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ON THE EXISTENCE AND STABILITY OF FILIFORM NILSOLITONS

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

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Dedicated to

*Yahweh, the God who Raises the Dead*

# Acknowledgments

I made it a special point to leave this section till all the mathematical hay was in the barn, lest I be tempted to only work on it to the detriment of my other work, because in short, there are far too many people to thank. In truth, if were to give proper honor to each person who has encourage, instructed, or invest in me would take a full other volume. However, I hope this short list will serve to honor those who have contributed in measures and means I will never be able to repay. If any feel they are omitted, I hope it is a testament to the debt of gratitude I owe to so many. Thank you!

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# Contents

# Abstract

This dissertation is concerned with the existence and stability of nilsoliton metrics on filiform Lie algebras. The results are presented in two major components. First, we give new results which preclude the existence of soliton metrics on rank 1 filiform algebras. Most notably, these methods circumvent the need to classify the algebras in each dimension (a major obstacle to the study, to this point). Second, we demonstrate that all soliton metrics on rank 2 filiform algebras are stable. In the course of this, we will develop new approximation techniques, and compute the full curvature tensor of rank 2 filiform algebras. The tables for these curvature tensors are contained in Appendix A.

# Introduction

One of the classical questions of Riemannian geometry is “Given a smooth manifold $M$, what is its ‘best’ metric?” along with its analogous question “Given a distinguished metric, which smooth manifolds admit such a metric?” The simplest choice is to consider the case of constant sectional curvature. It turns out this is an extremely restrictive condition, and few spaces admit such metrics. In particular, if a space has constant sectional curvature, then its universal cover is isometric to a sphere, Euclidean space, or hyperbolic space [**?** ], determined by whether the curvature is positive, zero, or negative, respectively. Since, this question has been resolved, modulo a classification of quotients of these space, attention has turned to other distinguished metrics which may be admitted by spaces which are not quotients of space forms.

Einstein metrics are natural generalizations of constant curvature. Namely, the case where the ric = $cg$, where $g$ is the Riemannian metric, and $c$ is constant. Necessary and sufficient conditions for the existence of Einstein metrics has been an area of fruitful study for the past 75 years. For a thorough introduction to this topic, we refer the reader to [**?** ]. In particular, we are interested in the homogeneous setting, (that is, the manifold admits a transitive group of isometries). With this focus, we are able to use a full arsenal of algebraic tools to approach this geometric problem.

With the assumption of homogeneity, the cases breakdown into positive, zero, or scalar curvature. In the case of positive scalar curvature, Myers Theorem implies that the such a homogeneous Einstein manifold must be the quotient of a simply-connected compact space. Further, it has finite fundamental group. In the case of zero Ricci curvature, it is shown in [**?** ] that such a manifold actually has constant sectional curvature and thus is a quotient of Euclidean space, with the usual Euclidean metric. Our interest is in understanding the case where scalar curvature is negative. The bulk of study in the last 50 years has been put towards the following conjecture.

**Theorem 0.1** (Strong Alekseevskiĭ Conjecture)**.** A homogeneous Einstein manifold with negative scalar curvature is isometric to a solvmanifold. That is, a solvable Lie group with left invariant metric.

A consequence of this is that homogeneous Einstein solvmanifolds are diffeomorphic to Euclidean space, and a proof has been put forward in [**?** ]. This strongly restricts the structure of homogeneous Einstein manifolds with negative scalar curvature. Further, these metrics are of special geometric interest as [**?** ] demonstrates that Einstein solvmanifolds have maximal symmetry. That is, their isometry group is as large as possible. It is within this frame of reference that we will focus. This gives rise to the following question, which is far from resolved. See [**?** ] for more information on the current state of the field.

**Question.** Which solvable Lie groups admit left invariant Einstein metrics?

Since our interest is in simply-connected solvable Lie groups with left invariant metrics, all the pertinent information is contained in its Lie algebra. In particular, the existence of an Einstein metric on a solvable Lie algebra means the restriction to its nilradical (largest nilpotent ideal) is of a very special form. We call such nilpotent Lie algebras Einstein nilradicals, and the corresponding metrics nilsolitons. We examine this connection more fully in Section 1.2.

Nilsoliton metrics are far from classified. One of the few invariants for nilpotent Lie algebras is the length of their derived series. One example is when the algebra is two-step (See Section 1.3 for definition). That is, the non-abelian algebras whose derived series is of minimal length. These nilpotent algebras are as close to abelian as possible. It is known from ([**?** ], Theorem 7.25) that a generic two-step nilpotent group admits a nilsoliton metric. Further, work is done in [**?** ] and [**?** ]. We are interested in the 'other end' of nilpotency; the so-called *filiform* (thread-like) algebras. That is, the those whose derived series is as long as possible. This leads us to the first question which this work will address.

**Question.** Which filifrom nilpotent algebras admit a soliton metric?

As stated in the title of this work, we are also interested the so-called 'stability' of filiform nilsolitons. The name soliton is borrowed from the fact that nilsolitons are (up to diffeomorphism and scaling) fixed points of the Ricci flow on the space of Riemannian metrics. Thus, the study of the stability of nilsoliton metrics is natural. More explicitly, do the metrics sufficiently close to a nilsoliton metric converge to the nilsoliton metric under Ricci flow? Again, in the two-step case, the question of stability is in the affirmative, as in ([**?** ], Theorem 1.10) and ([**?** ], Theorem 1.2). So, as with determining the existence of metrics, we have the following question.

**Question.** Are filiform nilsolitons stable?

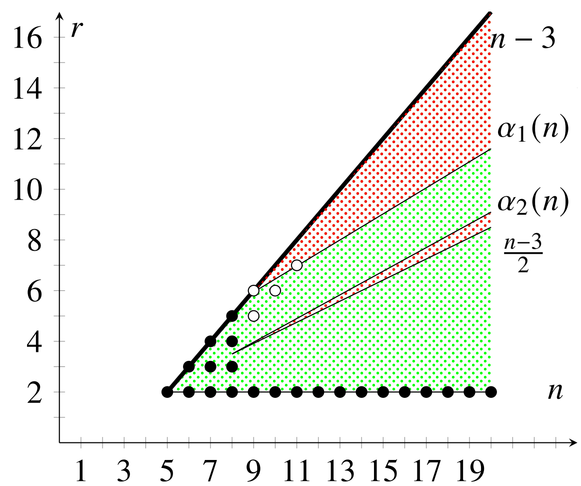
This work is organized as follows. In Chapter 1, we give the basic notions of Riemannian Geometry necessary for our study. We further examine the deep connection between Einstein solvmanifolds and nilsolitons. From there, we give examples and definitions of 'filiform' (thread-like) nilpotent Lie algebras. These filiform algebras are 'as far from abelian as possible', and will garner the attention of our study. Lastly, we consider nilsolitons as stable points of the Ricci flow, up to scaling and diffeomorphisms, and discuss sufficient conditions for their stability.

In Chapter 2, we study the existence of nilsoliton metrics on filiform Lie algebras. These algebras trifurcate into ranks 0, 1, 2, as in [**?** ], where rank refers the dimension of the maximal torus of derivations. Rank 0 filiform algebras do not admit solitons, as they are characteristically nilpotent, and thus do not admit a positive symmetric derivation. In rank 2, there are two families of algebras, which both admit nilsoliton metrics, as shown in [**?** ] and [**?** ].

Rank 1 filiform algebras further split into two families $A_{n,r}$ and $B_{n,r}$ (denoted $A_r, B_r$, respectively, in [**?** ] and others), where $2 \leq r \leq n-3$, for natural numbers $n, r$. In [**?** ], the algebras which admit solitons are classified for $A_{n,2}$ and $B_{n,2}$, that is $r = 2$. However, these results rely strongly on existing classifications of filiform algebras for the families $A_{n,2}$ and $B_{n,2}$, which do not exist for large $r$.

In low dimensions, [**?** ] uses the existing classification of nilpotent Lie algebras in dimensions 6 and below, to classify and give metrics for all nilsolitons of those dimensions. [**?** ] takes a similar approach and extends this work to classify nilsolitons and give metrics for dimension 7, again using existing classification of 7 dimensional nilpotent Lie algebras, while [**?** ] classifies filiform algebras in dimension 8, and determines which algebras admit solitons, though metrics are not computed. In this work, we study the existence of solitons for large $r$, and provide conditions to preclude the existence of a soliton which do not rely on existing classifications of filiform algebras. In particular, we have the following (partial) answer to the first question.

**Theorem 2.11.** Suppose $n > 8$, and $\mu \in A_{n,r}$. There exists functions $\alpha_1 > \alpha_2$ of $n$ such that if $\frac{n-3}{2} \leq r \leq \alpha_2(n)$ or $\alpha_1(n) \leq r \leq n-3$, then $\mu$ does not admit a soliton.

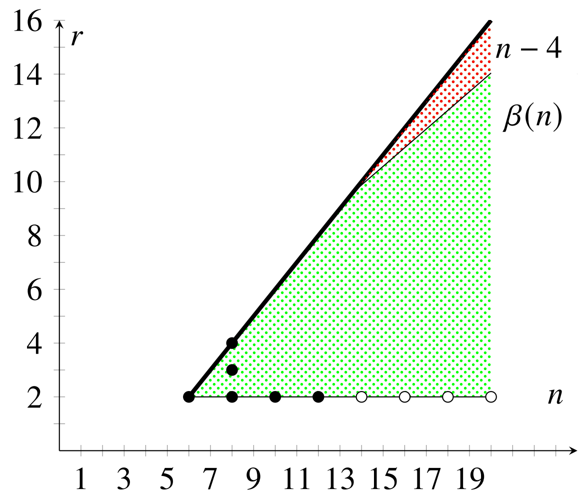



The curves $\alpha_1, \alpha_2$ are curves of solutions to cubic equations, which depend on $n$, so their formulas, given in Equations (2.2) and (2.3) are unwieldy. However, we can say $\alpha_1(n)$ is asymptotic to $\frac{n}{\sqrt{3}}$, and $\alpha_2(n)$ is asymptotic to $\frac{n}{2}$.

In this picture, the black dots represent pairs $(n, r)$ which, from [**?** ], [**?** ], [**?** ], and [**?** ], admit soliton metrics. That is, there is an algebra in $A_{n,r}$ which admits a soliton. However, there may be several other isomorphism classes which do not admit soliton metrics, as in ([**?** ], Section 4). The clear dots represent specific combinations of $(n, r)$ for which no algebra in $A_{n,r}$ admits a soliton. The red regions are pairs of $(n, r)$ which this theorem shows 'cannot' admit solitons, while in the green regions, without dots, the question of existence is still open. We further give results in the special case of $A_{n,n-3}$, $A_{n,n-4}$ and show that for these maximal values of $r$, there are no solitons for $n \geq 8$.

The situation for $B_{n,r}$ is analogous, but with some important distinctions. First, we give the sister result to Theorem 2.11.

**Theorem 2.23.** Suppose $n > 8$ is even, $\nu \in B_{n,r}$. There exists a function $\beta$ such that if $\beta(n) \leq r \leq n-4$, then $\nu$ does not admit a soliton.




Again, $\beta$ is a curve of solutions to a cubic, which depend on $n$. The equation is given in Equation (2.4). $\beta$ is asymptotic $\frac{n}{\sqrt{3}}$.

As in the previous picture, the black dots represent pairs $(n, r)$ which, from [**?** ], [**?** ], and [**?** ], which have an algebra that admits a soliton. The results of [**?** ] do not apply as $n$ must be even. The clear dots represent specific combinations of $(n, r)$ for which no algebra in $B_{n,r}$ admits a soliton. The red regions are pairs of $(n, r)$ which this theorem shows 'cannot' admit solitons, while in the green regions, without dots, the question of existence is still open.

The first major distinction in analysis of $B_{n,r}$ and $A_{n,r}$ is that $n$ is required to be even for $B_{n,r}$.

Further, an algebra in $B_{n,r}$ is a central extension of an algebra in $A_{n-1,r}$, which in general may not exist, as they often fail to satisfy the Jacobi condition. For example, ([**?** ], Section 4) shows that $B_{n,2} = \emptyset$ for $n > 12$. To this point, the study of algebras in $B_{n,r}$ has required an especially tedious classification problem, to show they even exist. These results again circumvent that challenge. The reasons for the difference in Theorem 2.11 and Theorem 2.23 becomes clear upon comparison of their proofs. We analyze this distinction after the proof of Theorem 2.23.

In Chapter 3, we consider the stability of nilsolitons. Along with considering solitons as an algebraic object, they may also be analyzed as fixed points of the Ricci flow. Thus, questions of fixed-point analysis and stability are apropos. We develop the notions of linear stability in [**?** ], to give approximations to determine the stability of a given metric, which according to [**?** ] is equivalent to dynamic stability. In particular, we have the following result.

**Theorem 3.7** Let $(N, \mathfrak{n}, \lambda, D)$ be a nilsoliton with fixed basis $\mathcal{B}$. If

$$2n \max_{ijkl} |R_{ijkl}| \Delta(\mathcal{B}, \mathfrak{n}) + \rho(\mathrm{Ric}) < \frac{1}{2} \mathrm{tr}\, D$$

then the soliton is linearly stable.

Here, $\Delta(\mathcal{B}, \mathfrak{n})$ denotes the *curvature density of* $\mathfrak{n}$ with respect to the basis, which we define in Chapter 3. Morally, it is a way to track the number of non-zero terms for the action of the curvature tensor on the space of symmetric two-tensors. Applying this estimate to rank 2 filiform nilsolitons, and applying ([**?** ], Theorem 1.2) yields the following theorem, which answers to the question above, in the rank 2 case.

**Theorem 3.18.** Rank 2 filiform nilsolitons are dynamically stable.

Finally, Appendix A gives the full curvature tensor of rank 2 filiform algebras, which are used in Chapter 3. Appendix B gives an outline of the programming techniques used in Chapter 3. While the estimates proven in Chapter 3 yield stability for $n$ sufficiently large, a program was developed to calculate the remaining cases.

# Chapter 1

# Preliminaries

## 1.1. Riemannian Geometry

Here we give some of the basic definitions and results from Riemannian Geometry. For a more extensive treatment, we refer you to [**?** ] and [**?** ].

Let $M^n$ be a smooth manifold of dimension $n$ (we often suppress the $n$ when it is clear), and let $\mathfrak{X}(M)$ denote the space of smooth vector fields on $M$. A *Riemannian metric* $g$ is an assignment of inner product (symmetric, postive-definite, bilinear form) $g_p(-,-)$ to each tangent space, $T_pM$ that is smooth in the sense that for all $X, Y \in \mathfrak{X}(M)$, the function $g(X,Y) : M \to \mathbb{R}$ given by

$$g(X,Y)(p) := g_p(X,Y)$$

is smooth. We often denote the pair $(M, g)$, and call it a *Riemannian manifold*.

Each Riemannian manifold $(M, g)$ comes with unique symmetric affine connection $\nabla$

compatible with the metric, given by Kozul's formula. Namely,

$$g_p(Z, \nabla_X Y) = \frac{1}{2}\Big(Xg_p(Y,Z) + Yg_p(Z,X) - Zg_p(X,Y)$$
$$- g_p([X,Z],Y) - g_p([Y,Z],X) - g_p([X,Y],Z)\Big)$$

where $[X,Y]$ is the bracket of vector fields.

Riemannian manifolds have various invariants. Of particular interest to us is the *curvature* of a manifold. We follow the sign convention of [**?** ] and define the (3,1) *curvature tensor* $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ to be a multilinear map given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Be advised that [**?** ] uses the opposite sign convention. Using the Riemannian metric to change the tensor type, we have the (4,0) curvature tensor given by $R(X,Y,Z,W) = g(R(X,Y)Z,W)$. The full curvature tensor has various symmetries, for which we refer to reader to the references above.

Often, the full curvature tensor is intractable, so we examine various reductions. Let $\sigma \in T_pM$ be a 2-plane spanned by $X, Y \in \sigma$. The *sectional curvature* of $\sigma$, is given by $K(\sigma) = R(X,Y,Y,X)$. It is a routine exercise to show this does not depend on the choice of $X, Y$.

We define the *Ricci curvature*, a symmetric, bilinear (2,0)- tensor as follows. Let $\{E_i\}$ be an orthonormal frame about $p \in M^n$. Then,

$$\mathrm{ric}_p(X, Y) = \sum_{i=1}^{n} R(X, E_i, E_i, Y).$$

Equivalently, we have the corresponding symmetric linear map Ric defined implicitly via $g_p(\mathrm{Ric}_p X, Y) = \mathrm{ric}_p(X, Y)$. Finally, we define the scalar curvature at $p \in M$ by $\mathrm{scal}_p = \mathrm{tr}\,\mathrm{Ric}_p$.

We say $g$ is an *Einstein metric* if $\mathrm{ric} = cg$ for some $c \in \mathbb{R}$. Accordingly, we say $(M, g)$ is a *Einstein manifold*. This condition has attracted much study in Riemannian geometry over the past century. It is too restrictive to allow for general existence results, yet too relaxed to produce results on obstructions. For information in dimension 4, we refer the reader to [**?** ]. Much progress has been made in the past 50 years in the case where $(M, g)$ is a *homogeneous space*. That is, the isometry group of $(M, g)$ acts transitively.

We are particularly interested in the case where the homogeneous space is, in fact, a Lie group $G$, and $g$ is a *left-invariant metric*. That is, $g$ is invariant under translation by left multiplication. This is equivalent to a choice of inner product $\langle\ ,\ \rangle$ at the identity, which is identified with $\mathfrak{g}$, the space of left-invariant vector fields on $G$. We identify a simply connected Lie group with left-invariant $(G, g)$, with its Lie algebra and corresponding inner product $(\mathfrak{g}, \langle\ ,\ \rangle)$. The pair $(\mathfrak{g}, \langle\ ,\ \rangle)$ is called a *metric Lie algebra*. Given a metric Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$, we can similarly construct it's corresponding simply connected Lie group with left-invariant metric $(G, g)$, via exponentiation. Further, we say $(\mathfrak{g}, \langle\ ,\ \rangle)$ is Einstein when $(G, g)$ is Einstein, and vice versa.

One major reason to restrict to left-variant metrics, is the various tensors of interest simplify to pointwise objects. When $X, Y \in \mathfrak{g}$ and $\langle\ ,\ \rangle$ is the restriction to $\mathfrak{g}$, of a left-invariant metric, the formula for $\nabla$ simplifies to

$$\nabla_X Y = \frac{1}{2}(\mathrm{ad}_X Y - \mathrm{ad}_X^* Y - \mathrm{ad}_Y^*)$$

where $\mathrm{ad}_X^*$ is the metric adjoint of $\mathrm{ad}_X$ relative to $\langle\ ,\ \rangle$. Further, the Ricci tensor may be calculated from purely algebraic data. From, ([**?** ], Corollary 7.38), when $(\mathfrak{g}, \langle\ ,\ \rangle)$ is a metric Lie algebra

$$\mathrm{ric}(X, X) = -\frac{1}{2}\sum_k |[X, X_k]|^2 - \frac{1}{2}B(X, X) + \frac{1}{4}\sum_{kl}\langle[X_k, X_l], X\rangle^2 - \langle[Z, X], X\rangle,$$

where $B$ is the Killing form and $Z$ is the mean curvature vector. We will be most interested in studying when $\mathfrak{n}$ is nilpotent. In this case, the Killing form and mean curvature vector both vanish. Thus, the formula reduces to

$$\mathrm{ric}(X, Y) = -\frac{1}{2}\sum_{kl}\langle[X, X_k], X_l\rangle\langle[Y, X_k], X_l\rangle + \frac{1}{4}\sum_{ij}\langle[X_k, X_l], X\rangle\langle[X_k, X_l], Y\rangle. \quad (1.1)$$

If we fix an orthonormal basis, $\mathcal{B} = \{X_i\}$, and consider the structure constants $\alpha_{ij}^k = \langle[X_i, X_j], X_k\rangle$, then we may rephrase this as

$$\mathrm{ric}(X_i, X_j) = -\frac{1}{2}\sum_{kl}\alpha_{ik}^l\alpha_{jk}^l + \frac{1}{4}\sum_{kl}\alpha_{kl}^i\alpha_{kl}^j \quad (1.2)$$

These two descriptions of the Ricci tensor will be used extensively.

## 1.2. Einstein Solvmanifolds and Nilsolitons

In the case of a homogeneous Einstein space $M = G/H$, the study splits into scalar curvature postive, negative, and zero. For positive scalar curvature, Myer's theorem implies that $M$ is compact. [**?** ] shows that, $G$ must be a compact semi-simple group. If scalar curvature is zero, then in fact, sectional curvature is zero, as shown in [**?** ], and so $G/K = \mathbb{R}^{n-k} \times T^k$. Negative scalar curvature implies $M$ is non-compact, by Bochner's Theorem (See [**?** ], Chapter 8). This is the case which will receive our attention, but for the positive case, we refer the reader to [**?** ] for a survey of the field.

We will now sketch the theory for negative scalar curvature and direct the reader to [**?** ] and [**?** ] for a more detailed introduction. Much of the work in the past several decades has been devoted to the following theorem, known to this point as the Strong Alekseevskiĭ Conjecture, for which a proof is put forward in [**?** ].

**Theorem 0.1.** A homogeneous Einstein manifold with negative scalar curvature is isometric to a solvmanifold. That is, a solvable Lie group with left-invariant metric.

Since this result greatly restricts the possible spaces admitting an Einstein metric, it gives rise to the following questions.

**Question 1.** Which solvmanifolds admit Einstein metrics?
**Question 2.** How many unique Einstein metrics can a given solvmanifold admit?

Question 2 is resolved through the notion of a standard solvmanifold. Let $(\mathfrak{s}, \langle\ ,\ \rangle)$ be the metric Lie algebra associated to a solvmanifold $(S, Q)$. We say $(S, Q)$, likewise $(\mathfrak{s}, \langle\ ,\ \rangle)$, is *standard* if $[\mathfrak{s}, \mathfrak{s}]^\perp$ is abelian, where the orthogonal decomposition is taken with respect to $\langle\ ,\ \rangle$. In ([**?** ], Theorem 3.1) it is shown that all Einstein solvmanifolds are standard. Further, in ([**?** ], Corollary 5.5) it is shown that on a given solvmanifold standard Einstein metrics are unique up to scaling and isometry. Thus, if a solvmanifold admits an Einstein metric, it is unique up to scaling.

Question 1 is unresolved and one that we will be considering in great detail through the course of our study. Let $\mathfrak{n}$ be a nilpotent Lie algebra. We say $\mathfrak{n}$ is an *Einstein nilradical* if it is the nilradical (i.e., the largest nilpotent ideal of $\mathfrak{n}$) of an Einstein solvmanifold $\mathfrak{s}$, where the inner product on $\mathfrak{n}$ is given by restriction. According to ([**?** ], Theorem 3.7), the restriction of the Ricci tensor of an Einstein solvmanifold to its nilradical a very particular form. Namely, if $(\mathfrak{s}, \langle\ ,\ \rangle)$ is a Einstein metric Lie algebra, then the Ricci tensor for $(\mathfrak{n}, \langle\ ,\ \rangle|_{\mathfrak{n}\times\mathfrak{n}})$ is given by

$$\mathrm{Ric}_{\mathfrak{n}} = cI + D, \tag{1.3}$$

where $c$ is Einstein constant of $(\mathfrak{s}, \langle\ ,\ \rangle)$, and $D \in \mathrm{Der}(\mathfrak{n})$. In fact, $D = \mathrm{ad}_H|_{\mathfrak{n}}$, where $H$ is the mean curvature vector of $(\mathfrak{s}, \langle\ ,\ \rangle)$.

Metrics of this form are given a special name. We say a metric is a *soliton metric* if $\mathrm{Ric} = cI + D$, for some derivation $D$. When $\mathfrak{n}$ is nilpotent, we say it is a *nilsoliton metric*. Further, $D$ is called the *soliton derivation*. From ([**?** ], Theorem 4.14), it follows that, up to scaling, $D$ is positive with rational eigenvalues. Further, from ([**?** ], Lemma 3.4), it follows that Ric is orthogonal to any symmetric derivation, under the inner product given by the trace pairing. Applying this fact to Equation 1.3 by multiplying by either Ric or $D$, respectively, and taking the trace, yields

$$c = \frac{\operatorname{tr} \mathrm{Ric}^2}{\operatorname{tr} \mathrm{Ric}} = -\frac{\operatorname{tr} D^2}{\operatorname{tr} D}. \tag{1.4}$$

By ([**?** ], Theorem 3.7), $\mathfrak{n}$ being an Einstein nilradical is equivalent to admitting a nilsoliton metric. In particular, every nilsoliton admits a rank 1 extension to an Einstein solvmanifold, by identifying solvmanifolds and nilsolitons, the search for Einstein solvmanifolds is equivalent to the following question.

**Question 3.** Given a metric nilpotent Lie algebra, when does it admit a soliton metric?

Much fruit has been gained in this program through means of geometric invariant theory. We will sketch some of the machinery necessary for our purposes, but refer the reader to ([**?** ], Section 3) for more details.

Let $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, which we identify with the space of vector-valued, antisymmetric, bilinear linear maps, and fix an inner product $\langle\ ,\ \rangle$, with orthonormal basis $\{X_i\}$. Notice that the Jacobi and nilopotent conditions are polynomial. Thus, we may consider $\mathcal{N} \subset V$, the variety of nilpotent Lie algebras of dimension $n$. Given, $\mu \in V$, we have a natural action of $\mathrm{GL}_n(\mathbb{R})$ on $V$ given by "change of basis", as follows. Let $\mu \in V$, $g \in \mathrm{GL}_n(\mathbb{R})$, $X, Y \in \mathbb{R}^n$. We define $g.\mu$ via

$$(g.\mu)(X, Y) = g\mu(g^{-1}X, g^{-1}Y).$$

Notice, that $\mathcal{N}$ is invariant under this action and the orbits of the action are precisely the isomorphism classes. That is $\mu, \nu \in \mathcal{N}$ are isomorphic if and only if there exists $g \in \mathrm{GL}_n(\mathbb{R})$ such that $g.\mu = \nu$. Further, from [**?** ], two nilpotent algebras are isometric if they are in the same $O(n)$ orbit. This perspective is especially useful because of the following fact:

$$(g.\mu, \langle\ ,\ \rangle) \text{ is isometric to } (\mu, \langle g\cdot, g\cdot\rangle).$$

So, instead of searching for inner products on a given Lie algebra, we may fix the inner product and vary the bracket structure isomorphically to find metrics of interest. This is the perspective we will take for the remainder of this work.

A necessary condition for the existence of a nilsoliton metric is the existence of a positive symmetric derivation as above. In practice, a nilpotent algebra may admit more than one symmetric derivation. A condition for determining which derivation is the candidate for the soliton derivation is given in ([**?** ], Theorem 1), as follows. We say a derivation $\phi$ of $\mathfrak{n}$ is a *pre-Einstein*, if it is semisimple, with all real eigenvalues and

$$\operatorname{tr}(\phi\psi) = \operatorname{tr}(\psi), \text{ for any } \psi \in \operatorname{Der}(\mathfrak{n}).$$

By ([**?** ], Theorem 1), every Lie algebra admits a pre-Einstein derivation which is unique up to conjugation. Further, if $\mathfrak{n}$ is a nilsoliton, its soliton derivation is a pre-Einstein derivation up to scaling and conjugation. Finding a pre-Einstein derivation of a given algebra greatly simplifies the search for a soliton metric, as shown in the following theorem, from ([**?** ], Theorem 3.2) and ([**?** ], Lemma 1).

**Theorem 1.1.** Let $\varphi$ be the pre-Einstein derivation of $\mathfrak{n}$, and identify $\mathfrak{n}$ with $\mu$. If $\mathfrak{n}$ admits a soliton, then there exists $g \in Z(\varphi)$, the centralizer of $\varphi$, such that $(g.\mu, \langle\ ,\ \rangle)$ is a soliton.

This theorem allows us to reduce our study of to so-called *admissible* algebras (cf. Definition 1.7), when testing to see if an algebra $\mathfrak{n}$ admits a soliton.

## 1.3. Filiform Lie Algebras

Here we give the basic definitions and set the notation for our study of Lie algebras. For more, we direct the reader to [**?** ]. Let $\mathfrak{g}$ be a Lie algebra. We define its *commutator series* $\mathfrak{g}^k$ by

$$\mathfrak{g}_0 = \mathfrak{g}, \qquad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}_k].$$

We say $\mathfrak{g}$ is *solvable* if, for some $k$, $\mathfrak{g}^k = 0$. Similarly, we define the *lower central series* $\mathfrak{g}_k$ by

$$\mathfrak{g}_0 = \mathfrak{g}, \qquad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k].$$

We say $\mathfrak{g}$ is *nilpotent* if, for some $k$, $\mathfrak{g}_k = 0$. Notice, nilpotent implies solvable. We say a nilpotent Lie algebra $\mathfrak{g}$ is $k$*-step* if $\mathfrak{g}_k = 0$, but $\mathfrak{g}_{k-1} \neq 0$. We say an $n$-dimensional Lie algebra is *filiform* if it's lower central series is as long as possible. That is, it is an $(n-1)$-step nilpotent Lie algebra. Here we give two examples of filiform algebras as well as a basis for their corresponding maximal torus of derivations.

**Example 1.2.** Let $\{X_1, ..., X_n\}$ be a basis for a Lie algebra with non-zero brackets given by $[X_1, X_i] = X_{i+1}$ for $i = 2, ..., n-1$, and antisymmetry. We denote this algebra $L_n$. It is the model space for filiform Lie algebras. A basis for the maximal torus of derivations is given by:

$$D_1 = \mathrm{diag}(0, 1, ..., 1)$$

$$D_2 = \mathrm{diag}(1, 2, ..., n)$$

Here we identify the derivation with its matrix associated to the basis given above.

**Example 1.3.** Let $\{X_1, ..., X_n\}$, with $n$ even, be a basis for a Lie algebra with non-zero brackets given by $[X_1, X_i] = X_{i+1}$ for $i = 2, ..., n-2$, and $[X_i, X_{n-i+1}] = (-1)^i X_n$ for

$i = 2, ..., n-1$ and antisymmetry. We denote this algebra $Q_n$. A basis for the maximal torus of derivations is given by:

$$D_1 = \mathrm{diag}(0, 1, ..., 1, 2)$$
$$D_2 = \mathrm{diag}(1, 2, ..., n-1, 0)$$

Again, we associate the derivation with its matrix associated to the basis given above.

Recall from Equation (1.3), a necessary condition for existence of a soliton metric is the presence of a positive symmetric derivation. As any filiform algebra is generated by 2 elements, its *rank* (the dimension of a maximal torus of derivations) is at most two. However, from ([**?** ], Théorème 4), for $n > 7$, in the variety of $n$-dimensional Lie algebras, there is a non-empty Zariski open set of *characteristically nilpotent*. That is, their derivation algebra is nilpotent, and thus do not admit any non-trivial semisimple derivations. Thus, we must only consider the case of rank 2 and rank 1, with structure results also from ([**?** ], Théorème 2). They are recorded in the theorems below. Note this result in [**?** ] gives another rank 1 filiform algebra, but its derivation it is not positive, and thus cannot admit a nilsoliton metric.

**Theorem 1.4.** Any rank 2 filiform algebra is isomorphic to $L_n$ or $Q_n$.

**Theorem 1.5.** Any rank 1 filiform algebra, with a postive derivation, is isomorphic to $A_{n,r}$ ($2 \leq r \leq n-3$) or $B_{n,r}$ ($n$ even, and $2 \leq r \leq n-4$), where $n, r$ are natural numbers, and the non-zero brackets, up to anti-symmetry, are given by:

$$\begin{array}{lll} A_{n,r}, 2 \leq r \leq n-3: & [X_1, X_i] = X_{i+1} & i = 2, ..., n-1 \\ & [X_i, X_j] = c_{i,j} X_{i+j+r-2} & i, j \geq 2, i+j+r-2 \leq n \end{array}$$

$$
\begin{aligned}
B_{n,r}, 2 \le r \le n-4: \quad & [X_1, X_i] = X_{i+1} && i = 2, \ldots, n-2 \\
& [X_i, X_j] = c_{i,j} X_{i+j+r-2} && i, j \ge 2, i+j+r-2 \le n-1 \\
& [X_i, X_{n-i+1}] = (-1)^i X_n && i = 2, \ldots, n-1
\end{aligned}
$$

The unique, up to scaling, semisimple derivations are given respectively by:

$$
\begin{aligned}
D_{A_{n,r}} &= \mathrm{diag}(1, r, r+1, \ldots, n+r-3, n+r-2) \\
D_{B_{n,r}} &= \mathrm{diag}(1, r, r+1, \ldots, n+r-3, n+2r-3)
\end{aligned}
$$

**Warning.** Theorem 1.5 is merely a 'description' of rank 1 filiform algebras, not a classification. It is not guaranteed that every object on this list is even a Lie algebra, as it may not satisfy the Jacobi condition. Further, there may be isomorphic Lie algebras with distinct descriptions.

**Remark.** The class of algebras $A_{n,r}$ is rich. For every pair $(n, r)$, in the appropriate range, $A_{n,r}$ is non-empty. For example,

$$
\begin{aligned}
\mu(X_1, X_i) &= X_{i+1} && i = 2, \ldots, n-1 \\
\mu(X_2, X_i) &= X_{i+r} && i = 3, \ldots, n-r
\end{aligned}
$$

However, every algebra in $B_{n,r}$ is a central extension of an algebra in $A_{n-1,r}$ which in general may not exist. For example, $A_{n,2}$ has multiple families of non-isomorphic Lie algebras in every dimension, whereas $B_{n,2}$ is empty for $n > 12$, as shown in ([**?** ], Theorem 2).

**Remark.** Since the rank one algebras only have one semisimple derivation, it is in fact the pre-Einstein derivation, thus simplifying the search for a soliton. Further, since the class of filiform algebras admitting positive derivations has such a strict form, we will exploit this to define our preferred inner product and vary the bracket accordingly.

**Definition 1.6.** Let $\mu \in A_{n,r}$ and we say the metric Lie algebra $(\mu, \langle\ ,\ \rangle)$ is *suitable* if the basis $\{X_i\}$ in Theorem 1.5 is orthonormal.

**Remark.** Notice, $\{X_i\}$ is a basis of eigenvectors for $D$ (which must of necessity be the pre-Einstein derivation), so $D$ is symmetric with respect to $\langle\ ,\ \rangle$. Thus, in the search for soliton, it is sufficient to restrict our search to $Z(D)$, motivating our next definition.

**Definition 1.7.** Let $\nu \in A_{n,r}$. We say the metric Lie algebra $(\nu, \langle\ ,\ \rangle)$ is *admissible* if there exists a $g \in Z(D)$ such that $\nu = g.\mu$, for some suitable metric Lie algebra $(\mu, \langle\ ,\ \rangle)$. When the metric is obvious, we will say $\mu$ is admissible.

**Remark.** As the Ricci tensor is symmetric, it it may be diagonalized. A important question is finding a basis which diagonalizes Ric. One answer, which is sufficient in our case is the so called 'nice' basis condition, as given in [**?** ], and ([**?** ], Definition 3).

**Definition 1.8.** Let $\mathfrak{n}$ be a nilpotent Lie algebra, and $\{X_i\}$ a basis of $\mathfrak{n}$, with $[X_i, X_j] = \sum_k \alpha_{ij}^k X_k$. The basis $\{X_i\}$ is called nice if for every $i, j$ $\#\{k | \alpha_{ij}^k \neq 0\} \leq 1$ and for every $i, k$ $\#\{j | \alpha_{ij}^k \neq 0\} \leq 1$.

**Lemma 1.9.** If $\{X_i\}$ is a nice basis for $(\mathfrak{n}, \langle\ ,\ \rangle)$, then ric is diagonal with respect to this basis.

*Proof.* The proof follows immediately by examining the formula for ric in Equation (1.1) and observing that the definition of a nice basis forces ric to be zero, unless $i = j$. That is,

$$\mathrm{ric}(X_i, X_j) = -\frac{1}{2}\sum_{kl} \alpha^l_{ik}\alpha^l_{jk} + \frac{1}{4}\sum_{kl} \alpha^i_{kl}\alpha^j_{kl}.$$

Thus, ric is diagonal with respect to this basis. In particular,

$$\mathrm{ric}(X_i, X_i) = -\frac{1}{2}\sum_{kl}(\alpha^l_{ik})^2 + \frac{1}{4}\sum_{kl}(\alpha^i_{kl})^2$$

*As Required.*

Finally, we observe that the rank 1 and rank 2 filiform algebras have a nice basis. Thus, as we study them, we need only calculate $\mathrm{ric}(X_i, X_i)$.

**Remark.** For a general nilpotent Lie algebra, with left-invariant metric, the geometry and the algebra are united by the following proposition, which we will exploit later.

**Proposition 1.10.** Let $(\mu, \langle\ ,\ \rangle)$ be a nilpotent metric Lie algebra. Then,

$$\mathrm{scal}_\mu = -\frac{1}{4}\|\mu\|^2$$

where $\|\mu\|$ is the norm on $V = \Lambda^2(\mathbb{R}^{n*}) \otimes \mathbb{R}^n$ induced from $\langle\ ,\ \rangle$.

*Proof.* By definition of scalar curvature and description of ric in Equation (1.1),

$$\begin{aligned}\mathrm{scal} &= \mathrm{tr}(\mathrm{Ric})\\ &= \sum_i \mathrm{ric}(X_i, X_i)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_i \sum_{jk} (\langle [X_i, X_j] X_k \rangle)^2 + \frac{1}{4}\sum_i \sum_{jk} (\langle [X_j, X_k] X_i \rangle)^2 \\
&= -\frac{1}{4}\|\mu\|^2
\end{aligned}
$$

The final equality follows from the fact that all the indices are present, so the second sum may be reindexed to match the first.

*As Required.*

## 1.4. Stability

The designation of the homogeneous metrics of interest as soliton metrics borrows from language common in the Ricci flow and dynamical systems. Ricci flow is a standard tool in geometry and topology, most notably, by Perelman in proof of of the Poincaré conjecture. In particular, Ricci flow is a dynamical system on the space of Riemannian metrics, of a given smooth manifold. The soliton metrics are precisely the ones which correspond to fixed points of this dynamical system, modulo diffeomorphism and dilation. In particular, if for a given manifold $M$, $\mathrm{ric} = \lambda g + \mathcal{L}_X g$, where $\mathcal{L}_X$ is the Lie derivative of the metric, we say $(M, g, \lambda, X)$ is a Ricci soliton.

One of the fundamental questions in dynamical systems and fixed point analysis is stability. That is, given a fixed point, is there a ball around the fixed point such that metrics within that ball converge to the fixed point? In this case, as in the study of differential equations, this reduces to a study of the eigenvalues of a linearizaton. We follow the notion of linear stability given in [**?** ] for a general manifold and then specify to nilsolitons. To motivate the study of linear stability, we give the result from ([**?** ], Theorem 1.2).

**Theorem 1.11.** Suppose $M$ is a simply connected solvable Lie group and $g$ is strictly linearly stable soliton. Then, for $R > 0$, $\rho \in (0, 1)$, there exists $\eta \in (\rho, 1)$ such that the following is true. There exists a neighborhood $U$ of $g$ in the $\mathfrak{h}^{1+\eta}(B_R)$-topology such that for all initial data $\tilde{g}(0) \in U$, the unique solution $\tilde{g}(t)$ of the curvature-normalized Ricci flow exists for all $t \geq 0$, and converges exponentially fast in the $\mathfrak{h}^{2+\rho}(B_R)$-norm to $g$.

At this point, we develop the notion of linear stability for a general manifold and then specify to a homogeneous space. Let $(M, g, \lambda, X)$ be a Ricci soliton, and $h$ a symmetric 2-tensor on $M$. The linearization of the flow is

$$\partial_t h = \mathbf{L}h := \Delta_L h + 2\lambda h + \mathcal{L}_X h$$

In this case $\Delta_L h$ is the Lichernerowicz Laplacian acting on symmetric 2-tensors. As we will not use them in the work, we refer the reader to ([**?** ], Chapter 9.3) for information.

$(M, g, \lambda, X)$ is said to be *strictly (resp. weakly) linearly stable*, if $\mathbf{L}$ has negative (resp. non-postive) spectrum. That is, there is an $\epsilon < 0$ (resp. $\epsilon = 0$) such that $(\mathbf{L}h, h) \leq -\epsilon \|h\|^2$ for all symmetric 2-tensors, where $(\cdot\,,\,\cdot)$ is the metric on the space of symmetric 2-tensors, induced by $g$.

To estimate this spectrum, we define the action of the curvature tensor, $\mathring{R}.\_$ on the space of symmetric two tensors via

$$(\mathring{R}.h)_{ij} = \sum_{pq} R_{ipqj} h^{pq}$$

Similarly, we define the action of the Ricci tensor, Ric._ on the space of symmetric two tensors via

$$(\mathrm{Ric}.h)_{ij} = \sum_k \mathrm{Ric}_i^k h_{kj} + \mathrm{Ric}_j^k h_{ki}$$

Combining these two expression, we define the following quadratic form on the space of two tensors,

$$Q(h) := ((\mathring{R} + \frac{1}{2}\mathrm{Ric}).h, h)$$

From ([**?** ], Proposition 1.5), this quadratic form may be used to estimate linear stability, as stated in the following proposition.

**Proposition 1.12.** Let $(M, g, \lambda, X)$ be a Ricci soliton with constant scalar curvature. The metric $g$ is strictly linearly stable, if

$$Q(h) < \frac{1}{2}\mathrm{div}(X)\|h\|^2$$

for all symmetric 2-tensors $h$.

**Remark.** For our purposes, we are only interested in homogeneous metrics, particularly nilpotent Lie groups, with left-invariant soliton metrics. Since we are interested in a point-wise approach, we have two simplifications. First, $\mathrm{div}(X) = \mathrm{tr}(D)$. Second, we need only consider the estimates for $Q$ with a single inner product on the metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$. Namely, $Q(h) = \langle(\mathring{R} + \frac{1}{2}\mathrm{Ric}).h, h\rangle$. With this, we may reformulate the above proposition as follows, which is given in ([**?** ], Corollary 2.12).

**Corollary 1.13.** Let $(\mathcal{S}, g, \lambda, D)$ be an algebraic Ricci soliton. $g$ is strictly linearly stable if

$$Q(h) < \frac{1}{2}\mathrm{tr}(D)|h|^2$$

for all symmetric 2-tensors $h$.

**Remark.** This will be the notion of stability we look to exploit in Chapter 3.

## 1.5. Matrix Norms and Eigenvalue Estimates

In the preceding section, we observed that stability may be determined through estimating a quadratic form. In particular, every quadratic form $Q$ has an associated symmetric linear operator, $\tilde{Q}$. Finding the maximum of a quadratic form is equivalent to determining the largest eigenvector of $\tilde{Q}$, which we denote $\rho(\tilde{Q})$. Once a basis is chosen, the largest eigenvector maybe estimated via analysis of the matrix coefficients. Note: The theory of estimating the maximum eigenvalue of a matrix is robust. We will sketch some of the theory, but refer the reader to ([**?** ], Chapter 4.2) for more information. The tool of choice for the remainder of this section will be a so-called *matrix norm*. We begin by recalling the definition of a vector norm.

**Definition 1.14.** Let $V$ be a vector space. We say $|\cdot| : V \to \mathbb{R}$ is a norm if it has the following properties:

1. $|v| \geq 0$ for every $v \in V$, with equality if and only if $v = 0$.
2. $|\lambda v| = |\lambda||v|$, for every $\lambda \in \mathbb{R}$, $v \in V$.
3. $|v + w| \leq |v| + |w|$, for every $v, w \in V$.

**Example.** A vector norm is a way of measuring the lengths of vectors. Common examples are the $\ell_p$ norms defined via:

$$|v|_p = \left( \sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}.$$

When $p = 2$, this is the usual Euclidean norm induced from the dot product.

**Remark.** In this case, we abuse the notation $|\cdot|$ to denote the absolute value of scalars and the norm of vector. However, it will be clear from context which is being denoted.

**Definition 1.15.** We say $\|\cdot\| : M_n(\mathbb{R}) \to \mathbb{R}$ is a Matrix Norm if it has the following properties:

1. $\|A\| \geq 0$ for every $A \in M_n(\mathbb{R})$, with equality if and only if $A = 0$.
2. $\|\lambda A\| = |\lambda|\|A\|$, for every $\lambda \in \mathbb{R}$, $A \in M_n(\mathbb{R})$
3. $\|A + B\| \leq \|A\| + \|B\|$, for every $A, B \in M_n(\mathbb{R})$
4. $\|AB\| \leq \|A\|\|B\|$, for every $A, B \in M_n(\mathbb{R})$

**Remark.** Be advised that some references use the name 'matrix norm' for properties (1-3), and use the term 'subadditive matrix norm' to refer for a norm a satisfying (1-4). The utility of matrix norms for our purpose of estimating eigenvalues becomes clear with the following lemma.

**Lemma 1.16.** Let $\|\cdot\|$ be a matrix norm and $A \in M_n(\mathbb{R})$, then $\rho(A) \leq \|A\|$, where $\rho(A)$ is the largest eigenvalue of $A$.

*Proof.* Let $A \in M_n(\mathbb{R})$ be a matrix and $x \neq 0$, an eigenvector with eigenvalue $\rho(A)$. Consider the matrix $X$ in which every column is $x$, that is

$$X := \left[ x \middle| \cdots \middle| x \right]$$

Then, $AX = \rho(A)X$. So,

$$|\rho(A)|\|X\| = \|\rho(A)X\| = \|AX\| \leq \|A\|\|X\|.$$

As $\|X\| > 0$, $\rho(A) \leq \|A\|$ *As Required.*

One way to gain examples of a matrix norms is to induce them from vector norms. That is, given an innner product space $V$ with vector norm $|\cdot|$, we may consider a matrix $A$ as an operator on $V$, and define the operator norm $\|\cdot\|$ induced from $|\cdot|$ by

$$\|A\| := \sup_{x \neq 0} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax|.$$

The first 3 properties of a matrix norm are trivial. Property 4 follows from

$$|Ax| \leq \|A\||x| \Rightarrow |ABx| \leq \|A\||Bx| \leq \|A\|\|B\||x|$$

**Lemma 1.17.** Let $\|\cdot\|$ be the operator norm induced from the $\ell_1$ norm. Then,

$$\|A\| = \max_l \sum_{k=1}^n |A_{kl}|.$$

*Proof.* Observe,

$$\begin{aligned}
\|A\| &= \sup_{|x|=1} |Ax| \\
&= \sup_{|x|=1} \sum_{k=1}^n |(Ax)_k| \\
&= \sup_{|x|=1} \sum_{k=1}^n \left| \sum_{l=1}^n A_{kl}x_l \right| \\
&\leq \sup_{|x|=1} \sum_{l=1}^n |x_l| \left( \sum_{k=1}^n |A_{kl}| \right)
\end{aligned}$$

$$\leq \sup_{|x|=1} \sum_{l=1}^{n} |x_l| \max_l \left( \sum_{k=1}^{n} |A_{kl}| \right)$$
$$= \max_l \left( \sum_{k=1}^{n} |A_{kl}| \right)$$

The first 3 equalities are by definition. The next inequality come from the triangle inequality and the distributivity of sums. The next from taking the max and the final one from the fact that $|x| = 1$.

If we take $e_L$ to be the basis element corresponding to the largest column. That is, $\left( \sum_{k=1}^{n} |A_{kL}| \right) = \max_l \left( \sum_{k=1}^{n} |A_{kl}| \right)$, then

$$\max_l \left( \sum_{k=1}^{n} |A_{kl}| \right) = |Ae_L| \leq \sup_{|x|=1} |Ax| = \|A\|$$

The result follows from combining these two inequalities. *As Required.*

**Remark.** The norm $\|A\|$ in the previous proof is often called the *column norm* of a matrix, which is the name we will refer to it by.

# Chapter 2

# Filiform Nilsolitons

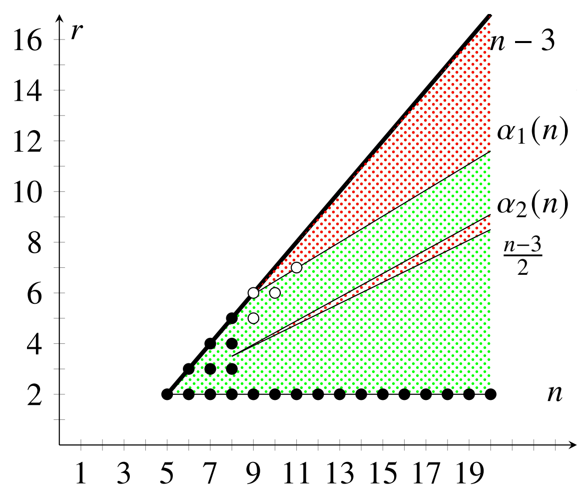
In this chapter, we turn our attention to our first question. Namely, "Which filiform nilpotent Lie algebras admit soliton metrics?" We begin with an account of the current state of the field.

From [**?** ], Filiform algebras trifurcate in rank 0, 1, 2. Rank 0 algebras do not admit solitons, as they are characteristically nilpotent, and thus do not have a positive derivation. Rank 2 has two classes: $L_n$ and $Q_n$. Their soliton metrics are given in [**?** ] and [**?** ], respectively. We will examine these metrics when we explore their stability in Chapter 3.

In rank 1, there are two families $A_{n,r}$, $B_{n,r}$ (denoted $A_r$, $B_r$, respectively, in [**?** ], [**?** ], [**?** ], and elsewhere. cf. Section 1.3 and Theorem 1.5 for their definition and properties). Using the classification of nilpotent Lie algebras in dimensions less than 6, [**?** ] shows that all such filiform algebras admit solitons and gives their metrics. Similarly, using the classification of nilpotent Lie algebras in dimension 7, [**?** ] classifies which filiform nilpotent Lie algebras admit solitons, giving metrics for the ones which are not in a one parameter family. Further, [**?** ] classifies 8 dimensional filiform algebras, and determines which of these admit a soliton and which do not, though metrics are not calculated. This exhausts the low dimensional examples currently in the literature.

In higher dimensions, [**?** ] considers the classes $A_{n,2}$ and $B_{n,2}$. Of the families in $A_{n,2}$, only one of them admits a soliton in each dimension. Meanwhile, $B_{n,2}$ contains only finitely many algebras, each of which admit a soliton. The major obstacle to these approaches is they rely heavily on the existence of classifications for filiform algebras, which do not exist for $n > 9$, or for $r > 3$. The results presented here are novel, in that they avoid this trouble, by leveraging only the relationship between $(n, r)$, and circumvent the need for classification results. With this, we present the main theorem of the chapter.

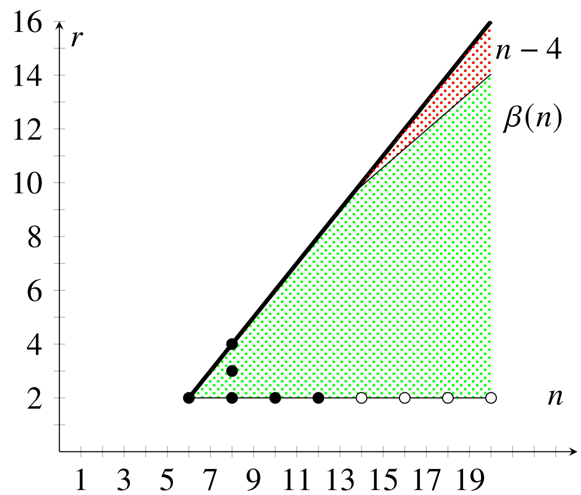
**Theorem 2.11.** Suppose $n > 8$, and $\mu \in A_{n,r}$. There exists functions $\alpha_1 > \alpha_2$ of $n$ such that if $\frac{n-3}{2} \leq r \leq \alpha_2(n)$ or $\alpha_1(n) \leq r \leq n-3$, then $\mu$ does not admit a soliton.




The curves $\alpha_1, \alpha_2$ are curves of solutions to cubic equations, which depend on $n$, so their formulas, given in Equations (2.2) and (2.3) are unwieldy. However, $\alpha_1(n)$ is asymptotic to $\frac{n}{\sqrt{3}}$, and $\alpha_2(n)$ is asymptotic to $\frac{n}{2}$.

In this picture, the black dots represent pairs $(n, r)$ which from [**?** ], [**?** ], [**?** ], and [**?** ] admit soliton metrics. That is, there is an algebra in $A_{n,r}$ which admits a soliton. However, there may be several other isomorphism classes which do not, as in ([**?** ], Chapter 4). The clear dots represent specific combinations of $(n, r)$ for which no algebra in $A_{n,r}$ admits a soliton. The red regions are pairs of $(n, r)$ which this theorem shows 'cannot' admit solitons, while in the green regions the question of existence is still open. We further give results in the special case of $A_{n,n-3}, A_{n,n-4}$ and show that for these maximal values of $r$, there are no solitons for $n \geq 8$.

**Theorem 2.23.** Suppose $n > 8$ is even, $\nu \in B_{n,r}$. There exists a function $\beta$ such that if $\beta(n) \leq r \leq n-4$, then $\nu$ does not admit a soliton.




As in the previous picture, the black dots represent pairs $(n, r)$ which, from [**?** ],[**?** ], and [**?** ], which have an algebra that admits a soliton. The results of [**?** ] do not apply as $n$ must be even. The clear dots represent specific combinations of $(n, r)$ for which no algebra in $B_{n,r}$ admits a soliton. The red regions are pairs of $(n, r)$ which this theorem shows 'cannot' admit solitons, while in the green regions the question of existence is open.

## 2.1. Finding Nilsolitons on Filiform Lie Algerbas

The theory for finding solitons on filiform algebras is rich. In fact, given a particular filiform algebra, the existence of a nilsoliton is actually a fairly simple linear problem. In what follows, we sketch some of the ideas critical to our study.

Lemma 1.1 says in the search for a soliton, it is sufficient to consider the centralizer of the pre-Einstein derivation, which we denote by $D$, in what follows. From Theorem 1.5, the pre-Einstein derivation is simple. That is, there are no-repeating eigenvalues. Thus, the eigenspaces are all one dimensional. Therefore, $Z(D) = (\mathbb{R}^{\times})^n$, which is identified with the invertible diagonal matrices. Equivalently, the diagonal matrices with no zero entries on the diagonal. In this case, the following lemma, shows that the action of a diagonal matrix on the bracket has a simple form.

**Lemma 2.1.** Let $\mu \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. If $g \in \mathrm{GL}_n(\mathbb{R})$ is diagonal, then the structure constants of $g.\mu$ are scalar multiples of the structure constants of $\mu$.

*Proof.* Let $\beta_{ij}^k$ be the structure constants associated to $g.\mu$ and $\alpha_{ij}^k$ the structure constants associated to $\mu$, and let $\{X_i\}$ be an orthonormal basis associated to $\langle\ ,\ \rangle$. Then, as $g$ is diagonal, $gX_i = g_i X_i$ and $g^t = g$. Thus,

$$\begin{aligned}\beta_{ij}^k &= \langle g.\mu(X_i, X_j), X_k\rangle \\ &= \langle \mu(g^{-1}X_i, g^{-1}X_j), g^t X_k\rangle \\ &= \langle \mu(\frac{1}{g_i}X_i, \frac{1}{g_j}X_j, g_k X_k\rangle \\ &= \frac{g_k}{g_i g_j}\langle \mu(X_i, X_j), X_k\rangle \\ &= \frac{g_k}{g_i g_j}\alpha_{ij}^k\end{aligned}$$

*As Required.*

When one is studying a particular filiform algebra determining whether or not it admits a soliton is a fairly simple exercise thanks to the following notions from [**?** ].

Let $e_i$ be a basis of eigenvectors for the pre-einstein derivation $\varphi$, and $\alpha_{ij}^k$ the stucture constants with respect to this basis. For computational purposes, we consider a fresh $\mathbb{R}^n$ with the inner product $(\cdot\,,\cdot)$, and orthonormal basis $f_1, ..., f_n$, define the subset $\mathbf{F} = \{\gamma_{ij}^k = f_i + f_j - f_k | \alpha_{ij}^k \neq 0\}$ and let $L$ be the affine span of $\mathbf{F}$, the smallest affine subspace of $R^n$ containing $\mathbf{F}$.

**Theorem 2.2** (Theorem 1 of [**?** ])**.** Let $\mathfrak{n}$ be a nilpotent Lie algebra whose pre-Einstein derivation has all eigenvalues simple. $\mathfrak{n}$ is an Einstein nilradical if and only if the projection to the origin of $\mathbb{R}^n$ to $L$ lies in the interior of the convex hull of $\mathbf{F}$.

This theorem maybe reformulated in terms of a linear problem. Denote $N = \#\mathbf{F}$, $[1]_N = (1, 1, ..., 1) \in R^N$ , and let $Y$ be a $N \times n$ matrix whose vector rows are $\beta_{ij}^k$ in some order.

**Corollary 2.3** (Corollary 1 of [**?** ])**.** A nilpotent Lie algebra $\mathfrak{n}$ whose pre-Einstein derivation has all its Eigenvalues simple is an Einstein nilradical if and only if there is a vector $v \in \mathbb{R}^N$ all of whose coordinates are positive such that $YY^t = [1]_N$

For an algebra $\mathfrak{n}$, the matrix $Y$ is called the root matrix. The matrix $YY^t$ is called the Gram matrix and is denoted by $U$.

## 2.2. The Non-Existence on Solitons in $A_{n,r}$

In this section, we prove Theorem 2.11, which shows the non-existence of a soliton when $n, r$ are in particular range.

As discussed previously in Section 1.2, we think of a Lie bracket as living inside the vector space $V = \Lambda^2(\mathbb{R}^{n)} * \otimes \mathbb{R}^n$ Thus, for $n \geq 5$, $\mu \in A_{n,r} \subset V$, we have:

$$
\begin{array}{lll}
A_{n,r}, 2 \leq r \leq n-3: & \mu(X_1, X_i) = X_{i+1} & i = 2, ..., n-1 \\
& \mu(X_i, X_j) = c_{i,j} X_{i+j+r-2} & i, j \geq 2, i + j + r - 2 \leq n
\end{array}
$$

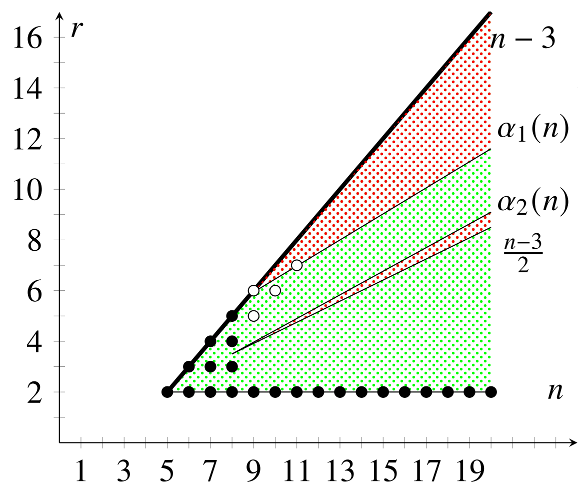
Throughout this section, we will assume $\mu \in A_{n,r}$ is accompanied with an inner product such that $(\mu, \langle\ ,\ \rangle)$ is admissible (cf. Defintion 1.7).

One feature of these rank 1 filiform algebras designating them as prime objects of study for soliton metric is that they admit only one symmetric derivation, up to scaling, as shown in Theorem 1.5. Thus, it is the Pre-Einstein derivation in the sense of from ([**?** ]). Namely:

$$D = \mathrm{diag}(1, r, r+1, r+2, ..., n+r-2)$$

Therefore, if an algebra admits a soliton metric, this $D$ must be the soliton derivation, up to scaling. Acting on the bracket by $g = \lambda I$ scales $D$ by a factor of $\frac{1}{\lambda^2}$. In this case, we fix the scale so that the derivation is exactly what is listed above. Lastly, in [**?** ], [**?** ] and [**?** ], it is determined which filiform algebras admit a soliton for $n \leq 8$. So, we restrict to $n > 8$. From Theorem 1.1, since $D$ is symmetric with no repeating eigenvalues, all the elements in the centralizers of $D$ are diagonal. Thus, this change of basis amounts to rescaling the structure constants, as shown in Lemma 2.1. To this end, we obtain a description of all possible Ricci tensors given by rescaling structure constants. The following theorem gives a condition which precludes the existence of a soliton. This culminates in the following theorem, which we now set about to prove.

**Theorem 2.11.** Suppose $n > 8$, and $\mu \in A_{n,r}$. There exists functions $\alpha_1 > \alpha_2$ of $n$ such that if $\frac{n-3}{2} \leq r \leq \alpha_2(n)$ or $\alpha_1(n) \leq r \leq n-3$, then $\mu$ does not admit a soliton.




This picture, as described in the Introduction and preamble to this section, gives a graphical representation of the regions where the existence of solitons is impossible.

**Remark.** The main idea of the proof is that we are able to write the scalar curvature as a linear combination of Ric, up to an error term, which must be positive. We then leverage the fact that the Pre-Einstein derivation is known, and calculable, to derive a contradiction. We now build up a sequence of technical lemmas to that end. Note that throughout, $\mu$ is not a soliton, unless explicitly specified.

**Notation.** Suppose $\mu \in A_{n,r}$ and $(\mu, \langle\ ,\ \rangle)$ is an admissible metric Lie algebra. While formulas for the Ricci tensor may be given in terms of structure constants $\alpha_{i,j}^k$, for clarity of presentation, we designate them as follows:

$$c_i := \langle [X_1, X_i], X_{i+1}\rangle = \langle \mu(X_1, X_i), X_{i+1}\rangle = \alpha_{1,i}^{i+1} \qquad \text{for } i = 2, \ldots, n-1$$

$$d_{i,j} := \langle [X_i, X_j], X_{i+j+r-2}\rangle = \langle \mu(X_i, X_j), X_{i+j+r-2}\rangle = \alpha_{i,j}^{i+j+r-2} \qquad \text{for } 2 \le i, j, \; i+j+r-2 \le n$$

As $(\mu, \langle\ ,\ \rangle)$ is admissible, these are the the only non-zero structure constants. When $r = 2$, there is potential overlap with $c_i$ and $d_{1,i}$. So, we further define that $d_{1,i} = 0$, and thus $d_{j,1} = 0$, by antisymmetry. Similarly, $c_1 = c_n = 0$ and $d_{i,j} = 0$, if $i + j + r - 2 > n$.

**Lemma 2.4.** Suppose $\mu \in A_{n,r}$ is admissible. Then, $c_i \neq 0$ for $i = 2, \ldots, n-1$.

*Proof.* Recall, $\mu$ admissible implies there is a suitable $(\nu, \langle\ ,\ \rangle)$ (cf. Definition 1.6) such that $\mu = g.\nu$ for some $g \in Z(D) = (\mathbb{R}^\times)^n$. Thus, analagous to the computation in Lemma 2.4,

$$c_i = \mu(X_1, X_i), X_{i+1}\rangle = \langle g.\nu(X_1, X_i), X_{i+1}\rangle = \frac{g_{i+1}}{g_1 g_i}\langle \nu(X_1, X_i), X_{i+1}\rangle = \frac{g_{i+1}}{g_1 g_i}$$

This is only zero if $g_{i+1} = 0$. However, $g_{i+1} \in \mathbb{R}^\times$. Thus, $c_i \neq 0$.

*As Required.*

**Remark.** Observe, if $d_{i,j} = 0$ for every $i, j$, then the algebra is isomorphic to $L_n$, and thus not in $A_{n,r}$. Thus, at least on $d_{i,j}$ must be non-zero.

We now describe the Ricci tensors of admissible metric Lie algebras.

**Lemma 2.5.** Suppose $\mu \in A_{n,r}$ is admissible,

$$\mathrm{ric}_\mu(X_i, X_i) = \begin{cases} -\frac{1}{2}\sum_{j=1}^{n}(c_j)^2 & \text{for } i = 1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2) - \frac{1}{2}\sum_{j=2}^{n} d_{i,j}^2 + \frac{1}{4}\sum_{j=2}^{n} d_{j,i-j-r+2}^2 & \text{for } i = 2, \dots, n \end{cases}$$

*Proof.* Recall, from Equation (1.1), for a nilpotent Lie algebra,

$$\mathrm{ric}_\mu(X, Y) = -\frac{1}{2}\sum_{jk}\langle \mu(X, X_j), X_k\rangle\langle \mu(Y, X_j), X_k\rangle + \frac{1}{4}\sum_{jk}\langle \mu(X_j, X_k), X\rangle\langle \mu(X_j, X_k), Y)\rangle$$

Recall, from Section (1.3), this basis is nice, so we need only the case where $X = Y$. In particular,

$$\begin{aligned} \mathrm{ric}_\mu(X_1, X_1) &= -\frac{1}{2}\sum_{jk}\langle \mu(X_1, X_j), X_k\rangle^2 + \frac{1}{4}\sum_{jk}\langle \mu(X_j, X_k), X_1\rangle^2 \\ &= -\frac{1}{2}\sum_{jk} c_j^2 \langle X_{j+1}, X_k\rangle \\ &= -\frac{1}{2}\sum_{jk} c_j^2 \delta_{j+1,k} \\ &= -\frac{1}{2}\sum_{j} c_j^2 \end{aligned}$$

The first equality follows by definition, the second by definition of $c_j$ and that fact that $X_1 \perp [\mathfrak{n}, \mathfrak{n}]$. For $i > 1$, the bracket relations above simplify the computations significantly. Observe,

$$\mathrm{ric}_\mu(X_i, X_i) = -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \mu(X_i, X_j), X_k)\rangle^2 + \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \mu(X_j, X_k), X_i\rangle^2$$

$$= -\frac{1}{2}\left(\langle \mu(X_i, X_1, X_k\rangle^2 + \sum_{j=2}^{n} \langle \mu(X_i, X_j), X_{i+j+r-2})\rangle^2\right)$$

$$+ \frac{1}{4}\left(\langle \mu(X_1, X_{i-1}), X_i\rangle^2 + \langle \mu(X_{i-1}, X_1), X_i\rangle^2 + \sum_{j=2}^{n} \langle \mu(X_j, X_{i-j-r+2}), X_i\rangle^2\right)$$

$$= -\frac{1}{2}\left((-c_i)^2 + \sum_{j=2}^{n} d_{i,j}^2\right) + \frac{1}{4}\left(c_{i-1}^2 + (-c_{i-1})^2 + \sum_{j=2}^{n} d_{i,i-j-r+2}^2\right)$$

$$= \frac{1}{2}(c_{i-1}^2 - c_i^2) - \frac{1}{2}\sum_{j=2}^{n} d_{i,j}^2 + \frac{1}{4}\sum_{j=2}^{n} d_{j,i-j-r+2}^2$$

*As Required.*

When $r$ is large, the formula for ric simplifies, as shown in the next lemma.

**Lemma 2.6.** If $\mu \in A_{n,r}$ is admissible, and $\frac{n-3}{2} < r \leq n-3$.

$$\mathrm{ric}_\mu(X_i, X_i) = \begin{cases} -\frac{1}{2}\sum_{j=1}^{n}(c_j)^2 & \text{for } i = 1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2) - \frac{1}{2}\sum_{j=2}^{n} d_{i,j}^2 & \text{for } i = 2, ..., n-r \\ \frac{1}{2}(c_{i-1}^2 - c_i^2) & \text{for } i = n-r+1, ..., r+2 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2) + \frac{1}{4}\sum_{j=2}^{n} d_{j,i-j-r+2}^2 & \text{for } i = r+3, ..., n \end{cases}$$

*Proof.* The condition that $\frac{n-3}{2} < r$ insures that $n - r < r + 3$. Note that for the purposes of this lemma, it is acceptable that $\{n - r + 1, ..., r + 2\} = \emptyset$. That is, $n - r + 1 = r + 3$. We use the expression for ric obtain in the previous lemma.

Suppose $i > n-r$, $j \geq 2$. Then, $i+j+r-2 > n+j-2 \geq n$. Thus, $d_{i,j} = 0$. So, $\sum_{j=2}^{n} d_{i,j}^2 = 0$.

Likewise, suppose $i < r+2$, and $j \geq 2$. Then $i-j-r+2 < -j+4 \leq 2$. Thus, $i-j-r+2 = 1$.

Since $d_{j,1} = 0$, $d_{j,i-j-r+2} = 0$. Thus, $\sum\limits_{j=2}^{n} d^2_{j,i-j-r+2} = 0$.

In the case $i = r + 2$, this necessitates $j = 1, 2, 3$. If $j = 1$, $d_{1,i-j-r+2} = 0$. If $j = 2$, $i - j - r + 2 = 2$, so $d_{j,i-j-r+2} = d_{2,2} = 0$, by antisymmetry. If $j = 3$, $i - j - r + 2 = 1$, and $d_{i,1} = 0$. Thus, $\sum_{j=2}^{n} d^2_{j,i-j-r+2} = 0$. *As Required.*

**Lemma 2.7.** Let $\mu \in A_{n,r}$ be admissible. If $\frac{n-3}{2} < r$, and $k \in \{n - r + 1, ..., r + 3\}$, then

$$\mathrm{ric}_{11} - \sum_{i=k}^{n} \mathrm{ric}_{ii} = \mathrm{scal} - \frac{1}{2} c^2_{k-1}$$

*Proof.* Recall, from Proposition 1.10, $\mathrm{scal}_\mu = -\frac{1}{4}\|\mu\|^2$, for nilpotent Lie algebras. Calculating, with the formula for Lemma 2.6, we have

$$\mathrm{ric}_{11} - \sum_{i=k}^{n} \mathrm{ric}_{ii} = -\frac{1}{2} \sum_{i=1}^{n} c_i^2 - \frac{1}{4} \sum_{i=k}^{n} \sum_{j=2}^{n} d^2_{j,i-j-r+2} - \frac{1}{2} c^2_{k-1}$$

The norm of the bracket is the sum over all the structure constants. In our case, we have defined $c_i := \alpha^{i+1}_{1,i} = -\alpha^{i+1}_{i,1}$. Thus,

$$\frac{1}{2} \sum_{i=1}^{n} c_i^2 = \frac{1}{4} \sum_{i=1}^{n} (\alpha^{i+1}_{1,i})^2 + \frac{1}{4} \sum_{i=1}^{n} (\alpha^{i+1}_{i,1})^2$$

Observe, all of the non-zero $d_{i,j}$ occur in the second part of the sum. Notice, that $j$ is free, so all that is left, is to ensure that every $l = i - j - r + 2$ occurs. This occurs when

$$5 \leq i + (j - i - r + 2) \leq n - r + 2 \iff r + 3 \leq i \leq n$$

The choice of $n - r + 1 \leq k \leq r + 3$ means that that all of these $i$ occur, with no repetition. The final $\frac{1}{2} c^2_{k-1}$ follows from the telescoping of the $c_i^2$ in the formula for ric, and $c_n = 0$.

*As Required.*

**Corollary 2.8.** Let $\mu \in A_{n,r}$ be admissible. Let $\frac{n-3}{2} < r$, and $k \in \{n-r+1, ..., r+3\}$, then

$$\mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii} > 0$$

*Proof.* Rearranging the expression in Lemma 2.7, yields

$$\mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii} = \frac{1}{2} c_{k-1}^2.$$

Applying Lemma 2.4, yields the result. *As Required.*

**Remark.** This corollary gives a strong necessary condition the structure of an algebra. We will leverage this in the proof of our main theorem. At this point, we assume the existence of a soliton and derive explicit formulae for each of these terms. For ease of notation, we denote the above quantity of interest by $\mathrm{scal}_\mu^k$. That is, we define

$$\mathrm{scal}_\mu^k := \mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii}.$$

**Lemma 2.9.** If $\mu \in A_{n,r}$ is a soliton where $\mathrm{Ric} = cI + D$, then

$$c = -\frac{(n-1)(2n^2 + n(6r-7) + 6(r-1)^2) + 6}{3((n-1)(n+2r-2)+2)}$$

*Proof.* Let $D_\mu$ be the soliton derivation for $\mathrm{Ric}_\mu$. Recall, from Equation (1.4), $c = -\frac{\mathrm{tr}(D^2)}{\mathrm{tr}(D)}$. In this case,

$$\mathrm{tr}(D) = 1 + \sum_{i=2}^{n} (i+r-2) = 1 + \frac{1}{2}(n-1)(n+2r-2)$$
$$\mathrm{tr}(D^2) = 1 + \sum_{i=2}^{n} (i+r-2)^2 = 1 + \frac{1}{6}(n-1)(2n^2 + n(6r-7) + 6(r-1)^2)$$

Thus, the expression for $c$ follows. *As Required.*

**Lemma 2.10.** If $\mu \in A_{n,r}$ is admissible and a soliton. Then for $k > 1$,

$$\mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{kk} = c(2n-k) + \frac{1}{2}\big((n-1)(n+2r-2) + (n-k+1)(k+n+2r-4)\big)$$

*Proof.* Since $\mu$ is a soliton, $\mathrm{ric}_{ii} = c + D_{ii}$. That is,

$$\mathrm{ric}_{ii} = \begin{cases} c+1 & \text{for } i = 1 \\ c+i+r-2 & \text{for } i = 2, \dots, n \end{cases}$$

The following computations were done with a computer algebra system, see Appendix B for more information.

$$\begin{aligned} \mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{kk} =& cn + \mathrm{tr}(D) - (c+1) + \sum_{i=k}^{n} (c+i+r-2) \\ =& cn + 1 + \frac{1}{2}(n-1)(n+2r-2) - (c+1) \\ & + (n-k+1)c + \frac{1}{2}(n-k+1)(k+n+2r-4) \\ =& c(2n-k) \\ & + \frac{1}{2}\big((n-1)(n+2r-2) + (n-k+1)(n+k+2r-4)\big) \end{aligned}$$

*As Required.*

We now turn to the proof the main theorem.

**Theorem 2.11.** Suppose $n > 8$, and $\mu \in A_{n,r}$. There exists functions $\alpha_1 > \alpha_2$ of $n$ such that if $\frac{n-3}{2} \le r \le \alpha_1(n)$ or $\alpha_2(n) \le r \le n-3$, then $\mu$ does not admit a soliton.

*Proof.* Suppose $\mu \in A_{n,r}$ such that $\frac{n-3}{2} \le r \le n-3$, and that $\mu$ admits a soliton. By choosing the basis as in Theorem 1.5, and applying Lemma 1.1, it is sufficient to consider the case where $\mu$ is admissible.

Since, $\frac{n-3}{2} \leq r$, we may apply Corollary 2.8. Thus, for $k \in \{n-r+1, ..., r+3\}$,

$$\mathrm{scal}^k = \mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{kk} > 0.$$

We declare $k = n-r+1$, and use the previous lemmas to derive conditions for its negativity. The reason for the choice of $k$ to be minimal in the lemma is that $\mathrm{ric}_{kk} < 0$, for small $k$, thus this choice optimizes the contradiction. Since $\mu$ is a soliton, $\mathrm{tr}(D) > 0$. Thus, $\mathrm{scal}_\mu^k > 0$ if and only if $2\mathrm{tr}(D)\mathrm{scal}_\mu^k > 0$. Applying Lemma 2.10, and computing, yields

$$\begin{aligned} 2\mathrm{tr}(D)\mathrm{scal}_\mu^k &= 2\mathrm{tr}(D)\left(\mathrm{scal} - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{kk}\right) \\ &= -2\mathrm{tr}(D^2)(n+r-1) + \mathrm{tr}(D)\big((n-1)(n+2r-2) + r(2n+r-3)\big) \\ &= 4 - \frac{14n}{3} + \frac{n^2}{6} + \frac{2n^3}{3} - \frac{n^4}{6} + \left(\frac{n^3}{3} - \frac{3n^2}{2} + \frac{37n}{6} - 8\right)r \\ &\quad + \left(\frac{n^2}{2} - \frac{n}{2} + 1\right)r^2 - (n-1)r^3. \end{aligned} \tag{2.1}$$

Again, note that Corollary 2.8 says that if $\mu$ admits a soliton, then Equation (2.1) must be positive. For ease of notation, we define

$$f_n(r) = 4 - \frac{14n}{3} + \frac{n^2}{6} + \frac{2n^3}{3} - \frac{n^4}{6} + \left(\frac{n^3}{3} - \frac{3n^2}{2} + \frac{37n}{6} - 8\right)r + \left(\frac{n^2}{2} - \frac{n}{2} + 1\right)r^2 - (n-1)r^3$$

Notice, each $n$ gives a cubic in $r$ with discriminant:

$$\begin{aligned} \mathrm{disc}(f_n) = \frac{1}{432}(&4n^{10} - 340n^9 + 11931n^8 - 99528n^7 + 399642n^6 - 896088n^5 \\ &+ 1263635n^4 - 1443764n^3 + 1616844n^2 - 1318464n + 470016), \end{aligned}$$

which is positive for $n > 0$. Thus there are 3 real roots for each $n$. These may be found using the cubic formula, which yields:

$$\alpha_1(n) = \frac{n^2-n+2}{6(n-1)} - \frac{A}{9\cdot 2^{1/3}(n-1)\sqrt[3]{B+\sqrt{4A^3+B^2}}} + \frac{\sqrt[3]{B+\sqrt{4A^3+B^2}}}{18\cdot 2^{1/3}(n-1)}$$

$$\alpha_2(n) = \frac{n^2-n+2}{6(n-1)} - \frac{(1-i\sqrt{3})A}{18\cdot 2^{1/3}(n-1)\sqrt[3]{B+\sqrt{4A^3+B^2}}} + \frac{(1+i\sqrt{3})\sqrt[3]{B+\sqrt{4A^3+B^2}}}{36\cdot 2^{1/3}(n-1)}$$

$$\alpha_3(n) = \frac{n^2-n+2}{6(n-1)} + \frac{(1+i\sqrt{3})A}{18\cdot 2^{1/3}(n-1)\sqrt[3]{B+\sqrt{4A^3+B^2}}} - \frac{(1-i\sqrt{3})(\sqrt[3]{B+\sqrt{4A^3+B^2}}}{36\cdot 2^{1/3}(n-1)}$$

where

$$A = -45n^4 + 216n^3 - 873n^2 + 1566n - 900$$

$$B = -594n^6 + 3564n^5 + 2592n^4 - 50760n^3 + 116154n^2 - 109836n + 39312$$

These $\alpha_1, \alpha_2$ are as in the statement of the theorem, while $\alpha_3(n) < 0$. To show this, consider $f_n(r)$ along the curves, $r = n-3, r = \frac{11n}{20}, r = \frac{n-3}{2}, r = 0, r = -n$. This will allow us to determine where the curves of zeros lie.

$$f_n(n-3) = \frac{30 + 37n - 50n^2 + 14n^3 - n^4}{3}$$

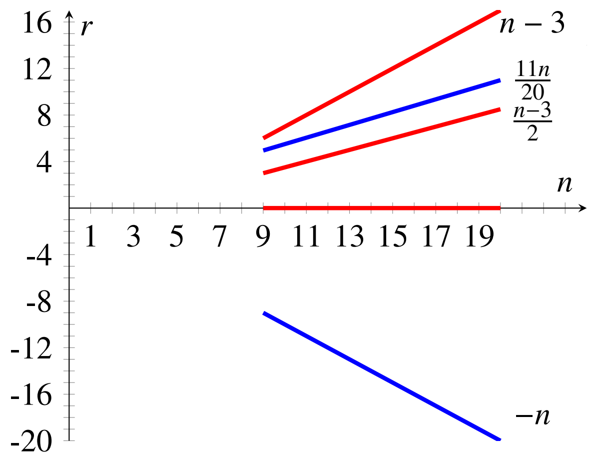
$$f_n\left(\frac{11n}{20}\right) = \frac{96000 - 217600n + 92660n^2 - 3437n^3 + 37n^4}{24000}$$

$$f_n\left(\frac{n-3}{2}\right) = \frac{357 - 331n + 75n^2 - 5n^3}{24}$$

$$f_n(0) = \frac{24 - 28n + n^2 + 4n^3 - n^4}{6}$$

$$f_n(-n) = \frac{12 + 10n - 15n^2 + 2n^3 + 3n^4}{3}$$

It's quick to check that for $n > 8$, $f_n(n-3) < 0$, $f_n(\frac{11n}{20}) > 0$, $f_n(\frac{n-3}{2}) < 0$, $f_n(0) < 0$, and $f_n(-n) > 0$. This is illustrated in the following graph, where red and blue represents that, on that curve, $f_n(r) < 0$, $f_n(r) > 0$, respectively.




Let $r_{n,1} < r_{n,2} < r_{n,3}$ be the 3 real roots for a given $n$. A quick application of the Intermediate Value Theorem shows

$$-n < r_{n,1} < 0 < \frac{n-3}{2} < r_{n,2} < \frac{11n}{20} < r_{n,3} < n-3.$$

Evaluating at $n = 9$, yields

$$\alpha_3(9) < 0 < \frac{n-3}{2} < \alpha_2(9) < \alpha_1(9) < n-3$$

Thus, by continuity,

$$\alpha_1(n) = \frac{n^2-n+2}{6(n-1)} - \frac{A}{9\cdot 2^{1/3}(n-1)\sqrt[3]{B+\sqrt{4A^3+B^2}}} + \frac{\sqrt[3]{B+\sqrt{4A^3+B^2}}}{18\cdot 2^{1/3}(n-1)} \tag{2.2}$$

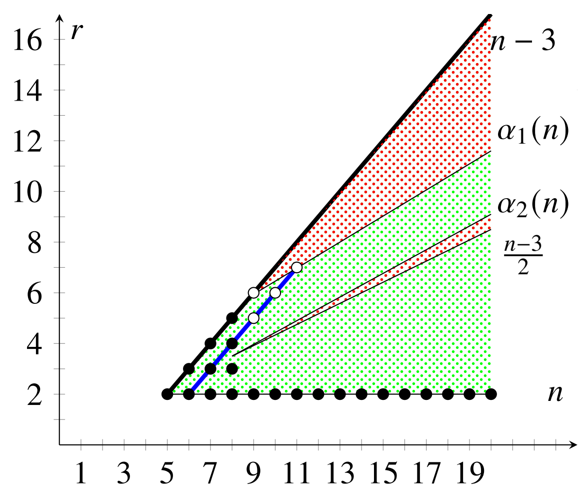
$$\alpha_2(n) = \frac{n^2-n+2}{6(n-1)} - \frac{(1-i\sqrt{3})A}{18\cdot 2^{1/3}(n-1)\sqrt[3]{B+\sqrt{4A^3+B^2}}} + \frac{(1+i\sqrt{3})\sqrt[3]{B+\sqrt{4A^3+B^2}}}{36\cdot 2^{1/3}(n-1)} \tag{2.3}$$

are as in the statement of the theorem.

*As Required.*

**Remark.** Though the formulas for $\alpha_1, \alpha_2$ are not illuminating in and of themselves, through the use of a computer algebra system, it's easy to show that $\alpha_1(n)$ is asymptotic to $\frac{n}{2}$, and $\alpha_2(n)$ is asymptotic to $\frac{\sqrt{3}n}{3}$. See Appendix B for information on the function used.

**Remark.** We now turn our attention to the particular strips $r = n-3$ and $r = n-4$. Solving $f_n(n-3), f_n(n-4)$ explicitly allows us to classify all filiform nilsolitons on these lines, as shown in the graph below.




To demonstrate these methods, we first show there is only one isomorphism class in each dimension. These algebraic results follow the proofs in the 8 dimensional case of ([**?** ], Lemmas 3.5,3.6).

**Lemma 2.12.** If $\lambda \in A_{n,n-3}$, then $\lambda$ is isomorphic to $\mu$, where, up to anti-symmetry, the non-zero brackets are

$$\mu(X_1, X_i) = X_{i+1} \text{ for } i = 2, \ldots n-1$$

$$\mu(X_2, X_3) = X_n$$

*Proof.* Recall, from Theorem 1.5, for $\lambda \in A_{n,n-3}$,

$$\lambda(X_1, X_i) = X_{i+1} \text{ for } i = 2, \ldots n-1$$
$$\lambda(X_2, X_3) = aX_n$$

It is quick to observe that $g = (1, \frac{1}{a}, \ldots, \frac{1}{a})$ gives $g.\lambda = \mu$. *As Required.*

**Lemma 2.13.** If $\lambda \in A_{n,n-4}$, then $\lambda$ is isomorphic to $\mu$, where, up to antisymmetry, the non-zero brackets are

$$\mu(X_1, X_i) = X_{i+1} \text{ for } i = 2, \ldots n-1$$
$$\mu(X_2, X_3) = X_{n-1}, \mu(X_2, X_4) = X_n$$

*Proof.* Recall, from Theorem 1.5, for $\lambda \in A_{n,n-4}$,

$$\lambda(X_1, X_i) = X_{i+1} \qquad \text{for } i = 2, \ldots n-1$$
$$\lambda(X_2, X_3) = aX_{n-1} \qquad \lambda(X_2, X_4) = bX_n$$

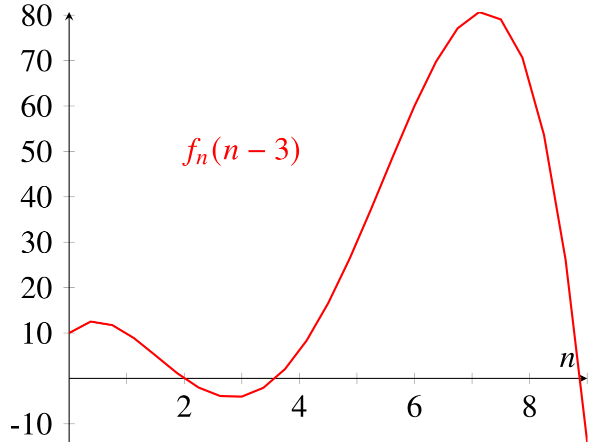
Applying the Jacobi identity to $X_1, X_2, X_3$, yields $a = b$. As in previous lemma, $g = (1, \frac{1}{a}, \ldots, \frac{1}{a})$ gives $g.\lambda = \mu$. *As Required.*

**Theorem 2.14.** Let $\mu \in A_{n,n-3}$. $\mu$ admits a soliton if and only if $5 \leq n \leq 8$.

*Proof.* Recall, the course of the proof of Theorem 2.11, we calculated

$$f_n(n-3) = -\frac{n^4}{3} + \frac{14n^3}{3} - \frac{50n^2}{3} + \frac{37n}{3} + 10$$

a simple analysis of critical values of this function shows that is negative for $n > 8$. Thus, from Corollary 2.8, $\mu$ cannot admit a soliton for $n > 8$. The graph of this function is seen here.




To show existence, we note that nilsolitons metrics are classified and given for dimension 5,6 in [**?** ]. In that work, the unique algebra in $A_{5,2}$ is denoted $N^\circ = 2$ on *Table 3* and the unique algebra in $A_{6,3}$ is denoted $N^\circ = 5$ on *Table 4*. Dimension 7 is given in [**?** ], and corresponds to (142). We also give the metric for dimension 8, which to this point has not appeared in the literature, though existence is shown in [**?** ], where it is denoted by $\mathfrak{h}_1(8)$. Note, these metrics are scaled so that $D = \mathrm{diag}(1, n-3, ..., 2n-5)$, so they may differ by a factor from the reference.

$$n = 5 : c_2 = 2, c_3 = 2, c_4 = \frac{2\sqrt{3}}{3}, d_{2,3} = \frac{2\sqrt{3}}{3}$$

$$n = 6 : c_2 = \frac{2\sqrt{91}}{13}, c_3 = \frac{\sqrt{390}}{13}, c_4 = \frac{6\sqrt{13}}{13}, c_5 = \frac{4\sqrt{13}}{13}, d_{2,3} = \frac{\sqrt{390}}{13}$$

$$n = 7 : c_2 = \sqrt{2}, c_3 = \sqrt{2}, c_4 = \frac{3\sqrt{10}}{5}, c_5 = \frac{4\sqrt{5}}{5}, c_6 = \frac{2\sqrt{5}}{5},$$
$$d_{2,3} = \frac{3\sqrt{10}}{5}$$

$$n = 8 : c_2 = \frac{\sqrt{570}}{19}, c_3 = \frac{\sqrt{418}}{19}, c_4 = \frac{\sqrt{1406}}{19}, c_5 = \frac{2\sqrt{418}}{19} c_6 = \frac{8\sqrt{19}}{19},$$
$$c_7 = \frac{\sqrt{38}}{19}, d_{2,3} = \frac{7\sqrt{38}}{19}$$

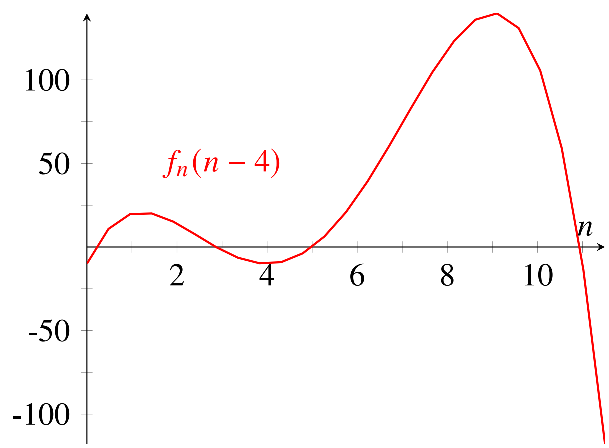
Since there is only one algebra in each dimension and $r = n - 3 \geq 2$, this completes the list. *As Required.*

**Theorem 2.15.** Let $\mu \in A_{n,n-4}$. $\mu$ admits a soliton if and only if $6 \leq n \leq 8$

*Proof.* In the course of the proof of Theorem 2.11, we computed $f_n(r)$. Now, we evaluate at $r = n - 4$.

$$f_n(n-4) = -\frac{n^4}{3} + \frac{19n^3}{3} - \frac{104n^2}{3} + \frac{176n}{3} - 12$$

Examining the graph shows that $f_n(n-4) < 0$ for $n \geq 11$.




This puts a very sharp bound on the dimensions which can admit solitons. However, examining the remain dimension yields bounds which are still sharper still. We exploit the results from Corollary 2.3, and apply them here.

Let $U_n$ be the Gram matrix of the unique algebra in $A_{n,n-4}$. Where $U_n = Y_n^t Y_n$ and $Y_n$ is the root matrix $A_{n,n-4}$. This yields:

$$U_9 = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 3 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}, U_{10} = \begin{bmatrix} 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 3 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 3 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 3 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

However, the equations $U_9 v_9 = [1], U_{10} v_{10} = [1]$ do not have solutions with postive coordinates as

$$v_9 = \begin{bmatrix} 0 \\ \frac{-1}{9} + x_9 \\ \frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9} \\ \frac{1}{9} \\ \frac{1}{3} - x_9 \\ \frac{4}{9} - x_9 \\ x_9 \end{bmatrix} \qquad v_{10} = \begin{bmatrix} \frac{-63}{703} \\ \frac{-184}{703} + x_{11} \\ \frac{-3}{703} \\ \frac{120}{703} \\ \frac{5}{19} \\ \frac{192}{703} \\ \frac{141}{703} \\ \frac{32}{703} \\ \frac{225}{703} - x_{11} \\ \frac{360}{703} - x_{11} \\ x_{11} \end{bmatrix}$$

From Corollary 2.3 this means $A_{9,5}, A_{10,4}$ do not admit solitons.

To show the solitons exist in the asserted dimensions, we note that nilsolitons metrics are classified and given for dimension 6 in [**?** ]. In that work, the unique algebra $A_{6,2}$ is denoted by $N^\circ = 4$. The unique algebra in $A_{7,3}$ is given in [**?** ], and is (144) in the list. However there is an error in the metric given, as listed, the structure constant corresponding to $d_{2,3} = \frac{\sqrt{5124}}{122}$. In actuality, with respect to that normalization $d_{2,3} = \frac{\sqrt{1281}}{122}$. We also give the metric for dimension 8, which to this point has not appeared in the literature, though existence is shown in [**?** ], where it corresponds to to the algebra $\mathfrak{d}_1(8)$. Note that structure constants given below are normalized so that $D = (1, , n-4, ...., 2n-6)$. Thus, the constants may be off by a factor from the reference.

$$
\begin{aligned}
n = 6 :\ & c_2 = \sqrt{22}, c_3 = 6, c_4 = \sqrt{22}, c_5 = \sqrt{30}, \\
& d_{2,3} = \sqrt{30}, d_{2,4} = 5 \\
n = 7 :\ & c_2 = \frac{2\sqrt{85}}{17}, c_3 = \frac{\sqrt{714}}{17}, c_4 = \frac{6\sqrt{17}}{17}, c_5 = \frac{4\sqrt{34}}{17}, c_6 = \frac{6\sqrt{17}}{17}, \\
& d_{2,3} = \frac{\sqrt{714}}{17}, d_{2,4} = \frac{6\sqrt{17}}{17} \\
n = 8 :\ & c_2 = \frac{3\sqrt{2}}{5}, c_3 = \frac{\sqrt{267330}}{335}, c_4 = \frac{\sqrt{58}}{5}, c_5 = \frac{4\sqrt{5}}{5}, c_6 = \frac{2\sqrt{13}}{5}, c_7 = \frac{8\sqrt{3819}}{335}, \\
& d_{2,3} = \frac{7\sqrt{7370}}{335}, d_{2,4} = \frac{8\sqrt{5159}}{335}
\end{aligned}
$$

Since there is only one algebra in each dimension, and $r = n - 4 \geq 2$, this completes the proof.

*As Required.*

## 2.3. The Non-Existence of Solitons on $B_{n,r}$

In this section, we turn our attention to the other family of Rank 1 filiform nilpotent Lie algebras, $B_{n,r}$. We apply the approach from Section 2.2 to prove Theorem 2.23. This theorem is analogous to Theorem 2.11, and the outline of the exposition is the same. While the results are analogous, there is one discrepancy, which we analyze at presenting the proof of Theorem 2.23.

Again, we think of the Lie bracket as living inside the vector space $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ with $n$ even. For $n \geq 6$, $\nu \in B_{n,r} \subset V$, we have:

$$\begin{array}{lll} B_{n,r}, 2 \leq r \leq n-4: & \nu(X_1, X_i) = X_{i+1} & i = 2, \ldots, n-2 \\ & \nu(X_i, X_j) = c_{i,j} X_{i+j+r-2} & i, j \geq 2, i+j+r-2 \leq n-1 \\ & \nu(X_i, X_{n-i+1}) = (-1)^i X_n & i = 2, \ldots, n-1 \end{array}$$

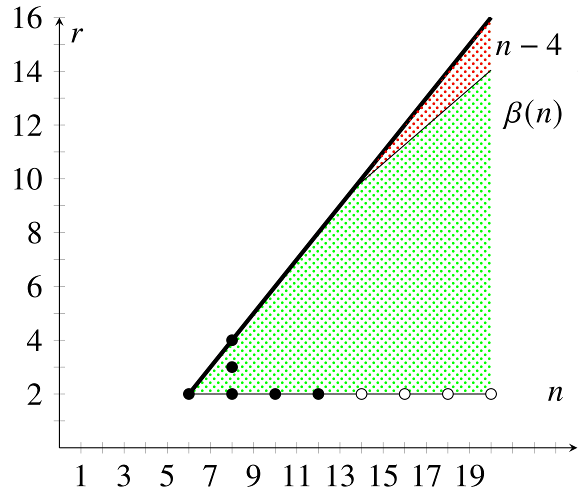
Throughout this section, we will assume $\nu \in B_{n,r}$ is accompanied with an inner product so that $(\nu, \langle\ ,\ \rangle)$ is admissible (cf. Definition 1.7).

Up to scaling, the only positive derivation of $B_{n,r}$ is

$$D = \text{diag} = (1, r, r+1, \ldots, n+r-3, n+2r-3),$$

which of necessity must be the pre-Einstein derivation. Again, as in Section 2.2, we fix the scale so that $D$ is in fact the pre-Einstein derivation, and we restrict our interest to $n > 8$, as $n \leq 8$ is well understood. The following theorem, analogous to Theorem 2.11, gives conditions on $(n, r)$, which preclude the existence of a soliton.

**Theorem 2.23.** Suppose $n > 8$, $\nu \in B_{n,r}$. There exists a function $\beta$ such that if $\beta(n) \leq r \leq n-4$, then $\nu$ does not admit a soliton.




This picture, as described in the Introduction and preamble to this section, gives a graphical representation of the regions where the existence of solitons is impossible.

**Remark.** The proof again proceeds as a series of lemmas which allows us write scalar curvature as a linear combination of Ric, up to an error term, which must be positive, and then relate this the pre-Einstein derivation $D$ to derive a contradiction. First, we make an observation that will allow us reuse our computations for $A_{n,r}$.

Observe that the definition $B_{n,r}$ is very similar to the defintion of $A_{n,r}$. In fact, viewed as a element of $V$, $\nu$ splits into a $A_{n-1,r}$ piece and a 'Heisenberg' piece in a way we now make precise. Let $\nu \in B_{n,r}$ with basis as above. Let $\mu, \lambda \in V$, with non-zero brackets given by:

$$\mu(X_1, X_i) := \nu(X_1, X_i) = X_{i+1} \qquad i = 2, \ldots, n-2$$

$$\mu(X_i, X_j) := \nu(X_i, X_j) = c_{i,j} X_{i+j+r-2} \qquad i, j \geq 2, i+j+r-2 \leq n-1$$

$$\lambda(X_i, X_{n-i+1}) := \nu(X_i, X_{n-i+1}) = (-1)^i X_n \qquad i = 2, \ldots, n-1.$$

As elements of $V$, $\nu = \mu + \lambda$, by definition. Denote the restriction of $\mu$ to the $\mathbb{R}$ span of $\{X_1, ..., X_{n-1}\}$ by $\mu'$, and the restriction of $\lambda$ to the $\mathbb{R}$ span of $\{X_2, ..., X_n\}$ by $\lambda'$. Then, $\mathfrak{n}_\mu \simeq \mathfrak{n}_{\mu'} \oplus \mathbb{R}$, $\mathfrak{n}_\lambda \simeq \mathbb{R} \oplus \mathfrak{n}_{\lambda'}$ where $\mathfrak{n}_\mu$ is the Lie algebra associated to the bracket $\mu$, and $\mathbb{R}$ is the abelian factor associated to $X_1, X_n$ respectively. In particular, $\mathfrak{n}_{\mu'} \in A_{n-1,r}$ and $\mathfrak{n}_{\lambda'} \simeq \mathrm{Heis}_{n-1}$, the $(n-1)$-dimensional Heisenberg algebra. The utility of this observation is given in the following lemma.

**Lemma 2.16.** Let $\lambda, \mu \in V$, and $\{X_i\}$ be a basis which is nice with respect to $\lambda, \mu$, abd $\lambda + \mu$. Let $\Psi_\mu = \{(i,j,k) | \langle \mu(X_i, X_j), X_k \rangle \neq 0\}$. If $\Psi_\lambda \cap \Psi_\mu = \emptyset$, $\mathrm{ric}_{\lambda+\mu} = \mathrm{ric}_\lambda + \mathrm{ric}_\mu$.

*Proof.* Recall, $\mathrm{ric}_\mu$ may be formally defined, as in Equation (1.1), by

$$\begin{aligned}\mathrm{ric}_\mu(X,Y) = &-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \mu(X, X_j), X_k\rangle\langle \mu(Y, X_j), X_k\rangle \\ &+ \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \mu(X_j, X_k), X\rangle\langle \mu(X_j, X_k), Y\rangle\end{aligned}$$

Since we are considering nice bases, it is sufficient to verify this identity on $\mathrm{ric}_\mu(X_i, X_i)$.

$$\mathrm{ric}_\mu(X_i, X_i) = -\frac{1}{2}\sum_{jk}\langle \mu(X_i, X_j), X_k\rangle^2 + \frac{1}{4}\sum_{jk}\langle \mu(X_j, X_k), X_i\rangle^2$$

Calculating on a sum of brackets yields,

$$\begin{aligned}\mathrm{ric}_{\mu+\lambda}(X_i, X_i) = &-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle (\mu+\lambda)(X_i, X_j), X_k\rangle^2 + \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle (\mu+\lambda)(X_j, X_k), X_i\rangle^2 \\ = &-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\Big(\langle \mu(X_i, X_j), X_k\rangle^2 + 2\langle(\mu(X_i, X_j), X_k\rangle\langle(\lambda(X_i, X_j), X_k\rangle \\ &+ \langle \lambda(X_i, X_j), X_k\rangle^2\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\Big(\langle\mu(X_j,X_k),X_i\rangle^2+2\langle(\mu(X_j,X_k),X_i\rangle\langle\lambda(X_j,X_k),X_i\rangle \\
&\qquad\qquad +\langle\lambda(X_j,X_k),X_i\rangle^2\Big) \\
=&\mathrm{ric}_\mu(X_i,X_i)\rangle+\mathrm{ric}_\lambda(X_i,X_i) \\
&-\sum_{j=1}^{n}\sum_{k=1}^{n}\Big(\langle(\mu(X_i,X_j),X_k\rangle\langle(\lambda(X_i,X_j),X_k\rangle \\
&\qquad\qquad +\frac{1}{2}\langle(\mu(X_j,X_k),X_i\rangle\langle\lambda(X_j,X_k),X_i\rangle\Big) \\
=&\mathrm{ric}_\mu(X_i,X_i)\rangle+\mathrm{ric}_\lambda(X_i,X_i)
\end{aligned}
$$

The last equality follows from $\Psi_\mu \cap \Psi_\lambda = \emptyset$. *As Required.*

For clarity of presentation, we define, as before:

$$
\begin{aligned}
c_i &:= \langle\nu(X_1,X_i),X_{i+1}\rangle && i=2,\dots,n-2 \\
d_{i,j} &:= \langle\nu(X_i,X_j),X_{i+j+r-2}\rangle && i,j\geq 2, i+j+r-2\leq n-1 \\
b_i &:= \langle\nu(X_i,X_{n-i+1}),X_n\rangle && i=2,\dots,n-1
\end{aligned}
$$

Similarly, we define $c_i = d_{i,j} = b_i = 0$, when $i, j$ are outside of the ranges listed above.

**Lemma 2.17.** Let $\lambda \in V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ be given by $\lambda(X_i, X_{n-i+1}) = b_i X_n$. Then,

$$
\mathrm{ric}_\lambda(X_i,X_i)=\begin{cases} 0 & \text{for } i=1 \\ -\frac{1}{2}b_i^2 & \text{for } i=2,\dots,n-1 \\ \frac{1}{4}\sum_{j=2}^{n-1} b_j^2 & \text{for } i=n \end{cases}
$$

*Proof.* Notice, $X_i$ is a nice basis for $\lambda$, so we need only compute $\mathrm{ric}_\lambda(X_i, X_i)$. Recall, from

Equation (1.1), $\mathrm{ric}_\lambda$ is given by

$$\mathrm{ric}_\lambda(X_i, X_i) = -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \mu(X_i, X_j), X_k\rangle^2 + \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \mu(X_j, X_k), X_i\rangle^2,$$

For $i = 1$, $\lambda(X_1, X_i) = 0$, by definition. Further, $\langle \lambda(X_i, X_j), X_1\rangle = 0$, as $X_1 \perp X_n$, thus $\mathrm{ric}_\lambda(X_1, X_1) = 0$.

For $i = 2, \ldots, n-1$,

$$\begin{aligned}
\mathrm{ric}_\lambda(X_i, X_i) &= -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \lambda(X_i, X_j), X_k\rangle^2 + \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \lambda(X_j, X_k), X_i\rangle^2 \\
&= -\frac{1}{2}\langle \lambda(X_i, X_{n-i-1}), X_n\rangle^2 \\
&= -\frac{1}{2}b_i^2
\end{aligned}$$

The positive sum vanishes because, for $i \neq n$, $X_i \perp [\mathfrak{n}, \mathfrak{n}]$.

For $i = n$, recall $\lambda(X_n, X_i) = 0$. Thus,

$$\begin{aligned}
\mathrm{ric}_\lambda(X_i, X_i) &= -\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \lambda(X_n, X_j), X_k\rangle^2 + \frac{1}{4}\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \lambda(X_j, X_k), X_n\rangle^2 \\
&= \frac{1}{4}\sum_{j=1}^{n}\langle \lambda(X_j, X_{n-i+1}), X_n\rangle^2 \\
&= \frac{1}{4}\sum_{j=2}^{n-1} b_j^2
\end{aligned}$$

This completes the proof. *As Required.*

**Corollary 2.18.** Let $\nu \in B_{n,r}$ be admissible.

$$\mathrm{ric}_\nu(X_i, X_i) = \begin{cases} -\frac{1}{2}\sum\limits_{j=2}^{n-2} c_j^2 & \text{for } i = 1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2 - b_i^2) - \frac{1}{2}\sum\limits_{j=2}^{n-1} d_{i,j}^2 + \frac{1}{4}\sum\limits_{j=2}^{n-1} d_{j,i-j-r+2}^2 & \text{for } i = 2, \ldots, n-1 \\ \frac{1}{4}\sum\limits_{j=2}^{n-1} b_j^2 & \text{for } i = n \end{cases}$$

*Proof.* Notice, $\nu = \mu + \lambda$, where $\mu \in A_{n-1,r}$, and $\lambda$ is as above. Further, upon inspection, $\Psi_\mu \cap \Psi_\lambda = \emptyset$. Applying Lemma 2.16 to the formulae for ric in Lemmas 2.5 and 2.17, yields the formula above. *As Required.*

**Corollary 2.19.** Let $\nu \in B_{n,r}$ be admissible. If $\frac{n-4}{2} < r \leq n-4$,

$$\mathrm{ric}_\nu(X_i, X_i) = \begin{cases} -\frac{1}{2}\sum\limits_{j=2}^{n-2} c_j^2 & \text{for } i = 1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2 - b_i^2) - \frac{1}{2}\sum\limits_{j=2}^{n-1} d_{i,j}^2 & \text{for } i = 2, \ldots, n-r-1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2 - b_i^2) & \text{for } i = n-r, \ldots, r+2 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2 - b_i^2) + \frac{1}{4}\sum\limits_{j=2}^{n-1} d_{j,i-j-r+2}^2 & \text{for } i = r+3, \ldots, n-1 \\ \frac{1}{4}\sum\limits_{j=2}^{n-1} b_j^2 & \text{for } i = n \end{cases}$$

*Proof.* Notice, $\nu = \mu + \lambda$, where $\mu \in A_{n-1,r}$, and $\lambda$ is as above. Again, $\Psi_\mu \cap \Psi_\lambda = \emptyset$. Applying 2.16 the formulae for ric as in Lemmas 2.6 and 2.17, yields the formula above. *As Required.*

**Lemma 2.20.** Let $\nu \in B_{n,r}$ be admissible. Then, for $k \in \{n-r, ..., r+3\}$,

$$\mathrm{scal}_\nu - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii} + 2\mathrm{ric}_{nn} \geq 0$$

*Proof.* The proof follows Lemma 2.7. The only difference is in the computation of the $b_i$ component, as $\mathrm{ric}_\nu$ (resp. $\mathrm{scal}_\nu$) split into the $\lambda, \mu$ piece. Here, we denote $\mathrm{ric}_{ii} = \mathrm{ric}_\nu(X_i, X_i)$. Notice,

$$\begin{aligned}
\mathrm{ric}_{11} - \sum_{i=k}^{n} \mathrm{ric}_{ii} - 2\mathrm{ric}_{nn} &= -\frac{1}{2}\sum_{j=2}^{n-2} c_j^2 - \sum_{i=k}^{n-1}\left(\frac{1}{2}(c_{i-1}^2 - c_i^2 - b_i^2) + \frac{1}{4}\sum_{j=2}^{n-1} d_{j,i-j-r+2}^2\right) - \frac{3}{4}\sum_{j=2}^{n-1} b_j^2 \\
&= -\frac{1}{2}\sum_{j=2}^{n-2} c_j^2 - \frac{1}{4}\sum_{i=k}^{n}\sum_{j=2}^{n-1} d_{j,i-j-r+2}^2 - \frac{1}{4}\sum_{j=2}^{n-1} b_j^2 - \frac{1}{2}\sum_{j=2}^{k-1} b_j^2 - \frac{1}{2}c_{k-1}^2 \\
&= \mathrm{scal}_\nu - \frac{1}{2}c_{k-1}^2 - \frac{1}{2}\sum_{j=2}^{k-1} b_j^2
\end{aligned}$$

The final equality here follows from $\mathrm{scal}_\nu = -\frac{1}{4}\|\nu\|^2$. Rearranging yields the result. See the proof of Lemma 2.7 for a thorough argument of the $c_i^2, d_{i,j}^2$ components of $-\frac{1}{4}\|\nu\|^2$, as the reasoning is identical. *As Required.*

**Remark.** As in Corollary 2.8, this gives a strong condition on the structure of an algebra in $B_{n,r}$. We will leverage this in the proof of our main theorem. At this point, we assume the existence of a soliton and use the pre-Einstein derivation to derive explicit formulae for each of these terms. For ease of notation, we denote the above quantity of interest by $\mathrm{scal}_\nu^k$. That is, we define

$$\mathrm{scal}_\nu^k := \mathrm{scal}_\nu - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii} + 2\mathrm{ric}_{nn}.$$

**Lemma 2.21.** If $\nu \in B_{n,r}$ is a soliton, where Ric $= cI + D$, then

$$c = -\frac{n^2(6r-9) + 2n^3 + n\left(6r^2 - 6r + 1\right) + 6\left(2r^2 - 6r + 5\right)}{3\left(n^2 + n(2r-3) + 2\right)}$$

*Proof.* Let $D$ be the soliton derivation for Ric. Recall from Equation (1.4), $c = -\frac{\operatorname{tr}(D^2)}{\operatorname{tr}(D)}$. In this case, $D = (1, r, r+1, \ldots, n+r-3, n+2r-3)$. Thus, using a computer algebra system, as outlined in Appendix B.

$$\begin{aligned}
\operatorname{tr}(D) &= \frac{1}{2}\left(n^2 + n(2r-3) + 2\right) \\
\operatorname{tr}(D^2) &= n^2\left(r - \frac{3}{2}\right) + \frac{n^3}{3} + n\left(r^2 - r + \frac{1}{6}\right) + 2r^2 - 6r + 5
\end{aligned}$$

Thus, the expression for $c$ follows. *As Required.*

**Lemma 2.22.** Let $\nu \in B_{n,r}$ be a soliton. For $k > 1$,

$$\operatorname{scal}_\nu^k = c(2n-k+2) + \frac{1}{2}\left((n^2 + n(2r-3) + 2) + (n-k)(k+n+2r-5) + 6(n+2r-3) - 2\right)$$

*Proof.* Since $\mu$ is a soliton, $\operatorname{ric}_{ii} = c + D_i$. That is,

$$\operatorname{ric}_{ii} = \begin{cases} c+1 & \text{for } i = 1 \\ c+i+r-2 & \text{for } i = 2, \ldots, n-1 \\ c+n+2r-3 & \text{for } i = n \end{cases}$$

The following computations were preformed via a computer algebra system. For more information, see Appendix B.

$$
\begin{aligned}
\mathrm{scal}_\nu^k &= \mathrm{scal}_\nu - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii} + 2\mathrm{ric}_{nn} \\
&= cn + \mathrm{tr}(D) - (c+1) + \sum_{i=k}^{n-1}(c+i+r-2) + 3(c+n+2r-3) \\
&= c(2n-k+2) + \mathrm{tr}(D) - 1 + \sum_{i=k}^{n-1}(i+r-2) + 3(n+2r-3) \\
&= c(2n-k+2) + \frac{1}{2}\big((n^2 + n(2r-3) + 2) \\
&\qquad\qquad + (n-k)(k+n+2r-5) + 6(n+2r-3) - 2\big)
\end{aligned}
$$

*As Required.*

**Theorem 2.23.** Suppose $n > 8$, and $\nu \in B_{n,r}$. There exists a function $\beta$, of $n$, such that if $\beta(n) \leq r \leq n-4$, then $\nu$ does not admit a soliton.

*Proof.* Suppose that $\nu \in B_{n,r}$, such that $\frac{n-4}{2} < r \leq n-4$, and that $\nu$ admits a soliton. Arguing as in the proof of Theorem 2.11, it is sufficient consider admissible metrics. Since $r > \frac{n-4}{2}$, we may apply Lemma 2.20. Thus for $k \in \{n-r, \ldots, r+3\}$

$$
\mathrm{scal}_\nu^k = \mathrm{scal}_\nu - \mathrm{ric}_{11} + \sum_{i=k}^{n} \mathrm{ric}_{ii} + 2\mathrm{ric}_{nn} \geq 0.
$$

We declare $k = n-r$, and use the previous lemmas to derive a contradiction. Since $\nu$ is a soliton, $\mathrm{tr}(D) > 0$. Thus, $\mathrm{scal}_\nu^k > 0$ if and only if $12\mathrm{tr}(D)\mathrm{scal}_\nu^k > 0$. Thus, applying Lemma 2.22, we have the following computation:

$$
\begin{aligned}
12\mathrm{tr}(D)\mathrm{scal}_\nu^k &= 12\mathrm{tr}(D)\Big(\mathrm{scal}_\nu - \mathrm{ric}_{11} + \sum_{i=k}^{n}\mathrm{ric}_{ii} + 2\mathrm{ric}_{nn}\Big)\\
&=12\mathrm{tr}(D)\Big(c(n+r+2) + \frac{1}{2}\big((n^2+n(2r-3)+2)+r(2n+r-5)\\
&\qquad\qquad + 6(n+2r-3)-2\big)\Big)\\
&\le -12\mathrm{tr}(D^2)(n+r+2)\\
&\quad + 6\mathrm{tr}(D)\left((n^2+n(2r-3)+2)+r(2n+r-5)+6(n+2r-3)-2\right)\\
&=-(6n+24)r^3 + \left(3n^2-3n+30\right)r^2 + \left(2n^3+9n^2-53n+126\right)r\\
&\qquad + \left(-n^4+10n^3-41n^2+116n-228\right)
\end{aligned}
$$

For clarity, we define

$$
\begin{aligned}
g_n(r) =& -(6n+24)r^3 + \left(3n^2-3n+30\right)r^2 + \left(2n^3+9n^2-53n+126\right)r\\
&+ \left(-n^4+10n^3-41n^2+116n-228\right)
\end{aligned}
$$

Observe, this is a cubic in $r$, with discriminant

$$
\begin{aligned}
\mathrm{disc}\big(g_n(r)\big) = 3(&4n^{10}+4084n^9-15525n^8-13704n^7+216054n^6-837612n^5\\
&+4339955n^4-12897856n^3+39597384n^2-33629760n-68397264)
\end{aligned}
$$

which is positive for $n>2$. Thus, for said $n$ there are 3 real roots, and we may apply the cubic formula to find them. The computations above and the following cubic solutions were aided via use of a computer algebra system, outlined in Appendix B. The three curves of solutions are:

$$\beta_1(n) = \frac{n^2 - n + 10}{6(n+4)} - \frac{A}{9\, 2^{2/3}(n+4)\sqrt[3]{\sqrt{4A^3 + B^2} + B}} + \frac{\sqrt[3]{\sqrt{4A^3 + B^2} + B}}{18\sqrt[3]{2}(n+4)} \tag{2.4}$$

$$\beta_2(n) = \frac{n^2 - n + 10}{6(n+4)} + \frac{\left(1 - i\sqrt{3}\right) A}{18\, 2^{2/3}(n+4)\sqrt[3]{\sqrt{4A^3 + B^2} + B}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{\sqrt{4A^3 + B^2} + B}}{36\sqrt[3]{2}(n+4)}$$

$$\beta_3(n) = \frac{n^2 - n + 10}{6(n+4)} + \frac{\left(1 + i\sqrt{3}\right) A}{18\, 2^{2/3}(n+4)\sqrt[3]{\sqrt{4A^3 + B^2} + B}} - \frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{\sqrt{4A^3 + B^2} + B}}{36\sqrt[3]{2}(n+4)}$$

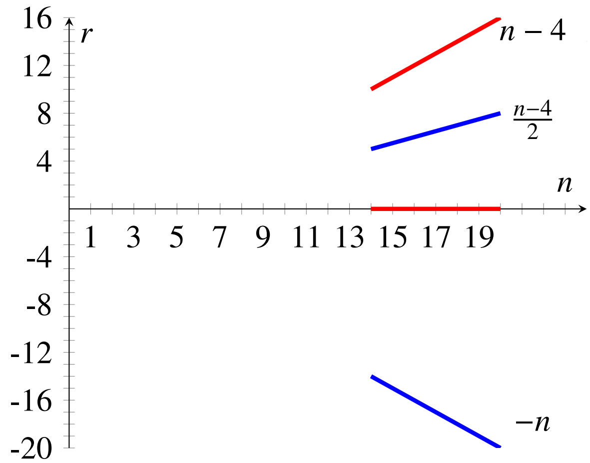
$$A = -45n^4 - 288n^3 + 117n^2 + 1728n - 9972$$

$$B = -2675376 - 206064n + 128628n^2 - 37476n^3 + 21870n^4 + 4212n^5 - 594n^6$$

We employ the same methods as in the proof of Theorem 2.11, to show that $\beta_1 = \beta$, as promised in the theorem. Evaluating along the curves $r = n - 4, r = \frac{n-4}{2}, r = 0, r = -n$, we find

$$g_n(n-4) = \frac{-n^4 + 16n^3 - 14n^2 - 301n + 642}{3}$$
$$g_n\left(\frac{n-4}{2}\right) = \frac{13n^3 - 80n^2 + 156n - 224}{8}$$
$$g_n(0) = \frac{24 - 28n + n^2 + 4n^3 - n^4}{6}$$
$$g_n(-n) = \frac{3n^4 + 11n^3 + 21n^2 - 5n - 114}{3}$$

Using computer graphing systems, as is Appendix B, it is quick to check that for $n \geq 14$, $g_n(n-3) < 0$, $g_n(\frac{n-4}{2}) > 0$, $g_n(0) < 0$, and $g_n(-n) > 0$. This is illustrated in the following graph, where red and blue represents that, on that strip $g_n(r) < 0, g_n(r) > 0$, respectively.




Let $r_{n,1} < r_{n,2} < r_{n,3}$ be the 3 real roots for a given $n$. A quick application of the Intermediate Value Theorem shows

$$-n < r_{n,1} < 0 < r_{n,2} < \frac{n-4}{2} < r_{n,3} < n-4.$$

Evaluating at $n = 14$, yields

$$-n < \beta_3(14) < 0 < \beta_2(14) < \frac{n-4}{2} < \beta_1(14) < n-4.$$

Thus, by continuity,

$$\beta = \beta_1(n) = \frac{n^2 - n + 10}{6(n+4)} - \frac{A}{9\,2^{2/3}(n+4)\sqrt[3]{\sqrt{4A^3 + B^2} + B}} + \frac{\sqrt[3]{\sqrt{4A^3 + B^2} + B}}{18\sqrt[3]{2}(n+4)}$$

is as in the statement of the theorem. *As Required.*

**Remark.** Using a computer algebra system, as in Appendix B, it is quick to show that $\beta$ is asymptotic to $\frac{n}{\sqrt{3}}$

**Remark.** Theorem 2.11 differs from Theorem 2.23 in that Theorem 2.11 has two regions which are excluded, where as Theorem 2.23 only has one. The difference is the fact that the curve $\beta_2(n)$ falls outside of the range $\{\frac{r-4}{2}, ..., r-4\}$, and thus the sequence of lemmas can't be applied to derive a contradiction. On the other hand, the curve $\alpha_2(n)$ from Theorem 2.11 happens to fall with the range $\{\frac{n-3}{2}, ..., n-3\}$.

**Remark.** The essential difference between the function $f_n(r)$ from the proof of Theorem 2.11 and $g_n(r)$ from 2.23 is the presence of the $2\mathrm{ric}_{nn}$ term. This is exactly what prevents $\beta_2$ from falling with in an applicable range. Future study may yield another clever expression of scal as a linear combination of Ric, which mitigates the effect of this term.

# Chapter 3

# Stability of Filiform Nilsolitons

In this chapter, we turn our attention from the existence of filiform nilsolitons, to their stability. Beyond the interest in their algebraic structure, Ricci solitons are, up to dilation and diffeomorphism, fixed points of the Ricci flow, one of the most important flows in geometric analysis (See [**?** ] or [**?** ] for more information). In fixed point analysis, one of the most natural questions is stability. More specifically, in our context, is there a neighborhood about the soliton metric such at all metrics in the neighborhood converge to the fixed point under the Ricci flow?

In the case of homogeneous algebraic nilsolitons, ([**?** ], Theorem 1.2) says that dynamical stability is equivalent to strict linear stability. Linear stability is defined in 1.4 and will be explored further in Section 3.1. From ([**?** ], Theorem 1.10), two-step nilsolitons are linearly stable. Our study is interested in stability at the 'other-end' of nilpotency. Namely, the filiform algebras.

In this chapter, we will choose a particular basis for the space of symmetric two tensors and apply machinery from matrix theory to estimate the eigenvalues of the linearization. This leads to the following theorem.

**Theorem 3.7.** Let $(N, \mathfrak{n}, \lambda, D)$ be a nilsoliton, with a fixed basis $\mathcal{B}$. If

$$2n \max_{ijkl} |R_{ijkl}| \Delta(\mathcal{B}, \mathfrak{n}) + \rho(\mathrm{Ric}) < \frac{1}{2}\mathrm{tr} D$$

then the soliton is strictly linearly stable.

**Notation.** In this theorem, we transition to the notation used in [**?** ] and [**?** ], where $N$ refers to the simply connected nilpotent Lie group with left-invariant soliton metric, $\mathfrak{n}$ is the Lie algebra of $N$ where the inner product is given by restriction, $\lambda$ is the soliton constant, and $D$ is the soliton derivation. At times, we will denote the nilsoliton by $(\mathfrak{n}, \langle\ ,\ \rangle)$, where $\mathfrak{n}$ is the Lie algebra, and $\langle\ ,\ \rangle$ is the soliton inner product.

**Remark.** We apply this theorem to the case of Rank 2 filiform solitons and exploit the fact that the metrics are known to derive stability as stated in the following theorem.

**Theorem 3.12.** The nilsoliton metric on $L_n$ is strictly linearly stable.

**Theorem 3.17.** The nilsoliton metric on $Q_n$ is strictly linearly stable.

The following theorem is a corollary to these results, and ([**?** ], Theorem 1.2).

**Theorem 3.18.** Rank 2 filiform nilsolitons are dynamically stable.

## 3.1. Nilsoliton Stability Approximations

Recall, from Section 1.4, a sufficient condition for the soliton to be strictly linearly stable is

$$Q(h) < \frac{1}{2}\mathrm{tr}(D)|h|^2, \tag{3.1}$$

where

$$Q(h) := ((\mathring{R} + \frac{1}{2}\mathrm{Ric}).h, h).$$

In this case, we are interested in nilpotent Lie groups with a left-invariant metric. In particular, filiform algebras. We declare the basis, $\{X_i\}$, with respect to which we defined the structure constants to be orthonormal. Since the basis we are taking is orthonormal, changing the tensor type from upper to lower indices does not change the coefficients. Thus, we may write, the actions of the Curvature and Ricci tensors as follows:

$$(\mathring{R}.h)_{ij} = (\mathring{R}.h)(X_i, X_j) = \sum_{p,q} R_{ipqj}h^{pq} = \sum_{p,q} R_{ipqj}h_{pq}$$
$$(\mathrm{Ric}.h)_{ij} = (\mathrm{Ric}.h)(X_i, X_j) = \sum_{k} \mathrm{Ric}_i^k h_{kj} + \mathrm{Ric}_j^k h_{ki} = \sum_{k} \mathrm{Ric}_{ik}h_{kj} + \mathrm{Ric}_{jk}h_{kl}$$

By way of convention, $h_{ij}$, $\mathrm{Ric}_{ij}$, $R_{ipqj}$ indicates evaluation of the tensors on the fixed orthonormal basis $\{X_i\}$.

**Notation.** Let $A$ be a diagonalizable operator. Let $\rho(A)$ denote the maximum eigenvalue of $A$.

In terms of attaining estimates on $Q$, we note that since $Q$ is a quadratic form, we my consider its associated symmetric linear operator, which we denote it by $\widetilde{Q}$. In this case, Equation (3.1) becomes

$$\rho(\widetilde{Q}) < \frac{1}{2}\mathrm{tr}\, D. \tag{3.2}$$

**Notation.** Let $W = \mathrm{Sym}^2(\mathfrak{n})$, the space of symmetric 2-tensors on $\mathfrak{n}$. In what follows, it will be convenient to think of the actions of the curvature and Ricci tensors, as linear maps acting on the whole space. Thus, we denote the action of the curvature tensor by $\mathring{R}.\_ : W \to W$. Similarly, we denote the action of the Ricci tensor by $\mathrm{Ric}.\_ : W \to W$. With this notation, we may write

$$\widetilde{Q} = \mathring{R}.\_ + \frac{1}{2}\mathrm{Ric}.\_.$$

With this set-up, we turn to estimating $\rho(\widetilde{Q})$, beginning with a simple proposition.

**Proposition 3.1.** The maximum eigenvalue of the sum of two matrices satisfies the triangle inequality. That is, $\rho(A + B) \leq \rho(A) + \rho(B)$.

*Proof.* Recall, $\rho(A) := \sup_{|v|=1}(v^t A v)$. So,

$$\begin{aligned} \rho(A + B) &= \sup_{|v|=1} v^t (A + B) v \\ &= \sup_{|v|=1} (v^t A v + v^t B v) \\ &\leq \sup_{|v|=1} (v^t A v) + \sup_{|v|=1} (v^t B v) \\ &= \rho(A) + \rho(B) \end{aligned}$$

The second equality follows from linearity of matrix multiplication, and the third inequality follows from subadditivity of the supremum. *As Required.*

**Corollary 3.2.** $\rho(\widetilde{Q}) \leq \rho(\mathring{R}) + \frac{1}{2}\rho(\mathrm{Ric}.\_)$.

*Proof.* Follows immediately from the lemma. *As Required.*

**Remark.** Since this allows us to calculate $\rho(\widetilde{Q})$ in terms of $\rho(\mathrm{Ric}.\_)$ and $\rho(\mathring{R}.\_)$, individually, we begin by computing $\rho(\mathrm{Ric}.\_)$.

**Notation.** Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a nilsoliton with a fixed basis $\{X_i\}$. We fix a basis $\{h_{ij}\}$ for $W = \mathrm{Sym}^2(\mathfrak{n})$, the space of symmetric (2,0) tensors on $\mathfrak{n}$, where

$$h_{ij}(X_k, X_l) = \begin{cases} 1 & \text{if } \{i, j\} = \{k, l\} \\ 0 & \text{else} \end{cases} \tag{3.3}$$

Note that this basis is orthogonal with respect to the inner product on $W$ induced from $(\mathfrak{n}, \langle\ ,\ \rangle)$ but we do not choose it to be orthonormal, in order to simplify computations later.

**Lemma 3.3.** Suppose $\mathrm{Ric} : \mathfrak{n} \to \mathfrak{n}$ is diagonal in the basis $\{X_i\}$. Then, for $\mathrm{Ric}.\_ : W \to W$, the basis $\{h_{ij}\}$, specified above, is a basis of eigenvectors, with eigenvalues $\mathrm{Ric}_{ii} + \mathrm{Ric}_{jj}$.

*Proof.* We need only to show that each basis element is indeed an eigenvector.

$$\begin{aligned}(\mathrm{Ric}.h_{ij})(X_l, X_m) &= \sum_k (\mathrm{Ric}_l^k h_{ij}(X_k, X_m) + \mathrm{Ric}_m^k h_{ij}(X_k, X_l)) \\ &= \mathrm{Ric}_l^l h_{ij}(X_l, X_m) + \mathrm{Ric}_m^m h_{ij}(X_m, X_l) \\ &= (\mathrm{Ric}_l^l + \mathrm{Ric}_m^m) h_{ij}(X_l, X_m)\end{aligned}$$

The first equality is by the definition, the second equality follows from the fact that Ric is diagonal with respect to $\{X_i\}$, and the third from symmetry of $h_{ij}$. By definition of $h_{ij}$,

$$(\mathrm{Ric}.h_{ij})_{lm} = \begin{cases} \mathrm{Ric}_{ii} + \mathrm{Ric}_{jj} & \text{if } \{i,j\} = \{l,m\} \\ 0 & \text{else.} \end{cases}$$

That is, $\mathrm{Ric}.h_{ij} = (\mathrm{Ric}_{ii} + \mathrm{Ric}_{jj})h_{ij}$.

*As Required.*

**Corollary 3.4.** For $\mathrm{Ric}.\_ : W \to W$, $\rho(\mathrm{Ric}.\_) = 2\rho(\mathrm{Ric})$.

*Proof.* From the previous lemma, the eigenvalues of $\mathrm{Ric}.\_$ are $\{\mathrm{Ric}_{ii} + \mathrm{Ric}_{jj} | 1 \le i, j \le n\}$, taking the max over $i, j$, yields $i = j$, where $\mathrm{Ric}_{ii} = \rho(\mathrm{Ric})$. Thus, $\rho(\mathrm{Ric}.\_) = 2\rho(\mathrm{Ric})$.

*As Required.*

**Remark.** In general, $\rho(\mathring{R}.\_)$ is much larger than $\rho(\mathrm{Ric})$, so the content of this approximation is really in coming up with an approximation for $\rho(\mathring{R}.\_)$, which is the task we turn to now. We first calculate the action of $\mathring{R}.\_$ on our preferred basis above. Recall, by definition of $\mathring{R}.\_$,

$$(\mathring{R}h_{kl})_{ij} = (\mathring{R}h_{kl})(X_i, X_j) = \sum_{pq} R_{ipqj}\big(h_{kl}(X_p, X_q)\big) = R_{iklj} + R_{ilkj} \tag{3.4}$$

for $k \neq l$, and

$$(\mathring{R}h_{kk})_{ij} = (\mathring{R}h_{kk})(X_i, X_j) = R_{ikkj} \tag{3.5}$$

The motivation for choosing a basis of $W$ and calculating $\mathring{R}.\_$ on that basis is to think of $\mathring{R}.\_$ as a matrix and leverage the results from matrix theory given in Section 1.5. We begin with a definition to aid in the computation.

**Definition 3.5.** Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a nilpotent metric Lie algebra, with fixed basis $\mathcal{B} = \{X_i\}$. Consider $R(X_i, X_j) : \mathfrak{n} \to \mathfrak{n}$, given by $R(X_i, X_j)X_k$. We define the Curvature Density $\Delta$ of $R(X_i, X_j)X_k$, with respect to the basis $\mathcal{B}$, via

$$\Delta(\mathcal{B}, R(X_i, X_j)X_k) := \#\{X_l | R_{ijkl} \neq 0\}.$$

If we consider the matrix associated to $R(X_i, X_j)$, in the basis $\mathcal{B}$, this is the number of non-zero entries in the $k$th column.

We further define the Curvature Density of $\mathfrak{n}$, with respect to the basis $\mathcal{B}$, to be

$$\Delta(\mathcal{B}, \mathfrak{n}) := \max_{ijk} \Delta(\mathcal{B}, R(X_i, X_j)X_k)$$

**Remark.** We apply the notions of curvature density and matrix norms to give a condition for linear stability which will we use to prove the stability of Rank 2 filiform nilsolitons.

**Lemma 3.6.** Let $\mathring{R}.\_ : W \to W$ be the linear map assosicated to the symmetric bilinear form $\mathring{R}$ and let $\mathcal{B}$ be a fixed basis. Then,

$$\rho(\mathring{R}.\_) \leq 2n(\max_{ijkl} |R_{ijkl}|)\Delta(\mathcal{B}, \mathfrak{n})$$

*Proof.* Recall from Lemma 1.16, $\rho(\mathring{R}.\_) \leq \|\mathring{R}.\_\|$. Thus, we consider the action of $\mathring{R}.\_$ on the basis in Equation (3.3), and find an upper bound on the 'column norm', which bounds the largest eigenvalue. If we consider the norm of column corresponding to the basis element

$h_{kl}$, from Equations (3.4) and (3.5) we have

$$\begin{aligned}
\sum_{i\leq j} |(\mathring{R}h_{kl})_{ij}| &= \sum_{i\leq j} |R_{iklj} + R_{ilkj}| \\
&\leq \sum_i \sum_j (|R_{iklj}| + |R_{ilkj}|) \\
&\leq \sum_i \big( \max_j |R_{ikjl}|)\Delta(\mathcal{B}, R(X_i, X_k)X_l) + \max_j (|R_{iljk}|)\Delta(\mathcal{B}, R(X_i, X_l)X_k)\big) \\
&\leq n(\max_{ij} |R_{ikjl}| + \max_{ij} |R_{iljk}|)\Delta(\mathcal{B}, \mathfrak{n})
\end{aligned}$$

The first equality follows from calculating $\mathring{R}._$ in this basis, the second from triangle inequality, and positivity of $|\cdot|$. The next from the definition of $\Delta$, and the final from applying the max over $i$. Finally, we observe that the choice of $kl$ was arbitrary. Thus, taking maxes over $k, l$, gives a bound on the column norm. Thus, we have that

$$\|\mathring{R}._\| \leq 2n \max_{ijkl} |R_{ijkl}|\Delta(\mathcal{B}, \mathfrak{n}),$$

and the result follows from Lemma 1.16. *As Required.*

**Remark.** Note that this bound may be sharpened, by counting ‘only’ the $i \leq j$, instead of all $i, j$.

**Theorem 3.7.** Let $(N, \mathfrak{n}, \lambda, D)$ be a nilsoliton, with a fixed basis $\mathcal{B}$. If

$$2n \max_{ijkl} |R_{ijkl}|\Delta(\mathcal{B}, \mathfrak{n}) + \rho(\mathrm{Ric}) < \frac{1}{2}\mathrm{tr}\, D$$

then the soliton is strictly linearly stable.

*Proof.* Recall, from Equation (3.2), it is enough to show $\rho(\tilde{Q}) < \frac{1}{2}\mathrm{tr}\, D$. So, we will calculate a series of upper bounds in our estimate of $\rho(\tilde{Q})$. First, applying Corollary 3.2, and the Lemma 3.6,

$$\rho(\tilde{Q}) \leq \rho(\mathring{R}._\_) + \frac{1}{2}\rho(\mathrm{Ric}._\_) \leq 2n \max_{ijkl} |R_{ijkl}| \Delta(\mathcal{B}, \mathfrak{n}) + \rho(\mathrm{Ric})$$

Thus, if this final quality is smaller than $\frac{1}{2}\mathrm{tr}\, D$, Equation (3.2) holds. *As Required.*

In the following section, we will apply Theorem 3.7 to prove the linear stability of both families of Rank 2 filiform nilsolitons.

## 3.2. Stability of Rank 2 Filiform Nilsolitons

We apply the results from the previous section to the rank 2 filiform families, give their soliton metrics, and show they are strictly linearly stable. Recall, the two families of rank 2 algebras are $L_n$ ($n \geq 3$), and $Q_n$ ($n \geq 4$ and even). Up to antisymmetry, their non-zero stucture constants are given as follows:

$$\begin{aligned}
L_n : \quad & [X_1, X_i] = X_{i+1} && i = 2, ..., n-1 \\
Q_n : \quad & [X_1, X_i] = X_{i+1} && i = 2, ..., n-2 \\
& [X_i, X_{n-i+1}] = (-1)^i X_n && i = 2, ..., n-1
\end{aligned}$$

We take the same approach as in Chapter 2. Namely, fix the bases above to be orthonormal, and apply Theorem 1.1 to act on the bracket by an element diagonal element. From Lemma 2.1, this amounts to a rescaling of the structure constants. It is shown in ([**?** ], Theorem 4.2) and ([**?** ], Theorem 10) that both $L_n$ and $Q_n$ admit nilsolitons, respectively, and they given the bracket relations. The structure constants for the soliton metrics, on $L_n, Q_n$ are given by

$$\begin{aligned}
L_n : c_i =& \sqrt{(i-1)(n-i)} && c_i = \langle [X_1, X_i], X_{i+1} \rangle. \\
Q_n : c_i =& \sqrt{(n-1-i)(i-1)\left(\frac{n+2}{4}\right)} && c_i = \langle [X_1, X_i], X_{i+1} \rangle \\
b_i =& (-1)^i \sqrt{\frac{(n-1)(n-2)(n-3)}{12} + 1} && b_i = \langle [X_i, X_{n-i+1}] X_n \rangle
\end{aligned}$$

Since the metrics for the two rank 2 families are known, we can explicitly compute the relevant tensors.

**Remark.** This section relies extensively on the use of computer algebra systems to compute and verify calculations. Further, stability is calculated for low-dimensional through use of matrix programming software. The functions used for algebraic manipulation, as well as code for the programming are included in Appendix $B$. Throughout this section, we will highlight the places either is used.

## Stability of $L_n$

Fix the basis $\mathcal{B}$ to be the standard one given in Section 1.3. We choose the normalization of the family $L_n$ so that the soliton metric is $[X_1, X_i] = c_i X_{i+1}$, where $c_i = \sqrt{\frac{(n-i)(i-1)}{n-2}}$, for $i = 2, ..., n-1$, with all other brackets zero. This gives a soliton as in ([**?** ], Theorem 4.2). The basis and metric are fixed throughout the section.

**Lemma 3.8.** For Ric : $L_n \to L_n$, $\rho(\text{Ric}) = 1/2$, and $\text{scal}_{L_n} = -\frac{n(n-1)}{12}$.

*Proof.* From ([**?** ], Theorem 4.5) the Ricci Tensor is diagonal with respect to this basis. In particular,

$$\text{ric}(X_i, X_i) = \begin{cases} -\frac{1}{2}\sum_{k=2}^{n-1} c_k^2 & \text{if } i = 1 \\ \frac{1}{2}(c_{i-1}^2 - c_i^2) & \text{if } i = 2, ..., n \end{cases}$$

We proceed by computation. For $n = 1$,

$$\langle \text{Ric}\, X_1, X_1 \rangle = -\frac{1}{2}\sum_{k=2}^{n-1} c_k^2 = -\frac{1}{2}\sum_{k=2}^{n-1}(k-1)\frac{n-k}{n-2} = -\frac{n(n-1)}{12}$$

Note, this is negative for $n > 1$. Observe, for $k = 2, ..., n$

$$\langle \text{Ric}\, X_k, X_k \rangle = \frac{1}{2}(c_{k-1}^2 - c_k^2) = \frac{1}{2}((k-2)\frac{n-(k-1)}{n-2} - (k-1)\frac{n-k}{n-2}) = \frac{2k-n-2}{2(n-2)}$$

is increasing in $k$. So, taking the maximal $k$, namely $k = n$, $\rho(\text{Ric}) = \frac{1}{2}$.

The expression for scalar curvature comes from observing that $\sum_{k=2}^{n} \mathrm{ric}_{kk} = 0$, as the sum telescopes (cf. Lemma 2.7). Thus, scal=ric$_{11}$. *As Required.*

**Proposition 3.9.** For the soliton metric on $L_n$, $\Delta(\mathcal{B}, L_n) = 3$ and $\max_{ijkl} |R_{ijkl}| \leq \dfrac{n^2 - 2n + 4}{2(n-2)}$

*Proof.* The curvature tensor is given in Appendix A, and $\Delta(\mathcal{B}, L_n) = 3$ is clearly seen there. Checking the table, the coefficients of the curvature tensor are of 2 types. Namely $\frac{1}{4}c_i c_j$ and $\frac{1}{4}(3c_i^2 - c_{i-1}^2)$.

Case 1: $\dfrac{1}{4}c_i c_j$. Our goal is to find the maximum of

$$f(x, y) = c_x c_y = \sqrt{\frac{(n-x)(x-1)}{n-2}}\sqrt{\frac{(n-y)(y-1)}{n-2}}$$

for $1 \leq x, y \leq n$. Notice, $c_x$ and $c_y$ are independent and symmetric, so they will be maximized at the same value. Further, for a fixed $n$, the quantity under the radical is a quadratic with negative leading term, whose roots are at $1, n$. Thus, by symmetry, $f$ is maximized at $x = y = \frac{n+1}{2}$.

Case 2: $\dfrac{1}{4}(3c_i^2 - c_{i-1}^2)$. We once again consider the function

$$g(x) = 3c_x^2 - c_{x-1}^2 = 3\frac{(n-x)(x-1)}{n-2} - \frac{(n-x+1)(x-2)}{n-2}$$

This function has max at $x = \frac{n}{2}$.

Comparing, we find that $g(\frac{n}{2}) > f(\frac{n+1}{2}, \frac{n+1}{2})$. So, we obtain a bound on the maximal entry of $R$. Namely, $g(\frac{n}{2}) = \dfrac{n^2 - 2n + 4}{2(n-2)}$. *As Required.*

**Corollary 3.10.** For the soliton metric described above, $\rho(\mathring{R}) \leq (6n)\dfrac{n^2-2n+4}{2(n-2)}$

*Proof.* This follows from the Proposition 3.9 and Lemma 3.6.

*As Required.*

**Remark.** We now calculate the remaining piece of Theorem 3.7, the trace of the soliton derivation.

**Proposition 3.11.** Let $D$ be the soliton derivation with respect to the structure constants above.

$$\mathrm{tr}D = \frac{n(n-1)^2}{12} + \frac{n}{n-2}$$

*Proof.* First, we derive the soliton constant. As we saw above,

$$\begin{aligned}
\langle \mathrm{Ric}X_1, X_1\rangle &= -\frac{n(n-1)}{12} \\
\langle \mathrm{Ric}X_k, X_k\rangle &= \frac{2k-n-2}{2(n-2)}
\end{aligned}$$

Given that the metric is a soliton, from Equation (1.4), $c = \frac{\mathrm{tr}\,\mathrm{Ric}^2}{\mathrm{tr}\,\mathrm{Ric}}$. Thus, using a computer algebra system,

$$\begin{aligned}
c &= \frac{\mathrm{tr}\,\mathrm{Ric}^2}{\mathrm{tr}\,\mathrm{Ric}} \\
&= \frac{\left(-\dfrac{n(n-1)}{12}\right)^2 + \sum\limits_{k=2}^{n}\left(\dfrac{2k-n-2}{2(n-2)}\right)^2}{-\dfrac{n(n-1)}{12}} \\
&= -\frac{12}{n(n-1)}\left(\frac{n^2(n-1)^2}{144} + \frac{n(n-1)}{12(n-2)}\right) \\
&= -\left(\frac{n(n-1)}{12} + \frac{1}{n-2}\right)
\end{aligned}$$

Further, we may calculate $\operatorname{tr} D = \operatorname{tr}\operatorname{Ric} - n\frac{\operatorname{tr}\operatorname{Ric}^2}{\operatorname{tr}\operatorname{Ric}}$. Thus,

$$\operatorname{tr} D = -\frac{n(n-1)}{12} + n\frac{n(n-1)}{12} + \frac{n}{n-2} = \frac{n(n-1)^2}{12} + \frac{n}{n-2}$$

*As Required.*

**Theorem 3.12.** The nisoliton metric on $L_n$ is strictly linearly stable.

*Proof.* Applying the Theorem 3.7, it is sufficient to show that

$$\frac{1}{2} + 6n\frac{n^2 - 2n + 4}{2(n-2)} < \frac{1}{2}\left(\frac{n(n-1)^2}{12} + \frac{n}{n-2}\right)$$

Using a computer algebra system, we find this inequality holds for $n > 74$. For the cases where $n \leq 74$, the sufficient estimate was verified by a computer program, which calculates the appropriate values. Consult Appendix B for an outline of the program, as well as the code.

*As Required.*

## Stability of $Q_n$

Next, we turn to the the family of rank 2 filifom algebras. Fix the basis $\mathcal{B}$ to be the one given in Section 1.3, and declare it to be orthonormal. From [**?** ], the following is a soliton metric on $Q_n$:

$$c_i = \sqrt{(n-1-i)(i-1)\left(\frac{n+2}{4}\right)} \text{ and } b_i = (-1)^i\sqrt{\frac{(n-1)(n-2)(n-3)}{12} + 1} \qquad (3.6)$$

where $[X_1, X_i] = c_i X_{i+1}$ for $i = 2, \ldots, n-2$, $[X_i, X_{n-i+1}] = b_k X_n$ for $k = 2, \ldots, n-1$, and all other brackets vanish.

**Proposition 3.13.** $\text{scal}_{Q_n} = \text{ric}(X_1, X_1) - \text{ric}(X_n, X_n)$.

*Proof.* This proof mirrors the results in chapter 2. Using the description in Section 1.1, it's quick to see ric is diagonal. Further,

$$\text{ric}(X_i, X_i) = \begin{cases} -\dfrac{1}{2}\sum\limits_{k=2}^{n-2} c_k^2 & \text{for } i = 1 \\ \dfrac{1}{2}(c_{i-1}^2 - c_i^2 - b_i^2) & \text{for } i = 2, ..., n-1 \\ \dfrac{1}{4}\sum\limits_{k=2}^{n-1} b_k^2 & \text{for } i = n \end{cases} \tag{3.7}$$

Thus,

$$\begin{aligned} \text{scal} &= \sum_{i=1}^{n} \text{ric}(X_i, X_i) \\ &= -\frac{1}{2}\sum_{k=1}^{n-2} c_k^2 + \frac{1}{2}\sum_{i=2}^{n-1}(c_{i-1}^2 - c_i^2 - b_i^2) + \frac{1}{4}\sum_{k=2}^{n-1} b_k^2 \\ &= -\frac{1}{2}\sum_{k=1}^{n-2} c_k^2 - \frac{1}{4}\sum_{k=2}^{n-1} b_k^2 \\ &= \text{ric}(X_1, X_1) - \text{ric}(X_n, X_n) \end{aligned}$$

*As Required.*

We now turn to estimate $\rho(\text{Ric}\_\_)$ and $\rho(\mathring{R}\_\_)$.

**Proposition 3.14.** For $\text{Ric} : Q_n \to Q_n$, $\rho(\text{Ric}\_\_) = \frac{1}{2}(n-2)(\frac{(n-1)(n-2)(n-3)}{12} + 1)$.

*Proof.* Applying the constants in Equation (3.6) to the description of ric in Equation (3.7),

$$\text{ric}(X_i, X_i) = \begin{cases} -\frac{(n-1)(n-2)(n-3)(n+2)}{48} & \text{if } i = 1 \\ \frac{1}{2}(\frac{n+2}{4}(-n+2i-1) - (\frac{(n-1)(n-2)(n-3)}{12} + 1)) & \text{if } i = 2, ..., n-1 \\ \frac{1}{4}(n-2)(\frac{(n-1)(n-2)(n-3)}{12} + 1) & \text{if } i = n \end{cases}$$

Applying Lemma 3.4 yields,

$$\rho(\mathrm{Ric}._\_) = 2\rho(\mathrm{ric}) = \frac{1}{2}(n-2)(\frac{(n-1)(n-2)(n-3)}{12} + 1)$$

*As Required.*

**Proposition 3.15.** For the soliton metric on $Q_n$, $\Delta(\mathcal{B}, Q_n) = 3$ and $\max_{ijkl} |R_{ijkl}| \leq (\frac{n}{2} - 1)^2(n+2)$

*Proof.* Consulting Appendix A, it it's quick to see that $\Delta(\mathcal{B}, Q_n) = 3$. Further, a quick analysis of the coefficients of $R$ reveal 7 types, as follows:

$$4R_{ijkl} = \begin{cases} 3c_\alpha^2 - c_{\alpha-1}^2 & (1) \\ c_\alpha c_\beta & (2) \\ c_\alpha b_\beta & (3) \\ c_\alpha b_\beta \pm c_\gamma b_\delta & (4) \\ c_\alpha b_\beta + 2c_\gamma b_\delta & (5) \\ b_\alpha b_\beta + c_\alpha^2 - 2b_\alpha^2 & (6) \\ 3b_\alpha^2 & (7) \end{cases}$$

Observe $|b_k| \equiv |b|$. By taking moduli and maxes over the constants, we have one of the following cases:

$$4\max_{ijkl} |R_{ijkl}| \leq 4\max_i c_i^2 = 4c_{n/2}^2 = 4\left(\frac{n}{2} - 1\right)^2 (n+2)$$
$$4\max_{ijkl} |R_{ijkl}| \leq 3\max_i |c_i b| = 3\left(\frac{n}{2} - 1\right)\sqrt{\frac{n+2}{4}}\sqrt{\frac{(n-1)(n-2)(n-3)}{12} + 1}$$
$$4\max_{ijkl} |R_{ijkl}| \leq \max_i 3|b| + c_i^2 = 3\sqrt{\frac{(n-1)(n-2)(n-3)}{12} + 1} + \left(\frac{n}{2} - 1\right)\sqrt{\frac{n+2}{4}}$$

It is quick to check that for $n > 3$, the largest of these is the top. Thus, $|R_{ijkl}| \leq (n/2-1)^2(n+2)$ gives a is a uniform bound in every dimension.

*As Required.*

Finally, we calculate the trace of the soliton derivation.

**Proposition 3.16.** Let $D$ be the soliton derivation for $Q_n$. Then,

$$\operatorname{tr} D = -\frac{(n-1)(n-2)(n-3)(n+2)}{48} + \frac{n^2+n+2}{4}\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right)$$

*Proof.* From ([**?** ], Theorem 3), the soliton constant for $Q_n$ is

$$c = -\left(\frac{n+2}{4}\right)\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right)$$

From the definition of soliton, applying a trace, and leveraging a computer algebra system, yields

$$\begin{aligned}
\operatorname{tr} D &= \operatorname{scal} - c \cdot n \\
&= \operatorname{ric}(X_1, X_1) - \operatorname{ric}(X_n, X_n) - c\dot{n} \\
&= -\frac{(n-1)(n-2)(n-3)(n+2)}{48} - \frac{n-2}{4}\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right) \\
&\quad + \frac{n(n+2)}{4}\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right) \\
&= -\frac{(n-1)(n-2)(n-3)(n+2)}{48} + \frac{n^2+n+2}{4}\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right)
\end{aligned}$$

*As Required.*

**Theorem 3.17.** The soliton metric on $Q_n$ is strictly linearly stable.

*Proof.* From Theorem 3.7, it is sufficient to show,

$$\begin{aligned}
&\frac{1}{4}(n-2)\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right)+6n(\frac{n}{2}-1)^2(n+2)\\
&<-\frac{(n-1)(n-2)(n-3)(n+2)}{48}-\frac{n-2}{4}\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right)\\
&\quad+\frac{n(n+2)}{4}\left(\frac{(n-1)(n-2)(n-3)}{12}+1\right)
\end{aligned}$$

Using a computer algebra system, as in Appendix A, one may verify that this equality holds for $n > 74$. For the case of $n \leq 74$, we verify this using a matrix algebra system as described in Appendix B, with code in B.3. *As Required.*

At last, we have come to proof of the main theorem of this chapter.

**Theorem 3.18.** Rank 2 filiform nilsolitons are dynamically stable.

*Proof.* Since the soliton metrics on $L_n$, $Q_n$ are strictly linear stable, ([**?** ], Theorem 1.2) says they are, in fact, dynamically stable. *As Required.*

# Appendix A

# Tables of Curvature Tensors

Showing stability for an algebraic soliton can be achieved through the use of approximation the action of the curvature tensor's action on the space of symmetric two tensors. To this end, it becomes extremely helpful to have the computation for the full curvature tensor, which we reference above. The full curavture tensor for $L_n$ appears in [**?** ], but we record here for completeness. The computation for the full curvature tensor of $Q_n$ has not appeared to this point. Throughout, we use the sign convention $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. The computations are long, so we don't reproduce them. Further, we consider metrics which are a rescaling of the structure constants presented. Since this must be the case for a nilsoliton, calculating these are sufficient for the study of stability as described above.

**Curvature of $L_n$.** For $1 < i, j, k$, and $i < j$ (as $R$ is determined up to anti-symmetry), the non-zeros values of the curvature tensor are given as follows.

$$R(X_1, X_j)X_1 = \frac{1}{4}(c_{j-2}c_{j-1}X_{j-2} + (3c_j^2 - c_{j-1}^2)X_j + c_jc_{j+1}X_{j+2}$$

$$R(X_1, X_j)X_k = \begin{cases} -\frac{1}{4}(c_{j-1}c_{j-2})X_1 & \text{for } k = j-2 > 1 \\ \frac{1}{4}(c_{j-1}^2 - 3c_j^2)X_1 & \text{for } k = j > 1 \\ -\frac{1}{4}(c_jc_{j+1})X_1 & \text{for } k = j+2 \end{cases}$$

$$R(X_i, X_j)X_k = \begin{cases} -\frac{1}{4}(c_{i-1}c_jX_{i-1} + c_ic_jX_{i+1}) & \text{for } k = j+1 \text{ and } j \neq i-2 \\ \frac{1}{4}(c_{i-1}c_{i-3}X_{i-3} - c_ic_{i-2}X_{i+1}) & \text{for } k = j+1 \text{ and } j = i-2 \\ -\frac{1}{4}(c_{i-1}c_{i+1}X_{i-1} + c_ic_{i+2}X_{i+3}) & \text{for } k = j-1 \text{ and } j = i+2 \\ -\frac{1}{4}(c_{i-1}c_{j-1}X_{i-1} + c_ic_{j-1}X_{i+1}) & \text{for } k = j-1 \text{ and } j \neq i+2 \\ \frac{1}{4}(c_ic_{j-1}X_{j-1} + c_ic_jX_{j+1}) & \text{for } k = i+1 \text{ and } j \neq k \pm 1 \\ \frac{1}{4}(c_{i-1}c_{j-1}X_{j-1} + c_ic_{j-1}X_{j+1}) & \text{for } k = i-1 \text{ and } j \neq k \pm 1 \end{cases}$$

**Curvature of $Q_n$.** For, $1 < i, j, k < n$, and $i < j$ (as $R$ is determined up to anti-symmetry), the non-zero values of the curvature tensor are as follows

$$
\begin{aligned}
R(X_1, X_j)X_1 &= \frac{1}{4}(c_j c_{j+1} X_{j+2} + (3c_j^2 - c_{j-1}^2)X_j + c_{j-1}c_{j-2}X_{j-2}) \\
R(X_1, X_j)X_n &= \frac{1}{4}((2c_j b_{j+1} + b_j c_{n-j})X_{n-j} - b_j c_{n-j+1}X_{n-j+2}) \\
R(X_1, X_n)X_1 &= 0 \\
R(X_1, X_n)X_k &= \frac{1}{4}(c_{n-k}b_k + c_k b_{k+1})X_{n-k} - \frac{1}{4}(b_k c_{n-k+1} + c_{k-1}b_{k-1})X_{n-k+2} \\
R(X_1, X_n)X_n &= 0 \\
R(X_i, X_j)X_1 &= \frac{1}{4}\delta_{j,n-i}(b_j c_i - b_i c_j)X_n + \frac{1}{4}\delta_{j,n-i+2}(b_j c_{i-1} - c_{j-1}b_i)X_n \\
R(X_i, X_j)X_n &= \frac{1}{4}\delta_{n-i,j}(b_i c_j - b_j c_i)X_1 + \frac{1}{4}\delta_{j,n-i+2}(c_{j-1}b_i - b_j c_{i-1})X_1 \\
R(X_i, X_n)X_1 &= -\frac{1}{4}(c_i b_{i+1})X_{n-i} - \frac{1}{4}(c_{i-1}b_{i-1})X_{n-i+2} \\
R(X_i, X_n)X_k &= -\frac{1}{4}\delta_{i,k}b_i^2 X_n - \frac{1}{4}(\delta_{i,n-k}b_k c_i + \delta_{i,n-k+2}c_{i-1}b_k)X_1 \\
R(X_i, X_n)X_n &= \frac{1}{4}b_i^2 X_i
\end{aligned}
$$

$$
\begin{aligned}
R(X_1, X_j)X_k = &-\frac{1}{4}c_k(\delta_{k,n-j}d_j X_n + \delta_{j,k}c_j X_1 + \delta_{j,k+2}c_{k+1}X_1) \\
&+ \frac{1}{4}c_{k-1}(\delta_{k,n-j+2}d_j X_n + \delta_{k,j+2}c_j X_1 + \delta_{j,k}c_{k-1}X_1) \\
&- \frac{1}{2}c_j(\delta_{k,n-j}d_{j+1}X_n + \delta_{k,j+2}c_{j+1}X_1 + \delta_{j,k}c_k X_1)
\end{aligned}
$$

Thus, $R(X_1, X_j)X_k = 0$ unless at least one of the following holds $k = n - j$, $j = k$, $j = k + 2$, $k = n - j + 2$, $k = j + 2$.

$$R(X_1,X_j)X_k=\begin{cases}-\frac{1}{4}(c_{j+2}b_j+2c_jb_{j+1})X_n-\frac{1}{4}c_jc_{j+1}X_1 & \text{if } k=j+2,\, j=\frac{n-2}{2}\\ -\frac{1}{4}(c_jb_j+2c_jb_{j+1})X_n-\frac{1}{4}(3c_j^2-c_j^2)X_1 & \text{if } k=j,\, j=\frac{n}{2}\\ -\frac{1}{4}(c_{j-2}b_j+2c_jb_{j+1})X_n-\frac{1}{4}c_{j-2}c_{j-1}X_1 & \text{if } k=j-2,\, \frac{n+2}{2}\\ -\frac{1}{4}(c_{n-j}b_j+2c_jb_{j+1})X_n & \text{if } k=n-j,\, j\neq\frac{n}{2},\frac{n\pm2}{2}\\ \frac{1}{4}(c_{j+1}b_jX_n-c_jc_{j+1}X_1) & \text{if } k=j+2,\, j=\frac{n}{2}\\ \frac{1}{4}(c_{j-1}b_jX_n-(3c_j^2-c_j^2)X_1) & \text{if } k=j,\, j=\frac{n+2}{2}\\ \frac{1}{4}(c_{j-3}b_jX_n-c_{j-1}c_{j-2}X_1) & \text{if } k=j-2,\, j=\frac{n+4}{2}\\ \frac{1}{4}c_{n-j+1}b_jX_n & \text{if } k=n-j+2,\\ & \quad j\neq\frac{n}{2},\frac{n+2}{2},\frac{n+4}{2}\\ -\frac{1}{4}c_jc_{j+1}X_1 & \text{if } k=j+2,\, j\neq\frac{n-2}{2},\frac{n}{2}\\ -\frac{1}{4}(3c_j^2-c_{j-1}^2)X_1 & \text{if } k=j,\, j\neq\frac{n}{2},\frac{n+2}{2}\\ -\frac{1}{4}c_{j-1}c_{j-2}X_1 & \text{if } k=j-2,\, j\neq\frac{n+4}{2},\frac{n+2}{2}\end{cases}$$

For $R(X_i, X_j)X_k$, we assume, $i < j$, as $R$ determined up to anti-symmetry.

$$
\begin{aligned}
R(X_i, X_j)X_k = &-\frac{1}{4}\big(\delta_{j,n-k+1} d_j d_i X_{n-i+1} + \delta_{k,j+1} c_j (c_i X_{i+1} + c_{i-1} X_{i-1}) \\
&\quad + \delta_{j,k+1} c_k (c_i X_{i+1} + c_{i-1} X_{i-1})) \\
&+ \frac{1}{4}(\delta_{i,n-k+1} d_i d_j X_{n-j+1} + \delta_{k,i+1} c_i (c_j X_{j+1} + c_{j-1} X_{j-1}) \\
&\quad + \delta_{i,k+1} c_k (c_j X_{j+1} + c_{j-1} X_{j-1})) \\
&+ \frac{1}{2}\delta_{i,n-j+1} d_i d_k X_{n-k+1}
\end{aligned}
$$

Thus, $R(X_i, X_j, X_k) = 0$ unless one of the following holds $j = n-k+1$, $k = j+1$, $j = k+1$, $i = n-k+1$, $k = i+1$, $i = k+1$, $i = n-j+1$.

For $k = n-j+1$,

$$
R(X_i, X_j)X_k = \begin{cases}
-\frac{1}{4}((c_i^2 - 3b_i^2)X_{i+1} + c_i c_{i-1} X_{i-1}) & \text{if } i = k, k = j-1 \\
\frac{1}{4}(3b_i^2)X_j & \text{if } i = k, k \neq j-1 \\
-\frac{1}{4}(b_j b_i X_{n-i+1} + c_j c_i X_{i+1} + c_j c_{i-1} X_{i-1}) & \text{if } j = \frac{n}{2} \\
-\frac{1}{4}((b_{i+2} b_i - c_i c_{i+2})X_{i+3} + c_{i+1} c_{i-1} X_{i-1}) & \text{if } k = \frac{n}{2}, j = i+2 \\
-\frac{1}{4}(b_j b_i X_{n-i+1} + c_{j-1} c_i X_{i+1} + c_{j-1} c_{i-1} X_{i-1}) & \text{if } k = j-1, i \neq \frac{n-2}{2} \\
-\frac{1}{4}((b_j b_i - c_i c_j)X_{j+1} - c_i c_{j-1} X_{j-1})) & \text{if } k = i+1, i \neq j-2 \\
-\frac{1}{4}((b_j b_i - c_{i-1} c_{j-1})X_{j-1} - c_{i-1} c_j X_{j+1}) & \text{if } k = i-1 \\
-\frac{1}{4} b_j b_i X_{n-i+1} & \text{if } k \neq i, j \pm 1, i \pm 1
\end{cases}
$$

For $k = n - i + 1$,

$$R(X_i, X_j)X_k = \begin{cases} \frac{1}{4}((c_i^2 - 3b_i^2)X_i + c_i c_{i+1} X_{i+2}) & \text{if } k = j, k = i+1 \\ -\frac{1}{4}(3b_i^2)X_i & \text{if } k = j, k \neq i+1 \\ \frac{1}{4}((b_i b_j - c_j c_i)X_{i+1} - c_j c_{i-1} X_{i-1}) & \text{if } k = j+1 \\ \frac{1}{4}((b_i b_j - c_{i+1} c_{i-1})X_{i-1} + c_i c_{i+2} X_{i+3}) & \text{if } k = \frac{n-2}{2}, j = i+2 \\ \frac{1}{4}(b_i b_j - c_{j-1} c_{i-1})X_{i-1} - c_{j-1} c_i X_{i+1}) & \text{if } k = j-1, j \neq i+2 \\ \frac{1}{4}((b_i b_j X_{n-j+1} + c_i c_j X_{j+1} + c_i c_{j-1} X_{j-1}) & \text{if } k = i+1, k \neq j, j-1 \\ \frac{1}{4}(b_i b_j X_{n-j+1} + c_{i-1} c_j X_{j+1} + c_{i-1} c_{j-1} X_{j-1}) & \text{if } k = i-1 \\ \frac{1}{4} b_i b_j X_{n-j+1} & \text{if } k \neq j, j \pm 1, i \pm 1 \end{cases}$$

Finally, if $k \neq n-i+1$ and $k \neq n-j+1$ we have

$$R(X_i, X_j)X_k = \begin{cases} \frac{1}{4}((2b_ib_{j+1} - c_jc_{i-1})X_{i-1} - c_jc_iX_{i+1}) & \text{if } i = n-j+1, k = j+1 \\ \frac{1}{4}((2b_ib_{j-1} - c_{j-1}c_i)X_{i+1} - c_{j-1}c_{i-1}X_{i-1}) & \text{if } i = n-j+1, k = j-1 \\ \frac{1}{4}((2b_ib_{i+1} + c_ic_{j-1})X_{j-1} + c_ic_jX_{j+1}) & \text{if } i = n-j+1, k = i+1 \\ \frac{1}{4}((2b_ib_{i-1} + c_jc_{i-1})X_{j+1} + c_{i-1}c_{j-1}X_{j-1}) & \text{if } i = n-j+1, k = i-1 \\ \frac{1}{2}b_ib_kX_{n-k+1} & \text{if } i = n-j+1, k \neq i+1, \\ & \quad k \neq j+1 \\ -\frac{1}{4}(c_jc_iX_{i+1} + c_jc_{i-1}X_{i-1}) & \text{if } k = j+1, \\ & \quad i \neq n-j+1 \\ -\frac{1}{4}(c_{i+1}c_{i-1}X_{i-1} - c_ic_{i+2}X_{i+3}) & \text{if } k = j-1, k = i+1 \\ -\frac{1}{4}(c_{j-1}c_iX_{i+1} + c_{j-1}c_{i-1}X_{i-1}) & \text{if } k = j-1, k \neq i+1, \\ & \quad i \neq n-j+1 \\ \frac{1}{4}(c_ic_jX_{j+1} + c_ic_{j-1}X_{j-1}) & \text{if } k = i+1, k \neq j-1, \\ & \quad i \neq n-j+1 \\ \frac{1}{4}(c_{i-1}c_jX_{j+1} + c_{i-1}c_{j-1}X_{j-1}) & \text{if } k = i-1, i \neq n-j+1 \end{cases}$$

# Appendix B

# Programming

This work was greatly aided through the use of Computer Algebra systems for large computations and solving cubic equations, in Chapter 2. These computations carried out using Mathematica® (See [? ]). In Chapter 3, the stability of low-dimensional filiform nilsolitons was carried out using MATLAB® (See [? ]). In order to increase testability, transparency, and share the methods for the mathematical community at large, this Appendix will showcase the tools/methods used.

## B.1. Computer Algebra

The following table the lists of Mathematica® commands and functions utilized in this work. The first column give the command, the next a description, and the final column gives a link to the documentation, for those using a compatible .pdf viewer. Along with the computational commands, this table includes the graphing functions used extensively in the course of this work.

| Command | Description | Link |
|---|---|---|
| **Sum** | Calculates a finite sum over given indices. | *Sum Documentation* |
| **Expand** | Expands out products and positive integer powers in a given expression. | *Expand Documentation* |
| **Factor** | Factors a polynomial over the integers. | *Factor Documentation* |
| **Simplify** | Performs a sequence transformations an expression and returns the simplest form it finds. | *Simplify Documentation* |
| **Collect** | Collects together terms involving the same powers of objects matching a given variable. | *Collect Documentation* |
| **D** | Gives partial derivative of function with respect to a given variable. | *D Documentation* |
| **Solve** | Solves a an equation for a given variable. | *Solve Documentation* |
| **Plot** | Plots a curve on a given domain. | *Plot Documentation* |
| **Plot3D** | Plots the graph of a 2-variable function on a given domain. | *Plot3D Documentation* |
| **Discriminant** | Computes the discriminant of a given polynomial with respect to a specified variable. | *Discriminant Documentation* |
| **Limit** | Compute the limit of a function at a given point or at infinity. | *Limit Documentation* |

## B.2. Matrix Computations

Since left-invariant metrics on a nilpotent Lie group may be identified with inner products on their Lie algebra, all of the relevant data may be encoded in matrices. In particular, Equation (3.2) is simply a problem on determining eigenvalues; a problem which computers are well-equipped to handle. Here we sketch an algorithm to compute the necessary quantities. Numbers indicates steps in the algorithm, while the bullets indicate comments. The computations were completed in MATLAB® ([**?** ]). The code is included in the next section.

1. Choose a Lie algebra $\mathfrak{n}$ to consider.

2. Fix an orthonormal basis $\mathcal{B}$ for $\mathfrak{n}$.

3. Build the Structure Constant Array $A$ with respect to the basis $\mathcal{B}$.

    - Let $A(i, j, k) = \alpha_{i,j}^k$

    - The structure constants contain all of the necessary data. In fact, they are the only data required.

4. Build the (4,0) Curvature tensor $R$ using the structure constants.

    - Calculate the expression for $R$ in terms of the structure constants.

    - Identify $R$ with the $n^4$ array, where $R(i, j, k, l) = R_{ijkl}$.

5. Calculate the Ricci tensor.

    - This may similarly be computed via structure constants as in Equation (1.2).

    - Identify Ric with the matrix associated to it.

6. Calculate the soliton constant $c$ and soliton derivation $D$.

- Since Ric has already been calculated, one may exploit $c = \dfrac{\operatorname{tr}\operatorname{Ric}^2}{\operatorname{tr}\operatorname{Ric}}$ and $D = \operatorname{Ric} - cI$.

7. Fix a basis $\mathcal{B}'$ for $W = \operatorname{Sym}^2(\mathfrak{n})$.

   - Identify each element of the basis with a symmetric matrix.
   - Basis maybe chosen as in Equation (3.3).

8. Compute $\mathring{R}.\_$, $\operatorname{Ric}.\_$ on $W$.

   - Use a computation analogous to Equation (3.5) or (3.4) and Lemma 3.3.
   - In this work, we used $n^4$ arrays $S, T$ where $S(i, j, k, l) = (\mathring{R}h_{kl})_{ij}$ and $T(i, j, k, l) = (\operatorname{Ric}.h_{kl})_{ij}$, where $h_{kl}$ is as in Equation (3.3).

9. If necessary, transform the $n^4$ array into the $n^2$ matrix with respect to the $\mathcal{B}'$ for $\tilde{Q} = \mathring{R}.\_ + \dfrac{1}{2}\operatorname{Ric}.\_$.

10. Compute $\rho(\tilde{Q})$, and $\operatorname{tr} D$.

11. If $\rho(\tilde{Q}) < \frac{1}{2}\operatorname{tr} D$, the soliton is strictly linearly stable. If not, the test is inconclusive.

**Remark.** Though the stability analysis only applies to nilsolitons, the first 6 steps make sense for any nilpotent metric Lie algebra.

## B.3. Matrix Computation Code

Having outlined the program in previous appendix, we now give the MATLAB ® functions and programs used. We first give the functions used.

```
function N = Nabla(const,n)
%%% Function Description
% Computes the connection of a metric Lie algebra
% N(i,j,k) = (\Nabla_{e_k})_{ij} = < \Nabla_{e_k} e_j, e_i >
%%% Input Description
% n is the dimension of the Lie algebra
% const is an nxnxn array of structure constants
for i=1:n
    for j=1:n
        for k=1:n
N(i,j,k) = 1/2*(const(k,j,i)-const(k,i,j)-const(j,i,k));
        end
    end
end

end
```

```
function r = curv(N,n,const)
%%% Function Description
% Computes the curvature array of a metric Lie algebra Particularly, this returns the matrices associated to R(e_i,e_j) viewed as a linear map.
%%% Input Description
% n is the dimension of the Lie group
% N is the nxnxn Connection array,
% struc is the nxnxn array of structure constants a
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Warning: This means that r(i,j,k,l)=<R(e_i,e_j)e_l,e_k)>
% Usually, (4,0) is written as R_{ijkl}=<R(e_i,e_j)e_k,e_l>.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This calculates the the Bracket Term of curvature.
% A(:,:,i,j) is the matrix associated to \nabla_{[e_i,e_j]}.

%Here N(:,:,k) is the matrix associated to the connection with e_k in the lower slot. N(:,:,k)=\nabla_{e_k}. N(i,j,k)=<N_{e_k}e_j,e_i>.

A=zeros(n,n,n,n);

for i=1:n
    for j=1:n
        for k=1:n
        A(:,:,i,j)=const(i,j,k)*N(:,:,k)+A(:,:,i,j);
        end
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This calculate the curvature array
```

```
r=zeros(n,n,n,n);

% By symmetry in the 12 and 34 slots
% r(:,:,i,j) is the matrix associated to R(e_i,e_j),
% as is r(i,j,:,:)

for i=1:n
    for j=1:n
r(:,:,i,j)=N(:,:,i)*N(:,:,j)-N(:,:,j)*N(:,:,i)-A(:,:,i,j);
    end
end

end
```

```
function ric= Ric(r1,n)
%%% Description
% Computes the matrix associated the to Ricci operator
% for a given metric Lie Algebra.
%%% Input description
% n is the dimension of the Lie algebra
% r1 is a nxnxnxn curvature array
% ric(i,j) = < Ric (e_i), e_j> (because Ric is symmetric)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ric=zeros(n);
for i=1:n
    for j=1:n
        for k=1:n
              ric(i,j)=r1(i,k,j,k)+ric(i,j);
        end
    end
end

end
```

```
function R = R0(r,n)
%%% Function Description
% R0 calculates the action of the curvature tensor on a basis of
    symmetric matrices for a metric Lie algebra. R(i,j,k,l) is the
    result of acting by the curvature tensor on the symmetric matrix E_{
    kl}+E_{lk} for l\neq k, and E_{kk} for k=l.
% NOTE-WARNING: This calculation only works for the basis with 1's in
    the (i,j) spot.
%%% Input Description
% n is the dimension of the Lie algebra
% r is an nxnxnxn curvature array.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
R=zeros(n,n,n,n);
for i=1:n
    for j=1:n
        for k=1:n
            for l=1:n
                R(i,j,k,l)=r(i,l,j,k)+r(i,k,j,l);
            end
        end
    end
end

%%%% This step normalizes the action on E_{kk}
for i=1:n
    R(:,:,i,i)=1/2*R(:,:,i,i);
end

end
```

```
function K = NewSymmBasis(n)
%%% Description
% Generates an orthogonal NOT orthonormal basis for the space of
    symmetric two-tensors on a metric Lie Algebra. Particularly, this
    generates  symmetric matrices with ones in the (i,j) and (j,i)
    position. For, i=j, there is a one in that diagonal entry.
%%%Input description
% n is the dimension of the Lie algebra
% WARNING: this array will have some zero elements,
% in order to avoid redundancies (i.e, if i>j K(:,:i,j)=0).
% Note: eq is a MATLAB function which gives the Kronecker delta of 2
    numbers.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
      for i=1:n
         for j=1:n
             for k=1:n
                 for l=k:n % note the index here!
                 K(i,j,k,l)=eq(i,k)*eq(j,l)+eq(i,l)*eq(j,k);
                 end
             end
         end
      end
for i=1:n
K(:,:,i,i)=1/2*K(:,:,i,i);
end

end
```

```
function Rich = RicAction(ric,n)
%%% Function Description
$ RicAaction calculates the action of the Ricci tensor on a basis of
    symmetric matrices. Rich(:,:,i,j) is the matrix obtained by the
    action of the Ricci operator on the matrix E_{ij}+E_{ji}, for i\neq
    j and E_{ii} for i=j.
%%% Input description
% n is the dimension of the Lie algebra
% ric is an nxn matrix associated to Ric
% The array Rich has a lot of blank entries.
% In particular if i>j rich(:,:,i,j)=0.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Rich=zeros(n,n,n,n);

H=NewSymmBasis(n);

for i=1:n
    for j=i:n %note the index in the first slot of j
        Rich(:,:,i,j)=(ric*H(:,:,i,j)+H(:,:,i,j)*ric);
    end
end

end
```

```
function T = symtoreg(A,n)
%%% Function Description
% Takes an n^4 array which represents the action of a map on the space of symmetric 2-tensors and rewrites it as a (n^2+n)/2x(n^2+n)/2 matrix in the chosen basis of symmetric 2-tensors.
%%% Input description
% n is the dimension of the Lie algebra
% A(:,:,a,b) = resulting symmetric matrix of A applied to the symmetric matrix with 1's in  (a,b) and (b,a).So, A(c,d,a,b,) is the coefficient appearing in  the (c,d) entry of the output of A applied to  symmetric matrix with 1's in (a,b) and (b,a).
% Note: symmetric matrices are determined by their  upper triangle and diagonal. This enumerates the  basis of sym(n) by starting in the top right corner  and tracing down the super diagonals. i.e., if {X_i}_{i=1}^{(n^2+n)/2} is the new basis  for sym(n), X_1=a_{1,n},X_2=a_{1,n-1},X_3=a_{2,n}, X_4=a_{1,n-2},...,X_{(n^2-n)/2}=X_{1,1},...,X_{(n^2+n)/2}=X{n,n}
% Note: The k,l are "counters" to track which super diagonal you are on.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

T=zeros(n*(n+1)/2,n*(n+1)/2);
for i=1:n*(n+1)/2
    for j=1:n*(n+1)/2
        for k=1:n
            for l=1:n
                if k*(k-1)/2<i && i<k*(k+1)/2+1 && l*(l-1)/2<j && j<l*(l+1 )/2+1;
                    T(i,j)=A(i-k*(k-1)/2,n+i-k*(k+1)/2,j-l*(l-1)/2,n+j-l*(l+1)/2);
                end
            end
```

```
        end
    end
end

end
```

```
function [Qmax,TraceD] = Linear_Stability_Test_Function(A,n)
%%% Function Description
% Computes the stability of a nilsoliton metric. Outputs the max
    eigenvalue of the linearization Qmax and trace of the soliton
    derivation
%%% Input description
% n is the dimension of the Lie algebra.
% A is an nxnxn structure constant array. Where A(i,j,k)=<[X_i,X_j],X_k
    >. Note, only include i<j, as the next step will give skew symmetry.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Skew-symmetry of structure constants
% This section takes the structure constants entered above and makes
    them skew-symmetric
  for k=1:n
    A(:,:,k)=(A(:,:,k)-A(:,:,k)');
  end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Connection
%This calculates the metric connection based on the structure constants
%entered, by calling the nabla function. See function for more
    information.
N1=Nabla(A,n);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Curvature Tensor
% This calculates the full curvature tensor. See function for more
```

```
information.
r1=curv(N1,n,A);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Ricci Tensor
% This calculates the Ricci tensor. See function for more information.
ric=Ric(r1,n);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculate Ric Derivation and C.
% This calculates the soliton constant and the soliton derivation.
C=trace(ric*ric)/trace(ric);
D=ric-C*eye(n);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculating the action of the curvature tensor on the space of symmetric 2 tensors
% This calculates the curvature tensor's action on the space of symmetric (2-0)tensors. See function for more details.
R=R0(r1,n);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculating Ric\circ H+H\circ Ric
% This calculates the Ricci tensor's action on the space of symmetric (2-0)tensors. See function for more details.
RicH=RicAction(ric,n);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculate Symmetric Linear Map Associated to Quadratic form Q.
% Q(h)=<\dot{R}h+Ric\circ h,h>. The symmetrization of A, is 1/2(A+A^t).
% As \dot{R} is symmetric, this tells  us the symmetrization of Q is
% \dot{R}+1/2(Ric\circ h+(Ric\circ h)^t)=\dot{R}+1/2(RicAction).
```

```
Q=R+1/2*RicH;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Change the Basis type for Q.
% This takes the n^4 array for Symmetric map associated to Q and changes
    it to an (n(n+1)/2)x(n(n+1)/2) matrix so that eigenvalues can be
    computed. See ‘symtoreg’ function for more information.
T=symtoreg(Q,n);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Calculate the eigenvalues of Q.
% Calculates the eigenvalues of the linear map associated to Q, thus
    finding the max of Q. X is an array all the eig values, eig,trace,
    and max are all functions contained in MATLAB.
X=eig(T);
TraceD=trace(D);
Qmax=max(X);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Stability Test.
% Calculates the max eigenvalue of Q, and compares to 1/2 trace(D).
% Prints if the soliton is stable, or this test is inconclusive. Also
    prints the trace of D, and the max eigenvalue of Q
if max(X)<1\2*trace(D) disp('Congratulations! The Soliton is stable')
else disp('The test is inconclusive')
end
disp('The maximum eigenvalue of Q is')
max(X)
disp('The trace of D is')
trace(D)
```