AN INVESTIGATION OF STUDENTS' APPLICATION OF CRITICAL THINKING TO SOLVING RELATED RATES PROBLEMS

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Abstract: This study aims to explore students' critical thinking novel to the related rates problem. Students are having hard time to understand the concepts mainly because they are not understanding the meaning while they solve relates rates problem. Facione, Peter A suggested important critical thinking skills and subskills are essential for success in each endeavor. I leveraged these skills to inform the design of task-based questions that I used when I interviewed with students to understand how their reflective thinking makes the problem progress or how wrong reasoning made them incorrect answer. The result of my study indicates that failing to engage in critical thinking skills such as Interpretation, Analysis, Evaluation, Inference, Explanation, and Self Correction will demonstrate wrong conceptualize meaning of related rates and derivatives. Also, the result suggest that students did not understand the meaning of implicit differentiation and the rates and showing difficulty of solving tasks. Thus, Instructors should develop conceptual understanding of related rates to encourage them to engage in critical thinking while they solve problems and provide them variety of problems to improve the skills.

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CHAPTER I

INTRODUCTION

Critical thinking entails the complex mental processes of analyzing or evaluating information. Critical thinking, or reflective thinking, plays a crucial role of students' learning and application of mathematics. Since mathematics is a concept-oriented subject, it is important to know how students engage in critical thinking while applying concepts to novel problems and contexts.

Calculus is a foundational mathematical subject for all students who intend to pursue STEM majors in college. Related rates are a central topic in single-variable calculus, and a context for students to apply the central concept of instantaneous rate of change in applied settings. The research literature has documented students' difficulties with related rate problems (e.g., Engelke, 2004; Martin, 1996, 2000; Selden, Selden, & Mason, 1994). Primary among these difficulties are students' interpretations of the problem context, which affects students' ability ot model the context with an appropriate formula relating relevant quantities and their rates of change.

The main steps involving in related rates problems are,

1. identify constant and varying quantities in an applied context.
2. define an equation that relates the values of these quantities.
3. differentiate the resulting equation with respect to elapsed time; and
4. use given information to solve the resulting related rate equation for the requested rate of change.

Enacting this general procedure for solving related rates problems requires a sequence of strategic inferences, coordinations, deductions, abstractions, and generalizations that originate in students' ability to think critically about the context. More specifically, accomplishing the first step requires students to conceptualize the context dynamically and identify rates of change specified in the problem statement and the rate of change the task prompts them to compute. On the basis of these abstractions, in step 2 students can either associate or construct an appropriate formula relating the values of varying quantities, specifically those whose rates are either known or to be determined. Engaging in this type of thinking will improve students' reasoning abilities, which will further advance students' understanding of the central concepts. Not all productive thinking is necessarily critical thinking, however. It is one among a family of closely related forms of thinking relevant to mathematical ability.

Almost all calculus teachers have observed students' difficulties solving related rates problems. It is therefore instructors' responsibility to support students' ability to analyze a problem context correctly, and to foster the kind of reasoning that enables students to strategically and purposefully engage in the process of solving related rate problems.

This study investigates students' engagement in critical thinking while reasoning about related rates problems and examines the role of critical thinking for developing a conceptual understanding of related rates in general. Many studies have discussed the relevance of applying critical thinking skills when solving problems in mathematics. For example, Facione (1990) described the cognitive aspect of critical thinking skills and sub-skills. However, this study mainly focuses on related rates and how two undergraduate calculus students applied critical thinking skills when reasoning about this important concept.

CHAPTER II

LITERATURE REVIEW

Critical Thinking

Thompson (2013) argued that teachers' inability to teach for understanding contributes to students' inability to develop mathematical meanings that support their critical thinking, interest, and future learning. Understanding many ideas of calculus requires students to engage in critical thinking and complex reasoning. Solving related rate problems, for example, requires students to reason quantitatively about complex scenarios to make informed decisions that guide their problem-solving process. Students often wonder why related rates are essential in mathematics and what the practical advantages of related rates are. Sometimes students pose these kinds of questions in classrooms. Teachers should encourage students' critical thinking to create interest in mathematics. Also, reasoning plays a vital role in solving problems in mathematics in general.

Facione (1990) provided a list of critical thinking skills and sub-skills necessary in mathematics. The below are the primary skills and sub-skills that are the focus of this research, and which collectively comprise the analytical framework.

Table 1

Critical Thinking Skills and Subskills (Facione, 1990)

Critical Thinking Skills	Critical Thinking Subskills
1. Interpretation	• Categorization • Decoding significance • Clarifying meaning
2. Analysis	• Examining ideas • Identifying arguments • Analyzing arguments
3. Evaluation	• Assessing claims • Assessing arguments
4. Inference	• Querying evidence • Conjecturing alternatives • Drawing conclusions
5. Explanation	• Stating results • Justifying procedures • Presenting arguments
6. Self-Regulation	• Self-examination • Self-correction

Facione (1990) argued that persons who can integrate successful execution of various critical thinking skills with confidence and sound judgment are more inclined to apply these powerful tools in their other studies and their everyday lives. Similarly, Rott (2021) concluded that students with sophisticated mathematical reasoning abilities seem to be more likely to choose study programs with higher mathematical demands. Several researchers have demonstrated that mathematics is one of the subjects that can develop students' critical thinking skills (Rajendran, 2010; Aizikovitsh and Amit, 2010). Critical thinking in mathematics is closely related to knowledge of the substantive and syntactic structures of mathematics, mathematical reasoning, mathematical epistemology, and mathematical proof and argumentation (Krulik and Rudnick, 1995).

Critical thinking skills, which I mentioned above, are needed when we understand the concept that will trigger ideas (Ennis, 1996). Krulik and Rudnick (1995) argue that critical thinking is tantamount to thinking analytically, which involves questioning, reasoning, testing activities, and evaluating the information or problem context. Similarly, critical thinking requires a student to use new information or manipulate existing knowledge and information to obtain reasonable responses to new situations (Lewis & Smith, 1993). Finally, educational psychologists describe critical thinking as the strategic use of reasoning skills for developing a form of reflective thinking that ultimately optimizes itself and includes decision-making and fluency in problem-solving (Jablonka, 2013).

This research study aims to understand how students engage in critical thinking activities as they solve related rates problems in calculus.

Related Rates: The Concept of Identifying Constant and Varying Quantities

Related rates problems require students to have a working knowledge of rates—as a proportional relationship between corresponding changes in the measures of continuously covarying quantities—and students should need an idea of how they can represent changing quantities with variables. Also, it is essential to conceptualize an applied context in terms of constant and varying quantities, and the relationships between them, or what Thompson (1990) called a *quantitative structure*. Sometimes students make a mistake when substituting variable values prior to implicitly differentiating a formula relating the values of covarying quantities, which will lead to incorrect solutions. This common error reflects unknowingly turning a dynamic situation into a static situation, where rates of change cannot be meaningfully related because nothing is *changing*. Finally, construct a robust image of the context is essential to students' selection or construction of an appropriate formula that relates the measures of quantities in an applied context, and which when differentiated implicitly with respect to elapsed time, yields a related rate formula that expresses the relationship between rates of change in an applied context.

Research suggests that students have difficulty with the concept of variable, which stems from their learning of algebra where variables are often treated as something to "solve for" (Trigueros, 2008). To solve related rate problems, students need to interpret what the problem is asking, analyze the information, evaluate the information, and make strategic inferences to guide their problem-solving activity. Jacobs (2002) suggested that students who had a mature understanding of variables, as representations of the continuum of values through which a quantitie's measure can vary, demonstrated a capacity to express quantitative relationships in addition to simply performing computational tasks fluently.

Related Rate Formulas and Implicit Differentiation

Many students in calculus attempt to apply implicit differentiation while solving related rates problems without ever having constructed a conceptual understanding of what a derivative represents. For example, some students do not interpret dy/dx as the constant of proportionality that relates corresponding infinitesimal changes in x and y.

Moreover, in related rate problems, it is important for students to understand the product structure of the chain rule, and to conceptualize relations between covarying quantities that vary not only with respect to each other, but also with respect to a common time elapsed. Thus, it is essential to know how students engage in critical thinking while connecting the various ideas on which a productive understanding of related rates depends. We cannot understand their reflective thinking abilities by examining their solution alone.

The study from Clark et al. (1997) mentioned that students have difficulty executing implicit differentiation while solving a related rate problem. For example, only 39% solved it correctly. But this study did not indicate whether students' difficulties were rooted in their critical thinking skills while solving these problems. Relatedly, Mitchelmore (1990) had mentioned that students write *dy*/*dx* while solving problems in calculus without much attention to what the expression represents.

Thus, solving related rates problems requires critical thinking skills, which will encourage students' robust understanding and coordination of the foundational concepts that need to be applied to fluently reason about novel related rate problems. Therefore, this research study mainly focuses on essential skills of critical thinking and how students engage in critical thinking when they learn a new concept or solve problems.

CHAPTER III

THEORETICAL FRAMING

Radical constructivism:

Radical constructivism was defined by Von Glasersfeld as a theory of knowing and which has similarity on Piaget constructivism and ideas about epistemology. Piaget and Glasersfeld argued that we construct out concepts and understanding of the world developmentally.

Concept of student developmental learning and hypothetical learning pathways are main focuses on radical constructivism (Von Glasersfield, 2007: Steffe 2007). Educational Psychologist Jean Piaget, notion of genetic epistemology was very influential in Glasersfeld’s theory of radical constructivism, and it provides an explanation of how educators develop new knowledge, and which can help students learn and understand about the concepts.

The Implications of Radical Constructivism:

Radical constructivism mainly focuses on how students learn and think when they learn new concept. Von Glasersfeld suggest that using classroom conversations and asking students verbalize their conceptual understanding will encourage two-way learning between student and the teacher. So, asking to verbalize their thought process in each concept will improve students critical thinking. Students conceptual learning and understanding includes:

- Creating opportunities for making students think and hence will improve their knowledge.
- It encourages students' creation of concepts
- It encourages students to think rationally and make judgements and it recognize and support their efforts learn and motivate students.
- It also encourages students to verbalize and construction of knowledge and their thought process which enables them to think critically (Glasersfeld 2001a).

radical constructivism opened the door for teachers and students to free themselves from very rigid approaches to teaching and learning, particularly in science education.

Quantitative Reasoning:

This study draws on the theory of quantitative reasoning on the context of related rates problem. The quantitative reasoning is the act of analyzing a problem situation in terms of the quantities and relationships among the quantities involved in the situation (Thompson, 1993). What is important in quantitative reasoning is making sense of relationships between quantities (Smith III & Thompson, 2007; Thompson, 1993). In related rates, quantitative reasoning refers to how students: (1) Interpreted and analyze the variable and making a relationship among them, (2) identified or created formulas that relate quantities in different related rates tasks, (3) evaluated these formulas to determine numeric values for quantities, (4) reasoned about time as an implicit or explicit quantity, and (5) reasoned about units or meanings of quantities. Related to the current study, the critical thinking skills also leads to quantitative reasoning.

A quantity is a measurable attribute of an object (Thompson,1994). Thompson (1993) distinguished between a quantity and a numerical value-the former has a unit of measurement and the latter does not. Thompson (1993) remarked that “quantities, when measured, have numerical values, but we need not measure them or know their measures to reason about them” (pp. 165–166). The research suggests that quantitative reasoning will particularly enhance student’s thinking level, and hence it makes more flexible mathematics concepts (Oehrtman, Carlson: Michael A Tallman). According to Thompson (1993), quantitative reasoning is analyzing a problem situation in terms of the quantities and relationships between quantities involved in the problem situation. Thompson argued that what is important in quantitative reasoning is not quantification (i.e., the process of assigning numeric measures to quantities), but rather reasoning about relationships between two or more quantities. Quantitative reasoning seems a particularly important lens for these types of problems, like the covariational reasoning lens used by Engelke (2007), since they inherently deal with quantities-measurable attributes of an object. In fact, it seems that some of the students’ issues seen in related rates studies (e.g., Martin, 2000; Monk, 1992) may deal directly with difficulties involving quantitative reasoning. That may lead them a wrong conceptual understanding of specific situation. While covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) is certainly a key construct when dealing with related rates problems, we note that there may be quantitative ideas, such as the role and use of diagrams to represent relationships between quantities, at play.

Critical Thinking Skills.

Facione(1990) has mentioned there are 6 cognitive aspects of critical thinking skills which are important in many different context. While characterizing each skill and every-subskill to become conceptually discrete from all the other ones. Following are descriptions of CT skills mentioned in Fascione(1990)

1. Interpretation: To comprehend and express the meaning of the problem, make a prediction and procedures how to solve the problem

   1.1 Categorization: Appropriately formulate the distinctions or describing characterizing the given problem.

   1.2 Decoding significance: Describe the information on specific to the content, explain it's purposes and significance to the problem, make a drawing, graphs charts etc.

   1.3 Clarifying meaning: To paraphrase or make explicit through stipulation, analogy, or figurative expression.

2. Analysis: To identify the intended and relationship among statements or make a judgement, reason, and opinion of the specific content.

   2.1. Examining ideas: To determine the role of various statements or arguments. It involves the define terms, compare or contrast ideas, concepts, or statement. Also, ability to make smaller ideas when we have complicated problems or statements.

2.2. Analyzing the arguments: It contains the intended main conclusion the premises and the reasons advanced in support. Also, this means that additional unexpressed elements of the reasoning given several reasons or chains of in support of a particular claim.

3. Evaluation: Evaluation assess the credibility of statements or make a logical connection using the arguments and assess the contextual relevance of the question, information's, and principles.

3.1. Assessing arguments: This justifies one's accepting as true (deductively certain), or very probably true (inductively justified, and that leads conclusion of that arguments. Also, one's can judge between reasonable inference and to determine the extent to which possible addition of an information might strengthen or weaken an argument.

4. Inference: To make a reasonable or secure statements needed to draw conclusion. In particular to recognize and formulate the strategy for seeking information. Also, one's able to determine which of the several possible conclusions most strongly warranted.

5. Explanation: In this skill particularly focusing to state the results of one's reasoning and justify that reasoning in terms of the evidential and contextual consideration. Stating results includes representations of the results of one's reasoning activities to analyze, evaluate, or monitor those results. Finally, presenting arguments means to give reasons for accepting some claims.

6. Self-Regulation: Self-correcting or evaluating one's reasoning or one's results. Self-examination of the work reflect one's own reasoning and verifying the correct application

and execution of cognitive skills involved. Self-correction involves the reasonable procedures to correct or how to resolve the mistakes and their causes.

CHAPTER IV

METHODOLOGY

To investigate calculus students' application of critical thinking while solving related rate problems, I conducted task-based clinical interviews (Clement, 2000) during the fall 2021 semester with two undergraduate students who were enrolled in Calculus 1 (MATH 2144) class at Oklahoma State University. The task sequence aimed at exposing how students engage in critical thinking in various types of related rate problems. (Appendix A contains the task sequence.) The course coordinator of MATH 2144, Dr. Tallman, and I decided to select five questions of increasing complexity from the related rates instructional materials for the course. To understand how students engage in critical thinking, I analyzed students responses to various problems and their interaction with me during the interviews.

Since I was curious about the critical thinking skills students demonstrated when solving the related rate problems in the task sequence, I developed the following general interview protocol that I followed during each students' engagement with each of the tasks in the sequence:

- Describe what is happening in this situation? (This question aims to expose how students *interpret* the problem.)
- What quantities are varying, and what quantities are constant? (This question enabled me to determine if the student conceptualized the context correctly, conceived the situation dynamically, and constructed a quantitative structure that related the quantities described in the context).
- What are you asked to compute? (Students' responses to this question allowed me to explore the *analytical* and *interpretation* aspects of critical thinking.)
- Describe how you might go about solving this problem? (This question, and the following one, enabled insight into the *inferences* students made and their *evaluation* of the context to strategize a plan for solving the problem.)
- Describe the general strategy that you will employ to solve this problem?
- Take a few minutes to solve the problem, and feel free to verbalize your thought process as you proceed.
- Summarize your process for solving this problem (after obtaining a solution),
- How confident are you that your solution is correct? Why? (This question allowed students to provide *explanations* and give insight into their *self-regulation*.)

The last four parts of the questions will evaluate the critical thinking skills, *evaluation, inferences*, and *self-regulation*. I am focusing here on how students' reasoning and thinking informs their actions while solving each related rates problem, rather simply determining whether they obtained an accurate answer. The rationale behind these questions is to evaluate students' critical thinking in various related rates problems.

I conducted two interview sessions with Meghan. Two interviews lasted one hour, and I recorded Meghan's interviews, both written and audio work. During the first interview, I covered three related rates problems, and Meghan was able to share what she thought and provided her thoughts, interpretations, and rationale for her actions. The second interview covered only one question from the task sequence, which was more complicated than the first three problems. But although she only completed one task, her second interview contributed substantial insight into Meghan’s critical thinking, including her interpretation of, and strategy for solving, related rate problems.

I conducted two interview sessions with Karl. Both interviews lasted about one hour. Karl has attempted five questions from related rates problem altogether, and he was able to attempt one more problem than Meghan. But, like Meghan, Karl was also wholly engaged in tasks and was effective at articulating his thoughts while solving problems from the task sequence.

CHAPTER V

RESULTS AND CONCLUSION

I selected five related rate problems of increasing difficulty for the task-based clinical interviews (see Appendix A). My aim was to evaluate the participants' critical thinking skills as they engaged with these five problems. The increasing difficulty of these tasks provided students the opportunity to think critically when they attempted each question. Students who participated in the interviews got a chance to analyze their work, regardless of whether it was correct. Also, I did not expect students to solve each problem correctly; the content of the questions and their progressive difficulty enabled me to focus on students' thought processes, including how they critically evaluated the validity of their work.

Interviews with Meghan

Meghan's' first interview lasted about an hour, and she attempted the first three questions listed in Appendix A. The first question stated, **"If the radius of a spherical balloon increases at a rate of two inches per minute, at what rate is volume increasing when the radius is three inches?"** Meghan was confident in her solution to this first problem. Excerpt 1 documents our discussion of this task.

Excerpt 1

Interviewer: Describe what is happening in this situation?

Meghan: I wanted to find the rate, and two is my derivative. So, I would use the volume of a sphere and then take the derivative with respect to the time *t*.

Interviewer: Can you explain what quantities are varying and what quantities are constant?

Meghan: Rate of change of radius is constant, and the radius is a varying quantity. (She demonstrated confusion about whether the radius is constant or not.)

Interviewer: Feel free to solve the problem and feel free to verbalize your thought process here.

***Meghan*:** Okay, we know $V = \frac{4}{3}\pi r^3$, and we want to make derivative on both sides, we get $\frac{dV}{dt} = \frac{4\pi}{3}\left(\frac{dr}{dt}\right)^3$. Now substitute $\frac{dr}{dt} = 2$, and that value is the rate of change of volume.

Interviewer: When you look back to your solution again, how confident are you?

Meghan: I went wrong on the step I took the derivative …

Interviewer: Why do you think that?

Meghan: Because I do not have any "*r*" in my rate equation to plug the value of the radius

r = 3. I think,

$$\frac{dv}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt} = \frac{4}{3}\pi\,(3)2^2(2) = 24\,\pi.$$

Interviewer: How confident you are currently?

Meghan: I am very confident now.

From her responses on the first task, Meghan demonstrated a correct interpretation of the first problem. When she started solving the problem, she did not recognize that she needed to use the chain rule to construct the correct related rate formula. She represented the derivative of r^3 with respect to t as $\left(\frac{dr}{dt}\right)^3$, but she then re-evaluated her work, identified where she went wrong, and finally, completed the problem correctly. From here, we could say Meghan was able to think critically about whether she was correct or not in the contest of the problem. She was also able to think critically brought up the right answer for the question.

The second task was a little more involved than the first one. The second question that Meghan attempted was, **"A 10-foot ladder is leaning against a wall, the bottom is pulled away at a constant speed of 1 feet/sec. How fast is the top of ladder moving when it is 5 feet from the ground?"** Meghan thinks this problem is more complicated than the first question.

Excerpt 2

Interviewer: How do you look at this problem?

Meghan: This problem is more challenging than the first one. For the first problem, I have the formula to use, but here I do not know.

Interviewer: What are you being asked to compute?

Meghan: I want to find a derivative at which the top of the ladder moving. One foot per second represents another derivative, and I must use it when I solve the problem. Also, the height of the ladder is constant, which is 10 feet height ladder.

Interviewer: Describe how you might go about solving this problem?

Meghan: This is another triangle problem, since I want to find the speed of the top of the ladder is moving, so I might use the equation speed = distance/time. Where speed = $\frac{ds}{dt} = 1$= 1 feet/sec and distance = 5 feet.

Interviewer: Is this your answer?

Meghan: I am not sure! I did not use the 10 feet in my answer, I think I want use that as well.

Interviewer: Okay, you are mixed up with a different concept. I will show you how the visual representation of this problem looks like. (Dr. Tallman showed a dynamic animation of a moving ladder). Now think how you would solve the problem.

Meghan: I would use Pythagorean theorem, $5^2 + c^2 = 102$

$$c^2 = \sqrt{}(100 - 25)$$

$$c = \sqrt{}75.$$

This would be the base length for the right triangle. So, I know the bottom of the ladder moving at a constant rate $\frac{dx}{dt}$ =1 feet/sec, and I want to find the speed at which top of the ladder falling. So, I can use the Pythagorean theorem, $\frac{dy}{dt}^2 + \frac{dx}{dt}^2 = \frac{d}{dt}^2$ but I don't have any speed corresponding to the side hypotenuse of my triangle.

Interviewer: So, do you think Pythagorean theorem is also correct for the rates corresponding to each side?

Meghan: Yes.

Interviewer: Can you explain me what is $\frac{dy}{dt}$, and $\frac{dx}{dt}$ represent?

Meghan: $\frac{dy}{dt}$ is the speed at which the top of the ladder falling, $\frac{dx}{dt}$ is the speed at which the bottom of the ladder pulled away, and which is 1 feet/sec.

Interviewer: How confident you are on the equation connecting the rates?

Meghan: I am very confident, because that equation contains $\frac{dy}{dt}$ and I can solve that.

When I analyzed Meghan's' interview, she was confused to proceed the problem as she did not interpret the problem correctly. She was thinking initially she can use $speed = \frac{distance}{time}$ to solve the problem. After seeing the visual representation, she understood that she wanted to evaluate the rate at which the ladder falling when the height is 5 feet from the ground. At this point Meghan's interpretation was right. Also, she was right at the marking sides of the right triangle, and she was right when she used the Pythagorean theorem to find the length of the base of the triangle. So, she was able to make an equation relating the variables, but she did not consider differentiating her equation $x^2 + y^2 = 10^2$. Visual representations were able made her to think critically about making accurate Pythagorean equation. More specifically, she did not make an actual inferential relationship among statements, and hence she did not critically analyze the problem situation correctly. For example, after $x^2 + y^2 = 10^2$, she did not assess the credibility of the statements "$\frac{dy}{dt}^2 + \frac{dx}{dt}^2 = \frac{d}{dt}^2$." Instead, her reasoning was she use Pythagorean theorem for the equations of rates. Also, the meaning of the derivates completely ignored when we write "$\frac{d}{dt}^2$."

When I asked her how confident she was in her solution, I wanted to know whether she could justify her reasoning. Here she is much confident about what she wrote, and she was confident that she can solve $\frac{dy}{dt}$, if she knows the rate that relate to the side hypotenuse of the triangle. The absence of the analyzing skills, Evaluation skill and the self-evaluations skills, made her unable to solve the task.

Meghan's third task was more complicated than first two tasks. We decided increasing complexity of task as it is important know how students are engage in critical thinking in various context. "A water tank in the shape of an inverted cone has a radius of 2 meters and a height of 4 meters. If water is pumped in at a rate of 3 m^3/min. Find the rate at which the water level is rising when the water is 3 meters deep?" In this task student want to think deeply about the situation before they attempted the problem. Also, needed the prerequisites for finding the volume of the cone.

Excerpt 3

Interviewer: How do you interpret the problem?

Meghan: Here I can say that the radius of a cone is 2 meters, which is constant, and height is 4 meters.

Interviewer: What is 3 m^3/min here?

Meghan: 3 m^3/min refers to the rate of the volume of the water pumped in.

Interviewer: How can you represent it?

Meghan: $\frac{dw}{dt} = \frac{3m^3}{minute}$.

Meghan correctly interpreted the rate of change of volume of water with respect the time. Also, she analyzed the problem in such a way that she knew she wanted to use the equation of the cone, but she forgets it. Also, Meghan was thinking the rate at which the water is pumped in is same as the rate which is asking to find on the problem. She did not interpret the meaning of the different rates described on the problem, and that is why she was unable to proceed the problem.

Excerpt 4

Interviewer: The equation of the volume of a cone is, $V = \frac{1}{3}\ \pi r^2 h$. Now how can you proceed here?

Meghan: I know *r* equals two, and height is four meters.

Meghan drew a picture of an inverted cone and she realized that the two rates are different. She concluded that we want to find the rate of change of height at which the water level rising with respect to the time, and $\frac{dw}{dt} = \frac{3m^3}{minute}$ is the rate of change of volume of water with respect to the time. At this point, Meghan was critically evaluated her previous statement and she realized that the two rates are different. So, it is important to think critically, and interpret, analyze, and evaluate our response before we proceed the problem. If she did not critically evaluate her statements, her thought regarding distinguish between two rates would lead her wrong answer to the problem.

Meghan did not get enough time to work on the problem, as we are reached 1 hour for the meeting. But, at last she was concluded that she wanted to find the rate at the water level is rising, also analyzed “water level is rising” related to the height. So, she concluded that she would proceed the problem to take the derivative on both sides of the equation, $V = \frac{1}{3}\,\pi r^2 h$.

Meghan’s second interview for task 4 lasted one hour. We constructed the task 4 as more complicated than first three tasks. I wanted to see how student’s critical thinking can help to solve complex problems. The task 4 addressed the problem “A man starts walking north 4 feet/sec from a point *P*. Five minutes later a woman starts walking south at 7 feet/sec from a point 500 feet due east to *P*. At what rate are the people moving apart 15 minutes after the woman start walking?” This problem required to think a lot before the student attempt to solve.

Excerpt 5

Interviewer: How do you look at this problem.?

Meghan: The problem gives the rates which man and woman are walking. So the rate at which man walking , $\frac{dm}{dt} = 4$ feet/second, and the rate at which woman walking $\frac{dw}{dt} = 7$ feet/sec. So, we can draw a right triangle connecting these informations.

Meghan tried to draw the picture, but she felt that’s not correct. But she did mark the directions correctly. Meghan identified the point *P* at which the man and woman are walking correctly, and then she marked $\frac{d}{dt}$ as the rates at which people are moving after 15 minutes. Initially, Meghan thought the man and woman are walking at the same rate after 15 minutes, but she self-evaluated her response, and told they are walking at different rates. Hence, it is important to know why she thought about this.

Excerpt 6

Interviewer: Tell me more about why do you think the possibility of two related rates?

Meghan: Because there are two different people, and they are walking at two different rates. So, I can use the Pythagorean theorem in the following way:

$$\frac{dm}{dt}^2 + \frac{dw}{dt}^2 = \frac{d}{dt}^2 = 4^2 + 7^2.$$

Meghan was not confident about the equation she wrote, as she felt she did not use much information. Especially, the woman's distance 500 feet from the point *P*. We could understand from here that Meghan was evaluated her work critically as it made her think I did not use all the information's here.

Excerpt 7

Interviewer: When you applied the Pythagorean theorem here, are you confident that you can use it for the rates as well rather than the length of the sides of the right triangle?

Meghan: Yes

From here we could understand that her understanding or meaning of the related rates are not clear. She did not interpret and analyze the problem correctly. After some point Meghan was thinking her triangle picture, she represented early is not correct, the intention for her thinking was she wanted to use 500 feet in her figure. So, she drew her picture again and marked hypo tenuous as 500 feet. But she evaluated her picture again and she made her comments that since the woman walking 500 feet east of the point *P*, 500 feet should be marked on the right of *P*. Also, she analyzed the problem again and she was thinking to need another equation connecting the variables and then take the derivatives on both sides. The reason why she thought about this is she can plug in, $\frac{dm}{dt}$ and $\frac{dw}{dt}$ at some point.

Excerpt 8

Interviewer: Why do you think you are missing an equation here?

Meghan: When we solve related rates problem, we will have an equation first and then we will take it's derivative. I am missing that step here.

When I analysed her response I understood that students mostly follow the steps involved in solving related rates problem without much further thinking of their work. I wanted to know how her reasoning will change if we show the dianamic representation of the problem.

Excerpt 9

Interviewer: Why do you think I draw the woman walking down here (see Figure 1)?

Meghan: Okay, now I realized that woman is walking south at 7 feet/sec. So, it should be the opposite direction of man walking.

Interviewer: what is asking to solve here?

Meghan: The speed at which man and woman are walking. Oh no. On the question it says they are moving apart. So, there should a line connecting those two points. (Meghan was right at this point. So, creating a visual representation of the problem made to think her in a right direction.)

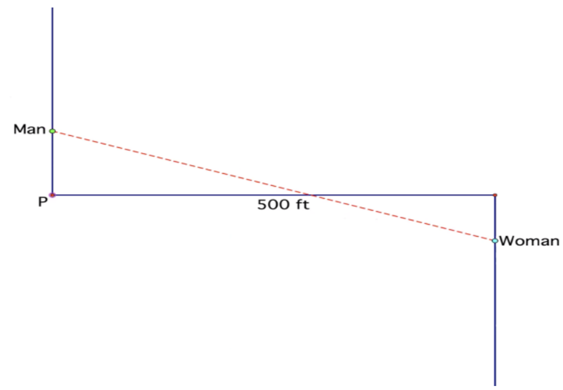
Interviewer: Very good! So, I should connect them using a straight line. Now, what do you think who is going to walk first? Man, or woman?

Meghan: The man starts first.

Interviewer: Okay, so here man going to walk first and then five minutes later woman is going to walk.

Figure 1

Interviewer's Sketch of the Context from Problem 4



Meghan agreed that she wanted to find the rate at which the red line moving after seeing the visual representation. We asked her to rethink about her work after seeing this, and Meghan did approach the problem in following way

$$\frac{dm}{dt} = 4\frac{feet}{sec} + 5\,min$$

$$\frac{dw}{dt} = 7\frac{feet}{sec}$$

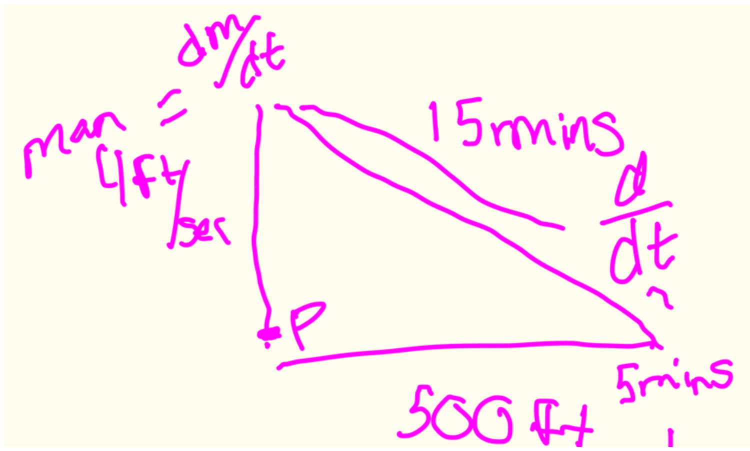
$$d = 500\,feet$$

Her assumption was she wanted to find $\frac{dd}{dt}$ at 15 minutes. At this point we could also understand that student was not able to interpret the meaning of what does it mean when finding the rate at which 15 minutes after the woman start walking. She explained why she had difficulty of solving this problem. Since this picture is having two triangles it is also difficult for her to identify the suitable right triangle appropriate to this situation. So, this means that she did not analyze the problem the way it is. So, we wanted to explain little more about the problem using the visual representation. Dr. Tallman created an animation which man walking and after five minutes later woman starts walking, and their walk followed by a traced line (see Figure 1).

After this Meghan draw her triangle as shown in Figure 2.

Figure 2

Meghan's Drawing While Responding to Task 4



Meghan was so confused why she did not have regular numbers here rather than the rates. At this point she did not realize to connect the sides of the triangle as distance of man or woman are walking. But she critically self-evaluated her work and made the assumptions that she could draw her right triangle with $m + w$, 500, and d as their length, and she said she can use the Pythagorean theorem in following way $(m+w)^2 + 500^2 = d^2$ and she knew that she wanted to take the derivative on both sides. But she did not analyze her work as if she wanted to use the chain rule while taking the derivatives. She completely ignored that, and she tried it following way,

$$\left(\frac{dm}{dt} + \frac{dw}{dt}\right)^2 + t^2 = \frac{dd^2}{dt}.$$

When I asked about why did she use the variable t in her derivative equation, her reasoning was we wanted plug in 15 minutes and 7 minutes at some point, so we want a variable t in our equation. In this situation, Meghan really thinks about the situation, but she did not come up with the inference, or did not able to make a reasonable conclusion of her work.

Excerpt 10

Interviewer: Ok on your last equation, you said you took the derivative on both sides, do you mean the derivative of 500^2 is t^2?

Meghan: No, I think I can think the following way.

$$\left(\frac{dm}{dt}+\frac{dw}{dt}\right)^2+0=\frac{dd^2}{dt}\ t$$

Interviewer: Why is that '*t*' on the right side

Meghan: Because I am taking the derivative with respect to time, but there is no actual time on numerator.

At last, she had concluded that,

$$\left(\frac{dm}{dt}+\frac{dw}{dt}\right)^2+0=\frac{d}{dt}\ t.$$

Meghan also stated that she could divide by *t* on both sides to find the derivative. She expressed a high degree of confidence in her work at this point.

When I analyzed Meghan's' second interview, the task was little hard. When I evaluated her work each time, I understood that she did not understand the meaning of the related rates completely. She failed to think critically most of her work, that's why she ended up a wrong reasoning. When an interviewer provided the visual representation of the problem, she provided more meaningful reasoning and were able to think critically than before. Also, she did not think critically evaluated her work when she took the derivative each time, and which made her reasoning and work wrong.

Table 2

Meghan's explanation and Critical thinking skills

Meghan's explanations in various tasks	Critical Thinking skills/Subskills
"I wanted to find the rate, and two is my derivative. So, I would use the volume of a sphere and then take the derivative with respect to the time *t*".	Meghan Interpreted the task 1 Analyzed the task that she wanted to use the volume of a sphere and then do the differentiation
$V = \frac{4}{3}\pi r^3$ $\frac{dV}{dt} = \frac{4\pi}{3}\left(\frac{dr}{dt}\right)^3$	Meghan did not evaluate the problem at first as she did not have a good idea about implicit differentiation
I do not have any "*r*" in my rate equation to plug the value of the radius $r = 3$.	She self-evaluated her work and that made her to rethink the differentiation again

$\frac{dv}{dt} = \frac{4}{3}\pi r^2 \frac{dr}{dt} = \frac{4}{3}\pi\,(3\,)2^2(\,2)$ $= 24\,\pi.$ "I know the bottom of the ladder moving at a constant rate $\frac{dx}{dt}$ =1 feet/sec, and I want to find the speed at which top of the ladder falling"	She made an inference and correctly evaluated the task. She did interpret the problem after seeing the dynamic representation of the problem
I can use the Pythagorean theorem, $\frac{dy^2}{dt} + \frac{dx^2}{dt} = \frac{dz^2}{dt}$, If the equation connection to the side is $x^2 + y^2 = z^2$	Did not evaluate the validity of the statement and failed to evaluate critically here!
"The rate at which the water is pumped in is same as the rate at which the water level is rising"	She analyzed and interpreted the meaning of $\frac{dW}{dt}$ and she did not analyze the problem that she wanted to evaluate $\frac{dh}{dt}$
The rate at which man walking , $\frac{dm}{dt} = 4$ feet/second, and the rate at which woman walking $\frac{dw}{dt} = 7$ feet/sec	Interpreted the arguments.
On the question it says they are moving apart. So, there should a line connecting those two points	Meghan Interpreted and Analyzed the task in a right direction and the visual representation helped her to reevaluate her previous work

$$\frac{dm}{dt} = 4\frac{feet}{sec} + 5\ min$$ And I wanted to find $\frac{dd}{dt}$ at 15 minutes, when d = 500 feet.	Meghan failed to think critically what it means when finding the rate at which 15 minutes after the woman start walking
$$(m + w)^2 + 500^2 = d^2$$ $$\left(\frac{dm}{dt} + \frac{dw}{dt}\right)^2 + t^2 = \frac{dd}{dt}^2.$$	Wrong evaluation of the problem. Did not have an idea about the implicit differentiation.

Interviews with Karl

We had two interview sessions with Karl. Both interviews were video recorded, and all his data was transcribed. Karl completed one more problem than Meghan. We did interview with Karl with the same tasks' series.

Task one was about the radius of the spherical balloon problem. Karl thought task 1 was an easy related rates problem, and he was able to solve the problem correctly and provided reasoning correctly. When asked to interpret the problem, Karl responded, "I want to find at what rate the volume of a spherical balloon is increasing when radius is exactly 3 inches. Radius is increasing at a rate of two incher per second." Karl's response demonstrates that he was able to interpret the problem correctly. Identified what he wants to find and what are the information's provided to find the unknown quantity. To answer the second question Karl wanted to analyze the problem correctly, so I wanted to make sure he understood about the quantities.

Excerpt 11

Interviewer: What quantities are varying here and what quantities are constant?

Karl: Volume is changing, radius is also varying. But when radius is 3 inches, which is constant we can spot on the volume. So, these are the constants.

Interviewer: What are you being asked to compute?

Karl: I want to find the rate at which the volume is increasing when the radius is 3 inches.

Interviewer: Okay, how do you represent that?

Karl: I want to find $\frac{dv}{dt}$ at $r = 3$ inches.

Interviewer: Describe how you might go about solving the problem?

Karl: We know what the volume of a sphere is, and then I can take the derivative with respect to the time *t*.

Interviewer: Take a few minutes to solve the problem and try to verbalize your thought while you solve.

Karl: Okay, I can solve the problem in following way: $V = \frac{4}{3}\pi r^3$. I want to find $\frac{dV}{dt}$ when r = 3 inches. When r = 3, V = 36π. Then, $\frac{dv}{dt} = 0$.

Interviewer: How do you evaluate your problem?

Karl: I am sure about the way I approached the problem. But the equation $\frac{dV}{dt} = 0$ does not make sense!

Interviewer: Why do you think like that?

Karl: Because I know that the rate of change of volume increases when r increases.

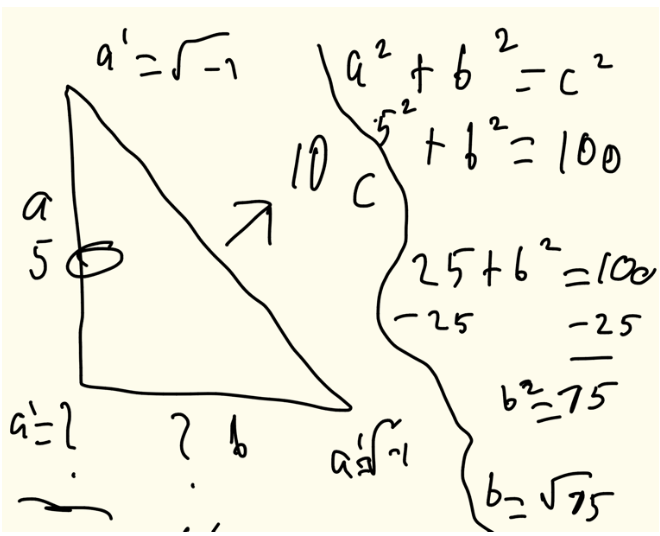
Karl was more confident about these questions. He was able to spot on what he wanted to compute and analyze the problem in a proper way. Here Karl was able to critically evaluate his equation of the related rates and that made him to rethink what should be the right way of approach the problem. Karl then said since we are taking the derivative of the volume with respect to t, it could be 72π. Because we have an exponent r^3. Karl's critical thinking on this problem varies differently. He was able to interpret the problem and analyze the problem correctly. He critically evaluated his statement $\frac{dV}{dt} = 0$, and that helped to solve the problem correctly.

Karls' next task is about the ladder problem. The ladder problem was little involved than the first task. Karl was sure that the ladder problem is a triangle problem. So, Karl was able to interpret and analyze the situation correctly. He correctly marked each side as the length and was able to identify variables and the rates are related. He was sure that $\frac{db}{dt} = 1$, and he wants to find $\frac{da}{dt}$.

When I compared Meghan's work with Karl's, I could see that Karl is having better understanding of rates and their meaning. Also, Karl is critically analyzed the problem here. Karl's written work is displayed in Figure 3

Figure 3

Karl's Written Work on Task 2



Notice that Karl expressed the relationship between the side lengths and the hypotenuse with the equation $a^2 + b^2 = 10^2$. He then plugged 5 in for *a* and solved for *b*, to obtain $b = \sqrt{75}$.

Karl was right here, or he could make the relationship between *a* and *b*. He was sure that he wanted to find a' here. He started solving following way,

$$a'^2 + b'^2 = c'^2$$

$$a'^2 + 1 = c'^2$$

Excerpt 12

Interviewer: What is 'c' here?

Karl: *c* is constant here, so *c'* is never change and hence *c'* = 0. (Even though Karl above equation is wrong, his reasoning for understanding related rates make sense! Karl solved the above equation as $a' = \sqrt{-1}$.)

Interviewer: Are there any part of your solution you are not confident?

Karl: I am confident.

Interviewer: When you look at your solution, you have not used the value $b = \sqrt{75}$ anywhere in the problem. What do you think about it?

Karl: I think we may not need that information.

Karl did not critically evaluate his work when he took the derivates. Also, he did not check his validity of the equation $a'^2 + b'^2 = c'^2$, whether the Pythagorean theorem is also applicable for derivative. It had really interested me that both Meghan and Karl think they can use the right triangle law for the rates.

The third task is about the inverted cone problem.

Excerpt 13

Interviewer: How do you look at the problem?

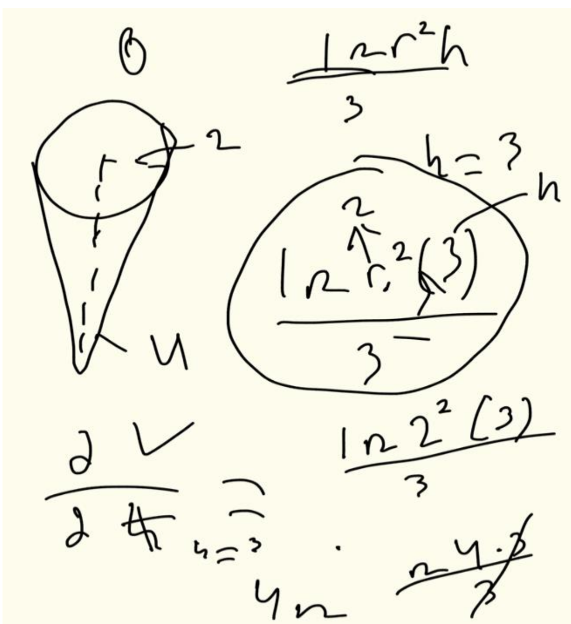
Karl: (Sketched the cone displayed in Figure 4) The volume of the cone is, $V = \frac{1}{3}\pi r^2 h$. Here, *r* = 2 and *h* = 3. So, when I substitute these here, I would get *V* = 4π.

Interviewer: Okay, what is 4π represent?

Karl: The volume of the cone. Then you need to find the derivative of the volume with respect to the time, and which is zero.

Figure 4

Karl's Written work for Problem 3



Karl did not realize here that he did not use all the information to solve the problem, that is he needs $\frac{dV}{dt} = 3$, and he did not identify the relationship between the radius r and h, which is $r = \frac{h}{2}$. Also, he was believing that he wanted to find the $\frac{dV}{dh}$ instead of $\frac{dV}{dt}$ as the questions asks the rate at which the water level is rising when the water is 3 meters deep. So, it is important to know whether he understood what's the difference is between $\frac{dV}{dh}$ and $\frac{dV}{dt}$ on the contest of problem.

Excerpt 14

Interviewer: What is the difference between $\frac{dV}{dh}$ and $\frac{dV}{dt}$ here?

Karl: Okay, here we want to find $\frac{dV}{dt}$, because the water is pumped in a rate of $3\ m^3/min$.

Interviewer: In which part of your work are you not confident here?

Karl: I have not used the rate at which water is pumped in, which is $3\ m^3/min$.

Interviewer: How confident you are about your work?

Karl: Compared to first two tasks, I am not that confident.

Karl did not analyze the problem that he wanted to find the rate $\frac{dh}{dt}$, and he said that he was more confident on the ladder problem than this. Karl self-evaluated his work, and he could not identify where he went wrong. He believed that he wanted to find the rate of change of volume with respect to the time when the height of the water, *h* =3. Also, he thinks once he solves $\frac{dV}{dt}$, there might be a formula where we can plug *h* = 3.

Karl was not able to correctly interpret the rates $\frac{dV}{dh}$ and $\frac{dV}{dt}$. He was thinking he wanted to solve $\frac{dV}{dt}$, and then we must substitute $h = 3$ in that equation. This is because Karl did not critically analyze the meaning of the sentence "find the rate at which the water level is rising when the water is 3 meters deep."

Since Karl was able attempt the first three tasks in the first clinical interview, the second interview focused on more complicated problems which need strong critical thinking skills before they attempt the problem. We decided the task series increasing difficulty, so that we could analyze how students engage in critical thinking, how their wrong reasoning led to solve the task. The statement of Problem 4 is as follows: "A man starts walking north at 4 feet/sec from a point *P*. Five minutes later a woman starts walking south at 7 feet/sec from a point 500 feet due east to *P*. At what rate are the people moving apart 15 minutes after the woman started walking?"

Excerpt 15

Interviewer: Describe what is happening in this situation?

Karl: We want to find the rate people are moving 15 minutes after the woman start walking by using the rates which man and woman are walking:

Interviewer: What are the quantities here are varying and what are the quantities are constant?

Karl: The distance 500 feet is constant; I can draw the situation here and explain. By using this triangle, I can make one right triangle, which will be (draws right triangle, see Figure 5). So, we want to find *c'*, the rate at which people are moving.

Interviewer: How did you get $a' = 11$?

Karl: I can add two rates together and can make a one side of the triangle. Since 500 feet is not changing so $b' = 0$. I can use the Pythagorean theorem to solve the *c'*.

$$a'^2 + b'^2 = c'^2$$

$$c'^2 = 121$$

$$c' = 11$$

Interviewer: Is this your final answer?

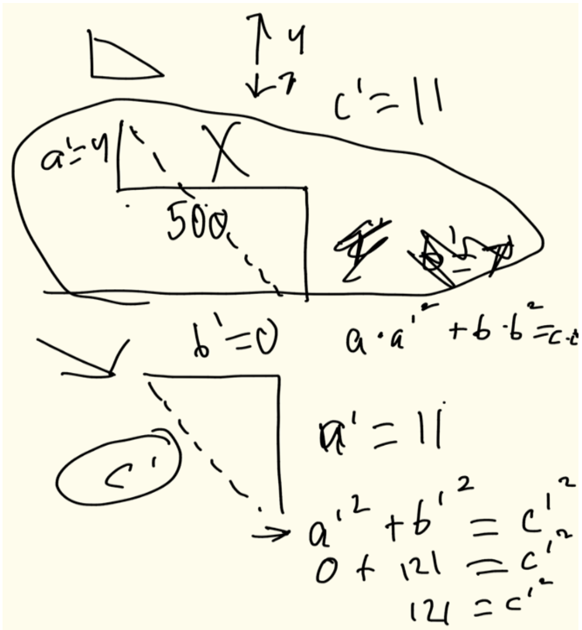
Karl: Yes.

Interviewer: How did you come up with the equation $a'^2 + b'^2 = c'^2$?

Karl: I can use the Pythagorean theorem for rates.

Figure 5

Karl's Written work on Problem 4



When I analyze the Karl’s work on Problem 4, he can interpret the problem and analyze the problem. But when started solving the problem, he did not realize that he wanted to first make a relationship between variables. His pictorial representation of the problem was correct. Karl said his first picture represent the given situation is not correct, as he thought man and woman are moving at different direction. He did not realize that Pythagorean theorem is applicable if he treats the sides of triangle as its length. Also, He was not able to make an inference about his conclusion.

When we analyzed Meghan’s and Karl’s work. They both were using Pythagorean theorem for the derivatives, and they were confident that they can use the theorem for rates. Karl was able to draw the situation correctly without seeing the virtual representation of the problem, and finally they both were failed to think critically about the correctness of their work at last.

Task 5 is a related rates problem involving trigonometry. The question is, "A hot air balloon raising straight up from a level field is tracked by an inclinometer 500 feet from the lift off point. At the moment the inclinometers angle of elevation is $\frac{\pi}{4}$ radians. The angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at this moment?" Karl was able to attempt this, which Meghan did not.

Excerpt 16

Interviewer: How can you interpret the problem?

Karl: I want to find how fast is balloon rising and given that the angle is increasing at the rate of $\frac{\pi}{4}$ radians.

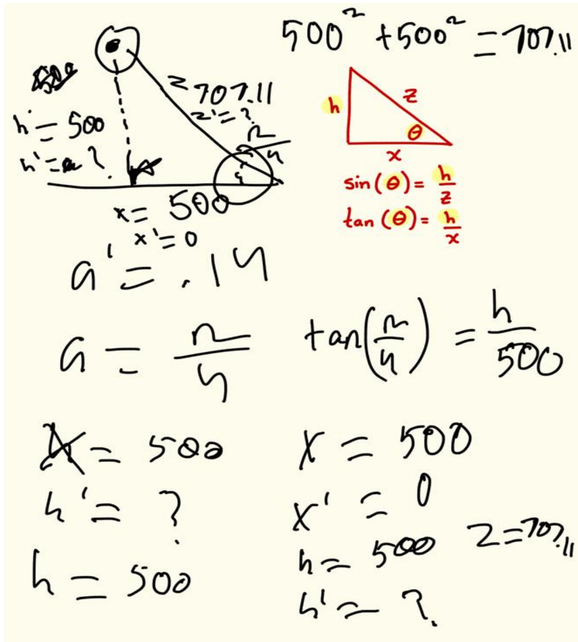
Interviewer: How do you represent the sentence "angle is increasing at the rate of $\frac{\pi}{4}$ radians" mathematically? (Karl drew a right triangle that represent the problem. See Figure 6)

Interviewer: What is the 500 represent here?

Karl: That is the initial height.

Figure 6

Karl's Written Work on Problem 5



Karl was confused about how to connect all the information to solve this problem. But he did try to let *a* represent the angle which is $\frac{\pi}{4}$, since it is given that the angle is increasing at a rate 0.14 radians per minute, $a' = 0.14$. When he made this assumption, Karl did not evaluate his response critically that his assumption is logically incorrect. Also, he proceeded by recognizing the need to compute the rate of change of *h* with respect to elapsed time, knowing that *h* = 500 feet. Karl was not sure how to connect the information provided in the task statement to construct an equation.

At this point, the interviewer supplied Karl with some information about trigonometric rations that might have enabled him to make progress on the task (see the red text in Figure 6). We just wanted to know whether he could make any equation using either of above trigonometric equation. After, providing this I just wanted to know whether Karl understand what *x*, *h* and *z* represent in this situation. After evaluating his work Karl was realized that 500 feet represent *x* here. So, the first time he was wrong, and this is because he did not correctly interpret the meaning of "500 feet from the lift off point." Karl then proceeded by writing $\tan\left(\frac{\pi}{4}\right) = \frac{h}{500}$. Then, *h* = 500, and *x* = 500. He stated, "I want to find *h* prime here."

Karl's next attempt was whether he could substitute $a' = 0.14$ into the trigonometric equation $\tan(\mathrm{a}) = \frac{h}{x}$, and then he will be able to find $\frac{dh}{dt}$. It's really interested that Karl thinks if he could replace the variable with rates, the trig equation still works.

So, he proceeded by writing the following:

$$\tan(a') = \frac{h'}{x'}$$

$$\tan(a') = \frac{0}{0}$$

At this point Karl realized that he might need to use other trig equation. But he wanted to first find what is *z*. He explained, "Okay, now I am thinking $500^2 + 500^2 = z^2$. So, *z* =707.11, and I am stuck here." The interviewer replied, "Replacing variables with its derivative in trig equation is not true, it happens only very limited circumstances." During his last task, Karl was able to interpret the problem critically and analyze it, but he was failed to think critically about his work.

Table 3

Karl's explanation and critical thinking skills

Karl’s explanations in various tasks	Critical Thinking skills/Subskills
I want to find $\frac{dv}{dt}$ at $r = 3$ inches.	Interpret the problem and analyze the arguments.
We know what the volume of a sphere is, and then I can take the derivative with respect to the time *t*	Karl did make an inference how he procced the problem
I want to find $\frac{dV}{dt}$ when *r* = 3 inches. When *r* = 3, *V* = 36π. Then, $\frac{dv}{dt} = 0$.	Did not critically evaluate the task at first
But the equation $\frac{dV}{dt} = 0$ does not make sense! I know that the rate of change of volume increases when *r* increases	Karl self-evaluated his work again/Critically evaluated his last statement and this helped to solve the problem finally!
$a^2 + b^2 = 10^2$. And $b = \sqrt{75}$.	Interpreted and analyzed the ladder problem in a right direction
$a'^2 + b'^2 = c'^2$ $a'^2 + 1 = c'^2$	Had an idea about to take the derivatives but evaluated it in a wrong way
We want to find the rate people are moving 15 minutes after the woman start	Correctly interpreted and analyzed what is asking and the drew the picture perfectly

walking by using the rates which man and woman are walking:	
$$a'^2 + b'^2 = c'^2$$ $$c'^2 = 121$$ $c' = 11.$ Pythagorean theorem is also applicable for rates	Failed to think critically/evaluated critically Not good understanding of rates and it's meaning.
$V = \frac{1}{3}\pi r^2 h$. Here, r = 2 and h = 3. So, when I substitute these here, I would get $V = 4\pi$.	Did not interpreted and analyzed the inverted cone problem. Also, ignored much information while solving problems. Did not analyze the arguments $\frac{dv}{dt}$ and $\frac{dh}{dt}$
$\tan\left(\frac{\pi}{4}\right) = \frac{h}{500}$. Then, h = 500, and x = 500. He stated, I want to find h prime here.	Interpreted to use trig identity but failed to evaluate critically on the contest.
$$\tan(a') = \frac{h'}{x'}$$ $$\tan(a') = \frac{0}{0}$$ Replacing variables with rates in trig equation still work	Wrong interpretation and evaluation:

Discussion and Conclusions

When I analyzed the two students' task-based clinical interviews, I revealed that students were engaged in critical thinking in various ways. The most challenging task required a high level of thinking, and students were facing difficulty in solving the problem. When students solve related rates problems, they are knowingly or unknowingly engaged in critical thinking. In some tasks, they were able to interpret, analyze, and evaluate their situation correctly critically. However, in the most challenging task, both students were facing the difficulty of providing reasoning and were failed to engage in critical thinking. Also, I understood the common mistakes that happen when they take derivatives on related rates problems. They just think $\frac{dy}{dx}$ as a fraction. This is mainly because students think of the function $\frac{dy}{dx}$ as a notation or misconceptions about the rates. Therefore, this led them to a wrong answer. Following are the main points I summarized when I interviewed Meghan and Karl.

- When I asked them about "how they interpret the problem," they both were able to provide me with what they wanted to find, what quantities vary, and what quantities are constant. So, they understood the meaning of the problem correctly.
- Since some of the tasks needed higher-level thinking, students were having difficulty coordinating the information. For example, on task 3, Meghan could not draw the picture exactly from where man and woman are starting initially. But Karl was able to visualize and draw the situation correctly.
- Both did not make logical inferences for their rate equation, that is Meghan was wrong at if she can write $x^2 + y^2 = c^2$, then she said this equation followed

$\frac{dx^2}{dt} + \frac{dy^2}{dt} = \frac{dc^2}{dt}$. On the other hand, Karl wrote if, $(x + y)^2 + z^2 = m^2$, then $\left(\frac{dx}{dt} + \frac{dy}{dt}\right)^2 + \frac{dz^2}{dt} = \frac{dm^2}{dt}$. So, both Meghan and Karl were failed to evaluate their equation critically and they did not think they did want to use the chain rule for the differentiation.

- Karl was able to attend the trigonometric problem, but he was failed to decode which trigonometric identity he wanted to use in that problem. Also, he did not correctly interpret the meaning "the angle is increasing at the rate of 0.14 radians per second.
- Both students were successfully able to solve the first task. When the problem was difficult, students were having difficulty of solving the tasks and they were not able to give the reason correctly.
- Both students were failed to evaluate their difficult tasks like trig problem and the problem of Man and women walking. Both had a wrong solution, and they did not have a good understanding of the concepts of rates and differentiation.
- Karl was wrong at critically evaluating his statement about interchanging variable in the trig identity with its derivative. That is, he wrote

$$\tan(a) = \frac{h}{x}$$

$$\tan(a') = \frac{h'}{x'} = \frac{0}{0}.$$

This was incorrect, and Karl demonstrated confusion about his answer.

During the interview with Meghan and Karl, they were confused about many concepts, and they did not wholly understand taking derivatives. But, when they critically evaluated and analyzed their work, I felt they both were confident in solving the task. So, engaging in critical thinking made their good understanding, and hence they were able to solve the problem correctly.

Recommendation for Instructors

The result of my study indicates that students having a hard time to understand the related rates problem. As critical thinking skills evoke students cognitive reasoning, it is important to make sure that students are engaged in critical thinking while they solve problems. Instructors should encourage students to think critically in classroom. It is also important to know each time how students interpret the problem, analyze the arguments and how they are making inference in given situation. Most of the time instructors won't understand their reasoning for wrong statements until we talk with them, so we have to make sure that we are providing enough time to think about each problem. When it's come to the complex problems like related rates and the implicit differentiation, most students are attempting to solve the problems as they are more curious to get the answers rather than understanding what they are doing. So, it is important to give some time to think what's going on the problem before they solve it. When students make false reason like "Pythagorean theorem is also applicable for rates", as an instructor we want to remind them it's only possible in a very rare circumstances, possibly with an example. Also, when student think if the equation $x^2 + y^2 = z^2$, and ask them what does happen if we take the first derivative. Like Karl and Meghan, they thought the rates equation is, $\frac{dx^2}{dt} + \frac{dy^2}{dt} = \frac{dz^2}{dt}$. Student did not think about the role of implicit differentiation. So, when instructor introduced the topics it's important that whether student's get an idea what's going there and what’s it’s meaning. Maybe, just ask couple of questions before students start doing differentiation, in order to understand whether they get the idea or not.

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APPENDICES

Task-based clinical interview problems:

1. If the radius of a spherical balloon is increasing at a rate of 2 inches per minute, at what rate is the volume increasing when the radius is 3 inches? (Enter your answer accurate to two decimal places.)
2. A 10-foot-long ladder is leaning against a wall. The bottom is pulled away at a constant speed of 1 feet/sec. How fast is the top of the ladder moving when it is 5 feet from the ground?
3. A water tank in the shape of an inverted cone has a radius of 2 meters and a height of 4 meters. If water is pumped in at a rate of 3 m^3/min. Find the rate at which the water level is rising when the water is 3 meters deep?

4. A man starts walking north 4 feet/sec from a point P. Five minutes later a woman starts walking south at 7 feet/sec from a point 500 feet due east of P. At what rate are the people moving apart 15 minutes after the woman started walking?

5. A hot air balloon raising straight up from a level field is tracked by an inclinometer 500 feet from the lift off point. At the moment the inclinometers angle of elevation is $\pi/4$ radians. The angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

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VITA

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