A MECHANICAL METHOD OF ANALYSIS FOR STATICALLY INDETERMINATE STRUCTURES

By

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The result of investigation of use of paper models in design of indeterminate structures.

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A MECHANICAL METHOD OF ANALYSIS FOR STATICALLY INDETERMINATE STRUCTURES

I. INTRODUCTION.

1. Preliminary. --- With the exception of a comparatively few specialists in the engineering field, the engineer has up to the present time not bothered with the analysis of complicated statically indeterminate structures. He has contended that the saving involved due to building rigid structures was more than offset by the amount of designing work required. This may have been true more or less during the past in America where the extra amount of material used was of little consequence. But building material is fast becomming a factor of great importance to our national growth and saving that may be affected by the use of rigid structures is of consequence.

Certain building materials lend themselves naturally to the construction of rigid structures. The reinforced concrete building frame is in itself rigid and indeterminate.

These structures are designed at present in the ordinary engineering office by making certain assumptions which are little more than guesses, some good and some bad. This is not true engineering. Such structures so designed may be amply strong to carry the loads for which they are designed; but they will probably not be economically designed with unit stresses equal in all their members. The carpenter does this sort of engineering when he chooses a joist size because that same joist size has proved satisfactory on a previous job.

2. Methods of Analysis.--- The analysis of indeterminate structures is at present done by means of the theory of least work and by slope deflections. The method of slope deflections is the more practicable and is the one in general use by American engineers. This method is developed and described in detail in Volume II of "Modern Framed Structures" by Johnson, Bryan, and Turneaure, and in "Reinforced Concrete Construction" by Hool. A method of analysis by the use of the Ellipse of Elasticity and elastic

weights is proposed by Mr. A.C.Janni, C.E., which is described in a discussion by Mr. Janni of "New Principle in Theory of Structures" By George F. Swain, Past President A.S.C.E., and given in Transactions of the A.S.C.E., Vol. LXXXII - 1919 - 1920 ; and as applied to concrete arches in Volumn III of "Reinforced Concrete Construction" by Hool. Any analytical method yet discovered for indeterminate structures is long and tedious, involving long mathematical calculations.

In the spring of 1922 a mechanical method for the analysis of statically indeterminate structures was proposed by George Erle Beggs, Associate Professor of Civil Engineering. Princeton University, in which paper models and special guages are used. He claims for the new method speed, accuracy, and simplicity.

3. Object and Scope of Investigation. --- This investigation of the newly proposed method of design was made for the purpose of further establishing the truth as to the accuracy, simplicity, and practicability of the method.

It was desired to know whether resułts obtained by this method would check with those found by the theoretical methods of exact design, and also to know how these results would compare with those found by actual tests on full size frames of concrete.

Several models of indeterminate frames were made and analyzed by this method and the results compared with those found by the method of slope deflections. An application pf this method to reinforced concrete frames was then investigated to prove its usefulness for the design of this important class of structures. In order to compare this method with the ordinary designing method for simplicity, an influence table for the springing section of a reinforced concrete arch ring was prepared by the ordinary method and by the mechanical method and these compared both as to mathematical results and ease of attainment.

In preparation for the Doctor's degree, Dr. Mikishi Abe, made and tested several reinforced concrete building frames.

The result of these tests are published in Bulletin No. 107 of the Engineering Experiment Station of the University of Illinois. A model of one of these frames was made and analyzed by the new method and the results compared with those found by Dr. Abe in the test and by the method of least work.

This work was done under the supervision of Professor Preston M. Geren, Head of the Department of Architecture and Architectural Engineering.

II. DESCRIPTION OF NEW METHOD

4. Basic Theory. --- The principle upon which this method depends is Maxwell's Law of Reciprocal Deflections, which simply stated is:

The displacement in any given direction a' of any point A of a structure, due to a load P applied at some other point B in a direction b', is equal to the displacement of the point B in the direction b' which would be caused by the application of the load P at the point A in direction a'.

If a load P (Figure I) be applied at A causing a deflection d' at A and d" at B, the work causing such deformation is Pd'/2 and is resisted by the internal work. But this displacement d' could be produced at A by applying a force H at B causing displacement d" at B and d' at A and the work done would be equal to Hd"/2 resisted by the internal work. But the internal work is equal in both cases since the same deformation is induced in the structure. Then Hd"/2 equals Pd'/2 or H equals Pd'/d".

This deflection d' at A could be produced by a moment at B as shown in Figure II. The work done in this case in causing deflection d' equals Wd"/2 when d" equals the distance W moves down in producing the deflection d'. The moment of W equals Wr. If r equals 1 then the moment equals W and since Wd" equals Pd', then M equals Pd'/d".

All that is needed to find a thrust, moment, or shear at a section B for a load P at A is the ratio d'/d".

5. Practical Test of Theory.--- This theory was put to a practical test by Professor Beggs by constructing the rigid frame shown in Figure III. The frame was made of wood sticks about 5/8 by 7/8 inches in section and columnsand main horizontal members 50 inches long. The members were fastened together by means of metal gusset plates and screws.

The 3-hinged frame was floated horizontally on ball bearings on the floor. No weights were applied to the structure while measuring deflections in determining the values of horizontal and vertical reactions at hinge 3 for a load of 12 lbs. applied horizontally at B. To determine by this theory the value of V_3 at hinge 3, the pin at 3 was removed and the ratio of deflection at B, horizontally, to that of 3, vertically, found by forcing point 3 to deflect upward an amount equal to d"/2 and then downward a like amount. The motion of B was measured and the ratio d'/d" established. According to the theory then V_3 equals d'/d" times 12. This was calculated to be 3.4 lbs. The value of H_3 was determined in a similar manner to be 14.6 lbs. The hinge reaction is according to the theory the square root of the square of 14.6 plus the square of 3.4 or 15 lbs for a load of 12 lbs. at B.

The pin was then replaced at 3 and a weight of 12 lbs. attached at B by a cord which was wound over a bicycle wheel as shown in the figure. A weight of 15 lbs. was attached in a similar manner at an angle such as to cause components at 3 equal to V_3 of 3.4 lbs. and H_3 of 14.6 lbs. The pin at 3 was then removed and no deflection

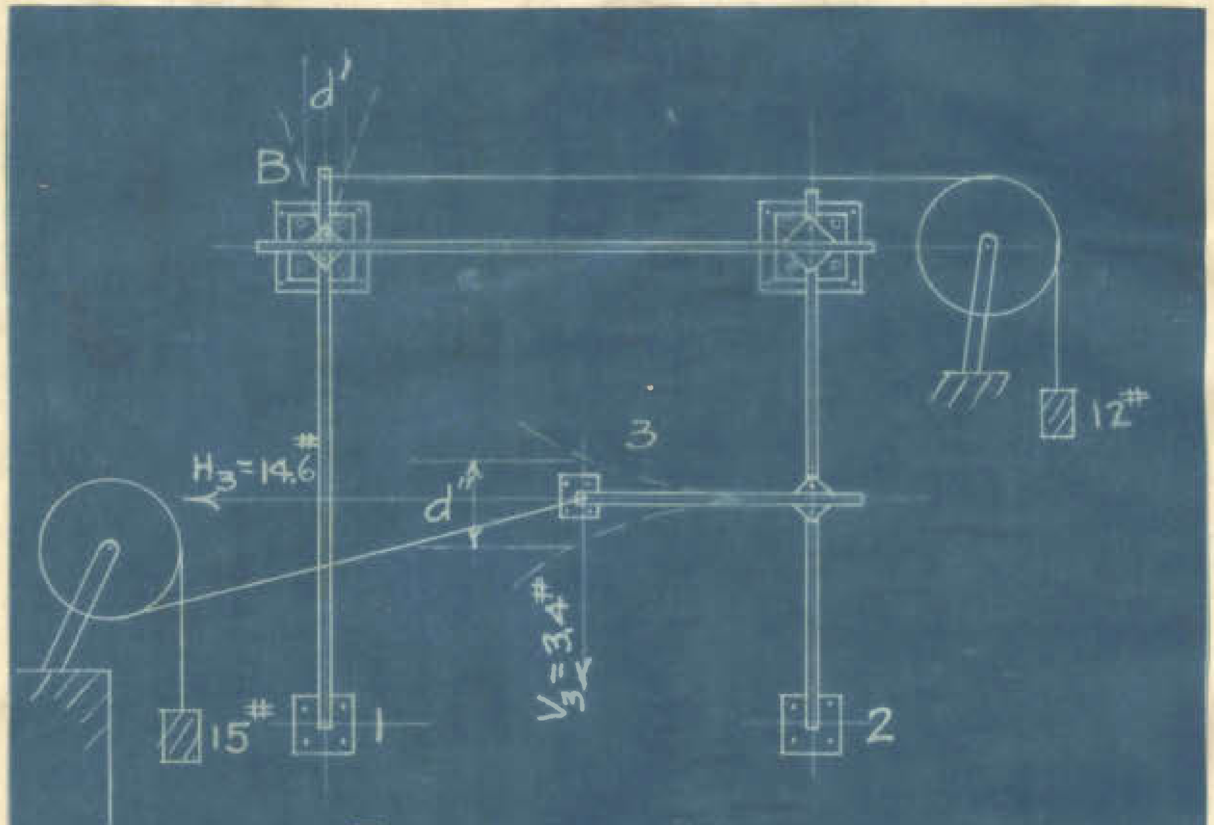


FIGURE III

occured which indicates that the value of the reaction at 3 was correct as determined mechanically since there was produced static and elastic equilibrium.

6. Application of Theory.---- The internal work of resilience in any structure is equal to

$\int S^2 ds/2EA$ for direct stress on the section and to $\int M^2 ds/2EI$ fpr bemdèmg stress, when

S equals stress at section;

A equals area of section,

E equals modulus of elasticity,

ds equals increment of length of axis,

I equals moment of inertia,

M equals bending moment at the section.

This neglects the work of shear which is small. Using the same notation as that used in explain-ing the theory and equating internal and external work:

$$Hd'/2 \text{ equals } \int S^2 ds/2AE + \int M^2 ds/2EI$$

and $$Pd''/2 \text{ equals } \int S_1^2 ds/2AE + \int M_1^2 ds/2EI$$

and since $Hd'/2$ equals $Pd''/2$,

then $$\int S^2 ds/2AE + \int M^2 ds/2EI = \int S_1^2 ds/2AE + \int M_1^2 ds/2EI$$

or considering E constant throughout the sturcture,

$$\int S^2 ds/A + \int M^2 ds/I = \int S_1^2 ds/A + \int M_1^2 ds/I$$

It can be seen then that when E is constant throughout the structure, the relation between H, P? and the deflections is not dependent upon the value of E.

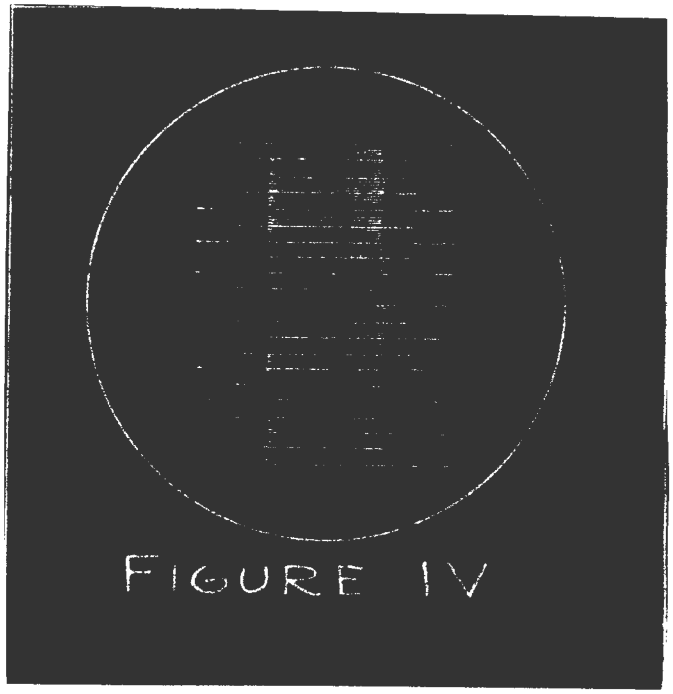
If we build a model then of the same shape as the structure, and of some other material, the same relation of Hd' to Pd" will hold in the model as in the full size frame so long as that material is not stressed beyond the elastic limit, since the value of d'/d" is simply a ratio and not dependent upon the values of the actual deflections. These models may be made to as small a scale as desired and the results obtained will be just as accurate as those obtained with a full size frame so long as the measurements are all made with the requir- ed degree of accuracy. By using cardboard heavy enough that it will not buckle under a very small deflection for models and measuring the displacements by means of a high powered microscope, this mechanical method may be used in the drafting room.

7. Apparatus.--- The apparatus used in the investigation of this method consisted of a microscope for reading deflections at load points produced by a small guage at the section where the value of the shear, thrust, or moment

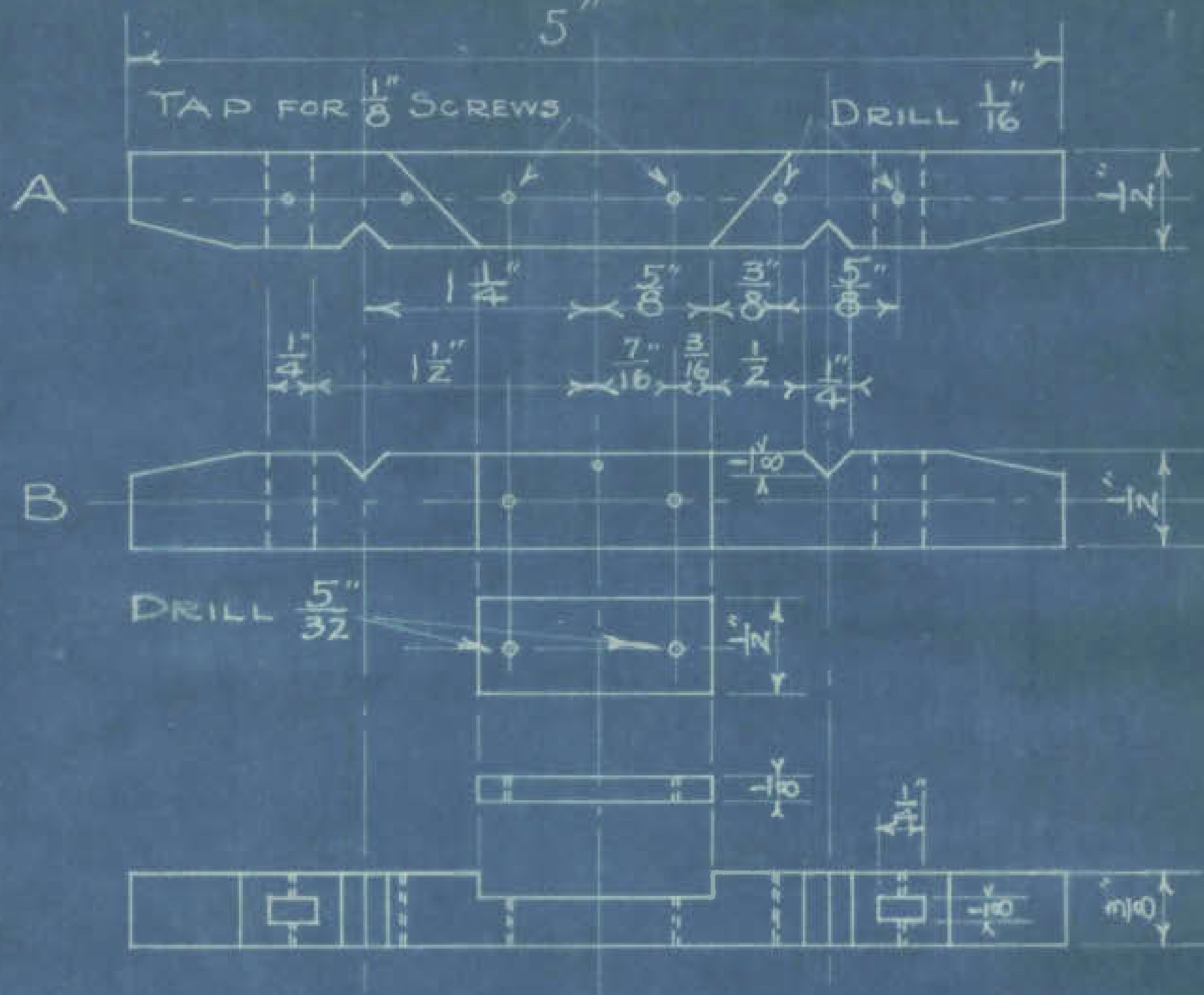
is desired. The microscope was of the ordinary type made by Bausch and Lomb Co. for use in Bacteriological work. For this work the telescope and adjusting mechanismwere dismounted from the pedestal that is between the base and table just under the telescope. This pedestal and table containing the apparatus for holding and lighting the specimen were taken off and the telescope turned thru 180 degrees so that it may be focused upon a model lying flat onthe drawing board. The only tools required for dismounting and remounting the microscope were a screw driver anda special wrench.

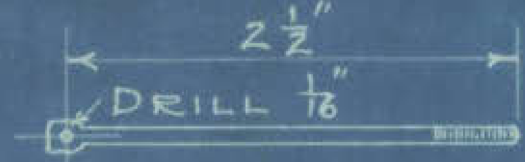
A view of the microscope fitted for this work is given in the illustration.

For measuring the deflection a micrometer eyepiece,which is a regular equipment for the microscope, is used in the regular eye-piece of the microscope. The field of vision appears as shown in Figure IV and can be turned in the telescope so that deflection in any direction may be read without moving the microscope.

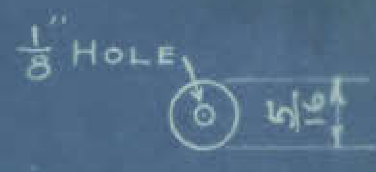


Deflection is produced at the section desired by means of small guages as shown in the illustration on page 12. Four of these guages were made in the O.A.M.C. shops. The material used was aluminium. The drawings on the next page give all the dimensions. The difference in the sizes of thrust and shear plugs was made approximately such that the dispalcement produced would be equal to the limits of measurement of the microscope. The guage may be used for fixing a section by clamping the model in B and



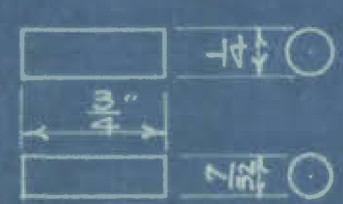


3/32" BOLT



WASHER

SHEAR PLUG



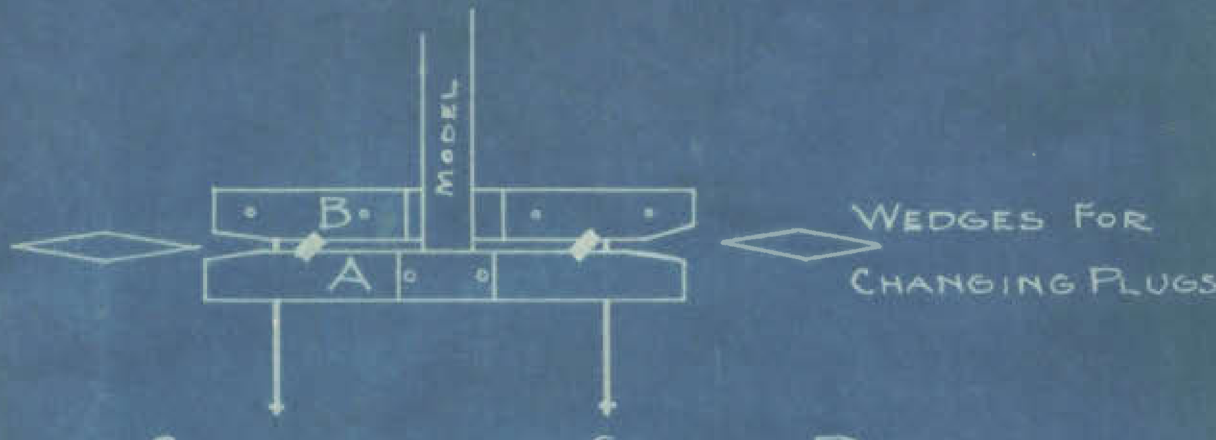
THRUST & MOMENT PLUGS

REQUIRED FOR 1 GAUGE

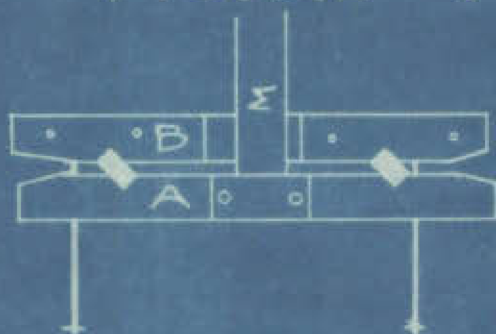
1 A
1 B
2 COVER PLATES
4 $\frac{1}{8} \times \frac{3}{8}$ MACHINE SCREWS
2 $\frac{3}{32} \times 2\frac{1}{2}$ BOLTS
2 WASHERS
2 SHEAR PLUGS
2 THRUST & MOMENT PLUGS
2 $1\frac{1}{2} \times \frac{3}{32}$ COIL SPRINGS

pinning the guage to the drawing board. If a hinge is desired at some point, the model is not clampwd but merely fastened in the guage with a single pin at the point where the hinge is desired and the guage pinned to the drawing board.

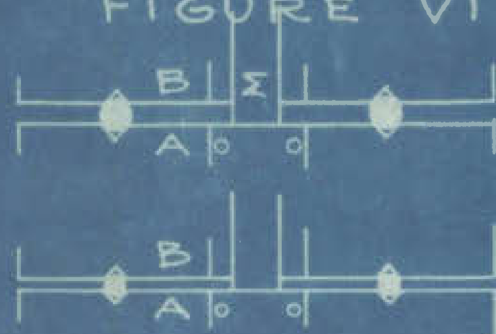
To produce a displacement for shear at a section, the model is clamped in A as shown in Figure V with the thrust plugs inserted as shown and B pinned to the drawing board. The plugs are then removed and inserted as shown in Figure VI producing a displacement normal to the axis at the section. To produce a thrust displacement; i.e., normal to the section and parallel to the axis, the larger plugs are placed in the guage as shown in Figure VII and the model clamped in A. B is then pinned to the drawing board, the larger plugs removed by means of the wedges shown and the smaller ones inserted. To produce a displacement due to moment, a large plug is placed on one side and a small one on the



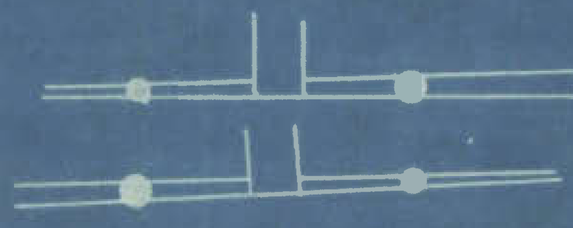
GUAGE WITH SHEAR PLUGS
FIGURE V



SHEAR PLUGS CHANGED
FIGURE VI



THRUST PLUGS IN PLACE
FIGURE VII



MOMENT PLUGS IN PLACE
FIGURE VIII

other, as in FigureVIII, and the model is clamped in A. B is then fastened to the drawing board and the plugs interchanged as shown in the figure, producing a moment at the section.

8. Method of Analysis.--- If the value of the displacement for shear, moment, and thrust produced by the guages as just expłained be known in terms of the divisions on the micrometer eye-piece, it is possible to determine the shear, thrust, or moment at the section of a structure for any loading by the following means:

A model of the structure to be analyzed is made of stiff cardboard to scale. The value of the scale is immaterial so long as it is of a size proportional to the deformation produced by the guages. This model is fastened in the guages at the points of support with the supports either fixed or hinged as previously described, andthe guages fastened to the board. Then the microscope is focused on the point at which the loadis assumed to be concentrated and the micrometer scale oriented to read deflection in the direction in which the load is assumed to be acting. The thrust plugs are changed as described and the displacement of the load point observed onthe micrometer scale. According to the theory just explained the thrust is equal to the load assumed times the ratio of the deflection of the load point tothat produced by the guage, or

$T = Pd_m/d_g$, when d_m is the displacement measured by the micrometer and d_g is the displacement produced by the guage. As d_g is constant for all points of loading, it is a simple matter to obtain values of T for plotting the influence line for thrust at the section simply by measuring the deflection at different load points. In the same manner influence lines for moment and shear may be obtained by producing deformation of moment and shear respectively by using the moment and shear plugs as explained earlier.

III. APPLICATION TO HOMOGENEOUS MATERIALS.

9. Calibrating the Guages.--- Before it is possible to use the apparatus described in the solution of problems, the value of the displacement produced by the guage must be known in terms of the divisions of the micrometer eyepiece. In determining this relation for the shear plugs, it was found that the motion was too extensive to be read directly on the scale. A white architects scale was then fastened parallel to the motion and the number of divisions of the micrometer in a division of the scale was determined. It was then possible to determine the motion in terms of the micrometer divisions. The motion due to changing thrust plugs was determined in the same manner with the scale turned parallel to the motion.

To determine the moment coefficient a stick was fastened in the guage and a point on the drawing board a distance of 26 inches from the guage center was marked. The plugs were changed in the guages and the motion of the point on the stick from its first position was measured. This motion divided by the distance from the center of rotationgives the rotation in radians.

Considerable difficulty was experienced in getting the guages adjusted so that there was not moment produced by the thrust plugs and no thrust nor moment by the shear plugs. Three of the guages were finally abandoned. The remaining one was tested and found to produce only shear when the shear plugs were changes, only thrust when the thrust plugs were changed, and only moment when the moment plugs were interchanged. This guage was used in all the mechanical solutions which follow, the other guages being used only for hinges and for fixing the ends of members.

The values of d" for this guage were determined to be as follows:

shear 12 divisions

thrust 13 divisions

moment 1.7 divisions

10. Continuous Girders.--- The analysis of continuous girders is, like all statically indeterminate structures, dependent upon the shape of thestructure itself,so before an analysis is made certain proportions must be assumed. If the cintinuous girder is of uniform section, the problem becomes a comparatively simple one and may be readily solved for any loading by means of the three moment equation. If, however, there are moving loads on the spanas in the case of bridges it is convenient to plot influence lines in which case the problem is at least lengthened and,for the case of a varying section, an exact solution is so laborious that it is seldom accomplished.

But with the proposed mechanical method, the solution is the same in any case. A girder of varying section may be analyzed as readily and in exactly the same manner as the girder of uniform section. In fact from the discussion of the theory in Section II, it can be seen that, if the method will solve one, it will the other.

In order to check the results of the new method against theoretical values, the ordinate at P to the influence lines for reactions at R_1, R_2, R_3, and R_4 as obtained by both methods will be compared.

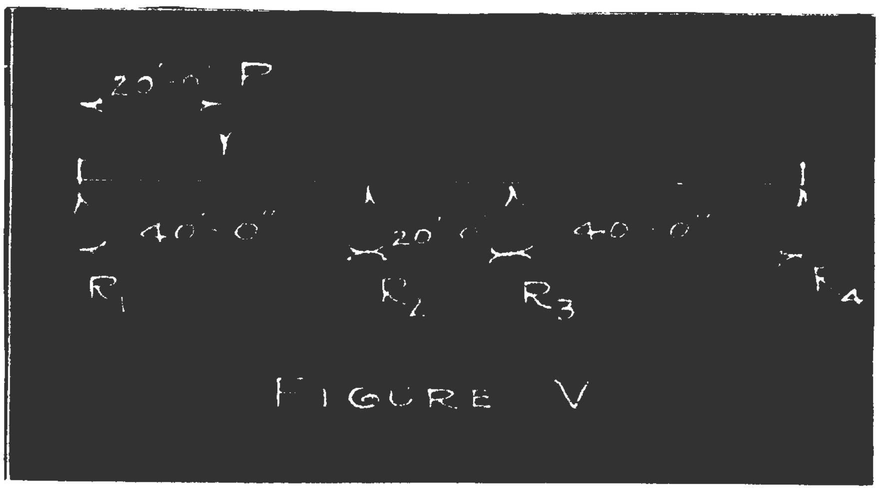


FIGURE V

Solving for these values by means of the three moment equation,

$$120\ M_2 + 20\ M_3 = -1600\ (1/2-1/8) = -600$$

and

$$20\ M_2 + 120\ M_3 = 0$$

Solving simultaneously

$$M_3 = 6/7, \text{ or } .857 \text{ ft. lb.}$$

from which

$$R_4 = .857/40 = .0214 \text{ upward}$$

$$R_3 = .3213 \text{ downward}$$

$R_2 = .928$

$R_1 = .372$

Checking these values

$.372 + .928 + (-.3213) + .0214 = 1.0001$

To obtain these values by the mechanical method the model was first made of stiff cardboard 1/2 in. wide and the length to a scale of 1 in. equals 6 ft. For the value of R_4, the model was placed in the four guages with the pins through the center of the girder at R_4, R_3, R_2, and R_1, forming hinges at these points. The guages were then pinned to the drawing board and the shear plugs changed at R_4 producing a motion of R_4 upward. The deflection of point P was read as .2upward. The value of R_4 then according to this method is .2/12 or .0167.

The shear plugs at R_4 were changed back and the value of R_3 found by changing the plugs in the guage at that point and reading the deflection of point P, which was 3.0. This divided by the factor for the guage gives a value of .25. R_2 and R_1 were obtained similarly as .77 and .37

Comparison of results.

Reaction.	Theory	Model
R_1	.372	.37
R_2	.928	.77
R_3	- .321	- .25
R_4	.0214	.0167

These results possibly donot check closer because the paper from which the model was cut was not of sufficient thickness to resist a slight buckling especially for the investigation of R_2 . Another reason might be because the pins forming the hinges had to be removed each time the shear plugs were changed and then replaced. Altho the deflection usually returned to zero upon releasing the displacement, there is good chance for error. This method is possibly not so well suited to the solution of continuous beams as for some other forms of indeterminate structures. The cardboard should be quite heavy and comparatively large models should be used. Better results would probably have been obtained with a little larger model. A great deal of care should be exercised in determining the reactions of continuous beams by this method.

11. Rectangular Frame with Vertical Legs. Ends of Columns Hinged. --- The only requirement for making this frame statically determinate is the value of the horizontal thrust at the bottoms of the columns.

Assume the frame for analysis as that shown in Figure VI. The deflection of point B with respect to the tangent at A in a horizontal direction is equal to $\int Myds/EI$; but $M = M' + Hy$ when M' is the moment due to vertical loads alone. The deflection then is

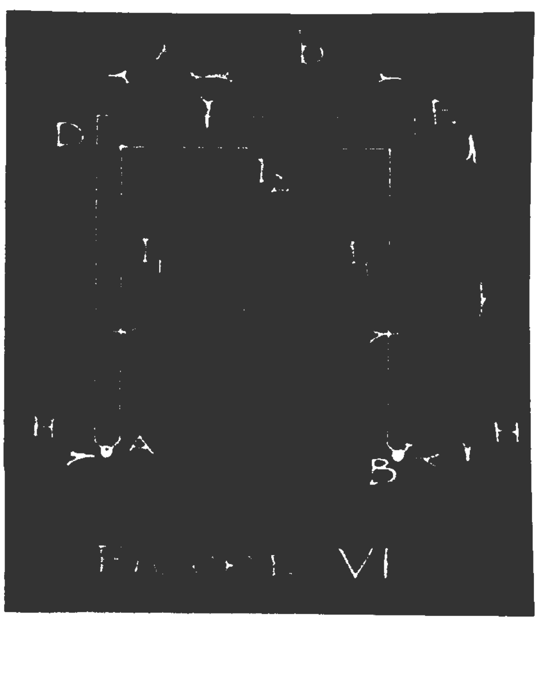


FIGURE VI

equal to $\int_A^B M'yds/EI + H\int_A^B y^2 ds/EI$ and is also equal to zero since the points A and B cannot move laterally nor vertically.

Then $$H = \frac{\int_A^B M'yds/I}{\int_A^B y^2 ds/I}$$

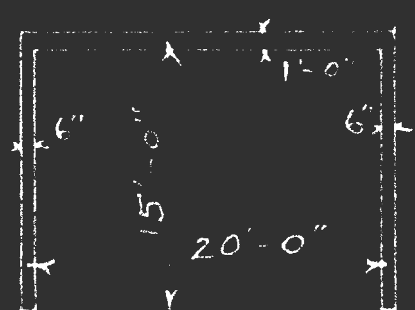
Integrating this for thisnframe,we have the general equation

$$H = \frac{3P\ a\ b\ I_1}{4\ h^2 I_2 + 6\ h\ l\ I_1}$$

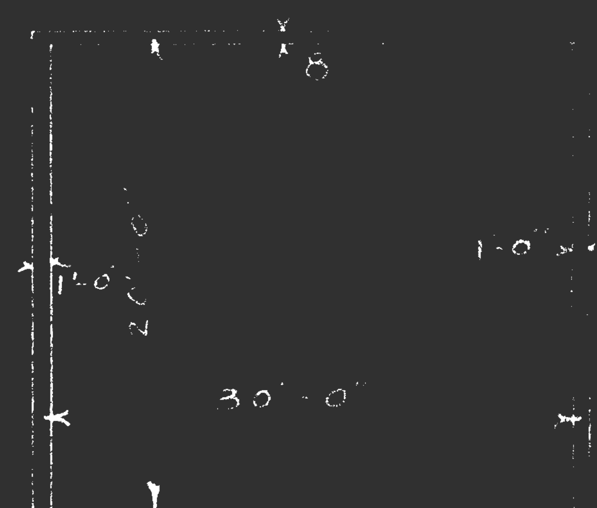
The five frames shown in Figure VII on the next page were analysed with loading at center and quartef points of the horizontal member. These frames were chosen with the several different ratios of the moments of inertia of columns and top members in order to see whether this method and theory agree in all cases. Frame C was made just like frame B except that the size of members is half that of the previous one with the same ratio of moments of inertia in order to see whether the mechanical method

FRAME A

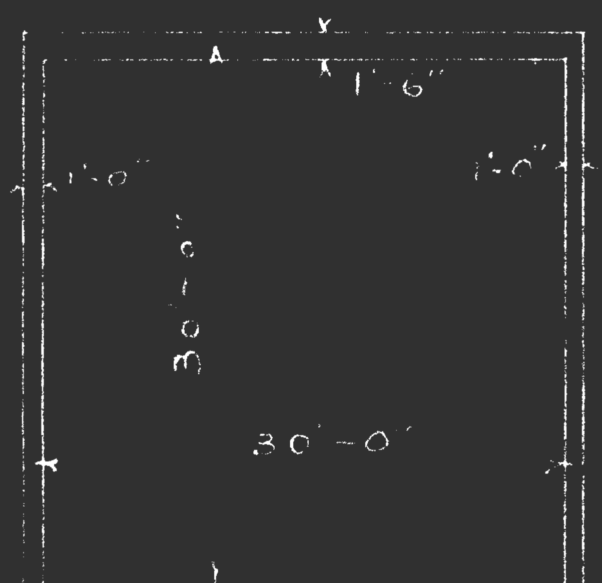
FRAME B



FRAME C



FRAME D



FRAME E

FIGURE VII

would bear out theory in that the size of members is immaterial so long as the ratio of moment of inertia is the same for a frame of the same size, measuring along the axis.

For frame A loaded at the center

$$H = \frac{(3)\ (10)\ (10)}{(4)\ (15)\ (15) + (6)\ (15)\ (20)} = .111$$

For quarter point loaded

$$H' = \frac{(3)\ (5)\ (15)}{(4)\ (15)\ (15)\ + (6)\ (15)\ (20)} = .0833$$

Since this frame is of the same cross section throughout, I_1 equals I_2 and cancel in the equation.

In frame B the value of I_2 (top member) equals 8/12 and I_1 equals 1/12 or a ratio of I_1 to I_2 of 1 to 8.

For center loading

$$H = \frac{(3)\ (10)\ (10)}{(4)\ (15)\ (15)(8) + (6)(15)(20)} = .033$$

For quarter point loaded

$$H = \frac{(3)(5)(15)}{(4)(15)(15)(8)(+ (6)(15)(20)} = .025$$

The same ratio of moments of inertia holds for frame C, therefore, the values of H will be the same for this frame as for frame B.

The values of I_2 and I_1 for frame D are 27/758 and 1/12 respectively or a ratio of 1 to 2.37.

For the center loaded

$$H = \frac{(3)(15)(15)(2.37)}{(4)(20)(20)(+ (6)(20)(30)(2.37)} = .158$$

For load at quarter point

$$H = \frac{(3)(7.5)(22.5)(2.37)}{(4)(20)(20) + (6)(20)(30)(2.37)} = .118$$

For frame E the values of I_1 and I_2 are 1/12 and 9/32 respectively or a ratio of 1 to 3.38

For center loading

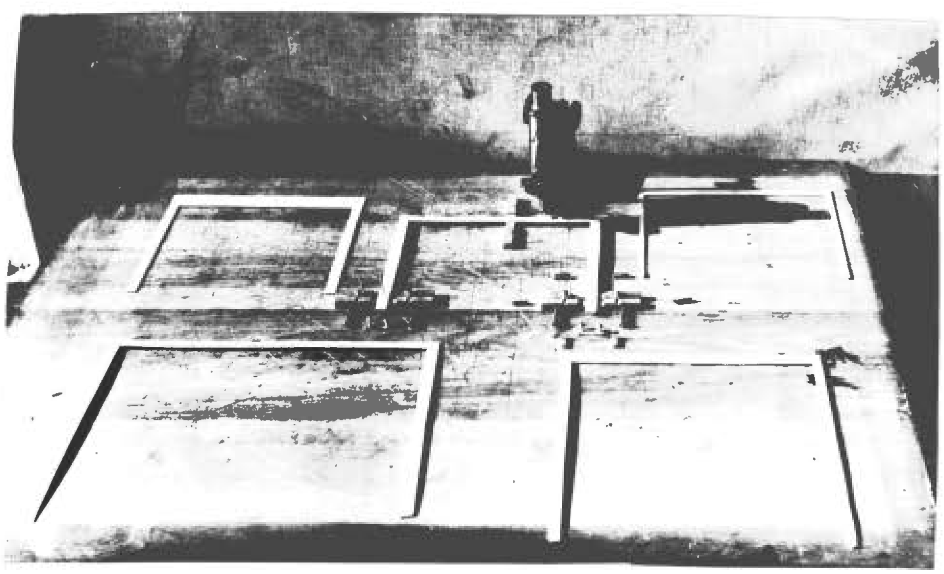
$$H = \frac{(3)(15)(15)}{(4)(30)(30)(3.38) + (6)(30)(30)} = .0385$$

For quarter point loaded

$$H = \frac{(3)(7.5)(22.5)}{(4)(30)(30)(3.38) + (6)(30)(30)} = .0288$$

To obtain the values of H by the mechanical method, a model was made of cardboard of each of the framesshown in Figure VI. It was found, however, that the ordinary weight cardboard was too thin allowing the frame to buckle when the deflectionswere applied with the guages. Two models were cut out for each frame and pasted together making the model twice as thick as the cardboard. This was found to be satisfactory.

Frame A was made to a scale of 1/2 in. equals 1 ft. It was fastened in the guages with pins for hinges as shown in the illustration.



When the shear plugs were changed producing horizontal deflection at the hinge, the deflection of the center was read as 1.4 divisions of the micrometer in the microscope. The value of H then as determined by this method is 1.4/12 or .116. The deflection of the quarter point was found tobe 1.0 on both sides of the center making H equal to 1/12 or .0834.

The thrust plugs were placed in the guages and the deflection found to be 6.4 making the vertical reactions equal to 6.4/13 or .493 for the center and 3.1/13 or .24 for the far quarter point and 9.8/13 or .75 for the near quarter point loaded.

A model pf frame B was made to a scale of 1/2 in. to the foot and placed in the guages in the same manneras frame A. The deflection when the shear plugs were changed, was read as .4 at the center and .5 for both the quarter points. The value of H then for a load of unity at the center is .4/12 of .033 and for the same load at the quarter point .3/12 or .025. When the thrust plugs were changed, a deflection of 6.5 was noted at the cehter of the top member and 9.8 at the near quarter point and 3.3 at the far quarter point. The reactions for the center loaded are 6.5/13 or .5 and, for the quarter points loaded, 9.8/13 or .75 and 3.3/13 or .25

Frame C was made to the same scale 1/2 in. equals 1 ft. and placed in the guages in the same way. The value of H for center loaded with a load of unity was found to be .4/12 or .0333 and .3/12 or .025 for the quarter points.

The same values for reactions were found for frame D as for frame C . The model was made th the same scale as the other two.

A model of frame D was made to the same scale as the others and placed in the guages in the same manner. The values of H for load at the center was 1.9/12 or .158 and 1.3/12 or .108 for aload at the quarter point.

The model for frame E was made to a scale of 3/8 in. equals 1 ft. and placed in the guages. The value of H for the center loaded for this frame was .4/12 or .0333 and for the quarter points .3/12 or .025.

Comparing these results;

	A		B		C		D		E	
	Theory	M	Th.	M	Th.	M	Th.	M	Th.	M
Center loaded										
H	.11	.11	.033	.033	.033	.03	.158	.158	.038	.03
R_L	.5	.49	.5	.5	.5	.5	.5	.5	.5	.5
R_R	.5	.49	.5	.5	.5	.5	.5	.5	.5	.5
Quarter point loaded										
H	.083	.083	.025	.025	.025	.025	.118	.108	.028	.025
R_L	.75	.75	.75	.75	.75	.75	.75	.75	.75	.75
R_R	.25	.24	.25	.25	.25	.25	.25	.25	.25	.25

From a study of these values, it can be seen that the mechanical method checks the theoretical method of analysis for this class of structures and indicates that the model automatically takes care of the varying moment of inertia.

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12. Redtangular Frame with Vertical Legs. Column Ends Fixed.--- All the following rectangular frames are calculated according to the following method of slope deflections. The fundamental equations for the solution by this method for any member AB are:

$$M_{AB} = 2EK(2\theta_A + \theta_B - 3R) \mp C_{AB}$$

$$M_{BA} = 2EK(2\theta_B + \theta_A - 3R) \pm C_{BA}$$

when E is the modulus of elasticity, K is I/l, R is deflection divided by l, and C_{AB} is the resisting moment at the end of a fixed beam with an equal span and carrying the same system of intermediate loads.

FIGURE VIII

These equations are fully developed in Bulletin 108 of the Engineering Experiment

Station of the University of Illinois.

In the solution of these problems R will be assumed as zero since it is so small as to be of no consequence. Assuming symmetrical loading and applying these equations to the frame in Figure IX :

$$M_{AB} = 2EK_1\theta_B$$

and

$$M_{BA} = 4EK_1\theta_B$$

since $\theta = 0$ and $C_{AB} = 0$

$$M_{BC} = 2EK_2(2\theta_B + \theta_C) - C_{BC}$$

Since the loading and frame are symmetrical, the distortion will be symmetrical and $\theta_B = -\theta_C$

Then

$$M_{BC} = 2EK_2\theta_B - C_{BC}$$

Since joint B is in equilibrium

$$M_{BA} + M_{BC} = 0$$

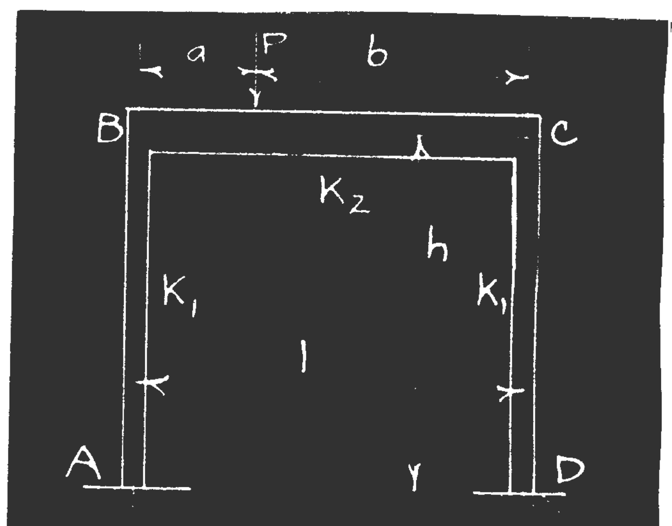


FIGURE. IX

Substituting

$$4EK_1\theta_B + 2EK_2\theta_B - C_{BC} = 0$$

$$\theta_B(4EK_1 + 2EK_2) = C_{BC}$$

$$\theta_B = \frac{C_{BC}}{(4EK_1 + 2K_2E)}$$

Substituting this value of θ_B in the equation for M_{BA}

$$M_{BA} = \frac{2C_{BC}K_1}{2K_1 + K_2}$$

For symmetrical loading

$$C_{BC} = Pab^2/l^2 + Pa^2b/l^2 = P\,a\,b/l$$

$$M_{BA} = \frac{2PabK_1}{l(2K_1 + K_2)}$$

and

$$M_{AB} = \frac{PabK_1}{l(2K_1 + K_2)}$$

$K_1 = I_1/h$ and $K_2 = I_2/l$

Substituting these values in the above formulas;

$$M_{BA} = \frac{2PabI_1}{2lI_1 + hI_2}$$

$$M_{AB} = \frac{PabI_1}{2lI_1 + hI_2}$$

The same frames will be analyzed for fixed ends that were used in the investigation with hinged ends.

For frame A the value of $I_1 = I_2$ and

$$M_{BA} = \frac{2Pab}{40 + 15} = 2/55 \text{ ab}$$

For center loaded with load of unity,

$M_{BA} = 200/55 = 3.64$

$M_{AB} = 1.82$

$2H = 5.46/15 = .364$

$H = .182$

For quarter point loaded,

$$M_{BA} = \frac{(2)\ (5)(15)}{55} = 2.72$$

$M_{AB} = 1.36$

$2H = 4.08/15 = .272$

$H = .136$

For frame B the ratio of I_1 to I_2 is 1 to 8

$$M_{BA} = \frac{2Pab}{2(20) + (15)(8)} = Pab/80$$

With center loaded

$$M_{BA} = 100/80 = 1.25$$

$$M_{AB} = .625$$

$$2H = 1.87/15 = .125$$

$$H = .0625$$

With quarter points loaded

$$M_{BA} = \frac{5(15)}{80} = .94$$

$$M_{AB} = .47$$

$$2H = 1.41/15 = .094$$

$$H = .047$$

These same results are true for frame C theoretically.

For frame D the ratio of I_1 to I_2 is 2.37 to 1

$$M_{AB} = \frac{2pab2.37}{2(30)(2.37) + 20} = Pab/34.2$$

With center loaded

$$M_{AB} = 112.5/34.2 = 3.29$$

$$M_{BA} = 6.58$$

$2H = 9.87/20 = .494$

$H = .247$

With quarter points loaded

$$M_{BA} = \frac{75(22.5)}{34.2} = 4.84$$

$M_{AB} = 2.42$

$2H = 7.26/20 = .363$

$H = .181$

For frame E the ratio of I_1 to I_2 is 1 to 3.38

$$M_{BA} = \frac{2Pab}{2(30) + 30(3.38)} = Pab/80.7$$

With center loaded

$$M_{BA} = \frac{15\ (15)}{80.7} = 2.79$$

$M_{AB} = 1.39$

$2H = 4.18/30 = .139$

$H = .069$

With quarter points loaded

$$M_{BA} = \frac{15(15)}{80.7} = 2.09$$

$M_{AB} = 1.05$

$2H = 3.14/30 = .1045$

$H = .052$

The mechanical analysis was carried out with the same models as were used with two hinged frames; but were clamped in the guages with fixed ends instead of with hinges.

The following values were obtained;

Frame A center loading

$H = 2.5/12 = .208$

$R = 6.5/13 = .5$

$M_{AB} = 5/1.7 = 2.94$

Near quarter point loading

$H = 1.6/12 = .133$

$R = 1.0/13 = .77$

$M_{AB} = 1.3/1.7 = .765$

Far quarter point loading

$H = 1.7/12 = .141$

$R = 2.8/13 = .216$

$M_{AB} = 6.0/1.7 = 3.5$

Frame B

Center loading

$H = 1.2/12 = .1$

$R = 6.5/13 = .5$

$M_{AB} = 2.0/1.7 = 1.17$

Near quarter point loading

$H = .5/12 = .0416$

$R = 9.7/12 = .75$

$M_{AB} = 1.6 = .94$

Far quarter point loading

$H = .8/12 = .0665$

$R = 3.3/13 = .254$

$MAB = 1.8/1.7 = 1.06$

Frame C

Center loading

$H = .6/12 = .05$

$R = 6.5/13 = .5$

$M_{AB} = 1.5/1.7 = .88$

Near quarter point loading

$H = .5/12 = .0416$

$R = 9.4/13 = .725$

$M_{AB} = 1.0/1.7 = .587$

Far quarter point loading

$H = .5/12 = .0416$

$R = 3/13 = .23$

$M_{AB} = 1.0/1.7 = .587$

Frame D

Center loading

$H = 3.0/12 = .25$

$R = 6.5/13 = .5$

$M_{AB} = 8.0/1.7 = 4.7$

Near quarter point loading

$H = 2.1/12 = .175$

$R = 10/13 = .77$

$M_{AB} = 3.0/1.7 = 1.76$

Far quarter point loading

$H = 2.1/12 = .175$

$R = 2.6/13 = .20$

$M_{AB} = 9/1.7 = 5.3$

Frame E

Center loading

$H = .8/12 = .066$

$R = 6.5/13 = .5$

$M_{AB} = 2.5/1.7 = 1.47$

Near quarter point

$H = .6/12 = .05$

$R = 9.5/13 = .73$

$M_{AB} = 1.5/1.7 = .88$

Far quarter point loading

$H = .5/12 = .0416$

$R = 3.2/13 = .246$

$M_{AB} = 2.0/1.7 = 1.18$

Comparison of results

	A		B		C		D		E	
	Th.	M.	Th.	M.	Th.	M.	Th.	M.	Th.	M.
Load at center										
H	.182	.208	.063	.1	.063	.05	.247	.25	.069	.066
R	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5
M_{AB}	1.82	2.9	1.25	1.17	.625	.88	3.29	4.7	1.39	1.47
Load at near quarter point										
H	.136	.13	.047	.041	.047	.041	.181	.175	.052	.05
R		.77		.75		.73		.77		.73
M_{AB}		.765		.94		.59		1.76		.88
Load at far quarter point										
H	.136	.141	.047	.06	.047	.041	.181	.17	.052	.041
R		.216		.25		.23		.20		.246
M_{AB}		3.52		1.06		.587		5.3		1.18
	2.72	4.23	.94	2.0	.94	1.17	4.84	7.05	2.09	2.06

The bottom column of this table is the sum of M_{AB} for both quarter points loaded.

From this comparisonm it can be seem that the values obtained by theory and with the models agree fairly well especially for the value of H. The moment as obtained bythe mechanical method seems to be large in nearly every case. It was thought that the reason for this was because the moment plugs produced more deformation than the others causing the model to buckle. In order to find out if this was the case, a model of the frame A was made of wall board. The results of the mechanical analysis of this frame were as follows:

	Center		Near quar. Pt		Far quar. Pt.	
	Th	M	Th	M	Th	M
H	.182	.125	.13	.11	.13	.11
R	.5	.5		.77		.23
M_{AB}	1.82	1.77		1.47		1.42

The sum of the moments at the foot of the columns theoretically is 2.72 and by the mechanical method is 2.87.

The assumption that the madel was buckling seems to be justified by these results.

13 Frame with Inclined Legs. Ends of Columns Hinged.---- Analyzing by the method of slope deflection for the general case shown in Figure X and assuming symmetrical loading.

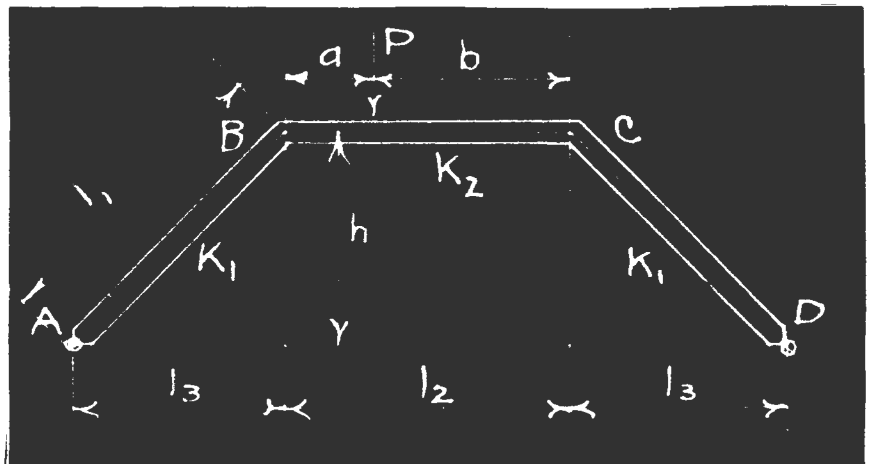


FIGURE X

$$M_{AB} = 0$$

$$M_{BA} = 3EK_1\theta_B$$

$$M_{BC} = 2EK_2(2\theta_B + \theta_C) - C_{BC}$$

Since the frame is symmetricaland is symmetrically loaded

$$\theta_B = -\theta_C$$

$$M_{BA} + M_{BC} = 0$$

$$3EK_1\theta_B + 2EK_2\theta_B - C_{BC} = 0$$

$$\theta_B = \frac{C_{BC}}{3K_1E + 2EK_2}$$

Substituting in the general equation for M_{BA}

$$M_{BA} = \frac{3C_{BC}K_1}{3K_1 + 2K_2}$$

From this equation we see that the value of M_{BA} is independent of the inclination of the legs. If the section is constant throughout, then I is constant and

$$K_1 = I/l_1 \quad \text{and } K_2 = I/l_2$$

Substituting in the equation for M_{BA} just derived

$$M_{BA} = \frac{3\ C_{BC}}{3 + 2\ l/l_2}$$

From this equation we see that theoretically the valuen of M_{BA} is independent of the size of the members so long as they are the same size throughout.

In order to compare the results as found by theory and by models, and to see whether the mechanical method bears out the theory in obtaining the same value of M_{BA}

regardless of the inclination of the legs and the size of the members of the frame, the three frames shown in Figure XI on the next page were analyzed by the two methods. The lengths of members are equal in all three cases and the depth of members in frame G is twice that in the others.

Assuming loads at B and C

$C_{BC} = 0$ and $M_{BA} = 0$ for all three frames.

For frames F and G, since for symmetrical loading, the vertical reactions equal 1, then $H = 1$

For load at either B or C

$H = .5$

Assuming loads at center of span,

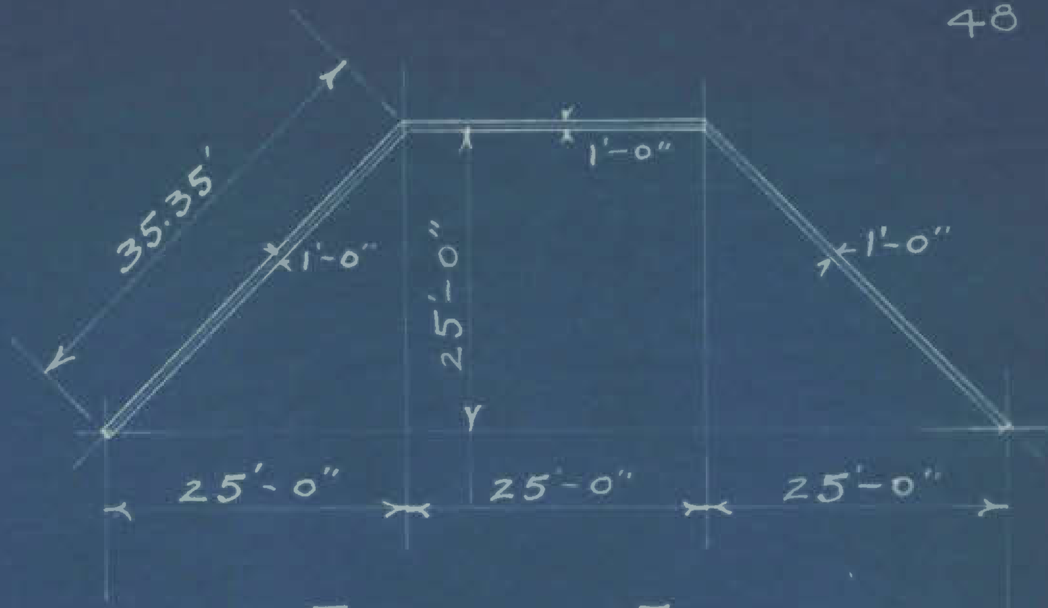
$C_{BC} = 1/4 = 25/4$

$$M_{BA} = \frac{3(25)}{4(3 + (2)\ 35.35/25)} = 3.22$$

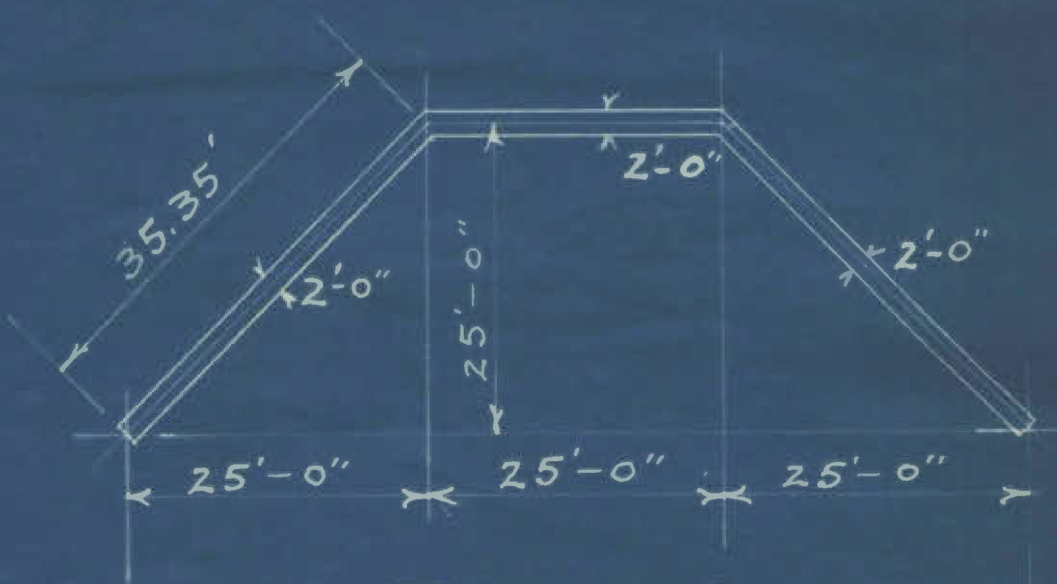
For frames F and G

$$H = \frac{3.22(25)}{25} = 1.13 \text{ for 2 loads of unity.}$$

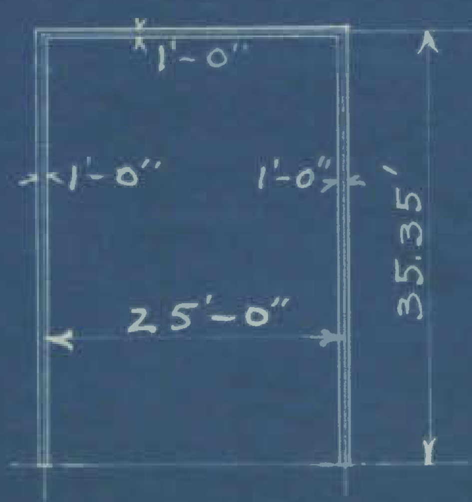
$H = .565$ for 1 load of unity



FRAME F



FRAME G



FRAME H

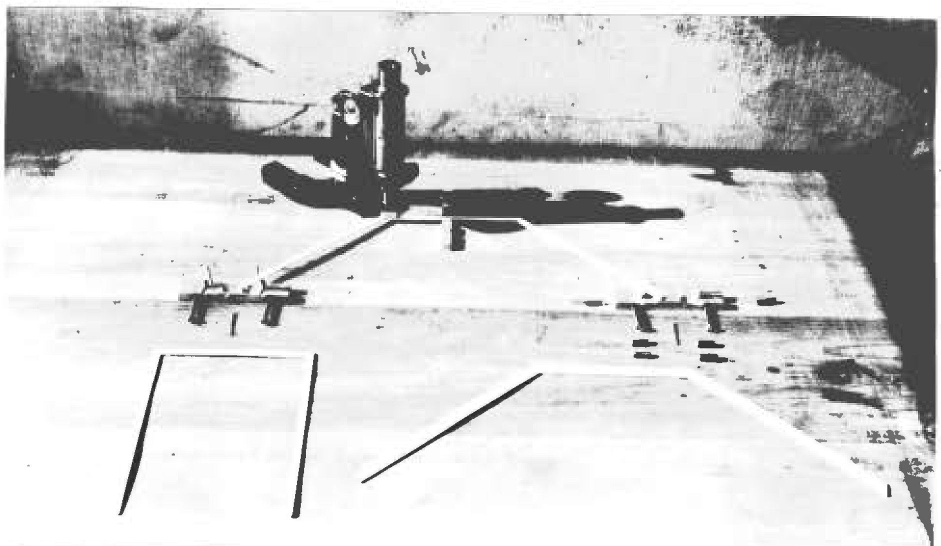
FIGURE XI

For frame H

$H = 3.22/35.35 = .0913$ for double load.

$H = .0456$ for single load.

For the mechanical analysis of these frames, models of cardboard to a scale of 1/4 in. equals 1 ft. They were placed in the guages with hinges at A and D as shown in the illustration.



With frame F in the guages, the deflection of point B was read as 6 divisions and point C as 6 divisions when the shear plugs were changed making $H = 6/12 = .5$. Deflection of middle point was 6.7 making H equal 6.7/12 or .56

Frame E was then place in the guages and the values of H found to be .5 for points B or C loaded and .54 for middle loaded.

Frame H was next placed in the guages and the deflection of points Band C read as 0. The deflection of the middle point was .55 making H equal to .046

Comparing results for H

Load	Frame F		Frame G		Frame H	
pt.	Theory	Model	Theory	Model	Theory	Model
B orC	.5	.5	.5	.5	0	0
Cen.	.565	.56	.565	.54	.0456	.046

Comparing results of frames F and G it can be seen that the model bears out the theory for loads at joints B and C. But for the heavier frame, and a load at the center a smaller value of H is obtained by the mechanical method than for the light frame. This seems the more reasonable result than that obtained by theory since it can be seen that as the depth of members approaches infinity, the value of H approaches zero. In view of this fact, there is as much reason for accepting the values obtained by the models as those obtained by theory. In comparing the values obtained for frames F and H it can be seen that the values of H for center loading bears about the same ratio to the theoretical values as for frames F and G. The mechanical method bears out the assumption that M_{BA} is independent of the inclination of the legs. For joints B and C loaded M_{BA} both theoretically and mechanically is equal to zero. For the center loading by the mechanical method, the value of M_{BA} for frame F equals .56(25) - 25/2 equals 1.5 and for frame H, M_{BA} equals .046(35.35) equals 1.6. This furnishes a fairly good check for the moment. The results on the whole agree with theoretical results very closely.

14. Frame with Inclined Legs. Ends of Columns Fixed. ---- We will assume symmetrical loading inbthis case as before.

Then $\theta_B = -\theta_C$

$$M_{AB} = 2EK_1 \theta_B$$

$$M_{BA} = 4EK_1\theta_B$$

$$MBC = 2EK_2(2\theta_B + \theta_C) - C_{BC}$$
$$= 2EK_2\theta_B - C_{BC}$$

$$4EK_1\theta_B + 2EK_2\theta_B - C_{BC} = 0$$

$$\theta_B = \frac{C_{BC}}{4EK_1\theta_B + 2EK_2\theta_B}$$

$$M_{AB} = \frac{C_{BC}K_1}{2K_1 + K_2}$$

$$M_{BA} = \frac{2\ C_{BC}K_1}{2K_1 + K_2}$$

From these equations we see that the moments are independent of the inclination of the legs the same as for hinged ends.

For uniform section throughout, the formulas become

$$M_{AB} = \frac{C_{BC}}{2 + l_1/l_2}$$

and $$M_{BA} = \frac{2C_{BC}}{2 + l_1/l_2}$$

As for the case of hinged ends the value of the moments and H are independent theoretically of the size of the section.

The same three frames as used in the previous article will be analyzed by both methods.

For joints B and C loaded $C_{BC} = 0$

and $M_{AB} = 0$ and also $M_{BA} = 0$

For frames F and G

$H = 1$ for double loading

$H = .5$ for one joint loaded.

For frame H

$H = 0$

For symmetrical loads at center

$C_{BC} = 25/4$

$$M_{AB} = \frac{25}{4(2 + 35.35/25)} = 1.83$$

$M_{BA} = 2(1.83) = 3.66$

For frames F and G

$H = 26.83/25 = 1.08$ for two loads

$H = .54$ for one load of unity

For frame H

$H = 1.83/35.35 = .053$

For the mechanical solution, the frames were fastened in their turn in the guages with the ends of the columns fixed by fastening to the guages with the plates provided for the purpose.

In the analysis of frame F, the deflection of points B and C were found to be 6 when the shear plugs were changed. $H = 6/12$ or .5 Deflection of center was 7. $H = 7/12$ or .583 When thrust plugs were changed, there was a deflection of 9.5 for point B, 3.6 for C, and 6.5 for the center point. Thus the vertical reactions for load over joints are 9.5/13 or .732 and 3.3/13 or .278 , and for center loaded 6.5/13 or .5 When the moment plugs were changed, deflection of B was 5.9 and of C was 6.0 in the opposite direction and of the center was 1.7

making M_{AB} equal 5.9/1.7 or 3.47 for B loaded, 1.7/1.7 or 1.0 for center loaded, and 6.0/1.7 or-3.52 for C loaded.

Thus the condition for a load at both B and C giving zero moment at A and D is fulfilled by the mechanical method.

Frame G was next placed in the gauges with ends fixed and the following values found in a similar manner to th t used for frame F.

For a load of unity at B

$H = 6.0/12 = .5$

$R = 9.1/13 = .7$

$M_{AB} = 5.5/1.7 = 3.24$

For load at C

$H = 6.0/12 = .5$

$R = 3.8/13 = .29$

$M_{AB} = 6.0/1.7 = -3.52$

For load at center

$H = 6.7/12 = .558$

$R = 6.5/13 = .5$

$M_{AB} = 2.0/1.7 = 1.17$

In this case for a load at both B and C the moment at B is not quite zero due possibly to some inaccuracy in cutting the model or in fastening in the gauges. The conditions of equilibrium are fairly well satisfied for the vertical and horizontal reactions.

For frame H the following results were obtained:

For load at B

$H = 0$

$R = 13/13 = 1$

$M_{AB} = 0$

For load at C

$H = 0$

$R = 0$

$M_{AB} = 0$

For load at center

$H = .6/12 = .05$

$R = 6.4/13 = .494$

$M_{AB} = 1.5/1.7 = .885$

Comparison of results

	Frame F		Frame G		Frame H	
Load B	Theory	Model	Theory	Model	Theory	Model
H	.5	.5	.5	.5	0	0
R_L		.732		.7	1	1
R_R		.278		.29	0	0
M_{AB}		3.47		3.24	0	0
Load	at center					
H	.54	.583	.54	.558	.052	.05
R	.5	.5	.5	.5	.5	.494
M_{AB}	.99	1.0	.99	1.17	.99	.885

The values compare very well for this class of frames.

15. Two Hinged Arch. ---- In order to simplify the mathematical calculations, a circular arch of constant cross section will be analyzed.

From the equations for the deflection of a curved beams, the formula for H for any two hinged arch is derived.

$$H = \frac{\int M'yds/I}{\int y^2ds/I}$$

when M' is the moment due to vertical loads only.

For the arch of constant cross section, I is constant and the formula becomes

$$H = \frac{\int M'yds}{\int y^2ds}$$

Assume symmetrical loading as shown in Figure XII.

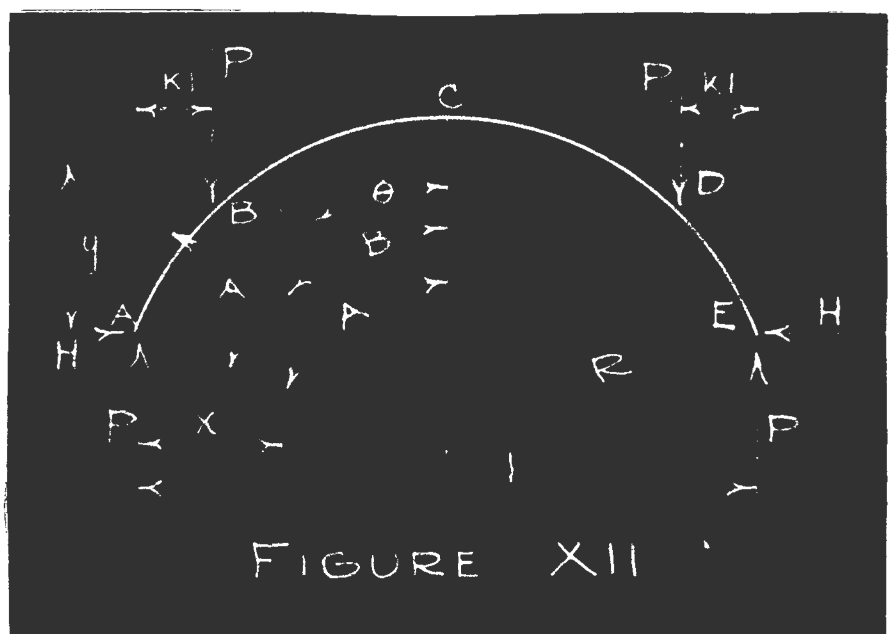


FIGURE XII

Expressing in polar coordinates

$$x = R\ (\sin A - \sin \theta)$$

$$kl = R\ (\sin A - \sin B)$$

$$y = R\ (\cos \theta - \cos A)$$

$$ds = R\ d\theta$$

For θ less than B

$$M' = Pkl = PR(\sin A - \sin B)$$

For θ greater than B

$$M' = Px = PR\ (\sin A - \sin \theta)$$

Substituting these values in the formula for H

$$H = \frac{2\int_0^B PR^3(\sin A - \sin B)(\cos \theta - \cos A)\ d\theta}{2\int_0^A R^3(\cos \theta - \cos A)^2\ d\theta} + \frac{2\int_B^A R^3(\sin A - \sin\theta)(\cos \theta - \cos A)\ d\theta}{2\int_0^A R^3(\cos\theta - \cos A)^2\ d\theta}$$

Integrating and dividing by 2 for single load

$$H = \frac{\underset{Y}{P}R^2\left[\sin^2 A - \sin^2 B - 2\cos A(A \sin A - B \sin B - \cos B + \cos B)\right]}{2\ R^2\ (A - 3 \cos A \sin A + 2\ A \cos^2 A)}$$

and adding the thrust shortening

$$H = \frac{\text{Denominator}}{\text{Numerator} + \frac{2 A I \cos A}{\text{area of section}}}$$

If the arch is semicircular, A is 90°
and sin A becomes 1 and cos A becomes 0.
Then

$$H = \frac{P \cos^2 B}{\pi}$$

The thrust shortening will be 0.

Two arches will be analyzed, the semicircular one shown in Figure XIII and one with A equal to 60°. Both will be loaded at the center and at quarter points measured along the arch axismaking B equal to 45° for the semicircular arch and 30° for the segmental arch.

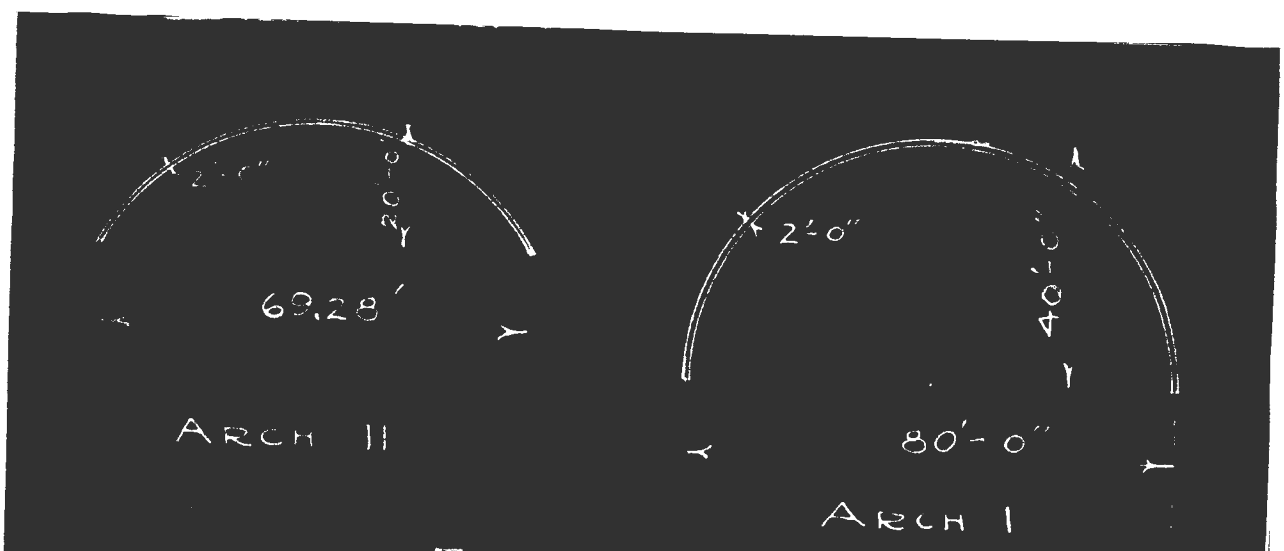


FIGURE XIII

Solving arch I

Load at quarter point

$B = 45°$, $\cos^2 B = 1/2$

$H = 1/2\pi = .17$

Load at center

$B = 0$, $\cos^2 B = 1$

$H = .34$

Solving arch II

$A = 60° = \pi/3$, $\sin A = \sqrt{3}/2$, $\cos A = 1/2$

Area of section $= 2$, $I = 2/3$

For load point half way between springing and crown along axis,

$B = 30° = \pi/6$, $\sin B = 1/2$, $\cos B = \sqrt{3}/2$

$$H = \frac{1200 \left[3/4 - 1/4 - (\pi/3 \cdot \sqrt{3}/2 - \pi/12 - \sqrt{3}/2 + 1/2) \right]}{2400 \, (\pi/3 - 3\sqrt{3}/4 + 2\pi/12) + \frac{4\pi/18}{2}}$$

$= .39$

For load at center

$B = 0$, $\sin B = 0$, $\cos B = 1$

$$H = \frac{1200 \left[3/4 - (\pi\sqrt{3}/6 - 1 + 1/2) \right]}{675.35}$$

$= -.622$

For the mechanical analysis, an arch model of arch I was made of two thicknesses of cardboard to a scale of 1/4 in. equals 1 ft. This was placed in the gauges with hinges at the springing and the deflection of the quarter points found to be 2 divisions for both when the shear plugs were changed.

H for quarter points = 2/12 = .166

H for center loaded = 4/12 = .333

The vertical reactions were found to be 1.8/13 or .138 and 11/13 or .85 for the quarter points and 6.7/13 or .515 for the center loaded.

For the analysis of arch II, the same model was used as for I with the gauges moved up 30° from the center. The deflection of the quarter points for this case was found to be 4.5 and for the center 6.6.

H for quarter points = 4.5/12 = .375

H for center = 6.6/12 = .55

The vertical reactions were found to be .185 and .81 for the quarter points and .485 for the center loading.

Comparison of results

	Arch I		Arch II	
Load at center	Theory	Model	Theory	Model
H	.34	.33	.622	.55
R_L	.5	.515	.5	.485
R_R	.5	.515	.5	.485
Load at quarter point				
H	.17	.166	.39	.375
R_L	.148	.138	.21	.165
R_R	.852	.85	.79	.81

These results compare favorably and indicate that the mechanical method might be used with about as much assurance as the theoretical method, especially for those types of two hinged arches which do not permit of easy integration.

16. Hingless Arch.--- In the analysis of the hingless arch, the arch is considered to be cut in the center and equations derived for the unknowns indicated at the crown in Figure XIV required to hold the half arch in equilibrium by equation the deflection of the right half to that of the left half. Thus

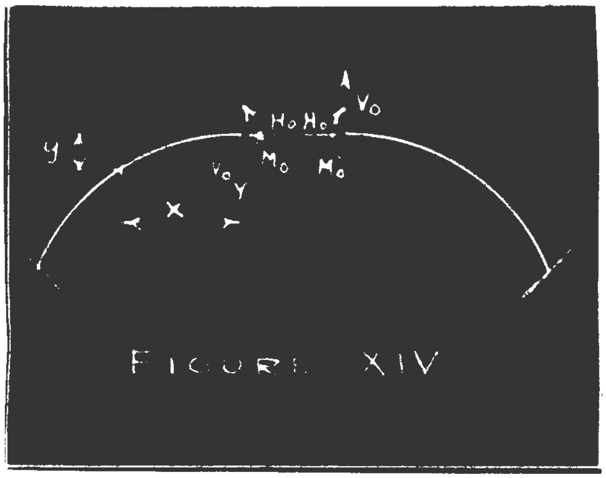


FIGURE XIV

$$\Delta x = -\Delta x'$$

$$\Delta y = \Delta y'$$

$$\Delta \phi = -\Delta \phi'$$

when Δx is the horizontal deflection of the left half at the crown, Δy is the vertical deflection of the left half at the crown, and $\Delta \phi$ is the angular deflection of the crown from the tangent at the springing of the left half. $\Delta x'$, $\Delta y'$, and $\Delta \phi'$ are the corresponding deflections for the right half.

Substituting for these terms their expressions as derived for the deflection of curved beams, and substituting for M at any point its value $M' + M_o + H_o y \pm V_o x$, the following equations for H_o, V_o, and M_o are derived.

$$H_o = \frac{\int \frac{ds}{I} \int \frac{M' y\, ds}{I} - \int \frac{M'\, ds}{I} \int \frac{y\, ds}{I}}{\left(\int \frac{y\, ds}{I} \right)^2 - \int \frac{ds}{I} \int \frac{y^2 ds}{I}}$$

$$V_o = \frac{\int \frac{M'_l\, x\, ds}{I} - \int \frac{M'_r\, x\, ds}{I}}{\int \frac{x^2\, ds}{I}}$$

$$M_o = \frac{H_o \int \frac{y\, ds}{I} + \int \frac{M'\, ds}{I}}{\int \frac{ds}{I}}$$

For this analysis, arch I of the previous article will be used. For constant cross section, the I's cancel in each of the three equations above and for the segmehtal or semicircular arch,

$$x^2 = 2Ry - y^2$$

$$y = R - \sqrt{R^2 - x^2}$$

$$dy = \frac{x\, dx}{\sqrt{R^2 - x^2}}$$

$$ds = \frac{R\, dx}{\sqrt{R^2 - x^2}}$$

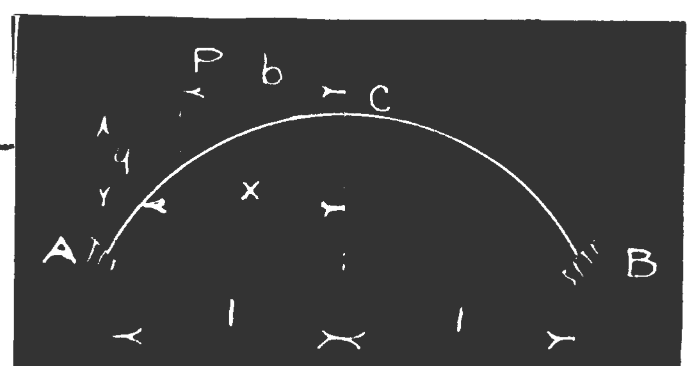


FIGURE XV

Assume a single load P at some point b distance to the left of the center.

Then M' between A and P (Figure XV) equals

P (x - b)) and equals zero over the remainder of the arch.

Evaluating the different terms for circular arch of constant cross section, considering l as half the span as shown in Figure XV.

$$\int ds = \int_0^l \frac{dx}{\sqrt{R^2 - x^2}} = 2R \left(\sin^{-1} x/R \right)_0^l = 2R \sin^{-1} l/R$$

for semicircular arch

$$= \pi l$$

$$\int y \, ds = 2R \int_0^l (R - \sqrt{R^2 - x^2}) \frac{dx}{\sqrt{R^2 - x^2}}$$

$$= (2R^2 \sin^{-1} x/R - 2R x)_0^l$$

$$= 2R^2 \sin^{-1} l/R - 2 R l$$

for semicircular arch

$$= l^2(\pi - 2) = 1.1416(l)$$

$$\int y^2 ds = 2R\int_0^L (R - \sqrt{R^2 - x^2})^2 \frac{dx}{\sqrt{R^2 - x^2}}$$

$$= 2R\,(3/2\;R^2 \sin^{-1} x/R - 2R\,x + x/2\,\sqrt{R^2 - x^2})\Big|_0^L$$

$$= 2\,R\,(2\,R^2 \sin^{-1} L/R - 2\,R\,L + L/2\,\sqrt{R^2 - L^2})$$

for semicircular arch

$$= 2\,L\,(\,3/4\,(\pi L^2) - 2\,L^2\,) = .7124\,L^3$$

$$\int M'\,y\,ds = R\int_b^L P(x - b)(R - \sqrt{R^2-x^2})\,\frac{dx}{\sqrt{R^2 - x^2}}$$

$$= PR(bx - R\sqrt{R^2 - x^2} - x^2/2 - Rb\,\sin^{-1}x/R + bx\,)\Big|_b^L$$

$$= PR(bL - b^2 - R\sqrt{R^2 - L^2} + R\sqrt{R^2 - b^2} - L^2/2 + b^2/2$$

$$- Rb\,\sin^{-1}L/R + Rb\,\sin^{-1}b/R\,)$$

for semicircular arch and b equal to L/2

$$= PL^3/24\,(12\sqrt{3} - 3 - 4\pi) = 5.2\,PL^3/24$$

for b equal to zero

$$= PL^3/2$$

$$\int L' \, ds = PR \int_b^L (x - b) \frac{dx}{\sqrt{R^2 - x^2}}$$

$$= PR \left(-\sqrt{R^2 - x^2} - b \sin^{-1} x/R \right)_b^L$$

$$= PR \left(-\sqrt{R^2 - L^2} + \sqrt{R^2 - b^2} - b \sin^{-1} L/R + b \sin^{-1} b/R \right)$$

for semicircular arch and b equal to L/2

$$= PL(L\sqrt{3}/2 - L\pi/4 + L\pi/12)$$

$$= PL^2 \, 4.1/12$$

for b equal to zero

$$= PL^2$$

$$\int x^2 \, ds = 2R \int_o^L \frac{x^2 dx}{\sqrt{R^2 - x^2}}$$

$$= 2R \left(R^2/2 \sin^{-1} x/R - x/2 \sqrt{R^2 - x^2} \right)_o^L$$

for semicircular arch

$$= L^3 \pi/2$$

$$\int M_1' \, x \, ds = PR\int_b^L (x - b)\frac{x\,dx}{\sqrt{R^2 - x^2}}$$

$$= PR\left(R^2/2 \sin^{-1} x/R - x/2\sqrt{R^2 - x^2} + b\sqrt{R^2 - x^2}\right)_b^L$$

$$= PR\left(R^2/2 \sin^{-1} L/R - R^2/2 \sin^{-1} b/R - L/2\sqrt{R^2 - L^2} + b/2\sqrt{R^2 - b^2} + b\sqrt{R^2 - L^2} - b\sqrt{R^2 - b^2}\right)$$

for semicircular arch and b equal to L /2

$$= L^3/24\,(4\pi - 3\sqrt{3}) = 7.3764\,L^3/24$$

Substitutingthese values inthe general equations for load at quarter point

$$H_o = \frac{\frac{\pi L^4(5.2)}{24} - \frac{L^4(4.1)(1.1416)}{12}}{(L^2\,1.1416)^2 - \pi L^4\,(.7124)}$$

$$= \frac{.682 - .39}{1.3 - 2.24} = -.31$$

$$V_o = \frac{\frac{7.3764\,L^3}{12}}{\pi L^3} = .195$$

$$M_o = -\frac{L^2\,4.1/12 - (.31)(1.1416)\,L^2}{\pi L}$$

$$= -\frac{(40)(4.1)/12 - .31\,(40)(1.1416)}{\pi}$$

$$= .159$$

$$M_B = .31\ (40) - .195\ (40) + .159 = 4.759$$

For center loaded

$$H_o = \frac{\pi L^4/2 - P\ L^4(1.1416)}{- L^4\ .94} = -.455$$

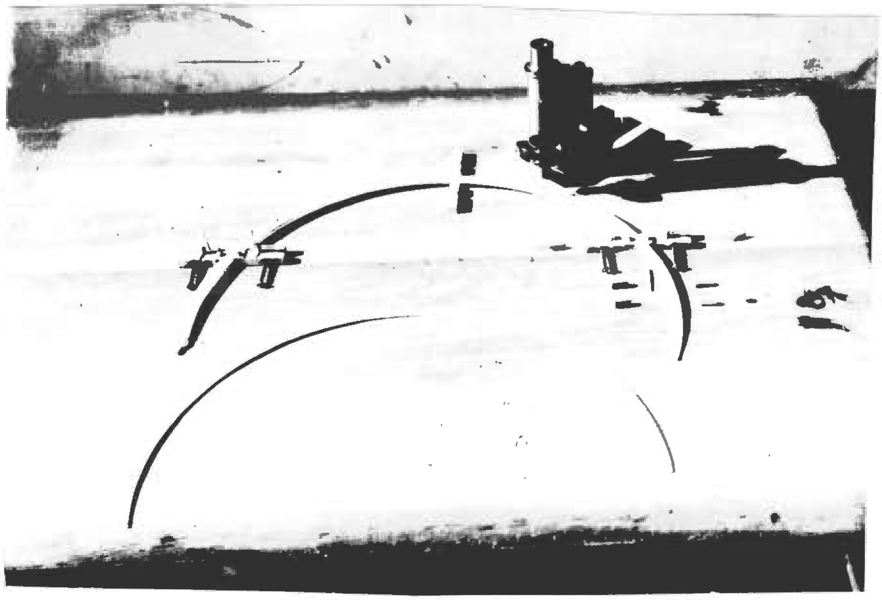
$$V_o = \frac{2L^3\pi}{4L^3\pi} = .5$$

$$M_o = \frac{2\ L^2 - .455\ L^2(1.1416)}{\pi L}$$

$$= \frac{40 - 21.25}{3.1416} = -5.64$$

$$M_B = .455(40) - .5(40) + 5.64 = 3.84$$

The value of I does not enter into the calculations so these values obtained by theory are the same for any depth of arch ring. In the mechanical analysis arch I of article 13 was fastened in the gauges with fixed ends as shown in the illustration. The illustration shows arch II of article 13 fastened in the gauges with fixed ends. Arch I is the same except that it is fastened on the diameter.



The load points were taken as the quarter point of span and not along the arch axis as was the case with the two hinged arch. The values obtained are as follows for quarter point loaded.

$$H = 3.6/12 = .3$$

$$V_B = 2.7/13 = .207$$

$$M_B = 8/1.7 = 4.7$$

and for center loaded

$$H = 4.2/12 = .35$$

$$V_B = 6.5/13 = .5$$

$$M_B = 7.8/1.7 = 4.6$$

Since the theoretical values are independent of I, another semicircular arch model was made withmthe same span but with half the thickness, and placed in the gauges in the same way. The values obtained were:

for quarter point loaded

$$H = 3.7/12 = .308$$

$$V_B = 2.5/13 = .192$$

$$M_B = 8/1.7 = 4.7$$

for center loaded

$$H = 4.5/12 = .375$$

$$V_B = 6.5/13 = .5$$

$$M_B = 7.8/1.7 = 4.6$$

Comparison of results

	Theory	Arch I	Arch with 1/2 d.
Quarter point loaded			
H	.31	.3	.308
V_B	.195	.207	.192
M_B	4.759	4.7	4.7
Center loaded			
H	.45	.35	.375
V_B	.5	.5	.5
M_B	3.84	4.6	4.6

In comparing these values it will be noted that there is a little variation for the theoretical values but considering thegreat amount of work required to obtain these values, those obtained with the models are probably as nearly correct as those obtained by theory. It is believed that these values obtained by the models are as safe to use in the design of such structures as those obtained by theoretical methods and there is not as much danger of error.

IV. APPLICATION TO REINFORCED CONCRETE.

17. Discussion of Theory.--- From article 6, we know that for any structure with a constant modulus of elasticity; i.e., made of the same material throughout, the value of E does not enter into the equation for internal work. The only terms in the equation which is different for reinforced condrete frames from those of homogeneous material are the values of A and I. Since the concrete and steel must have the same deformation under stress, the equivalent area of

steel for concreté is A_s (area of steel) times E_s/E_c (ratio of modulus of elasticity of steel to modulus of elasticity of concrete). The value of I for reinforced concrete section will be the I of the concreteabout the neutral axis plus the equivalent area of steel times the square of its distance from the neutral axis. The value of A can be stated inlterms of the depth anI so that for any structure the analysis depends upon the value of I. Since the models are of a homogeneous material, they will have to be made with a moment of inertia equal to that for the concrete section. Since the different members of the model cannot be varied in thickness, the depth must be varied to give an equivalent moment of inertia.But since the different values depend theoretically upon the ratio of the moments of inertia of the different members and not upon the value of the moments of inertia, it is thought that if the size of members is kept as nearly as possible to the size of the concrete section and the moments of inertia varied accordingly, that the results will bé more nearly their true value.

18. Rectangular Frame with Hinged Column Ends.--- The frame shown in Figure XVI is one analyzed and tested by Dr. Abe at the University of Illinois as part of his work for the Doctor's degree. The ratio of E_s/E_c was determined by separate tests on the concrete and steel as 14.3. The hinge was placed below the bottom of the column ends making the moment arm of H equal to 4.63 ft. The value of k (proportionality factor for location ofneutral axis) for section A-A is .41, A_s is .78 sq.in., and j is .863.

The value of H as computed by Dr. Abe was .082 P when P is the sum of equal loads on the third points. In the test, when a load of 9000 lb. was applied at each of the third points, the value of f_s (unit steel stress) was determined by extensometers to be 17,900 lb. per sq. in. This stress is caused by a bending moment and a horizontal thrust in the beam equal to H. Assuming that H will be distributed over the equivalent area of the beam, the amount of this compressive stress carried by the steel is $\frac{14.3\ (.78)\ H}{80 + 13.3\ (.78)}$ or .123 H. The observed value of the resisting moment at section A - A is then equal to(17,900 (.78) + .115 H) .863(10) and the external moment is equal to (9000 (24) - H 4.63(12)). Equating the external and resisting moment at section A -A and solving for H

$$120,000 + 1.06\,H = 216,000 - 55.5H$$

$$H = 1690 \text{ lb.}$$

$$\text{or } H = .094\,P$$

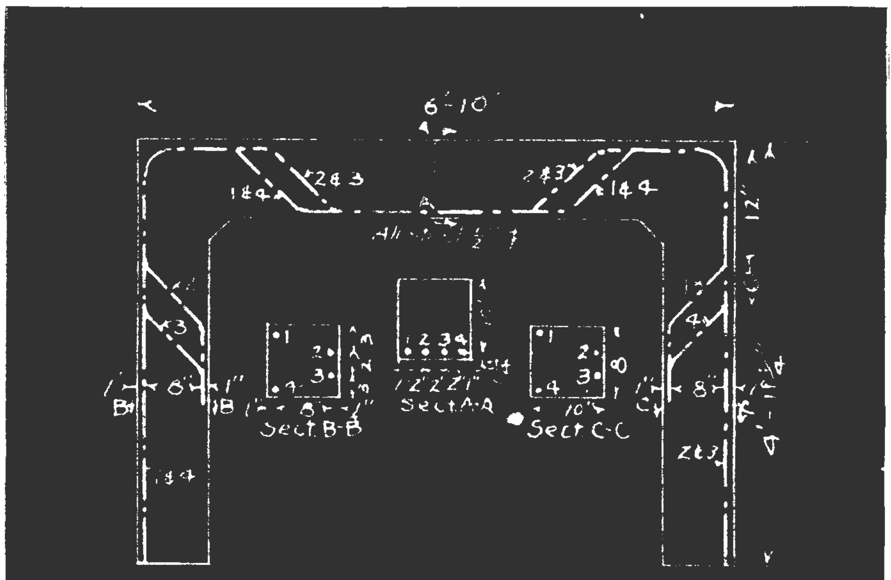


FIGURE XVI

The computed value of k is .41 for the top member and j is .863

$$I_{n.a.} \text{ at center} = \frac{8(1000)(.9)^2}{12} + 14(.78)(8.6)^2$$

$$= 540 + 808 = 1148$$

I in the column just below the joint is about the same as for the center of the beam. The bottom half (approximately) of the column has a value of k equal to .329 and j equal to .89

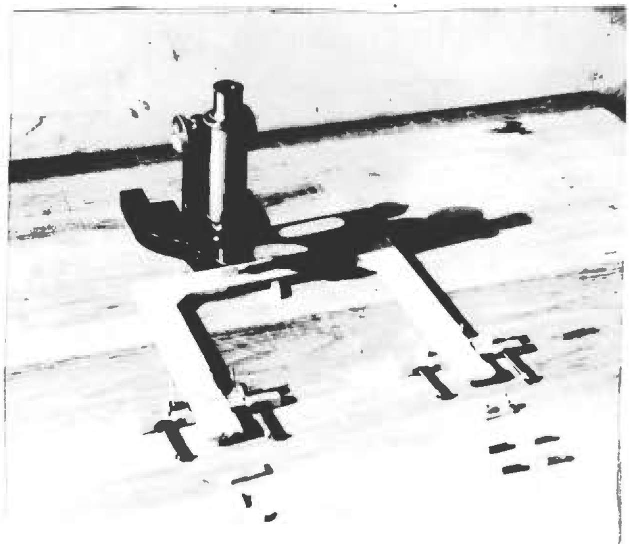
$$I = \frac{8(9)(9)(9)(1.5)^2}{12} + .39(64)$$

$$= 1090 + 25 = 1115$$

If we assume a depth of 10 in. for the top member, the thickness b of a homogeneous section for a value of I of 1148 is equal to 13.8. The value of d (depth to center of steel) for this same breadth of column is for a value of I of 1115 equal to

$\sqrt[3]{\frac{1115 (12}{13.8}}$ or 9.9 in.

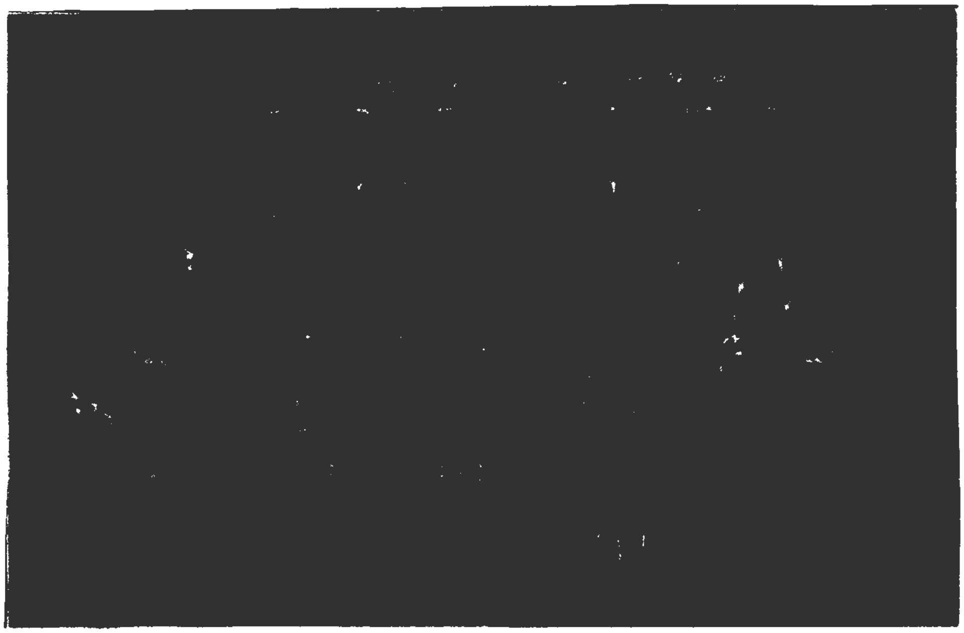
Cardboard was found to be too limber for a model of a frame of these dimensions, allowing the members to buckle when the deformation was produced. a model was made of wall board to a scale of 3/2 in. equal 1 ft. and the value of H found to be, for each of the load points, equal to .0416. This makes a value of .0832 for both third points loaded. This value agrees remarkably well with Dr. Abe's value of .082.



Model of concrete frame in gauges

19. Hingless Reinforced Concrete Arch.--- The object of this analysis is to compare the results obtained by the ordinary method of analysis with those obtained by model for plotting influence lines for horizontal reactions, vertical reactions, thrust, shear, and moment at the springing section of the reinforced concrete arch ring shown in Figure XVII.

This arch is one analyzed in Hool's "Reinforced Concrete Construction" Vol. III. It was analyzed by dividing the ring into sections such that s/I for each section is constant.



This was done by computing the value of I at several points on the arch ring. The arch axis is then laid off as abscissa and the values of I laid off as ordinates. These points located, they are connected by a smooth curve and the axis divided by trial into bases (usually ten) of isosceles triangles with apex in the curve for I. Using the value of s thus found as the length along the axes for that particular section of arch ring and measuring x and y to the center of gravity of the sections with the crown as the origin, the following equations for H_o, V_o, and M_o at the crown are obtained from the general equations

Figure showing method of dividing arch axis into constant s/I.

as given for the hingless arch in article 16. Since this type of arch cannot be integrated, ds becomes s and the integral becomes a summation of parts. Since s/I is constant

$$H_o = \frac{n\,\Sigma(mL + mR)\,y \;-\; \Sigma(mL + mR)\,\Sigma y}{2\left[n\Sigma y^2 - (\Sigma y)^2\right]}$$

$$V_o = \frac{\Sigma(mL - mR)\,x}{2\Sigma x^2}$$

$$M_o = \frac{\Sigma(mL + mR) - 2\,H_o\Sigma y}{2\,n}$$

when

s equals length of division of arch axis,

n equals number of divisions in one half of arch,

mL equals moment at any point on left half of arch axis of all external loads between the point and the crown,

mR equals same for right half of arch.

A table of these terms used in the above equations is prepared and the values of H_o, V_o, and M_o calculated by substituting values from the table in the equations. This table as taken from "Reinforced Concrete Construction" is given on the next page. The following values are then found.

Unit load at L_1

$$H_o = \frac{10(13.13) - (1.08)(29.90)}{2\left[10(222.94) - (29.90)^2\right]} = .037$$

$$V_o = \frac{34.2}{2(2611.6)} = -.007$$

$$M_o = \frac{1.08 - 2(.037)(29.90)}{2(10)} = -.057$$

TABLE FROM "REINFORCED CONCRETE CONSTRUCTION"

Pt.	x	y	x^2	y^2	Unit load at L_1			Unit load at L_2			Unit load at L_3			Unit load at L_4		
					m	mx	my	m	mx	my	m	mx	my	m	mx	my
L_1	30.62															
L_2	21.87															
L_3	13.12															
L_4	4.37															
1	31.70	12.16	1004.9	147.87	1.08	34.2	13.13	9.83	311.6	119.53	18.58	589.0	225.93	27.33	866.4	332.33
2	28.18	6.63	584.7	43.96				2.31	55.9	15.32	11.06	267.4	73.33	19.8[illegible]	479.0	131.34
3	19.61	4.25	384.6	18.06							6.49	127.3	27.58	15.24	298.9	64.77
4	16.08	2.80	258.6	7.84							2.96	47.6	8.29	11.71	188.3	32.79
5	13.06	1.85	170.6	3.42										8.69	113.5	16.08
6	10.33	1.13	106.7	1.28										5.96	61.6	6.73
7	7.80	.64	60.8	.41										3.43	26.7	2.20
8	5.43	.30	29.5	.09										1.06	5.8	.32
9	3.18	.12	10.1	.01												
10	1.05	.02	1.1	.00												
Σ		29.90	2611.6	222.9[illegible]	1.08	34.2	13.13	12.14	367.5	134.85	33.09	1031.3	335.13	93.23	2,040.2	586.[illegible]

Unit load at L_2

$$H_o = \frac{10(134.85) - (12.14)(29.90)}{2670.8} = .369$$

$$V_o = \frac{367.5}{5223.2} = .07$$

$$M_o = \frac{12.14 - 2(.369)(29.90)}{20} = -.496$$

Unit load at L_3

$$H_o = \frac{10(335.13) - 39.09(29.90)}{2670.8} = .818$$

$$V_o = \frac{1031.3}{5223.2} = .197$$

$$M_o = \frac{39.09 - 2(.818)(29.90)}{20} = -.491$$

Unit load at L_4

$$H_o = \frac{10(586.56) - 93.23(29.90)}{2670.8} = 1.151$$

$$V_o = \frac{2040.2}{5223.2} = .391$$

$$M_o = \frac{93.23 - 2(1.151)(29.90)}{20} = +1.22$$

After these values are found the arch is statically determinate and the values of H, T, V, and M at the springing line may be determined either graphically or algebraically.
The values of these terms are given in the table on the next page.

The following method was used for the mechanical analysis,

The value of I at the crown is

$$16^3 + 15\,(.994)(2)(5.5)^2 = 5002$$

and I at the springing line is

$$36^3 + 15(.994)(2)(15.5)^2 = 53{,}760$$

If we keep the depth of crown the same for the model as the concrete section, the value of b giving an equal I is $\frac{5002\,(12)}{16^3}$ or 14.65.

The equivalent depth at springing is

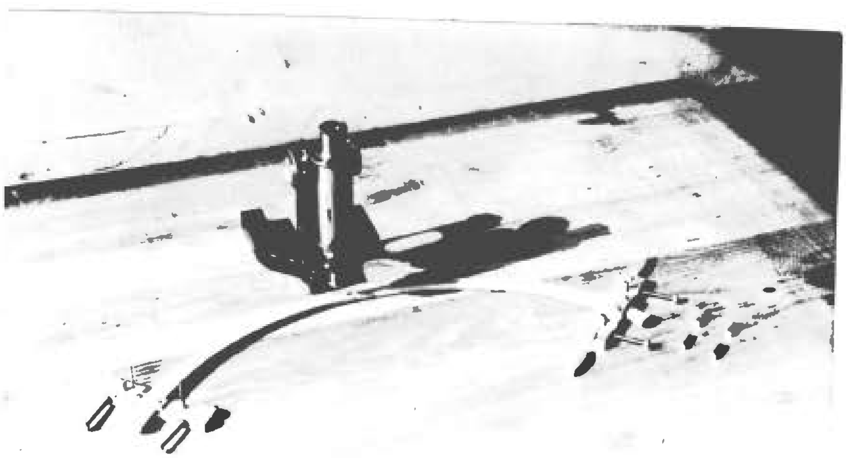
$$\sqrt[3]{\frac{53{,}760\,(12)}{14.65}} \quad \text{or } 35.4 \text{ in.}$$

A model was made of wall board with a depth of crown equal to 16 in. and depth of

INFLUENCE TABLE LEFT SPRINGING

	L_1 LEFT		L_2 LEFT		L_3 LEFT		L_4 LEFT		L_4 RIGHT		L_3 RIGHT		L_2 RIGHT		L_1 RIGHT	
	Theory	Model	Theory	Model	Theory	Model	Theory	Model	Theory	Model	Theory	Model	Theory	Model	Theory	Model
H	.037	.041	.369	.38	.818	.79	1.15	1.08	1.15	1.08	.818	.79	.369	.38	.037	.041
R	.993	—	.93	.925	.803	.75	.609	.55	.391	.36	.197	.177	.07	.07	.007	.01
V	+.61	+.57	+.327	+.33	-.085	-.09	-.458	-.41	-.599	-.46	-.592	-.417	-.228	-.21	-.0229	.035
T	.783	.78	.448	.92	1.142	1.1	1.211	1.2	1.04	1.0	.68	.64	.293	.3	.03	.05
M	-4.65	-5.0	-6.004	-6.18	-2.63	-2.5	+2.82	+2.9	+6.33	+6.35	+5.99	+5.9	+3.12	+3.2	+.319	+.55

springing equal to 35.4 in. This model was placed in the gauges with the gauge horizontal to produce horizontal deflection at the springing for determining the value of H and R. The gauges were then changed to produce displacements parallel to the axis and also normal to the axis for determining T, V, and M at the springing. These values are given on page 86. The model fastened in the gauges is shown in the illustration.



The values found by the mechanical met hod are seen to compare very well with those found by the ordinary method and may be used in the design of such structures. There is as much reason for thinking that the results of the mechanical method are as near the truth as those found by theoretical methods, as the later method is a summation and not exact. At the same time it is not possible to say just what the equivalent depth should be taken to give the correct result. The results of the mechanical method are certainly much easier of attainment and there is not the danger of making mistakes as there are in the mathematical calculations.

V. CONCLUSION

20. Difficulties Encountered.--- There were several difficulties encountered in the investigation, the most outstanding one being with the gauges. If the gauges are to give satisfaction, they must be made with a great deal of care. The notches in the gauges should be made with an angle of exactly 90° and with the diagonal of the square formed exactly parallel to the main parts of the gauge. They must also be exactly the same distance apart or the gauge will rock back and forth as the pressure is shifted from one end of the gauge to the other. The shear plugs should be made with all the faces making exact right angles and they should be exactly the same size, otherwise the same displacement will not be produced unless they are placed in the gauges in exactly the same manner each time, which is a practical impossibility. The thrust and moment plugs must be exactly round or the same displacement will not be produced each time the plugs are changed.

Another difficulty was encountered in reading the deflection in the micrometer eyepiece. A point was tried for marking the load point at which the deflection was measured but without success. The load point not only has vertical motion b ut also a horizontal displacement when the plugs are changed in the gauge. This is sometimes great enough to cause the point to pass entirely off the scale and always from one side of the scale to the other. If, when this happenes, the scale is not exactly oriented, the proper deflection is not read by the difference in readings on the scale. A point when seen through the mecroscope is not a point and it is difficult to determine the deflection. Also apoint furnishes no means of orienting the scale to read deflection in the direction in which the load is acting. It was not found possible to use a pencil line on the model for orienting because it is too ragged. This difficulty was finally solved by making several small blocks of aluminium about 1/4 in. square by 3/4 in. long and smoothing the faces till the edges appeared as smooth straight lines when seen under the microscope. These were fastened to the model with paste at the load points with the edges normal to the direction in which the load was assumed to be acting. This furnished a means of orienting the scale and the line was always on the scale permitting easy and correct reading of the deflection. To read the deflection, the scale was set with zero line along the edge of the block and the deflection produced in a direction such that the edge of the block moved back along the scale. The divisions of deflection could then be easily counted as they were clear of the confusing lines on the surface of the aluminium.

21. Care to be Exercised.*** A great deal of care should be exercised if comparatively accurate results are expected. If the models are too thin in comparison with the depth and

length of members, they will buckle; and the investigation shows that results will be obtained under these circumstances which are far from accurate. The ordinary thickness of cardboard was found to be too thin for most models, two thicknesses of the comparatively heavy cardboard were found to give good results for most models; but wall board 3/16 in. thick gave excellent results in every case in which it was tried. Wall board is of course more difficult to cut into models than cardboard; but with a sharp knife may be done very well.

The gauges must be placed exactly normal to the axis at the section if accurate results are expected for shear and thrust. In determining the moment the section must be placed in the gauge with the axis at that point exactly midway between the plugs; or thrust displacement will be produced in connection with the moment and erroneous results will be obtained.

22. Determination of Signs . --- The mechanical method automatically determines the direction of the shear, thrust, or moment at the section. It can readily be seen that if a shear, or thrust displacement at the gauge produces a deflection of the load point apposite in direction to that in which the load is assumed to be acting, then the displacement at the gauge is in the direction of the reaction at the section and vice versa. From a study of Figure II, it can be seen that if a produced moment at B in the same direction as that caused by P about B, causes an upward deflection atA, then a downward deflection at A will cause a moment at B opposite in direction to that of the load about B ; i.e. tension in the lower fiber. We may say then that if the turning effect in the gauge is in the same direction as that of the load about B and produces a deflection opposite in direction to thatin which the load is acting, then the moment at the section is in such a direction as to cause tension in the bottom fiber and vice versa.

23. Determination of Thrust, Shear , and Moment at a Section Other Than Support.--
The theoretical discussion in the introduction holds for any sectio of the structure. To determine the shear, thrust or moment at any section other than the support, fasten the model at the supports in exactly the same manner as before. Then at the sectio where the thrust, shear or moment is desired, place a gauge and clamp the model in both A and B of the gauge and cut the model between them. A displacement can then be produced with the plugs at that section. The gauge should be floated on glass and ball bearings in this case to allow free motion of the model.

This wasnot tried in this investigation because of lack of time and since it is not as greatly important as the determination of the reactions. For any but very complicated structures the stresses are all determinate as soon as the reactions are determined nd, since the gauge would restrict the deflection ofn the model unless great care was exercised, it is thought that better results will be obtained by determining stresses from the reactions in every case where it is possible to do so.

24. Field of Usefulness.--- In the light of this investigation, a great field of usefulness is predicted for this method of analysis because of its simplicity. Even though it may not be used for the original analysis of indeterminate structures, it may be used as a valuable check upon the mathematical calculations until its reliability is further established.

It would seem to have a special field of usefulness in the design of concrete arches. There is a great saving affected in the construction of a series of arches if the piers are made elastic. This is seldom done due to the great amount of designing work required. The new method offers a simple solution for such structures. This is also true with regard to unsymmetrical arches.