	0.9087	0.8958	DDL	0.9123
	D	0.9048	0.9087	0.8954	DDT	0.8967
	IM	0.9093	0.9145	0.9123	GM	0.8924
1.4	M	0.9897	0.9899	0.9856	DDL	0.9902
	D	0.9896	0.9899	0.9856	DDT	0.9880
	IM	0.9904	0.9905	0.9903	GM	0.9853
1.5	M	0.9992	0.9991	0.9990	DDL	0.9991
	D	0.9992	0.9991	0.9990	DDT	0.9992
	IM	0.9992	0.9991	0.9991	GM	0.9990
1.6	M	1.0000	0.9999	0.9999	DDL	0.9999
	D	1.0000	0.9999	0.9999	DDT	0.9999
	IM	1.0000	0.9999	0.9999	GM	0.9999

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXIX

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 10$, $n_2 = n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0645	0.0570	0.0344	DDL	0.0683
	D	0.0642	0.0566	0.0342	DDT	0.0405
	IM	0.0763	0.0753	0.0572	GM	0.0324
1.1	M	0.1222	0.0899	0.0458	DDL	0.1074
	D	0.1220	0.0897	0.0455	DDT	0.0862
	IM	0.1401	0.1233	0.0849	GM	0.0481
1.2	M	0.2403	0.1906	0.1015	DDL	0.2208
	D	0.2401	0.1902	0.1012	DDT	0.1775
	IM	0.2675	0.2451	0.1811	GM	0.1117
1.3	M	0.4151	0.3617	0.2149	DDL	0.4016
	D	0.4144	0.3614	0.2143	DDT	0.3236
	IM	0.4465	0.4314	0.3470	GM	0.2344
1.4	M	0.5898	0.5328	0.3546	DDL	0.5757
	D	0.5885	0.5325	0.3543	DDT	0.4858
	IM	0.6252	0.6051	0.5165	GM	0.3843
1.5	M	0.7528	0.7017	0.5085	DDL	0.7385
	D	0.7515	0.7016	0.5081	DDT	0.6623
	IM	0.7776	0.7655	0.6840	GM	0.5433
1.6	M	0.8711	0.8318	0.6371	DDL	0.8601
	D	0.8701	0.8317	0.6368	DDT	0.7966
	IM	0.8897	0.8794	0.8170	GM	0.6779

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXX

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 20$, $n_2 = 10$, $n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0645	0.0570	0.0344	DDL	0.0683
	D	0.0642	0.0566	0.0342	DDT	0.0405
	IM	0.0763	0.0753	0.0572	GM	0.0324
1.1	M	0.1041	0.1069	0.0726	DDL	0.1237
	D	0.1040	0.1065	0.0724	DDT	0.0650
	IM	0.1187	0.1310	0.1118	GM	0.0693
1.2	M	0.2445	0.2657	0.2063	DDL	0.2902
	D	0.2435	0.2655	0.2057	DDT	0.1614
	IM	0.2689	0.3036	0.2811	GM	0.1992
1.3	M	0.4574	0.5018	0.4238	DDL	0.5287
	D	0.4564	0.5012	0.4232	DDT	0.3419
	IM	0.4905	0.5430	0.5202	GM	0.4129
1.4	M	0.6858	0.7228	0.6502	DDL	0.7462
	D	0.6852	0.7225	0.6496	DDT	0.5703
	IM	0.7127	0.7574	0.7395	GM	0.6337
1.5	M	0.8555	0.8788	0.8231	DDL	0.8907
	D	0.8547	0.8785	0.8225	DDT	0.7680
	IM	0.8727	0.8975	0.8864	GM	0.8097
1.6	M	0.9425	0.9541	0.9256	DDL	0.9601
	D	0.9421	0.9538	0.9252	DDT	0.8995
	IM	0.9506	0.9624	0.9577	GM	0.9181

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 20$, $n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0645	0.0570	0.0344	DDL	0.0683
	D	0.0642	0.0566	0.0342	DDT	0.0405
	IM	0.0763	0.0753	0.0572	GM	0.0324
1.1	M	0.0612	0.0901	0.0806	DDL	0.1067
	D	0.0608	0.0897	0.0800	DDT	0.0355
	IM	0.0729	0.1068	0.0994	GM	0.0731
1.2	M	0.1233	0.1982	0.1862	DDL	0.2264
	D	0.1227	0.1977	0.1855	DDT	0.0774
	IM	0.1470	0.2240	0.2162	GM	0.1696
1.3	M	0.2407	0.3714	0.3503	DDL	0.4061
	D	0.2393	0.3704	0.3496	DDT	0.1621
	IM	0.2793	0.4046	0.3923	GM	0.3241
1.4	M	0.4084	0.5627	0.5324	DDL	0.5957
	D	0.4061	0.5621	0.5312	DDT	0.3057
	IM	0.4586	0.5930	0.5812	GM	0.5026
1.5	M	0.5783	0.7299	0.6929	DDL	0.7582
	D	0.5763	0.7288	0.6923	DDT	0.4686
	IM	0.6331	0.7594	0.7450	GM	0.6643
1.6	M	0.7400	0.8489	0.8137	DDL	0.8670
	D	0.7381	0.8486	0.8131	DDT	0.6442
	IM	0.7838	0.8679	0.8582	GM	0.7913

M = McKay's Approximation
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TABLE LXXII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 10$, $n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0668	0.0582	0.0331	DDL	0.0731
	D	0.0666	0.0579	0.0328	DDT	0.0374
	IM	0.0809	0.0804	0.0616	GM	0.0300
1.1	M	0.1274	0.0834	0.0230	DDL	0.1054
	D	0.1273	0.0829	0.0228	DDT	0.0760
	IM	0.1480	0.1233	0.0783	GM	0.0327
1.2	M	0.2506	0.1865	0.0446	DDL	0.2212
	D	0.2504	0.1861	0.0444	DDT	0.1622
	IM	0.2806	0.2498	0.1702	GM	0.0802
1.3	M	0.4148	0.3288	0.0972	DDL	0.3756
	D	0.4141	0.3284	0.0969	DDT	0.2891
	IM	0.4496	0.4172	0.2998	GM	0.1618
1.4	M	0.6062	0.5191	0.1859	DDL	0.5705
	D	0.6051	0.5188	0.1859	DDT	0.4630
	IM	0.6426	0.6099	0.4869	GM	0.2908
1.5	M	0.7606	0.6829	0.3053	DDL	0.7307
	D	0.7602	0.6825	0.3048	DDT	0.6361
	IM	0.7881	0.7634	0.6480	GM	0.4305
1.6	M	0.8676	0.8086	0.4348	DDL	0.8438
	D	0.8670	0.8085	0.4347	DDT	0.7751
	IM	0.8845	0.8698	0.7835	GM	0.5727

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 10$, $n_2 = 20$, $n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0668	0.0582	0.0331	DDL	0.0731
	D	0.0666	0.0579	0.0328	DDT	0.0374
	IM	0.0809	0.0804	0.0616	GM	0.0300
1.1	M	0.0887	0.0790	0.0517	DDL	0.1009
	D	0.0885	0.0789	0.0513	DDT	0.0520
	IM	0.1066	0.1102	0.0890	GM	0.0442
1.2	M	0.1395	0.1444	0.1120	DDL	0.1771
	D	0.1388	0.1439	0.1116	DDT	0.0842
	IM	0.1630	0.1899	0.1619	GM	0.0964
1.3	M	0.2360	0.2681	0.2239	DDL	0.3101
	D	0.2354	0.2674	0.2231	DDT	0.1556
	IM	0.2766	0.3236	0.2950	GM	0.1920
1.4	M	0.3528	0.4054	0.3380	DDL	0.4538
	D	0.3513	0.4044	0.3368	DDT	0.2443
	IM	0.4063	0.4705	0.4394	GM	0.2951
1.5	M	0.4918	0.5514	0.4529	DDL	0.5991
	D	0.4897	0.5503	0.4515	DDT	0.3705
	IM	0.5470	0.6170	0.5788	GM	0.4145
1.6	M	0.6279	0.6745	0.5653	DDL	0.7201
	D	0.6258	0.6739	0.5644	DDT	0.4975
	IM	0.6809	0.7365	0.6982	GM	0.5267

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXIV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 20$, $n_2 = n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0668	0.0582	0.0331	DDL	0.0731
	D	0.0666	0.0579	0.0328	DDT	0.0374
	IM	0.0809	0.0804	0.0616	GM	0.0300
1.1	M	0.0576	0.0920	0.0856	DDL	0.1102
	D	0.0572	0.0912	0.0852	DDT	0.0285
	IM	0.0733	0.1137	0.1057	GM	0.0697
1.2	M	0.0972	0.2073	0.2052	DDL	0.2340
	D	0.0959	0.2059	0.2045	DDT	0.0458
	IM	0.1244	0.2316	0.2284	GM	0.1751
1.3	M	0.1912	0.3722	0.3718	DDL	0.4074
	D	0.1895	0.3713	0.3711	DDT	0.0964
	IM	0.2389	0.4025	0.4034	GM	0.3322
1.4	M	0.3318	0.5566	0.5451	DDL	0.5965
	D	0.3276	0.5553	0.5440	DDT	0.1925
	IM	0.3954	0.5890	0.5864	GM	0.5076
1.5	M	0.5035	0.7237	0.7041	DDL	0.7549
	D	0.5010	0.7231	0.7035	DDT	0.3328
	IM	0.5768	0.7508	0.7456	GM	0.6723
1.6	M	0.6690	0.8423	0.8195	DDL	0.8603
	D	0.6655	0.8413	0.8190	DDT	0.5045
	IM	0.7296	0.8584	0.8520	GM	0.7941

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 20$, $n_2 = n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0543	0.0535	0.0404	DDL	0.0592
	D	0.0542	0.0535	0.0403	DDT	0.0436
	IM	0.0611	0.0630	0.0554	GM	0.0381
1.1	M	0.1336	0.1158	0.0790	DDL	0.1286
	D	0.1335	0.1157	0.0790	DDT	0.1126
	IM	0.1464	0.1391	0.1154	GM	0.0819
1.2	M	0.3321	0.3149	0.2372	DDL	0.3336
	D	0.3319	0.3146	0.2368	DDT	0.2948
	IM	0.3512	0.3501	0.3137	GM	0.2446
1.3	M	0.6136	0.5953	0.4917	DDL	0.6162
	D	0.6130	0.5952	0.4913	DDT	0.5758
	IM	0.6321	0.6343	0.5931	GM	0.5062
1.4	M	0.8226	0.8113	0.7193	DDL	0.8271
	D	0.8222	0.8112	0.7190	DDT	0.7954
	IM	0.8377	0.8382	0.8080	GM	0.7342
1.5	M	0.9406	0.9313	0.8750	DDL	0.9391
	D	0.9406	0.9312	0.8749	DDT	0.9275
	IM	0.9471	0.9451	0.9292	GM	0.8851
1.6	M	0.9860	0.9824	0.9532	DDL	0.9845
	D	0.9860	0.9823	0.9531	DDT	0.9810
	IM	0.9870	0.9868	0.9810	GM	0.9587

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXVI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 30$, $n_2 = 20$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0543	0.0535	0.0404	DDL	0.0592
	D	0.0542	0.0535	0.0403	DDT	0.0436
	IM	0.0611	0.0630	0.0554	GM	0.0381
1.1	M	0.1296	0.1395	0.1164	DDL	0.1492
	D	0.1293	0.1393	0.1164	DDT	0.1081
	IM	0.1412	0.1535	0.1459	GM	0.1134
1.2	M	0.3560	0.3800	0.3411	DDL	0.3978
	D	0.3557	0.3796	0.3408	DDT	0.3151
	IM	0.3752	0.4056	0.3958	GM	0.3289
1.3	M	0.6726	0.6925	0.6494	DDL	0.7089
	D	0.6719	0.6922	0.6492	DDT	0.6320
	IM	0.6889	0.7170	0.7051	GM	0.6392
1.4	M	0.8843	0.8957	0.8704	DDL	0.9028
	D	0.8842	0.8957	0.8703	DDT	0.8616
	IM	0.8937	0.9070	0.9021	GM	0.8621
1.5	M	0.9719	0.9757	0.9674	DDL	0.9791
	D	0.9719	0.9757	0.9674	DDT	0.9651
	IM	0.9750	0.9799	0.9783	GM	0.9647
1.6	M	0.9959	0.9967	0.9946	DDL	0.9973
	D	0.9959	0.9967	0.9946	DDT	0.9938
	IM	0.9966	0.9976	0.9971	GM	0.9939

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXVII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 30$, $n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0543	0.0535	0.0404	DDL	0.0592
	D	0.0542	0.0535	0.0403	DDT	0.0436
	IM	0.0611	0.0630	0.0554	GM	0.0381
1.1	M	0.0895	0.1181	0.1131	DDL	0.1303
	D	0.0893	0.1177	0.1129	DDT	0.0725
	IM	0.0992	0.1320	0.1284	GM	0.1040
1.2	M	0.2552	0.3221	0.3090	DDL	0.3402
	D	0.2546	0.3215	0.3086	DDT	0.2240
	IM	0.2775	0.3402	0.3372	GM	0.2908
1.3	M	0.5394	0.6147	0.5925	DDL	0.6335
	D	0.5387	0.6139	0.5918	DDT	0.4976
	IM	0.5631	0.6344	0.6292	GM	0.5724
1.4	M	0.7715	0.8258	0.8064	DDL	0.8375
	D	0.7706	0.8254	0.8059	DDT	0.7352
	IM	0.7901	0.8395	0.8339	GM	0.7918
1.5	M	0.9186	0.9431	0.9302	DDL	0.9472
	D	0.9183	0.9430	0.9300	DDT	0.9028
	IM	0.9270	0.9477	0.9463	GM	0.9210
1.6	M	0.9748	0.9816	0.9763	DDL	0.9838
	D	0.9744	0.9816	0.9762	DDT	0.9689
	IM	0.9783	0.9843	0.9822	GM	0.9737

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXVIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 20$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0542	0.0536	0.0401	DDL	0.0607
	D	0.0540	0.0532	0.0401	DDT	0.0418
	IM	0.0608	0.0645	0.0576	GM	0.0379
1.1	M	0.1322	0.1197	0.0726	DDL	0.1319
	D	0.1319	0.1195	0.0726	DDT	0.1119
	IM	0.1446	0.1421	0.1206	GM	0.0795
1.2	M	0.3330	0.3103	0.2164	DDL	0.3319
	D	0.3325	0.3103	0.2162	DDT	0.2919
	IM	0.3521	0.3480	0.3077	GM	0.2325
1.3	M	0.6004	0.5744	0.4464	DDL	0.5974
	D	0.6003	0.5742	0.4462	DDT	0.5532
	IM	0.6206	0.6157	0.5695	GM	0.4685
1.4	M	0.8254	0.8021	0.6936	DDL	0.8208
	D	0.8249	0.8019	0.6935	DDT	0.7889
	IM	0.8410	0.8351	0.7993	GM	0.7155
1.5	M	0.9432	0.9354	0.8693	DDL	0.9426
	D	0.9430	0.9351	0.8689	DDT	0.9293
	IM	0.9501	0.9489	0.9333	GM	0.8808
1.6	M	0.9844	0.9807	0.9523	DDL	0.9830
	D	0.9844	0.9806	0.9522	DDT	0.9785
	IM	0.9860	0.9857	0.9803	GM	0.9566

M = McKay's Approximation
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TABLE LXXIX

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 20$, $n_2 = 30$, $n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0542	0.0536	0.0401	DDL	0.0607
	D	0.0540	0.0532	0.0401	DDT	0.0418
	IM	0.0608	0.0645	0.0576	GM	0.0379
1.1	M	0.0994	0.1075	0.0860	DDL	0.1184
	D	0.0988	0.1075	0.0859	DDT	0.0798
	IM	0.1121	0.1237	0.1156	GM	0.0799
1.2	M	0.2433	0.2636	0.2272	DDL	0.2848
	D	0.2426	0.2633	0.2266	DDT	0.2079
	IM	0.2617	0.2947	0.2818	GM	0.2121
1.3	M	0.4729	0.5071	0.4524	DDL	0.5337
	D	0.4720	0.5062	0.4520	DDT	0.4250
	IM	0.4994	0.5418	0.5284	GM	0.4338
1.4	M	0.6995	0.7294	0.6714	DDL	0.7501
	D	0.6990	0.7286	0.6704	DDT	0.6555
	IM	0.7232	0.7597	0.7443	GM	0.6530
1.5	M	0.8697	0.8826	0.8326	DDL	0.8941
	D	0.8690	0.8820	0.8324	DDT	0.8447
	IM	0.8835	0.8996	0.8886	GM	0.8221
1.6	M	0.9512	0.9544	0.9236	DDL	0.9594
	D	0.9509	0.9544	0.9234	DDT	0.9387
	IM	0.9564	0.9629	0.9570	GM	0.9168

M = McKay's Approximation
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IM = Iglewicz and Myers' Approximation
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DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXX

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 30$, $n_2 = n_3 = 20$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0542	0.0536	0.0401	DDL	0.0607
	D	0.0540	0.0532	0.0401	DDT	0.0418
	IM	0.0608	0.0645	0.0576	GM	0.0379
1.1	M	0.0886	0.1249	0.1193	DDL	0.1365
	D	0.0885	0.1245	0.1192	DDT	0.0682
	IM	0.1027	0.1369	0.1350	GM	0.1061
1.2	M	0.2431	0.3195	0.3124	DDL	0.3399
	D	0.2426	0.3192	0.3116	DDT	0.2009
	IM	0.2667	0.3887	0.3401	GM	0.2919
1.3	M	0.5056	0.5954	0.5836	DDL	0.6143
	D	0.5045	0.5948	0.5829	DDT	0.4485
	IM	0.5338	0.6142	0.6133	GM	0.5601
1.4	M	0.7554	0.8232	0.8046	DDL	0.8358
	D	0.7544	0.8229	0.8041	DDT	0.7125
	IM	0.7775	0.8370	0.8332	GM	0.7879
1.5	M	0.9049	0.9377	0.9293	DDL	0.9442
	D	0.9044	0.9374	0.9292	DDT	0.8845
	IM	0.9167	0.9453	0.9416	GM	0.9214
1.6	M	0.9716	0.9828	0.9775	DDL	0.9846
	D	0.9714	0.9827	0.9775	DDT	0.9629
	IM	0.9757	0.9850	0.9841	GM	0.9755

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TABLE LXXXI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 10$, $n_2 = n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0714	0.0552	0.0309	DDL	0.0661
	D	0.0710	0.0551	0.0307	DDT	0.0443
	IM	0.0805	0.0723	0.0437	GM	0.0375
1.1	M	0.1466	0.0945	0.0443	DDL	0.1110
	D	0.1466	0.0943	0.0442	DDT	0.1003
	IM	0.1631	0.1282	0.0723	GM	0.0552
1.2	M	0.3095	0.2245	0.1125	DDL	0.2553
	D	0.3094	0.2242	0.1123	DDT	0.2281
	IM	0.3344	0.2838	0.1829	GM	0.1432
1.3	M	0.5132	0.4207	0.2470	DDL	0.4587
	D	0.5127	0.4206	0.2465	DDT	0.4064
	IM	0.5400	0.4908	0.3591	GM	0.3006
1.4	M	0.7146	0.6342	0.4169	DDL	0.6702
	D	0.7144	0.6342	0.4163	DDT	0.6115
	IM	0.7379	0.6998	0.5665	GM	0.4936
1.5	M	0.8558	0.7978	0.5884	DDL	0.8239
	D	0.8555	0.7977	0.5880	DDT	0.7793
	IM	0.8719	0.8436	0.7402	GM	0.6674
1.6	M	0.9423	0.9072	0.7450	DDL	0.9217
	D	0.9422	0.9071	0.7446	DDT	0.8982
	IM	0.9507	0.9338	0.8683	GM	0.8683

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE LXXXII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 30$, $n_2 = 10$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0714	0.0552	0.0309	DDL	0.0661
	D	0.0710	0.0551	0.0307	DDT	0.0443
	IM	0.0805	0.0723	0.0437	GM	0.0375
1.1	M	0.1427	0.1384	0.1001	DDL	0.1553
	D	0.1423	0.1381	0.0998	DDT	0.0877
	IM	0.1587	0.1633	0.1305	GM	0.1089
1.2	M	0.3798	0.3962	0.3260	DDL	0.4218
	D	0.3790	0.3956	0.3253	DDT	0.2551
	IM	0.4058	0.4360	0.3924	GM	0.3359
1.3	M	0.6853	0.7084	0.6465	DDL	0.7274
	D	0.6844	0.7078	0.6464	DDT	0.5452
	IM	0.7096	0.7374	0.7115	GM	0.6489
1.4	M	0.8780	0.8915	0.8601	DDL	0.8998
	D	0.8779	0.8912	0.8595	DDT	0.7966
	IM	0.8900	0.9036	0.8947	GM	0.8565
1.5	M	0.9744	0.9780	0.9670	DDL	0.9813
	D	0.9744	0.9779	0.9670	DDT	0.9422
	IM	0.9773	0.9825	0.9795	GM	0.9652
1.6	M	0.9952	0.9961	0.9928	DDL	0.9968
	D	0.9951	0.9961	0.9927	DDT	0.9857
	IM	0.9961	0.9970	0.9964	GM	0.9923

M = McKay's Approximation
D = David's Approximation
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TABLE LXXXIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 30$, $n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0714	0.0552	0.0309	DDL	0.0661
	D	0.0710	0.0551	0.0307	DDT	0.0443
	IM	0.0805	0.0723	0.0437	GM	0.0375
1.1	M	0.0726	0.0998	0.0791	DDL	0.1144
	D	0.0725	0.0994	0.0786	DDT	0.0365
	IM	0.0835	0.1141	0.0916	GM	0.0868
1.2	M	0.1492	0.2415	0.2214	DDL	0.2667
	D	0.1482	0.2407	0.2199	DDT	0.0828
	IM	0.1703	0.2585	0.2281	GM	0.2287
1.3	M	0.3012	0.4507	0.4136	DDL	0.4838
	D	0.2999	0.4497	0.4125	DDT	0.1977
	IM	0.3368	0.4726	0.4338	GM	0.4237
1.4	M	0.5126	0.6651	0.6229	DDL	0.6910
	D	0.5108	0.6639	0.6217	DDT	0.3830
	IM	0.5562	0.6842	0.6490	GM	0.6283
1.5	M	0.7114	0.8251	0.7906	DDL	0.8416
	D	0.7099	0.8250	0.7901	DDT	0.5899
	IM	0.7466	0.8384	0.8157	GM	0.7923
1.6	M	0.8519	0.9161	0.8905	DDL	0.9255
	D	0.8506	0.9159	0.8899	DDT	0.7587
	IM	0.8746	0.9249	0.9100	GM	0.8910

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE LXXXIV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$ FOR OVERALL $R = 0.1$, $n_1 = n_2 = 10$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0802	0.0545	0.0322	DDL	0.0700
	D	0.0801	0.0544	0.0320	DDT	0.0430
	IM	0.0935	0.0790	0.0474	GM	0.0314
1.1	M	0.1750	0.0996	0.0132	DDL	0.1221
	D	0.1747	0.0996	0.0132	DDT	0.1032
	IM	0.1965	0.1432	0.0614	GM	0.0302
1.2	M	0.3351	0.1978	0.0119	DDL	0.2348
	D	0.3348	0.1977	0.0117	DDT	0.2083
	IM	0.3612	0.2758	0.1248	GM	0.0673
1.3	M	0.5325	0.3692	0.0229	DDL	0.4200
	D	0.5322	0.3692	0.0229	DDT	0.3828
	IM	0.5654	0.4704	0.2557	GM	0.1587
1.4	M	0.7178	0.5604	0.0524	DDL	0.6156
	D	0.7177	0.5602	0.0523	DDT	0.5710
	IM	0.7441	0.6673	0.4152	GM	0.2779
1.5	M	0.8589	0.7426	0.1079	DDL	0.7843
	D	0.8586	0.7426	0.1079	DDT	0.7485
	IM	0.8745	0.8220	0.6072	GM	0.4399
1.6	M	0.9437	0.8720	0.1934	DDL	0.9020
	D	0.9437	0.8720	0.1936	DDT	0.8802
	IM	0.9541	0.9240	0.7698	GM	0.6047

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXXV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL R = 0.1, $n_1 = 10$, $n_2 = 30$, $n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0802	0.0545	0.0322	DDL	0.0700
	D	0.0801	0.0544	0.0320	DDT	0.0430
	IM	0.0935	0.0790	0.0474	GM	0.0314
1.1	M	0.1096	0.0814	0.0532	DDL	0.1012
	D	0.1093	0.0812	0.0526	DDT	0.0606
	IM	0.1255	0.1110	0.0747	GM	0.0510
1.2	M	0.1619	0.1466	0.1199	DDL	0.1756
	D	0.1616	0.1464	0.1189	DDT	0.0935
	IM	0.1866	0.1864	0.1487	GM	0.1045
1.3	M	0.2543	0.2566	0.2163	DDL	0.2982
	D	0.2534	0.2557	0.2150	DDT	0.1603
	IM	0.2871	0.3136	0.2698	GM	0.1921
1.4	M	0.3718	0.4017	0.3487	DDL	0.4495
	D	0.3703	0.4002	0.3481	DDT	0.2459
	IM	0.4177	0.4679	0.4201	GM	0.3193
1.5	M	0.5112	0.5436	0.4684	DDL	0.5922
	D	0.5102	0.5248	0.4670	DDT	0.3668
	IM	0.5638	0.6090	0.5638	GM	0.4273
1.6	M	0.6409	0.6778	0.5846	DDL	0.7228
	D	0.6386	0.6771	0.5834	DDT	0.4918
	IM	0.6876	0.7356	0.6934	GM	0.5448

M = McKay's Approximation
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DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE LXXXVI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 30$, $n_2 = n_3 = 10$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0802	0.0545	0.0322	DDL	0.0700
	D	0.0801	0.0544	0.0320	DDT	0.0430
	IM	0.0935	0.0790	0.0474	GM	0.0314
1.1	M	0.0561	0.1001	0.1006	DDL	0.1192
	D	0.0557	0.0999	0.0994	DDT	0.0261
	IM	0.0698	0.1159	0.1026	GM	0.0928
1.2	M	0.0891	0.2374	0.2484	DDL	0.2649
	D	0.0885	0.2363	0.2467	DDT	0.0313
	IM	0.1172	0.2555	0.2369	GM	0.2369
1.3	M	0.1809	0.4423	0.4456	DDL	0.4764
	D	0.1789	0.4407	0.4445	DDT	0.0712
	IM	0.2365	0.4558	0.4365	GM	0.4411
1.4	M	0.3396	0.6440	0.6388	DDL	0.6733
	D	0.3374	0.6429	0.6369	DDT	0.1587
	IM	0.4154	0.6590	0.6331	GM	0.6385
1.5	M	0.5365	0.8011	0.7886	DDL	0.8215
	D	0.5333	0.8008	0.7883	DDT	0.3104
	IM	0.6140	0.8123	0.7880	GM	0.7920
1.6	M	0.7044	0.8982	0.8864	DDL	0.9125
	D	0.7022	0.8977	0.8858	DDT	0.4767
	IM	0.7622	0.9050	0.8889	GM	0.8914

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TABLE LXXXVII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 30$, $n_2 = n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0516	0.0515	0.0419	DDL	0.0560
	D	0.0515	0.0514	0.0419	DDT	0.0444
	IM	0.0556	0.0577	0.0509	GM	0.0426
1.1	M	0.1849	0.1621	0.1079	DDL	0.1694
	D	0.1849	0.1620	0.1078	DDT	0.1639
	IM	0.1942	0.1790	0.1537	GM	0.1251
1.2	M	0.5110	0.4684	0.3622	DDL	0.4827
	D	0.5106	0.4683	0.3621	DDT	0.4764
	IM	0.5237	0.5020	0.4502	GM	0.3963
1.3	M	0.8293	0.8015	0.7048	DDL	0.8131
	D	0.8293	0.8015	0.7047	DDT	0.8076
	IM	0.8364	0.8257	0.7841	GM	0.7383
1.4	M	0.9679	0.9576	0.9146	DDL	0.9612
	D	0.9677	0.9576	0.9146	DDT	0.9598
	IM	0.9702	0.9651	0.9518	GM	0.9298
1.5	M	0.9956	0.9932	0.9839	DDL	0.9960
	D	0.9956	0.9932	0.9839	DDT	0.9951
	IM	0.9960	0.9951	0.9916	GM	0.9916
1.6	M	0.9996	0.9993	0.9982	DDL	0.9993
	D	0.9996	0.9993	0.9982	DDT	0.9995
	IM	0.9997	0.9997	0.9990	GM	0.9985

M = McKay's Approximation
D = David's Approximation
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DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE LXXXVIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 50$, $n_2 = 30$, $n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0516	0.0515	0.0419	DDL	0.0560
	D	0.0515	0.0514	0.0419	DDT	0.0444
	IM	0.0556	0.0577	0.0509	GM	0.0426
1.1	M	0.1998	0.2028	0.1815	DDL	0.2129
	D	0.1995	0.2028	0.1814	DDT	0.1801
	IM	0.2077	0.2182	0.2082	GM	0.1826
1.2	M	0.5919	0.6052	0.5790	DDL	0.6180
	D	0.5915	0.6050	0.5786	DDT	0.5613
	IM	0.6037	0.6242	0.6136	GM	0.5767
1.3	M	0.8995	0.9041	0.8893	DDL	0.9085
	D	0.8994	0.9040	0.8892	DDT	0.8860
	IM	0.9040	0.9115	0.9069	GM	0.8870
1.4	M	0.9893	0.9907	0.9876	DDL	0.9916
	D	0.9893	0.9907	0.9875	DDT	0.9880
	IM	0.9905	0.9921	0.9915	GM	0.9871
1.5	M	0.9990	0.9993	0.9990	DDL	0.9994
	D	0.9989	0.9993	0.9990	DDT	0.9988
	IM	0.9992	0.9995	0.9994	GM	0.9990
1.6	M	1.0000	1.0000	1.0000	DDL	1.0000
	D	1.0000	1.0000	1.0000	DDT	1.0000
	IM	1.0000	1.0000	1.0000	GM	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE LXXXIX

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 50$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0516	0.0515	0.0419	DDL	0.0560
	D	0.0515	0.0514	0.0419	DDT	0.0444
	IM	0.0556	0.0577	0.0509	GM	0.0426
1.1	M	0.1314	0.1635	0.1597	DDL	0.1725
	D	0.1311	0.1634	0.1594	DDT	0.1169
	IM	0.1395	0.1696	0.1675	GM	0.1566
1.2	M	0.4312	0.4929	0.4796	DDL	0.5059
	D	0.4304	0.4926	0.4792	DDT	0.4073
	IM	0.4454	0.4997	0.4961	GM	0.4726
1.3	M	0.7697	0.8145	0.8032	DDL	0.8234
	D	0.7691	0.8144	0.8029	DDT	0.7493
	IM	0.7806	0.8213	0.8157	GM	0.7979
1.4	M	0.9475	0.9588	0.9531	DDL	0.9613
	D	0.9473	0.9588	0.9531	DDT	0.9388
	IM	0.9515	0.9614	0.9592	GM	0.9510
1.5	M	0.9931	0.9943	0.9928	DDL	0.9950
	D	0.9931	0.9943	0.9928	DDT	0.9918
	IM	0.9936	0.9949	0.9945	GM	0.9922
1.6	M	0.9992	0.9993	0.9995	DDL	0.9993
	D	0.9992	0.9993	0.9995	DDT	0.9991
	IM	0.9992	0.9993	0.9993	GM	0.9993

M = McKay's Approximation
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TABLE XC

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = n_2 = 30$, $n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0535	0.0522	0.0413	DDL	0.0567
	D	0.0534	0.0522	0.0413	DDT	0.0453
	IM	0.0579	0.0593	0.0522	GM	0.0420
1.1	M	0.1799	0.1547	0.0986	DDL	0.1657
	D	0.1796	0.1546	0.0986	DDT	0.1616
	IM	0.1897	0.1735	0.1427	GM	0.1165
1.2	M	0.5160	0.4746	0.3665	DDL	0.4917
	D	0.5158	0.4745	0.3664	DDT	0.4806
	IM	0.5294	0.5090	0.4542	GM	0.4054
1.3	M	0.8295	0.8033	0.7151	DDL	0.8133
	D	0.8293	0.8032	0.7151	DDT	0.8066
	IM	0.8377	0.8241	0.7897	GM	0.7460
1.4	M	0.9675	0.9573	0.9160	DDL	0.9609
	D	0.9675	0.9573	0.9160	DDT	0.9616
	IM	0.9704	0.9654	0.9508	GM	0.9298
1.5	M	0.9956	0.9937	0.9859	DDL	0.9945
	D	0.9956	0.9937	0.9859	DDT	0.9946
	IM	0.9958	0.9951	0.9930	GM	0.9884
1.6	M	0.9993	0.9992	0.9974	DDL	0.9993
	D	0.9993	0.9992	0.9974	DDT	0.9992
	IM	0.9993	0.9993	0.9992	GM	0.9977

M = McKay's Approximation
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TABLE XCI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 30$, $n_2 = 50$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0535	0.0522	0.0413	DDL	0.0567
	D	0.0534	0.0522	0.0413	DDT	0.0453
	IM	0.0579	0.0593	0.0522	GM	0.0420
1.1	M	0.1292	0.1354	0.1190	DDL	0.1445
	D	0.1290	0.1353	0.1189	DDT	0.1145
	IM	0.1388	0.1479	0.1407	GM	0.1151
1.2	M	0.3714	0.3926	0.3559	DDL	0.4090
	D	0.3711	0.3916	0.3557	DDT	0.3447
	IM	0.3890	0.4154	0.4041	GM	0.3464
1.3	M	0.6823	0.7014	0.6634	DDL	0.7141
	D	0.6820	0.7011	0.6633	DDT	0.6557
	IM	0.6964	0.7191	0.7115	GM	0.6525
1.4	M	0.8895	0.8966	0.8719	DDL	0.9041
	D	0.8892	0.8965	0.8719	DDT	0.8769
	IM	0.8958	0.9079	0.9021	GM	0.8671
1.5	M	0.9727	0.9735	0.9629	DDL	0.9754
	D	0.9726	0.9734	0.9629	DDT	0.9676
	IM	0.9750	0.9765	0.9744	GM	0.9603
1.6	M	0.9954	0.9949	0.9913	DDL	0.9952
	D	0.9954	0.9949	0.9913	DDT	0.9944
	IM	0.9959	0.9956	0.9949	GM	0.9903

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TABLE XCII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.1$, $n_1 = 50$, $n_2 = n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0535	0.0522	0.0413	DDL	0.0567
	D	0.0534	0.0522	0.0413	DDT	0.0453
	IM	0.0579	0.0593	0.0522	GM	0.0420
1.1	M	0.1265	0.1650	0.1654	DDL	0.1751
	D	0.1261	0.1647	0.1653	DDT	0.1081
	IM	0.1346	0.1719	0.1695	GM	0.1593
1.2	M	0.4195	0.5015	0.4974	DDL	0.5143
	D	0.4187	0.5008	0.4969	DDT	0.3869
	IM	0.4369	0.5131	0.5042	GM	0.4885
1.3	M	0.7588	0.8156	0.8076	DDL	0.8232
	D	0.7583	0.8154	0.8073	DDT	0.7314
	IM	0.7717	0.8205	0.8171	GM	0.8024
1.4	M	0.9395	0.9588	0.9563	DDL	0.9626
	D	0.9394	0.9588	0.9563	DDT	0.9303
	IM	0.9459	0.9622	0.9596	GM	0.9549
1.5	M	0.9912	0.9947	0.9935	DDL	0.9951
	D	0.9912	0.9947	0.9935	DDT	0.9884
	IM	0.9919	0.9950	0.9945	GM	0.9934
1.6	M	0.9993	0.9996	0.9992	DDL	0.9996
	D	0.9993	0.9996	0.9992	DDT	0.9989
	IM	0.9993	0.9996	0.9995	GM	0.9992

M = McKay's Approximation
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TABLE XCIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0479	0.0503	0.0408	DDL	0.0545
	D	0.0471	0.0496	0.0404	DDT	0.0409
	IM	0.0534	0.0586	0.0544	GM	0.0372
1.05	M	0.0683	0.0755	0.0654	DDL	0.0812
	D	0.0674	0.0751	0.0650	DDT	0.0612
	IM	0.0754	0.0852	0.0803	GM	0.0599
1.1	M	0.1190	0.1282	0.1123	DDL	0.1374
	D	0.1178	0.1269	0.1119	DDT	0.1076
	IM	0.1265	0.1420	0.1355	GM	0.1053
1.15	M	0.2132	0.2347	0.2076	DDL	0.2477
	D	0.2117	0.2338	0.2063	DDT	0.1996
	IM	0.2264	0.2537	0.2475	GM	0.1974
1.2	M	0.3405	0.3692	0.3375	DDL	0.3840
	D	0.3388	0.3680	0.3367	DDT	0.3250
	IM	0.3580	0.3918	0.3832	GM	0.3250
1.25	M	0.4885	0.5210	0.4838	DDL	0.5376
	D	0.4861	0.5202	0.4825	DDT	0.4696
	IM	0.5065	0.5472	0.5350	GM	0.4700
1.3	M	0.6376	0.6716	0.6301	DDL	0.6874
	D	0.6349	0.6706	0.6286	DDT	0.6197
	IM	0.6560	0.6942	0.6838	GM	0.6176

M = McKay's Approximation
D = David's Approximation
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TABLE XCIV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0494	0.0501	0.0430	DDL	0.0527
	D	0.0488	0.0499	0.0429	DDT	0.0448
	IM	0.0521	0.0547	0.0522	GM	0.0415
1.05	M	0.0811	0.0859	0.0797	DDL	0.0894
	D	0.0806	0.0857	0.0792	DDT	0.0765
	IM	0.0846	0.0915	0.0902	GM	0.0757
1.1	M	0.1787	0.1930	0.1816	DDL	0.1999
	D	0.1782	0.1928	0.1813	DDT	0.1713
	IM	0.1875	0.2040	0.2012	GM	0.1760
1.15	M	0.3454	0.3619	0.3452	DDL	0.3723
	D	0.3443	0.3610	0.3442	DDT	0.3357
	IM	0.3554	0.3765	0.3710	GM	0.3375
1.2	M	0.5620	0.5756	0.5572	DDL	0.5885
	D	0.5603	0.5753	0.5561	DDT	0.5503
	IM	0.5707	0.5928	0.5869	GM	0.5497
1.25	M	0.7428	0.7571	0.7377	DDL	0.7649
	D	0.7420	0.7567	0.7373	DDT	0.7348
	IM	0.7508	0.7687	0.7637	GM	0.7319
1.3	M	0.8787	0.8855	0.8690	DDL	0.8890
	D	0.8779	0.8851	0.8689	DDT	0.8715
	IM	0.8840	0.8914	0.8888	GM	0.8655

M = McKay's Approximation
D = David's Approximation
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DDL = Doornbos and Dijkstra's LR Test
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TABLE XCV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = n_3 = 100$

Effect Size		Wald	LR	Score	Alternate Tests	
1	M	0.0525	0.0535	0.0512	DDL	0.0553
	D	0.0521	0.0533	0.0510	DDT	0.0504
	IM	0.0545	0.0566	0.0555	GM	0.0501
1.05	M	0.1154	0.1211	0.1145	DDL	0.1230
	D	0.1150	0.1207	0.1145	DDT	0.1134
	IM	0.1189	0.1246	0.1226	GM	0.1128
1.1	M	0.3424	0.3515	0.3428	DDL	0.3568
	D	0.3420	0.3509	0.3421	DDT	0.3388
	IM	0.3472	0.3584	0.3561	GM	0.3397
1.15	M	0.6532	0.6584	0.6485	DDL	0.6641
	D	0.6524	0.6579	0.6481	DDT	0.6491
	IM	0.6585	0.6670	0.6631	GM	0.6450
1.2	M	0.8773	0.8816	0.8732	DDL	0.8850
	D	0.8767	0.8813	0.8729	DDT	0.8755
	IM	0.8801	0.8865	0.8839	GM	0.8713
1.25	M	0.9732	0.9755	0.9729	DDL	0.9763
	D	0.9731	0.9754	0.9729	DDT	0.9728
	IM	0.9742	0.9761	0.9755	GM	0.9724
1.3	M	0.9969	0.9970	0.9966	DDL	0.9971
	D	0.9969	0.9970	0.9966	DDT	0.9969
	IM	0.9972	0.9971	0.9971	GM	0.9965

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D = David's Approximation
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TABLE XCVI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 30$, $n_2 = n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0540	0.0526	0.0418	DDL	0.0560
	D	0.0536	0.0521	0.0414	DDT	0.0475
	IM	0.0577	0.0584	0.0518	GM	0.0429
1.05	M	0.0844	0.0739	0.0503	DDL	0.0789
	D	0.0841	0.0736	0.0502	DDT	0.0738
	IM	0.0894	0.0833	0.0704	GM	0.0555
1.1	M	0.1744	0.1520	0.1130	DDL	0.1622
	D	0.1732	0.1516	0.1129	DDT	0.1603
	IM	0.1816	0.1691	0.1437	GM	0.1217
1.15	M	0.3159	0.2929	0.2256	DDL	0.3047
	D	0.3148	0.2923	0.2253	DDT	0.2981
	IM	0.3281	0.3149	0.2777	GM	0.2431
1.2	M	0.4797	0.4581	0.3736	DDL	0.4734
	D	0.4791	0.4577	0.3730	DDT	0.4587
	IM	0.4926	0.4847	0.4435	GM	0.3998
1.25	M	0.6632	0.6349	0.5429	DDL	0.6485
	D	0.6625	0.6345	0.5426	DDT	0.6424
	IM	0.6753	0.6636	0.6164	GM	0.5708
1.3	M	0.8030	0.7855	0.6973	DDL	0.7966
	D	0.8024	0.7852	0.6969	DDT	0.7899
	IM	0.8127	0.8073	0.7706	GM	0.7254

M = McKay's Approximation
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IM = Iglewicz and Myers' Approximation
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TABLE XCVII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 50$, $n_2 = 30$, $n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0540	0.0526	0.0418	DDL	0.0560
	D	0.0536	0.0521	0.0414	DDT	0.0475
	IM	0.0577	0.0584	0.0518	GM	0.0429
1.05	M	0.0827	0.0854	0.0712	DDL	0.0909
	D	0.0821	0.0850	0.0711	DDT	0.0754
	IM	0.0865	0.0938	0.0842	GM	0.0727
1.1	M	0.1794	0.1922	0.1718	DDL	0.2000
	D	0.1785	0.1914	0.1711	DDT	0.1642
	IM	0.1880	0.2038	0.1951	GM	0.1710
1.15	M	0.3524	0.3724	0.3465	DDL	0.3831
	D	0.3504	0.3712	0.3457	DDT	0.3328
	IM	0.3633	0.3875	0.3759	GM	0.3454
1.2	M	0.5685	0.5895	0.5588	DDL	0.6005
	D	0.5673	0.5877	0.5582	DDT	0.5477
	IM	0.5805	0.6061	0.5947	GM	0.5551
1.25	M	0.7475	0.7627	0.7365	DDL	0.7704
	D	0.7464	0.7618	0.7359	DDT	0.7300
	IM	0.7556	0.7747	0.7657	GM	0.7326
1.3	M	0.8782	0.8891	0.8738	DDL	0.8932
	D	0.8777	0.8884	0.8734	DDT	0.8677
	IM	0.8846	0.8963	0.8920	GM	0.8717

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TABLE XCVIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = 50$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0540	0.0526	0.0418	DDL	0.0560
	D	0.0536	0.0521	0.0414	DDT	0.0475
	IM	0.0577	0.0584	0.0518	GM	0.0429
1.05	M	0.0636	0.0768	0.0727	DDL	0.0827
	D	0.0630	0.0762	0.0718	DDT	0.0582
	IM	0.0679	0.0814	0.0797	GM	0.0712
1.1	M	0.1211	0.1567	0.1515	DDL	0.1644
	D	0.1196	0.1559	0.1505	DDT	0.1120
	IM	0.1285	0.1651	0.1600	GM	0.1487
1.15	M	0.2367	0.3001	0.2960	DDL	0.3126
	D	0.2351	0.2987	0.2952	DDT	0.2273
	IM	0.2496	0.3096	0.3062	GM	0.2907
1.2	M	0.3875	0.4590	0.4525	DDL	0.4712
	D	0.3853	0.4573	0.4510	DDT	0.3744
	IM	0.4035	0.4701	0.4649	GM	0.4447
1.25	M	0.5877	0.6529	0.6432	DDL	0.6648
	D	0.5851	0.6517	0.6417	DDT	0.5759
	IM	0.6033	0.6627	0.6587	GM	0.6348
1.3	M	0.7445	0.7972	0.7840	DDL	0.8053
	D	0.7428	0.7962	0.7832	DDT	0.7333
	IM	0.7580	0.8055	0.7980	GM	0.7783

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE XCIX

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = 30$, $n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0529	0.0516	0.0409	DDL	0.0572
	D	0.0524	0.0512	0.0405	DDT	0.0452
	IM	0.0571	0.0595	0.0521	GM	0.0406
1.05	M	0.0891	0.0765	0.0494	DDL	0.0826
	D	0.0885	0.0765	0.0491	DDT	0.0785
	IM	0.0961	0.0893	0.0717	GM	0.0565
1.1	M	0.1769	0.1542	0.0990	DDL	0.1637
	D	0.1759	0.1535	0.0985	DDT	0.1616
	IM	0.1863	0.1732	0.1399	GM	0.1138
1.15	M	0.3173	0.2848	0.1977	DDL	0.3013
	D	0.3167	0.2842	0.1973	DDT	0.2916
	IM	0.3303	0.3139	0.2661	GM	0.2271
1.2	M	0.4887	0.4548	0.3452	DDL	0.4709
	D	0.4879	0.4545	0.3448	DDT	0.4640
	IM	0.5045	0.4853	0.4312	GM	0.3826
1.25	M	0.6534	0.6203	0.5056	DDL	0.6360
	D	0.6530	0.6201	0.5051	DDT	0.6286
	IM	0.6661	0.6515	0.5951	GM	0.5446
1.3	M	0.8007	0.7765	0.6716	DDL	0.7890
	D	0.7999	0.7761	0.6712	DDT	0.7817
	IM	0.8099	0.8003	0.7554	GM	0.7092

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
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GM = Gupta and Ma's Score Test

TABLE C

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 30$, $n_2 = 50$, $n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0529	0.0516	0.0409	DDL	0.0572
	D	0.0524	0.0512	0.0405	DDT	0.0452
	IM	0.0571	0.0595	0.0521	GM	0.0406
1.05	M	0.0708	0.0715	0.0588	DDL	0.0782
	D	0.0705	0.0709	0.0585	DDT	0.0628
	IM	0.0767	0.0821	0.0717	GM	0.0586
1.1	M	0.1225	0.1329	0.1100	DDL	0.1410
	D	0.1216	0.1324	0.1092	DDT	0.1114
	IM	0.1307	0.1471	0.1343	GM	0.1070
1.15	M	0.2157	0.2277	0.2045	DDL	0.2406
	D	0.2145	0.2271	0.2029	DDT	0.2021
	IM	0.2288	0.2474	0.2357	GM	0.1974
1.2	M	0.3548	0.3785	0.3444	DDL	0.3945
	D	0.3531	0.3768	0.3431	DDT	0.3364
	IM	0.3704	0.4001	0.3868	GM	0.3352
1.25	M	0.4993	0.5267	0.4924	DDL	0.5444
	D	0.4964	0.5256	0.4899	DDT	0.4818
	IM	0.5165	0.5502	0.5377	GM	0.4789
1.3	M	0.6491	0.6725	0.6306	DDL	0.6870
	D	0.6467	0.6719	0.6297	DDT	0.6315
	IM	0.6666	0.6942	0.6793	GM	0.6225

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE CI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL R = 0.2, $n_1 = 50$, $n_2 = n_3 = 30$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0529	0.0516	0.0409	DDL	0.0572
	D	0.0524	0.0512	0.0405	DDT	0.0452
	IM	0.0571	0.0595	0.0521	GM	0.0406
1.05	M	0.0581	0.0747	0.0709	DDL	0.0796
	D	0.0573	0.0744	0.0702	DDT	0.0524
	IM	0.0630	0.0807	0.0774	GM	0.0672
1.1	M	0.1140	0.1581	0.1621	DDL	0.1685
	D	0.1124	0.1572	0.1610	DDT	0.1037
	IM	0.1222	0.1677	0.1659	GM	0.1566
1.15	M	0.2256	0.2987	0.2992	DDL	0.3094
	D	0.2234	0.2974	0.2977	DDT	0.2116
	IM	0.2385	0.3079	0.3059	GM	0.2908
1.2	M	0.3873	0.4777	0.4734	DDL	0.4904
	D	0.3838	0.4766	0.4710	DDT	0.3689
	IM	0.4017	0.4885	0.4844	GM	0.4626
1.25	M	0.5637	0.6542	0.6488	DDL	0.6652
	D	0.5608	0.6526	0.6473	DDT	0.5451
	IM	0.5801	0.6627	0.6592	GM	0.6418
1.3	M	0.7240	0.7956	0.7831	DDL	0.8043
	D	0.7217	0.7948	0.7813	DDT	0.7072
	IM	0.7388	0.8031	0.7965	GM	0.7781

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE CII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 50$, $n_2 = n_3 = 100$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0512	0.0507	0.0401	DDL	0.0525
	D	0.0509	0.0506	0.0401	DDT	0.0479
	IM	0.0531	0.0537	0.0448	GM	0.0458
1.05	M	0.1170	0.1012	0.0715	DDL	0.1049
	D	0.1169	0.1011	0.0715	DDT	0.1093
	IM	0.1216	0.1092	0.0851	GM	0.0845
1.1	M	0.2811	0.2538	0.1943	DDL	0.2610
	D	0.2808	0.2537	0.1940	DDT	0.2699
	IM	0.2869	0.2690	0.2267	GM	0.2256
1.15	M	0.5270	0.4901	0.4033	DDL	0.4999
	D	0.5267	0.4899	0.4031	DDT	0.5123
	IM	0.5361	0.5112	0.4497	GM	0.4482
1.2	M	0.7588	0.7336	0.6557	DDL	0.7422
	D	0.7587	0.7335	0.6554	DDT	0.7473
	IM	0.7659	0.7481	0.7029	GM	0.7005
1.25	M	0.9109	0.8955	0.8449	DDL	0.8981
	D	0.9109	0.8956	0.8447	DDT	0.9059
	IM	0.9149	0.9038	0.8768	GM	0.8743
1.3	M	0.9768	0.9708	0.9441	DDL	0.9720
	D	0.9768	0.9708	0.9441	DDT	0.9747
	IM	0.9784	0.9740	0.9599	GM	0.9581

M = McKay's Approximation
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IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE CIII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 100$, $n_2 = 50$, $n_3 = 100$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0512	0.0507	0.0401	DDL	0.0525
	D	0.0509	0.0506	0.0401	DDT	0.0479
	IM	0.0531	0.0537	0.0448	GM	0.0458
1.05	M	0.1123	0.1135	0.1012	DDL	0.1172
	D	0.1122	0.1133	0.1010	DDT	0.1069
	IM	0.1153	0.1190	0.1094	GM	0.1073
1.1	M	0.3377	0.3480	0.3259	DDL	0.3536
	D	0.3368	0.3476	0.3258	DDT	0.3281
	IM	0.3444	0.3561	0.3422	GM	0.3328
1.15	M	0.6517	0.6631	0.6468	DDL	0.6685
	D	0.6508	0.6626	0.6467	DDT	0.6396
	IM	0.6571	0.6706	0.6614	GM	0.6502
1.2	M	0.8808	0.8865	0.8790	DDL	0.8897
	D	0.8805	0.8864	0.8788	DDT	0.8763
	IM	0.8838	0.8913	0.8869	GM	0.8800
1.25	M	0.9752	0.9763	0.9732	DDL	0.9770
	D	0.9752	0.9763	0.9732	DDT	0.9730
	IM	0.9759	0.9777	0.9764	GM	0.9734
1.3	M	0.9959	0.9963	0.9959	DDL	0.9966
	D	0.9959	0.9962	0.9959	DDT	0.9949
	IM	0.9962	0.9969	0.9964	GM	0.9958

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE CIV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = 100$, $n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0512	0.0507	0.0401	DDL	0.0525
	D	0.0509	0.0506	0.0401	DDT	0.0479
	IM	0.0531	0.0537	0.0448	GM	0.0458
1.05	M	0.0770	0.0979	0.0939	DDL	0.1014
	D	0.0765	0.0971	0.0936	DDT	0.0737
	IM	0.0797	0.0991	0.0925	GM	0.1017
1.1	M	0.2146	0.2591	0.2511	DDL	0.2649
	D	0.2137	0.2587	0.2502	DDT	0.2096
	IM	0.2209	0.2617	0.2475	GM	0.2607
1.15	M	0.4489	0.5101	0.4966	DDL	0.5177
	D	0.4471	0.5089	0.4955	DDT	0.4434
	IM	0.4582	0.5128	0.4976	GM	0.5089
1.2	M	0.6906	0.7359	0.7245	DDL	0.7424
	D	0.6898	0.7352	0.7232	DDT	0.6861
	IM	0.6977	0.7387	0.7244	GM	0.7320
1.25	M	0.8697	0.8959	0.8882	DDL	0.8989
	D	0.8688	0.8951	0.8879	DDT	0.8666
	IM	0.8736	0.8972	0.8884	GM	0.8944
1.3	M	0.9650	0.9727	0.9687	DDL	0.9740
	D	0.9649	0.9724	0.9686	DDT	0.9640
	IM	0.9660	0.9735	0.9702	GM	0.9707

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE CV

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = n_2 = 50$, $n_3 = 100$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0517	0.0506	0.0403	DDL	0.0527
	D	0.0515	0.0504	0.0400	DDT	0.0470
	IM	0.0542	0.0546	0.0458	GM	0.0451
1.05	M	0.1179	0.0969	0.0590	DDL	0.0999
	D	0.1178	0.0967	0.0589	DDT	0.1091
	IM	0.1219	0.1064	0.0803	GM	0.0783
1.1	M	0.2818	0.2408	0.1592	DDL	0.2503
	D	0.2818	0.2407	0.1591	DDT	0.2658
	IM	0.2899	0.2627	0.2027	GM	0.2021
1.15	M	0.5250	0.4775	0.3642	DDL	0.4889
	D	0.5248	0.4776	0.3642	DDT	0.5050
	IM	0.5334	0.5006	0.4286	GM	0.4256
1.2	M	0.7627	0.7234	0.6111	DDL	0.7312
	D	0.7625	0.7233	0.6111	DDT	0.7472
	IM	0.7682	0.7426	0.6796	GM	0.6778
1.25	M	0.9098	0.8875	0.8128	DDL	0.8917
	D	0.9096	0.8877	0.8129	DDT	0.9020
	IM	0.9129	0.8980	0.8620	GM	0.8571
1.3	M	0.9772	0.9673	0.9352	DDL	0.9694
	D	0.9772	0.9673	0.9352	DDT	0.9742
	IM	0.9782	0.9727	0.9554	GM	0.9526

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
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TABLE CVI

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 50$, $n_2 = 100$, $n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0517	0.0506	0.0403	DDL	0.0527
	D	0.0515	0.0504	0.0400	DDT	0.0470
	IM	0.0542	0.0546	0.0458	GM	0.0451
1.05	M	0.0786	0.0801	0.0692	DDL	0.0846
	D	0.0784	0.0797	0.0690	DDT	0.0744
	IM	0.0824	0.0864	0.0769	GM	0.0721
1.1	M	0.1821	0.1941	0.1793	DDL	0.2012
	D	0.1816	0.1938	0.1788	DDT	0.1753
	IM	0.1891	0.2030	0.1923	GM	0.1815
1.15	M	0.3499	0.3661	0.3457	DDL	0.3761
	D	0.3488	0.3653	0.3446	DDT	0.3387
	IM	0.3597	0.3783	0.3682	GM	0.3472
1.2	M	0.5693	0.5862	0.5580	DDL	0.5970
	D	0.5685	0.5854	0.5566	DDT	0.5573
	IM	0.5811	0.5984	0.5875	GM	0.5584
1.25	M	0.7537	0.7672	0.7379	DDL	0.7742
	D	0.7528	0.7668	0.7372	DDT	0.7440
	IM	0.7379	0.7619	0.7775	GM	0.7382
1.3	M	0.8842	0.8903	0.8687	DDL	0.8940
	D	0.8836	0.8900	0.8684	DDT	0.8783
	IM	0.8893	0.8975	0.8891	GM	0.8674

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
GM = Gupta and Ma's Score Test

TABLE CVII

REJECTION RATES FOR ONE-FACTOR TEST AT $\alpha = 0.05$
FOR OVERALL $R = 0.2$, $n_1 = 100$, $n_2 = n_3 = 50$

Effect Size		Wald	LR	Score	Alternate Tests	
1**	M	0.0517	0.0506	0.0403	DDL	0.0527
	D	0.0515	0.0504	0.0400	DDT	0.0470
	IM	0.0542	0.0546	0.0458	GM	0.0451
1.05	M	0.0766	0.0998	0.0984	DDL	0.1041
	D	0.0761	0.0994	0.0975	DDT	0.0713
	IM	0.0803	0.1026	0.0964	GM	0.1026
1.1	M	0.1973	0.2565	0.2530	DDL	0.2632
	D	0.1962	0.2551	0.2519	DDT	0.1903
	IM	0.2057	0.2583	0.2446	GM	0.2620
1.15	M	0.4206	0.4955	0.4902	DDL	0.5021
	D	0.4185	0.4944	0.4883	DDT	0.4117
	IM	0.4286	0.4975	0.4815	GM	0.5018
1.2	M	0.6776	0.7450	0.7364	DDL	0.7490
	D	0.6759	0.7437	0.7348	DDT	0.6685
	IM	0.6863	0.7435	0.7317	GM	0.7490
1.25	M	0.8512	0.8920	0.8842	DDL	0.8949
	D	0.8504	0.8915	0.8838	DDT	0.8458
	IM	0.8578	0.8924	0.8827	GM	0.8924
1.3	M	0.9494	0.9655	0.9616	DDL	0.9662
	D	0.9487	0.9654	0.9615	DDT	0.9464
	IM	0.9516	0.9657	0.9613	GM	0.9645

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation
DDL = Doornbos and Dijkstra's LR Test
DDT = Doornbos and Dijkstra's t Test
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TABLE CVIII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{12} = n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0478	0.0460	0.0335
	D	0.0463	0.0444	0.0329
	IM	0.0599	0.0596	0.0531
1.1	M	0.1038	0.1006	0.0783
	D	0.1007	0.0972	0.0754
	IM	0.1230	0.1232	0.1160
1.2	M	0.2816	0.2752	0.2293
	D	0.2757	0.2704	0.2250
	IM	0.3176	0.3186	0.3035
1.3	M	0.5179	0.5110	0.4506
	D	0.5111	0.5045	0.4444
	IM	0.5605	0.5630	0.5445
1.4	M	0.7418	0.7358	0.6759
	D	0.7356	0.7293	0.6710
	IM	0.7734	0.7746	0.7583
1.5	M	0.8881	0.8849	0.8291
	D	0.8849	0.8826	0.8283
	IM	0.9065	0.9077	0.8872
1.6	M	0.9574	0.9560	0.8782
	D	0.9557	0.9541	0.8848
	IM	0.9678	0.9682	0.9264

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CIX

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{12} = n_{21} = n_{22} = 20$

Effect Size		Wald	LR	Score
1	M	0.0445	0.0437	0.0388
	D	0.0440	0.0429	0.0381
	IM	0.0504	0.0504	0.0478
1.1	M	0.1786	0.1760	0.1608
	D	0.1763	0.1725	0.1582
	IM	0.1918	0.1921	0.1872
1.2	M	0.5471	0.5437	0.5170
	D	0.5442	0.5398	0.5132
	IM	0.5652	0.5660	0.5605
1.3	M	0.8445	0.8420	0.8292
	D	0.8422	0.8397	0.8273
	IM	0.8570	0.8574	0.8538
1.4	M	0.9712	0.9707	0.9668
	D	0.9707	0.9703	0.9662
	IM	0.9741	0.9741	0.9732
1.5	M	0.9964	0.9964	0.9954
	D	0.9963	0.9963	0.9954
	IM	0.9968	0.9969	0.9961
1.6	M	0.9997	0.9997	0.9949
	D	0.9997	0.9997	0.9961
	IM	0.9997	0.9997	0.9972

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CX

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{12} = n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0412	0.0408	0.0363
	D	0.0406	0.0401	0.0359
	IM	0.0452	0.0453	0.0446
1.1	M	0.2589	0.2566	0.2430
	D	0.2568	0.2549	0.2416
	IM	0.2709	0.2712	0.2672
1.2	M	0.7213	0.7190	0.7060
	D	0.7190	0.7175	0.7028
	IM	0.7317	0.7321	0.7283
1.3	M	0.9608	0.9607	0.9571
	D	0.9606	0.9599	0.9564
	IM	0.9641	0.9643	0.9630
1.4	M	0.9975	0.9975	0.9972
	D	0.9975	0.9975	0.9972
	IM	0.9977	0.9977	0.9977
1.5	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.6	M	1.0000	1.0000	0.9999
	D	1.0000	1.0000	0.9999
	IM	1.0000	1.0000	0.9999

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXI

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{12} = n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0423	0.0419	0.0400
	D	0.0419	0.0415	0.0397
	IM	0.0448	0.0448	0.0438
1.05	M	0.1306	0.1302	0.1262
	D	0.1303	0.1295	0.1256
	IM	0.1350	0.1352	0.1341
1.1	M	0.3935	0.3928	0.3864
	D	0.3929	0.3918	0.3853
	IM	0.4050	0.4054	0.4024
1.15	M	0.7336	0.7328	0.7256
	D	0.7330	0.7317	0.7250
	IM	0.7408	0.7410	0.7389
1.2	M	0.9189	0.9184	0.9147
	D	0.9184	0.9178	0.9142
	IM	0.9225	0.9228	0.9215
1.25	M	0.9855	0.9854	0.9846
	D	0.9853	0.9851	0.9844
	IM	0.9863	0.9863	0.9860
1.3	M	0.9975	0.9975	0.9974
	D	0.9975	0.9975	0.9974
	IM	0.9977	0.9977	0.9976

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{12} = n_{21} = n_{22} = 100$

Effect Size		Wald	LR	Score
1	M	0.0416	0.0414	0.0405
	D	0.0414	0.0413	0.0404
	IM	0.0432	0.0432	0.0430
1.05	M	0.2345	0.2344	0.2310
	D	0.2344	0.2340	0.2305
	IM	0.2397	0.2399	0.2386
1.1	M	0.6918	0.6912	0.6870
	D	0.6913	0.6906	0.6864
	IM	0.6982	0.6984	0.6971
1.15	M	0.9567	0.9565	0.9555
	D	0.9565	0.9561	0.9553
	IM	0.9582	0.9582	0.9578
1.2	M	0.9984	0.9984	0.9984
	D	0.9984	0.9984	0.9984
	IM	0.9984	0.9984	0.9984
1.25	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000
1.3	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXIII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL R = 0.6, $n_{11} = n_{12} = n_{21} = n_{22} = 10$

Effect Size		Wald	LR	Score
1	M	0.0133	0.0160	0.0065
	D	0.0128	0.0141	0.0056
	IM	0.0232	0.0237	0.0155
1.1	M	0.0377	0.0442	0.0195
	D	0.0347	0.0394	0.0171
	IM	0.0619	0.0643	0.0403
1.2	M	0.1087	0.1302	0.0646
	D	0.1015	0.1201	0.0616
	IM	0.1659	0.1740	0.1179
1.3	M	0.2616	0.3047	0.1519
	D	0.2486	0.2859	0.1485
	IM	0.3609	0.3779	0.2553
1.4	M	0.4623	0.5280	0.2655
	D	0.4486	0.5040	0.2733
	IM	0.5886	0.6083	0.4026
1.5	M	0.6696	0.7318	0.3671
	D	0.6615	0.7139	0.3886
	IM	0.7779	0.7934	0.5003
1.6	M	0.8110	0.8669	0.3959
	D	0.8102	0.8526	0.4342
	IM	0.8940	0.9037	0.5093

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXIV

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{12} = n_{21} = n_{22} = 20$

Effect Size		Wald	LR	Score
1	M	0.0140	0.0160	0.0110
	D	0.0121	0.0152	0.0100
	IM	0.0212	0.0235	0.0189
1.1	M	0.0794	0.0883	0.0729
	D	0.0766	0.0833	0.0692
	IM	0.1065	0.1111	0.1001
1.2	M	0.3144	0.3332	0.2956
	D	0.3052	0.3233	0.2857
	IM	0.3745	0.3846	0.3599
1.3	M	0.6591	0.6814	0.6344
	D	0.6499	0.6687	0.6245
	IM	0.7190	0.7307	0.7017
1.4	M	0.8867	0.8980	0.8485
	D	0.8816	0.8922	0.8482
	IM	0.9194	0.9222	0.8866
1.5	M	0.9759	0.9785	0.9002
	D	0.9743	0.9775	0.9154
	IM	0.9825	0.9835	0.9065
1.6	M	0.9969	0.9971	**0.8167**
	D	0.9968	0.9970	**0.8501**
	IM	0.9980	0.9982	**0.8025**

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXV

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{12} = n_{21} = n_{22} = 30$

Effect Size		Wald	LR	Score
1	M	0.0164	0.0172	0.0157
	D	0.0157	0.0165	0.0152
	IM	0.0213	0.0223	0.0203
1.1	M	0.1317	0.1390	0.1253
	D	0.1277	0.1342	0.1212
	IM	0.1602	0.1641	0.1553
1.2	M	0.5113	0.5258	0.4990
	D	0.5031	0.5160	0.4907
	IM	0.5640	0.5704	0.5547
1.3	M	0.8658	0.8732	0.8594
	D	0.8614	0.8683	0.8544
	IM	0.8935	0.8961	0.8897
1.4	M	0.9826	0.9845	0.9774
	D	0.9817	0.9835	0.9774
	IM	0.9876	0.9882	0.9820
1.5	M	0.9991	0.9991	0.9779
	D	0.9989	0.9991	0.9829
	IM	0.9994	0.9994	0.9724
1.6	M	1.0000	1.0000	1.0000
	D	1.0000	1.0000	1.0000
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXVI

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{12} = n_{21} = n_{22} = 50$

Effect Size		Wald	LR	Score
1	M	0.0189	0.0198	0.0180
	D	0.0183	0.0190	0.0174
	IM	0.0243	0.0251	0.0235
1.05	M	0.0678	0.0697	0.0651
	D	0.0657	0.0682	0.0634
	IM	0.0824	0.0840	0.0805
1.1	M	0.2390	0.2449	0.2343
	D	0.2352	0.2402	0.2308
	IM	0.2733	0.2771	0.2700
1.15	M	0.5104	0.5199	0.5062
	D	0.5063	0.5137	0.5005
	IM	0.5563	0.5599	0.5525
1.2	M	0.7794	0.7852	0.7747
	D	0.7756	0.7811	0.7705
	IM	0.8093	0.8123	0.8064
1.25	M	0.9342	0.9364	0.9325
	D	0.9330	0.9349	0.9313
	IM	0.9449	0.9465	0.9435
1.3	M	0.9881	0.9890	0.9878
	D	0.9878	0.9886	0.9871
	IM	0.9924	0.9927	0.9921

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXVII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{12} = n_{21} = n_{22} = 100$

Effect Size		Wald	LR	Score
1	M	0.0209	0.0217	0.0204
	D	0.0204	0.0212	0.0200
	IM	0.0269	0.0272	0.0264
1.05	M	0.1180	0.1203	0.1167
	D	0.1170	0.1188	0.1153
	IM	0.1377	0.1391	0.1360
1.1	M	0.4936	0.4961	0.4907
	D	0.4911	0.4943	0.4888
	IM	0.5317	0.5347	0.5298
1.15	M	0.8613	0.8629	0.8604
	D	0.8602	0.8617	0.8594
	IM	0.8790	0.8798	0.8780
1.2	M	0.9832	0.9836	0.9828
	D	0.9829	0.9833	0.9828
	IM	0.9861	0.9862	0.9861
1.25	M	0.9992	0.9992	0.9992
	D	0.9992	0.9992	0.9992
	IM	0.9994	0.9994	0.9994
1.3	M	1.0000	1.0000	0.9999
	D	1.0000	1.0000	0.9999
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXVIII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{22} = 10$, $n_{12} = n_{21} = 20$

Effect Size		Wald	LR	Score
1*	M	0.0485	0.0443	0.0211
	D	0.0479	0.0434	0.0202
	IM	0.0577	0.0556	0.0371
1.1	M	0.0925	0.1124	0.1054
	D	0.0881	0.1080	0.1006
	IM	0.1105	0.1220	0.1135
1.2	M	0.3125	0.3546	0.3464
	D	0.3021	0.3454	0.3362
	IM	0.3437	0.3741	0.3589
1.3	M	0.6044	0.6437	0.6362
	D	0.5945	0.6355	0.6274
	IM	0.6333	0.6601	0.6471
1.4	M	0.8319	0.8557	0.8512
	D	0.8268	0.8513	0.8471
	IM	0.8502	0.8665	0.8585
1.5	M	0.9410	0.9511	0.9495
	D	0.9387	0.9494	0.9475
	IM	0.9491	0.9542	0.9515
1.6	M	0.9836	0.9867	0.9853
	D	0.9827	0.9860	0.9849
	IM	0.9858	0.9885	0.9870

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXIX

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{22} = 20$, $n_{12} = n_{21} = 10$

Effect Size		Wald	LR	Score
1*	M	0.0485	0.0443	0.0211
	D	0.0479	0.0434	0.0202
	IM	0.0577	0.0556	0.0371
1.1	M	0.1610	0.1333	0.0413
	D	0.1609	0.1331	0.0411
	IM	0.1850	0.1674	0.1040
1.2	M	0.4195	0.3680	0.1536
	D	0.4190	0.3674	0.1528
	IM	0.4575	0.4278	0.3127
1.3	M	0.7128	0.6582	0.3722
	D	0.7124	0.6581	0.3721
	IM	0.7470	0.7202	0.5931
1.4	M	0.8986	0.8667	0.6109
	D	0.8982	0.8662	0.6116
	IM	0.9143	0.9016	0.8117
1.5	M	0.9759	0.9652	0.7566
	D	0.9759	0.9649	0.7617
	IM	0.9809	0.9774	0.9023
1.6	M	0.9954	0.9923	0.7487
	D	0.9954	0.9922	0.7572
	IM	0.9964	0.9957	**0.8695**

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXX

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{22} = 50$, $n_{12} = n_{21} = 100$

Effect Size		Wald	LR	Score
1*	M	0.0417	0.0410	0.0277
	D	0.0418	0.0408	0.0275
	IM	0.0447	0.0436	0.0313
1.05	M	0.1400	0.1496	0.1369
	D	0.1385	0.1478	0.1353
	IM	0.1438	0.1512	0.1343
1.1	M	0.4884	0.5079	0.4850
	D	0.4853	0.5051	0.4822
	IM	0.4963	0.5103	0.4808
1.15	M	0.8353	0.8477	0.8328
	D	0.8335	0.8459	0.8311
	IM	0.8397	0.8490	0.8297
1.2	M	0.9658	0.9692	0.9649
	D	0.9651	0.9687	0.9643
	IM	0.9669	0.9696	0.9638
1.25	M	0.9973	0.9976	0.9971
	D	0.9972	0.9976	0.9971
	IM	0.9974	0.9977	0.9970
1.3	M	0.9998	0.9998	0.9998
	D	0.9998	0.9998	0.9996
	IM	0.9998	0.9998	0.9996

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXXI

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.3$, $n_{11} = n_{22} = 100$, $n_{12} = n_{21} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0417	0.0410	0.0277
	D	0.0418	0.0408	0.0275
	IM	0.0447	0.0436	0.0313
1.05	M	0.1841	0.1716	0.1162
	D	0.1846	0.1723	0.1166
	IM	0.1904	0.1831	0.1369
1.1	M	0.5435	0.5228	0.4219
	D	0.5446	0.5235	0.4227
	IM	0.5538	0.5412	0.4660
1.15	M	0.8623	0.8502	0.7791
	D	0.8630	0.8503	0.7804
	IM	0.8681	0.8612	0.8126
1.2	M	0.9777	0.9741	0.9535
	D	0.9778	0.9744	0.9538
	IM	0.9790	0.9774	0.9634
1.25	M	0.9984	0.9982	0.9968
	D	0.9984	0.9982	0.9968
	IM	0.9985	0.9983	0.9971
1.3	M	1.0000	1.0000	0.9999
	D	1.0000	1.0000	0.9999
	IM	1.0000	1.0000	1.0000

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXXII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{22} = 10$, $n_{12} = n_{21} = 20$

Effect Size		Wald	LR	Score
1*	M	0.0200	0.0164	0.0045
	D	0.0198	0.0150	0.0036
	IM	0.0287	0.0262	0.0104
1.1	M	0.0187	0.0458	0.0392
	D	0.0162	0.0376	0.0322
	IM	0.0367	0.0582	0.0529
1.2	M	0.0925	0.1907	0.1765
	D	0.0835	0.1675	0.1558
	IM	0.1604	0.2223	0.2138
1.3	M	0.2700	0.4388	0.4210
	D	0.2494	0.4049	0.3868
	IM	0.3906	0.4802	0.4698
1.4	M	0.5387	0.7069	0.6889
	D	0.5158	0.6772	0.6615
	IM	0.6648	0.7406	0.7297
1.5	M	0.7714	0.8827	0.8667
	D	0.7580	0.8667	0.8530
	IM	0.8584	0.8996	0.8840
1.6	M	0.9047	0.9618	0.9393
	D	0.9027	0.9545	0.9414
	IM	0.9513	0.9693	0.9361

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXXIII

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{22} = 20$, $n_{12} = n_{21} = 10$

Effect Size		Wald	LR	Score
1*	M	0.0200	0.0164	0.0045
	D	0.0198	0.0150	0.0036
	IM	0.0287	0.0262	0.0104
1.1	M	0.0883	0.0602	0.0085
	D	0.0884	0.0600	0.0087
	IM	0.1177	0.0934	0.0289
1.2	M	0.2577	0.1932	0.0289
	D	0.2581	0.1932	0.0304
	IM	0.3216	0.2686	0.0888
1.3	M	0.5185	0.4353	0.0813
	D	0.5187	0.4357	0.0844
	IM	0.5931	0.5343	0.2085
1.4	M	0.7556	0.6813	0.1256
	D	0.7554	0.6812	0.1309
	IM	0.8163	0.7680	0.2817
1.5	M	0.9021	0.8562	0.1604
	D	0.9017	0.8560	0.1733
	IM	0.9335	0.9113	0.3075
1.6	M	0.9634	0.9450	0.1555
	D	0.9636	0.9449	0.1699
	IM	0.9806	0.9693	**0.2584**

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXXIV

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{22} = 50$, $n_{12} = n_{21} = 100$

Effect Size		Wald	LR	Score
1*	M	0.0195	0.0202	0.0127
	D	0.0197	0.0197	0.0121
	IM	0.0252	0.0250	0.0178
1.05	M	0.0637	0.0806	0.0682
	D	0.0605	0.0758	0.0650
	IM	0.0795	0.0939	0.0822
1.1	M	0.2808	0.3209	0.2981
	D	0.2714	0.3116	0.2870
	IM	0.3173	0.3475	0.3242
1.15	M	0.6249	0.6651	0.6420
	D	0.6139	0.6559	0.6302
	IM	0.6615	0.6885	0.6676
1.2	M	0.8797	0.8999	0.8868
	D	0.8752	0.8949	0.8824
	IM	0.8977	0.9105	0.8999
1.25	M	0.9776	0.9831	0.9803
	D	0.9759	0.9820	0.9790
	IM	0.9824	0.9850	0.9829
1.3	M	0.9983	0.9986	0.9983
	D	0.9980	0.9986	0.9983
	IM	0.9986	0.9989	0.9986

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

TABLE CXXV

REJECTION RATES FOR INTERACTION TEST FOR GAMMA-DISTRIBUTED DATA AT $\alpha = 0.05$ FOR OVERALL $R = 0.6$, $n_{11} = n_{22} = 100$, $n_{12} = n_{21} = 50$

Effect Size		Wald	LR	Score
1*	M	0.0195	0.0202	0.0127
	D	0.0197	0.0197	0.0121
	IM	0.0252	0.0250	0.0178
1.05	M	0.1044	0.0920	0.0534
	D	0.1054	0.0932	0.0544
	IM	0.1243	0.1112	0.0720
1.1	M	0.3617	0.3327	0.2319
	D	0.3645	0.3361	0.2341
	IM	0.4032	0.3765	0.2833
1.15	M	0.7053	0.6771	0.5585
	D	0.7082	0.6798	0.5614
	IM	0.7450	0.7180	0.6252
1.2	M	0.9151	0.9035	0.8405
	D	0.9164	0.9051	0.8423
	IM	0.9319	0.9221	0.8772
1.25	M	0.9858	0.9825	0.9629
	D	0.9862	0.9831	0.9638
	IM	0.9895	0.9877	0.9753
1.3	M	0.9988	0.9984	0.9954
	D	0.9989	0.9984	0.9954
	IM	0.9991	0.9991	0.9971

M = McKay's Approximation
D = David's Approximation
IM = Iglewicz and Myers' Approximation

APPENDIX B

SAS PROGRAM TO CALCULATE EXACT AND APPROXIMATE QUANTILES

```
/****************************************************
   This SAS program generates selected quantiles from the exact
   distribution of the sample CV for data drawn from a normal
   population using the method of Owen (1968).  Corresponding
   quantiles from McKay's, David's, and Iglewicz and Myers'
   approximations are also calculated.
 ****************************************************/

DATA CVQUANT;
 DO R=0.1,0.2,0.33; /* population CVs */
  DO N=10,50,100; /* sample sizes */
   DO P=0.01,0.05,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,0.95,0.99; /* quantiles */

    NC=SQRT(N)/R; /* calculate non-centrality parameter */
    TQ=1-P+PROBT(0,N-1,NC); /* calculate relevent quantile
                              from non-central t */
    RQ=SQRT(N)/TINV(TQ,N-1,NC); /* compute p-th exact quantile of r */
    RSUBNQ=SQRT((N-1)/N)*RQ; /* compute p-th exact quantile of r sub n */

    C0=(R**2/(1+R**2))*CINV(P,N-1)/(N-1);
    C1=(N-1)/N*C0;

    RSUBNQM=SQRT(C1/(1-C1)); /* calculate p-th quantile from McKay */
    RQD=SQRT(C0/(1-C0)); /* calculate p-th quantile from David */
    RQIM=R+PROBIT(P)*SQRT(R**2/N*(R**2+0.5)); /* calculate p-th quantile
                                               from Iglewicz and Myers */
    OUTPUT;
   END;
  END;
 END;

PROC PRINT NOOBS DATA=CVQUANT; /* print exact and approximate quantiles */
 VAR R N P RSUBNQ RSUBNQM RQ RQD RQIM;
RUN;
```

APPENDIX C

SAS PROGRAM TO SIMULATE THE INTERACTION TEST

```
/*****************************************************
  This SAS program simulates the test of interaction using data
   from normal populations arranged in a 2 x 2 factorial having
   CVs determined by the model

      R = exp(rstar + a + b + ab),

   where exp(rstar) is the overall population CV, exp(a) is the
   effect of factor A, exp(b) is the effect of factor B, and
   exp(ab) is the effect of interaction between A and B. For
   simplicity, both main effects in the generating model are set
   to zero.
*****************************************************/

PROC IML;

 START;
 NUMSAMP=10000;MAXITER=1000;ALPHA=.05; /* calculate 10,000 sets */
 OVERALLR=0.1;FACTAB=1.3;N11=10;N12=10;N21=10;N22=10;

   /* as an example, exp(rstar) is set at 0.1, exp(ab) is set at 1.3,
      and all sample sizes are set at 10 */

 DWALDREJ=0;DLRREJ=0;DSCREJ=0;
 MWALDREJ=0;MLRREJ=0;MSCREJ=0;
 IWALDREJ=0;ILRREJ=0;ISCREJ=0;
 STEP=0.5;BOUND=1E-6; /* set step length and convergence criterion */
 DO COUNT=1 TO NUMSAMP;

   /* generate a set of samples from a 2 x 2 factorial and compute
      sample CVs using (n-1) divisor for sample variance */

  SUM11=0;SUMSQ11=0;
  DO OBSCOUNT=1 TO N11;
   Y11=1 + RANNOR(0)*(OVERALLR*FACTAB);
   SUM11=SUM11+Y11; SUMSQ11=SUMSQ11+Y11**2;
  END;
   CV11=SQRT((SUMSQ11-SUM11**2/N11)/(N11-1))/(SUM11/N11);

  SUM12=0;SUMSQ12=0;
  DO OBSCOUNT=1 TO N12;
   Y12=1 + RANNOR(0)*(OVERALLR*INV(FACTAB));
   SUM12=SUM12+Y12; SUMSQ12=SUMSQ12+Y12**2;
  END;
   CV12=SQRT((SUMSQ12-SUM12**2/N12)/(N12-1))/(SUM12/N12);
```

```
 SUM21=0;SUMSQ21=0;
 DO OBSCOUNT=1 TO N21;
  Y21=1 + RANNOR(0)*(OVERALLR*INV(FACTAB));
  SUM21=SUM21+Y21; SUMSQ21=SUMSQ21+Y21**2;
 END;
  CV21=SQRT((SUMSQ21-SUM21**2/N21)/(N21-1))/(SUM21/N21);

 SUM22=0;SUMSQ22=0;
 DO OBSCOUNT=1 TO N22;
  Y22=1 + RANNOR(0)*(OVERALLR*FACTAB);
  SUM22=SUM22+Y22; SUMSQ22=SUMSQ22+Y22**2;
 END;
  CV22=SQRT((SUMSQ22-SUM22**2/N22)/(N22-1))/(SUM22/N22);

R=CV11//CV12//CV21//CV22;
N=N11//N12//N21//N22;
RSTAR=R##2/(1+R##2);
Z=LOG(SQRT(RSTAR/(1-RSTAR))); /* estimate saturated model using David's */
W=DIAG(2#(N-1)#(1-RSTAR)##2);   /* approximation                        */

XB={1  1  1  1,
     1  1 -1 -1,
     1 -1  1 -1,
     1 -1 -1  1};

B=INV(T(XB)*XB)*T(XB)*Z;
COVB=INV(T(XB)*W*XB);
C={0 0 0 1};
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic for David's */
PWALD=1-PROBCHI(CHISQ,1);          /* approximation                      */
IF PWALD < ALPHA THEN DWALDREJ=DWALDREJ + 1;

X={1  1  1, /* estimate main effects model using David's approximation */
   1  1 -1,
   1 -1  1,
   1 -1 -1};

B=INV(T(X)*W*X)*T(X)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((EXP(X*B))##2)/(1+((EXP(X*B))##2));
 Z=LOG(SQRT(RSTARHAT/(1-RSTARHAT)))+
     STEP*((RSTAR-RSTARHAT)/(2#RSTARHAT#(1-RSTARHAT)));
```

```
 W=DIAG(2#(N-1)#(1-RSTARHAT)##2);
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS=-(N-1)#(LOG(RSTAR/RSTARHAT)-
          ((RSTAR- RSTARHAT)/RSTARHAT));
DEV=SUM(DEVOBS); /* compute LR statistic for David's approximation */
PLR=1-PROBCHI(DEV,1);
IF PLR < ALPHA THEN DLRREJ=DLRREJ + 1;

G=DIAG(1/(2#RSTARHAT#(1-RSTARHAT)));
ESTEQ=T(XB)*G*W*(RSTAR-RSTARHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ; /* compute score
                                                            statistic */
PSCORE=1-PROBCHI(CHISQ,1);                          /* for David's
                                                            approx. */
IF PSCORE < ALPHA THEN DSCREJ=DSCREJ + 1;

RN=SQRT((N-1)/N)#R; /* estimate saturated model using McKay's approximation */
RSTARN=(N/(N-1))#(RN##2/(1+RN##2));
Z=LOG(SQRT(RSTARN/(1-RSTARN)));
W=DIAG(2#(N-1)#(1-RSTARN)##2);

B=INV(T(XB)*XB)*T(XB)*Z;
COVB=INV(T(XB)*W*XB);
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic for McKay's */
PWALD=1-PROBCHI(CHISQ,1);          /* approximation                      */
IF PWALD < ALPHA THEN MWALDREJ=MWALDREJ + 1;

B=INV(T(X)*W*X)*T(X)*W*Z; /* estimate main effects model using */
OLDB=B+1;                 /* McKay's approximation             */

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((EXP(X*B))##2)/(1+((EXP(X*B))##2));
 Z=LOG(SQRT(RSTARHAT/(1-RSTARHAT)))+
     STEP*((RSTARN-RSTARHAT)/(2#RSTARHAT#(1-RSTARHAT)));
 W=DIAG(2#(N-1)#(1-RSTARHAT)##2);
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS=-(N-1)#(LOG(RSTARN/RSTARHAT)-
          ((RSTARN-RSTARHAT)/RSTARHAT));
DEV=SUM(DEVOBS); /* compute LR statistic for McKay's approximation */
PLR=1-PROBCHI(DEV,1);
```

```
IF PLR < ALPHA THEN MLRREJ=MLRREJ + 1;

G=DIAG(1/(2#RSTARHAT#(1-RSTARHAT)));
ESTEQ=T(XB)*G*W*(RSTARN-RSTARHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ; /* compute score
                                                              statistic */
PSCORE=1-PROBCHI(CHISQ,1);                            /* for McKay's
                                                              approx. */
IF PSCORE < ALPHA THEN MSCREJ=MSCREJ + 1;

Z=LOG(R); /* estimate saturated model using Iglewicz and Myers' approximation */
W=DIAG(N/(R##2+0.5));

B=INV(T(XB)*XB)*T(XB)*Z;
COVB=INV(T(XB)*W*XB);
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic using IM */
PWALD=1-PROBCHI(CHISQ,1);          /* approximation                   */
IF PWALD < ALPHA THEN IWALDREJ=IWALDREJ + 1;

B=INV(T(X)*W*X)*T(X)*W*Z; /* estimate main effects model using IM approx. */
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RHAT=EXP(X*B);
 Z=LOG(RHAT)+STEP*((R-RHAT)/RHAT);
 W=DIAG(N/(RHAT##2+0.5));
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS=N#(2#((SQRT(2)#R#(ATAN(SQRT(2)#R) - ATAN(SQRT(2)#RHAT)))-
        ((R- RHAT)/RHAT))+
        LOG(((R##2)#(RHAT##2+.5))/((RHAT##2)#(R##2+.5))));
DEV=-2*SUM(DEVOBS); /* compute LR statistic for IM approximation */
PLR=1-PROBCHI(DEV,1);
IF PLR < ALPHA THEN ILRREJ=ILRREJ + 1;

G=DIAG(1/RHAT);
ESTEQ=T(XB)*G*W*(R-RHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ; /* compute score
                                                              statistic */
PSCORE=1-PROBCHI(CHISQ,1);                          /* using IM approx. */
IF PSCORE < ALPHA THEN ISCREJ=ISCREJ + 1;

END;
```

```
DWALDPWR=DWALDREJ/NUMSAMP; /* calculate observed powers and print */
                           /* results                            */
DLRPWR=DLRREJ/NUMSAMP;
DSCPWR=DSCREJ/NUMSAMP;

MWALDPWR=MWALDREJ/NUMSAMP;
MLRPWR=MLRREJ/NUMSAMP;
MSCPWR=MSCREJ/NUMSAMP;

IWALDPWR=IWALDREJ/NUMSAMP;
ILRPWR=ILRREJ/NUMSAMP;
ISCPWR=ISCREJ/NUMSAMP;

PRINT DWALDPWR DLRPWR DSCPWR MWALDPWR MLRPWR MSCPWR
      IWALDPWR ILRPWR ISCPWR;
PRINT OVERALLR FACTAB;
PRINT N11 N12 N21 N22;
FINISH;

RUN;
```

APPENDIX D

SAS PROGRAM TO SIMULATE THE MAIN-EFFECT TEST

```
/**************************************************
  This SAS program simulates the test of a main effect using
  data from normal populations arranged in a 2 X 2 factorial
  having CVs determined by the model

     R = exp(rstar + a + b),

  where exp(rstar) is the overall population CV, exp(a) is
  the effect of factor A, and exp(b) is the effect of factor
  B.  In order to examine the capability of the tests in a
  proper setting, no interaction is included in the
  generating model.  For simplicity, one main effect, say A,
  is also set to zero in the generating model.
**************************************************/

PROC IML;

 START;
 NUMSAMP=10000;MAXITER=1000;ALPHA=0.05; /* calculate 10,000 sets */
 OVERALLR=0.2;FACTB=1.15;N11=10;N12=10;N21=10;N22=10;

   /* as an example, exp(rstar) is set at 0.2, exp(b) is set at 1.15,
      and all sample sizes are set at 10 */

 DWALDREJ=0;DLRREJ=0;DSCREJ=0;
 MWALDREJ=0;MLRREJ=0;MSCREJ=0;
 IWALDREJ=0;ILRREJ=0;ISCREJ=0;
 STEP=0.5;BOUND=1E-6; /* set step length and convergence criterion */
 DO COUNT=1 TO NUMSAMP;

  /* generate a set of samples from a 2 x 2 factorial and compute
     sample CVs using (n-1) divisor for sample variance */

  SUM11=0;SUMSQ11=0;
  DO OBSCOUNT=1 TO N11;
   Y11=1 + RANNOR(0)*(OVERALLR*FACTB);
   SUM11=SUM11+Y11; SUMSQ11=SUMSQ11+Y11**2;
  END;
   CV11=SQRT((SUMSQ11-SUM11**2/N11)/(N11-1))/(SUM11/N11);

  SUM12=0;SUMSQ12=0;
  DO OBSCOUNT=1 TO N12;
   Y12=1 + RANNOR(0)*(OVERALLR*INV(FACTB));
   SUM12=SUM12+Y12; SUMSQ12=SUMSQ12+Y12**2;
  END;
```

```
 CV12=SQRT((SUMSQ12-SUM12**2/N12)/(N12-1))/(SUM12/N12);

SUM21=0;SUMSQ21=0;
DO OBSCOUNT=1 TO N21;
 Y21=1 + RANNOR(0)*(OVERALLR*FACTB);
 SUM21=SUM21+Y21; SUMSQ21=SUMSQ21+Y21**2;
END;
 CV21=SQRT((SUMSQ21-SUM21**2/N21)/(N21-1))/(SUM21/N21);

SUM22=0;SUMSQ22=0;
DO OBSCOUNT=1 TO N22;
 Y22=1 + RANNOR(0)*(OVERALLR*INV(FACTB));
 SUM22=SUM22+Y22; SUMSQ22=SUMSQ22+Y22**2;
END;
 CV22=SQRT((SUMSQ22-SUM22**2/N22)/(N22-1))/(SUM22/N22);

R=CV11//CV12//CV21//CV22;
N=N11//N12//N21//N22;
RSTAR=R##2/(1+R##2);
Z=LOG(SQRT(RSTAR/(1-RSTAR))); /* estimate main effects model using */
W=DIAG(2#(N-1)#(1-RSTAR)##2);  /* David's approximation            */

XB={1  1  1,
      1  1 -1,
      1 -1  1,
      1 -1 -1};

B=INV(T(XB)*W*XB)*T(XB)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((EXP(XB*B))##2)/(1+((EXP(XB*B))##2));
 Z=LOG(SQRT(RSTARHAT/(1-RSTARHAT)))+
    STEP*((RSTAR-RSTARHAT)/(2#RSTARHAT#(1-RSTARHAT)));
 W=DIAG(2#(N-1)#(1-RSTARHAT)##2);
 B=INV(T(XB)*W*XB)*T(XB)*W*Z;
END;

COVB=INV(T(XB)*W*XB);
C={0 0 1};
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic for 'B' */
PWALD=1-PROBCHI(CHISQ,1);            /* effect for David's approx.     */
IF PWALD < ALPHA THEN DWALDREJ=DWALDREJ + 1;
```

```
DEVOBS1=-(N-1)#(LOG(RSTAR/RSTARHAT)-
              ((RSTAR- RSTARHAT)/RSTARHAT));
DEV1=SUM(DEVOBS1); /* scaled deviance for main effects model */

Z=LOG(SQRT(RSTAR/(1-RSTAR))); /* estimate model with 'A' effect only */
W=DIAG(2#(N-1)#(1-RSTAR)##2);  /* using David's approximation          */

X={1  1,
   1  1,
   1 -1,
   1 -1};

B=INV(T(X)*W*X)*T(X)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((EXP(X*B))##2)/(1+((EXP(X*B))##2));
 Z=LOG(SQRT(RSTARHAT/(1-RSTARHAT)))+
     STEP*((RSTAR-RSTARHAT)/(2#RSTARHAT#(1-RSTARHAT)));
 W=DIAG(2#(N-1)#(1-RSTARHAT)##2);
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS2=-(N-1)#(LOG(RSTAR/RSTARHAT)-
              ((RSTAR-RSTARHAT)/RSTARHAT));
DEV2=SUM(DEVOBS2);  /* scaled deviance for 'A' effect model */
PLR=1-PROBCHI(DEV2-DEV1,1);                /* compute LR statistic for 'B' */
IF PLR < ALPHA THEN DLRREJ=DLRREJ + 1; /* effect for David's approx.   */

G=DIAG(1/(2#RSTARHAT#(1-RSTARHAT)));
ESTEQ=T(XB)*G*W*(RSTAR-RSTARHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ;
   /* compute score statistic for ‘B’ effect for David’s approximation */
PSCORE=1-PROBCHI(CHISQ,1);
IF PSCORE < ALPHA THEN DSCREJ=DSCREJ + 1;

RN=SQRT((N-1)/N)#R;
RSTARN=(N/(N-1))#(RN##2/(1+RN##2));
Z=LOG(SQRT(RSTARN/(1-RSTARN)));  /* estimate main effects model using */
W=DIAG(2#(N-1)#(1-RSTARN)##2);      /* McKay's approximation              */

B=INV(T(XB)*W*XB)*T(XB)*W*Z;
OLDB=B+1;
```

```
DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((EXP(XB*B))##2)/(1+((EXP(XB*B))##2));
 Z=LOG(SQRT(RSTARHAT/(1-RSTARHAT)))+
     STEP*((RSTARN-RSTARHAT)/(2#RSTARHAT#(1-RSTARHAT)));
 W=DIAG(2#(N-1)#(1-RSTARHAT)##2);
 B=INV(T(XB)*W*XB)*T(XB)*W*Z;
END;

COVB=INV(T(XB)*W*XB);
C={0 0 1};
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic for 'B' */
PWALD=1-PROBCHI(CHISQ,1);                /* effect for McKay's approx.     */
IF PWALD < ALPHA THEN MWALDREJ=MWALDREJ + 1;

DEVOBS1=-(N-1)#(LOG(RSTARN/RSTARHAT)-
             ((RSTARN-RSTARHAT)/RSTARHAT));
DEV1=SUM(DEVOBS1); /* compute scaled deviance for main effects model */

Z=LOG(SQRT(RSTARN/(1-RSTARN))); /* estimate model with 'A' effect only */
W=DIAG(2#(N-1)#(1-RSTARN)##2);     /* using McKay's approximation        */

B=INV(T(X)*W*X)*T(X)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((EXP(X*B))##2)/(1+((EXP(X*B))##2));
 Z=LOG(SQRT(RSTARHAT/(1-RSTARHAT)))+
     STEP*((RSTARN-RSTARHAT)/(2#RSTARHAT#(1-RSTARHAT)));
 W=DIAG(2#(N-1)#(1-RSTARHAT)##2);
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS2=-(N-1)#(LOG(RSTARN/RSTARHAT)-
             ((RSTARN-RSTARHAT)/RSTARHAT));
DEV2=SUM(DEVOBS2); /* scaled deviance for 'A' effect model */
PLR=1-PROBCHI(DEV2-DEV1,1);                /* compute LR statistic for 'B' */
IF PLR < ALPHA THEN MLRREJ=MLRREJ + 1; /* effect for McKay's approx.   */

G=DIAG(1/(2#RSTARHAT#(1-RSTARHAT)));
ESTEQ=T(XB)*G*W*(RSTARN-RSTARHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ;
  /* compute score statistic for 'B' effect for McKay's approximation */
PSCORE=1-PROBCHI(CHISQ,1);
```

```
IF PSCORE < ALPHA THEN MSCREJ=MSCREJ + 1;

Z=LOG(R);                  /* estimate main effects model using   */
W=DIAG(N/(R##2+0.5));   /* Iglewicz and Myers' approximation */

B=INV(T(XB)*W*XB)*T(XB)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RHAT=EXP(XB*B);
 Z=LOG(RHAT)+STEP*((R-RHAT)/RHAT);
 W=DIAG(N/(RHAT##2+0.5));
 B=INV(T(XB)*W*XB)*T(XB)*W*Z;
END;

COVB=INV(T(XB)*W*XB);
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic for 'B' */
PWALD=1-PROBCHI(CHISQ,1);              /* effect using IM approximation  */
IF PWALD < ALPHA THEN IWALDREJ=IWALDREJ + 1;

DEVOBS1=N#(2#((SQRT(2)#R#(ATAN(SQRT(2)#R) - ATAN(SQRT(2)#RHAT)))-
            ((R-RHAT)/RHAT))+
            LOG(((R##2)#(RHAT##2+.5))/((RHAT##2)#(R##2+.5))));
DEV1=-2*SUM(DEVOBS1); /* scaled deviance for main effects model */

Z=LOG(R);                  /* estimate model with 'A' effect only */
W=DIAG(N/(R##2+0.5));     /* using IM approximation           */

B=INV(T(X)*W*X)*T(X)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RHAT=EXP(X*B);
 Z=LOG(RHAT)+STEP*((R-RHAT)/RHAT);
 W=DIAG(N/(RHAT##2+0.5));
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS2=N#(2#((SQRT(2)#R#(ATAN(SQRT(2)#R) - ATAN(SQRT(2)#RHAT)))-
            ((R-RHAT)/RHAT))+
            LOG(((R##2)#(RHAT##2+.5))/((RHAT##2)#(R##2+.5))));
DEV2=-2*SUM(DEVOBS2);       /* scaled deviance for 'A' effect model */
PLR=1-PROBCHI(DEV2-DEV1,1);              /* compute LR statistic for 'B' */
```

```
  IF PLR < ALPHA THEN ILRREJ=ILRREJ + 1; /* effect for IM approx.          */

  G=DIAG(1/RHAT);
  ESTEQ=T(XB)*G*W*(R-RHAT);
  CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ;
    /*compute score statistic for 'B' effect for IM approximation */
  PSCORE=1-PROBCHI(CHISQ,1);
  IF PSCORE < ALPHA THEN ISCREJ=ISCREJ + 1;

 END;

 DWALDPWR=DWALDREJ/NUMSAMP; /* calculate observed powers and print */
 DLRPWR=DLRREJ/NUMSAMP;          /* results                            */
 DSCPWR=DSCREJ/NUMSAMP;

 MWALDPWR=MWALDREJ/NUMSAMP;
 MLRPWR=MLRREJ/NUMSAMP;
 MSCPWR=MSCREJ/NUMSAMP;

 IWALDPWR=IWALDREJ/NUMSAMP;
 ILRPWR=ILRREJ/NUMSAMP;
 ISCPWR=ISCREJ/NUMSAMP;

 PRINT DWALDPWR DLRPWR DSCPWR MWALDPWR MLRPWR MSCPWR
       IWALDPWR ILRPWR ISCPWR;
 PRINT OVERALLR FACTB;
 PRINT N11 N12 N21 N22;
 FINISH;

RUN;
```

APPENDIX E

SAS PROGRAM TO SIMULATE THE ONE-FACTOR TEST

```
/*****************************************************
  This SAS program simulates the test of the single factor in
  a one-factor experiment using data from normal populations
  having CVs determined by the model

     R = exp(rstar + a),

  where exp(rstar) is the overall population CV and exp(a) is
  the effect of the single factor.  Fitted models are additive.
*****************************************************/

PROC IML;

 START;
 NUMSAMP=10000;MAXITER=1000;ALPHA=0.05; /* calculate 10,000 sets */
 OVERALLR=0.1;FACTA=1.2;N1=20;N2=20;N3=20;

  /* as an example, exp(rstar) is set at 0.1, exp(a) is set at
     1.2, and all sample sizes are set at 20 */

 DWALDREJ=0;DLRREJ=0;DSCREJ=0;
 MWALDREJ=0;MLRREJ=0;MSCREJ=0;
 IWALDREJ=0;ILRREJ=0;ISCREJ=0;
 DDLRREJ=0;GSCREJ=0;DDTREJ=0;
 STEP=0.5;BOUND=1E-6; /* set step length and convergence criterion */

 DO COUNT=1 TO NUMSAMP;

  /* generate a set of samples from a one-factor model with three
     levels and compute sample CVs using (n-1) divisor for sample
     variance */

  SUM1=0;SUMSQ1=0;
  DO OBSCOUNT=1 TO N1;
   Y1=1 + RANNOR(0)*(OVERALLR*INV(FACTA));
   SUM1=SUM1+Y1; SUMSQ1=SUMSQ1+Y1**2;
  END;
   SSQR1=(SUMSQ1-SUM1**2/N1)/(N1-1);XBAR1=SUM1/N1;
   CV1=SQRT(SSQR1)/XBAR1;

  SUM2=0;SUMSQ2=0;
  DO OBSCOUNT=1 TO N2;
   Y2=1 + RANNOR(0)*(OVERALLR);
   SUM2=SUM2+Y2; SUMSQ2=SUMSQ2+Y2**2;
  END;
```

```
  SSQR2=(SUMSQ2-SUM2**2/N2)/(N2-1);XBAR2=SUM2/N2;
  CV2=SQRT(SSQR2)/XBAR2;

SUM3=0;SUMSQ3=0;
DO OBSCOUNT=1 TO N3;
  Y3=1 + RANNOR(0)*(OVERALLR*FACTA);
  SUM3=SUM3+Y3; SUMSQ3=SUMSQ3+Y3**2;
END;
  SSQR3=(SUMSQ3-SUM3**2/N3)/(N3-1);XBAR3=SUM3/N3;
  CV3=SQRT(SSQR3)/XBAR3;

R=CV1//CV2//CV3;
N=N1//N2//N3;
RSTAR=R##2/(1+R##2);
Z=SQRT(RSTAR/(1-RSTAR));                  /* estimate saturated model using */
W=DIAG(2#(N-1)#(1-RSTAR)##3/RSTAR);  /* David's approximation             */

XB={1  1  0,
      1  0  1,
      1 -1 -1};

B=INV(T(XB)*XB)*T(XB)*Z;
COVB=INV(T(XB)*W*XB);
C={0 1 0,
   0 0 1};
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic for David's */
PWALD=1-PROBCHI(CHISQ,2);              /* approximation                         */
IF PWALD < ALPHA THEN DWALDREJ=DWALDREJ + 1;

X={1,  /* estimate null model using David's approximation */
   1,
   1};

B=INV(T(X)*W*X)*T(X)*W*Z;
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
  OLDB=B;
  RSTARHAT=((X*B)##2)/(1+(X*B)##2);
  Z=SQRT(RSTARHAT/(1-RSTARHAT))+
        STEP*((RSTAR-RSTARHAT)/
              (2#SQRT(RSTARHAT#(1-RSTARHAT)##3)));
  W=DIAG(2#(N-1)#(1-RSTARHAT)##3/RSTARHAT);
  B=INV(T(X)*W*X)*T(X)*W*Z;
END;
```

```
DEVOBS=-(N-1)#(LOG(RSTAR/RSTARHAT)-
          ((RSTAR-RSTARHAT)/RSTARHAT));
DEV=SUM(DEVOBS); /* compute LR statistic for David's approximation */
PLR=1-PROBCHI(DEV,2);
IF PLR < ALPHA THEN DLRREJ=DLRREJ + 1;

G=DIAG(1/(2#SQRT(RSTARHAT#(1-RSTARHAT)##3)));
ESTEQ=T(XB)*G*W*(RSTAR-RSTARHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ;
  /* compute score statistic for David's approximation */
PSCORE=1-PROBCHI(CHISQ,2);
IF PSCORE < ALPHA THEN DSCREJ=DSCREJ + 1;

RN=SQRT((N-1)/N)#R; /* estimate saturated model using McKay's approx. */
RSTARN=(N/(N-1))#(RN##2/(1+RN##2));
Z=SQRT(RSTARN/(1-RSTARN));
W=DIAG(2#(N-1)#(1-RSTAR)##3/RSTAR);

B=INV(T(XB)*XB)*T(XB)*Z;
COVB=INV(T(XB)*W*XB);
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B;
  /* compute Wald statistic for McKay's approximation */
PWALD=1-PROBCHI(CHISQ,2);
IF PWALD < ALPHA THEN MWALDREJ=MWALDREJ + 1;

B=INV(T(X)*W*X)*T(X)*W*Z; /* estimate null model using McKay's approx. */
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RSTARHAT=((X*B)##2)/(1+(X*B)##2);
 Z=SQRT(RSTARHAT/(1-RSTARHAT))+
     STEP*((RSTARN-RSTARHAT)/
          (2#SQRT(RSTARHAT#(1-RSTARHAT)##3)));
 W=DIAG(2#(N-1)#(1-RSTARHAT)##3/RSTARHAT);
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS=-(N-1)#(LOG(RSTARN/RSTARHAT)-
          ((RSTARN-RSTARHAT)/RSTARHAT));
DEV=SUM(DEVOBS); /* compute LR statistic for McKay's approximation */
PLR=1-PROBCHI(DEV,2);
IF PLR < ALPHA THEN MLRREJ=MLRREJ + 1;

G=DIAG(1/(2#SQRT(RSTARHAT#(1-RSTARHAT)##3)));
```

```
ESTEQ=T(XB)*G*W*(RSTARN-RSTARHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ;
   /* compute score statistic for McKay's approximation */
PSCORE=1-PROBCHI(CHISQ,2);
IF PSCORE < ALPHA THEN MSCREJ=MSCREJ + 1;

Z=R; /* estimate saturated model using Iglewicz and Myers' approx. */
W=DIAG(N/(R##2#(R##2+0.5)));

B=INV(T(XB)*XB)*T(XB)*Z;
COVB=INV(T(XB)*W*XB);
CHISQ=T(C*B)*INV(C*COVB*T(C))*C*B; /* compute Wald statistic using IM */
PWALD=1-PROBCHI(CHISQ,2);              /* approximation                 */
IF PWALD < ALPHA THEN IWALDREJ=IWALDREJ + 1;

B=INV(T(X)*W*X)*T(X)*W*Z; /* estimate null model using IM approximation */
OLDB=B+1;

DO ITER=1 TO MAXITER WHILE(MAX(ABS(B-OLDB)) > BOUND);
 OLDB=B;
 RHAT=X*B;
 W=DIAG(N/(RHAT##2#(RHAT##2+0.5)));
 B=INV(T(X)*W*X)*T(X)*W*Z;
END;

DEVOBS=N#(2#((SQRT(2)#R#(ATAN(SQRT(2)#R) - ATAN(SQRT(2)#RHAT)))-
          ((R-RHAT)/RHAT)) +
          LOG(((R##2)#(RHAT##2+.5))/((RHAT##2)#(R##2+.5))));
DEV=-2*SUM(DEVOBS); /* compute LR statistic for IM approximation */
PLR=1-PROBCHI(DEV,2);
IF PLR < ALPHA THEN ILRREJ=ILRREJ + 1;

ESTEQ=T(XB)*W*(R-RHAT);
CHISQ=T(C*ESTEQ)*INV(C*T(XB)*W*XB*T(C))*C*ESTEQ;
   /*compute score statistic using IM approximation */
PSCORE=1-PROBCHI(CHISQ,2);
IF PSCORE < ALPHA THEN ISCREJ=ISCREJ + 1;


SMALLM=MIN(R);  /* calculate ML estimate of R in (2.4) using */
LARGEM=MAX(R); /* Gupta and Ma's solution                    */
RTILDA=(SMALLM+LARGEM)/2;
G=SUM(N#(1+SQRT(1+4#(1+R##2)#RTILDA##2))/(2#(1+R##2)))-SUM(N);
DO ITER=1 TO MAXITER WHILE(ABS(G)>BOUND);
 IF G<=0 THEN SMALLM=RTILDA;
```

```
   ELSE LARGEM=RTILDA;
  RTILDA=(SMALLM+LARGEM)/2;
  G=SUM(N#(1+SQRT(1+4#(1+R##2)#RTILDA##2))/(2#(1+R##2)))-SUM(N);
 END;

 XBAR=XBAR1//XBAR2//XBAR3;
 SSQR=SSQR1//SSQR2//SSQR3;
 SUMSQ=SUMSQ1//SUMSQ2//SUMSQ3;
 MU=((2#(1+R##2))#XBAR)/(1+SQRT(1+4#(1+R##2)#RTILDA##2));
  /* calculate ML estimates of mu's in (2.5) */
 LR=SUM(N#LOG((MU#RTILDA)##2/SSQR)); /* compute Doornbos and
                                          Dijkstra's LR statistic */
 PVAL=1-PROBCHI(LR,2);
 IF PVAL<ALPHA THEN DDLRREJ=DDLRREJ+1;
 TEMPVEC1=SUMSQ-2#N#MU#XBAR+N#MU##2; /* compute Gupta and Ma's
                                          score statistic */
 TEMPVEC2=MU##2#RTILDA##3;
 AVEC=TEMPVEC1/TEMPVEC2-(N/RTILDA);
 AVEC1=AVEC##2/N;
 SCORE=0.5#RTILDA##2#(2#RTILDA##2+1)#SUM(AVEC1);
 PVAL=1-PROBCHI(SCORE,2);
 IF PVAL<ALPHA THEN GSCREJ=GSCREJ+1;

 B=1/R; /* calculate Doornbos and Dijkstra's non-central t statistic */
 BIGN=SUM(N);
 BBAR=SUM(N#B)/BIGN;
 T=SUM(N#(B-BBAR)##2);
 RTILDA=(SUM(N#B##2)-SUM((N-1)/(N-3)))/SUM(N#(N-1)/(N-3));
 EP=SQRT((N-1)/2)#GAMMA((N-2)/2)/GAMMA((N-1)/2);
 EXPT=SUM((BIGN-N)#(N-1)/(BIGN#(N-3)))+
     RTILDA#(SUM(N#(BIGN-N)#(N-1)/(BIGN#(N-3)))+
     ((SUM((N#EP)##2)-(SUM(N#EP))##2)/BIGN));
 D=2*T/EXPT;
 PVAL=1-PROBCHI(D,2);
 IF PVAL<ALPHA THEN DDTREJ=DDTREJ+1;

END;
DWALDPWR=DWALDREJ/NUMSAMP; /* calculate observed powers and print
                                results */
DLRPWR=DLRREJ/NUMSAMP;
DSCPWR=DSCREJ/NUMSAMP;

MWALDPWR=MWALDREJ/NUMSAMP;
MLRPWR=MLRREJ/NUMSAMP;
MSCPWR=MSCREJ/NUMSAMP;
```

```
IWALDPWR=IWALDREJ/NUMSAMP;
ILRPWR=ILRREJ/NUMSAMP;
ISCPWR=ISCREJ/NUMSAMP;

DDLRPWR=DDLRREJ/NUMSAMP;
GSCPWR=GSCREJ/NUMSAMP;
DDTPWR=DDTREJ/NUMSAMP;

PRINT DWALDPWR DLRPWR DSCPWR MWALDPWR MLRPWR MSCPWR
      IWALDPWR ILRPWR ISCPWR;
PRINT DDLRPWR GSCPWR DDTPWR;
PRINT OVERALLR FACTA;
PRINT N1 N2 N3;
FINISH;

RUN;
```

APPENDIX F

SAS CODE TO GENERATE GAMMA-DISTRIBUTED DATA FOR THE INTERACTION TEST

```
/*******************************************************
  The following SAS code should be inserted in place of the data
  generation code in the test-of-interaction program in order to
  obtain data from gamma distributions with those same CVs.
 *******************************************************/

  SUM11=0;SUMSQ11=0;
  DO OBSCOUNT=1 TO N11;
   Y11=RANGAM(0,(OVERALLR*FACTAB)**-2)*(OVERALLR*FACTAB)**2;
   SUM11=SUM11+Y11; SUMSQ11=SUMSQ11+Y11**2;
  END;
   CV11=SQRT((SUMSQ11-SUM11**2/N11)/(N11-1))/(SUM11/N11);

  SUM12=0;SUMSQ12=0;
  DO OBSCOUNT=1 TO N12;
   Y12=RANGAM(0,(OVERALLR*INV(FACTAB))**-2)*
                 (OVERALLR*INV(FACTAB))**2;
   SUM12=SUM12+Y12; SUMSQ12=SUMSQ12+Y12**2;
  END;
   CV12=SQRT((SUMSQ12-SUM12**2/N12)/(N12-1))/(SUM12/N12);

  SUM21=0;SUMSQ21=0;
  DO OBSCOUNT=1 TO N21;
   Y21=RANGAM(0,(OVERALLR*INV(FACTAB))**-2)*
                 (OVERALLR*INV(FACTAB))**2;
   SUM21=SUM21+Y21; SUMSQ21=SUMSQ21+Y21**2;
  END;
   CV21=SQRT((SUMSQ21-SUM21**2/N21)/(N21-1))/(SUM21/N21);

  SUM22=0;SUMSQ22=0;
  DO OBSCOUNT=1 TO N22;
   Y22=RANGAM(0,(OVERALLR*FACTAB)**-2)*(OVERALLR*FACTAB)**2;
   SUM22=SUM22+Y22; SUMSQ22=SUMSQ22+Y22**2;
  END;
   CV22=SQRT((SUMSQ22-SUM22**2/N22)/(N22-1))/(SUM22/N22);
```

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