Application of Autocorrelation Principles to Characterize Line Source Radiation

By

Christopher Daniel Wilson

Bachelor of Science in Aerospace Engineering
Master of Science in Aerospace Engineering
Master of Science in Engineering Management
University of Missouri - Rolla
Rolla, MO
2001, 2002, 2002

Master of Science in Electrical Engineering
University of Idaho
Moscow, ID
2015

Doctor of Philosophy in Aerospace Engineering
Washington University in St. Louis
St. Louis, MO
2010

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Dissertation Approved:

Dr. Jeffrey L. Young

Dissertation Advisor

Dr. Charles F. Bunting

Dr. James C. West

Dr. Andrew S. Arena

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Name: Christopher Daniel Wilson

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Abstract:

The problem of line source radiation is examined through the application of autocorrelation principles. These principles enable the radiation from a line source to be characterized with only knowledge of the current distribution — *a priori* knowledge of the antenna pattern is not required. First, the radiated power from a broadside line source radiator is examined. Exact closed-form analytical expressions for the radiated power are developed for several canonical current distributions. These expressions are validated using numerical integration of the radiated power equation. Next, the statistical concept of variance is applied to characterize the antenna pattern performance. As with the radiated power, closed-form expressions for the "beamwidth variance" are determined through application of autocorrelation principles for several canonical current distributions, without knowledge of the antenna pattern. These results are also validated by numerically integrating the antenna pattern to calculate the variance. Finally, the methodologies developed for characterizing broadside line source radiation are extended to characterize scanning beams from line source radiators. Exact closed-form expressions for the radiated power are developed for several canonical current distributions and are validated through numerical integration of the antenna pattern. The concept of "pattern mean" is introduced in addition to the beamwidth variance, since the mean of a scanning beam will not necessarily be along the direction of the scan angle. Exact closed-form expressions for both statistical performance measures are developed for canonical current distributions. As with the other results, numerical integration of the first and second moments of the antenna pattern are performed to validate the analytical expressions.

TABLE OF CONTENTS

Chapter Page

Chapter Page

List of Figures

Figure Page

Figure Page

Chapter I

Introduction

Characterization of line source radiation has received extensive attention throughout the evolution of electromagnetics and antenna theory. Traditionally, the performance of an antenna can be characterized by determining the radiated pattern from the current distribution and then analyzing the pattern to yield specific performance metrics (e.g., total radiated power, radiation resistance, main beam scan angle, side lobe level, 3-dB beamwidth, peak directivity, etc.). The process for calculating antenna pattern performance metrics is accomplished almost exclusively using numerical methods. Likewise, the current distributions for more complex radiating structures (e.g., antenna arrays) are also calculated using computational techniques (e.g., method of moments). Even in light of the extensive and successful application of numerical methods to analyze radiating structures, it can sometimes be enlightening to revisit the foundational problems from which these analysis techniques have been developed. This dissertation revisits the fundamental problem of line source radiation and presents the development of a methodology by which the performance of a line source radiator can be characterized through the application of autocorrelation principles. The methodology presented herein will enable, among other things, the ability to determine the total radiated power from a line source radiator without *a priori* knowledge of the antenna pattern.

The dissertation first presents a brief historical review of the key developments in electromagnetics that established the foundation for antenna theory, the evolution of antenna theory in the early 20th century, and how the advent of the computer led to the development of numerical methods for modern antenna analysis. Next, the line source radiation problem under consideration in this dissertation is defined, including a brief review of the corresponding governing equations. Additionally, a review is conducted of several statistical concepts that will be leveraged throughout the dissertation (i.e., autocorrelation functions and moments). Following the review are four chapters that present the evolutionary application of autocorrelation functions to the line source radiation problem.

First, autocorrelation principles are applied to develop a methodology that enables determination of the radiated power directly from the current distribution without the need to intermediately determine the antenna pattern. The methodology is applied to several canonical current distributions to develop closed-form analytical expressions for the radiated power. These expressions are then validated by numerically calculating the radiated power. Second, the successful application of autocorrelation principles to determine the radiated power revealed the possibility of determining the variance of the antenna pattern using a similar approach. The statistical concept of variance is applied to the power pattern function and autocorrelation principles are utilized to develop a corresponding methodology, which is then applied to determine closed-form analytical expressions for the variance for the same canonical current distributions. As before, these expressions are validated numerically. Third, the methodologies developed in the two preceding chapters applied exclusively to broadside, non-scanning line source radiation. The admissibility of scanning line source radiators to the application of autocorrelation principles is considered as the next evolutionary step. The principles are applied to develop a methodology for determining the radiated power from a scanning line source, which is then applied to several canonical current distributions to develop closed-form, analytical expressions for the radiated power. Again, these expressions are validated numerically. Fourth, in light of the success of applying autocorrelation principles to scanning line sources, the application of variance to characterize the pattern from a line source radiator is extended to include the first-order moment (i.e., mean). The introduction of the mean is necessary since the main beam scans away from boresight, which results in an asymmetric antenna pattern. Autocorrelation principles are again applied to successfully develop a methodology for determining the mean and variance of a scanning beam with no *a priori* knowledge of the antenna pattern. Closed-form analytical expressions for both the mean and the variance are derived for the same canonical current distributions, which are validated numerically. Finally, a summary of the work presented in the dissertation is provided, along with an extensive discussion of other potential applications of autocorrelation principles.

It is important to note that the application of autocorrelation principles to line source radiation is novel. As a result of this novel approach, closed-form analytical expressions for radiated power, mean, and variance for both broadside and scanning beams are developed for a variety of canonical current distributions. All of these expressions — except for two — have never been presented previously in the open literature prior to the research conducted in support of this dissertation. The corresponding validations demonstrate that the expressions are exact, which is evidenced by the practically perfect agreement between the closed-form equations and the numerical results. The exactness of the expressions obtained using this methodology is claimed within the context of the assumptions made for the problem under consideration (i.e., radiation from a line source with an infinitesimally small diameter and an assumed current distribution), not in the context of an actual antenna.

It should be noted that the fundamental theory presented in this dissertation is straightforward to implement. As seen in the Appendices, which provide detailed derivations, the closed-form expressions are obtained using basic techniques from differential and integral calculus. At best, the difficulty in obtaining these expressions is limited to the tedium that one must endure while keeping track of all the terms that arise throughout the course of performing the derivations. Otherwise, highly specialized mathematical skills and knowledge are not needed. Hence, the theory is not only robust, but highly implementable.

The body of work presented in this dissertation demonstrates the robustness of the application of autocorrelation principles and the potentially limitless extensibility to other classes of problems. The potential impact of this work will reveal itself through the evolutionary process of research as new problems are evaluated in the context of the autocorrelation principles presented in this dissertation.

Chapter II

Foundation

This chapter reviews the historical development of antenna theory in order to establish a motivation for the research presented in this dissertation. Subsequently, some of the fundamental concepts associated with line source radiation and statistical distributions are reviewed, which will, in part, serve as the foundation for the theories presented in this dissertation. It is assumed that the reader has a basic understanding of the theory and nomenclature of electromagnetics, antenna theory, probability, and statistics.

2.1 Development of Antenna Theory

Historical developments and advancements in a given field of practice are sometimes viewed as a fully encapsulated body of knowledge that has preceded the current state of the art. While the sum total of developments and advancements of a given field did contribute to its present state, they did not contribute equally, nor did they necessarily occur along a single linear continuous process. In many cases, parallel processes were occurring, with contributions being made between processes, and with some processes experiencing discrete, measurable, and sometimes extensive gaps of time between major advancements. Regardless of the field of practice, it is important to revisit work that may seem decades or centuries old, because there may exist a hidden gem of knowledge that was overshadowed or suppressed by notable advancements in a parallel process. Some examples of these phenomena include the gap in time between Maxwell's presentation of the complete set of equations governing electromagnetics to the Royal Society of London in 1864 [1] and Hertz's experimental discovery of electromagnetic waves in 1888 [2]. During that gap, many incremental advances were occurring, including Heaviside's restructuring of Maxwell's equations using fields instead of potentials and reducing the number of equations from 20 to four [3]. Experimentation with electromagnetic waves continued until Marconi's transatlantic transmission from England to Newfoundland in 1901, which marked the beginning of the practical application of the antenna for long distance communication [4]. In parallel, Carson was examining electromagnetic radiation in the context of transmission line theory [5], which explored the concept of radiation from a transmission line in terms of resistance. Additionally, Carson explored the concept of reciprocity in radio communication [6], which was leveraged by Carter to develop a methodology for determining input impedances based on resonant and non-resonant current distributions [7]. Analysis of and experimentation with wire antennas continued into the 1930's and 1940's. Of significant note is the extensive compendium of works produced by King, specifically wire antennas, which contributed to his seminal work *Theory of Linear Antennas* [8]. Schelkunoff also applied Carson's work on radiation from transmission lines to the field of antenna theory in 1941 [9].

Another parallel development was that of the computer. Scientists and engineers were able to use computers to apply existing numerical methods, and to develop new methods, for analyzing complex mathematical and physical problems. Computers enabled the application of numerical methods to find solutions to equations with no known analytical solutions and were also used to reduce the laboriousness of performing hand calculations (e.g., Euler's method or the Newton-Raphson method). These newfound computational capabilities were used to advance the fields of electromagnetics and antenna theory through the development of the method of moments. Harrington published a comprehensive treatment of the method of moments [10], in addition to his masterpiece work on electromagnetic theory [11].

As summarized above, the problem of radiation from a line source current distribution has received thorough treatment throughout the history of electromagnetics. The conventional methodology for calculating performance metrics (e.g., radiated power, gain, directivity, etc.) associated with the line source radiator primarily involves determining the radiated pattern from the current distribution through Fourier transform relationships. The aforementioned performance metrics can then be determined through well-defined mathematical operations. For example, the total radiated power can be determined by integrating the power radiation pattern over 4π steradians. Taking into account appropriate axes or planes of symmetry can simplify the calculations. However, as current distributions become more complex or line sources are discretized into arrays, the ability to analyze antenna performance becomes entirely predicated on numerical calculations of the radiated pattern. From those numerical results, additional numerical calculations are performed to determine the aforementioned antenna performance metrics.

Given that numerically derived antenna patterns and performance metrics are eminently obtainable, it is reasonable to question why an alternative method for calculating closed-form results for canonical line source current distributions is even necessary. There exist many rational answers to that question. First, mathematics is the language of physics — if a physical phenomenon can be completely described through rigorous application of mathematics, then further insight into that phenomenon could be gained — which is often not a consideration of the brute force (and sometimes the corresponding ignorance) that is often associated with numerical calculations alone. In this case, the complete mathematical description of a simple scanning beam using the statistical concepts of mean and variance could shed new light on the behavior of more complex radiating structures. Furthermore, concepts like mean and variance can be used to relate the spread of the current distribution to the spread of the antenna pattern via the Heisenberg Uncertainty Principle [12]. Second, successfully analyzing a physical problem using an alternate mathematical approach could open the door to other breakthroughs — sometimes those breakthroughs cannot be foreseen and only become evident after a critical foundational development is unearthed. In this case, the successful application of autocorrelation principles to line source radiators could open the door to reductions in computational complexities of the method of moments or possibly a novel approach to antenna synthesis. Finally, even if the two aforementioned possibilities never come to fruition, research is a necessary and exciting step on the path of discovery.

As presented in this dissertation, the application of autocorrelation principles to line source radiation has successfully led to the development of exact expressions for the radiated power from both a broadside and scanning line source. In addition, the success of the methodology has enabled definition of new metrics for the performance of line source radiators (i.e., pattern mean and beamwidth variance). These metrics are based on traditional statistical concepts. Application of autocorrelation principles enables the obtainment of closed-form analytical expressions when applied to both broadside and scanning line sources. These foundational concepts will be reviewed in the following sections.

2.2 Line Source Radiation

The problem considered in this dissertation is that of radiation from a line source with an infinitesimally small diameter. The line source has a total length L and is coincident with the z-axis on the interval $[-L/2, L/2]$. A current distribution $I(z)$ is impressed along the extent of the line source. The resulting antenna pattern is measured from the z-axis in the θ-direction and can be shown to be ϕ-symmetric. The basic constructs of the line source radiator are shown in Figure (2.1). Taking advantage of the ϕ-symmetry, the radiated power from the line source can be determined by integrating the θ-component of the electric field over 4π steradians,

$$P_{\text{rad}} = \frac{1}{2\eta} \int_0^{2\pi} \int_0^{\pi} |E_\theta|^2 \, r^2 \sin\theta \; d\theta \; d\phi. \tag{2.1}$$

From Equation (2.1), it is obvious that is only necessary to determine the θ-component of

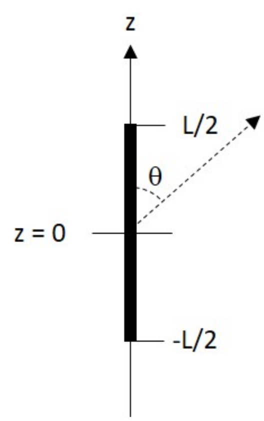

Figure 2.1: Line Source Radiator

the electric field in order to calculate the radiated power. Fortunately, it is known that the electric far-field of a line source radiator situated symmetrically on the z-axis can be shown to be,

$$\mathbf{E} = j\omega\mu w(r) G(u) \sin\theta \; \mathbf{a}_\theta, \tag{2.2}$$

where $w(r)$ is the spherical wave function,

$$w(r) = \frac{e^{-jkr}}{4\pi r}, \tag{2.3}$$

and $G(u)$ is the field pattern function [13]. By definition,

$$u = u_0 \cos\theta, \tag{2.4}$$

where the electric length of the line source u_0 can be determined from,

$$u_0 = \frac{kL}{2\pi} = \frac{L}{\lambda}. \tag{2.5}$$

Substituting Equation (2.2) into Equation (2.1), recalling $k = \omega\sqrt{\mu\epsilon}$ and $\eta = \sqrt{\mu/\epsilon}$, and simplifying yields,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi} \int_0^{\pi} G^2(u) \sin^3\theta \; d\theta. \tag{2.6}$$

At this point, it is useful to invoke a transformation to simplify the bookkeeping of the subsequent analysis by letting

$$p = \frac{2\pi z}{L}. \tag{2.7}$$

The normalized current distribution $g(p)$ is then defined as,

$$g(p) = \left.\frac{LI(z)}{2\pi}\right|_{z=\frac{pL}{2\pi}} \tag{2.8}$$

in which case,

$$G(u) = \int_{-\pi}^{\pi} g(p) e^{-jpu} \; dp. \tag{2.9}$$

Substituting $\sin^2\theta = 1 - \cos^2\theta$ into Equation (2.6) yields

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi} \int_0^{\pi} G^2(u) \left(1 - \cos^2\theta\right) \sin\theta \; d\theta. \tag{2.10}$$

Furthermore, taking the derivative of Equation (2.4) gives,

$$du = -u_0 \sin\theta \; d\theta, \tag{2.11}$$

and substituting Equations (2.4) and (2.11) into Equation (2.10) and reevaluating the limits of integration yields,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3} \int_{-u_0}^{u_0} G^2(u) \left(u_0^2 - u^2\right) \, du. \tag{2.12}$$

Equation (2.12) will serve as the springboard for a majority of the developments presented in this dissertation.

2.3 Statistical Principles

Though not imminently evident, the reason for reviewing some basic concepts of probability and statistics will reveal itself in subsequent chapters. Autocorrelation functions will be reviewed since they are the primary instrument used to develop the methodologies presented in this dissertation. As a direct consequence of utilizing autocorrelation functions, the opportunity to leverage some well known statistical concepts became apparent. In particular, the concepts of the mean and variance (i.e., first and second order moments) will be reviewed.

2.3.1 Autocorrelation Functions

While convolution is regularly used in a wide variety of mathematical and physical processes, readers may not be as acquainted with the concept or application of autocorrelation functions. The autocorrelation of a function $w(p)$ is given by,

$$R_w(p) = \int_{-\infty}^{\infty} w^*(\tau)w(\tau - p) \, d\tau, \tag{2.13}$$

where $w^*(\tau)$ represents the complex conjugate [14]. When the autocorrelation function is evaluated at $p = 0$ the result is the average normalized power in the signal,

$$R_w(0) = E\left[w^2(p)\right], \tag{2.14}$$

which is written in terms of the expected value [15]. The autocorrelation function of a continuous signal is shown to be related to its power spectral density through the Wiener-Khinchin Theorem,

$$R_w(p) = \mathcal{F}^{-1}\left\{W^2(u)\right\}, \tag{2.15}$$

where $W(u)$ is the Fourier transform of $w(p)$. Equation (2.15) can also be expressed as the Fourier transform pair,

$$R_w(p) \Leftrightarrow \mathscr{P}_w(f), \tag{2.16}$$

where $\mathscr{P}_w(f)$ is the power spectral density. Parseval's Theorem, which is closely related to the Wiener-Khinchin Theorem, provides a relationship between the signal power in the time domain and the spectral power in the frequency domain,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |W(u)|^2 \ du = \int_{-\infty}^{\infty} |w(p)|^2 \ dp. \tag{2.17}$$

Parseval's Theorem will be leveraged regularly in the formulation development presented in the subsequent chapters of this dissertation. One final property of the autocorrelation function worth noting is its inherent evenness,

$$R_w(p) = R_w(-p), \tag{2.18}$$

which will also be used extensively in the derivations presented in this dissertation.

2.3.2 Moments

Moments will serve as the basis for the development of the methodologies presented in two chapters of this dissertation. Moments are a mathematical construct that is often used in any field that considers distribution functions (e.g., mass properties, probability density functions, etc.). In general, the n^{th}-order raw moment of a continuous function $f(x)$ is defined as [16],

$$m_n = \int_{-\infty}^{\infty} x^n f(x) \ dx. \tag{2.19}$$

The first-order raw moment is commonly referred to as the mean, μ, and is identically zero when the function is even symmetric; by definition $\mu = m_1$. The second-order moment is commonly referred to as the variance, σ^2, such that $\sigma^2 = m_2$. The generalized moments are constructed by evaluating the moment about some value, a,

$$\mu_{g,n} = \int_{-\infty}^{\infty} (x-a)^n \, f(x) \ dx. \tag{2.20}$$

The generalized moments become the central moments when a is equal to the mean (i.e., $a = \mu$),

$$\mu_n = \int_{-\infty}^{\infty} (x - \mu)^n f(x) \, dx, \tag{2.21}$$

where the mean is the first-order raw moment, which can be determined from Equation (2.19). For a distribution with a non-zero mean, such as a antenna pattern scanning away from broadside, the variance is determined from the second-order central moment. Using a mass properties analogy, the mean represents the center of gravity and the variance represents the mass moment of inertia about the center of gravity.

Chapter III

Radiated Power of a Broadside Line Source

This chapter presents the development of a methodology, based on autocorrelation principles, for determining the radiated power of a broadside line source radiator directly from the current distribution without *a priori* knowledge of the antenna pattern. The methodology enables determining exact closed-form expressions for the radiated power and is applied to six canonical current distributions — half-wave dipole, cosine, cosine-squared, generalized dipole, uniform, and triangular. The exact closed-form results are then compared to results obtained by numerically integrating the power-pattern functions in order to validate the newly developed closed-form expressions. The validation demonstrates perfect agreement between the closed-form expressions and the numerically integrated results. The theory, results, and validation have been published in the *IEEE Transactions on Antennas and Propagation* [13].

3.1 Formulation

As shown previously in Equation (2.12), the radiated power can be determined by integrating the power-pattern function, defined in u-space, on the interval $[-u_0, u_0]$,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3} \int_{-u_0}^{u_0} \left(u_0^2 - u^2\right) G^2\left(u\right)\ du. \tag{3.1}$$

Recalling the definition of the pulse function,

$$\Pi(u) = \begin{cases} 1, & |u| \leq \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}, \tag{3.2}$$

and making the appropriate substitution into Equation (3.2) then,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3} \int_{-\infty}^{\infty} \Pi^2 \left(\frac{u}{2u_0}\right) \left(u_0^2 - u^2\right) G^2 \left(u\right) \; du. \tag{3.3}$$

Defining the function,

$$H^2(u) = F^2(u)\Pi^2 \left(\frac{u}{2u_0}\right), \tag{3.4}$$

where,

$$F^2(u) = \left(u_0^2 - u^2\right) G^2(u), \tag{3.5}$$

enables Equation (3.3) to be written as,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3} \int_{-\infty}^{\infty} H^2(u) \; du. \tag{3.6}$$

Using Parseval's identity, given previously in Equation (2.17), $H(u)$ can be related to its inverse Fourier transform $h(p)$ using,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(u)|^2 \; du = \int_{-\infty}^{\infty} |h(p)|^2 \; dp. \tag{3.7}$$

Equation (3.7) can be applied to Equation (3.6),

$$P_{\text{rad}} = \frac{k^2\eta}{8u_0^3} \int_{-\infty}^{\infty} |h(u)|^2 \; du. \tag{3.8}$$

Invoking the definition of the stationary autocorrelation function, as defined by Equation (2.13),

$$R_h(p) = \int_{-\infty}^{\infty} h(\tau)h^*(\tau - p) \; d\tau, \tag{3.9}$$

then evaluating for $p = 0$,

$$R_h(0) = \int_{-\infty}^{\infty} h(\tau)h^*(\tau)\ d\tau = \int_{-\infty}^{\infty} |h(\tau)|^2\ d\tau, \tag{3.10}$$

and substituting into Equation (4.8) yields the expression for the radiated power expressed solely in terms of an autocorrelation function,

$$P_{\text{rad}} = \frac{k^2 \eta R_h(0)}{8u_0^3}. \tag{3.11}$$

The autocorrelation function $R_h(0)$ can be determined by first taking the inverse Fourier transform of Equation (3.4) such that,

$$\mathcal{F}^{-1}\left\{H^2(u)\right\} = \mathcal{F}^{-1}\left\{F^2(u)\Pi^2\left(\frac{u}{2u_0}\right)\right\}. \tag{3.12}$$

The product in Equation (3.12) can be written as the convolution of two inverse Fourier transforms:

$$R_h(p) = \mathcal{F}^{-1}\left\{F^2(u)\right\} * \mathcal{F}^{-1}\left\{\Pi^2\left(\frac{u}{2u_0}\right)\right\}, \tag{3.13}$$

which then results in the convolution of two autocorrelation functions after performing the inverse Fourier transforms,

$$R_h(p) = R_f(p) * R_s(p). \tag{3.14}$$

Applying the convolution definition to Equation (3.14) gives,

$$R_h(p) = \int_{-\infty}^{\infty} R_f(\tau) R_s(p - \tau)\ d\tau. \tag{3.15}$$

The autocorrelation function $R_s(p)$ can be defined in the same manner as Equation (3.9) namely,

$$R_s(p) = \int_{-\infty}^{\infty} s(\tau)s^*(\tau - p)\ d\tau, \tag{3.16}$$

where $s(p)$ is the inverse Fourier transform of the pulse function, which is the sinc function:

$$s(p) = \mathcal{F}^{-1}\left\{\Pi^2\left(\frac{u}{2u_0}\right)\right\} = \frac{u_0}{\pi}\frac{\sin(u_0 p)}{u_0 p}. \tag{3.17}$$

It is well known that the autocorrelation of a sinc function produces a sinc function. Therefore, substituting Equation (3.17) into Equation (3.13) yields the autocorrelation function $R_s(p)$:

$$R_s(p) = \frac{u_0}{\pi} \frac{\sin(u_0 p)}{u_0 p}. \tag{3.18}$$

Substituting Equation (3.18) into Equation (3.15) gives,

$$R_h(p) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin[u_0(p-\tau)]}{u_0(p-\tau)}\, d\tau. \tag{3.19}$$

Evaluting Equation (3.19) at $p = 0$ yields the stationary autocorrelation function $R_h(0)$:

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0 \tau)}{u_0 \tau}\, d\tau. \tag{3.20}$$

The autocorrelation function $R_f(p)$ can be found by first taking the inverse Fourier transform of Equation (3.5) such that,

$$\mathcal{F}^{-1}\left\{F^2(u)\right\} = \mathcal{F}^{-1}\left\{\left(u_0^2 - u^2\right) G^2(u)\right\}. \tag{3.21}$$

Using Fourier transform identities, Equation (3.21) can be written in terms of a differential Helmholtz operator acting on the inverse Fourier transform of $G^2(u)$, which is $R_g(p)$. That is,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{3.22}$$

Finally, the autocorrelation function $R_g(p)$ can be determined in a manner similar to Equations (3.9) and (3.16), except the complex conjugate is not required since the current distribution is real-valued:

$$R_g(p) = \int_{-\infty}^{\infty} g(\tau) g(\tau - p)\, d\tau. \tag{3.23}$$

In summary, the radiated power can be determined by successively evaluating Equations (3.23), (3.22), (3.20), and (3.11), which will be demonstrated for several current distributions in the next section.

3.2 Examples

The theory developed in the preceding section has been applied to several canonical current distributions to develop exact closed-form expressions for the stationary autocorrelation function $R_h(0)$. These closed-form expressions can then be used directly in conjunction with Equation (3.11) to yield exact closed-form expressions for the radiated power generated from that current distribution. Considered in this chapter are the half-wave dipole, cosine, cosine-squared, generalized dipole, uniform, and triangular current distributions. These distributions were chosen, in part, because: i) the cosine distribution is a good approximation of the current distribution for a resonant dipole; ii) the triangular distribution is a good approximation for electrically short dipoles; iii) the uniform distribution is a good distribution for understanding basic antenna radiation; and iv) the generalized dipole distribution is a good approximation for a dipole of arbitrary electric length. All of these distributions are studied extensively in most antenna texts. Only the final results are presented in this chapter, while the detailed derivations for each current distribution are presented in Appendix A.

3.2.1 Half-Wave Dipole Distribution

The half-wave dipole current distribution is associated with a resonant structure of length $\lambda/2$ and is given by,

$$g(p) = A_m \cos\left(\frac{p}{2}\right) \tag{3.24}$$

on the interval $-\pi \leq p \leq \pi$ and $g(p) = 0$ otherwise. The autocorrelation function $R_g(p)$ can be found by applying Equation (3.23) to Equation (3.24):

$$R_g(p) = A_m^2 \begin{cases} \left(\frac{2\pi + p}{2}\right) \cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{2\pi - p}{2}\right) \cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{3.25}$$

The autocorrelation function $R_f(p)$ can then be found by applying Equation (3.22) to Equation (3.25):

$$R_f(p) = \frac{A_m^2}{2} \begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{3.26}$$

Finally, Equation (3.20) can be applied to Equation (3.26) to yield the stationary autocorrelation function $R_h(0)$ for the half-wave dipole distribution such that,

$$R_h(0) = A_m^2 \left[\frac{\text{Cin}(2\pi)}{2\pi}\right], \tag{3.27}$$

where $\text{Cin}(x)$ is the modified cosine integral [17]. The radiation resistance at the feed point of the line source (i.e., $z = 0$) can be shown to be [18],

$$R_{\text{rad}} = \frac{2P_{\text{rad}}}{I^2(0)}. \tag{3.28}$$

Using Equation (2.8), the current at $z = 0$ is determined to be,

$$I(0) = \frac{2\pi g(0)}{L}. \tag{3.29}$$

and can be substituted into Equation (3.28) to obtain,

$$R_{\text{rad}} = \left(\frac{L}{2\pi}\right)^2 \left[\frac{2P_{\text{rad}}}{g^2(0)}\right]. \tag{3.30}$$

Substituting Equations (2.5) and (3.11) into Equation (3.30) yields the radiation resistance expressed solely in terms of the autocorrelation function $R_h(0)$:

$$R_{\text{rad}} = \frac{\eta R_h(0)}{4u_0 g^2(0)}. \tag{3.31}$$

Equation (3.24) can be evaluated at $z = 0$ and, along with Equation (3.27), can be substituted into Equation (3.31) to obtain $R_{\text{rad}} = 73.1\ \Omega$ for the half-wave dipole, which is the correct result [18]. The fact that this classical result could be replicated through the application of autocorrelation principles provides a preliminary and encouraging validation of the methodology.

3.2.2 Cosine Distribution

The cosine current distribution is the same as for the half-wave dipole, presented in the previous subsection. Therefore, the autocorrelation function $R_g(p)$ for the cosine distribution is the same as the result derived for the half-wave dipole, given previously in Equation (3.25). The varying electric length u_0 of the cosine distribution is introduced when applying Equation (3.22) to (3.25) to develop the expression for the autocorrelation function $R_f(p)$, namely,

$$R_f(p) = A_m^2 \begin{cases} \left(u_0^2 - \frac{1}{4}\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right) - \left(u_0^2 + \frac{1}{4}\right)\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(u_0^2 - \frac{1}{4}\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right) + \left(u_0^2 + \frac{1}{4}\right)\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{3.32}$$

Equation (3.20) can be applied to Equation (3.32) to yield the stationary autocorrelation function $R_h(0)$ for the cosine distribution:

$$\begin{aligned} R_h(0) = \frac{A_m^2}{\pi}\Bigg\{ &\left(u_0^2 + \frac{1}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \frac{1}{2}\right)\right]\right] \\ &+ \pi\left(u_0^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \frac{1}{2}\right)\right]\right] \\ &- u_0\left[\cos\left(2\pi u_0\right) + 1\right]\Bigg\}, \end{aligned} \tag{3.33}$$

where $\mathrm{Cin}(x)$ is the modified cosine integral and $\mathrm{Si}(x)$ is the sine integral [17].

3.2.3 Cosine-Squared Distribution

The cosine-squared current distribution is given by

$$g(p) = A_m \cos^2\left(\frac{p}{2}\right) \tag{3.34}$$

on the interval $-\pi \leq p \leq \pi$ and $g(p) = 0$ otherwise. The autocorrelation function $R_g(p)$ can be found by applying Equation (3.23) to Equation (3.34) such that,

$$R_g(p) = \frac{A_m^2}{8} \begin{cases} (2\pi + p)\left[2 + \cos(p)\right] - 3\sin(p), & -2\pi \leq p \leq 0 \\ (2\pi - p)\left[2 + \cos(p)\right] + 3\sin(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{3.35}$$

The autocorrelation function $R_f(p)$ can then be found by applying Equation (3.22) to Equation (3.35):

$$R_f(p) = \frac{A_m^2}{8} \begin{cases} (2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p), & -2\pi \leq p \leq 0 \\ (2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{3.36}$$

Finally, Equation (3.20) can be applied to Equation (3.36) to yield the stationary autocorrelation function $R_h(0)$ for the cosine-squared distribution:

$$\begin{aligned} R_h(0) = \frac{A_m^2}{8\pi} \Big\{ & 2\pi\left(u_0^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + 1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - 1\right)\right]\right] \\ & - \left(1 - 3u_0^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + 1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - 1\right)\right]\right] \\ & + 8\pi u_0^2 \mathrm{Si}\left(2\pi u_0\right) + 6u_0\left[\cos\left(2\pi u_0\right) - 1\right] \Big\}. \end{aligned} \tag{3.37}$$

3.2.4 Generalized Dipole Distribution

The generalized dipole current distribution is given by

$$g(p) = A_m \begin{cases} \sin\left[u_0\left(\pi + p\right)\right], & -2\pi \leq p \leq 0 \\ \sin\left[u_0\left(\pi - p\right)\right], & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{3.38}$$

The autocorrelation function $R_g(p)$ can be found by applying Equation (3.23) to Equation (3.38). Specifically,

$$R_g(p) = A_m^2 \begin{cases} R_{g1}(p), & -2\pi \leq p \leq -\pi \\ R_{g2}(p), & -\pi \leq p \leq 0 \\ R_{g3}(p), & 0 \leq p \leq \pi \\ R_{g4}(p), & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}, \tag{3.39}$$

where

$$R_{g1}(p) = \frac{1}{2}\left[\frac{1}{u_0}\sin\left[u_0\left(2\pi + p\right)\right] - \left(2\pi + p\right)\cos\left[u_0\left(2\pi + p\right)\right]\right], \tag{3.40}$$

$$\begin{aligned} R_{g2}(p) = \frac{1}{2}\Bigg[& 2\left(\pi + p\right)\cos\left(u_0 p\right) - \frac{2}{u_0}\sin\left(u_0 p\right) \\ & - \frac{1}{u_0}\sin\left[u_0\left(2\pi + p\right)\right] + p\cos\left[u_0\left(2\pi + p\right)\right]\Bigg], \end{aligned} \tag{3.41}$$

$$\begin{aligned} R_{g3}(p) = \frac{1}{2}\Bigg[& 2\left(\pi - p\right)\cos\left(u_0 p\right) + \frac{2}{u_0}\sin\left(u_0 p\right) \\ & - \frac{1}{u_0}\sin\left[u_0\left(2\pi - p\right)\right] - p\cos\left[u_0\left(2\pi - p\right)\right]\Bigg], \end{aligned} \tag{3.42}$$

and

$$R_{g4}(p) = \frac{1}{2}\left[\frac{1}{u_0}\sin\left[u_0\left(2\pi - p\right)\right] - \left(2\pi - p\right)\cos\left[u_0\left(2\pi - p\right)\right]\right]. \tag{3.43}$$

The autocorrelation function $R_f(p)$ can then be found by applying Equation (3.22) to Equation (3.39):

$$R_f(p) = A_m^2 u_0 \begin{cases} \sin\left[u_0\left(2\pi + p\right)\right], & -2\pi \leq p \leq -\pi \\ -2\sin\left(u_0 p\right) - \sin\left[u_0\left(2\pi + p\right)\right], & -\pi \leq p \leq 0 \\ 2\sin\left(u_0 p\right) - \sin\left[u_0\left(2\pi - p\right)\right], & 0 \leq p \leq \pi \\ \sin\left[u_0\left(2\pi - p\right)\right], & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{3.44}$$

Finally, Equation (3.20) can be applied to Equation (3.44) to yield the stationary autocorrelation function $R_h(0)$ for the generalized dipole distribution:

$$R_h(0) = \frac{A_m^2 u_0}{\pi}\left\{2\mathrm{Cin}\left(2\pi u_0\right) - \cos\left(2\pi u_0\right)\left[\mathrm{Cin}\left(4\pi u_0\right) - 2\mathrm{Cin}\left(2\pi u_0\right)\right]\right.$$
$$\left. + \sin\left(2\pi u_0\right)\left[\mathrm{Si}\left(4\pi u_0\right) - 2\mathrm{Si}\left(2\pi u_0\right)\right]\right\}. \tag{3.45}$$

The previous result, when used in conjunction with the definitions of the radiated power or the radiation resistance, is the classical result presented previously by Harrington [11] and Stutzman and Thiele [18].

3.2.5 Uniform Distribution

The uniform current distribution is given by

$$g(p) = A_m \tag{3.46}$$

on the interval $-\pi \leq p \leq \pi$ and $g(p) = 0$ otherwise. The autocorrelation function $R_g(p)$ can be found by applying Equation (3.23) to Equation (3.46) to yield,

$$R_g(p) = A_m^2\left[R\left(p + 2\pi\right) - 2R(p) + R\left(p - 2\pi\right)\right], \tag{3.47}$$

where $R(p)$ is the ramp function. The autocorrelation function $R_f(p)$ can then be found by applying Equation (3.22) to Equation (3.47):

$$R_f(p) = A_m^2\left[\delta\left(p + 2\pi\right) - 2\delta(p) + \delta\left(p - 2\pi\right)\right] + u_0^2 R_g(p). \tag{3.48}$$

Finally, Equation (3.20) can be applied to Equation (3.48) to yield the stationary autocorrelation function $R_h(0)$ for the uniform distribution. Specifically,

$$R_h(0) = \frac{2A_m^2 u_0}{\pi}\left[\frac{\sin\left(2\pi u_0\right)}{2\pi u_0} + 2\pi u_0 \mathrm{Si}\left(2\pi u_0\right) + \cos\left(2\pi u_0\right) - 2\right]. \tag{3.49}$$

Of particular note regarding the result presented in Equation (3.49) is that the classical approximation for the peak directivity of an electrically long line source with a uniform current distribution does not include the "−2" bias [18]. In the limit, when $u_0 \gg 1$, the effect of the bias becomes negligible. However, the result presented in Equation (3.49) is exact — which differs from the results obtained by utilizing the large argument approximation of the sine integral and neglecting the $\sin^2(\theta)$ element factor. Discovery of the "−2" bias has not been presented previously in the open literature and could not have been obtained using traditional methods.

3.2.6 Triangular Distribution

The triangular current distribution is given by

$$g(p) = A_m \begin{cases} 1 + \dfrac{p}{\pi}, & -2\pi \le p \le 0 \\ 1 - \dfrac{p}{\pi}, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{3.50}$$

The autocorrelation function $R_g(p)$ can be found by applying Equation (3.23) to Equation (3.50):

$$R_g(p) = \frac{A_m^2}{6\pi^2} \begin{cases} (2\pi + p)^3, & -2\pi \le p \le -\pi \\ 4\pi^3 - 6\pi p^2 - 3p^3, & -\pi \le p \le 0 \\ 4\pi^3 - 6\pi p^2 + 3p^3, & 0 \le p \le \pi \\ (2\pi - p)^3, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{3.51}$$

The autocorrelation function $R_f(p)$ can then be found by applying Equation (3.22) to Equation (3.51) to yield,

$$R_f(p) = \frac{A_m^2}{6\pi^2} \begin{cases} u_0^2 (2\pi + p)^3 + 6(2\pi + p), & -2\pi \le p \le -\pi \\ u_0^2 \left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p, & -\pi \le p \le 0 \\ u_0^2 \left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p, & 0 \le p \le \pi \\ u_0^2 (2\pi - p)^3 + 6(2\pi - p), & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{3.52}$$

Finally, Equation (3.20) can be applied to Equation (3.52) to yield the stationary autocorrelation function $R_h(0)$ for the triangular distribution:

$$\begin{aligned} R_h(0) = \frac{A_m^2}{6\pi^3 u_0} \Big\{ & \left(16\pi^3 u_0^3 + 24\pi u_0\right) \mathrm{Si}\left(2\pi u_0\right) - \left(8\pi^3 u_0^3 + 48\pi u_0\right) \mathrm{Si}\left(\pi u_0\right) \\ & + \left(8\pi^2 u_0^2 + 8\right) \cos\left(2\pi u_0\right) - \left(8\pi^2 u_0^2 + 32\right) \cos\left(\pi u_0\right) \\ & + 4\pi u_0 \sin\left(2\pi u_0\right) - 8\pi u_0 \sin\left(\pi u_0\right) + 24 \Big\}. \end{aligned} \tag{3.53}$$

3.3 Validation

The results of the preceding section can be validated by inserting each result for the stationary autocorrelation function $R_h(0)$ into Equation (3.11) and plotting the radiated power P_{rad} as a function of electric length u_0. It is important to note that the results for the broadside radiated power formulation are contained within the results for the scanning radiated power formulation when the phase progression constant is identically zero (i.e., $\alpha = 0$). For purposes of comparison, the amplitude of each current distribution is set to unity (i.e., $A_m = 1$). In addition, the radiated power can be calculated by numerically integrating Equation (3.1) with the power-pattern function $G(u)$ for each current distribution. The power-pattern functions can be found by taking the Fourier transform of each current distribution, which produces the following results:

$$\begin{aligned} G(u) &= A_m \cos(\pi u) \left[\frac{4}{1 - 4u^2}\right] && \text{Cosine} \\ &= A_m \sin(\pi u) \left[\frac{1}{u(1-u^2)}\right] && \text{Cosine} - \text{Squared} \\ &= 2A_m u_0 \left[\frac{\cos\left(\pi u_0\right) - \cos\left(\pi u\right)}{u^2 - u_0^2}\right] && \text{Generalized} \\ &= 2\pi A_m \left[\frac{\sin(\pi u)}{\pi u}\right] && \text{Uniform} \\ &= \pi A_m \left[\frac{\sin(\pi u/2)}{\pi u/2}\right]^2 && \text{Triangular} \end{aligned} \tag{3.54}$$

The results are compared for each current distribution in Figures (3.1) through (3.5). The comparisons demonstrate effectively perfect agreement between the closed-form equations and the numerically integrated results. (Obviously, the claim of perfect agreement is subject to the precision of the calculations performed for both the numerical integrations and evaluation of the closed-form expressions, which resulted in an average relative error on the order of 10^{-10}. This definition of "perfect" will also be germane in subsequent chapters.)

Also included with the comparisons are the electrically short and electrically long approximations that can be obtained from the closed-form analytical results. As mentioned previously, the electrically long approximation for the uniform current distribution, as derived from its corresponding closed-form analytical expression, revealed a missing "-2" bias from the previously published classical results. The discovery of that new result, along with the electrically short and long approximations for the remaining current distributions, have been presented previously [13]. With the exception of Equations (3.27) and (3.45), the results presented in this chapter are entirely new. Additionally, all of the expressions are exact in the context of a line source radiator with an infinitesimally small radius. The validation of these equations indicates that the methodology is robust in its ability to obtain closed-form results for almost any current distribution — not just those discussed in this chapter. The only hurdle seems to be the tedium associated with performing the derivations.

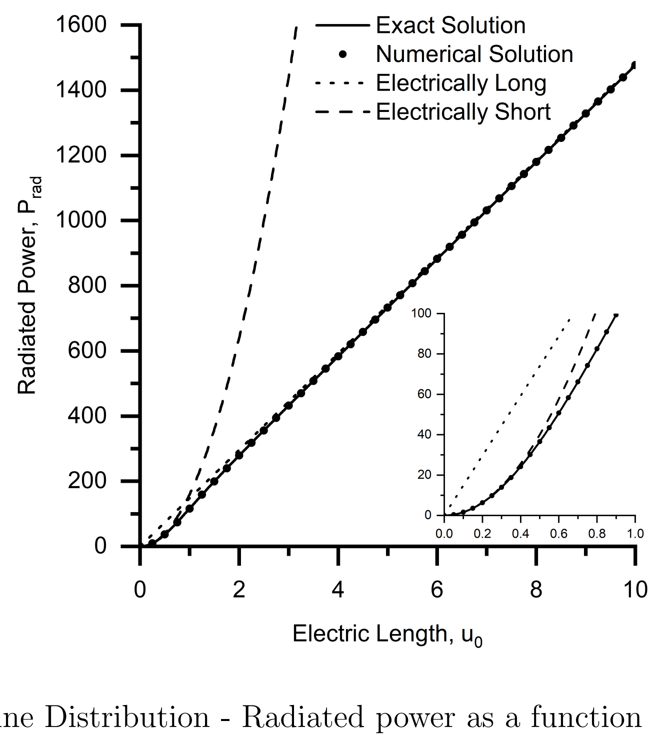

Figure 3.1: Cosine Distribution - Radiated power as a function of electrical length.

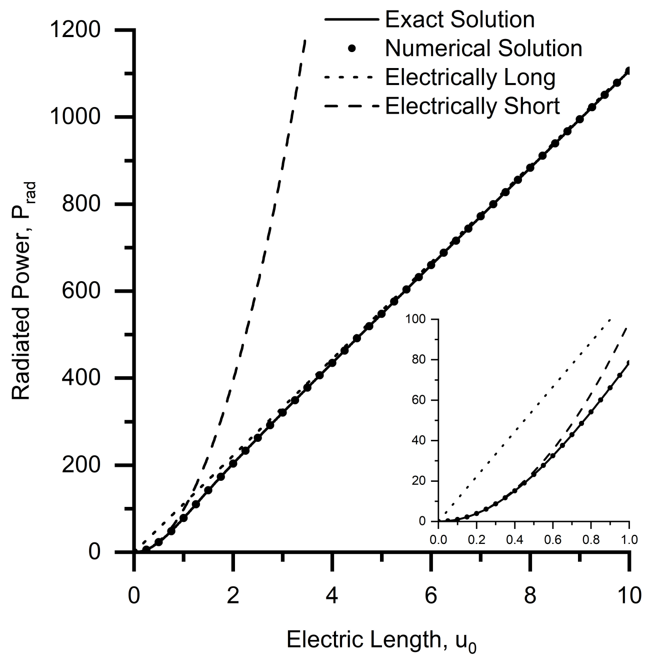

Figure 3.2: Cosine-Squared Distribution - Radiated power as a function of electrical length.

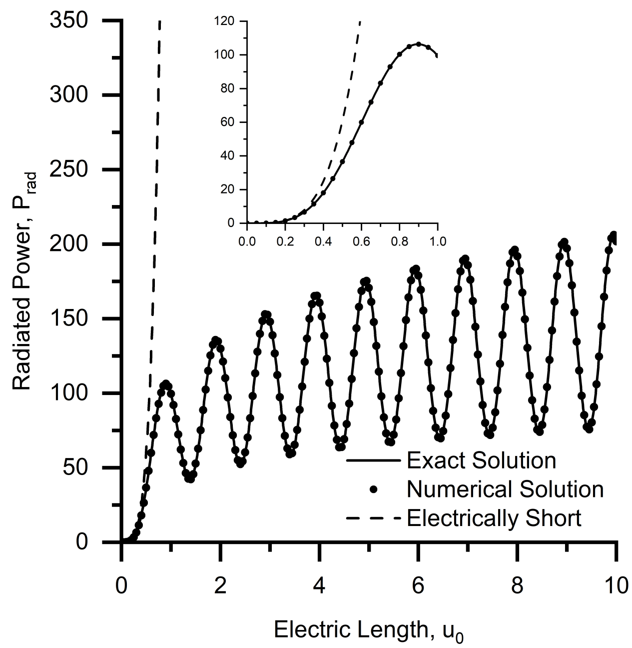

Figure 3.3: Generalized Dipole Distribution - Radiated power as a function of electrical length.

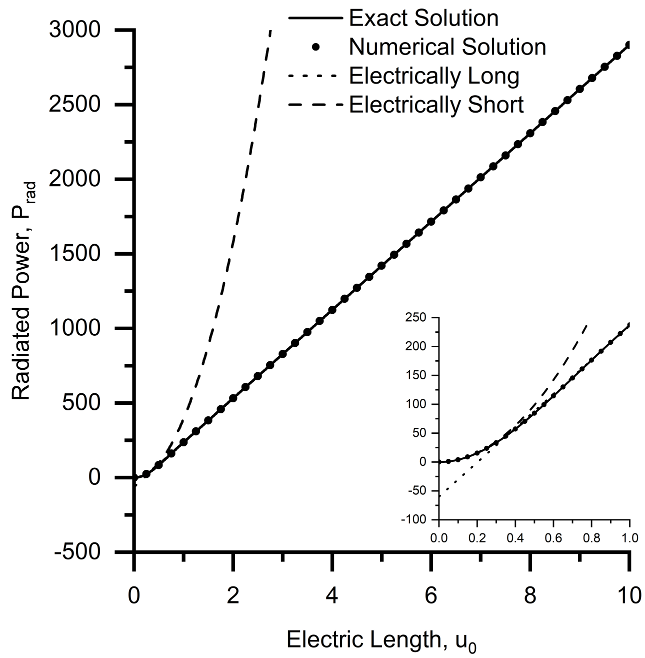

Figure 3.4: Uniform Distribution - Radiated power as a function of electrical length.

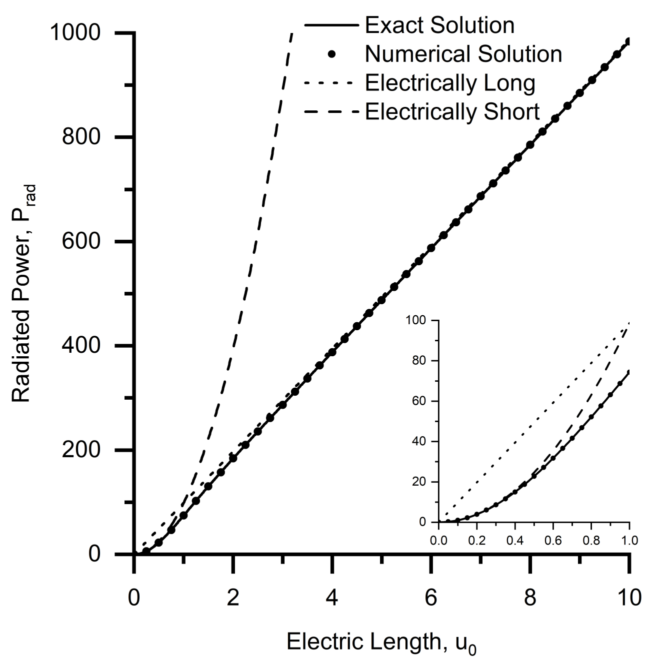

Figure 3.5: Triangular Distribution - Radiated power as a function of electrical length.

Chapter IV

Beamwidth Variance of a Broadside Line Source

This chapter presents a new concept for characterizing antenna pattern performance using the well defined statistical concept of variance. The variance is suggested as a proxy for antenna pattern beamwidth, which is termed "beamwidth variance." The benefit of using beamwidth variance to characterize antenna pattern performance is that it can be determined using the same autocorrelation principles used in the previous chapter to develop the radiated power methodology. As before, the autocorrelation principles are applied to a broadside line source radiator and the resulting methodology is then used to develop exact closed-form expressions for the beamwidth variance for six canonical current distributions — half-wave dipole, cosine, cosine-squared, generalized dipole, uniform, and triangular. The exact closed-form results are then compared to results obtained by numerically integrating the second moment of the power-pattern functions in order to validate the newly developed expressions. The validation demonstrates effectively perfect agreement between the closed-form expressions and the numerically integrated results. The theory, results, and validation have been published in the *IEEE Transactions on Antennas and Propagation* [19].

4.1 Formulation

A constant phase line source radiator with an even current distribution has a pattern mean equal to zero because the pattern produced by an even current distribution is also even. Therefore, the variance can be found by constructing the second moment around that zero mean,

$$\sigma_r^2 = \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} u^2 \left(u_0^2 - u^2\right) G^2\left(u\right)\ du, \tag{4.1}$$

which is the second raw moment described in Equation (2.19). Recalling the definition of the pulse function,

$$\Pi(u) = \begin{cases} 1, & |u| \leq \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}, \tag{4.2}$$

and making the appropriate substitution into Equation (4.2) then,

$$\sigma_r^2 = \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} \Pi^2 \left(\frac{u}{2u_0}\right) u^2 \left(u_0^2 - u^2\right) G^2\left(u\right)\ du. \tag{4.3}$$

Defining the function,

$$M^2(u) = N^2(u)\Pi^2\left(\frac{u}{2u_0}\right), \tag{4.4}$$

where,

$$N^2(u) = u^2\left(u_0^2 - u^2\right) G^2(u), \tag{4.5}$$

enables Equation (4.3) to be written as,

$$\sigma_r^2 = \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} M^2(u)\ du. \tag{4.6}$$

Again, using Parseval's identity, $M(u)$ can be related to its inverse Fourier transform $m(p)$ using,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left|M(u)\right|^2\ du = \int_{-\infty}^{\infty} \left|m(p)\right|^2\ dp. \tag{4.7}$$

Equation (4.7) can be applied to Equation (4.6),

$$\sigma_r^2 = \frac{k^2 \eta}{8 u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} \left|m(p)\right|^2\ dp. \tag{4.8}$$

Invoking the definition of the stationary autocorrelation function, as defined by Equation (2.13),

$$R_m(p) = \int_{-\infty}^{\infty} m(\tau) m^*(\tau - p)\ d\tau, \tag{4.9}$$

then evaluating for $p = 0$,

$$R_m(0) = \int_{-\infty}^{\infty} m(\tau) m^*(\tau)\ d\tau = \int_{-\infty}^{\infty} |m(\tau)|^2\ d\tau, \tag{4.10}$$

and substituting into Equation (4.8),

$$\sigma_r^2 = \frac{k^2 \eta R_m(0)}{8 u_0^3 P_{\text{rad}}}. \tag{4.11}$$

Recalling the definition for the radiated power P_{rad} given previously in Equation (3.11), substituting into Equation (4.11), and canceling terms yields the expression for the second moment of the power pattern function expressed solely in terms of autocorrelation functions,

$$\sigma_r^2 = \frac{R_m(0)}{R_h(0)}, \tag{4.12}$$

where $R_h(0)$ is the stationary autocorrelation function determined using the radiated power formulation discussed in the preceding chapter. Therefore, it is only necessary to find the autocorrelation function $R_m(0)$, which can be determined by first recalling Equation (3.5) and substituting into Equation (4.5),

$$N^2(u) = u^2 F^2(u). \tag{4.13}$$

Taking the inverse Fourier transform of Equation (4.13),

$$\mathcal{F}^{-1}\left\{N^2(u)\right\} = \mathcal{F}^{-1}\left\{u^2 F^2(u)\right\}, \tag{4.14}$$

and applying the Fourier transform identity

$$\frac{d^n}{dp^n} f(p) \Leftrightarrow (-ju)^n F(u), \tag{4.15}$$

enables Equation (4.13) to be written as

$$R_n(p) = -\frac{d^2}{dp^2} R_f(p). \tag{4.16}$$

Additionally, taking the inverse Fourier transform of Equation (4.4) such that,

$$\mathcal{F}^{-1}\left\{M^2(u)\right\} = \mathcal{F}^{-1}\left\{N^2(u)\Pi\left(\frac{u}{2u_0}\right)\right\}. \tag{4.17}$$

The product in Equation (4.17) can be written as the convolution of two Fourier transforms:

$$R_m(p) = \mathcal{F}^{-1}\left\{N^2(u)\right\} * \mathcal{F}^{-1}\left\{\Pi^2\left(\frac{u}{2u_0}\right)\right\}, \tag{4.18}$$

which then results in the convolution of two autocorrelation functions after performing the inverse inverse Fourier transforms,

$$R_m(p) = R_n(p) * R_s(p). \tag{4.19}$$

Applying the convolution definition to Equation (4.19) gives,

$$R_m(p) = \int_{-\infty}^{\infty} R_n(\tau) R_s\left(p - \tau\right)\ d\tau. \tag{4.20}$$

Recalling the autocorrelation function $R_s(p)$ given in Equation (3.18) and inserting into Equation (4.20),

$$R_m(p) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_n(\tau) \frac{\sin\left[u_0\left(p - \tau\right)\right]}{u_0\left(p - \tau\right)}\ d\tau. \tag{4.21}$$

Evaluating Equation (4.21) at $p = 0$ yields the stationary autocorrelation function $R_m(0)$:

$$R_m(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_n(\tau) \frac{\sin\left(u_0 \tau\right)}{u_0 \tau}\ d\tau. \tag{4.22}$$

In order to finalize the concept of beamwidth variance, it is necessary to recall that the antenna pattern is defined on the interval $[-u_0, u_0]$ in u-space, which corresponds to the interval $[-\pi, \pi]$ in θ-space, using the transformation $u = u_0 \cos\theta$. Also recall that θ is the angle along the antenna pattern measured from the z-axis, which is aligned with the endfire axis of the line source. Therefore, u must be scaled by u_0 (i.e., u/u_0) in order for a given value of u to represent a specific angle around the antenna pattern. This scaling enables the comparison of patterns from antennas with different electric lengths. Since the variance is the second moment, the variance σ_r^2 must be normalized using u_0^2 to represent the variance of the pattern on the interval $[-\pi, \pi]$. This normalized variance is termed the beamwidth variance,

$$\sigma_{\mathrm{BW}}^2 = \frac{\sigma_r^2}{u_0^2}. \tag{4.23}$$

The variance formulation is an extension of the radiated power formulation, which leverages the results obtained for $R_g(p)$, $R_f(p)$, and $R_h(0)$ presented in the previous chapter. Since those results are already known, the variance problem is already partially solved.

4.2 Examples

The theory developed in the preceding section has been applied to several canonical current distributions to develop exact closed-form expressions for the stationary autocorrelation function $R_m(0)$. These closed-form expressions can then be combined with the results for $R_h(0)$, presented in the preceding chapter, using Equations (4.12) and (4.23) to yield exact closed-form expressions for the beamwidth variance. Considered in this chapter are the half-wave dipole, cosine, cosine-squared, generalized dipole, triangular, and uniform current distributions. Only the final results are presented in this chapter, while the detailed derivations for each current distribution are presented in Appendix B.

4.2.1 Half-Wave Dipole Distribution

Recall the expression for $R_f(p)$ for the half-wave dipole distribution, presented in the previous chapter as Equation (3.26),

$$R_f(p) = \frac{A_m^2}{2} \sin\left(\frac{p}{2}\right) \left[-H\left(p + 2\pi\right) + 2H\left(p\right) - H\left(p - 2\pi\right)\right], \tag{4.24}$$

where $H(p)$ is the Heaviside step function. Equation (4.24) can be used with Equation (4.16) to derive the expression for $R_n(p)$:

$$\begin{aligned} R_n(p) = \frac{A_m^2}{4} \Bigg\{ & \frac{1}{2} \sin\left(\frac{p}{2}\right) \left[-H\left(p + 2\pi\right) + 2H\left(p\right) - H\left(p - 2\pi\right)\right] \\ & - \delta\left(p + 2\pi\right) - 2\delta\left(p\right) - \delta\left(p - 2\pi\right) \Bigg\}. \end{aligned} \tag{4.25}$$

Finally, Equation (4.25) can be inserted into Equation (4.22) to find the stationary autocorrelation function $R_m(0)$ for the half-wave dipole distribution such that,

$$R_m(0) = \frac{A_m^2}{8\pi}\left[\mathrm{Cin}\left(2\pi\right) - 2\right]. \tag{4.26}$$

To complete the analysis for the half-wave dipole, Equations (3.27) and (4.26) can be combined in Equation (4.12) and the result inserted into Equation (4.23) to yield the beamwidth variance,

$$\sigma_{\mathrm{BW}}^2 = 1 - \frac{2}{\mathrm{Cin}\left(2\pi\right)} \approx 0.1795. \tag{4.27}$$

Subsequent examples in this chapter will suppress these last few steps due to the complexity of the resulting equations.

4.2.2 Cosine Distribution

Recall the expression for $R_f(p)$ for the cosine distribution, presented in the previous chapter as Equation (3.32),

$$\begin{aligned} R_f(p) = A_m^2 \Bigg\{ &\left[-\left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ &+ \left[\left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Bigg\}. \end{aligned} \tag{4.28}$$

Equation (4.28) can be used with Equation (4.16) to derive the expression for $R_n(p)$:

$$\begin{aligned} R_n(p) = \frac{A_m^2}{4} \Bigg\{ &\left[-\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ &+ \left[\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \\ &- \delta\left(p + 2\pi\right) - 2\delta\left(p\right) - \delta\left(p - 2\pi\right)\Bigg\}. \end{aligned} \tag{4.29}$$

Finally, Equation (4.29) can be inserted into Equation (4.22) to find the stationary autocorrelation function $R_m(0)$ for the cosine distribution:

$$\begin{aligned} R_m(0) = \frac{A_m^2}{4\pi} \Bigg\{ & \left(\frac{3}{4} - u_0^2\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \frac{1}{2}\right)\right]\right] \\ & - \pi\left(\frac{1}{4} - u_0^2\right) \left[\mathrm{Si}\left[2\pi\left(u_0 + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \frac{1}{2}\right)\right]\right] \\ & - u_0 \cos\left(2\pi u_0\right) - \frac{\sin\left(2\pi u_0\right)}{\pi} - 3u_0 \Bigg\}. \end{aligned} \tag{4.30}$$

4.2.3 Cosine-Squared Distribution

Recall the expression for $R_f(p)$ for the cosine-squared distribution, presented in the previous chapter as Equation (3.36),

$$\begin{aligned} R_f(p) = \frac{A_m^2}{8} \Bigg\{ & \left[(2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p)\right] \left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ & + \left[(2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p)\right] \left[H\left(p\right) - H\left(p - 2\pi\right)\right] \Bigg\}. \end{aligned} \tag{4.31}$$

Equation (4.31) can be used with Equation (4.16) to derive the expression for $R_n(p)$ such that,

$$\begin{aligned} R_n(p) = \frac{A_m^2}{8} \Bigg\{ & \left[(2\pi + p)\left(u_0^2 - 1\right)\cos(p) - \left(u_0^2 + 1\right)\sin(p)\right] \left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ & + \left[(2\pi - p)\left(u_0^2 - 1\right)\cos(p) + \left(u_0^2 + 1\right)\sin(p)\right] \left[H\left(p\right) - H\left(p - 2\pi\right)\right] \Bigg\}. \end{aligned} \tag{4.32}$$

Finally, Equation (4.32) can be inserted into Equation (4.22) to find the stationary autocorrelation function $R_m(0)$ for the cosine-squared distribution:

$$\begin{aligned} R_m(0) = \frac{A_m^2}{8\pi} \Bigg\{ & 2\pi\left(u_0^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + 1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - 1\right)\right]\right] \\ & + \left(u_0^2 + 1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + 1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - 1\right)\right]\right] \\ & + 2u_0\left[\cos\left(2\pi u_0\right) - 1\right] \Bigg\}. \end{aligned} \tag{4.33}$$

4.2.4 Generalized Dipole Distribution

Recall the expression for $R_f(p)$ for the generalized dipole distribution, presented in the previous chapter as Equation (3.44),

$$\begin{aligned} R_f(p) = A_m^2 u_0 \Big\{ & \sin\left[u_0\left(2\pi + p\right)\right]\left[H\left(p + 2\pi\right) - H\left(p + \pi\right)\right] \\ & - \left[2\sin\left(u_0 p\right) + \sin\left[u_0\left(2\pi + p\right)\right]\right]\left[H\left(p + \pi\right) - H\left(p\right)\right] \\ & + \left[2\sin\left(u_0 p\right) - \sin\left[u_0\left(2\pi - p\right)\right]\right]\left[H\left(p\right) - H\left(p - \pi\right)\right] \\ & + \sin\left[u_0\left(2\pi - p\right)\right]\left[H\left(p - \pi\right) - H\left(p - 2\pi\right)\right] \Big\}. \end{aligned} \tag{4.34}$$

Equation (4.34) can be used with Equation (4.16) to derive the expression for $R_n(p)$. Specifically,

$$\begin{aligned} R_n(p) = A_m^2 u_0 \Big\{ & u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\left[H\left(p + 2\pi\right) - H\left(p + \pi\right)\right] \\ & - \left[2u_0^2\sin\left(u_0 p\right) + u_0^2\sin\left[u_0\left(2\pi + p\right)\right]\right]\left[H\left(p + \pi\right) - H\left(p\right)\right] \\ & + \left[2u_0^2\sin\left(u_0 p\right) - u_0^2\sin\left[u_0\left(2\pi - p\right)\right]\right]\left[H\left(p\right) - H\left(p - \pi\right)\right] \\ & + u_0^2\sin\left[u_0\left(2\pi - p\right)\right]\left[H\left(p - \pi\right) - H\left(p - 2\pi\right)\right] \\ & - u_0\delta\left(p + 2\pi\right) + 4u_0\cos\left(\pi u_0\right)\delta\left(p + \pi\right) \\ & - \left[4u_0 + 2u_0\cos\left(2\pi u_0\right)\right]\delta\left(p\right) \\ & + 4u_0\cos\left(\pi u_0\right)\delta\left(p - \pi\right) - u_0\delta\left(p - 2\pi\right) \Big\}. \end{aligned} \tag{4.35}$$

Finally, Equation (4.35) can be inserted into Equation (4.22) to find the stationary autocorrelation function $R_m(0)$ for the generalized dipole distribution:

$$\begin{aligned} R_m(0) = \frac{A_m^2 u_0^3}{\pi} \Big\{ & 2\mathrm{Cin}\left(2\pi u_0\right) + 8\cos\left(\pi u_0\right)\frac{\sin\left(\pi u_0\right)}{\pi u_0} - 4 \\ & - \cos\left(2\pi u_0\right)\left[\mathrm{Cin}\left(4\pi u_0\right) - 2\mathrm{Cin}\left(2\pi u_0\right) + 2\right] \\ & + \sin\left(2\pi u_0\right)\left[\mathrm{Si}\left(4\pi u_0\right) - 2\mathrm{Si}\left(2\pi u_0\right) - \frac{1}{\pi u_0}\right] \Big\}. \end{aligned} \tag{4.36}$$

4.2.5 Triangular Distribution

Recall the expression for $R_f(p)$ for the triangular distribution, presented in the previous chapter as Equation (3.52),

$$R_f(p) = \frac{A_m^2}{6\pi^2}\Bigg\{ \left[u_0^2\left(2\pi+p\right)^3 + 6\left(2\pi+p\right)\right]\left[H\left(p+2\pi\right) - H\left(p+\pi\right)\right]$$
$$+ \left[u_0^2\left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p\right]\left[H\left(p+\pi\right) - H\left(p\right)\right]$$
$$+ \left[u_0^2\left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p\right]\left[H\left(p\right) - H\left(p-\pi\right)\right]$$
$$+ \left[u_0^2\left(2\pi-p\right)^3 + 6\left(2\pi-p\right)\right]\left[H\left(p-\pi\right) - H\left(p-2\pi\right)\right]\Bigg\}. \tag{4.37}$$

Equation (4.37) can be used with Equation (4.16) to derive the expression for $R_n(p)$:

$$R_n(p) = \frac{A_m^2}{\pi^2}\Bigg\{ - u_0^2\left(2\pi+p\right)\left[H\left(p+2\pi\right) - H\left(p+\pi\right)\right]$$
$$+ u_0^2\left(2\pi+3p\right)\left[H\left(p+\pi\right) - H\left(p\right)\right]$$
$$+ u_0^2\left(2\pi-3p\right)\left[H\left(p\right) - H\left(p-\pi\right)\right]$$
$$- u_0^2\left(2\pi-p\right)\left[H\left(p-\pi\right) - H\left(p-2\pi\right)\right]$$
$$- \delta\left(p+2\pi\right) + 4\delta\left(p+\pi\right) - 6\delta\left(p\right) + 4\delta\left(p-\pi\right) - \delta\left(p-2\pi\right)\Bigg\}. \tag{4.38}$$

Finally, Equation (4.38) can be inserted into Equation (4.22) to find the stationary autocorrelation function $R_m(0)$ for the triangular distribution such that,

$$R_m(0) = \frac{A_m^2 u_0}{\pi^3}\Bigg\{8\pi u_0 \mathrm{Si}\left(\pi u_0\right) - 4\pi u_0 \mathrm{Si}\left(2\pi u_0\right)$$
$$+ \frac{8\sin\left(\pi u_0\right)}{\pi u_0} - \frac{2\sin\left(2\pi u_0\right)}{2\pi u_0}$$
$$+ 8\cos\left(\pi u_0\right) - 2\cos\left(2\pi u_0\right) - 12\Bigg\}. \tag{4.39}$$

4.2.6 Uniform Distribution

Recall the expression for $R_f(p)$ for the uniform distribution, presented in the previous chapter as Equation (3.48),

$$R_f(p) = A_m^2 \Big\{ \left[\delta\left(p+2\pi\right) - 2\delta\left(p\right) + \delta\left(p-2\pi\right)\right] \\ + u_0^2 \left[R\left(p+2\pi\right) - 2R\left(p\right) + R\left(p-2\pi\right)\right] \Big\}. \tag{4.40}$$

Equation (4.40) can be used with Equation (4.16) to derive the expression for $R_n(p)$:

$$R_n(p) = -A_m^2 \Big\{ \left[\delta''\left(p+2\pi\right) - 2\delta''\left(p\right) + \delta''\left(p-2\pi\right)\right] \\ + u_0^2 \left[\delta\left(p+2\pi\right) - 2\delta\left(p\right) + \delta\left(p-2\pi\right)\right] \Big\}. \tag{4.41}$$

Finally, Equation (4.41) can be inserted into Equation (4.22) to find the stationary autocorrelation function $R_m(0)$ for the uniform distribution. Specifically,

$$R_m(0) = \frac{2A_m^2 u_0}{\pi} \left\{ \frac{\cos\left(2\pi u_0\right)}{2\pi^2} - \frac{\sin\left(2\pi u_0\right)}{4\pi^3 u_0} + \frac{2u_0^2}{3} \right\}. \tag{4.42}$$

4.3 Validation

The results of the preceding section can be validated by inserting each result for the stationary autocorrelation function $R_h(0)$, developed in the previous chapter, and $R_m(0)$ into Equation (4.12), normalizing by u_0^2 in accordance with Equation (4.23), and plotting the beamwidth variance σ_{BW}^2 as a function of electric length u_0. For purposes of comparison, the amplitude of each current distribution is set to unity (i.e., $A_m = 1$). In addition, the beamwidth variance can be calculated by numerically integrating Equation (4.1) with the power-pattern function $G(u)$ for each current distribution and normalizing by u_0^2, in accordance with Equation (4.23). As before, the power-pattern functions can be found by taking the Fourier transform of each current distribution:

$$
\begin{aligned}
G(u) &= A_m \cos(\pi u) \left[\frac{4}{1-4u^2}\right] && \text{Cosine} \\
&= A_m \sin(\pi u) \left[\frac{1}{u(1-u^2)}\right] && \text{Cosine} - \text{Squared} \\
&= 2A_m u_0 \left[\frac{\cos(\pi u_0) - \cos(\pi u)}{u^2 - u_0^2}\right] && \text{Generalized} \qquad (4.43) \\
&= \pi A_m \left[\frac{\sin(\pi u/2)}{\pi u/2}\right]^2 && \text{Triangular} \\
&= 2\pi A_m \left[\frac{\sin(\pi u)}{\pi u}\right] && \text{Uniform}
\end{aligned}
$$

The results are compared for each current distribution in Figures (4.1) through (4.5). The comparisons demonstrate effectively perfect agreement between the closed-form equations and the numerically integrated results. Also depicted are electrically long approximations derived from the exact closed-form analytical expressions [19]. The results presented in this chapter are entirely new and the equations are exact.

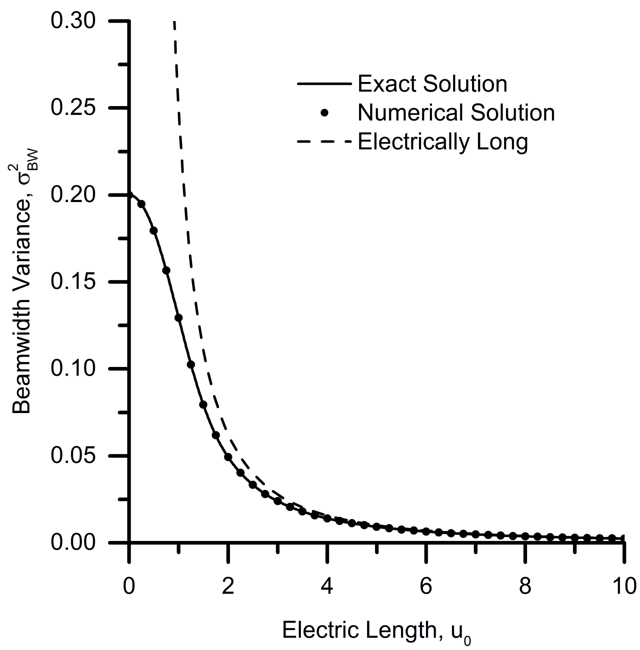

Figure 4.1: Cosine Distribution - Beamwidth variance as a function of electrical length.

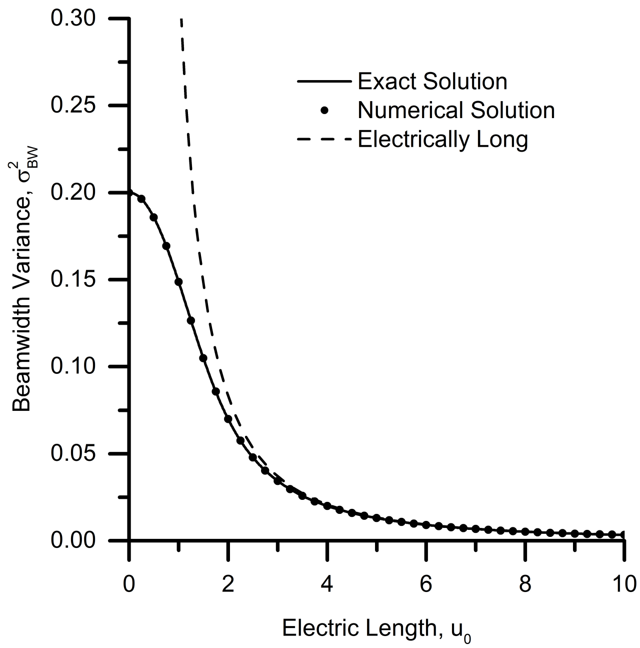

Figure 4.2: Cosine-Squared Distribution - Beamwidth variance as a function of electrical length.

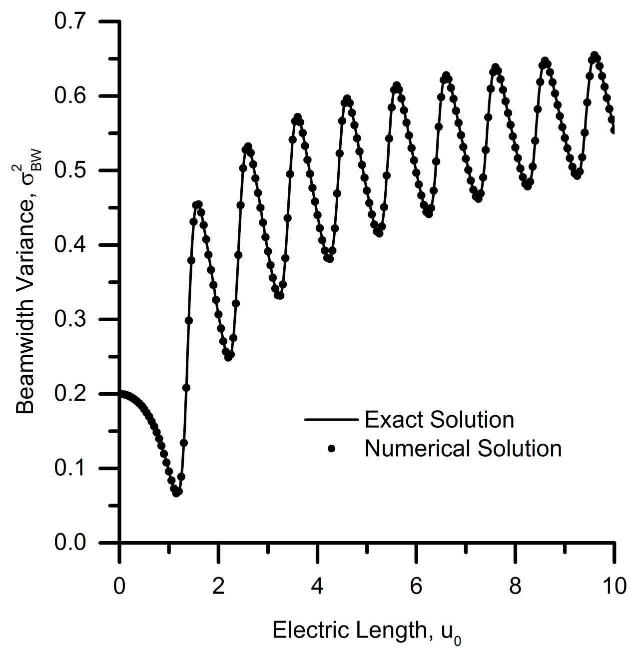

Figure 4.3: Generalized Dipole Distribution - Beamwidth variance as a function of electrical length.

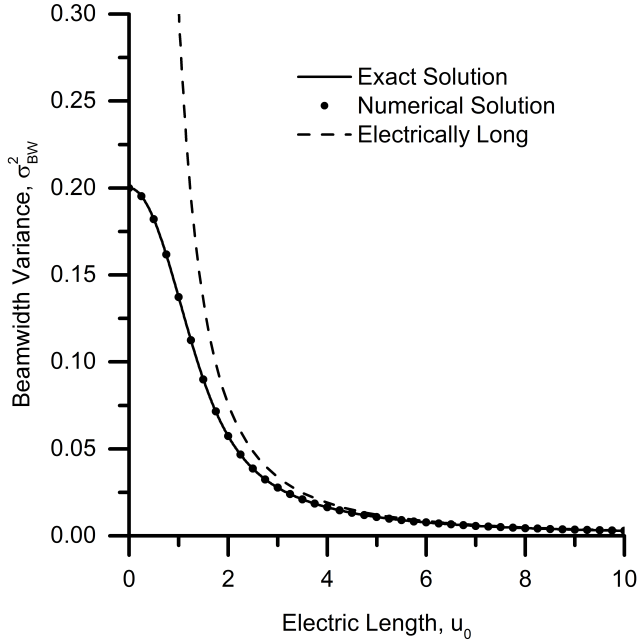

Figure 4.4: Triangular Distribution - Beamwidth variance as a function of electrical length.

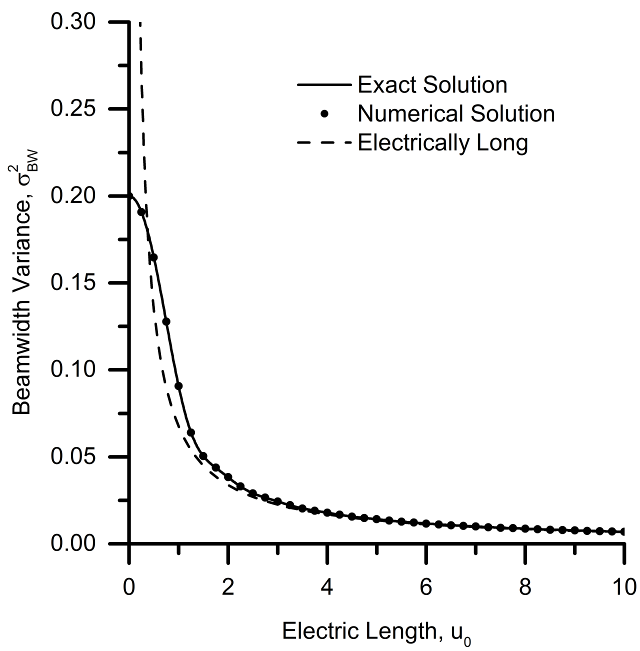

Figure 4.5: Uniform Distribution - Beamwidth variance as a function of electrical length.

Chapter V

Radiated Power of a Scanning Line Source

This chapter presents the development of a methodology, based on autocorrelation principles, for determining the radiated power of a scanning line source radiator directly from the current distribution without *a priori* knowledge of the antenna pattern. The methodology enables determining exact closed-form expressions for the radiated power and is applied to five canonical current distributions — half-wave dipole, cosine, cosine-squared, triangular, and uniform. The closed-form results are then compared to results obtained by numerically integrating the power-pattern functions in order to validate the newly developed closed-form expressions. The validation demonstrates effectively perfect agreement between the exact closed-form expressions and the numerically integrated results. The theory, results, and validation have been published in the *IEEE Transactions on Antennas and Propagation* [20].

5.1 Formulation

Consider the real-valued current distribution, $g(p)$, used in the preceding formulations, but now modified with a linear phase progression,

$$\widetilde{g}(p) = g(p)e^{-j\alpha p}, \tag{5.1}$$

where α is the linear phase progression constant. Taking the Fourier transform of Equation (5.1) yields,

$$\mathcal{F}\left\{\widetilde{g}(p)\right\} = \mathcal{F}\left\{g(p)e^{-j\alpha p}\right\} = G(u-\alpha) \tag{5.2}$$

where

$$G(u) = \mathcal{F}\left\{g(p)\right\}. \tag{5.3}$$

When $\alpha = 0$ the radiated power is given by Equation (3.1),

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0}^{u_0} G^2(u)\left(u_0^2 - u^2\right)\, du. \tag{5.4}$$

Substituting Equation (5.2) into Equation (5.4) yields the expression for the radiated power from a line source radiator that includes a linearly varying phase,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0}^{u_0} G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du. \tag{5.5}$$

Making the substitutions $y = u - \alpha$ and $dy = du$ in Equation (5.5) yields,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0-\alpha}^{u_0-\alpha} G^2(y)\left[u_0^2 - (y+\alpha)^2\right]\, dy. \tag{5.6}$$

Expanding terms in Equation (5.6) results in:

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0-\alpha}^{u_0-\alpha} G^2(y)\left(u_0^2 - y^2 - 2\alpha y - \alpha^2\right)\, dy. \tag{5.7}$$

Obviously, a straightforward change of variables, $y = u$ and $dy = du$, can be made in Equation (5.7),

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-u_0-\alpha}^{u_0-\alpha} G^2(u)\left(u_0^2 - u^2 - 2\alpha u - \alpha^2\right)\, du. \tag{5.8}$$

Recalling the definition of the pulse function:

$$\Pi(u) = \begin{cases} 1, & |u| \leq \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}, \tag{5.9}$$

and making the appropriate substitution into Equation (5.8),

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-\infty}^{\infty} \Pi^2\left(\frac{u+\alpha}{2u_0}\right) G^2(u)\left(u_0^2 - u^2 - 2\alpha u - \alpha^2\right)\, du. \tag{5.10}$$

Defining the function,

$$\widetilde{H}^2(u) = \widetilde{F}^2(u)\Pi^2\left(\frac{u+\alpha}{2u_0}\right), \tag{5.11}$$

where,

$$\widetilde{F}^2(u) = \left(u_0^2 - u^2 - 2\alpha u - \alpha^2\right) G^2(u), \tag{5.12}$$

enables Equation (5.10) to be written as,

$$P_{\text{rad}} = \frac{k^2\eta}{16\pi u_0^3}\int_{-\infty}^{\infty} \widetilde{H}^2(u)\ du. \tag{5.13}$$

Using Parseval's identity, $\widetilde{H}(u)$ can be related to its inverse Fourier transform $\widetilde{h}(p)$ using,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widetilde{H}(u)\right|^2\ du = \int_{-\infty}^{\infty} \left|\widetilde{h}(p)\right|^2\ dp. \tag{5.14}$$

Equation (5.14) can be applied to Equation (5.13) such that,

$$P_{\text{rad}} = \frac{k^2\eta}{8u_0^3}\int_{-\infty}^{\infty} \left|\widetilde{h}(p)\right|^2\ dp. \tag{5.15}$$

Invoking the definition of the stationary autocorrelation function:

$$\widetilde{R}_h(p) = \int_{-\infty}^{\infty} \widetilde{h}(\tau)\widetilde{h}^*(\tau - p)\ d\tau, \tag{5.16}$$

then evaluating for $p = 0$,

$$\widetilde{R}_h(0) = \int_{-\infty}^{\infty} \widetilde{h}(\tau)\widetilde{h}^*(\tau)\ d\tau = \int_{-\infty}^{\infty} \left|\widetilde{h}(\tau)\right|^2\ d\tau, \tag{5.17}$$

and substituting into Equation (5.15) yields an expression for the radiated power from a scanning line source expressed solely in terms of an autocorrelation function,

$$P_{\text{rad}} = \frac{k^2\eta\widetilde{R}_h(0)}{8u_0^3}. \tag{5.18}$$

Taking the inverse Fourier transform of Equation (5.11) and applying the convolution theorem yields,

$$\mathcal{F}^{-1}\left\{\widetilde{H}^2(u)\right\} = \mathcal{F}^{-1}\left\{\widetilde{F}^2(u)\Pi^2\left(\frac{u+\alpha}{2u_0}\right)\right\} = \mathcal{F}^{-1}\left\{\widetilde{F}^2(u)\right\} * \mathcal{F}^{-1}\left\{\Pi^2\left(\frac{u+\alpha}{2u_0}\right)\right\}. \quad (5.19)$$

Evaluating the inverse Fourier transforms in Equation (5.19) to yield,

$$\widetilde{R}_h(p) = \widetilde{R}_f(p) * \widetilde{R}_s(p). \quad (5.20)$$

Applying the frequency or phase shifting property of the Fourier transform to the stationary autocorrelation function for the pulse function produces the following relationship:

$$\widetilde{R}_s(p) = R_s(p)e^{j\alpha p}, \quad (5.21)$$

since the pulse function is shifted in u-space for the linearly varying phase case. Substituting Equation (5.21) into Equation (5.20),

$$\widetilde{R}_h(p) = \widetilde{R}_f(p) * \left[R_s(p)e^{j\alpha p}\right]. \quad (5.22)$$

Applying the definition of convolution to Equation (5.22),

$$\widetilde{R}_h(p) = \int_{-\infty}^{\infty} \widetilde{R}_f(\tau)R_s(p-\tau)e^{j\alpha(p-\tau)}\ d\tau. \quad (5.23)$$

For the case where $p = 0$, Equation (5.23) can be written as

$$\widetilde{R}_h(0) = \int_{-\infty}^{\infty} \widetilde{R}_f(\tau)R_s(-\tau)e^{-j\alpha\tau}\ d\tau. \quad (5.24)$$

Recalling the definition for $R_s(p)$, previously given in Equation (3.18),

$$R_s(p) = \frac{u_0}{\pi}\frac{\sin(u_0 p)}{u_0 p}, \quad (5.25)$$

and substituting into Equation (5.24),

$$\widetilde{R}_h(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \widetilde{R}_f(\tau)\frac{\sin\left(u_0\tau\right)}{u_0\tau}e^{-j\alpha\tau}\ d\tau. \tag{5.26}$$

Next, taking the inverse Fourier transform of Equation (5.12),

$$\mathcal{F}^{-1}\left\{\widetilde{F}^2(u)\right\} = \mathcal{F}^{-1}\left\{\left(u_0^2 - u^2 - 2\alpha u - \alpha^2\right)G^2(u)\right\}, \tag{5.27}$$

and applying the Fourier transform identity,

$$\frac{d^n}{dp^n}f(p) \Leftrightarrow \left(-ju\right)^n F(u), \tag{5.28}$$

yields an extension of the Helmholtz operator, previously presented in Equation (3.22), to now include linear phase terms,

$$\widetilde{R}_f(p) = \left[u_0^2 + \frac{d^2}{dp^2} - 2j\alpha\frac{d}{dp} - \alpha^2\right]R_g(p). \tag{5.29}$$

Separating the operator in Equation (5.29),

$$\widetilde{R}_f(p) = \left[u_0^2 + \frac{d^2}{dp^2}\right]R_g(p) - \left[2j\alpha\frac{d}{dp} + \alpha^2\right]R_g(p), \tag{5.30}$$

and substituting the definition for $R_f(p)$ from Equation (3.22),

$$R_f(p) = \left[u_0^2 + \frac{d^2}{dp^2}\right]R_g(p), \tag{5.31}$$

yields the simplified form,

$$\widetilde{R}_f(p) = R_f(p) - \left[2j\alpha\frac{d}{dp} + \alpha^2\right]R_g(p). \tag{5.32}$$

Equation (5.32) can then be substituted into Equation (5.26),

$$\widetilde{R}_h(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty}\left\{R_f(\tau) - \left[2j\alpha\frac{d}{d\tau} + \alpha^2\right]R_g(\tau)\right\}\frac{\sin\left(u_0\tau\right)}{u_0\tau}e^{-j\alpha\tau}\ d\tau. \tag{5.33}$$

Distributing terms in Equation (5.33),

$$\widetilde{R}_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \left[R_f(\tau) - \alpha^2 R_g(\tau) - 2j\alpha \frac{d}{d\tau} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} e^{-j\alpha\tau} \, d\tau. \tag{5.34}$$

Separating the integrals in Equation (5.34),

$$\begin{aligned} \widetilde{R}_h(0) = \frac{u_0}{\pi} \Bigg\{ & \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0\tau)}{u_0\tau} e^{-j\alpha\tau} \, d\tau \\ & - \alpha^2 \int_{-\infty}^{\infty} R_g(\tau) \frac{\sin(u_0\tau)}{u_0\tau} e^{-j\alpha\tau} \, d\tau \\ & - 2j\alpha \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} e^{-j\alpha\tau} \, d\tau \Bigg\}. \end{aligned} \tag{5.35}$$

Recalling Euler's formula,

$$e^{-j\alpha\tau} = \cos(\alpha\tau) - j\sin(\alpha\tau), \tag{5.36}$$

and substituting into Equation(5.35),

$$\begin{aligned} \widetilde{R}_h(0) = \frac{u_0}{\pi} \Bigg\{ & \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0\tau)}{u_0\tau} \left[\cos(\alpha\tau) - j\sin(\alpha\tau)\right] \, d\tau \\ & - \alpha^2 \int_{-\infty}^{\infty} R_g(\tau) \frac{\sin(u_0\tau)}{u_0\tau} \left[\cos(\alpha\tau) - j\sin(\alpha\tau)\right] \, d\tau \\ & - 2j\alpha \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} \left[\cos(\alpha\tau) - j\sin(\alpha\tau)\right] \, d\tau \Bigg\}. \end{aligned} \tag{5.37}$$

Distributing terms and separating the integrals in Equation (5.37),

$$\begin{aligned}\widetilde{R}_h(0) = \frac{u_0}{\pi}\Bigg\{ &\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\ d\tau \\ &- j\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\sin(\alpha\tau)\ d\tau \\ &- \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\ d\tau \\ &+ j\alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\sin(\alpha\tau)\ d\tau \\ &- 2j\alpha\int_{-\infty}^{\infty} \left[\frac{d}{d\tau}R_g(\tau)\right]\frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\ d\tau \\ &- 2\alpha\int_{-\infty}^{\infty} \left[\frac{d}{d\tau}R_g(\tau)\right]\frac{\sin(u_0\tau)}{u_0\tau}\sin(\alpha\tau)\ d\tau\Bigg\}. \end{aligned} \tag{5.38}$$

Recalling that i) the autocorrelation function is even; ii) the derivative of an even function is odd; and iii) the integral of an odd function over symmetric limits of integration is zero, enables the elimination of integrals with odd integrands in Equation (5.38),

$$\begin{aligned}\widetilde{R}_h(0) = \frac{u_0}{\pi}\Bigg\{ &\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\ d\tau \\ &- \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\ d\tau \\ &- 2\alpha\int_{-\infty}^{\infty} \left[\frac{d}{d\tau}R_g(\tau)\right]\frac{\sin(u_0\tau)}{u_0\tau}\sin(\alpha\tau)\ d\tau\Bigg\}. \end{aligned} \tag{5.39}$$

Applying product-to-sum trigonometric identities to Equation (5.39),

$$\begin{aligned}\widetilde{R}_h(0) = \frac{u_0}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin[(u_0+\alpha)\tau] + \sin[(u_0-\alpha)\tau]}{u_0\tau}\ d\tau \\ &- \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin[(u_0+\alpha)\tau] + \sin[(u_0-\alpha)\tau]}{u_0\tau}\ d\tau \\ &- 2\alpha\int_{-\infty}^{\infty} \left[\frac{d}{d\tau}R_g(\tau)\right]\frac{\cos[(u_0-\alpha)\tau] - \cos[(u_0+\alpha)\tau]}{u_0\tau}\ d\tau\Bigg\}. \end{aligned} \tag{5.40}$$

Distributing terms and separating the integrals in Equation (5.40) yields the stationary autocorrelation function for the radiated power from a line source radiator with a linearly varying phase,

$$\widetilde{R}_h(0) = \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_f(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau + \int_{-\infty}^{\infty} R_f(\tau)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g(\tau)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau + 2\alpha \int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau - 2\alpha \int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau \Bigg\}, \tag{5.41}$$

where $R_g'(\tau)$ is the first derivative of $R_g(\tau)$ in prime notation. In summary, the radiated power for a scanning line source can be found by utilizing the results for $R_g(p)$ and $R_f(p)$ determined previously from the broadside radiated power formulation, calculating the first derivative of $R_g(p)$, and substituting all three results into Equation (5.41). The integrations in Equation (5.41) can then be performed to yield the result for $\widetilde{R}_h(0)$ which, when combined with Equation (5.18), yields the radiated power for a scanning line source. This formulation will be exercised for several canonical current distributions in the following sections.

5.2 Examples

The theory developed in the preceding section has been applied to several canonical current distributions to develop exact closed-form expressions for the stationary autocorrelation function $\widetilde{R}_h(0)$. These closed-form expressions can then be used directly in conjunction with Equation (5.18) to yield exact closed-form expressions for the radiated power generated from that current distribution. Considered in this chapter are the half-wave dipole, cosine, cosine-squared, triangular, and uniform current distributions. Only the final results are presented in this chapter, while the detailed derivations for each current distribution are presented in Appendix C.

5.2.1 Half-Wave Dipole Distribution

Recalling the autocorrelation function $R_g(p)$ for the half-wave dipole distribution, developed previously for a broadside line source,

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{5.42}$$

Then finding the first-derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{2} \begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{5.43}$$

Also, recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{2} \begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{5.44}$$

Equations (5.42), (5.43), and (5.44) can be used in conjunction with Equation (5.41) to find the stationary autocorrelation function $\widetilde{R}_h(0)$ for the half-wave dipole distribution with a linearly varying phase:

$$\begin{aligned} \widetilde{R}_h(0) = \frac{A_m^2}{4\pi} \Big\{ & \left(1 - 2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Cin}\left(2\pi\alpha\right) + \mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & - 2\pi\alpha^2\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & - 2\pi\alpha\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Si}\left(2\pi\alpha\right) - \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & + 2\left[\cos\left(2\pi\alpha\right) - 1\right] \Big\}. \end{aligned} \tag{5.45}$$

To complete the analysis for the half-wave dipole with a linearly varying phase, Equation (5.45) can be substituted into Equation (5.18) to yield the exact expression for the radiated power:

$$\begin{aligned} P_{\text{rad}} = \frac{A_m^2 k^2 \eta}{4\pi} \Big\{ & \left(1 - 2\alpha^2\right) \left[\text{Cin}\left[2\pi\left(1+\alpha\right)\right] - 2\text{Cin}\left(2\pi\alpha\right) + \text{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & - 2\pi\alpha^2 \left[\text{Si}\left[2\pi\left(1+\alpha\right)\right] + \text{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & - 2\pi\alpha \left[\text{Si}\left[2\pi\left(1+\alpha\right)\right] - 2\text{Si}\left(2\pi\alpha\right) - \text{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & + 2\left[\cos\left(2\pi\alpha\right) - 1\right] \Big\}. \end{aligned} \tag{5.46}$$

Subsequent examples in this chapter will suppress this last step due to the complexity of the resulting equations.

5.2.2 Cosine Distribution

Recalling the autocorrelation function $R_g(p)$ for the cosine distribution, developed previously for a broadside line source,

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.47}$$

Then finding the first-derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{2} \begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.48}$$

Also, recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2 \begin{cases} -\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.49}$$

Equations (5.47), (5.48), and (5.49) can be used in conjunction with Equation (5.41) to find the stationary autocorrelation function $\widetilde{R}_h(0)$ for the cosine distribution with a linearly varying phase:

$$\begin{aligned} \widetilde{R}_h(0) = \frac{A_m^2}{2\pi}\Bigg\{ & \left(u_0^2-\alpha^2+\frac{1}{4}\right)\left[\text{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right] - \text{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\ & + \left(u_0^2-\alpha^2+\frac{1}{4}\right)\left[\text{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right] - \text{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & + \pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\left[\text{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right] + \text{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & + \pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\left[\text{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right] + \text{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\ & - (u_0-\alpha)\cos\left[2\pi(u_0+\alpha)\right] - (u_0+\alpha)\cos\left[2\pi(u_0-\alpha)\right] - 2u_0 \Bigg\}. \end{aligned} \tag{5.50}$$

5.2.3 Cosine-Squared Distribution

Recalling the autocorrelation function $R_g(p)$ for the cosine-squared distribution, developed previously for a broadside line source,

$$R_g(p) = \frac{A_m^2}{8} \begin{cases} (2\pi+p)\left[2+\cos(p)\right] - 3\sin(p), & -2\pi \le p \le 0 \\ (2\pi-p)\left[2+\cos(p)\right] + 3\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.51}$$

Then finding the first-derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{8} \begin{cases} -(p+2\pi)\sin(p) - 2\cos(p) + 2, & -2\pi \le p \le 0 \\ (p-2\pi)\sin(p) + 2\cos(p) - 2, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.52}$$

Also, recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{8} \begin{cases} (2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ (2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \quad (5.53)$$

Equations (5.51), (5.52), and (5.53) can be used in conjunction with Equation (5.41) to find the stationary autocorrelation function $\widetilde{R}_h(0)$ for the cosine-squared distribution with a linearly varying phase:

$$\begin{aligned} \widetilde{R}_h(0) = \frac{A_m^2}{16\pi} \Big\{ & 8\alpha \left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha\right)\right]\right] \\ & + \left(3u_0^2 - 3\alpha^2 - 4\alpha - 1\right)\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + 1\right)\right] \\ & - \left(3u_0^2 - 3\alpha^2 + 4\alpha - 1\right)\mathrm{Cin}\left[2\pi\left(u_0 + \alpha - 1\right)\right] \\ & + \left(3u_0^2 - 3\alpha^2 + 4\alpha - 1\right)\mathrm{Cin}\left[2\pi\left(u_0 - \alpha + 1\right)\right] \\ & - \left(3u_0^2 - 3\alpha^2 - 4\alpha - 1\right)\mathrm{Cin}\left[2\pi\left(u_0 - \alpha - 1\right)\right] \\ & + 8\pi\left(u_0^2 - \alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha\right)\right]\right] \\ & + 2\pi\left(u_0^2 - \alpha^2 - 2\alpha - 1\right)\mathrm{Si}\left[2\pi\left(u_0 + \alpha + 1\right)\right] \\ & + 2\pi\left(u_0^2 - \alpha^2 + 2\alpha - 1\right)\mathrm{Si}\left[2\pi\left(u_0 + \alpha - 1\right)\right] \\ & + 2\pi\left(u_0^2 - \alpha^2 + 2\alpha - 1\right)\mathrm{Si}\left[2\pi\left(u_0 - \alpha + 1\right)\right] \\ & + 2\pi\left(u_0^2 - \alpha^2 - 2\alpha - 1\right)\mathrm{Si}\left[2\pi\left(u_0 - \alpha - 1\right)\right] \\ & + 6\left(u_0 - \alpha\right)\cos\left[2\pi\left(u_0 + \alpha\right)\right] + 6\left(u_0 + \alpha\right)\cos\left[2\pi\left(u_0 - \alpha\right)\right] - 12u_0 \Big\}. \end{aligned} \quad (5.54)$$

5.2.4 Triangular Distribution

Recalling the autocorrelation function $R_g(p)$ for the triangular distribution, developed previously for a broadside line source,

$$R_g(p) = \frac{A_m^2}{6\pi^2} \begin{cases} (2\pi + p)^3, & -2\pi \le p \le -\pi \\ 4\pi^3 - 6\pi p^2 - 3p^3, & -\pi \le p \le 0 \\ 4\pi^3 - 6\pi p^2 + 3p^3, & 0 \le p \le \pi \\ (2\pi - p)^3, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.55}$$

Then finding the first-derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{6\pi^2} \begin{cases} 3(2\pi + p)^2, & -2\pi \le p \le -\pi \\ -12\pi p - 9p^2, & -\pi \le p \le 0 \\ -12\pi p + 9p^2, & 0 \le p \le \pi \\ -3(2\pi - p)^2, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.56}$$

Also, recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{6\pi^2} \begin{cases} u_0^2 (2\pi + p)^3 + 12\pi + 6p, & -2\pi \le p \le -\pi \\ u_0^2 \left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p, & -\pi \le p \le 0 \\ u_0^2 \left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p, & 0 \le p \le \pi \\ u_0^2 (2\pi - p)^3 + 12\pi - 6p, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.57}$$

Equations (5.55), (5.56), and (5.57) can be used in conjunction with Equation (5.41) to find the stationary autocorrelation function $\widetilde{R}_h(0)$ for the triangular distribution with a linearly varying phase:

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & 48\pi^2\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& - 48\pi^2\alpha\left[\mathrm{Cin}\left[\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right) + 24\pi\right]\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& - \left[8\pi^3\left(u_0^2-\alpha^2\right) + 48\pi\right]\left[\mathrm{Si}\left[\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& + \left[8\pi^2\left(u_0-\alpha\right) + \frac{4\left(2u_0+\alpha\right)}{\left(u_0+\alpha\right)^2}\right]\cos\left[2\pi\left(u_0+\alpha\right)\right] \\
& + \left[8\pi^2\left(u_0+\alpha\right) + \frac{4\left(2u_0-\alpha\right)}{\left(u_0-\alpha\right)^2}\right]\cos\left[2\pi\left(u_0-\alpha\right)\right] \\
& - \left[8\pi^2\left(u_0-\alpha\right) + \frac{16\left(2u_0+\alpha\right)}{\left(u_0+\alpha\right)^2}\right]\cos\left[\pi\left(u_0+\alpha\right)\right] \\
& - \left[8\pi^2\left(u_0+\alpha\right) + \frac{16\left(2u_0-\alpha\right)}{\left(u_0-\alpha\right)^2}\right]\cos\left[\pi\left(u_0-\alpha\right)\right] \\
& + 4\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right)\sin\left[2\pi\left(u_0+\alpha\right)\right] \\
& + 4\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right)\sin\left[2\pi\left(u_0-\alpha\right)\right] \\
& - 8\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right)\sin\left[\pi\left(u_0+\alpha\right)\right] \\
& - 8\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right)\sin\left[\pi\left(u_0-\alpha\right)\right] \\
& + 12\left[\frac{2u_0+\alpha}{\left(u_0+\alpha\right)^2} + \frac{2u_0-\alpha}{\left(u_0-\alpha\right)^2}\right]\Bigg\}.
\end{aligned}
\tag{5.58}
$$

5.2.5 Uniform Distribution

Recalling the autocorrelation function $R_g(p)$ for the uniform distribution, developed previously for a broadside line source,

$$
R_g(p) = A_m^2\begin{cases} 2\pi + p, & -2\pi \le p \le 0 \\ 2\pi - p, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{5.59}
$$

Then finding the first-derivative of $R_g(p)$,

$$R_g'(p) = A_m^2 \begin{cases} 1, & -2\pi \le p \le 0 \\ -1, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{5.60}$$

Also, recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2 \left[\delta(p+2\pi) - 2\delta(p) + \delta(p-2\pi)\right] + u_0^2 R_g(p) \tag{5.61}$$

for $-2\pi \le p \le 2\pi$ and $R_f(p) = 0$ otherwise. Equations (5.59), (5.60), and (5.61) can be used in conjunction with Equation (5.41) to find the stationary autocorrelation function $\widetilde{R}_h(0)$ for the uniform distribution with a linearly varying phase:

$$\begin{aligned} \widetilde{R}_h(0) = \frac{A_m^2}{\pi} \Bigg\{ & 2\alpha \left[\mathrm{Cin}\left[2\pi(u_0+\alpha)\right] - \mathrm{Cin}\left[2\pi(u_0-\alpha)\right]\right] \\ & + 2\pi\left(u_0^2 - \alpha^2\right)\left[\mathrm{Si}\left[2\pi(u_0+\alpha)\right] + \mathrm{Si}\left[2\pi(u_0-\alpha)\right]\right] \\ & + \frac{\sin\left[2\pi(u_0+\alpha)\right]}{2\pi} + \frac{\sin\left[2\pi(u_0-\alpha)\right]}{2\pi} \\ & + (u_0-\alpha)\cos\left[2\pi(u_0+\alpha)\right] + (u_0+\alpha)\cos\left[2\pi(u_0-\alpha)\right] - 4u_0 \Bigg\}. \end{aligned} \tag{5.62}$$

5.3 Validation

The results of the preceding section can be validated by inserting each result for the stationary autocorrelation function $\widetilde{R}_h(0)$ into Equation (5.18) and plotting the radiated power P_{rad} as a function of i) electric length u_0 for various phase coefficients α, and 2) phase coefficient for various electric lengths. Obviously, the half-wave dipole distribution results are only presented as a function of phase coefficient. For purposes of comparison, the amplitude of each current distribution is set to unity (i.e., $A_m = 1$). In addition, the radiated power can be calculated by numerically integrating Equation (5.1) with the power-pattern function $G(u)$ for each current distribution. The power-pattern functions can be found by taking the

Fourier transform of each current distribution, which produces the following results:

$$
\begin{aligned}
G(u) &= A_m \cos(\pi u) \left[\frac{4}{1 - 4u^2} \right] && \text{Cosine} \\
&= A_m \sin(\pi u) \left[\frac{1}{u(1 - u^2)} \right] && \text{Cosine} - \text{Squared} \\
&= \pi A_m \left[\frac{\sin(\pi u/2)}{\pi u/2} \right]^2 && \text{Triangular} \\
&= 2\pi A_m \left[\frac{\sin(\pi u)}{\pi u} \right] && \text{Uniform}
\end{aligned} \tag{5.63}
$$

The results are compared for each current distribution in Figures (5.1) through (5.9). The comparisons demonstrate effectively perfect agreement between the exact closed-form equations and the numerically integrated results. The results presented in this chapter are entirely new and the equations are exact.

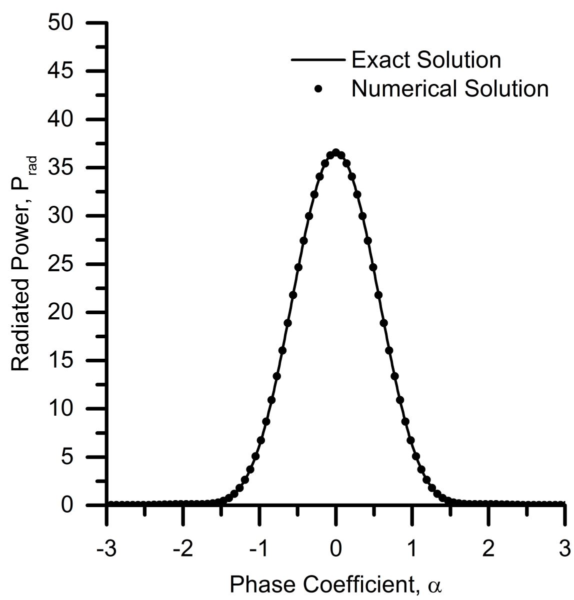

Figure 5.1: Half-Wave Dipole Distribution - Radiated power as a function of phase coefficient.

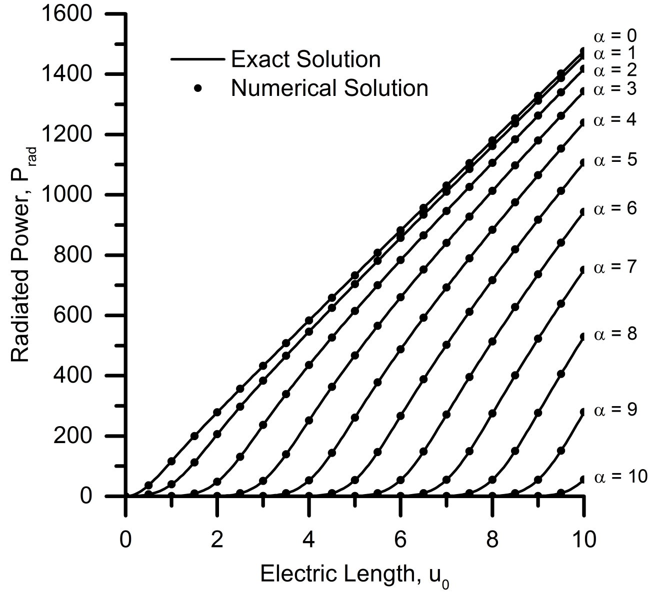

Figure 5.2: Cosine Distribution - Radiated power as a function of electrical length for various phase coefficients.

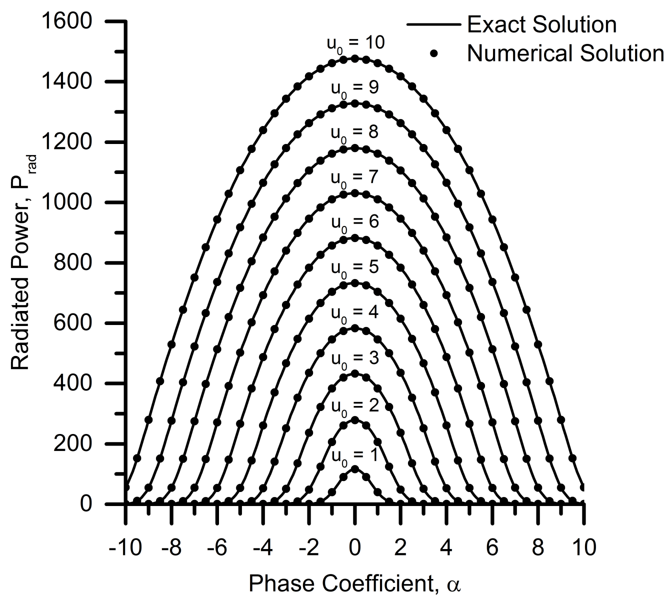

Figure 5.3: Cosine Distribution - Radiated power as a function of phase coefficient for various electrical lengths.

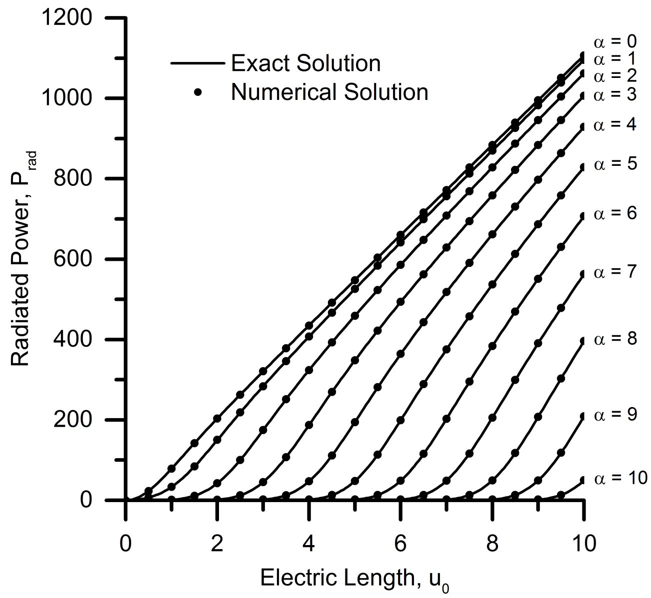

Figure 5.4: Cosine-Squared Distribution - Radiated power as a function of electrical length for various phase coefficients.

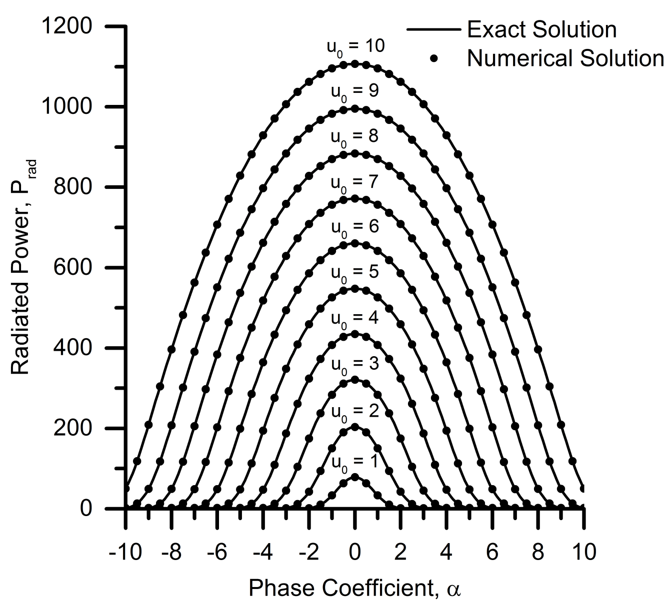

Figure 5.5: Cosine-Squared Distribution - Radiated power as a function of phase coefficient for various electrical lengths.

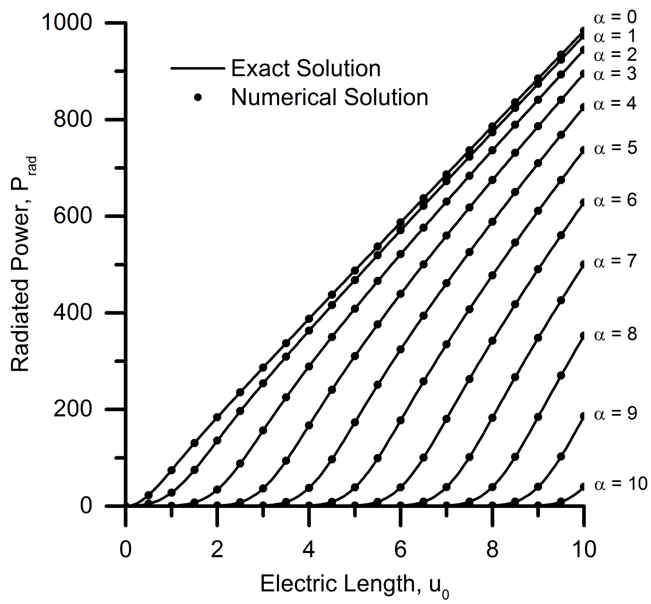

Figure 5.6: Triangular Distribution - Radiated power as a function of electrical length for various phase coefficients.

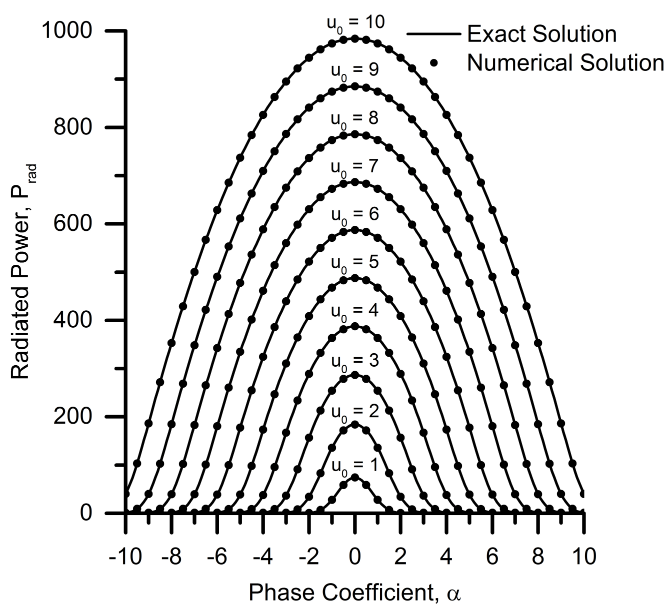

Figure 5.7: Triangular Distribution - Radiated power as a function of phase coefficient for various electrical lengths.

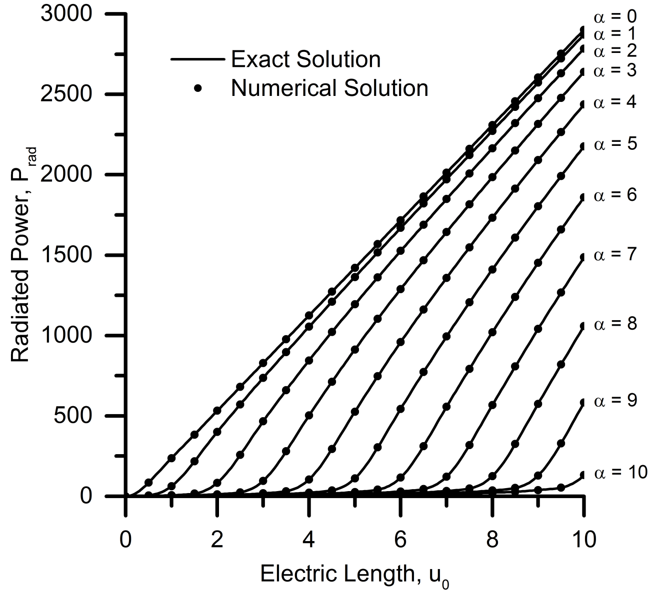

Figure 5.8: Uniform Distribution - Radiated power as a function of electrical length for various phase coefficients.

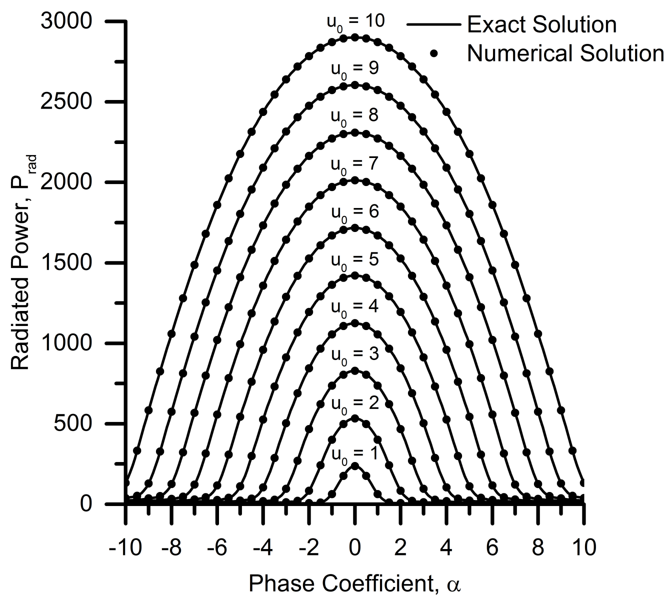

Figure 5.9: Uniform Distribution - Radiated power as a function of phase coefficient for various electrical lengths.

Chapter VI

Pattern Mean and Beamwidth Variance of a Scanning Line Source

This chapter presents the extension of the beamwidth variance concept to scanning line source radiators. In addition to the development of the formulation for the beamwidth variance, the concept of the "pattern mean" is introduced. Like the beamwidth variance, the pattern mean is based on the well defined statistical concept of the mean, or first raw moment, of a continuous distribution. The pattern mean and beamwidth variance can be used as metrics for characterizing the antenna pattern of a scanning line source radiator. Again, both the pattern mean and beamwidth variance can be determined using the same autocorrelation principles used in the previous chapters. Those principles are applied to a line source radiator with a linearly varying phase and the resulting methodology is then used to develop exact closed-form expressions for both the pattern mean and the beamwidth variance for five canonical current distributions — half-wave dipole, cosine, cosine-squared, triangular, and uniform. The exact closed-form equations are then compared to results obtained by numerically integrating both the first and second moments of the power-pattern functions in order to validate the newly developed expressions. The validations demonstrate effectively perfect agreement between the exact closed-form expressions and the numerically integrated results.

6.1 Formulation

The variance for a scanning beam can be found by applying the definition of the second central moment about the pattern mean, as represented in Equation (2.21),

$$\sigma_r^2 = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} (u-\mu)^2 G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du, \tag{6.1}$$

where μ is the mean. The mean is also the first raw moment of the power pattern function, which can be defined as

$$\mu = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} u G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du. \tag{6.2}$$

By expanding terms, Equation (6.1) can be written as the summation of three distinct integrals,

$$\begin{aligned}\sigma_r^2 = &\left[\frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} u^2 G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du\right] \\ &- 2\mu\left[\frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} u G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du\right] \\ &+ \mu^2\left[\frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du\right].\end{aligned} \tag{6.3}$$

The first term in Equation (6.3) is the second raw moment of the power pattern function, σ_0^2. The second term is simply the mean as previously defined in Equation (6.2). The third term is the radiated power, P_{rad}, given previously in Equation (5.5). Recognizing these relationships and making the appropriate substitutions enables Equation (6.3) to be written as

$$\sigma_r^2 = \sigma_0^2 - \mu^2, \tag{6.4}$$

where

$$\sigma_0^2 = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0}^{u_0} u^2 G^2(u-\alpha)\left(u_0^2 - u^2\right)\, du. \tag{6.5}$$

6.1.1 First Moment Formulation

The substitutions $y = u - \alpha$ and $dy = du$ can be made in Equation (6.2),

$$\mu = \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} (y+\alpha)\, G^2(y) \left[u_0^2 - (y+\alpha)^2\right]\, dy. \tag{6.6}$$

Expanding terms in Equation (6.6),

$$\mu = \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} (y+\alpha)\, G^2(y) \left(u_0^2 - y^2 - 2y\alpha - \alpha^2\right)\, dy. \tag{6.7}$$

Obviously, u can be directly substituted for y to obtain,

$$\mu = \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} (u+\alpha)\, G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right)\, du. \tag{6.8}$$

Distributing terms and separating the integrals in Equation (6.8),

$$\begin{aligned}\mu = &\left[\frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} uG^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right)\, du\right] \\ &+ \alpha \left[\frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right)\, du\right].\end{aligned} \tag{6.9}$$

The second term in Equation (6.9) is simply the radiated power, given previously in Equation (5.8). Therefore, Equation (6.9) can be written as

$$\mu = \alpha + \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} uG^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right)\, du. \tag{6.10}$$

The pulse function can be incorporated in Equation (6.10),

$$\mu = \alpha + \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} \Pi^2\left(\frac{u+\alpha}{2u_0}\right) uG^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right)\, du. \tag{6.11}$$

Equation (6.11) can be written as

$$\mu = \alpha + \frac{k^2 \eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} \widetilde{M}_1^2(u)\, du, \tag{6.12}$$

where the subscript "1" represents the first moment and the wide tilde represents varying phase,

$$\widetilde{M}_1^2(u) = \widetilde{N}_1^2(u)\Pi^2\left(\frac{u+\alpha}{2u_0}\right), \tag{6.13}$$

and

$$\widetilde{N}_1^2(u) = u\left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right)G^2(u). \tag{6.14}$$

Using Parseval's identity, $\widetilde{M}_1(u)$ can be related to its inverse Fourier transform $\widetilde{m}_1(p)$ using,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{M}_1(u)\right|^2\ du = \int_{-\infty}^{\infty}\left|\widetilde{m}_1(p)\right|^2\ dp. \tag{6.15}$$

Equation (6.15) can be applied to Equation (6.12),

$$\mu = \alpha + \frac{k^2\eta}{8u_0^3 P_{\text{rad}}}\int_{-\infty}^{\infty}\left|\widetilde{m}_1(p)\right|^2\ dp. \tag{6.16}$$

Invoking the definition of the stationary autocorrelation function,

$$\widetilde{R}_{m1}(p) = \int_{-\infty}^{\infty}\widetilde{m}_1(\tau)\widetilde{m}_1^*(\tau - p)\ d\tau, \tag{6.17}$$

then evaluating for $p = 0$,

$$\widetilde{R}_{m1}(0) = \int_{-\infty}^{\infty}\widetilde{m}_1(\tau)\widetilde{m}_1^*(\tau)\ d\tau = \int_{-\infty}^{\infty}\left|\widetilde{m}_1(\tau)\right|^2\ d\tau, \tag{6.18}$$

and substituting into Equation (6.16),

$$\mu = \alpha + \frac{k^2\eta\widetilde{R}_{m1}(0)}{8u_0^3 P_{\text{rad}}}. \tag{6.19}$$

Substituting the definition for the radiated power P_{rad}, given previously in Equation (5.18), into Equation (6.19) and canceling terms yields the final expression for the pattern mean expressed solely in terms of autocorrelation functions,

$$\mu = \alpha + \frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)}. \tag{6.20}$$

The autocorrelation function $\widetilde{R}_{m1}(0)$ can be determined by first recalling Equation (5.12) and substituting into Equation (6.14),

$$\widetilde{N}_1^2(u) = u\widetilde{F}^2(u). \tag{6.21}$$

Taking the inverse Fourier transform of Equation (6.21),

$$\mathcal{F}^{-1}\left\{\widetilde{N}_1^2(u)\right\} = \mathcal{F}^{-1}\left\{u\widetilde{F}^2(u)\right\}, \tag{6.22}$$

and applying the Fourier transform identity

$$\frac{d^n}{dp^n}f(p) \Leftrightarrow (-ju)^n F(u), \tag{6.23}$$

enables Equation (6.21) to be written as

$$\widetilde{R}_{n1}(p) = j\frac{d}{dp}\widetilde{R}_f(p). \tag{6.24}$$

Additionally, taking the inverse Fourier transform of Equation (6.13),

$$\mathcal{F}^{-1}\left\{\widetilde{M_1^2}(u)\right\} = \mathcal{F}^{-1}\left\{\widetilde{N}_1^2(u)\Pi^2\left(\frac{u+\alpha}{2u_0}\right)\right\}. \tag{6.25}$$

The product in Equation (6.25) can be written as the convolution of two inverse Fourier transforms,

$$\widetilde{R}_{m1}(p) = \mathcal{F}^{-1}\left\{\widetilde{N}_1^2(u)\right\} * \mathcal{F}^{-1}\left\{\Pi^2\left(\frac{u+\alpha}{2u_0}\right)\right\}, \tag{6.26}$$

which then results in the convolution of two autocorrelation functions after performing the inverse Fourier transforms,

$$\widetilde{R}_{m1}(p) = \widetilde{R}_{n1}(p) * \left[R_s(p)e^{j\alpha p}\right]. \tag{6.27}$$

Applying the convolution definition to Equation (6.27),

$$\widetilde{R}_{m1}(p) = \int_{-\infty}^{\infty} \widetilde{R}_{n1}(\tau)R_s(p-\tau)\, e^{j\alpha(p-\tau)}\, d\tau. \tag{6.28}$$

Recalling the autocorrelation function $R_s(p)$ given in Equation (3.18) and inserting into Equation (6.28),

$$\widetilde{R}_{m1}(p) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \widetilde{R}_{n1}(\tau) \frac{\sin\left[u_0\left(p-\tau\right)\right]}{u_0\left(p-\tau\right)} e^{j\alpha(p-\tau)} \ d\tau. \tag{6.29}$$

Evaluating Equation (6.29) at $p = 0$ yields the stationary autocorrelation function $\widetilde{R}_{m1}(0)$,

$$\widetilde{R}_{m1}(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \widetilde{R}_{n1}(\tau) \frac{\sin\left(u_0\tau\right)}{u_0\tau} e^{-j\alpha\tau} \ d\tau. \tag{6.30}$$

Substituting the previously determined definition of $\widetilde{R}_f(p)$, given in Equation (5.29), into Equation (6.24),

$$\widetilde{R}_{n1}(p) = j\frac{d}{dp}\left[\left(u_0^2 + \frac{d^2}{dp^2} - 2j\alpha\frac{d}{dp} - \alpha^2\right) R_g(p)\right]. \tag{6.31}$$

Applying the definition of $R_f(p)$, given in Equation (5.31), to Equation (6.31),

$$\widetilde{R}_{n1}(p) = j\frac{d}{dp}\left[R_f(p) - \left(2j\alpha\frac{d}{dp} + \alpha^2\right) R_g(p)\right]. \tag{6.32}$$

Distributing terms in Equation (6.32),

$$\widetilde{R}_{n1}(p) = j\frac{d}{dp}R_f(p) - j\alpha^2\frac{d}{dp}R_g(p) + 2\alpha\frac{d^2}{dp^2}R_g(p). \tag{6.33}$$

Substituting Equation (6.33) into Equation (6.30),

$$\widetilde{R}_{m1}(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \left[j\frac{d}{d\tau}R_f(\tau) - j\alpha^2\frac{d}{d\tau}R_g(\tau) + 2\alpha\frac{d^2}{d\tau^2}R_g(\tau)\right] \frac{\sin\left(u_0\tau\right)}{u_0\tau} e^{-j\alpha\tau} \ d\tau. \tag{6.34}$$

Applying Euler's formula to Equation (6.34),

$$\begin{aligned} \widetilde{R}_{m1}(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} &\left[j\frac{d}{d\tau}R_f(\tau) - j\alpha^2\frac{d}{d\tau}R_g(\tau) + 2\alpha\frac{d^2}{d\tau^2}R_g(\tau)\right] \\ &\times \frac{\sin\left(u_0\tau\right)}{u_0\tau} \left[\cos\left(\alpha\tau\right) - j\sin\left(\alpha\tau\right)\right] \ d\tau. \end{aligned} \tag{6.35}$$

Distributing terms and separating the integrals in Equation (6.35),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{u_0}{\pi}\Bigg\{ & j \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_f(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \;\; d\tau \\ & + \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_f(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \;\; d\tau \\ & - j\alpha^2 \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \;\; d\tau \\ & - \alpha^2 \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \;\; d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2} R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \;\; d\tau \\ & - 2j\alpha \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2} R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \;\; d\tau \Bigg\}. \end{aligned} \tag{6.36}$$

Again, knowing that i) the autocorrelation functions are even; ii) the derivative of an even function produces an odd function, and vice versa; and iii) the integral of an odd function over symmetric limits of integration is zero, the odd integrands can be eliminated and Equation (6.36) can be simplified,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{u_0}{\pi}\Bigg\{ & \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_f(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \;\; d\tau \\ & - \alpha^2 \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \;\; d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2} R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \;\; d\tau \Bigg\}. \end{aligned} \tag{6.37}$$

Applying product-to-sum trigonometric identities to Equation (6.37),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{u_0}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_f(\tau)\right] \frac{\cos[(u_0-\alpha)\tau] - \cos[(u_0+\alpha)\tau]}{u_0\tau} \;\; d\tau \\ & - \alpha^2 \int_{-\infty}^{\infty} \left[\frac{d}{d\tau} R_g(\tau)\right] \frac{\cos[(u_0-\alpha)\tau] - \cos[(u_0+\alpha)\tau]}{u_0\tau} \;\; d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2} R_g(\tau)\right] \frac{\sin[(u_0+\alpha)\tau] + \sin[(u_0-\alpha)\tau]}{u_0\tau} \;\; d\tau \Bigg\}. \end{aligned} \tag{6.38}$$

Distributing terms and separating the integrals in Equation (6.38) yields the stationary autocorrelation function for the pattern mean,

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{u_0}{2\pi}\Bigg\{ &- \int_{-\infty}^{\infty} R_f'(\tau)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau \\
&+ \int_{-\infty}^{\infty} R_f'(\tau)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau \\
&+ \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau \\
&- \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau \\
&+ 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau \\
&+ 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau \Bigg\},
\end{aligned} \tag{6.39}$$

where $R_f'(\tau)$ is the first derivative of $R_f(\tau)$ and $R_g''(\tau)$ is the second derivative of $R_g(\tau)$. In summary, the mean for a scanning line source can be found by utilizing the results for $R_f(p)$ and $R_g'(p)$ determined previously from the broadside radiated power and beamwidth variance formulations, calculating the first derivative of $R_f(p)$, calculating the second derivative of $R_g(p)$, and substituting the required results into Equation (6.39). The integrations in Equation (6.39) can be performed to yield the result for $\widetilde{R}_{m1}(0)$, which can then be combined in Equation (6.20) with the result for $\widetilde{R}_h(0)$ from the scanning radiated power formulation to yield the mean for a scanning line source.

6.1.2 Second Moment Formulation

The substitutions $y = u - \alpha$ and $dy = du$ can be made in Equation (6.5),

$$\sigma_0^2 = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} (y+\alpha)^2\, G^2(y)\left[u_0^2 - (y+\alpha)^2\right]\; dy. \tag{6.40}$$

Expanding terms in Equation (6.40),

$$\sigma_0^2 = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} \left(y^2 + 2y\alpha + \alpha^2\right) G^2(y)\left(u_0^2 - y^2 - 2y\alpha - \alpha^2\right)\; dy. \tag{6.41}$$

Obviously, u can be directly substituted for y to obtain,

$$\sigma_0^2 = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} \left(u^2 + 2u\alpha + \alpha^2\right) G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) \, du. \tag{6.42}$$

Distributing terms and separating the integrals in Equation (6.42),

$$\begin{aligned} \sigma_0^2 = \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \Bigg\{ & \int_{-u_0-\alpha}^{u_0-\alpha} u^2 G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) \, du \\ & + 2\alpha \int_{-u_0-\alpha}^{u_0-\alpha} G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) \, du \\ & + \alpha^2 \int_{-u_0-\alpha}^{u_0-\alpha} G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) \, du \Bigg\}. \end{aligned} \tag{6.43}$$

The second term in Equation (6.43) was determined from Equation (6.10) and the third term is simply the radiated power P_{rad}. Therefore, Equation (6.43) can be written as

$$\sigma_0^2 = 2\alpha \frac{k^2\eta \widetilde{R}_{m1}(0)}{8u_0^3 P_{\text{rad}}} + \alpha^2 + \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-u_0-\alpha}^{u_0-\alpha} u^2 G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) \, du. \tag{6.44}$$

Incorporating the pulse function in Equation (6.44),

$$\begin{aligned} \sigma_0^2 = {} & 2\alpha \frac{k^2\eta \widetilde{R}_{m1}(0)}{8u_0^3 P_{\text{rad}}} + \alpha^2 \\ & + \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} \Pi^2 \left(\frac{u+\alpha}{2u_0}\right) u^2 G^2(u) \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) \, du. \end{aligned} \tag{6.45}$$

Equation (6.45) can be written as

$$\sigma_0^2 = 2\alpha \frac{k^2\eta \widetilde{R}_{m1}(0)}{8u_0^3 P_{\text{rad}}} + \alpha^2 + \frac{k^2\eta}{16\pi u_0^3 P_{\text{rad}}} \int_{-\infty}^{\infty} \widetilde{M}_2^2(u) \, du, \tag{6.46}$$

where the subscript "2" represents the second moment,

$$\widetilde{M}_2^2(u) = \widetilde{N}_2^2(u) \Pi^2 \left(\frac{u+\alpha}{2u_0}\right), \tag{6.47}$$

and

$$\widetilde{N}_2^2(u) = u^2 \left(u_0^2 - u^2 - 2u\alpha - \alpha^2\right) G^2(u). \tag{6.48}$$

Again, using Parseval's identity, $\widetilde{M}_2(u)$ can be related to its inverse Fourier transform $\widetilde{m}_2(p)$ using,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{M}_2(u)\right|^2\,du = \int_{-\infty}^{\infty}\left|\widetilde{m}_2(p)\right|^2\,dp. \tag{6.49}$$

Equation (6.49) can be applied to Equation (6.46),

$$\sigma_0^2 = 2\alpha\frac{k^2\eta\widetilde{R}_{m1}(0)}{8u_0^3P_{\text{rad}}} + \alpha^2 + \frac{k^2\eta}{8u_0^3P_{\text{rad}}}\int_{-\infty}^{\infty}\left|\widetilde{m}_2(p)\right|^2\,dp, \tag{6.50}$$

Again, invoking the definition of the stationary autocorrelation function,

$$\widetilde{R}_{m2}(p) = \int_{-\infty}^{\infty}\widetilde{m}_2(\tau)\widetilde{m}_2^*(\tau - p)\,d\tau, \tag{6.51}$$

then evaluating for $p = 0$,

$$\widetilde{R}_{m2}(0) = \int_{-\infty}^{\infty}\widetilde{m}_2(\tau)\widetilde{m}_2^*(\tau)\,d\tau = \int_{-\infty}^{\infty}\left|\widetilde{m}_2(\tau)\right|^2\,d\tau, \tag{6.52}$$

and substituting into Equation (6.50),

$$\sigma_0^2 = \frac{k^2\eta\widetilde{R}_{m2}(0)}{8u_0^3P_{\text{rad}}} + 2\alpha\frac{k^2\eta\widetilde{R}_{m1}(0)}{8u_0^3P_{\text{rad}}} + \alpha^2, \tag{6.53}$$

Again, recalling the definition for the radiated power P_{rad} given previously in Equation (5.18), substituting into Equation (6.53), and canceling terms yields the expression for the second raw moment of the power pattern function expressed solely in terms of autocorrelation functions,

$$\sigma_0^2 = \frac{\widetilde{R}_{m2}(0)}{\widetilde{R}_h(0)} + 2\alpha\frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)} + \alpha^2. \tag{6.54}$$

Recalling the original definition of the second central moment of the power pattern function given in Equation (6.4) and the previously derived expression for the mean given in Equation (6.20) and substituting into Equation (6.54),

$$\sigma_r^2 = \frac{\widetilde{R}_{m2}(0)}{\widetilde{R}_h(0)} + 2\alpha\frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)} + \alpha^2 - \left(\alpha + \frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)}\right)^2. \tag{6.55}$$

Expanding and subsequently canceling terms in Equation (6.55) yields the expression for the second central moment of the power pattern function expressed solely in terms of autocorrelation functions,

$$\sigma_r^2 = \frac{\widetilde{R}_{m2}(0)}{\widetilde{R}_h(0)} - \left(\frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)}\right)^2. \tag{6.56}$$

The autocorrelation function $\widetilde{R}_{m2}(0)$ can be determined by first recalling Equation (5.12)) and substituting into Equation (6.48),

$$\widetilde{N}_2^2(u) = u^2\widetilde{F}^2(u). \tag{6.57}$$

Taking the inverse Fourier transform of Equation (6.57),

$$\mathcal{F}^{-1}\left\{\widetilde{N}_2^2(u)\right\} = \mathcal{F}^{-1}\left\{u^2\widetilde{F}^2(u)\right\}, \tag{6.58}$$

and applying the Fourier transform identity

$$\frac{d^n}{dp^n}f(p) \Leftrightarrow (-ju)^n F(u), \tag{6.59}$$

enables Equation (6.57) to be written as

$$\widetilde{R}_{n2}(p) = -\frac{d^2}{dp^2}\widetilde{R}_f(p). \tag{6.60}$$

Additionally, taking the inverse Fourier transform of Equation (6.47),

$$\mathcal{F}^{-1}\left\{\widetilde{M}_2^2(u)\right\} = \mathcal{F}^{-1}\left\{\widetilde{N}_2^2(u)\Pi\left(\frac{u+\alpha}{2u_0}\right)\right\}. \tag{6.61}$$

The product in Equation (6.61) can be written as the convolution of two inverse Fourier transforms,

$$\widetilde{R}_{m2}(p) = \mathcal{F}^{-1}\left\{\widetilde{N}_2^2(u)\right\} * \mathcal{F}^{-1}\left\{\Pi^2\left(\frac{u+\alpha}{2u_0}\right)\right\}, \tag{6.62}$$

which then results in the convolution of two autocorrelation functions after performing the inverse Fourier transforms,

$$\widetilde{R}_{m2}(p) = \widetilde{R}_{n2}(p) * \left[R_s(p)e^{j\alpha p}\right]. \tag{6.63}$$

Applying the convolution definition to Equation (6.63),

$$\widetilde{R}_{m2}(p) = \int_{-\infty}^{\infty} \widetilde{R}_{n2}(\tau)R_s\left(p-\tau\right)e^{j\alpha(p-\tau)}\ d\tau. \tag{6.64}$$

Recalling the autocorrelation function $R_s(p)$ given in Equation (3.18) and inserting into Equation (6.64),

$$\widetilde{R}_{m2}(p) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \widetilde{R}_{n2}(\tau)\frac{\sin\left[u_0\left(p-\tau\right)\right]}{u_0\left(p-\tau\right)}e^{j\alpha(p-\tau)}\ d\tau. \tag{6.65}$$

Evaluating Equation (6.65) at $p = 0$ yields the stationary autocorrelation function $\widetilde{R}_{m2}(0)$,

$$\widetilde{R}_{m2}(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} \widetilde{R}_{n2}(\tau)\frac{\sin\left(u_0\tau\right)}{u_0\tau}e^{-j\alpha\tau}\ d\tau. \tag{6.66}$$

Substituting the previously determined definition of $\widetilde{R}_f(p)$, given in Equation (5.29), into Equation (6.60),

$$\widetilde{R}_{n2}(p) = -\frac{d^2}{dp^2}\left[\left(u_0^2 + \frac{d^2}{dp^2} - 2j\alpha\frac{d}{dp} - \alpha^2\right)R_g(p)\right]. \tag{6.67}$$

Applying the definition of $R_f(p)$, given in Equation (5.31), to Equation (6.67),

$$\widetilde{R}_{n2}(p) = -\frac{d^2}{dp^2}\left[R_f(p) - \left(2j\alpha\frac{d}{dp} + \alpha^2\right)R_g(p)\right]. \tag{6.68}$$

Distributing terms in Equation (6.68),

$$\widetilde{R}_{n2}(p) = -\frac{d^2}{dp^2}R_f(p) + \alpha^2\frac{d^2}{dp^2}R_g(p) + 2j\alpha\frac{d^3}{dp^3}R_g(p). \tag{6.69}$$

Recalling the definition for $R_n(p)$, previously given in Equation (4.16), and substituting into Equation (6.69),

$$\widetilde{R}_{n2}(p) = R_n(p) + \alpha^2 \frac{d^2}{dp^2} R_g(p) + 2j\alpha \frac{d^3}{dp^3} R_g(p). \tag{6.70}$$

Substituting Equation (6.70) into Equation (6.66),

$$\widetilde{R}_{m2}(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \left[R_n(\tau) + \alpha^2 \frac{d^2}{d\tau^2} R_g(\tau) + 2j\alpha \frac{d^3}{d\tau^3} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} e^{-j\alpha\tau} \; d\tau. \tag{6.71}$$

Applying Euler's formula to Equation (6.71),

$$\begin{aligned} \widetilde{R}_{m2}(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} & \left[R_n(\tau) + \alpha^2 \frac{d^2}{d\tau^2} R_g(\tau) + 2j\alpha \frac{d^3}{d\tau^3} R_g(\tau) \right] \\ & \times \frac{\sin(u_0\tau)}{u_0\tau} \left[\cos(\alpha\tau) - j \sin(\alpha\tau) \right] \; d\tau. \end{aligned} \tag{6.72}$$

Distributing terms and separating the integrals in Equation (6.72),

$$\begin{aligned} \widetilde{R}_{m2}(0) = \frac{u_0}{\pi} \Bigg\{ & \int_{-\infty}^{\infty} R_n(\tau) \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \; d\tau \\ & - j \int_{-\infty}^{\infty} R_n(\tau) \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \; d\tau \\ & + \alpha^2 \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \; d\tau \\ & - j\alpha^2 \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \; d\tau \\ & + 2j\alpha \int_{-\infty}^{\infty} \left[\frac{d^3}{d\tau^3} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} \cos(\alpha\tau) \; d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} \left[\frac{d^3}{d\tau^3} R_g(\tau) \right] \frac{\sin(u_0\tau)}{u_0\tau} \sin(\alpha\tau) \; d\tau \Bigg\}. \end{aligned} \tag{6.73}$$

As before, knowing that i) the autocorrelation functions are even; ii) the derivative of an even function produces an odd function, and vice versa; and iii) the integral of an odd function over symmetric limits of integration is zero, the odd integrands can be eliminated and Equation

(6.73) can be simplified,

$$\widetilde{R}_{m2}(0) = \frac{u_0}{\pi}\Bigg\{ \int_{-\infty}^{\infty} R_n(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\; d\tau + \alpha^2 \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2}R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau}\cos(\alpha\tau)\; d\tau + 2\alpha \int_{-\infty}^{\infty} \left[\frac{d^3}{d\tau^3}R_g(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau}\sin(\alpha\tau)\; d\tau \Bigg\}. \tag{6.74}$$

Applying product-to-sum trigonometric identities to Equation (6.74),

$$\widetilde{R}_{m2}(0) = \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right] + \sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau + \alpha^2 \int_{-\infty}^{\infty} \left[\frac{d^2}{d\tau^2}R_g(\tau)\right] \frac{\sin\left[(u_0+\alpha)\,\tau\right] + \sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau + 2\alpha \int_{-\infty}^{\infty} \left[\frac{d^3}{d\tau^3}R_g(\tau)\right] \frac{\cos\left[(u_0-\alpha)\,\tau\right] - \cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau \Bigg\}. \tag{6.75}$$

Distributing terms and separating the integrals in Equation (6.75) yields the stationary autocorrelation function for the beamwidth variance,

$$\widetilde{R}_{m2}(0) = \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau + \int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau + \alpha^2 \int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau + \alpha^2 \int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau - 2\alpha \int_{-\infty}^{\infty} R_g'''(\tau)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\; d\tau + 2\alpha \int_{-\infty}^{\infty} R_g'''(\tau)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\; d\tau \Bigg\}, \tag{6.76}$$

where $R_g'''(\tau)$ is the third derivative of $R_g(\tau)$. In summary, the variance for a scanning line source can be found by utilizing the results for $R_n(p)$ determined previously from the broadside beamwidth variance formulation, the results for $R_g''(p)$ from the first moment formulation, calculating the third derivative of $R_g(p)$, and substituting the results into Equation (6.76).

The integrations in Equation (6.76) can be performed to yield the result for $\widetilde{R}_{m2}(0)$, which can then be combined in Equation (6.56) with the results for $\widetilde{R}_h(0)$ from the scanning radiated power formulation and $\widetilde{R}_{m1}(0)$ from the first moment formulation to yield the variance for a scanning line source.

6.1.3 Formulation Summary

The mean is given by Equation (6.20),

$$\mu = \alpha + \frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)}, \tag{6.77}$$

where $\widetilde{R}_{m1}(0)/\widetilde{R}_h(0)$ represents the difference between the scanning direction and the direction of the pattern mean. It is important to note that the mean must be normalized by u_0 in a manner similar to the variance (i.e., μ/u_0) in order to make a consistent comparison between line sources of different electric lengths. However, since the mean is the first moment, u_0 to the first power is used instead of u_0^2, which was used to determine the beamwidth variance. The normalized mean can be dubbed the "pattern mean" and formally expressed as,

$$\mu_{\text{pat}} = \frac{\mu}{u_0}. \tag{6.78}$$

It should be noted that the exponent for u_0 corresponds to the order of the moment (i.e., u_0^n for the n^{th}-order moment). The variance is given by Equation (6.56),

$$\sigma_r^2 = \frac{\widetilde{R}_{m2}(0)}{\widetilde{R}_h(0)} - \left(\frac{\widetilde{R}_{m1}(0)}{\widetilde{R}_h(0)}\right)^2. \tag{6.79}$$

As for the constant phase case, the beamwidth variance is determined by normalizing the variance by u_0^2,

$$\sigma_{\text{BW}}^2 = \frac{\sigma_r^2}{u_0^2}. \tag{6.80}$$

The stationary autocorrelation functions $\widetilde{R}_{m1}(0)$ and $\widetilde{R}_{m2}(0)$ are determined using Equations (6.39) and (6.76), respectively. The autocorrelation function $\widetilde{R}_h(0)$ was determined from the radiated power formulation for a scanning line source, presented in the preceding chapter.

6.2 Examples

The theory developed in the preceding sections has been applied to several canonical current distributions to develop exact closed-form expressions for the stationary autocorrelation functions $\widetilde{R}_{m1}(0)$ and $\widetilde{R}_{m2}(0)$. These closed-form expressions can then be combined with the results for $\widetilde{R}_h(0)$, presented in the preceding chapter, using Equations (6.77) and (6.78) to yield exact closed-form expressions for the pattern mean and using Equations (6.79) and (6.80) to yield exact closed-form expressions for the beamwidth variance. These steps will not be included in the following examples due to the substantial complexity of the resulting equations. Considered in this chapter are the half-wave dipole, cosine, cosine-squared, triangular, and uniform current distributions. Only the final results are presented in this chapter, while the detailed derivations for each current distribution are presented in Appendix D.

6.2.1 Half-Wave Dipole Distribution

The autocorrelation function $R_g(p)$ for the half-wave dipole is,

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.81}$$

Recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{2} \begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.82}$$

Finding the second derivative of $R_g(p)$,

$$R_g''(p) = \frac{A_m^2}{4} \begin{cases} \left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.83}$$

And, finding the third derivative of $R_g(p)$,

$$R_g'''(p) = \frac{A_m^2}{4} \begin{cases} \frac{1}{2}\left(\frac{p}{2}+\pi\right)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ -\frac{1}{2}\left(\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.84}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{2} \begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.85}$$

Finding the first derivative of $R_f(p)$,

$$R_f'(p) = \frac{A_m^2}{4} \begin{cases} -\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{6.86}$$

Recalling the autocorrelation function $R_n(p)$,

$$R_n(p) = \frac{A_m^2}{4}\left[\frac{1}{2}\sin\left(\frac{|p|}{2}\right) - \delta(p+2\pi) - 2\delta(p) - \delta(p-2\pi)\right] \tag{6.87}$$

for $-2\pi \le p \le 2\pi$ and $R_n(p) = 0$ otherwise. Equations (6.82), (6.83), and (6.86) can be used in conjunction with Equation (6.39) to yield the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the half-wave dipole distribution,

$$\begin{aligned} \widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Big\{ & \mathrm{Cin}\left[2\pi(1+\alpha)\right] - \mathrm{Cin}\left[2\pi(1-\alpha)\right] \\ & + 2\alpha\left[\mathrm{Cin}\left[2\pi(1+\alpha)\right] - 2\mathrm{Cin}(2\pi\alpha) + \mathrm{Cin}\left[2\pi(1-\alpha)\right]\right] \\ & - 2\pi\alpha^2\left[\mathrm{Si}\left[2\pi(1+\alpha)\right] - 2\mathrm{Si}(2\pi\alpha) - \mathrm{Si}\left[2\pi(1-\alpha)\right]\right] \\ & - 2\pi\alpha\left[\mathrm{Si}\left[2\pi(1+\alpha)\right] + \mathrm{Si}\left[2\pi(1-\alpha)\right]\right]\Big\}. \end{aligned} \tag{6.88}$$

Likewise, Equations (6.83), (6.84), and (6.87) can be used in conjunction with Equation (6.76) to yield the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the half-wave dipole distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \frac{1}{2}\left(1+2\alpha^2\right)\left[\text{Cin}\left[2\pi(1+\alpha)\right] - 2\text{Cin}(2\pi\alpha) + \text{Cin}\left[2\pi(1-\alpha)\right]\right] \\ & + 2\alpha\left[\text{Cin}\left[2\pi(1+\alpha)\right] - \text{Cin}\left[2\pi(1-\alpha)\right]\right] \\ & - \pi\alpha\left[\text{Si}\left[2\pi(1+\alpha)\right] - 2\text{Si}(2\pi\alpha) - \text{Si}\left[2\pi(1-\alpha)\right]\right] \\ & - \pi\alpha^2\left[\text{Si}\left[2\pi(1+\alpha)\right] + \text{Si}\left[2\pi(1-\alpha)\right]\right] + \cos(2\pi\alpha) - 3\Bigg\}. \end{aligned} \tag{6.89}$$

6.2.2 Cosine Distribution

Recalling the autocorrelation function $R_g(p)$ for the cosine distribution,

$$R_g(p) = \frac{A_m^2}{2}\begin{cases}(2\pi+p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ (2\pi-p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{6.90}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{2}\begin{cases}\left(-\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{6.91}$$

Finding the second derivative of $R_g(p)$,

$$R_g''(p) = \frac{A_m^2}{4}\begin{cases}\left(-\frac{p}{2}-\pi\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2}-\pi\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{6.92}$$

And, finding the third derivative of $R_g(p)$,

$$R_g'''(p) = \frac{A_m^2}{4} \begin{cases} \frac{1}{2}\left(\frac{p}{2}+\pi\right)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ -\frac{1}{2}\left(\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.93}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2 \begin{cases} -\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.94}$$

Finding the first derivative of $R_f(p)$,

$$R_f'(p) = \frac{A_m^2}{4} \begin{cases} \left(\frac{1}{4}-u_0^2\right)(2\pi+p)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{1}{4}-u_0^2\right)(2\pi-p)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.95}$$

Recalling the autocorrelation function $R_n(p)$,

$$\begin{aligned} R_n(p) = A_m^2 \Bigg\{ & \frac{1}{8}\sin\left(\frac{|p|}{2}\right) + \frac{1}{2}\left(\frac{1}{4}-u_0^2\right)\left[\frac{1}{2}\sin\left(\frac{|p|}{2}\right) - \left(\frac{2\pi-|p|}{4}\right)\cos\left(\frac{p}{2}\right)\right] \\ & - \frac{\delta\left(p+2\pi\right)}{4} - \frac{\delta\left(p\right)}{2} - \frac{\delta\left(p-2\pi\right)}{4} \Bigg\} \end{aligned} \tag{6.96}$$

for $-2\pi \leq p \leq 2\pi$ and $R_n\left(p\right) = 0$ otherwise. Equations (6.91), (6.92), and (6.95) can be used in conjunction with Equation (6.39) to yield the stationary autocorrelation function $\widetilde{R}_{m1}(0)$

for the cosine distribution,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & -2\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\ & +(2\alpha+1)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & +(2\alpha-1)\left[\mathrm{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\ & +\cos\left[2\pi\left(u_0+\alpha\right)\right]-\cos\left[2\pi\left(u_0-\alpha\right)\right]\Bigg\}. \end{aligned} \tag{6.97}$$

Likewise, Equations (6.92), (6.93), and (6.96) can be used in conjunction with Equation (6.76) to yield the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the cosine distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) = & \frac{A_m^2}{8\pi} \\ & \times\Bigg\{-\left(u_0^2-2\alpha-\alpha^2-\frac{3}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & -\left(u_0^2+2\alpha-\alpha^2-\frac{3}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\ & +\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & +\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\ & -\left(u_0-\alpha\right)\cos\left[2\pi\left(u_0+\alpha\right)\right]-\left(u_0+\alpha\right)\cos\left[2\pi\left(u_0-\alpha\right)\right] \\ & -\frac{1}{\pi}\left[\left[2\pi\left(u_0+\alpha\right)\right]+\sin\left[2\pi\left(u_0-\alpha\right)\right]\right]-6u_0\Bigg\}. \end{aligned} \tag{6.98}$$

6.2.3 Cosine-Squared Distribution

Recalling the autocorrelation function $R_g(p)$ for the cosine-squared distribution,

$$R_g(p) = \frac{A_m^2}{8}\begin{cases}(2\pi+p)\left[2+\cos(p)\right]-3\sin(p), & -2\pi \le p \le 0 \\ (2\pi-p)\left[2+\cos(p)\right]+3\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{6.99}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{8} \begin{cases} -(p+2\pi)\sin(p) - 2\cos(p) + 2, & -2\pi \le p \le 0 \\ (p-2\pi)\sin(p) + 2\cos(p) - 2, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.100}$$

Finding the second derivative of $R_g(p)$,

$$R_g''(p) = \frac{A_m^2}{8} \begin{cases} -(p+2\pi)\cos(p) + \sin(p), & -2\pi \le p \le 0 \\ (p-2\pi)\cos(p) - \sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.101}$$

And, finding the third derivative of $R_g(p)$,

$$R_g'''(p) = \frac{A_m^2}{8} \begin{cases} (p+2\pi)\sin(p), & -2\pi \le p \le 0 \\ -(p-2\pi)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.102}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{8} \begin{cases} (2\pi+p)\left[2u_0^2 + \left(u_0^2-1\right)\cos(p)\right] + \left(1-3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ (2\pi-p)\left[2u_0^2 + \left(u_0^2-1\right)\cos(p)\right] - \left(1-3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.103}$$

Finding the first derivative of $R_f(p)$,

$$R_f'(p) = \frac{A_m^2}{8} \begin{cases} -\left(u_0^2-1\right)(2\pi+p)\sin(p) + 2u_0^2\left[1-\cos(p)\right], & -2\pi \le p \le 0 \\ -\left(u_0^2-1\right)(2\pi-p)\sin(p) - 2u_0^2\left[1-\cos(p)\right], & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.104}$$

Recalling the autocorrelation function $R_n(p)$,

$$R_n(p) = \frac{A_m^2}{8} \begin{cases} \left(1-u_0^2\right)\left[-2\sin(p)-(2\pi+p)\cos(p)\right]+\left(1-3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ \left(1-u_0^2\right)\left[2\sin(p)-(2\pi-p)\cos(p)\right]-\left(1-3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \quad (6.105)$$

Equations (6.100), (6.101), and (6.104) can be used in conjunction with Equation (6.39) to yield the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the cosine-squared distribution,

$$\begin{aligned} \widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi}\Big\{ & 2\pi\left(u_0^2-2\alpha-\alpha^2-1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+1\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\ & -2\pi\left(u_0^2+2\alpha-\alpha^2-1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-1\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & +2\left(u_0^2-\alpha-\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\ & +2\left(u_0^2+\alpha-\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & -4\left(u_0^2-\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & -2\cos\left[2\pi\left(u_0+\alpha\right)\right]+2\cos\left[2\pi\left(u_0-\alpha\right)\right]\Big\}. \end{aligned} \quad (6.106)$$

Likewise, Equations (6.101), (6.102), and (6.105) can be used in conjunction with Equation (6.76) to yield the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the cosine-squared distribution,

$$\begin{aligned} \widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi}\Big\{ & \left(u_0^2-\alpha^2+1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right]\right] \\ & +\left(u_0^2-\alpha^2+1\right)\left[\mathrm{Cin}\left[2\pi\left(\mathrm{u}_0-\alpha+1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\ & +2\pi\left(u_0^2-2\alpha-\alpha^2-1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+1\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\ & +2\pi\left(u_0^2+2\alpha-\alpha^2-1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-1\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & +2\left(u_0-\alpha\right)\cos\left(u_0+\alpha\right)+2\left(u_0+\alpha\right)\cos\left(u_0-\alpha\right)-4u_0\Big\}. \end{aligned} \quad (6.107)$$

6.2.4 Triangular Distribution

Recalling the autocorrelation function $R_g(p)$ for the triangular distribution,

$$R_g(p) = \frac{A_m^2}{6\pi^2} \begin{cases} (2\pi + p)^3, & -2\pi \leq p \leq -\pi \\ 4\pi^3 - 6\pi p^2 - 3p^3, & -\pi \leq p \leq 0 \\ 4\pi^3 - 6\pi p^2 + 3p^3, & 0 \leq p \leq \pi \\ (2\pi - p)^3, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.108}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{6\pi^2} \begin{cases} 3(2\pi + p)^2, & -2\pi \leq p \leq -\pi \\ -\left(12\pi p + 9p^2\right), & -\pi \leq p \leq 0 \\ -\left(12\pi p - 9p^2\right), & 0 \leq p \leq \pi \\ -3(2\pi - p)^2, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.109}$$

Finding the second derivative of $R_g(p)$,

$$R_g''(p) = \frac{A_m^2}{6\pi^2} \begin{cases} 12\pi + 6p, & -2\pi \leq p \leq -\pi \\ -(12\pi + 18p), & -\pi \leq p \leq 0 \\ -(12\pi - 18p), & 0 \leq p \leq \pi \\ 12\pi - 6p, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{6.110}$$

And, finding the third derivative of $R_g(p)$,

$$R_g'''(p) = \frac{A_m^2}{6\pi^2}\begin{cases} 6, & -2\pi \le p \le -\pi \\ -18, & -\pi \le p \le 0 \\ 18, & 0 \le p \le \pi \\ -6, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.111}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{6\pi^2}\begin{cases} u_0^2\left(2\pi+p\right)^3 + 6\left(2\pi+p\right), & -2\pi \le p \le -\pi \\ u_0^2\left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p, & -\pi \le p \le 0 \\ u_0^2\left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p, & 0 \le p \le \pi \\ u_0^2\left(2\pi-p\right)^3 + 6\left(2\pi-p\right), & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.112}$$

Finding the first derivative of $R_f(p)$,

$$R_f'(p) = \frac{A_m^2}{6\pi^2}\begin{cases} 3u_0^2\left(2\pi+p\right)^2 + 6, & -2\pi \le p \le -\pi \\ -u_0^2\left(12\pi p + 9p^2\right) - 18, & -\pi \le p \le 0 \\ -u_0^2\left(12\pi p - 9p^2\right) + 18, & 0 \le p \le \pi \\ -3u_0^2\left(2\pi-p\right)^2 - 6, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.113}$$

Recalling the autocorrelation function $R_n(p)$,

$$R_n(p) = R_{n1}(p) + R_{n2}(p) - \frac{6A_m^2}{\pi^2}\delta(p) + \frac{4A_m^2}{\pi^2}\delta\left(|p| - \pi\right) - \frac{A_m^2}{\pi^2}\delta\left(|p| - 2\pi\right), \tag{6.114}$$

where

$$R_{n1}(p) = \frac{A_m^2 u_0^2}{\pi^2}\left(2\pi - 3|p|\right) \tag{6.115}$$

for $0 \le |p| \le \pi$ and 0 otherwise, and

$$R_{n2}(p) = \frac{A_m^2 u_0^2}{\pi^2} \left(|p| - 2\pi\right) \tag{6.116}$$

for $\pi \le |p| \le 2\pi$ and 0 otherwise. Equations (6.109), (6.110), and (6.113) can be used in conjunction with Equation (6.39) to yield the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the triangular distribution,

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3} \Bigg\{ & -12\pi \left(u_0 - \alpha\right) \sin\left[2\pi \left(u_0 + \alpha\right)\right] + 12\pi \left(u_0 + \alpha\right) \sin\left[2\pi \left(u_0 - \alpha\right)\right] \\
& + 24\pi \left(u_0 - \alpha\right) \sin\left[\pi \left(u_0 + \alpha\right)\right] - 24\pi \left(u_0 + \alpha\right) \sin\left[\pi \left(u_0 - \alpha\right)\right] \\
& + 6 \left(\frac{u_0 + 3\alpha}{u_0 + \alpha}\right) \cos\left[2\pi \left(u_0 + \alpha\right)\right] - 6 \left(\frac{u_0 - 3\alpha}{u_0 - \alpha}\right) \cos\left[2\pi \left(u_0 - \alpha\right)\right] \\
& - 24 \left(\frac{u_0 + 3\alpha}{u_0 + \alpha}\right) \cos\left[\pi \left(u_0 + \alpha\right)\right] + 24 \left(\frac{u_0 - 3\alpha}{u_0 - \alpha}\right) \cos\left[\pi \left(u_0 - \alpha\right)\right] \\
& - \left[24\pi^2 \left(u_0^2 - \alpha^2\right) + 12\right] \left[\mathrm{Cin}\left[2\pi \left(u_0 + \alpha\right)\right] - \mathrm{Cin}\left[2\pi \left(u_0 - \alpha\right)\right]\right] \\
& + \left[24\pi^2 \left(u_0^2 - \alpha^2\right) + 48\right] \left[\mathrm{Cin}\left[\pi \left(u_0 + \alpha\right)\right] - \mathrm{Cin}\left[\pi \left(u_0 - \alpha\right)\right]\right] \\
& + 48\pi\alpha \left[\mathrm{Si}\left[2\pi \left(u_0 + \alpha\right)\right] + \mathrm{Si}\left[2\pi \left(u_0 - \alpha\right)\right]\right] \\
& - 96\pi\alpha \left[\mathrm{Si}\left[\pi \left(u_0 + \alpha\right)\right] + \mathrm{Si}\left[\pi \left(u_0 - \alpha\right)\right]\right] + \frac{72 u_0 \alpha}{u_0^2 - \alpha^2} \Bigg\}.
\end{aligned} \tag{6.117}$$

Likewise, Equations (6.110), (6.111), and (6.114) can be used in conjunction with Equation (6.76) to yield the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the triangular distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{12\pi^3}\Big\{ & -24\alpha \left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \operatorname{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 96\alpha \left[\operatorname{Cin}\left[\pi\left(u_0+\alpha\right)\right] - \operatorname{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & - 24\pi\left(u_0^2-\alpha^2\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \operatorname{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 48\pi\left(u_0^2-\alpha^2\right)\left[\operatorname{Si}\left[\pi\left(u_0+\alpha\right)\right] + \operatorname{Si}\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & - \frac{6}{\pi}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right] + \sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + \frac{48}{\pi}\left[\sin\left[\pi\left(u_0+\alpha\right)\right] + \sin\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & - 12\left(u_0-\alpha\right)\cos\left[2\pi\left(u_0+\alpha\right)\right] - 12\left(u_0+\alpha\right)\cos\left[2\pi\left(u_0-\alpha\right)\right] \\ & + 48\left(u_0-\alpha\right)\cos\left[\pi\left(u_0+\alpha\right)\right] + 48\left(u_0+\alpha\right)\cos\left[\pi\left(u_0-\alpha\right)\right] - 144u_0\Big\}.\end{aligned} \tag{6.118}$$

6.2.5 Uniform Distribution

Recalling the autocorrelation function $R_g(p)$ for the uniform distribution,

$$R_g(p) = A_m^2 \begin{cases} 2\pi + p, & -2\pi \le p \le 0 \\ 2\pi - p, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{6.119}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = A_m^2 \left[H(p+2\pi) - 2H(p) + H(p-2\pi)\right]. \tag{6.120}$$

Finding the second derivative of $R_g(p)$,

$$R_g''(p) = A_m^2 \left[\delta(p+2\pi) - 2\delta(p) + \delta(p-2\pi)\right]. \tag{6.121}$$

And, finding the third derivative of $R_g(p)$,

$$R_g'''(p) = A_m^2 \frac{d}{dp} \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right]. \tag{6.122}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2 \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right] + u_0^2 R_g(p). \tag{6.123}$$

Finding the first derivative of $R_f(p)$,

$$R_f'(p) = A_m^2 \frac{d}{dp} \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right] + u_0^2 R_g'(p), \tag{6.124}$$

which can also be written as

$$R_f'(p) = R_g'''(p) + u_0^2 R_g'(p). \tag{6.125}$$

Recalling the autocorrelation function $R_n(p)$,

$$\begin{aligned} R_n(p) = &- A_m^2 \frac{d^2}{dp^2} \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right] \\ &- A_m^2 u_0^2 \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right], \end{aligned} \tag{6.126}$$

which can also be written as

$$R_n(p) = -\frac{d}{dp} R_g'''(p) - u_0^2 R_g''(p) = -R_g^{(4)}(p) - u_0^2 R_g''(p). \tag{6.127}$$

Equations (6.120), (6.121), and (6.124) can be used in conjunction with Equation (6.39) to yield the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the uniform distribution,

$$\begin{aligned} \widetilde{R}_{m1}(0) = \frac{A_m^2}{2\pi} \Bigg\{ &- 2\left(u_0^2 - \alpha^2\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha\right)\right] \right] \\ &- \frac{1}{\pi} \left[\left(u_0 - \alpha\right) \sin\left[2\pi\left(u_0 + \alpha\right)\right] - \left(u_0 + \alpha\right) \sin\left[2\pi\left(u_0 - \alpha\right)\right] \right] \\ &- \frac{1}{2\pi^2} \left[\cos\left[2\pi\left(u_0 + \alpha\right)\right] - \cos\left[2\pi\left(u_0 - \alpha\right)\right] \right] - 4u_0\alpha \Bigg\}. \end{aligned} \tag{6.128}$$

Likewise, Equations (6.121), (6.122), and (6.126) can be used in conjunction with Equation (6.76) to yield the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the uniform distribution,

$$\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\left\{ -\frac{1}{2\pi^3}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right]+\sin\left[2\pi\left(u_0-\alpha\right)\right]\right] + \frac{u_0}{\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right]+\cos\left[2\pi\left(u_0-\alpha\right)\right]\right]+\frac{8}{3}u_0^3\right\}. \tag{6.129}$$

6.3 Validation

The results of the preceding section can be validated by inserting each result for the stationary autocorrelation functions $\widetilde{R}_h(0)$ and $\widetilde{R}_{m1}(0)$ into Equation (6.77) and plotting the normalized pattern mean μ/u_0 as a function of i) electric length u_0 for various phase coefficients α, and 2) phase coefficient for various electric lengths. Obviously, the half-wave dipole distribution results are only presented as a function of phase coefficient. For purposes of comparison, the amplitude of each current distribution is set to unity (i.e., $A_m = 1$). In addition, the normalized pattern mean can be calculated by numerically integrating Equation (6.2) with the power-pattern function $G(u)$ for each current distribution. The power-pattern functions can be found by taking the Fourier transform of each current distribution, which produces the following results:

$$\begin{aligned} G(u) &= A_m\cos(\pi u)\left[\frac{4}{1-4u^2}\right] && \text{Cosine} \\ &= A_m\sin(\pi u)\left[\frac{1}{u(1-u^2)}\right] && \text{Cosine}-\text{Squared} \\ &= \pi A_m\left[\frac{\sin(\pi u/2)}{\pi u/2}\right]^2 && \text{Triangular} \\ &= 2\pi A_m\left[\frac{\sin(\pi u)}{\pi u}\right]. && \text{Uniform} \end{aligned} \tag{6.130}$$

Similarly, each result for the stationary autocorrelation functions $\widetilde{R}_h(0)$, $\widetilde{R}_{m1}(0)$, and $\widetilde{R}_{m2}(0)$ can be inserted into Equation (6.79) to determine the variance. The beamwidth variance can then be calculated by normalizing the result, in accordance with Equation (6.80), and plotting the beamwidth variance σ_{BW}^2 as a function of i) electric length u_0 for various phase coefficients α, and 2) phase coefficient for various electric lengths. Again, the half-wave dipole distribution results are only presented as a function of phase coefficient. The amplitude of each current distribution is set to unity (i.e., $A_m = 1$). In addition, the beamwidth variance can be calculated by numerically integrating Equation (6.1) with the power-pattern function $G(u)$ for each current distribution given above in Equation (6.130). All of the results are presented in Figures (6.1) through (6.26). Included are the comparisons between the analytical and numerical results. In addition, contour plots are included to illustrate the overall behavior of the pattern mean and beamwidth variance and to emphasize the significant complexity of the exact closed-form analytical equations obtained in this chapter. The comparisons demonstrate effectively perfect agreement between the exact closed-form equations and the numerically integrated results. The results presented in this chapter are entirely new and the equations are exact.

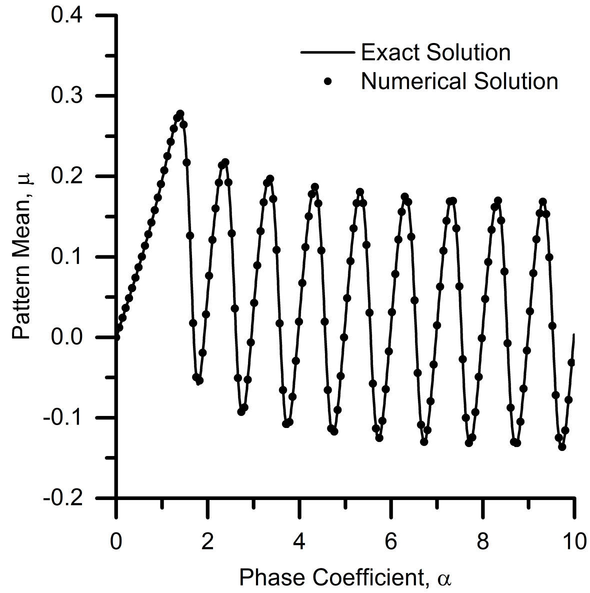

Figure 6.1: Half-Wave Dipole Distribution - Pattern mean as a function of phase coefficient.

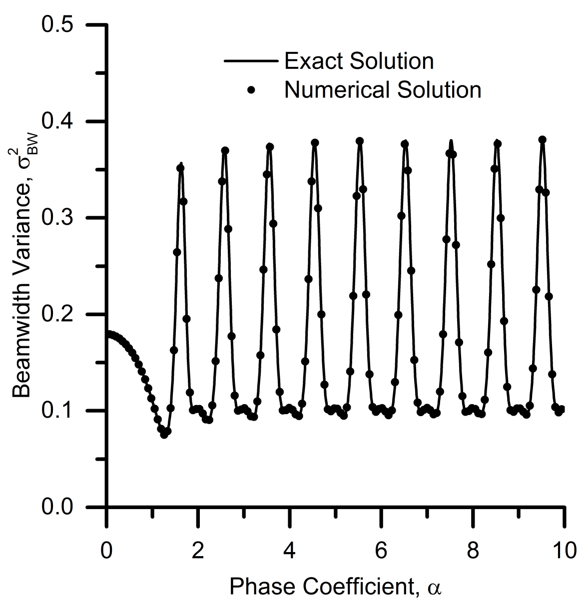

Figure 6.2: Half-Wave Dipole Distribution - Beamwidth variance as a function of phase coefficient.

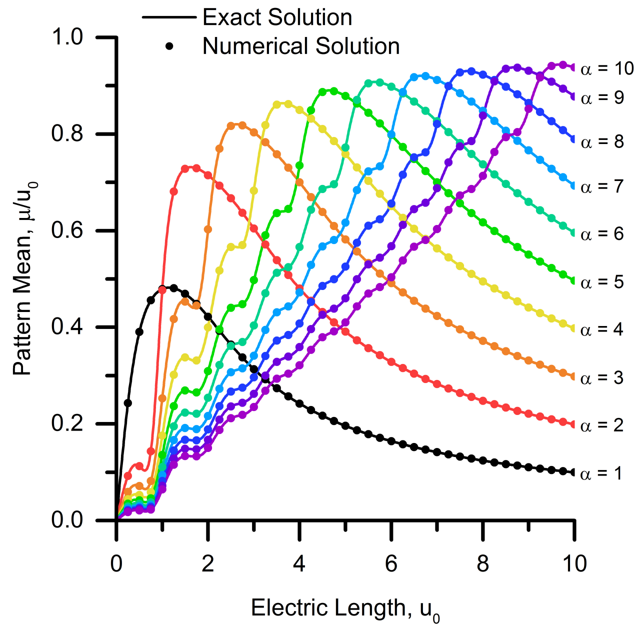

Figure 6.3: Cosine Distribution - Pattern mean as a function of electrical length for various phase coefficients.

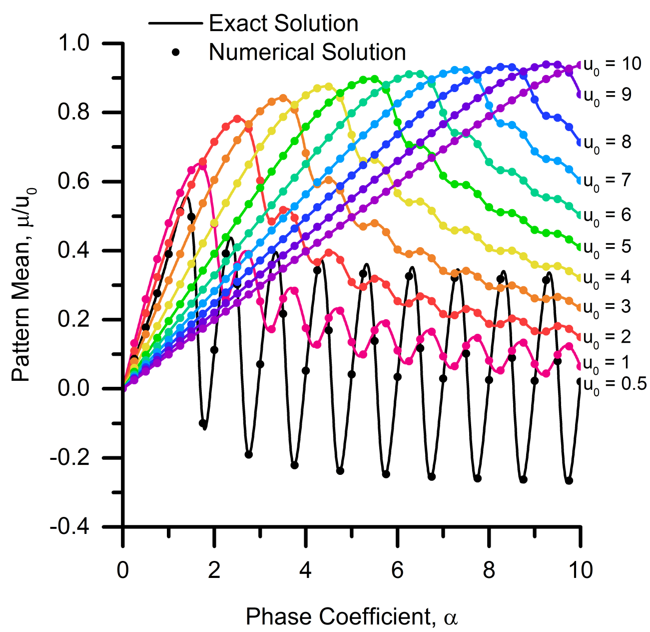

Figure 6.4: Cosine Distribution - Pattern mean as a function of phase coefficient for various electrical lengths.

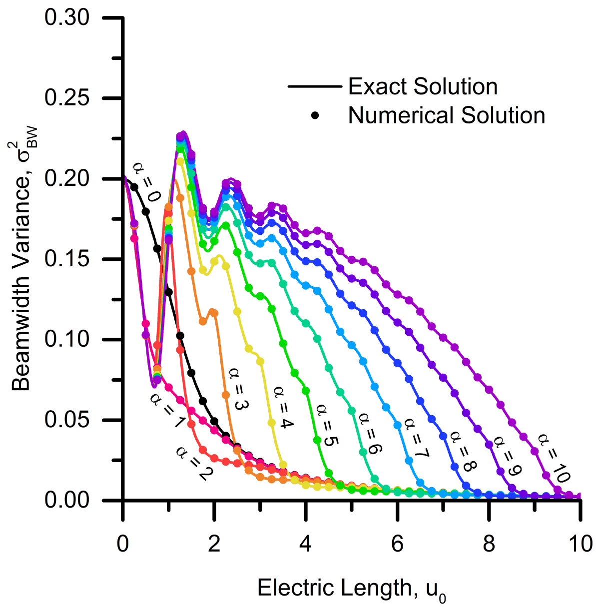

Figure 6.5: Cosine Distribution - Beamwidth variance as a function of electrical length for various phase coefficients.

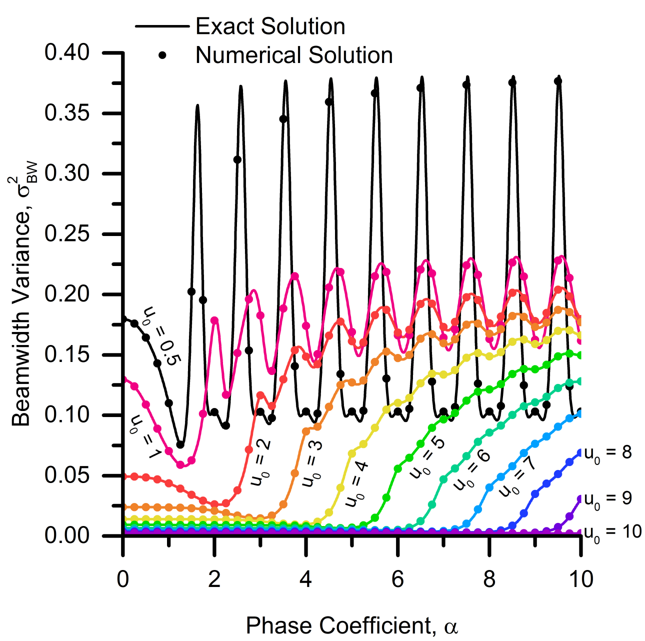

Figure 6.6: Cosine Distribution - Beamwidth variance as a function of phase coefficient for various electrical lengths.

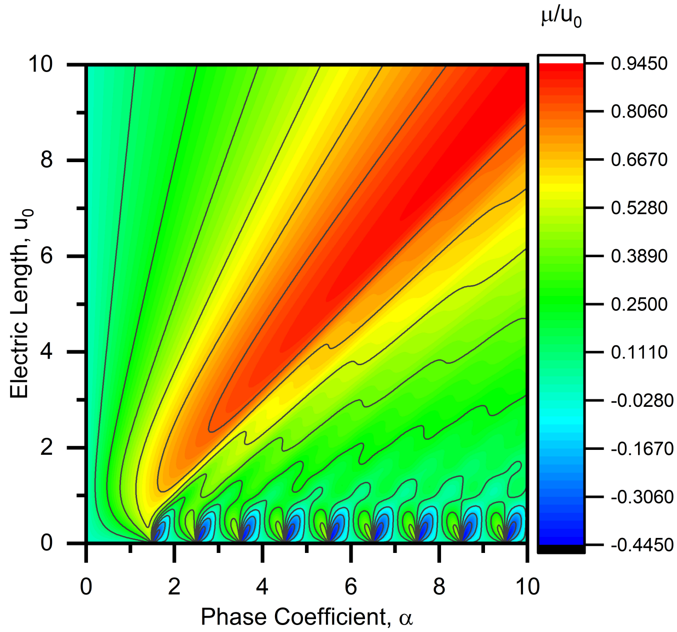

Figure 6.7: Cosine Distribution – Pattern mean contours as a function of electrical length and phase coefficient.

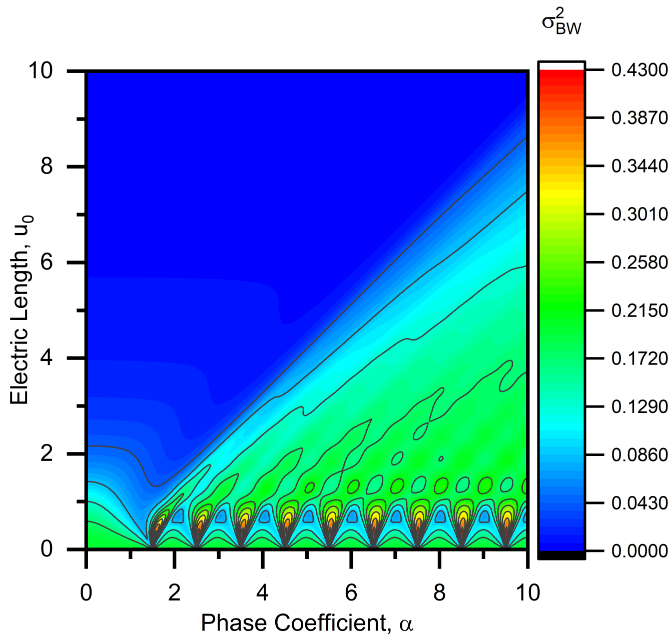

Figure 6.8: Cosine Distribution – Beamwidth variance contours as a function of electrical length and phase coefficient.

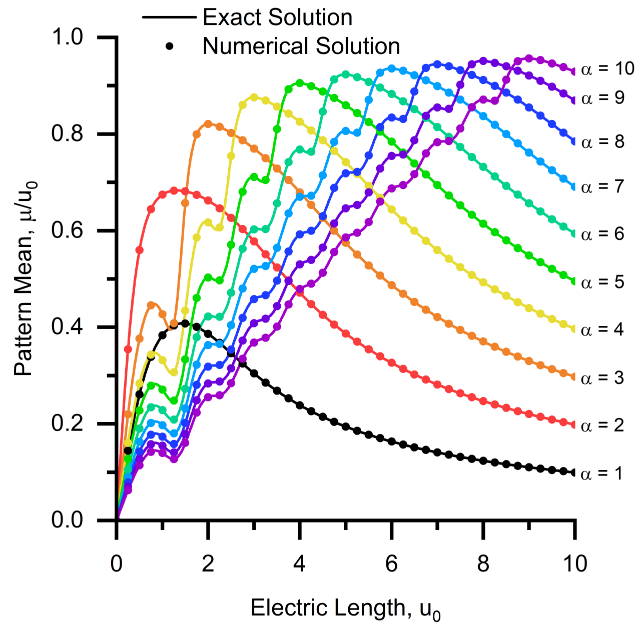

Figure 6.9: Cosine-squared Distribution - Pattern mean as a function of electrical length for various phase coefficients.

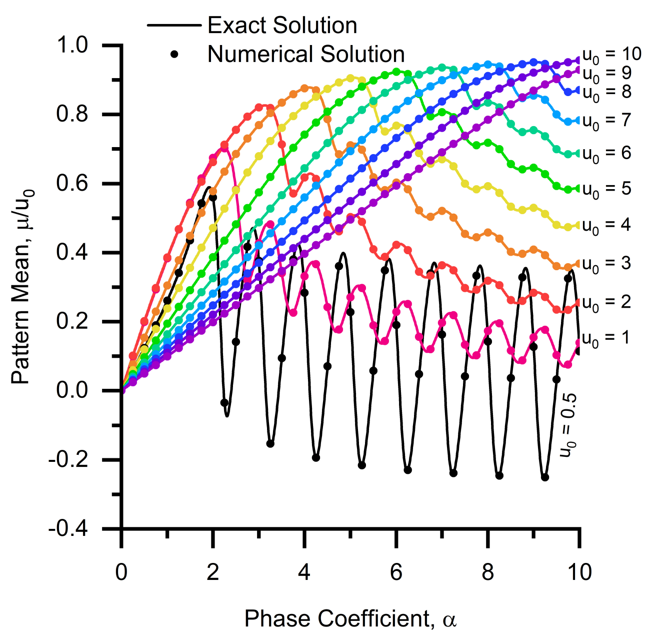

Figure 6.10: Cosine-Squared Distribution - Pattern mean as a function of phase coefficient for various electrical lengths.

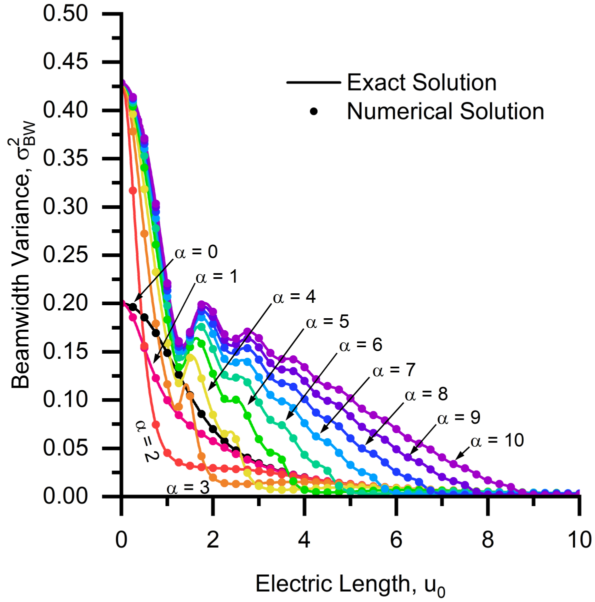

Figure 6.11: Cosine-Squared Distribution - Beamwidth variance as a function of electrical length for various phase coefficients.

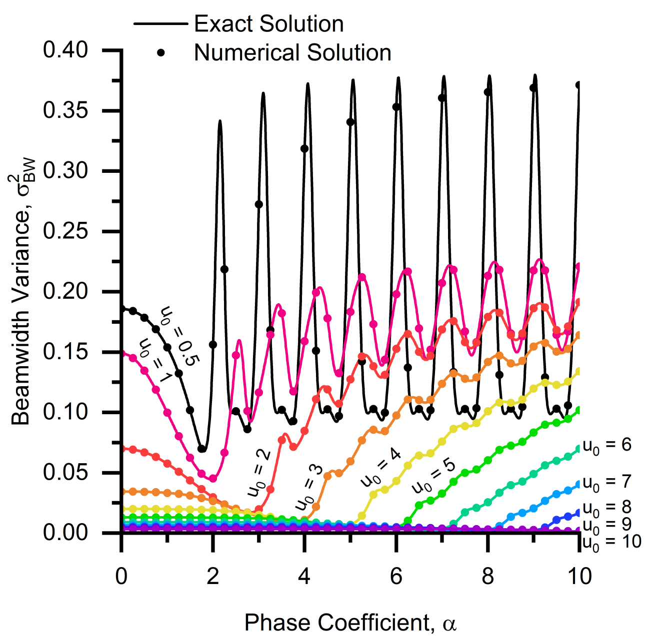

Figure 6.12: Cosine-Squared Distribution - Beamwidth variance as a function of phase coefficient for various electrical lengths.

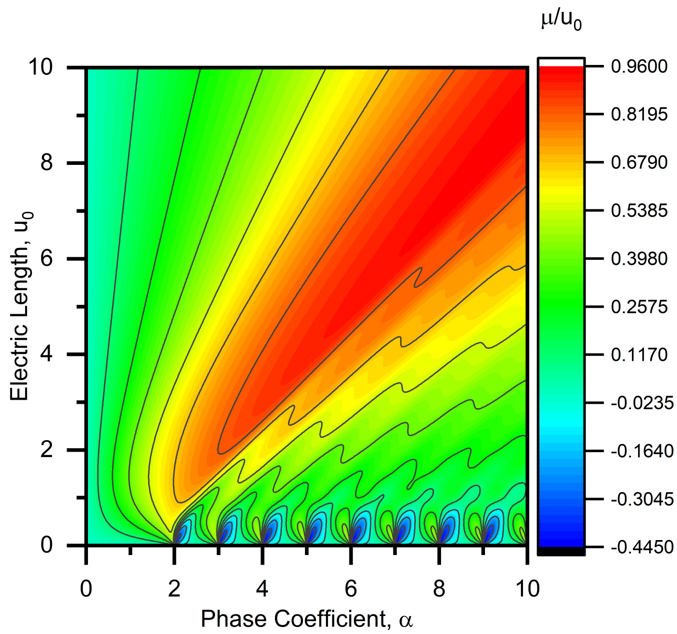

Figure 6.13: Cosine-Squared Distribution – Pattern mean contours as a function of electrical length and phase coefficient.

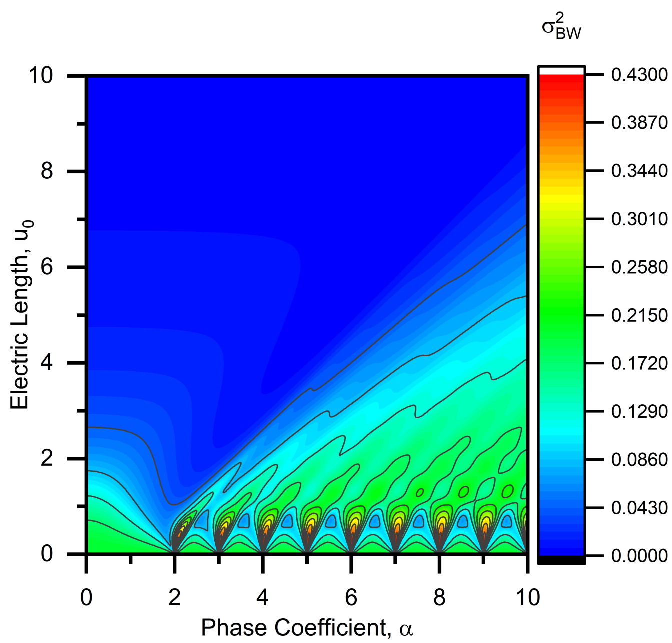

Figure 6.14: Cosine-Squared Distribution – Beamwidth variance contours as a function of electrical length and phase coefficient.

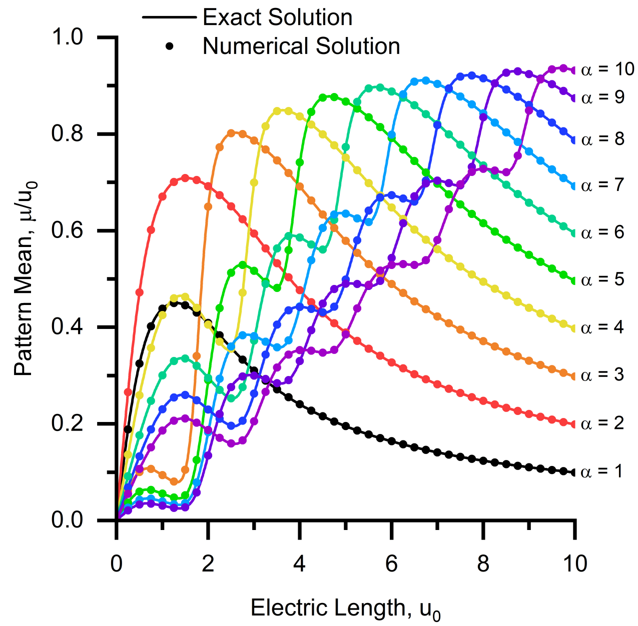

Figure 6.15: Triangular Distribution - Pattern mean as a function of electrical length for various phase coefficients.

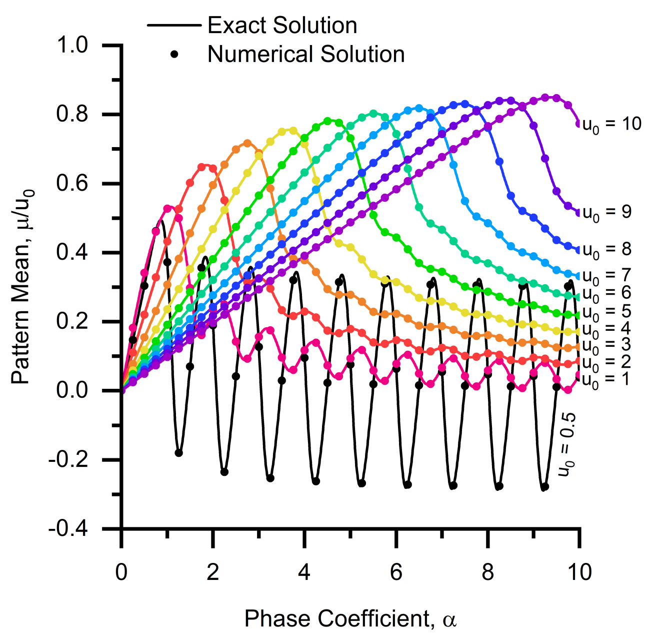

Figure 6.16: Triangular Distribution - Pattern mean as a function of phase coefficient for various electrical lengths.

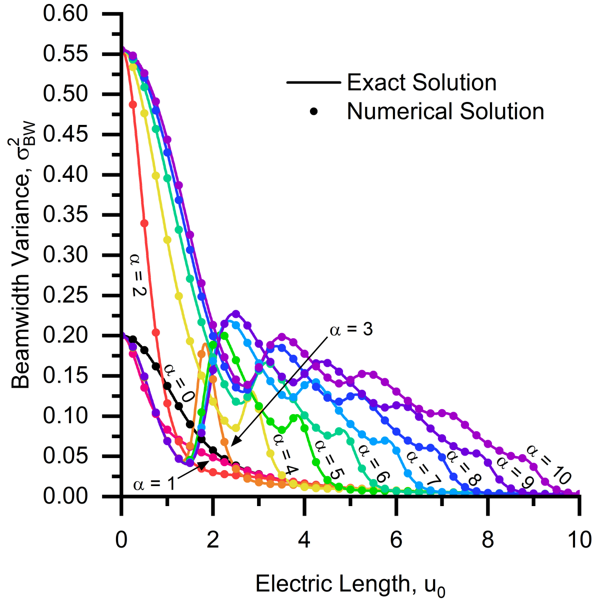

Figure 6.17: Triangular Distribution - Beamwidth variance as a function of electrical length for various phase coefficients.

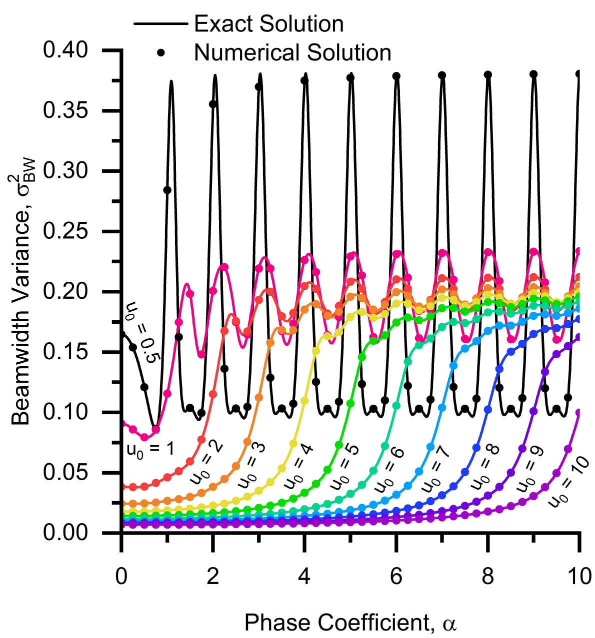

Figure 6.18: Triangular Distribution - Beamwidth variance as a function of phase coefficient for various electrical lengths.

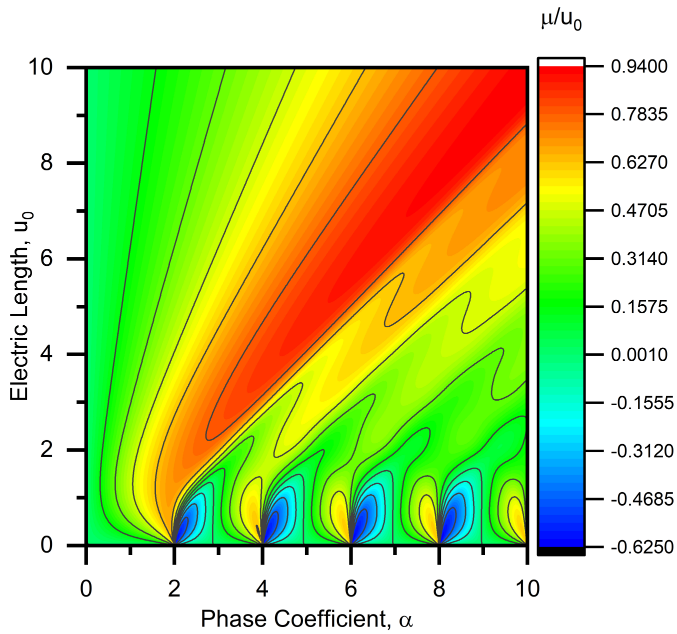

Figure 6.19: Triangular Distribution – Pattern mean contours as a function of electrical length and phase coefficient.

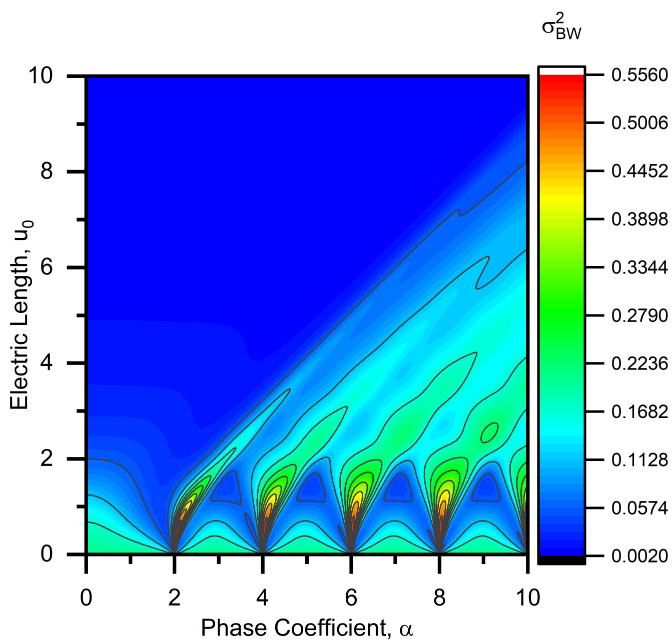

Figure 6.20: Triangular Distribution – Beamwidth variance contours as a function of electrical length and phase coefficient.

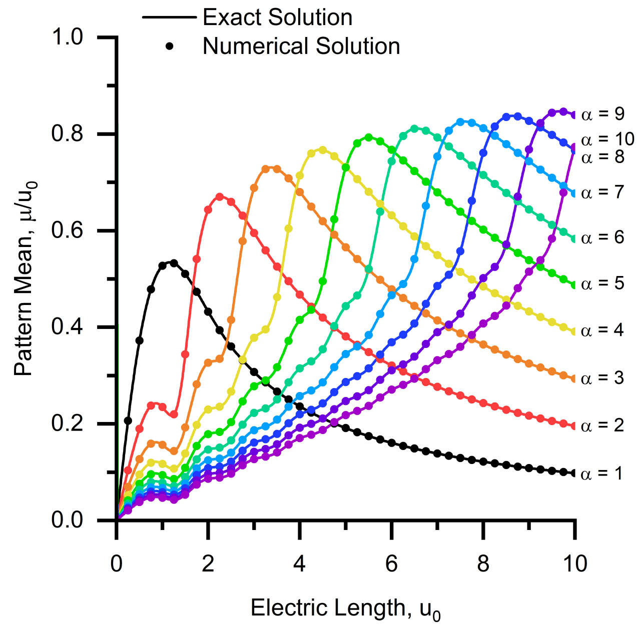

Figure 6.21: Uniform Distribution - Pattern mean as a function of electrical length for various phase coefficients.

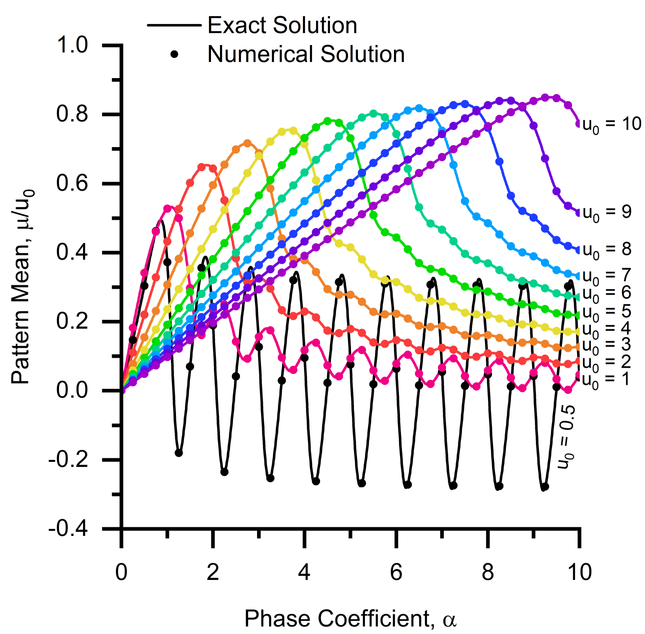

Figure 6.22: Uniform Distribution - Pattern mean as a function of phase coefficient for various electrical lengths.

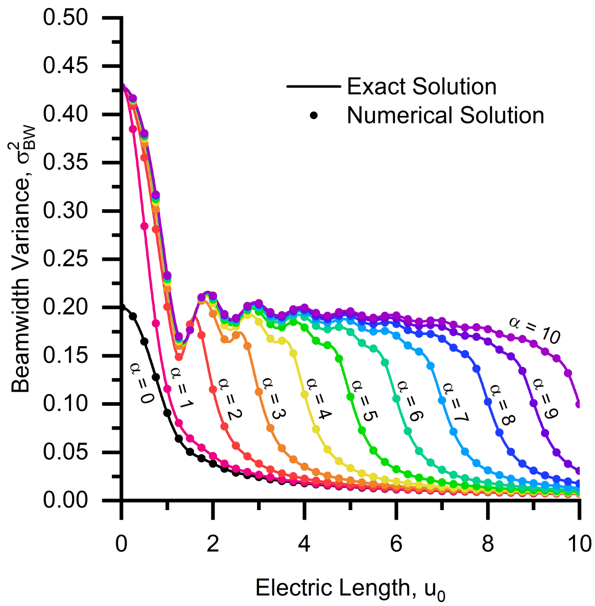

Figure 6.23: Uniform Distribution - Beamwidth variance as a function of electrical length for various phase coefficients.

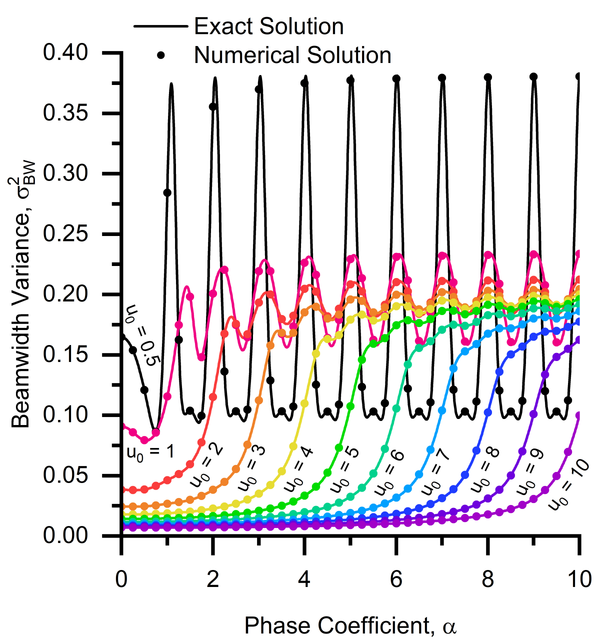

Figure 6.24: Uniform Distribution - Beamwidth variance as a function of phase coefficient for various electrical lengths.

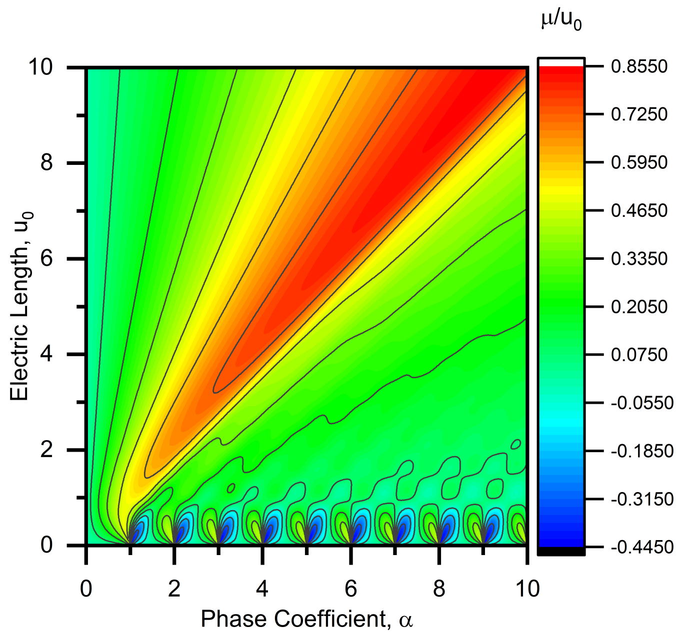

Figure 6.25: Uniform Distribution – Pattern mean contours as a function of electrical length and phase coefficient.

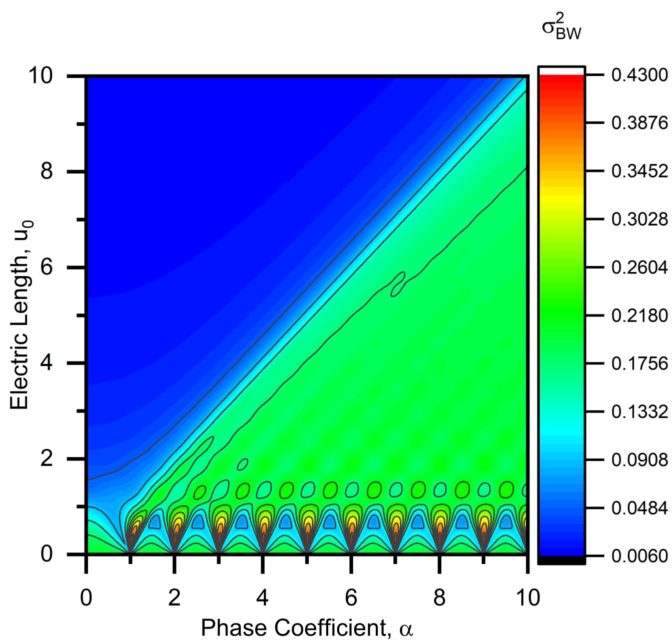

Figure 6.26: Uniform Distribution – Beamwidth variance contours as a function of electrical length and phase coefficient.

Chapter VII

Conclusion

A solid foundation for analyzing line source radiation has been established using a novel approach based on autocorrelation principles and common statistical concepts. Application of autocorrelation principles has enabled the development of a methodology to characterize the radiated power, pattern mean, and beamwidth variance for both broadside and scanning beams. The methodology has been applied to develop exact closed-form analytical expressions for the radiated power and beamwidth variance for the half-wave dipole, cosine, cosine-squared, generalized dipole, uniform, and triangular current distributions of constant phase. A new methodology was developed for line source radiators with a linearly progressive phase. Exact analytical expressions for the radiated power, pattern mean, and beamwidth variance for a scanning line source with a linearly varying phase were derived for the half-wave dipole, cosine, cosine-squared, uniform, and triangular current distributions. The expressions for radiated power were validated by numerically integrating the power pattern function corresponding to each current distribution. Likewise, the expressions for pattern mean and beamwidth variance were validated by numerically integrating the first or second moment of the power pattern function. All of the validations demonstrated effectively perfect agreement between the newly developed closed-form analytical expressions and the numerically integrated results. As stated in each chapter, except for the radiated power expression for the broadside half-wave dipole and generalized dipole current distributions, all of the exact analytical expressions are new and have not been presented previously in the annals of antenna theory.

The fact that new methodologies were developed to analyze both broadside and scanning beams from line source radiators, and to characterize not only the radiated power, but newly proposed metrics for antenna radiation (i.e., pattern mean and beamwidth variance), truly demonstrates the power of applying autocorrelation principles. As noted in the Introduction, application of these principles is really blazing an alternate path of analysis that departs from traditionally applied numerical techniques. This new development path will most likely not supplant the commonly used numerical methods of the practicing antenna engineer. However, through evolutionary research, these methodologies could lead to alternate means of characterizing antenna performance or lead to new methodologies for antenna analysis that could accelerate numerical analysis techniques of complex radiating structures. Specifically of interest is the potential applicability to the method of moments that is used to determine induced current distributions — this work considered impressed current distributions. As described by Harrington, the impedance matrix of the method of moments formulation can be obtained by performing two integrals and one derivative, which are calculated numerically [10]. Elements of this matrix are simply self impedances of each element of the antenna and mutual impedances between elements. The populated impedance matrix is then inverted to calculate the current on each element of the antenna. These impedances are identical to those of an antenna array, which can also be calculated using autocorrelation techniques. It is possible that a methodology could be developed using autocorrelation principles to populate the impedance matrix with only analytical equations, which would leave the inversion of the impedance matrix as the only task to be completed numerically. The power and efficiency of the method of moments could be increased drastically by removing the computationally intensive requirement of numerically estimating a large number of integrals and derivatives for each element of the impedance matrix.

The application of statistical measures (e.g., pattern mean and beamwidth variance) could also lead to new ways of comparing the performance of radiating structures. In fact, there is no reason why the autocorrelation principles could not be applied to higher order moments (e.g., skewness and kurtosis). Furthermore, it might be instructive to evaluate the applicability of the Hausdorff moment problem, which states that the complete collection of moments uniquely defines a distribution supported on a finite interval [21]. In this case, the distribution of interest is the antenna pattern, which is defined on a finite interval. Similar to Fourier analysis, it would be interesting to explore how much additional information about the antenna pattern is contributed by each successive higher order moment. Potentially, a finite number of moments could provide a sufficient amount of information to know "enough" to adequately characterize the performance of the antenna pattern.

In addition, the problems considered herein are strictly one-dimensional (i.e., the current distribution is defined only along the z-axis). The one-dimensional limitation enables several simplifications — cross-correlations for the current distribution are not considered and ϕ-symmetry can be assumed. In the case of a multi-dimensional current distribution, cross-correlations would need to be taken into account in the development of an expression for radiated power. Similarly, the variance definition would need to be extended to include variances in each dimension and also include the corresponding covariances.

The current distributions considered in this dissertation consist entirely of those that produce sum-type patterns (i.e., even-symmetric current distributions). Equally admissible to the procedure are current distributions that produce difference-type patterns (i.e., odd-symmetric current distributions). Also of interest is the linkage between "equivalent" sum and difference patterns through application of Hilbert Transform principles.

Also of significant interest is the possibility that the application of autocorrelation principles could be applied to the problem of antenna synthesis. At this point, it is possible that a synthesis procedure could be developed by first solving the differential equation represented by the Helmholtz operator. At a minimum, the approach could yield a "family of functions" that would produce the desired radiated power. Additionally, the mean, variance, and higher order moments could be used as constraints to refine the solution space.

Finally, in general, the application of autocorrelation principles has been applied successfully to develop a new methodology for characterizing antenna performance. Specifically, the integral of the power pattern function, and its corresponding moments, can be determined directly from the current distribution — without *a priori* knowledge of the antenna pattern. However, from a strictly mathematical point of view, it would be fascinating to explore whether these principles are generally applicable to other problems that are based on a distribution function. These types of problems are plentiful across a variety of fields — probability and statistics, quantum mechanics, fluid mechanics, and rarefied gas dynamics, to name a few. The true power and extensibility of this methodology can only be discovered by continuing to make incremental advancements through evolutionary research.

Bibliography

[1] James Clerk Maxwell. A dynamical theory of the electromagnetic field. *Philosophical Transactions of the Royal Society of London*, 155:459–512, January 1865.

[2] Heinrich Hertz. *Electric Waves, Being Researches on the Propagation of Electric Action with Finite Velocity through Space. Authorised English Translation by D.E. Jones, with a Preface by Lord Kelvin.* Macmillan, England, 1893.

[3] Oliver Heaviside. *Electrical Papers.* Macmillan and Co., New York and London, 1894.

[4] Guglielmo Marconi and P.T. McGrath. Marconi and his transatlantic signal. *Century Illustrated Magazine (1881-1906)*, LXIII(5):769, March 1902.

[5] John R. Carson. Radiation from transmission lines. *Journal of the American Institute of Electrical Engineers*, 40(10):789–790, October 1921.

[6] John R. Carson. Reciprocal theorems in radio communication. *Proceedings of the Institute of Radio Engineers*, 17(6):952–956, June 1929.

[7] P. S. Carter. Circuit relations in radiating systems and applications to antenna problems. *Proceedings of the Institute of Radio Engineers*, 20(6):1004–1041, June 1932.

[8] Ronold W. P. King. *Theory of Linear Antennas.* Harvard University Press, Cambridge, Massachusetts, 1955.

[9] S.A. Schelkunoff. Theory of antennas of arbitrary size and shape. *Journal of the Institute of Radio Engineers*, 29(9):493–521, September 1941.

[10] Roger F. Harrington. *Field Computation by Moment Methods.* John-Wiley & Sons, Inc., New York, New York, 2000.

[11] Roger F. Harrington. *Time-Harmonic Electromagnetic Fields.* John-Wiley & Sons, Inc., New York, New York, 2001.

[12] Jeffrey L. Young and Christopher D. Wilson. An application of Heisenberg's Uncertainty principle to line source radiation. In *2015 IEEE International Symposium on Antennas & Propagation and USNC/URSI National Radio Science Meeting*, pages 1060–1061, 2015.

[13] Jeffrey L. Young and Christopher D. Wilson. A general theory to determine the exact radiated power, directivity, and radiation resistance of a line-source radiator. *IEEE Transactions on Antennas and Propagation*, 64(6):2283–2292, June 2016.

[14] Leon W. Couch III. *Digital and Analog Communication Systems.* Pearson Education, Inc., Boston, Massachusetts, 2013.

[15] Touraj Assefi. *Stochastic Processes and Estimation Theory with Applications.* John-Wiley & Sons, Inc., New York, New York, 1979.

[16] Athanasios Papoulis and S. Unnikrishna Pillai. *Propability, Random Variables and Stochastic Processes.* McGraw-Hill, Boston, Massachusetts, 2002.

[17] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Forumlas, Graphs, and Mathematical Tables.* Dover, New York, New York, 1972.

[18] Warren L. Stutzman and Gary A. Thiele. *Antenna Theory and Design, Second Edition.* John-Wiley & Sons, Inc., Hoboken, NJ, 1998.

[19] Jeffrey L. Young and Christopher D. Wilson. Variance as a proxy for line source beamwidth. *IEEE Transactions on Antennas and Propagation*, 65(3):1003–1014, March 2017.

[20] Christopher D. Wilson and Jeffrey L. Young. The exact radiated power and directivity of a scanning line source radiator. *IEEE Transactions on Antennas and Propagation*, 65(6):2880–2889, June 2017.

[21] Robert M. Mnatsakanov. Hausdorff moment problem: Reconstruction of probability density functions. *Statistics & Probability Letters*, 78(13):1869–1877, February 2008.

Appendix A

Derivations for the Radiated Power of a Broadside Line Source

This appendix presents the detailed derivations of the autocorrelation functions used to determine the radiated power for a broadside line source radiator. The derivations were performed for the half-wave dipole, cosine, cosine-squared, generalized dipole, triangular, and uniform distributions.

A.1 Half-Wave Dipole Distribution

The half-wave dipole current distribution is given by

$$g(p) = A_m \cos\left(\frac{p}{2}\right) \tag{A.1}$$

on the interval $-\pi \leq p \leq \pi$ and $g(p) = 0$ otherwise. Recalling the autocorrelation function $R_g(p)$,

$$R_g(p) = \int_{-\infty}^{\infty} g(\tau)g(\tau - p)\ d\tau. \tag{A.2}$$

Substituting Equation (A.1) into Equation (A.2) and applying the piecewise definition of $g(p)$ yields

$$R_g(p) = A_m^2 \begin{cases} \displaystyle\int_{-\pi}^{p+\pi} \cos\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau-p}{2}\right)\, d\tau, & -2\pi \le p \le 0 \\ \displaystyle\int_{p-\pi}^{\pi} \cos\left(\frac{\tau}{2}\right)\cos\left(\frac{\tau-p}{2}\right)\, d\tau, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.3}$$

Product-to-sum trigonometric identities can be applied to Equation (A.3),

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} \displaystyle\int_{-\pi}^{p+\pi} \left[\cos\left(\frac{p}{2}\right) + \cos\left(\tau - \frac{p}{2}\right)\right] d\tau, & -2\pi \le p \le 0 \\ \displaystyle\int_{p-\pi}^{\pi} \left[\cos\left(\frac{p}{2}\right) + \cos\left(\tau - \frac{p}{2}\right)\right] d\tau, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.4}$$

Performing the integrals in Equation (A.4),

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} \left[\tau\cos\left(\frac{p}{2}\right) + \sin\left(\tau - \frac{p}{2}\right)\right]\Bigg|_{-\pi}^{p+\pi}, & -2\pi \le p \le 0 \\ \left[\tau\cos\left(\frac{p}{2}\right) + \sin\left(\tau - \frac{p}{2}\right)\right]\Bigg|_{p-\pi}^{\pi}, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.5}$$

Evaluating the limits of integration in Equation (A.5),

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} \left[(p+\pi)\cos\left(\frac{p}{2}\right) + \sin\left(p+\pi-\frac{p}{2}\right) + \pi\cos\left(\frac{p}{2}\right) - \sin\left(-\pi-\frac{p}{2}\right)\right], & -2\pi \le p \le 0 \\ \left[\pi\cos\left(\frac{p}{2}\right) + \sin\left(\pi-\frac{p}{2}\right) - (p-\pi)\cos\left(\frac{p}{2}\right) - \sin\left(p-\pi-\frac{p}{2}\right)\right], & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.6}$$

Simplifying Equation (A.6) yields the autocorrelation function $R_g(p)$ for the half-wave dipole distribution,

$$R_g(p) = A_m^2 \begin{cases} \left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.7}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{A.8}$$

The electric length of a half-wave dipole is $u_0 = \frac{1}{2}$, which can be substituted into Equation (A.8),

$$R_f(p) = \left[\frac{d^2}{dp^2} + \frac{1}{4}\right] R_g(p). \tag{A.9}$$

Applying Equation (A.9) to $R_g(p)$ for the interval $-2\pi \leq p \leq 0$, given previously in Equation (A.7),

$$R_f(p)\Big|_{-2\pi \leq p \leq 0} = A_m^2 \left[\frac{d^2}{dp^2} + \frac{1}{4}\right] \left[\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right)\right]. \tag{A.10}$$

Distributing terms in Equation (A.10) and performing the first derivative,

$$\begin{aligned} R_f(p)\Big|_{-2\pi \leq p \leq 0} = A_m^2 \Bigg\{ & \frac{d}{dp}\left[\frac{1}{2}\cos\left(\frac{p}{2}\right) - \left(\frac{2\pi + p}{4}\right)\sin\left(\frac{p}{2}\right) - \frac{1}{2}\cos\left(\frac{p}{2}\right)\right] \\ & + \left(\frac{2\pi + p}{8}\right)\cos\left(\frac{p}{2}\right) - \frac{1}{4}\sin\left(\frac{p}{2}\right) \Bigg\}. \end{aligned} \tag{A.11}$$

Canceling terms in Equation (A.11) and performing the second derivative,

$$\begin{aligned} R_f(p)\Big|_{-2\pi \leq p \leq 0} = A_m^2 \Bigg\{ & -\left(\frac{2\pi + p}{8}\right)\cos\left(\frac{p}{2}\right) - \frac{1}{4}\sin\left(\frac{p}{2}\right) \\ & + \left(\frac{2\pi + p}{8}\right)\cos\left(\frac{p}{2}\right) - \frac{1}{4}\sin\left(\frac{p}{2}\right) \Bigg\}. \end{aligned} \tag{A.12}$$

Collecting and canceling terms in Equation (A.12),

$$R_f(p)\Big|_{-2\pi\le p\le 0} = -\frac{A_m^2}{2}\sin\left(\frac{p}{2}\right). \tag{A.13}$$

Applying Equation (A.9) to $R_g(p)$ for the interval $0 \le p \le 2\pi$, given previously in Equation (A.7),

$$R_f(p)\Big|_{0\le p\le 2\pi} = A_m^2\left[\frac{d^2}{dp^2}+\frac{1}{4}\right]\left[\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)+\sin\left(\frac{p}{2}\right)\right]. \tag{A.14}$$

Distributing terms in Equation (A.14) and performing the first derivative,

$$\begin{aligned} R_f(p)\Big|_{0\le p\le 2\pi} = A_m^2\Bigg\{&\frac{d}{dp}\left[-\frac{1}{2}\cos\left(\frac{p}{2}\right)-\left(\frac{2\pi-p}{4}\right)\sin\left(\frac{p}{2}\right)+\frac{1}{2}\cos\left(\frac{p}{2}\right)\right] \\ &+\left(\frac{2\pi-p}{8}\right)\cos\left(\frac{p}{2}\right)+\frac{1}{4}\sin\left(\frac{p}{2}\right)\Bigg\}. \end{aligned} \tag{A.15}$$

Canceling terms in Equation (A.15) and performing the second derivative,

$$\begin{aligned} R_f(p)\Big|_{0\le p\le 2\pi} = A_m^2\Bigg\{&-\left(\frac{2\pi-p}{8}\right)\cos\left(\frac{p}{2}\right)+\frac{1}{4}\sin\left(\frac{p}{2}\right) \\ &+\left(\frac{2\pi-p}{8}\right)\cos\left(\frac{p}{2}\right)+\frac{1}{4}\sin\left(\frac{p}{2}\right)\Bigg\}. \end{aligned} \tag{A.16}$$

Collecting and canceling terms in Equation (A.16),

$$R_f(p)\Big|_{0\le p\le 2\pi} = \frac{A_m^2}{2}\sin\left(\frac{p}{2}\right). \tag{A.17}$$

Combining Equations (A.13) and (A.17) yields the autocorrelation function $R_f(p)$ for the half-wave dipole distribution,

$$R_f(p) = \frac{A_m^2}{2}\begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi\le p\le 0 \\ \sin\left(\frac{p}{2}\right), & 0\le p\le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.18}$$

Recalling the definition for $R_h(0)$,

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0 \tau)}{u_0 \tau} \, d\tau. \tag{A.19}$$

Substituting Equation (A.18) into Equation (A.19),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{2\pi} \Bigg\{ &- \int_{-2\pi}^{0} \sin\left(\frac{\tau}{2}\right) \frac{\sin(u_0 \tau)}{u_0 \tau} \, d\tau \\ &+ \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin(u_0 \tau)}{u_0 \tau} \, d\tau \Bigg\}. \end{aligned} \tag{A.20}$$

Substituting the electric length for the half-wave dipole, $u_0 = \frac{1}{2}$, into Equation (A.20),

$$\begin{aligned} R_h(0) = \frac{A_m^2}{2\pi} \Bigg\{ &- \int_{-2\pi}^{0} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left(\frac{\tau}{2}\right)}{\tau} \, d\tau \\ &+ \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left(\frac{\tau}{2}\right)}{\tau} \, d\tau \Bigg\}. \end{aligned} \tag{A.21}$$

Recognizing even and odd functions in Equation (A.21) and combining terms,

$$R_h(0) = \frac{A_m^2}{\pi} \left\{ \int_{0}^{2\pi} \frac{\sin^2\left(\frac{\tau}{2}\right)}{\tau} \, d\tau \right\}. \tag{A.22}$$

Applying the power-reduction trigonometric identity to Equation (A.22) yields

$$R_h(0) = \frac{A_m^2}{2\pi} \int_{0}^{2\pi} \frac{1 - \cos(\tau)}{\tau} \, d\tau. \tag{A.23}$$

Recalling the definition of the modified cosine integral,

$$\mathrm{Cin}\,(ba) = \int_{0}^{b} \frac{1 - \cos(at)}{t} \, dt. \tag{A.24}$$

Substituting Equation (A.24) into Equation (A.23) yields the stationary autocorrelation function $R_h(0)$ for the half-wave dipole distribution,

$$R_h(0) = A_m^2 \left[\frac{\mathrm{Cin}\,(2\pi)}{2\pi} \right]. \tag{A.25}$$

A.2 Cosine Distribution

The cosine distribution is identical to the half-wave dipole distribution. Therefore, the autocorrelation function $R_g(p)$ for the cosine current distribution is repeated directly from Equation (A.7),

$$R_g(p) = A_m^2 \begin{cases} \left(\dfrac{2\pi + p}{2}\right)\cos\left(\dfrac{p}{2}\right) - \sin\left(\dfrac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\dfrac{2\pi - p}{2}\right)\cos\left(\dfrac{p}{2}\right) + \sin\left(\dfrac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.26}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{A.27}$$

Applying Equation (A.27) to $R_g(p)$ for the interval $-2\pi \le p \le 0$, given previously in Equation (A.26),

$$R_f(p)\Big|_{-2\pi \le p \le 0} = A_m^2 \left[\frac{d^2}{dp^2} + u_0^2\right] \left[\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right)\right]. \tag{A.28}$$

Distributing terms in Equation (A.28) and performing the first derivative,

$$\begin{aligned} R_f(p)\Big|_{-2\pi \le p \le 0} = A_m^2 \Bigg\{ & \frac{d}{dp}\left[-\left(\frac{2\pi + p}{4}\right)\sin\left(\frac{p}{2}\right) + \frac{1}{2}\cos\left(\frac{p}{2}\right) - \frac{1}{2}\cos\left(\frac{p}{2}\right)\right] \\ & + u_0^2 \left[\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right)\right]\Bigg\}. \end{aligned} \tag{A.29}$$

Canceling terms in Equation (A.29) and performing the second derivative,

$$\begin{aligned} R_f(p)\Big|_{-2\pi \le p \le 0} = A_m^2 \Bigg\{ & \left[-\left(\frac{2\pi + p}{8}\right)\cos\left(\frac{p}{2}\right) - \frac{1}{4}\sin\left(\frac{p}{2}\right)\right] \\ & + u_0^2 \left[\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right)\right]\Bigg\}. \end{aligned} \tag{A.30}$$

Collecting terms in Equation (A.30),

$$R_f(p)\bigg|_{-2\pi\le p\le 0} = A_m^2\left[\left(u_0^2-\frac{1}{4}\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right)-\left(u_0^2+\frac{1}{4}\right)\sin\left(\frac{p}{2}\right)\right] \tag{A.31}$$

Applying Equation (A.27) to $R_g(p)$ for the interval $0\le p\le 2\pi$, given previously in Equation (A.26),

$$R_f(p)\bigg|_{0\le p\le 2\pi} = A_m^2\left[\frac{d^2}{dp^2}+u_0^2\right]\left[\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)+\sin\left(\frac{p}{2}\right)\right]. \tag{A.32}$$

Distributing terms in Equation (A.32) and performing the first derivative,

$$\begin{aligned} R_f(p)\bigg|_{0\le p\le 2\pi} = A_m^2\Bigg\{&\frac{d}{dp}\left[-\left(\frac{2\pi-p}{4}\right)\sin\left(\frac{p}{2}\right)-\frac{1}{2}\cos\left(\frac{p}{2}\right)+\frac{1}{2}\cos\left(\frac{p}{2}\right)\right] \\ &+u_0^2\left[\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)+\sin\left(\frac{p}{2}\right)\right]\Bigg\}. \end{aligned} \tag{A.33}$$

Canceling terms in Equation (A.33) and performing the second derivative,

$$\begin{aligned} R_f(p)\bigg|_{0\le p\le 2\pi} = A_m^2\Bigg\{&\left[-\left(\frac{2\pi-p}{8}\right)\cos\left(\frac{p}{2}\right)+\frac{1}{4}\sin\left(\frac{p}{2}\right)\right] \\ &+u_0^2\left[\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)+\sin\left(\frac{p}{2}\right)\right]\Bigg\}. \end{aligned} \tag{A.34}$$

Collecting and canceling terms in Equation (A.34),

$$R_f(p)\bigg|_{0\le p\le 2\pi} = A_m^2\left[\left(u_0^2-\frac{1}{4}\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)+\left(u_0^2+\frac{1}{4}\right)\sin\left(\frac{p}{2}\right)\right]. \tag{A.35}$$

Combining Equations (A.31) and (A.35) yields the autocorrelation function $R_f(p)$ for the cosine distribution,

$$R_f(p) = A_m^2\begin{cases}\left(u_0^2-\frac{1}{4}\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right)-\left(u_0^2+\frac{1}{4}\right)\sin\left(\frac{p}{2}\right), & -2\pi\le p\le 0\\ \left(u_0^2-\frac{1}{4}\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)+\left(u_0^2+\frac{1}{4}\right)\sin\left(\frac{p}{2}\right), & 0\le p\le 2\pi\\ 0, & \text{otherwise}\end{cases}. \tag{A.36}$$

Recalling the definition for $R_h(0)$,

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0 \tau)}{u_0 \tau}\, d\tau. \tag{A.37}$$

Substituting Equation (A.36) into Equation (A.37),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & \int_{-2\pi}^{0} \left[\left(u_0^2 - \frac{1}{4}\right) \left(\frac{2\pi + \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) - \left(u_0^2 + \frac{1}{4}\right) \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ & + \int_{0}^{2\pi} \left[\left(u_0^2 - \frac{1}{4}\right) \left(\frac{2\pi - \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) + \left(u_0^2 + \frac{1}{4}\right) \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \Bigg\}. \end{aligned} \tag{A.38}$$

Recognizing even and odd functions in Equation (A.38) and combining terms,

$$R_h(0) = \frac{2A_m^2 \cancel{u_0}}{\pi} \int_{0}^{2\pi} \left[\left(u_0^2 - \frac{1}{4}\right) \left(\frac{2\pi - \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) + \left(u_0^2 + \frac{1}{4}\right) \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin(u_0\tau)}{\cancel{u_0}\tau}\, d\tau. \tag{A.39}$$

Separating integrals in Equation (A.39),

$$\begin{aligned} R_h(0) = \frac{2A_m^2}{\pi} \Bigg\{ & \pi \left(u_0^2 - \frac{1}{4}\right) \int_{0}^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\sin(u_0\tau)}{\tau}\, d\tau \\ & - \frac{1}{2} \left(u_0^2 - \frac{1}{4}\right) \int_{0}^{2\pi} \cos\left(\frac{\tau}{2}\right) \sin(u_0\tau)\, d\tau \\ & + \left(u_0^2 + \frac{1}{4}\right) \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin(u_0\tau)}{\tau}\, d\tau \Bigg\}. \end{aligned} \tag{A.40}$$

Applying product-to-sum trigonometric identities to Equation (A.40),

$$\begin{aligned} R_h(0) = \frac{A_m^2}{\pi} \Bigg\{ & \pi \left(u_0^2 - \frac{1}{4}\right) \int_{0}^{2\pi} \frac{\sin\left[\left(u_0 + \frac{1}{2}\right)\tau\right] + \sin\left[\left(u_0 - \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\ & - \frac{1}{2} \left(u_0^2 - \frac{1}{4}\right) \int_{0}^{2\pi} \left[\sin\left[\left(u_0 + \frac{1}{2}\right)\tau\right] + \sin\left[\left(u_0 - \frac{1}{2}\right)\tau\right] \right] d\tau \\ & + \left(u_0^2 + \frac{1}{4}\right) \int_{0}^{2\pi} \frac{1 - \cos\left[\left(u_0 + \frac{1}{2}\right)\tau\right] - 1 + \cos\left[\left(u_0 - \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \Bigg\}. \end{aligned} \tag{A.41}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\operatorname{Si}(ba) = \int_{0}^{b} \frac{\sin(at)}{t}\, dt \tag{A.42}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos\,(at)}{t}\;dt. \tag{A.43}$$

Applying Equations (A.42) and (A.43) to Equation (A.41) and performing the remaining integrals,

$$\begin{aligned}R_h(0) = \frac{A_m^2}{\pi}\Bigg\{ &\left(u_0^2+\frac{1}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &+ \pi\left(u_0^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &+ \frac{1}{2}\left(\frac{u_0^2-\frac{1}{4}}{u_0+\frac{1}{2}}\right)\cos\left[\left(u_0+\frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi} \\ &+ \frac{1}{2}\left(\frac{u_0^2-\frac{1}{4}}{u_0-\frac{1}{2}}\right)\cos\left[\left(u_0-\frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi}\Bigg\}. \end{aligned} \tag{A.44}$$

Evaluating the limits of integration in Equation (A.44),

$$\begin{aligned}R_h(0) = \frac{A_m^2}{\pi}\Bigg\{ &\left(u_0^2+\frac{1}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &+ \pi\left(u_0^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &+ \frac{1}{2}\left(\frac{u_0^2-\frac{1}{4}}{u_0+\frac{1}{2}}\right)\left[\cos\left[2\pi\left(u_0+\frac{1}{2}\right)\right]-1\right] \\ &+ \frac{1}{2}\left(\frac{u_0^2-\frac{1}{4}}{u_0-\frac{1}{2}}\right)\left[\cos\left[2\pi\left(u_0-\frac{1}{2}\right)\right]-1\right]\Bigg\}. \end{aligned} \tag{A.45}$$

Simplifying Equation (A.45) yields the stationary autocorrelation function $R_h(0)$ for the cosine distribution,

$$\begin{aligned}R_h(0) = \frac{A_m^2}{\pi}\Bigg\{ &\left(u_0^2+\frac{1}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &+ \pi\left(u_0^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &- u_0\left[\cos\left(2\pi u_0\right)+1\right]\Bigg\}. \end{aligned} \tag{A.46}$$

A.3 Cosine-Squared Distribution

The cosine-squared current distribution is given by

$$g(p) = A_m \cos^2\left(\frac{p}{2}\right) \tag{A.47}$$

on the interval $-\pi \leq p \leq \pi$ and $g(p) = 0$ otherwise. Recalling the autocorrelation function $R_g(p)$,

$$R_g(p) = \int_{-\infty}^{\infty} g(\tau)g(\tau - p)\ d\tau. \tag{A.48}$$

Substituting Equation (A.47) into Equation (A.48) and applying the piecewise definition of $g(p)$ yields

$$R_g(p) = A_m^2 \begin{cases} \displaystyle\int_{-\pi}^{p+\pi} \cos^2\left(\frac{\tau}{2}\right)\cos^2\left(\frac{\tau-p}{2}\right)\ d\tau, & -2\pi \leq p \leq 0 \\ \displaystyle\int_{p-\pi}^{\pi} \cos^2\left(\frac{\tau}{2}\right)\cos^2\left(\frac{\tau-p}{2}\right)\ d\tau, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.49}$$

Applying power-reduction trigonometric identities to Equation (A.49),

$$R_g(p) = \frac{A_m^2}{4} \begin{cases} \displaystyle\int_{-\pi}^{p+\pi} \left[1+\cos(\tau)\right]\left[1+\cos(\tau-p)\right]\ d\tau, & -2\pi \leq p \leq 0 \\ \displaystyle\int_{p-\pi}^{\pi} \left[1+\cos(\tau)\right]\left[1+\cos(\tau-p)\right]\ d\tau, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.50}$$

Distributing terms in Equation (A.50),

$$R_g(p) = \frac{A_m^2}{4} \begin{cases} \displaystyle\int_{-\pi}^{p+\pi} \left[1+\cos(\tau)+\cos(\tau-p)+\cos(\tau)\cos(\tau-p)\right]\ d\tau, & -2\pi \leq p \leq 0 \\ \displaystyle\int_{p-\pi}^{\pi} \left[1+\cos(\tau)+\cos(\tau-p)+\cos(\tau)\cos(\tau-p)\right]\ d\tau, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.51}$$

Applying product-to-sum trigonometric identities to Equation (A.51),

$$R_g(p) = \frac{A_m^2}{4} \begin{cases} \int_{-\pi}^{p+\pi} \left[1 + \cos(\tau) + \cos(\tau - p) + \frac{\cos(p)}{2} + \frac{\cos(2\tau - p)}{2}\right] d\tau, & -2\pi \leq p \leq 0 \\ \int_{p-\pi}^{\pi} \left[1 + \cos(\tau) + \cos(\tau - p) + \frac{\cos(p)}{2} + \frac{\cos(2\tau - p)}{2}\right] d\tau, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.52}$$

Performing the integrals in Equation (A.52),

$$R_g(p) = \frac{A_m^2}{4} \begin{cases} \left[\left[1 + \frac{\cos(p)}{2}\right] \tau + \sin(\tau) + \sin(\tau - p) + \frac{\sin(2\tau - p)}{4}\right] \Big|_{-\pi}^{p+\pi}, & -2\pi \leq p \leq 0 \\ \left[\left[1 + \frac{\cos(p)}{2}\right] \tau + \sin(\tau) + \sin(\tau - p) + \frac{\sin(2\tau - p)}{4}\right] \Big|_{p-\pi}^{\pi}, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.53}$$

Evaluating the limits of integration in Equation (A.53) and simplifying,

$$R_g(p) = \frac{A_m^2}{4} \begin{cases} \left[1 + \frac{\cos(p)}{2}\right] (p + 2\pi) - 2\sin(p) + \frac{\sin(p)}{2}, & -2\pi \leq p \leq 0 \\ \left[1 + \frac{\cos(p)}{2}\right] (2\pi - p) + 2\sin(p) - \frac{\sin(p)}{2}, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.54}$$

Simplifying Equation (A.54) yields the autocorrelation function $R_g(p)$ for the cosine-squared distribution,

$$R_g(p) = \frac{A_m^2}{8} \begin{cases} (2\pi + p) [2 + \cos(p)] - 3\sin(p), & -2\pi \leq p \leq 0 \\ (2\pi - p) [2 + \cos(p)] + 3\sin(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.55}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{A.56}$$

Applying Equation (A.56) to $R_g(p)$ for the interval $-2\pi \leq p \leq 0$, given previously in Equation (A.55),

$$R_f(p)\Big|_{-2\pi \leq p \leq 0} = \frac{A_m^2}{8} \left[\frac{d^2}{dp^2} + u_0^2\right] [(2\pi + p) [2 + \cos(p)] - 3\sin(p)] . \tag{A.57}$$

Distributing terms in Equation (A.57) and performing the first derivative,

$$R_f(p)\Big|_{-2\pi\le p\le 0} = \frac{A_m^2}{8}\Bigg\{\frac{d}{dp}\left[2+\cos(p)-(2\pi+p)\sin(p)-3\cos(p)\right]$$
$$+\,u_0^2\left[(2\pi+p)\left[2+\cos(p)\right]-3\sin(p)\right]\Bigg\}. \tag{A.58}$$

Combining terms in Equation (A.58) and performing the second derivative,

$$R_f(p)\Big|_{-2\pi\le p\le 0} = \frac{A_m^2}{8}\Bigg\{2\sin(p)-\sin(p)-(2\pi+p)\cos(p)$$
$$+\,u_0^2\left[(2\pi+p)\left[2+\cos(p)\right]-3\sin(p)\right]\Bigg\}. \tag{A.59}$$

Collecting terms in Equation (A.59),

$$R_f(p)\Big|_{-2\pi\le p\le 0} = \frac{A_m^2}{8}\Bigg\{(2\pi+p)\left[2u_0^2+\left(u_0^2-1\right)\cos(p)\right]+\left(1-3u_0^2\right)\sin(p)\Bigg\}. \tag{A.60}$$

Applying Equation (A.56) to $R_g(p)$ for the interval $0\le p\le 2\pi$, given previously in Equation (A.55),

$$R_f(p)\Big|_{0\le p\le 2\pi} = \frac{A_m^2}{8}\left[\frac{d^2}{dp^2}+u_0^2\right]\left[(2\pi-p)\left[2+\cos(p)\right]+3\sin(p)\right]. \tag{A.61}$$

Distributing terms in Equation (A.61) and performing the first derivative,

$$R_f(p)\Big|_{0\le p\le 2\pi} = \frac{A_m^2}{8}\Bigg\{\frac{d}{dp}\left[-2-\cos(p)-(2\pi-p)\sin(p)+3\cos(p)\right]$$
$$+\,u_0^2\left[(2\pi-p)\left[2+\cos(p)\right]+3\sin(p)\right]\Bigg\}. \tag{A.62}$$

Combining terms in Equation (A.62) and performing the second derivative,

$$R_f(p)\Big|_{0\le p\le 2\pi} = \frac{A_m^2}{8}\Bigg\{-2\sin(p)+\sin(p)-(2\pi-p)\cos(p)$$
$$+\,u_0^2\left[(2\pi-p)\left[2+\cos(p)\right]+3\sin(p)\right]\Bigg\}. \tag{A.63}$$

Collecting terms in Equation (A.63),

$$R_f(p)\Big|_{0\le p\le 2\pi} = \frac{A_m^2}{8}\Big\{ (2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p) \Big\}. \tag{A.64}$$

Combining Equations (A.60) and (A.64) yields the autocorrelation function $R_f(p)$ for the cosine-squared distribution,

$$R_f(p) = \frac{A_m^2}{8}\begin{cases} (2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ (2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.65}$$

Recalling the definition for $R_h(0)$,

$$R_h(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau. \tag{A.66}$$

Substituting Equation (A.65) into Equation (A.66),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{8\pi}\Big\{ &\int_{-2\pi}^{0} \left[(2\pi + \tau)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(\tau)\right] + \left(1 - 3u_0^2\right)\sin(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ &+ \int_{0}^{2\pi} \left[(2\pi - \tau)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(\tau)\right] - \left(1 - 3u_0^2\right)\sin(\tau)\right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \Big\}. \end{aligned} \tag{A.67}$$

Recognizing even and odd functions in Equation (A.67) and combining terms,

$$R_h(0) = \frac{A_m^2 \cancel{u_0}}{4\pi}\int_{0}^{2\pi} \left[(2\pi - \tau)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(\tau)\right] - \left(1 - 3u_0^2\right)\sin(\tau)\right] \frac{\sin(u_0\tau)}{\cancel{u_0}\tau}\, d\tau. \tag{A.68}$$

Separating integrals in Equation (A.68),

$$\begin{aligned} R_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & 4\pi u_0^2 \int_0^{2\pi} \frac{\sin(u_0\tau)}{\tau}\ d\tau \\ & - 2u_0^2 \int_0^{2\pi} \sin(u_0\tau)\ d\tau \\ & + 2\pi\left(u_0^2 - 1\right) \int_0^{2\pi} \cos(\tau)\frac{\sin(u_0\tau)}{\tau}\ d\tau \\ & - \left(u_0^2 - 1\right) \int_0^{2\pi} \cos(\tau)\sin(u_0\tau)\ d\tau \\ & - \left(1 - 3u_0^2\right) \int_0^{2\pi} \sin(\tau)\frac{\sin(u_0\tau)}{\tau}\ d\tau \Bigg\}. \end{aligned} \tag{A.69}$$

Applying product-to-sum trigonometric identities to Equation (A.69),

$$\begin{aligned} R_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & 4\pi u_0^2 \int_0^{2\pi} \frac{\sin(u_0\tau)}{\tau}\ d\tau \\ & - 2u_0^2 \int_0^{2\pi} \sin(u_0\tau)\ d\tau \\ & + \pi\left(u_0^2 - 1\right) \int_0^{2\pi} \frac{\sin[(u_0+1)\tau] + \sin[(u_0-1)\tau]}{\tau}\ d\tau \\ & - \frac{1}{2}\left(u_0^2 - 1\right) \int_0^{2\pi} \left[\sin[(u_0+1)\tau] + \sin[(u_0-1)\tau]\right]\ d\tau \\ & - \frac{1}{2}\left(1 - 3u_0^2\right) \int_0^{2\pi} \frac{1 - \cos[(u_0+1)\tau] - 1 + \cos[(u_0-1)\tau]}{\tau}\ d\tau \Bigg\}. \end{aligned} \tag{A.70}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}(ba) = \int_0^b \frac{\sin(at)}{t}\ dt \tag{A.71}$$

and

$$\mathrm{Cin}(ba) = \int_0^b \frac{1 - \cos(at)}{t}\ dt. \tag{A.72}$$

Applying Equations (A.71) and (A.72) to Equation (A.70) and performing the remaining integrals,

$$\begin{aligned}R_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & 4\pi u_0^2 \mathrm{Si}\left(2\pi u_0\right) \\ & + \pi\left(u_0^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-1\right)\right]\right] \\ & - \frac{1}{2}\left(1 - 3u_0^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-1\right)\right]\right] \\ & + 2u_0 \cos\left(u_0\tau\right)\Big|_0^{2\pi} \\ & + \frac{1}{2}\left(\frac{u_0^2-1}{u_0+1}\right)\cos\left[\left(u_0+1\right)\tau\right]\Big|_0^{2\pi} \\ & + \frac{1}{2}\left(\frac{u_0^2-1}{u_0-1}\right)\cos\left[\left(u_0-1\right)\tau\right]\Big|_0^{2\pi}\Bigg\}. \end{aligned} \tag{A.73}$$

Evaluating the limits of integration in Equation (A.73),

$$\begin{aligned}R_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & 4\pi u_0^2 \mathrm{Si}\left(2\pi u_0\right) \\ & + \pi\left(u_0^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-1\right)\right]\right] \\ & - \frac{1}{2}\left(1 - 3u_0^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-1\right)\right]\right] \\ & + 2u_0\left[\cos\left(2\pi u_0\right) - 1\right] \\ & + \frac{1}{2}\left(u_0-1\right)\left[\cos\left[2\pi\left(u_0+1\right)\right] - 1\right] \\ & + \frac{1}{2}\left(u_0+1\right)\left[\cos\left[2\pi\left(u_0-1\right)\right] - 1\right]\Bigg\}. \end{aligned} \tag{A.74}$$

Simplifying Equation (A.74) yields the stationary autocorrelation function $R_h(0)$ for the cosine-squared distribution,

$$\begin{aligned}R_h(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-1\right)\right]\right] \\ & - \left(1 - 3u_0^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-1\right)\right]\right] \\ & + 8\pi u_0^2 \mathrm{Si}\left(2\pi u_0\right) + 6u_0\left[\cos\left(2\pi u_0\right) - 1\right]\Bigg\}. \end{aligned} \tag{A.75}$$

A.4 Generalized Dipole Distribution

The generalized dipole current distribution is given by

$$g(p) = A_m \begin{cases} \sin\left[u_0\left(\pi + p\right)\right], & -2\pi \le p \le 0 \\ \sin\left[u_0\left(\pi - p\right)\right], & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.76}$$

Recalling the autocorrelation function $R_g(p)$,

$$R_g(p) = \int_{-\infty}^{\infty} g(\tau)g(\tau - p)\; d\tau. \tag{A.77}$$

Substituting Equation (A.76) into Equation (A.77) and applying the piecewise definition of $g(p)$ yields

$$R_g(p) = A_m^2 \begin{cases} R_{g1}(p), & -2\pi \le p \le -\pi \\ R_{g2}(p), & -\pi \le p \le 0 \\ R_{g3}(p), & 0 \le p \le \pi \\ R_{g4}(p), & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}, \tag{A.78}$$

where each element of the autocorrelation function can be determined by integrating Equation (A.77). The first element $R_{g1}(p)$ is defined on the interval $-2\pi \le p \le -\pi$,

$$R_{g1}(p) = \int_{-\pi}^{p+\pi} \sin\left[u_0\left(\pi + \tau\right)\right]\sin\left[u_0\left(\pi - \tau + p\right)\right]\; d\tau. \tag{A.79}$$

Applying product-to-sum trigonometric identities to Equation (A.79),

$$R_{g1}(p) = \frac{1}{2}\int_{-\pi}^{p+\pi} \left[\cos\left[u_0\left(2\tau - p\right)\right] - \cos\left[u_0\left(2\pi + p\right)\right]\right]\; d\tau. \tag{A.80}$$

Performing the integrals in Equation (A.80),

$$R_{g1}(p) = \frac{1}{2}\left[\frac{1}{2u_0}\sin\left[u_0\left(2\tau - p\right)\right] - \tau\cos\left[u_0\left(2\pi + p\right)\right]\right]\Bigg|_{-\pi}^{p+\pi}. \tag{A.81}$$

Evaluating the limits of integration in Equation (A.81) and simplifying,

$$R_{g1}(p) = \frac{1}{2}\left[\frac{1}{u_0}\sin\left[u_0\left(2\pi+p\right)\right] - \left(2\pi+p\right)\cos\left[u_0\left(2\pi+p\right)\right]\right]. \tag{A.82}$$

The second element $R_{g2}(p)$ is defined on the interval $-\pi \leq p \leq 0$,

$$\begin{aligned} R_{g2}(p) =& \left\{ \int_{-\pi}^{p} \sin\left[u_0\left(\pi+\tau\right)\right]\sin\left[u_0\left(\pi+\tau-p\right)\right]\ d\tau \right.\\ &+ \int_{p}^{0} \sin\left[u_0\left(\pi+\tau\right)\right]\sin\left[u_0\left(\pi-\tau+p\right)\right]\ d\tau \\ &\left.+ \int_{0}^{p+\pi} \sin\left[u_0\left(\pi-\tau\right)\right]\sin\left[u_0\left(\pi-\tau+p\right)\right]\ d\tau \right\}. \end{aligned} \tag{A.83}$$

Applying product-to-sum trigonometric identities to Equation (A.83),

$$\begin{aligned} R_{g2}(p) = \frac{1}{2}& \left\{ \int_{-\pi}^{p} \left[\cos\left(u_0 p\right) - \cos\left[u_0\left(2\pi+2\tau-p\right)\right]\right]\ d\tau \right.\\ &+ \int_{p}^{0} \left[\cos\left[u_0\left(2\tau-p\right)\right] - \cos\left[u_0\left(2\pi+p\right)\right]\right]\ d\tau \\ &\left.+ \int_{0}^{p+\pi} \left[\cos\left(u_0 p\right) - \cos\left[u_0\left(2\pi-2\tau+p\right)\right]\right]\ d\tau \right\}. \end{aligned} \tag{A.84}$$

Performing the integrals in Equation (A.84),

$$\begin{aligned} R_{g2}(p) = \frac{1}{2}& \left\{ \left[\tau\cos\left(u_0 p\right) - \frac{1}{2u_0}\sin\left[u_0\left(2\pi+2\tau-p\right)\right]\right]\Bigg|_{-\pi}^{p} \right.\\ &+ \left[\frac{1}{2u_0}\sin\left[u_0\left(2\tau-p\right)\right] - \tau\cos\left[u_0\left(2\pi+p\right)\right]\right]\Bigg|_{p}^{0} \\ &\left.+ \left[\tau\cos\left(u_0 p\right) + \frac{1}{2u_0}\sin\left[u_0\left(2\pi-2\tau+p\right)\right]\right]\Bigg|_{0}^{p+\pi} \right\}. \end{aligned} \tag{A.85}$$

Evaluating the limits of integration in Equation (A.85) and simplifying,

$$\begin{aligned}R_{g2}(p) = \frac{1}{2}\Bigg\{ &\left[(p+\pi)\cos(u_0 p) - \frac{1}{2u_0}\sin[u_0(2\pi+p)] - \frac{1}{2u_0}\sin(u_0 p)\right] \\ &+ \left[-\frac{1}{2u_0}\sin(u_0 p) - \frac{1}{2u_0}\sin(u_0 p) + p\cos[u_0(2\pi+p)]\right] \\ &+ \left[(p+\pi)\cos(u_0 p) - \frac{1}{2u_0}\sin(u_0 p) - \frac{1}{2u_0}\sin[u_0(2\pi+p)]\right]\Bigg\}. \end{aligned} \tag{A.86}$$

Further simplifying Equation (A.86),

$$\begin{aligned}R_{g2}(p) = \frac{1}{2}\Bigg[&2(\pi+p)\cos(u_0 p) - \frac{2}{u_0}\sin(u_0 p) \\ &- \frac{1}{u_0}\sin[u_0(2\pi+p)] + p\cos[u_0(2\pi+p)]\Bigg]. \end{aligned} \tag{A.87}$$

The third element $R_{g3}(p)$ is defined on the interval $0 \leq p \leq \pi$,

$$\begin{aligned}R_{g3}(p) = \Bigg\{ &\int_{p-\pi}^{0} \sin[u_0(\pi+\tau)]\sin[u_0(\pi+\tau-p)]\ d\tau \\ &+ \int_{0}^{p} \sin[u_0(\pi-\tau)]\sin[u_0(\pi+\tau-p)]\ d\tau \\ &+ \int_{p}^{\pi} \sin[u_0(\pi-\tau)]\sin[u_0(\pi-\tau+p)]\ d\tau\Bigg\}. \end{aligned} \tag{A.88}$$

Applying product-to-sum trigonometric identities to Equation (A.88),

$$\begin{aligned}R_{g3}(p) = \frac{1}{2}\Bigg\{ &\int_{p-\pi}^{0} [\cos(u_0 p) - \cos[u_0(2\pi+2\tau-p)]]\ d\tau \\ &+ \int_{0}^{p} [\cos[u_0(2\tau-p)] - \cos[u_0(2\pi-p)]]\ d\tau \\ &+ \int_{p}^{\pi} [\cos(u_0 p) - \cos[u_0(2\pi-2\tau+p)]]\ d\tau\Bigg\}. \end{aligned} \tag{A.89}$$

Performing the integrals in Equation (A.89),

$$R_{g3}(p) = \frac{1}{2}\Bigg\{ \left[\tau\cos\left(u_0 p\right) - \frac{1}{2u_0}\sin\left[u_0\left(2\pi + 2\tau - p\right)\right]\right]\Bigg|_{p-\pi}^{0} + \left[\frac{1}{2u_0}\sin\left[u_0\left(2\tau - p\right)\right] - \tau\cos\left[u_0\left(2\pi - p\right)\right]\right]\Bigg|_{0}^{p} + \left[\tau\cos\left(u_0 p\right) + \frac{1}{2u_0}\sin\left[u_0\left(2\pi - 2\tau + p\right)\right]\right]\Bigg|_{p}^{\pi}\Bigg\}. \tag{A.90}$$

Evaluating the limits of integration in Equation (A.90) and simplifying,

$$R_{g3}(p) = \frac{1}{2}\Bigg\{ \left[(\pi - p)\cos\left(u_0 p\right) - \frac{1}{2u_0}\sin\left[u_0\left(2\pi - p\right)\right] + \frac{1}{2u_0}\sin\left(u_0 p\right)\right] + \left[\frac{1}{2u_0}\sin\left(u_0 p\right) + \frac{1}{2u_0}\sin\left(u_0 p\right) - p\cos\left[u_0\left(2\pi - p\right)\right]\right] + \left[(\pi - p)\cos\left(u_0 p\right) + \frac{1}{2u_0}\sin\left(u_0 p\right) - \frac{1}{2u_0}\sin\left[u_0\left(2\pi - p\right)\right]\right]\Bigg\}. \tag{A.91}$$

Further simplifying Equation (A.91),

$$R_{g3}(p) = \frac{1}{2}\Bigg[2\left(\pi - p\right)\cos\left(u_0 p\right) + \frac{2}{u_0}\sin\left(u_0 p\right) - \frac{1}{u_0}\sin\left[u_0\left(2\pi - p\right)\right] - p\cos\left[u_0\left(2\pi - p\right)\right]\Bigg]. \tag{A.92}$$

The fourth element $R_{g4}(p)$ is defined on the interval $\pi \leq p \leq 2\pi$,

$$R_{g4}(p) = \int_{p-\pi}^{\pi} \sin\left[u_0\left(\pi - \tau\right)\right]\sin\left[u_0\left(\pi + \tau - p\right)\right]\, d\tau. \tag{A.93}$$

Applying product-to-sum trigonometric identities to Equation (A.93),

$$R_{g4}(p) = \frac{1}{2}\int_{p-\pi}^{\pi} \left[\cos\left[u_0\left(2\tau - p\right)\right] - \cos\left[u_0\left(2\pi - p\right)\right]\right]\, d\tau. \tag{A.94}$$

Performing the integrals in Equation (A.94),

$$R_{g4}(p) = \frac{1}{2}\left[\frac{1}{2u_0}\sin\left[u_0\left(2\tau - p\right)\right] - \tau\cos\left[u_0\left(2\pi - p\right)\right]\right]\Bigg|_{p-\pi}^{\pi}. \tag{A.95}$$

Evaluating the limits of integration in Equation (A.95) and simplifying,

$$R_{g4}(p) = \frac{1}{2}\left[\frac{1}{u_0}\sin\left[u_0\left(2\pi - p\right)\right] - \left(2\pi - p\right)\cos\left[u_0\left(2\pi - p\right)\right]\right]. \tag{A.96}$$

Equations (A.82), (A.87), (A.92), and (A.96) can be combined with Equation (A.78) to yield the autocorrelation function $R_g(p)$ for the generalized dipole distribution. Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{A.97}$$

Equation (A.97) can be expressed as,

$$R_f(p) = A_m^2 \begin{cases} R_{f1}(p), & -2\pi \leq p \leq -\pi \\ R_{f2}(p), & -\pi \leq p \leq 0 \\ R_{f3}(p), & 0 \leq p \leq \pi \\ R_{f4}(p), & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.98}$$

Applying Equation (A.97) to $R_{g1}(p)$, given previously in Equation (A.82),

$$R_{f1}(p) = \frac{1}{2}\left[\frac{d^2}{dp^2} + u_0^2\right]\left[\frac{1}{u_0}\sin\left[u_0\left(2\pi + p\right)\right] - \left(2\pi + p\right)\cos\left[u_0\left(2\pi + p\right)\right]\right]. \tag{A.99}$$

Distributing terms in Equation (A.99) and performing the first derivative,

$$\begin{aligned} R_{f1}(p) = \frac{1}{2}\Bigg\{ & \frac{d}{dp}\left[\cos\left[u_0\left(2\pi + p\right)\right] - \cos\left[u_0\left(2\pi + p\right)\right] + u_0\left(2\pi + p\right)\sin\left[u_0\left(2\pi + p\right)\right]\right] \\ & + u_0^2\left[\frac{1}{u_0}\sin\left[u_0\left(2\pi + p\right)\right] - \left(2\pi + p\right)\cos\left[u_0\left(2\pi + p\right)\right]\right]\Bigg\}. \end{aligned} \tag{A.100}$$

Canceling terms in Equation (A.100) and performing the second derivative,

$$R_{f1}(p) = \frac{1}{2}\Bigg\{ \left[u_0 \sin\left[u_0\left(2\pi + p\right)\right] + u_0^2 \left(2\pi + p\right) \cos\left[u_0\left(2\pi + p\right)\right]\right] + u_0^2 \left[\frac{1}{u_0} \sin\left[u_0\left(2\pi + p\right)\right] - \left(2\pi + p\right) \cos\left[u_0\left(2\pi + p\right)\right]\right] \Bigg\}. \quad \text{(A.101)}$$

Collecting and canceling terms in Equation (A.101),

$$R_{f1}(p) = u_0 \sin\left[u_0\left(2\pi + p\right)\right]. \quad \text{(A.102)}$$

Applying Equation (A.97) to $R_{g2}(p)$, given previously in Equation (A.87),

$$R_{f2}(p) = \frac{1}{2}\left[\frac{d^2}{dp^2} + u_0^2\right]\Bigg[2\left(\pi + p\right)\cos\left(u_0 p\right) - \frac{2}{u_0}\sin\left(u_0 p\right) - \frac{1}{u_0}\sin\left[u_0\left(2\pi + p\right)\right] + p\cos\left[u_0\left(2\pi + p\right)\right]\Bigg]. \quad \text{(A.103)}$$

Distributing terms in Equation (A.99) and performing the first derivative,

$$\begin{aligned} R_{f2}(p) = \frac{1}{2}\Bigg\{ & \frac{d}{dp}\Bigg[2\cos\left(u_0 p\right) - 2u_0\left(\pi + p\right)\sin\left(u_0 p\right) - 2\cos\left(u_0 p\right) \\ & - \cos\left[u_0\left(2\pi + p\right)\right] + \cos\left[u_0\left(2\pi + p\right)\right] - u_0 p \sin\left[u_0\left(2\pi + p\right)\right]\Bigg] \\ & + u_0^2\Bigg[2\left(\pi + p\right)\cos\left(u_0 p\right) - \frac{2}{u_0}\sin\left(u_0 p\right) \\ & - \frac{1}{u_0}\sin\left[u_0\left(2\pi + p\right)\right] + p\cos\left[u_0\left(2\pi + p\right)\right]\Bigg]\Bigg\}. \end{aligned} \quad \text{(A.104)}$$

Combining terms in Equation (A.104) and performing the second derivative,

$$R_{f2}(p) = \frac{1}{2}\Bigg\{\Bigg[-2u_0 \sin(u_0 p) - 2u_0^2(\pi + p)\cos(u_0 p) - u_0 \sin[u_0(2\pi + p)] - u_0^2 p \cos[u_0(2\pi + p)]\Bigg] + u_0^2\Bigg[2(\pi + p)\cos(u_0 p) - \frac{2}{u_0}\sin(u_0 p) - \frac{1}{u_0}\sin[u_0(2\pi + p)] + p\cos[u_0(2\pi + p)]\Bigg]\Bigg\}. \tag{A.105}$$

Collecting and canceling terms in Equation (A.105),

$$R_{f2}(p) = -2u_0 \sin(u_0 p) - u_0 \sin[u_0(2\pi + p)]. \tag{A.106}$$

Applying Equation (A.97) to $R_{g3}(p)$, given previously in Equation (A.92),

$$R_{f3}(p) = \frac{1}{2}\left[\frac{d^2}{dp^2} + u_0^2\right]\Bigg[2(\pi - p)\cos(u_0 p) + \frac{2}{u_0}\sin(u_0 p) - \frac{1}{u_0}\sin[u_0(2\pi - p)] - p\cos[u_0(2\pi - p)]\Bigg]. \tag{A.107}$$

Distributing terms in Equation (A.107) and performing the first derivative,

$$R_{f3}(p) = \frac{1}{2}\Bigg\{\frac{d}{dp}\Bigg[-2\cos(u_0 p) - 2u_0(\pi - p)\sin(u_0 p) + 2\cos(u_0 p) + \cos[u_0(2\pi - p)] - \cos[u_0(2\pi - p)] - u_0 p \sin[u_0(2\pi - p)]\Bigg] + u_0^2\Bigg[2(\pi - p)\cos(u_0 p) + \frac{2}{u_0}\sin(u_0 p) - \frac{1}{u_0}\sin[u_0(2\pi - p)] - p\cos[u_0(2\pi - p)]\Bigg]\Bigg\}. \tag{A.108}$$

Combining terms in Equation (A.108) and performing the second derivative,

$$R_{f3}(p) = \frac{1}{2}\Bigg\{\Bigg[2u_0 \sin(u_0 p) - 2u_0^2(\pi - p)\cos(u_0 p)$$
$$- u_0 \sin[u_0(2\pi - p)] + u_0^2 p \cos[u_0(2\pi - p)]\Bigg]$$
$$+ u_0^2\Bigg[2(\pi - p)\cos(u_0 p) + \frac{2}{u_0}\sin(u_0 p)$$
$$- \frac{1}{u_0}\sin[u_0(2\pi - p)] - p\cos[u_0(2\pi - p)]\Bigg]\Bigg\}. \tag{A.109}$$

Collecting and canceling terms in Equation (A.109),

$$R_{f3}(p) = 2u_0 \sin(u_0 p) - u_0 \sin[u_0(2\pi - p)]\,. \tag{A.110}$$

Applying Equation (A.97) to $R_{g4}(p)$, given previously in Equation (A.96),

$$R_{f4}(p) = \frac{1}{2}\left[\frac{d^2}{dp^2} + u_0^2\right]\left[\frac{1}{u_0}\sin[u_0(2\pi - p)] - (2\pi - p)\cos[u_0(2\pi - p)]\right]. \tag{A.111}$$

Distributing terms in Equation (A.111) and performing the first derivative,

$$R_{f4}(p) = \frac{1}{2}\Bigg\{\frac{d}{dp}\left[-\cos[u_0(2\pi - p)] + \cos[u_0(2\pi - p)] - u_0(2\pi - p)\sin[u_0(2\pi - p)]\right]$$
$$+ u_0^2\left[\frac{1}{u_0}\sin[u_0(2\pi - p)] - (2\pi - p)\cos[u_0(2\pi - p)]\right]\Bigg\}. \tag{A.112}$$

Canceling terms in Equation (A.112) and performing the second derivative,

$$R_{f4}(p) = \frac{1}{2}\Bigg\{\left[u_0 \sin[u_0(2\pi - p)] + u_0^2(2\pi - p)\cos[u_0(2\pi - p)]\right]$$
$$+ u_0^2\left[\frac{1}{u_0}\sin[u_0(2\pi - p)] - (2\pi - p)\cos[u_0(2\pi - p)]\right]\Bigg\}. \tag{A.113}$$

Collecting and canceling terms in Equation (A.113),

$$R_{f4}(p) = u_0 \sin[u_0(2\pi - p)]\,. \tag{A.114}$$

Equations (A.102), (A.106), (A.110), and (A.114) can be combined with Equation (A.98) to yield the autocorrelation function $R_f(p)$ for the generalized dipole distribution. Recalling the definition for $R_h(0)$,

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau. \tag{A.115}$$

Substituting Equation (A.98), along with Equations (A.102), (A.106), (A.110), and (A.114, into Equation (A.115),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & u_0 \int_{-2\pi}^{-\pi} \sin\left[u_0\left(2\pi+\tau\right)\right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ & - u_0 \int_{-\pi}^{0} \left[2\sin(u_0\tau) + \sin\left[u_0\left(2\pi+\tau\right)\right]\right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ & + u_0 \int_{0}^{\pi} \left[2\sin(u_0\tau) - \sin\left[u_0\left(2\pi-\tau\right)\right]\right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ & + u_0 \int_{\pi}^{2\pi} \sin\left[u_0\left(2\pi-\tau\right)\right] \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \Bigg\}. \end{aligned} \tag{A.116}$$

Recognizing even and odd functions in Equation (A.116) and combining terms,

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & 2\cancel{u_0} \int_{0}^{\pi} \left[2\sin(u_0\tau) - \sin\left[u_0\left(2\pi-\tau\right)\right]\right] \frac{\sin(u_0\tau)}{\cancel{u_0}\tau}\, d\tau \\ & + 2\cancel{u_0} \int_{\pi}^{2\pi} \sin\left[u_0\left(2\pi-\tau\right)\right] \frac{\sin(u_0\tau)}{\cancel{u_0}\tau}\, d\tau \Bigg\}. \end{aligned} \tag{A.117}$$

Separating integrals in Equation (A.117) and combining terms,

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & 4 \int_{0}^{\pi} \sin(u_0\tau) \frac{\sin(u_0\tau)}{\tau}\, d\tau \\ & - 4 \int_{0}^{\pi} \sin\left[u_0\left(2\pi-\tau\right)\right] \frac{\sin(u_0\tau)}{\tau}\, d\tau \\ & + 2 \int_{0}^{2\pi} \sin\left[u_0\left(2\pi-\tau\right)\right] \frac{\sin(u_0\tau)}{\tau}\, d\tau \Bigg\}. \end{aligned} \tag{A.118}$$

Applying angle difference trigonometric identities to Equation (A.118),

$$R_h(0) = \frac{A_m^2 u_0}{\pi}\left\{4\int_0^{\pi} \sin(u_0\tau)\,\frac{\sin(u_0\tau)}{\tau}\,d\tau \right.$$
$$-4\int_0^{\pi}\left[\sin(2\pi u_0)\cos(u_0\tau)-\cos(2\pi u_0)\sin(u_0\tau)\right]\frac{\sin(u_0\tau)}{\tau}\,d\tau$$
$$\left.+2\int_0^{2\pi}\left[\sin(2\pi u_0)\cos(u_0\tau)-\cos(2\pi u_0)\sin(u_0\tau)\right]\frac{\sin(u_0\tau)}{\tau}\,d\tau\right\}. \quad \text{(A.119)}$$

Separating integrals in Equation (A.119),

$$R_h(0) = \frac{A_m^2 u_0}{\pi}\left\{4\int_0^{\pi} \sin(u_0\tau)\,\frac{\sin(u_0\tau)}{\tau}\,d\tau \right.$$
$$-4\sin(2\pi u_0)\int_0^{\pi}\cos(u_0\tau)\,\frac{\sin(u_0\tau)}{\tau}\,d\tau$$
$$+4\cos(2\pi u_0)\int_0^{\pi}\sin(u_0\tau)\,\frac{\sin(u_0\tau)}{\tau}\,d\tau$$
$$+2\sin(2\pi u_0)\int_0^{2\pi}\cos(u_0\tau)\,\frac{\sin(u_0\tau)}{\tau}\,d\tau$$
$$\left.-2\cos(2\pi u_0)\int_0^{2\pi}\sin(u_0\tau)\,\frac{\sin(u_0\tau)}{\tau}\,d\tau\right\}. \quad \text{(A.120)}$$

Applying product-to-sum trigonometric identities to Equation (A.120),

$$R_h(0) = \frac{A_m^2 u_0}{\pi}\left\{2\int_0^{\pi}\frac{1-\cos(2u_0\tau)}{\tau}\,d\tau \right.$$
$$-2\sin(2\pi u_0)\int_0^{\pi}\frac{\sin(2u_0\tau)}{\tau}\,d\tau$$
$$+2\cos(2\pi u_0)\int_0^{\pi}\frac{1-\cos(2u_0\tau)}{\tau}\,d\tau$$
$$+\sin(2\pi u_0)\int_0^{2\pi}\frac{\sin(2u_0\tau)}{\tau}\,d\tau$$
$$\left.-\cos(2\pi u_0)\int_0^{2\pi}\frac{1-\cos(2u_0\tau)}{\tau}\,d\tau\right\}. \quad \text{(A.121)}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^{b}\frac{\sin(at)}{t}\,dt \quad \text{(A.122)}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos\,(at)}{t}\ dt. \tag{A.123}$$

Applying Equations (A.122) and (A.123) to Equation (A.121),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi}\Big\{ & 2\mathrm{Cin}\,(2\pi u_0) \\ & - 2\sin\,(2\pi u_0)\,\mathrm{Si}\,(2\pi u_0) \\ & + 2\cos\,(2\pi u_0)\,\mathrm{Cin}\,(2\pi u_0) \\ & + \sin\,(2\pi u_0)\,\mathrm{Si}\,(4\pi u_0) \\ & - \cos\,(2\pi u_0)\,\mathrm{Cin}\,(4\pi u_0)\Big\}. \end{aligned} \tag{A.124}$$

Rearranging Equation (A.124) yields the stationary autocorrelation function $R_h(0)$ for the generalized dipole distribution,

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi}\Big\{ & 2\mathrm{Cin}\,(2\pi u_0) - \cos\,(2\pi u_0)\,[\mathrm{Cin}\,(4\pi u_0) - 2\mathrm{Cin}\,(2\pi u_0)] \\ & + \sin\,(2\pi u_0)\,[\mathrm{Si}\,(4\pi u_0) - 2\mathrm{Si}\,(2\pi u_0)]\Big\}. \end{aligned} \tag{A.125}$$

A.5 Uniform Distribution

The uniform current distribution is given by

$$g(p) = A_m \tag{A.126}$$

on the interval $-\pi \le p \le \pi$ and $g(p) = 0$ otherwise. Recalling the autocorrelation function $R_g(p)$,

$$R_g(p) = \int_{-\infty}^{\infty} g(\tau)g(\tau - p)\ d\tau. \tag{A.127}$$

Substituting Equation (A.126) into Equation (A.127) and applying the piecewise definition of $g(p)$ yields

$$R_g(p) = \begin{cases} \displaystyle\int_{-\pi}^{p+\pi} A_m^2 \, d\tau, & -2\pi \leq p \leq 0 \\ \displaystyle\int_{p-\pi}^{\pi} A_m^2 \, d\tau, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.128}$$

Integrating Equation (A.128),

$$R_g(p) = A_m^2 \begin{cases} \tau \Big|_{-\pi}^{p+\pi}, & -2\pi \leq p \leq 0 \\ \tau \Big|_{p-\pi}^{\pi}, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.129}$$

Evaluating the limits of integration in Equation (A.129) and simplifying yields the autocorrelation function $R_g(p)$ for the uniform distribution,

$$R_g(p) = A_m^2 \begin{cases} p + 2\pi, & -2\pi \leq p \leq 0 \\ -p + 2\pi, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.130}$$

Equation (A.130) can be written in terms of ramp functions,

$$R_g(p) = A_m^2 \left[R\left(p + 2\pi\right) - 2R(p) + R\left(p - 2\pi\right) \right] . \tag{A.131}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2 \right] R_g(p). \tag{A.132}$$

Applying Equation (A.132) to (A.131),

$$R_f(p) = A_m^2 \left[\frac{d^2}{dp^2} + u_0^2 \right] \left[R\left(p + 2\pi\right) - 2R(p) + R\left(p - 2\pi\right) \right] . \tag{A.133}$$

Distributing terms in Equation (A.133) and performing the first derivative, and recognizing that the derivative of the ramp function is the Heaviside step function,

$$R_f(p) = A_m^2 \Bigg\{ \frac{d}{dp} \left[H\left(p+2\pi\right) - 2H(p) + H\left(p-2\pi\right)\right] + u_0^2 \left[R\left(p+2\pi\right) - 2R(p) + R\left(p-2\pi\right)\right] \Bigg\}. \quad \text{(A.134)}$$

Performing the second derivative in Equation (A.134), recognizing that the derivative of the Heaviside step function is the delta function,

$$R_f(p) = A_m^2 \Bigg\{ \left[\delta\left(p+2\pi\right) - 2\delta(p) + \delta\left(p-2\pi\right)\right] + u_0^2 \left[R\left(p+2\pi\right) - 2R(p) + R\left(p-2\pi\right)\right] \Bigg\}. \quad \text{(A.135)}$$

Equation (A.135) can also be written in terms of $R_g(p)$, which yields the autocorrelation function $R_f(p)$ for the uniform distribution,

$$R_f(p) = A_m^2 \left[\delta\left(p+2\pi\right) - 2\delta(p) + \delta\left(p-2\pi\right)\right] + u_0^2 R_g(p). \quad \text{(A.136)}$$

Recalling the definition for $R_h(0)$,

$$R_h(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin\left(u_0\tau\right)}{u_0\tau} \, d\tau. \quad \text{(A.137)}$$

Substituting Equation (A.136) into Equation (A.137),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & \int_{-\infty}^{\infty} \delta\left(\tau+2\pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau} \, d\tau \\ & - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left(u_0\tau\right)}{u_0\tau} \, d\tau \\ & + \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left(u_0\tau\right)}{u_0\tau} \, d\tau \\ & + u_0^2 \int_{-2\pi}^{0} \left(\tau+2\pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau} \, d\tau \\ & - u_0^2 \int_{0}^{2\pi} \left(\tau-2\pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau} \, d\tau \Bigg\}. \end{aligned} \quad \text{(A.138)}$$

Recalling the sifting property of the delta function,

$$\int_{-\infty}^{\infty} \delta\left(x-a\right) f(x)\; dx = f(a). \tag{A.139}$$

Applying Equation (A.139) to Equation (A.138),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & \frac{\sin\left(\cancel{-}2\pi u_0\right)}{\cancel{-}2\pi u_0} + \frac{\sin\left(2\pi u_0\right)}{2\pi u_0} - 2 \\ & + u_0^2 \int_{-2\pi}^{0} \left(\tau + 2\pi\right) \frac{\sin\left(u_0 \tau\right)}{u_0 \tau}\; d\tau \\ & - u_0^2 \int_{0}^{2\pi} \left(\tau - 2\pi\right) \frac{\sin\left(u_0 \tau\right)}{u_0 \tau}\; d\tau \Bigg\}. \end{aligned} \tag{A.140}$$

Recognizing even and odd functions in Equation (A.140) and combining terms,

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & \frac{2\sin\left(2\pi u_0\right)}{2\pi u_0} - 2 \\ & - 2u_0^{\cancel{2}} \int_{0}^{2\pi} \left(\tau - 2\pi\right) \frac{\sin\left(u_0 \tau\right)}{\cancel{u_0} \tau}\; d\tau \Bigg\}. \end{aligned} \tag{A.141}$$

Separating integrals in Equation (A.141),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{\pi} \Bigg\{ & \frac{2\sin\left(2\pi u_0\right)}{2\pi u_0} - 2 \\ & - 2u_0 \int_{0}^{2\pi} \sin\left(u_0 \tau\right)\; d\tau \\ & + 4\pi u_0 \int_{0}^{2\pi} \frac{\sin\left(u_0 \tau\right)}{\tau}\; d\tau \Bigg\}. \end{aligned} \tag{A.142}$$

Recalling the definition for the sine integral,

$$\mathrm{Si}\left(ba\right) = \int_{0}^{b} \frac{\sin\left(at\right)}{t}\; dt. \tag{A.143}$$

Applying Equation (A.143) to Equation (A.142) and performing the remaining integral,

$$R_h(0) = \frac{A_m^2 u_0}{\pi}\left\{\frac{2\sin(2\pi u_0)}{2\pi u_0} - 2 \right. \\ + 2\cos(u_0\tau)\Big|_0^{2\pi} \\ \left. + 4\pi u_0 \mathrm{Si}(2\pi u_0)\right\}. \tag{A.144}$$

Evaluating the limits of integration in Equation (A.144),

$$R_h(0) = \frac{A_m^2 u_0}{\pi}\left\{\frac{2\sin(2\pi u_0)}{2\pi u_0} - 2 \right. \\ + 2\left[\cos(2\pi u_0) - 1\right] \\ \left. + 4\pi u_0 \mathrm{Si}(2\pi u_0)\right\}. \tag{A.145}$$

Simplifying Equation (A.145) yields the stationary autocorrelation function $R_h(0)$ for the uniform distribution,

$$R_h(0) = \frac{2A_m^2 u_0}{\pi}\left[\frac{\sin(2\pi u_0)}{2\pi u_0} + 2\pi u_0 \mathrm{Si}(2\pi u_0) + \cos(2\pi u_0) - 2\right]. \tag{A.146}$$

A.6 Triangular Distribution

The triangular current distribution is given by

$$g(p) = A_m \begin{cases} 1 + \frac{p}{\pi}, & -2\pi \le p \le 0 \\ 1 - \frac{p}{\pi}, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{A.147}$$

Recalling the autocorrelation function $R_g(p)$,

$$R_g(p) = \int_{-\infty}^{\infty} g(\tau) g(\tau - p)\ d\tau. \tag{A.148}$$

Substituting Equation (A.147) into Equation (A.148) and applying the piecewise definition of $g(p)$ yields

$$R_g(p) = A_m^2 \begin{cases} R_{g1}(p), & -2\pi \leq p \leq -\pi \\ R_{g2}(p), & -\pi \leq p \leq 0 \\ R_{g3}(p), & 0 \leq p \leq \pi \\ R_{g4}(p), & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}, \tag{A.149}$$

where each element of the autocorrelation function can be determined by integrating Equation (A.148). The first element $R_{g1}(p)$ is defined on the interval $-2\pi \leq p \leq -\pi$,

$$R_{g1}(p) = \int_{-\pi}^{p+\pi} \left(1 + \frac{\tau}{\pi}\right)\left(1 - \frac{\tau - p}{\pi}\right) \, d\tau. \tag{A.150}$$

Distributing terms in Equation (A.150),

$$R_{g1}(p) = \int_{-\pi}^{p+\pi} \left[1 + \frac{\tau}{\pi} - \frac{\tau - p}{\pi} - \frac{\tau\left(\tau - p\right)}{\pi^2}\right] \, d\tau. \tag{A.151}$$

Collecting and canceling terms in Equation (A.151),

$$R_{g1}(p) = \int_{-\pi}^{p+\pi} \left[1 + \frac{p}{\pi} + \frac{p\tau}{\pi^2} - \frac{\tau^2}{\pi^2}\right] \, d\tau. \tag{A.152}$$

Performing the integrals in Equation (A.152),

$$R_{g1}(p) = \left[\left(1 + \frac{p}{\pi}\right)\tau + \frac{p\tau^2}{2\pi^2} - \frac{\tau^3}{3\pi^2}\right]\Bigg|_{-\pi}^{p+\pi}. \tag{A.153}$$

Evaluating the limits of integration in Equation (A.153) and simplifying,

$$R_{g1}(p) = \left[\left(1 + \frac{p}{\pi}\right)(p + 2\pi) + \frac{p\left(p + \pi\right)^2}{2\pi^2} - \frac{p}{2} - \frac{\left(p + \pi\right)^3}{3\pi^2} - \frac{\pi}{3}\right]. \tag{A.154}$$

Further simplying Equation (A.154),

$$R_{g1}(p) = \frac{1}{6\pi^2}\left(2\pi + p\right)^3. \tag{A.155}$$

The second element $R_{g2}(p)$ is defined on the interval $-\pi \leq p \leq 0$,

$$\begin{aligned} R_{g2}(p) =& \Bigg\{ \int_{-\pi}^{p} \left(1 + \frac{\tau}{\pi}\right) \left(1 + \frac{\tau - p}{\pi}\right) \, d\tau \\ &+ \int_{p}^{0} \left(1 + \frac{\tau}{\pi}\right) \left(1 - \frac{\tau - p}{\pi}\right) \, d\tau \\ &+ \int_{0}^{p+\pi} \left(1 - \frac{\tau}{\pi}\right) \left(1 - \frac{\tau - p}{\pi}\right) \, d\tau \Bigg\}. \end{aligned} \tag{A.156}$$

Distributing terms in Equation (A.156),

$$\begin{aligned} R_{g2}(p) =& \Bigg\{ \int_{-\pi}^{p} \left[1 + \frac{\tau}{\pi} + \frac{\tau - p}{\pi} + \frac{\tau\left(\tau - p\right)}{\pi^2}\right] \, d\tau \\ &+ \int_{p}^{0} \left[1 + \frac{\tau}{\pi} - \frac{\tau - p}{\pi} - \frac{\tau\left(\tau - p\right)}{\pi^2}\right] \, d\tau \\ &+ \int_{0}^{p+\pi} \left[1 - \frac{\tau}{\pi} - \frac{\tau - p}{\pi} + \frac{\tau\left(\tau - p\right)}{\pi^2}\right] \, d\tau \Bigg\}. \end{aligned} \tag{A.157}$$

Collecting and canceling terms in Equation (A.157),

$$\begin{aligned} R_{g2}(p) =& \Bigg\{ \int_{-\pi}^{p} \left[1 - \frac{p}{\pi} + \frac{2\tau}{\pi} - \frac{p\tau}{\pi^2} + \frac{\tau^2}{\pi^2}\right] \, d\tau \\ &+ \int_{p}^{0} \left[1 + \frac{p}{\pi} + \frac{p\tau}{\pi^2} - \frac{\tau^2}{\pi^2}\right] \, d\tau \\ &+ \int_{0}^{p+\pi} \left[1 + \frac{p}{\pi} - \frac{2\tau}{\pi} - \frac{p\tau}{\pi^2} + \frac{\tau^2}{\pi^2}\right] \, d\tau \Bigg\}. \end{aligned} \tag{A.158}$$

Performing the integrals in Equation (A.158),

$$\begin{aligned} R_{g2}(p) =& \Bigg\{ \left[\left(1 - \frac{p}{\pi}\right)\tau + \left(\frac{1}{\pi} - \frac{p}{2\pi^2}\right)\tau^2 + \frac{\tau^3}{3\pi^2}\right] \Bigg|_{-\pi}^{p} \\ &+ \left[\left(1 + \frac{p}{\pi}\right)\tau + \frac{p\tau^2}{2\pi^2} - \frac{\tau^3}{3\pi^2}\right] \Bigg|_{p}^{0} \\ &+ \left[\left(1 + \frac{p}{\pi}\right)\tau - \left(\frac{1}{\pi} + \frac{p}{2\pi^2}\right)\tau^2 + \frac{\tau^3}{3\pi^2}\right] \Bigg|_{0}^{p+\pi} \Bigg\}. \end{aligned} \tag{A.159}$$

Evaluating the limits of integration in Equation (A.159) and simplifying,

$$
\begin{aligned}
R_{g2}(p) =& \Bigg\{ \left[\left(1 - \frac{p}{\pi}\right)(p+\pi) + \left(\frac{1}{\pi} - \frac{p}{2\pi^2}\right)\left(p^2 - \pi^2\right) + \frac{p^3}{3\pi^2} + \frac{\pi}{3} \right] \\
&+ \left[- \left(1 + \frac{p}{\pi}\right) p - \frac{p^3}{2\pi^2} + \frac{p^3}{3\pi^2} \right] \\
&+ \left[\left(1 + \frac{p}{\pi}\right)(p+\pi) - \left(\frac{1}{\pi} + \frac{p}{2\pi^2}\right)(p+\pi)^2 + \frac{(p+\pi)^3}{3\pi^2} \right] \Bigg\}. \qquad \text{(A.160)}
\end{aligned}
$$

Further simplying Equation (A.160),

$$
R_{g2}(p) = \frac{1}{6\pi^2}\left(4\pi^3 - 6\pi p^2 - 3p^3\right). \qquad \text{(A.161)}
$$

The third element $R_{g3}(p)$ is defined on the interval $0 \leq p \leq \pi$,

$$
\begin{aligned}
R_{g3}(p) =& \Bigg\{ \int_{p-\pi}^{0} \left(1 + \frac{\tau}{\pi}\right)\left(1 + \frac{\tau - p}{\pi}\right) \, d\tau \\
&+ \int_{0}^{p} \left(1 - \frac{\tau}{\pi}\right)\left(1 + \frac{\tau - p}{\pi}\right) \, d\tau \\
&+ \int_{p}^{\pi} \left(1 - \frac{\tau}{\pi}\right)\left(1 - \frac{\tau - p}{\pi}\right) \, d\tau \Bigg\}. \qquad \text{(A.162)}
\end{aligned}
$$

Distributing terms in Equation (A.162),

$$
\begin{aligned}
R_{g3}(p) =& \Bigg\{ \int_{p-\pi}^{0} \left[1 + \frac{\tau}{\pi} + \frac{\tau - p}{\pi} + \frac{\tau(\tau - p)}{\pi^2}\right] \, d\tau \\
&+ \int_{0}^{p} \left[1 - \frac{\tau}{\pi} + \frac{\tau - p}{\pi} - \frac{\tau(\tau - p)}{\pi^2}\right] \, d\tau \\
&+ \int_{p}^{\pi} \left[1 - \frac{\tau}{\pi} - \frac{\tau - p}{\pi} + \frac{\tau(\tau - p)}{\pi^2}\right] \, d\tau \Bigg\}. \qquad \text{(A.163)}
\end{aligned}
$$

Collecting and canceling terms in Equation (A.163),

$$\begin{aligned} R_{g3}(p) = & \Bigg\{ \int_{p-\pi}^{0} \left[1 - \frac{p}{\pi} + \frac{2\tau}{\pi} - \frac{p\tau}{\pi^2} + \frac{\tau^2}{\pi^2} \right] \, d\tau \\ & + \int_{0}^{p} \left[1 - \frac{p}{\pi} + \frac{p\tau}{\pi^2} - \frac{\tau^2}{\pi^2} \right] \, d\tau \\ & + \int_{p}^{\pi} \left[1 + \frac{p}{\pi} - \frac{2\tau}{\pi} - \frac{p\tau}{\pi^2} + \frac{\tau^2}{\pi^2} \right] \, d\tau \Bigg\}. \end{aligned} \tag{A.164}$$

Performing the integrals in Equation (A.164),

$$\begin{aligned} R_{g3}(p) = & \Bigg\{ \left[\left(1 - \frac{p}{\pi} \right) \tau + \left(\frac{1}{\pi} - \frac{p}{2\pi^2} \right) \tau^2 + \frac{\tau^3}{3\pi^2} \right] \Bigg|_{p-\pi}^{0} \\ & + \left[\left(1 - \frac{p}{\pi} \right) \tau + \frac{p\tau^2}{2\pi^2} - \frac{\tau^3}{3\pi^2} \right] \Bigg|_{0}^{p} \\ & + \left[\left(1 + \frac{p}{\pi} \right) \tau - \left(\frac{1}{\pi} + \frac{p}{2\pi^2} \right) \tau^2 + \frac{\tau^3}{3\pi^2} \right] \Bigg|_{p}^{\pi} \Bigg\}. \end{aligned} \tag{A.165}$$

Evaluating the limits of integration in Equation (A.165) and simplifying,

$$\begin{aligned} R_{g3}(p) = & \Bigg\{ \left[\left(1 - \frac{p}{\pi} \right) (\pi - p) - \left(\frac{1}{\pi} - \frac{p}{2\pi^2} \right) (p - \pi)^2 - \frac{(p - \pi)^3}{3\pi^2} \right] \\ & + \left[\left(1 - \frac{p}{\pi} \right) p + \frac{p^3}{2\pi^2} - \frac{p^3}{3\pi^2} \right] \\ & + \left[\left(1 + \frac{p}{\pi} \right) (\pi - p) - \left(\frac{1}{\pi} + \frac{p}{2\pi^2} \right) \left(\pi^2 - p^2 \right) - \frac{p^3}{3\pi^2} + \frac{\pi}{3} \right] \Bigg\}. \end{aligned} \tag{A.166}$$

Further simplying Equation (A.166),

$$R_{g3}(p) = \frac{1}{6\pi^2} \left(4\pi^3 - 6\pi p^2 + 3p^3 \right). \tag{A.167}$$

The fourth element $R_{g4}(p)$ is defined on the interval $\pi \leq p \leq 2\pi$,

$$R_{g4}(p) = \int_{p-\pi}^{\pi} \left(1 - \frac{\tau}{\pi} \right) \left(1 + \frac{\tau - p}{\pi} \right) \, d\tau. \tag{A.168}$$

Distributing terms in Equation (A.168),

$$R_{g4}(p) = \int_{p-\pi}^{\pi} \left[1 - \frac{\tau}{\pi} + \frac{\tau - p}{\pi} - \frac{\tau\left(\tau - p\right)}{\pi^2}\right] \, d\tau. \tag{A.169}$$

Collecting and canceling terms in Equation (A.169),

$$R_{g4}(p) = \int_{p-\pi}^{\pi} \left[1 - \frac{p}{\pi} + \frac{p\tau}{\pi^2} - \frac{\tau^2}{\pi^2}\right] \, d\tau. \tag{A.170}$$

Performing the integrals in Equation (A.170),

$$R_{g4}(p) = \left[\left(1 - \frac{p}{\pi}\right)\tau + \frac{p\tau^2}{2\pi^2} - \frac{\tau^3}{3\pi^2}\right]\Bigg|_{p-\pi}^{\pi}. \tag{A.171}$$

Evaluating the limits of integration in Equation (A.171) and simplifying,

$$R_{g4}(p) = \left[\left(1 - \frac{p}{\pi}\right)(2\pi - p) + \frac{p}{2} - \frac{p\left(p - \pi\right)^2}{2\pi^2} + \frac{\left(p - \pi\right)^3}{3\pi^2} - \frac{\pi}{3}\right]. \tag{A.172}$$

Further simplying Equation (A.172),

$$R_{g4}(p) = \frac{1}{6\pi^2}\left(2\pi - p\right)^3. \tag{A.173}$$

Equations (A.155), (A.161), (A.167), and (A.173) can be combined with Equation (A.149) to yield the autocorrelation function $R_g(p)$ for the generalized dipole distribution. Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \left[\frac{d^2}{dp^2} + u_0^2\right] R_g(p). \tag{A.174}$$

Equation (A.174) can be expressed as,

$$R_f(p) = A_m^2 \begin{cases} R_{f1}(p), & -2\pi \le p \le -\pi \\ R_{f2}(p), & -\pi \le p \le 0 \\ R_{f3}(p), & 0 \le p \le \pi \\ R_{f4}(p), & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{A.175}$$

Applying Equation (A.174) to $R_{g1}(p)$, given previously in Equation (A.155),

$$R_{f1}(p) = \frac{1}{6\pi^2}\left[\frac{d^2}{dp^2} + u_0^2\right](2\pi + p)^3 . \tag{A.176}$$

Distributing terms in Equation (A.176) and performing the derivatives,

$$R_{f1}(p) = \frac{1}{6\pi^2}\left[u_0^2 (2\pi + p)^3 + 6(2\pi + p)\right]. \tag{A.177}$$

Applying Equation (A.174) to $R_{g2}(p)$, given previously in Equation (A.161),

$$R_{f2}(p) = \frac{1}{6\pi^2}\left[\frac{d^2}{dp^2} + u_0^2\right]\left(4\pi^3 - 6\pi p^2 - 3p^3\right). \tag{A.178}$$

Distributing terms in Equation (A.178) and performing the derivatives,

$$R_{f2}(p) = \frac{1}{6\pi^2}\left[u_0^2\left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p\right]. \tag{A.179}$$

Applying Equation (A.174) to $R_{g3}(p)$, given previously in Equation (A.167),

$$R_{f3}(p) = \frac{1}{6\pi^2}\left[\frac{d^2}{dp^2} + u_0^2\right]\left(4\pi^3 - 6\pi p^2 + 3p^3\right). \tag{A.180}$$

Distributing terms in Equation (A.180) and performing the derivatives,

$$R_{f3}(p) = \frac{1}{6\pi^2}\left[u_0^2\left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p\right]. \tag{A.181}$$

Applying Equation (A.174) to $R_{g4}(p)$, given previously in Equation (A.173),

$$R_{f4}(p) = \frac{1}{6\pi^2}\left[\frac{d^2}{dp^2} + u_0^2\right](2\pi - p)^3. \tag{A.182}$$

Distributing terms in Equation (A.181) and performing the derivatives,

$$R_{f4}(p) = \frac{1}{6\pi^2}\left[u_0^2(2\pi - p)^3 + 6(2\pi - p)\right]. \tag{A.183}$$

Equations (A.177), (A.179), (A.181), and (A.112) can be combined with Equation (A.175) to yield the autocorrelation function $R_f(p)$ for the generalized dipole distribution. Recalling the definition for $R_h(0)$,

$$R_h(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau. \tag{A.184}$$

Substituting Equation (A.175), along with Equations (A.177), (A.179), (A.181), and (A.183, into Equation (A.184),

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{6\pi^3}\Bigg\{ &\int_{-2\pi}^{-\pi}\left[u_0^2(2\pi+\tau)^3 + 6(2\pi+\tau)\right]\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ &+ \int_{-\pi}^{0}\left[u_0^2\left(4\pi^3 - 6\pi\tau^2 - 3\tau^3\right) - 12\pi - 18\tau\right]\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ &+ \int_{0}^{\pi}\left[u_0^2\left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) - 12\pi + 18\tau\right]\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ &+ \int_{\pi}^{2\pi}\left[u_0^2(2\pi-\tau)^3 + 6(2\pi-\tau)\right]\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau\Bigg\}. \end{aligned} \tag{A.185}$$

Recognizing even and odd functions in Equation (A.185) and combining terms,

$$\begin{aligned} R_h(0) = \frac{A_m^2 u_0}{6\pi^3}\Bigg\{ &2\int_{0}^{\pi}\left[u_0^2\left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) - 12\pi + 18\tau\right]\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\ &+ 2\int_{\pi}^{2\pi}\left[u_0^2(2\pi-\tau)^3 + 6(2\pi-\tau)\right]\frac{\sin(u_0\tau)}{u_0\tau}\, d\tau\Bigg\}. \end{aligned} \tag{A.186}$$

Separating integrals in Equation (A.186) and combining terms,

$$\begin{aligned} R_h(0) = \frac{A_m^2}{6\pi^3} \Bigg\{ & 2\left(8\pi^3 u_0^2 + 12\pi\right) \int_0^{2\pi} \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & - 2\left(4\pi^3 u_0^2 + 24\pi\right) \int_0^{\pi} \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & + 36 \int_0^{\pi} \sin\left(u_0\tau\right)\ d\tau \\ & - 12\pi u_0^2 \int_0^{\pi} \tau \sin\left(u_0\tau\right)\ d\tau \\ & + 6u_0^2 \int_0^{\pi} \tau^2 \sin\left(u_0\tau\right)\ d\tau \\ & - 2\left(12\pi^2 u_0^2 + 6\right) \int_{\pi}^{2\pi} \sin\left(u_0\tau\right)\ d\tau \\ & + 12\pi u_0^2 \int_{\pi}^{2\pi} \tau \sin\left(u_0\tau\right)\ d\tau \\ & - 2u_0^2 \int_{\pi}^{2\pi} \tau^2 \sin\left(u_0\tau\right)\ d\tau \Bigg\}. \end{aligned} \tag{A.187}$$

Recalling the definition for the sine cosine integral,

$$\mathrm{Si}\left(ba\right) = \int_0^b \frac{\sin\left(at\right)}{t}\ dt. \tag{A.188}$$

Applying Equation (A.188) to Equation (A.187) and performing the remaining integrals,

$$
\begin{aligned}
R_h(0) = \frac{A_m^2}{6\pi^3}\Bigg\{ & 2\left(8\pi^3 u_0^2 + 12\pi\right)\mathrm{Si}\left(2\pi u_0\right) - 2\left(4\pi^3 u_0^2 + 24\pi\right)\mathrm{Si}\left(\pi u_0\right) \\
& - \frac{36}{u_0}\cos\left(u_0\tau\right)\Bigg|_0^{\pi} \\
& - 12\pi u_0^2\left[\frac{\sin\left(u_0\tau\right) - u_0\tau\cos\left(u_0\tau\right)}{u_0^2}\right]\Bigg|_0^{\pi} \\
& + 6u_0^2\left[\frac{-u_0^2\tau^2\cos\left(u_0\tau\right) + 2u_0\tau\sin\left(u_0\tau\right) + 2\cos\left(u_0\tau\right)}{u_0^3}\right]\Bigg|_0^{\pi} \\
& + \frac{2}{u_0}\left(12\pi^2 u_0^2 + 6\right)\cos\left(u_0\tau\right)\Bigg|_{\pi}^{2\pi} \\
& + 12\pi u_0^2\left[\frac{\sin\left(u_0\tau\right) - u_0\tau\cos\left(u_0\tau\right)}{u_0^2}\right]\Bigg|_{\pi}^{2\pi} \\
& - 2u_0^2\left[\frac{-u_0^2\tau^2\cos\left(u_0\tau\right) + 2u_0\tau\sin\left(u_0\tau\right) + 2\cos\left(u_0\tau\right)}{u_0^3}\right]\Bigg|_{\pi}^{2\pi}\Bigg\}.
\end{aligned}
\tag{A.189}
$$

Evaluating the limits of integration in Equation (A.189),

$$
\begin{aligned}
R_h(0) = \frac{A_m^2}{6\pi^3}\Bigg\{ & 2\left(8\pi^3 u_0^2 + 12\pi\right)\mathrm{Si}\left(2\pi u_0\right) - 2\left(4\pi^3 u_0^2 + 24\pi\right)\mathrm{Si}\left(\pi u_0\right) \\
& - \frac{36}{u_0}\left[\cos\left(\pi u_0\right) - 1\right] \\
& - 12\pi u_0^2\left[\frac{\sin\left(\pi u_0\right) - \pi u_0\cos\left(\pi u_0\right)}{u_0^2}\right] \\
& + 6u_0^2\left[\frac{-\pi^2 u_0^2\cos\left(\pi u_0\right) + 2\pi u_0\sin\left(\pi u_0\right) + 2\cos\left(\pi u_0\right) - 2}{u_0^3}\right] \\
& + \frac{2}{u_0}\left(12\pi^2 u_0^2 + 6\right)\left[\cos\left(2\pi u_0\right) - \cos\left(\pi u_0\right)\right] \\
& + 12\pi u_0^2\left[\frac{\sin\left(2\pi u_0\right) - 2\pi u_0\cos\left(2\pi u_0\right) - \sin\left(\pi u_0\right) + \pi u_0\cos\left(\pi u_0\right)}{u_0^2}\right] \\
& - 2u_0^2\left[\frac{-4\pi^2 u_0^2\cos\left(2\pi u_0\right) + 4\pi u_0\sin\left(2\pi u_0\right) + 2\cos\left(2\pi u_0\right)}{u_0^3}\right] \\
& + 2u_0^2\left[\frac{-\pi^2 u_0^2\cos\left(\pi u_0\right) + 2\pi u_0\sin\left(\pi u_0\right) + 2\cos\left(\pi u_0\right)}{u_0^3}\right]\Bigg\}.
\end{aligned}
\tag{A.190}
$$

Collecting terms and simplifying Equation (A.190),

$$
\begin{aligned}
R_h(0) = \frac{A_m^2}{6\pi^3}\Bigg\{ & 2\left(8\pi^3 u_0^2 + 12\pi\right)\mathrm{Si}\left(2\pi u_0\right) - 2\left(4\pi^3 u_0^2 + 24\pi\right)\mathrm{Si}\left(\pi u_0\right) \\
& + \left(24\pi^2 u_0 - 24\pi^2 u_0 + 8\pi^2 u_0 + \frac{12}{u_0} - \frac{4}{u_0}\right)\cos\left(2\pi u_0\right) \\
& + \left(12\pi^2 u_0 - 24\pi^2 u_0 - 6\pi^2 u_0 + 12\pi^2 u_0 - 2\pi^2 u_0 - \frac{36}{u_0} + \frac{12}{u_0} - \frac{12}{u_0} + \frac{4}{u_0}\right)\cos\left(\pi u_0\right) \\
& + \left(12\pi - 8\pi\right)\sin\left(2\pi u_0\right) \\
& + \left(12\pi - 12\pi - 12\pi + 4\pi\right)\sin\left(\pi u_0\right) \\
& + \frac{36}{u_0} - \frac{12}{u_0}\Bigg\}.
\end{aligned}
\qquad \text{(A.191)}
$$

Simplifying Equation (A.191) yields the stationary autocorrelation function $R_h(0)$ for the triangular distribution,

$$
\begin{aligned}
R_h(0) = \frac{A_m^2}{6\pi^3 u_0}\Bigg\{ & \left(16\pi^3 u_0^3 + 24\pi u_0\right)\mathrm{Si}\left(2\pi u_0\right) - \left(8\pi^3 u_0^3 + 48\pi u_0\right)\mathrm{Si}\left(\pi u_0\right) \\
& + \left(8\pi^2 u_0^2 + 8\right)\cos\left(2\pi u_0\right) - \left(8\pi^2 u_0^2 + 32\right)\cos\left(\pi u_0\right) \\
& + 4\pi u_0 \sin\left(2\pi u_0\right) - 8\pi u_0 \sin\left(\pi u_0\right) + 24\Bigg\}.
\end{aligned}
\qquad \text{(A.192)}
$$

Appendix B

Derivations for the Beamwidth Variance of a Broadside Line Source

This appendix presents the detailed derivations of the autocorrelation functions used to determine the beamwidth variance for a broadside line source radiator. The derivations were performed for the half-wave dipole, cosine, cosine-squared, generalized dipole, triangular, and uniform distributions.

B.1 Half-Wave Dipole Distribution

The autocorrelation function $R_f(p)$ for the half-wave dipole is,

$$R_f(p) = \frac{A_m^2}{2} \begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{B.1}$$

Equation (B.1) can also be written in terms of Heaviside step functions,

$$R_f(p) = \frac{A_m^2}{2} \sin\left(\frac{p}{2}\right) \left[-H\left(p+2\pi\right) + 2H\left(p\right) - H\left(p-2\pi\right)\right]. \tag{B.2}$$

Recalling the definition for $R_n(p)$,

$$R_n(p) = -\frac{d^2 R_f(p)}{dp^2}. \tag{B.3}$$

Finding the first derivative of Equation (B.2),

$$\frac{dR_f(p)}{dp} = \frac{A_m^2}{2}\frac{d}{dp}\left\{\sin\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right]\right\}, \tag{B.4}$$

which can be shown to be

$$\begin{aligned}\frac{dR_f(p)}{dp} = \frac{A_m^2}{2}\Bigg\{&\frac{1}{2}\cos\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right]\\ &+\sin\left(\frac{p}{2}\right)\left[-\delta\left(p+2\pi\right)+2\delta\left(p\right)-\delta\left(p-2\pi\right)\right]\Bigg\}.\end{aligned} \tag{B.5}$$

Recognizing that the sine function is identically zero when evaluated at $p = \{-2\pi, 0, 2\pi\}$ enables Equation (B.5) to be simplifed to,

$$\frac{dR_f(p)}{dp} = \frac{A_m^2}{4}\cos\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right]. \tag{B.6}$$

The second derivative of Equation (B.2) can be found by taking the derivative of Equation (B.6),

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{4}\frac{d}{dp}\left\{\cos\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right]\right\}, \tag{B.7}$$

which can be shown to be

$$\begin{aligned}\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{4}\Bigg\{&-\frac{1}{2}\sin\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right]\\ &+\cos\left(\frac{p}{2}\right)\left[-\delta\left(p+2\pi\right)+2\delta\left(p\right)-\delta\left(p-2\pi\right)\right]\Bigg\}.\end{aligned} \tag{B.8}$$

The cosine function in Equation (B.8) is one when evaluated at $p = 0$ and negative one when evaluated at $p = \{-2\pi, 2\pi\}$. Therefore, Equation (B.8) can be simplified to

$$\begin{aligned}\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{4}\Bigg\{&-\frac{1}{2}\sin\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right]\\ &+\delta\left(p+2\pi\right)+2\delta\left(p\right)+\delta\left(p-2\pi\right)\Bigg\}.\end{aligned} \tag{B.9}$$

Substituting Equation (B.9) into Equation (B.3) yields

$$R_n(p) = \frac{A_m^2}{4}\Bigg\{\frac{1}{2}\sin\left(\frac{p}{2}\right)\left[-H\left(p+2\pi\right)+2H\left(p\right)-H\left(p-2\pi\right)\right] \\ -\delta\left(p+2\pi\right)-2\delta\left(p\right)-\delta\left(p-2\pi\right)\Bigg\}. \tag{B.10}$$

Recalling the definition for $R_m(0)$,

$$R_m(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau. \tag{B.11}$$

Substituting Equation (B.10) into Equation (B.11),

$$R_m(0) = \frac{A_m^2 u_0}{4\pi}\Bigg\{-\frac{1}{2}\int_{-2\pi}^{0}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau \\ +\frac{1}{2}\int_{0}^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau \\ -\int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau \\ -2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau \\ -\int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\,d\tau\Bigg\}. \tag{B.12}$$

Recognizing even and odd functions in Equation (B.12), substituting $u_0 = \frac{1}{2}$, and combining terms,

$$R_m(0) = \frac{A_m^2 \cancel{u_0}}{4\pi}\Bigg\{\int_{0}^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left(\frac{\tau}{2}\right)}{\cancel{u_0}\tau}\,d\tau \\ -\int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left(\frac{\tau}{2}\right)}{\cancel{u_0}\tau}\,d\tau \\ -2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left(\frac{\tau}{2}\right)}{\cancel{u_0}\tau}\,d\tau \\ -\int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left(\frac{\tau}{2}\right)}{\cancel{u_0}\tau}\,d\tau\Bigg\}. \tag{B.13}$$

Recalling the sifting property of the delta function,

$$\int_{-\infty}^{\infty} \delta\left(x - a\right) f(x)\ dx = f(a). \tag{B.14}$$

Applying Equation (B.14) to Equation (B.13),

$$R_m(0) = \frac{A_m^2 \cancel{\nu_0}}{4\pi}\left\{ \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left(\frac{\tau}{2}\right)}{\cancel{\nu_0}\tau}\ d\tau - \frac{\sin\left(\frac{\cancel{-}2\pi}{\cancel{2}}\right)}{\cancel{-}2\pi} - \frac{\sin\left(\frac{\cancel{2}\pi}{\cancel{2}}\right)}{2\pi} - 1 \right\}. \tag{B.15}$$

Simplifying Equation (B.15) and applying product-to-sum trigonometric identities,

$$R_m(0) = \frac{A_m^2}{4\pi}\left\{\frac{1}{2}\int_0^{2\pi} \frac{1 - \cos\left(\tau\right)}{\tau}\ d\tau - 1\right\}. \tag{B.16}$$

Recalling the definition for the modified cosine integral,

$$\mathrm{Cin}\left(ba\right) = \int_0^{b} \frac{1 - \cos\left(at\right)}{t}\ dt. \tag{B.17}$$

Substituting Equation (B.17) into Equation (B.16),

$$R_m(0) = \frac{A_m^2}{4\pi}\left[\frac{1}{2}\mathrm{Cin}\left(2\pi\right) - 1\right]. \tag{B.18}$$

Rearranging Equation (B.18) yields the stationary autocorrelation function $R_m(0)$ for the half-wave dipole,

$$R_m(0) = \frac{A_m^2}{8\pi}\left[\mathrm{Cin}\left(2\pi\right) - 2\right]. \tag{B.19}$$

B.2 Cosine Distribution

The autocorrelation function $R_f(p)$ for the cosine distribution is,

$$R_f(p) = A_m^2 \begin{cases} -\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{B.20}$$

Equation (B.1) can also be written in terms of Heaviside step functions,

$$\begin{aligned} R_f(p) = A_m^2 \Bigg\{ & \left[-\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p+2\pi\right)-H\left(p\right)\right] \\ & + \left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right)-H\left(p-2\pi\right)\right]\Bigg\}. \end{aligned} \tag{B.21}$$

Recalling the definition for $R_n(p)$,

$$R_n(p) = -\frac{d^2 R_f(p)}{dp^2}. \tag{B.22}$$

Finding the first derivative of Equation (B.21),

$$\begin{aligned} \frac{dR_f(p)}{dp} = A_m^2 \frac{d}{dp} \Bigg\{ & \left[-\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p+2\pi\right)-H\left(p\right)\right] \\ & + \left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right)-H\left(p-2\pi\right)\right]\Bigg\}, \end{aligned} \tag{B.23}$$

which can be shown to be

$$\begin{aligned}\frac{dR_f(p)}{dp} = A_m^2 \Bigg\{ &\left[-\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ &+ \left[-\left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[\delta\left(p + 2\pi\right) - \delta\left(p\right)\right] \\ &+ \left[\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \\ &+ \left[\left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[\delta\left(p\right) - \delta\left(p - 2\pi\right)\right] \Bigg\}.\end{aligned} \tag{B.24}$$

Evaluating the products including delta functions reveals that for $p = \{-2\pi, 0, 2\pi\}$, terms with sine functions are zero, and terms with cosine functions cancel for $p = 0$ and are zero for $p = \{-2\pi, 2\pi\}$. Therefore, Equation (B.24) can be simplified to

$$\begin{aligned}\frac{dR_f(p)}{dp} = A_m^2 \Bigg\{ &\left[-\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ &+ \left[\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \Bigg\}.\end{aligned} \tag{B.25}$$

The second derivative of Equation (B.21) can be found by taking the derivative of Equation (B.25),

$$\begin{aligned}\frac{d^2R_f(p)}{dp^2} = A_m^2 \frac{d}{dp}\Bigg\{ &\left[-\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ &+ \left[\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \Bigg\},\end{aligned} \tag{B.26}$$

which can be shown to be

$$\begin{aligned}\frac{d^2R_f(p)}{dp^2} = A_m^2 \Bigg\{ &\left[\frac{1}{4}\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) + \frac{1}{4}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ &+ \left[-\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[\delta\left(p + 2\pi\right) - \delta\left(p\right)\right] \\ &+ \left[-\frac{1}{4}\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) + \frac{1}{4}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \\ &+ \left[\frac{1}{4}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\sin\left(\frac{p}{2}\right)\right]\left[\delta\left(p\right) - \delta\left(p - 2\pi\right)\right] \Bigg\}.\end{aligned} \tag{B.27}$$

As before, evaluating the products including delta functions reveals that for $p = \{-2\pi, 0, 2\pi\}$ terms with sine functions are zero and terms with cosine functions can be simplified. Therefore, Equation (B.27) can be simplified to

$$\begin{aligned}\frac{d^2 R_f(p)}{dp^2} = \frac{A_m^2}{4}\Bigg\{ & \left[\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) + \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ & + \left[-\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) + \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \\ & + \delta\left(p + 2\pi\right) + 2\delta\left(p\right) + \delta\left(p - 2\pi\right)\Bigg\}. \end{aligned} \tag{B.28}$$

Equation (B.28) can be substituted into Equation (B.22),

$$\begin{aligned}R_n(p) = \frac{A_m^2}{4}\Bigg\{ & \left[-\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ & + \left[\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right] \\ & - \delta\left(p + 2\pi\right) - 2\delta\left(p\right) - \delta\left(p - 2\pi\right)\Bigg\}. \end{aligned} \tag{B.29}$$

Recalling the definition for $R_m(0)$,

$$R_m(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau. \tag{B.30}$$

Substituting Equation (B.29) into Equation (B.30),

$$\begin{aligned}R_m(0) = \frac{A_m^2 u_0}{4\pi}\Bigg\{ & \int_{-2\pi}^{0}\left[-\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + \tau}{2}\right)\cos\left(\frac{\tau}{2}\right)\right]\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\ & + \int_{0}^{2\pi}\left[\left(\frac{3}{4} - u_0^2\right)\sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - \tau}{2}\right)\cos\left(\frac{\tau}{2}\right)\right]\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\ & - \int_{-\infty}^{\infty}\delta\left(\tau + 2\pi\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\ & - 2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\ & - \int_{-\infty}^{\infty}\delta\left(\tau - 2\pi\right)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau\Bigg\}. \end{aligned} \tag{B.31}$$

Recognizing even and odd functions in Equation (B.31) and combining terms,

$$R_m(0) = \frac{A_m^2 \cancel{u_0}}{4\pi}\Bigg\{2\int_0^{2\pi}\left[\left(\frac{3}{4}-u_0^2\right)\sin\left(\frac{\tau}{2}\right)-\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-\tau}{2}\right)\cos\left(\frac{\tau}{2}\right)\right]\frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\;d\tau$$
$$-\int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\;d\tau$$
$$-2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\;d\tau$$
$$-\int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\;d\tau\Bigg\}. \tag{B.32}$$

Recalling the sifting property of the delta function,

$$\int_{-\infty}^{\infty}\delta\left(x-a\right)f(x)\;dx = f(a). \tag{B.33}$$

Applying Equation (B.33) to Equation (B.32) and separating integrals,

$$R_m(0) = \frac{A_m^2}{4\pi}\Bigg\{2\left(\frac{3}{4}-u_0^2\right)\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left(u_0\tau\right)}{\tau}\;d\tau$$
$$-2\pi\left(\frac{1}{4}-u_0^2\right)\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left(u_0\tau\right)}{\tau}\;d\tau$$
$$+\left(\frac{1}{4}-u_0^2\right)\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left(u_0\tau\right)\;d\tau$$
$$-\frac{\sin\left(\cancel{-}2\pi u_0\right)}{\cancel{-}2\pi}-\frac{\sin\left(2\pi u_0\right)}{2\pi}-2u_0\Bigg\}. \tag{B.34}$$

Simplifying Equation (B.15) and applying product-to-sum trigonometric identities,

$$R_m(0) = \frac{A_m^2}{4\pi}\Bigg\{\left(\frac{3}{4}-u_0^2\right)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\frac{1}{2}\right)\tau\right]-1+\cos\left[\left(u_0-\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau$$
$$-\pi\left(\frac{1}{4}-u_0^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau$$
$$+\frac{1}{2}\left(\frac{1}{4}-u_0^2\right)\int_0^{2\pi}\left[\sin\left[\left(u_0+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\frac{1}{2}\right)\tau\right]\right]\;d\tau$$
$$-\frac{\sin\left(2\pi u_0\right)}{\pi}-2u_0\Bigg\}. \tag{B.35}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^b \frac{\sin{(at)}}{t}\;dt \tag{B.36}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos{(at)}}{t}\;dt. \tag{B.37}$$

Applying Equations (B.36) and (B.37) to Equation (B.35) and performing the remaining integrals,

$$\begin{aligned} R_m(0) = \frac{A_m^2}{4\pi}\Bigg\{ &\left(\frac{3}{4}-u_0^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &-\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &-\frac{1}{2}\left(\frac{\frac{1}{4}-u_0^2}{\frac{1}{2}+u_0}\right)\cos\left[\left(u_0+\frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi} \\ &+\frac{1}{2}\left(\frac{\frac{1}{4}-u_0^2}{\frac{1}{2}-u_0}\right)\cos\left[\left(u_0-\frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi} \\ &-\frac{\sin{(2\pi u_0)}}{\pi}-2u_0\Bigg\}. \end{aligned} \tag{B.38}$$

Evaluating the limits of integration and simplifying Equation (B.38),

$$\begin{aligned} R_m(0) = \frac{A_m^2}{4\pi}\Bigg\{ &\left(\frac{3}{4}-u_0^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &-\pi\left(\frac{1}{4}-u_0^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\frac{1}{2}\right)\right]\right] \\ &-\frac{1}{2}\left(\frac{1}{2}-u_0\right)\left[\cos\left[2\pi\left(u_0+\frac{1}{2}\right)\right]-1\right] \\ &+\frac{1}{2}\left(\frac{1}{2}+u_0\right)\left[\cos\left[2\pi\left(u_0-\frac{1}{2}\right)\right]-1\right] \\ &-\frac{\sin{(2\pi u_0)}}{\pi}-2u_0\Bigg\}. \end{aligned} \tag{B.39}$$

Simplifying Equation (B.39) yields the stationary autocorrelation function $R_m(0)$ for the cosine distribution,

$$R_m(0) = \frac{A_m^2}{4\pi}\Bigg\{ \left(\frac{3}{4} - u_0^2\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \frac{1}{2}\right)\right]\right] \\ - \pi\left(\frac{1}{4} - u_0^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \frac{1}{2}\right)\right]\right] \\ - u_0\cos\left(2\pi u_0\right) - \frac{\sin\left(2\pi u_0\right)}{\pi} - 3u_0\Bigg\}. \tag{B.40}$$

B.3 Cosine-Squared Distribution

The autocorrelation function $R_f(p)$ for the cosine-squared distribution is,

$$R_f(p) = \frac{A_m^2}{8}\begin{cases}(2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ (2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{B.41}$$

Equation (B.41) can also be written in terms of Heaviside step functions,

$$R_f(p) = \frac{A_m^2}{8}\Bigg\{ \left[(2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ + \left[(2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Bigg\}. \tag{B.42}$$

Recalling the definition for $R_n(p)$,

$$R_n(p) = -\frac{d^2R_f(p)}{dp^2}. \tag{B.43}$$

Finding the first derivative of Equation (B.42),

$$\frac{dR_f(p)}{dp} = \frac{A_m^2}{8}\frac{d}{dp}\Bigg\{ \left[(2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] \\ + \left[(2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Bigg\}, \tag{B.44}$$

which can be shown to be

$$\frac{dR_f(p)}{dp} = \frac{A_m^2}{8}\Bigg\{ \left[-\left(2\pi + p\right)\left(u_0^2 - 1\right)\sin(p) - 2u_0^2\cos(p) + 2u_0^2\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right]$$
$$+ \left[\left(2\pi + p\right)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p)\right]\left[\delta\left(p + 2\pi\right) - \delta\left(p\right)\right]$$
$$+ \left[-\left(2\pi - p\right)\left(u_0^2 - 1\right)\sin(p) + 2u_0^2\cos(p) - 2u_0^2\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]$$
$$+ \left[\left(2\pi - p\right)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p)\right]\left[\delta\left(p\right) - \delta\left(p - 2\pi\right)\right]\Bigg\}. \quad \text{(B.45)}$$

Evaluating the products including delta functions reveals that all terms are zero for $p = \{-2\pi, 2\pi\}$ and terms are zero or cancel for $p = 0$. Therefore, Equation (B.45) can be simplified to

$$\frac{dR_f(p)}{dp} = \frac{A_m^2}{8}\Bigg\{ \left[-\left(2\pi + p\right)\left(u_0^2 - 1\right)\sin(p) - 2u_0^2\cos(p) + 2u_0^2\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right]$$
$$+ \left[-\left(2\pi - p\right)\left(u_0^2 - 1\right)\sin(p) + 2u_0^2\cos(p) - 2u_0^2\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Bigg\}. \quad \text{(B.46)}$$

The second derivative of Equation (B.42) can be found by taking the derivative of Equation (B.46),

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{8}\frac{d}{dp}\Bigg\{ \left[-\left(2\pi + p\right)\left(u_0^2 - 1\right)\sin(p) - 2u_0^2\cos(p) + 2u_0^2\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right]$$
$$+ \left[-\left(2\pi - p\right)\left(u_0^2 - 1\right)\sin(p) + 2u_0^2\cos(p) - 2u_0^2\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Bigg\}, \quad \text{(B.47)}$$

which can be shown to be

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{8}\Bigg\{ \left[-\left(2\pi + p\right)\left(u_0^2 - 1\right)\cos(p) + \left(u_0^2 + 1\right)\sin(p)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right]$$
$$+ \left[-\left(2\pi + p\right)\left(u_0^2 - 1\right)\sin(p) - 2u_0^2\cos(p) + 2u_0^2\right]\left[\delta\left(p + 2\pi\right) - \delta\left(p\right)\right]$$
$$+ \left[-\left(2\pi - p\right)\left(u_0^2 - 1\right)\cos(p) - \left(u_0^2 + 1\right)\sin(p)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]$$
$$+ \left[-\left(2\pi - p\right)\left(u_0^2 - 1\right)\sin(p) + 2u_0^2\cos(p) - 2u_0^2\right]\left[\delta\left(p\right) - \delta\left(p - 2\pi\right)\right]\Bigg\}. \quad \text{(B.48)}$$

As before, evaluating the products including delta functions reveals that for $p = \{-2\pi, 0, 2\pi\}$ terms with sine functions are zero and terms with cosine functions or constants can ultimately be canceled. Therefore, Equation (B.27) can be simplified to

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{8}\Big\{ \left[-\left(2\pi + p\right)\left(u_0^2 - 1\right)\cos(p) + \left(u_0^2 + 1\right)\sin(p)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] + \left[-\left(2\pi - p\right)\left(u_0^2 - 1\right)\cos(p) - \left(u_0^2 + 1\right)\sin(p)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Big\}. \tag{B.49}$$

Equation (B.49) can be substituted into Equation (B.43),

$$R_n(p) = \frac{A_m^2}{8}\Big\{ \left[\left(2\pi + p\right)\left(u_0^2 - 1\right)\cos(p) - \left(u_0^2 + 1\right)\sin(p)\right]\left[H\left(p + 2\pi\right) - H\left(p\right)\right] + \left[\left(2\pi - p\right)\left(u_0^2 - 1\right)\cos(p) + \left(u_0^2 + 1\right)\sin(p)\right]\left[H\left(p\right) - H\left(p - 2\pi\right)\right]\Big\}. \tag{B.50}$$

Recalling the definition for $R_m(0)$,

$$R_m(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau. \tag{B.51}$$

Substituting Equation (B.50) into Equation (B.51),

$$R_m(0) = \frac{A_m^2 u_0}{8\pi}\Big\{ \int_{-2\pi}^{0} \left[\left(2\pi + \tau\right)\left(u_0^2 - 1\right)\cos(\tau) - \left(u_0^2 + 1\right)\sin(\tau)\right]\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau + \int_{0}^{2\pi} \left[\left(2\pi - \tau\right)\left(u_0^2 - 1\right)\cos(\tau) + \left(u_0^2 + 1\right)\sin(\tau)\right]\frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau\Big\}. \tag{B.52}$$

Recognizing even and odd functions in Equation (B.52) and combining terms,

$$R_m(0) = \frac{A_m^2 \cancel{u_0}}{8\pi}\Big\{2\int_{0}^{2\pi} \left[\left(2\pi - \tau\right)\left(u_0^2 - 1\right)\cos(\tau) + \left(u_0^2 + 1\right)\sin(\tau)\right]\frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau\Big\}. \tag{B.53}$$

Separating integrals in Equation (B.32),

$$R_m(0) = \frac{A_m^2}{8\pi}\Big\{4\pi\left(u_0^2 - 1\right)\int_{0}^{2\pi}\cos(\tau)\frac{\sin\left(u_0\tau\right)}{\tau}\, d\tau - 2\left(u_0^2 - 1\right)\int_{0}^{2\pi}\cos(\tau)\sin\left(u_0\tau\right)\, d\tau + 2\left(u_0^2 + 1\right)\int_{0}^{2\pi}\sin(\tau)\frac{\sin\left(u_0\tau\right)}{\tau}\, d\tau\Big\}. \tag{B.54}$$

Applying product-to-sum trigonometric identities to Equation (B.54),

$$R_m(0) = \frac{A_m^2}{8\pi}\Bigg\{2\pi\left(u_0^2-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+1\right)\tau\right]+\sin\left[\left(u_0-1\right)\tau\right]}{\tau}\,d\tau$$
$$-\left(u_0^2-1\right)\int_0^{2\pi}\left[\sin\left[\left(u_0+1\right)\tau\right]+\sin\left[\left(u_0-1\right)\tau\right]\right]\,d\tau$$
$$+\left(u_0^2+1\right)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+1\right)\tau\right]-1+\cos\left[\left(u_0-1\right)\tau\right]}{\tau}\,d\tau\Bigg\}. \quad \text{(B.55)}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\left(ba\right) = \int_0^b \frac{\sin\left(at\right)}{t}\,dt \quad \text{(B.56)}$$

and

$$\mathrm{Cin}\left(ba\right) = \int_0^b \frac{1-\cos\left(at\right)}{t}\,dt. \quad \text{(B.57)}$$

Applying Equations (B.56) and (B.57) to Equation (B.55) and performing the remaining integrals,

$$R_m(0) = \frac{A_m^2}{8\pi}\Bigg\{2\pi\left(u_0^2-1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+1\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-1\right)\right]\right]$$
$$+\left(u_0^2+1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-1\right)\right]\right]$$
$$+\left(\frac{u_0^2-1}{u_0+1}\right)\cos\left[\left(u_0+1\right)\tau\right]\Big|_0^{2\pi}$$
$$+\left(\frac{u_0^2-1}{u_0-1}\right)\cos\left[\left(u_0-1\right)\tau\right]\Big|_0^{2\pi}\Bigg\}. \quad \text{(B.58)}$$

Evaluating the limits of integration and simplifying Equation (B.58),

$$R_m(0) = \frac{A_m^2}{8\pi}\Bigg\{2\pi\left(u_0^2-1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+1\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-1\right)\right]\right]$$
$$+\left(u_0^2+1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-1\right)\right]\right]$$
$$+\left(u_0-1\right)\left[\cos\left[2\pi\left(u_0+1\right)\right]-1\right]$$
$$+\left(u_0+1\right)\left[\cos\left[2\pi\left(u_0-1\right)\right]-1\right]\Bigg\}. \quad \text{(B.59)}$$

Simplifying Equation (B.59) yields the stationary autocorrelation function $R_m(0)$ for the cosine-squared distribution,

$$R_m(0) = \frac{A_m^2}{8\pi}\Big\{2\pi\left(u_0^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-1\right)\right]\right]$$
$$+ \left(u_0^2 + 1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-1\right)\right]\right]$$
$$+ 2u_0\left[\cos\left(2\pi u_0\right) - 1\right]\Big\}. \tag{B.60}$$

B.4 Generalized Dipole Distribution

The autocorrelation function $R_f(p)$ for the generalized dipole distribution is,

$$R_f(p) = A_m^2 u_0 \begin{cases} \sin\left[u_0\left(2\pi+p\right)\right], & -2\pi \le p \le -\pi \\ -2\sin\left(u_0 p\right) - \sin\left[u_0\left(2\pi+p\right)\right], & -\pi \le p \le 0 \\ 2\sin\left(u_0 p\right) - \sin\left[u_0\left(2\pi-p\right)\right], & 0 \le p \le \pi \\ \sin\left[u_0\left(2\pi-p\right)\right], & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{B.61}$$

Equation (B.61) can also be written in terms of Heaviside step functions,

$$R_f(p) = A_m^2 u_0\Big\{\sin\left[u_0\left(2\pi+p\right)\right]\left[H\left(p+2\pi\right) - H\left(p+\pi\right)\right]$$
$$- \left[2\sin\left(u_0 p\right) + \sin\left[u_0\left(2\pi+p\right)\right]\right]\left[H\left(p+\pi\right) - H\left(p\right)\right]$$
$$+ \left[2\sin\left(u_0 p\right) - \sin\left[u_0\left(2\pi-p\right)\right]\right]\left[H\left(p\right) - H\left(p-\pi\right)\right]$$
$$+ \sin\left[u_0\left(2\pi-p\right)\right]\left[H\left(p-\pi\right) - H\left(p-2\pi\right)\right]\Big\}. \tag{B.62}$$

Recalling the definition for $R_n(p)$,

$$R_n(p) = -\frac{d^2 R_f(p)}{dp^2}. \tag{B.63}$$

Finding the first derivative of Equation (B.62),

$$\begin{aligned}\frac{dR_f(p)}{dp} = A_m^2 u_0 \frac{d}{dp}\Bigg\{ & \sin\left[u_0\left(2\pi+p\right)\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right] \\ & -\left[2\sin\left(u_0 p\right)+\sin\left[u_0\left(2\pi+p\right)\right]\right]\left[H\left(p+\pi\right)-H\left(p\right)\right] \\ & +\left[2\sin\left(u_0 p\right)-\sin\left[u_0\left(2\pi-p\right)\right]\right]\left[H\left(p\right)-H\left(p-\pi\right)\right] \\ & +\sin\left[u_0\left(2\pi-p\right)\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]\Bigg\},\end{aligned} \tag{B.64}$$

which can be shown to be

$$\begin{aligned}\frac{dR_f(p)}{dp} = A_m^2 u_0 \Bigg\{ & u_0\cos\left[u_0\left(2\pi+p\right)\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right] \\ & +\sin\left[u_0\left(2\pi+p\right)\right]\left[\delta\left(p+2\pi\right)-\delta\left(p+\pi\right)\right] \\ & -\left[2u_0\cos\left(u_0 p\right)+u_0\cos\left[u_0\left(2\pi+p\right)\right]\right]\left[H\left(p+\pi\right)-H\left(p\right)\right] \\ & -\left[2\sin\left(u_0 p\right)+\sin\left[u_0\left(2\pi+p\right)\right]\right]\left[\delta\left(p+\pi\right)-\delta\left(p\right)\right] \\ & +\left[2u_0\cos\left(u_0 p\right)+u_0\cos\left[u_0\left(2\pi-p\right)\right]\right]\left[H\left(p\right)-H\left(p-\pi\right)\right] \\ & +\left[2\sin\left(u_0 p\right)-\sin\left[u_0\left(2\pi-p\right)\right]\right]\left[\delta\left(p\right)-\delta\left(p-\pi\right)\right] \\ & -u_0\cos\left[u_0\left(2\pi-p\right)\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right] \\ & +\sin\left[u_0\left(2\pi-p\right)\right]\left[\delta\left(p-\pi\right)-\delta\left(p-2\pi\right)\right]\Bigg\}.\end{aligned} \tag{B.65}$$

Evaluating the products including delta functions reveals that all terms are zero or cancel for $p = \{-2\pi, 0, 2\pi\}$ and terms cancel for $p = \{-\pi, \pi\}$. Therefore, Equation (B.65) can be simplified to

$$\begin{aligned}\frac{dR_f(p)}{dp} = A_m^2 u_0 \Bigg\{ & u_0\cos\left[u_0\left(2\pi+p\right)\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right] \\ & -\left[2u_0\cos\left(u_0 p\right)+u_0\cos\left[u_0\left(2\pi+p\right)\right]\right]\left[H\left(p+\pi\right)-H\left(p\right)\right] \\ & +\left[2u_0\cos\left(u_0 p\right)+u_0\cos\left[u_0\left(2\pi-p\right)\right]\right]\left[H\left(p\right)-H\left(p-\pi\right)\right] \\ & -u_0\cos\left[u_0\left(2\pi-p\right)\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]\Bigg\}.\end{aligned} \tag{B.66}$$

The second derivative of Equation (B.62) can be found by taking the derivative of Equation (B.66),

$$\begin{aligned}\frac{d^2R_f(p)}{dp^2} = A_m^2 u_0 \frac{d}{dp}\Bigg\{ & u_0 \cos\left[u_0\left(2\pi + p\right)\right]\left[H\left(p + 2\pi\right) - H\left(p + \pi\right)\right] \\ & - \left[2u_0 \cos\left(u_0 p\right) + u_0 \cos\left[u_0\left(2\pi + p\right)\right]\right]\left[H\left(p + \pi\right) - H\left(p\right)\right] \\ & + \left[2u_0 \cos\left(u_0 p\right) + u_0 \cos\left[u_0\left(2\pi - p\right)\right]\right]\left[H\left(p\right) - H\left(p - \pi\right)\right] \\ & - u_0 \cos\left[u_0\left(2\pi - p\right)\right]\left[H\left(p - \pi\right) - H\left(p - 2\pi\right)\right]\Bigg\}, \end{aligned} \tag{B.67}$$

which can be shown to be

$$\begin{aligned}\frac{d^2R_f(p)}{dp^2} = A_m^2 u_0 \Bigg\{ & - u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\left[H\left(p + 2\pi\right) - H\left(p + \pi\right)\right] \\ & + u_0 \cos\left[u_0\left(2\pi + p\right)\right]\left[\delta\left(p + 2\pi\right) - \delta\left(p + \pi\right)\right] \\ & + \left[2u_0^2 \sin\left(u_0 p\right) + u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\right]\left[H\left(p + \pi\right) - H\left(p\right)\right] \\ & - \left[2u_0 \cos\left(u_0 p\right) + u_0 \cos\left[u_0\left(2\pi + p\right)\right]\right]\left[\delta\left(p + \pi\right) - \delta\left(p\right)\right] \\ & - \left[2u_0^2 \sin\left(u_0 p\right) - u_0^2 \sin\left[u_0\left(2\pi - p\right)\right]\right]\left[H\left(p\right) - H\left(p - \pi\right)\right] \\ & + \left[2u_0 \cos\left(u_0 p\right) + u_0 \cos\left[u_0\left(2\pi - p\right)\right]\right]\left[\delta\left(p\right) - \delta\left(p - \pi\right)\right] \\ & - u_0^2 \sin\left[u_0\left(2\pi - p\right)\right]\left[H\left(p - \pi\right) - H\left(p - 2\pi\right)\right] \\ & - u_0 \cos\left[u_0\left(2\pi - p\right)\right]\left[\delta\left(p - \pi\right) - \delta\left(p - 2\pi\right)\right]\Bigg\}. \end{aligned} \tag{B.68}$$

As before, evaluating the products including delta functions reveals that for $p = \{-2\pi, 0, 2\pi\}$ terms with sine functions are zero and terms with cosine functions or constants can ultimately be canceled. Therefore, Equation (B.68) can be simplified to

$$\begin{aligned}\frac{d^2 R_f(p)}{dp^2} = A_m^2 u_0 \Bigg\{ & - u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\left[H\left(p + 2\pi\right) - H\left(p + \pi\right)\right] \\ & + \left[2u_0^2 \sin\left(u_0 p\right) + u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\right]\left[H\left(p + \pi\right) - H\left(p\right)\right] \\ & - \left[2u_0^2 \sin\left(u_0 p\right) - u_0^2 \sin\left[u_0\left(2\pi - p\right)\right]\right]\left[H\left(p\right) - H\left(p - \pi\right)\right] \\ & - u_0^2 \sin\left[u_0\left(2\pi - p\right)\right]\left[H\left(p - \pi\right) - H\left(p - 2\pi\right)\right] \\ & + u_0 \delta\left(p + 2\pi\right) - 4u_0 \cos\left(\pi u_0\right)\delta\left(p + \pi\right) \\ & + \left[4u_0 + 2u_0 \cos\left(2\pi u_0\right)\right]\delta\left(p\right) \\ & - 4u_0 \cos\left(\pi u_0\right)\delta\left(p - \pi\right) + u_0 \delta\left(p - 2\pi\right) \Bigg\}. \end{aligned} \tag{B.69}$$

Equation (B.69) can be substituted into Equation (B.63),

$$\begin{aligned} R_n(p) = A_m^2 u_0 \Bigg\{ & u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\left[H\left(p + 2\pi\right) - H\left(p + \pi\right)\right] \\ & - \left[2u_0^2 \sin\left(u_0 p\right) + u_0^2 \sin\left[u_0\left(2\pi + p\right)\right]\right]\left[H\left(p + \pi\right) - H\left(p\right)\right] \\ & + \left[2u_0^2 \sin\left(u_0 p\right) - u_0^2 \sin\left[u_0\left(2\pi - p\right)\right]\right]\left[H\left(p\right) - H\left(p - \pi\right)\right] \\ & + u_0^2 \sin\left[u_0\left(2\pi - p\right)\right]\left[H\left(p - \pi\right) - H\left(p - 2\pi\right)\right] \\ & - u_0 \delta\left(p + 2\pi\right) + 4u_0 \cos\left(\pi u_0\right)\delta\left(p + \pi\right) \\ & - \left[4u_0 + 2u_0 \cos\left(2\pi u_0\right)\right]\delta\left(p\right) \\ & + 4u_0 \cos\left(\pi u_0\right)\delta\left(p - \pi\right) - u_0 \delta\left(p - 2\pi\right) \Bigg\}. \end{aligned} \tag{B.70}$$

Recalling the definition for $R_m(0)$,

$$R_m(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_n(\tau) \frac{\sin\left(u_0 \tau\right)}{u_0 \tau}\, d\tau. \tag{B.71}$$

Substituting Equation (B.70) into Equation (B.71),

$$
\begin{aligned}
R_m(0) = \frac{A_m^2 u_0^2}{\pi} \Bigg\{ & u_0^2 \int_{-2\pi}^{-\pi} \sin\left[u_0\left(2\pi + \tau\right)\right] \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& - u_0^2 \int_{-\pi}^{0} \left[2\sin\left(u_0\tau\right) + \sin\left[u_0\left(2\pi + \tau\right)\right]\right] \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& + u_0^2 \int_{0}^{\pi} \left[2\sin\left(u_0\tau\right) - \sin\left[u_0\left(2\pi - \tau\right)\right]\right] \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& + u_0^2 \int_{\pi}^{2\pi} \sin\left[u_0\left(2\pi - \tau\right)\right] \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& - u_0 \int_{-\infty}^{\infty} \delta\left(\tau + 2\pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& + 4u_0 \cos\left(\pi u_0\right) \int_{-\infty}^{\infty} \delta\left(\tau + \pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& - \left[4u_0 + 2u_0\cos\left(2\pi u_0\right)\right] \int_{-\infty}^{\infty} \delta\left(\tau\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& + 4u_0 \cos\left(\pi u_0\right) \int_{-\infty}^{\infty} \delta\left(\tau - \pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& - u_0 \int_{-\infty}^{\infty} \delta\left(\tau - 2\pi\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{B.72}
$$

Recognizing even and odd functions in Equation (B.72) and combining terms,

$$
\begin{aligned}
R_m(0) = \frac{A_m^2 u_0^2}{\pi} \Bigg\{ & 2u_0^{\cancel{2}} \int_{0}^{\pi} \left[2\sin\left(u_0\tau\right) - \sin\left[u_0\left(2\pi - \tau\right)\right]\right] \frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau \\
& + 2u_0^{\cancel{2}} \int_{\pi}^{2\pi} \sin\left[u_0\left(2\pi - \tau\right)\right] \frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau \\
& - \cancel{u_0} \int_{-\infty}^{\infty} \delta\left(\tau + 2\pi\right) \frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau \\
& + 4\cancel{u_0} \cos\left(\pi u_0\right) \int_{-\infty}^{\infty} \delta\left(\tau + \pi\right) \frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau \\
& - \left[4u_0 + 2u_0\cos\left(2\pi u_0\right)\right] \int_{-\infty}^{\infty} \delta\left(\tau\right) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau \\
& + 4\cancel{u_0} \cos\left(\pi u_0\right) \int_{-\infty}^{\infty} \delta\left(\tau - \pi\right) \frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau \\
& - \cancel{u_0} \int_{-\infty}^{\infty} \delta\left(\tau - 2\pi\right) \frac{\sin\left(u_0\tau\right)}{\cancel{u_0}\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{B.73}
$$

Recalling the sifting property of the delta function,

$$\int_{-\infty}^{\infty} \delta\left(x-a\right) f(x)\ dx = f(a). \tag{B.74}$$

Applying Equation (B.74) to Equation (B.73) and separating integrals,

$$\begin{aligned} R_m(0) = \frac{A_m^2 u_0^2}{\pi} \Bigg\{ & 4u_0 \int_0^{\pi} \frac{\sin^2\left(u_0\tau\right)}{\tau}\ d\tau \\ & - 4u_0 \int_0^{\pi} \sin\left[u_0\left(2\pi-\tau\right)\right] \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & + 2u_0 \int_0^{2\pi} \sin\left[u_0\left(2\pi-\tau\right)\right] \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & - \frac{\sin\left(\cancel{-}2\pi u_0\right)}{\cancel{-}2\pi} - \frac{\sin\left(2\pi u_0\right)}{2\pi} \\ & + 4\cos\left(\pi u_0\right) \frac{\sin\left(\cancel{-}\pi u_0\right)}{\cancel{-}\pi} + 4\cos\left(\pi u_0\right) \frac{\sin\left(\pi u_0\right)}{\pi} \\ & - \left[4u_0 + 2u_0\cos\left(2\pi u_0\right)\right] \Bigg\}. \end{aligned} \tag{B.75}$$

Applying power reduction, angle difference, and product-to-sum trigonometric identities to Equation (B.75),

$$\begin{aligned} R_m(0) = \frac{A_m^2 u_0^2}{\pi} \Bigg\{ & 2u_0 \int_0^{\pi} \frac{1-\cos\left(2u_0\tau\right)}{\tau}\ d\tau \\ & - 2u_0 \sin\left(2\pi u_0\right) \int_0^{\pi} \frac{\sin\left(2u_0\tau\right)}{\tau}\ d\tau \\ & + 2u_0 \cos\left(2\pi u_0\right) \int_0^{\pi} \frac{1-\cos\left(2u_0\tau\right)}{\tau}\ d\tau \\ & + u_0 \sin\left(2\pi u_0\right) \int_0^{2\pi} \frac{\sin\left(2u_0\tau\right)}{\tau}\ d\tau \\ & - u_0 \cos\left(2\pi u_0\right) \int_0^{2\pi} \frac{1-\cos\left(2u_0\tau\right)}{\tau}\ d\tau \\ & - \frac{\sin\left(2\pi u_0\right)}{\pi} \\ & + 8\cos\left(\pi u_0\right) \frac{\sin\left(\pi u_0\right)}{\pi} \\ & - \left[4u_0 + 2u_0\cos\left(2\pi u_0\right)\right] \Bigg\}. \end{aligned} \tag{B.76}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^b \frac{\sin(at)}{t}\,dt \tag{B.77}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos(at)}{t}\,dt. \tag{B.78}$$

Applying Equations (B.77) and (B.78) to Equation (B.76) and rearranging yields the stationary autocorrelation function $R_m(0)$ for the generalized dipole distribution,

$$\begin{aligned} R_m(0) = \frac{A_m^2 u_0^3}{\pi} \Bigg\{ & 2\mathrm{Cin}\,(2\pi u_0) + 8\cos(\pi u_0)\frac{\sin(\pi u_0)}{\pi u_0} - 4 \\ & - \cos(2\pi u_0)\left[\mathrm{Cin}\,(4\pi u_0) - 2\mathrm{Cin}\,(2\pi u_0) + 2\right] \\ & + \sin(2\pi u_0)\left[\mathrm{Si}\,(4\pi u_0) - 2\mathrm{Si}\,(2\pi u_0) - \frac{1}{\pi u_0}\right]\Bigg\}. \end{aligned} \tag{B.79}$$

B.5 Triangular Distribution

The autocorrelation function $R_f(p)$ for the triangular distribution is,

$$R_f(p) = \frac{A_m^2}{6\pi^2}\begin{cases} u_0^2\,(2\pi+p)^3 + 6\,(2\pi+p)\,, & -2\pi \le p \le -\pi \\ u_0^2\left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p, & -\pi \le p \le 0 \\ u_0^2\left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p, & 0 \le p \le \pi \\ u_0^2\,(2\pi-p)^3 + 6\,(2\pi-p)\,, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{B.80}$$

Equation (B.80) can also be written in terms of Heaviside step functions,

$$\begin{aligned} R_f(p) = \frac{A_m^2}{6\pi^2}\Bigg\{ & \left[u_0^2\,(2\pi+p)^3 + 6\,(2\pi+p)\right]\left[H\,(p+2\pi) - H\,(p+\pi)\right] \\ & + \left[u_0^2\left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p\right]\left[H\,(p+\pi) - H\,(p)\right] \\ & + \left[u_0^2\left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p\right]\left[H\,(p) - H\,(p-\pi)\right] \\ & + \left[u_0^2\,(2\pi-p)^3 + 6\,(2\pi-p)\right]\left[H\,(p-\pi) - H\,(p-2\pi)\right]\Bigg\}. \end{aligned} \tag{B.81}$$

Recalling the definition for $R_n(p)$,

$$R_n(p) = -\frac{d^2 R_f(p)}{dp^2}. \tag{B.82}$$

Finding the first derivative of Equation (B.81),

$$\begin{aligned}\frac{dR_f(p)}{dp} = \frac{A_m^2}{6\pi^2}\frac{d}{dp}\bigg\{ & \left[u_0^2\left(2\pi+p\right)^3+6\left(2\pi+p\right)\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right] \\ & +\left[u_0^2\left(4\pi^3-6\pi p^2-3p^3\right)-12\pi-18p\right]\left[H\left(p+\pi\right)-H\left(p\right)\right] \\ & +\left[u_0^2\left(4\pi^3-6\pi p^2+3p^3\right)-12\pi+18p\right]\left[H\left(p\right)-H\left(p-\pi\right)\right] \\ & +\left[u_0^2\left(2\pi-p\right)^3+6\left(2\pi-p\right)\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]\bigg\},\end{aligned} \tag{B.83}$$

which can be shown to be

$$\begin{aligned}\frac{dR_f(p)}{dp} = \frac{A_m^2}{6\pi^2}\bigg\{ & \left[3u_0^2\left(2\pi+p\right)^2+6\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right] \\ & +\left[u_0^2\left(2\pi+p\right)^3+6\left(2\pi+p\right)\right]\left[\delta\left(p+2\pi\right)-\delta\left(p+\pi\right)\right] \\ & +\left[u_0^2\left(-12\pi p-9p^2\right)-18\right]\left[H\left(p+\pi\right)-H\left(p\right)\right] \\ & +\left[u_0^2\left(4\pi^3-6\pi p^2-3p^3\right)-12\pi-18p\right]\left[\delta\left(p+\pi\right)-\delta\left(p\right)\right] \\ & +\left[u_0^2\left(-12\pi p+9p^2\right)+18\right]\left[H\left(p\right)-H\left(p-\pi\right)\right] \\ & +\left[u_0^2\left(4\pi^3-6\pi p^2+3p^3\right)-12\pi+18p\right]\left[\delta\left(p\right)-\delta\left(p-\pi\right)\right] \\ & +\left[-3u_0^2\left(2\pi-p\right)^2-6\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right] \\ & +\left[u_0^2\left(2\pi-p\right)^3+6\left(2\pi-p\right)\right]\left[\delta\left(p-\pi\right)-\delta\left(p-2\pi\right)\right]\bigg\}.\end{aligned} \tag{B.84}$$

Evaluating the products including delta functions reveals that all terms are zero or cancel for $p = \{-2\pi, 0, 2\pi\}$ and terms cancel for $p = \{-\pi, \pi\}$. Therefore, Equation (B.65) can be simplified to

$$\frac{dR_f(p)}{dp} = \frac{A_m^2}{6\pi^2}\Big\{ \left[3u_0^2\left(2\pi+p\right)^2+6\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right]$$
$$- \left[u_0^2\left(12\pi p+9p^2\right)+18\right]\left[H\left(p+\pi\right)-H\left(p\right)\right]$$
$$+ \left[u_0^2\left(-12\pi p+9p^2\right)+18\right]\left[H\left(p\right)-H\left(p-\pi\right)\right]$$
$$- \left[3u_0^2\left(2\pi-p\right)^2+6\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]\Big\}. \tag{B.85}$$

The second derivative of Equation (B.81) can be found by taking the derivative of Equation (B.85),

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{6\pi^2}\frac{d}{dp}\Big\{ \left[3u_0^2\left(2\pi+p\right)^2+6\right]\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right]$$
$$- \left[u_0^2\left(12\pi p+9p^2\right)+18\right]\left[H\left(p+\pi\right)-H\left(p\right)\right]$$
$$+ \left[u_0^2\left(-12\pi p+9p^2\right)+18\right]\left[H\left(p\right)-H\left(p-\pi\right)\right]$$
$$- \left[3u_0^2\left(2\pi-p\right)^2+6\right]\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]\Big\}, \tag{B.86}$$

which can be shown to be

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{6\pi^2}\Big\{6u_0^2\left(2\pi+p\right)\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right]$$
$$+ \left[3u_0^2\left(2\pi+p\right)^2+6\right]\left[\delta\left(p+2\pi\right)-\delta\left(p+\pi\right)\right]$$
$$- u_0^2\left(12\pi+18p\right)\left[H\left(p+\pi\right)-H\left(p\right)\right]$$
$$- \left[u_0^2\left(12\pi p+9p^2\right)+18\right]\left[\delta\left(p+\pi\right)-\delta\left(p\right)\right]$$
$$- u_0^2\left(12\pi-18p\right)\left[H\left(p\right)-H\left(p-\pi\right)\right]$$
$$+ \left[u_0^2\left(-12\pi p+9p^2\right)+18\right]\left[\delta\left(p\right)-\delta\left(p-\pi\right)\right]$$
$$+ 6u_0^2\left(2\pi-p\right)\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]$$
$$- \left[3u_0^2\left(2\pi-p\right)^2+6\right]\left[\delta\left(p-\pi\right)-\delta\left(p-2\pi\right)\right]\Big\}. \tag{B.87}$$

Evaluating the products including delta functions reveals that terms are zero or cancel for $p = \{-2\pi, 0, 2\pi\}$ and terms cancel for $p = \{-\pi, \pi\}$. Therefore, Equation (B.65) can be simplified to

$$\frac{d^2R_f(p)}{dp^2} = \frac{A_m^2}{6\pi^2}\Bigg\{6u_0^2\left(2\pi+p\right)\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right]$$
$$-u_0^2\left(12\pi+18p\right)\left[H\left(p+\pi\right)-H\left(p\right)\right]$$
$$-u_0^2\left(12\pi-18p\right)\left[H\left(p\right)-H\left(p-\pi\right)\right]$$
$$+6u_0^2\left(2\pi-p\right)\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]$$
$$+6\delta\left(p+2\pi\right)-24\delta\left(p+\pi\right)+36\delta\left(p\right)-24\delta\left(p-\pi\right)+6\delta\left(p-2\pi\right)\Bigg\}. \tag{B.88}$$

Equation (B.88) can be substituted into Equation (B.82) and simplified,

$$R_n(p) = \frac{A_m^2}{\pi^2}\Bigg\{-u_0^2\left(2\pi+p\right)\left[H\left(p+2\pi\right)-H\left(p+\pi\right)\right]$$
$$+u_0^2\left(2\pi+3p\right)\left[H\left(p+\pi\right)-H\left(p\right)\right]$$
$$+u_0^2\left(2\pi-3p\right)\left[H\left(p\right)-H\left(p-\pi\right)\right]$$
$$-u_0^2\left(2\pi-p\right)\left[H\left(p-\pi\right)-H\left(p-2\pi\right)\right]$$
$$-\delta\left(p+2\pi\right)+4\delta\left(p+\pi\right)-6\delta\left(p\right)+4\delta\left(p-\pi\right)-\delta\left(p-2\pi\right)\Bigg\}. \tag{B.89}$$

Recalling the definition for $R_m(0)$,

$$R_m(0) = \frac{u_0}{\pi}\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left(u_0\tau\right)}{u_0\tau}\ d\tau. \tag{B.90}$$

Substituting Equation (B.89) into Equation (B.90),

$$
\begin{aligned}
R_m(0) = \frac{A_m^2 u_0}{\pi^3} \Bigg\{ & - u_0^2 \int_{-2\pi}^{-\pi} (2\pi + \tau) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& + u_0^2 \int_{-\pi}^{0} (2\pi + 3\tau) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& + u_0^2 \int_{0}^{\pi} (2\pi - 3\tau) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& - u_0^2 \int_{\pi}^{2\pi} (2\pi - \tau) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& + 4 \int_{-\infty}^{\infty} \delta(\tau + \pi) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& - 6 \int_{-\infty}^{\infty} \delta(\tau) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& + 4 \int_{-\infty}^{\infty} \delta(\tau - \pi) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \, \frac{\sin(u_0\tau)}{u_0\tau} \; d\tau \Bigg\}. \qquad \text{(B.91)}
\end{aligned}
$$

Recognizing even and odd functions in Equation (B.91) and combining terms,

$$
\begin{aligned}
R_m(0) = \frac{A_m^2 \cancel{u_0}}{\pi^3} \Bigg\{ & 2u_0^2 \int_{0}^{\pi} (2\pi - 3\tau) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \\
& - 2u_0^2 \int_{\pi}^{2\pi} (2\pi - \tau) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \\
& + 4 \int_{-\infty}^{\infty} \delta(\tau + \pi) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \\
& - 6 \int_{-\infty}^{\infty} \delta(\tau) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \\
& + 4 \int_{-\infty}^{\infty} \delta(\tau - \pi) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \, \frac{\sin(u_0\tau)}{\cancel{u_0}\tau} \; d\tau \Bigg\}. \qquad \text{(B.92)}
\end{aligned}
$$

Recalling the sifting property of the delta function,

$$\int_{-\infty}^{\infty} \delta\left(x-a\right) f(x)\ dx = f(a). \tag{B.93}$$

Applying Equation (B.93) to Equation (B.92) and separating integrals,

$$\begin{aligned} R_m(0) = \frac{A_m^2}{\pi^3}\Bigg\{ & 4\pi u_0^2 \int_0^{\pi} \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & - 6u_0^2 \int_0^{\pi} \sin\left(u_0\tau\right)\ d\tau \\ & - 4\pi u_0^2 \int_0^{2\pi} \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & + 4\pi u_0^2 \int_0^{\pi} \frac{\sin\left(u_0\tau\right)}{\tau}\ d\tau \\ & + 2u_0^2 \int_{\pi}^{2\pi} \sin\left(u_0\tau\right)\ d\tau \\ & - \frac{\sin\left(\cancel{-}2\pi u_0\right)}{\cancel{-}2\pi} - \frac{\sin\left(2\pi u_0\right)}{2\pi} \\ & + 4\frac{\sin\left(\cancel{-}\pi u_0\right)}{\cancel{-}\pi} + 4\frac{\sin\left(\pi u_0\right)}{\pi} - 6u_0 \Bigg\}. \end{aligned} \tag{B.94}$$

Recalling the definition for the sine integral,

$$\mathrm{Si}\left(ba\right) = \int_0^{b} \frac{\sin\left(at\right)}{t}\ dt. \tag{B.95}$$

Applying Equation (B.95) to Equation (B.94), performing the remaining integrals, and simplifying,

$$\begin{aligned} R_m(0) = \frac{A_m^2}{\pi^3}\Bigg\{ & 8\pi u_0^2 \mathrm{Si}\left(\pi u_0\right) - 4\pi u_0^2 \mathrm{Si}\left(2\pi u_0\right) \\ & + 6u_0 \cos\left(u_0\tau\right)\Big|_0^{\pi} \\ & - 2u_0 \cos\left(u_0\tau\right)\Big|_{\pi}^{2\pi} \\ & - \frac{2\sin\left(2\pi u_0\right)}{2\pi} + \frac{8\sin\left(\pi u_0\right)}{\pi} - 6u_0 \Bigg\}. \end{aligned} \tag{B.96}$$

Evaluating the limits of integration in Equation (B.96),

$$R_m(0) = \frac{A_m^2}{\pi^3}\Bigg\{8\pi u_0^2 \mathrm{Si}\left(\pi u_0\right) - 4\pi u_0^2 \mathrm{Si}\left(2\pi u_0\right)$$
$$+ 6u_0\left[\cos\left(\pi u_0\right) - 1\right]$$
$$- 2u_0\left[\cos\left(2\pi u_0\right) - \cos\left(\pi u_0\right)\right]$$
$$- \frac{2\sin\left(2\pi u_0\right)}{2\pi} + \frac{8\sin\left(\pi u_0\right)}{\pi} - 6u_0\Bigg\}. \tag{B.97}$$

Simplifying Equation (B.97) yields the stationary autocorrelation function $R_m(0)$ for the triangular distribution,

$$R_m(0) = \frac{A_m^2 u_0}{\pi^3}\Bigg\{8\pi u_0 \mathrm{Si}\left(\pi u_0\right) - 4\pi u_0 \mathrm{Si}\left(2\pi u_0\right)$$
$$+ \frac{8\sin\left(\pi u_0\right)}{\pi u_0} - \frac{2\sin\left(2\pi u_0\right)}{2\pi u_0}$$
$$+ 8\cos\left(\pi u_0\right) - 2\cos\left(2\pi u_0\right) - 12\Bigg\}. \tag{B.98}$$

B.6 Uniform Distribution

The autocorrelation function $R_f(p)$ for the uniform distribution, written in terms of Heaviside and ramp functions, is,

$$R_f(p) = A_m^2\Bigg\{\left[\delta\left(p + 2\pi\right) - 2\delta\left(p\right) + \delta\left(p - 2\pi\right)\right]$$
$$+ u_0^2\left[R\left(p + 2\pi\right) - 2R\left(p\right) + R\left(p - 2\pi\right)\right]\Bigg\}. \tag{B.99}$$

Recalling the definition for $R_n(p)$,

$$R_n(p) = -\frac{d^2 R_f(p)}{dp^2}. \tag{B.100}$$

Finding the first derivative of Equation (B.99),

$$\frac{dR_f(p)}{dp} = A_m^2 \frac{d}{dp}\Big\{ \left[\delta\left(p+2\pi\right) - 2\delta\left(p\right) + \delta\left(p-2\pi\right)\right] + u_0^2 \left[R\left(p+2\pi\right) - 2R\left(p\right) + R\left(p-2\pi\right)\right] \Big\}, \tag{B.101}$$

which can be shown to be

$$\frac{dR_f(p)}{dp} = A_m^2 \Big\{ \left[\delta'\left(p+2\pi\right) - 2\delta'\left(p\right) + \delta'\left(p-2\pi\right)\right] + u_0^2 \left[H\left(p+2\pi\right) - 2H\left(p\right) + H\left(p-2\pi\right)\right] \Big\}. \tag{B.102}$$

The second derivative of Equation (B.99) can be found by taking the derivative of Equation (B.102),

$$\frac{d^2R_f(p)}{dp^2} = A_m^2 \frac{d}{dp}\Big\{ \left[\delta'\left(p+2\pi\right) - 2\delta'\left(p\right) + \delta'\left(p-2\pi\right)\right] + u_0^2 \left[H\left(p+2\pi\right) - 2H\left(p\right) + H\left(p-2\pi\right)\right] \Big\}, \tag{B.103}$$

which can be shown to be

$$\frac{d^2R_f(p)}{dp^2} = A_m^2 \Big\{ \left[\delta''\left(p+2\pi\right) - 2\delta''\left(p\right) + \delta''\left(p-2\pi\right)\right] + u_0^2 \left[\delta\left(p+2\pi\right) - 2\delta\left(p\right) + \delta\left(p-2\pi\right)\right] \Big\}. \tag{B.104}$$

Equation (B.104) can be substituted into Equation (B.100),

$$R_n(p) = -A_m^2 \Big\{ \left[\delta''\left(p+2\pi\right) - 2\delta''\left(p\right) + \delta''\left(p-2\pi\right)\right] + u_0^2 \left[\delta\left(p+2\pi\right) - 2\delta\left(p\right) + \delta\left(p-2\pi\right)\right] \Big\}. \tag{B.105}$$

Recalling the definition for $R_m(0)$,

$$R_m(0) = \frac{u_0}{\pi} \int_{-\infty}^{\infty} R_n(\tau) \frac{\sin\left(u_0\tau\right)}{u_0\tau}\, d\tau. \tag{B.106}$$

Substituting Equation (B.105) into Equation (B.106),

$$
\begin{aligned}
R_m(0) = -\frac{A_m^2 u_0}{\pi} \Bigg\{ & \int_{-\infty}^{\infty} \delta''(\tau + 2\pi) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta''(\tau) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\
& + \int_{-\infty}^{\infty} \delta''(\tau - 2\pi) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\
& + u_0^2 \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\
& - 2u_0^2 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \\
& + u_0^2 \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin(u_0\tau)}{u_0\tau}\, d\tau \Bigg\}.
\end{aligned} \tag{B.107}
$$

Recalling the sifting property of the delta function,

$$\int_{-\infty}^{\infty} \delta(x - a) f(x)\, dx = f(a). \tag{B.108}$$

Also, integrating a function with the second derivative of the delta function yields the following sifting property,

$$\int_{-\infty}^{\infty} \delta''(x - a) f(x)\, dx = f''(a). \tag{B.109}$$

As such, it is necessary to determine the second derivative of the sinc funciton, which can be shown to be

$$\frac{d^2}{d\tau^2}\left[\frac{\sin(u_0\tau)}{u_0\tau}\right] = \frac{(2 - u_0^2\tau^2)\sin(u_0\tau) - 2u_0\tau\cos(u_0\tau)}{u_0\tau^3}. \tag{B.110}$$

Applying Equations (B.108), (B.109), and (B.110) to Equation (B.107),

$$
\begin{aligned}
R_m(0) = -\frac{A_m^2 u_0}{\pi} \Bigg\{ & \left[\frac{(2 - 4\pi^2 u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] \\
& - 2\left[\frac{(2 - u_0^2\tau^2)\sin(u_0\tau) - 2u_0\tau\cos(u_0\tau)}{u_0\tau^3}\right]\Bigg|_{\tau=0} \\
& + \left[\frac{(2 - 4\pi^2 u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] \\
& + u_0^2 \frac{\sin(\not{-}2\pi u_0)}{\not{-}2\pi u_0} + u_0^2 \frac{\sin(2\pi u_0)}{2\pi u_0} - 2u_0^2 \Bigg\}.
\end{aligned} \tag{B.111}
$$

Combining terms in Equation (B.111),

$$R_m(0) = -\frac{A_m^2 u_0}{\pi}\Bigg\{2\left[\frac{(2-4\pi^2u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] - 2\left[\frac{(2-u_0^2\tau^2)}{u_0\tau^3}\sin(u_0\tau) - \frac{2}{\tau^2}\cos(u_0\tau)\right]\Bigg|_{\tau=0} + 2u_0^2\frac{\sin(2\pi u_0)}{2\pi u_0} - 2u_0^2\Bigg\}. \tag{B.112}$$

Recalling the Taylor series expansions for sine and cosine,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{B.113}$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots . \tag{B.114}$$

Applying Equations (B.113) and (B.114) to Equation (B.112),

$$R_m(0) = -\frac{A_m^2 u_0}{\pi}\Bigg\{2\left[\frac{(2-4\pi^2u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] - 2\left[\frac{(2-u_0^2\tau^2)}{u_0\tau^3}\left[u_0\tau - \frac{u_0^3\tau^3}{3!} + \frac{u_0^5\tau^5}{5!} - \cdots\right]\right]\Bigg|_{\tau=0} + 2\left[\frac{2}{\tau^2}\left[1 - \frac{u_0^2\tau^2}{2!} + \frac{u_0^4\tau^4}{4!} - \cdots\right]\right]\Bigg|_{\tau=0} + 2u_0^2\frac{\sin(2\pi u_0)}{2\pi u_0} - 2u_0^2\Bigg\}. \tag{B.115}$$

Canceling terms in Equation (B.115),

$$R_m(0) = -\frac{A_m^2 u_0}{\pi}\Bigg\{2\left[\frac{(2-4\pi^2u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] - 2\left[\frac{(2-u_0^2\tau^2)}{u_0}\left[\frac{u_0}{\tau^2} - \frac{u_0^3}{3!} + \frac{u_0^5\tau^2}{5!} - \cdots\right]\right]\Bigg|_{\tau=0} + 4\left[\frac{1}{\tau^2} - \frac{u_0^2}{2!} + \frac{u_0^2\tau^4}{4!} - \cdots\right]\Bigg|_{\tau=0} + 2u_0^2\frac{\sin(2\pi u_0)}{2\pi u_0} - 2u_0^2\Bigg\}. \tag{B.116}$$

Excluding higher order terms in Equation (B.116) and combining the remaining terms,

$$R_m(0) = -\frac{A_m^2 u_0}{\pi}\Bigg\{2\left[\frac{(2-4\pi^2u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] \\ + 2u_0^2 - 2u_0^2 + \frac{2}{3}u_0^2 + \left[\frac{4}{\tau^2} - \frac{4}{\tau^2} - \frac{u_0^4\tau^2}{3}\right]\Bigg|_{\tau=0} \\ + 2u_0^2\frac{\sin(2\pi u_0)}{2\pi u_0} - 2u_0^2\Bigg\}. \tag{B.117}$$

Canceling terms in Equation (B.117) and performing the remaining evaluations for $\tau = 0$,

$$R_m(0) = -\frac{A_m^2 u_0}{\pi}\Bigg\{2\left[\frac{(2-4\pi^2u_0^2)\sin(2\pi u_0) - 4\pi u_0\cos(2\pi u_0)}{8\pi^3 u_0}\right] \\ + 2u_0^2\frac{\sin(2\pi u_0)}{2\pi u_0} - \frac{4u_0^2}{3}\Bigg\}. \tag{B.118}$$

Simplifying Equation (B.117) yields the stationary autocorrelation function $R_m(0)$ for the uniform distribution,

$$R_m(0) = \frac{2A_m^2 u_0}{\pi}\left\{\frac{\cos(2\pi u_0)}{2\pi^2} - \frac{\sin(2\pi u_0)}{4\pi^3 u_0} + \frac{2u_0^2}{3}\right\}. \tag{B.119}$$

Appendix C

Derivations for the Radiated Power of a Scanning Line Source

This appendix presents the detailed derivations of the autocorrelation functions used to determine the radiated power for a scanning line source radiator. The derivations were performed for the half-wave dipole, cosine, cosine-squared, uniform, and triangular distributions.

C.1 Half-Wave Dipole Distribution

The autocorrelation function $R_g(p)$ for the half-wave dipole is,

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{C.1}$$

Finding the first derivative of $R_g(p)$:

$$R_g'(p) = \frac{A_m^2}{2}\frac{d}{dp} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{C.2}$$

where

$$R_g'(p) = \frac{A_m^2}{2}\begin{cases} \cos\left(\frac{p}{2}\right) - \left(\frac{2\pi + p}{2}\right)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ -\cos\left(\frac{p}{2}\right) - \left(\frac{2\pi - p}{2}\right)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{C.3}$$

which can be simplified to be

$$R_g'(p) = \frac{A_m^2}{2}\begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{C.4}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{2}\begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{C.5}$$

Recalling the definition for $\widetilde{R}_h(0)$,

$$\begin{aligned} \widetilde{R}_h(0) =& \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_f(\tau)\frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_f(\tau)\frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\ &- \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau - \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\ &+ 2\alpha\int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau - 2\alpha\int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau\Bigg\}. \end{aligned} \tag{C.6}$$

Substituting Equations (C.1), (C.4), and (C.5) into Equation (C.6),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2 u_0}{4\pi} \Bigg\{ &- \int_{-2\pi}^{0} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau + \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \int_{-2\pi}^{0} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau + \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \alpha^2 \int_{-2\pi}^{0} \left[(2\pi+\tau)\cos\left(\frac{\tau}{2}\right) - 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \alpha^2 \int_{0}^{2\pi} \left[(2\pi-\tau)\cos\left(\frac{\tau}{2}\right) + 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \alpha^2 \int_{-2\pi}^{0} \left[(2\pi+\tau)\cos\left(\frac{\tau}{2}\right) - 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \alpha^2 \int_{0}^{2\pi} \left[(2\pi-\tau)\cos\left(\frac{\tau}{2}\right) + 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- 2\alpha \int_{-2\pi}^{0} \left(\frac{\tau}{2}+\pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ 2\alpha \int_{0}^{2\pi} \left(\frac{\tau}{2}-\pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ 2\alpha \int_{-2\pi}^{0} \left(\frac{\tau}{2}+\pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- 2\alpha \int_{0}^{2\pi} \left(\frac{\tau}{2}-\pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \Bigg\}. \qquad \text{(C.7)}
\end{aligned}
$$

Recognizing even and odd functions in Equation (C.7) and combining terms,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2 \cancel{u_0}}{4\pi} \Bigg\{ &2 \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
&+ 2 \int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
&- 2\alpha^2 \int_{0}^{2\pi} \left[(2\pi-\tau)\cos\left(\frac{\tau}{2}\right) + 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
&- 2\alpha^2 \int_{0}^{2\pi} \left[(2\pi-\tau)\cos\left(\frac{\tau}{2}\right) + 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
&+ 4\alpha \int_{0}^{2\pi} \left(\frac{\tau}{2}-\pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
&- 4\alpha \int_{0}^{2\pi} \left(\frac{\tau}{2}-\pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}. \qquad \text{(C.8)}
\end{aligned}
$$

Combining terms in Equation (C.8) and inserting $u_0 = \frac{1}{2}$ for the half-wave dipole,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & 2\left(1-2\alpha^2\right)\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 2\left(1-2\alpha^2\right)\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 4\pi\alpha^2\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 4\pi\alpha^2\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 2\alpha^2\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]\,d\tau \\
& + 2\alpha^2\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]\,d\tau \\
& - 4\pi\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 4\pi\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 2\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right]\,d\tau \\
& - 2\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right]\,d\tau\Bigg\}.
\end{aligned}
\tag{C.9}
$$

Applying product-to-sum trigonometric identities to Equation (C.9),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{4\pi}\bigg\{ & \left(1-2\alpha^2\right)\int_0^{2\pi} \frac{\cos(\alpha\tau)-\cos[(1+\alpha)\tau]}{\tau}\ d\tau \\
& + \left(1-2\alpha^2\right)\int_0^{2\pi} \frac{\cos(\alpha\tau)-\cos[(1-\alpha)\tau]}{\tau}\ d\tau \\
& - 2\pi\alpha^2\int_0^{2\pi} \frac{\sin[(1+\alpha)\tau]+\sin(\alpha\tau)}{\tau}\ d\tau \\
& - 2\pi\alpha^2\int_0^{2\pi} \frac{\sin[(1-\alpha)\tau]-\sin(\alpha\tau)}{\tau}\ d\tau \\
& + \alpha^2\int_0^{2\pi} \left[\sin[(1+\alpha)\tau]+\sin(\alpha\tau)\right]\ d\tau \\
& + \alpha^2\int_0^{2\pi} \left[\sin[(1-\alpha)\tau]-\sin(\alpha\tau)\right]\ d\tau \\
& - 2\pi\alpha\int_0^{2\pi} \frac{\sin[(1+\alpha)\tau]-\sin(\alpha\tau)}{\tau}\ d\tau \\
& + 2\pi\alpha\int_0^{2\pi} \frac{\sin[(1-\alpha)\tau]+\sin(\alpha\tau)}{\tau}\ d\tau \\
& + \alpha\int_0^{2\pi} \left[\sin[(1+\alpha)\tau]-\sin(\alpha\tau)\right]\ d\tau \\
& - \alpha\int_0^{2\pi} \left[\sin[(1-\alpha)\tau]+\sin(\alpha\tau)\right]\ d\tau \bigg\}. \qquad \text{(C.10)}
\end{aligned}
$$

Collecting and canceling terms in Equation (C.10),

$$\begin{aligned}\widetilde{R}_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & \left(1-2\alpha^2\right)\int_0^{2\pi} \frac{1-\cos\left[(1+\alpha)\,\tau\right]-1+\cos(\alpha\tau)}{\tau}\ d\tau \\ & + \left(1-2\alpha^2\right)\int_0^{2\pi} \frac{1-\cos\left[(1-\alpha)\,\tau\right]-1+\cos(\alpha\tau)}{\tau}\ d\tau \\ & - 2\pi\alpha^2 \int_0^{2\pi} \frac{\sin\left[(1+\alpha)\,\tau\right]+\sin\left[(1-\alpha)\,\tau\right]}{\tau}\ d\tau \\ & + \alpha^2 \int_0^{2\pi} \sin\left[(1+\alpha)\,\tau\right]\ d\tau \\ & + \alpha^2 \int_0^{2\pi} \sin\left[(1-\alpha)\,\tau\right]\ d\tau \\ & - 2\pi\alpha \int_0^{2\pi} \frac{\sin\left[(1+\alpha)\,\tau\right]-2\sin(\alpha\tau)-\sin\left[(1-\alpha)\,\tau\right]}{\tau}\ d\tau \\ & + \alpha \int_0^{2\pi} \sin\left[(1+\alpha)\,\tau\right]\ d\tau \\ & - 2\alpha \int_0^{2\pi} \sin(\alpha\tau)\ d\tau \\ & - \alpha \int_0^{2\pi} \sin\left[(1-\alpha)\,\tau\right]\ d\tau \Bigg\}. \end{aligned} \tag{C.11}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}(ba) = \int_0^b \frac{\sin(at)}{t}\ dt \tag{C.12}$$

and

$$\mathrm{Cin}(ba) = \int_0^b \frac{1-\cos(at)}{t}\ dt. \tag{C.13}$$

Applying Equations (C.12) and (C.13) to Equation (C.11) and performing the remaining integrals,

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & \left(1-2\alpha^2\right)\int_0^{2\pi}\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Cin}\left(2\pi\alpha\right) + \mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - 2\pi\alpha^2\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - 2\pi\alpha\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Si}\left(2\pi\alpha\right) - \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - \frac{\alpha^2}{1+\alpha}\cos\left[\left(1+\alpha\right)\tau\right]\Big|_0^{2\pi} \\
& - \frac{\alpha^2}{1-\alpha}\cos\left[\left(1-\alpha\right)\tau\right]\Big|_0^{2\pi} \\
& - \frac{\alpha}{1+\alpha}\cos\left[\left(1+\alpha\right)\tau\right]\Big|_0^{2\pi} \\
& + 2\cos\left(\alpha\tau\right)\Big|_0^{2\pi} \\
& + \frac{\alpha}{1-\alpha}\cos\left[\left(1-\alpha\right)\tau\right]\Big|_0^{2\pi}\Bigg\}.
\end{aligned} \tag{C.14}$$

Evaluating the limits of integration and simplifying Equation (C.14),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{4\pi}\Bigg\{ & \left(1-2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Cin}\left(2\pi\alpha\right) + \mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - 2\pi\alpha^2\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - 2\pi\alpha\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Si}\left(2\pi\alpha\right) - \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - \frac{\alpha^2}{1+\alpha}\left[\cos\left[2\pi\left(1+\alpha\right)\right] - 1\right] \\
& - \frac{\alpha^2}{1-\alpha}\left[\cos\left[2\pi\left(1-\alpha\right)\right] - 1\right] \\
& - \frac{\alpha}{1+\alpha}\left[\cos\left[2\pi\left(1+\alpha\right)\right] - 1\right] \\
& + 2\left[\cos\left(2\pi\alpha\right) - 1\right] \\
& + \frac{\alpha}{1-\alpha}\left[\cos\left[2\pi\left(1-\alpha\right)\right] - 1\right]\Bigg\}.
\end{aligned} \tag{C.15}$$

Simplifying Equation (C.15) yields the stationary autocorrelation function $\widetilde{R}_h(0)$ for the half-wave dipole,

$$\begin{aligned}\widetilde{R}_h(0) = \frac{A_m^2}{4\pi}\Big\{ & \left(1-2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Cin}\left(2\pi\alpha\right) + \mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & - 2\pi\alpha^2\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & - 2\pi\alpha\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right] - 2\mathrm{Si}\left(2\pi\alpha\right) - \mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\ & + 2\left[\cos\left(2\pi\alpha\right) - 1\right]\Big\}. \end{aligned} \tag{C.16}$$

C.2 Cosine Distribution

The autocorrelation function $R_g(p)$ for the cosine distribution is,

$$R_g(p) = \frac{A_m^2}{2}\begin{cases}(2\pi+p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ (2\pi-p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{C.17}$$

Finding the first derivative of $R_g(p)$:

$$R_g'(p) = \frac{A_m^2}{2}\frac{d}{dp}\begin{cases}(2\pi+p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ (2\pi-p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}; \tag{C.18}$$

where

$$R_g'(p) = \frac{A_m^2}{2}\begin{cases}\cos\left(\frac{p}{2}\right) - \left(\frac{2\pi+p}{2}\right)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ -\cos\left(\frac{p}{2}\right) - \left(\frac{2\pi-p}{2}\right)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}; \tag{C.19}$$

which can be simplified to be

$$R'_g(p) = \frac{A_m^2}{2}\begin{cases}\left(-\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0\\ \left(\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi\\ 0, & \text{otherwise}\end{cases}. \tag{C.20}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2\begin{cases}-\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right)-\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi+p}{2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0\\ \left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{p}{2}\right)-\left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-p}{2}\right)\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi\\ 0, & \text{otherwise}\end{cases}. \tag{C.21}$$

Recalling the definition for $\widetilde{R}_h(0)$,

$$\begin{aligned}\widetilde{R}_h(0) =& \frac{u_0}{2\pi}\left\{\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_f(\tau)\frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau\right.\\ &- \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau - \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau\\ &\left.+ 2\alpha\int_{-\infty}^{\infty} R'_g(\tau)\frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau - 2\alpha\int_{-\infty}^{\infty} R'_g(\tau)\frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau\right\}.\end{aligned} \tag{C.22}$$

Substituting Equations (C.17), (C.20), and (C.21) into Equation (C.22),

$$
\begin{aligned}
\widetilde{R}_h(0) = {} & \frac{A_m^2 u_0}{2\pi} \\
& \times \left\{ \int_{-2\pi}^{0} \left[-\left(\frac{1}{4} + u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4} - u_0^2\right) \left(\frac{2\pi + \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 + \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \right. \\
& + \int_{0}^{2\pi} \left[\left(\frac{1}{4} + u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4} - u_0^2\right) \left(\frac{2\pi - \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 + \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& + \int_{-2\pi}^{0} \left[-\left(\frac{1}{4} + u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4} - u_0^2\right) \left(\frac{2\pi + \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 - \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& + \int_{0}^{2\pi} \left[\left(\frac{1}{4} + u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4} - u_0^2\right) \left(\frac{2\pi - \tau}{2}\right) \cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 - \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& - \frac{\alpha^2}{2} \int_{-2\pi}^{0} \left[(2\pi + \tau) \cos\left(\frac{\tau}{2}\right) - 2 \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 + \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& - \frac{\alpha^2}{2} \int_{0}^{2\pi} \left[(2\pi - \tau) \cos\left(\frac{\tau}{2}\right) + 2 \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 + \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& - \frac{\alpha^2}{2} \int_{-2\pi}^{0} \left[(2\pi + \tau) \cos\left(\frac{\tau}{2}\right) - 2 \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 - \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& - \frac{\alpha^2}{2} \int_{0}^{2\pi} \left[(2\pi - \tau) \cos\left(\frac{\tau}{2}\right) + 2 \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 - \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& - \alpha \int_{-2\pi}^{0} \left(\frac{\tau}{2} + \pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0 + \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& + \alpha \int_{0}^{2\pi} \left(\frac{\tau}{2} - \pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0 + \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& + \alpha \int_{-2\pi}^{0} \left(\frac{\tau}{2} + \pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0 - \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \\
& \left. - \alpha \int_{0}^{2\pi} \left(\frac{\tau}{2} - \pi\right) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0 - \alpha)\,\tau\right]}{u_0 \tau} \, d\tau \right\}. \qquad \text{(C.23)}
\end{aligned}
$$

Recognizing even and odd functions in Equation (C.23) and combining terms,

$$
\begin{aligned}
\widetilde{R}_h(0) = {} & \frac{A_m^2 \cancel{u_0}}{2\pi} \\
& \times \Bigg\{ 2\int_0^{2\pi} \left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-\tau}{2}\right)\cos\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\int_0^{2\pi} \left[\left(\frac{1}{4}+u_0^2\right)\sin\left(\frac{\tau}{2}\right) - \left(\frac{1}{4}-u_0^2\right)\left(\frac{2\pi-\tau}{2}\right)\cos\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \alpha^2 \int_0^{2\pi} \left[(2\pi-\tau)\cos\left(\frac{\tau}{2}\right) + 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \alpha^2 \int_0^{2\pi} \left[(2\pi-\tau)\cos\left(\frac{\tau}{2}\right) + 2\sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha \int_0^{2\pi} \left(\frac{\tau}{2}-\pi\right)\sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\alpha \int_0^{2\pi} \left(\frac{\tau}{2}-\pi\right)\sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}. \qquad \text{(C.24)}
\end{aligned}
$$

Combining terms in Equation (C.24),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{2\pi}\Bigg\{ & 2\left(\frac{1}{4}+u_0^2-\alpha^2\right)\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + 2\left(\frac{1}{4}+u_0^2-\alpha^2\right)\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& - 2\pi\left(\frac{1}{4}-u_0^2+\alpha^2\right)\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau \\
& - 2\pi\left(\frac{1}{4}-u_0^2+\alpha^2\right)\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + \left(\frac{1}{4}-u_0^2+\alpha^2\right)\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[(u_0+\alpha)\,\tau\right]\;d\tau \\
& + \left(\frac{1}{4}-u_0^2+\alpha^2\right)\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[(u_0-\alpha)\,\tau\right]\;d\tau \\
& - 2\pi\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + 2\pi\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + \alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[(u_0+\alpha)\,\tau\right]\;d\tau \\
& - \alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[(u_0-\alpha)\,\tau\right]\;d\tau\Bigg\}.
\end{aligned}
\tag{C.25}
$$

Applying product-to-sum trigonometric identities to Equation (C.25),

$$
\begin{aligned}
\widetilde{R}_h\left(0\right) = \frac{A_m^2}{2\pi}\Bigg\{ & \left(\frac{1}{4} + u_0^2 - \alpha^2\right) \int_0^{2\pi} \frac{\cos\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right] - \cos\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right]}{\tau}\ d\tau \\
& + \left(\frac{1}{4} + u_0^2 - \alpha^2\right) \int_0^{2\pi} \frac{\cos\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right] - \cos\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right]}{\tau}\ d\tau \\
& - \pi\left(\frac{1}{4} - u_0^2 + \alpha^2\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right] + \sin\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\ d\tau \\
& - \pi\left(\frac{1}{4} - u_0^2 + \alpha^2\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right] + \sin\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\ d\tau \\
& + \frac{1}{2}\left(\frac{1}{4} - u_0^2 + \alpha^2\right) \int_0^{2\pi} \left[\sin\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right] + \sin\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]\right]\ d\tau \\
& + \frac{1}{2}\left(\frac{1}{4} - u_0^2 + \alpha^2\right) \int_0^{2\pi} \left[\sin\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right] + \sin\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]\right]\ d\tau \\
& - \pi\alpha \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right] - \sin\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\ d\tau \\
& + \pi\alpha \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right] - \sin\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\ d\tau \\
& + \frac{\alpha}{2} \int_0^{2\pi} \left[\sin\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right] - \sin\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]\right]\ d\tau \\
& - \frac{\alpha}{2} \int_0^{2\pi} \left[\sin\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right] - \sin\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]\right]\ d\tau \Bigg\}. \qquad \text{(C.26)}
\end{aligned}
$$

Collecting terms in Equation (C.26),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{2\pi} \Bigg\{ & \left(u_0^2 - \alpha^2 + \frac{1}{4}\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right] - 1 + \cos\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\
& + \left(u_0^2 - \alpha^2 + \frac{1}{4}\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right] - 1 + \cos\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\
& + \pi \left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\
& + \pi \left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\
& + \pi \left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\
& + \pi \left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]}{\tau}\, d\tau \\
& - \frac{1}{2} \left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \sin\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right]\, d\tau \\
& - \frac{1}{2} \left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \sin\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]\, d\tau \\
& - \frac{1}{2} \left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \sin\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right]\, d\tau \\
& - \frac{1}{2} \left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right) \int_0^{2\pi} \sin\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]\, d\tau \Bigg\}.
\end{aligned} \tag{C.27}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^b \frac{\sin(at)}{t}\, dt \tag{C.28}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1 - \cos(at)}{t}\, dt. \tag{C.29}$$

Applying Equations (C.28) and (C.29) to Equation (C.27) and performing the remaining integrals,

$$
\begin{aligned}
\widetilde{R}_h\left(0\right) = \frac{A_m^2}{2\pi}\Bigg\{ & \left(u_0^2 - \alpha^2 + \frac{1}{4}\right)\left[\operatorname{Cin}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] - \operatorname{Cin}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right]\right] \\
& + \left(u_0^2 - \alpha^2 + \frac{1}{4}\right)\left[\operatorname{Cin}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right] - \operatorname{Cin}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)\left[\operatorname{Si}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] + \operatorname{Si}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)\left[\operatorname{Si}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right] + \operatorname{Si}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right]\right] \\
& + \frac{1}{2}\frac{\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 + \alpha + \frac{1}{2}\right)}\cos\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi} \\
& + \frac{1}{2}\frac{\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 + \alpha - \frac{1}{2}\right)}\cos\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi} \\
& + \frac{1}{2}\frac{\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 - \alpha + \frac{1}{2}\right)}\cos\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi} \\
& + \frac{1}{2}\frac{\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 - \alpha - \frac{1}{2}\right)}\cos\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]\Bigg|_0^{2\pi}\Bigg\}. \qquad \text{(C.30)}
\end{aligned}
$$

Evaluating the limits of integration and simplifying Equation (C.30),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{2\pi} \Bigg\{ & \left(u_0^2 - \alpha^2 + \frac{1}{4}\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right]\right] \\
& + \left(u_0^2 - \alpha^2 + \frac{1}{4}\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right) \left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right) \left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right]\right] \\
& - \frac{1}{2}\left(u_0 - \alpha - \frac{1}{2}\right)\left[\cos\left[2\pi\left(u_0 + \alpha\right)\right] + 1\right] \\
& - \frac{1}{2}\left(u_0 - \alpha + \frac{1}{2}\right)\left[\cos\left[2\pi\left(u_0 + \alpha\right)\right] + 1\right] \\
& - \frac{1}{2}\left(u_0 + \alpha - \frac{1}{2}\right)\left[\cos\left[2\pi\left(u_0 - \alpha\right)\right] + 1\right] \\
& - \frac{1}{2}\left(u_0 + \alpha + \frac{1}{2}\right)\left[\cos\left[2\pi\left(u_0 - \alpha\right)\right] + 1\right] \Bigg\}. \qquad \text{(C.31)}
\end{aligned}
$$

Simplifying Equation (C.31) yields the stationary autocorrelation function $\widetilde{R}_h(0)$ for the cosine distribution,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{2\pi} \Bigg\{ & \left(u_0^2 - \alpha^2 + \frac{1}{4}\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right]\right] \\
& + \left(u_0^2 - \alpha^2 + \frac{1}{4}\right) \left[\mathrm{Cin}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right) \left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right) \left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right]\right] \\
& - \left(u_0 - \alpha\right)\cos\left[2\pi\left(u_0 + \alpha\right)\right] - \left(u_0 + \alpha\right)\cos\left[2\pi\left(u_0 - \alpha\right)\right] - 2u_0 \Bigg\}. \qquad \text{(C.32)}
\end{aligned}
$$

C.3 Cosine-Squared Distribution

The autocorrelation function $R_g(p)$ for the cosine-squared distribution is,

$$R_g(p) = \frac{A_m^2}{8} \begin{cases} (2\pi + p)\,[2 + \cos(p)] - 3\sin(p), & -2\pi \leq p \leq 0 \\ (2\pi - p)\,[2 + \cos(p)] + 3\sin(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{C.33}$$

Finding the first derivative of $R_g(p)$:

$$R_g'(p) = \frac{A_m^2}{8}\frac{d}{dp} \begin{cases} (2\pi + p)\,[2 + \cos(p)] - 3\sin(p), & -2\pi \leq p \leq 0 \\ (2\pi - p)\,[2 + \cos(p)] + 3\sin(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{C.34}$$

where

$$R_g'(p) = \frac{A_m^2}{8} \begin{cases} 2 + \cos(p) - (2\pi + p)\sin(p) - 3\cos(p), & -2\pi \leq p \leq 0 \\ -2 - \cos(p) - (2\pi - p)\sin(p) + 3\cos(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{C.35}$$

which can be simplified to be

$$R_g'(p) = \frac{A_m^2}{8} \begin{cases} -(p + 2\pi)\sin(p) - 2\cos(p) + 2, & -2\pi \leq p \leq 0 \\ (p - 2\pi)\sin(p) + 2\cos(p) - 2, & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{C.36}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{8} \begin{cases} (2\pi + p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] + \left(1 - 3u_0^2\right)\sin(p), & -2\pi \leq p \leq 0 \\ (2\pi - p)\left[2u_0^2 + \left(u_0^2 - 1\right)\cos(p)\right] - \left(1 - 3u_0^2\right)\sin(p), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{C.37}$$

Recalling the definition for $\widetilde{R}_h(0)$,

$$\begin{aligned}\widetilde{R}_h(0) =& \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau + \int_{-\infty}^{\infty} R_f(\tau) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- \alpha^2 \int_{-\infty}^{\infty} R_g(\tau) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g(\tau) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &+ 2\alpha \int_{-\infty}^{\infty} R_g'(\tau) \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau - 2\alpha \int_{-\infty}^{\infty} R_g'(\tau) \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \Bigg\}.\end{aligned} \tag{C.38}$$

Substituting Equations (C.33), (C.36), and (C.37) into Equation (C.38),

$$\begin{aligned}\widetilde{R}_h(0) =& \frac{A_m^2 u_0}{16\pi} \\ &\times \Bigg\{ \int_{-2\pi}^{0} \left[(2\pi+\tau)\left[2u_0^2 + \left(u_0^2-1\right)\cos(\tau)\right] + \left(1-3u_0^2\right)\sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &+ \int_{0}^{2\pi} \left[(2\pi-\tau)\left[2u_0^2 + \left(u_0^2-1\right)\cos(\tau)\right] - \left(1-3u_0^2\right)\sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &+ \int_{-2\pi}^{0} \left[(2\pi+\tau)\left[2u_0^2 + \left(u_0^2-1\right)\cos(\tau)\right] + \left(1-3u_0^2\right)\sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &+ \int_{0}^{2\pi} \left[(2\pi-\tau)\left[2u_0^2 + \left(u_0^2-1\right)\cos(\tau)\right] - \left(1-3u_0^2\right)\sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- \alpha^2 \int_{-2\pi}^{0} \left[(2\pi+\tau)\left[2+\cos(\tau)\right] - 3\sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- \alpha^2 \int_{0}^{2\pi} \left[(2\pi-\tau)\left[2+\cos(\tau)\right] + 3\sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- \alpha^2 \int_{-2\pi}^{0} \left[(2\pi+\tau)\left[2+\cos(\tau)\right] - 3\sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- \alpha^2 \int_{0}^{2\pi} \left[(2\pi-\tau)\left[2+\cos(\tau)\right] + 3\sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- 2\alpha \int_{-2\pi}^{0} \left[(\tau+2\pi)\sin(\tau) + 2\cos(\tau) - 2\right] \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &+ 2\alpha \int_{0}^{2\pi} \left[(\tau-2\pi)\sin(\tau) + 2\cos(\tau) - 2\right] \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &+ 2\alpha \int_{-2\pi}^{0} \left[(\tau+2\pi)\sin(\tau) + 2\cos(\tau) - 2\right] \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\ &- 2\alpha \int_{0}^{2\pi} \left[(\tau-2\pi)\sin(\tau) + 2\cos(\tau) - 2\right] \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \Bigg\}.\end{aligned} \tag{C.39}$$

Recognizing even and odd functions in Equation (C.39) and combining terms,

$$
\begin{aligned}
\widetilde{R}_h(0) = {} & \frac{A_m^2 \cancel{u_0}}{16\pi} \\
& \times \Bigg\{ 2 \int_0^{2\pi} \left[(2\pi - \tau) \left[2u_0^2 + \left(u_0^2 - 1 \right) \cos(\tau) \right] - \left(1 - 3u_0^2 \right) \sin(\tau) \right] \frac{\sin\left[(u_0 + \alpha) \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& + 2 \int_0^{2\pi} \left[(2\pi - \tau) \left[2u_0^2 + \left(u_0^2 - 1 \right) \cos(\tau) \right] - \left(1 - 3u_0^2 \right) \sin(\tau) \right] \frac{\sin\left[(u_0 - \alpha) \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} \left[(2\pi - \tau) \left[2 + \cos(\tau) \right] + 3 \sin(\tau) \right] \frac{\sin\left[(u_0 + \alpha) \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} \left[(2\pi - \tau) \left[2 + \cos(\tau) \right] + 3 \sin(\tau) \right] \frac{\sin\left[(u_0 - \alpha) \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[(\tau - 2\pi) \sin(\tau) + 2 \cos(\tau) - 2 \right] \frac{\cos\left[(u_0 + \alpha) \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& - 4\alpha \int_0^{2\pi} \left[(\tau - 2\pi) \sin(\tau) + 2 \cos(\tau) - 2 \right] \frac{\cos\left[(u_0 - \alpha) \tau \right]}{\cancel{u_0} \tau} \, d\tau \Bigg\}.
\end{aligned}
\tag{C.40}
$$

Combining terms in Equation (C.40),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 8\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 8\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 4\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\cos(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 4\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\cos(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 2\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\cos(\tau)\sin\left[\left(u_0+\alpha\right)\tau\right]\,d\tau \\
& - 2\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\cos(\tau)\sin\left[\left(u_0-\alpha\right)\tau\right]\,d\tau \\
& + 2\left(3u_0^2-3\alpha^2-1\right)\int_0^{2\pi}\sin(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 2\left(3u_0^2-3\alpha^2-1\right)\int_0^{2\pi}\sin(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 4\alpha\int_0^{2\pi}\sin(\tau)\cos\left[\left(u_0+\alpha\right)\tau\right]\,d\tau \\
& - 4\alpha\int_0^{2\pi}\sin(\tau)\cos\left[\left(u_0-\alpha\right)\tau\right]\,d\tau \\
& - 8\pi\alpha\int_0^{2\pi}\sin(\tau)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 8\pi\alpha\int_0^{2\pi}\sin(\tau)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 8\alpha\int_0^{2\pi}\cos(\tau)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 8\alpha\int_0^{2\pi}\cos(\tau)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha\right)\tau\right]\,d\tau \\
& - 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha\right)\tau\right]\,d\tau \\
& - 8\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 8\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau\Bigg\}. \qquad \text{(C.41)}
\end{aligned}
$$

Applying product-to-sum trigonometric identities to Equation (C.41),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 8\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 8\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi} \frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]+\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi} \frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]+\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\, d\tau \\
& - \left(u_0^2-\alpha^2-1\right)\int_0^{2\pi} \left[\sin\left[\left(u_0+\alpha+1\right)\tau\right]+\sin\left[\left(u_0+\alpha-1\right)\tau\right]\right]\, d\tau \\
& - \left(u_0^2-\alpha^2-1\right)\int_0^{2\pi} \left[\sin\left[\left(u_0-\alpha+1\right)\tau\right]+\sin\left[\left(u_0-\alpha-1\right)\tau\right]\right]\, d\tau \\
& + \left(3u_0^2-3\alpha^2-1\right)\int_0^{2\pi} \frac{\cos\left[\left(u_0+\alpha-1\right)\tau\right]-\cos\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\, d\tau \\
& + \left(3u_0^2-3\alpha^2-1\right)\int_0^{2\pi} \frac{\cos\left[\left(u_0-\alpha-1\right)\tau\right]-\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\alpha\int_0^{2\pi} \left[\sin\left[\left(u_0+\alpha+1\right)\tau\right]-\sin\left[\left(u_0+\alpha-1\right)\tau\right]\right]\, d\tau \\
& - 2\alpha\int_0^{2\pi} \left[\sin\left[\left(u_0-\alpha+1\right)\tau\right]-\sin\left[\left(u_0-\alpha-1\right)\tau\right]\right]\, d\tau \\
& - 4\pi\alpha\int_0^{2\pi} \frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]-\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\, d\tau \\
& + 4\pi\alpha\int_0^{2\pi} \frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]-\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\, d\tau \\
& + 4\alpha\int_0^{2\pi} \frac{\cos\left[\left(u_0+\alpha-1\right)\tau\right]+\cos\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\, d\tau \\
& - 4\alpha\int_0^{2\pi} \frac{\cos\left[\left(u_0-\alpha-1\right)\tau\right]+\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\, d\tau \\
& - 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \sin\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& - 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \sin\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& - 8\alpha\int_0^{2\pi} \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 8\alpha\int_0^{2\pi} \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}. \qquad \text{(C.42)}
\end{aligned}
$$

Collecting terms in Equation (C.42),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 8\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 8\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 2\pi\left(u_0^2-\alpha^2-2\alpha-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& + 2\pi\left(u_0^2-\alpha^2+2\alpha-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& + 2\pi\left(u_0^2-\alpha^2+2\alpha-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& + 2\pi\left(u_0^2-\alpha^2-2\alpha-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& - \left(3u_0^2-3\alpha^2-1\right)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha-1\right)\tau\right]-1+\cos\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& - \left(3u_0^2-3\alpha^2-1\right)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0-\alpha-1\right)\tau\right]-1+\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& - 4\alpha\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha+1\right)\tau\right]-1+\cos\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& - 4\alpha\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha-1\right)\tau\right]-1+\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& + 8\alpha\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha\right)\tau\right]-1+\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& - \left(u_0^2-\alpha^2-2\alpha-1\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha+1\right)\tau\right]\ d\tau \\
& - \left(u_0^2-\alpha^2+2\alpha-1\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha-1\right)\tau\right]\ d\tau \\
& - \left(u_0^2-\alpha^2+2\alpha-1\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha+1\right)\tau\right]\ d\tau \\
& - \left(u_0^2-\alpha^2-2\alpha-1\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha-1\right)\tau\right]\ d\tau \\
& - 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& - 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau\Bigg\}. \qquad \text{(C.43)}
\end{aligned}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^b \frac{\sin(at)}{t}\;dt \tag{C.44}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos(at)}{t}\;dt. \tag{C.45}$$

Applying Equations (C.44) and (C.45) to Equation (C.43) and performing the remaining integrals,

$$\begin{aligned}\widetilde{R}_h(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 8\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 2\pi\left(u_0^2-\alpha^2-2\alpha-1\right)\mathrm{Si}\left[2\pi\left(u_0+\alpha+1\right)\right] \\ & + 2\pi\left(u_0^2-\alpha^2+2\alpha-1\right)\mathrm{Si}\left[2\pi\left(u_0+\alpha-1\right)\right] \\ & + 2\pi\left(u_0^2-\alpha^2+2\alpha-1\right)\mathrm{Si}\left[2\pi\left(u_0-\alpha+1\right)\right] \\ & + 2\pi\left(u_0^2-\alpha^2-2\alpha-1\right)\mathrm{Si}\left[2\pi\left(u_0-\alpha-1\right)\right] \\ & - \left(3u_0^2-3\alpha^2-1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right]\right] \\ & - \left(3u_0^2-3\alpha^2-1\right)\left[\mathrm{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & - 4\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+1\right)-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]\right]\right] \\ & - 4\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & + 8\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + \frac{\left(u_0^2-\alpha^2-2\alpha-1\right)}{\left(u_0+\alpha+1\right)}\cos\left[\left(u_0+\alpha+1\right)\tau\right]\Big|_0^{2\pi} \\ & + \frac{\left(u_0^2-\alpha^2+2\alpha-1\right)}{\left(u_0+\alpha-1\right)}\cos\left[\left(u_0+\alpha-1\right)\tau\right]\Big|_0^{2\pi} \\ & + \frac{\left(u_0^2-\alpha^2+2\alpha-1\right)}{\left(u_0-\alpha+1\right)}\cos\left[\left(u_0-\alpha+1\right)\tau\right]\Big|_0^{2\pi} \\ & + \frac{\left(u_0^2-\alpha^2-2\alpha-1\right)}{\left(u_0-\alpha-1\right)}\cos\left[\left(u_0-\alpha-1\right)\tau\right]\Big|_0^{2\pi} \\ & + 4\frac{\left(u_0^2-\alpha^2\right)}{u_0+\alpha}\cos\left[\left(u_0+\alpha\right)\tau\right]\Big|_0^{2\pi} + 4\frac{\left(u_0^2-\alpha^2\right)}{u_0-\alpha}\cos\left[\left(u_0-\alpha\right)\tau\right]\Big|_0^{2\pi}\Bigg\}.\end{aligned} \tag{C.46}$$

Evaluating the limits of integration and simplifying Equation (C.46),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{16\pi} \Big\{ & 8\alpha \left[\mathrm{Cin}\left[2\pi(u_0+\alpha)\right] - \mathrm{Cin}\left[2\pi(u_0-\alpha)\right]\right] \\
& + \left(3u_0^2 - 3\alpha^2 - 4\alpha - 1\right) \mathrm{Cin}\left[2\pi(u_0+\alpha+1)\right] \\
& - \left(3u_0^2 - 3\alpha^2 + 4\alpha - 1\right) \mathrm{Cin}\left[2\pi(u_0+\alpha-1)\right] \\
& + \left(3u_0^2 - 3\alpha^2 + 4\alpha - 1\right) \mathrm{Cin}\left[2\pi(u_0-\alpha+1)\right] \\
& - \left(3u_0^2 - 3\alpha^2 - 4\alpha - 1\right) \mathrm{Cin}\left[2\pi(u_0-\alpha-1)\right] \\
& + 8\pi\left(u_0^2 - \alpha^2\right) \left[\mathrm{Si}\left[2\pi(u_0+\alpha)\right] + \mathrm{Si}\left[2\pi(u_0-\alpha)\right]\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 - 2\alpha - 1\right) \mathrm{Si}\left[2\pi(u_0+\alpha+1)\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 + 2\alpha - 1\right) \mathrm{Si}\left[2\pi(u_0+\alpha-1)\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 + 2\alpha - 1\right) \mathrm{Si}\left[2\pi(u_0-\alpha+1)\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 - 2\alpha - 1\right) \mathrm{Si}\left[2\pi(u_0-\alpha-1)\right] \\
& + (u_0 - \alpha - 1)\left[\cos\left[2\pi(u_0+\alpha)\right] - 1\right] \\
& + (u_0 - \alpha + 1)\left[\cos\left[2\pi(u_0+\alpha)\right] - 1\right] \\
& + (u_0 + \alpha - 1)\left[\cos\left[2\pi(u_0-\alpha)\right] - 1\right] \\
& + (u_0 + \alpha + 1)\left[\cos\left[2\pi(u_0-\alpha)\right] - 1\right] \\
& + 4(u_0 - \alpha)\left[\cos\left[2\pi(u_0+\alpha)\right] - 1\right] \\
& + 4(u_0 + \alpha)\left[\cos\left[2\pi(u_0-\alpha)\right] - 1\right] \Big\}. \qquad \text{(C.47)}
\end{aligned}$$

Simplifying Equation (C.47) yields the stationary autocorrelation function $\widetilde{R}_h(0)$ for the cosine-squared distribution,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{16\pi} \Big\{ & 8\alpha \left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + \left(3u_0^2 - 3\alpha^2 - 4\alpha - 1\right) \mathrm{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right] \\
& - \left(3u_0^2 - 3\alpha^2 + 4\alpha - 1\right) \mathrm{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right] \\
& + \left(3u_0^2 - 3\alpha^2 + 4\alpha - 1\right) \mathrm{Cin}\left[2\pi\left(u_0-\alpha+1\right)\right] \\
& - \left(3u_0^2 - 3\alpha^2 - 4\alpha - 1\right) \mathrm{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right] \\
& + 8\pi\left(u_0^2 - \alpha^2\right) \left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 - 2\alpha - 1\right) \mathrm{Si}\left[2\pi\left(u_0+\alpha+1\right)\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 + 2\alpha - 1\right) \mathrm{Si}\left[2\pi\left(u_0+\alpha-1\right)\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 + 2\alpha - 1\right) \mathrm{Si}\left[2\pi\left(u_0-\alpha+1\right)\right] \\
& + 2\pi\left(u_0^2 - \alpha^2 - 2\alpha - 1\right) \mathrm{Si}\left[2\pi\left(u_0-\alpha-1\right)\right] \\
& + 6\left(u_0-\alpha\right)\cos\left[2\pi\left(u_0+\alpha\right)\right] + 6\left(u_0+\alpha\right)\cos\left[2\pi\left(u_0-\alpha\right)\right] - 12u_0 \Big\}. \quad \text{(C.48)}
\end{aligned}
$$

C.4 Triangular Distribution

The autocorrelation function $R_g(p)$ for the triangular distribution is,

$$
R_g(p) = \frac{A_m^2}{6\pi^2} \begin{cases} (2\pi+p)^3, & -2\pi \le p \le -\pi \\ 4\pi^3 - 6\pi p^2 - 3p^3, & -\pi \le p \le 0 \\ 4\pi^3 - 6\pi p^2 + 3p^3, & 0 \le p \le \pi \\ (2\pi-p)^3, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \quad \text{(C.49)}
$$

Finding the first derivative of $R_g(p)$:

$$R_g'(p) = \frac{A_m^2}{6\pi^2}\frac{d}{dp}\begin{cases} (2\pi+p)^3, & -2\pi \le p \le -\pi \\ 4\pi^3 - 6\pi p^2 - 3p^3, & -\pi \le p \le 0 \\ 4\pi^3 - 6\pi p^2 + 3p^3, & 0 \le p \le \pi \\ (2\pi-p)^3, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{C.50}$$

which can be shown to be

$$R_g'(p) = \frac{A_m^2}{6\pi^2}\begin{cases} 3(2\pi+p)^2, & -2\pi \le p \le -\pi \\ -12\pi p - 9p^2, & -\pi \le p \le 0 \\ -12\pi p + 9p^2, & 0 \le p \le \pi \\ -3(2\pi-p)^2, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{C.51}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{6\pi^2}\begin{cases} u_0^2(2\pi+p)^3 + 12\pi + 6p, & -2\pi \le p \le -\pi \\ u_0^2\left(4\pi^3 - 6\pi p^2 - 3p^3\right) - 12\pi - 18p, & -\pi \le p \le 0 \\ u_0^2\left(4\pi^3 - 6\pi p^2 + 3p^3\right) - 12\pi + 18p, & 0 \le p \le \pi \\ u_0^2(2\pi-p)^3 + 12\pi - 6p, & \pi \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{C.52}$$

Recalling the definition for $\widetilde{R}_h(0)$,

$$\begin{aligned} \widetilde{R}_h(0) = &\frac{u_0}{2\pi}\Bigg\{\int_{-\infty}^{\infty} R_f(\tau)\frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_f(\tau)\frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &- \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau - \alpha^2\int_{-\infty}^{\infty} R_g(\tau)\frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+ 2\alpha\int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau - 2\alpha\int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau\Bigg\}. \end{aligned} \tag{C.53}$$

Substituting Equations (C.49), (C.51), and (C.52) into Equation (C.53),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2 u_0}{12\pi^3} \Bigg\{ & \int_{-2\pi}^{-\pi} \left[u_0^2 (2\pi+\tau)^3 + 12\pi + 6\tau\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{-\pi}^{0} \left[u_0^2 \left(4\pi^3 - 6\pi\tau^2 - 3\tau^3\right) - 12\pi - 18\tau\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{0}^{\pi} \left[u_0^2 \left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) - 12\pi + 18\tau\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{\pi}^{2\pi} \left[u_0^2 (2\pi-\tau)^3 + 12\pi - 6\tau\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{-2\pi}^{-\pi} \left[u_0^2 (2\pi+\tau)^3 + 12\pi + 6\tau\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{-\pi}^{0} \left[u_0^2 \left(4\pi^3 - 6\pi\tau^2 - 3\tau^3\right) - 12\pi - 18\tau\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{0}^{\pi} \left[u_0^2 \left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) - 12\pi + 18\tau\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \\
& + \int_{\pi}^{2\pi} \left[u_0^2 (2\pi-\tau)^3 + 12\pi - 6\tau\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \qquad \text{(C.54)} \\
& - \alpha^2 \int_{-2\pi}^{-\pi} \left[(2\pi+\tau)^3\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau - \alpha^2 \int_{-\pi}^{0} \left[4\pi^3 - 6\pi\tau^2 - 3\tau^3\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{0}^{\pi} \left[4\pi^3 - 6\pi\tau^2 + 3\tau^3\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau - \alpha^2 \int_{\pi}^{2\pi} \left[(2\pi-\tau)^3\right] \frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{-2\pi}^{-\pi} \left[(2\pi+\tau)^3\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau - \alpha^2 \int_{-\pi}^{0} \left[4\pi^3 - 6\pi\tau^2 - 3\tau^3\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{0}^{\pi} \left[4\pi^3 - 6\pi\tau^2 + 3\tau^3\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau - \alpha^2 \int_{\pi}^{2\pi} \left[(2\pi-\tau)^3\right] \frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{-2\pi}^{-\pi} \left[3(2\pi+\tau)^2\right] \frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau + 2\alpha \int_{-\pi}^{0} \left[-12\pi\tau - 9\tau^2\right] \frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{0}^{\pi} \left[-12\pi\tau + 9\tau^2\right] \frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau + 2\alpha \int_{\pi}^{2\pi} \left[-3(2\pi-\tau)^2\right] \frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\, d\tau \\
& - 2\alpha \int_{-2\pi}^{-\pi} \left[(2\pi-\tau)^3\right] \frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau - 2\alpha \int_{-\pi}^{0} \left[-12\pi\tau - 9\tau^2\right] \frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \\
& - 2\alpha \int_{0}^{\pi} \left[-12\pi\tau + 9\tau^2\right] \frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau - 2\alpha \int_{\pi}^{2\pi} \left[-3(2\pi-\tau)^2\right] \frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\, d\tau \Bigg\}.
\end{aligned}
$$

Recognizing even and odd functions in Equation (C.54) and combining terms,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2 \cancel{u_0}}{12\pi^3} \Bigg\{ & 2\int_0^{\pi} \left[u_0^2\left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) - 12\pi + 18\tau\right] \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\int_{\pi}^{2\pi} \left[u_0^2\left(8\pi^3 - 12\pi^2\tau + 6\pi\tau^2 - \tau^3\right) + 12\pi - 6\tau\right] \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\int_0^{\pi} \left[u_0^2\left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) - 12\pi + 18\tau\right] \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\int_{\pi}^{2\pi} \left[u_0^2\left(8\pi^3 - 12\pi^2\tau + 6\pi\tau^2 - \tau^3\right) + 12\pi - 6\tau\right] \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\alpha^2\int_0^{\pi} \left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\alpha^2\int_{\pi}^{2\pi} \left(8\pi^3 - 12\pi^2\tau + 6\pi\tau^2 - \tau^3\right) \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\alpha^2\int_0^{\pi} \left(4\pi^3 - 6\pi\tau^2 + 3\tau^3\right) \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\alpha^2\int_{\pi}^{2\pi} \left(8\pi^3 - 12\pi^2\tau + 6\pi\tau^2 - \tau^3\right) \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 4\alpha\int_0^{\pi} \left(-12\pi\tau + 9\tau^2\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 12\alpha\int_{\pi}^{2\pi} \left(4\pi^2 - 4\pi\tau + \tau^2\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 4\alpha\int_0^{\pi} \left(-12\pi\tau + 9\tau^2\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 12\alpha\int_{\pi}^{2\pi} \left(4\pi^2 - 4\pi\tau + \tau^2\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{C.55}
$$

Combining terms in Equation (C.55),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & \left[8\pi^3\left(u_0^2-\alpha^2\right)-24\pi\right]\int_0^{\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 36\int_0^{\pi}\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau - 12\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi}\tau\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + 6\left(u_0^2-\alpha^2\right)\int_0^{\pi}\tau^2\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right)+24\pi\right]\int_{\pi}^{2\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& - \left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\int_{\pi}^{2\pi}\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + 12\pi\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi}\tau\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau - 2\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi}\tau^2\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + \left[8\pi^2\left(u_0^2-\alpha^2\right)-24\pi\right]\int_0^{\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 36\int_0^{\pi}\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau - 12\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi}\tau\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& + 6\left(u_0^2-\alpha^2\right)\int_0^{\pi}\tau^2\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right)+24\pi\right]\int_{\pi}^{2\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& - \left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\int_{\pi}^{2\pi}\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& + 12\pi\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi}\tau\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau - 2\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi}\tau^2\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& - 48\pi\alpha\int_0^{\pi}\cos\left[\left(u_0+\alpha\right)\tau\right]\ d\tau + 36\alpha\int_0^{\pi}\tau\cos\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + 48\pi\alpha\int_{\pi}^{2\pi}\cos\left[\left(u_0+\alpha\right)\tau\right]\ d\tau - 12\alpha\int_{\pi}^{2\pi}\tau\cos\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + 48\pi\alpha\int_0^{\pi}\cos\left[\left(u_0-\alpha\right)\tau\right]\ d\tau - 36\alpha\int_0^{\pi}\tau\cos\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& - 48\pi\alpha\int_{\pi}^{2\pi}\cos\left[\left(u_0-\alpha\right)\tau\right]\ d\tau + 12\alpha\int_{\pi}^{2\pi}\tau\cos\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& - 48\pi^2\alpha\int_{\pi}^{2\pi}\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau + 48\pi^2\alpha\int_{\pi}^{2\pi}\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau\Bigg\}. \qquad \text{(C.56)}
\end{aligned}
$$

Performing the integrals in Equation (C.56),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -\frac{36}{(u_0+\alpha)}\cos\left[(u_0+\alpha)\tau\right]\Bigg|_0^{\pi} + \frac{\left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]}{(u_0+\alpha)}\cos\left[(u_0+\alpha)\tau\right]\Bigg|_{\pi}^{2\pi} \\
& - 12\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0+\alpha)\tau\right]-(u_0+\alpha)\tau\cos\left[(u_0+\alpha)\tau\right]}{(u_0+\alpha)^2}\right]\Bigg|_0^{\pi} \\
& + 6\left(u_0^2-\alpha^2\right)\left[\frac{2(u_0+\alpha)\tau\sin\left[(u_0+\alpha)\tau\right]-\left[(u_0+\alpha)^2\tau^2-2\right]\cos\left[(u_0+\alpha)\tau\right]}{(u_0+\alpha)^3}\right]\Bigg|_0^{\pi} \\
& + 12\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0+\alpha)\tau\right]-(u_0+\alpha)\tau\cos\left[(u_0+\alpha)\tau\right]}{(u_0+\alpha)^2}\right]\Bigg|_{\pi}^{2\pi} \\
& - 2\left(u_0^2-\alpha^2\right)\left[\frac{2(u_0+\alpha)\tau\sin\left[(u_0+\alpha)\tau\right]-\left[(u_0+\alpha)^2\tau^2-2\right]\cos\left[(u_0+\alpha)\tau\right]}{(u_0+\alpha)^3}\right]\Bigg|_{\pi}^{2\pi} \\
& - \frac{36}{(u_0-\alpha)}\cos\left[(u_0-\alpha)\tau\right]\Bigg|_0^{\pi} + \frac{\left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]}{(u_0-\alpha)}\cos\left[(u_0-\alpha)\tau\right]\Bigg|_{\pi}^{2\pi} \\
& - 12\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0-\alpha)\tau\right]-(u_0-\alpha)\tau\cos\left[(u_0-\alpha)\tau\right]}{(u_0-\alpha)^2}\right]\Bigg|_0^{\pi} \\
& + 6\left(u_0^2-\alpha^2\right)\left[\frac{2(u_0-\alpha)\tau\sin\left[(u_0-\alpha)\tau\right]-\left[(u_0-\alpha)^2\tau^2-2\right]\cos\left[(u_0-\alpha)\tau\right]}{(u_0-\alpha)^3}\right]\Bigg|_0^{\pi} \\
& + 12\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0-\alpha)\tau\right]-(u_0-\alpha)\tau\cos\left[(u_0-\alpha)\tau\right]}{(u_0-\alpha)^2}\right]\Bigg|_{\pi}^{2\pi} \\
& - 2\left(u_0^2-\alpha^2\right)\left[\frac{2(u_0-\alpha)\tau\sin\left[(u_0-\alpha)\tau\right]-\left[(u_0-\alpha)^2\tau^2-2\right]\cos\left[(u_0-\alpha)\tau\right]}{(u_0-\alpha)^3}\right]\Bigg|_{\pi}^{2\pi} \\
& - \frac{48\pi\alpha}{(u_0+\alpha)}\sin\left[(u_0+\alpha)\tau\right]\Bigg|_0^{\pi} + 36\alpha\left[\frac{(u_0+\alpha)\tau\sin\left[(u_0+\alpha)\tau\right]+\cos\left[(u_0+\alpha)\tau\right]}{(u_0+\alpha)^2}\right]\Bigg|_0^{\pi} \\
& + \frac{48\pi\alpha}{(u_0+\alpha)}\sin\left[(u_0+\alpha)\tau\right]\Bigg|_{\pi}^{2\pi} - 12\alpha\left[\frac{(u_0+\alpha)\tau\sin\left[(u_0+\alpha)\tau\right]+\cos\left[(u_0+\alpha)\tau\right]}{(u_0+\alpha)^2}\right]\Bigg|_{\pi}^{2\pi} \\
& + \frac{48\pi\alpha}{(u_0-\alpha)}\sin\left[(u_0-\alpha)\tau\right]\Bigg|_0^{\pi} - 36\alpha\left[\frac{(u_0-\alpha)\tau\sin\left[(u_0-\alpha)\tau\right]+\cos\left[(u_0-\alpha)\tau\right]}{(u_0-\alpha)^2}\right]\Bigg|_0^{\pi} \\
& - \frac{48\pi\alpha}{(u_0-\alpha)}\sin\left[(u_0-\alpha)\tau\right]\Bigg|_{\pi}^{2\pi} + 12\alpha\left[\frac{(u_0-\alpha)\tau\sin\left[(u_0-\alpha)\tau\right]+\cos\left[(u_0-\alpha)\tau\right]}{(u_0-\alpha)^2}\right]\Bigg|_{\pi}^{2\pi} \\
& + \left[8\pi^3\left(u_0^2-\alpha^2\right)-24\pi\right]\int_0^{\pi}\frac{\sin\left[(u_0+\alpha)\tau\right]}{\tau}\,d\tau + \left[16\pi^3\left(u_0^2-\alpha^2\right)+24\pi\right]\int_{\pi}^{2\pi}\frac{\sin\left[(u_0+\alpha)\tau\right]}{\tau}\,d\tau \\
& + \left[8\pi^2\left(u_0^2-\alpha^2\right)-24\pi\right]\int_0^{\pi}\frac{\sin\left[(u_0-\alpha)\tau\right]}{\tau}\,d\tau + \left[16\pi^3\left(u_0^2-\alpha^2\right)+24\pi\right]\int_{\pi}^{2\pi}\frac{\sin\left[(u_0-\alpha)\tau\right]}{\tau}\,d\tau \\
& - 48\pi^2\alpha\int_{\pi}^{2\pi}\frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau}\,d\tau + 48\pi^2\alpha\int_{\pi}^{2\pi}\frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau}\,d\tau\Bigg\}. \qquad \text{(C.57)}
\end{aligned}
$$

Evaluating the limits of integration in Equation (C.57) and simplifying,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3} \Bigg\{ & \left[8\pi^2(u_0-\alpha) + \frac{4(2u_0+\alpha)}{(u_0+\alpha)^2}\right] \cos[2\pi(u_0+\alpha)] \\
& - \left[8\pi^2(u_0-\alpha) + \frac{16(2u_0+\alpha)}{(u_0+\alpha)^2}\right] \cos[\pi(u_0+\alpha)] \\
& + 4\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right) \sin[2\pi(u_0+\alpha)] \\
& - 8\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right) \sin[\pi(u_0+\alpha)] \\
& + \frac{36}{(u_0+\alpha)} - \frac{36\alpha}{(u_0+\alpha)^2} - \frac{12(u_0-\alpha)}{(u_0+\alpha)^2} \\
& + \left[8\pi^2(u_0+\alpha) + \frac{4(2u_0-\alpha)}{(u_0-\alpha)^2}\right] \cos[2\pi(u_0-\alpha)] \\
& - \left[8\pi^2(u_0+\alpha) + \frac{16(2u_0-\alpha)}{(u_0-\alpha)^2}\right] \cos[\pi(u_0-\alpha)] \\
& + 4\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right) \sin[2\pi(u_0-\alpha)] \\
& - 8\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right) \sin[\pi(u_0-\alpha)] \\
& + \frac{36}{(u_0-\alpha)} + \frac{36\alpha}{(u_0-\alpha)^2} - \frac{12(u_0+\alpha)}{(u_0-\alpha)^2} \\
& + \left[8\pi^3\left(u_0^2-\alpha^2\right) - 24\pi\right] \int_0^{\pi} \frac{\sin[(u_0+\alpha)\tau]}{\tau}\, d\tau \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right) + 24\pi\right] \int_{\pi}^{2\pi} \frac{\sin[(u_0+\alpha)\tau]}{\tau}\, d\tau \\
& + \left[8\pi^2\left(u_0^2-\alpha^2\right) - 24\pi\right] \int_0^{\pi} \frac{\sin[(u_0-\alpha)\tau]}{\tau}\, d\tau \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right) + 24\pi\right] \int_{\pi}^{2\pi} \frac{\sin[(u_0-\alpha)\tau]}{\tau}\, d\tau \\
& - 48\pi^2\alpha \int_{\pi}^{2\pi} \frac{\cos[(u_0+\alpha)\tau]}{\tau}\, d\tau + 48\pi^2\alpha \int_{\pi}^{2\pi} \frac{\cos[(u_0-\alpha)\tau]}{\tau}\, d\tau \Bigg\}. \quad \text{(C.58)}
\end{aligned}
$$

Further simplifying and rearranging Equation (C.58),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3} \Bigg\{ & \left[8\pi^2 (u_0 - \alpha) + \frac{4(2u_0 + \alpha)}{(u_0 + \alpha)^2}\right] \cos[2\pi (u_0 + \alpha)] \\
& + \left[8\pi^2 (u_0 + \alpha) + \frac{4(2u_0 - \alpha)}{(u_0 - \alpha)^2}\right] \cos[2\pi (u_0 - \alpha)] \\
& - \left[8\pi^2 (u_0 - \alpha) + \frac{16(2u_0 + \alpha)}{(u_0 + \alpha)^2}\right] \cos[\pi (u_0 + \alpha)] \\
& - \left[8\pi^2 (u_0 + \alpha) + \frac{16(2u_0 - \alpha)}{(u_0 - \alpha)^2}\right] \cos[\pi (u_0 - \alpha)] \\
& + 4\pi \left(\frac{u_0 + 5\alpha}{u_0 + \alpha}\right) \sin[2\pi (u_0 + \alpha)] \\
& + 4\pi \left(\frac{u_0 - 5\alpha}{u_0 - \alpha}\right) \sin[2\pi (u_0 - \alpha)] \\
& - 8\pi \left(\frac{u_0 + 5\alpha}{u_0 + \alpha}\right) \sin[\pi (u_0 + \alpha)] \\
& - 8\pi \left(\frac{u_0 - 5\alpha}{u_0 - \alpha}\right) \sin[\pi (u_0 - \alpha)] \\
& + 12 \left[\frac{2u_0 + \alpha}{(u_0 + \alpha)^2} + \frac{2u_0 - \alpha}{(u_0 - \alpha)^2}\right] \\
& + \left[16\pi^3 \left(u_0^2 - \alpha^2\right) + 24\pi\right] \int_0^{2\pi} \frac{\sin[(u_0 + \alpha)\tau]}{\tau}\, d\tau \\
& + \left[16\pi^3 \left(u_0^2 - \alpha^2\right) + 24\pi\right] \int_0^{2\pi} \frac{\sin[(u_0 - \alpha)\tau]}{\tau}\, d\tau \\
& - \left[8\pi^3 \left(u_0^2 - \alpha^2\right) + 48\pi\right] \int_0^{\pi} \frac{\sin[(u_0 + \alpha)\tau]}{\tau}\, d\tau \\
& - \left[8\pi^3 \left(u_0^2 - \alpha^2\right) + 48\pi\right] \int_0^{\pi} \frac{\sin[(u_0 - \alpha)\tau]}{\tau}\, d\tau \\
& + 48\pi^2 \alpha \int_0^{2\pi} \frac{1 - \cos[(u_0 + \alpha)\tau] - 1 + \cos[(u_0 - \alpha)\tau]}{\tau}\, d\tau \\
& - 48\pi^2 \alpha \int_0^{\pi} \frac{1 - \cos[(u_0 + \alpha)\tau] - 1 + \cos[(u_0 - \alpha)\tau]}{\tau}\, d\tau \Bigg\}. \qquad \text{(C.59)}
\end{aligned}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}(ba) = \int_0^b \frac{\sin(at)}{t}\, dt \qquad \text{(C.60)}$$

and

$$\operatorname{Cin}(ba) = \int_0^b \frac{1-\cos(at)}{t}\ dt. \tag{C.61}$$

Applying Equations (C.60) and (C.61) to Equation (C.59),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & \left[8\pi^2(u_0-\alpha) + \frac{4(2u_0+\alpha)}{(u_0+\alpha)^2}\right]\cos[2\pi(u_0+\alpha)] \\
& + \left[8\pi^2(u_0+\alpha) + \frac{4(2u_0-\alpha)}{(u_0-\alpha)^2}\right]\cos[2\pi(u_0-\alpha)] \\
& - \left[8\pi^2(u_0-\alpha) + \frac{16(2u_0+\alpha)}{(u_0+\alpha)^2}\right]\cos[\pi(u_0+\alpha)] \\
& - \left[8\pi^2(u_0+\alpha) + \frac{16(2u_0-\alpha)}{(u_0-\alpha)^2}\right]\cos[\pi(u_0-\alpha)] \\
& + 4\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right)\sin[2\pi(u_0+\alpha)] \\
& + 4\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right)\sin[2\pi(u_0-\alpha)] \\
& - 8\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right)\sin[\pi(u_0+\alpha)] \\
& - 8\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right)\sin[\pi(u_0-\alpha)] \\
& + 12\left[\frac{2u_0+\alpha}{(u_0+\alpha)^2} + \frac{2u_0-\alpha}{(u_0-\alpha)^2}\right] \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right) + 24\pi\right]\operatorname{Si}[2\pi(u_0+\alpha)] \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right) + 24\pi\right]\operatorname{Si}[2\pi(u_0-\alpha)] \\
& - \left[8\pi^3\left(u_0^2-\alpha^2\right) + 48\pi\right]\operatorname{Si}[\pi(u_0+\alpha)] \\
& - \left[8\pi^3\left(u_0^2-\alpha^2\right) + 48\pi\right]\operatorname{Si}[\pi(u_0-\alpha)] \\
& + 48\pi^2\alpha\left[\operatorname{Cin}[2\pi(u_0+\alpha)] - \operatorname{Cin}[2\pi(u_0-\alpha)]\right] \\
& - 48\pi^2\alpha\left[\operatorname{Cin}[\pi(u_0+\alpha)] - \operatorname{Cin}[\pi(u_0-\alpha)]\right]\Bigg\}.
\end{aligned} \tag{C.62}$$

Rearranging Equation (C.62) yields the stationary autocorrelation function $\widetilde{R}_h(0)$ for the triangular distribution,

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{12\pi^3} \Bigg\{ & 48\pi^2\alpha\left[\text{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \text{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& - 48\pi^2\alpha\left[\text{Cin}\left[\pi\left(u_0+\alpha\right)\right] - \text{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& + \left[16\pi^3\left(u_0^2-\alpha^2\right) + 24\pi\right]\left[\text{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \text{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& - \left[8\pi^3\left(u_0^2-\alpha^2\right) + 48\pi\right]\left[\text{Si}\left[\pi\left(u_0+\alpha\right)\right] + \text{Si}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& + \left[8\pi^2\left(u_0-\alpha\right) + \frac{4\left(2u_0+\alpha\right)}{\left(u_0+\alpha\right)^2}\right]\cos\left[2\pi\left(u_0+\alpha\right)\right] \\
& + \left[8\pi^2\left(u_0+\alpha\right) + \frac{4\left(2u_0-\alpha\right)}{\left(u_0-\alpha\right)^2}\right]\cos\left[2\pi\left(u_0-\alpha\right)\right] \\
& - \left[8\pi^2\left(u_0-\alpha\right) + \frac{16\left(2u_0+\alpha\right)}{\left(u_0+\alpha\right)^2}\right]\cos\left[\pi\left(u_0+\alpha\right)\right] \\
& - \left[8\pi^2\left(u_0+\alpha\right) + \frac{16\left(2u_0-\alpha\right)}{\left(u_0-\alpha\right)^2}\right]\cos\left[\pi\left(u_0-\alpha\right)\right] \\
& + 4\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right)\sin\left[2\pi\left(u_0+\alpha\right)\right] \\
& + 4\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right)\sin\left[2\pi\left(u_0-\alpha\right)\right] \\
& - 8\pi\left(\frac{u_0+5\alpha}{u_0+\alpha}\right)\sin\left[\pi\left(u_0+\alpha\right)\right] \\
& - 8\pi\left(\frac{u_0-5\alpha}{u_0-\alpha}\right)\sin\left[\pi\left(u_0-\alpha\right)\right] \\
& + 12\left[\frac{2u_0+\alpha}{\left(u_0+\alpha\right)^2} + \frac{2u_0-\alpha}{\left(u_0-\alpha\right)^2}\right]\Bigg\}.
\end{aligned} \tag{C.63}$$

C.5 Uniform Distribution

The autocorrelation function $R_g(p)$ for the uniform distribution is,

$$R_g(p) = A_m^2 \begin{cases} 2\pi + p, & -2\pi \le p \le 0 \\ 2\pi - p, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{C.64}$$

Finding the first derivative of $R_g(p)$:

$$R_g'(p) = A_m^2 \frac{d}{dp} \begin{cases} 2\pi + p, & -2\pi \le p \le 0 \\ 2\pi - p, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{C.65}$$

which can be simplified to be

$$R_g'(p) = A_m^2 \begin{cases} 1, & -2\pi \le p \le 0 \\ -1, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{C.66}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2 \left[\delta\left(p + 2\pi\right) - 2\delta\left(p\right) + \delta\left(p - 2\pi\right)\right] + u_0^2 R_g(p) \tag{C.67}$$

for $-2\pi \le p \le 2\pi$ and $R_f(p) = 0$ otherwise. Recalling the definition for $\widetilde{R}_h(0)$,

$$\begin{aligned} \widetilde{R}_h\left(0\right) = & \frac{u_0}{2\pi} \Bigg\{ \int_{-\infty}^{\infty} R_f\left(\tau\right) \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau} \, d\tau + \int_{-\infty}^{\infty} R_f\left(\tau\right) \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau} \, d\tau \\ & - \alpha^2 \int_{-\infty}^{\infty} R_g\left(\tau\right) \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau} \, d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g\left(\tau\right) \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau} \, d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} R_g'\left(\tau\right) \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau} \, d\tau - 2\alpha \int_{-\infty}^{\infty} R_g'\left(\tau\right) \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau} \, d\tau \Bigg\}. \end{aligned} \tag{C.68}$$

Substituting Equations (C.64), (C.66), and (C.67) into Equation (C.68),

$$\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2 u_0}{2\pi} \Bigg\{ & \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau \\
& + u_0^2 \int_{-2\pi}^{0} (2\pi + \tau) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau + u_0^2 \int_{0}^{2\pi} (2\pi - \tau) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau \\
& + u_0^2 \int_{-2\pi}^{0} (2\pi + \tau) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau + u_0^2 \int_{0}^{2\pi} (2\pi - \tau) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau \\
& - \alpha^2 \int_{-2\pi}^{0} (2\pi + \tau) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau - \alpha^2 \int_{0}^{2\pi} (2\pi - \tau) \frac{\sin[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau \\
& - \alpha^2 \int_{-2\pi}^{0} (2\pi + \tau) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau - \alpha^2 \int_{0}^{2\pi} (2\pi - \tau) \frac{\sin[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau \\
& + 2\alpha \int_{-2\pi}^{0} \frac{\cos[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau - 2\alpha \int_{0}^{2\pi} \frac{\cos[(u_0+\alpha)\tau]}{u_0 \tau}\, d\tau \\
& - 2\alpha \int_{-2\pi}^{0} \frac{\cos[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau + 2\alpha \int_{0}^{2\pi} \frac{\cos[(u_0-\alpha)\tau]}{u_0 \tau}\, d\tau \Bigg\}. \qquad \text{(C.69)}
\end{aligned}$$

Recognizing even and odd functions in Equation (C.69) and combining terms,

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2 \cancel{u_0}}{2\pi} \Bigg\{ & \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin[(u_0 + \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& - 2 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin[(u_0 + \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin[(u_0 + \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin[(u_0 - \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& - 2 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin[(u_0 - \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin[(u_0 - \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& + 2u_0^2 \int_0^{2\pi} (2\pi - \tau) \frac{\sin[(u_0 + \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& + 2u_0^2 \int_0^{2\pi} (2\pi - \tau) \frac{\sin[(u_0 - \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} (2\pi - \tau) \frac{\sin[(u_0 + \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} (2\pi - \tau) \frac{\sin[(u_0 - \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& - 4\alpha \int_0^{2\pi} \frac{\cos[(u_0 + \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \frac{\cos[(u_0 - \alpha)\tau]}{\cancel{u_0}\tau} \, d\tau \Bigg\}. \qquad \text{(C.70)}
\end{aligned}
$$

Combining terms in Equation (C.70),

$$
\begin{aligned}
\widetilde{R}_h(0) = \frac{A_m^2}{2\pi} \Bigg\{ & \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin[(u_0 + \alpha)\tau]}{\tau} \, d\tau \\
& - 2 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin[(u_0 + \alpha)\tau]}{\tau} \, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin[(u_0 + \alpha)\tau]}{\tau} \, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin[(u_0 - \alpha)\tau]}{\tau} \, d\tau \\
& - 2 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin[(u_0 - \alpha)\tau]}{\tau} \, d\tau \\
& + \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin[(u_0 - \alpha)\tau]}{\tau} \, d\tau \\
& + 2\left(u_0^2 - \alpha^2\right) \int_0^{2\pi} (2\pi - \tau) \frac{\sin[(u_0 + \alpha)\tau]}{\tau} \, d\tau \\
& + 2\left(u_0^2 - \alpha^2\right) \int_0^{2\pi} (2\pi - \tau) \frac{\sin[(u_0 - \alpha)\tau]}{\tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \frac{1 - \cos[(u_0 + \alpha)\tau] - 1 + \cos[(u_0 - \alpha)\tau]}{\tau} \, d\tau \Bigg\}. \qquad \text{(C.71)}
\end{aligned}
$$

Recalling the sifting property of the delta function,

$$
\int_{-\infty}^{\infty} \delta(x - a) f(x) \, dx = f(a). \qquad \text{(C.72)}
$$

Applying Equation (C.72) to Equation (C.71) and separating the integrals including the sine function,

$$
\begin{aligned}
\widetilde{R}_h\left(0\right) = \frac{A_m^2}{2\pi}\Bigg\{ & \frac{\sin\left[\not{-}2\pi\left(u_0+\alpha\right)\right]}{\not{-}2\pi} + \frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} \\
& + \frac{\sin\left[\not{-}2\pi\left(u_0-\alpha\right)\right]}{\not{-}2\pi} + \frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} \\
& - 2\left(u_0+\alpha\right) - 2\left(u_0-\alpha\right) \\
& - 2\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \sin\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& - 2\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \sin\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& + 4\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 4\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 4\alpha\int_0^{2\pi} \frac{1-\cos\left[\left(u_0+\alpha\right)\tau\right]-1+\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}.
\end{aligned} \tag{C.73}
$$

Recalling the definitions for the sine and modified cosine integrals,

$$
\mathrm{Si}\left(ba\right) = \int_0^b \frac{\sin\left(at\right)}{t}\, dt \tag{C.74}
$$

and

$$
\mathrm{Cin}\left(ba\right) = \int_0^b \frac{1-\cos\left(at\right)}{t}\, dt. \tag{C.75}
$$

Applying Equations (C.74) and (C.75) to Equation (C.73) and simplfying,

$$\begin{aligned}\widetilde{R}_h(0) = \frac{A_m^2}{2\pi}\Bigg\{ & \frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{\pi} + \frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{\pi} - 4u_0 \\ & - 2\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\ & - 2\left(u_0^2-\alpha^2\right)\int_0^{2\pi} \sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\ & + 4\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 4\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right]\Bigg\}. \end{aligned} \tag{C.76}$$

Performing the integrals in Equation (C.76),

$$\begin{aligned}\widetilde{R}_h(0) = \frac{A_m^2}{2\pi}\Bigg\{ & \frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{\pi} + \frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{\pi} - 4u_0 \\ & + 2\left(\frac{u_0^2-\alpha^2}{u_0+\alpha}\right)\cos\left[\left(u_0+\alpha\right)\tau\right]\Big|_0^{2\pi} \\ & + 2\left(\frac{u_0^2-\alpha^2}{u_0-\alpha}\right)\cos\left[\left(u_0-\alpha\right)\tau\right]\Big|_0^{2\pi} \\ & + 4\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 4\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right]\Bigg\}. \end{aligned} \tag{C.77}$$

Evaluating the limits of integration and simplifying Equation (C.77),

$$\begin{aligned}\widetilde{R}_h(0) = \frac{A_m^2}{2\pi}\Bigg\{ & \frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{\pi} + \frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{\pi} - 4u_0 \\ & + 2\left(u_0-\alpha\right)\left[\cos\left[2\pi\left(u_0+\alpha\right)\right]-1\right] \\ & + 2\left(u_0+\alpha\right)\left[\cos\left[2\pi\left(u_0-\alpha\right)\right]-1\right] \\ & + 4\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 4\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right]\Bigg\}. \end{aligned} \tag{C.78}$$

Simplifying and rearranging Equation (C.78) yields the stationary autocorrelation function $\widetilde{R}_h(0)$ for the uniform distribution,

$$\begin{aligned}\widetilde{R}_h\left(0\right) = \frac{A_m^2}{\pi}\Bigg\{ & 2\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 2\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + \frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} + \frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} \\ & + \left(u_0-\alpha\right)\cos\left[2\pi\left(u_0+\alpha\right)\right] + \left(u_0+\alpha\right)\cos\left[2\pi\left(u_0-\alpha\right)\right] - 4u_0\Bigg\}. \end{aligned} \tag{C.79}$$

Appendix D

Derivations for the Pattern Mean and Beamwidth Variance of a Scanning Line Source

This appendix presents the detailed derivations of the autocorrelation functions used to determine the pattern mean and beamwidth variance for a scanning line source radiator. The derivations were performed for the half-wave dipole, cosine, cosine-squared, triangular, and uniform distributions.

D.1 Half-Wave Dipole Distribution

The autocorrelation function $R_g(p)$ for the half-wave dipole is,

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.1}$$

Recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{2} \begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.2}$$

Finding the second derivative of $R_g(p)$:

$$R_g''(p) = \frac{A_m^2}{2}\frac{d}{dp}\begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.3}$$

where

$$R_g''(p) = \frac{A_m^2}{2}\begin{cases} \frac{1}{2}\left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \frac{1}{2}\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \frac{1}{2}\left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \frac{1}{2}\sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.4}$$

which can be simplified to be

$$R_g''(p) = \frac{A_m^2}{4}\begin{cases} \left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.5}$$

Finding the third derivative of $R_g(p)$:

$$R_g'''(p) = \frac{A_m^2}{4}\frac{d}{dp}\begin{cases} \left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.6}$$

where

$$R_g'''(p) = \frac{A_m^2}{4}\begin{cases} -\frac{1}{2}\left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right) - \frac{1}{2}\cos\left(\frac{p}{2}\right) - \frac{1}{2}\cos\left(\frac{1}{2}\right), & -2\pi \le p \le 0 \\ -\frac{1}{2}\left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right) + \frac{1}{2}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.7}$$

which can be simplified to be

$$R_g'''(p) = \frac{A_m^2}{4}\begin{cases} \frac{1}{2}\left(\frac{p}{2}+\pi\right)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ -\frac{1}{2}\left(\frac{p}{2}-\pi\right)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.8}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{2}\begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.9}$$

Finding the first derivative of $R_f(p)$:

$$R_f'(p) = \frac{A_m^2}{2}\frac{d}{dp}\begin{cases} -\sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.10}$$

where

$$R_f'(p) = \frac{A_m^2}{2}\begin{cases} -\frac{1}{2}\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \frac{1}{2}\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.11}$$

which can be simplified to be

$$R_f'(p) = \frac{A_m^2}{4}\begin{cases} -\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.12}$$

Recalling the autocorrelation function $R_n(p)$,

$$R_n(p) = \frac{A_m^2}{4}\left[\frac{1}{2}\sin\left(\frac{|p|}{2}\right) - \delta\left(p+2\pi\right) - 2\delta(p) - \delta\left(p-2\pi\right)\right] \tag{D.13}$$

for $-2\pi \leq p \leq 2\pi$ and $R_n(p) = 0$ otherwise. Recalling the definition for $\widetilde{R}_{m1}(0)$,

$$\begin{aligned}\widetilde{R}_{m1}(0) =& \frac{u_0}{2\pi}\Bigg\{ -\int_{-\infty}^{\infty} R_f'(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_f'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ \alpha^2\int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau - \alpha^2\int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ 2\alpha\int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + 2\alpha\int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.14}$$

Substituting Equations (D.12), (D.2), and (D.4) into Equation (D.14),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2 u_0}{8\pi}\Bigg\{ &\int_{-2\pi}^{0} \cos\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- \int_{0}^{2\pi} \cos\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- \int_{-2\pi}^{0} \cos\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ \int_{0}^{2\pi} \cos\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- \alpha^2\int_{-2\pi}^{0} (\tau+2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ \alpha^2\int_{0}^{2\pi} (\tau-2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ \alpha^2\int_{-2\pi}^{0} (\tau+2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- \alpha^2\int_{0}^{2\pi} (\tau-2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- 2\alpha\int_{-2\pi}^{0} \left[\left(\frac{\tau}{2}+\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ 2\alpha\int_{0}^{2\pi} \left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- 2\alpha\int_{-2\pi}^{0} \left[\left(\frac{\tau}{2}+\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ 2\alpha\int_{0}^{2\pi} \left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.15}$$

Recognizing even and odd functions in Equation (D.15) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 \cancel{u_0}}{8\pi} \Bigg\{ & - 2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} (\tau - 2\pi) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} (\tau - 2\pi) \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[\left(\frac{\tau}{2} - \pi\right) \cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[\left(\frac{\tau}{2} - \pi\right) \cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau} \, d\tau \Bigg\}. \qquad \text{(D.16)}
\end{aligned}
$$

Combining terms in Equation (D.79) and inserting $u_0 = \frac{1}{2}$ for the half-wave dipole,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi} \Bigg\{ & -2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right] d\tau \\
& - 2\alpha^2 \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right] d\tau \\
& - 4\pi\alpha^2 \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 4\pi\alpha^2 \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 2\alpha \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right] d\tau \\
& + 2\alpha \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right] d\tau \\
& - 4\pi\alpha \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 4\pi\alpha \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}. \qquad \text{(D.17)}
\end{aligned}
$$

Applying product-to-sum trigonometric identities to Equation (D.17),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & -\int_0^{2\pi} \frac{\cos(\alpha\tau) + \cos[(1+\alpha)\tau]}{\tau}\ d\tau \\
& + \int_0^{2\pi} \frac{\cos(\alpha\tau) + \cos[(1-\alpha)\tau]}{\tau}\ d\tau \\
& + \alpha^2 \int_0^{2\pi} [\sin[(1+\alpha)\tau] - \sin(\alpha\tau)]\ d\tau \\
& - \alpha^2 \int_0^{2\pi} [\sin[(1-\alpha)\tau] + \sin(\alpha\tau)]\ d\tau \\
& - 2\pi\alpha^2 \int_0^{2\pi} \frac{\sin[(1+\alpha)\tau] - \sin(\alpha\tau)}{\tau}\ d\tau \\
& + 2\pi\alpha^2 \int_0^{2\pi} \frac{\sin[(1-\alpha)\tau] + \sin(\alpha\tau)}{\tau}\ d\tau \\
& + \alpha \int_0^{2\pi} [\sin[(1+\alpha)\tau] + \sin(\alpha\tau)]\ d\tau \\
& + \alpha \int_0^{2\pi} [\sin[(1-\alpha)\tau] - \sin(\alpha\tau)]\ d\tau \\
& - 2\pi\alpha \int_0^{2\pi} \frac{\sin[(1+\alpha)\tau] + \sin(\alpha\tau)}{\tau}\ d\tau \\
& - 2\pi\alpha \int_0^{2\pi} \frac{\sin[(1-\alpha)\tau] - \sin(\alpha\tau)}{\tau}\ d\tau \\
& + 2\alpha \int_0^{2\pi} \frac{\cos(\alpha\tau) - \cos[(1+\alpha)\tau]}{\tau}\ d\tau \\
& + 2\alpha \int_0^{2\pi} \frac{\cos(\alpha\tau) - \cos[(1-\alpha)\tau]}{\tau}\ d\tau \Bigg\}. \qquad \text{(D.18)}
\end{aligned}$$

Collecting terms in Equation (D.51),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \int_0^{2\pi} \frac{1-\cos\left[(1+\alpha)\,\tau\right] - 1 + \cos\left[(1-\alpha)\,\tau\right]}{\tau}\ d\tau \\
& + \alpha^2 \int_0^{2\pi} \sin\left[(1+\alpha)\,\tau\right]\ d\tau \\
& - 2\alpha^2 \int_0^{2\pi} \sin(\alpha\tau)\ d\tau \\
& - \alpha^2 \int_0^{2\pi} \sin\left[(1-\alpha)\,\tau\right]\ d\tau \\
& - 2\pi\alpha^2 \int_0^{2\pi} \frac{\sin\left[(1+\alpha)\,\tau\right] - \sin(\alpha\tau)}{\tau}\ d\tau \\
& + 2\pi\alpha^2 \int_0^{2\pi} \frac{\sin\left[(1-\alpha)\,\tau\right] + \sin(\alpha\tau)}{\tau}\ d\tau \\
& + \alpha \int_0^{2\pi} \sin\left[(1+\alpha)\,\tau\right]\ d\tau \\
& + \alpha \int_0^{2\pi} \sin\left[(1-\alpha)\,\tau\right]\ d\tau \\
& - 2\pi\alpha \int_0^{2\pi} \frac{\sin\left[(1+\alpha)\,\tau\right] + \sin\left[(1-\alpha)\,\tau\right]}{\tau}\ d\tau \\
& + 2\alpha \int_0^{2\pi} \frac{1-\cos\left[(1+\alpha)\,\tau\right] - 1 + \cos(\alpha\tau)}{\tau}\ d\tau \\
& + 2\alpha \int_0^{2\pi} \frac{1-\cos\left[(1-\alpha)\,\tau\right] - 1 + \cos(\alpha\tau)}{\tau}\ d\tau \Bigg\}.
\end{aligned}
\tag{D.19}
$$

Recalling the definitions for the sine and modified cosine integrals,

$$
\mathrm{Si}(ba) = \int_0^b \frac{\sin(at)}{t}\ dt \tag{D.20}
$$

and

$$
\mathrm{Cin}(ba) = \int_0^b \frac{1-\cos(at)}{t}\ dt. \tag{D.21}
$$

Applying Equations (D.20) and (D.21) to Equation (D.19) and performing the remaining integrals,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \left[\mathrm{Cin}\left[2\pi(1+\alpha)\right] - \mathrm{Cin}\left[2\pi(1-\alpha)\right]\right] \\
& - \alpha^2 \left[\frac{\cos\left[(1+\alpha)\tau\right]}{1+\alpha}\right]_0^{2\pi} \\
& + 2\alpha^2 \left[\frac{\cos(\alpha\tau)}{\alpha}\right]_0^{2\pi} \\
& + \alpha^2 \left[\frac{\cos\left[(1-\alpha)\tau\right]}{1-\alpha}\right]_0^{2\pi} \\
& - 2\pi\alpha^2 \left[\mathrm{Si}\left[2\pi(1+\alpha)\right] - 2\mathrm{Si}(2\pi\alpha) - \mathrm{Si}\left[2\pi(1-\alpha)\right]\right] \\
& - \alpha \left[\frac{\cos\left[(1+\alpha)\tau\right]}{1+\alpha}\right]_0^{2\pi} \\
& - \alpha \left[\frac{\cos\left[(1-\alpha)\tau\right]}{1-\alpha}\right]_0^{2\pi} \\
& - 2\pi\alpha \left[\mathrm{Si}\left[2\pi(1+\alpha)\right] + \mathrm{Si}\left[2\pi(1-\alpha)\right]\right] \\
& + 2\alpha \left[\mathrm{Cin}\left[2\pi(1+\alpha)\right] - 2\mathrm{Cin}(2\pi\alpha) + \mathrm{Cin}\left[2\pi(1-\alpha)\right]\right] \Bigg\}.
\end{aligned}
\tag{D.22}
$$

Evaluating the limits of integration and simplifying Equation (D.22),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & [\mathrm{Cin}[2\pi(1+\alpha)] - \mathrm{Cin}[2\pi(1-\alpha)]] \\ & - \alpha^2\left[\frac{\cos[2\pi(1+\alpha)]-1}{1+\alpha}\right] \\ & + 2\alpha^2\left[\frac{\cos(2\pi\alpha)-1}{\alpha}\right] \\ & + \alpha^2\left[\frac{\cos[2\pi(1-\alpha)]-1}{1-\alpha}\right] \\ & - 2\pi\alpha^2[\mathrm{Si}[2\pi(1+\alpha)] - 2\mathrm{Si}(2\pi\alpha) - \mathrm{Si}[2\pi(1-\alpha)]] \\ & - \alpha\left[\frac{\cos[2\pi(1+\alpha)]-1}{1+\alpha}\right] \\ & - \alpha\left[\frac{\cos[2\pi(1-\alpha)]-1}{1-\alpha}\right] \\ & - 2\pi\alpha[\mathrm{Si}[2\pi(1+\alpha)] + \mathrm{Si}[2\pi(1-\alpha)]] \\ & + 2\alpha[\mathrm{Cin}[2\pi(1+\alpha)] - 2\mathrm{Cin}(2\pi\alpha) + \mathrm{Cin}[2\pi(1-\alpha)]]\Bigg\}. \qquad \text{(D.23)}\end{aligned}$$

Simplifying Equation (D.23) yields the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the half-wave dipole,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \mathrm{Cin}[2\pi(1+\alpha)] - \mathrm{Cin}[2\pi(1-\alpha)] \\ & + 2\alpha[\mathrm{Cin}[2\pi(1+\alpha)] - 2\mathrm{Cin}(2\pi\alpha) + \mathrm{Cin}[2\pi(1-\alpha)]] \\ & - 2\pi\alpha^2[\mathrm{Si}[2\pi(1+\alpha)] - 2\mathrm{Si}(2\pi\alpha) - \mathrm{Si}[2\pi(1-\alpha)]] \\ & - 2\pi\alpha[\mathrm{Si}[2\pi(1+\alpha)] + \mathrm{Si}[2\pi(1-\alpha)]]\Bigg\}. \qquad \text{(D.24)}\end{aligned}$$

Recalling the definition for $\widetilde{R}_{m2}(0)$,

$$\begin{aligned}\widetilde{R}_{m2}(0) =& \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_n(\tau)\,\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_n(\tau)\,\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ \alpha^2 \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \alpha^2 \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &- 2\alpha \int_{-\infty}^{\infty} R_g'''(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + 2\alpha \int_{-\infty}^{\infty} R_g'''(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.25}$$

Substituting Equations (D.76), (D.69), and (D.72) into Equation (D.25),

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2 u_0}{8\pi}\Bigg\{ &-\frac{1}{2}\int_{-2\pi}^{0} \sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \frac{1}{2}\int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-\frac{1}{2}\int_{-2\pi}^{0} \sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \frac{1}{2}\int_{0}^{2\pi} \sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-\int_{-\infty}^{\infty} \delta(\tau+2\pi)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau - \int_{-\infty}^{\infty} \delta(\tau+2\pi)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-2\int_{-\infty}^{\infty} \delta(\tau)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau - 2\int_{-\infty}^{\infty} \delta(\tau)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-\int_{-\infty}^{\infty} \delta(\tau-2\pi)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau - \int_{-\infty}^{\infty} \delta(\tau-2\pi)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-\alpha^2\int_{-2\pi}^{0}\left[\left(\frac{\tau}{2}+\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+\alpha^2\int_{0}^{2\pi}\left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-\alpha^2\int_{-2\pi}^{0}\left[\left(\frac{\tau}{2}+\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+\alpha^2\int_{0}^{2\pi}\left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-2\alpha\int_{-2\pi}^{0}\left[\frac{1}{2}\left(\frac{\tau}{2}+\pi\right)\sin\left(\frac{\tau}{2}\right)-\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+2\alpha\int_{0}^{2\pi}\left[\frac{1}{2}\left(\frac{\tau}{2}-\pi\right)\sin\left(\frac{\tau}{2}\right)-\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+2\alpha\int_{-2\pi}^{0}\left[\frac{1}{2}\left(\frac{\tau}{2}+\pi\right)\sin\left(\frac{\tau}{2}\right)-\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &-2\alpha\int_{0}^{2\pi}\left[\frac{1}{2}\left(\frac{\tau}{2}-\pi\right)\sin\left(\frac{\tau}{2}\right)-\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.26}$$

Recognizing even and odd functions in Equation (D.59) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2 \cancel{u_0}}{8\pi} \Bigg\{ & \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau + \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[\frac{1}{2}\left(\frac{\tau}{2}-\pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right)\right] \frac{\cos\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \qquad \text{(D.27)} \\
& - 4\alpha \int_0^{2\pi} \left[\frac{1}{2}\left(\frac{\tau}{2}-\pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right)\right] \frac{\cos\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau+2\pi) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau+2\pi) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}.
\end{aligned}
$$

Combining terms in Equation (D.27)and inserting $u_0 = \frac{1}{2}$ for the half-wave dipole,

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi} \Bigg\{ & \left(1+2\alpha^2\right) \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + \left(1+2\alpha^2\right) \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + \alpha^2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right] d\tau \\
& + \alpha^2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right] d\tau \\
& - 2\pi\alpha^2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 2\pi\alpha^2 \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 4\alpha \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} \cos\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \qquad \text{(D.28)} \\
& + \alpha \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right] d\tau \\
& - \alpha \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right] d\tau \\
& - 2\pi\alpha \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\alpha \int_0^{2\pi} \sin\left(\frac{\tau}{2}\right) \frac{\cos\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau+2\pi) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau+2\pi) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}.
\end{aligned}$$

Applying product-to-sum trigonometric identities to Equation (D.28),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \frac{1}{2}\left(1+2\alpha^2\right)\int_0^{2\pi} \frac{\cos(\alpha\tau)-\cos\left[(1+\alpha)\tau\right]}{\tau}\, d\tau \\
& + \frac{1}{2}\left(1+2\alpha^2\right)\int_0^{2\pi} \frac{\cos(\alpha\tau)-\cos\left[(1-\alpha)\tau\right]}{\tau}\, d\tau \\
& + \frac{\alpha^2}{2}\int_0^{2\pi} \left[\sin\left[(1+\alpha)\tau\right]+\sin(\alpha\tau)\right] d\tau \\
& + \frac{\alpha^2}{2}\int_0^{2\pi} \left[\sin\left[(1-\alpha)\tau\right]-\sin(\alpha\tau)\right] d\tau \\
& - \pi\alpha^2\int_0^{2\pi} \frac{\sin\left[(1+\alpha)\tau\right]+\sin(\alpha\tau)}{\tau}\, d\tau \\
& - \pi\alpha^2\int_0^{2\pi} \frac{\sin\left[(1-\alpha)\tau\right]-\sin(\alpha\tau)}{\tau}\, d\tau \\
& - 2\alpha\int_0^{2\pi} \frac{\cos(\alpha\tau)+\cos\left[(1+\alpha)\tau\right]}{\tau}\, d\tau \\
& + 2\alpha\int_0^{2\pi} \frac{\cos(\alpha\tau)+\cos\left[(1-\alpha)\tau\right]}{\tau}\, d\tau \\
& + \frac{\alpha}{2}\int_0^{2\pi} \left[\sin\left[(1+\alpha)\tau\right]-\sin(\alpha\tau)\right] d\tau \\
& - \frac{\alpha}{2}\int_0^{2\pi} \left[\sin\left[(1-\alpha)\tau\right]+\sin(\alpha\tau)\right] d\tau \\
& - \pi\alpha\int_0^{2\pi} \frac{\sin\left[(1+\alpha)\tau\right]-\sin(\alpha\tau)}{\tau}\, d\tau \\
& + \pi\alpha\int_0^{2\pi} \frac{\sin\left[(1-\alpha)\tau\right]+\sin(\alpha\tau)}{\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau+2\pi)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau+2\pi)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - 2\int_{-\infty}^{\infty} \delta(\tau)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau-2\pi)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau-2\pi)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{D.29}
$$

Collecting terms in Equation (D.62),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \frac{1}{2}\left(1+2\alpha^2\right)\int_0^{2\pi} \frac{1-\cos\left[(1+\alpha)\tau\right]-1+\cos(\alpha\tau)}{\tau}\, d\tau \\
& + \frac{1}{2}\left(1+2\alpha^2\right)\int_0^{2\pi} \frac{1-\cos\left[(1-\alpha)\tau\right]-1+\cos(\alpha\tau)}{\tau}\, d\tau \\
& + \frac{\alpha^2}{2}\int_0^{2\pi} \sin\left[(1+\alpha)\tau\right]\, d\tau \\
& + \frac{\alpha^2}{2}\int_0^{2\pi} \sin\left[(1-\alpha)\tau\right]\, d\tau \\
& - \pi\alpha^2\int_0^{2\pi} \frac{\sin\left[(1+\alpha)\tau\right]+\sin\left[(1-\alpha)\tau\right]}{\tau}\, d\tau \\
& + 2\alpha\int_0^{2\pi} \frac{1-\cos\left[(1+\alpha)\tau\right]-1+\cos\left[(1-\alpha)\tau\right]}{\tau}\, d\tau \\
& + \frac{\alpha}{2}\int_0^{2\pi} \sin\left[(1+\alpha)\tau\right]\, d\tau \\
& - \alpha\int_0^{2\pi} \sin(\alpha\tau)\, d\tau \\
& - \frac{\alpha}{2}\int_0^{2\pi} \sin\left[(1-\alpha)\tau\right]\, d\tau \\
& - \pi\alpha\int_0^{2\pi} \frac{\sin\left[(1+\alpha)\tau\right]-2\sin(\alpha\tau)-\sin\left[(1-\alpha)\tau\right]}{\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau+2\pi)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau+2\pi)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - 2\int_{-\infty}^{\infty} \delta(\tau)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau-2\pi)\frac{\sin\left[\left(\frac{1}{2}+\alpha\right)\tau\right]}{\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau-2\pi)\frac{\sin\left[\left(\frac{1}{2}-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{D.30}
$$

Applying Equations (D.20) and (D.21), and the sifting property of the delta function, to Equation (D.30) and performing the remaining integrals,

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \frac{1}{2}\left(1+2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi(1+\alpha)\right]-\mathrm{Cin}(2\pi\alpha)\right] \\
& + \frac{1}{2}\left(1+2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi(1-\alpha)\right]-\mathrm{Cin}(2\pi\alpha)\right] \\
& - \frac{\alpha^2}{2}\left[\frac{\cos\left[(1+\alpha)\tau\right]}{1+\alpha}\right]_0^{2\pi} \\
& - \frac{\alpha^2}{2}\left[\frac{\cos\left[(1-\alpha)\tau\right]}{1-\alpha}\right]_0^{2\pi} \\
& - \pi\alpha^2\left[\mathrm{Si}\left[2\pi(1+\alpha)\right]+\mathrm{Si}\left[2\pi(1-\alpha)\right]\right] \\
& + 2\alpha\left[\mathrm{Cin}\left[2\pi(1+\alpha)\right]-\mathrm{Cin}\left[2\pi(1-\alpha)\right]\right] \\
& - \frac{\alpha}{2}\left[\frac{\cos\left[(1+\alpha)\tau\right]}{1+\alpha}\right]_0^{2\pi} \\
& + \alpha\left[\frac{\cos(\alpha\tau)}{\alpha}\right]_0^{2\pi} \\
& + \frac{\alpha}{2}\left[\frac{\cos\left[(1-\alpha)\tau\right]}{1-\alpha}\right]_0^{2\pi} \\
& - \pi\alpha\left[\mathrm{Si}\left[2\pi(1+\alpha)\right]-2\mathrm{Si}(2\pi\alpha)-\mathrm{Si}\left[2\pi(1-\alpha)\right]\right] \\
& - \frac{\sin\left[-2\pi\left(\frac{1}{2}+\alpha\right)\right]}{-2\pi} - \frac{\sin\left[-2\pi\left(\frac{1}{2}-\alpha\right)\right]}{-2\pi} \\
& - 2\left(\frac{1}{2}+\alpha\right) - 2\left(\frac{1}{2}-\alpha\right) \\
& - \frac{\sin\left[2\pi\left(\frac{1}{2}+\alpha\right)\right]}{2\pi} - \frac{\sin\left[2\pi\left(\frac{1}{2}-\alpha\right)\right]}{2\pi}\Bigg\}.
\end{aligned} \tag{D.31}$$

Evaluating the limits of integration and simplifying Equation (D.31),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \frac{1}{2}\left(1+2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right]-\mathrm{Cin}\left(2\pi\alpha\right)\right] \\
& + \frac{1}{2}\left(1+2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]-\mathrm{Cin}\left(2\pi\alpha\right)\right] \\
& - \frac{\alpha^2}{2}\left[\frac{\cos\left[2\pi\left(1+\alpha\right)\right]-1}{1+\alpha}\right] \\
& - \frac{\alpha^2}{2}\left[\frac{\cos\left[2\pi\left(1-\alpha\right)\right]-1}{1-\alpha}\right] \\
& - \pi\alpha^2\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right]+\mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& + 2\alpha\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - \frac{\alpha}{2}\left[\frac{\cos\left[2\pi\left(1+\alpha\right)\right]-1}{1+\alpha}\right] \\
& + \alpha\left[\frac{\cos\left(2\pi\alpha\right)-1}{\alpha}\right] \\
& + \frac{\alpha}{2}\left[\frac{\cos\left[2\pi\left(1-\alpha\right)\right]-1}{1-\alpha}\right] \\
& - \pi\alpha\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right]-2\mathrm{Si}\left(2\pi\alpha\right)-\mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - \frac{\sin\left[2\pi\left(\frac{1}{2}+\alpha\right)\right]}{2\pi} - \frac{\sin\left[2\pi\left(\frac{1}{2}-\alpha\right)\right]}{2\pi} \\
& - 2\left(\frac{1}{2}+\alpha\right) - 2\left(\frac{1}{2}-\alpha\right) \\
& - \frac{\sin\left[2\pi\left(\frac{1}{2}+\alpha\right)\right]}{2\pi} - \frac{\sin\left[2\pi\left(\frac{1}{2}-\alpha\right)\right]}{2\pi}\Bigg\}.
\end{aligned} \tag{D.32}$$

Simplifying Equation (D.32) yields the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the half-wave dipole,

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \frac{1}{2}\left(1+2\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right]-2\mathrm{Cin}\left(2\pi\alpha\right)+\mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& + 2\alpha\left[\mathrm{Cin}\left[2\pi\left(1+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - \pi\alpha\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right]-2\mathrm{Si}\left(2\pi\alpha\right)-\mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right] \\
& - \pi\alpha^2\left[\mathrm{Si}\left[2\pi\left(1+\alpha\right)\right]+\mathrm{Si}\left[2\pi\left(1-\alpha\right)\right]\right]+\cos\left(2\pi\alpha\right)-3\Bigg\}.
\end{aligned} \tag{D.33}$$

D.2 Cosine Distribution

Recalling the autocorrelation function $R_g(p)$ for the cosine distribution,

$$R_g(p) = \frac{A_m^2}{2} \begin{cases} (2\pi + p)\cos\left(\frac{p}{2}\right) - 2\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ (2\pi - p)\cos\left(\frac{p}{2}\right) + 2\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.34}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{2} \begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.35}$$

Finding the second derivative of $R_g(p)$:

$$R_g''(p) = \frac{A_m^2}{2}\frac{d}{dp} \begin{cases} \left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.36}$$

where

$$R_g''(p) = \frac{A_m^2}{2} \begin{cases} \frac{1}{2}\left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \frac{1}{2}\sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \frac{1}{2}\left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \frac{1}{2}\sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.37}$$

which can be simplified to be

$$R_g''(p) = \frac{A_m^2}{4} \begin{cases} \left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \leq p \leq 0 \\ \left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.38}$$

Finding the third derivative of $R_g(p)$:

$$R_g'''(p) = \frac{A_m^2}{4}\frac{d}{dp}\begin{cases} \left(-\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) - \sin\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{p}{2} - \pi\right)\cos\left(\frac{p}{2}\right) + \sin\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.39}$$

where

$$R_g'''(p) = \frac{A_m^2}{4}\begin{cases} -\frac{1}{2}\left(-\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right) - \frac{1}{2}\cos\left(\frac{p}{2}\right) - \frac{1}{2}\cos\left(\frac{1}{2}\right), & -2\pi \le p \le 0 \\ -\frac{1}{2}\left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right) + \frac{1}{2}\cos\left(\frac{p}{2}\right) + \frac{1}{2}\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.40}$$

which can be simplified to be

$$R_g'''(p) = \frac{A_m^2}{4}\begin{cases} \frac{1}{2}\left(\frac{p}{2} + \pi\right)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ -\frac{1}{2}\left(\frac{p}{2} - \pi\right)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.41}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2\begin{cases} -\left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.42}$$

Finding the first derivative of $R_f(p)$:

$$R_f'(p) = A_m^2\frac{d}{dp}\begin{cases} -\left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{1}{4} + u_0^2\right)\sin\left(\frac{p}{2}\right) - \left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.43}$$

where

$$R'_f(p) = A_m^2 \begin{cases} -\frac{1}{2}\left(\frac{1}{4} + \cancel{u_0^2}\right)\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi + p}{2}\right)\sin\left(\frac{p}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - \cancel{u_0^2}\right)\cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \frac{1}{2}\left(\frac{1}{4} + \cancel{u_0^2}\right)\cos\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left(\frac{2\pi - p}{2}\right)\sin\left(\frac{p}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - \cancel{u_0^2}\right)\cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.44}$$

which can be simplified to be

$$R'_f(p) = \frac{A_m^2}{4} \begin{cases} \left(\frac{1}{4} - u_0^2\right)(2\pi + p)\sin\left(\frac{p}{2}\right) - \cos\left(\frac{p}{2}\right), & -2\pi \le p \le 0 \\ \left(\frac{1}{4} - u_0^2\right)(2\pi - p)\sin\left(\frac{p}{2}\right) + \cos\left(\frac{p}{2}\right), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.45}$$

Recalling the autocorrelation function $R_n(p)$,

$$\begin{aligned} R_n(p) = A_m^2 \Bigg\{ & \frac{1}{8}\sin\left(\frac{|p|}{2}\right) + \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)\left[\frac{1}{2}\sin\left(\frac{|p|}{2}\right) - \left(\frac{2\pi - |p|}{4}\right)\cos\left(\frac{p}{2}\right)\right] \\ & - \frac{\delta(p + 2\pi)}{4} - \frac{\delta(p)}{2} - \frac{\delta(p - 2\pi)}{4} \Bigg\} \end{aligned} \tag{D.46}$$

for $-2\pi \le p \le 2\pi$ and $R_n(p) = 0$ otherwise. Recalling the definition for $\widetilde{R}_{m1}(0)$,

$$\begin{aligned} \widetilde{R}_{m1}(0) = & \frac{u_0}{2\pi}\Bigg\{ - \int_{-\infty}^{\infty} R'_f(\tau)\frac{\cos[(u_0 + \alpha)\tau]}{u_0\tau}\, d\tau + \int_{-\infty}^{\infty} R'_f(\tau)\frac{\cos[(u_0 - \alpha)\tau]}{u_0\tau}\, d\tau \\ & + \alpha^2 \int_{-\infty}^{\infty} R'_g(\tau)\frac{\cos[(u_0 + \alpha)\tau]}{u_0\tau}\, d\tau - \alpha^2 \int_{-\infty}^{\infty} R'_g(\tau)\frac{\cos[(u_0 - \alpha)\tau]}{u_0\tau}\, d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} R''_g(\tau)\frac{\sin[(u_0 + \alpha)\tau]}{u_0\tau}\, d\tau + 2\alpha \int_{-\infty}^{\infty} R''_g(\tau)\frac{\sin[(u_0 - \alpha)\tau]}{u_0\tau}\, d\tau \Bigg\}. \end{aligned} \tag{D.47}$$

Substituting Equations (D.45), (D.35), and (D.37) into Equation (D.47),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 u_0}{8\pi}\Bigg\{ & -\int_{-2\pi}^{0}\left[\left(\frac{1}{4}-u_0^2\right)(2\pi+\tau)\sin\left(\frac{\tau}{2}\right)-\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& -\int_{0}^{2\pi}\left[\left(\frac{1}{4}-u_0^2\right)(2\pi-\tau)\sin\left(\frac{\tau}{2}\right)+\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& +\int_{-2\pi}^{0}\left[\left(\frac{1}{4}-u_0^2\right)(2\pi+\tau)\sin\left(\frac{\tau}{2}\right)-\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& +\int_{0}^{2\pi}\left[\left(\frac{1}{4}-u_0^2\right)(2\pi-\tau)\sin\left(\frac{\tau}{2}\right)+\cos\left(\frac{\tau}{2}\right)\right]\frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& -\alpha^2\int_{-2\pi}^{0}(\tau+2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& +\alpha^2\int_{0}^{2\pi}(\tau-2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& +\alpha^2\int_{-2\pi}^{0}(\tau+2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& -\alpha^2\int_{0}^{2\pi}(\tau-2\pi)\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& -2\alpha\int_{-2\pi}^{0}\left[\left(\frac{\tau}{2}+\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& +2\alpha\int_{0}^{2\pi}\left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& -2\alpha\int_{-2\pi}^{0}\left[\left(\frac{\tau}{2}+\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau \\
& +2\alpha\int_{0}^{2\pi}\left[\left(\frac{\tau}{2}-\pi\right)\cos\left(\frac{\tau}{2}\right)+\sin\left(\frac{\tau}{2}\right)\right]\frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\,d\tau\Bigg\}. \qquad \text{(D.48)}
\end{aligned}
$$

Recognizing even and odd functions in Equation (D.48) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 \cancel{u_0}}{8\pi} \Bigg\{ & -2 \int_0^{2\pi} \left[\left(\frac{1}{4} - u_0^2 \right) (2\pi - \tau) \sin\left(\frac{\tau}{2} \right) + \cos\left(\frac{\tau}{2} \right) \right] \frac{\cos\left[(u_0 + \alpha)\, \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& -2 \int_0^{2\pi} \left[\left(\frac{1}{4} - u_0^2 \right) (2\pi - \tau) \sin\left(\frac{\tau}{2} \right) + \cos\left(\frac{\tau}{2} \right) \right] \frac{\cos\left[(u_0 - \alpha)\, \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} (\tau - 2\pi) \sin\left(\frac{\tau}{2} \right) \frac{\cos\left[(u_0 + \alpha)\, \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} (\tau - 2\pi) \sin\left(\frac{\tau}{2} \right) \frac{\cos\left[(u_0 - \alpha)\, \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[\left(\frac{\tau}{2} - \pi \right) \cos\left(\frac{\tau}{2} \right) + \sin\left(\frac{\tau}{2} \right) \right] \frac{\sin\left[(u_0 + \alpha)\, \tau \right]}{\cancel{u_0} \tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[\left(\frac{\tau}{2} - \pi \right) \cos\left(\frac{\tau}{2} \right) + \sin\left(\frac{\tau}{2} \right) \right] \frac{\sin\left[(u_0 - \alpha)\, \tau \right]}{\cancel{u_0} u_0 \tau} \, d\tau \Bigg\}.
\end{aligned}
\tag{D.49}
$$

Combining terms in Equation (D.49),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & \left[-4\pi\left(\frac{1}{4}-u_0^2\right)-4\pi\alpha^2\right]\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + \left[4\pi\left(\frac{1}{4}-u_0^2\right)+4\pi\alpha^2\right]\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + \left[2\left(\frac{1}{4}-u_0^2\right)+2\alpha^2\right]\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& - \left[2\left(\frac{1}{4}-u_0^2\right)+2\alpha^2\right]\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& - 2\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 2\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 2\alpha\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& + 2\alpha\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& - 4\pi\alpha\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& - 4\pi\alpha\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 4\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 4\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau\Bigg\}. \qquad \text{(D.50)}
\end{aligned}
$$

Applying product-to-sum trigonometric identities to Equation (D.50),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& -2\pi\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& -\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]\right]\,d\tau \\
& +\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]\right]\,d\tau \\
& -\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]+\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]+\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +\alpha\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\right]\right]\,d\tau \\
& +\alpha\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\right]\right]\,d\tau \\
& -2\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& -2\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +2\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]-\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +2\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]-\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau\Bigg\}. \qquad \text{(D.51)}
\end{aligned}$$

Collecting terms in Equation (D.51),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau \\
& -2\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau \\
& -2\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau \\
& +2\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau \\
& +(2\alpha+1)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]-1+\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau \\
& +(2\alpha-1)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]-1+\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\;d\tau \\
& -\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]\;d\tau \\
& +\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]\;d\tau \\
& +\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]\;d\tau \\
& -\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]\;d\tau\Bigg\}
\end{aligned} \tag{D.52}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^b \frac{\sin(at)}{t}\;dt \tag{D.53}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos(at)}{t}\;dt. \tag{D.54}$$

Applying Equations (D.53) and (D.54) to Equation (D.52) and performing the remaining integrals,

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& - 2\pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right]\right] \\
& + (2\alpha + 1)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
& + (2\alpha - 1)\left[\mathrm{Cin}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right]\right] \\
& + \frac{\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 + \alpha + \frac{1}{2}\right)}\left[\cos\left[\left(u_0 + \alpha + \frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& - \frac{\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 + \alpha - \frac{1}{2}\right)}\left[\cos\left[\left(u_0 + \alpha - \frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& - \frac{\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 - \alpha + \frac{1}{2}\right)}\left[\cos\left[\left(u_0 - \alpha + \frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& + \frac{\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)}{\left(u_0 - \alpha - \frac{1}{2}\right)}\left[\cos\left[\left(u_0 - \alpha - \frac{1}{2}\right)\tau\right]\right]_0^{2\pi}\Bigg\}.
\end{aligned} \tag{D.55}$$

Evaluating the limits of integration and simplifying Equation (D.55),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & -2\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\ & +(2\alpha+1)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & +(2\alpha-1)\left[\mathrm{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\ & +\left(u_0-\alpha-\frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0+\alpha\right)\right]-1\right] \\ & -\left(u_0-\alpha+\frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0+\alpha\right)\right]-1\right] \\ & -\left(u_0+\alpha-\frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0-\alpha\right)\right]-1\right] \\ & +\left(u_0+\alpha+\frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0-\alpha\right)\right]-1\right]\Bigg\}. \end{aligned} \tag{D.56}$$

Simplifying Equation (D.56) yields the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the cosine distribution,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & -2\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\ & +(2\alpha+1)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ & +(2\alpha-1)\left[\mathrm{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\ & +\cos\left[2\pi\left(u_0+\alpha\right)\right]-\cos\left[2\pi\left(u_0-\alpha\right)\right]\Bigg\}. \end{aligned} \tag{D.57}$$

Recalling the definition for $\widetilde{R}_{m2}(0)$,

$$\begin{aligned}\widetilde{R}_{m2}\left(0\right) =& \frac{u_0}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} R_n\left(\tau\right) \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\, d\tau + \int_{-\infty}^{\infty} R_n\left(\tau\right) \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\, d\tau \\ &+ \alpha^2 \int_{-\infty}^{\infty} R_g''\left(\tau\right) \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\, d\tau + \alpha^2 \int_{-\infty}^{\infty} R_g''\left(\tau\right) \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\, d\tau \\ &- 2\alpha \int_{-\infty}^{\infty} R_g'''\left(\tau\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\, d\tau + 2\alpha \int_{-\infty}^{\infty} R_g'''\left(\tau\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\, d\tau \Bigg\}.\end{aligned} \tag{D.58}$$

Substituting Equations (D.46), (D.38), and (D.41) into Equation (D.58),

$$\begin{aligned}
\widetilde{R}_{m2}(0) &= \frac{A_m^2 u_0}{8\pi} \\
&\times \left\{ \int_{-2\pi}^{0} \left[-\left(\frac{3}{4} - u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)(2\pi + \tau)\cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \right. \\
&+ \int_{0}^{2\pi} \left[\left(\frac{3}{4} - u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)(2\pi - \tau)\cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ \int_{-2\pi}^{0} \left[-\left(\frac{3}{4} - u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)(2\pi + \tau)\cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ \int_{0}^{2\pi} \left[\left(\frac{3}{4} - u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)(2\pi - \tau)\cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \alpha^2 \int_{-2\pi}^{0} \left[\left(\frac{\tau}{2} + \pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ \alpha^2 \int_{0}^{2\pi} \left[\left(\frac{\tau}{2} - \pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- \alpha^2 \int_{-2\pi}^{0} \left[\left(\frac{\tau}{2} + \pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ \alpha^2 \int_{0}^{2\pi} \left[\left(\frac{\tau}{2} - \pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&- 2\alpha \int_{-2\pi}^{0} \left[\frac{1}{2}\left(\frac{\tau}{2} + \pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right) \right] \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ 2\alpha \int_{0}^{2\pi} \left[\frac{1}{2}\left(\frac{\tau}{2} - \pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right) \right] \frac{\cos\left[(u_0+\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&+ 2\alpha \int_{-2\pi}^{0} \left[\frac{1}{2}\left(\frac{\tau}{2} + \pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right) \right] \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \\
&\left. - 2\alpha \int_{0}^{2\pi} \left[\frac{1}{2}\left(\frac{\tau}{2} - \pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right) \right] \frac{\cos\left[(u_0-\alpha)\tau\right]}{u_0\tau}\, d\tau \right\}. \qquad \text{(D.59)}
\end{aligned}$$

Recognizing even and odd functions in Equation (D.59) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = {} & \frac{A_m^2 \cancel{u_0}}{8\pi} \\
& \times \Bigg\{ 2\int_0^{2\pi} \left[\left(\frac{3}{4} - u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)(2\pi - \tau)\cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\int_0^{2\pi} \left[\left(\frac{3}{4} - u_0^2\right) \sin\left(\frac{\tau}{2}\right) - \frac{1}{2}\left(\frac{1}{4} - u_0^2\right)(2\pi - \tau)\cos\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[\left(\frac{\tau}{2} - \pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[\left(\frac{\tau}{2} - \pi\right)\cos\left(\frac{\tau}{2}\right) + \sin\left(\frac{\tau}{2}\right) \right] \frac{\sin\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[\frac{1}{2}\left(\frac{\tau}{2} - \pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right) \right] \frac{\cos\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 4\alpha \int_0^{2\pi} \left[\frac{1}{2}\left(\frac{\tau}{2} - \pi\right)\sin\left(\frac{\tau}{2}\right) - \cos\left(\frac{\tau}{2}\right) \right] \frac{\cos\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau - 2\int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau - \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}. \qquad \text{(D.60)}
\end{aligned}
$$

Combining terms in Equation (D.60),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & 2\left[\left(\frac{3}{4}-u_0^2\right)+\alpha^2\right]\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + 2\left[\left(\frac{3}{4}-u_0^2\right)+\alpha^2\right]\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + 2\left[-\pi\left(\frac{1}{4}-u_0^2\right)-\pi\alpha^2\right]\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + 2\left[-\pi\left(\frac{1}{4}-u_0^2\right)-\pi\alpha^2\right]\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + 2\left[\frac{1}{2}\left(\frac{1}{4}-u_0^2\right)+\frac{\alpha^2}{2}\right]\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[(u_0+\alpha)\,\tau\right]\;d\tau \\
& + 2\left[\frac{1}{2}\left(\frac{1}{4}-u_0^2\right)+\frac{\alpha^2}{2}\right]\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\sin\left[(u_0-\alpha)\,\tau\right]\;d\tau \\
& - 4\alpha\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau + 4\alpha\int_0^{2\pi}\cos\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& - 2\pi\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau + 2\pi\alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& + \alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[(u_0+\alpha)\,\tau\right]\;d\tau - \alpha\int_0^{2\pi}\sin\left(\frac{\tau}{2}\right)\cos\left[(u_0-\alpha)\,\tau\right]\;d\tau \\
& - \int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau - \int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& - 2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau - 2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
& - \int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau - \int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau\Bigg\}. \quad \text{(D.61)}
\end{aligned}
$$

Applying product-to-sum trigonometric identities to Equation (D.61),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{8\pi}\Bigg\{ & -\left(u_0^2-\alpha^2-\frac{3}{4}\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]-\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& -\left(u_0^2-\alpha^2-\frac{3}{4}\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]-\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +\pi\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +\pi\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& -\frac{1}{2}\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]\right]\,d\tau \\
& -\frac{1}{2}\left(u_0^2-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]\right]\,d\tau \\
& -2\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]+\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +2\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]+\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& -\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau \\
& +\frac{\alpha}{2}\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]\right]\,d\tau \\
& -\frac{\alpha}{2}\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]-\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]\right]\,d\tau \\
& -\int_{-\infty}^{\infty}\delta(\tau+2\pi)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau-\int_{-\infty}^{\infty}\delta(\tau+2\pi)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& -2\int_{-\infty}^{\infty}\delta(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau-2\int_{-\infty}^{\infty}\delta(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& -\int_{-\infty}^{\infty}\delta(\tau-2\pi)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau-\int_{-\infty}^{\infty}\delta(\tau-2\pi)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau\Bigg\}. \qquad \text{(D.62)}
\end{aligned}$$

Collecting terms in Equation (D.62),

$$
\begin{aligned}
&\widetilde{R}_{m2}\left(0\right)=\frac{A_m^2}{8\pi}\\
&\times\Bigg\{-\left(u_0^2-2\alpha-\alpha^2-\frac{3}{4}\right)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]-1+\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau\\
&-\left(u_0^2+2\alpha-\alpha^2-\frac{3}{4}\right)\int_0^{2\pi}\frac{1-\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]-1+\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau\\
&+\pi\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau\\
&+\pi\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]+\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]}{\tau}\,d\tau\\
&-\frac{1}{2}\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]\,d\tau\\
&-\frac{1}{2}\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]\,d\tau\\
&-\frac{1}{2}\left(u_0^2+\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]\,d\tau\\
&-\frac{1}{2}\left(u_0^2-\alpha-\alpha^2-\frac{1}{4}\right)\int_0^{2\pi}\sin\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]\,d\tau\\
&-\int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau-\int_{-\infty}^{\infty}\delta\left(\tau+2\pi\right)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau\\
&-2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau-2\int_{-\infty}^{\infty}\delta\left(\tau\right)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau\\
&-\int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau-\int_{-\infty}^{\infty}\delta\left(\tau-2\pi\right)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau\Bigg\}.
\end{aligned}
\tag{D.63}
$$

Applying Equations (D.53) and (D.54), and the sifting property of the delta function, to Equation (D.63) and performing the remaining integrals,

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = {} & \frac{A_m^2}{8\pi} \\
& \times \Bigg\{ -\left(u_0^2 - 2\alpha - \alpha^2 - \frac{3}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\
& - \left(u_0^2 + 2\alpha - \alpha^2 - \frac{3}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\
& + \pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\
& + \frac{\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)}{2\left(u_0+\alpha+\frac{1}{2}\right)}\left[\cos\left[\left(u_0+\alpha+\frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& + \frac{\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)}{2\left(u_0+\alpha-\frac{1}{2}\right)}\left[\cos\left[\left(u_0+\alpha-\frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& + \frac{\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)}{2\left(u_0-\alpha+\frac{1}{2}\right)}\left[\cos\left[\left(u_0-\alpha+\frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& + \frac{\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)}{2\left(u_0-\alpha-\frac{1}{2}\right)}\left[\cos\left[\left(u_0-\alpha-\frac{1}{2}\right)\tau\right]\right]_0^{2\pi} \\
& - \frac{\sin\left[\not{-}2\pi\left(u_0+\alpha\right)\right]}{\not{-}2\pi} - \frac{\left[\not{-}2\pi\left(u_0-\alpha\right)\right]}{\not{-}2\pi} \\
& - 2\left(u_0+\alpha\right) - 2\left(u_0-\alpha\right) \\
& - \frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} - \frac{\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi}\Bigg\}. \qquad \text{(D.64)}
\end{aligned}
$$

Evaluating the limits of integration and simplifying Equation (D.64),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = &\frac{A_m^2}{8\pi} \\
&\times \Bigg\{ -\left(u_0^2 - 2\alpha - \alpha^2 - \frac{3}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
&- \left(u_0^2 + 2\alpha - \alpha^2 - \frac{3}{4}\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right]\right] \\
&+ \pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha + \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha - \frac{1}{2}\right)\right]\right] \\
&+ \pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha - \frac{1}{2}\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha + \frac{1}{2}\right)\right]\right] \\
&+ \frac{1}{2}\left(u_0 - \alpha - \frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0 + \alpha\right)\right] - 1\right] \\
&+ \frac{1}{2}\left(u_0 - \alpha + \frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0 + \alpha\right)\right] - 1\right] \\
&+ \frac{1}{2}\left(u_0 + \alpha - \frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0 - \alpha\right)\right] - 1\right] \\
&+ \frac{1}{2}\left(u_0 + \alpha + \frac{1}{2}\right)\left[-\cos\left[2\pi\left(u_0 - \alpha\right)\right] - 1\right] \\
&- \frac{1}{\pi}\left[\left[2\pi\left(u_0 + \alpha\right)\right] + \sin\left[2\pi\left(u_0 - \alpha\right)\right]\right] - 4u_0 \Bigg\}. \qquad \text{(D.65)}
\end{aligned}
$$

Simplifying Equation (D.65) yields the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the cosine distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) &= \frac{A_m^2}{8\pi} \\ &\times \Bigg\{ -\left(u_0^2 - 2\alpha - \alpha^2 - \frac{3}{4}\right)\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right] - \operatorname{Cin}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ &- \left(u_0^2 + 2\alpha - \alpha^2 - \frac{3}{4}\right)\left[\operatorname{Cin}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right] - \operatorname{Cin}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right]\right] \\ &+ \pi\left(u_0^2 - \alpha - \alpha^2 - \frac{1}{4}\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha+\frac{1}{2}\right)\right] + \operatorname{Si}\left[2\pi\left(u_0-\alpha-\frac{1}{2}\right)\right]\right] \\ &+ \pi\left(u_0^2 + \alpha - \alpha^2 - \frac{1}{4}\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha-\frac{1}{2}\right)\right] + \operatorname{Si}\left[2\pi\left(u_0-\alpha+\frac{1}{2}\right)\right]\right] \\ &- (u_0-\alpha)\cos\left[2\pi\left(u_0+\alpha\right)\right] - (u_0+\alpha)\cos\left[2\pi\left(u_0-\alpha\right)\right] \\ &- \frac{1}{\pi}\left[\left[2\pi\left(u_0+\alpha\right)\right] + \sin\left[2\pi\left(u_0-\alpha\right)\right]\right] - 6u_0 \Bigg\}. \qquad \text{(D.66)}\end{aligned}$$

D.3 Cosine-Squared Distribution

Recalling the autocorrelation function $R_g(p)$ for the cosine-squared distribution,

$$R_g(p) = \frac{A_m^2}{8}\begin{cases}(2\pi+p)\left[2+\cos(p)\right] - 3\sin(p), & -2\pi \le p \le 0 \\ (2\pi-p)\left[2+\cos(p)\right] + 3\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \qquad \text{(D.67)}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{8}\begin{cases}-(p+2\pi)\sin(p) - 2\cos(p) + 2, & -2\pi \le p \le 0 \\ (p-2\pi)\sin(p) + 2\cos(p) - 2, & 0 \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \qquad \text{(D.68)}$$

Finding the second derivative of $R_g(p)$:

$$R_g''(p) = \frac{A_m^2}{8}\frac{d}{dp}\begin{cases} -(p+2\pi)\sin(p) - 2\cos(p) + 2, & -2\pi \le p \le 0 \\ (p-2\pi)\sin(p) + 2\cos(p) - 2, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{D.69}$$

where

$$R_g''(p) = \frac{A_m^2}{8}\begin{cases} -\sin(p) - (p+2\pi)\cos(p) + 2\sin(p), & -2\pi \le p \le 0 \\ \sin(p) + (p-2\pi)\cos(p) - 2\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{D.70}$$

which can be simplified to be

$$R_g''(p) = \frac{A_m^2}{8}\begin{cases} -(p+2\pi)\cos(p) + \sin(p), & -2\pi \le p \le 0 \\ (p-2\pi)\cos(p) - \sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{D.71}$$

Finding the third derivative of $R_g(p)$:

$$R_g'''(p) = \frac{A_m^2}{8}\frac{d}{dp}\begin{cases} -(p+2\pi)\cos(p) + \sin(p), & -2\pi \le p \le 0 \\ (p-2\pi)\cos(p) - \sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{D.72}$$

where

$$R_g'''(p) = \frac{A_m^2}{8}\begin{cases} -\cancel{\cos(p)} + (p+2\pi) + \cancel{\cos(p)}, & -2\pi \le p \le 0 \\ \cancel{\cos(p)} - (p-2\pi)\sin(p) - \cancel{\cos(p)}, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{D.73}$$

which can be simplified to be

$$R_g'''(p) = \frac{A_m^2}{8} \begin{cases} (p+2\pi)\sin(p), & -2\pi \le p \le 0 \\ -(p-2\pi)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.74}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{8} \begin{cases} (2\pi+p)\left[2u_0^2 + \left(u_0^2-1\right)\cos(p)\right] + \left(1-3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ (2\pi-p)\left[2u_0^2 + \left(u_0^2-1\right)\cos(p)\right] - \left(1-3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.75}$$

Finding the first derivative of $R_f(p)$:

$$R_f'(p) = \frac{A_m^2}{8}\frac{d}{dp} \begin{cases} (2\pi+p)\left[2u_0^2 + \left(u_0^2-1\right)\cos(p)\right] + \left(1-3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ (2\pi-p)\left[2u_0^2 + \left(u_0^2-1\right)\cos(p)\right] - \left(1-3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.76}$$

where

$$R_f'(p) = \frac{A_m^2}{8} \begin{cases} \left[2u_0^2 + \left(u_0^2-\not{1}\right)\cos(p)\right] - (2\pi+p)\left(u_0^2-1\right)\sin(p) + \left(\not{1}-3u_0^2\right)\cos(p), & -2\pi \le p \le 0 \\ -\left[2u_0^2 + \left(u_0^2-\not{1}\right)\cos(p)\right] - (2\pi-p)\left(u_0^2-1\right)\sin(p) - \left(\not{1}-3u_0^2\right)\cos(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} ; \tag{D.77}$$

which can be simplified to be

$$R_f'(p) = \frac{A_m^2}{8} \begin{cases} -\left(u_0^2-1\right)(2\pi+p)\sin(p) + 2u_0^2\left[1-\cos(p)\right], & -2\pi \le p \le 0 \\ -\left(u_0^2-1\right)(2\pi-p)\sin(p) - 2u_0^2\left[1-\cos(p)\right], & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.78}$$

Recalling the autocorrelation function $R_n(p)$,

$$R_n(p) = \frac{A_m^2}{8} \begin{cases} \left(1-u_0^2\right)\left[-2\sin(p) - (2\pi+p)\cos(p)\right] + \left(1-3u_0^2\right)\sin(p), & -2\pi \le p \le 0 \\ \left(1-u_0^2\right)\left[2\sin(p) - (2\pi-p)\cos(p)\right] - \left(1-3u_0^2\right)\sin(p), & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases} . \tag{D.79}$$

Recalling the definition for $\widetilde{R}_{m1}(0)$,

$$\begin{aligned}\widetilde{R}_{m1}(0) =& \frac{u_0}{2\pi}\Bigg\{ -\int_{-\infty}^{\infty} R_f'(\tau)\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_f'(\tau)\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+ \alpha^2\int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau - \alpha^2\int_{-\infty}^{\infty} R_g'(\tau)\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+ 2\alpha\int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau + 2\alpha\int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.80}$$

Substituting Equations (D.78), (D.68), and (D.71) into Equation (D.80),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2 u_0}{16\pi}\Bigg\{ &-\int_{-2\pi}^{0}\left[-\left(u_0^2-1\right)(2\pi+\tau)\sin(\tau)+2u_0^2[1-\cos(\tau)]\right]\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau \\ &-\int_{0}^{2\pi}\left[-\left(u_0^2-1\right)(2\pi-\tau)\sin(\tau)-2u_0^2[1-\cos(\tau)]\right]\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau \\ &+\int_{-2\pi}^{0}\left[-\left(u_0^2-1\right)(2\pi+\tau)\sin(\tau)+2u_0^2[1-\cos(\tau)]\right]\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+\int_{0}^{2\pi}\left[-\left(u_0^2-1\right)(2\pi-\tau)\sin(\tau)-2u_0^2[1-\cos(\tau)]\right]\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+\alpha^2\int_{-2\pi}^{0}[-(\tau+2\pi)\sin(\tau)-2\cos(\tau)+2]\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau \\ &+\alpha^2\int_{0}^{2\pi}[(\tau-2\pi)\sin(\tau)+2\cos(\tau)-2]\frac{\cos[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau \\ &-\alpha^2\int_{-2\pi}^{0}[-(\tau+2\pi)\sin(\tau)-2\cos(\tau)+2]\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &-\alpha^2\int_{0}^{2\pi}[(\tau-2\pi)\sin(\tau)+2\cos(\tau)-2]\frac{\cos[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+2\alpha\int_{-2\pi}^{0}[-(\tau+2\pi)\cos(\tau)+\sin(\tau)]\frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau \\ &+2\alpha\int_{0}^{2\pi}[(\tau-2\pi)\cos(\tau)-\sin(\tau)]\frac{\sin[(u_0+\alpha)\tau]}{u_0\tau}\,d\tau \\ &+2\alpha\int_{-2\pi}^{0}[-(\tau+2\pi)\cos(\tau)+\sin(\tau)]\frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau \\ &+2\alpha\int_{0}^{2\pi}[(\tau-2\pi)\cos(\tau)-\sin(\tau)]\frac{\sin[(u_0-\alpha)\tau]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.81}$$

Recognizing even and odd functions in Equation (D.81) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 \cancel{u_0}}{16\pi} \Bigg\{ & -2 \int_0^{2\pi} \left[-\left(u_0^2 - 1\right)(2\pi - \tau)\sin(\tau) - 2u_0^2 \left[1 - \cos(\tau)\right] \right] \frac{\cos\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2 \int_0^{2\pi} \left[-\left(u_0^2 - 1\right)(2\pi - \tau)\sin(\tau) - 2u_0^2 \left[1 - \cos(\tau)\right] \right] \frac{\cos\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[(\tau - 2\pi)\sin(\tau) + 2\cos(\tau) - 2\right] \frac{\cos\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 2\alpha^2 \int_0^{2\pi} \left[(\tau - 2\pi)\sin(\tau) + 2\cos(\tau) - 2\right] \frac{\cos\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[(\tau - 2\pi)\cos(\tau) - \sin(\tau)\right] \frac{\sin\left[(u_0 + \alpha)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{2\pi} \left[(\tau - 2\pi)\cos(\tau) - \sin(\tau)\right] \frac{\sin\left[(u_0 - \alpha)\tau\right]}{\cancel{u_0}\tau} \, d\tau \Bigg\}.
\end{aligned}
\tag{D.82}
$$

Combining terms in Equation (D.82),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 4\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\sin(\tau)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -4\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\sin(\tau)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\cos(\tau)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& +4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\cos(\tau)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& +4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -2\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\sin(\tau)\cos\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& +2\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\sin(\tau)\cos\left[\left(u_0-\alpha\right)\tau\right]\ d\tau \\
& -8\pi\alpha\int_0^{2\pi}\cos(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -8\pi\alpha\int_0^{2\pi}\cos(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -4\alpha\int_0^{2\pi}\sin(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\ d\tau \\
& -4\alpha\int_0^{2\pi}\sin(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& +4\alpha\int_0^{2\pi}\cos(\tau)\sin\left[\left(u_0+\alpha\right)\tau\right]\ d\tau \\
& +4\alpha\int_0^{2\pi}\cos(\tau)\sin\left[\left(u_0-\alpha\right)\tau\right]\ d\tau\Bigg\}. \qquad \text{(D.83)}
\end{aligned}
$$

Applying product-to-sum trigonometric identities to Equation (D.83),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 2\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]-\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& - 2\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]-\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& - \left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+1\right)\tau\right]-\sin\left[\left(u_0+\alpha-1\right)\tau\right]\right]\ d\tau \\
& + \left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+1\right)\tau\right]-\sin\left[\left(u_0-\alpha-1\right)\tau\right]\right]\ d\tau \\
& - 2\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-1\right)\tau\right]+\cos\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& + 2\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-1\right)\tau\right]+\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& + 4\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]-\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\ d\tau \\
& + 2\alpha\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+1\right)\tau\right]+\sin\left[\left(u_0+\alpha-1\right)\right]\right]\ d\tau \\
& + 2\alpha\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+1\right)\tau\right]+\sin\left[\left(u_0-\alpha-1\right)\right]\right]\ d\tau \\
& - 4\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]+\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& - 4\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]+\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\ d\tau \\
& - 2\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-1\right)\tau\right]-\cos\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\ d\tau \\
& - 2\alpha\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-1\right)\tau\right]-\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\ d\tau\Bigg\}.
\end{aligned} \tag{D.84}$$

Collecting terms in Equation (D.84),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi} \Bigg\{ & 2\pi \left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha + 1\right)\tau\right]}{\tau}\, d\tau \\
& - 2\pi \left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& - 2\pi \left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha + 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi \left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2 \left(u_0^2 - \alpha - \alpha^2\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 + \alpha + 1\right)\tau\right] - 1 + \cos\left[\left(u_0 - \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2 \left(u_0^2 + \alpha - \alpha^2\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 + \alpha - 1\right)\tau\right] - 1 + \cos\left[\left(u_0 - \alpha + 1\right)\tau\right]}{\tau}\, d\tau \\
& - 4 \left(u_0^2 - \alpha^2\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 + \alpha\right)\tau\right] - 1 + \cos\left[\left(u_0 - \alpha\right)\tau\right]}{\tau}\, d\tau \\
& - \left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 + \alpha + 1\right)\tau\right]\, d\tau \\
& + \left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 + \alpha - 1\right)\tau\right]\, d\tau \\
& + \left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 - \alpha + 1\right)\tau\right]\, d\tau \\
& - \left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 - \alpha - 1\right)\tau\right]\, d\tau \Bigg\}
\end{aligned}
\tag{D.85}
$$

Recalling the definitions for the sine and modified cosine integrals,

$$
\mathrm{Si}\left(ba\right) = \int_0^b \frac{\sin\left(at\right)}{t}\, dt
\tag{D.86}
$$

and

$$
\mathrm{Cin}\left(ba\right) = \int_0^b \frac{1 - \cos\left(at\right)}{t}\, dt.
\tag{D.87}
$$

Applying Equations (D.86) and (D.87) to Equation (D.85) and performing the remaining integrals,

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 2\pi\left(u_0^2 - 2\alpha - \alpha^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha + 1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha - 1\right)\right]\right] \\
& - 2\pi\left(u_0^2 + 2\alpha - \alpha^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha - 1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha + 1\right)\right]\right] \\
& + 2\left(u_0^2 - \alpha - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + 1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha - 1\right)\right]\right] \\
& + 2\left(u_0^2 + \alpha - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha - 1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha + 1\right)\right]\right] \\
& - 4\left(u_0^2 - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha\right)\right]\right] \\
& + \frac{\left(u_0^2 - 2\alpha - \alpha^2 - 1\right)}{\left(u_0 + \alpha + 1\right)}\Bigg[\cos\left[\left(u_0 + \alpha + 1\right)\tau\right]\Bigg]_0^{2\pi} \\
& - \frac{\left(u_0^2 + 2\alpha - \alpha^2 - 1\right)}{\left(u_0 + \alpha - 1\right)}\Bigg[\cos\left[\left(u_0 + \alpha - 1\right)\tau\right]\Bigg]_0^{2\pi} \\
& - \frac{\left(u_0^2 + 2\alpha - \alpha^2 - 1\right)}{\left(u_0 - \alpha + 1\right)}\Bigg[\cos\left[\left(u_0 - \alpha + 1\right)\tau\right]\Bigg]_0^{2\pi} \\
& + \frac{\left(u_0^2 - 2\alpha - \alpha^2 - 1\right)}{\left(u_0 - \alpha - 1\right)}\Bigg[\cos\left[\left(u_0 - \alpha - 1\right)\tau\right]\Bigg]_0^{2\pi}\Bigg\}. \qquad \text{(D.88)}
\end{aligned}$$

Evaluating the limits of integration and simplifying Equation (D.88),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 2\pi\left(u_0^2 - 2\alpha - \alpha^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha + 1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha - 1\right)\right]\right] \\
& - 2\pi\left(u_0^2 + 2\alpha - \alpha^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0 + \alpha - 1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0 - \alpha + 1\right)\right]\right] \\
& + 2\left(u_0^2 - \alpha - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha + 1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha - 1\right)\right]\right] \\
& + 2\left(u_0^2 + \alpha - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha - 1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha + 1\right)\right]\right] \\
& - 4\left(u_0^2 - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0 + \alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0 - \alpha\right)\right]\right] \\
& + \left(u_0 - \alpha - 1\right)\left[\cos\left[2\pi\left(u_0 + \alpha\right)\right] - 1\right] \\
& - \left(u_0 - \alpha + 1\right)\left[\cos\left[2\pi\left(u_0 + \alpha\right)\right] - 1\right] \\
& - \left(u_0 + \alpha - 1\right)\left[\cos\left[2\pi\left(u_0 - \alpha\right)\right] - 1\right] \\
& + \left(u_0 + \alpha + 1\right)\left[\cos\left[2\pi\left(u_0 - \alpha\right)\right] - 1\right]\Bigg\}. \qquad \text{(D.89)}
\end{aligned}$$

Simplifying Equation (D.89) yields the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the cosine-squared distribution,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & 2\pi\left(u_0^2 - 2\alpha - \alpha^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha+1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\ & - 2\pi\left(u_0^2 + 2\alpha - \alpha^2 - 1\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha-1\right)\right] + \mathrm{Si}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & + 2\left(u_0^2 - \alpha - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\ & + 2\left(u_0^2 + \alpha - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\ & - 4\left(u_0^2 - \alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & - 2\cos\left[2\pi\left(u_0+\alpha\right)\right] + 2\cos\left[2\pi\left(u_0-\alpha\right)\right]\Bigg\}. \end{aligned} \tag{D.90}$$

Recalling the definition for $\widetilde{R}_{m2}(0)$,

$$\begin{aligned}\widetilde{R}_{m2}(0) = & \frac{u_0}{2\pi}\Bigg\{\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau \\ & + \alpha^2\int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau + \alpha^2\int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau \\ & - 2\alpha\int_{-\infty}^{\infty} R_g'''(\tau)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau + 2\alpha\int_{-\infty}^{\infty} R_g'''(\tau)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau\Bigg\}. \end{aligned} \tag{D.91}$$

Substituting Equations (D.71), (D.74), and (D.79) into Equation (D.91),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2 u_0}{16\pi}\Bigg\{ & \int_{-2\pi}^{0} \left[-\left(1+u_0^2\right)\sin(\tau) - \left(1-u_0^2\right)(2\pi+\tau)\cos(\tau)\right] \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{0}^{2\pi} \left[\left(1+u_0^2\right)\sin(\tau) - \left(1-u_0^2\right)(2\pi-\tau)\cos(\tau)\right] \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{-2\pi}^{0} \left[-\left(1+u_0^2\right)\sin(\tau) - \left(1-u_0^2\right)(2\pi+\tau)\cos(\tau)\right] \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{0}^{2\pi} \left[\left(1+u_0^2\right)\sin(\tau) - \left(1-u_0^2\right)(2\pi-\tau)\cos(\tau)\right] \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{-2\pi}^{0} \left[-(\tau+2\pi)\cos(\tau) + \sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{0}^{2\pi} \left[(\tau-2\pi)\cos(\tau) - \sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{-2\pi}^{0} \left[-(\tau+2\pi)\cos(\tau) + \sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{0}^{2\pi} \left[(\tau-2\pi)\cos(\tau) - \sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 2\alpha \int_{-2\pi}^{0} (\tau+2\pi)\sin(\tau) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{0}^{2\pi} (\tau-2\pi)\sin(\tau) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{-2\pi}^{0} (\tau+2\pi)\sin(\tau) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 2\alpha \int_{0}^{2\pi} (\tau-2\pi)\sin(\tau) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{D.92}
$$

Recognizing even and odd functions in Equation (D.92) and combining terms,

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2 \cancel{u_0}}{16\pi} \Bigg\{ & 2\int_0^{2\pi} \left[\left(1+u_0^2\right)\sin(\tau) - \left(1-u_0^2\right)(2\pi-\tau)\cos(\tau)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\int_0^{2\pi} \left[\left(1+u_0^2\right)\sin(\tau) - \left(1-u_0^2\right)(2\pi-\tau)\cos(\tau)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[(\tau-2\pi)\cos(\tau) - \sin(\tau)\right] \frac{\sin\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha^2 \int_0^{2\pi} \left[(\tau-2\pi)\cos(\tau) - \sin(\tau)\right] \frac{\sin\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} (\tau-2\pi)\sin(\tau) \frac{\cos\left[(u_0+\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 4\alpha \int_0^{2\pi} (\tau-2\pi)\sin(\tau) \frac{\cos\left[(u_0-\alpha)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}.
\end{aligned} \tag{D.93}$$

Combining terms in Equation (D.93),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi} \Bigg\{ & 2\left(u_0^2-\alpha^2+1\right) \int_0^{2\pi} \sin(\tau) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\tau}\, d\tau \\
& + 2\left(u_0^2-\alpha^2+1\right) \int_0^{2\pi} \sin(\tau) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\tau}\, d\tau \\
& + 4\pi\left(u_0^2-\alpha^2-1\right) \int_0^{2\pi} \cos(\tau) \frac{\sin\left[(u_0+\alpha)\tau\right]}{\tau}\, d\tau \\
& + 4\pi\left(u_0^2-\alpha^2-1\right) \int_0^{2\pi} \cos(\tau) \frac{\sin\left[(u_0-\alpha)\tau\right]}{\tau}\, d\tau \\
& - 2\left(u_0^2-\alpha^2-1\right) \int_0^{2\pi} \cos(\tau)\sin\left[(u_0+\alpha)\tau\right]\, d\tau \\
& - 2\left(u_0^2-\alpha^2-1\right) \int_0^{2\pi} \cos(\tau)\sin\left[(u_0-\alpha)\tau\right]\, d\tau \\
& - 8\pi\alpha \int_0^{2\pi} \sin(\tau) \frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau}\, d\tau \\
& + 8\pi\alpha \int_0^{2\pi} \sin(\tau) \frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau}\, d\tau \\
& + 4\alpha \int_0^{2\pi} \sin(\tau)\cos\left[(u_0+\alpha)\tau\right]\, d\tau \\
& - 4\alpha \int_0^{2\pi} \sin(\tau)\cos\left[(u_0-\alpha)\tau\right]\, d\tau \Bigg\}.
\end{aligned} \tag{D.94}$$

Applying product-to-sum trigonometric identities to Equation (D.94),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & \left(u_0^2-\alpha^2+1\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0+\alpha-1\right)\tau\right]-\cos\left[\left(u_0+\alpha+1\right)\tau\right]}{\tau}\,d\tau \\
& + \left(u_0^2-\alpha^2+1\right)\int_0^{2\pi}\frac{\cos\left[\left(u_0-\alpha-1\right)\tau\right]-\cos\left[\left(u_0-\alpha+1\right)\tau\right]}{\tau}\,d\tau \\
& + 2\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]+\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\,d\tau \\
& + 2\pi\left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]+\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\,d\tau \\
& - \left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+1\right)\tau\right]+\sin\left[\left(u_0+\alpha-1\right)\tau\right]\right]\,d\tau \\
& - \left(u_0^2-\alpha^2-1\right)\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+1\right)\tau\right]+\sin\left[\left(u_0-\alpha-1\right)\tau\right]\right]\,d\tau \\
& - 4\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha+1\right)\tau\right]-\sin\left[\left(u_0+\alpha-1\right)\tau\right]}{\tau}\,d\tau \\
& + 4\pi\alpha\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha+1\right)\tau\right]-\sin\left[\left(u_0-\alpha-1\right)\tau\right]}{\tau}\,d\tau \\
& + 2\alpha\int_0^{2\pi}\left[\sin\left[\left(u_0+\alpha+1\right)\tau\right]-\sin\left[\left(u_0+\alpha-1\right)\tau\right]\right]\,d\tau \\
& - 2\alpha\int_0^{2\pi}\left[\sin\left[\left(u_0-\alpha+1\right)\tau\right]-\sin\left[\left(u_0-\alpha-1\right)\tau\right]\right]\,d\tau\Bigg\}. \qquad \text{(D.95)}
\end{aligned}$$

Collecting terms in Equation (D.95),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & \left(u_0^2 - \alpha^2 + 1\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 + \alpha + 1\right)\tau\right] - 1 + \cos\left[\left(u_0 + \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& + \left(u_0^2 - \alpha^2 + 1\right) \int_0^{2\pi} \frac{1 - \cos\left[\left(u_0 - \alpha + 1\right)\tau\right] - 1 + \cos\left[\left(u_0 - \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha + 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 + \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha + 1\right)\tau\right]}{\tau}\, d\tau \\
& + 2\pi\left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \frac{\sin\left[\left(u_0 - \alpha - 1\right)\tau\right]}{\tau}\, d\tau \\
& - \left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 + \alpha + 1\right)\tau\right]\, d\tau \\
& - \left(u_0^2 + 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 + \alpha - 1\right)\tau\right]\, d\tau \\
& - \left(u_0^2 + \alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 - \alpha + 1\right)\tau\right]\, d\tau \\
& - \left(u_0^2 - 2\alpha - \alpha^2 - 1\right) \int_0^{2\pi} \sin\left[\left(u_0 - \alpha - 1\right)\tau\right]\, d\tau \Bigg\}. \qquad \text{(D.96)}
\end{aligned}$$

Applying Equations (D.86) and (D.87), and the sifting property of the delta function, to Equation (D.96) and performing the remaining integrals,

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & \left(u_0^2-\alpha^2+1\right)\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right]\right] \\
& + \left(u_0^2-\alpha^2+1\right)\left[\operatorname{Cin}\left[2\pi\left(\mathrm{u}_0-\alpha+1\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\
& + 2\pi\left(u_0^2-2\alpha-\alpha^2-1\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha+1\right)\right]+\operatorname{Si}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\
& + 2\pi\left(u_0^2+2\alpha-\alpha^2-1\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha-1\right)\right]+\operatorname{Si}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\
& + \frac{\left(u_0^2-2\alpha-\alpha^2-1\right)}{\left(u_0+\alpha+1\right)}\Bigg[\cos\left[\left(u_0+\alpha+1\right)\tau\right]\Bigg]_0^{2\pi} \\
& + \frac{\left(u_0^2+2\alpha-\alpha^2-1\right)}{\left(u_0+\alpha-1\right)}\Bigg[\cos\left[\left(u_0+\alpha-1\right)\tau\right]\Bigg]_0^{2\pi} \\
& + \frac{\left(u_0^2+\alpha-\alpha^2-1\right)}{\left(u_0-\alpha+1\right)}\Bigg[\cos\left[\left(u_0-\alpha+1\right)\tau\right]\Bigg]_0^{2\pi} \\
& + \frac{\left(u_0^2-2\alpha-\alpha^2-1\right)}{\left(u_0-\alpha-1\right)}\Bigg[\cos\left[\left(u_0-\alpha-1\right)\tau\right]\Bigg]_0^{2\pi}\Bigg\}.
\end{aligned} \tag{D.97}$$

Evaluating the limits of integration and simplifying Equation (D.97),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi}\Bigg\{ & \left(u_0^2-\alpha^2+1\right)\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha+1\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0+\alpha-1\right)\right]\right] \\
& + \left(u_0^2-\alpha^2+1\right)\left[\operatorname{Cin}\left[2\pi\left(\mathrm{u}_0-\alpha+1\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\
& + 2\pi\left(u_0^2-2\alpha-\alpha^2-1\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha+1\right)\right]+\operatorname{Si}\left[2\pi\left(u_0-\alpha-1\right)\right]\right] \\
& + 2\pi\left(u_0^2+2\alpha-\alpha^2-1\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha-1\right)\right]+\operatorname{Si}\left[2\pi\left(u_0-\alpha+1\right)\right]\right] \\
& + \left(u_0-\alpha-1\right)\left[\cos\left[\left(u_0+\alpha\right)\right]-1\right] \\
& + \left(u_0-\alpha+1\right)\left[\cos\left[\left(u_0+\alpha\right)\right]-1\right] \\
& + \left(u_0+\alpha-1\right)\left[\cos\left[\left(u_0-\alpha\right)\right]-1\right] \\
& + \left(u_0+\alpha+1\right)\left[\cos\left[\left(u_0-\alpha\right)\right]-1\right]\Bigg\}.
\end{aligned} \tag{D.98}$$

Simplifying Equation (D.98) yields the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the cosine-squared distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{16\pi}\Big\{ & \left(u_0^2 - \alpha^2 + 1\right)\left[\text{Cin}\left[2\pi\left(u_0 + \alpha + 1\right)\right] - \text{Cin}\left[2\pi\left(u_0 + \alpha - 1\right)\right]\right] \\ & + \left(u_0^2 - \alpha^2 + 1\right)\left[\text{Cin}\left[2\pi\left(\text{u}_0 - \alpha + 1\right)\right] - \text{Cin}\left[2\pi\left(u_0 - \alpha - 1\right)\right]\right] \\ & + 2\pi\left(u_0^2 - 2\alpha - \alpha^2 - 1\right)\left[\text{Si}\left[2\pi\left(u_0 + \alpha + 1\right)\right] + \text{Si}\left[2\pi\left(u_0 - \alpha - 1\right)\right]\right] \\ & + 2\pi\left(u_0^2 + 2\alpha - \alpha^2 - 1\right)\left[\text{Si}\left[2\pi\left(u_0 + \alpha - 1\right)\right] + \text{Si}\left[2\pi\left(u_0 - \alpha + 1\right)\right]\right] \\ & + 2\left(u_0 - \alpha\right)\cos\left(u_0 + \alpha\right) + 2\left(u_0 + \alpha\right)\cos\left(u_0 - \alpha\right) - 4u_0\Big\}. \end{aligned} \tag{D.99}$$

D.4 Triangular Distribution

Recalling the autocorrelation function $R_g(p)$ for the triangular distribution,

$$R_g(p) = \frac{A_m^2}{6\pi^2}\begin{cases}(2\pi + p)^3, & -2\pi \le p \le -\pi \\ 4\pi^3 - 6\pi p^2 - 3p^3, & -\pi \le p \le 0 \\ 4\pi^3 - 6\pi p^2 + 3p^3, & 0 \le p \le \pi \\ (2\pi - p)^3, & \pi \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{D.100}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = \frac{A_m^2}{6\pi^2}\begin{cases}3(2\pi + p)^2, & -2\pi \le p \le -\pi \\ -\left(12\pi p + 9p^2\right), & -\pi \le p \le 0 \\ -\left(12\pi p - 9p^2\right), & 0 \le p \le \pi \\ -3(2\pi - p)^2, & \pi \le p \le 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{D.101}$$

Finding the second derivative of $R_g(p)$:

$$R_g''(p) = \frac{A_m^2}{6\pi^2}\frac{d}{dp}\begin{cases} 3\left(2\pi+p\right)^2, & -2\pi \leq p \leq -\pi \\ -\left(12\pi p + 9p^2\right), & -\pi \leq p \leq 0 \\ -\left(12\pi p - 9p^2\right), & 0 \leq p \leq \pi \\ -3\left(2\pi - p\right)^2, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{D.102}$$

which can be simplified to be

$$R_g''(p) = \frac{A_m^2}{6\pi^2}\begin{cases} 12\pi + 6p, & -2\pi \leq p \leq -\pi \\ -\left(12\pi + 18p\right), & -\pi \leq p \leq 0 \\ -\left(12\pi - 18p\right), & 0 \leq p \leq \pi \\ 12\pi - 6p, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{D.103}$$

Finding the third derivative of $R_g(p)$:

$$R_g'''(p) = \frac{A_m^2}{6\pi^2}\frac{d}{dp}\begin{cases} 12\pi + 6p, & -2\pi \leq p \leq -\pi \\ -\left(12\pi + 18p\right), & -\pi \leq p \leq 0 \\ -\left(12\pi - 18p\right), & 0 \leq p \leq \pi \\ 12\pi - 6p, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}; \tag{D.104}$$

which can be simplified to be

$$R_g'''(p) = \frac{A_m^2}{6\pi^2}\begin{cases} 6, & -2\pi \leq p \leq -\pi \\ -18, & -\pi \leq p \leq 0 \\ 18, & 0 \leq p \leq \pi \\ -6, & \pi \leq p \leq 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{D.105}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = \frac{A_m^2}{6\pi^2}\begin{cases} u_0^2\left(2\pi+p\right)^3+6\left(2\pi+p\right), & -2\pi\leq p\leq -\pi \\ u_0^2\left(4\pi^3-6\pi p^2-3p^3\right)-12\pi-18p, & -\pi\leq p\leq 0 \\ u_0^2\left(4\pi^3-6\pi p^2+3p^3\right)-12\pi+18p, & 0\leq p\leq \pi \\ u_0^2\left(2\pi-p\right)^3+6\left(2\pi-p\right), & \pi\leq p\leq 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{D.106}$$

Finding the first derivative of $R_f(p)$:

$$R_f'(p) = \frac{A_m^2}{6\pi^2}\frac{d}{dp}\begin{cases} u_0^2\left(2\pi+p\right)^3+6\left(2\pi+p\right), & -2\pi\leq p\leq -\pi \\ u_0^2\left(4\pi^3-6\pi p^2-3p^3\right)-12\pi-18p, & -\pi\leq p\leq 0 \\ u_0^2\left(4\pi^3-6\pi p^2+3p^3\right)-12\pi+18p, & 0\leq p\leq \pi \\ u_0^2\left(2\pi-p\right)^3+6\left(2\pi-p\right), & \pi\leq p\leq 2\pi \\ 0, & \text{otherwise}\end{cases}; \tag{D.107}$$

which can be simplified to be

$$R_f'(p) = \frac{A_m^2}{6\pi^2}\begin{cases} 3u_0^2\left(2\pi+p\right)^2+6, & -2\pi\leq p\leq -\pi \\ -u_0^2\left(12\pi p+9p^2\right)-18, & -\pi\leq p\leq 0 \\ -u_0^2\left(12\pi p-9p^2\right)+18, & 0\leq p\leq \pi \\ -3u_0^2\left(2\pi-p\right)^2-6, & \pi\leq p\leq 2\pi \\ 0, & \text{otherwise}\end{cases}. \tag{D.108}$$

Recalling the autocorrelation function $R_n(p)$,

$$R_n(p) = R_{n1}(p)+R_{n2}(p)-\frac{6A_m^2}{\pi^2}\delta(p)+\frac{4A_m^2}{\pi^2}\delta\left(|p|-\pi\right)-\frac{A_m^2}{\pi^2}\delta\left(|p|-2\pi\right), \tag{D.109}$$

where

$$R_{n1}(p) = \frac{A_m^2u_0^2}{\pi^2}\left(2\pi-3|p|\right) \tag{D.110}$$

for $0 \leq |p| \leq \pi$ and 0 otherwise, and

$$R_{n2}(p) = \frac{A_m^2 u_0^2}{\pi^2} \left(|p| - 2\pi\right) \tag{D.111}$$

for $\pi \leq |p| \leq 2\pi$ and 0 otherwise. Recalling the definition for $\widetilde{R}_{m1}(0)$,

$$\begin{aligned} \widetilde{R}_{m1}(0) = & \frac{u_0}{2\pi}\Bigg\{ - \int_{-\infty}^{\infty} R_f'(\tau) \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau + \int_{-\infty}^{\infty} R_f'(\tau) \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\ & + \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau) \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau) \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau) \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau) \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \Bigg\}. \end{aligned} \tag{D.112}$$

Substituting Equations (D.108), (D.101), and (D.103) into Equation (D.112),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 u_0}{12\pi^3} \Bigg\{ & - \int_{-2\pi}^{-\pi} \left(3u_0\tau^2 + 12\pi u_0^2\tau + 12\pi^2 u_0^2 + 6\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{-\pi}^{0} \left(9u_0^2\tau^2 + 12\pi u_0^2\tau + 18\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \int_{0}^{\pi} \left(9u_0^2\tau^2 - 12\pi u_0^2\tau + 18\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{\pi}^{2\pi} \left(3u_0^2\tau^2 - 12\pi u_0^2\tau + 12\pi^2 u_0^2 + 6\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{-2\pi}^{-\pi} \left(3u_0\tau^2 + 12\pi u_0^2\tau + 12\pi^2 u_0^2 + 6\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \int_{-\pi}^{0} \left(9u_0^2\tau^2 + 12\pi u_0^2\tau + 18\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \int_{0}^{\pi} \left(9u_0^2\tau^2 - 12\pi u_0^2\tau + 18\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \int_{\pi}^{2\pi} \left(3u_0^2\tau^2 - 12\pi u_0^2\tau + 12\pi^2 u_0^2 + 6\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{-2\pi}^{-\pi} \left(3\tau^2 + 12\pi\tau + 12\pi^2\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{-\pi}^{0} \left(9\tau^2 + 12\pi\tau\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau + \alpha^2 \int_{0}^{\pi} \left(9\tau^2 - 12\pi\tau\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{\pi}^{2\pi} \left(3\tau^2 - 12\pi\tau + 12\pi^2\right) \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{-2\pi}^{-\pi} \left(3\tau^2 + 12\pi\tau + 12\pi^2\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{-\pi}^{0} \left(9\tau^2 + 12\pi\tau\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - \alpha^2 \int_{0}^{\pi} \left(9\tau^2 - 12\pi\tau\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{\pi}^{2\pi} \left(3\tau^2 - 12\pi\tau + 12\pi^2\right) \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{-2\pi}^{-\pi} (6\tau + 12\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 2\alpha \int_{-\pi}^{0} (18\tau + 12\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{0}^{\pi} (18\tau - 12\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 2\alpha \int_{\pi}^{2\pi} (6\tau - 12\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{-2\pi}^{-\pi} (6\tau + 12\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 2\alpha \int_{-\pi}^{0} (18\tau + 12\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 2\alpha \int_{0}^{\pi} (18\tau - 12\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 2\alpha \int_{\pi}^{2\pi} (6\tau - 12\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{D.113}
$$

Recognizing even and odd functions in Equation (D.113) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 \cancel{u_0}}{12\pi^3} \Bigg\{ & -2 \int_0^{\pi} \left(9u_0^2\tau^2 - 12\pi u_0^2\tau + 18\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2 \int_{\pi}^{2\pi} \left(3u_0^2\tau^2 - 12\pi u_0^2\tau + 12\pi^2 u_0^2 + 6\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2 \int_0^{\pi} \left(9u_0^2\tau^2 - 12\pi u_0^2\tau + 18\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 2 \int_{\pi}^{2\pi} \left(3u_0^2\tau^2 - 12\pi u_0^2\tau + 12\pi^2 u_0^2 + 6\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2\alpha^2 \int_0^{\pi} \left(9\tau^2 - 12\pi\tau\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 2\alpha^2 \int_{\pi}^{2\pi} \left(3\tau^2 - 12\pi\tau + 12\pi^2\right) \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 2\alpha^2 \int_0^{\pi} \left(9\tau^2 - 12\pi\tau\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 2\alpha^2 \int_{\pi}^{2\pi} \left(3\tau^2 - 12\pi\tau + 12\pi^2\right) \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{\pi} \left(18\tau - 12\pi\right) \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 4\alpha \int_{\pi}^{2\pi} \left(6\tau - 12\pi\right) \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& + 4\alpha \int_0^{\pi} \left(18\tau - 12\pi\right) \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \\
& - 4\alpha \int_{\pi}^{2\pi} \left(6\tau - 12\pi\right) \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau} \, d\tau \Bigg\}.
\end{aligned}
\tag{D.114}
$$

Combining terms in Equation (D.114),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -18\left(u_0^2-\alpha^2\right)\int_0^{\pi} \tau^{\cancel{2}}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& + 24\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi} \cancel{\tau}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& - 36\int_0^{\pi} \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 6\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi} \tau^{\cancel{2}}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& - 24\pi\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi} \cancel{\tau}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& + \left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\int_{\pi}^{2\pi} \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 18\left(u_0^2-\alpha^2\right)\int_0^{\pi} \tau^{\cancel{2}}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& - 24\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi} \cancel{\tau}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& + 36\int_0^{\pi} \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 6\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi} \tau^{\cancel{2}}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& + 24\pi\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi} \cancel{\tau}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& - \left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\int_{\pi}^{2\pi} \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 72\alpha\int_0^{\pi} \cancel{\tau}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]d\tau - 48\pi\alpha\int_0^{\pi} \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 24\alpha\int_{\pi}^{2\pi} \cancel{\tau}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau + 48\pi\alpha\int_{\pi}^{2\pi} \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 72\alpha\int_0^{\pi} \cancel{\tau}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau - 48\pi\alpha\int_0^{\pi} \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 24\alpha\int_{\pi}^{2\pi} \cancel{\tau}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau + 48\pi\alpha\int_{\pi}^{2\pi} \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau\Bigg\}. \qquad \text{(D.115)}
\end{aligned}
$$

Canceling terms and rearranging Equation (D.115),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & - 18\left(u_0^2 - \alpha^2\right)\int_0^{\pi} \tau\cos\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& + 6\left(u_0^2 - \alpha^2\right)\int_{\pi}^{2\pi} \tau\cos\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& + 18\left(u_0^2 - \alpha^2\right)\int_0^{\pi} \tau\cos\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& - 6\left(u_0^2 - \alpha^2\right)\int_{\pi}^{2\pi} \tau\cos\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& + 24\pi\left(u_0^2 - \alpha^2\right)\int_0^{\pi} \cos\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& - 24\pi\left(u_0^2 - \alpha^2\right)\int_{\pi}^{2\pi} \cos\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& - 24\pi\left(u_0^2 - \alpha^2\right)\int_0^{\pi} \cos\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& + 24\pi\left(u_0^2 - \alpha^2\right)\int_{\pi}^{2\pi} \cos\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& - 36\int_0^{\pi} \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 36\int_0^{\pi} \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + \left[24\pi^2\left(u_0^2 - \alpha^2\right) + 12\right]\int_{\pi}^{2\pi} \frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - \left[24\pi^2\left(u_0^2 - \alpha^2\right) + 12\right]\int_{\pi}^{2\pi} \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \\
& + 72\alpha\int_0^{\pi} \sin\left[\left(u_0+\alpha\right)\tau\right] d\tau - 24\alpha\int_{\pi}^{2\pi} \sin\left[\left(u_0+\alpha\right)\tau\right]\, d\tau \\
& + 72\alpha\int_0^{\pi} \sin\left[\left(u_0-\alpha\right)\tau\right]\, d\tau - 24\alpha\int_{\pi}^{2\pi} \sin\left[\left(u_0-\alpha\right)\tau\right]\, d\tau \\
& - 48\pi\alpha\int_0^{\pi} \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau + 48\pi\alpha\int_{\pi}^{2\pi} \frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\, d\tau \\
& - 48\pi\alpha\int_0^{\pi} \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau + 48\pi\alpha\int_{\pi}^{2\pi} \frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\, d\tau \Bigg\}. \quad \text{(D.116)}
\end{aligned}
$$

Integrating and rearranging terms in Equation (D.116),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ &- 18\left(u_0^2-\alpha^2\right)\left[\frac{(u_0+\alpha)\,\tau\sin\left[(u_0+\alpha)\,\tau\right]+\cos\left[(u_0+\alpha)\,\tau\right]}{(u_0+\alpha)^2}\right]_0^{\pi} \\
&+ 6\left(u_0^2-\alpha^2\right)\left[\frac{(u_0+\alpha)\,\tau\sin\left[(u_0+\alpha)\,\tau\right]+\cos\left[(u_0+\alpha)\,\tau\right]}{(u_0+\alpha)^2}\right]_{\pi}^{2\pi} \\
&+ 18\left(u_0^2-\alpha^2\right)\left[\frac{(u_0-\alpha)\,\tau\sin\left[(u_0-\alpha)\,\tau\right]+\cos\left[(u_0-\alpha)\,\tau\right]}{(u_0-\alpha)^2}\right]_0^{\pi} \\
&- 6\left(u_0^2-\alpha^2\right)\left[\frac{(u_0-\alpha)\,\tau\sin\left[(u_0-\alpha)\,\tau\right]+\cos\left[(u_0-\alpha)\,\tau\right]}{(u_0-\alpha)^2}\right]_{\pi}^{2\pi} \\
&+ 24\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0+\alpha}\right]_0^{\pi} - 24\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0+\alpha}\right]_{\pi}^{2\pi} \\
&- 24\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0-\alpha}\right]_0^{\pi} + 24\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0-\alpha}\right]_{\pi}^{2\pi} \\
&+ 36\int_0^{\pi}\frac{1-\cos\left[(u_0+\alpha)\,\tau\right]-1+\cos\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau \\
&- \left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\left[\int_0^{2\pi}\frac{1-\cos\left[(u_0+\alpha)\,\tau\right]-1+\cos\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau\right] \\
&+ \left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\left[\int_0^{\pi}\frac{1-\cos\left[(u_0+\alpha)\,\tau\right]-1+\cos\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau\right] \\
&- 72\alpha\left[\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0+\alpha}\right]_0^{\pi} + 24\alpha\left[\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0+\alpha}\right]_{\pi}^{2\pi} \\
&- 72\alpha\left[\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0-\alpha}\right]_0^{\pi} + 24\alpha\left[\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0-\alpha}\right]_{\pi}^{2\pi} \\
&- 96\pi\alpha\int_0^{\pi}\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau + 48\pi\alpha\int_0^{2\pi}\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\tau}\;d\tau \\
&- 96\pi\alpha\int_0^{\pi}\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau + 48\pi\alpha\int_0^{2\pi}\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\tau}\;d\tau\Bigg\}.
\end{aligned} \tag{D.117}$$

Recalling the definitions for the sine and modified cosine integrals,

$$\mathrm{Si}\,(ba) = \int_0^b \frac{\sin\,(at)}{t}\;dt \tag{D.118}$$

and

$$\mathrm{Cin}\,(ba) = \int_0^b \frac{1-\cos\,(at)}{t}\;dt. \tag{D.119}$$

Applying Equations (D.118) and (D.119) to Equation (D.117) and evaluating the limits of integration,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -18\left(u_0^2-\alpha^2\right)\left[\frac{\pi\left(u_0+\alpha\right)\sin\left[\pi\left(u_0+\alpha\right)\right]+\cos\left[\pi\left(u_0+\alpha\right)\right]-1}{\left(u_0+\alpha\right)^2}\right] \\
& +6\left(u_0^2-\alpha^2\right)\left[\frac{2\pi\left(u_0+\alpha\right)\sin\left[2\pi\left(u_0+\alpha\right)\right]+\cos\left[2\pi\left(u_0+\alpha\right)\right]}{\left(u_0+\alpha\right)^2}\right] \\
& -6\left(u_0^2-\alpha^2\right)\left[\frac{\pi\left(u_0+\alpha\right)\sin\left[\pi\left(u_0+\alpha\right)\right]+\cos\left[\pi\left(u_0+\alpha\right)\right]}{\left(u_0+\alpha\right)^2}\right] \\
& +18\left(u_0^2-\alpha^2\right)\left[\frac{\pi\left(u_0-\alpha\right)\sin\left[\pi\left(u_0-\alpha\right)\right]+\cos\left[\pi\left(u_0-\alpha\right)\right]-1}{\left(u_0-\alpha\right)^2}\right] \\
& -6\left(u_0^2-\alpha^2\right)\left[\frac{2\pi\left(u_0-\alpha\right)\sin\left[2\pi\left(u_0-\alpha\right)\right]+\cos\left[2\pi\left(u_0-\alpha\right)\right]}{\left(u_0-\alpha\right)^2}\right] \\
& +6\left(u_0^2-\alpha^2\right)\left[\frac{\pi\left(u_0-\alpha\right)\sin\left[\pi\left(u_0-\alpha\right)\right]+\cos\left[\pi\left(u_0-\alpha\right)\right]}{\left(u_0-\alpha\right)^2}\right] \\
& +24\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[\pi\left(u_0+\alpha\right)\right]-\sin\left[2\pi\left(u_0+\alpha\right)\right]+\sin\left[\pi\left(u_0+\alpha\right)\right]}{u_0+\alpha}\right] \\
& -24\pi\left(u_0^2-\alpha^2\right)\left[\frac{\sin\left[\pi\left(u_0-\alpha\right)\right]-\sin\left[2\pi\left(u_0-\alpha\right)\right]+\sin\left[\pi\left(u_0-\alpha\right)\right]}{u_0-\alpha}\right] \\
& +36\left[\mathrm{Cin}\left[\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& -\left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& +\left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\left[\mathrm{Cin}\left[\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& -72\alpha\left[\frac{\cos\left[\pi\left(u_0+\alpha\right)\right]-1}{u_0+\alpha}\right] \\
& +24\alpha\left[\frac{\cos\left[2\pi\left(u_0+\alpha\right)\right]-\cos\left[\pi\left(u_0+\alpha\right)\right]}{u_0+\alpha}\right] \\
& -72\alpha\left[\frac{\cos\left[\pi\left(u_0-\alpha\right)\right]-1}{u_0-\alpha}\right] \\
& +24\alpha\left[\frac{\cos\left[2\pi\left(u_0-\alpha\right)\right]-\cos\left[\pi\left(u_0-\alpha\right)\right]}{u_0-\alpha}\right] \\
& -96\pi\alpha\mathrm{Si}\left[\pi\left(u_0+\alpha\right)\right]+48\pi\alpha\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right] \\
& -96\pi\alpha\mathrm{Si}\left[\pi\left(u_0-\alpha\right)\right]+48\pi\alpha\mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\Bigg\}.
\end{aligned}
\tag{D.120}
$$

Collecting terms in Equation (D.120),

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & (12\pi - 24\pi)(u_0 - \alpha)\sin[2\pi(u_0 + \alpha)] \\
& - (12\pi - 24\pi)(u_0 + \alpha)\sin[2\pi(u_0 - \alpha)] \\
& - (18\pi + 6\pi - 48\pi)(u_0 - \alpha)\sin[\pi(u_0 + \alpha)] \\
& + (18\pi + 6\pi - 48\pi)(u_0 + \alpha)\sin[\pi(u_0 - \alpha)] \\
& + \frac{6(u_0 - \alpha) + 24\alpha}{u_0 + \alpha}\cos[2\pi(u_0 + \alpha)] \\
& - \frac{6(u_0 + \alpha) - 24\alpha}{u_0 - \alpha}\cos[2\pi(u_0 - \alpha)] \\
& - \frac{24(u_0 - \alpha) + 96\alpha}{u_0 + \alpha}\cos[\pi(u_0 + \alpha)] \\
& + \frac{24(u_0 + \alpha) - 96\alpha}{u_0 - \alpha}\cos[\pi(u_0 - \alpha)] \\
& - \left[24\pi^2\left(u_0^2 - \alpha^2\right) + 12\right]\left[\mathrm{Cin}[2\pi(u_0 + \alpha)] - \mathrm{Cin}[2\pi(u_0 - \alpha)]\right] \\
& + \left[24\pi^2\left(u_0^2 - \alpha^2\right) + 48\right]\left[\mathrm{Cin}[\pi(u_0 + \alpha)] - \mathrm{Cin}[\pi(u_0 - \alpha)]\right] \\
& + 48\pi\alpha\left[\mathrm{Si}[2\pi(u_0 + \alpha)] + \mathrm{Si}[2\pi(u_0 - \alpha)]\right] \\
& - 96\pi\alpha\left[\mathrm{Si}[\pi(u_0 + \alpha)] + \mathrm{Si}[\pi(u_0 - \alpha)]\right] \\
& + 18\left(\frac{u_0 - \alpha}{u_0 + \alpha}\right) - 18\left(\frac{u_0 + \alpha}{u_0 - \alpha}\right) + \frac{72\alpha}{u_0 + \alpha} + \frac{72\alpha}{u_0 - \alpha}\Bigg\}. \qquad \text{(D.121)}
\end{aligned}
$$

Simplifying Equation (D.121) yields the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the triangular distribution,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -12\pi\left(u_0-\alpha\right)\sin\left[2\pi\left(u_0+\alpha\right)\right]+12\pi\left(u_0+\alpha\right)\sin\left[2\pi\left(u_0-\alpha\right)\right] \\ & +24\pi\left(u_0-\alpha\right)\sin\left[\pi\left(u_0+\alpha\right)\right]-24\pi\left(u_0+\alpha\right)\sin\left[\pi\left(u_0-\alpha\right)\right] \\ & +6\left(\frac{u_0+3\alpha}{u_0+\alpha}\right)\cos\left[2\pi\left(u_0+\alpha\right)\right]-6\left(\frac{u_0-3\alpha}{u_0-\alpha}\right)\cos\left[2\pi\left(u_0-\alpha\right)\right] \\ & -24\left(\frac{u_0+3\alpha}{u_0+\alpha}\right)\cos\left[\pi\left(u_0+\alpha\right)\right]+24\left(\frac{u_0-3\alpha}{u_0-\alpha}\right)\cos\left[\pi\left(u_0-\alpha\right)\right] \\ & -\left[24\pi^2\left(u_0^2-\alpha^2\right)+12\right]\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & +\left[24\pi^2\left(u_0^2-\alpha^2\right)+48\right]\left[\mathrm{Cin}\left[\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & +48\pi\alpha\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & -96\pi\alpha\left[\mathrm{Si}\left[\pi\left(u_0+\alpha\right)\right]+\mathrm{Si}\left[\pi\left(u_0-\alpha\right)\right]\right]+\frac{72u_0\alpha}{u_0^2-\alpha^2}\Bigg\}. \end{aligned} \tag{D.122}$$

Recalling the definition for $\widetilde{R}_{m2}(0)$,

$$\begin{aligned}\widetilde{R}_{m2}(0) = & \frac{u_0}{2\pi}\Bigg\{\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau+\int_{-\infty}^{\infty} R_n(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau \\ & +\alpha^2\int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau+\alpha^2\int_{-\infty}^{\infty} R_g''(\tau)\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau \\ & -2\alpha\int_{-\infty}^{\infty} R_g'''(\tau)\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau+2\alpha\int_{-\infty}^{\infty} R_g'''(\tau)\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau\Bigg\}. \end{aligned} \tag{D.123}$$

Substituting Equations (D.109), (D.103), and (D.105) into Equation (D.123),

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2 u_0}{12\pi^3}\Bigg\{ & 12u_0^2 \int_0^{\pi} (2\pi - 3\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 12u_0^2 \int_{\pi}^{2\pi} (\tau - 2\pi)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 12u_0^2 \int_0^{\pi} (2\pi - 3\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 12u_0^2 \int_{\pi}^{2\pi} (\tau - 2\pi)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 36 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 36 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 24 \int_{-\infty}^{\infty} \delta(\tau+\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau + 24 \int_{-\infty}^{\infty} \delta(\tau-\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 6 \int_{-\infty}^{\infty} \delta\left(\tau+2\pi\right) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 6 \int_{-\infty}^{\infty} \delta\left(\tau-2\pi\right) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 24 \int_{-\infty}^{\infty} \delta(\tau+\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau + 24 \int_{-\infty}^{\infty} \delta(\tau-\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 6 \int_{-\infty}^{\infty} \delta(\tau+2\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 6 \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{-2\pi}^{-\pi} (12\pi + 6\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau - \alpha^2 \int_{-\pi}^{0} (12\pi + 18\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{0}^{\pi} (12\pi - 18\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau + \alpha^2 \int_{\pi}^{2\pi} (12\pi - 6\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + \alpha^2 \int_{-2\pi}^{-\pi} (12\pi + 6\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - \alpha^2 \int_{-\pi}^{0} (12\pi + 18\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - \alpha^2 \int_{0}^{\pi} (12\pi - 18\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau + \alpha^2 \int_{\pi}^{2\pi} (12\pi - 6\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 12\alpha \int_{-2\pi}^{-\pi} \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau + 36\alpha \int_{-\pi}^{0} \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& - 36\alpha \int_{0}^{\pi} \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau + 12\alpha \int_{\pi}^{2\pi} \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 12\alpha \int_{-2\pi}^{-\pi} \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 36\alpha \int_{-\pi}^{0} \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \\
& + 36\alpha \int_{0}^{\pi} \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau - 12\alpha \int_{\pi}^{2\pi} \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\, d\tau \Bigg\}.
\end{aligned} \tag{D.124}$$

Recognizing even and odd functions in Equation (D.124) and combining terms,

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2 \cancel{u_0}}{12\pi^3} \Bigg\{ & 12u_0^2 \int_0^{\pi} (2\pi - 3\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& + 12u_0^2 \int_{\pi}^{2\pi} (\tau - 2\pi)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& + 12u_0^2 \int_0^{\pi} (2\pi - 3\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& + 12u_0^2 \int_{\pi}^{2\pi} (\tau - 2\pi)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& - 36 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau - 36 \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& + 24 \int_{-\infty}^{\infty} \delta(\tau+\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau + 24 \int_{-\infty}^{\infty} \delta(\tau-\pi) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& - 6 \int_{-\infty}^{\infty} \delta\left(\tau+2\pi\right) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau - 6 \int_{-\infty}^{\infty} \delta\left(\tau-2\pi\right) \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& + 24 \int_{-\infty}^{\infty} \delta(\tau+\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau + 24 \int_{-\infty}^{\infty} \delta(\tau-\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& - 6 \int_{-\infty}^{\infty} \delta(\tau+2\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau - 6 \int_{-\infty}^{\infty} \delta(\tau-2\pi) \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& - 2\alpha^2 \int_0^{\pi} (12\pi - 18\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau + 2\alpha^2 \int_{\pi}^{2\pi} (12\pi - 6\tau)\, \frac{\sin\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& - 2\alpha^2 \int_0^{\pi} (12\pi - 18\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau + 2\alpha^2 \int_{\pi}^{2\pi} (12\pi - 6\tau)\, \frac{\sin\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& - 72\alpha \int_0^{\pi} \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau + 24\alpha \int_{\pi}^{2\pi} \frac{\cos\left[(u_0+\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \\
& + 72\alpha \int_0^{\pi} \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau - 24\alpha \int_{\pi}^{2\pi} \frac{\cos\left[(u_0-\alpha)\,\tau\right]}{\cancel{u_0}\tau}\; d\tau \Bigg\}. \qquad \text{(D.125)}
\end{aligned}
$$

Applying the sifting property of the delta function and collecting terms in Equation (D.125),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & 48\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& -36\left(u_0^2-\alpha^2\right)\int_0^{\pi}\cancel{\tau}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& -24\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& +12\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi}\cancel{\tau}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& +48\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& -36\left(u_0^2-\alpha^2\right)\int_0^{\pi}\cancel{\tau}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& -24\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& +12\left(u_0^2-\alpha^2\right)\int_{\pi}^{2\pi}\cancel{\tau}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{\tau}}\right]\,d\tau \\
& -36\left(u_0+\alpha\right)-36\left(u_0-\alpha\right) \\
& +24\frac{\sin\left[\cancel{-}\pi\left(u_0+\alpha\right)\right]}{\cancel{-}\pi}+24\frac{\sin\left[\pi\left(u_0+\alpha\right)\right]}{\pi} \\
& -6\frac{\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}2\pi}-6\frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} \\
& +24\frac{\sin\left[\cancel{-}\pi\left(u_0-\alpha\right)\right]}{\cancel{-}\pi}+24\frac{\sin\left[\pi\left(u_0-\alpha\right)\right]}{\pi} \\
& -6\frac{\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\right]}{\cancel{-}2\pi}-6\frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} \\
& +96\alpha\int_0^{\pi}\frac{1-\cos\left[\left(u_0+\alpha\right)\tau\right]-1+\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& -24\alpha\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha\right)\tau\right]-1+\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau. \qquad \text{(D.126)}
\end{aligned}
$$

Further simplifying Equation (D.126) and performing integrations,

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -72u_0 + \frac{48}{\pi}\sin\left[\pi\left(u_0+\alpha\right)\right] + \frac{48}{\pi}\sin\left[\pi\left(u_0-\alpha\right)\right] \\
& - \frac{6}{\pi}\sin\left[2\pi\left(u_0+\alpha\right)\right] - \frac{6}{\pi}\sin\left[2\pi\left(u_0-\alpha\right)\right] \\
& + 36\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{u_0+\alpha}\right]_0^{\pi} \\
& - 12\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{u_0+\alpha}\right]_{\pi}^{2\pi} \\
& + 36\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{u_0-\alpha}\right]_0^{\pi} \\
& - 12\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{u_0-\alpha}\right]_{\pi}^{2\pi} \\
& + 48\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 24\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 48\pi\left(u_0^2-\alpha^2\right)\int_0^{\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 24\pi\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& + 96\alpha\int_0^{\pi}\frac{1-\cos\left[\left(u_0+\alpha\right)\tau\right]-1+\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau \\
& - 24\alpha\int_0^{2\pi}\frac{1-\cos\left[\left(u_0+\alpha\right)\tau\right]-1+\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\,d\tau. \qquad \text{(D.127)}
\end{aligned}
$$

Applying Equations (D.118) and (D.119) to Equation (D.127) and evaluating the limits of integration,

$$\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -72u_0 + \frac{48}{\pi}\sin\left[\pi\left(u_0+\alpha\right)\right] + \frac{48}{\pi}\sin\left[\pi\left(u_0-\alpha\right)\right] \\
& - \frac{6}{\pi}\sin\left[2\pi\left(u_0+\alpha\right)\right] - \frac{6}{\pi}\sin\left[2\pi\left(u_0-\alpha\right)\right] \\
& + 36\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[\pi\left(u_0+\alpha\right)\right]-1}{u_0+\alpha}\right] \\
& - 12\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[2\pi\left(u_0+\alpha\right)\right]-\cos\left[\pi\left(u_0+\alpha\right)\right]}{u_0+\alpha}\right] \\
& + 36\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[\pi\left(u_0-\alpha\right)\right]-1}{u_0-\alpha}\right] \\
& - 12\left(u_0^2-\alpha^2\right)\left[\frac{\cos\left[2\pi\left(u_0-\alpha\right)\right]-\cos\left[\pi\left(u_0-\alpha\right)\right]}{u_0-\alpha}\right] \\
& + 48\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[\pi\left(u_0+\alpha\right)\right]+\mathrm{Si}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& - 24\pi\left(u_0^2-\alpha^2\right)\left[\mathrm{Si}\left[2\pi\left(u_0+\alpha\right)\right]+\mathrm{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + 96\alpha\left[\mathrm{Cin}\left[\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\
& - 24\alpha\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right]\Bigg\}. \qquad \text{(D.128)}
\end{aligned}$$

Simplifying Equation (D.128) and rearranging yields the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the triangular distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{12\pi^3}\Bigg\{ & -24\alpha\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha\right)\right] - \operatorname{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 96\alpha\left[\operatorname{Cin}\left[\pi\left(u_0+\alpha\right)\right] - \operatorname{Cin}\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & - 24\pi\left(u_0^2-\alpha^2\right)\left[\operatorname{Si}\left[2\pi\left(u_0+\alpha\right)\right] + \operatorname{Si}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + 48\pi\left(u_0^2-\alpha^2\right)\left[\operatorname{Si}\left[\pi\left(u_0+\alpha\right)\right] + \operatorname{Si}\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & - \frac{6}{\pi}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right] + \sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & + \frac{48}{\pi}\left[\sin\left[\pi\left(u_0+\alpha\right)\right] + \sin\left[\pi\left(u_0-\alpha\right)\right]\right] \\ & - 12\left(u_0-\alpha\right)\cos\left[2\pi\left(u_0+\alpha\right)\right] - 12\left(u_0+\alpha\right)\cos\left[2\pi\left(u_0-\alpha\right)\right] \\ & + 48\left(u_0-\alpha\right)\cos\left[\pi\left(u_0+\alpha\right)\right] + 48\left(u_0+\alpha\right)\cos\left[\pi\left(u_0-\alpha\right)\right] - 144u_0\Bigg\}.\end{aligned} \tag{D.129}$$

D.5 Uniform Distribution

Recalling the autocorrelation function $R_g(p)$ for the uniform distribution,

$$R_g(p) = A_m^2 \begin{cases} 2\pi + p, & -2\pi \le p \le 0 \\ 2\pi - p, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{D.130}$$

Also recalling the first derivative of $R_g(p)$,

$$R_g'(p) = A_m^2 \begin{cases} 1, & -2\pi \le p \le 0 \\ -1, & 0 \le p \le 2\pi \\ 0, & \text{otherwise} \end{cases}. \tag{D.131}$$

Alternatively, the first derivative of $R_g(p)$ can be written in terms of the Heaviside step function,

$$R_g'(p) = A_m^2\left[H(p+2\pi) - 2H(p) + H(p-2\pi)\right]. \tag{D.132}$$

Finding the second derivative of $R_g(p)$:

$$R_g''(p) = A_m^2 \frac{d}{dp} \left[H(p + 2\pi) - 2H(p) + H(p - 2\pi) \right]; \tag{D.133}$$

recalling that the derivative of the Heaviside step function is the delta function; then

$$R_g''(p) = A_m^2 \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right]. \tag{D.134}$$

The third derivative of $R_g(p)$ can be written in terms of the first derivative of the delta function,

$$R_g'''(p) = A_m^2 \frac{d}{dp} \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right]. \tag{D.135}$$

Recalling the autocorrelation function $R_f(p)$,

$$R_f(p) = A_m^2 \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right] + u_0^2 R_g(p). \tag{D.136}$$

Finding the first derivative of $R_f(p)$:

$$R_f'(p) = A_m^2 \frac{d}{dp} \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right] + u_0^2 R_g'(p); \tag{D.137}$$

which can also be written as

$$R_f'(p) = R_g'''(p) + u_0^2 R_g'(p). \tag{D.138}$$

Recalling the autocorrelation function $R_n(p)$,

$$\begin{aligned} R_n(p) = &- A_m^2 \frac{d^2}{dp^2} \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right] \\ &- A_m^2 u_0^2 \left[\delta(p + 2\pi) - 2\delta(p) + \delta(p - 2\pi) \right], \end{aligned} \tag{D.139}$$

which can also be written as

$$R_n(p) = -\frac{d}{dp} R_g'''(p) - u_0^2 R_g''(p) = -R_g^{(4)}(p) - u_0^2 R_g''(p). \tag{D.140}$$

Recalling the definition for $\widetilde{R}_{m1}(0)$,

$$\begin{aligned}\widetilde{R}_{m1}(0) =& \frac{u_0}{2\pi}\Bigg\{ -\int_{-\infty}^{\infty} R_f'(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_f'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ &+ 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.141}$$

Substituting Equation (D.138) into Equation (D.141),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{u_0}{2\pi}\Bigg\{ & -\int_{-\infty}^{\infty} \left[R_g'''(\tau) + u_0^2 R_g'(\tau)\right]\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ & + \int_{-\infty}^{\infty} \left[R_g'''(\tau) + u_0^2 R_g'(\tau)\right]\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ & + \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau - \alpha^2 \int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.142}$$

Rearranging Equation (D.142),

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{u_0}{2\pi}\Bigg\{ & -\left(u_0^2-\alpha^2\right)\int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ & + \left(u_0^2-\alpha^2\right)\int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ & + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + 2\alpha \int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau \\ & - \int_{-\infty}^{\infty} R_g'''(\tau)\,\frac{\cos\left[(u_0+\alpha)\,\tau\right]}{u_0\tau}\,d\tau + \int_{-\infty}^{\infty} R_g'(\tau)\,\frac{\cos\left[(u_0-\alpha)\,\tau\right]}{u_0\tau}\,d\tau\Bigg\}.\end{aligned} \tag{D.143}$$

Substituting Equations (D.132), (D.134), and (D.135) into Equation (D.143),

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 u_0}{2\pi} \Bigg\{ & -\left(u_0^2 - \alpha^2\right) \int_{-2\pi}^{0} \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau + \left(u_0^2 - \alpha^2\right) \int_{0}^{2\pi} \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\
& + \left(u_0^2 - \alpha^2\right) \int_{-2\pi}^{0} \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau - \left(u_0^2 - \alpha^2\right) \int_{0}^{2\pi} \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\
& + 2\alpha \int_{-\infty}^{\infty} \left[\delta(\tau + 2\pi) - 2\delta(\tau) + \delta(\tau - 2\pi)\right] \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\
& + 2\alpha \int_{-\infty}^{\infty} \left[\delta(\tau + 2\pi) - 2\delta(\tau) + \delta(\tau - 2\pi)\right] \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \left[\delta'(\tau + 2\pi) - 2\delta'(\tau) + \delta'(\tau - 2\pi)\right] \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \\
& + \int_{-\infty}^{\infty} \left[\delta'(\tau + 2\pi) - 2\delta'(\tau) + \delta'(\tau - 2\pi)\right] \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{u_0 \tau}\, d\tau \Bigg\}.
\end{aligned} \qquad \text{(D.144)}$$

Recognizing odd functions in Equation (D.144) and distributing terms,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2 \cancel{u_0}}{2\pi} \Bigg\{ & 2\left(u_0^2 - \alpha^2\right) \int_0^{2\pi} \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 2\left(u_0^2 - \alpha^2\right) \int_0^{2\pi} \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 4\alpha \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha \int_{-\infty}^{\infty} \delta(\tau + 2\pi) \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - 4\alpha \int_{-\infty}^{\infty} \delta(\tau) \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2\alpha \int_{-\infty}^{\infty} \delta(\tau - 2\pi) \frac{\sin\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta'(\tau + 2\pi) \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + 2 \int_{-\infty}^{\infty} \delta'(\tau) \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& - \int_{-\infty}^{\infty} \delta'(\tau - 2\pi) \frac{\cos\left[\left(u_0 + \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \\
& + \int_{-\infty}^{\infty} \delta'(\tau + 2\pi) \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}. \\
& - 2 \int_{-\infty}^{\infty} \delta'(\tau) \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}. \\
& + \int_{-\infty}^{\infty} \delta'(\tau - 2\pi) \frac{\cos\left[\left(u_0 - \alpha\right)\tau\right]}{\cancel{u_0}\tau}\, d\tau \Bigg\}.
\end{aligned}
\tag{D.145}
$$

Recalling the sifting property of the delta function and its derivative,

$$\int_{-\infty}^{\infty} \delta\left(x - a\right) f(x)\, dx = f(a) \tag{D.146}$$

and

$$\int_{-\infty}^{\infty} \delta'\left(x - a\right) f(x)\, dx = -f'(a). \tag{D.147}$$

Applying Equations (D.146) and (D.147) to Equation (D.145) and rearranging,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -2\left(u_0^2-\alpha^2\right)\int_0^{2\pi}\frac{1-\cos\left[(u_0+\alpha)\tau\right]-1+\cos\left[(u_0-\alpha)\tau\right]}{\tau}\,d\tau \\ & +2\alpha\left[\frac{\sin\left[\not-2\pi(u_0+\alpha)\right]}{\not-2\pi}\right]+2\alpha\left[\frac{\sin\left[2\pi(u_0+\alpha)\right]}{2\pi}\right] \\ & +2\alpha\left[\frac{\sin\left[\not-2\pi(u_0-\alpha)\right]}{\not-2\pi}\right]+2\alpha\left[\frac{\sin\left[2\pi(u_0-\alpha)\right]}{2\pi}\right] \\ & -4\alpha(u_0+\alpha)-4\alpha(u_0-\alpha) \\ & +\frac{d}{d\tau}\left[\frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau}\right]\Bigg|_{\tau=-2\pi} \\ & -2\frac{d}{d\tau}\left[\frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau}\right]\Bigg|_{\tau=0} \\ & +\frac{d}{d\tau}\left[\frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau}\right]\Bigg|_{\tau=2\pi} \\ & -\frac{d}{d\tau}\left[\frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau}\right]\Bigg|_{\tau=-2\pi} \\ & +2\frac{d}{d\tau}\left[\frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau}\right]\Bigg|_{\tau=0} \\ & -\frac{d}{d\tau}\left[\frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau}\right]\Bigg|_{\tau=2\pi}\Bigg\}. \end{aligned} \tag{D.148}$$

Recalling the definition of the modified cosine integral,

$$\operatorname{Cin}(ba) = \int_0^b \frac{1-\cos(at)}{t}\,dt. \tag{D.149}$$

Performing the derivatives in Equation (D.148),

$$\frac{d}{d\tau}\left[\frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau}\right] = -\frac{(u_0+\alpha)\sin\left[(u_0+\alpha)\tau\right]}{\tau}-\frac{\cos\left[(u_0+\alpha)\tau\right]}{\tau^2} \tag{D.150}$$

and

$$\frac{d}{d\tau}\left[\frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau}\right] = -\frac{(u_0-\alpha)\sin\left[(u_0-\alpha)\tau\right]}{\tau}-\frac{\cos\left[(u_0-\alpha)\tau\right]}{\tau^2}. \tag{D.151}$$

Substituting Equations (D.149), (D.150), and (D.151) into Equation (D.148) and simplifying,

$$
\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -2\left(u_0^2-\alpha^2\right)\left[\mathrm{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\mathrm{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + \frac{2\alpha}{\pi}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right]+\sin\left[2\pi\left(u_0-\alpha\right)\right]\right]-8\alpha u_0 \\
& - \left[\frac{\left(u_0+\alpha\right)\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}+\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=-2\pi} \\
& + 2\left[\frac{\left(u_0+\alpha\right)\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}+\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0} \\
& - \left[\frac{\left(u_0+\alpha\right)\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}+\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=2\pi} \\
& + \left[\frac{\left(u_0-\alpha\right)\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}+\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=-2\pi} \\
& - 2\left[\frac{\left(u_0-\alpha\right)\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}+\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0} \\
& + \left[\frac{\left(u_0-\alpha\right)\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}+\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=2\pi}\Bigg\}. \qquad \text{(D.152)}
\end{aligned}
$$

Performing the evaluations in Equation (D.152) and simplifying,

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -2\left(u_0^2-\alpha^2\right)\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + \frac{2\alpha}{\pi}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right]+\sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& - \left[\frac{\left(u_0+\alpha\right)\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}2\pi}+\frac{\cos\left[-2\pi\left(u_0+\alpha\right)\right]}{4\pi^2}\right] \\
& - \left[\frac{\left(u_0+\alpha\right)\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi}+\frac{\cos\left[2\pi\left(u_0+\alpha\right)\right]}{4\pi^2}\right] \\
& + \left[\frac{\left(u_0-\alpha\right)\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\right]}{\cancel{-}2\pi}+\frac{\cos\left[-2\pi\left(u_0-\alpha\right)\right]}{4\pi^2}\right] \\
& + \left[\frac{\left(u_0-\alpha\right)\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi}+\frac{\cos\left[2\pi\left(u_0-\alpha\right)\right]}{4\pi^2}\right] \\
& + 2\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}-\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0} \\
& + 2\left(u_0+\alpha\right)^2-2\left(u_0-\alpha\right)^2-8\alpha u_0\Bigg\}.
\end{aligned} \tag{D.153}$$

Collecting terms in Equation (D.153) and simplifying,

$$\begin{aligned}
\widetilde{R}_{m1}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -2\left(u_0^2-\alpha^2\right)\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& - \frac{1}{\pi}\left(u_0-\alpha\right)\sin\left[2\pi\left(u_0+\alpha\right)\right] \\
& + \frac{1}{\pi}\left(u_0+\alpha\right)\sin\left[2\pi\left(u_0-\alpha\right)\right] \\
& - \frac{1}{2\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right]-\cos\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + 2\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}-\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0}\Bigg\}.
\end{aligned} \tag{D.154}$$

It can be shown that

$$\lim_{\tau\to 0}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]-\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right] = -2u_0\alpha. \tag{D.155}$$

Substituting Equation (D.155) into Equation (D.154) and simplifying yields the stationary autocorrelation function $\widetilde{R}_{m1}(0)$ for the uniform distribution,

$$\begin{aligned}\widetilde{R}_{m1}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -2\left(u_0^2-\alpha^2\right)\left[\operatorname{Cin}\left[2\pi\left(u_0+\alpha\right)\right]-\operatorname{Cin}\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & -\frac{1}{\pi}\left[\left(u_0-\alpha\right)\sin\left[2\pi\left(u_0+\alpha\right)\right]-\left(u_0+\alpha\right)\sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ & -\frac{1}{2\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right]-\cos\left[2\pi\left(u_0-\alpha\right)\right]\right]-4u_0\alpha\Bigg\}. \end{aligned} \tag{D.156}$$

Recalling the definition for $\widetilde{R}_{m2}(0)$,

$$\begin{aligned}\widetilde{R}_{m2}(0) =& \frac{u_0}{2\pi}\Bigg\{\int_{-\infty}^{\infty} R_n(\tau)\,\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau+\int_{-\infty}^{\infty} R_n(\tau)\,\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau \\ & +\alpha^2\int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau+\alpha^2\int_{-\infty}^{\infty} R_g''(\tau)\,\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau \\ & -2\alpha\int_{-\infty}^{\infty} R_g'''(\tau)\,\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{u_0\tau}\,d\tau+2\alpha\int_{-\infty}^{\infty} R_g'''(\tau)\,\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{u_0\tau}\,d\tau\Bigg\}. \end{aligned} \tag{D.157}$$

Substituting Equations (D.139), (D.134), and (D.135) into Equation (D.157),

$$\begin{aligned}\widetilde{R}_{m2}(0) =& \frac{A_m^2\cancel{u_0}}{2\pi}\Bigg\{-\int_{-\infty}^{\infty}\left[\delta''(\tau+2\pi)-2\delta''(\tau)+\delta''(\tau-2\pi)\right]\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\,d\tau \\ & -\int_{-\infty}^{\infty}\left[\delta''(\tau+2\pi)-2\delta''(\tau)+\delta''(\tau-2\pi)\right]\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\,d\tau \\ & -\left(u_0^2-\alpha^2\right)\int_{-\infty}^{\infty}\left[\delta(\tau+2\pi)-2\delta(\tau)+\delta(\tau-2\pi)\right]\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\,d\tau \\ & -\left(u_0^2-\alpha^2\right)\int_{-\infty}^{\infty}\left[\delta(\tau+2\pi)-2\delta(\tau)+\delta(\tau-2\pi)\right]\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\,d\tau \\ & -2\alpha\int_{-\infty}^{\infty}\left[\delta'(\tau+2\pi)-2\delta'(\tau)+\delta'(\tau-2\pi)\right]\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\cancel{u_0}\tau}\,d\tau \\ & +2\alpha\int_{-\infty}^{\infty}\left[\delta'(\tau+2\pi)-2\delta'(\tau)+\delta'(\tau-2\pi)\right]\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\cancel{u_0}\tau}\,d\tau\Bigg\}. \end{aligned} \tag{D.158}$$

Recalling Equations (D.146) and (D.147) and the sifting property for higher order derivatives of the delta function,

$$\int_{-\infty}^{\infty}\delta^{(n)}(x-a)f(x)\;dx=(-1)^n f^{(n)}(a). \tag{D.159}$$

Applying Equations (D.146), (D.147), and (D.159) to Equation (D.158) and simplifying,

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -\left(u_0^2-\alpha^2\right)\frac{\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}2\pi} - \left(u_0^2-\alpha^2\right)\frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} \\
& -\left(u_0^2-\alpha^2\right)\frac{\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\tau\right]}{\cancel{-}2\pi} - \left(u_0^2-\alpha^2\right)\frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} \\
& + 2\left(u_0^2-\alpha^2\right)\left(u_0-\alpha\right) + 2\left(u_0^2-\alpha^2\right)\left(u_0+\alpha\right) \\
& + 2\alpha\frac{d}{d\tau}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=-2\pi} \\
& - 4\alpha\frac{d}{d\tau}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=0} \\
& + 2\alpha\frac{d}{d\tau}\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=2\pi} \\
& - 2\alpha\frac{d}{d\tau}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=-2\pi} \\
& + 4\alpha\frac{d}{d\tau}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=0} \\
& - 2\alpha\frac{d}{d\tau}\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=2\pi} \\
& - \frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=-2\pi} \\
& + 2\frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=0} \\
& - \frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=2\pi} \\
& - \frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=-2\pi} \\
& + 2\frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=0} \\
& - \frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right]\Bigg|_{\tau=2\pi}\Bigg\}.
\end{aligned}
\tag{D.160}
$$

Recalling the derivatives performed previoiusly in Equations (D.150) and (D.151) and performing the second derivatives in Equation (D.160),

$$\frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau}\right] = -\frac{\left(u_0+\alpha\right)^2\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau} - \frac{2\left(u_0+\alpha\right)\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2} + \frac{2\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau^3} \tag{D.161}$$

and

$$\frac{d^2}{d\tau^2}\left[\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau}\right] = -\frac{\left(u_0-\alpha\right)^2\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau} - \frac{2\left(u_0-\alpha\right)\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2} + \frac{2\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau^3}. \tag{D.162}$$

Substituting Equations (D.150), (D.151), (D.161), and (D.162) into Equation (D.160),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -\left(u_0^2-\alpha^2\right)\frac{\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}2\pi} - \left(u_0^2-\alpha^2\right)\frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} \\
& -\left(u_0^2-\alpha^2\right)\frac{\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\right]}{\cancel{-}2\pi} - \left(u_0^2-\alpha^2\right)\frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} \\
& + 2\left(u_0^2-\alpha^2\right)\left(u_0-\alpha\right) + 2\left(u_0^2-\alpha^2\right)\left(u_0+\alpha\right) \\
& + 4\alpha\left(u_0+\alpha\right)^2 - 4\alpha\left(u_0-\alpha\right)^2 - 2\left(u_0+\alpha\right)^3 - 2\left(u_0-\alpha\right)^3 \\
& - 2\alpha\left[\frac{\left(u_0+\alpha\right)\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}2\pi} + \frac{\cos\left[-2\pi\left(u_0+\alpha\right)\right]}{4\pi^2}\right] \\
& - 2\alpha\left[\frac{\left(u_0+\alpha\right)\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} + \frac{\cos\left[2\pi\left(u_0+\alpha\right)\right]}{4\pi^2}\right] \\
& + 2\alpha\left[\frac{\left(u_0-\alpha\right)\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\right]}{\cancel{-}2\pi} + \frac{\cos\left[-2\pi\left(u_0-\alpha\right)\right]}{4\pi^2}\right] \\
& + 2\alpha\left[\frac{\left(u_0-\alpha\right)\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} + \frac{\cos\left[2\pi\left(u_0-\alpha\right)\right]}{4\pi^2}\right] \\
& + \left[\frac{\left(u_0+\alpha\right)^2\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}2\pi}\right] + \left[\frac{\left(u_0+\alpha\right)^2\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi}\right] \\
& + \left[\frac{2\left(u_0+\alpha\right)\cos\left[-2\pi\left(u_0+\alpha\right)\right]}{4\pi^2}\right] + \left[\frac{2\left(u_0+\alpha\right)\cos\left[2\pi\left(u_0+\alpha\right)\right]}{4\pi^2}\right] \\
& - \left[\frac{2\sin\left[\cancel{-}2\pi\left(u_0+\alpha\right)\right]}{\cancel{-}8\pi^3}\right] - \left[\frac{2\sin\left[2\pi\left(u_0+\alpha\right)\right]}{8\pi^3}\right] \\
& + \left[\frac{\left(u_0-\alpha\right)^2\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi}\right] + \left[\frac{\left(u_0-\alpha\right)^2\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\right]}{\cancel{-}2\pi}\right] \\
& + \left[\frac{2\left(u_0-\alpha\right)\cos\left[2\pi\left(u_0-\alpha\right)\right]}{4\pi^2}\right] + \left[\frac{2\left(u_0-\alpha\right)\cos\left[-2\pi\left(u_0-\alpha\right)\right]}{4\pi^2}\right] \\
& - \left[\frac{2\sin\left[2\pi\left(u_0-\alpha\right)\right]}{8\pi^3}\right] - \left[\frac{2\sin\left[\cancel{-}2\pi\left(u_0-\alpha\right)\right]}{\cancel{-}8\pi^3}\right] \\
& + 4\alpha\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2} - \frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0} \\
& - 2\left[\frac{2\left(u_0-\alpha\right)\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2} - \frac{2\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau^3}\right]\Bigg|_{\tau=0} \\
& - 2\left[\frac{2\left(u_0+\alpha\right)\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2} - \frac{2\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau^3}\right]\Bigg|_{\tau=0}\Bigg\}. \qquad \text{(D.163)}
\end{aligned}
$$

Simplifying Equation (D.163),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & \left[-2\left(u_0^2-\alpha^2\right)+2\left(u_0+\alpha\right)^2-4\alpha\left(u_0+\alpha\right)-\frac{1}{\pi^2}\right]\frac{\sin\left[2\pi\left(u_0+\alpha\right)\right]}{2\pi} \\
& + \left[-2\left(u_0^2-\alpha^2\right)+2\left(u_0-\alpha\right)^2+4\alpha\left(u_0-\alpha\right)-\frac{1}{\pi^2}\right]\frac{\sin\left[2\pi\left(u_0-\alpha\right)\right]}{2\pi} \\
& + \left[-4\alpha+4\left(u_0+\alpha\right)\right]\frac{\cos\left[2\pi\left(u_0+\alpha\right)\right]}{4\pi^2} \\
& + \left[4\alpha+4\left(u_0-\alpha\right)\right]\frac{\cos\left[2\pi\left(u_0-\alpha\right)\right]}{4\pi^2} \\
& + 4\alpha\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}-\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0} \\
& - 2\left[\frac{2\left(u_0+\alpha\right)\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}-\frac{2\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau^3}\right]\Bigg|_{\tau=0} \\
& - 2\left[\frac{2\left(u_0-\alpha\right)\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}-\frac{2\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau^3}\right]\Bigg|_{\tau=0}\Bigg\}. \qquad \text{(D.164)}
\end{aligned}
$$

Applying Equation (D.155) and further simplifying Equation (D.164),

$$
\begin{aligned}
\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ & -\frac{1}{2\pi^3}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right]+\sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\
& + \frac{u_0}{\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right]+\cos\left[2\pi\left(u_0-\alpha\right)\right]\right]+8u_0\alpha^2 \\
& + 4\left[\frac{\sin\left[\left(u_0+\alpha\right)\tau\right]}{\tau^3}+\frac{\sin\left[\left(u_0-\alpha\right)\tau\right]}{\tau^3}\right]\Bigg|_{\tau=0} \\
& - 4\left(u_0+\alpha\right)\left[\frac{\cos\left[\left(u_0+\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0} \\
& - 4\left(u_0-\alpha\right)\left[\frac{\cos\left[\left(u_0-\alpha\right)\tau\right]}{\tau^2}\right]\Bigg|_{\tau=0}\Bigg\}. \qquad \text{(D.165)}
\end{aligned}
$$

Recalling the Taylor series expansions for sine and cosine,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{(D.166)}$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots . \tag{D.167}$$

Substituting Equations (D.166) and (D.167) to Equation (D.165),

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ &- \frac{1}{2\pi^3}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right] + \sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ &+ \frac{u_0}{\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right] + \cos\left[2\pi\left(u_0-\alpha\right)\right]\right] - 8u_0\alpha^2 \\ &+ 4\left[\frac{\left(u_0+\alpha\right)}{\tau^2} - \frac{\left(u_0+\alpha\right)^3}{3!} + \frac{\left(u_0+\alpha\right)^5\tau^2}{5!} - \cdots\right]\Bigg|_{\tau=0} \\ &+ 4\left[\frac{\left(u_0-\alpha\right)}{\tau^2} - \frac{\left(u_0-\alpha\right)^3}{3!} + \frac{\left(u_0-\alpha\right)^3\tau^2}{5!} - \cdots\right]\Bigg|_{\tau=0} \\ &- 4\left(u_0+\alpha\right)\left[\frac{1}{\tau^2} - \frac{\left(u_0+\alpha\right)^2}{2!} + \frac{\left(u_0+\alpha\right)^4\tau^2}{4!} - \cdots\right]\Bigg|_{\tau=0} \\ &- 4\left(u_0-\alpha\right)\left[\frac{1}{\tau^2} - \frac{\left(u_0-\alpha\right)^2}{2!} + \frac{\left(u_0-\alpha\right)^4\tau^2}{4!} - \cdots\right]\Bigg|_{\tau=0}\Bigg\}. \end{aligned} \tag{D.168}$$

Canceling terms and simplifying Equation (D.168),

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ &- \frac{1}{2\pi^3}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right] + \sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ &+ \frac{u_0}{\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right] + \cos\left[2\pi\left(u_0-\alpha\right)\right]\right] - 8u_0\alpha^2 \\ &- \frac{2}{3}\left(u_0+\alpha\right)^3 - \frac{2}{3}\left(u_0-\alpha\right)^3 + 2\left(u_0+\alpha\right)^2 + 2\left(u_0-\alpha\right)^2\Bigg\}. \end{aligned} \tag{D.169}$$

Simplifying Equation (D.169) yields the stationary autocorrelation function $\widetilde{R}_{m2}(0)$ for the uniform distribution,

$$\begin{aligned}\widetilde{R}_{m2}(0) = \frac{A_m^2}{2\pi}\Bigg\{ &- \frac{1}{2\pi^3}\left[\sin\left[2\pi\left(u_0+\alpha\right)\right] + \sin\left[2\pi\left(u_0-\alpha\right)\right]\right] \\ &+ \frac{u_0}{\pi^2}\left[\cos\left[2\pi\left(u_0+\alpha\right)\right] + \cos\left[2\pi\left(u_0-\alpha\right)\right]\right] + \frac{8}{3}u_0^3\Bigg\}. \end{aligned} \tag{D.170}$$

VITA

Christopher Daniel Wilson

Candidate for the Degree of:

Doctor of Philosophy

Dissertation: APPLICATION OF AUTOCORRELATION PRINCIPLES TO CHARACTERIZE LINE SOURCE RADIATION

Major Field: Electrical Engineering

Biographical:

Education:

Completed the requirements for Doctor of Philosophy in Electrical Engineering at Oklahoma State University, Stillwater, Oklahoma in December, 2021.

Completed the requirements for Master of Science in Electrical Engineering at University of Idaho, Moscow, Idaho in 2015.

Completed the requirements for Doctor of Philosophy in Aerospace Engineering at Washington University in St. Louis, St. Louis, Missouri in 2010.

Completed the requirements for Master of Science in Aerospace Engineering and Master of Science in Engineering Management at Univeristy of Missouri – Rolla, Rolla, Missouri in 2002.

Completed the requirements for Bachelor of Science in Aerospace Engineering at Univeristy of Missouri – Rolla, Rolla, Missouri in 2001.