STABILITY AND REGULARITY PROBLEMS ON THE MAGNETOHYDRODYNAMIC SYSTEM

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Abstract: The stabilizing and damping phenomenon of a background magnetic field on electrically conducting fluids has been observed in various physical experiments and numerical simulations. The first chapter here establishes this observation as mathematically rigorous facts on a 2D magnetohydrodynamic (MHD) system with only partial dissipation. Without the magnetic field, the fluid velocity obeys a 2D anisotropic Navier-Stokes equation and is not known to be stable in the Sobolev setting H^2 due to the potential double exponential growth of its H^2-norm in time. Under the influence of a background magnetic field, the velocity field is shown here to stabilize and decay in time through the coupling and the interaction. Mathematically we reduce the MHD system concerned here to a system of degenerate and damped wave equations and exploit the smoothing and stabilizing effects of the wave structure. We are able to prove that any perturbation near a background magnetic field remains asymptotically stable. In addition, certain explicit large time behavior is also established. In chapter 2, we show that the H^2-norm of any perturbation near a background magnetic field actually decays algebraically in time. Mathematically this result along with its proof offers a new and effective approach to the large-time behavior on partially dissipated systems of partial differential equations (PDEs). Existing methods are mostly designed for systems with full dissipation and do not apply when the dissipation is anisotropic. In chapter 3, we focuses on a 2D MHD system with only horizontal dissipation in the domain $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a periodic box. We solve the desired stability problem by simultaneously exploiting two smoothing and stabilizing mechanisms: the enhanced dissipation due to the coupling between the velocity and the magnetic fields, and the strong Poincaré type inequalities for the oscillation part of the solution, namely the difference between the solution and its horizontal average. In addition, the oscillation part of the solution is shown to converge exponentially to zero in H^1 as $t \to \infty$. As a consequence, the solution converges to its horizontal average asymptotically.

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CHAPTER I

INFLUENCE OF A BACKGROUND MAGNETIC FIELD ON A 2D MAGNETOHYDRODYNAMIC FLOW

1.1 Introduction

The stabilization and smoothing effect of a background magnetic field on electrically conducting fluids has been observed in physical experiments and numerical simulations, and demonstrated in theoretical analysis (see, e.g., [1, 2, 3, 6, 27, 28]). In addition, the stabilization effect of a strong magnetic field has been employed in the development of magnetic polymers and paints (see, e.g., [35]). One goal of this paper is to understand the mechanism of the stabilization and establish the observations as a mathematically rigorous fact on a system modeling the electrically conducting fluids. We consider the following 2D incompressible magnetohydrodynamic (MHD) system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u + b \cdot \nabla b + \partial_1 b, \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \ \ b(x,0) = b_0(x). \end{cases} \tag{1.1.1}$$

where u denotes the velocity field, b the magnetic field and P the pressure, and $\nu > 0$ and η are the viscosity and the damping coefficient, respectively. Here the velocity u obeys a degenerate Navier-Stokes equation with only vertical dissipation $\nu \partial_{22} u$ and with a Lorentz forcing term. The magnetic field b satisfies the induction equation. The extra two terms $\partial_1 b$ and $\partial_1 u$ are created when we write the original magnetic field as the sum of a background magnetic field and a perturbation, namely $(1,0) + b$. The system focused here governs the motion of the perturbation near a background magnetic field.

The justification for including only one-directional dissipation in (1.1.1) is two fold. The first is that the Laplacian dissipation in some partial differential equation systems modeling fluids reduces to the degenerate case in certain physical regimes and after suitable scaling. One prominent example is Prandtl's boundary layer equation. The second justification is to demonstrate the smoothing and stabilization effect of the magnetic field. Mathematically only one directional dissipation in the Navier-Stokes equations makes the stability problem much more difficult. Without the coupling with the magnetic field, the velocity of the Navier-Stokes equation with only vertical dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u, \quad x \in \mathbb{R}^2, \, t > 0, \\ \nabla \cdot u = 0. \end{cases} \tag{1.1.2}$$

is not known to be stable near the trivial solution. Some physically relevant infinite energy solutions of (1.1.2) can grow rather rapidly [13]. One expects the solution of (1.1.2) in the Sobolev space setting to be unstable, but a proof is currently lacking. When there is no dissipation at all, the 2D Euler equation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = 0, \quad x \in \mathbb{R}^2,\, t > 0, \\ \nabla \cdot u = 0. \end{cases}$$

can generate solutions that grow exponentially or even double exponentially in time (see, e.g., [17, 34, 73]). In contrast, solutions to the 2D Navier-Stokes equations with full dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u, \quad x \in \mathbb{R}^2,\, t > 0, \\ \nabla \cdot u = 0. \end{cases}$$

in the Sobolev spaces are always asymptotically stable with explicit decay rates (see [47, 49]).

Since the partially dissipated Navier-Stokes equation itself alone is not known to be stable, we must seek the stabilizing effect from the magnetic field in order to achieve any stability. The two terms in (1.1.1) related to the magnetic field, namely $b \cdot \nabla b$ and $\partial_1 b$, do not appear to be helpful at first glance, but the smoothing and damping effect would emerge when we convert the MHD system in (1.1.1) into an equivalent form. To do so, we first apply the Helmholtz-Leray projection operator

$$\mathbb{P} := I - \nabla \Delta^{-1} \nabla \cdot$$

to eliminate the pressure term to obtain

$$\partial_t u = \nu \partial_{22} u + \partial_1 b + N_1, \qquad N_1 = \mathbb{P}(-u \cdot \nabla u + b \cdot \nabla b). \tag{1.1.3}$$

By separating the linear terms from the nonlinear ones in (1.1.1), the equation of b can be written as

$$\partial_t b = -\eta b + \partial_1 u + N_2, \qquad N_2 = -u \cdot \nabla b + b \cdot \nabla u. \tag{1.1.4}$$

Differentiating (1.1.3) and (1.1.4) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt} u - (\nu \partial_{22} - \eta)\partial_t u - (\partial_{11} u + \eta \nu \partial_{22} u) = N_3, \\ \partial_{tt} b - (\nu \partial_{22} - \eta)\partial_t b - (\partial_{11} b + \eta \nu \partial_{22} b) = N_4, \end{cases} \tag{1.1.5}$$

where N_3 and N_4 are given by

$$N_3 = (\partial_t + \eta) N_1 + \partial_1 N_2, \qquad N_4 = (\partial_t - \nu \partial_{22}) N_2 + \partial_1 N_1.$$

Surprisingly, both u and b are found to satisfy nonhomogeneous wave equations with exactly the same linear parts. Clearly, (1.1.5) exhibits much more regularization than its original counterpart in (1.1.1). Similarly, the equations of the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$ given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega + b \cdot \nabla j + \partial_1 j, \\ \partial_t j + u \cdot \nabla j + \eta\, j = b \cdot \nabla \omega + Q + \partial_1 \omega, \end{cases} \tag{1.1.6}$$

with

$$Q = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)$$

can also be converted into the following system of wave equations

$$\begin{cases} \partial_{tt}\omega - (\nu\partial_{22} - \eta)\partial_t\omega - (\partial_{11}\omega + \eta\nu\partial_{22}\omega) = N_5, \\ \partial_{tt}j - (\nu\partial_{22} - \eta)\partial_t j - (\partial_{11}j + \eta\nu\partial_{22}j) = N_6, \end{cases} \tag{1.1.7}$$

where N_5 and N_6 are given by

$$N_5 = (\partial_t + \eta)(-u\cdot\nabla\omega + b\cdot\nabla j) + \partial_1(b\cdot\nabla\omega - u\cdot\nabla j + Q),$$
$$N_6 = (\partial_t - \nu\partial_{22})(b\cdot\nabla\omega - u\cdot\nabla j + Q) + \partial_1(-u\cdot\nabla\omega + b\cdot\nabla j).$$

Again ω and j share the same wave structure as that for u and b. In particular, (1.1.5) and (1.1.7) brings in the much-need horizontal regularization even though it is lacking in the original system (1.1.1).

Our first effort is devoted to understanding how the wave structure affects the regularity and large-time behavior. For simplicity, we consider the linearized portion of (1.1.5), namely

$$\begin{cases} \partial_{tt}u - (\nu\partial_{22} - \eta)\partial_t u - (\partial_{11}u + \eta\nu\partial_{22}u) = 0, \\ \partial_{tt}b - (\nu\partial_{22} - \eta)\partial_t b - (\partial_{11}b + \eta\nu\partial_{22}b) = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \end{cases} \tag{1.1.8}$$

or equivalently, the linearization of the original system

$$\begin{cases} \partial_t u = \nu\partial_{22}u + \partial_1 b, \\ \partial_t b = -\eta b + \partial_1 u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{1.1.9}$$

The goal here is to obtain all possible regularization due to the dissipation and dispersion effects and to provide a sharp large-time decay rate. To give a precise statement of our result, we define a Fourier multiplier operator Φ,

$$\widehat{\Phi f}(\xi) = \widehat{\Psi}(\xi)\,\widehat{f}(\xi). \tag{1.1.10}$$

A very important class of Φ is the fractional Laplacian operator $(-\Delta)^\gamma$ with $\gamma\in\mathbb{R}$, which can be defined in terms of the Fourier transform,

$$\widehat{(-\Delta)^\gamma f}(\xi) = |\xi|^{2\gamma}\,\widehat{f}(\xi).$$

It is clear that the norm in the standard homogeneous Sobolev space $\mathring{H}^s$ with $s\in\mathbb{R}$ is given by

$$\|f\|_{\mathring{H}^s} = \|(-\Delta)^{\frac{s}{2}}f\|_{L^2}.$$

For the sake of conciseness, we shall write $\|(f,g)\|_{L^2}^2$ for $\|f\|_{L^2}^2 + \|g\|_{L^2}^2$.

Theorem 1.1.1 *Consider the linearized system (1.1.8). Let Φ be a given Fourier multiplier operator. Assume the initial data (u_0, b_0) satisfies*

$$\Phi u_0,\ \Phi b_0,\ \nabla\Phi u_0,\ \nabla\Phi b_0,\ \partial_{22}\Phi u_0 \in L^2, \quad \nabla\cdot u_0 = \nabla\cdot b_0 = 0.$$

Let (u, b) be the corresponding solution of (1.1.8). Then (u, b) obeys the following regularization and decay estimates.

(1) (u, b) is uniformly bounded for all time with the following explicit bounds,

$$\begin{aligned}
&\|\partial_t\Phi b(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(t)\|_{L^2}^2 + 2\|\partial_1\Phi b(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\,\|\partial_2\Phi b(t)\|_{L^2}^2 \\
&\quad + \int_0^t \Big(4\nu\|\partial_2\partial_\tau\Phi b\|_{L^2}^2 + 3\eta\|\partial_\tau\Phi b\|_{L^2}^2 + \eta\|\partial_1\Phi b\|_{L^2}^2 + \nu\eta^2\|\partial_2\Phi b\|_{L^2}^2\Big)\,d\tau \\
&\le C(\nu,\eta)\,\left(\|\Phi b_0\|_{L^2}^2 + \|\partial_1\Phi(u_0,b_0)\|_{L^2}^2 + \|\partial_2\Phi b_0\|_{L^2}^2\right). \qquad (1.1.11)
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_t\Phi u(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi u(t)\|_{L^2}^2 + 2\|\partial_1\Phi u(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\,\|\partial_2\Phi u(t)\|_{L^2}^2 \\
&\quad + \int_0^t \Big(4\nu\|\partial_2\partial_\tau\Phi u\|_{L^2}^2 + 3\eta\|\partial_\tau\Phi u\|_{L^2}^2 + \eta\|\partial_1\Phi u\|_{L^2}^2 + \nu\eta^2\|\partial_2\Phi u\|_{L^2}^2\Big)\,d\tau \\
&\le C(\nu,\eta)\,\left(\|\Phi u_0\|_{L^2}^2 + \|\partial_1\Phi(u_0,b_0)\|_{L^2}^2 + \|\partial_2\Phi u_0\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right). \qquad (1.1.12)
\end{aligned}$$

Especially, for any $s\in\mathbb{R}$ and for $\Phi = (-\Delta)^{\frac{s}{2}}$, we have the uniform bounds in Sobolev (or L^2) spaces,

$$\begin{aligned}
&\|\partial_t b(t)\|_{\mathring{H}^s}^2 + \frac{\eta^2}{4}\|b(t)\|_{\mathring{H}^s}^2 + 2\|\partial_1 b(t)\|_{\mathring{H}^s}^2 + \frac{5}{2}\eta\nu\,\|\partial_2 b(t)\|_{\mathring{H}^s}^2 \\
&\quad + \int_0^t \Big(4\nu\|\partial_2\partial_\tau b\|_{\mathring{H}^s}^2 + 3\eta\|\partial_\tau b\|_{\mathring{H}^s}^2 + \eta\|\partial_1 b\|_{\mathring{H}^s}^2 + \nu\eta^2\|\partial_2 b\|_{\mathring{H}^s}^2\Big)\,d\tau \\
&\le C(\nu,\eta)\,\left(\|b_0\|_{\mathring{H}^s}^2 + \|\partial_1(u_0,b_0)\|_{\mathring{H}^s}^2 + \|\partial_2 b_0\|_{\mathring{H}^s}^2\right). \qquad (1.1.13)
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_t u(t)\|_{\mathring{H}^s}^2 + \frac{\eta^2}{4}\|u(t)\|_{\mathring{H}^s}^2 + 2\|\partial_1 u(t)\|_{\mathring{H}^s}^2 + \frac{5}{2}\eta\nu\,\|\partial_2 u(t)\|_{\mathring{H}^s}^2 \\
&\quad + \int_0^t \Big(4\nu\|\partial_2\partial_\tau u\|_{\mathring{H}^s}^2 + 3\eta\|\partial_\tau u\|_{\mathring{H}^s}^2 + \eta\|\partial_1 u\|_{\mathring{H}^s}^2 + \nu\eta^2\|\partial_2 u\|_{\mathring{H}^s}^2\Big)\,d\tau \\
&\le C(\nu,\eta)\,\left(\|u_0\|_{\mathring{H}^s}^2 + \|\partial_1(u_0,b_0)\|_{\mathring{H}^s}^2 + \|\partial_2 u_0\|_{\mathring{H}^s}^2 + \|\partial_{22} u_0\|_{\mathring{H}^s}^2\right). \qquad (1.1.14)
\end{aligned}$$

(2) (u, b) obeys the following decay properties, as $t\to\infty$,

$$\begin{aligned}
&(1+t)\left(\|\partial_t\Phi b(t)\|_{L^2}^2 + \|\Phi b(t)\|_{L^2}^2 + \|\nabla\Phi b(t)\|_{L^2}^2\right) \to 0, \\
&(1+t)\|\nabla\Phi u(t)\|_{L^2}^2 \to 0.
\end{aligned}$$

In particular, the following pointwise estimates hold,

$$\begin{aligned}&\|\partial_t \Phi b(t)\|_{L^2} + \|\Phi b(t)\|_{L^2} + \|\nabla \Phi b(t)\|_{L^2}\\ &\le C(\nu,\eta)\ (\|\Phi b_0\|_{L^2} + \|\partial_1 \Phi(u_0,b_0)\|_{L^2} + \|\partial_2 \Phi b_0\|_{L^2})\,(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{1.1.15}$$

and

$$\begin{aligned}&\|\nabla \Phi u(t)\|_{L^2} \le C(\nu,\eta)\\ &\times (\|\Phi(u_0,b_0)\|_{L^2} + \|\Phi(\nabla u_0, \nabla b_0)\|_{L^2} + \|\partial_{22}\Phi u_0\|_{L^2})\ (1+t)^{-\frac{1}{2}}.\end{aligned} \tag{1.1.16}$$

When $\Phi = (-\Delta)^{\frac{s}{2}}$*, we obtain, as* $t \to \infty$*,*

$$\begin{aligned}&(1+t)\left(\|\partial_t b(t)\|^2_{\mathring{H}^s} + \|b(t)\|^2_{\mathring{H}^s} + \|\nabla b(t)\|^2_{\mathring{H}^s}\right) \to 0,\\ &(1+t)\|\nabla u(t)\|^2_{\mathring{H}^s} \to 0,\end{aligned}$$

which especially imply

$$\begin{aligned}&\|\partial_t b(t)\|_{\mathring{H}^s} + \|b(t)\|_{\mathring{H}^s} + \|\nabla b(t)\|_{\mathring{H}^s}\\ &\qquad \le C(\nu,\eta)\ (\|b_0\|_{\mathring{H}^s} + \|\partial_1(u_0,b_0)\|_{\mathring{H}^s} + \|\partial_2 b_0\|_{\mathring{H}^s})\,(1+t)^{-\frac{1}{2}}.\end{aligned}$$

and

$$\begin{aligned}&\|\nabla u(t)\|_{\mathring{H}^s} \le C(\nu,\eta)\\ &\qquad \times (\|(u_0,b_0)\|_{\mathring{H}^s} + \|(\nabla u_0, \nabla b_0)\|_{\mathring{H}^s} + \|\partial_{22} u_0\|_{\mathring{H}^s})\ (1+t)^{-\frac{1}{2}}.\end{aligned}$$

We notice from the statement of Theorem 1.1.1 that u and b obey slightly different regularization upper bounds and exhibits slightly different large-time behavior. When (u_0, b_0), $(\nabla u_0, \nabla b_0)$ and $\partial_{22} u_0$ are all in the homogeneous Sobolev space $\mathring{H}^s$ for a real number s, then b, ∇b and $\partial_t b$ are all bounded uniformly in $\mathring{H}^s$ and their $\mathring{H}^s$-norms are all square time integrable. The $\mathring{H}^s$-norms of b, ∇b and $\partial_t b$ all decay faster than the rate $(1+t)^{-\frac{1}{2}}$. However, the $\mathring{H}^s$-norm of u itself is not known to be square time integrable and we do not have a decay rate for it. Another remark is that, if the initial data is more regular, we can establish higher time regularity estimates and decay bounds for $\|\partial_t \nabla u(t)\|_{\mathring{H}^s}$.

Next we explore the large-time behavior of the frequency piece of the solution (u, b) to (1.1.8) that is supported away from the origin. We take advantage of the wave structure in (1.1.8) to derive energy inequalities that imply an exponential decay rate for the frequency piece away from the origin. These inequalities also allow us to conclude that if the Fourier transform of the initial data (u_0, b_0) is supported away from the origin, then the solution (u, b) decay exponentially in time. To state our result precisely, we define a Fourier cutoff function. Let $a > 0$ be arbitrarily fixed and define

$$\widehat{\phi}(\xi) = \begin{cases} 1 & if \quad |\xi| \ge a,\\ 0 & if \quad |\xi| < a.\end{cases} \tag{1.1.17}$$

Theorem 1.1.2 *Consider the linearized system in (1.1.8). Assume that the initial data* (u_0, b_0) *satisfies*

$$u_0,\ b_0,\ \nabla u_0,\ \nabla b_0,\ \partial_{22} u_0 \in L^2, \quad \nabla \cdot u_0 = \nabla \cdot b_0 = 0.$$

Then (u, b) *decays exponentially in time in the following sense*

$$\begin{aligned} &\|\partial_t(\phi * u)(t)\|_{L^2} + \|(\phi * u)(t)\|_{H^1} \\ &\quad \le C(\nu, \eta)\left(\|((\phi * u_0), (\phi * b_0))\|_{H^1} + \|\partial_{22}(\phi * u_0)\|_{L^2}\right) e^{-c_0 t}, \qquad (1.1.18) \\ &\|\partial_t(\phi * b)(t)\|_{L^2} + \|(\phi * b)(t)\|_{H^1} \\ &\quad \le C(\nu, \eta)\, \|((\phi * u_0), (\phi * b_0))\|_{H^1}\, e^{-c_0 t}, \qquad (1.1.19) \end{aligned}$$

where H^1 *denotes the inhomogeneous* H^1*-norm and* $c_0 > 0$ *is a constant.*

Theorem 1.1.1 and Theorem 1.1.2 tell us about how much regularity we can extract from the wave structure and how fast the solution decays. To deal with the full nonlinear system in (1.1.1), we take full advantage of the smoothing and stabilization effect generated by the wave structure to control the nonlinearity. We are able to establish the following nonlinear stability and large-time behavior result.

Theorem 1.1.3 *Let* η *and* $\nu > 0$*. Consider* (1.1.1) *with the initial data* $(u_0, b_0) \in H^2(\mathbb{R}^2)$*, and* $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$*. Then there exists a constant* $\varepsilon = \varepsilon(\nu, \eta) > 0$ *such that, if*

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \le \varepsilon,$$

then (1.1.1) *has a unique global classical solution* (u, b) *satisfying, for any* $t > 0$*,*

$$\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \int_0^t \left(\|\partial_1 u\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|b\|_{H^2}^2\right) d\tau \le C\,\varepsilon^2$$

for some universal constant. In addition, the solution obeys the following large-time decay estimates, for some constant C*,*

$$\|\nabla u(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \le C\,(1 + t)^{-\frac{1}{2}}. \qquad (1.1.20)$$

Theorem 1.1.3 is a consequence of the smoothing and stabilization effect of the magnetic field. In particular, the time integrability

$$\int_0^\infty \|\partial_1 u(t)\|_{L^2}^2\, dt \le C\,\varepsilon^2$$

is not a consequence of the vertical dissipation in the velocity equation, but an exhibition of the smoothing effect of the magnetic field. We explain why the stability for the 2D Navier-Stokes equation with only vertical dissipation, namely (1.1.2) remains open and what makes the stability problem for the MHD system solvable. It follows from (1.1.2) and the corresponding vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega \qquad (1.1.21)$$

that the H^1-norm of u is uniformly bounded,

$$\|u(t)\|_{H^1} \leq \|u_0\|_{H^1}.$$

The difficulty is how to control the H^2-norm of u or $\|\nabla\omega\|_{L^2}$. When we estimate $\|\nabla\omega\|_{L^2}$ via (1.1.21), the nonlinear term becomes an insurmountable hurdle. In fact, it follows from (1.1.21) that

$$\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + 2\nu\|\partial_2\nabla\omega\|_{L^2}^2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx.$$

The right-hand side can be further decomposed into four terms

$$\begin{aligned}\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx &= \int \partial_1 u_1\,(\partial_1\omega)^2\,dx \\ &+ \int \partial_1 u_2 \partial_1\omega\,\partial_2\omega\,dx + \int \partial_2 u_1\,\partial_1\omega\,\partial_2\omega\,dx + \int \partial_2 u_2 (\partial_2\omega)^2\,dx.\end{aligned} \tag{1.1.22}$$

Due to the lack of the horizontal dissipation, the first two terms can not be suitably bounded. When we deal with the stability problem on the MHD system (1.1.1), we need to control exactly the same nonlinearity. It is the coupling and interaction in the MHD system that allows us to have more maneuver. When we estimate the H^2-norm via the equations of the vorticity and current density in (1.1.6), we also encounter the term (1.1.22). The idea of bounding the first two terms in (1.1.22) is to replace $\partial_1\omega$ by the equation of j,

$$\partial_1\omega = \partial_t j + u\cdot\nabla j + \eta\, j - b\cdot\nabla\omega - Q.$$

For example, the first term on the right of (1.1.22) would become

$$\int \partial_1 u_1\,(\partial_1\omega)^2\,dx = \int \partial_1 u_1\,\partial_1\omega(\partial_t j + u\cdot\nabla j + \eta\, j - b\cdot\nabla\omega - Q)\,dx. \tag{1.1.23}$$

We further shift the time derivative in the first term in (1.1.23), namely

$$\begin{aligned}\int \partial_1 u_1\,\partial_1\omega\,\partial_t j\,dx &= \frac{d}{dt}\int \partial_1 u_1\,\partial_1\omega\, j\,dx \\ &\quad - \int \partial_t\partial_1 u_1\,\partial_1\omega\, j\,dx - \int \partial_1 u_1\, j\,\partial_t\partial_1\omega\,dx.\end{aligned} \tag{1.1.24}$$

By substituting $\partial_t u_1$ and $\partial_t\omega$ by their corresponding equations in (**??**), we find that the first term in (1.1.22) is then converted to

$$\begin{aligned}\int \partial_1 u_1\,(\partial_1\omega)^2\,dx &= \frac{d}{dt}\int \partial_1 u_1\,\partial_1\omega\, j\,dx \\ &- \int \partial_1\omega\, j\,\partial_1(-u\cdot\nabla u_1 - \partial_1 P + \nu\partial_{22}u_1 + b\cdot\nabla b_1 + \partial_1 b_1)\,dx \\ &- \int \partial_1 u_1\, j\,\partial_1(-u\cdot\nabla\omega + \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j)\,dx \\ &+ \int \partial_1 u_1\,\partial_1\omega\,(u\cdot\nabla j + \eta\, j - b\cdot\nabla\omega - Q)\,dx.\end{aligned} \tag{1.1.25}$$

Even though the original one term is converted into fourteen terms, but all of the terms can be bounded suitably by applying anisotropic inequalities such as the one stated in the following lemma. The second terms on the right of (1.1.22) can be treated similarly. Estimating all these terms is a tedious and long process.

Lemma 1.1.1 *Assume that f, g, $\partial_2 g$, h and $\partial_1 h$ are all in $L^2(\mathbb{R}^2)$. Then, for some constant $C > 0$,*

$$\int_{\mathbb{R}^2} |fgh| \; dx \le C\|f\|_{L^2}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_1 h\|_{L^2}^{\frac{1}{2}}.$$

This lemma is taken from [10]. It is very useful in dealing with partial differential equations with anisotropic dissipation and allows us to selectively put directional derivatives on the components of a triple product.

To prove the stability part of Theorem 1.1.3, we use the bootstrapping argument (see, e.g., [52, p.21]). It starts with the definition of a suitable energy functional $E(t)$. We set

$$E(t) := \sup_{0\le\tau\le t}\{\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2\} + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2 \; d\tau + 2\eta\int_0^t \|b\|_{H^2}^2 \; d\tau.$$

The main efforts are then devoted to proving that for some constants C,

$$E(t) \le E(0) + CE^{\frac{3}{2}}(0) + CE^2(t) + CE^{\frac{3}{2}}(t). \tag{1.1.26}$$

This is a long process including estimating the term (1.1.22) and making the substitution as in (1.1.25). The bootstrapping argument applied to (1.1.26) allows us to conclude that, if $E(0)$ or $\|(u_0, b_0)\|_{H^2}$ is sufficiently small, say

$$E(0) \le \varepsilon^2 \quad \text{or} \quad \|(u_0, b_0)\|_{H^2} \le \varepsilon$$

for some sufficiently small $\varepsilon > 0$, then $E(t)$ remains small for all time $t > 0$ and

$$E(t) \le C\,\varepsilon^2 \tag{1.1.27}$$

for some constant $C > 0$.

In order to prove the large-time decay estimates stated in Theorem 1.1.3, we further show that the solution (u, b) obtained above has the following properties,

$$\int_0^\infty \|\partial_1 u(t)\|_{L^2}^2 \, dt \le C\,\varepsilon^2. \tag{1.1.28}$$

and

$$\|(\nabla u(t), \nabla b(t))\|_{L^2} \le C\,\|(\nabla u(s), \nabla b(s))\|_{L^2} \quad \text{for any } 0 \le s \le t \tag{1.1.29}$$

(1.1.28) is not a direct consequence of the dissipation in the velocity equation. It is shown by taking into account of the coupling of the system. We replace $\partial_1 u$ by

$$\partial_1 u = \partial_t b + u\cdot\nabla b + \eta b - b\cdot\nabla u$$

in the L^2-norm,

$$\int_0^\infty \int \partial_1 u \cdot (\partial_t b + u \cdot \nabla b + \eta b - b \cdot \nabla u)\, dx\, dt.$$

By shifting the time derivative and applying various anisotropic inequalities, we are able to prove (1.1.28). The generalized monotonicity in (1.1.29) is established by estimating $\|\omega\|_{L^2}$ and $\|j\|_{L^2}$ via (1.1.6). Then (1.1.27) and (1.1.28) together leads to the time integrability of

$$\int_0^\infty \left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\right)\, dt \le C\,\varepsilon^2. \tag{1.1.30}$$

(1.1.29) and (1.1.30) then fulfill the two conditions of Lemma 1.2.1 and the desired decay estimate in (1.1.20) follows as a consequence.

Finally we remark that there are substantial recent developments on fundamental issues concerning the MHD equations such as the global regularity and stability problems. One recent focus is on the MHD equations with only partial or fractional dissipation. Significant progress has been made (see, e.g., [6, 8, 9, 10, 11, 18, 20, 21, 22, 26, 29, 30, 31, 32, 33, 36, 39, 37, 43, 45, 46, 48, 51, 53, 54, 57, 58, 59, 60, 63, 64, 65, 66, 67, 69, 70, 71, 72]).

The rest of this paper is divided into two sections. Section 1.2 presents the proofs of Theorem 1.1.1 and Theorem 1.1.2 while Section 1.3 proves Theorem 1.1.3.

1.2 Regularization and Large-Time Decay Rate Result

This section is devoted to proving Theorem 1.1.1 and Theorem 1.1.2. The proof of Theorems 1.1.1 makes use of the wave structure to construct a suitable Lyapunov functional, which allows us to eliminate some unfavorable terms. The decay estimates are obtained by using a tool lemma stated below and the key components are the verification on the conditions of the lemma. The proof of Theorem 1.1.2 also involves the combination of energy estimates to form a suitable Lyapunov functional. The frequency part of the solution that is supported away from the origin allows the application of Poincare type inequalities.

The following lemma provides a precise decay rate for a nonnegative integrable function, which is also monotonic in a generalized sense.

Lemma 1.2.1 *Let $f = f(t)$ be a nonnegative function satisfying, for two constants $a_0 > 0$ and $a_1 > 0$,*

$$\int_0^\infty f(\tau)\, d\tau \le a_0 < \infty \quad \text{and} \quad f(t) \le a_1 f(s) \quad \text{for any } 0 \le s < t. \tag{1.2.1}$$

Then $f(t)$ decays at a rate faster than $(1+t)^{-1}$, or

$$(1+t)\, f(t) \to 0 \quad \text{as } t \to \infty.$$

In particular, for $a_2 = \max\{2a_1 f(0), 4a_0 a_1\}$ and for any $t > 0$,

$$f(t) \le a_2 (1+t)^{-1}.$$

Proof of Theorem 1.1.1. We start with the estimates on the norms of b. Let Φ be the Fourier multiplier defined in (1.1.10). Applying Φ to the equation of b in (1.1.8) and then taking the L^2-inner product with $\partial_t \Phi b$, we obtain after integrating by parts and invoking $\nabla \cdot b = 0$,

$$\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t \Phi b\|_{L^2}^2 + \|\partial_1 \Phi b\|_{L^2}^2 + \eta\nu\|\partial_2 \Phi b\|_{L^2}^2\Big) + \nu\|\partial_2\partial_t \Phi b\|_{L^2}^2 + \eta\|\partial_t \Phi b\|_{L^2}^2 = 0 \tag{1.2.2}$$

Applying Φ to the equation of b in (1.1.8) and then taking the L^2-inner product with Φb, we have

$$\frac{1}{2}\frac{d}{dt}(\eta\|\Phi b\|_{L^2}^2 + \nu\|\partial_2 \Phi b\|_{L^2}^2) + \|\partial_1 \Phi b\|_{L^2}^2 + \eta\nu\|\partial_2 \Phi b\|_{L^2}^2 + \int \partial_{tt}\Phi b \cdot \Phi b \, dx = 0.$$

We further rewrite the last term as

$$\begin{aligned}\int \partial_{tt}\Phi b \cdot \Phi b \, dx &= \int \big(\partial_t(\partial_t \Phi b \cdot \Phi b) - |\partial_t \Phi b|^2\big) \, dx \\ &= \frac{d}{dt}(\partial_t \Phi b, \Phi b) - \|\partial_t \Phi b\|_{L^2}^2,\end{aligned}$$

where we have introduced the notation for the L^2-inner product,

$$(f, g) = \int_{\mathbb{R}^2} f \cdot g \, dx.$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\eta\|\Phi b\|_{L^2}^2 + \nu\|\partial_2 \Phi b\|_{L^2}^2 + 2(\partial_t \Phi b, \Phi b)\Big) \\ &\quad + \|\partial_1 \Phi b\|_{L^2}^2 + \eta\nu\|\partial_2 \Phi b\|_{L^2}^2 - \|\partial_t \Phi b\|_{L_2}^2 = 0.\end{aligned} \tag{1.2.3}$$

Let $\lambda > 0$ be a parameter to be determined later. Then, (1.2.2)+λ(1.2.3) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t \Phi b\|_{L^2}^2 + \|\partial_1 \Phi b\|_{L^2}^2 + (\lambda\nu + \eta\nu)\|\partial_2 \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 + 2\lambda(\partial_t \Phi b, \Phi b)\Big) \\ &+\nu\|\partial_2\partial_t \Phi b\|_{L^2}^2 + (\eta - \lambda)\|\partial_t \Phi b\|_{L^2}^2 + \lambda\|\partial_1 \Phi b\|_{L^2}^2 + \lambda\eta\nu\|\partial_2 \Phi b\|_{L^2}^2 = 0..\end{aligned} \tag{1.2.4}$$

By Hölder's and Young's inequality,

$$\begin{aligned}&\|\partial_t \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 + 2\lambda(\partial_t \Phi b, \Phi b) \\ &\quad \geq \|\partial_t \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 - 2\lambda\|\partial_t \Phi b(t)\|_{L^2}\|\Phi b\|_{L^2} \\ &\quad \geq \|\partial_t \Phi b\|_{L^2}^2 + \lambda\eta\|\Phi b\|_{L^2}^2 - \left(\frac{1}{2}\|\partial_t \Phi b\|_{L^2}^2 + 2\lambda^2\|\Phi b\|_{L^2}^2\right) \\ &\quad \geq \frac{1}{2}\|\partial_t \Phi b\|_{L^2}^2 + (\lambda\eta - 2\lambda^2)\|\Phi b\|_{L^2}^2.\end{aligned} \tag{1.2.5}$$

In particular, for $\lambda = \frac{\eta}{4}$, (1.2.5) becomes

$$\|\partial_t \Phi b\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b\|_{L^2}^2 + \frac{\eta}{2}(\partial_t \Phi b, \Phi b) \geq \frac{1}{2}\|\partial_t \Phi b\|_{L^2}^2 + \frac{\eta^2}{8}\|\Phi b\|_{L^2}^2. \tag{1.2.6}$$

Integrating (1.2.4) in time and invoking (1.2.6), we find, for any $0 \le s \le t$,

$$\begin{aligned} &\|\partial_t \Phi b(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(t)\|_{L^2}^2 + 2\|\partial_1 \Phi b(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\, \|\partial_2 \Phi b(t)\|_{L^2}^2 \\ &\quad + \int_s^t \Big(4\nu\|\partial_2\partial_\tau \Phi b\|_{L^2}^2 + 3\eta\|\partial_\tau \Phi b\|_{L^2}^2 + \eta\|\partial_1 \Phi b\|_{L^2}^2 + \nu\eta^2\|\partial_2 \Phi b\|_{L^2}^2\Big)\, d\tau \\ &\le 3\|(\partial_t \Phi b)(s)\|_{L^2}^2 + \frac{3\eta^2}{4}\|\Phi b(s)\|_{L^2}^2 + 2\|\partial_1 \Phi b(s)\|_{L^2}^2 + \frac{5}{2}\eta\nu\, \|\partial_2 \Phi b(s)\|_{L^2}^2, \end{aligned} \tag{1.2.7}$$

where we have used the following upper bound to obtain the right-hand side

$$\|(\partial_t \Phi b)(s)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(s)\|_{L^2}^2 + \frac{\eta}{2}(\partial_t \Phi b, \Phi b)(s) \le \frac{3}{2}\|(\partial_t \Phi b)(s)\|_{L^2}^2 + \frac{3\eta^2}{8}\|\Phi b(s)\|_{L^2}^2.$$

Since u and b satisfy exactly the same wave equation, the bound above also holds for u,

$$\begin{aligned} &\|\partial_t \Phi u(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi u(t)\|_{L^2}^2 + 2\|\partial_1 \Phi u(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\, \|\partial_2 \Phi u(t)\|_{L^2}^2 \\ &\quad + \int_s^t \Big(4\nu\|\partial_2\partial_\tau \Phi u\|_{L^2}^2 + 3\eta\|\partial_\tau \Phi u\|_{L^2}^2 + \eta\|\partial_1 \Phi u\|_{L^2}^2 + \nu\eta^2\|\partial_2 \Phi u\|_{L^2}^2\Big)\, d\tau \\ &\le 3\|(\partial_t \Phi u)(s)\|_{L^2}^2 + \frac{3\eta^2}{4}\|\Phi u(s)\|_{L^2}^2 + 2\|\partial_1 \Phi u(s)\|_{L^2}^2 \\ &\quad + \frac{5}{2}\eta\nu\, \|\partial_2 \Phi u(s)\|_{L^2}^2. \end{aligned} \tag{1.2.8}$$

Invoking the original linearized system of (u, b), namely (1.1.9) and letting $t \to 0$, we obtain

$$(\partial_t u)(0) = \nu\partial_{22} u_0 + \partial_1 b_0, \qquad (\partial_t b)(0) = -\eta b_0 + \partial_1 u_0. \tag{1.2.9}$$

By setting $s = 0$ in (1.2.7) and (1.2.8), and using (1.2.9), we obtain the desired global bound in (1.1.11) and (1.1.12). By taking the Fourier multiplier operator Φ to be the fractional Laplacian operator,

$$\Phi f = (-\Delta)^{\frac{s}{2}} f$$

and identifying the homogeneous $\mathring{H}^s$-norm as the following L^2-norm,

$$\|f\|_{\mathring{H}^s} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^2},$$

we can then reduce (1.1.11) and (1.1.12) to (1.1.13) and (1.1.14), respectively.

Next we show the decay rates in (1.1.15) and (1.1.16). The idea is to apply Lemma 1.2.1. We set

$$F(t) := \|\partial_t \Phi b(t)\|_{L^2}^2 + \frac{\eta^2}{4}\|\Phi b(t)\|_{L^2}^2 + 2\|\partial_1 \Phi b(t)\|_{L^2}^2 + \frac{5}{2}\eta\nu\, \|\partial_2 \Phi b(t)\|_{L^2}^2$$

and verify that $F(t)$ obeys the conditions in (1.2.1). It is clear from (1.2.7) that, for any $0 \le s \le t < \infty$, there is a constant C independent of s and t satisfying

$$F(t) \le C\, F(s). \tag{1.2.10}$$

In addition, by taking $s = 0$ in (1.2.7) and invoking (1.2.9), we have

$$\int_0^\infty \left(3\eta\|\partial_t\Phi b\|_{L^2}^2 + \eta\|\partial_1\Phi b\|_{L^2}^2 + \nu\eta^2\|\partial_2\Phi b\|_{L^2}^2\right) dt$$
$$\leq \frac{15\eta^2}{4}\|\Phi b_0\|_{L^2}^2 + 3\|\partial_1\Phi(u_0, b_0)\|_{L^2}^2 + \frac{5}{2}\nu\eta\|\partial_2\Phi b_0\|_{L^2}^2. \tag{1.2.11}$$

In addition, a simple L^2-energy estimate on (1.1.9) leads to

$$\|\Phi(u(t), b(t))\|_{L^2}^2 + 2\int_0^t (\nu\|\Phi\partial_2 u\|_{L^2}^2 + \eta\|\Phi b\|_{L^2}^2)\, d\tau = \|\Phi(u_0, b_0)\|_{L^2}^2.$$

In particular,

$$\eta \int_0^\infty \|\Phi b\|_{L^2}^2\, dt \leq \|\Phi(u_0, b_0)\|_{L^2}^2. \tag{1.2.12}$$

Adding (1.2.11) and (1.2.12) yields

$$\int_0^\infty F(t)\, dt \leq C(\nu, \eta) \left(\|\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_1\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_2\Phi b_0\|_{L^2}^2\right). \tag{1.2.13}$$

(1.2.10) and (1.2.13) then verify (1.2.1). Lemma 1.2.1 then implies

$$(1+t)\, F(t) \to 0 \quad \text{as } t \to \infty. \tag{1.2.14}$$

As a special consequence,

$$F(t) \leq C(\nu, \eta) \left(\|\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_1\Phi(u_0, b_0)\|_{L^2}^2 + \|\partial_2\Phi b_0\|_{L^2}^2\right)(1+t)^{-1},$$

which is (1.1.15). The process of showing the decay rate for b does not work for u. The reason is that we do not have the corresponding time integrability bound (1.2.12) for u. We do not know if $\|\Phi u(t)\|_{L^2}$ decays or not. What we can obtain is an explicit decay rate for $\|\nabla\Phi u(t)\|_{L^2}$. According to (1.1.9) and (1.2.14), we have

$$(1+t)\|\partial_1\Phi u(t)\|_{L^2}^2 \leq C\,(1+t)\left(\|\partial_t\Phi b(t)\|_{L^2}^2 + \eta\|\Phi b(t)\|_{L^2}^2\right) \to 0 \quad \text{as } t \to \infty$$

and

$$\|\partial_1\Phi u(t)\|_{L^2} \leq \|\partial_t\Phi b(t)\|_{L^2} + \eta\|\Phi b(t)\|_{L^2}$$
$$\leq C(\nu, \eta)\, (\|\Phi(u_0, b_0)\|_{L^2} + \|\partial_1\Phi(u_0, b_0)\|_{L^2} + \|\partial_2\Phi b_0\|_{L^2})\,(1+t)^{-\frac{1}{2}}. \tag{1.2.15}$$

To obtain the decay rate for $\partial_2\Phi u$, we apply $\partial_2\Phi$ to (1.1.9) and then dot with $(\partial_2\Phi u, \partial_2\Phi b)$ to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\partial_2\Phi u\|_{L^2}^2 + \|\partial_2\Phi b\|_{L^2}^2) + \nu\|\partial_{22}\Phi u\|_{L^2}^2 + \eta\|\partial_2\Phi b\|_{L^2}^2 = 0.$$

Therefore, for $0 \leq s \leq t$,

$$\|\partial_2\Phi u(t)\|_{L^2}^2 + \|\partial_2\Phi b(t)\|_{L^2}^2 \leq \|\partial_2\Phi u(s)\|_{L^2}^2 + \|\partial_2\Phi b(s)\|_{L^2}^2.$$

Furthermore, (1.2.8) with $s = 0$, together with (1.2.9), gives

$$\int_0^\infty \|\partial_2 \Phi u(t)\|_{L^2}^2\, dt \\ \le C(\nu,\eta)\, \left(\|\Phi u_0\|_{L^2}^2 + \|\nabla \Phi u_0\|_{L^2}^2 + \|\partial_1 \Phi b_0\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right) \tag{1.2.16}$$

Combining (1.2.11) and (1.2.16) leads to

$$\int_0^\infty \left(\|\partial_2 \Phi u(t)\|_{L^2}^2 + \|\partial_2 \Phi b(t)\|_{L^2}^2\right)\, dt \\ \le C(\nu,\eta)\, \left(\|\Phi(u_0,b_0)\|_{L^2}^2 + \|\Phi(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right).$$

It then follows from Lemma 1.2.1 that

$$(1+t)\left(\|\partial_2 \Phi u(t)\|_{L^2}^2 + \|\partial_2 \Phi b(t)\|_{L^2}^2\right) \to 0 \quad \text{as } t \to \infty$$

and

$$\|\partial_2 \Phi u(t)\|_{L^2}^2 + \|\partial_2 \Phi b(t)\|_{L^2}^2 \\ \le C(\nu,\eta)\, \left(\|\Phi(u_0,b_0)\|_{L^2}^2 + \|\Phi(\nabla u_0, \nabla b_0)\|_{L^2}^2 + \|\partial_{22}\Phi u_0\|_{L^2}^2\right)\, (1+t)^{-1}. \tag{1.2.17}$$

(1.2.15) and (1.2.17) yield (1.1.16). This completes the proof of Theorem 1.1.1. ■

We now turn to the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. We make use of some of the estimates from the proof of Theorem 1.1.1. Recall the definition of ϕ in (1.1.17). By taking Φ to be the convolution operator ϕ, namely

$$\Phi f = \phi * f \quad \text{or} \quad \widehat{\Phi f}(\xi) = \widehat{\phi}(\xi)\, \widehat{f}(\xi),$$

we obtain from (1.2.4) that

$$\frac{d}{dt} G(t) + 2\nu \|\partial_2 \partial_t (\phi * b)\|_{L^2}^2 + 2(\eta - \lambda)\|\partial_t(\phi * b)\|_{L^2}^2 \\ +2\lambda \|\partial_1(\phi * b)\|_{L^2}^2 + 2\lambda\eta\nu \|\partial_2(\phi * b)\|_{L^2}^2 = 0,$$

where

$$\begin{aligned} G(t) &= \|\partial_t(\phi * b)\|_{L^2}^2 + \|\partial_1(\phi * b)\|_{L^2}^2 + (\lambda\nu + \eta\nu)\|\partial_2(\phi * b)\|_{L^2}^2 \\ &\quad +\lambda\eta \|(\phi * b)\|_{L^2}^2 + 2\lambda(\partial_t(\phi * b), (\phi * b)). \end{aligned}$$

By setting $\lambda = \frac{\eta}{4}$, we find

$$\frac{d}{dt} G(t) + \frac{3\eta}{2}\|\partial_t(\phi * b)\|_{L^2}^2 + \frac{\eta}{2}\|\partial_1(\phi * b)\|_{L^2}^2 + \frac{\nu\eta^2}{2}\|\partial_2(\phi * b)\|_{L^2}^2 \le 0. \tag{1.2.18}$$

In particular, if we set

$$C_1 = \min\left\{\frac{3\eta}{2}, \frac{\eta}{2}, \frac{\nu\eta^2}{2}\right\},$$

then (1.2.18) yields

$$\frac{d}{dt}G(t) + C_1 \left(\|\partial_t(\phi * b)\|_{L^2}^2 + \|\nabla(\phi * b)\|_{L^2}^2\right) \le 0.$$

By Plancherel's Theorem and the definition of $\widehat{\phi}$,

$$\begin{aligned}\|\phi * b\|_{L^2}^2 = \|\widehat{\phi} \cdot \widehat{b}\|_{L^2}^2 = \int_{|\xi|\ge a} \left|\widehat{\phi}\right|^2 \left|\widehat{b}\right|^2 \, dx \\ \le \int_{|\xi|\ge a} \frac{|\xi|^2}{a^2} \left|\widehat{\phi}\right|^2 \left|\widehat{b}\right|^2 \, dx \le \frac{1}{a^2}\|\nabla(\phi * b)\|_{L^2}^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}G(t) + C_1 \left(\|\partial_t(\phi * b)\|_{L^2}^2 + \frac{1}{2}\|\nabla(\phi * b)\|_{L^2}^2 + \frac{a^2}{2}\|(\phi * b)\|_{L^2}^2\right) \le 0.$$

If we write

$$C_2 = \min\left\{\frac{C_1}{2}, \frac{C_1}{2}a^2\right\},$$

then

$$\frac{d}{dt}G(t) + C_2(\|\partial_t(\phi * b)\|_{L^2}^2 + \|\nabla(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{L^2}^2) \le 0. \tag{1.2.19}$$

Clearly, for $\lambda = \frac{\eta}{4}$,

$$G(t) \le \frac{3}{2}\|\partial_t(\phi * b)\|_{L^2}^2 + \frac{3\eta^2}{8}\|(\phi * b)\|_{L^2}^2 + \|\partial_1(\phi * b)\|_{L^2}^2 + \frac{5\eta\nu}{4}\|\partial_2(\phi * b)\|_{L^2}^2. \tag{1.2.20}$$

For any constant C_0 satisfying

$$0 < C_0 \le \min\left\{\frac{2C_2}{3}, \frac{8C_2}{3\eta^2}, \frac{4C_2}{5\nu\eta}\right\},$$

(1.2.20) implies

$$C_2(\|\partial_t(\phi * b)\|_{L^2}^2 + \|\nabla(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{L^2}^2) \ge C_0 G(t).$$

(1.2.19) then implies

$$\frac{d}{dt}G(t) + C_0 G \le 0 \quad \text{or} \quad G(t) \le G(0)\, e^{-C_0 t}. \tag{1.2.21}$$

By the definition of G,

$$\begin{aligned}G(0) = \|(\partial_t(\phi * b))(0)\|_{L^2}^2 + \|\partial_1(\phi * b_0)\|_{L^2}^2 + \frac{5\eta\nu}{4}\|\partial_2(\phi * b_0)\|_{L^2}^2 \\ + \frac{\eta^2}{4}\|(\phi * b_0)\|_{L^2}^2) + \frac{\eta}{2}((\partial_t(\phi * b))(0), (\phi * b_0)).\end{aligned}$$

By (1.1.9),

$$(\partial_t(\phi * b))(0) = -\eta(\phi * b_0) + \partial_1(\phi * u_0).$$

Setting $\lambda = \frac{\eta}{4}$ and applying Hölder's and Young's inequalities, we have

$$\begin{aligned} G(0) \le\ & \frac{3}{2}\|(\partial_t(\phi * b))(0)\|_{L^2}^2 + \frac{3\eta^2}{8}\|(\phi * b_0)\|_{L^2}^2 \\ & + \|\partial_1(\phi * b_0)\|_{L^2}^2 + \frac{5\eta\nu}{4}\|\partial_2(\phi * b_0)\|_{L^2}^2 \\ \le\ & C_3(\nu,\eta)\, \|(\phi * u_0, \phi * b_0)\|_{H^1}^2. \end{aligned} \tag{1.2.22}$$

Clearly, for $\lambda = \frac{\eta}{4}$, $G(t)$ admits the lower bound

$$G(t) \ge C_4(\|\partial_t(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{H^1}^2), \tag{1.2.23}$$

where $C_4 = C_4(\nu, \eta)$ is a constant. Hence, (1.2.21), (1.2.22) and (1.2.23) lead to

$$\|\partial_t(\phi * b)\|_{L^2}^2 + \|(\phi * b)\|_{H^1}^2 \le C_5(\|(\phi * u_0)\|_{H^1}^2 + \|(\phi * b_0)\|_{H^1}^2)e^{-C_0 t},$$

which is (1.1.19). The proof for (1.1.18) is very similar and we omit the details. This completes the proof of Theorem 1.1.2. ■

1.3 Nonlinear Stability and Large-Time Behavior Result

This section proves Theorem 1.1.3. This theorem consists of two main parts, the stability and the large-time behavior estimate. Naturally our proof is divided into two main parts with the first devoted to the stability and the second to the proof of (1.1.20). Due to the lack of the horizontal dissipation in the velocity equation, the main difficulty in the proof of the stability is how to bound the velocity nonlinear term, namely (1.1.22). This is the reason that the 2D Navier-Stokes with degenerate dissipation is not known to be stable. We fully exploit the smoothing and stabilization effect of the magnetic field to overcome this difficulty.

The proof of the decay estimate (1.1.20) focuses on the time integrability

$$\int_0^\infty \|\partial_1 u\|_{L^2}^2 \, dt \le C\, \varepsilon^2,$$

which is not a consequence of the vertical dissipation in the velocity equation. It is established by making use of the regularization effect of the magnetic field through the coupling and interaction.

In order to make efficient use of the anisotropic dissipation, we employ several anisotropic tools to control the nonlinear terms. One of them is Lemma 1.1.1 stated in the introduction. Another anisotropic inequality we also use extensively is given in the following lemma. A proof is also presented for the convenience of readers.

Lemma 1.3.1 *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

Consequently,

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_1 f\|_{H^1}^{\frac{1}{2}},$$

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_2 f\|_{H^1}^{\frac{1}{2}}.$$

Proof. We recall the following inequality, for a one-dimensional function $g \in H^1(\mathbb{R})$,

$$\|g\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\,\|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\,\|g'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{1.3.1}$$

By (1.3.1) and Minkowski's inequality,

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{R}^2)} &= \Big\| \|f\|_{L^\infty_{x_2}(\mathbb{R})} \Big\|_{L^\infty_{x_1}(\mathbb{R})} \\
&\le \sqrt{2}\Big\| \|f\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \|\partial_2 f\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \Big\|_{L^\infty_{x_1}(\mathbb{R})} \\
&\le \sqrt{2}\Big\| \|f\|_{L^2_{x_2}(\mathbb{R})} \Big\|_{L^\infty_{x_1}(\mathbb{R})}^{\frac{1}{2}} \ \Big\| \|\partial_2 f\|_{L^2_{x_2}(\mathbb{R})} \Big\|_{L^\infty_{x_1}(\mathbb{R})}^{\frac{1}{2}} \\
&\le \sqrt{2}\Big\| \|f\|_{L^\infty_{x_1}(\mathbb{R})} \Big\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \ \Big\| \|\partial_2 f\|_{L^\infty_{x_1}(\mathbb{R})} \Big\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \\
&\le 2\ \Big\| \|f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}}\ \|\partial_1 f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}} \Big\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \ \Big\| \|\partial_2 f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}}\ \|\partial_1\partial_2 f\|_{L^2_{x_1}(\mathbb{R})}^{\frac{1}{2}} \Big\|_{L^2_{x_2}(\mathbb{R})}^{\frac{1}{2}} \\
&\le 2\,\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.
\end{aligned}$$

Here we have written $\|f\|_{L^\infty_{x_j}(\mathbb{R})}$ with $j = 1, 2$ to denote the L^∞-norm of f in terms of x_j on $\mathbb{R}$, and, similarly, $\|f\|_{L^2_{x_j}(\mathbb{R})}$ denotes the L^2-norm. ∎

We are ready to prove Theorem 1.1.3.

Proof of Theorem 1.1.3. The framework of the proof is the bootstrapping argument. We define the energy functional to be

$$E(t) = \sup_{0\le\tau\le t}\{\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2\} + 2\nu\int_0^t \|\partial_2 u\|_{H^2}^2\ d\tau + 2\eta\int_0^t \|b\|_{H^2}^2\ d\tau \tag{1.3.2}$$

an show that

$$E(t) \le E(0) + C_1E^{\frac{3}{2}}(0) + C_2E^2(t) + C_3E^{\frac{3}{2}}(t). \tag{1.3.3}$$

(1.3.3) is established by estimating the H^2-norm of (u, b). As aforementioned in the introduction, it is extremely difficult to obtain suitable upper bounds for some of the terms such as the nonlinear term in the momentum equation. We can only control them through the coupling with the equation of the magnetic field. Equivalently we exploit the regularization and damping effects of the wave structure derived from the coupling and interaction of the velocity and the magnetic fields. The estimates of $\|(u,b)\|_{H^2}$ will involves various operations such as repeated substitutions to take the full advantage of the wave structure.

Due to the equivalence of the inhomogeneous norm $\|(u,b)\|_{H^2}$ with the sum of the L^2-norm and the homogeneous norm $\|(u,b)\|_{\mathring{H}^2}$, it suffices to bound the homogeneous norm $\|(u,b)\|_{\mathring{H}^2}$. The uniform L^2-bound is an easy consequence of the system in (1.1.1) itself. Taking the inner product of (1.1.1) with (u,b), we obtain, after integrating by parts and using $\nabla\cdot u = \nabla\cdot b = 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_2 u\|_{L^2}^2\, d\tau + 2\eta\int_0^t \|b\|_{L^2}^2\, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{1.3.4}$$

To estimate the homogeneous norm $\|(u,b)\|_{\mathring{H}^2}$, we make use of the equations of the vorticity $\omega = \nabla\times u$ and the current density $j = \nabla\times b$, namely (1.1.6),

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j, \\ \partial_t j + u\cdot\nabla j + \eta j = b\cdot\nabla\omega + Q + \partial_1\omega, \end{cases} \tag{1.3.5}$$

where $Q := 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)$. Due to

$$\|(u,b)\|_{\mathring{H}^2} = \|(\nabla\omega,\nabla j)\|_{L^2},$$

we focus on $\|(\nabla\omega,\nabla j)\|_{L^2}$. Applying the gradient ∇ to (1.3.5) and taking the inner product of the resultant with $(\nabla\omega,\nabla j)$, we find, after integration by parts and the divergence-free conditions,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \nu\|\partial_2\nabla\omega\|_{L^2}^2 + \eta\|\nabla j\|_{L^2}^2 \\ &\quad = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\, dx - \int \nabla j\cdot\nabla u\cdot\nabla j\, dx + \int \nabla\omega\cdot\nabla b\cdot\nabla j\, dx \\ &\qquad + \int \nabla j\cdot\nabla b\cdot\nabla\omega\, dx + \int \nabla Q\cdot\nabla j\, dx \\ &\quad := J + K + L + M + N. \end{aligned} \tag{1.3.6}$$

J is the most difficult term and its estimate is long and tedious. We start with the easy terms. Even though L and M are not exactly the same, they obviously admit the same upper bound. To bound L, we further decompose it into four terms in order to make use of the anisotropic dissipation,

$$\begin{aligned} L &= \int \nabla\omega\cdot\nabla b\cdot\nabla j\, dx \\ &= \int \partial_1\omega\,\partial_1 b_1\,\partial_1 j\, dx + \int \partial_1\omega\,\partial_1 b_2\,\partial_2 j\, dx \\ &\quad + \int \partial_2\omega\,\partial_2 b_1\,\partial_1 j\, dx + \int \partial_2\omega\,\partial_2 b_2\,\partial_2 j\, dx \\ &:= L_1 + L_2 + L_3 + L_4. \end{aligned}$$

By Lemma 1.1.1,

$$
\begin{aligned}
L_1 &= \int \partial_1\omega\,\partial_1 b_1\,\partial_1 j\ dx \le C\|\partial_1 j\|_{L^2}\|\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_1\|_{L^2}^{2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\\
&\le C\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}},\\
L_2 &= \int \partial_1\omega\,\partial_1 b_2\,\partial_2 j\ dx \le C\|\partial_2 j\|_{L^2}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_2\|_{L^2}^{2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\\
&\le C\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}},
\end{aligned}
$$

where we have used the basic facts,

$$
\begin{aligned}
&\|\partial_1 j\|_{L^2} = \|\partial_1\nabla b\|_{L^2} \le \|b\|_{H^2}, \quad \|\partial_1\omega\|_{L^2} = \|\partial_1\nabla u\|_{L^2} \le \|u\|_{H^2},\\
&\|\partial_2\partial_1\omega\|_{L^2} = \|\partial_2\partial_1\nabla u\|_{L^2} \le \|\partial_2 u\|_{H^2}.
\end{aligned}
$$

L_3 and L_4 can be bounded similarly. Therefore,

$$
L \le C\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}. \tag{1.3.7}
$$

Similarly,

$$
M \le C\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}. \tag{1.3.8}
$$

We now turn to K. Again, in order to make efficient use of the anisotropic dissipation, we decompose K into four terms,

$$
\begin{aligned}
K &= -\int \nabla j\cdot\nabla u\cdot\nabla j\ dx\\
&= -\int \partial_1 j\,\partial_1 u_1\,\partial_1 j\ dx - \int \partial_1 j\,\partial_1 u_2\,\partial_2 j\ dx\\
&\quad -\int \partial_2 j\,\partial_2 u_1\,\partial_1 j\ dx - \int \partial_2 j\,\partial_2 u_2\,\partial_2 j\ dx\\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

By Hölder's inequality and Lemma 1.3.1,

$$
\begin{aligned}
K_1 &= -\int \partial_1 j\,\partial_1 u_1\,\partial_1 j\ dx \le \|\partial_1 u_1\|_{L^\infty}\|\partial_1 j\|_{L^2}\|\partial_1 j\|_{L^2}\\
&\le \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}}\|b\|_{H^2}^{2} \le \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

The other three terms K_2, K_3 and K_4 all admit the same upper bound. Therefore,

$$
K \le C\,\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}. \tag{1.3.9}
$$

We now bound N. We write out all the component terms in Q explicitly,

$$
\begin{aligned}
N &= \int \nabla Q \cdot \nabla j \; dx \\
&= 2 \int \Big(\partial_1^2 b_1 \partial_2 u_1 \partial_1^2 b_2 \; + \partial_1^2 b_1 \partial_1 u_2 \partial_1^2 b_1 - \partial_1^2 b_1 \partial_2 u_1 \partial_1 \partial_2 b_1 - \partial_2^2 b_1 \partial_1 u_2 \partial_1 \partial_2 b_1 \\
&\quad + \partial_1 b_1 \partial_1 \partial_2 u_1 \partial_1^2 b_2 + \partial_1 b_1 \partial_1^2 u_1 \partial_1^2 b_2 - \partial_1 b_1 \partial_1 \partial_2 u_1 \partial_1 \partial_2 b_1 - \partial_1 b_1 \partial_1^2 u_2 \partial_1 \partial_2 b_1 \\
&\quad - \partial_1^2 u_1 \partial_2 b_1 \partial_1^2 b_2 - \partial_1^2 u_2 \partial_1 b_2 \partial_1^2 b_2 + \partial_1^2 u_1 \partial_2 b_1 \partial_1 \partial_2 b_1 + \partial_1^2 u_1 \partial_1 b_2 \partial_1 \partial_2 b_1 \\
&\quad - \partial_1 u_1 \partial_1 \partial_2 b_1 \partial_1^2 b_2 - \partial_1 u_1 \partial_1^2 b_2 \partial_1^2 b_2 + \partial_1 u_1 \partial_1 \partial_2 b_1 \partial_1 \partial_2 b_1 + \partial_1 u_1 \partial_1^2 b_2 \partial_1 \partial_2 b_1 \Big) \, dx.
\end{aligned}
$$

Even though N contains sixteen terms, but all of them can be bounded suitably using Hölder's inequality, Lemma 1.1.1 and Lemma 1.3.1. Since the details are quite similar to those in the estimates of K, we omit them for conciseness. The upper bound is

$$
N \leq C \|b\|_{H^2}^2 \|u\|_{H^2}^{\frac{1}{2}} \|\partial_2 u\|_{H^2}^{\frac{1}{2}}. \tag{1.3.10}
$$

We now turn to the most difficult term J. Again, we split J into four terms,

$$
\begin{aligned}
J &= - \int \nabla \omega \cdot \nabla u \cdot \nabla \omega \; dx \\
&= - \int \partial_1 \omega \, \partial_1 u_1 \, \partial_1 \omega \; dx - \int \partial_1 \omega \, \partial_1 u_2 \, \partial_2 \omega \; dx \\
&\quad - \int \partial_2 \omega \, \partial_2 u_1 \, \partial_1 \omega \; dx - \int \partial_2 \omega \, \partial_2 u_2 \, \partial_2 \omega \; dx \\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{1.3.11}
$$

As we have explained in the introduction, due to the lack of the horizontal dissipation, J_1 and J_2 can not be bounded suitably. It is the smoothing and stabilization effect of the magnetic field that makes it possible to control these two terms. To incorporate this effect, we make use of the equation for the magnetic field. By replacing $\partial_1 \omega$ by the second equation in (1.3.5), namely

$$
\partial_1 \omega = \partial_t j + u \cdot \nabla j + \eta j - b \cdot \nabla \omega - Q,
$$

then J_1 is converted into five terms,

$$
\begin{aligned}
J_1 \quad &= \quad - \int \partial_1 u_1 \left(\partial_t j + u \cdot \nabla j + \eta j - b \cdot \nabla \omega - Q \right) \partial_1 \omega \; dx \\
:= \quad & J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}.
\end{aligned}
$$

We shift the time derivative in $J_{1,1}$, namely

$$
\begin{aligned}
J_{1,1} &= - \int \partial_1 u_1 \, \partial_t j \, \partial_1 \omega \; dx \\
&= - \frac{d}{dt} \int \partial_1 u_1 \, j \, \partial_1 \omega \; dx + \int \partial_1 (\partial_t u_1) \, j \, \partial_1 \omega \; dx + \int \partial_1 u_1 \, j \, \partial_1 (\partial_t \omega) \; dx \\
&:= J_{1,1,1} + J_{1,1,2} + J_{1,1,3}.
\end{aligned}
$$

Replacing $\partial_t u_1$ by the first equation of (1.1.1), we have

$$\begin{aligned} J_{1,1,2} &= \int j\partial_1\omega\left(-\partial_1(u\cdot\nabla u_1) - \partial_{11}P + \nu\partial_{221}u_1 + \partial_1(b\cdot\nabla b_1) + \partial_{11}b_1\right)\,dx \\ &:= J_{1,1,2,1} + J_{1,1,2,2} + J_{1,1,2,3} + J_{1,1,2,4} + J_{1,1,2,5}. \end{aligned}$$

Similarly, we replace $\partial_t\omega$ by the first equation in (1.3.5),

$$\begin{aligned} J_{1,1,3} &= \int \partial_1 u_1 j\left(-\partial_1(u\cdot\nabla\omega) + \nu\partial_{221}\omega + \partial_1(b\cdot\nabla j) + \partial_{11}j\right)\,dx \\ &= J_{1,1,3,1} + J_{1,1,3,2} + J_{1,1,3,3} + J_{1,1,3,4} \end{aligned}$$

We thus have rewritten J_1 as

$$\begin{aligned} J_1 &= J_{1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5} \\ &= J_{1,1,1} + J_{1,1,2} + J_{1,1,3} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5} \\ &= J_{1,1,2,1} + J_{1,1,2,2} + J_{1,1,2,3} + J_{1,1,2,4} + J_{1,1,2,5} \\ &\quad + J_{1,1,3,1} + J_{1,1,3,2} + J_{1,1,3,3} + J_{1,1,3,4} \\ &\quad + J_{1,1,1} + J_{1,2} + J_{1,3} + J_{1,4} + J_{1,5}. \end{aligned} \tag{1.3.12}$$

We estimate the fourteen terms on the right of (1.3.12). By Hölder's inequality, Lemma 1.1.1 and Lemma 1.3.1,

$$\begin{aligned} J_{1,1,2,1} &= -\int j\partial_1\omega\partial_1 u\cdot\nabla u_1\,dx - \int j\partial_1\omega u\cdot\nabla\partial_1 u_1\,dx \\ &\lesssim \|\partial_1 u\|_{L^\infty}\|\partial_1\omega\|_{L^2}\|\nabla u_1\|_{L^2}^{\frac12}\|\partial_2\nabla u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12} \\ &\quad + \|u\|_{L^\infty}\|\partial_1\omega\|_{L^2}\|\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12} \\ &\lesssim \|\partial_1 u\|_{H^1}^{\frac12}\|\partial_2\partial_1 u\|_{H^1}^{\frac12}\|\partial_1\omega\|_{L^2}\|\nabla u_1\|_{L^2}^{\frac12}\|\partial_2\nabla u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12} \\ &\quad + \|u\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^1}^{\frac12}\|\partial_1\omega\|_{L^2}\|\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_1 u_1\|_{L^2}^{\frac12}\|j\|_{L^2}^{\frac12}\|\partial_1 j\|_{L^2}^{\frac12} \\ &\lesssim \|u\|_{H^2}^2\|\partial_2 u\|_{H^2}\|b\|_{H^2}. \end{aligned}$$

Applying the divergence operator $\nabla\cdot$ to the first equation of (1.1.1) and invoking $\nabla\cdot u = 0$, we have

$$\partial_{11}P = \partial_{11}(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u - b\cdot\nabla b) \tag{1.3.13}$$

By substituting (1.3.13) into $J_{1,1,2,2}$,

$$\begin{aligned}
J_{1,1,2,2} &= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u - b\cdot\nabla b)\ dx \\
&= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_1(u\cdot\nabla u_1)\ dx - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_2(u\cdot\nabla u_2)\ dx \\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_1(b\cdot\nabla b_1)\ dx + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}\partial_2(b\cdot\nabla b_2)\ dx \\
&= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u\cdot\nabla u_1)\ dx - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(u\cdot\nabla\partial_1 u_1)\ dx \\
&\quad - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 u\cdot\nabla u_2)\ dx - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(u\cdot\nabla\partial_2 u_2)\ dx \\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b\cdot\nabla b_1)\ dx + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(b\cdot\nabla\partial_1 b_1)\ dx \\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 b\cdot\nabla b_2)\ dx + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(b\cdot\nabla\partial_2 b_2)\ dx \\
&= -\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u\cdot\nabla u_1)\ dx - \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 u\cdot\nabla u_2)\ dx \\
&\quad + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b\cdot\nabla b_1)\ dx + \int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_2 b\cdot\nabla b_2)\ dx \\
&= -2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u_1\partial_1 u_1)\ dx - 2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 u_2\partial_2 u_1)\ dx \\
&\quad + 2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b_1\partial_1 b_1)\ dx + 2\int j\partial_1\omega\partial_{11}(-\Delta)^{-1}(\partial_1 b_2\partial_2 b_1)\ dx.
\end{aligned}$$

The four terms on the right can be estimated as follows. We use Hölder's inequality, Lemma 1.1.1 and Lemma 1.3.1, and the fact that the double Riesz transform $\partial_{11}\Delta^{-1}$ is bounded on

L^p for any $1 < p < \infty$.

$$
\begin{aligned}
J_{1,1,2,2} &\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_{11}(-\Delta)^{-1}(\partial_1 u_1\partial_1 u_1)\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_{11}(-\Delta)^{-1}(\partial_1 u_2\partial_2 u_1)\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_{11}(-\Delta)^{-1}(\partial_1 b_1\partial_1 b_1)\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_{11}(-\Delta)^{-1}(\partial_1 b_2\partial_2 b_1)\|_{L^2} \\
&\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \Big(\|\partial_1 u_1\partial_1 u_1\|_{L^2} + \|\partial_1 u_2\partial_2 u_1\|_{L^2} + \|\partial_1 b_1\partial_1 b_1\|_{L^2} + \|\partial_1 b_2\partial_2 b_1\|_{L^2}\Big) \\
&\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{L^\infty}\, \|\partial_2 u\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^4}^2 \\
&\lesssim \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{H^1}^{\frac{1}{2}}\, \|\partial_2\partial_1 u\|_{H^1}^{\frac{1}{2}}\, \|\partial_2 u\|_{L^2} \\
&\quad + \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\, \|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}\, \|\Delta b\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}\|b||_{H^2}\|u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}.
\end{aligned}
$$

By Lemma 1.1.1,

$$
\begin{aligned}
J_{1,1,2,3} &= \nu\int j\partial_1\omega\partial_{221}u_1\ dx \lesssim \|\partial_{221}u_1\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}
\end{aligned}
$$

By Lemma 1.1.1 and Sobolev's inequality,

$$
\begin{aligned}
J_{1,1,2,4} &= \int j\partial_1\omega\partial_1(b\cdot\nabla b_1)\ dx = \int j\partial_1\omega\partial_1 b\cdot\nabla b_1\ dx + \int j\partial_1\omega b\cdot\partial_1\nabla b_1\ dx \\
&\lesssim \|\partial_1 b\cdot\nabla b_1\|_{L^2}\, \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|b\cdot\partial_1\nabla b_1\|_{L^2}\, \|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}}\, \|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim (\|\nabla b\|_{L^4}^2 + \|b\|_{L^\infty}\|\partial_1\nabla b_1\|_{L^2})\|b\|_{H^2}\, \|u\|_{H^2}^{\frac{1}{2}}\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^2}^3\, \|u\|_{H^2}^{\frac{1}{2}}\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

The last term $J_{1,1,2,5}$ can also be bounded via Lemma 1.1.1,

$$
\begin{aligned}
J_{1,1,2,5} &= \int j\partial_1\omega\partial_{11}b_1\ dx \lesssim \|\partial_{11}b_1\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

We rewrite $J_{1,1,3,1}$ as

$$
\begin{aligned}
J_{1,1,3,1} &= -\int \partial_1 u_1 j\partial_1 u\cdot\nabla\omega\ dx - \int \partial_1 u_1 j u\cdot\partial_1\nabla\omega\ dx \\
&= J_{1,1,3,1,1} + J_{1,1,3,1,2}.
\end{aligned}
$$

By Hölder's inequality, Lemmas 1.1.1 and 1.3.1,

$$
\begin{aligned}
J_{1,1,3,1,1} &= \int \partial_1 u_1 j \partial_1 u \nabla\omega \, dx \\
&\lesssim \|\partial_1 u\|_{L^\infty} \|\nabla\omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\nabla\omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
\end{aligned}
$$

By integration by parts, Lemma 1.1.1 and Lemma 1.3.1,

$$
\begin{aligned}
J_{1,1,3,1,2} &= -\int \partial_1 u_1 j u \cdot \partial_1 \nabla\omega \; dx \\
&= \int \partial_{11} u_1 j u \cdot \nabla\omega \; dx + \int \partial_1 u_1 \partial_1 j u \cdot \nabla\omega \; dx + \int \partial_1 u_1 j \partial_1 u \cdot \nabla\omega \; dx \\
&\lesssim \quad \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2} \|\nabla\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla\omega\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\qquad + \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_1 j\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\nabla\omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla\omega\|_{L^2}^{\frac{1}{2}} \\
&\qquad + \|\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u\|_{H^1}^{\frac{1}{2}} \|\nabla\omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_{1,1,3,2} &= \nu \int \partial_1 u_1 j \partial_{221}\omega \; dx = -\nu \int \partial_{11} u_1 j \partial_{22}\omega \; dx - \nu \int \partial_1 u_1 \partial_1 j \partial_{22}\omega \; dx \\
&\lesssim \|\partial_{22}\omega\|_{L^2} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_1 j\|_{L^2} \|\partial_{22}\omega\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \|b\|_{H^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

By integration by parts, Lemma 1.1.1 and Lemma 1.3.1,

$$
\begin{aligned}
J_{1,1,3,3} &= \int \partial_1 u_1 j \partial_1 (b \cdot \nabla j) \; dx \\
&= -\int \partial_{11} u_1 j b \cdot \nabla j \; dx - \int \partial_1 u_1 \partial_1 j b \cdot \nabla j \; dx \\
&\lesssim \|b\|_{L^\infty} \|\nabla j\|_{L^2} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1 u_1\|_{L^\infty} \|b\|_{L^\infty} \|\partial_1 j\|_{L^2} \|\nabla j\|_{L^2} \\
&\lesssim \|b\|_{H^1}^{\frac{1}{2}} \|\partial_1 b\|_{H^1}^{\frac{1}{2}} \|\nabla j\|_{L^2} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|b\|_{H^1}^{\frac{1}{2}} \|\partial_2 b\|_{H^1}^{\frac{1}{2}} \|\partial_1 j\|_{L^2} \|\nabla j\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}}.
\end{aligned}
$$

By integration by parts and Lemma 1.1.1,

$$
\begin{aligned}
J_{1,1,3,4} &= \int \partial_1 u_1 j \partial_{11} j \ dx = - \int \partial_{11} u_1 j \partial_1 j \ dx - \int \partial_1 u_1 \partial_1 j \partial_1 j \ dx \\
&\lesssim \|\partial_1 j\|_{L^2} \|\partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_{11} u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} + \|\partial_1 u_1\|_{L^\infty} \|\partial_1 j\|_{L^2}^2 \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} + \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_2 \partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|b\|_{H^2}^2 \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

The next term in (1.3.12) is $J_{1,1,1}$, which involves the time derivative. Its handling is easy and it will be bounded after we take the time integral. We turn to the next term in (1.3.12), namely $J_{1,2}$. By Lemma 1.1.1 and then Lemma 1.3.1,

$$
\begin{aligned}
J_{1,2} &= \int \partial_1 u_1 u \cdot \nabla j \partial_1 \omega \ dx \\
&\lesssim \|u\|_{L^\infty} \|\nabla j\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\nabla j\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
\end{aligned}
$$

By integration by parts, $\nabla \cdot u = 0$, and Lemma 1.1.1,

$$
\begin{aligned}
J_{1,3} &= - \int \eta \partial_1 u_1 j \partial_1 \omega \ dx \\
&= \int \eta \partial_1^2 u_1 j \omega \ dx + \int \eta \partial_2 u_2 \partial_1 j \omega \ dx \\
&\lesssim \|j\|_{L^2} \|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 u_1\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1 j\|_{L^2} \|\partial_2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 u_2\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}.
\end{aligned}
$$

By Lemmas 1.1.1 and 1.3.1,

$$
\begin{aligned}
J_{1,4} &= \int \partial_1 u_1 b \cdot \nabla \omega \partial_1 \omega \ dx \\
&\lesssim \|\partial_1 u_1\|_{L^\infty} \|\nabla \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|b\|_{L^2}^{\frac{1}{2}} \|\partial_1 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\partial_2 \partial_1 u_1\|_{H^1}^{\frac{1}{2}} \|\nabla \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|b\|_{L^2}^{\frac{1}{2}} \|\partial_1 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
\end{aligned}
$$

$J_{1,5}$ is written more explicitly into four pieces by the definition of Q,

$$
\begin{aligned}
J_{1,5} &= \int \partial_1 u_1 Q \partial_1 \omega \ dx \\
&= \int \partial_1 u_1 \partial_1 \omega \left(2 \partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - 2 \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)\right) \ dx \\
&:= J_{1,5,1} + J_{1,5,2} + J_{1,5,3} + J_{1,5,4}.
\end{aligned}
$$

By Lemmas 1.1.1 and 1.3.1,

$$
\begin{aligned}
J_{1,5,1} &= 2\int \partial_1 u_1 \partial_1 b_1 \partial_1 u_2 \partial_1 \omega \; dx \\
&\lesssim \|\partial_1 u_2\|_{L^\infty} \|\partial_1 \omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u_2\|_{H^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_2\|_{H^2}^{\frac{1}{2}} \|\partial_1 \omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
\end{aligned}
$$

It is easy to check that $J_{1,5,2}$, $J_{1,5,3}$ and $J_{1,5,4}$ all obey the same bound. Therefore,

$$
|J_{1,5}| \lesssim \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2.
$$

We have completed the estimates of the terms of J_1 in (1.3.12). Collecting the upper bounds leads to

$$
\begin{aligned}
J_1 \le & -\frac{d}{dt}\int \partial_1 u_1 \, j \, \partial_1 \omega \; dx + C\, \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2}^2 \\
& + C\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} + C\, \|\partial_2 u\|_{H^2} \|b\|_{H^2} \|u\|_{H^2} \\
& + C\, \|\partial_2 u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{3}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}. \qquad (1.3.14)
\end{aligned}
$$

We now turn to the second term J_2 in (1.3.11). As we have explained before, we need to invoke the smoothing and regularization effect of the magnetic field in order to bound this term suitably. By replacing $\partial_1 u_2$ by (1.1.1), namely

$$
\partial_1 u_2 = \partial_t b_2 + u \cdot \nabla b_2 + \eta b_2 - b \cdot \nabla u_2,
$$

we can write

$$
\begin{aligned}
J_2 &= -\int \partial_1 \omega \,(\partial_t b_2 + u \cdot \nabla b_2 + \eta b_2 - b \cdot \nabla u_2)\, \partial_2 \omega \; dx \\
&:= J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4}.
\end{aligned}
$$

We bound $J_{2,2}$, $J_{2,3}$ and $J_{2,4}$ first. By Lemmas 1.1.1 and 1.3.1,

$$
\begin{aligned}
J_{2,2} &= -\int \partial_1 \omega u \cdot \nabla b_2 \partial_2 \omega \; dx \\
&\lesssim \|u\|_{L^\infty} \|\partial_2 \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\nabla b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\partial_2 \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\nabla b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^2 \|b\|_{H^2} \|u\|_{H^2}.
\end{aligned}
$$

By Lemma 1.1.1,

$$
\begin{aligned}
J_{2,3} &= -\eta \int \partial_1 \omega \, b_2 \, \partial_2 \omega \; dx \\
&\lesssim \|\partial_2 \omega\|_{L^2} \|\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_1 \omega\|_{L^2}^{\frac{1}{2}} \|b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}} \|b\|_{H^2}^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned}
$$

Again, by Lemmas 1.1.1 and 1.3.1,

$$\begin{aligned} J_{2,4} &= \int \partial_1\omega b\cdot\nabla u_2\partial_2\omega \ dx \\ &\lesssim \|b\|_{L^\infty}\|\partial_2\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\nabla u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u_2\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|b\|_{H^1}^{\frac{1}{2}}\|\partial_2 b\|_{H^1}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\nabla u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u_2\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2 u\|_{H^2}^{\frac{3}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{3}{2}}\|b\|_{H^2}^{\frac{1}{2}}. \end{aligned}$$

To deal with $J_{2,1}$, we shift the time derivative,

$$\begin{aligned} J_{2,1} &= -\int \partial_1\omega\partial_t b_2\partial_2\omega \ dx \\ &= -\frac{d}{dt}\int \partial_1\omega b_2\partial_2\omega \ dx + \int \partial_1(\partial_t\omega)b_2\partial_2\omega \ dx + \int \partial_1\omega b_2\partial_2(\partial_t\omega) \ dx \\ &= J_{2,1,1} + J_{2,1,2} + J_{2,1,3}. \end{aligned}$$

By invoking the vorticity equation in (1.3.5), we can write

$$\begin{aligned} J_{2,1,2} &= \int \partial_1(-u\cdot\nabla\omega + \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j)b_2\partial_2\omega \ dx \\ &:= J_{2,1,2,1} + J_{2,1,2,2} + J_{2,1,2,3} + J_{2,1,2,4}. \end{aligned}$$

By integration by parts, and Lemmas 1.1.1 and 1.3.1,

$$\begin{aligned} J_{2,1,2,1} &= \int u\cdot\nabla\omega\partial_1 b_2\partial_2\omega \ dx + u\cdot\nabla\omega b_2\partial_1\partial_2\omega \ dx \\ &\lesssim \|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|u\|_{L^\infty}\|\partial_1\partial_2\omega\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla\omega\|_{L^2}^{\frac{1}{2}}\|b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\|b\|_{H^2}\|u\|_{H^2}, \end{aligned}$$

and

$$\begin{aligned} J_{2,1,2,2} &= -\nu\int \partial_{22}\omega\partial_1 b_2\partial_2\omega \ dx - \nu\int \partial_{22}\omega b_2\partial_1\partial_2\omega \ dx \\ &\lesssim \|\partial_{22}\omega\|_{L^2}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}} \\ &\quad + \|b_2\|_{L^\infty}\|\partial_{22}\omega\|_{L^2}\|\partial_1\partial_2\omega\|_{L^2} \\ &\lesssim \|\partial_2 u\|_{H^2}^2\|b\|_{H^2} + \|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_2\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^2 \\ &\lesssim \|\partial_2 u\|_{H^2}^2\|b\|_{H^2}. \end{aligned}$$

Similarly,

$$\begin{aligned}J_{2,1,2,3} &= -\int b\cdot\nabla j\partial_1 b_2\partial_2\omega\ dx - \int b\cdot\nabla j b_2\partial_1\partial_2\omega\ dx\\ &\lesssim \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_2\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_1^2 b_2\|_{L^2}^{\frac12}\\ &\quad + \|b\|_{L^\infty}\|b_2\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_1\partial_2\omega\|_{L^2}\\ &\lesssim \|b\|_{H^1}^{\frac12}\|\partial_1 b\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^2}\|b\|_{H^2}^2 + \|b\|_{H^1}\|\partial_1 b\|_{H^1}\|b\|_{H^2}\|\partial_2 u\|_{H^2}\\ &\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}^3\end{aligned}$$

and

$$\begin{aligned}J_{2,1,2,4} &= \int \partial_{11} j b_2\partial_2\omega\ dx\\ &= -\int \partial_1 j\partial_1 b_2\partial_2\omega\ dx - \int \partial_1 j b_2\partial_1\partial_2\omega\ dx\\ &\lesssim \|\partial_1 j\|_{L^2}\|\partial_2\omega\|_{L^2}^{\frac12}\|\partial_2\partial_2\omega\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_1^2 b_2\|_{L^2}^{\frac12}\\ &\quad + \|b_2\|_{L^\infty}\|\partial_1 j\|_{L^2}\|\partial_1\partial_2\omega\|_{L^2}\\ &\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}^2 + \|b_2\|_{H^1}^{\frac12}\|\partial_1 b_2\|_{H^1}^{\frac12}\|b\|_{H^2}\|\partial_2 u\|_{H^2}\\ &\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}^2.\end{aligned}$$

To bound $J_{2,1,3}$, we invoke the vorticity equation in (1.3.5) again,

$$\begin{aligned}J_{2,1,3} &= \int \partial_2(-u\cdot\nabla\omega + \nu\partial_{22}\omega + b\cdot\nabla j + \partial_1 j) b_2\partial_1\omega\ dx\\ &:= J_{2,1,3,1} + J_{2,1,3,2} + J_{2,1,3,3} + J_{2,1,3,4}.\end{aligned}$$

By integration by parts, and Lemmas 1.1.1 and 1.3.1,

$$\begin{aligned}J_{2,1,3,1} &= \int u\cdot\nabla\omega\partial_2 b_2\partial_1\omega + u\cdot\nabla\omega b_2\partial_2\partial_1\omega\ dx\\ &\lesssim \|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac12}\\ &\quad + \|u\|_{L^\infty}\|\partial_2\partial_1\omega\|_{L^2}\|\nabla\omega\|_{L^2}^{\frac12}\|\partial_2\nabla\omega\|_{L^2}^{\frac12}\|b_2\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\|b\|_{H^2}\|u\|_{H^2},\end{aligned}$$

$$\begin{aligned}J_{2,1,3,2} &= -\nu\int \partial_{22}\omega\partial_2 b_2\partial_1\omega\ dx - \nu\int \partial_{22}\omega b_2\partial_2\partial_1\omega\, dx\\ &\lesssim \|\partial_{22}\omega\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2\partial_1\omega\|_{L^2}^{\frac12}\|\partial_2 b_2\|_{L^2}^{\frac12}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac12}\\ &\quad + \|b_2\|_{L^\infty}\|\partial_{22}\omega\|_{L^2}\|\partial_2\partial_1\omega\|_{L^2}\\ &\lesssim \|\partial_2 u\|_{H^2}^{\frac32}\|b\|_{H^2}^{\frac12}\|u\|_{H^2}^{\frac12}\|b\|_{H^2}^{\frac12} + \|b_2\|_{H^1}^{\frac12}\|\partial_1 b_2\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^2}^2,\end{aligned}$$

$$
\begin{aligned}
J_{2,1,3,3} &= -\int b\cdot\nabla j\partial_2 b_2\partial_1\omega\ dx - \int b\cdot\nabla j b_2\partial_2\partial_1\omega\ dx \\
&\lesssim \|b\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|b\|_{L^\infty}\|b_2\|_{L^\infty}\|\nabla j\|_{L^2}\|\partial_2\partial_1\omega\|_{L^2} \\
&\lesssim \|b\|_{H^1}^{\frac{1}{2}}\|\partial_1 b\|_{H^1}^{\frac{1}{2}}\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^2 + \|b\|_{H^1}\|\partial_1 b\|_{H^1}\|b\|_{H^2}\|\partial_2 u\|_{H^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}} + \|\partial_2 u\|_{H^2}\|b\|_{H^2}^3,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{2,1,3,4} &= \int \partial_{12} j b_2\partial_1\omega\ dx \\
&= -\int \partial_1 j\partial_2 b_2\partial_1\omega\ dx - \int \partial_1 j b_2\partial_2\partial_1\omega\ dx \\
&\lesssim \|\partial_1 j\|_{L^2}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2 b_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|b_2\|_{L^\infty}\|\partial_1 j\|_{L^2}\|\partial_2\partial_1\omega\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^2 + \|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_2\|_{H^1}^{\frac{1}{2}}\|b\|_{H^2}\|\partial_2 u\|_{H^2} \\
&\lesssim \|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}} + \|\partial_2 u\|_{H^2}\|b\|_{H^2}^2.
\end{aligned}
$$

Collecting the estimates for J_2, we obtain

$$
\begin{aligned}
J_2 \le & -\frac{d}{dt}\int \partial_1\omega b_2\partial_2\omega\ dx + C\,\|\partial_2 u\|_{H^2}^2\|b\|_{H^2}\|u\|_{H^2} + C\,\|\partial_2 u\|_{H^2}^2\|b\|_{H^2} \\
& + C\,\|\partial_2 u\|_{H^2}^2\|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_1 b_2\|_{H^1}^{\frac{1}{2}} + C\,\|\partial_2 u\|_{H^2}^{\frac{3}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}} \\
& + C\,\|\partial_2 u\|_{H^2}^{\frac{3}{2}}\|b\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{3}{2}}\|b\|_{H^2}^{\frac{1}{2}} + C\,\|\partial_2 u\|_{H^2}\|b\|_{H^2}\|u\|_{H^2}^2 \\
& + C\,\|\partial_2 u\|_{H^2}\|b\|_{H^2}^3 + C\,\|\partial_2 u\|_{H^2}\|b\|_{H^2}^2 \\
& + C\,\|\partial_2 u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|b\|_{H^2}^{\frac{1}{2}}.
\end{aligned} \tag{1.3.15}
$$

The last two terms in (1.3.11) are J_3 and J_4. We now evaluate them. By Lemma 1.1.1,

$$
\begin{aligned}
J_3 &= -\int \partial_2\omega\partial_2 u_1\partial_1\omega\ dx \\
&\lesssim \|\partial_2\omega\|_{L^2}\|\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^2\|u\|_{H^2}
\end{aligned} \tag{1.3.16}
$$

and

$$
\begin{aligned}
J_4 &= -\int \partial_2\omega\partial_2 u_2\partial_2\omega\ dx \\
&\lesssim \|\partial_2\omega\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\omega\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2\omega\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 u\|_{H^2}^2\|u\|_{H^2}.
\end{aligned} \tag{1.3.17}
$$

Adding (1.3.4) and (1.3.6), integrating in time, and recalling the definition of E in (1.3.2), we have

$$E(t) \leq E(0) + \int_0^t (J + K + L + M + N)\, d\tau.$$

Collecting the upper bounds in (1.3.7), (1.3.8), (1.3.9), (1.3.10), (1.3.14), (1.3.15), (1.3.16) and (1.3.17), we find

$$E(t) \leq E(0) - \int \partial_1 u_1\, j\, \partial_1 \omega \; dx + \int \partial_1 u_{01}\, j_0\, \partial_1 \omega_0 \; dx \tag{1.3.18}$$

$$- \int \partial_1 \omega\, b_2\, \partial_2 \omega dx + \int \partial_1 \omega_0\, b_{02}\, \partial_2 \omega_0 \; dx \tag{1.3.19}$$

$$+ C\, E^2(t) + C\, E^{\frac{3}{2}}(t). \tag{1.3.20}$$

The last two terms on (1.3.18) come from the time integral of the first term in (1.3.14), and the two terms on (1.3.19) are from the time integral of the first term in (1.3.15). The two terms on (1.3.20) are obtained by integrating the aforementioned upper bounds and applying Hölder's inequality. For example, when we integrate one of the upper bounds in (1.3.14), say $C\, \|\partial_2 u\|_{H^2}\, \|b\|_{H^2}\, \|u\|_{H^2}^2$,

$$\begin{aligned}\int_0^t C\, \|\partial_2 u\|_{H^2}\, \|b\|_{H^2}\, \|u\|_{H^2}^2\, d\tau &\leq C \sup_{0\leq\tau\leq t} \|u(\tau)\|_{H^2}^2 \int_0^t \|\partial_2 u\|_{H^2}\, \|b\|_{H^2}\, d\tau \\ &\leq C\, E(t) \left(\int_0^t \|\partial_2 u\|_{H^2}^2\, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \|b\|_{H^2}^2\, d\tau\right)^{\frac{1}{2}} \\ &\leq C\, E^2(t).\end{aligned}$$

The four terms on (1.3.18) and (1.3.19) can be further bounded as follows. By Hölder's inequality and Lemma 1.1.1,

$$\begin{aligned}&- \int \partial_1 u_1\, j\, \partial_1 \omega\, dx + \int \partial_1 u_{01}\, j_0\, \partial_1 \omega_0\, dx \\ &\leq C \|\partial_1 \omega\|_{L^2} \|\partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 u_1\|_{L^2}^{\frac{1}{2}} \|j\|_{L^2}^{\frac{1}{2}} \|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_1 \omega_0\|_{L^2} \|\partial_1 u_{01}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 u_{01}\|_{L^2}^{\frac{1}{2}} \|j_0\|_{L^2}^{\frac{1}{2}} \|\partial_1 j_0\|_{L^2}^{\frac{1}{2}} \\ &\leq C\, E^{\frac{3}{2}}(t) + C\, E^{\frac{3}{2}}(0).\end{aligned}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}&- \int \partial_1 \omega\, b_2\, \partial_2 \omega\, dx + \int \partial_1 \omega_0\, b_{02}\, \partial_2 \omega_0\, dx \\ &\leq C\, \|b_2\|_{L^\infty} \|\partial_1 \omega\|_{L^2} \|\partial_2 \omega\|_{L^2} + C\, \|b_{02}\|_{L^\infty} \|\partial_1 \omega_0\|_{L^2} \|\partial_2 \omega_0\|_{L^2} \\ &\leq C\, \|b\|_{H^2}\, \|\partial_1 \omega\|_{L^2} \|\partial_2 \omega\|_{L^2} + C\, \|b_0\|_{H^2}\, \|\partial_1 \omega_0\|_{L^2} \|\partial_2 \omega_0\|_{L^2} \\ &\leq C\, E^{\frac{3}{2}}(t) + C\, E^{\frac{3}{2}}(0).\end{aligned}$$

We have finally obtained (1.3.3), namely

$$E(t) \le E(0) + C_1 E^{\frac{3}{2}}(0) + C_2 E^2(t) + C_3 E^{\frac{3}{2}}(t). \tag{1.3.21}$$

A bootstrapping argument applied to (1.3.21) would lead to the desired stability. We show, by the bootstrapping argument, that if the initial data is sufficiently small, say

$$\|(u_0, b_0)\|_{H^2} \le \varepsilon,$$

with ε satisfying

$$4\,\varepsilon^2 + 4C_1\varepsilon^3 \le \delta_0 := \min\left\{\frac{1}{4C_2}, \frac{1}{(4C_3)^2}\right\},$$

then, for any $0 \le t \le \infty$,

$$\|(u(t), b(t))\|_{H^2}^2 \le E(t) \le \delta_0.$$

In fact, if we make the ansatz that, for $0 \le t \le T$,

$$E(t) \le \delta_0,$$

then (1.3.21) implies

$$\begin{aligned} E(t) &\le \varepsilon^2 + C_1\varepsilon^3 + C_2 E(t)\, E(t) + C_3 E^{\frac{1}{2}}(t)\, E(t) \\ &\le \varepsilon^2 + C_1\varepsilon^3 + C_2\,\frac{1}{4C_2} E(t) + C_3\,\frac{1}{4C_3}\, E(t) \end{aligned}$$

or

$$\frac{1}{2}E(t) \le \varepsilon^2 + C_1\varepsilon^3 \quad \text{or} \quad E(t) \le 2\varepsilon^2 + 2C_1\varepsilon^3 = \frac{1}{2}\delta_0.$$

The bootstrapping argument then implies that $T = \infty$ and $E(t) \le \delta_0$. This completes the proof for the stability part of Theorem 1.1.3.

Next we prove the large-time behavior estimates stated in Theorem 1.1.3, namely (1.1.20). We make use of Lemma 1.2.1. The main efforts are devoted to verifying that

$$f(t) := \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2$$

satisfies (1.2.1), namely

$$\int_0^\infty f(t)\, dt \le C\,\varepsilon^2 < \infty \tag{1.3.22}$$

and, for any $0 \le s < t$,

$$f(t) \le C f(s). \tag{1.3.23}$$

The proof of (1.3.23) is relatively easy while the proof of (1.3.22) is more complex. Since

$$\|\nabla u(t)\|_{L^2} = \|\omega\|_{L^2} \quad \text{and} \quad \|\nabla b(t)\|_{L^2} = \|j\|_{L^2},$$

we resort to the equations of ω and j in (1.3.5). By taking the inner product of (1.3.5) with (ω, j), we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \nu\|\partial_2\omega\|_{L^2}^2 + \eta\|j\|_{L^2}^2 = \int Qj\ dx \\
&= 2\int \partial_1 b_1\partial_2 u_1 j\ dx + 2\int \partial_1 b_1 \partial_1 u_2 j dx \\
&\quad - 2\int \partial_1 u_1 \partial_2 b_1 j\ dx - 2\int \partial_1 u_1 \partial_1 b_2 j\ dx. \qquad (1.3.24)
\end{aligned}$$

The four terms on the right-hand side can be estimated very similarly. We bound the second term as an example. By Lemma 1.1.1,

$$\begin{aligned}
2\int \partial_1 b_1 \partial_1 u_2 j dx &\le C\|\partial_1 b_1\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_1 j\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\nu}{16}\|\partial_2\partial_1 u_2\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}} \\
&\le \frac{\nu}{16}\|\partial_2\omega\|_{L^2}^2 + C\|j\|_{L^2}^2\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}}.
\end{aligned}$$

The other three terms obey the same bound. Invoking these bounds in (1.3.24) yields

$$\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \frac{3\nu}{2}\|\partial_2\omega\|_{L^2}^2 + 2\left(\eta - C\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}}\right)\|j\|_{L^2}^2 \le 0. \qquad (1.3.25)$$

According to the first part of our proof, if the initial data (u_0, b_0) satisfies

$$\|(u_0, b_0)\|_{H^2} \le \varepsilon$$

for sufficiently small $\varepsilon > 0$, the solution (u, b) remains small,

$$\|(u(t), b(t))\|_{H^2} \le C\,\varepsilon.$$

When $\varepsilon > 0$ is taken to be small enough such that

$$\eta - C\|\omega\|_{L^2}^{\frac{2}{3}}\|\partial_1 j\|_{L^2}^{\frac{2}{3}} \ge \eta - C\varepsilon^{\frac{4}{3}} \ge 0,$$

then (1.3.25) implies, for any $0 \le s < t$,

$$\|\omega(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 \le \|\omega(s)\|_{L^2}^2 + \|j(s)\|_{L^2}^2 \quad \text{or} \quad f(t) \le f(s).$$

We now prove (1.3.22). We have shown in the previous part that

$$\int_0^\infty \|\partial_2 u(t)\|_{H^2}^2\ dt \le C\,\varepsilon^2, \qquad \int_0^\infty \|b(t)\|_{H^2}^2\ dt \le C\,\varepsilon^2. \qquad (1.3.26)$$

To prove (1.3.22), it remains to prove

$$\int_0^\infty \|\partial_1 u\|_{L^2}^2\ dt \le C\,\varepsilon^2. \qquad (1.3.27)$$

The proof for this upper bound is not trivial. We need to take advantage of the regularization of the magnetic field. We replace one of $\partial_1 u$ in (1.3.27) by the equation of the magnetic field

$$\partial_1 u = \partial_t b + u\cdot\nabla b + \eta b - b\cdot\nabla u$$

and obtain

$$\begin{aligned}\|\partial_1 u\|_{L^2}^2 &= \int \partial_1 u\cdot\partial_1 u\ dx = \int \partial_1 u\cdot(\partial_t b + u\cdot\nabla b + \eta b - b\cdot\nabla u)\ dx\\ &:= N_1+N_2+N_3+N_4. \end{aligned}\tag{1.3.28}$$

We first estimate N_2, N_3 and N_4 and then come back to N_1. We write out all component terms explicitly,

$$\begin{aligned}N_2 &= \int \partial_1 u\cdot(u\cdot\nabla b)\ dx = \int (\partial_1 u_1(u\cdot\nabla)b_1 + \partial_1 u_2 u\cdot\nabla b_2)\,dx\\ &= \int ((-\partial_2 u_2)(u\cdot\nabla)b_1 + \partial_1 u_2(u_1\partial_1 b_2 + u_2\partial_2 b_2))\,dx\\ &= N_{2,1}+N_{2,2}+N_{2,3}.\end{aligned}$$

By Lemma 1.1.1,

$$\begin{aligned}N_{2,1} &= \int (-\partial_2 u_2)(u\cdot\nabla)b_1\ dx\\ &\le C\|\partial_2 u_2\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\nabla b_1\|_{L^2}^{\frac12}\|\partial_1\nabla b_1\|_{L^2}^{\frac12} \le C\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|b\|_{H^2}.\end{aligned}$$

By integration by parts and Lemma 1.1.1,

$$\begin{aligned}N_{2,2} &= -\int (u_2\partial_1 u_1\partial_1 b_2\ dx + u_2 u_1\partial_{11} b_2)\ dx\\ &\le C\,\|u_2\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1 b_2\|_{L^2}^{\frac12}\|\partial_1\partial_1 b_2\|_{L^2}^{\frac12}\\ &\quad + C\,\|\partial_{11} b_2\|_{L^2}\|\partial_2 u_2\|_{L^2}^{\frac12}\|u_2\|_{L^2}^{\frac12}\|u_1\|_{L^2}^{\frac12}\|\partial_1 u_1\|_{L^2}^{\frac12}\\ &\le C\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|b\|_{H^2}.\end{aligned}$$

Again, by integration by parts and Lemma 1.1.1,

$$\begin{aligned}N_{2,3} &= \int \partial_1 u_2 u_2\partial_2 b_2\ dx = \int \partial_1(\frac{u_2^2}{2})\partial_2 b_2\ dx\\ &= \int \partial_2(\frac{u_2^2}{2})\partial_1 b_2\ dx = \int u_2\partial_2 u_2\partial_1 b_2\ dx\\ &\le C\|u\|_{H^2}\|\partial_2 u\|_{H^2}\|b\|_{H^2}.\end{aligned}$$

Clearly,

$$N_3 = \eta\int \partial_1 u\cdot b\ dx \le C\|\partial_1 u\|_{L^2}\|b\|_{L^2} \le \frac14\|\partial_1 u\|_{L^2}^2 + C\|b\|_{L^2}^2.$$

To bound N_4, we again write out the component terms explicitly,

$$\begin{aligned}
N_4 &= -\int \partial_1 u \cdot (b \cdot \nabla u)\ dx = -\int \partial_1 u_1 (b\cdot\nabla) u_1\ dx - \int \partial_1 u_2 (b \cdot \nabla) u_2\ dx \\
&= \int \partial_2 u_2 (b\cdot\nabla) u_1\ dx - \int \partial_1 u_2 b_1 \partial_1 u_2\ dx - \int \partial_1 u_2 b_2 \partial_2 u_2\ dx \\
&= N_{4,1} + N_{4,2} + N_{4,3}.
\end{aligned}$$

By Lemma 1.1.1,

$$\begin{aligned}
N_{4,1} + N_{4,3} &= \int \partial_2 u_2 (b\cdot\nabla) u_1\ dx - \int \partial_1 u_2 b_2 \partial_2 u_2\ dx \\
&\le C\, \|\partial_2 u_2\|_{L^2} \|b\|_{L^2}^{\frac12} \|\partial_1 b\|_{L^2}^{\frac12} \|\nabla u_1\|_{L^2}^{\frac12} \|\partial_2 \nabla u_1\|_{L^2}^{\frac12} \\
&\quad + C\, \|\partial_2 u_2\|_{L^2} \|b_2\|_{L^2}^{\frac12} \|\partial_1 b_2\|_{L^2}^{\frac12} \|\partial_1 u_2\|_{L^2}^{\frac12} \|\partial_2 \partial_1 u_2\|_{L^2}^{\frac12} \\
&\le C \|u\|_{H^2} \|\partial_2 u\|_{H^2} \|b\|_{H^2}.
\end{aligned}$$

By Sobolev's inequality,

$$\begin{aligned}
N_{4,2} &= -\int (\partial_1 u_2)^2 b_1\ dx \le C \|b\|_{L^2} \|\partial_1 u_2\|_{L^4}^2 \\
&\le C \|b\|_{L^2} \|\partial_1 u_2\|_{L^2} \|\nabla \partial_1 u_2\|_{L^2} \le \frac{1}{4} \|\partial_1 u_2\|_{L^2}^2 + C \|b\|_{H^2}^2 \|u\|_{H^2}^2.
\end{aligned}$$

We now return to estimate N_1. Shifting the time integral and invoking (1.1.1), we obtain

$$\begin{aligned}
N_1 &= \int \partial_1 u \cdot \partial_t b\ dx \\
&= \frac{d}{dt} \int \partial_1 u \cdot b\ dx + \int \partial_1 (u\cdot\nabla u + \nabla P - \nu \partial_{22} u - b \cdot \nabla b - \partial_1 b) \cdot b\ dx \\
&:= N_{1,1} + N_{1,2} + N_{1,3} + N_{1,4} + N_{1,5} + N_{1,6}.
\end{aligned}$$

$N_{1,1}$ is the time derivative term and we bound it later after we integrate it in time. To estimate $N_{1,2}$, we rewrite it into sums of component terms to reveal the terms with favorable partial derivative such as $\partial_2 u$,

$$\begin{aligned}
N_{1,2} &= \int \partial_1 (u\cdot\nabla u)\cdot b\ dx = \int (\partial_1 u \cdot \nabla u) \cdot b\ dx + \int (u \cdot \nabla \partial_1 u) \cdot b\ dx \\
&= \int \partial_1 u_1 \partial_1 u \cdot b\, dx + \int \partial_1 u_2 \partial_2 u \cdot b\, dx + \int u_1 \partial_{11} u_1 b_1\, dx \\
&\quad + \int u_1 \partial_{11} u_2 b_2\, dx + \int u_2 \partial_2 \partial_1 u \cdot b\, dx \\
&= \int (-\partial_2 u_2) \partial_1 u \cdot b\, dx + \int \partial_1 u_2 \partial_2 u \cdot b\, dx + \int u_1 (-\partial_{21} u_2) b_1\, dx \\
&\quad + \int (\partial_1 (u_1 \partial_1 u_2)\, b_2 - \partial_1 u_1 \partial_1 u_2 b_2)\, dx + \int u_2 \partial_2 \partial_1 u \cdot b\, dx \\
&= \int (-\partial_2 u_2) \partial_1 u \cdot b\, dx + \int \partial_1 u_2 \partial_2 u \cdot b\, dx + \int u_1 (-\partial_{21} u_2) b_1\, dx \\
&\quad - \int (u_1 \partial_1 u_2) \partial_1 b_2\, dx + \int \partial_2 u_2 \partial_1 u_2 b_2\, dx + \int u_2 \partial_2 \partial_1 u \cdot b\, dx.
\end{aligned}$$

By Sobolev's inequality and Lemma 1.1.1,

$$
\begin{aligned}
N_{1,2} &\le \|\partial_2 u_2\|_{L^2}\,\|\partial_1 u\|_{L^4}\,\|b\|_{L^4} + \|\partial_1 u_2\|_{L^4}\|\partial_2 u\|_{L^2}\,\|b\|_{L^4} \\
&\quad + \|u_1\|_{L^4}\|\partial_{21} u_2\|_{L^2}\|b_1\|_{L^4} + \|u_1\|_{L^2}^{\frac{1}{2}}\,\|\partial_1 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\,\|\partial_2\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\,\|\partial_1 b_2\|_{L^2} \\
&\quad + \|\partial_2 u_2\|_{L^2}\,\|\partial_1 u_2\|_{L^4}\|b_2\|_{L^4} + \|u_2\|_{L^4}\|\partial_2\partial_1 u\|_{L^2}\|b\|_{L^4} \\
&\le C\,\|\partial_2 u\|_{H^2}\,\|b\|_{H^2}\,\|u\|_{H^2}.
\end{aligned}
$$

N_{13} contains the pressure term P. By (1.3.13),

$$
\begin{aligned}
N_{1,3} &= \int \partial_1 \nabla P \cdot b\; dx = -\int \nabla P \cdot \partial_1 b\, dx \\
&= \int \nabla(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u - b\cdot\nabla b)\cdot\partial_1 b\; dx \\
&= -\int \nabla(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u)\cdot\partial_1 b\; dx + \int \nabla(-\Delta)^{-1}\nabla\cdot(b\cdot\nabla b)\cdot\partial_1 b\; dx \\
&= -\int \nabla(-\Delta)^{-1}(\partial_1\partial_1(u_1^2) + 2\partial_1\partial_2(u_1 u_2) + \partial_2\partial_2(u_2^2))\cdot\partial_1 b\; dx \\
&\quad + \int \nabla(-\Delta)^{-1}\nabla\cdot(b\cdot\nabla b)\cdot\partial_1 b\; dx.
\end{aligned}
$$

By Hölder's inequality and using the fact that the singular integral operators are bounded on L^p for $1 < p < \infty$, namely

$$
\|\nabla(-\Delta)^{-1}\partial_1 f\|_{L^p} \le C\,\|f\|_{L^p}, \quad \|\nabla(-\Delta)^{-1}\partial_2 f\|_{L^p} \le C\,\|f\|_{L^p},
$$

we have

$$
\begin{aligned}
N_{1,3} &\le C\,(\|\partial_1(u_1^2)\|_{L^2} + \|\partial_2(u_1 u_2)\|_{L^2} + \|\partial_2(u_2^2)\|_{L^2})\,\|\partial_1 b\|_{L^2} \\
&\quad + C\,\|b\cdot\nabla b\|_{L^2}\,\|\partial_1 b\|_{L^2} \\
&\le C\,(\|u_1\|_{L^\infty}\|\partial_1 u_1\|_{L^2} + \|u\|_{L^\infty}\|\partial_2 u\|_{L^2})\,\|\partial_1 b\|_{L^2} \\
&\quad + C\,\|b\|_{L^\infty}\,\|\nabla b\|_{L^2}\,\|\partial_1 b\|_{L^2} \\
&\le C\,\|\partial_2 u_2\|_{H^2}\|u\|_{H^2}\|b\|_{H^2} + C\,\|b\|_{H^2}^3.
\end{aligned}
$$

We now estimate the rest of the terms. By integration by parts,

$$
\begin{aligned}
N_{1,4} &= -\nu\int \partial_1\partial_{22} u\cdot b\; dx = \nu\int \partial_{22} u\cdot\partial_1 b\; dx \\
&\le C\|\partial_2 u\|\|_{H^1}\|b\|_{H^2}, \\
N_{1,5} &= -\int \partial_1(b\cdot\nabla b)\cdot b\; dx = -\int (\partial_1 b\cdot\nabla b)\cdot b\; dx - (b\cdot\partial_1\nabla b)\cdot b\; dx \\
&\le C\|b\|_{H^2}^3, \\
N_{1,6} &= -\int \partial_{11} b\cdot b\; dx \le C\|b\|_{H^2}^2.
\end{aligned}
$$

Collecting the upper bounds for N_1 through N_4 and inserting them in (1.3.28), we find

$$\begin{aligned}\|\partial_1 u(t)\|_{L^2}^2 \le \frac{d}{dt}\int \partial_1 u \cdot b\, dx + C\,\|\partial_2 u\|_{H^2}\,\|b\|_{H^2}\,\|u\|_{H^2} + C\,\|b\|_{H^2}^3 \\ + C\,\|\partial_2 u\|_{H^1}\|b\|_{H^2} + C\,\|b\|_{H^2}^2 + \frac{1}{2}\|\partial_1 u\|_{L^2}^2 + C\,\|b\|_{H^2}^2\,\|u\|_{H^2}^2.\end{aligned}$$

Combining some of the terms and integrating in time, we obtain, for any $T > 0$,

$$\begin{aligned}\int_0^T \|\partial_1 u(t)\|_{L^2}^2\, dt &\le 2\int (\partial_1 u \cdot b)(x,T)\, dx - 2\int \partial_1 u_0 \cdot b_0\, dx \\ &\quad + C\int_0^T \Big(\|\partial_2 u\|_{H^2}\,\|b\|_{H^2}\,\|u\|_{H^2} + \|b\|_{H^2}^3 + \|\partial_2 u\|\|_{H^1}\|b\|_{H^2} \\ &\qquad\qquad + \|b\|_{H^2}^2 + \|b\|_{H^2}^2\,\|u\|_{H^2}^2\Big)\, dt \\ &\le 2\|\partial_1 u(T)\|_{L^2}\,\|b(T)\|_{L^2} + 2\|\partial_1 u_0\|_{L^2}\,\|b_0\|_{L^2} \\ &\quad + C\sup_{0\le t\le T}\|(u,b)(t)\|_{H^2}\int_0^T (\|\partial_2 u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2)\, dt \\ &\quad + C\,(1 + \sup_{0\le t\le T}\|u(t)\|_{H^2}^2)\int_0^T (\|\partial_2 u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2)\, dt \\ &\le C\,(\varepsilon^2 + \varepsilon^3 + \varepsilon^4). \qquad (1.3.29)\end{aligned}$$

Since the upper bound in (1.3.29) is uniform in time, we have thus verified (1.3.27), which, together with (1.3.26), confirms (1.3.22). This completes the proof of Theorem 1.1.3. ■

CHAPTER II

STABILITY AND LARGE-TIME BEHAVIOR OF 2D MHD SYSTEM WITH VERTICAL DISSIPATION

2.1 Introduction

This paper intends to understand the stability problem and especially the precise large-time behavior on the perturbations near a background magnetic field governed by the incompressible magnetohydrodynakic (MHD) system. This study is partially motivated by a remarkable stabilizing phenomenon exhibited by electrically conducting fluids. Extensive physical experiments and numerical simulations have performed to understand the influence of the magnetic field on the bulk turbulence involving various electrically conducting fluids such as liquid metals. These experiments and simulations have observed a remarkable phenomenon that a background magnetic field can smooth and stabilize turbulent electrically conducting fluids (see, e.g., [2, 3, 7, 14, 15, 16, 27, 28]).

We focus on a very special 2D incompressible MHD system with anisotropic dissipation,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u + B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B + \eta B = B \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot B = 0, \end{cases} \tag{2.1.1}$$

where u represents the velocity field, P the total pressure and B the magnetic field, and ν and η denote the viscosity and the magnetic damping coefficients, respectively. The MHD systems, the center piece of the magnetohydrodynamics initiated by H. Alfvén [3], models electrically conducting fluid such as plasmas, liquid metals and electrolytes, and have a very wide range of applications in astrophysics, geophysics, cosmology and engineering (see, e.g.,[4, 16, 44]). The MHD equations are also mathematically important. They not only share many crucial features with the Euler or the Navier-Stokes equations, but also exhibit many more fascinating characteristics such as various wave phenomena that the Euler or the Navier-Stokes equations lack.

Clearly, (2.1.1) admits a special class of steady-state solutions represented by the background magnetic field. Attention is focused on the steady-state solution

$$u^{(0)} = (0, 0), \quad B^{(0)}(x) = e_1 = (1, 0).$$

The perturbation (u, b) around this steady solution with $b = B - e_1$ obeys

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{22} u + b \cdot \nabla b + \partial_1 b, \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x), \ b(x, 0) = b_0(x). \end{cases} \tag{2.1.2}$$

The system (2.1.2) differs from the original system (2.1.1) by two extra terms, $\partial_1 b$ and $\partial_1 u$. As we shall see later, these two terms generated due to the background magnetic field play an important role in the stability properties of the perturbation as well as in the large-time behavior. These terms reflect the influence of the background magnetic field on the behavior of the fluids.

Our goal has been to understand the stability problem and the large-time behavior of solutions to (2.1.2). Due to the lack of the horizontal dissipation, these problems are not trivial. even when the magnetic field is identically zero, $b \equiv 0$, the velocity u satisfies the 2D anisotropic Navier-Stokes equation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu \partial_{22} u, \quad x \in \mathbb{R}^2, \, t > 0, \\ \nabla \cdot u = 0 \end{cases} \tag{2.1.3}$$

or, in terms of the vorticity $\omega = \nabla \times u$,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \nu \partial_{22} \omega, \quad x \in \mathbb{R}^2, \, t > 0, \\ u = \nabla^{\perp} \Delta^{-1} \omega := (-\partial_2, \partial_1) \Delta^{-1} \omega. \end{cases} \tag{2.1.4}$$

The stability problem on (2.1.4) in the Sobolev setting H^2 remains an open problem in the whole space case, although this problem in some other domains such as $\mathbb{R} \times \mathbb{T}$ has been resolved [19]. In the case of the whole space domain, the dissipation in one direction is insufficient to control the nonlinearity when we estimate the H^2-norm of u or the H^1-norm of ω. In fact, in the estimate of $\|\nabla \omega\|_{L^2}$,

$$\frac{d}{dt} \|\nabla \omega(t)\|_{L^2}^2 + 2\nu \|\partial_2 \nabla \omega(t)\|_{L^2}^2 = -2 \int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx,$$

the nonlinear part contains four component terms

$$\begin{aligned} \text{Hard} := & -\int_{\mathbb{R}^2} \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dx \\ = & -\int_{\mathbb{R}^2} \partial_1 u_1 \, (\partial_1 \omega)^2 \, dx - \int_{\mathbb{R}^2} \partial_1 u_2 \, \partial_1 \omega \, \partial_2 \omega \, dx \\ & -\int_{\mathbb{R}^2} \partial_2 u_1 \, \partial_1 \omega \, \partial_2 \omega \, dx - \int_{\mathbb{R}^2} \partial_2 u_2 \, (\partial_2 \omega)^2 \, dx \end{aligned} \tag{2.1.5}$$

and the first two terms in (2.1.5) do not admit any time-integrable upper bound. As a consequence, the best upper bound for the gradient of the vorticity $\|\nabla \omega(t)\|_{L^q}$ with $1 \le q \le \infty$ is double exponentially in time,

$$\|\nabla \omega(t)\|_{L^q} \le (\|\nabla \omega(0)\|_{L^q})^{e^{C\|\omega(0)\|_{L^\infty} t}}. \tag{2.1.6}$$

Indeed in the case of the 2D Euler equation in a unit disk, Kiselev and Sverak were able to construct an explicit vorticity solution whose gradient grows double exponentially [34]. Furthermore, classical approaches on the MHD well-posedness problem treat the magnetic field related terms as bad terms. As a consequence, the stability problem and large-time behavior concerned here in the classical framework appear to be hopeless.

The novel idea here is to treat the magnetic field related terms as good terms and to explore the smoothing and stabilizing effects of the magnetic field through coupling and interaction. In a previous work [25], the authors were successful in implementing this strategy to establish the stability of solutions to (2.1.2). For the sake of convenience of later references, we reproduce Theorem 1.3 from [25] here.

Theorem 2.1.1 (Theorem 1.3, [25]) *Let $\nu > 0$ and $\eta > 0$. Consider (2.1.2) with the initial data $(u_0, b_0) \in H^2(\mathbb{R}^2)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^2} + \|b_0\|_{H^2} \leq \varepsilon,$$

then (2.1.2) has a unique global classical solution (u, b) satisfying, for any $t > 0$,

$$\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \int_0^t \left(\|\partial_1 u\|_{L^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|b\|_{H^2}^2\right) \, d\tau \leq C\,\varepsilon^2$$

for some universal constant $C > 0$.

The goal of this paper is to give a precise account on the large-time behavior of these stable solutions. Clearly we need to continue to pursue the stabilizing and damping effect of the magnetic field. To do so, we combine the equations of u and b to derive an equivalent system of wave equations to reveal the stabilizing mechanism. We start by separating the linear terms in (2.1.2) from the nonlinear ones. Applying the Helmholtz-Leray projection operator

$$\mathbb{P} := I - \nabla\Delta^{-1}\nabla\cdot$$

to the velocity equation in (2.1.2), we eliminate the pressure to obtain

$$\partial_t u = \nu\partial_{22}u + \partial_1 b + N_1, \qquad N_1 = \mathbb{P}(-u \cdot \nabla u + b \cdot \nabla b). \tag{2.1.7}$$

By separating the linear terms from the nonlinear ones in (2.1.2), the equation of b can be written as

$$\partial_t b = -\eta b + \partial_1 u + N_2, \qquad N_2 = -u \cdot \nabla b + b \cdot \nabla u.$$

Thus, (2.1.2) can be written as

$$\begin{cases} \partial_t u = \nu\partial_{22}u + \partial_1 b + N_1, \\ \partial_t b = -\eta b + \partial_1 u + N_2, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x),\ b(x,0) = b_0(x). \end{cases} \tag{2.1.8}$$

Differentiating (2.1.8) in time and making several substitutions, we find

$$\begin{cases} \partial_{tt}u - (\nu\partial_{22} - \eta)\partial_t u - (\partial_{11}u + \eta\nu\partial_{22}u) = N_3, \\ \partial_{tt}b - (\nu\partial_{22} - \eta)\partial_t b - (\partial_{11}b + \eta\nu\partial_{22}b) = N_4, \end{cases} \tag{2.1.9}$$

where N_3 and N_4 are given by

$$N_3 = (\partial_t + \eta)N_1 + \partial_1 N_2, \qquad N_4 = (\partial_t - \nu\partial_{22})N_2 + \partial_1 N_1.$$

Surprisingly, both u and b are found to satisfy nonhomogeneous wave equations with exactly the same linear parts. Clearly, (2.1.9) exhibits much more regularization than its original counterpart in (2.1.2). The stabilizing and damping properties of (2.1.9) is a consequence of the background magnetic field and interactions within the MHD system. By exploiting these properties, we are able to establish the following theorem assessing the large-time behavior of the solutions of (2.1.2).

Theorem 2.1.2 *Let $\nu > 0$ and $\eta > 0$. Assume $(u_0, b_0) \in H^2 \cap L^1$ satisfies $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, $\|(u_0, b_0)\|_{H^2 \cap L^1} \leq \delta$ for sufficiently small $\delta > 0$. Let (u, b) be the corresponding solution obtained in Theorem 2.1.1, then, for a pure constant $c > 0$,*

$$\|(u(t), b(t))\|_{L^2} \leq c\delta(1+t)^{-\frac{1}{2}}, \tag{2.1.10}$$

$$\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \leq c\delta(1+t)^{-\frac{1}{2}},$$
$$\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} \leq c\delta(1+t)^{-1}. \tag{2.1.11}$$

In contrast to the potential double exponential growth rate in (2.1.6), Theorem 2.1.2 asserts that the solution of (2.1.2) actually decay algebraically in time. This result rigorously confirms the experimentally observed stabilizing and damping effect of the background magnetic field. The decay rates in (2.1.10) and (2.1.11) are the same as those for the fully dissipative heat equation, and reveal the stabilizing and damping effect of the magnetic field.

Theorem 2.1.2 is also mathematically important. It establishes the precise large-time behavior of a partially dissipated system. Many powerful classical methods designed for the large-time behavior of fully dissipated systems such as Schonbek's Fourier splitting scheme ([47, 48, 49]) may not apply to partially dissipated systems. The approach presented in this paper serves as a new method that work for some partially dissipated systems of partial differential equations.

Due to its physical applications and mathematical significance, the stability and large-time behavior problems on the MHD equations near a background magnetic field have recently attracted considerable interests. The stability problem on either the ideal MHD system or the fully dissipated MHD system with identical viscosity and resistivity has been thoroughly investigated and significant results have been obtained [3, 6, 8, 29]. The requirement that the viscosity coefficient be the same as the resistivity coefficient comes from the use of the Elsässer variables. [54] allows these two coefficients to be slightly different. The paper of Lin, Xu and Zhang [37] initiated the study on the stability problem of the 2D MHD system with only velocity dissipation. By using the Lagrangian approach and controlling all quantities in terms of the trajectory, they were able to establish the desired stability. The work of Ren, Xiang, Wu and Zhang [45] examined the stability and the large-time behavior simultaneously of the 2D MHD system without resistivity in an anisotropic Besov setting. The approach in [45] is Eulerian and establishes extensive anisotropic energy estimates. Instead of the velocity dissipation, Wu, Wu and Xu studied the stability of the 2D MHD system with only velocity damping and without resistivity [59]. Their paper exploits the wave structure of the system. More recent studies on the MHD stability problem focuses on the anisotropic MHD systems. The paper of Boardman, Lin and Wu [5] deals with the stability problem on the 2D MHD system with the fluid vorticity satisfying an Euler-like equation. Wu and Zhu established the stability of the 3D anisotropic MHD system with velocity dissipation in two directions and the magnetic diffusion in only one direction [60]. We remark that there are substantial recent developments on the well-posedness and stability problems on the MHD systems and many other important results are also available (see, e.g., [9, 10, 11, 18, 20, 21, 22, 26, 30, 31, 32, 33, 41, 36, 39, 40, 43, 46, 51, 53, 57, 58, 60, 63, 64, 65, 66, 67, 69, 70, 71, 72]). This list is by no means exhaustive.

We explain the main idea in the proof of Theorem 2.1.2. Clearly Theorem 2.1.2 can not be established via direct energy methods. Instead the approach here is to represent (2.1.2) in an integral form and then apply the bootstrapping argument. To convert (2.1.2) into an integral form, we first take the Fourier transform of (2.1.8) to obtain

$$\begin{cases} \partial_t \widehat{u} = -\nu \xi_2^2 \widehat{u} + i\xi_1 \widehat{b} + \widehat{N_1}, \\ \partial_t \widehat{b} = -\eta \widehat{b} + i\xi_1 \widehat{u} + \widehat{N_2}. \end{cases} \tag{2.1.12}$$

(2.1.12) can be written as a 2D system associated with a matrix A,

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} + \begin{pmatrix} \widehat{N_1} \\ \widehat{N_2} \end{pmatrix}.$$

where

$$A = \begin{pmatrix} -\nu\xi_2^2 & i\xi_1 \\ i\xi_1 & -\eta \end{pmatrix}.$$

By Duhamel's principle,

$$\begin{pmatrix} \widehat{u}(t) \\ \widehat{b}(t) \end{pmatrix} = e^{At} \begin{pmatrix} \widehat{u}_0 \\ \widehat{b}_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} \widehat{N_1(\tau)} \\ \widehat{N_2(\tau)} \end{pmatrix} d\tau, \tag{2.1.13}$$

The fundamental solution matrix e^{At} can be made more explicit via the eigenvalues and eigenvectors of A. In fact, if λ_1 and λ_2 are the roots of the characteristic polynomial associated with A,

$$\lambda^2 + (\eta + \nu\xi_2^2)\lambda + \xi_1^2 + \nu\eta\xi_2^2 = 0$$

or

$$\lambda_1 = \frac{-(\eta + \nu\xi_2^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2}$$

with

$$\Gamma = (\eta + \nu\xi_2^2)^2 - 4(\xi_1^2 + \nu\eta\xi_2^2),$$

then e^{At} can be written explicitly as

$$e^{At} = \begin{pmatrix} \widehat{K_1} & \widehat{K_2} \\ \widehat{K_2} & \widehat{K_3} \end{pmatrix},$$

where

$$
\begin{aligned}
\widehat{K_1} &= \eta \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \\
\widehat{K_2} &= i\xi_1 \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \\
\widehat{K_3} &= \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} - \eta \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}.
\end{aligned}
$$

Thus we have converted (2.1.2) into the integral form

$$
\begin{cases}
\widehat{u}(t) = \widehat{K_1}\widehat{u_0} + \widehat{K_2}\widehat{b_0} + \int_0^t \widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\ d\tau, \\
\widehat{b}(t) = \widehat{K_2}\widehat{u_0} + \widehat{K_3}\widehat{b_0} + \int_0^t \widehat{K_2}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{N_2}(\tau)\ d\tau.
\end{cases}
\tag{2.1.14}
$$

More technical details are provided in Proposition 2.2.1.

The next step is to extract the desired large-time decay estimates from the integral representation in (2.1.14). We use the bootstrapping argument (see, e.g., [52, p.21]). As a preparation, we first derive suitable upper bounds for the kernel functions. Clearly the kernel functions are anisotropic and frequency dependent. By dividing the frequency space $\mathbb{R}^2$ into suitable subsets, we are able to obtain definite upper bounds for the kernel functions in each subset. The details are given in Proposition 2.2.2. To implement the bootstrapping argument, we make the ansatz

$$
\begin{aligned}
\|(u(t), b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-\frac{1}{2}}, \\
\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-\frac{1}{2}}, \\
\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\le \tilde{c}\delta(1+t)^{-1},
\end{aligned}
\tag{2.1.15}
$$

where $\tilde{c}$ will be specified later. We show through the integral representation of $\widehat{u}$ and $\widehat{b}$ in (2.1.14) that

$$
\begin{aligned}
\|(u(t), b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}, \\
\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}, \\
\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\le \frac{\tilde{c}}{2}\delta(1+t)^{-1},
\end{aligned}
\tag{2.1.16}
$$

with the coefficients being half of the corresponding ones in (2.1.15). Then the bootstrapping argument implies that (2.1.16) holds for all $1 \le t < \infty$. The process of establishing upper bounds in (2.1.16) is very long and tedious, and the details are presented in three subsections in Section 2.3. We just want to mention some of the technical points. Due to the higher decay rate for the vertical derivative than the horizontal one, efforts have been made throughout to replace the horizontal derivatives by the vertical ones. One way to do so is to make use of the divergence-free condition, $\nabla \cdot u = \nabla \cdot b = 0$. Another helpful way is to invoke the anisotropic type inequalities such as

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_1 f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_2 f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}\|\partial_{12} f\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2)}.$$

These type of technicalities are used throughout the proof such as in (2.3.30) and many other places. The proof also employs many other helpful strategies such as dividing the time integral involving the nonlinear terms into two parts such as

$$\begin{aligned}&\int_0^t \|\widehat{K}_1(t-\tau)\,\widehat{u\cdot\nabla u}\|_{L^2(A_1)}\,d\tau\\ &= \int_0^{t/2} \|\widehat{K}_1(t-\tau)\,\widehat{u\cdot\nabla u}\|_{L^2(A_1)}\,d\tau + \int_{t/2}^t \|\widehat{K}_1(t-\tau)\,\widehat{u\cdot\nabla u}\|_{L^2(A_1)}\,d\tau.\end{aligned}$$

This division would help distinguish different properties of the integrand in different time intervals. The decay of the first piece relies on the kernel function while the decay of the second piece comes from the nonlinear term. We leave more technical details to Section 2.3.

The rest of this paper is divided into two main sections. Section 2.2 provides the details in the derivation of the integral representation (2.1.14). In addition, this section divides the frequency space $\mathbb{R}^2$ into suitable subdomains and establishes explicit upper bounds for the kernel functions in each subdomain. Section 2.3 presents the proof of Theorem 2.1.2 by applying the bootstrapping argument to (2.1.14). This is a very long and tedious process. For the sake of clarity, we divide this section into three subsections with each devoted to one of the inequalities in (2.1.16).

2.2 The integral representation and bounds for the kernels

This section details the derivation of the integral representation and establishes upper bounds for the kernel functions involved in the integral representation. These upper bounds will be used in the proof of Theorem 2.1.2. Proposition 2.2.1 and its proof are devoted to the integral representation while Proposition 2.2.2 focuses on the upper bounds for the kernel functions.

Proposition 2.2.1 *Let $\nu > 0$ and $\eta > 0$. Assume (u, b) is a solution of (2.1.2). Then (u, b) satisfies*

$$\begin{cases}\widehat{u}(t) = \widehat{K_1}\widehat{u_0} + \widehat{K_2}\widehat{b_0} + \displaystyle\int_0^t \widehat{K_1}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\ d\tau,\\ \widehat{b}(t) = \widehat{K_2}\widehat{u_0} + \widehat{K_3}\widehat{b_0} + \displaystyle\int_0^t \widehat{K_2}(t-\tau)\widehat{N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{N_2}(\tau)\ d\tau,\end{cases} \tag{2.2.1}$$

where the kernel functions $\widehat{K}_1$ through $\widehat{K}_3$ are given by

$$\begin{aligned}\widehat{K_1} &= \eta\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} := \eta\, G_1 + G_2,\\ \widehat{K_2} &= i\xi_1\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} := i\xi_1 G_1,\\ \widehat{K_3} &= \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} - \eta\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} := -\eta G_1 + G_3.\end{aligned}$$

with λ_1 *and* λ_2 *being the roots of*

$$\lambda^2 + (\eta + \nu\xi_2^2)\lambda + \xi_1^2 + \nu\eta\xi_2^2 = 0$$

or

$$\lambda_1 = \frac{-(\eta + \nu\xi_2^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2}, \quad \Gamma = (\eta + \nu\xi_2^2)^2 - 4(\xi_1^2 + \nu\eta\xi_2^2),$$

and G_1*,* G_2 *and* G_3 *given by*

$$G_1 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_2 = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_3 = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}.$$

In the case when $\lambda_1 = \lambda_2$ *or* $\Gamma = 0$*, the formulas of the kernel functions* $\widehat{K}_1$ *through* $\widehat{K}_3$ *are replaced by the corresponding limiting formulas*

$$\begin{aligned} \widehat{K_1} &= \eta \lim_{\lambda_2 \to \lambda_1} G_1 + \lim_{\lambda_2 \to \lambda_1} G_2 = \eta t\, e^{\lambda_1 t} + (1 + \lambda_1 t)e^{\lambda_1 t}, \\ \widehat{K_2} &= i\xi_1 t e^{\lambda_1 t}, \\ \widehat{K_3} &= -\eta t\, e^{\lambda_1 t} + (1 - \lambda_1 t)e^{\lambda_1 t}. \end{aligned} \tag{2.2.2}$$

Proof. As explained in the introduction, any solution (u, b) of (2.1.2) would solve (2.1.13), namely

$$\begin{pmatrix} \widehat{u}(t) \\ \widehat{b}(t) \end{pmatrix} = e^{At} \begin{pmatrix} \widehat{u}_0 \\ \widehat{b}_0 \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} \widehat{N_1(\tau)} \\ \widehat{N_2(\tau)} \end{pmatrix} d\tau \tag{2.2.3}$$

with

$$A = \begin{pmatrix} -\nu\xi_2^2 & i\xi_1 \\ i\xi_1 & -\eta \end{pmatrix}.$$

The characteristic polynomial of A is

$$\lambda^2 + (\eta + \nu\xi_2^2)\lambda + \xi_1^2 + \nu\eta\xi_2^2 = 0$$

and thus the eigenvalues of A are

$$\lambda_1 = \frac{-(\eta + \nu\xi_2^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2}, \quad \Gamma = (\eta + \nu\xi_2^2)^2 - 4(\xi_1^2 + \nu\eta\xi_2^2),$$

The eigenvectors corresponding to λ_1 and λ_2 are given by

$$v^{(1)} = \begin{pmatrix} \eta + \lambda_1 \\ i\xi_1 \end{pmatrix}, \qquad v^{(2)} = \begin{pmatrix} \eta + \lambda_2 \\ i\xi_1 \end{pmatrix},$$

respectively. Therefore,

$$\begin{aligned} A &= \begin{pmatrix} v^{(1)} & v^{(2)} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v^{(1)} & v^{(2)} \end{pmatrix}^{-1}. \\ e^{At} &= \frac{1}{i\xi_1(\lambda_1 - \lambda_2)} \begin{pmatrix} \eta + \lambda_1 & \eta + \lambda_2 \\ i\xi_1 & i\xi_1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} i\xi_1 & -(\eta + \lambda_2) \\ -i\xi_1 & \eta + \lambda_1 \end{pmatrix} \\ &:= \begin{pmatrix} \widehat{K_1} & \widehat{K_2} \\ \widehat{K_2} & \widehat{K_3} \end{pmatrix}. \end{aligned}$$

where

$$\widehat{K_1} = \eta \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2},$$
$$\widehat{K_2} = i\xi_1 \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2},$$
$$\widehat{K_3} = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} - \eta \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}.$$

To simplify the notation, we define

$$G_1 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_2 = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2}, \quad G_3 = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}$$

and write

$$e^{At} = \begin{pmatrix} \widehat{K_1} & \widehat{K_2} \\ \widehat{K_2} & \widehat{K_3} \end{pmatrix} = \begin{pmatrix} G_2 + \eta G_1 & i\xi_1 G_1 \\ i\xi_1 G_1 & G_3 - \eta G_1 \end{pmatrix}. \tag{2.2.4}$$

Inserting (2.2.4) in (3.2.5) yields (2.2.1). In the case when $\lambda_1 = \lambda_2$, the associated eigenvector of A is

$$v^{(1)} = \begin{pmatrix} \eta + \lambda_1 \\ i\xi_1 \end{pmatrix}$$

and the general solution of $\partial_t \widehat{V} = A\widehat{V}$ is given by

$$a_1 \, v^{(1)} \, e^{\lambda_1 t} + a_2 \, (v^{(1)} \, t + \sigma) e^{\lambda_1 t},$$

where a_1 and a_2 are to be determined by the initial data, and σ solves

$$(A - \lambda_1 I)\sigma = v^{(1)}.$$

After some simple computation, we find

$$\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We determine a_1 and a_2 by the initial data $\widehat{u}_0$ and $\widehat{b}_0$. This process leads to the kernel functions in (2.2.2) when $\lambda_1 = \lambda_2$. This completes the proof of Proposition 2.2.1. ■

The next proposition provides upper bounds for the kernel functions $\widehat{K}_1$ through $\widehat{K}_3$. It is clear that the kernel functions depend on the Fourier frequency and are anisotropic. Consequently we need to divide the frequency space $\mathbb{R}^2$ into suitable subsets so that the behavior of these kernel functions are definite. Our decomposition will be based on the second eigenvalue,

$$\lambda_2 = \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2}, \qquad \Gamma = (\eta + \nu\xi_2^2)^2 - 4(\xi_1^2 + \nu\eta\xi_2^2).$$

A natural choice is to separate the domain where λ_2 behaves like $-\frac{1}{4}(\eta+\nu\xi_2^2)$ from the rest. In particular, this occurs if

$$\sqrt{\Gamma} \le \frac{1}{2}(\eta+\nu\xi_2^2) \quad \text{or} \quad \nu\eta\xi_2^2+\xi_1^2 \ge \frac{3}{16}(\eta+\nu\xi_2^2)^2.$$

This explains the decomposition in the following proposition.

Proposition 2.2.2 *Let $\nu > 0$ and $\eta > 0$. We decompose $\mathbb{R}^2$ into two subsets A_1 and A_2 with*

$$A_1 = \{\xi \in \mathbb{R}^2,\ \nu\eta\xi_2^2+\xi_1^2 \ge \frac{3}{16}(\eta+\nu\xi_2^2)^2\},$$
$$A_2 = \{\xi \in \mathbb{R}^2,\ \nu\eta\xi_2^2+\xi_1^2 < \frac{3}{16}(\eta+\nu\xi_2^2)^2\}.$$

A_2 is further divided into A_{21} and A_{22} with

$$\begin{aligned} A_{21} &= \{\xi \in \mathbb{R}^2,\ \xi \in A_2,\ \nu\xi_2^2 \le \eta\},\\ A_{22} &= \{\xi \in \mathbb{R}^2,\ \xi \in A_2,\ \nu\xi_2^2 > \eta\}. \end{aligned} \tag{2.2.5}$$

Then

1. *For any $\xi \in A_1$, there is $c_0 > 0$ and $C > 0$ such that*

$$|\widehat{K_1}|,\ |\widehat{K_2}|,\ |\widehat{K_3}| \le C\, e^{-c_0(1+\xi_2^2)t}$$

2. *For any $\xi \in A_{21}$, there is $c_0 > 0$ and $C > 0$ such that*

$$|\widehat{K_1}|,\ |\widehat{K_2}|,\ |\widehat{K_3}| \le C\left(e^{-c_0(1+\xi_2^2)t} + e^{-c_0|\xi|^2 t}\right).$$

3. *For any $\xi \in A_{22}$, there is $c_0 > 0$ and $C > 0$ such that*

$$|\widehat{K_1}|,\ |\widehat{K_2}|,\ |\widehat{K_3}| \le C\left(e^{-c_0(1+\xi_2^2)t} + e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\right).$$

Proof of Proposition 2.2.2. We start with the case when $\xi \in A_1$. For any $\xi \in A_1$,

$$\Gamma = (\eta+\nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2+\xi_1^2) \le (\eta+\nu\xi_2^2)^2 - \frac{3}{4}(\eta+\nu\xi_2^2)^2 = \frac{1}{4}(\eta+\nu\xi_2^2)^2.$$

Therefore, either $\sqrt{\Gamma}$ is pure imaginary or $\sqrt{\Gamma} \le \frac{1}{2}(\eta+\nu\xi_2^2)$. Hence, the real parts $\Re(\lambda_1)$ and $\Re(\lambda_2)$ are bounded by

$$\Re(\lambda_1) \le -\frac{1}{2}(\eta+\nu\xi_2^2), \qquad \Re(\lambda_2) \le -\frac{1}{4}(\eta+\nu\xi_2^2).$$

To bound $\widehat{K_1}$, $\widehat{K_2}$ and $\widehat{K_3}$, we realize that they all involve only λ_1, λ_2 and G_1. In fact, since G_2 and G_3 can be written as

$$G_2 = \frac{\lambda_1 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} + \lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = e^{\lambda_2 t} + \lambda_1 G_1$$
$$G_3 = \frac{(\lambda_1 - \lambda_2) e^{\lambda_2 t} + \lambda_2 (e^{\lambda_2 t} - e^{\lambda_1 t})}{\lambda_1 - \lambda_2} = e^{\lambda_2 t} - \lambda_2 G_1,$$

we have

$$\widehat{K}_1 = e^{\lambda_2 t} + \lambda_1 G_1 + \eta G_1, \quad \widehat{K_2} = i\xi_1 G_1, \quad \widehat{K_3} = e^{\lambda_2 t} - \lambda_2 G_1 - \eta G_1. \tag{2.2.6}$$

When $\Gamma > 0$, both λ_1 and λ_2 are real. Then the mean-value theorem implies that there is $\frac{1}{4} < a < 1$ such that

$$G_1 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = t\, e^{-a(\eta + \nu \xi_2^2) t}.$$

When $\Gamma = 0$, $\lambda_1 = \lambda_2$ and (2.2.2) implies that G_1 is replaced by

$$G_1 = t\, e^{\lambda_1 t}.$$

When $\Gamma < 0$, both λ_1 and λ_2 are imaginary and

$$G_1 = e^{-\frac{1}{2}(\eta + \nu \xi_2^2) t}\ \frac{i \sin(-\frac{\sqrt{\Gamma}}{2} t)}{\frac{\sqrt{\Gamma}}{2}}$$

Therefore we always have

$$|G_1| \le t e^{-\frac{1}{4}(\eta + \nu \xi_2^2) t}.$$

Using the simple fact that $\rho e^{-C_1 \rho} \le C_2$ for any$\rho \ge 0$ and $C_1 > 0$ and suitable $C_2 > 0$, we have, for $c_0 > 0$ and $C > 0$,

$$|\widehat{K_1}|, |\widehat{K_3}| \le e^{-\frac{1}{4}(\eta + \nu \xi_2^2) t} + 2(\eta + \nu \xi_2^2) t e^{-\frac{1}{4}(\eta + \nu \xi_2^2) t} \le C e^{-c_0 (1 + \xi_2^2) t}.$$

To bound $\widehat{K_2}$, we divide the consideration into two cases:

$$\frac{|\xi_1|}{|\sqrt{\Gamma}|} \le 1 \quad \text{and} \quad \frac{|\xi_1|}{|\sqrt{\Gamma}|} > 1.$$

When $\frac{|\xi_1|}{|\sqrt{\Gamma}|} \le 1$, we write, due to $\lambda_1 - \lambda_2 = -\sqrt{\Gamma}$,

$$|\widehat{K_2}| \le \frac{|\xi_1|}{|\sqrt{\Gamma}|} (|e^{\lambda_1 t}| + |e^{\lambda_2 t}|) \le e^{-\frac{1}{2}(\eta + \nu \xi_2^2) t} + e^{-\frac{1}{4}(\eta + \nu \xi_2^2) t} \le C\, e^{-\frac{1}{4}(1 + \xi_2^2) t}.$$

When $\frac{|\xi_1|}{|\sqrt{\Gamma}|} > 1$, then

$$|(\eta + \nu \xi_2^2)^2 - 4(\nu \eta \xi_2^2 + \xi_1^2)| \le \xi_1^2$$

which is equivalent to

$$0 \le (\eta + \nu \xi_2^2)^2 - 4(\nu \eta \xi_2^2 + \xi_1^2) \le \xi_1^2 \tag{2.2.7}$$

or

$$0 \le 4(\nu\eta\xi_2^2 + \xi_1^2) - (\eta + \nu\xi_2^2)^2 \le \xi_1^2. \tag{2.2.8}$$

Clearly, (2.2.7) implies

$$-(\eta + \nu\xi_2^2)^2 \le -4(\nu\eta\xi_2^2 + \xi_1^2) \le -4\xi_1^2$$

while (2.2.8) yields

$$-(\eta + \nu\xi_2^2)^2 \le -4(\nu\eta\xi_2^2 + \xi_1^2) + \xi_1^2 \le -3\xi_1^2.$$

In either case, we have, for $c > 0$

$$-(\eta + \nu\xi_2^2) \le -c\,|\xi_1|.$$

Therefore,

$$\begin{aligned}|\widehat{K_2}| &\le |\xi_1| t e^{-a(\eta+\nu\xi_2^2)t} = |\xi_1| t e^{-\frac{a}{2}(\eta+\nu\xi_2^2)t} e^{-\frac{a}{2}(\eta+\nu\xi_2^2)t} \\ &\le |\xi_1| t e^{-\frac{a}{2}c|\xi_1|t} e^{-\frac{a}{2}(\eta+\nu\xi_2^2)t} \le C e^{-c_0(1+\xi_2^2)t} \quad \text{for } c_0 > 0 \text{ and } C > 0.\end{aligned}$$

We now turn to the case when $\xi \in A_2$. For $\xi \in A_2$,

$$\lambda_1 \le -\frac{1}{2}(\eta + \nu\xi_2^2).$$

By $\Gamma = (\eta + \nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2 + \xi_1^2) \le (\eta + \nu\xi_2^2)^2$,

$$\begin{aligned}\lambda_2 &= \frac{-(\eta + \nu\xi_2^2) + \sqrt{\Gamma}}{2} = \frac{(-(\eta + \nu\xi_2^2) + \sqrt{\Gamma})(-(\eta + \nu\xi_2^2) - \sqrt{\Gamma})}{-2((\eta + \nu\xi_2^2) + \sqrt{\Gamma})} \\ &= \frac{2(\nu\eta\xi_2^2 + \xi_1^2)}{-(\eta + \nu\xi_2^2 + \sqrt{\Gamma})} \le \frac{-2(\nu\eta\xi_2^2 + \xi_1^2)}{2(\eta + \nu\xi_2^2)} = -\frac{\nu\eta\xi_2^2 + \xi_1^2}{\eta + \nu\xi_2^2}.\end{aligned}$$

Since $\Gamma = (\eta + \nu\xi_2^2)^2 - 4(\nu\eta\xi_2^2 + \xi_1^2) \ge (\eta + \nu\xi_2^2)^2 - \frac{3}{4}(\eta + \nu\xi_2^2)^2 \ge \frac{1}{4}(\eta + \nu\xi_2^2)^2$, we obtain $\sqrt{\Gamma} \ge \frac{1}{2}(\eta + \nu\xi_2^2)$. It follows that

$$|G_1| = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\sqrt{\Gamma}} \le \frac{2}{\eta + \nu\xi_2^2}\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right). \tag{2.2.9}$$

Furthermore, for $\xi \in A_2$, we have

$$\xi_1^2 \le \frac{3}{16}(\eta + \nu\xi_2^2)^2 \quad \text{or} \quad \frac{|\xi_1|}{\eta + \nu\xi_2^2} \le \frac{\sqrt{3}}{4}$$

and thus

$$|\widehat{K_2}| \le |\xi_1||G_1| \le C\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right).$$

In addition, by (2.2.6),

$$|\widehat{K_1}| \le e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t} + 4\frac{\eta+\nu\xi_2^2}{\eta+\nu\xi_2^2}\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right)$$
$$\le C\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right).$$

$|\widehat{K_3}|$ admits the same upper bound. By further using the definitions of A_{21} and A_{22} in (2.2.5), we obtain the desired upper bounds. This completes the proof of Proposition 2.2.2. ■

2.3 Large-time Decay Estimates

This section is devoted to the proof of Theorem 2.1.2. The framework of the proof is the bootstrapping argument. The proof involves the estimates of many terms and is a long and tedious process. It will be divided into three subsections after we present several tool lemmas.

We need several basic tool lemmas. The first one provides the $L^p - L^q$ estimate for a general fractional Laplacian heat operator $e^{\nu t\Lambda^\alpha}$. The fractional Laplacian operator Λ^α with $\alpha \in \mathbb{R}$ is defined via the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

The proof of this $L^p - L^q$ estimate can be found in many references (see, e.g., [56]).

Lemma 2.3.1 *Let $\alpha > 0$, $\beta \ge 0$ and $1 \le p \le q \le \infty$. There is a constant $C > 0$ such that for $t > 0$,*

$$\|\Lambda^\beta e^{-c_0\Lambda^\alpha t} f\|_{L^q(\mathbb{R}^d)} \le C\, t^{-\frac{\beta}{\alpha}-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

The next lemma presents an 1D Sobolev inequality involving fractional derivatives. This 1D inequality is at the core of many higher dimensional anisotropic Sobolev inequalities. The proof of this lemma can be found in [68].

Lemma 2.3.2 *Assume that f is in $L^q(\mathbb{R})$,*

$$\|f\|_{L^q(\mathbb{R})} \le C\|f\|_{L^2}^{1-\frac{1}{s}(\frac{1}{2}-\frac{1}{q})}\|\Lambda^s f\|_{L^2}^{\frac{1}{s}(\frac{1}{2}-\frac{1}{q})},$$

where $2 \le q \le \infty$ and $\frac{1}{s}(\frac{1}{2}-\frac{1}{q}) \le 1$.

Anisotropic Sobolev inequalities have become a necessary tool in the study of anisotropic equations. The next lemma states a 2D anisotropic inequality, which can be seen as a consequence of the previous lemma.

Lemma 2.3.3 *The following estimates hold when the right-hand sides are all bounded.*

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

For the convenience of later reference, we also provide two standard inequalities. The first one is a Sobolev inequality while the second one is a calculus inequality on the fractional derivative of a product.

Lemma 2.3.4 *Assume that* $f \in L^q(\mathbb{R}^2)$ *with* $2 < q < \infty$. *Then*

$$\|f\|_{L^q} \leq C\|f\|_{L^2}^{\frac{2}{q}} \|\nabla f\|_{L^2}^{1-\frac{2}{q}}.$$

Lemma 2.3.5 *For any* $s > 0$, *then for all* $f, g \in H^s \cap L^\infty$, *and we have the estimates*

$$\|\Lambda^s(fg)\|_{L^p} \leq C\left(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_3}}\|\Lambda^s g\|_{L^{p_4}}\right),$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. *and* $p, p_2, p_3 \in (1, \infty)$. *In particular,*

$$\|\Lambda^s(fg)\|_{L^2} \leq C\left(\|\Lambda^s f\|_{L^2}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\Lambda^s g\|_{L^2}\right).$$

We are ready to prove Theorem 2.1.2.

Proof of Theorem 2.1.2. We prove Theorem 2.1.2 by the bootstrapping argument. We make the ansatz, for $1 \leq t < T$,

$$\begin{aligned} \|(u(t), b(t))\|_{L^2} &\leq \tilde{c}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\leq \tilde{c}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\leq \tilde{c}\delta(1+t)^{-1}. \end{aligned} \tag{2.3.1}$$

where $\tilde{c}$ will be specified later. We show through the integral representation of $\widehat{u}$ and $\widehat{b}$ in (2.1.14) that

$$\begin{aligned} \|(u(t), b(t))\|_{L^2} &\leq \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} &\leq \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}, \\ \|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} &\leq \frac{\tilde{c}}{2}\delta(1+t)^{-1}. \end{aligned} \tag{2.3.2}$$

Since the coefficients in (2.3.2) are just half of those in (2.3.1), the bootstrapping argument then implies (2.3.2) holds for all $1 \leq t < \infty$.

The main efforts are devoted to the inequalities in (2.3.2). This process involves the estimates of many terms and is very long. For the sake of clarity, we divide the rest of this section into three subsections with each subsection devoted to one of the inequalities in (2.3.2).

2.3.1 Estimates of $\|(u(t), b(t))\|_{L^2}$

This subsection proves the first inequality in (2.3.2). To estimate $\|(u(t), b(t))\|_{L^2(\mathbb{R}^2)}$, we estimate it in the three subdomains A_1, A_{21} and A_{22} defined in Proposition 2.2.2. By (2.1.14),

$$
\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_1)} \leq \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_1)} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_1)} + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_1)}\, d\tau \\
+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2(A_1)}\, d\tau \\
:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

By Part (1) in Proposition 2.2.2,

$$
\begin{aligned}
I_1 &= \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_1)} \leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(A_1)} \\
&\leq Ce^{-c_0 t}\|\widehat{u_0}\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2},
\end{aligned}
\tag{2.3.3}
$$

where we have used $e^{-c_0 t} \leq C(1+t)^{-\frac{1}{2}}$ for $t \geq 0$. Similarly,

$$
I_2 = \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_1)} \leq C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2}. \tag{2.3.4}
$$

Noticing that $\widehat{N_1} = (I - \frac{\xi\otimes\xi}{|\xi|^2})(-\widehat{u\cdot\nabla u} + \widehat{b\cdot\nabla b})$ and using the boundedness of the Riesz transform on L^2, we have

$$
\begin{aligned}
I_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_1)}\, d\tau \\
&\leq C\int_0^t \left(\|\widehat{K_1}\widehat{u\cdot\nabla u}\|_{L^2(A_1)} + \|\widehat{K_1}\widehat{b\cdot\nabla b}\|_{L^2(A_1)}\right) d\tau \\
&= I_{3,1} + I_{3,2}.
\end{aligned}
$$

$I_{3,1}$ is further decomposed into two parts,

$$
\begin{aligned}
I_{3,1} \leq C\int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla u}(\tau)\|_{L^2(A_1)}\, d\tau \\
+ C\int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla u}(\tau)\|_{L^2(A_1)}\, d\tau \\
= I_{3,1,1} + I_{3,1,2}.
\end{aligned}
$$

By Proposition 2.2.2, Hölder's inequality and Ladyzhenskaya's inequality,

$$
\begin{aligned}
I_{3,1,1} &\leq C\int_0^{t/2} e^{-c_0(t-\tau)}\|u\cdot\nabla u(\tau)\|_{L^2}\ d\tau \leq Ce^{-c_0\frac{t}{2}}\int_0^{t/2}\|u\|_{L^4}\|\nabla u\|_{L^4}\ d\tau \\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{1/2}\ d\tau \\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{1}{2}}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})(c\delta)^{\frac{1}{2}}\ d\tau \\
&\leq C\tilde{c}^{3/2}\delta^2 e^{-\frac{c_0}{2}t}\int_0^{t/2}(1+\tau)^{-3/4}\ d\tau \leq C\tilde{c}^{3/2}\delta^2 e^{-\frac{c_0 t}{2}}(1+t)^{1/4} \\
&\leq C\tilde{c}^{3/2}\delta^2(1+t)^{-1/2},
\end{aligned}
$$

where we have used the ansatz in (2.3.1) and the fact that $\|u\|_{H^2}\leq c\delta$. In addition, in the last step, we have used $e^{-\frac{c_0 t}{2}}\leq C(1+t)^{-3/4}$ for $C>0$. We estimate $I_{3,1,2}$.

$$
\begin{aligned}
I_{3,1,2} &= C\int_{t/2}^{t}\|\widehat{K_1}(t-\tau)\widehat{u\cdot\nabla u}(\tau)\|_{L^2(A_1)}\ d\tau \\
&\leq C\int_{t/2}^{t}\|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2(A_1)}\ d\tau \\
&\leq C\int_{t/2}^{t}e^{-c_0(t-\tau)}\|e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2(A_1)}\ d\tau \qquad (2.3.5) \\
&\leq C\int_{t/2}^{t}e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{3}{2}}\ d\tau \\
&\leq C\int_{t/2}^{t}e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\tilde{c}^2\delta^2(1+\tau)^{-\frac{1}{4}-\frac{3}{4}}\ d\tau \leq C\tilde{c}^2\delta^2(1+t)^{-1},
\end{aligned}
$$

where we have used $\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{4}}\, d\tau = C$ for $C > 0$ in the last inequality of (2.3.5), and invoked the following estimate in the fourth inequality of (2.3.5),

$$\begin{aligned}
\|e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2(A_1)}^2 &= \|\|e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^2_{\xi_2}}\|_{L^2_{\xi_1}}^2\\
&= \int\int |e^{-c_0\xi_2^2(t-\tau)}\widehat{u\cdot\nabla u}(\xi,\tau)|^2\, d\xi_2\, d\xi_1\\
&\le \int\int |e^{-c_0\xi_2^2(t-\tau)}|^2\, d\xi_2 \|\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^\infty_{\xi_2}}^2\, d\xi_1\\
&= \int\int (t-\tau)^{-\frac{1}{2}} e^{-c_0\eta_2^2} d\eta_2 \|\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^\infty_{\xi_2}}^2\, d\xi_1\\
&= C(t-\tau)^{-\frac{1}{2}} \int \|\widehat{u\cdot\nabla u}(\xi,\tau)\|_{L^\infty_{\xi_2}}^2\, d\xi_1\\
&\le C(t-\tau)^{-\frac{1}{2}} \|u\cdot\nabla u\|_{L^1_{x_2}L^2_{x_1}}^2\\
&\le C(t-\tau)^{-\frac{1}{2}} \|\|u\|_{L^2_{x_2}}\|\nabla u\|_{L^2_{x_2}}\|_{L^2_{x_1}}^2\\
&\le C(t-\tau)^{-\frac{1}{2}} \|u\|_{L^2_{x_2}L^\infty_{x_1}}^2 \|\nabla u\|_{L^2}^2\\
&\le C(t-\tau)^{-\frac{1}{2}} \|u\|_{L^2}\|\partial_1 u\|_{L^2}\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Similarly,

$$I_{3,2} \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}.$$

Therefore, for a constant $C > 0$,

$$I_3 \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.6}$$

By invoking N_2 in (2.1.7) and going through a very similar process, we have

$$I_4 \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.7}$$

Combining (2.3.3), (2.3.4), (2.3.6) and (2.3.7) yields

$$\|\widehat{u(t)}\|_{L^2(A_1)} \le C(1+t)^{-\frac{1}{2}}(\|u_0\|_{L^2} + \|b_0\|_{L^2}) + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.8}$$

We now turn to $\|(u(t), b(t))\|_{L^2(A_{21})}$. By (2.1.14),

$$\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_{21})} \le\ & \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{21})} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_{21})}\\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{21})}\, d\tau\\
&+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2(A_{21})}\, d\tau\\
:=\ & J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Part (2) in Proposition 2.2.2 and Lemma 2.3.1,

$$\begin{aligned}
J_1 = \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{21})} &\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(A_{21})} + C\|e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2(A_{21})} \\
&\le Ce^{-c_0 t}\|\widehat{u_0}\|_{L^2(\mathbb{R}^2)} + C\|e^{c_0\Delta t}u_0\|_{L^2(\mathbb{R}^2)} \\
&\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2(\mathbb{R}^2)} + Ct^{-\frac{2}{2}(1-\frac{1}{2})}\|u_0\|_{L^1(\mathbb{R}^2)} \\
&\le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2\cap L^1}.
\end{aligned}$$

where we have used $e^{-c_0 t} \le C(1+t)^{-\frac{1}{2}}$ for $t \ge 0$. Similarly,

$$J_2 \le C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2\cap L^1}.$$

By Proposition 2.2.2,

$$\begin{aligned}
J_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{21})}\, d\tau \\
&\le C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{b\cdot\nabla b}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{b\cdot\nabla b}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&= J_{3,1} + J_{3,2} + J_{3,3} + J_{3,4}.
\end{aligned}$$

By (2.3.6), for $C > 0$,

$$J_{3,1} + J_{3,2} \le C(\tilde{c}^2 + \tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}.$$

We further decompose $J_{3,3}$ as

$$\begin{aligned}
J_{3,3} &= C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_{t/2}^t \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&= J_{3,3,1} + J_{3,3,2}.
\end{aligned}$$

By Lemma 2.3.1, and the ansatz (2.3.1),

$$
\begin{aligned}
J_{3,3,1} &= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}|\xi||\widehat{u\otimes u}|(\xi,\tau)|\|_{L^2}\ d\tau \\
&\le C\int_0^{t/2} (t-\tau)^{-\frac{1}{2}-\frac{2}{2}(1-\frac{1}{2})}\|u\otimes u\|_{L^1}\ d\tau \\
&\le C(t/2)^{-1}\int_0^{t/2} \|u\otimes u\|_{L^1}\ d\tau \\
&\le Ct^{-1}\int_0^{t/2} \|u\|_{L^2}^2 d\tau \\
&\le Ct^{-1}\int_0^{t/2} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^2 d\tau \\
&\le C\tilde{c}^2\delta^2 t^{-1}\ln(1+t/2) \le C(\sigma)\tilde{c}^2\delta^2 t^{-1+\sigma},
\end{aligned}
$$

where we have used $t^{-\sigma}\ln(1+t/2) \le C(\sigma)$ for $\sigma > 0$ and for all $t \ge 1$. By Lemma 2.3.1, the ansatz (2.3.1) and Hölder's inequality,

$$
\begin{aligned}
J_{3,3,2} &= C\int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\cdot\nabla u}|(\xi,\tau)|\|_{L^2}\ d\tau \\
&\le C\int_{t/2}^{t} (t-\tau)^{-\frac{2}{2}(1-\frac{1}{2})}\|u\cdot\nabla u\|_{L^1}\ d\tau \\
&\le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}\|u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\ d\tau \\
&\le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}\tilde{c}\delta(1+\tau)^{-\frac{1}{2}}\tilde{c}\delta(1+\tau)^{-\frac{1}{2}}\ d\tau \\
&\le C\tilde{c}^2\delta^2\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}\ d\tau \\
&\le C\tilde{c}^2\delta^2(1+t/2)^{-1}\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}}\ d\tau \\
&\le C\tilde{c}^2\delta^2(1+t)^{-1}(t/2)^{1/2} \\
&\le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}.
\end{aligned}
$$

$J_{3,4}$ admits the same upper bound as $J_{3,3}$,

$$
J_{3,4} \le C(\sigma)\tilde{c}^2\delta^2 t^{-1+\sigma} + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}.
$$

J_4 admits the same bound as J_3. By taking σ sufficiently small, say $\sigma < \frac{1}{2}$, we have

$$
\|\widehat{u}(t)\|_{L^2(A_{21})} \le C(1+t)^{-\frac{1}{2}}\left(\|u_0\|_{L^2\cap L^1} + \|b_0\|_{L^2\cap L^1}\right) + C(\tilde{c}^2+\tilde{c}^{3/2})\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.9}
$$

We estimate $\|\widehat{u}\|_{L^2(A_{22})}$. By (2.1.14),

$$\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_{22})} &\le \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{22})} + \|\widehat{K_1}(t)\widehat{b_0}\|_{L^2(A_{22})} \\
&\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{22})}\,d\tau \\
&\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\|_{L^2(A_{22})}\,d\tau \\
&:= M_1 + M_2 + M_3 + M_4.
\end{aligned}$$

By Part (3) in Proposition 2.2.2,

$$\begin{aligned}
M_1 &= \|\widehat{K_1}(t)\widehat{u_0}\|_{L^2(A_{22})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(A_{22})} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{u_0}\|_{L^2(A_{22})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{u_0}\|_{L^2(\mathbb{R}^2)} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{u_0}\|_{L^2(\mathbb{R}^2)} \\
&\le Ce^{-c_0 t}\|u_0\|_{L^2} \le C(1+t)^{-\frac{1}{2}}\|u_0\|_{L^2}.
\end{aligned}$$

Similarly,

$$M_2 \le C(1+t)^{-\frac{1}{2}}\|b_0\|_{L^2}.$$

By Proposition 2.2.2,

$$\begin{aligned}
M_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2(A_{22})}\,d\tau \\
&\le C\int_0^t \|(e^{-c_0(1+\xi_2^2)(t-\tau)} + e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)})(|\widehat{u\cdot\nabla u}(\tau)| + |\widehat{b\cdot\nabla b}(\tau)|)\|_{L^2}\,d\tau \\
&\le C\int_0^t e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u(\tau)\|_{L^2} + \|b\cdot\nabla b(\tau)\|_{L^2}\right)\,d\tau \\
&\le C\int_0^{t/2} e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u\|_{L^2} + \|b\cdot\nabla b\|_{L^2}\right)\,d\tau \\
&\quad + C\int_{t/2}^t e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u\|_{L^2} + \|b\cdot\nabla b\|_{L^2}\right)\,d\tau = M_{3,1} + M_{3,2}.
\end{aligned}$$

We set

$$M_{3,1} = M_{3,1,1} + M_{3,1,2}.$$

For $2 < q < \infty$ and $\tilde{q}$ satisfying $\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{1}{2}$, we have, by Lemma 2.3.4,

$$\|u\|_{L^q} \le C\|u\|_{L^2}^{\frac{2}{q}}\|\nabla u\|_{L^2}^{1-\frac{2}{q}}, \qquad \|\nabla u\|_{L^{\tilde{q}}} \le C\|\nabla u\|_{L^2}^{1-\frac{2}{q}}\|\Delta u\|_{L^2}^{\frac{2}{q}}. \tag{2.3.10}$$

and thus

$$\begin{aligned}
M_{3,1,1} &\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\cdot\nabla u\|_{L^2(\mathbb{R}^2)}\ d\tau \leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\|_{L^q}\|\nabla u\|_{L^{\tilde{q}}}\ d\tau\\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}\|u\|_{L^2}^{\frac{2}{q}}\|\nabla u\|_{L^2}^{2(1-\frac{2}{q})}\|\Delta u\|_{L^2}^{\frac{2}{q}}d\tau\\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2}(\tilde{c}\delta(1+\tau)^{-1/2})^{2/q}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{2(1-2/q)}(c\delta)^{2/q}\ d\tau\\
&\leq Ce^{-\frac{c_0}{2}t}\tilde{c}^{2-2/q}\delta^2\int_0^{t/2}(1+\tau)^{-1+1/q}d\tau \leq C\tilde{c}^{2-2/q}\delta^2(1+t)^{-1/2},
\end{aligned}$$

where we have used $(1+t)^{\frac{1}{q}}e^{-\frac{c_0}{2}t} \leq C(1+t)^{-1/2}$. Similarly, $M_{3,1,2}$ obeys the same bound,

$$M_{3,1} \leq C\tilde{c}^{2-2/q}\delta^2(1+t)^{-1/2}.$$

$M_{3,2}$ is naturally divided into two parts,

$$M_{3,2} \leq C\int_{t/2}^t e^{-c_0(t-\tau)}\left(\|u\cdot\nabla u\|_{L^2} + \|b\cdot\nabla b\|_{L^2}\right)\ d\tau := M_{3,2,1} + M_{3,2,2}.$$

By Hölder's inequality and (2.3.10),

$$\begin{aligned}
M_{3,2,1} &\leq C\int_{t/2}^t e^{-c_0(t-\tau)}\|u(\tau)\|_{L^q}\|\nabla u(\tau)\|_{L^{\tilde{q}}}\ d\tau\\
&\leq C\int_{t/2}^t e^{-c_0(t-\tau)}\|u\|_{L^2}^{\frac{2}{q}}\|\nabla u\|_{L^2}^{2(1-\frac{2}{q})}\|\Delta u\|_{L^2}^{\frac{2}{q}}d\tau\\
&\leq C\int_{t/2}^t e^{-c_0(t-\tau)}(\tilde{c}\delta(1+\tau)^{-1/2})^{2/q}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{2(1-2/q)}(c\delta)^{2/q}\ d\tau\\
&\leq C\tilde{c}^{2-2/q}\delta^2(1+t/2)^{-1+1/q}\int_{t/2}^t e^{-c_0(t-\tau)}\ d\tau\\
&\leq C\tilde{c}^{2-2/q}\delta^2(1+t/2)^{-1+1/q}.
\end{aligned}$$

By taking $q=3$, we obtain

$$M_{3,2,1} \leq C\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.$$

$M_{3,2,2}$ admits the same bound,

$$M_3 \leq C\tilde{c}^2\delta^2(1+t)^{-1/2} + C\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.$$

Similarly, M_4 obeys the smae upper bound. Therefore,

$$\begin{aligned}
\|\widehat{u}(t)\|_{L^2(A_{22})} &\leq C(1+t)^{-\frac{1}{2}}(\|u_0\|_{L^2} + \|b_0\|_{L^2})\\
&\quad + C\tilde{c}^2\delta^2(1+t)^{-1/2} + C\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}. \qquad (2.3.11)
\end{aligned}$$

By (2.3.8), (2.3.9) and (2.3.11),

$$\begin{aligned}\|u(t)\|_{L^2} &\le C_1(1+t)^{-\frac{1}{2}}\|(u_0,b_0)\|_{L^1\cap L^2}\\ &\quad + C_2\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}} + C_3\tilde{c}^{3/2}\delta^2(1+t)^{-\frac{1}{2}} + C_4\tilde{c}^{4/3}\delta^2(1+t)^{-2/3}.\end{aligned} \tag{2.3.12}$$

Therefore, if we choose $\tilde{c}$ and δ satisfying

$$C_1 \le \frac{\tilde{c}}{8}, \qquad C_2\tilde{c}\delta \le \frac{1}{32}, \qquad C_3\tilde{c}^{\frac{1}{2}}\delta \le \frac{1}{32}, \qquad C_4\tilde{c}^{\frac{1}{3}}\delta \le \frac{1}{16},$$

then (2.3.12) implies

$$\begin{aligned}\|u(t)\|_{L^2} &\le \frac{\tilde{c}}{8}\delta(1+t)^{-\frac{1}{2}} + \frac{\tilde{c}}{16}\delta(1+t)^{-\frac{1}{2}} + \frac{\tilde{c}}{16}\delta(1+t)^{-\frac{1}{2}}\\ &= \frac{\tilde{c}}{4}\delta(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{2.3.13}$$

Similarly, $\|b\|_{L^2}$ obeys the same bound. Therefore,

$$\|(u(t),b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}.$$

This completes the proof of the first inequality in (2.3.2).

2.3.2 Estimates of $\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2}$

The goal of this subsection is to prove the third inequality in (2.3.2), namely

$$\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\delta(1+t)^{-1}.$$

Applying ∂_2 to (2.1.14),

$$\begin{cases}\widehat{\partial_2 u}(t) = \widehat{K_1}\widehat{\partial_2 u_0} + \widehat{K_2}\widehat{\partial_2 b_0} + \displaystyle\int_0^t \widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\, d\tau\\ \widehat{\partial_2 b}(t) = \widehat{K_2}\widehat{\partial_2 u_0} + \widehat{K_3}\widehat{\partial_2 b_0} + \displaystyle\int_0^t \widehat{K_2}(t-\tau)\widehat{\partial_2 N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{\partial_2 N_2}(\tau)\, d\tau.\end{cases} \tag{2.3.14}$$

We estimate $\|\partial_2 u\|_{L^2(A_1)}$, $\|\partial_2 u\|_{L^2(A_{21})}$ and $\|\partial_2 u\|_{L^2(A_{22})}$. We start with $\|\partial_2 u\|_{L^2(A_1)}$. By (2.3.14),

$$\begin{aligned}\|\widehat{\partial_2 u}(t)\|_{L^2(A_1)} &\le \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_1)} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_1)}\\ &\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_1)}\, d\tau\\ &\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_1)}\, d\tau\\ &:= O_1 + O_2 + O_3 + O_4.\end{aligned}$$

By Proposition 2.2.2,

$$O_1 \le \|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} \le e^{-c_0 t}\|\partial_2 u_0\|_{L^2} \le C\delta(1+t)^{-1},$$

where we have used $(1+t)e^{-c_0 t} \le C$. Similarly,

$$O_2 \le C\delta(1+t)^{-1}.$$

O_3 is naturally decomposed into two parts,

$$\begin{aligned} O_3 &\le \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2(b\cdot\nabla b)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= O_{3,1} + O_{3,2}. \end{aligned}$$

We further write

$$\begin{aligned} O_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_2(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= O_{3,1,1} + O_{3,1,2}. \end{aligned}$$

By Ladyzhenskaya's inequality, Proposition 2.2.2 and Lemma 2.3.3,

$$\begin{aligned} O_{3,1,1} &\le \int_0^{t/2} e^{-c_0(t-\tau)}\|\partial_2(u\cdot\nabla u)\|_{L^2}\, d\tau \\ &\le e^{-\frac{c_0}{2}t}\int_0^{t/2} (\|\partial_2 u\|_{L^4}\|\nabla u\|_{L^4} + \|u\|_{L^\infty}\|\partial_2\nabla u\|_{L^2})\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\Delta u\|^{\frac{5}{4}}\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \tilde{c}(1+\tau)^{-\frac{3}{4}}\delta^2 + \tilde{c}^{\frac{3}{4}}(1+\tau)^{-\frac{1}{2}}\delta^2\, d\tau \\ &\le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1}, \end{aligned} \tag{2.3.15}$$

where we used $e^{-\frac{c_0}{2}t}(1+t)^\gamma \le C(\gamma)$ for any $\gamma > 0$. To bound $O_{3,1,2}$, we write the norm in $O_{3,1,2}$ from the frequency space to be in the physical space, and then use Hölder's inequality, Lemma 2.3.1 and Lemma 2.3.2 to obtain

$$
\begin{aligned}
O_{3,1,2} &= \int_{t/2}^{t} e^{-c_0(t-\tau))} \|e^{-c_0\xi_2^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2}\ d\tau \\
&\le \int_{t/2}^{t} e^{-c_0(t-\tau))} \left\| \|\Lambda_2 e^{-c_0\Lambda_2^2(t-\tau)}(\widehat{u\cdot\nabla u})\|_{L^2_{x_2}} \right\|_{L^2_{x_1}}\ d\tau \\
&\le C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{3}{4}} \left\| \|u\cdot\nabla u\|_{L^1_{x_2}}\ \right\|_{L^2_{x_1}}\ d\tau \\
&\le C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{3}{4}} \left\| \|u\|_{L^2_{x_2}}\|\nabla u\|_{L^2_{x_2}} \right\|_{L^2_{x_1}}\ d\tau \\
&\le C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{3}{4}} \|u\|_{L^2_{x_2}L^\infty_{x_1}}\|\nabla u\|_{L^2}\ d\tau \\
&\le C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{3}{4}} \|u\|_{L^2}^{\frac{1}{2}}\|\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\ d\tau \\
&\le C\tilde{c}^2\delta^2(1+t)^{-1},
\end{aligned}
\tag{2.3.16}
$$

where we have used $\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac{3}{4}}\ d\tau < \infty$. Since $O_{3,2}$ admits the same bound as $O_{3,1}$,

$$
O_3 \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}.
$$

O_4 obeys the same bound as O_3. Therefore,

$$
\begin{aligned}
\|\widehat{\partial_2 u}(t)\|_{L^2(A_1)} \le\ & C\delta(1+t)^{-1} + C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} \\
&+ C\tilde{c}^2\delta^2(1+t)^{-1}.
\end{aligned}
\tag{2.3.17}
$$

Next we bound $\|\partial_2 u\|_{L^2(A_{21})}$. By (2.3.14),

$$
\begin{aligned}
\|\widehat{\partial_2 u}(t)\|_{L^2(A_{21})} \le\ & \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{21})} + \|\widehat{K_2}(t)\widehat{\partial_2 b_0}\|_{L^2(A_{21})} \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{21})}\ d\tau \\
&+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{21})}\ d\tau \\
:=\ & P_1 + P_2 + P_3 + P_4.
\end{aligned}
$$

By Part (2) in Proposition 2.2.2 and Lemma 2.3.1,

$$
\begin{aligned}
P_1 &= \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{21})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(A_{21})} + C\|e^{-c_0|\xi|^2 t}\widehat{\partial_2 u_0}\|_{L^2(A_{21})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(A_{21})} + C\|\xi e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2(A_{21})} \\
&\le Ce^{-c_0 t}\|\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} + C\|e^{c_0\Lambda^2 t}\Lambda u_0\|_{L^2(\mathbb{R}^2)} \\
&\le C(1+t)^{-1}\|\partial_2 u_0\|_{L^2(\mathbb{R}^2)} + Ct^{-1}\|u_0\|_{L^1(\mathbb{R}^2)} \\
&\le C(1+t)^{-1}\|u_0\|_{H^1\cap L^1} \le C\delta(1+t)^{-1}.
\end{aligned}
\tag{2.3.18}
$$

where we have used $e^{-c_0 t} \le C(1+t)^{-1}$ for $t \ge 0$. Similarly,

$$P_2 \le C(1+t)^{-1}\|b_0\|_{H^1\cap L^1} \le C\delta(1+t)^{-1}.$$

We rewrite P_3 as

$$\begin{aligned}
P_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{21})}\,d\tau \\
&\le C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\quad + C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&= P_{3,1}+P_{3,2}+P_{3,3}+P_{3,4}.
\end{aligned}$$

$P_{3,1}$ can be bounded similarly as O_3,

$$P_{3,1} \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{2.3.19}$$

$P_{3,2}$ admits the same bound as the one for $P_{3,1}$. To bound $P_{3,3}$, we divide it into two parts,

$$\begin{aligned}
P_{3,3} &= C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\quad + C\int_{t/2}^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&= P_{3,3,1}+P_{3,3,2}.
\end{aligned}$$

By Lemma 2.3.1 and Hölder's inequality,

$$\begin{aligned}
P_{3,3,1} &\le C\int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\le C\int_0^{t/2} \||\xi_2\xi|e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\le C\int_0^{t/2} \||\xi|^2 e^{-c_0|\xi|^2(t-\tau)}|\widehat{u\otimes u}|\|_{L^2(\mathbb{R}^2)}\,d\tau \\
&\le C\int_0^{t/2} (t-\tau)^{-\frac{3}{2}}\|u\otimes u\|_{L^1}\,d\tau \le C(t/2)^{-\frac{3}{2}}\int_0^{t/2}\|u\|_{L^2}^2\,d\tau \\
&\le Ct^{-\frac{3}{2}}\int_0^{t/2}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^2\,d\tau \le C\tilde{c}^2\delta^2 t^{-\frac{3}{2}}\ln(1+t/2) \\
&\le C\tilde{c}^2\delta^2(1+t)^{-1}.
\end{aligned}$$

where we have used $t^{-\frac{1}{2}}\ln(1+t/2) \le C$ for all $t \ge 1$. By Lemma 2.3.1, Lemma 2.3.4 and Hölder's inequality,

$$\begin{aligned}
P_{3,3,2} \le & C\int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\ d\tau \\
& \le C\int_{t/2}^{t} \||\xi|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_2(u\otimes u})\|_{L^2}\ d\tau \\
& \le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{q}-\frac{1}{2})}\|\partial_2(u\otimes u)\|_{L^q} d\tau \\
& \le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\|\partial_2 u\|_{L^2}\|u\|_{L^r} d\tau \qquad (1<q<2, \quad r>2) \\
& \le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\|\partial_2 u\|\|u\|_{L^2}^{\frac{2}{r}}\|\nabla u\|_{L^2}^{1-\frac{2}{r}}\ d\tau \\
& \le C\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}(\tilde{c}\delta(1+\tau)^{-1})(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})\ d\tau \\
& \le C\tilde{c}^2\delta^2(1+t)^{-\frac{3}{2}}\int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}}\ d\tau \\
& \le C\tilde{c}^2\delta^2(1+t)^{-\frac{3}{2}}t^{1-\frac{1}{q}} \le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}-\frac{1}{q}} \le C\tilde{c}^2\delta^2(1+t)^{-1}.
\end{aligned}$$

Therefore,

$$P_{3,3} \le C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{2.3.20}$$

Similarly, $P_{3,4}$ obeys the same bound. By (2.3.19) and (2.3.20),

$$P_3 \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{2.3.21}$$

Furthermore, P_4 admits the same bound as P_3 in (2.3.21). By collecting all the bounds for P_1, P_2, P_3 and P_4 from (2.3.18) to (2.3.21), we obtain

$$\|\widehat{\partial_2 u}(t)\|_{L^2(A_{21})} \le C\delta(1+t)^{-1} + C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1} + C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{2.3.22}$$

Next we estimate $\|\partial_2 u\|_{L^2(A_{22})}$. By (2.3.14),

$$\begin{aligned}
\|\widehat{\partial_2 u}(t)\|_{L^2(A_{22})} \le & \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{22})} + \|\widehat{K_2}(t)\widehat{\partial_2 b_0}\|_{L^2(A_{22})} \\
& + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\ d\tau \\
& + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})}\ d\tau \\
:= & Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned}$$

By Part (3) in Proposition 2.2.2,

$$\begin{aligned}
Q_1 &= \|\widehat{K_1}(t)\widehat{\partial_2 u_0}\|_{L^2(A_{22})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(A_{22})} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_2 u_0}\|_{L^2(A_{22})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^2)} \\
&\le Ce^{-c_0 t}\|\partial_2 u_0\|_{L^2} \le C(1+t)^{-1}\|\partial_2 u_0\|_{L^2} \le C\delta(1+t)^{-1}.
\end{aligned} \tag{2.3.23}$$

Similarly, Q_2 admits the same bound, namely,

$$Q_2 \le C(1+t)^{-1}\|\partial_2 b_0\|_{L^2} \le C\delta(1+t)^{-1}. \tag{2.3.24}$$

The bounds in Proposition 2.2.2 are not sufficient for estimating Q_3 and Q_4, so we drive some alternative upper bounds. Recall that

$$A_{22} = \{\xi \in \mathbb{R}^2,\ \nu\eta\xi_2^2 + \xi_1^2 < \frac{3}{16}(\eta + \nu\xi_2^2)^2,\ \nu\xi_2^2 > \eta\},$$

and G_2 and G_3 can be rewritten as

$$G_2 = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = \frac{\lambda_2(e^{\lambda_2 t} - e^{\lambda_1 t}) + (\lambda_2 - \lambda_1)e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} + \lambda_2 G_1.$$

$$G_3 = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = \frac{\lambda_1(e^{\lambda_1 t} - e^{\lambda_2 t}) + (\lambda_2 - \lambda_1)e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1.$$

Furthermore, by the statement of Proposition 2.2.1,

$$\widehat{K_1} = e^{\lambda_1 t} + \lambda_2 G_1 + \eta G_1, \qquad \widehat{K_2} = i\xi_1 G_1, \qquad \widehat{K_3} = e^{\lambda_1 t} - \lambda_1 G_1 - \eta G_1.$$

By (2.2.9), we obtain the new upper bounds for $\widehat{K_1}$ and $\widehat{K_2}$,

$$\begin{aligned}
|\widehat{K_1}| &\le e^{-c_0(1+\xi_2^2)t} + C(\frac{\nu\eta\xi_2^2 + \xi_1^2}{\eta + \nu\xi_2^2} + \eta)|G_1| \\
&\le e^{-c_0(1+\xi_2^2)t} + \frac{2C}{\eta + \nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2 + \xi_1^2}{\eta + \nu\xi_2^2} + \eta\right)\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right),
\end{aligned} \tag{2.3.25}$$

$$|\widehat{K_2}| \le \frac{|2\xi_1|}{\eta + \nu\xi_2^2}\left(e^{-\frac{1}{2}(\eta+\nu\xi_2^2)t} + e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}t}\right). \tag{2.3.26}$$

To bound Q_3, we first decompose it as

$$\begin{aligned}
Q_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&:= Q_{3,1} + Q_{3,2}.
\end{aligned}$$

Invoking the upper bounds in Part (3) in Proposition 2.2.2 and further dividing $Q_{3,1}$ into four parts, we can show via similar techniques as for $O_{3,1,1}$ in (2.3.15) that

$$\begin{aligned}
Q_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_2 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\
&\leq C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0((1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_2(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^{t/2} \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_2(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\leq C\int_0^{t/2} e^{-c_0(t-\tau)}(\|\partial_2(u\cdot\nabla u)\|_{L^2} + \|\partial_2(b\cdot\nabla b)\|_{L^2})\, d\tau \\
&\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2} (\|\partial_2(u\cdot\nabla u)\|_{L^2} + \|\partial_2(b\cdot\nabla b)\|_{L^2}) d\tau \\
&\leq C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1}.
\end{aligned} \tag{2.3.27}$$

To bound $Q_{3,2}$, we use the new bounds in (2.3.25) and (2.3.26). By Hölder's inequality and (2.3.25),

$$\begin{aligned}
Q_{3,2} &\leq C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\|_{L^2} + \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\
&+ C\int_{t/2}^{t} \left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right) e^{-c_0(1+\xi_2^2)(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla u)}| + |\widehat{\partial_2(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\
&+ C\int_{t/2}^{t} \left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right) e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla u)}| + |\widehat{\partial_2(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\
&:= Q_{3,2,1} + Q_{3,2,2} + Q_{3,2,3}.
\end{aligned}$$

We rewrite $Q_{3,2,1}$ into two parts,

$$\begin{aligned}
Q_{3,2,1} &= C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(u\cdot\nabla u)}|\|_{L^2}\, d\tau \\
&\quad + C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_2(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\
&= Q_{3,2,1,1} + Q_{3,2,1,2}.
\end{aligned}$$

Following similar estimates as those for $O_{3,1,2}$ in (2.3.16), we have

$$Q_{3,2,1,1} \leq C\tilde{c}^2\delta^2(1+t)^{-1}.$$

Clearly, $Q_{3,2,1,2}$ admits the same bound,

$$Q_{3,2,1} \leq C\tilde{c}^2\delta^2(1+t)^{-1}. \tag{2.3.28}$$

For $\xi \in A_{22}$, we have $\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} < \frac{3}{16}$. By (2.3.28),

$$\begin{aligned}
Q_{3,2,2} \leq{}& C\int_{t/2}^{t} \left\| \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} + 1 \right) e^{-c_0(1+\xi_2^2)(t-\tau)} |\widehat{\partial_2(u\cdot\nabla u)}| \right\|_{L^2} \, d\tau \\
&+ C\int_{t/2}^{t} \left\| \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} + 1 \right) e^{-c_0(1+\xi_2^2)(t-\tau)} |\widehat{\partial_2(b\cdot\nabla b)}| \right\|_{L^2} \, d\tau \\
\leq{}& C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_2(u\cdot\nabla u)} \|_{L^2} \, d\tau \\
&+ C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_2(b\cdot\nabla b)} \|_{L^2} \, d\tau \\
\leq{}& CQ_{3,2,1} \leq C\tilde{c}^2\delta^2(1+t)^{-1}.
\end{aligned}$$

$Q_{3,2,3}$ can be further rewritten as

$$\begin{aligned}
Q_{3,2,3} ={}& C\int_{t/2}^{t} \left\| \frac{1}{\eta+\nu\xi_2^2} \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2} + \eta \right) e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_2(u\cdot\nabla u)}| \right\|_{L^2} \, d\tau \\
&+ C\int_{t/2}^{t} \left\| \frac{1}{\eta+\nu\xi_2^2} \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2} + \eta \right) e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_2(b\cdot\nabla b)}| \right\|_{L^2} \, d\tau \\
={}& Q_{3,2,3,1} + Q_{3,2,3,2}.
\end{aligned}$$

We first estimate $Q_{3,2,3,1}$,

$$\begin{aligned}
Q_{3,2,3,1} \leq{}& C\int_{t/2}^{t} \left\| \frac{|\xi|^2}{(1+\xi_2^2)^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(u\cdot\nabla u)}| \right\|_{L^2} \, d\tau \\
&+ C\int_{t/2}^{t} \left\| \frac{|\xi|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \widehat{u\cdot\nabla u} \right\|_{L^2} \, d\tau \\
={}& Q_{3,2,3,1,1} + Q_{3,2,3,1,2}.
\end{aligned}$$

The process of controlling $Q_{3,2,3,1,1}$ is tedious, so we first estimate $Q_{3,2,3,1,2}$. By Lemma 2.3.3,

$$
\begin{aligned}
Q_{3,2,3,1,2} &\le C\int_{t/2}^{t}\left\|\frac{|\xi|}{\sqrt{1+\xi_2^2}}(t-\tau)^{\frac12}(t-\tau)^{-\frac12}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}\|e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{u\cdot\nabla u}\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\cdot\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac14}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}(\tilde c\delta(1+\tau)^{-\frac12})^{\frac32}(\tilde c\delta(1+\tau)^{-1})^{\frac14}(c\delta)^{\frac14}d\tau\\
&\le C\tilde c^2\delta^2(1+t/2)^{-1}\le C\tilde c^2\delta^2(1+t)^{-1},
\end{aligned}
$$

where we have used the facts that $\displaystyle\int_{t/2}^{t}(t-\tau)^{-\frac12}e^{-\frac{c_0}{4}(t-\tau)}\,d\tau<\infty$ and $\gamma\,e^{-\frac{c_0}{2}\gamma^2}\le C$ or more explicitly

$$
\frac{|\xi|}{\sqrt{1+\xi_2^2}}(t-\tau)^{\frac12}\,e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\le C.
$$

As (2.3.1) indicates, the decay rates associated with the horizontal and the vertical derivatives are different. To bound $Q_{3,2,3,1,1}$ properly, we need to distinguish the horizontal derivative from the vertical one. By $\nabla\cdot u=0$, we write

$$
\partial_2(u\cdot\nabla u)=\partial_1\partial_2(uu_1)+\partial_2\partial_2(uu_2)
$$

and divide $Q_{3,2,3,1,1}$ into two parts,

$$
\begin{aligned}
Q_{3,2,3,1,1} &\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(uu_1)+\partial_2\partial_2(uu_2)}|\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(uu_1)}|\right\|_{L^2}\,d\tau\\
&\quad+C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_2\partial_2(uu_2)}|\right\|_{L^2}\,d\tau\\
&=Q_{3,2,3,1,1,1}+Q_{3,2,3,1,1,2}.
\end{aligned}
$$

Since $\xi \in A_{22}$, we have $|\xi|^2 \le C(1+\xi_2^2)^2$. By Lemma 2.3.5 and Lemma 2.3.3,

$$
\begin{aligned}
Q_{3,2,3,1,1,1} \le & \int_{t/2}^{t} \left\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma} \left(\frac{|\xi|}{(1+\xi_2^2)}\right)^{2-\sigma} \right. \\
& \qquad \left. \times |\xi_1|^{1-\sigma}(t-\tau)^{-\sigma} e^{-c_0 \frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(uu_1)}| \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} \left\| (t-\tau)^{-\sigma} |\xi_1|^{1-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(uu_1)}| \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \left\| \widehat{\Lambda_1^{1-\sigma}(\partial_2(uu_1))} \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u\|_{L^2} \, d\tau \\
& + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 u\|_{L^2}^{\sigma} \|\partial_1 \partial_2 u\|_{L^2}^{1-\sigma} \\
& \qquad \times \|\partial_2 u\|_{L^2}^{\frac14} \|\partial_1 u\|_{L^2}^{\frac14} \|u\|_{L^2}^{\frac14} \|\Delta u\|^{\frac14} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac{2\sigma}{3}} (C\delta)^{1-\frac{2\sigma}{3}} \tilde{c}\delta(1+\tau)^{-1} \, d\tau \\
& + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-1})^{\sigma+\frac14} (C\delta)^{\frac54-\sigma} (\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac12} \, d\tau \\
\le & \, C(\sigma)\tilde{c}^{\frac{2\sigma}{3}+1}\delta^2(1+t)^{-1-\frac{\sigma}{3}} + C(\sigma)\tilde{c}^{\sigma+\frac34}\delta^2(1+t)^{-\sigma-\frac12} \\
\le & \, C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-1},
\end{aligned}
$$

where we have set $\sigma = \frac34$, and used $\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} < \infty$, and

$$
\frac{|\xi|}{1+\xi_2^2} \le C(\sigma), \qquad \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma} e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \le C(\sigma). \tag{2.3.29}
$$

In addition, we have used the following upper bound on $\|\Lambda_1^{1-\sigma}(\partial_2(uu_1))\|$ in the fourth inequality above, by Lemma 2.3.5,

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_2(uu_1))\|_{L^2} \le & \|\Lambda_1^{1-\sigma}(\partial_2 u u_1)\|_{L^2} + \|\Lambda_1^{1-\sigma}(u\partial_2 u_1)\|_{L^2} \\
\le & \, C\|\partial_2 u\|_{L^2}\|\Lambda_1^{1-\sigma}u_1\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u\|_{L^2}\|u_1\|_{L^\infty} \\
& + C\|\partial_2 u_1\|_{L^2}\|\Lambda_1^{1-\sigma}u\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u_1\|_{L^2}\|u\|_{L^\infty} \\
\le & \, C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u\|_{L^2} \\
& + C\|\partial_2 u\|_{L^2}^{\sigma}\|\partial_1\partial_2 u\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14}.
\end{aligned} \tag{2.3.30}
$$

Similarly, by (2.3.29),

$$
\begin{aligned}
Q_{3,2,3,1,1,2} &\leq \int_{t/2}^{t} \Bigg\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\sigma} \left(\frac{|\xi|}{(1+\xi_2^2)} \right)^{2-\sigma} \\
&\qquad\qquad \times |\xi_2|^{1-\sigma}(t-\tau)^{-\sigma} e^{-c_0 \frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(uu_2)}| \Bigg\|_{L^2} \, d\tau \\
&\leq C(\sigma) \int_{t/2}^{t} \Bigg\| (t-\tau)^{-\sigma} |\xi_2|^{1-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(uu_2)}| \Bigg\|_{L^2} \, d\tau \\
&\leq C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \left\| \widehat{\Lambda_2^{1-\sigma}(\partial_2(uu_2))} \right\|_{L^2} \, d\tau \\
&\leq C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u\|_{L^2} \, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 u\|_{L^2}^{\sigma} \|\partial_2 \partial_2 u\|_{L^2}^{1-\sigma} \\
&\qquad\qquad \times \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1 u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{1}{4}} \|\Delta u\|^{\frac{1}{4}} \, d\tau \\
&\leq C(\sigma) \tilde{c}^{\frac{3}{2}} \delta^2 (1+t)^{-1},
\end{aligned}
$$

where we have set $\sigma = \frac{3}{4}$, and used (2.3.30) and the following estimate

$$
\begin{aligned}
\|\Lambda_2^{1-\sigma}(\partial_2(uu_2))\|_{L^2} &\leq C\|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u\|_{L^2} \\
&\qquad + C\|\partial_2 u\|_{L^2}^{\sigma} \|\partial_2 \partial_2 u\|_{L^2}^{1-\sigma} \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\|_{L^2}^{\frac{1}{4}} \|\Delta u\|_{L^2}^{\frac{1}{4}}.
\end{aligned}
$$

Therefore,

$$
Q_{3,2,3,1} \leq C\tilde{c}^2 \delta^2 (1+t)^{-1} + C(\sigma) \tilde{c}^{\frac{3}{2}} \delta^2 (1+t)^{-1}.
$$

Similarly, $Q_{3,2,3,2}$ admits the same bound. Collecting the bounds for $Q_{3,2,1}, Q_{3,2,2}$ and $Q_{3,2,3}$ yields

$$
Q_{3,2} \leq C\tilde{c}^2 \delta^2 (1+t)^{-1} + C(\sigma) \tilde{c}^{\frac{3}{2}} \delta^2 (1+t)^{-1}. \tag{2.3.31}
$$

Combining the estimates for $Q_{3,1}$ and $Q_{3,2}$ in (2.3.27) and (2.3.31) respectively, we obtain

$$
Q_3 \leq (\tilde{c} + \tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}}) C \delta^2 (1+t)^{-1}. \tag{2.3.32}
$$

Next we bound Q_4. By (2.3.26), we rewrite Q_4 as

$$
\begin{aligned}
Q_4 &= \int_0^t \|\widehat{K_2}(t-\tau) \widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})} \, d\tau \\
&= \int_0^{t/2} \|\widehat{K_2}(t-\tau) \widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})} \, d\tau + \int_{t/2}^{t} \|\widehat{K_2}(t-\tau) \widehat{\partial_2 N_2}(\tau)\|_{L^2(A_{22})} \, d\tau \\
&:= Q_{4,1} + Q_{4,2}.
\end{aligned}
$$

By Part (3) in Proposition 2.2.2 and by (2.3.27), $Q_{4,1}$ obeys the same bound as $Q_{3,1}$, namely,

$$Q_{4,1} \le C\tilde{c}\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-1}. \tag{2.3.33}$$

Since the bound for $\widehat{K_2}$ in (2.3.26) is not the same as the bound for $\widehat{K_1}$ in (2.3.25), we need to estimate $Q_{4,2}$ differently from $Q_{3,2}$.

$$\begin{aligned}
Q_{4,2} \le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0(1+\xi_2^2)(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla b)}| + |\widehat{\partial_2(b\cdot\nabla u)}| \right) \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} \left(|\widehat{\partial_2(u\cdot\nabla b)}| + |\widehat{\partial_2(b\cdot\nabla u)}| \right) \right\|_{L^2} d\tau \\
:=\ & Q_{4,2,1} + Q_{4,2,2}.
\end{aligned}$$

Since $\xi \in A_{22}$, $|\xi|^2 \le C\,(1+\xi_2^2)^2$. By the same process as in (2.3.16),

$$\begin{aligned}
Q_{4,2,1} \le\ & C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_2(u\cdot\nabla b)} \|_{L^2}\, d\tau \\
& + C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_2(b\cdot\nabla u)} \|_{L^2}\, d\tau \\
\le\ & C\tilde{c}^2\delta^2(1+t)^{-1}.
\end{aligned}$$

We further decompose $Q_{4,2,2}$ into two parts,

$$\begin{aligned}
Q_{4,2,2} \le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_2(u\cdot\nabla b)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_2(b\cdot\nabla u)}| \right\|_{L^2} d\tau \\
:=\ & Q_{4,2,2,1} + Q_{4,2,2,2}.
\end{aligned}$$

As before, we write $\partial_2(u\cdot\nabla b) = \partial_1\partial_2(bu_1) + \partial_2\partial_2(bu_2)$ and thus decompose $Q_{4,2,2,1}$ into two parts,

$$\begin{aligned}
Q_{4,2,2,1} \le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(bu_1) + \partial_2\partial_2(bu_2)}| \right\|_{L^2} d\tau \\
\le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(bu_1)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2\partial_2(bu_2)}| \right\|_{L^2} d\tau \\
=\ & Q_{4,2,2,1,1} + Q_{4,2,2,1,2}.
\end{aligned}$$

The first part $Q_{4,2,2,1,1}$ can be bounded by

$$\begin{aligned}
Q_{4,2,2,1,1} &\le \int_{t/2}^{t} \left\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\frac{1+\sigma}{2}} |\xi_1|^{1-\sigma}(t-\tau)^{-\frac{1+\sigma}{2}} e^{-c_0 \frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(bu_1)}| \right\|_{L^2} d\tau \\
&\le C(\sigma) \int_{t/2}^{t} \left\| |\xi_1|^{1-\sigma}(t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(bu_1)}| \right\|_{L^2} d\tau \\
&\le C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}}(t-\tau) \left\| \widehat{\Lambda_1^{1-\sigma}(\partial_2(bu_1))} \right\|_{L^2} d\tau \\
&\le C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}}(t-\tau) \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 b\|_{L^2} \, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 b\|_{L^2}^{\sigma} \|\partial_1 \partial_2 b\|_{L^2}^{1-\sigma} \|\partial_2 u\|_{L^2}^{\frac14} \\
&\qquad\qquad \times \|\partial_1 u\|_{L^2}^{\frac14} \|u\|_{L^2}^{\frac14} \|\Delta u\|_{L^2}^{\frac14} \, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}}(t-\tau) \|b\|_{L^2}^{\frac{\sigma}{3}} \|\nabla b\|_{L^2}^{\frac{\sigma}{3}} \|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u_1\|_{L^2} \, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}}(t-\tau) \|\partial_2 u_1\|_{L^2}^{\sigma} \|\partial_1 \partial_2 u_1\|_{L^2}^{1-\sigma} \\
&\qquad\qquad \times \|\partial_2 b\|_{L^2}^{\frac14} \|\partial_1 b\|_{L^2}^{\frac14} \|b\|_{L^2}^{\frac14} \|\Delta b\|_{L^2}^{\frac14} \, d\tau \\
&\le C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac{2\sigma}{3}} (C\delta)^{1-\frac{2\sigma}{3}} \tilde{c}\delta(1+\tau)^{-1} \, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-1})^{\sigma+\frac14} (C\delta)^{\frac54-\sigma} (\tilde{c}\delta(1+\tau)^{-\frac12})^{\frac12} \, d\tau \\
&\le C(\sigma) \tilde{c}^{\frac{2\sigma}{3}+1} \delta^2 (1+t)^{-1-\frac{\sigma}{3}} + C(\sigma) \tilde{c}^{\sigma+\frac34} \delta^2 (1+t)^{-\sigma-\frac12} \\
&\le C \tilde{c}^{\frac32} \delta^2 (1+t)^{-1},
\end{aligned}$$

where we set $\sigma = \frac{3}{4}$, and have used $\int_{t/2}^{t} (t-\tau)^{-\frac{1+\sigma}{2}} e^{-\frac{c_0}{4}(t-\tau)} < \infty$ and

$$\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\frac{1+\sigma}{2}} e^{-\frac{c_0}{2} \frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \le C(\sigma).$$

In addition, we have also used the following upper bound on $\|\Lambda_1^{1-\sigma}(\partial_2(bu_1))\|$, by Lemma 2.3.5,

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_2(bu_1))\|_{L^2} &\leq \|\Lambda_1^{1-\sigma}(\partial_2 bu_1)\|_{L^2} + \|\Lambda_1^{1-\sigma}(b\partial_2 u_1)\|_{L^2} \\
&\leq C\|\partial_2 b\|_{L^2}\|\Lambda_1^{1-\sigma}u_1\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 b\|_{L^2}\|u_1\|_{L^\infty} \\
&\quad + C\|\partial_2 u_1\|_{L^2}\|\Lambda_1^{1-\sigma}b\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u_1\|_{L^2}\|b\|_{L^\infty} \\
&\leq C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 b\|_{L^2} \\
&\quad + C\|\partial_2 b\|_{L^2}^{\sigma}\|\partial_1\partial_2 b\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|u\|_{L^2}^{\frac{1}{4}}\|\Delta u\|_{L^2}^{\frac{1}{4}} \\
&\quad + C\|b\|_{L^2}^{\frac{\sigma}{3}}\|\nabla b\|_{L^2}^{\frac{\sigma}{3}}\|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u_1\|_{L^2} \\
&\quad + C\|\partial_2 u_1\|_{L^2}^{\sigma}\|\partial_1\partial_2 u_1\|_{L^2}^{1-\sigma}\|\partial_2 b\|_{L^2}^{\frac{1}{4}}\|\partial_1 b\|_{L^2}^{\frac{1}{4}}\|b\|_{L^2}^{\frac{1}{4}}\|\Delta b\|_{L^2}^{\frac{1}{4}}.
\end{aligned}
$$

Similarly,

$$Q_{4,2,2,1,2} \leq C\tilde{c}^{\frac{3}{2}}\delta^2(1+t)^{-1}.$$

Since $Q_{4,2,2,2}$ obeys the same bound as $Q_{4,2,2,1}$, we find

$$Q_{4,2,2} \leq C(\sigma)\tilde{c}^{\frac{3}{2}}\delta^2(1+t)^{-1}.$$

Therefore,

$$Q_{4,2} \leq C\tilde{c}^2\delta^2(1+t)^{-1} + C\tilde{c}^{\frac{3}{2}}\delta^2(1+t)^{-1}. \tag{2.3.34}$$

Putting (2.3.33) and (2.3.34) together yields

$$Q_4 \leq (\tilde{c} + \tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}})C\delta^2(1+t)^{-1}. \tag{2.3.35}$$

Combining (2.3.23), (2.3.24), (2.3.32) and (2.3.35), we have

$$\|\widehat{\partial_2 u}(t)\|_{L^2(A_{22})} \leq C\delta(1+t)^{-1} + (\tilde{c} + \tilde{c}^{\frac{3}{4}} + \tilde{c}^{\frac{3}{2}})C\delta^2(1+t)^{-1} \tag{2.3.36}$$

Collecting the estimates in (2.3.17), (2.3.22) and (2.3.36), we find

$$\|\widehat{\partial_2 u}(t)\|_{L^2} \leq C_1\delta(1+t)^{-1} + \tilde{c}C_2\delta^2(1+t)^{-1} + (\tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}})C_3\delta^2(1+t)^{-1}.$$

If we choose $\tilde{c}$ and δ satisfying

$$C_1 \leq \frac{\tilde{c}}{8}, \qquad C_2\delta \leq \frac{1}{16}, \qquad (\tilde{c}^{\frac{3}{4}} + \tilde{c}^2 + \tilde{c}^{\frac{3}{2}})C_3\delta \leq \frac{\tilde{c}}{16},$$

then we obtain

$$
\begin{aligned}
\|\partial_2 u(t)\|_{L^2} &\leq \frac{\tilde{c}}{8}\delta(1+t)^{-1} + \frac{\tilde{c}}{16}\delta(1+t)^{-1} + \frac{\tilde{c}}{16}\delta(1+t)^{-1} \\
&= \frac{\tilde{c}}{4}\delta(1+t)^{-1}.
\end{aligned}
$$

The same upper bound holds for $\|\partial_2 b\|_{L^2}$. Thus we have obtained

$$\|(\partial_2 u(t), \partial_2 b(t))\|_{L^2} \leq \frac{\tilde{c}}{2}\delta(1+t)^{-1}.$$

This completes the proof of the third inequality in (2.3.2).

2.3.3 Estimates of $\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2}$

This subsection establishes the second inequality in (2.3.2), namely

$$\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac{1}{2}}.$$

Applying ∂_1 to (2.1.14) yields

$$\begin{cases} \widehat{\partial_1 u}(t) = \widehat{K_1}\widehat{\partial_1 u_0} + \widehat{K_2}\widehat{\partial_1 b_0} + \displaystyle\int_0^t \widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau) + \widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\, d\tau \\ \widehat{\partial_1 b}(t) = \widehat{K_2}\widehat{\partial_1 u_0} + \widehat{K_3}\widehat{\partial_1 b_0} + \displaystyle\int_0^t \widehat{K_2}(t-\tau)\widehat{\partial_1 N_1}(\tau) + \widehat{K_3}(t-\tau)\widehat{\partial_1 N_2}(\tau)\, d\tau. \end{cases} \tag{2.3.37}$$

To estimate $\|\partial_1 u\|_{L^2(\mathbb{R}^2)}$, we estimate $\|\partial_1 u\|_{L^2(A_1)}$, $\|\partial_1 u\|_{L^2(A_{21})}$ and $\|\partial_1 u\|_{L^2(A_{22})}$. We start with $\|\partial_1 u\|_{L^2(A_1)}$. By (2.3.37),

$$\begin{aligned} \|\widehat{\partial_1 u}(t)\|_{L^2(A_1)} &\le \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_1)} + \|\widehat{K_2}(t)\widehat{b_0}\|_{L^2(A_1)} \\ &\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_1)}\, d\tau \\ &\quad + \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_1)}\, d\tau \\ &:= H_1 + H_2 + H_3 + H_4. \end{aligned}$$

By Proposition 2.2.2,

$$H_1 \le \|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)} \le e^{-c_0 t}\|\partial_1 u_0\|_{L^2} \le C\delta(1+t)^{-\frac{1}{2}},$$

where we have used $(1+t)e^{-c_0 t} \le C$. By the same technique, H_2 obeys the same bound, namely,

$$H_2 \le C\delta(1+t)^{-\frac{1}{2}}.$$

H_3 can be decomposed into two parts,

$$\begin{aligned} H_3 &\le \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &\quad + \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1(b\cdot\nabla b)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= H_{3,1} + H_{3,2}. \end{aligned}$$

We further divide $H_{3,1}$ into two parts,

$$\begin{aligned} H_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_1(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &\quad + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1(u\cdot\nabla u)}(\tau)\|_{L^2(A_1)}\, d\tau \\ &= H_{3,1,1} + H_{3,1,2}. \end{aligned}$$

By Ladyzhenskaya's inequality, Proposition 2.2.2 and Lemma 2.3.3,

$$\begin{aligned} H_{3,1,1} &\leq \int_0^{t/2} e^{-c_0(t-\tau)} \|\partial_1(u\cdot\nabla u)\|_{L^2}\ d\tau \\ &\leq e^{-\frac{c_0}{2}t}\int_0^{t/2} (\|\partial_1 u\|_{L^4}\|\nabla u\|_{L^4} + \|u\|_{L^\infty}\|\partial_1\nabla u\|_{L^2})\ d\tau \\ &\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \|\partial_1 u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12} + \|u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\Delta u\|^{\frac54}\ d\tau \\ &\leq Ce^{-\frac{c_0}{2}t}\int_0^{t/2} \tilde{c}(1+\tau)^{-\frac12}\delta^2 + \tilde{c}^{\frac34}(1+\tau)^{-\frac12}\delta^2\ d\tau \\ &\leq C\tilde{c}\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12}, \end{aligned} \tag{2.3.38}$$

where we have used $e^{-\frac{c_0}{2}t}(1+t)^\gamma \leq C(\gamma) < \infty$ for $\gamma > 0$. We write the norm in $H_{3,1,2}$ from frequency space to physical space, by Hölder's inequality, Lemma 2.3.1 and Lemma 2.3.2,

$$\begin{aligned} H_{3,1,2} &= \int_{t/2}^{t} e^{-c_0(t-\tau))}\|e^{-c_0\xi_2^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2}\ d\tau \\ &\leq \int_{t/2}^{t} e^{-c_0(t-\tau))}\left\|\|e^{-c_0\Lambda_2^2(t-\tau)}\partial_1(u\cdot\nabla u)\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}\ d\tau \\ &\leq C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\left\|\|\partial_1(u\cdot\nabla u)\|_{L^1_{x_2}}\ \right\|_{L^2_{x_1}}\ d\tau \\ &\leq C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\left(\left\|\|\partial_1 u\cdot\nabla u\|_{L^1_{x_2}} + \|u\cdot\partial_1\nabla u\|_{L^1_{x_2}}\ \right\|_{L^2_{x_1}}\right) d\tau \\ &\leq C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\left(\left\|\|\partial_1 u\|_{L^2_{x_2}}\|\nabla u\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}\right. \\ &\qquad\qquad\qquad\qquad\qquad \left. + \left\|\|u\|_{L^2_{x_2}}\|\partial_1\nabla u\|_{L^2_{x_2}}\right\|_{L^2_{x_1}}\right) d\tau \\ &\leq C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\left(\|\partial_1 u\|_{L^2_{x_2}L^\infty_{x_1}}\|\nabla u\|_{L^2} + \|u\|_{L^2_{x_2}L^\infty_{x_1}}\|\Delta u\|_{L^2}\right) d\tau \\ &\leq C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\|\partial_1 u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2} d\tau \\ &\quad + C\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\|u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\Delta u\|_{L^2} d\tau \\ &\leq C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12} + C\tilde{c}\delta^2(1+t)^{-\frac12} \end{aligned} \tag{2.3.39}$$

where we used $\int_{t/2}^{t} e^{-c_0(t-\tau)}(t-\tau)^{-\frac14}\ d\tau < \infty$. Since $H_{3,2}$ admits the same bound as $H_{3,1}$,

$$H_3 \leq C\tilde{c}\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12}.$$

H_4 obeys the same bound as H_3, hence,

$$\|\widehat{\partial_1 u}(t)\|_{L^2(A_1)} \leq C\delta(1+t)^{-\frac12} + C\tilde{c}\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12}. \tag{2.3.40}$$

Now we estimate $\|\partial_1 u\|_{L^2(A_{21})}$. By (2.3.37),

$$\begin{aligned}
\|\widehat{\partial_1 u}(t)\|_{L^2(A_{21})} \le& \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{21})} + \|\widehat{K_2}(t)\widehat{\partial_1 b_0}\|_{L^2(A_{21})} \\
&+ \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{21})}\, d\tau \\
&+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{21})}\, d\tau \\
:=& L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

By Part (2) in Proposition 2.2.2 and Lemma 2.3.1,

$$\begin{aligned}
L_1 &= \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{21})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(A_{21})} + C\|e^{-c_0|\xi|^2 t}\widehat{\partial_1 u_0}\|_{L^2(A_{21})} \\
&\le C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(A_{21})} + C\|\xi e^{-c_0|\xi|^2 t}\widehat{u_0}\|_{L^2(A_{21})} \\
&\le Ce^{-c_0 t}\|\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)} + C\|e^{c_0\Lambda^2 t}\Lambda u_0\|_{L^2(\mathbb{R}^2)} \\
&\le C(1+t)^{-1}\|\partial_1 u_0\|_{L^2(\mathbb{R}^2)} + Ct^{-1}\|u_0\|_{L^1(\mathbb{R}^2)} \\
&\le C(1+t)^{-1}\|u_0\|_{H^1\cap L^1} \le C\delta(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

where we have used $e^{-c_o t} \le C(1+t)^{-1}$ for $t \ge 0$. Similarly,

$$L_2 \le C(1+t)^{-1}\|b_0\|_{H^1\cap L^1} \le C\delta(1+t)^{-\frac{1}{2}}.$$

We divide L_3 into four parts,

$$\begin{aligned}
L_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{21})}\, d\tau \\
&\le C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^t \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&\quad + C\int_0^t \|e^{-c_0|\xi|^2(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\
&= L_{3,1} + L_{3,2} + L_{3,3} + L_{3,4}.
\end{aligned}$$

Clearly, $L_{3,1}$ can be bounded similarly as H_3, namely,

$$L_{3,1} \le C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.41}$$

$L_{3,2}$ admits the same bound as $L_{3,1}$. $L_{3,3}$ is decomposed into two parts,

$$\begin{aligned} L_{3,3} = C \int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)} \widehat{\partial_1 (u \cdot \nabla u)}\|_{L^2(\mathbb{R}^2)} \; d\tau \\ + C \int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)} \widehat{\partial_1 (u \cdot \nabla u)}\|_{L^2(\mathbb{R}^2)} \; d\tau \\ = L_{3,3,1} + L_{3,3,2} \end{aligned}$$

By Lemma 2.3.1 and Hölder's inequality,

$$\begin{aligned} L_{3,3,1} &\le C \int_0^{t/2} \|e^{-c_0|\xi|^2(t-\tau)} \widehat{\partial_1 (u \cdot \nabla u)}\|_{L^2(\mathbb{R}^2)} \; d\tau \\ &\le C \int_0^{t/2} \||\xi_1 \xi| e^{-c_0|\xi|^2(t-\tau)} |\widehat{u \otimes u}|\|_{L^2(\mathbb{R}^2)} \; d\tau \\ &\le C \int_0^{t/2} \||\xi|^2 e^{-c_0|\xi|^2(t-\tau)} |\widehat{u \otimes u}|\|_{L^2(\mathbb{R}^2)} \; d\tau \\ &\le C \int_0^{t/2} (t-\tau)^{-\frac{3}{2}} \|u \otimes u\|_{L^1} \; d\tau \le C (t/2)^{-\frac{3}{2}} \int_0^{t/2} \|u\|_{L^2}^2 \; d\tau \\ &\le C t^{-\frac{3}{2}} \int_0^{t/2} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^2 \; d\tau \le C\tilde{c}^2\delta^2 t^{-\frac{3}{2}} \ln(1+t/2) \le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

where we have used $t^{-\frac{1}{2}} \ln(1+t/2) \le C$ for all $t \ge 1$. By Lemma 2.3.1, Lemma 2.3.4 and Hölder's inequality,

$$\begin{aligned} L_{3,3,2} \le & C \int_{t/2}^{t} \|e^{-c_0|\xi|^2(t-\tau)} \widehat{\partial_1 (u \cdot \nabla u)}\|_{L^2(\mathbb{R}^2)} \; d\tau \\ &\le C \int_{t/2}^{t} \||\xi| e^{-c_0|\xi|^2(t-\tau)} \widehat{\partial_1 (u \otimes u)}\|_{L^2} \; d\tau \\ &\le C \int_{t/2}^{t} (t-\tau)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{q}-\frac{1}{2})} \|\partial_1 (u \otimes u)\|_{L^q} d\tau \\ &\le C \int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}} \|\partial_1 u\|_{L^2} \|u\|_{L^r} d\tau \qquad (1 < q < 2, \quad r > 2) \\ &\le C \int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}} \|\partial_1 u\| \|u\|_{L^2}^{\frac{2}{r}} \|\nabla u\|_{L^2}^{1-\frac{2}{r}} \; d\tau \\ &\le C \int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}}) \; d\tau \\ &\le C\tilde{c}^2\delta^2(1+t)^{-1} \int_{t/2}^{t} (t-\tau)^{-\frac{1}{q}} \; d\tau \\ &\le C\tilde{c}^2\delta^2(1+t)^{-1} t^{1-\frac{1}{q}} \le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{q}} \le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned}$$

Therefore,

$$L_{3,3} \le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.42}$$

Similarly, $L_{3,4}$ obeys the same bound. By (2.3.41) and (2.3.42),

$$L_3 \leq C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.43}$$

L_4 admits the same bound as L_3 in (2.3.43). By collecting all the bounds for L_1, L_2, L_3 and L_4, we obtain

$$\|\widehat{\partial_1 u}(t)\|_{L^2(A_{21})} \leq C\delta(1+t)^{-\frac{1}{2}} + C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.44}$$

Next we estimate $\|\partial_1 u\|_{L^2(A_{22})}$. By (2.3.14),

$$\begin{aligned}\|\widehat{\partial_1 u}(t)\|_{L^2(A_{22})} \leq{}& \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{22})} + \|\widehat{K_2}(t)\widehat{\partial_1 b_0}\|_{L^2(A_{22})}\\ &+ \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau\\ &+ \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})}\, d\tau\\ :={}& S_1 + S_2 + S_3 + S_4.\end{aligned}$$

By Part (3) in Proposition 2.2.2,

$$\begin{aligned}S_1 &= \|\widehat{K_1}(t)\widehat{\partial_1 u_0}\|_{L^2(A_{22})}\\ &\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(A_{22})} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_1 u_0}\|_{L^2(A_{22})}\\ &\leq C\|e^{-c_0(1+\xi_2^2)t}\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)} + \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})t}\widehat{\partial_1 u_0}\|_{L^2(\mathbb{R}^2)}\\ &\leq Ce^{-c_0 t}\|\partial_1 u_0\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}\|\partial_1 u_0\|_{L^2} \leq C\delta(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{2.3.45}$$

Similarly, S_2 admits the same bound, namely,

$$S_2 \leq C(1+t)^{-\frac{1}{2}}\|\partial_1 b_0\|_{L^2} \leq C\delta(1+t)^{-\frac{1}{2}}. \tag{2.3.46}$$

We decompose S_3 into two parts,

$$\begin{aligned}S_3 &= \int_0^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau\\ &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau + \int_{t/2}^t \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau\\ &:= S_{3,1} + S_{3,2}.\end{aligned}$$

To bound $S_{3,1}$, we first apply Part (3) in Proposition 2.2.2 to decompose it into four terms

$$\begin{aligned} S_{3,1} &= \int_0^{t/2} \|\widehat{K_1}(t-\tau)\widehat{\partial_1 N_1}(\tau)\|_{L^2(A_{22})}\, d\tau \\ &\le C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\quad + C\int_0^{t/2} \|e^{-c_0(1+\xi_2^2)(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\quad + C\int_0^{t/2} \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_1(u\cdot\nabla u)}\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\quad + C\int_0^{t/2} \|e^{-c_0(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\widehat{\partial_1(b\cdot\nabla b)}\|_{L^2(\mathbb{R}^2)}\, d\tau. \end{aligned}$$

Then we use the same techniques as in the estimates of $H_{3,1,1}$ in (2.3.38) to obtain

$$\begin{aligned} S_{3,1} &\le C\int_0^{t/2} e^{-c_0(t-\tau)}(\|\partial_1(u\cdot\nabla u)\|_{L^2} + \|\partial_1(b\cdot\nabla b)\|_{L^2})\, d\tau \\ &\le Ce^{-\frac{c_0}{2}t}\int_0^{t/2}(\|\partial_1(u\cdot\nabla u)\|_{L^2} + \|\partial_1(b\cdot\nabla b)\|_{L^2})d\tau \\ &\le C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{2.3.47}$$

Now we use the new bounds in (2.3.25) and (2.3.26) to estimate $S_{3,2}$. By Hölder's inequality and (2.3.25),

$$\begin{aligned} S_{3,2} &\le C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\|_{L^2} + \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\ &+ C\int_{t/2}^{t} \left\|\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2}+1\right)e^{-c_0(1+\xi_2^2)(t-\tau)}\left(|\widehat{\partial_1(u\cdot\nabla u)}| + |\widehat{\partial_1(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\ &+ C\int_{t/2}^{t} \left\|\frac{1}{\eta+\nu\xi_2^2}\left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}+\eta\right)e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)}\left(|\widehat{\partial_1(u\cdot\nabla u)}| + |\widehat{\partial_1(b\cdot\nabla b)}|\right)\right\|_{L^2} d\tau \\ &:= S_{3,2,1} + S_{3,2,2} + S_{3,2,3}. \end{aligned}$$

We further rewrite $S_{3,2,1}$ into two parts,

$$\begin{aligned} S_{3,2,1} &= C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(u\cdot\nabla u)}|\|_{L^2}\, d\tau \\ &\quad + C\int_{t/2}^{t} \|e^{-c_0(1+\xi_2^2)(t-\tau)}|\widehat{\partial_1(b\cdot\nabla b)}|\|_{L^2}\, d\tau \\ &= S_{3,2,1,1} + S_{3,2,1,2}. \end{aligned}$$

By the same estimates as for $H_{3,1,2}$ in (2.3.39),

$$S_{3,2,1,1} \le C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}$$

Clearly, $S_{3,2,1,2}$ admits the same bound, namely,

$$S_{3,2,1} \leq C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.48}$$

Since $\xi \in A_{22}$, we have $\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} < \frac{3}{16}$. By (2.3.48),

$$\begin{aligned}
S_{3,2,2} \leq\ & C\int_{t/2}^{t} \left\| \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} + 1 \right) e^{-c_0(1+\xi_2^2)(t-\tau)} |\widehat{\partial_1(u\cdot\nabla u)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{(\eta+\nu\xi_2^2)^2} + 1 \right) e^{-c_0(1+\xi_2^2)(t-\tau)} |\widehat{\partial_1(b\cdot\nabla b)}| \right\|_{L^2} d\tau \\
\leq\ & C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_1(u\cdot\nabla u)} \|_{L^2}\, d\tau \\
& + C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_1(b\cdot\nabla b)} \|_{L^2}\, d\tau \\
\leq\ & CS_{3,2,1} \leq C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

Furthermore, $S_{3,2,3}$ can be rewritten as

$$\begin{aligned}
S_{3,2,3} =\ & C\int_{t/2}^{t} \left\| \frac{1}{\eta+\nu\xi_2^2} \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2} + \eta \right) e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_1(u\cdot\nabla u)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{1}{\eta+\nu\xi_2^2} \left(\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2} + \eta \right) e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_1(b\cdot\nabla b)}| \right\|_{L^2} d\tau \\
=\ & S_{3,2,3,1} + S_{3,2,3,2}.
\end{aligned}$$

$S_{3,2,3,1}$ is naturally divided into two parts,

$$\begin{aligned}
S_{3,2,3,1} \leq\ & C\int_{t/2}^{t} \left\| \frac{|\xi|^2}{(1+\xi_2^2)^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1(u\cdot\nabla u)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \widehat{u\cdot\nabla u} \right\|_{L^2} d\tau \\
=\ & S_{3,2,3,1,1} + S_{3,2,3,1,2}.
\end{aligned}$$

The process of estimating $S_{3,2,3,1,1}$ is tedious, so we first estimate $S_{3,2,3,1,2}$. By Lemma 2.3.3,

$$\begin{aligned}
S_{3,2,3,1,2} &\le C\int_{t/2}^{t}\left\|\frac{|\xi|}{\sqrt{(1+\xi_2^2)}}(t-\tau)^{\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)}\|\widehat{u\cdot\nabla u}\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|u\cdot\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{4}}\,d\tau\\
&\le C\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}e^{-\frac{c_0}{4}(t-\tau)}(\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{3}{2}}(\tilde{c}\delta(1+\tau)^{-1})^{\frac{1}{4}}(c\delta)^{\frac{1}{4}}d\tau\\
&\le C\tilde{c}^2\delta^2(1+t/2)^{-1}\le C\tilde{c}^2\delta^2(1+t)^{-\frac{1}{2}},
\end{aligned}$$

where we have used $\gamma e^{-c_0\gamma^2}\le C$ and $\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}e^{-\frac{c_0}{4}(t-\tau)}\,d\tau<\infty$. To estimate $S_{3,2,3,1,1}$, we first write $\partial_1(u\cdot\nabla u)=\partial_1\partial_2(uu_2)+\partial_1\partial_1(uu_1)$,

$$\begin{aligned}
S_{3,2,3,1,1} &\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(uu_2)+\partial_1\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&\le C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_2(uu_2)}|\right\|_{L^2}\,d\tau\\
&\quad+C\int_{t/2}^{t}\left\|\frac{|\xi|^2}{(1+\xi_2^2)^2}e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)}|\widehat{\partial_1\partial_1(uu_1)}|\right\|_{L^2}\,d\tau\\
&= S_{3,2,3,1,1,1}+S_{3,2,3,1,1,2}.
\end{aligned}$$

Since $\xi \in A_{22}$, $|\xi|^2 \le C(1+\xi_2^2)^2$. By Lemma 2.3.5 and Lemma 2.3.3,

$$
\begin{aligned}
S_{3,2,3,1,1,1} \le & \int_{t/2}^{t} \left\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\sigma} \left(\frac{|\xi|}{(1+\xi_2^2)} \right)^{2-\sigma} \right. \\
& \qquad \left. \times |\xi_1|^{1-\sigma} (t-\tau)^{-\sigma} e^{-c_0 \frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(uu_2)}| \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} \left\| \left(\frac{|\xi|}{1+\xi_2^2} \right)^{2-\sigma} |\xi_1|^{1-\sigma} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(uu_2)}| \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} \left\| (t-\tau)^{-\sigma} |\xi_1|^{1-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(uu_2)}| \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \left\| \widehat{\Lambda_1^{1-\sigma}(\partial_2(uu_2))} \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u\|_{L^2} \, d\tau \\
& + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 u\|_{L^2}^{\sigma} \|\partial_1 \partial_2 u\|_{L^2}^{1-\sigma} \\
& \qquad \times \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1 u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{1}{4}} \|\Delta u\|^{\frac{1}{4}} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{2\sigma}{3}} (C\delta)^{1-\frac{2\sigma}{3}} \tilde{c}\delta(1+\tau)^{-1} \, d\tau \\
& + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-1})^{\sigma+\frac{1}{4}} (C\delta)^{\frac{5}{4}-\sigma} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{1}{2}} \, d\tau \\
\le & \, C(\sigma) \tilde{c}^{\frac{2\sigma}{3}+1} \delta^2 (1+t)^{-1-\frac{\sigma}{3}} + C(\sigma) \tilde{c}^{\sigma+\frac{3}{4}} \delta^2 (1+t)^{-\sigma-\frac{1}{2}} \\
\le & \, C(\sigma) \tilde{c}^{\frac{3}{2}} \delta^2 (1+t)^{-\frac{1}{2}},
\end{aligned}
$$

where we have set $\sigma = \frac{3}{4}$, and used $\int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} < \infty$, (2.3.29) and the following inequality from (2.3.30),

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_2(uu_2))\|_{L^2} \le & \, C\|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u\|_{L^2} \\
& + C\|\partial_2 u\|_{L^2}^{\sigma} \|\partial_1 \partial_2 u\|_{L^2}^{1-\sigma} \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1 u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{1}{4}} \|\Delta u\|_{L^2}^{\frac{1}{4}}.
\end{aligned}
$$

Similarly, by (2.3.29),

$$
\begin{aligned}
S_{3,2,3,1,1,2} \le & \int_{t/2}^{t} \Bigg\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)\right)^{\sigma} \left(\frac{|\xi|}{(1+\xi_2^2)}\right)^{2-\sigma} \\
& \qquad \times |\xi_1|^{1-\sigma}(t-\tau)^{-\sigma} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1(uu_1)}| \Bigg\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} \Bigg\| \left(\frac{|\xi|}{1+\xi_2^2}\right)^{2-\sigma} |\xi_1|^{1-\sigma}(t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_1^2})(t-\tau)} |\widehat{\partial_1(uu_1)}| \Bigg\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} \Bigg\| (t-\tau)^{-\sigma} |\xi_1|^{1-\sigma} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_1(uu_1)}| \Bigg\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \left\| \widehat{\Lambda_1^{1-\sigma}(\partial_1(uu_1))} \right\|_{L^2} \, d\tau \\
\le & \, C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_1 u\|_{L^2} \, d\tau \\
& + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\sigma} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 u\|_{L^2}^{\sigma} \|\partial_1\partial_1 u\|_{L^2}^{1-\sigma} \\
& \qquad \times \|\partial_2 u\|_{L^2}^{\frac14} \|\partial_1 u\|_{L^2}^{\frac14} \|u\|_{L^2}^{\frac14} \|\Delta u\|^{\frac14} \, d\tau \\
\le & \, C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12},
\end{aligned}
$$

where we have set $\sigma = \frac{3}{4}$, and used the inequality below following from (2.3.30),

$$
\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_1(uu_1))\|_{L^2} \le & \, C\|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_1 u\|_{L^2} \\
& + C\|\partial_1 u\|_{L^2}^{\sigma} \|\partial_1\partial_1 u\|_{L^2}^{1-\sigma} \|\partial_2 u\|_{L^2}^{\frac14} \|\partial_1 u\|_{L^2}^{\frac14} \|u\|_{L^2}^{\frac14} \|\Delta u\|_{L^2}^{\frac14}.
\end{aligned}
$$

Therefore,

$$
S_{3,2,3,1} \le C\tilde{c}^2\delta^2(1+t)^{-\frac12} + \tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12}.
$$

Similarly, $S_{3,2,3,2}$ admits the same bound. Collecting the bounds for $S_{3,2,1}, S_{3,2,2}$ and $S_{3,2,3}$ yields

$$
S_{3,2} \le C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12} + C\tilde{c}^2\delta^2(1+t)^{-\frac12} + C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12}. \tag{2.3.49}
$$

Combining the estimates for $S_{3,1}$ and $S_{3,2}$ in (2.3.47) and (2.3.49) respectively, we have

$$
S_3 \le (\tilde{c} + \tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C\delta^2(1+t)^{-\frac12}. \tag{2.3.50}
$$

Next we estimate S_4. We first divide S_4 into two parts according to (2.3.26),

$$
\begin{aligned}
S_4 = & \int_0^t \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})} \, d\tau \\
= & \int_0^{t/2} \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})} \, d\tau + \int_{t/2}^{t} \|\widehat{K_2}(t-\tau)\widehat{\partial_1 N_2}(\tau)\|_{L^2(A_{22})} \, d\tau \\
:= & \, S_{4,1} + S_{4,2}.
\end{aligned}
$$

By Part (3) in Proposition 2.2.2 and by (2.3.47), $S_{4,1}$ obeys the same bound as $S_{3,1}$, namely,

$$S_{4,1} \le C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}}. \tag{2.3.51}$$

Since the bound for $\widehat{K_2}$ in (2.3.26) is not the same as the bound for $\widehat{K_1}$ in (2.3.25), we need to estimate $S_{4,2}$ differently from $S_{3,2}$.

$$\begin{aligned}
S_{4,2} \le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0(1+\xi_2^2)(t-\tau)} \left(|\widehat{\partial_1(u\cdot\nabla b)}| + |\widehat{\partial_1(b\cdot\nabla u)}| \right) \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} \left(|\widehat{\partial_1(u\cdot\nabla b)}| + |\widehat{\partial_1(b\cdot\nabla u)}| \right) \right\|_{L^2} d\tau \\
:=\ & S_{4,2,1} + S_{4,2,2}.
\end{aligned}$$

Since $\xi \in A_{22}$, $\frac{\xi^2}{(1+\xi_2^2)^2} \le C$. By the same process as in (2.3.39), we write

$$\begin{aligned}
S_{4,2,1} \le\ & C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_1(u\cdot\nabla b)} \|_{L^2}\, d\tau \\
& + C\int_{t/2}^{t} \| e^{-c_0(1+\xi_2^2)(t-\tau)} \widehat{\partial_1(b\cdot\nabla u)} \|_{L^2}\, d\tau \\
\le\ & C\tilde{c}^{\frac{3}{4}}\delta^2(1+t)^{-\frac{1}{2}} + C\tilde{c}\delta^2(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

We rewrite $S_{4,2,2}$ as

$$\begin{aligned}
S_{4,2,2} \le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_1(u\cdot\nabla b)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-\frac{\nu\eta\xi_2^2+\xi_1^2}{\eta+\nu\xi_2^2}(t-\tau)} |\widehat{\partial_1(b\cdot\nabla u)}| \right\|_{L^2} d\tau \\
:=\ & S_{4,2,2,1} + S_{4,2,2,2}.
\end{aligned}$$

To bound $S_{4,2,2,1}$, we write $\partial_1(u\cdot\nabla b) = \partial_1\partial_2(bu_2) + \partial_1\partial_1(bu_1)$,

$$\begin{aligned}
S_{4,2,2,1} \le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(bu_2) + \partial_1\partial_1(bu_1)}| \right\|_{L^2} d\tau \\
\le\ & C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_2(bu_2)}| \right\|_{L^2} d\tau \\
& + C\int_{t/2}^{t} \left\| \frac{|\xi_1|}{1+\xi_2^2} e^{-c_0\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_1\partial_1(bu_1)}| \right\|_{L^2} d\tau \\
=\ & S_{4,2,2,1,1} + S_{4,2,2,1,2}.
\end{aligned}$$

The first piece is bounded by

$$
\begin{aligned}
S_{4,2,2,1,1} &\le \int_{t/2}^{t} \left\| \left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\frac{\sigma+1}{2}} |\xi_1|^{1-\sigma}(t-\tau)^{-\frac{\sigma+1}{2}} e^{-c_0 \frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} |\widehat{\partial_2(bu_2)}| \right\|_{L^2} d\tau \\
&\le C(\sigma) \int_{t/2}^{t} \left\| |\xi_1|^{1-\sigma}(t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(1+\frac{\xi_1^2}{\xi_2^2})(t-\tau)} |\widehat{\partial_2(bu_2)}| \right\|_{L^2} d\tau \\
&\le C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}}(t-\tau) \left\| \widehat{\Lambda_1^{1-\sigma}(\partial_2(bu_2))} \right\|_{L^2} d\tau \\
&\le C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} \|u\|_{L^2}^{\frac{\sigma}{3}} \|\nabla u\|_{L^2}^{\frac{\sigma}{3}} \|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 b\|_{L^2}\, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 b\|_{L^2}^{\sigma} \|\partial_1\partial_2 b\|_{L^2}^{1-\sigma} \\
&\qquad\qquad \times \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1 u\|_{L^2}^{\frac{1}{4}} \|u\|_{L^2}^{\frac{1}{4}} \|\Delta u\|_{L^2}^{\frac{1}{4}}\, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} \|b\|_{L^2}^{\frac{\sigma}{3}} \|\nabla b\|_{L^2}^{\frac{\sigma}{3}} \|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}} \|\partial_2 u_1\|_{L^2}\, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} \|\partial_2 u_1\|_{L^2}^{\sigma} \|\partial_1\partial_2 u_1\|_{L^2}^{1-\sigma} \\
&\qquad\qquad \times \|\partial_2 b\|_{L^2}^{\frac{1}{4}} \|\partial_1 b\|_{L^2}^{\frac{1}{4}} \|b\|_{L^2}^{\frac{1}{4}} \|\Delta b\|_{L^2}^{\frac{1}{4}}\, d\tau \\
&\le C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{2\sigma}{3}} (C\delta)^{1-\frac{2\sigma}{3}} \tilde{c}\delta(1+\tau)^{-1}\, d\tau \\
&\quad + C(\sigma) \int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} (\tilde{c}\delta(1+\tau)^{-1})^{\sigma+\frac{1}{4}} (C\delta)^{\frac{5}{4}-\sigma} (\tilde{c}\delta(1+\tau)^{-\frac{1}{2}})^{\frac{1}{2}}\, d\tau \\
&\le C(\sigma)\tilde{c}^{\frac{2\sigma}{3}+1}\delta^2(1+t)^{-1-\frac{\sigma}{3}} + C(\sigma)\tilde{c}^{\sigma+\frac{3}{4}}\delta^2(1+t)^{-\sigma-\frac{1}{2}} \\
&\le C\tilde{c}^{\frac{3}{2}}\delta^2(1+t)^{-\frac{1}{2}},
\end{aligned}
$$

where we have set $\sigma = \frac{3}{4}$, and used $\int_{t/2}^{t} (t-\tau)^{-\frac{\sigma+1}{2}} e^{-\frac{c_0}{4}(t-\tau)} < \infty$, and

$$
\left(\frac{|\xi|^2}{1+\xi_2^2}(t-\tau) \right)^{\frac{\sigma+1}{2}} e^{-\frac{c_0}{2}\frac{|\xi|^2}{1+\xi_2^2}(t-\tau)} \le C(\sigma).
$$

In addition, we have used the following estimate above, due to Lemma 2.3.5,

$$\begin{aligned}
\|\Lambda_1^{1-\sigma}(\partial_2(bu_2))\|_{L^2} &\le \|\Lambda_1^{1-\sigma}(\partial_2 b u_2)\|_{L^2} + \|\Lambda_1^{1-\sigma}(b\partial_2 u_2)\|_{L^2} \\
&\le C\|\partial_2 b\|_{L^2}\|\Lambda_1^{1-\sigma}u_2\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 b\|_{L^2}\|u_2\|_{L^\infty} \\
&\quad + C\|\partial_2 u_2\|_{L^2}\|\Lambda_1^{1-\sigma}b\|_{L^\infty} + C\|\Lambda_1^{1-\sigma}\partial_2 u_2\|_{L^2}\|b\|_{L^\infty} \\
&\le C\|u\|_{L^2}^{\frac{\sigma}{3}}\|\nabla u\|_{L^2}^{\frac{\sigma}{3}}\|\Delta u\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 b\|_{L^2} \\
&\quad + C\|\partial_2 b\|_{L^2}^{\sigma}\|\partial_1\partial_2 b\|_{L^2}^{1-\sigma}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_1 u\|_{L^2}^{\frac14}\|u\|_{L^2}^{\frac14}\|\Delta u\|_{L^2}^{\frac14} \\
&\quad + C\|b\|_{L^2}^{\frac{\sigma}{3}}\|\nabla b\|_{L^2}^{\frac{\sigma}{3}}\|\Delta b\|_{L^2}^{1-\frac{2\sigma}{3}}\|\partial_2 u_1\|_{L^2} \\
&\quad + C\|\partial_2 u_1\|_{L^2}^{\sigma}\|\partial_1\partial_2 u_1\|_{L^2}^{1-\sigma}\|\partial_2 b\|_{L^2}^{\frac14}\|\partial_1 b\|_{L^2}^{\frac14}\|b\|_{L^2}^{\frac14}\|\Delta b\|_{L^2}^{\frac14}.
\end{aligned}$$

Similarly,

$$S_{4,2,2,1,2} \le C\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12}.$$

Since $S_{4,2,2,2}$ obeys the same bound as $S_{4,2,2,1}$, we obtain

$$S_{4,2,2} \le C(\sigma)\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12}.$$

Therefore,

$$S_{4,2} \le C\tilde{c}^2\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac32}\delta^2(1+t)^{-\frac12} + C\tilde{c}^{\frac34}\delta^2(1+t)^{-\frac12}. \tag{2.3.52}$$

Collecting (2.3.51) and (2.3.52) yields

$$S_4 \le (\tilde{c} + \tilde{c}^2 + \tilde{c}^{\frac34} + \tilde{c}^{\frac32})C\delta^2(1+t)^{-\frac12}. \tag{2.3.53}$$

Combining (2.3.45), (2.3.46), (2.3.50) and (2.3.53), we obtain

$$\|\widehat{\partial_1 u}(t)\|_{L^2(A_{22})} \le C\delta(1+t)^{-\frac12} + (\tilde{c} + \tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C\delta^2(1+t)^{-\frac12} \tag{2.3.54}$$

Putting (2.3.40), (2.3.44) and (2.3.54) together leads to

$$\|\widehat{\partial_1 u}(t)\|_{L^2} \le C_1\delta(1+t)^{-\frac12} + \tilde{c}C_2\delta^2(1+t)^{-\frac12} + (\tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C_3\delta^2(1+t)^{-\frac12}.$$

If we choose $\tilde{c}$ and δ satisfying

$$C_1 \le \frac{\tilde{c}}{8}, \qquad C_2\delta \le \frac{1}{16}, \qquad (\tilde{c}^{\frac34} + \tilde{c}^2 + \tilde{c}^{\frac32})C_3\delta \le \frac{\tilde{c}}{16},$$

then we obtain

$$\begin{aligned}
\|\partial_1 u(t)\|_{L^2} &\le \frac{\tilde{c}}{4}\delta(1+t)^{-\frac12} + \frac{\tilde{c}}{8}\delta(1+t)^{-1} + \frac{\tilde{c}}{8}\delta(1+t)^{-\frac12} \\
&= \frac{\tilde{c}}{2}\delta(1+t)^{-\frac12}.
\end{aligned}$$

A similar bound holds for $\|\partial_1 b\|_{L^2}$. Therefore,

$$\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \le \frac{\tilde{c}}{2}\delta(1+t)^{-\frac12}.$$

This completes the proof of the second inequality in (2.3.2). ∎

CHAPTER III

STABILITY AND EXPONENTIAL DECAY FOR MAGNETOHYDRODYNAMIC EQUATIONS

3.1 Introduction

Let $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a one-dimensional (1D) periodic domain and $\mathbb{R}$ being the real line. Consider the 2D incompressible magnetohydrodynamic (MHD) equations with horizontal dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu\, \partial_{11} u + B \cdot \nabla B, \qquad x \in \Omega,\ t > 0, \\ \partial_t B + u \cdot \nabla B + \eta B = B \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\ u(x, 0) = u_0(x),\ B(x, 0) = B_0(x), \end{cases} \tag{3.1.1}$$

where u denotes the velocity field, B the magnetic field and P the pressure, and $\nu > 0$ and η are the viscosity and the damping coefficients, respectively. Here the velocity u obeys a degenerate Navier-Stokes equation with only horizontal dissipation $\nu\partial_{11}u$ and with a Lorentz forcing term. The magnetic field B satisfies the induction equation with a damping term. The goal of this paper is to understand the stability and the large-time behavior of perturbations near a background magnetic field.

This study is partially motivated by the stabilizing phenomenon of a background magnetic field on electrically conducting fluids that has been observed in physical experiments and numerical simulations (see, e.g., [2, 3, 7, 14, 15, 16, 27, 28]). Since the dynamics of electrically conducting fluids is governed by the MHD equations (see, e.g., [4, 44]), the aim here is to establish this remarkable observation as a mathematically rigorous fact on the MHD equations.

We take the background magnetic field to be the unit vector in the x_1-direction, $B^{(0)} = (1, 0)$. The corresponding steady-state solution of (3.1.1) is given by

$$u^{(0)} = (0, 0), \quad B^{(0)} = (1, 0).$$

We write (u, b) with $b = B - B^{(0)}$ for the perturbation near $(u^{(0)}, B^{(0)})$. Our attention will be focused on the following new system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu\partial_{11} u + b \cdot \nabla b + \partial_1 b, \qquad x \in \Omega,\ t > 0 \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x, 0) = u_0(x),\ b(x, 0) = b_0(x). \end{cases} \tag{3.1.2}$$

In comparison with the original system in (3.1.1), there are two extra terms $\partial_1 b$ and $\partial_1 u$ in (3.1.2). We aim to achieve a complete understanding on the stability of solutions to (3.1.2) in the Sobolev setting. In addition, we also attempt to obtain the precise large-time behavior of (u, b) and establish the eventual dynamics of (3.1.2).

Due to the lack of vertical dissipation in (3.1.2), the resolution of the stability problem is not direct. If we follow the standard energy method approach, the difficulty is immediate. The divergence-free conditions $\nabla\cdot u = \nabla\cdot b = 0$ allow us to obtain a suitable upper bound on the H^1-norm of (u, b), but it does not appear to be possible to control the H^2-norm directly. Even if we completely ignore the terms related to the magnetic field and simply consider the 2D anisotropic Navier-Stokes equations

$$\partial_t u + u\cdot\nabla u + \nabla P = \nu\partial_{11} u,$$

direct energy estimates fail to generate a suitable H^2-bound. In fact, when we resort to the corresponding vorticity formulation

$$\partial_t \omega + u\cdot\nabla\omega = \nu\partial_{11}\omega,$$

the one-directional dissipation is insufficient to bound the nonlinearity directly. In the estimate of $\nabla\omega$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + \nu\|\partial_1\nabla\omega\|_{L^2}^2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\, dx,$$

the right-hand side does not admit a suitable upper bound. In fact,

$$\begin{aligned}\int \nabla\omega\cdot\nabla u\cdot\nabla\omega\, dx = &\int \partial_1 u_1(\partial_1\omega)^2\, dx + \int \partial_1 u_2\partial_1\omega\partial_2\omega\, dx\\ &+ \int \partial_2 u_1\partial_1\omega\,\partial_2\omega\, dx + \int \partial_2 u_2(\partial_2\omega)^2\, dx\end{aligned} \tag{3.1.3}$$

and the two terms in (3.1.3) can not be controlled suitably.

One novel idea to overcome this difficulty is to explore the stabilizing effect of the magnetic field on the fluids as hinted by the aforementioned experimental results. Mathematically we make full use of the coupling and interaction in the MHD system in (3.1.2) to unearth the hidden smoothing and stabilizing properties. To do so, we first apply the Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation to eliminate the pressure,

$$\partial_t u = \nu\partial_{11} u + \partial_1 b + \mathbb{P}(b\cdot\nabla b - u\cdot\nabla u).$$

By differentiating the linearized system in time

$$\begin{cases}\partial_t u = \nu\partial_{11} u + \partial_1 b,\\ \partial_t b = -\eta b + \partial_1 u\end{cases} \tag{3.1.4}$$

and making several substitutions, we can convert (3.1.4) into a system of wave equations

$$\begin{cases}\partial_{tt} u + (\eta + \nu\partial_{11})\partial_t u - (1+\nu\eta)\partial_{11} u = 0,\\ \partial_{tt} b + (\eta + \nu\partial_{11})\partial_t b - (1+\nu\eta)\partial_{11} b = 0.\end{cases} \tag{3.1.5}$$

(3.1.5) allows us to decouple u and b and exhibits more smoothing and stabilizing properties than (3.1.4). In particular, both u and b gain weak horizontal dissipation as can be seen from the pieces $(1+\nu\eta)\partial_{11}u$ and $(1+\nu\eta)\partial_{11}b$. Unfortunately, this extra regularization does not appear to help with the deficiency of vertical dissipation in the velocity equation. As a consequence, this approach fails.

We remark that a previous work of Feng, Hafeez and Wu [25] explored the extra stabilizing and smoothing of the wave structure, and successfully resolved the stability problem on the same MHD system near the background magnetic field $B^{(0)} = (0,1)$. When the background magnetic field is $(0,1)$, the extra regularity is in the vertical direction and complements with the horizontal dissipation in the velocity equation. Therefore, the direction of the background magnetic field plays a crucial role in the stabilizing phenomenon on electrically conduction fluids.

This paper seeks a different approach to resolve the stability problem concerned here. The spatial domain here is $\Omega = \mathbb{T}\times\mathbb{R}$ and we take full advantage of the geometry of this domain. The horizontal direction is periodic and we can separate the zeroth Fourier mode from the non-zero ones. The zeroth Fourier mode corresponds to the horizontal average. This hints the decomposition of the physical quantities into the horizontal averages and the corresponding oscillation parts. More precisely, for a function f that is integrable in $x \in \mathbb{T}$, we define

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1,x_2)\,dx_1, \qquad f = \bar{f} + \widetilde{f}.$$

This decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any integer $k \geq 0$ (see Lemma 3.2.2 in Section 3.2). More crucially, the oscillation part $\widetilde{f}$ obeys a strong version of the Poincaré type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C\,\|\partial_1\widetilde{f}\|_{L^2(\Omega)}.$$

This inequality allows us to control some of the nonlinear parts in terms of the horizontal dissipation. By invoking the decompositions

$$u = \bar{u} + \widetilde{u}, \quad b = \bar{b} + \widetilde{b}$$

and applying the aforementioned Poincaré inequality together with various anisotropic inequalities, we are able to successfully bound the nonlinearity and establish the following stability result.

Theorem 3.1.1 *Let $\eta > 0$ and $\nu > 0$. Consider* (3.1.2) *with the initial data $(u_0, b_0) \in H^3(\Omega)$, and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu,\eta) > 0$ such that, if*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon,$$

then (3.1.2) *has a unique global classical solution (u,b) satisfying, for any $t > 0$,*

$$\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t \left(\|\partial_1 u\|_{H^3}^2 + \|b\|_{H^3}^2\right)\,d\tau \leq C\,\varepsilon^2,$$

where $C > 0$ is independent of ε and t.

Theorem 3.1.1 successfully resolves the stability problem on a partially dissipated MHD system near a background magnetic field even when the smoothing effect of the magnetic field is not sufficient to deal with the dissipation deficiency.

Efforts are also devoted to understanding the precise large-time behavior of the perturbation. We expect the horizontal average $(\bar{u}, \bar{b})$ to behave differently from the oscillation part $(\widetilde{u}, \widetilde{b})$. Intuitively $(\bar{u}, \bar{b})$ corresponds to the zeroth horizontal Fourier mode and the associated dissipation term vanishes. Thus $(\bar{u}, \bar{b})$ may not decay in time. In contrast, $(\widetilde{u}, \widetilde{b})$ consists of non-zero horizontal Fourier modes and the horizontal dissipation effectively plays the role of damping. As a consequence, $(\widetilde{u}, \widetilde{b})$ could decay exponentially in time. Our second theorem rigorously confirms this intuition.

Theorem 3.1.2 *Let $u_0, b_0 \in H^3(\Omega)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Assume that $\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon$ for sufficiently small $\varepsilon > 0$. Let (u, b) be the corresponding solution of (3.1.2). Then the H^1 norm of the oscillation part $(\widetilde{u}, \widetilde{b})$ decays exponentially in time,*

$$\|\widetilde{u}(t)\|_{H^1} + \|\widetilde{b}(t)\|_{H^1} \le (\|u_0\|_{H^1} + \|b_0\|_{H^1}) e^{-C_1 t}, \tag{3.1.6}$$

for some contant $C_1 > 0$ and for all $t > 0$.

We explain the main lines in the proof of Theorem 3.1.1. The local well-posedness of (3.1.2) in the Sobolev space $H^3(\Omega)$ can be shown via standard procedures such as the approach in the book of Majda and Bertozzi [42]. Our attention is focused on the global bound of (u, b) in $H^3(\Omega)$. One of the most suitable tools for this purpose is the bootstrapping argument [52]. To set up the argument, we first construct the energy functional. For the MHD system in (3.1.2), the energy functional $E(t)$ is naturally given by the H^3-norm of (u, b) together with the time integrals from dissipative and damping terms, namely

$$E(t) = \sup_{0 \le \tau \le t} \{\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\} + 2\nu \int_0^t \|\partial_1 u\|_{H^3}^2 \, d\tau + 2\eta \int_0^t \|b\|_{H^3}^2 \, d\tau.$$

The main effort is then devoted to proving the energy inequality

$$E(t) \le E(0) + \ C E^{\frac{3}{2}}(t). \tag{3.1.7}$$

Once (3.1.7) is at our disposal, the bootstrapping argument then implies that, if $E(0) := \|(u_0, b_0)\|_{H^3}^2$ is sufficiently small, say

$$\|(u_0, b_0)\|_{H^3} \le \varepsilon$$

for some suitable $\varepsilon > 0$, then $E(t)$ remains uniformly bounded for any $t > 0$,

$$E(t) \le C\,\varepsilon^2,$$

which gives us the desired global bound on $\|(u(t), b(t))\|_{H^3}$. To prove (3.1.7), we invoke the orthogonal decompositions $u = \bar{u} + \widetilde{u}$ and $b = \bar{b} + \widetilde{b}$, apply the Poincaré type inequalities and anisotropic upper bounds for triple products. More technical details are provided in Section 3.3.

To prove Theorem 3.1.2, we first take the horizontal average of (3.1.2) to obtain the equations of $(\bar{u}, \bar{b})$,

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \bar{p} \end{pmatrix} = \overline{b \cdot \nabla \widetilde{b}}, \\ \partial_t \bar{b} + \overline{u \cdot \nabla \widetilde{b}} + \eta \bar{b} = \overline{b \cdot \nabla \widetilde{u}}. \end{cases} \tag{3.1.8}$$

We then write the equations of $(\widetilde{u}, \widetilde{b})$ by taking the difference of (3.1.2) and (3.1.8),

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + u_2 \partial_2 \bar{u} + \nabla \widetilde{p} - \nu \partial_1^2 \widetilde{u} - \widetilde{b \cdot \nabla \widetilde{b}} - b_2 \partial_2 \bar{b} - \partial_1 \widetilde{b} = 0, \\ \partial_t \widetilde{b} + \widetilde{u \cdot \nabla \widetilde{b}} + u_2 \partial_2 \bar{b} + \eta \widetilde{b} - \widetilde{b \cdot \nabla \widetilde{u}} - b_2 \partial_2 \bar{u} - \partial_1 \widetilde{u} = 0. \end{cases} \tag{3.1.9}$$

The proof of (3.1.6) is divided into the estimates of $\|(\widetilde{u}, \widetilde{b})\|_{L^2}$ and $\|(\nabla \widetilde{u}, \nabla \widetilde{b})\|_{L^2}$. The efforts are devoted to bounding the nonlinearity in terms of the horizontal derivatives of $\widetilde{u}$. Poincaré's inequality and anisotropic upper bounds for the triple products are used extensively. After a tedious process of evaluating many terms, we obtain

$$\begin{aligned} \frac{d}{dt}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{b}\|_{H^1}^2) + (2\nu - C\|(u, b)\|_{H^3})\|\partial_1 \widetilde{u}\|_{H^1}^2 \\ + (2\eta - C\|(u, b)\|_{H^3})\|\widetilde{b}\|_{H^1} \leq 0, \end{aligned}$$

which yields the decay rate in (3.1.6). A detailed proof is provided in Section 3.4.

Finally we briefly summarize some of related results to provide a broader view on the studies of the MHD equations. Fundamental issues on the MHD equations such as well-posedness and stability problems have attracted a lot of attention. Substantial progress has recently been made on the well-posedness problem concerning the MHD equations with various partial or fractional dissipation (see, e.g., [9, 10, 11, 20, 21, 22, 26, 23, 24, 33, 36, 39, 48, 50, 53, 57, 61, 63, 64, 65, 66, 68, 67, 69]). Since the pioneering work of Alfvén [3], the stability problem on various MHD systems has recently gained renewed interests and there are substantial developments. By taking advantage of the Elsässer variables, several papers have successfully solved the stability problem on the ideal MHD equations or the fully dissipated MHD equations with identical (or almost identical) viscosity and magnetic diffusivity (see [6, 8, 29, 54]). The stability problem on the MHD equations with only kinematic dissipation in $\mathbb{R}^2$ or $\mathbb{R}^3$ have been solved via different approaches [18, 30, 31, 37, 38, 45, 46, 51, 58, 59, 70, 71]. The same problem in the periodic setting $\mathbb{T}^3$ has been investigated by [43]. The MHD equations with only magnetic diffusivity have recently been studied for the small data global well-posedness near the trivial solution or a background magnetic field [12, 32, 55, 62, 72], although a complete solution on the stability problem near a background magnetic field is currently lacking. When the velocity equation involves only horizontal or vertical dissipation, the velocity equation itself alone may not be stable and the stability problem relies on the enhanced dissipation resulting from the coupling and interaction. Several such MHD systems with degenerate velocity dissipation have been shown to be stable near suitable background magnetic fields [5, 25, 41, 40, 60].

The rest of this paper is divided into three sections. Section 3.2 states several properties on the aforementioned decomposition and provides several anisotropic inequalities. Section 3.3 proves Theorem 3.1.1 while Section 3.4 presents the proof of Theorem 3.1.2.

3.2 Preliminaries

This section states several properties on the decomposition defined in the introduction and provides several anisotropic inequalities to be used in the proofs of Theorems 3.1.1 and 3.1.2. Some of the materials presented here can be found in [10, 19].

We start by recalling the definition of the horizontal average and the oscillation part. Let $\Omega = \mathbb{T} \times \mathbb{R}$ and let $f = f(x_1, x_2)$ with $(x_1, x_2) \in \Omega$ be sufficiently smooth, say integrable in $x_1 \in \mathbb{T}$. The horizontal average $\overline{f}$ is given by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1. \tag{3.2.1}$$

We decompose f into $\overline{f}$ and the oscillation portion $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{3.2.2}$$

The following lemma is a direct consequence of (3.2.1) and (3.2.2).

Lemma 3.2.1 *The average operator and the oscillation operator commute with the partial derivatives, for $i = 1, 2$,*

$$\partial_i \overline{f} = \overline{\partial_i f}, \quad \partial_i \widetilde{f} = \widetilde{\partial_i f}, \quad \partial_1 \overline{f} = 0, \quad \overline{\widetilde{f}} = 0,$$

As a special consequence, if $\nabla \cdot f = 0$, then

$$\nabla \cdot \overline{f} = 0, \quad \nabla \cdot \widetilde{f} = 0.$$

The second lemma states that the decomposition in (3.2.2) is orthogonal in any Sobolev space $\dot{H}^k(\Omega)$.

Lemma 3.2.2 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. Let $k \geq 0$ be an integer. Let $f \in \dot{H}^k(\Omega)$. Then $\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k(\Omega)$, namely*

$$(\overline{f}, \widetilde{f})_{\dot{H}^k} := \int_\Omega D^k \overline{f} \cdot D^k \widetilde{f}\, dx = 0. \qquad \|f\|^2_{\dot{H}^k(\Omega)} = \|\overline{f}\|^2_{\dot{H}^k(\Omega)} + \|\widetilde{f}\|^2_{\dot{H}^k(\Omega)}$$

In particular, $\|\overline{f}\|_{\dot{H}^k} \leq \|f\|_{\dot{H}^k}$ and $\|\widetilde{f}\|_{\dot{H}^k} \leq \|f\|_{\dot{H}^k}$.

The oscillation part obeys the following Poincaré type inequalities.

Lemma 3.2.3 *If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}.$$

In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

Proof of Lemma 3.2.3. Since the horizontal average of $\widetilde{f}$ is zero, for any fixed $x_2 \in \mathbb{R}$, there is $a \in \mathbb{T}$ such that

$$\widetilde{f}(a, x_2) = 0.$$

Then, for any $(x_1, x_2) \in \Omega$,

$$\widetilde{f}(x_1, x_2) = \int_a^{x_1} \partial_z \widetilde{f}(z, x_2)\, dz \le \int_{\mathbb{T}} |\partial_z \widetilde{f}(z, x_2)|\, dz \le \|\partial_{x_1} \widetilde{f}\|_{L^2(\mathbb{T})}. \tag{3.2.3}$$

Squaring each side of (3.2.3) and integrating over Ω yields the first inequality. The second inequality is obtained by taking the $L^\infty(\Omega)$ in (3.2.3) and using the simple fact that $\|f\|_{L^\infty(\mathbb{R})} \le C\,\|f\|_{H^1(\mathbb{R})}$ for any 1D function $f \in H^1(\mathbb{R})$. ■

Next we present several anisotropic inequalities. Anisotropic upper bounds for triple products are frequently used to bound the nonlinear terms when only partial dissipation is present. In the case when the spatial domain is the whole space $\mathbb{R}^2$, Cao and Wu [10] showed and applied the following inequality

$$\left|\int_{\mathbb{R}^2} f\, g\, h\right| \le C\,\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\,\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\,\|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\,\|\partial_2 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\,\|h\|_{L^2(\mathbb{R}^2)}. \tag{3.2.4}$$

(3.2.4) is a consequence of the elementary 1D inequality

$$\|f\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\,\|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{3.2.5}$$

Another consequence of (3.2.5) is the following inequality

$$\|f\|_{L^\infty(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\,\|\partial_1\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

When the 1D spatial domain is a bounded domain, say $\mathbb{T}$,

$$\|f\|_{L^\infty(\mathbb{T})} \le C\,\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\left(\|f\|_{L^2(\mathbb{T})} + \|f'\|_{L^2(\mathbb{T})}\right)^{\frac{1}{2}}.$$

Since the oscillation part $\widetilde{f}$ has mean zero, for $\widetilde{f} \in H^1(\mathbb{T})$,

$$\|\widetilde{f}\|_{L^\infty(\mathbb{T})} \le C\|\widetilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|(\widetilde{f})'\|_{L^2(\mathbb{T})}^{\frac{1}{2}}.$$

As a consequence of these elementary inequalities, the following two lemmas hold.

Lemma 3.2.4 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\int_\Omega |fgh|\, dx \le C\|f\|_{L^2}^{\frac{1}{2}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}.$$

For any $f \in H^2(\Omega)$, we have

$$\begin{aligned}\|f\|_{L^\infty}(\Omega) \le C\|f\|_{L^2(\Omega)}^{\frac{1}{4}}(\|f\|_{L^2(\Omega)} + \|\partial_1 f\|_{L^2(\Omega)})^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\Omega)}^{\frac{1}{4}}\\ \times (\|\partial_2 f\|_{L^2(\Omega)} + \|\partial_1\partial_2 f\|_{L^2(\Omega)})^{\frac{1}{4}}.\end{aligned}$$

After replacing f by the oscillation part, we have the following inequalities.

Lemma 3.2.5 *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f, g, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\int_\Omega |\widetilde{f} g h| \, dx \leq C \|\widetilde{f}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{f}\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}.$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leq C \|\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\partial_2 \widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\partial_1 \partial_2 \widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}}.$$

3.3 Nonlinear Stability Result

This section is devoted to the proof of Theorem 3.1.1 on the stability of (3.1.2).

Proof of Theorem 3.1.1. The local well-posedness of (3.1.2) in the Sobolev space $H^3(\Omega)$ can be shown via standard procedures such as the approach in the book of Majda and Bertozzi [42]. Our attention is focused on the global bound of (u, b) in $H^3(\Omega)$.

The framework of the proof is the bootstrapping argument. To proceed, we define the energy functional as

$$E(t) = \sup_{0 \leq \tau \leq t} \{\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\} + 2\nu \int_0^t \|\partial_1 u\|_{H^3}^2 \, d\tau + 2\eta \int_0^t \|b\|_{H^3}^2 \, d\tau. \tag{3.3.1}$$

Our main efforts are devoted to proving the following energy inequality

$$E(t) \leq E(0) + \ C E^{\frac{3}{2}}(t). \tag{3.3.2}$$

As we explain later, a direct application of the bootstrapping argument to (3.3.2) implies the desired global uniform bound on $\|(u, b)\|_{H^3}$.

Attention is first focused on proving (3.3.1). Due to the equivalence of the inhomogeneous norm $\|(u, b)\|_{H^3}$ with the sum of the L^2-norm and the homogeneous norm $\|(u, b)\|_{\mathring{H}^3}$, it suffices to bound the homogeneous norm $\|(u, b)\|_{\mathring{H}^3}$. The uniform L^2-bound is an easy consequence of the system in (3.1.2) itself. Taking the inner product of (3.1.2) with (u, b), we obtain, after integrating by parts and using $\nabla \cdot u = \nabla \cdot b = 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu \int_0^t \|\partial_1 u\|_{L^2}^2 \, d\tau + 2\eta \int_0^t \|b\|_{L^2}^2 \, d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \tag{3.3.3}$$

To estimate the homogeneous norm $\|(u, b)\|_{\mathring{H}^3}$, we apply $\partial_i^3 (i = 1, 2)$ to (3.1.2) and then dot with $(\partial_i^3 u, \partial_i^3 b)$ to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \sum_{i=1}^2 (\|\partial_i^3 u\|_{L^2}^2 + \|\partial_i^3 b\|_{L^2}^2) + \sum_{i=1}^2 \nu \|\partial_i^3 \partial_1 u\|_{L^2}^2 + \sum_{i=1}^2 \eta \|\partial_i^3 b\|_{L^2}^2 \\ &\quad := J + K + L + M + N, \end{aligned} \tag{3.3.4}$$

where

$$J = \sum_{i=1}^{2} \int_\Omega \partial_i^3 \partial_1 b \cdot \partial_i^3 u + \partial_i^3 \partial_1 u \cdot \partial_i^3 b \; dx,$$

$$K = -\sum_{i=1}^{2} \int_\Omega \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u \; dx,$$

$$L = \sum_{i=1}^{2} \int_\Omega (\partial_i^3 (b \cdot \nabla b) - b \cdot \nabla \partial_i^3 b) \cdot \partial_i^3 u \; dx,$$

$$M = -\sum_{i=1}^{2} \int_\Omega \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \; dx,$$

$$N = \sum_{i=1}^{2} \int_\Omega (\partial_i^3 (b \cdot \nabla u) - b \cdot \nabla \partial_i^3 u) \cdot \partial_i^3 b \; dx.$$

By integration by parts, $J = 0$. The estimate of K is long and tedious, and is provided in the later part of the proof. To bound L, we decompose it into two parts,

$$\begin{aligned}
L &= \sum_{i=1}^{2} \left(\int_\Omega \partial_i^3 (b \cdot \nabla b) \cdot \partial_i^3 u \; dx - \int_\Omega b \cdot \nabla \partial_i^3 b \cdot \partial_i^3 u \; dx \right) \\
&= \sum_{i=1}^{2} \sum_{k=1}^{3} C_3^k \int_\Omega \partial_i^k b \cdot \partial_i^{3-k} \nabla b \cdot \partial_i^3 u \; dx \\
&= \sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k b \cdot \partial_1^{3-k} \nabla b \cdot \partial_1^3 u \; dx + \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k b \cdot \partial_2^{3-k} \nabla b \cdot \partial_2^3 u \; dx \\
&= L_1 + L_2,
\end{aligned}$$

where $C_3^k = \frac{3!}{k!(3-k)!}$ is the binomial coefficient. By Lemma 3.2.1 and Lemma 3.2.5,

$$\begin{aligned}
L_1 &= \sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k \widetilde{b} \cdot \partial_1^{3-k} \nabla \widetilde{b} \cdot \partial_1^3 \widetilde{u} \; dx \\
&\lesssim \sum_{k=1}^{2} \|\partial_1^{3-k} \nabla \widetilde{b}\|_{L^2} \|\partial_1^k \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1^k \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1^3 \widetilde{b}\|_{L^2} \|\nabla \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3}^2 \|\partial_1 u\|_{H^3}.
\end{aligned}$$

We further decompose L_2 into three terms,

$$\begin{aligned}
L_2 &= 3 \int_\Omega \partial_2 b \cdot \partial_2^2 \nabla b \cdot \partial_2^3 u \; dx + 3 \int_\Omega \partial_2^2 b \cdot \partial_2 \nabla b \cdot \partial_2^3 u \; dx + \int_\Omega \partial_2^3 b \cdot \nabla b \cdot \partial_2^3 u \; dx \\
&= L_{2,1} + L_{2,2} + L_{2,3}.
\end{aligned}$$

By Hölder's inequality and Lemma 3.2.4,

$$
\begin{aligned}
L_{2,1} &\lesssim \|\partial_2 b\|_{L^\infty}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 u\|_{L^2} \\
&\lesssim \|\partial_2 b\|_{L^2}^{\frac{1}{4}}(\|\partial_2 b\|_{L^2}+\|\partial_{12}b\|_{L^2})^{\frac{1}{4}}\|\partial_2^2 b\|_{L^2}^{\frac{1}{4}} \\
&\quad \times(\|\partial_2^2 b\|_{L^2}+\|\partial_1\partial_2^2 b\|_{L^2})^{\frac{1}{4}}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 u\|_{L^2} \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_{2,3} &\lesssim \ \|\nabla b\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 u\|_{L^2} \\
&\lesssim \|\nabla b\|_{L^2}^{\frac{1}{4}}\|\nabla b\|_{L^2}+\|\partial_1\nabla b\|_{L^2})^{\frac{1}{4}}\|\partial_2\nabla b\|_{L^2}^{\frac{1}{4}} \\
&\quad \times((\|\partial_2\nabla b\|_{L^2}+\|\partial_1\partial_2\nabla b\|_{L^2})^{\frac{1}{4}}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 u\|_{L^2} \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

By Lemma 3.2.1, $\partial_2\overline{b_2}=-\partial_1\overline{b_1}=0$ and Lemma 3.2.5,

$$
\begin{aligned}
L_{2,2} &= 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla b\cdot\partial_2^3 u\ dx \\
&= 3\left(\int_\Omega \partial_2^2\overline{b_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 u\ dx+\int_\Omega \partial_2^2\widetilde{b_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 u\ dx+\int_\Omega \partial_2^2\widetilde{b_2}\partial_2^2 b\cdot\partial_2^3 u\ dx\right) \\
&\lesssim \|\partial_2^3 u\|_{L^2}\|\partial_{21}\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{21}\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\overline{b_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2^2\overline{b_1}\|_{L^2}^{\frac{1}{2}} \\
&\quad +\|\partial_2^3 u\|_{L^2}\|\partial_{21}\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_{21}\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\widetilde{b_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2^2\widetilde{b_1}\|_{L^2}^{\frac{1}{2}} \\
&\quad +\|\partial_2^3 u\|_{L^2}\|\partial_2^2\widetilde{b_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^2\widetilde{b_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_2^2 b\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

Combining the estimates of L_1 and L_2, we obtain

$$
L\lesssim \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2+\|u\|_{H^3}\|b\|_{H^3}^2. \tag{3.3.5}
$$

Now we estimate M,

$$
\begin{aligned}
M &= -\sum_{i=1}^{2}\int_\Omega \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\ dx, \\
&= -\int_\Omega \partial_1^3(u\cdot\nabla b)\cdot\partial_1^3 b\ dx-\int_\Omega \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\ dx, \\
&= M_1+M_2.
\end{aligned}
$$

By Lemma 3.2.1,

$$
\begin{aligned}
M_1 &= -\sum_{k=1}^{3}C_3^k\int_\Omega \partial_1^k\widetilde{u}\cdot\partial_1^{3-k}\nabla\widetilde{b}\cdot\partial_1^3\widetilde{b}\ dx-\int_\Omega \widetilde{u}\cdot\partial_1^3\nabla\widetilde{b}\cdot\partial_1^3\widetilde{b}\ dx \\
&= M_{1,1}+M_{1,2}.
\end{aligned}
$$

By Lemma 3.2.5, Hölder's inequality, and Lemma 3.2.3,

$$\begin{aligned}
M_{1,1} &\lesssim \sum_{k=2}^{3} \|\partial_1^3\widetilde{b}\|_{L^2}\|\partial_1^k\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1^k\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1^{3-k}\nabla\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^{3-k}\nabla\widetilde{b}\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1\widetilde{u}\|_{L^\infty}\|\partial_1^2\nabla\widetilde{b}\|_{L^2}\|\partial_1^3\widetilde{b}\|_{L^2} \\
&\lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3} + \|\partial_1^2\widetilde{u}\|_{H^1}\|b\|_{H^3}^2 \\
&\lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3}.
\end{aligned}$$

By integration by parts and $\nabla\cdot\widetilde{u}=0$,

$$M_{1,2} = -\int_\Omega \widetilde{u}\cdot\partial_1^3\nabla\widetilde{b}\cdot\partial_1^3\widetilde{b}\,dx = -\frac{1}{2}\int_\Omega \widetilde{u}\cdot\nabla(\partial_1^3\widetilde{b})^2\,dx = 0.$$

To estimate M_2, we split it into four terms,

$$\begin{aligned}
M_2 &= -\int_\Omega \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\,dx, \\
&= -\sum_{k=1}^{3} C_3^k\int_\Omega \partial_2^k u\cdot\partial_2^{3-k}\nabla b\cdot\partial_2^3 b\,dx - \int_\Omega u\cdot\partial_2^3\nabla b\cdot\partial_2^3 b\,dx \\
&= M_{2,1}+M_{2,2}+M_{2,3}+M_{2,4}.
\end{aligned}$$

$M_{2,4}=0$ due to $\nabla\cdot u=0$. By Hölder's inequality and Lemma 3.2.4,

$$\begin{aligned}
M_{2,1} &= -3\int_\Omega \partial_2 u\cdot\partial_2^2\nabla b\cdot\partial_2^3 b\,dx \\
&\lesssim \|\partial_2 u\|_{L^\infty}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{L^2}^{\frac{1}{4}}(\|\partial_2 u\|_{L^2}+\|\partial_{12}u\|_{L^2})^{\frac{1}{4}}\|\partial_2^2 u\|_{L^2}^{\frac{1}{4}}(\|\partial_2^2 u\|_{L^2}+\|\partial_1\partial_2^2 u\|_{L^2})^{\frac{1}{4}}\|b\|_{H^3}^2 \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
M_{2,3} &= -\int_\Omega \partial_2^3 u\cdot\nabla b\cdot\partial_2^3 b\,dx \\
&\lesssim \ \|\nabla b\|_{L^\infty}\|\partial_2^3 u\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\lesssim \|\nabla b\|_{L^2}^{\frac{1}{4}}(\|\nabla b\|_{L^2}+\|\partial_1\nabla b\|_{L^2})^{\frac{1}{4}}\|\partial_2\nabla b\|_{L^2}^{\frac{1}{4}}(\|\partial_2\nabla b\|_{L^2} \\
&\quad +\|\partial_1\partial_2\nabla b\|_{L^2})^{\frac{1}{4}}\|\partial_2^3 u\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

By Lemma 3.2.1, $\partial_2\overline{u_2}=-\partial_1\overline{u_1}$ and Lemma 3.2.5,

$$\begin{aligned}
M_{2,2} &= -3\int_\Omega \partial_2^2 u\cdot\partial_2\nabla b\cdot\partial_2^3 b\,dx \\
&= -3\left(\int_\Omega \partial_2^2\overline{u_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 b\,dx + \int_\Omega \partial_2^2\widetilde{u}_1\partial_{21}\widetilde{b}\cdot\partial_2^3 b\,dx + \int_\Omega \partial_2^2\widetilde{u}_2\partial_2^2 b\cdot\partial_2^3 b\,dx\right) \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

Combining the estimates for M_1 and M_2, we obtain

$$M \lesssim \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2 + \|u\|_{H^3}\|b\|_{H^3}^2. \tag{3.3.6}$$

Now we estimate the term N,

$$\begin{aligned}
N &= \sum_{i=1}^{2}\left(\int_\Omega \partial_i^3(b\cdot\nabla u)\cdot\partial_i^3 b\,dx - \int_\Omega b\cdot\nabla\partial_i^3 u\cdot\partial_i^3 b\,dx\right)\\
&= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k \int_\Omega \partial_i^k b\cdot\partial_i^{3-k}\nabla u\cdot\partial_i^3 b\,dx\\
&= \sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k b\cdot\partial_1^{3-k}\nabla u\cdot\partial_1^3 b\,dx + \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k b\cdot\partial_2^{3-k}\nabla u\cdot\partial_2^3 b\,dx\\
&= N_1 + N_2.
\end{aligned}$$

By Lemma 3.2.1, Lemma 3.2.5, Hölder's inequality, and Lemma 3.2.3,

$$\begin{aligned}
N_1 &= \sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k\widetilde{b}\cdot\partial_1^{3-k}\nabla\widetilde{u}\cdot\partial_1^3\widetilde{b}\,dx\\
&\lesssim \sum_{k=1}^{2}\|\partial_1^3\widetilde{b}\|_{L^2}\|\partial_1^{3-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1^{3-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^k\widetilde{b}\|_{L^2}^{\frac12}\|\partial_2\partial_1^k\widetilde{b}\|_{L^2}^{\frac12}\\
&\quad + \|\nabla\widetilde{u}\|_{L^\infty}\|\partial_1^3\widetilde{b}\|_{L_2}\|\partial_1^3\widetilde{b}\|_{L^2}\\
&\lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3} + \|\partial_1\nabla\widetilde{u}\|_{H^1}\|b\|_{H^3}^2 \lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3}.
\end{aligned}$$

To bound N_2 we further decompose it into three terms as

$$\begin{aligned}
N_2 &= 3\int_\Omega \partial_2 b\cdot\partial_2^2\nabla u\cdot\partial_2^3 b\,dx + 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla u\cdot\partial_2^3 b\,dx + \int_\Omega \partial_2^3 b\cdot\nabla u\cdot\partial_2^3 b\,dx\\
&= N_{2,1} + N_{2,2} + N_{2,3}.
\end{aligned}$$

By Hölder's inequality and Lemma 3.2.4,

$$\begin{aligned}
N_{2,1} &\lesssim \|\partial_2 b\|_{L^\infty}\|\partial_2^2\nabla u\|_{L^2}\|\partial_2^3 b\|_{L^2}\\
&\lesssim \|\partial_2 b\|_{L^2}^{\frac14}(\|\partial_2 b\|_{L^2} + \|\partial_{12} b\|_{L^2})^{\frac14}\|\partial_2^2 b\|_{L^2}^{\frac14}\\
&\quad\times(\|\partial_2^2 b\|_{L^2} + \|\partial_1\partial_2^2 b\|_{L^2})^{\frac14}\|\partial_2^2\nabla u\|_{L^2}\|\partial_2^3 b\|_{L^2}\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_{2,3} &\lesssim \|\nabla u\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 b\|_{L^2}\\
&\lesssim \|\nabla u\|_{L^2}^{\frac14}(\|\nabla u\|_{L^2} + \|\partial_1\nabla u\|_{L^2})^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14}\\
&\quad\times(\|\partial_2\nabla u\|_{L^2} + \|\partial_1\partial_2\nabla u\|_{L^2})^{\frac14}\|\partial_2^3 b\|_{L^2}^2\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

By Lemma 3.2.1, $\partial_2\overline{b_2} = -\partial_1\overline{b_1} = 0$ and Lemma 3.2.5,

$$
\begin{aligned}
N_{2,2} &= 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla u\cdot\partial_2^3 b\ dx\\
&= 3\left(\int_\Omega \partial_2^2\overline{b_1}\partial_{21}\widetilde{u}\cdot\partial_2^3 b\ dx + \int_\Omega \partial_2^2\widetilde{b_1}\partial_{21}\widetilde{u}\cdot\partial_2^3 b\ dx + \int_\Omega \partial_2^2\widetilde{b_2}\partial_2^2 u\cdot\partial_2^3 b\ dx\right)\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

Combining estimates of N_1 and N_2, we have

$$
N \lesssim \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2 + \|u\|_{H^3}\|b\|_{H^3}^2. \tag{3.3.7}
$$

We now turn to the term K. We split K into two terms,

$$
\begin{aligned}
K &= -\int_\Omega \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\ dx - \int_\Omega \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\ dx,\\
&= K_1 + K_2.
\end{aligned}
$$

By integration by parts, Lemma 3.2.1, Lemma 3.2.5 and Lemma 3.2.3,

$$
\begin{aligned}
K_1 &= -\int_\Omega \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\ dx\\
&= \int_\Omega \partial_1^2(u\cdot\nabla u)\cdot\partial_1^4 u\ dx\\
&= \sum_{k=0}^{2} C_2^k\int_\Omega \partial_1^k u\cdot\partial_1^{2-k}\nabla u\cdot\partial_1^4 u\ dx\\
&= \sum_{k=0}^{2} C_2^k\int_\Omega \partial_1^k \widetilde{u}\cdot\partial_1^{2-k}\nabla \widetilde{u}\cdot\partial_1^4 \widetilde{u}\ dx\\
&\lesssim \sum_{k=1}^{2}\|\partial_1^4\widetilde{u}\|_{L^2}\|\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\\
&\quad + \|\partial_1^4\widetilde{u}\|_{L^2}\|\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}\\
&\lesssim \sum_{k=1}^{2}\|\partial_1^4\widetilde{u}\|_{L^2}\|\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\\
&\quad + \|\partial_1^4\widetilde{u}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}.
\end{aligned}
$$

Therefore,

$$
K_1 \lesssim \|\partial_1\widetilde{u}\|_{H^3}^2\|\widetilde{u}\|_{H^3}. \tag{3.3.8}
$$

To bound K_2, we further decompose it into four terms,

$$
\begin{aligned}
K_2 &= -\int_\Omega \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\ dx,\\
&= -\sum_{k=0}^{3} C_3^k\int_\Omega \partial_2^k u\cdot\partial_2^{3-k}\nabla u\cdot\partial_2^3 u\ dx\\
&= K_{2,1} + K_{2,2} + K_{2,3} + K_{2,4}.
\end{aligned} \tag{3.3.9}
$$

By integration by parts and $\nabla \cdot u = 0$,

$$K_{2,1} = -\int_\Omega u \cdot \partial_2^3 \nabla u \cdot \partial_2^3 u \, dx = 0.$$

Next we bound $K_{2,2}$. By Lemma 3.2.1 and $\nabla \cdot u = 0$,

$$\begin{aligned}
K_{2,2} &= -3 \int_\Omega \partial_2 u \cdot \partial_2^2 \nabla u \cdot \partial_2^3 u \, dx \\
&= -3 \left(\int_\Omega \partial_2 u_1 \partial_2^2 \partial_1 u \cdot \partial_2^3 u \, dx + \int_\Omega \partial_2 u_2 \partial_2^2 \partial_2 u \cdot \partial_2^3 u \, dx \right) \\
&= -3 \int_\Omega \partial_2 \overline{u_1} \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \overline{u} \, dx - 3 \int_\Omega \partial_2 \overline{u_1} \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \widetilde{u} \, dx \\
&\quad - 3 \int_\Omega \partial_2 \widetilde{u}_1 \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \overline{u} \, dx - 3 \int_\Omega \partial_2 \widetilde{u}_1 \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \widetilde{u} \, dx \\
&\quad + 3 \int_\Omega \partial_1 \widetilde{u}_1 \partial_2^2 \partial_2 \overline{u} \cdot \partial_2^3 \overline{u} \, dx + 3 \int_\Omega \partial_1 \widetilde{u}_1 \partial_2^2 \partial_2 \overline{u} \cdot \partial_2^3 \widetilde{u} \, dx \\
&\quad + 3 \int_\Omega \partial_1 \widetilde{u}_1 \partial_2^2 \partial_2 \widetilde{u} \cdot \partial_2^3 \overline{u} \, dx + 3 \int_\Omega \partial_1 \widetilde{u}_1 \partial_2^2 \partial_2 \widetilde{u} \cdot \partial_2^3 \widetilde{u} \, dx \\
&= K_{2,2,1} + K_{2,2,2} + K_{2,2,3} + K_{2,2,4} + K_{2,2,5} + K_{2,2,6} + K_{2,2,7} + K_{2,2,8}.
\end{aligned}$$

By integration by parts and Lemma 3.2.1,

$$\begin{aligned}
K_{2,2,1} &= -3 \int_\Omega \partial_2 \overline{u_1} \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \overline{u} \, dx \\
&= 3 \left(\int_\Omega \partial_1 \partial_2 \overline{u_1} \partial_2^2 \widetilde{u} \cdot \partial_2^3 \overline{u} \, dx + \int_\Omega \partial_2 \overline{u_1} \partial_2^2 \widetilde{u} \cdot \partial_1 \partial_2^3 \overline{u} \, dx \right) = 0.
\end{aligned}$$

Similarly, $K_{2,2,5} = 0$. By Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}
K_{2,2,2} &= -3 \int_\Omega \partial_2 \overline{u_1} \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \widetilde{u} \, dx \\
&\lesssim \|\partial_2 \overline{u_1}\|_{L^2} \|\partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 \overline{u_1}\|_{L^2} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}.
\end{aligned}$$

$K_{2,2,4}$ and $K_{2,2,8}$ can be bounded similarly as $K_{2,2,2}$. By Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}
K_{2,2,3} &= -3 \int_\Omega \partial_2 \widetilde{u}_1 \partial_2^2 \partial_1 \widetilde{u} \cdot \partial_2^3 \overline{u} \, dx \\
&\lesssim \|\partial_2^3 \overline{u}\|_{L^2} \|\partial_2 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^3 \overline{u}\|_{L^2} \|\partial_1 \partial_2 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2^2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}.
\end{aligned}$$

By Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}
K_{2,2,6} &= 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&\lesssim \|\partial_2^2\partial_2\overline{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^2\partial_2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

$$\begin{aligned}
K_{2,2,7} &= 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\widetilde{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&\lesssim \|\partial_2^2\partial_2\widetilde{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^2\partial_2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

Therefore,

$$K_{2,2} \lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.$$

Now we bound $K_{2,3}$. By Lemma 3.2.1 and the divergence free condition,

$$\begin{aligned}
K_{2,3} &= -3\int_\Omega \partial_2^2 u\cdot\partial_2\nabla u\cdot\partial_2^3 u\;dx \\
&= -3\left(\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u\cdot\partial_2^3 u\;dx + \int_\Omega \partial_2^2 u_2\partial_2\partial_2 u\cdot\partial_2^3 u\;dx\right) \\
&= -3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\;dx - 3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&\quad - 3\int_\Omega \partial_2^2\widetilde{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\;dx - 3\int_\Omega \partial_2^2\widetilde{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&\quad + 3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\overline{u}\cdot\partial_2^3\overline{u}\;dx + 3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&\quad + 3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\widetilde{u}\cdot\partial_2^3\overline{u}\;dx + 3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\widetilde{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&= K_{2,3,1}+K_{2,3,2}+K_{2,3,3}+K_{2,3,4}+K_{2,3,5}+K_{2,3,6}+K_{2,3,7}+K_{2,3,8}.
\end{aligned}$$

Clearly $K_{2,3,1}=0$ and $K_{2,3,5}=0$. To bound the remaining terms of $K_{2,3}$ we use Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}
K_{2,3,2} &= -3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\;dx \\
&\lesssim \|\partial_2^2\overline{u_1}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^2\overline{u_1}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

$K_{2,3,4}$ and $K_{2,3,7}$ can be bounded similarly,

$$\begin{aligned}
K_{2,3,3} &= -3\int_\Omega \partial_2^2\widetilde{u}_1\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx \\
&\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_2^2\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_1\partial_2^2\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

$$\begin{aligned}
K_{2,3,6} &= 3\int_\Omega \partial_2\partial_1\widetilde{u}_1\partial_2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\,dx \\
&\lesssim \|\partial_2^2\overline{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

$K_{2,3,8}$ can also be bounded similarly. Hence,

$$K_{2,3} \lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.$$

Now we bound the last term $K_{2,4}$ in (3.3.9). By Lemma 3.2.1 and $\nabla\cdot u = 0$,

$$\begin{aligned}
K_{2,4} &= \int_\Omega \partial_2^3 u\cdot\nabla u\cdot\partial_2^3 u\,dx \\
&= -\left(\int_\Omega \partial_2^3 u_1\partial_1 u\cdot\partial_2^3 u\,dx + \int_\Omega \partial_2^3 u_2\partial_2 u\cdot\partial_2^3 u\,dx\right) \\
&= -\int_\Omega \partial_2^3\overline{u_1}\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx - \int_\Omega \partial_2^3\overline{u_1}\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx \\
&\quad - \int_\Omega \partial_2^3\widetilde{u}_1\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx - \int_\Omega \partial_2^3\widetilde{u}_1\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx \\
&\quad + \int_\Omega \partial_2^2\partial_1\widetilde{u}_1\partial_2\overline{u}\cdot\partial_2^3\overline{u}\,dx + \int_\Omega \partial_2^2\partial_1\widetilde{u}_1\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\,dx \\
&\quad + \int_\Omega \partial_2^2\partial_1\widetilde{u}_1\partial_2\widetilde{u}\cdot\partial_2^3\overline{u}\,dx + \int_\Omega \partial_2^2\partial_1\widetilde{u}_1\partial_2\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx \\
&= K_{2,4,1} + K_{2,4,2} + K_{2,4,3} + K_{2,4,4} + K_{2,4,5} + K_{2,4,6} + K_{2,4,7} + K_{2,4,8}.
\end{aligned}$$

Again $K_{2,4,1} = 0$ and $K_{2,4,5} = 0$. To bound the remaining terms of $K_{2,4}$ we use Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}
K_{2,4,2} &= -\int_\Omega \partial_2^3\overline{u_1}\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx \\
&\lesssim \|\partial_2^3\overline{u_1}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^3\overline{u_1}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}$$

$K_{2,4,4}$ and $K_{2,4,7}$ can be bounded similarly.

$$\begin{aligned} K_{2,4,3} &= -\int_\Omega \partial_2^3\widetilde{u}_1\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx \\ &\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_2^3\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}. \end{aligned}$$

$$\begin{aligned} K_{2,4,6} &= \int_\Omega \partial_2^2\partial_1\widetilde{u}_1\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\,dx \\ &\lesssim \|\partial_2\overline{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^3\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2^2\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\partial_2^3\partial_1\widetilde{u}_1\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}. \end{aligned}$$

$K_{2,4,8}$ can be bounded similarly. Hence,

$$K_{2,4} \lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.$$

Putting together the upper bounds for $K_{2,1}$ through $K_{2,4}$, we find

$$K_2 \lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}. \tag{3.3.10}$$

Collecting the upper bounds in (3.3.8) and (3.3.10) yields

$$K \lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}. \tag{3.3.11}$$

Integrating (3.3.4) in time and then adding to (3.3.3), we have, after recalling the definition of E in (3.3.1),

$$E(t) \le E(0) + \int_0^t (J+K+L+M+N)\,d\tau.$$

Collecting the upper bounds in (3.3.5), (3.3.6), (3.3.7)and (3.3.11) leads to the desired inequality in (3.3.2),

$$\begin{aligned} E(t) &\lesssim E(0) + \int_0^t (\|\partial_1 u\|_{H^3}^2\|u\|_{H^3} + \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2 + \|u\|_{H^3}\|b\|_{H^3}^2)\,d\tau \\ &\le \quad E(0) + C\,E^{\frac{3}{2}}(t). \end{aligned} \tag{3.3.12}$$

We apply the bootstrapping argument to (3.3.12). The initial data is taken to be sufficiently small, say

$$\|(u_0,b_0)\|_{H^3} \le \varepsilon$$

with ε satisfying

$$4\,\varepsilon^2 \le \delta_0 := \frac{1}{4C^2}.$$

We make the ansatz that, for $0 \le t \le T$

$$E(t) \le \delta_0.$$

Then (3.3.12) implies

$$\begin{aligned} E(t) &\le \varepsilon^2 + C E^{\frac{1}{2}}(t)\, E(t) \\ &\le \varepsilon^2 + C\,\frac{1}{2C}\, E(t) \end{aligned}$$

or

$$\frac{1}{2}E(t) \le \varepsilon^2 \quad \text{or} \quad E(t) \le 2\varepsilon^2 = \frac{1}{2}\delta_0.$$

The bootstrapping argument then implies that $T = \infty$ and $E(t) \le \delta_0$. As a consequence, for any $0 \le t \le \infty$,

$$\|(u(t), b(t))\|_{H^3}^2 \le E(t) \le \delta_0.$$

This completes the proof for Theorem 3.1.1. ■

3.4 Exponential Decay of the oscillation part $(\widetilde{u}, \widetilde{b})$ Result

This section proves Theorem 3.1.2, which assesses that the oscillation part $(\widetilde{u}, \widetilde{b})$ decays exponentially to zero in the H^1-norm as $t \to \infty$. We consider the equations of $(\widetilde{u}, \widetilde{b})$ and apply the properties of the orthogonal decomposition and several anisotropic inequalities.

Proof of Theorem 3.1.2. We first write the equation of $(\bar{u}, \bar{b})$. By taking the average of (3.1.2), we have

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \bar{p} \end{pmatrix} = \overline{b \cdot \nabla \widetilde{b}}, \\ \partial_t \bar{b} + \overline{u \cdot \nabla \widetilde{b}} + \eta \bar{b} = \overline{b \cdot \nabla \widetilde{u}}. \end{cases} \tag{3.4.1}$$

Taking the difference of (3.1.2) and (3.4.1), we obtain

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + u_2 \partial_2 \bar{u} + \nabla \widetilde{p} - \nu \partial_1^2 \widetilde{u} - \widetilde{b \cdot \nabla \widetilde{b}} - b_2 \partial_2 \bar{b} - \partial_1 \widetilde{b} = 0, \\ \partial_t \widetilde{b} + \widetilde{u \cdot \nabla \widetilde{b}} + u_2 \partial_2 \bar{b} + \eta \widetilde{b} - \widetilde{b \cdot \nabla \widetilde{u}} - b_2 \partial_2 \bar{u} - \partial_1 \widetilde{u} = 0. \end{cases} \tag{3.4.2}$$

Taking the inner product of (3.4.2) with $(\widetilde{u}, \widetilde{b})$, after integration by parts and divergence-free conditions, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|_{L^2} + \|\widetilde{b}\|_{L^2}) + \nu \|\partial_1 \widetilde{u}\|_{L^2}^2 + \eta \|\widetilde{b}\|_{L^2}^2 \\ &= -\int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u}\, dx - \int u_2 \partial_2 \bar{u} \cdot \widetilde{u}\, dx - \int \widetilde{u \cdot \nabla \widetilde{b}} \cdot \widetilde{b}\, dx \\ &\quad + \int \widetilde{b \cdot \nabla \widetilde{b}} \cdot \widetilde{u}\, dx + \int b_2 \partial_2 \bar{b} \cdot \widetilde{u}\, dx - \int u_2 \partial_2 \bar{b} \cdot \widetilde{b}\, dx \\ &\quad + \int \widetilde{b \cdot \nabla \widetilde{u}} \cdot \widetilde{b}\, dx + \int b_2 \partial_2 \bar{u} \cdot \widetilde{b}\, dx \\ &:= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8. \end{aligned} \tag{3.4.3}$$

By Lemma 3.2.1,

$$A_1 = -\int u \cdot \nabla \widetilde{u} \cdot \widetilde{u}\ dx + \int \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u}\ dx = 0.$$

Similarly, $A_3 = 0$. By Lemma 3.2.5, Lemma 3.2.3 and the divergence-free conditions,

$$\begin{aligned}
A_2 &= -\int \widetilde{u_2}\partial_2 \bar{u} \cdot \widetilde{u}\ dx \\
&\lesssim \|\partial_2 \bar{u}\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 \bar{u}\|_{L^2} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^1} \|\partial_1 \widetilde{u}\|_{L^2}^2.
\end{aligned}$$

By Lemma 3.2.1 and the divergence-free conditions,

$$\begin{aligned}
A_4 + A_7 &= \int \widetilde{b \cdot \nabla \widetilde{b}} \cdot \widetilde{u}\ dx + \int \widetilde{b \cdot \nabla \widetilde{u}} \cdot \widetilde{b}\ dx \\
&= \int b \cdot \nabla \widetilde{b} \cdot \widetilde{u}\ dx + \int b \cdot \nabla \widetilde{u} \cdot \widetilde{b}\ dx - \int \overline{b \cdot \nabla \widetilde{b}} \cdot \widetilde{u}\ dx \\
&\qquad - \int \overline{b \cdot \nabla \widetilde{u}} \cdot \widetilde{b}\ dx \\
&= 0.
\end{aligned}$$

By Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}
A_5 &= \int b_2 \partial_2 \bar{b} \cdot \widetilde{u}\ dx = \int \widetilde{b_2} \partial_2 \bar{b} \cdot \widetilde{u}\ dx \\
&\lesssim \|\widetilde{b_2}\|_{L^2} \|\partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^2} \|\widetilde{b}\|_{L^2} \|\partial_1 \widetilde{u}\|_{L^2} \\
&\lesssim \|b\|_{H^2} (\|\widetilde{b}\|_{L^2}^2 + \|\partial_1 \widetilde{u}\|_{L^2}^2).
\end{aligned}$$

Similarly, by Lemma 3.2.5, Lemma 3.2.3 and the divergence-free conditions,

$$\begin{aligned}
A_6 &= -\int \widetilde{u_2} \partial_2 \bar{b} \cdot \widetilde{b}\ dx \\
&\lesssim \|\widetilde{b}\|_{L^2} \|\partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\widetilde{b}\|_{L^2} \|b\|_{H^2} \|\partial_1 \widetilde{u_2}\|_{L^2} \\
&\lesssim \|b\|_{H^2} (\|\widetilde{b}\|_{L^2}^2 + \|\partial_1 \widetilde{u}\|_{L^2}^2).
\end{aligned}$$

By Lemma 3.2.4 and Hölder's inequality,

$$\begin{aligned}
A_8 &= \int \widetilde{b_2} \partial_2 \bar{u} \cdot \widetilde{b}\ dx \lesssim \|\partial_2 \bar{u}\|_{L^\infty} \|\widetilde{b}\|_{L^2}^2 \\
&\lesssim \|\partial_2 \bar{u}\|_{L^2}^{\frac{1}{4}} (\|\partial_2 \bar{u}\|_{L^2} + \|\partial_1 \partial_2 \bar{u}\|_{L^2})^{\frac{1}{4}} \|\partial_2 \partial_2 \bar{u}\|_{L^2}^{\frac{1}{4}} \\
&\quad \times (\|\partial_2 \partial_2 \bar{u}\|_{L^2} + \|\partial_1 \partial_2^2 \bar{u}\|_{L^2})^{\frac{1}{4}} \|\widetilde{b}\|_{L^2}^2 \\
&\lesssim \|u\|_{H^3} \|\widetilde{b}\|_{L^2}^2.
\end{aligned} \tag{3.4.4}$$

Collecting the estimates for A_1 through A_8 in (3.4.3), we obtain

$$\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + (2\nu - C\|(u,b)\|_{H^3})\|\partial_1\widetilde{u}\|_{L^2}^2 \\ + (2\eta - C\|(u,b)\|_{H^3})\|\widetilde{b}\|_{L^2} \le 0.$$

According to Theorem 3.1.1, if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{H^3} + \|b_0\|_{H^3} \le \varepsilon$, then $\|u\|_{H^3} + \|b\|_{H^3} \le C\varepsilon$ and

$$2\nu - C\|(u,b)\|_{H^3} \ge \nu, \qquad 2\eta - C\|(u,b)\|_{H^3} \ge \eta.$$

By Lemma 3.2.3,

$$\|\widetilde{u}(t)\|_{L^2} + \|\widetilde{b}(t)\|_{L^2} \le (\|u_0\|_{L^2} + \|b_0\|_{L^2})e^{-C_1 t}, \tag{3.4.5}$$

where $C_1 = C_1(\nu, \eta) > 0$.

Next we consider the exponential decay for $\|(\nabla\widetilde{u}(t), \nabla\widetilde{b}(t))\|_{L^2}$. Taking the gradient of (3.4.2) yields

$$\begin{cases} \partial_t\nabla\widetilde{u} + \nabla(\widetilde{u\cdot\nabla\widetilde{u}}) + \nabla(\widetilde{u_2\partial_2\bar{u}}) + \nabla\nabla\widetilde{p} - \nu\partial_1^2\nabla\widetilde{u} \\ \qquad\qquad -\nabla(\widetilde{b\cdot\nabla\widetilde{b}}) - \nabla(b_2\partial_2\bar{b}) - \partial_1\nabla\widetilde{b} = 0, \\ \partial_t\nabla\widetilde{b} + \nabla(\widetilde{u\cdot\nabla\widetilde{b}}) + \nabla(u_2\partial_2\bar{b}) + \eta\nabla\widetilde{b} - \nabla(\widetilde{b\cdot\nabla\widetilde{u}}) \\ \qquad\qquad -\nabla(b_2\partial_2\bar{u}) - \partial_1\nabla\widetilde{u} = 0. \end{cases} \tag{3.4.6}$$

Dotting (3.4.6) with $(\nabla\widetilde{u}, \nabla\widetilde{b})$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2} + \|\nabla\widetilde{b}\|_{L^2}) + \nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\nabla\widetilde{b}\|_{L^2}^2 \\ &= -\int \nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx - \int \nabla(u_2\partial_2\bar{u})\cdot\nabla\widetilde{u}\,dx - \int \nabla(\widetilde{u\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{b}\,dx \\ &\quad + \int \nabla(\widetilde{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,d + \int \nabla(b_2\partial_2\bar{b})\cdot\nabla\widetilde{u}\,dx - \int \nabla(u_2\partial_2\bar{b})\cdot\nabla\widetilde{b}\,dx \\ &\quad + \int \nabla(\widetilde{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx + \int \nabla(b_2\partial_2\bar{u})\cdot\nabla\widetilde{b}\,dx \\ &:= B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8. \end{aligned} \tag{3.4.7}$$

By Lemma 3.2.1, B_1 can be written as

$$\begin{aligned} B_1 &= -\int \nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx + \int \nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx \\ &= -\int \partial_1 u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}\,dx - \int \partial_1 u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}\,dx \\ &\quad - \int \partial_2 u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}\,dx - \int \partial_2 u_2\partial_2\widetilde{u}\cdot\partial_2\widetilde{u}\,dx \\ &= B_{1,1} + B_{1,2} + B_{1,3} + B_{1,4}. \end{aligned}$$

By Lemma 3.2.5 and Lemma 3.2.3, $B_{1,1}$ can be bounded by

$$\begin{aligned} B_{1,1} &\lesssim \|\partial_1 u_1\|_{L^2} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_1 u_1\|_{L^2} \|\partial_1^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2. \end{aligned}$$

$B_{1,2}$ and $B_{1,3}$ can be bounded similarly and

$$B_{1,2},\ B_{1,3} \lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2.$$

For $B_{1,4}$, using the divergence-free condition of u and by Lemma 3.2.5, Lemma 3.2.1 and Lemma 3.2.3, we obtain

$$\begin{aligned} B_{1,4} &= \int \partial_1 u_1 \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u}\, dx = \int \partial_1 \widetilde{u_1} \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u}\, dx \\ &\lesssim \|\partial_2 \widetilde{u}\|_{L^2} \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2 \widetilde{u}\|_{L^2} \|\partial_1^2 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2. \end{aligned}$$

Hence, B_1 is bounded by

$$B_1 \lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2.$$

Similarly, we can bound B_3 by Lemma 3.2.4 and Hölder's inequality,

$$\begin{aligned} B_3 &= -\int \nabla \widetilde{(u \cdot \nabla \widetilde{b})} \cdot \nabla \widetilde{b}\, dx \\ &= -\int \nabla (u \cdot \nabla \widetilde{b}) \cdot \nabla \widetilde{b}\, dx + \int \nabla \overline{(u \cdot \nabla \widetilde{b})} \cdot \nabla \widetilde{b}\, dx \\ &= -\int \nabla u \cdot \nabla \widetilde{b} \cdot \nabla \widetilde{b}\, dx \lesssim \|u\|_{H^3} \|\nabla \widetilde{b}\|_{L^2}^2. \end{aligned}$$

In order to bound B_2, we rewrite it as

$$\begin{aligned} B_2 &= -\int \nabla (u_2 \partial_2 \bar{u}) \cdot \nabla \widetilde{u}\, dx \\ &= -\int \nabla u_2 \partial_2 \bar{u} \cdot \nabla \widetilde{u}\, dx - \int u_2 \nabla \partial_2 \bar{u} \cdot \nabla \widetilde{u}\, dx \\ &= -\int \partial_1 u_2 \partial_2 \bar{u} \cdot \partial_1 \widetilde{u}\, dx - \int u_2 \partial_1 \partial_2 \bar{u} \cdot \partial_1 \widetilde{u}\, dx \\ &\quad + \int \partial_1 u_1 \partial_2 \bar{u} \cdot \partial_2 \widetilde{u}\, dx - \int u_2 \partial_2 \partial_2 \bar{u} \cdot \partial_2 \widetilde{u}\, dx \\ &:= B_{2,1} + B_{2,2} + B_{2,3} + B_{2,4}. \end{aligned}$$

According to the definition of $\bar{u}$,

$$B_{2,2} = 0.$$

Using Lemma 3.2.3, Hölder's inequality and proceeding as in (3.4.4) for $\|\partial_2\bar{u}\|_{L^\infty}$, we find

$$\begin{aligned}B_{2,1} &= -\int \partial_1\widetilde{u_2}\partial_2\bar{u}\cdot\partial_1\widetilde{u}\ dx\\ &\lesssim \|\partial_2\bar{u}\|_{L^\infty}\|\partial_1\widetilde{u}\|_{L^2}^2 \lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.\end{aligned}$$

Similarly, $B_{2,3}$ has the same bound as $B_{2,1}$. By Lemma 3.2.5 and Lemma 3.2.3,

$$\begin{aligned}B_{2,4} &= -\int \widetilde{u_2}\partial_2\partial_2\bar{u}\cdot\partial_2\widetilde{u}\ dx\\ &\lesssim \|\partial_2\partial_2\bar{u}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2\partial_2\bar{u}\|_{L^2}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.\end{aligned}$$

Hence, the bound for B_2 is

$$B_2 \lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.$$

Similarly,

$$\begin{aligned}B_5 &= \int \nabla(b_2\partial_2\bar{b})\cdot\nabla\widetilde{u}\ dx\\ &= -\int \partial_1 b_2\partial_2\bar{b}\cdot\partial_1\widetilde{u}\ dx - \int b_2\partial_1\partial_2\bar{b}\cdot\partial_1\widetilde{u}\ dx\\ &\quad + \int \partial_1 b_1\partial_2\bar{b}\cdot\partial_2\widetilde{u}\ dx - \int b_2\partial_2\partial_2\bar{b}\cdot\partial_2\widetilde{u}\ dx\\ &= B_{5,1}+B_{5,2}+B_{5,3}+B_{5,4}.\end{aligned}$$

By the definition of $\bar{b}$, $B_{5,2}=0$. By Lemma 3.2.1, Lemma 3.2.4, Lemma 3.2.3, Hölder's inequality and Young's inequality,

$$\begin{aligned}B_{5,1} &= -\int \partial_1\widetilde{b_2}\partial_2\bar{b}\cdot\partial_1\widetilde{u}\ dx\\ &\lesssim \|\partial_2\bar{b}\|_{L^\infty}\|\partial_1\widetilde{b_2}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2}\\ &\lesssim \|\partial_2\bar{b}\|_{L^2}^{\frac14}(\|\partial_2\bar{b}\|_{L^2}+\|\partial_1\partial_2\bar{b}\|_{L^2})^{\frac14}\|\partial_2\partial_2\bar{b}\|_{L^2}^{\frac14}\\ &\qquad\times(\|\partial_2\partial_2\bar{b}\|_{L^2}+\|\partial_1\partial_2^2\bar{b}\|_{L^2})^{\frac14}\|\partial_1\widetilde{b_2}\|_{L^2}\|\partial_1\partial_1\widetilde{u}\|_{L^2}\\ &\lesssim \|b\|_{H^3}\|\nabla\widetilde{b}\|_{L^2}\|\partial_1\nabla\widetilde{u}\|_{L^2} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2+\|\partial_1\nabla\widetilde{u}\|_{L^2}^2).\end{aligned}$$

Similarly, $B_{5,3}$ obeys the same bound. By Lemma 3.2.5, Lemma 3.2.3 and Young's inequality,

$$\begin{aligned}B_{5,4} &= -\int \widetilde{b_2}\partial_2\partial_2\bar{b}\cdot\partial_2\widetilde{u}\ dx\\ &\lesssim \|\partial_2\partial_2\bar{b}\|_{L^2}\|\widetilde{b_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{b_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2\partial_2\bar{b}\|_{L^2}\|\partial_1\widetilde{b_2}\|_{L^2}^{\frac12}\|\partial_2\widetilde{b_2}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u}\|_{L^2}\\ &\lesssim \|b\|_{H^3}\|\nabla\widetilde{b}\|_{L^2}\|\partial_1\nabla\widetilde{u}\|_{L^2} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2+\|\partial_1\nabla\widetilde{u}\|_{L^2}^2).\end{aligned}$$

Collecting the bounds for $B_{5,1}$, $B_{5,2}$, $B_{5,3}$ and $B_{5,4}$,

$$B_5 \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).. \tag{3.4.8}$$

Similarly, B_6 and B_8 are bounded by

$$B_6,\ B_8 \lesssim (\|b\|_{H^3} + \|u\|_{H^3}) \times (\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2). \tag{3.4.9}$$

By Lemma 3.2.1 and the divergence-free condition $\nabla \cdot b = 0$,

$$\begin{aligned}
B_4 + B_7 &= \int \nabla(\widetilde{b\cdot\nabla\widetilde{b}}) \cdot \nabla\widetilde{u}\ dx + \int \nabla(\widetilde{b\cdot\nabla\widetilde{u}}) \cdot \nabla\widetilde{b}\ dx \\
&= \int \nabla(b\cdot\nabla\widetilde{b}) \cdot \nabla\widetilde{u}\ dx + \int \nabla(b\cdot\nabla\widetilde{u}) \cdot \nabla\widetilde{b}\ dx - \int \nabla(\overline{b\cdot\nabla\widetilde{b}}) \cdot \nabla\widetilde{u}\ dx \\
&\quad - \int \nabla(\overline{b\cdot\nabla\widetilde{u}}) \cdot \nabla\widetilde{b}\ dx \\
&= \int \nabla b\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{u}\ dx + \int \nabla b\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{b}\ dx + \int b\cdot\nabla\nabla\widetilde{b}\cdot\nabla\widetilde{u}\ dx \\
&\quad + \int b\cdot\nabla\nabla\widetilde{u}\cdot\nabla\widetilde{b}\ dx \\
&= \int \nabla b\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{u}\ dx + \int \nabla b\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{b}\ dx \\
&:= B_{4,1} + B_{4,2}.
\end{aligned}$$

We can rewrite $B_{4,1}$ as

$$\begin{aligned}
B_{4,1} &= \int \partial_1 b_1\partial_1\widetilde{b}\cdot\partial_1\widetilde{u}\ dx + \int \partial_1 b_2\partial_2\widetilde{b}\cdot\partial_1\widetilde{u}\ dx \\
&\quad + \int \partial_2 b_1\partial_1\widetilde{b}\cdot\partial_2\widetilde{u}\ dx + \int \partial_2 b_2\partial_2\widetilde{b}\cdot\partial_2\widetilde{u}\ dx \\
&:= B_{4,1,1} + B_{4,1,2} + B_{4,1,3} + B_{4,1,4}.
\end{aligned}$$

By Lemma 3.2.4, Lemma 3.2.3, Hölder's inequality and Young's inequality,

$$\begin{aligned}
B_{4,1,1} &\lesssim \|\partial_1 b_1\|_{L^\infty}\|\partial_1\widetilde{b}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2} \\
&\lesssim \|\partial_1 b_1\|_{L^2}^{\frac{1}{4}}(\|\partial_1 b_1\|_{L^2} + \|\partial_1\partial_1 b_1\|_{L^2})^{\frac{1}{4}}\|\partial_2\partial_1 b_1\|_{L^2}^{\frac{1}{4}} \\
&\quad \times (\|\partial_2\partial_1 b_1\|_{L^2} + \|\partial_1\partial_2\partial_1 b_1\|_{L^2})^{\frac{1}{4}}\|\partial_1\widetilde{u}\|_{L^2} \\
&\lesssim \|b\|_{H^3}\|\partial_1\widetilde{b}\|_{L^2}\|\partial_1\partial_1\widetilde{u}\|_{L^2} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).
\end{aligned}$$

$B_{4,1,2}, B_{4,1,3}$ and $B_{4,1,4}$ can be bounded similarly as $B_{4,1,1}$ and

$$B_{4,1,2},\ B_{4,1,3},\ B_{4,1,4} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).$$

Therefore, $B_{4,1}$ is bounded by

$$B_{4,1} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).$$

Similarly, $B_{4,2}$ obeys the same bound as $B_{4,1}$. Hence,

$$B_4 + B_7 \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).$$

Inserting the estimates for B_1 through B_8 in (3.4.7),

$$\frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2}^2 + \|\nabla\widetilde{b}\|_{L^2}^2) + (2\nu - C\|(u,b)\|_{H^3})\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 \\ + (2\eta - C\|(u,b)\|_{H^3})\|\nabla\widetilde{b}\|_{L^2} \leq 0.$$

Choosing $\varepsilon > 0$ sufficiently small and by Theorem 3.1.1, if $\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \varepsilon$, then $\|u\|_{H^3} + \|b\|_{H^3} \leq C\varepsilon$ and

$$2\nu - C\|(u,b)\|_{H^3} \geq \nu, \qquad 2\eta - C\|(u,b)\|_{H^3} \geq \eta.$$

By Lemma 3.2.3, we obtain the exponential decay result for $\|(\nabla\widetilde{u}(t), \nabla\widetilde{b}(t))\|_{L^2}$,

$$\|\nabla\widetilde{u}(t)\|_{L^2} + \|\nabla\widetilde{b}(t)\|_{L^2} \leq (\|\nabla u_0\|_{L^2} + \|\nabla b_0\|_{L^2})e^{-C_1 t}, \tag{3.4.10}$$

where $C_1 = C_1(\nu, \eta) > 0$. Cominbing the estimates in (3.4.5) and (3.4.10), we obtain the desired decay result in Theorem 3.1.2. ∎

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