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FORCES AND ENERGY TRANSFER INDUCED BY

RAREFIED PLASMA FLOWS PAST SOLID BODIES

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ABSTRACT

An analytic investigation has been made of the problem of determining the electric potential field and the aerodynamic moments for a steady rarefied plasma flow past a sphere by means of the Vlasov-Poisson set of equations. The equations are attacked by means of a perturbation method exploiting the features of an ionospheric satellite, i.e., the velocity of the satellite is much larger than the thermal velocity of ions, but much smaller than that of electrons, and the Debye length is much smaller than the characteristic length of the satellite. The effects of photoemission, the secondary emission of electrons, and the earth's magnetic field are neglected.

The electric potential, especially in the vicinity of the sphere, has been examined. This part of the investigation gives the nature of the electric field and provides a number of useful approximate formulas including the one for the Maxwell drag. The Maxwell drag is found to be negative and actually a thrust when the surface potential is large. The first-order approximation to the ion distribution function is obtained for the special case of a spherical potential, which is suitable for the sphere problem. The first-order approximate expressions of floating potential, slip velocity, molecular drag, and energy transfer are obtained as a function of the ion speed ratio, surface potential, and the Debye length. The molecular drag is found to increase with the increase of the Debye length.

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This work is dedicated to the memory of the author's father, Shou-Jen Liu, who first inspired his interest in the field of mechanical engineering, and to his beloved mother, Pao-Yu Liu, who has unselfishly given her love to him.

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NOMENCLATURE

A	$n_\infty(\frac{m}{2\pi kT_\infty})^{3/2} = n_\infty(\frac{B}{\pi})^{3/2}$
a	radius of the sphere
B	$\frac{m}{2kT_\infty}$
B_W	$\frac{m_O}{2kT_W}$
C_D	drag coefficient, $\frac{D}{\frac{1}{2}(m_i n_{i\infty}+m_o n_{o\infty})U_\infty^2 \pi a^2}$
C_q	heat transfer coefficient, $\frac{q}{\frac{1}{2}(m_i n_{i\infty}+m_o n_{o\infty})U_\infty^3}$
D	drag force on sphere
$D(S)$	Dawson function, $e^{-S^2}\int_0^S e^{x^2}dx$
E	total energy of a particle
$\hat{e}$	unit vector
e	electronic charge (4.803×10^{-10}esu)
$\mathcal{E}$	ionization potential per molecule
$\mathcal{E}^*$	$\frac{\mathcal{E}}{kT_{i\infty}}$
erf(x)	error function, $\frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$
erfc(x)	complementary error function, $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-z^2}dz = 1 - \mathrm{erf}(x)$
$\vec{F}$	force exerted on the body of the flowing plasma
f	distribution function
f_{io}	zeroth order approximation of ion distribution function

f_{i1}	first order approximation of ion distribution function
$\bar{f}_1$	part of the solution for f_{i1}
f_{eo}	zeroth order approximation of electron distribution function
f_{e1}	first order approximation of electron distribution function
$I_o(x)$	modified Bessel function of the first kind and of order zero
$I_1(x)$	modified Bessel function of the first kind and of order one
$I(\phi_o')$	$r\xi\int_\infty^r \frac{\frac{d\phi_o'}{dr}\,dr}{r\xi_r}$
I_∞	$\frac{L}{m_i}\int_\infty^r \frac{dr}{r^2\xi_r}$
J	contribution of current from sources other than plasma flux
k	Boltzmann's gas constant (8.616×10^{-5}eV per deg.)
K	$\delta \frac{m_i}{m_e} S_e^2$
L	angular momentum of a particle
m	particle mass
N_i	$n_{io}/n_{i\infty}$
N_e	$n_{eo}/n_{e\infty}$
$\hat{n}$	unit normal vector pointing outward from the solid surface
n	number density
n_{io}	zeroth order ion number density
n_{eo}	zeroth order electron number density
n_d	diffuse function
q	electric charge
q	local energy flux to the sphere
$\vec{r}$	position vector

$\vec{r}'$	$\vec{r}/a$
$\bar{r}$	r/a
r_m^2	$r^2 \frac{\xi_\theta^2+\xi_\psi^2}{\xi^2} = r^2 \sin^2\gamma$
r_o	position beyond which potential gradient is zero
S	surface area of the body
S	speed ratio, $\sqrt{\frac{m}{2kT_\infty}}\, U_\infty = \sqrt{B}\, U_\infty$
T	temperature
t	time
$\vec{U}_\infty$	free stream velocity
V	mean velocity
Y	$\phi_{os}^o - \phi_o^i$
Y_s	$\phi_{os}^o - \phi_s^*$
α	degree of ionization, $\frac{m_i n_{i\infty}}{m_i n_{i\infty} + m_o n_{o\infty}}$
δ	$\frac{-e\phi_s}{m_i U_\infty^2} = \frac{1}{2} \frac{T_{e\infty}}{T_{i\infty}} \frac{\phi_s^*}{S_i^2}$
ε	λ_D/a
ζ	$\frac{\bar{r}-1}{\varepsilon}$
θ_w	critical angle
λ_D	Debye length, $\sqrt{\frac{kT_{e\infty}}{4\pi e^2 n_{e\infty}}}$
$\vec{\xi}$	particle velocity relative to sphere
$\vec{\xi}'$	$\vec{\xi}/U_\infty$
$\vec{\xi}''$	$\sqrt{\frac{m_e}{2kT_{e\infty}}}\, \vec{\xi} = \sqrt{B_e}\, \vec{\xi}$

$\overleftrightarrow{\sigma}_f$	momentum flux stress tensor
$\overleftrightarrow{\sigma}_M$	electric Maxwell stress tensor
ϕ	electric potential
ϕ'	ϕ/ϕ_s
ϕ^*	$\frac{-e\phi_o}{kT_{e\infty}}$
ϕ_o^o	zeroth order outer expansion of ϕ^*
ϕ_1^o	first order outer expansion of ϕ^*
ϕ_o^i	zeroth order inner expansion of ϕ^*
ϕ_1^i	first order inner expansion of ϕ^*
$\bar{\phi}_s$	$-\frac{2e\phi_s}{m_i} = 2U_\infty^2\delta$
$\overleftrightarrow{\mathbb{I}}$	unit second order tensor
∇'	$a\nabla$

Coordinates:

x,y,z	rectangular spatial
r,θ,ψ	spherical spatial
$\xi_r,\xi_\theta,\xi_\psi$	rectangular velocity
ξ,γ,ε	spherical velocity

Subscripts:

d	diffuse emission
e	electron
emit	emitted from sphere
$\mathcal{E}$	contribution due to recombination of ions
f	fluid dynamic
i	ion

M	contribution due to Maxwell stress
m	turning point of trajectory
o	neutral particle
s	at body surface
s	refers to species
w	at body surface
ϕ	contribution due to potential
∞	free stream value
o	zeroth order approximation
1	first order approximation

Coordinate subscript refers to vector component in coordinate direction.

Superscripts:

'	nondimensional quantity
"	nondimensional quantity
o	outer expansion
i	inner expansion

CHAPTER I

INTRODUCTION

A detailed knowledge of flowfields around ionospheric satellites is required in many practical applications. The prediction of satellite decay times, as well as the use of measured rates of decay to gain information about the properties of the upper ionosphere requires accurate knowledge of the forces acting on a satellite. Owing to the high temperature generated during the re-entry into the earth's atmosphere, a space vehicle becomes enveloped by a sheath of partially ionized gas which affects radio transmission to and from the vehicle. Knowledge of the plasma flowfield around a re-entry vehicle is important in tracking and detecting it by radar. The aerodynamic effects of an ionized environment on a vehicle, such as drag and heat transfer, are also important in engineering applications. Owing to these applications, the problems of steady rarefied plasma flows past solid bodies have received considerable attention in recent years.

To clearly understand the problem, it is useful to examine briefly the properties of the ionosphere. The upper ionosphere is extremely rarefied and partially or fully ionized depending on the altitude. The mean free path of neutral particles is extremely large. It is about 80 m at the relatively low altitude of 200 km and increases rapidly to about 8000 km at an altitude of 1000 km. The mean free path of ions and electrons is also rather large. It is about 0.1 km at an altitude of 200 km and increases considerably more slowly to about 8 km at an altitude of 1000 km. The thermal velocities of neutral particles and of ions are of the same order of magnitude, 0.4 to 6 km/sec. On the other hand, the thermal velocity of electrons is extremely large, of the order of 100 to 400 km/sec. The Debye length (Debye shielding distance), i.e., the distance over which a static electric charge is screened by polarization of the plasma, is extremely small in the ionosphere, of the order of 0.1 to 4 cm. The characteristic length of artificial satellites moving through the upper ionosphere is of the order of one or a few meters. The velocity of the satellites is about 8 km/sec (the data are taken from Ref. 1).

From the brief data given above, it can be seen that the features of the motion of a body in the ionosphere are that the characteristic length of the body is smaller than the mean free path but larger than the Debye length, and that the velocity of the body is larger than the thermal velocity of neutral particles or ions but smaller than the thermal velocity of electrons. Because the thermal velocity of electrons is much higher than that of the ions, a net negative current will flow initially from the ambient atmosphere to an uncharged body. The flow will continue until the body accumulates an equilibrium charge sufficiently negative to repel enough low-energy electrons that the total electron flux to the body surface just equals the ion flux to the surface. The equilibrium condition is altered somewhat by the other sources of current, e.g., the photoemission of electrons by solar radiation, etc. The flowfield will be highly asymmetric, and significant electric fields will appear near the surface of the body and in the wake, which have effects on the trajectories of the charged particles and hence on the drag force experienced by the body. For a description of the motion of a body in a highly rarefied medium, the conventional methods of continuum fluid mechanics are inapplicable, and the problem must be attacked from the standpoint of kinetic theory. Some of the basic features of the physics of bodies in ionized atmospheres are discussed by Brundin [1963], Al'pert, Gurevich, and Pitaevskii [1965], Singer [1965], and in a recent review article by Liu [1969].

Jastrow and Pearse [1957] were among the first to consider the drag on a spherical satellite in the ionosphere. Using the Maxwellian distribution for the electron number density and the free stream values for the ion number density, they numerically integrated the resulting spherically symmetric Poisson's equation to obtain the electric potential. The charged drag, in addition to the neutral drag, was obtained numerically by computing the change in momentum flux of the cold ions (thermal motion neglected) as they traversed the potential field. The assumption of the Maxwellian distribution for the electron number density is not valid near the sphere surface where electrons are absorbed by the surface. Also the assumption of uniform ion density is not valid in the wake, since the ion density is very small aft of the sphere.

Extending the method used by Jastrow and Pearse, Davis and Harris [1961] tried to account for the asymmetry in the problem by using iterative procedure to solve the axisymmetric Poisson's equation. Starting from an assumed ion distribution, the procedure is repeated until a self-consistent solution is obtained. The wake they obtained is not completely realistic, because the basic scheme of Jastrow and Pearse does not account for the ion thermal motion in the structure of the wake. Hohl and Wood [1963] tried further improvements in this type of investigation by accounting for the absorptive effect of electrons at the surface of the sphere and also included some of the effects of the earth's geomagnetic field. Their electron density distribution, however, did not have the correct behavior at infinity.

Kiel, Gey, and Gustafson [1968] also numerically integrated the Poisson's equation by using more realistic models for the ion and electron number densities and were able to obtain results which are in agreement with the experimental measurements on the satellites. They found that upstream of the satellite the potential decays quickly and monotonically to the ambient potential and the electrostatic field is spherically symmetric, and that downstream of the satellite the potential will either increase first to a maximum value and then decrease, or decrease monotonically to the ambient potential depending on the value of the ratio of the Debye length to the radius of the satellite, and the electrostatic field is far from spherically symmetric. They did not consider the drag force.

The above investigations were principally numerical. Aside from various theoretical difficulties, they failed to bring out the functional relationships between the pertinent combinations of parameters that governs the problem. Except in the work of Kiel, Gey, and Gustafson [1968], the trends that result from variation of the governing parameters were not clearly exhibited, if at all. Pitts and Knechtel [1965], by means of the theory of Jastrow and Pearse, devised a semi-empirical correlation equation to relate the drag of the sphere to the parameters that governs the problem and used it to interpret experimental drag data they obtained. The effect of the ratio of surface potential energy to mean ion energy and the ratio of Debye length to sphere radius were shown to be key parameters that govern the drag. Their work was not "rational" in the sense that the results were not obtained from a well-posed boundary-value problem followed by a systematic analysis.

Al'pert, Gurevich, and Pitaevskii [1965] in their monograph studied rarefied plasma flow past a sphere by means of the Vlasov-Poisson set of equations and obtained various important results. They were interested primarily in the density and electric fields, however, as opposed to such aerodynamic features as drag and heat transfer. Prager and Rasmussen [1967] also studied the sphere problem by means of the Vlasov-Poisson equations, and, by means of an ad hoc approximation in the characteristics-solution of the Vlasov equation, were able to obtain analytic expressions for the drag and heat transfer. Whereas the drag formulas agreed in some respects with the available data of Pitts and Knechtel [1965], a desired dependence on the Debye length was not obtained.

Taylor [1967 a,b] in two papers considered two methods for analyzing rarefied plasma flow past a cylinder. He used the Maxwellian distribution for the electron number density and discussed a so-called heuristic method for determining the electric potential field in the first paper and a formal perturbation scheme in the second. His assumption that the body size and the Debye length are of the same order of magnitude is rarely true for the ionospheric satellite. He did not obtain a general expression for the drag. Flow past a cylinder is also studied by Pashchenko [1964] by means of singular perturbation methods.

One of the objects of the present investigation is to derive a formal perturbation method akin to that of Taylor and apply it to the sphere problem.

Lam and Greenblatt [1965] and Lam [1967] developed a theory for the flow of a collisionless plasma past an arbitrary body for which the mode of analysis is through the moment equations instead of the direct use of the kinetic equation. One of the main restrictions of the theory is that the ion temperature is much less than the electron temperature. In this cold-ion approximation the ion streamlines are the same as the ion trajectories, and thus the theory is not valid in the wake. Nevertheless various features of the far field and plasma sheath are exhibited. In the present investigation we wish to work directly with the kinetic equation and eliminate the cold-ion approximation. In this way, corresponding situations with free-molecule theory and accompanying boundary conditions are easy to perceive.

There are several investigations that deal primarily with the wake. Among these are those of Pan and Vaglio-Laurin [1967], Kiel, Gey and Gustafson [1968], and Vaglio-Laurin and Miller [1970]. These studies of the electric-potential field can be important in connection with determining the drag on a body, as will be clear later.

In view of the above discussion, the object of the present investigation is to formulate and treat the problem of rarefied plasma flow past a sphere in a systematic kinetic analysis. In particular, drag and heat transfer will be the primary goals in mind. Although the particular shape of a sphere will be studied, the methodology can be adapted for other shapes. In this connection some preliminary work has been done for a cone by Elliott and Rasmussen [1969]. It is hoped that the present analysis will reveal the basis of various approximations and serve as a foundation for further investigations.

CHAPTER II

FORMULATION OF THE PROBLEM

Vlasov-Poisson Equations

Consider a rarefied mixture of ions, electrons, and neutral particles such as exist in the ionosphere flowing past a solid body. If we assume that binary collisions are unimportant, the distribution function $f_s(\vec{r},\vec{\xi})$ for species s in steady flow is governed by the Vlasov equation:

$$\vec{\xi} \cdot \frac{\partial f_s}{\partial \vec{r}} - \frac{q_s}{m_s} \nabla\phi \cdot \frac{\partial f_s}{\partial \vec{\xi}} = 0 \tag{2.1}$$

where

$$\begin{aligned} q_s &= +\,e \quad \text{ions} \\ &= -\,e \quad \text{electrons} \\ &= 0 \quad \text{neutrals} \end{aligned}$$

The mass of species s is denoted by m_s, $\vec{\xi}$ is the molecular velocity, $\vec{r}$ is the position vector, $\phi(\vec{r})$ is the electric potential. Magnetic effects are neglected, which is a usual assumption in this type of problem. The electric potential is governed by the Poisson equation:

$$\nabla^2\phi = -\,4\pi e(n_i - n_e) \quad \text{(in Gaussian units)} \tag{2.2}$$

Here the ion and electron number densities are defined by the moments

$$\begin{aligned} n_i(\vec{r}) &= \int f_i d^3\xi \\ n_e(\vec{r}) &= \int f_e d^3\xi \end{aligned} \tag{2.3}$$

We have assumed that the ions are singly ionized. The Vlasov equations (2.1) are coupled nonlinearly to the Poisson equation (2.2) by means of the moments n_i and n_e.

Infinity Conditions

We assume that the plasma is neutral at infinity so that

$$\phi(\vec{r}) \to 0 \quad \text{as} \quad \vec{r} \to \infty \tag{2.4}$$

Further, each species will be considered to be in thermodynamic equilibrium with itself at infinity. Thus, the distribution functions, with respect to an observer on a fixed rigid body, will be Maxwellian displaced with the free stream velocity $\vec{U}_\infty$:

$$f_s(\vec{r},\vec{\xi}) = A_s e^{-B_s(\vec{\xi}-\vec{U}_\infty)^2} \tag{2.5}$$

where

$$A_s = n_{s\infty}\left(\frac{m_s}{2\pi kT_{s\infty}}\right)^{3/2}$$

$$B_s = \frac{m_s}{2kT_{s\infty}}$$

Here the temperatures of the different species $T_{s\infty}$ need not be equal and k is the Boltzmann constant.

Surface Boundary Conditions

There are several boundary conditions required at the surface of a solid body. First, the mean mass flux normal to the surface must vanish. Second, the distribution function for the emitted particles must be specified and be compatible with the zero normal mass flux condition. We assume that the ions and electrons are absorbed at the surface and emitted as neutrals. This is the usual assumption for this type of problem, and now only an emission distribution function for neutral particles is needed. We shall assume a diffuse emission at an arbitrary temperature, T_w, that is constant on the surface. Third, the current at the body surface or the surface potential must be specified. We shall assume that the body is perfectly conducting and hence that the electric potential is constant on the surface. Assuming that the net current to an isolated body is zero provides a condition for determining the so-called floating potential.

Zero Mass Flux Normal to Surface

The condition for zero mass flux normal to the surface can be written

$$m_i \int_{\vec{\xi}\cdot\hat{n}\leq 0} \vec{\xi}\cdot\hat{n} f_i d^3\xi + m_e \int_{\vec{\xi}\cdot\hat{n}\leq 0} \vec{\xi}\cdot\hat{n} f_e d^3\xi + m_o \int_{\vec{\xi}\cdot\hat{n}\leq 0} \vec{\xi}\cdot\hat{n} f_o d^3\xi + m_o \int_{\vec{\xi}\cdot\hat{n}\geq 0} \vec{\xi}\cdot\hat{n} f_d d^3\xi = 0 \qquad (2.6)$$

where $\hat{n}$ is the unit normal pointing outward from the solid surface. Here f_d is the diffuse distribution function defined by

$$f_d(\vec{r}_s,\vec{\xi}) = n_d(\vec{r}_s)\left(\frac{m_o}{2\pi kT_w}\right)^{3/2} \exp\left(\frac{-m_o}{2kT_w}\xi^2\right) \qquad (2.7)$$

where $\vec{r}_s$ is a position on the body surface, $n_d(\vec{r}_s)$ is the number density of the diffusely emitted neutral particles determined so that Eq. (2.6) is identically satisfied, and T_w is the temperature of the solid body. Substitution of expression (2.7) into (2.6) and evaluation of the integral gives

$$n_d(\vec{r}_s) = -\left(\frac{2\pi}{m_o kT_w}\right)^{\frac{1}{2}} \left[m_i \int_{\vec{\xi}\cdot n\leq 0} \vec{\xi}\cdot\hat{n} f_i d^3\xi + m_e \int_{\vec{\xi}\cdot n\leq 0} \vec{\xi}\cdot\hat{n} f_e d^3\xi + m_o \int_{\vec{\xi}\cdot n\leq 0} \vec{\xi}\cdot\hat{n} f_o d^3\xi\right] \qquad (2.8)$$

Current at the Surface

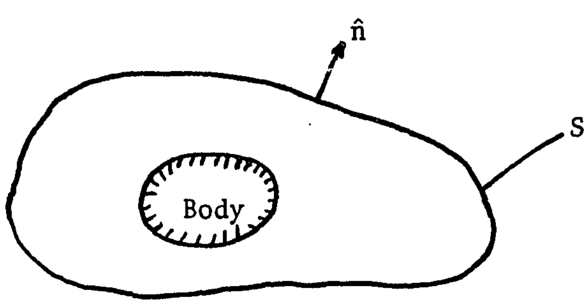
We assume that all incident electrons and ions are absorbed, recombine, and are emitted as neutrals. In addition the surface will be considered a perfect conductor. Because of steady-state conditions, the net current to an isolated body is zero:

$$\oint\!\!\!\oint_S \Big[e\int_{\vec{\xi}\cdot\hat{n}\leq 0}\vec{\xi}\cdot\hat{n}f_i\,d^3\xi \;-\; e\int_{\vec{\xi}\cdot\hat{n}\leq 0}\vec{\xi}\cdot\hat{n}f_e\,d^3\xi\Big]dS + J = 0 \qquad (2.9)$$

Here S is the surface area of the body and J is the contribution of current from sources such as photoemission, thermoelectric emission, and the "knocking out" of ions or electrons from the surface because of collisions with neutral particles. For some details of these additional sources, one should consult Brundin [1963], Hall, Kemp, and Sellen [1964], and Medved [1965]. Equations (2.9) determines the value of the electric potential on the surface of the body. For purposes of simplicity, it is convenient to set J equal to zero; Eq. (2.9) then yields a negative surface potential.

Force on a Solid Body

Let S denote an arbitrary surface that encloses a solid body but does not necessarily coincide with the surface of the solid body. Let $\hat{n}$ denote the unit outward normal. From momentum considerations, the force exerted



on the body by the flowing plasma is given by the following surface integrals:

$$\vec{F} = \oiint_S \hat{n}\cdot\overleftrightarrow{\sigma}_f dS + \oiint_S \hat{n}\cdot\overleftrightarrow{\sigma}_M dS \tag{2.10}$$

The first integral evaluates the contribution from the momentum-flux tensor

$$\overleftrightarrow{\sigma}_f \equiv - \sum_s^{i,e,o} m_s \int \vec{\xi}\vec{\xi} f_s(\vec{r},\vec{\xi}) d^3\xi \tag{2.11}$$

and the second evaluates the contribution from the electric Maxwell stresses

$$\overleftrightarrow{\sigma}_M \equiv \frac{1}{4\pi}[\nabla\phi\nabla\phi - \tfrac{1}{2}(\nabla\phi)^2 \overleftrightarrow{\mathrm{I}}] \tag{2.12}$$

where $\overleftrightarrow{\mathrm{I}}$ is the unit second-order tensor. Appropriate choices for the surface S in the evaluation of $\vec{F}$ are when S coincides with the actual surface of the solid body, or when S moves outward asymptotically to infinity. The case when S coincides with the body surface has the advantage that the distribution function for the emitted neutral particles is specified at the body surface. Conventional drag and lift formulas for free-molecule flow are obtained by means of this choice. We will also use this choice to determine the effect of the induced electric field.

Effects of the electric potential arise explicitly from the Maxwell stress tensor, $\overleftrightarrow{\sigma}_M$, but they arise implicitly also from the gas dynamic stress tensor because f_i and f_e depend on ϕ through the Vlasov equation. We wish to isolate this dependence on the electric potential.

CHAPTER III

APPROXIMATIONS FOR THE DISTRIBUTION FUNCTIONS

Flow Past a Sphere

Consider a rarefied plasma flow past a sphere of radius a. Let the free stream be in the z direction and introduce spherical coordinates r, θ, ψ as shown in Fig. 1.

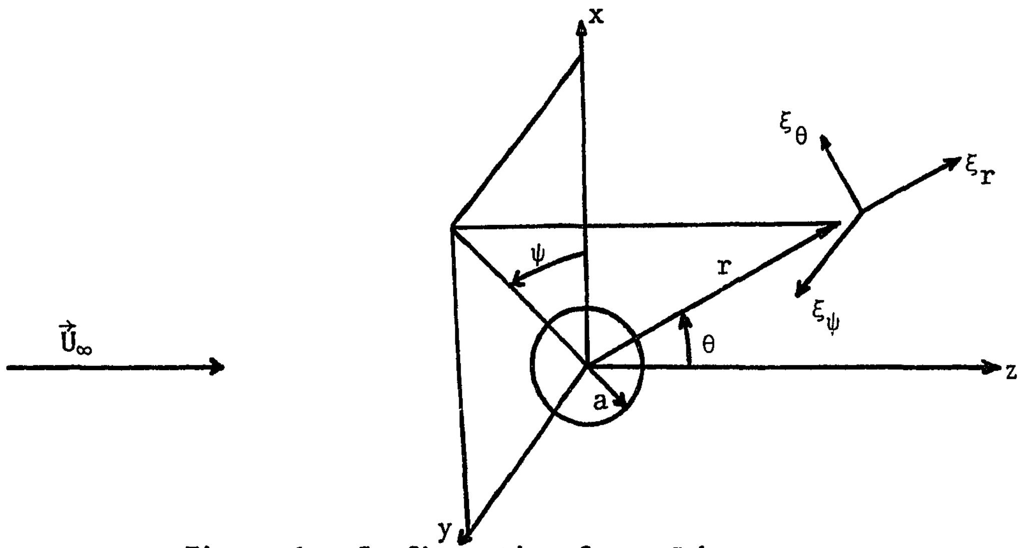


Figure 1. Configuration for a Sphere.

Let ξ_r, ξ_θ, ξ_ψ denote the rectangular components of the molecular velocity corresponding to r, θ, ψ.

Ion Distribution Function

Under the practical ionospheric conditions envisaged for a typical satellite problem, we assume that the mean speed of the free stream ions is much greater than the mean thermal speed of the ions. In terms of the ion speed ratio, S_i, we have

$$S_i \equiv \sqrt{B_i}\, U_\infty = \sqrt{\frac{m_i}{2kT_{i\infty}}}\, U_\infty >> 1 \qquad (3.1)$$

Typically, S_i lies in the range between 5 and 8, and thus can be considered asymptotically as a large parameter. The maximum potential energy of the electric field is found to be much less than the mean kinetic energy of the ions. If ϕ_s is the surface potential of the sphere and is regarded as the same order of magnitude as the maximum potential, then the ratio of the electric potential energy to the mean ion kinetic energy can be regarded as a small parameter for the problem

$$\delta \equiv \frac{-e\phi_s}{m_i U_\infty^2} << 1 \qquad (3.2)$$

The factor $\frac{1}{2}$ has been omitted from the ion kinetic energy for convenience, and the negative sign has been inserted to keep δ positive since ϕ_s is usually negative. Because S_i is large, most of the ions will have the characteristic speed U_∞. It is now convenient to normalize the variables in the ion Vlasov equation as follows:

$$\vec{\xi}' \equiv \frac{\vec{\xi}}{U_\infty}, \quad \vec{r}' \equiv \frac{\vec{r}}{a}, \quad \phi' \equiv \frac{\phi}{\phi_s}, \quad \nabla' = a\nabla$$

The ion Vlasov equation now takes the form

$$\vec{\xi}' \cdot \frac{\partial f_i}{\partial \vec{r}'} + \delta \nabla' \phi' \cdot \frac{\partial f_i}{\partial \vec{\xi}'} = 0 \qquad (3.3)$$

Consider now series expansions for f_i and ϕ' in terms of the small parameter δ. We assume

$$f_i = f_{io}[1+\delta f_{i1}+\delta^2 f_{i2}+\ldots] \tag{3.4a}$$

$$\phi' = \phi'_o + \delta\phi'_1 + \delta^2\phi'_2 + \ldots \tag{3.4b}$$

Substituting expansions (3.4) into Eq. (3.3), collecting like orders of δ, and requiring each order to vanish gives for the first two orders

$$\vec{\xi}'\cdot\frac{\partial f_{io}}{\partial\vec{r}'} = 0 \tag{3.5a}$$

$$\vec{\xi}'\cdot\frac{\partial f_{i1}}{\partial\vec{r}'} = -\nabla'\phi'_o\cdot\frac{\partial \ell n f_{io}}{\partial\vec{\xi}'} \tag{3.5b}$$

Equation (3.5a) is the same as for neutral particles. Thus we assume that the heavy ions have so much energy compared to the electric field that they move undisturbed, to a zeroth approximation, through the electric field as if it were not present. The perturbation caused by the electric field is described in Eq. (3.5b). Equation (3.5b) cannot be solved completely until the potential ϕ'_o is known. Analysis of this equation must be deferred until the electron Vlasov equation and the Poisson equation are treated.

The solution to Eq. (3.5a) that satisfies the infinity condition (2.5) is the same as for neutral particles in free-molecule flow. It is not difficult to verify that

$$f_{io}(\vec{r},\vec{\xi}) = A_i e^{-B_i(\vec{\xi}-\vec{U}_\infty)^2} \tag{3.6}$$

where

$$A_i \equiv n_{i\infty}\left(\frac{m_i}{2\pi kT_{i\infty}}\right)^{3/2} \text{ and } B_i \equiv \frac{m_i}{2kT_{i\infty}}$$

Substituting this result into (3.5b) gives for the first-order perturbation

$$\vec{\xi}'\cdot\frac{\partial f_{i1}}{\partial\vec{r}'} = 2S_i^2\left(\vec{\xi}' - \frac{\vec{U}_\infty}{U_\infty}\right)\cdot\nabla'\phi_o' \tag{3.7}$$

Part of the solution for f_{i1} is easy to determine. If we write

$$f_{i1} = 2S_i^2(\phi_o'+\bar{f}_1) \tag{3.8}$$

then the first-order problem reduces to solving the equation

$$\vec{\xi}'\cdot\frac{\partial\bar{f}_1}{\partial\vec{r}'} = -\frac{\vec{U}_\infty}{U_\infty}\cdot\nabla'\phi_o' \tag{3.9}$$

for the function $\bar{f}_1$. As mentioned previously, this equation cannot be treated completely until the potential ϕ_o' is known.

Zeroth Approximation for Ion Density

A zeroth approximation for the ion number density is obtained by integration of f_{io} over all ion velocities. Some ion velocities do not exist, however, because of collisions with the sphere. The configuration of the ion velocity space, according to the zeroth approximation, is shown in Fig. 2 (see Prager and Rasmussen [1967]).

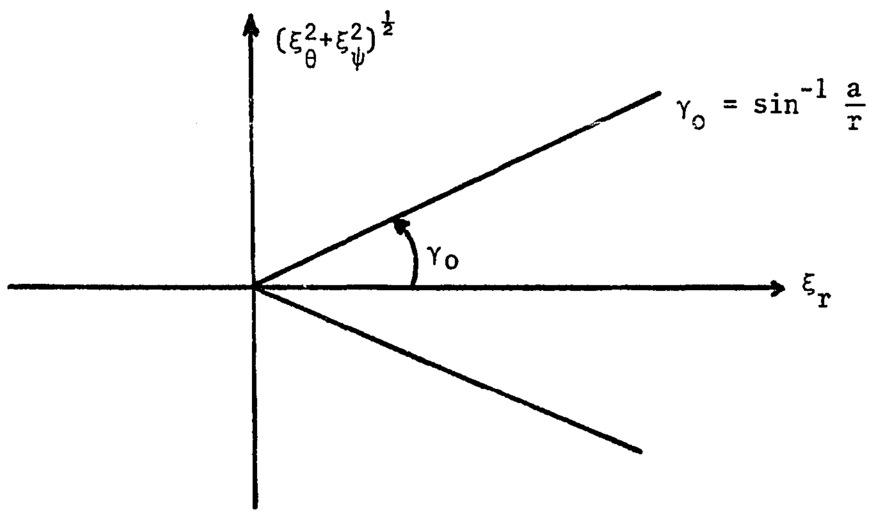


Figure 2. Ion Velocity Space (Zeroth Approximation)

The integration is to be taken over the region outside a cone of semi-vertex angle $\gamma_o = \sin^{-1}\frac{a}{r}$, that is,

$$n_{io} = \int_{\gamma \geq \gamma_o} f_{io} d^3\xi$$

As shown by Al'pert *et al.* [1965], Kiel *et al.* [1968], and Prager and Rasmussen [1967], as well as others, the ion number density n_{io} takes the form

$$n_{io}(\bar{r},\theta) = n_{i\infty} e^{-S_i^2 \sin^2\theta} \int_0^\infty x e^{-x^2} I_o(2S_i x \sin\theta)\,\mathrm{erfc}(S_i\cos\theta - x\sqrt{\bar{r}^2-1})\,dx \tag{3.10a}$$

or

$$n_{io}(\bar{r},\theta) = n_{i\infty}[\tfrac{1}{2} + e^{-S_i^2\sin^2\theta} \int_0^\infty x e^{-x^2} I_o(2S_i x\sin\theta)\,\mathrm{erf}(x\sqrt{\bar{r}^2-1} - S_i\cos\theta)\,dx] \tag{3.10b}$$

where $\bar{r} = r/a$, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$, $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$, and I_o is the modified Bessel function of the first kind and order zero. For the cases $\theta = \pi$ (upstream) and $\theta = 0$ (downstream) the integrals in (3.10) can be evaluated in terms of tabulated functions:

$\theta = \pi$ (upstream)

$$n_{io}(\bar{r},\pi) = \tfrac{1}{2}n_{i\infty}[\mathrm{erfc}(-S_i) + \sqrt{1-\frac{1}{\bar{r}^2}}\, e^{-S_i^2/\bar{r}^2}\,\mathrm{erfc}(S_i\sqrt{1-\frac{1}{\bar{r}^2}})] \tag{3.11a}$$

$\theta = 0$ (downstream)

$$n_{io}(\bar{r},0) = \tfrac{1}{2}n_{i\infty}[\mathrm{erfc}(S_i) + \sqrt{1-\frac{1}{\bar{r}^2}}\, e^{-S_i^2/\bar{r}^2}\,\mathrm{erfc}(-S_i\sqrt{1-\frac{1}{\bar{r}^2}})] \tag{3.11b}$$

Near the surface of the sphere ($\bar{r} \to 1$), the ion number density has the asymptotic behavior

$$n_{io}(\bar{r},\theta) \sim \tfrac{1}{2}n_{i\infty}[\mathrm{erfc}(S_i\cos\theta) + \frac{4}{\sqrt{\pi}}\, e^{-S_i^2}\sqrt{\bar{r}-1}\int_0^\infty x^2 e^{-x^2} I_o(2S_i x\sin\theta)\,dx] \qquad \bar{r}\to 1 \tag{3.12}$$

These expressions will be useful later.

Electron Distribution Function

Under the practical conditions of an ionospheric satellite, the mean speed of the free stream electrons is much less than the mean thermal speed of the electrons. In terms of the electron speed ratio, S_e, we have

$$S_e \equiv \sqrt{B_e}\, U_\infty = \sqrt{\frac{m_e}{2kT_{e\infty}}}\, U_\infty << 1 \tag{3.13}$$

The electron speed ratio is much smaller than the ion speed ratio because the electron mass is very much smaller than the ion mass. Thus S_e, or the ratio m_e/m_i, is to be regarded as a small parameter in the problem.

Because of the small mass of the electron, the characteristic speed is the thermal speed, and the appropriate nondimensional velocity is

$$\vec{\xi}'' = \sqrt{\frac{m_e}{2kT_{e\infty}}}\, \vec{\xi}$$

If the other variables remain as before, the electron Vlasov equation takes the form

$$\vec{\xi}'' \cdot \frac{\partial f_e}{\partial \vec{r}'} - (\delta \frac{m_i}{m_e} S_e^2) \nabla' \phi' \cdot \frac{\partial f_e}{\partial \vec{\xi}''} = 0 \tag{3.14}$$

The combination of parameters

$$\delta \frac{m_i}{m_e} S_e^2 = \frac{-e\phi_s}{2kT_{e\infty}} \equiv K$$

is now assumed to be of order unity, which is consistent with our present problem of ionospheric aerodynamics. It is thus necessary that the potential ϕ' enter into the zeroth approximation for the electron distribution function.

We now assume that the potential is expanded by the series (3.4b), that is

$$\phi' = \phi_o' + \delta\phi_1' + \delta^2\phi_2' + \ldots \tag{3.4b}$$

We further assume that the electron distribution function can be expanded in the series

$$f_e = f_{e_o}[1+\delta f_{e1}+\delta^2 f_{e2}+\ldots] \tag{3.15a}$$

where

$$f_{e_o} = A_e e^{-2K\phi_o'} e^{-(\vec{\xi}''-\vec{S}_e)^2} \tag{3.15b}$$

The zeroth-order function (3.15b) satisfies the infinity condition (2.5). Substituting (3.4b) and (3.15) into Eq. (3.14), retaining terms to order δ, and eliminating terms that cancel because of Eq. (3.15b), gives

$$-2Kf_{e_o}\vec{S}_e\cdot\nabla'\phi_o' + \delta[\vec{\xi}''\cdot\frac{\partial f_{eo}f_{e1}}{\partial\vec{r}'} - K\nabla'\phi_o'\cdot\frac{\partial f_{eo}f_{e1}}{\partial\vec{\xi}''} - K\nabla'\phi_1'\cdot\frac{\partial f_{eo}}{\partial\vec{\xi}''}] = 0 \tag{3.16}$$

A consistent ordering of terms can now proceed if we set $S_e = 0(\delta)$, in which case δ cancels out of Eq. (3.16). This means, however, that the zeroth-order function f_{e_o}, given by (3.15b), contains the parameter δ. This ad hoc approximation is made so that the infinity conditions are satisfied and the current at infinity will vanish uniformly in the zeroth approximation. If S_e is of order δ, then it follows, since K is of order unity, that m_e/m_i is of order δ^3. This ordering is consistent with practical values associated with ionospheric aerodynamics. The higher approximation f_{e1} can now be pursued by solving Eq. (3.16).

Zeroth Approximations for Electron Densities

In order to obtain a self-consistent zeroth approximation for the electron number density it is necessary to know the electric potential field $\phi_0(\vec{r})$. Because the electric field does not affect the ions to a zeroth approximation, the ion trajectories are straight lines. The electron trajectories, however, are curved lines that depend on the nature of the electric field. Thus the region of velocity space for which no electrons exist because of collisions with the sphere is not known ahead of time. In order to make progress, it is thus necessary to make several ad hoc approximations concerning the structure of the electron velocity space.

Spherically Symmetric Approximation

As will be shown later, the electric potential upstream of the sphere is approximately spherically symmetric. Downstream of the sphere a wake exists and the potential is far from spherically symmetric. Moreover, the potential field does not decrease monotonically in the wake. Nevertheless, it is a useful approximation to assume that the electric field is spherically symmetric and monotonically decreasing, at least for the region upstream of the sphere.

As shown by Prager and Rasmussen [1967], the surface that separates the region of absorbed electrons in velocity space for spherical-symmetric potential is a hyperboloid of revolution given by

$$\sin\gamma_1 \equiv \frac{(\xi_0^2+\xi_\psi^2)^{\frac{1}{2}}}{\xi} = \frac{a}{r}\left[1-\frac{\xi_e^2}{\xi^2}\right]^{\frac{1}{2}} \tag{3.17}$$

where

$$\xi_e^2 \equiv \left(-\frac{2e\phi_s}{m_e}\right) - \left(-\frac{2e\phi}{m_e}\right) > 0$$

This surface is shown in Fig. 3.

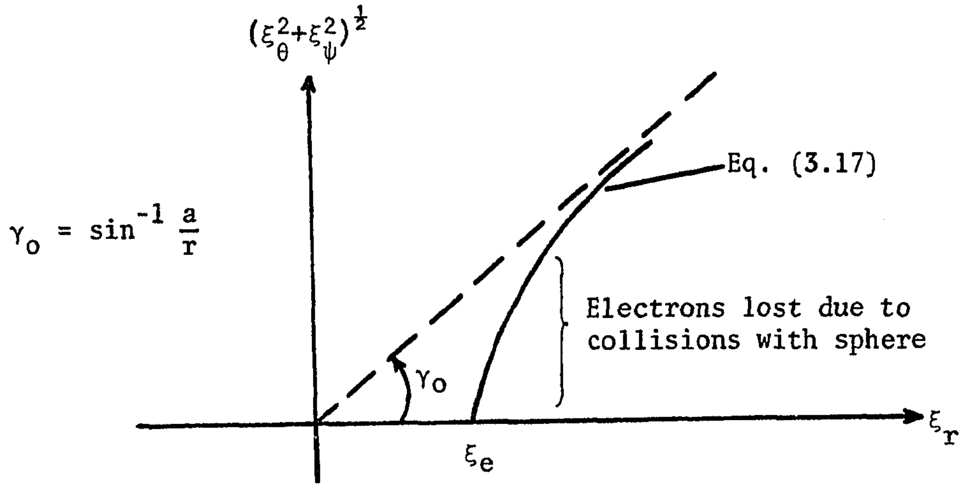


Figure 3. Electron Velocity Space

In terms of this configuration, the electron number density is

$$
\begin{aligned}
n_{e_o} &= \int_{\gamma>\gamma_1} f_{e_o} d^3\xi \\
&= \frac{n_{e\infty}}{2} e^{-\phi^*}\left[1+\mathrm{erf}\sqrt{\phi_s^*-\phi^*}+\sqrt{1-\frac{1}{\bar{r}^2}}\exp\left(-\frac{\phi_s^*-\phi^*}{\bar{r}^2-1}\right)\mathrm{erfc}\sqrt{\frac{\phi_s^*-\phi^*}{1-\frac{1}{\bar{r}^2}}}\right] \\
&\quad + O(S_e) \qquad (3.18)
\end{aligned}
$$

where

$$\phi^* \equiv \frac{-e\phi_o}{kT_{e\infty}}, \qquad \bar{r} \equiv \frac{r}{a}$$

and ϕ_s^* is the value of the potential at the sphere surface. Formula (3.18) was obtained by Al'pert *et al.* [1965], Kiel *et al.* [1968], and Prager and Rasmussen [1967], among others.

Linear-Trajectory Approximation

Because formula (3.18) does not allow for non-monotonic behavior, that is, ϕ^* cannot be greater than ϕ_s^*, it is not useful for the wake. Near the surface aft of the sphere, the electrons will be attracted rather than repelled to the sphere. We approximate the average behavior by assuming the electron trajectories are straight lines. The hyperboloid in Fig. 3 then reduces to the cone of semivertex angle $\gamma_o = \sin^{-1}\frac{a}{r}$. The electron density under this approximation is given by

$$n_{e_o} = \int_{\gamma>\gamma_o} f_{e_o} d^3\xi$$
$$= \frac{n_{e\infty}}{2} e^{-\phi^*}\left[1+\sqrt{1-\frac{1}{\bar{r}^2}} - \frac{2S_e}{\sqrt{\pi}}\frac{\cos\theta}{\bar{r}^2} + O(S_e^2)\right] \tag{3.19}$$

We shall neglect the small asymmetry introduced by the presence of the higher-order parameter S_e.

Maxwellian Approximation

A common approximation in ionospheric aerodynamics is so-called Maxwellian distribution

$$n_{e_o} = n_{e\infty} e^{-\phi^*} \tag{3.20}$$

This approximation is valid for an equilibrium situation and does not account for absorption of electrons by a solid body. It is thus only valid for the far field. It is of interest to compare this approximation with results obtained with the previous more accurate approximations.

CHAPTER IV

THE ELECTRIC FIELD

The electric potential field can now be obtained to a zeroth approximation in terms of the leading terms of the expansions utilized in the previous section. If we neglect terms of order δ, we can write Poisson's equation (2.2) in the following dimensionless form:

$$\varepsilon^2 \nabla'^2 \phi^* = N_i(\bar{r},\theta) - N_e(\phi^*,\bar{r},\theta) \tag{4.1}$$

where

$$N_i(\bar{r},\theta) \equiv n_{io}/n_{i\infty}$$

$$N_e(\phi^*,\bar{r},\theta) \equiv n_{eo}/n_{e\infty}$$

$$\phi^* \equiv - e\phi_o/kT_{e\infty}$$

$$\varepsilon \equiv \lambda_D/a$$

$$\lambda_D = (kT_{e\infty}/4\pi e^2 n_{e\infty})^{\frac{1}{2}}$$

The parameter λ_D is the Debye length. The ratio of Debye length to sphere radius, ε, is typically a small parameter for ionospheric conditions. Thus

$$\varepsilon \ll 1$$

and ε is another small parameter in the problem. The boundary conditions for Eq. (4.1) are

$$\begin{aligned} \phi^* &\to 0 \quad \text{as} \quad \bar{r} \to \infty \\ \phi^* &= \phi_s^* \quad \text{at} \quad \bar{r}=1 \end{aligned} \tag{4.2}$$

If we now seek a solution to Eq. (4.1) by means of a series expansion for small ε, we find because the highest derivative is multiplied by the small parameter ε^2 that the problem is a singular perturbation problem. Thus an electric boundary layer will be present and more than one series expansion will be required. The so-called inner and outer expansions must be matched together by means of asymptotic methods outlined in the book by Van Dyke [1964].

Outer Expansion or Quasi-Neutral Approximation

Let us seek a straightforward series expansion of the form

$$\phi^* \sim \phi_o^o + \varepsilon^2\phi_1^o + \varepsilon^4\phi_2^o + \cdots \tag{4.3}$$

Here it has been found that the appropriate expansion is in powers of ε^2. The electron number density $N_e(\phi^*,\bar{r},\theta)$ now has the expansion

$$N_e(\phi^*,\bar{r},\theta) = N_e(\phi_o^o,\bar{r},\theta) + \varepsilon^2\left(\frac{\partial N_e}{\partial\phi^*}\right)_{\phi_o^o}\phi_1^o + O(\varepsilon^4) \tag{4.4}$$

Substitution of expansions (4.3) and (4.4) into Eq. (4.1), collection of like orders of ε^2, and setting each order equal to zero gives for the first two orders

$$\varepsilon^o: \quad N_i(\bar{r},\theta) - N_e(\phi_o^o,\bar{r},\theta) = 0 \tag{4.5a}$$

$$\varepsilon^2: \quad \nabla'^2\phi_o^o = -\phi_1^o\left(\frac{\partial N_e}{\partial\phi^*}\right)_{\phi_o^o} \tag{4.5b}$$

Equation (4.5a) is an implicit equation for the zeroth-order function $\phi_o^o(\bar{r},\theta)$. It can be solved when the particular model for $N_e(\phi_o^o,\bar{r},\theta)$ is specified. When ϕ_o^o has been obtained, then the first-order function ϕ_1^o can be obtained from Eq. (4.5b). We shall concern ourselves with the zeroth-order function ϕ_o^o in this study.

Electron Density Model Based on Spherical Symmetry

The electron density according to the assumptions underlying expression (3.18) is

$$N_e(\phi_o^o,\bar{r}) = \tfrac{1}{2}e^{-\phi_o^o}[1+\text{erf}\sqrt{\phi_s^*-\phi_o^o} + \sqrt{1-\frac{1}{\bar{r}^2}}\exp(\frac{\phi_s^*-\phi_o^o}{\bar{r}^2-1})\text{erfc}\sqrt{\frac{\phi_s^*-\phi_o^o}{1-\frac{1}{\bar{r}^2}}}] \tag{4.6}$$

The ion number density is given by expression (3.10). It is not possible to solve explicitly for ϕ_o^o from Eq. (4.5a) by means of this model. It is useful, however, to obtain the asymptotic behavior as $\bar{r} \to \infty$ and as $\bar{r} \to 1$.

As $\bar{r} \to \infty$, it can be shown that $\phi_o^o \to 0$ is a solution to Eq. (4.5a). Thus the infinity condition is satisfied. If we assume that ϕ_o^o vanishes like $1/\bar{r}^2$ as $\bar{r} \to \infty$, then N_e as given by (4.6) behaves asymptotically like

$$N_e \sim 1 - \phi_o^o + \frac{1}{\bar{r}^2}\left\{\tfrac{1}{2}(\phi_s^*-\tfrac{1}{2})\text{erfc}\sqrt{\phi_s^*} - \frac{\sqrt{\phi_s^*}e^{-\phi_s^*}}{2\sqrt{\pi}}\right\} + 0(\frac{1}{\bar{r}^4}) \tag{4.7}$$

On the other hand, it can be shown that the ion number density N_i, as given by (3.10), varies asymptotically for $\bar{r} \to \infty$ like

$$N_i \sim 1 - \frac{1}{4\bar{r}^2}[\frac{2}{\sqrt{\pi}} S_i\cos\theta e^{-S_i^2} + (1+2S_i^2\cos^2\theta)e^{-S_i^2\sin^2\theta}\text{erfc}(-S_i\cos\theta)] + 0(\frac{1}{\bar{r}^3}) \tag{4.8}$$

Substituting (4.7) and (4.8) into Eq. (4.5a), and retaining terms to order $1/\bar{r}^2$, we obtain for large $\bar{r}$:

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[(2\phi_s^*-1)\text{erfc}\sqrt{\phi_s^*} - \frac{2\sqrt{\phi_s^*}\,e^{-\phi_s^*}}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}S_i\cos\theta e^{-S_i^2}$$
$$+ (1+2S_i^2\cos^2\theta)e^{-S_i^2\sin^2\theta}\text{erfc}(-S_i\cos\theta)] \tag{4.9}$$

On the axis upstream of the sphere ($\theta=\pi$), expression (4.9) takes the form

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[(2\phi_s^*-1)\mathrm{erfc}\sqrt{\phi_s^*} - \frac{2\sqrt{\phi_s^*}\, e^{-\phi_s^*}}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}} S_i e^{-S_i^2}$$

$$+ (1+2S_i^2)\mathrm{erfc}(S_i)] \qquad \bar{r}\to\infty \tag{4.10}$$

If we now recall that S_i is large and assume that $\sqrt{\phi_s^*}$ is large enough to make use of the asymptotic expansion

$$\mathrm{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}\, x}, \qquad x\to\infty \tag{4.11}$$

we find that expression (4.10) takes the approximate value for large ϕ_s^* and S_i

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[-\frac{e^{-\phi_s^*}}{\sqrt{\pi\phi_s^*}} + \frac{e^{-S_i^2}}{\sqrt{\pi}S_i}] \quad \theta=\pi, \quad \bar{r}\to\infty \tag{4.12}$$

Thus the potential ϕ_o^o far upstream of the sphere is very small <u>and negative</u> since S_i^2 is in general much larger than ϕ_s^*. For $\theta = \pi/2$, we find

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[-\frac{e^{-\phi_s^*}}{\sqrt{\pi\phi_s^*}} + e^{-S_i^2}] \quad \theta = \frac{\pi}{2}, \quad \bar{r}\to\infty \tag{4.13}$$

and again the potential is very small and negative. For large S_i the dependence on θ is very weak, and hence the potential is nearly spherically symmetric in front of the sphere.

On the axis in the wake of the sphere ($\theta=0$), expression (4.9) yields for large S_i and ϕ_s^*:

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[-\frac{e^{-\phi_s^*}}{\sqrt{\pi\phi_s^*}} + 2(1+2S_i^2)] \quad \theta = 0, \quad \bar{r}\to\infty \tag{4.14}$$

Here we find that the term involving ϕ_s^* is negligible and the potential is positive. Thus aft of the sphere, that is, in the wake, the potential depends strongly on θ.

Consider now the value of ϕ_o^o when $\bar{r} = 1$ according to the quasi-neutral approximation (4.5a). Setting $\phi_o^o = \phi_{os}^o$ when $\bar{r} = 1$, we obtain from (4.6)

$$N_e(\phi_{os}^o,1) = \tfrac{1}{2}e^{-\phi_{os}^o}[1+\mathrm{erf}\sqrt{\phi_s^*-\phi_{os}^o}] \qquad (4.15)$$

For the ion density $N_i(1,\theta)$, we get from (3.12)

$$N_i(1,\theta) = \tfrac{1}{2}\mathrm{erfc}(S_i\cos\theta) \qquad (4.16)$$

Substituting into (4.5a) then yields

$$e^{-\phi_{os}^o}[1+\mathrm{erf}\sqrt{\phi_s^*-\phi_{os}^o}] = \mathrm{erfc}(S_i\cos\theta) \qquad \bar{r} = 1 \qquad (4.17)$$

This expression does not satisfy the surface boundary condition $\phi^* = \phi_s^*$ when $\bar{r} = 1$. Nevertheless, it will be used as a matching condition for the outer limit of an inner expansion that does satisfy the surface boundary condition. Note that no real value of ϕ_{os}^o greater than ϕ_s^* satisfies Eq. (4.17). Thus no solutions exist when θ is less than a critical angle θ_w defined by

$$e^{-\phi_s^*} = \mathrm{erfc}(S_i\cos\theta_w) \qquad (4.18)$$

This situation arises because the potential ϕ^* increases from ϕ_s^* when $\theta<\theta_w$; thus the electron density N_e given by (4.6) is not valid in the near wake. Expression (4.18) is plotted in Fig. 4.

We further ascertain from (4.17) that ϕ_{os}^o is zero when θ has the value

$$\cos\theta = -\sqrt{\phi_s^*}/S_i \ . \qquad (4.19)$$

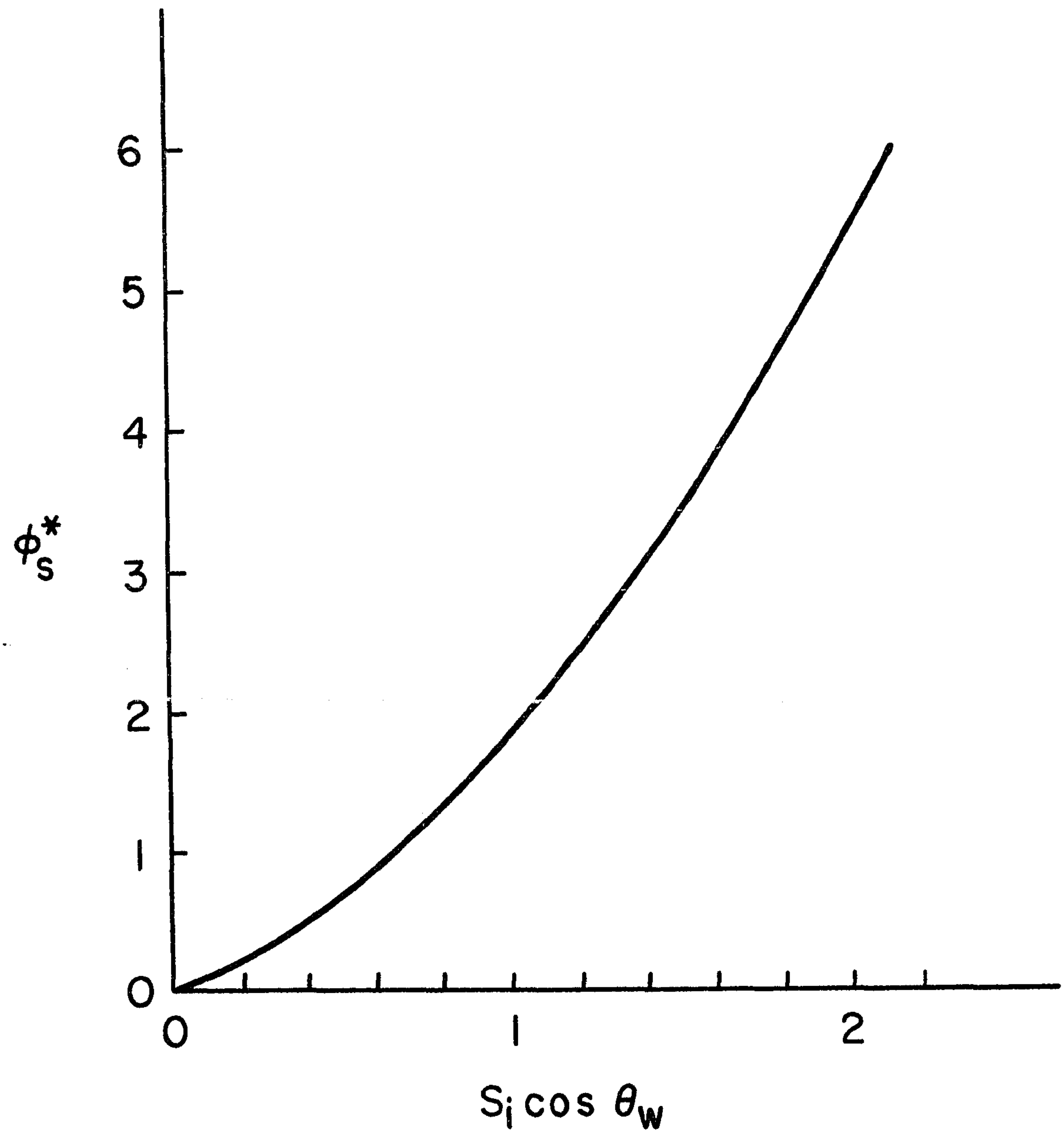


Fig. 4. Position of Zero Potential Gradient on Sphere Surface.

At this angle ϕ_{os}^{o} changes sign, and at $\theta = \pi$, ϕ_{os}^{o} is negative, which it is far upstream of the sphere according to (4.12). If we assume that ϕ_{os}^{o} is small at $\theta = \pi$, then we can obtain from (4.17) the approximate value

$$\phi_{os}^{o} = - \frac{\text{erf } S_i - \text{erf } \sqrt{\phi_s^*}}{1+\text{erf}\sqrt{\phi_s^*} + \dfrac{e^{-\phi_s^*}}{\sqrt{\pi}\,\sqrt{\phi_s^*}}} \tag{4.20}$$

If we assume that ϕ_s^* and S_i are large enough to use the asymptotic expansion (4.11), we get

$$\phi_{os}^{o} \simeq - \left[\frac{e^{-\phi_s^*}}{2\sqrt{\pi}\,\sqrt{\phi_s^*}} - \frac{e^{-S_i^2}}{2\sqrt{\pi}\,S_i}\right] \qquad \theta = \pi, \quad \bar{r} = 1 \tag{4.21}$$

The term involving S_i can be neglected, since S_i is usually much greater than $\sqrt{\phi_s^*}$.

Electron-Density Model Based on Straight-Line Trajectories

By means of the electron density model (3.19) we can solve Eq. (4.5a) for ϕ_o^o and get (neglecting terms of order S_e):

$$\phi_o^o = - \ln\left[\frac{2N_i(\bar{r},\theta)}{1+\sqrt{1-\frac{1}{\bar{r}^2}}}\right] \tag{4.22}$$

where $N_i \equiv n_{io}/n_{i\infty}$ is determined from expression (3.10). By means of expression (4.8), we expand for large $\bar{r}$ and obtain

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}\left[-1+ \frac{2}{\sqrt{\pi}} S_i\cos\theta e^{-S_i^2} + (1+2S_i^2\cos^2\theta)e^{-S_i^2\sin^2\theta}\,\text{erfc}(-S_i\cos\theta)\right]$$

$$\bar{r} \to \infty \tag{4.23}$$

On the axis upstream of the sphere ($\theta=\pi$), this expression takes the form for large S_i

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[-1+\frac{e^{-S_i^2}}{\sqrt{\pi}S_i}] \qquad \theta = \pi, \quad \bar{r}\to\infty \tag{4.24}$$

Thus the potential is negative for this model, but much more negative than the corresponding value (4.12) for the previous model. On the wake axis (θ=0), we get for large $\bar{r}$ and large S_i

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}[-1+2(1+2S_i^2)] \qquad \theta = 0, \quad \bar{r}\to\infty \tag{4.25}$$

For large S_i, this result is effectively the same as for the previous model.

Near the surface of the sphere, $\bar{r}\to 1$, we can use expression (3.12) for the ions and obtain

$$\phi_o^o \sim -\ln[\mathrm{erfc}(S_i\cos\theta)]$$
$$- \sqrt{\bar{r}^2-1}\,\frac{[\frac{4}{\sqrt{\pi}}\, e^{-S_i^2}\int_0^\infty x^2 e^{-x^2} I_o(2S_i x\sin\theta)dx - \mathrm{erfc}(S_i\cos\theta)]}{\mathrm{erfc}(S_i\cos\theta)}$$
$$\bar{r}\to 1 \tag{4.26}$$

This expression is valid all around the sphere, but note that it is not analytic at $\bar{r}$ = 1. The value on the sphere surface is

$$\phi_{os}^o = -\ln[\mathrm{erfc}(S_i\cos\theta)] \tag{4.27}$$

Upstream of the sphere at $\theta = \pi$, the potential ϕ_{os}^o is negative and has the approximate value $\phi_{os}^o = -\ln 2$ for large S_i. Aft of the sphere at $\theta = 0$, the potential ϕ_{os}^o is very large and for large S_i has the approximate value $\phi_{os}^o \simeq S_i^2 + \ln(\sqrt{\pi}S_i)$. These values, of course, do not satisfy the surface boundary condition, and thus must be regarded as the inner limit of the outer expansion.

The advantage of this particular model is that it is simple and yields results all around the sphere. On the other hand, it gives values of the potential outside the wake that are far too negative, as can be seen by comparison with the model in the previous section.

Maxwellian Electron Density

It is interesting to examine the results yielded by the Maxwellian model $N_e = \exp(-\phi_o^o)$. In this case we get

$$\phi_o^o = - \ln N_i \tag{4.28}$$

For large $\bar{r}$, we get by means of expression (4.8)

$$\phi_o^o \sim \frac{1}{4\bar{r}^2}\left[\frac{2}{\sqrt{\pi}} S_i \cos\theta e^{-S_i^2} + (1+2S_i^2\cos^2\theta) e^{-S_i^2\sin^2\theta} \operatorname{erfc}(-S_i\cos\theta)\right] \tag{4.29}$$

This value is positive all around the sphere. On the upstream and wake axis we get for large S_i

$$\theta = \pi: \quad \phi_o^o \sim \frac{1}{4\bar{r}^2}\left[\frac{e^{-S_i^2}}{\sqrt{\pi} S_i}\right] \qquad \bar{r} \to \infty \tag{4.30}$$

$$\theta = 0: \quad \phi_o^o \sim \frac{1}{4\bar{r}^2}\left[2(1+2S_i^2)\right] \qquad \bar{r} \to \infty \tag{4.31}$$

Near the sphere surface $\bar{r} \to 1$, we obtain from Eq. (3.12)

$$\phi_o^o \sim - \ln[\tfrac{1}{2}\operatorname{erfc}(S_i\cos\theta)] - \sqrt{\bar{r}^2-1}\, \frac{\frac{4}{\sqrt{\pi}} e^{-S_i^2} \int_0^\infty x^2 e^{-x^2} I_o(2S_i x\sin\theta)\, dx}{\operatorname{erfc}(S_i\cos\theta)} \qquad \bar{r} \to 1 \tag{4.32}$$

On the sphere surface, we have

$$\phi_{os}^o = - \ln[\tfrac{1}{2}\operatorname{erfc}(S_i\cos\theta)] \tag{4.33}$$

Upstream of the sphere ($\theta=\pi$), we have for large S_i

$$\phi^o_{os} \simeq + \frac{e^{-S_i^2}}{2\sqrt{\pi}S_i} \tag{4.34}$$

This value is positive, but exceedingly small for large S_i. Aft of the sphere ($\theta=0$), we have for large S_i

$$\phi^o_{os} \simeq S_i^2 + \ln(2\sqrt{\pi}S_i) \tag{4.35}$$

This value is nearly the same as obtained from expression (4.27) for the previous model.

Inner Expansion or the Boundary-Layer Approximation

As we have seen in the previous section, the quasi-neutral solution, or outer expansion, cannot satisfy the surface boundary condition. It is thus necessary to consider a new expansion that is valid near the surface of the sphere. Near the surface of the sphere, the second derivative $\partial^2\phi^*/\partial\bar{r}^2$ becomes large enough to compensate for the small parameter ε^2 that multiplies it. To account for this, it is useful to introduce a new variable, ζ, measured normal from the sphere surface:

$$\zeta = \frac{\bar{r}-1}{\varepsilon} \quad \text{or} \quad \bar{r} = 1 + \varepsilon\zeta \tag{4.36}$$

In terms of this variable the Laplacian and Poisson's equation (4.1) now expand to read

$$\frac{\partial^2\phi^*}{\partial\zeta^2} + \frac{2\varepsilon}{1+\varepsilon\zeta}\frac{\partial\phi^*}{\partial\zeta} + \frac{\varepsilon^2}{(1+\varepsilon\zeta)^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi^*}{\partial\theta}\right)$$

$$= N_i(1+\varepsilon\zeta,\theta) - N_e(\phi^*,1+\varepsilon\zeta,\theta) \tag{4.37}$$

We now wish to construct a series expansion for small ε in terms of the new variable ζ. To see how the expansion should proceed, note from expression (3.12) that N_i behaves in the following way for $\varepsilon \to 0$:

$$N_i(1+\varepsilon\zeta,\theta) = N_i(1,\theta) + \sqrt{\varepsilon}N_{i1}(\zeta,\theta) + O(\varepsilon) \tag{4.38}$$

where

$$N_i(1,\theta) = \tfrac{1}{2}\mathrm{erfc}(S_i\cos\theta)$$

$$N_{i1}(\zeta,\theta) = \sqrt{\frac{\zeta}{2}}\,\frac{4}{\sqrt{\pi}}\,e^{-S_i^2}\int_0^\infty x^2 e^{-x^2} I_o(2S_i x\sin\theta)\,dx$$

Thus N_i expands in terms of powers of the square root of ε.

Motivated by this result, we assume an inner expansion for ϕ^* in the following form:

$$\phi^*(1+\varepsilon\zeta,\theta) = \phi_o^i(\zeta,\theta) + \sqrt{\varepsilon}\phi_1^i(\zeta,\theta) + \varepsilon\phi_2^i(\zeta,\theta) + \cdots \tag{4.39}$$

The corresponding expansion for $N_e(\phi^*,1+\varepsilon\zeta,\theta)$ reads

$$N_e(\phi^*,1+\varepsilon\zeta,\theta) = N_e(\phi_o^i,1,\theta) + \sqrt{\varepsilon}\Big[\Big(\frac{\partial N_e}{\partial\phi^*}\Big)_{\varepsilon=0}\phi_1^i + \Big(\frac{\partial N_e}{\partial\sqrt{\varepsilon\zeta}}\Big)_{\varepsilon=0}\sqrt{\zeta}\Big] + O(\varepsilon) \tag{4.40}$$

Substituting (4.38), (4.39) and (4.40) into Eq. (4.37), collecting like orders of ε, and requiring each order to vanish yields for the two lowest orders:

$$\varepsilon^o:\quad \frac{\partial^2\phi_o^i}{\partial\zeta^2} = N_i(1,\theta) - N_e(\phi_o^i,1,\theta) \tag{4.41}$$

$$\varepsilon^{\frac{1}{2}}:\quad \frac{\partial^2\phi_1^i}{\partial\zeta^2} = N_{i1}(\zeta,\theta) - \Big(\frac{\partial N_e}{\partial\phi^*}\Big)_{\varepsilon=0}\phi_1^i - \Big(\frac{\partial N_e}{\partial\sqrt{\varepsilon\zeta}}\Big)_{\varepsilon=0}\sqrt{\zeta} \tag{4.42}$$

Note for the two lowest-order functions that θ appears as a parameter. Thus the equations for ϕ_o^i and ϕ_1^i are to be treated as ordinary differential equations.

The inner expansion must satisfy the surface boundary condition. Thus we have

$$\phi_o^i(0,\theta) = \phi_s^* \tag{4.43a}$$

$$\phi_1^i(0,\theta) = 0 \tag{4.43b}$$

The outer boundary condition must be obtained by matching with the inner limit of the outer expansion. We shall concern ourselves here with only the lowest-order solution.

Electron-Density Model Based on Spherical Symmetry

For the electron density (3.18), we have

$$N_e(\phi_o^i,1,\theta) = \tfrac{1}{2}e^{-\phi_o^i}[1+\mathrm{erf}\sqrt{\phi_s^*-\phi_o^i}] \tag{4.44}$$

and thus Eq. (4.41) becomes

$$\frac{\partial^2\phi_o^i}{\partial\zeta^2} = N_i(1,\theta) - \tfrac{1}{2}e^{-\phi_o^i}[1+\mathrm{erf}\sqrt{\phi_s^*-\phi_o^i}] \tag{4.45}$$

This equation is the same as given by Al'pert *et al.* [1965] except that $N_i(1,\theta)$ has been replaced by unity. Multiply Eq. (4.45) by $\partial\phi_o^i/\partial\zeta$ and integrating from $\zeta = 0$ yields

$$\left(\frac{\partial\phi_o^i}{\partial\zeta}\right)^2 - \left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s^2 = 2(\phi_o^i-\phi_s^*)N_i(1,\theta)$$

$$+ e^{-\phi_o^i}\{1+\mathrm{erf}\sqrt{\phi_s^*-\phi_o^i}\} - e^{-\phi_s^*}\{1+\frac{2}{\sqrt{\pi}}\sqrt{\phi_s^*-\phi_o^i}\} \tag{4.46}$$

Because the derivative $\partial\phi_o^i/\partial\zeta$ depends only on ϕ_o^i and not explicitly on ζ, we can integrate Eq. (4.46) by means of a quadrature

$$\zeta = +\int_{\phi_s^*}^{\phi_o^i} \frac{d\phi_o^i}{\left(\frac{\partial\phi_o^i}{\partial\zeta}\right)} \tag{4.47}$$

where

$$\frac{\partial\phi_o^i}{\partial\zeta} = -\ [2(\phi_o^i-\phi_s^*)N_i(1,\theta)+e^{-\phi_o^i}\{1+\operatorname{erf}\sqrt{\phi_s^*-\phi_o^i}\}$$

$$-\ e^{-\phi_s^*}\{1+\frac{2}{\sqrt{\pi}}\sqrt{\phi_s^*-\phi_o^i}\ \} + \left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s^2]^{\frac{1}{2}} \tag{4.48}$$

According to the so-called "limit matching principle" of Van Dyke [1965], the outer limit of the inner expansion equals the inner limit of the outer expansion, that is,

$$\lim_{\zeta\to\infty} \phi_o^i = \lim_{\bar{r}\to 1} \phi_o^o$$

$$= \phi_{os}^o$$

Thus from Eq. (4.47) we can see that for ζ to go to infinity when $\phi_o^i \to \phi_{os}^o$, the derivative $\partial\phi_o^i/\partial\zeta$ must vanish when $\phi_o^i = \phi_{os}^o$. This condition enables the gradient at the wall $(\partial\phi_o^i/\partial\zeta)_s$ to be evaluated:

$$\left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s = -\ [2(\phi_s^*-\phi_{os}^o-1)N_i(1,\theta)+e^{-\phi_s^*}\{1+\frac{2}{\sqrt{\pi}}\sqrt{\phi_s^*-\phi_{os}^o}\ \}]^{\frac{1}{2}} \tag{4.49}$$

where Eqs. (4.17) and (4.38) have been utilized. This value vanishes when $\phi_{os}^o = \phi_s^*$, that is when $\theta = \theta_w$ as defined by (4.18). No solution exists for this electron-density model when $\theta<\theta_w$. The value of ϕ_{os}^o in general is obtained from (4.17). Al'pert *et al.* [1965] obtained a similar expression to that of (4.49) except that they set $\phi_{os}^o = o$, which is not strictly correct according to asymptotic matching principles. Setting ϕ^o_{os} equal to zero is a good approximation only for $\theta = \pi$ and when ϕ^*_s is large, as can be seen from (4.21).

Electron-Density Model Based on Straight-Line Trajectories

For the electron density (3.19) we have

$$N_e(\phi^i_o,1,\theta) = \tfrac{1}{2}e^{-\phi^i_o} \tag{4.50}$$

and now Eq. (4.41) becomes

$$\frac{\partial^2\phi^i_o}{\partial\zeta^2} = N_i(1,\theta) - \tfrac{1}{2}e^{-\phi^i_o} \tag{4.51}$$

Integration as for the previous model yields

$$\left(\frac{\partial\phi^i_o}{\partial\zeta}\right)^2 - \left(\frac{\partial\phi^i_o}{\partial\zeta}\right)^2_s = 2(\phi^i_o-\phi^*_s)N_i(1,\theta) + e^{-\phi^i_o} - e^{-\phi^*_s} \tag{4.52}$$

Another integration yields (4.47) as before. The matching condition yields

$$\left(\frac{\partial\phi^i_o}{\partial\zeta}\right)_s = \pm[2(\phi^*_s-\phi^o_{os}-1)N_i(1,\theta)+e^{-\phi^*_s}]^{\frac{1}{2}} \tag{4.53}$$

where now ϕ^o_{os} is given by (4.27), and has been utilized in obtaining (4.53). In this case, however, values of the gradient exist for all θ. The gradient vanishes when $\theta = \theta_W$, as given by (4.18). The minus sign is to be used when $\theta>\theta_W$ and the plus sign when $\theta<\theta_W$.

Maxwellian Electron-Density Model

For the Maxwellian electron density, $N_e = \exp(-\phi^i_o)$, Eq. (4.41) becomes

$$\frac{\partial^2\phi^i_o}{\partial\zeta^2} = N_i(1,\theta) - e^{-\phi^i_o} \tag{4.54}$$

Integration and the matching condition give

$$\left(\frac{\partial\phi_o^i}{\partial\zeta}\right)^2 - \left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s^2 = 2(\phi_o^i-\phi_s^*)N_i(1,\theta) + 2(e^{-\phi_o^i}-e^{-\phi_s^*}) \tag{4.55}$$

$$\left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s = \pm[2(\phi_s^*-\phi_{os}^o-1)N_i(1,\theta)+2e^{-\phi_s^*}]^{\frac{1}{2}} \tag{4.56}$$

where ϕ_{os}^o is given by (4.33). This gradient vanishes when $\theta = \theta_w$ as before. In this case, however, θ_w is determined by

$$e^{-\phi_s^*} = \tfrac{1}{2}\text{erfc}(S_i\cos\theta_w) \tag{4.57}$$

The minus sign is to be used when $\theta>\theta_w$ and the plus sign when $\theta<\theta_w$.

Potential Gradient on the Sphere Surface

The potential gradient on the surface of the sphere is plotted as a function of θ in Fig. 5 for the straight-trajectory model (4.53) and the Maxwellian model (4.56). For this comparison typical values of $\phi_s^* = 3$ and $S_i = 5$ were used. The spherical-symmetry model (4.49) gives nearly the same values as the Maxwellian model, except that no solution exists aft of the sphere (θ<74°). Because the Maxwellian model is nearly the same as the spherical-symmetry model in front of the sphere, and the spherical-symmetry model is the best approximation in this region, it can be seen that the straight-trajectory model gives gradients that are too large in front of the sphere. Aft of the sphere, the straight-trajectory model is the best approximation since it allows for absorption of electrons, and it gives gradients that are smaller than the Maxwellian model in this region. Figure 6 shows the effect of changing S_i from 5 to 8 for the straight-trajectory model. The other models show qualitatively the same effects.

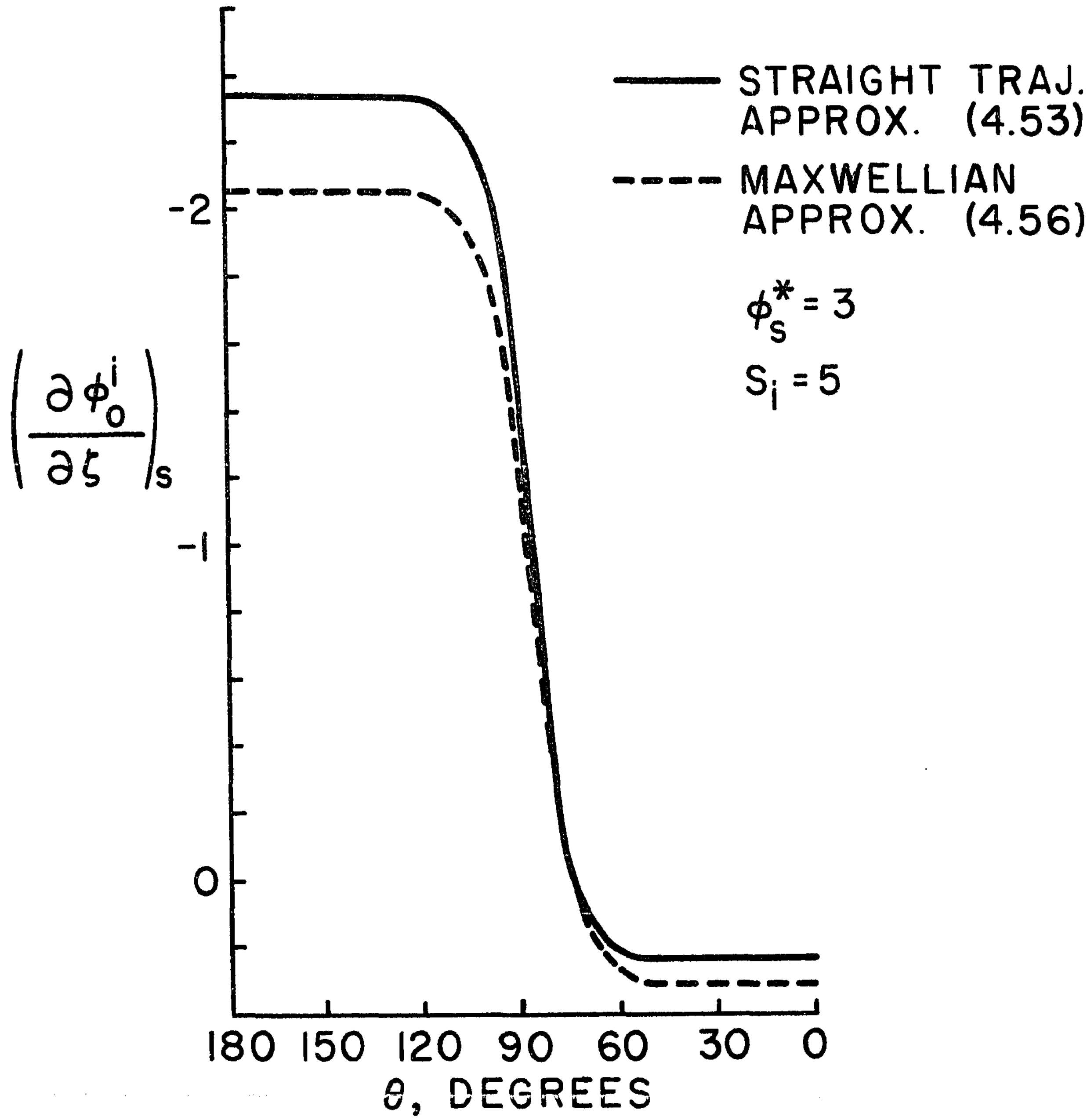


Fig. 5. Potential Gradient on Sphere Surface.

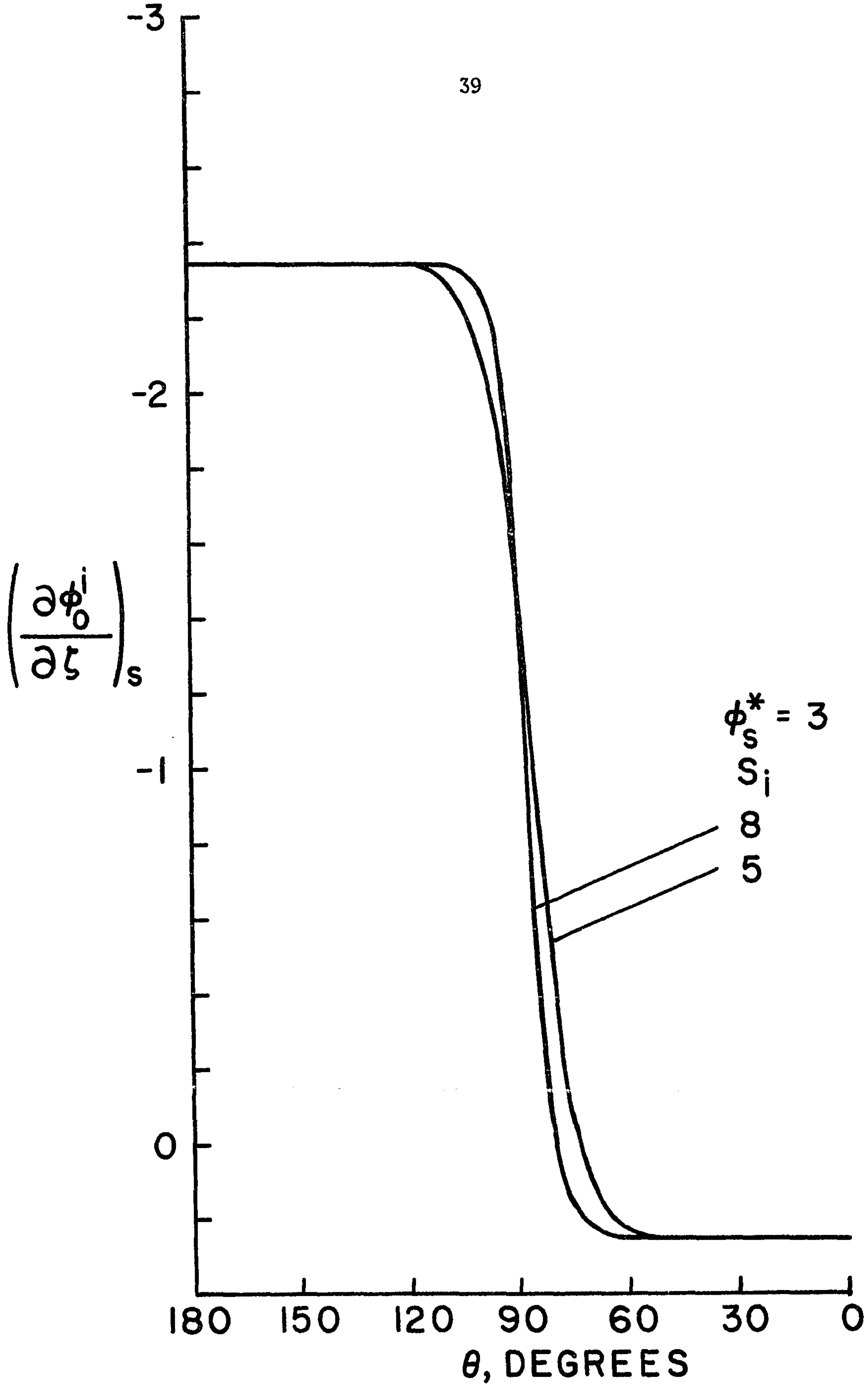


Fig. 6. Effect of S_i on Potential Gradient on Sphere Surface.

Maxwell Drag

The electric Maxwell drag is determined from the Maxwell stresses as given by expression (2.12). If $\hat{e}_z$ is a unit vector in the direction of the free stream, the Maxwell drag is

$$D_M = \frac{1}{8\pi} \oiint_{\text{sphere}} \left(\frac{\partial\phi}{\partial r}\right)^2 \hat{e}_r \cdot \hat{e}_z dS$$

$$= \frac{a^2}{4} \int_0^{\pi} \left(\frac{\partial\phi}{\partial r}\right)^2 \cos\theta \sin\theta \, d\theta$$

For the zeroth approximation we thus have

$$D_{M_o} = \pi a^2 n_{e\infty} k\, T_{e\infty} \int_0^{\pi} \left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s^2 \cos\theta \sin\theta \, d\theta \tag{4.58}$$

In terms of the drag coefficient

$$C_{D_M} = \frac{D_M}{\frac{1}{2} m_i (n_{i\infty} + n_{o\infty}) U_\infty^2 \pi a^2}$$

we have

$$C_{D_{M_o}} = \frac{\alpha}{S_i^2} \frac{T_{e\infty}}{T_{i\infty}} \int_0^{\pi} \left(\frac{\partial\phi_o^i}{\partial\zeta}\right)_s^2 \cos\theta \sin\theta \, d\theta \tag{4.59}$$

where $\alpha \equiv n_{e\infty}/(n_{i\infty} + n_{o\infty})$ is the degree of ionization.

For the straight-line trajectory model (4.53), substitution into (4.59) yields

$$C_{D_{M_o}} = \frac{\alpha}{S_i^2} \frac{T_{e\infty}}{T_{i\infty}} \left[-(\phi_s^* - 1) \left\{ \left(1 - \frac{1}{2S_i^2}\right) \operatorname{erf} S_i + \frac{e^{-S_i^2}}{\sqrt{\pi} S_i} \right\} \right.$$

$$\left. + \frac{1}{S_i^2} \int_{-S_i}^{S_i} x \operatorname{erfc} x \, \ln(\operatorname{erfc} x) dx \right] \tag{4.60}$$

while for the Maxwellian model (4.56), the Eq. (4.59) yields

$$C_{D_{M_o}} = \frac{\alpha}{S_i^2}\frac{T_{e\infty}}{T_{i\infty}}\left[-(\phi_s^*-1-\ell n2)\left\{(1-\frac{1}{2S_i^2})\text{erf}S_i + \frac{e^{-S_i^2}}{\sqrt{\pi}S_i}\right\} + \frac{1}{S_i^2}\int_{-S_i}^{S_i} x\,\text{erfc}\,x\;\ell n(\text{erfc}\,x)dx\right] \tag{4.61}$$

As mentioned in the previous section, the best approximation aft of the sphere is the straight-line trajectory model, and in front of the sphere the spherical-symmetry model, hence the best approximation of the Maxwell drag is the combination of these two models. Since the Maxwellian model is nearly the same as the spherical-symmetry model, this simpler model (4.56) is used in front of the sphere along with the straight-line trajectory model (4.53) used aft of the sphere to obtain

$$C_{D_{M_o}} = \frac{\alpha}{S_i^2}\frac{T_{e\infty}}{T_{i\infty}}\left[-(\phi_s^*-1-\tfrac{1}{2}\ell n2)\left\{(1-\frac{1}{2S_i^2})\text{erf}S_i + \frac{e^{-S_i^2}}{\sqrt{\pi}S_i}\right\} + \tfrac{1}{2}(\ell n2-e^{-\phi_s^*}) + \frac{1}{S_i^2}\int_{-S_i}^{S_i} x\,\text{erfc}\,x\;\ell n(\text{erfc}\,x)dx\right] \tag{4.62}$$

The last term in the brackets has the limiting value $-\ell n\ 2$ for large S_i. Hence for large S_i we obtain

$$C_{D_{M_o}} = -\frac{\alpha}{S_i^2}\frac{T_{e\infty}}{T_{i\infty}}(\tfrac{1}{2}e^{-\phi_s^*}+\phi_s^*-1) \tag{4.63}$$

The Maxwell drag is negative when $\phi_s^* \geq 1$, and is thus actually a thrust.

Potential Near the Wake Axis

The electric potential in the near wake can be examined by means of the straight-trajectory or Maxwellian models. For either of these models the zeroth-order inner expansion for the electric potential can be expressed by means of (4.47) in the form

$$\zeta = \pm \frac{1}{\sqrt{2N_i(1,\theta)}} \int_Y^{Y_s} \frac{dx}{\sqrt{e^x-(1+x)}} \tag{4.64}$$

where

$$Y \equiv \phi_{os}^o - \phi_o^i$$

$$Y_s \equiv \phi_{os}^o - \phi_s^*$$

The plus sign is to be used when

$$Y < Y_s \quad \text{or} \quad \phi_o^i > \phi_s^* \qquad (\theta<\theta_w)$$

and the negative sign when

$$Y > Y_s \quad \text{or} \quad \phi_o^i < \phi_s^* \qquad (\theta>\theta_w)$$

The inner limit of the outer expansion ϕ_{os}^o has the value $\phi_{os}^o = -\ln N_i(1,\theta)$ for the Maxwellian model and $\phi_{os}^o = -\ln 2N_i(1,\theta)$ for the straight-trajectory model.

When $S_i \cos\theta$ is large, ϕ_{os}^o is of the order of $S_i^2 \cos^2\theta$, and thus Y_s is very large when ϕ_s^* has the typical value of 3. Near the sphere surface in the region of the wake axis both Y_s and Y will be large, and we can gain an approximation for (4.64) as follows:

$$\zeta = \frac{1}{\sqrt{2N_i(1,\theta)}} \int_Y^{Y_s} e^{-x/2} dx$$

$$= \frac{2}{\sqrt{2N_i(1,\theta)}} (e^{-Y/2} - e^{-Y_s/2})$$

$$= \frac{2\, e^{-\phi_{os}^o/2}}{\sqrt{2N_i(1,\theta)}} (e^{\phi_o^i/2} - e^{\phi_s^*/2})$$

If we now solve for ϕ_o^i, we get

$$\phi_o^i = \phi_s^* + \ln[1 + \frac{\zeta}{2} e^{-\phi_s^*/2}]^2 \tag{4.65}$$

for the straight-trajectory model, and

$$\phi_o^i = \phi_s^* + \ln[1 + \frac{\zeta}{\sqrt{2}} e^{-\phi_s^*/2}]^2 \tag{4.66}$$

for the Maxwellian model. These expressions are valid near the sphere surface when θ is small and S_i is large, and are independent of both S_i and θ. The Maxwellian model gives larger values of ϕ_o^i than the straight-trajectory model.

It is interesting to compare the results of the present analyses with the numerical results of Kiel *et al.* [1968]. Because of the singular behavior of the Laplacian operator when $\sin\theta \to 0$, numerical difficulties arise near the axis of the wake. Because of this, the potential curve for $\theta = 4.5°$ is plotted by Kiel *et al.* In order to make a comparison of results, it is useful to obtain an approximation for the outer expansion (quasi-neutral) that is valid for small θ. It is possible to expand the general expression (3.10) for $N_i(\bar{r},\theta)$ for small θ and obtain, keeping only the largest term for large S_i,

$$N_i(\bar{r},\theta) \simeq \tfrac{1}{2}\sqrt{1-\frac{1}{\bar{r}^2}}\, e^{-S_i^2/\bar{r}^2} \operatorname{erfc}(-S_i\sqrt{1-\frac{1}{\bar{r}^2}})[1+\frac{S_i^4\sin^2\theta}{\bar{r}^2}(1-\frac{1}{\bar{r}^2})] \qquad \theta \to 0 \tag{4.67}$$

Keeping only the largest terms in S_i and $1/\bar{r}^2$, and solving ϕ_o^o by means of (4.22), we obtain

$$\phi_o^o \simeq \frac{S_i^2}{\bar{r}^2} - \ln[1 + \frac{S_i^4\sin^2\theta}{\bar{r}^2}] \qquad \theta \to 0 \tag{4.68}$$

This is an approximation valid near the wake axis for large S_i and away from the sphere surface.

The potential near the wake axis is plotted in Fig. 7 for the typical ionospheric values $\phi_s^* = 3$, $S_i = 8$, and $\varepsilon = 0.001$. The outer expansion (4.68) and inner expansion (4.65) are compared with the numerical results of Kiel *et al.* The results of the numerical integration by Kiel agree well with the quasi-neutral (outer) approximation (4.68) for large $\bar{r}$. Near the sphere, the present formula (4.65) gives values of ϕ^* that are larger than those of Kiel. This discrepancy arises because of the different model for the electron density used by Kiel in the near wake. The simple formulas for the inner and outer regions of the near wake allow rapid computations to be made. They agree qualitatively with the more sophisticated (and more complicated) results of numerical integration.

The effect of varying the ratio of Debye length to sphere radius is shown in Fig. 8. These curves are for the same values as the previous figure except that ε is ten times larger, $\varepsilon = 0.01$. The same qualitative agreement with the results of Kiel are found. The present theory gives a better approximation as ε becomes smaller.

Higher-order approximations in the inner expansion should improve the accuracy near the sphere surface, especially since the inner expansion proceeds in a series expansion in the square-root of ε. A composite formula would give a smooth variation between the inner and outer expansions, but such a composite expression does not give fruitful results unless ε is exceedingly small or unless higher-order terms are obtained for the inner expansion.

With some knowledge of the behavior of the potential field, we can now proceed to obtain a higher-order approximation for the ion distribution function.

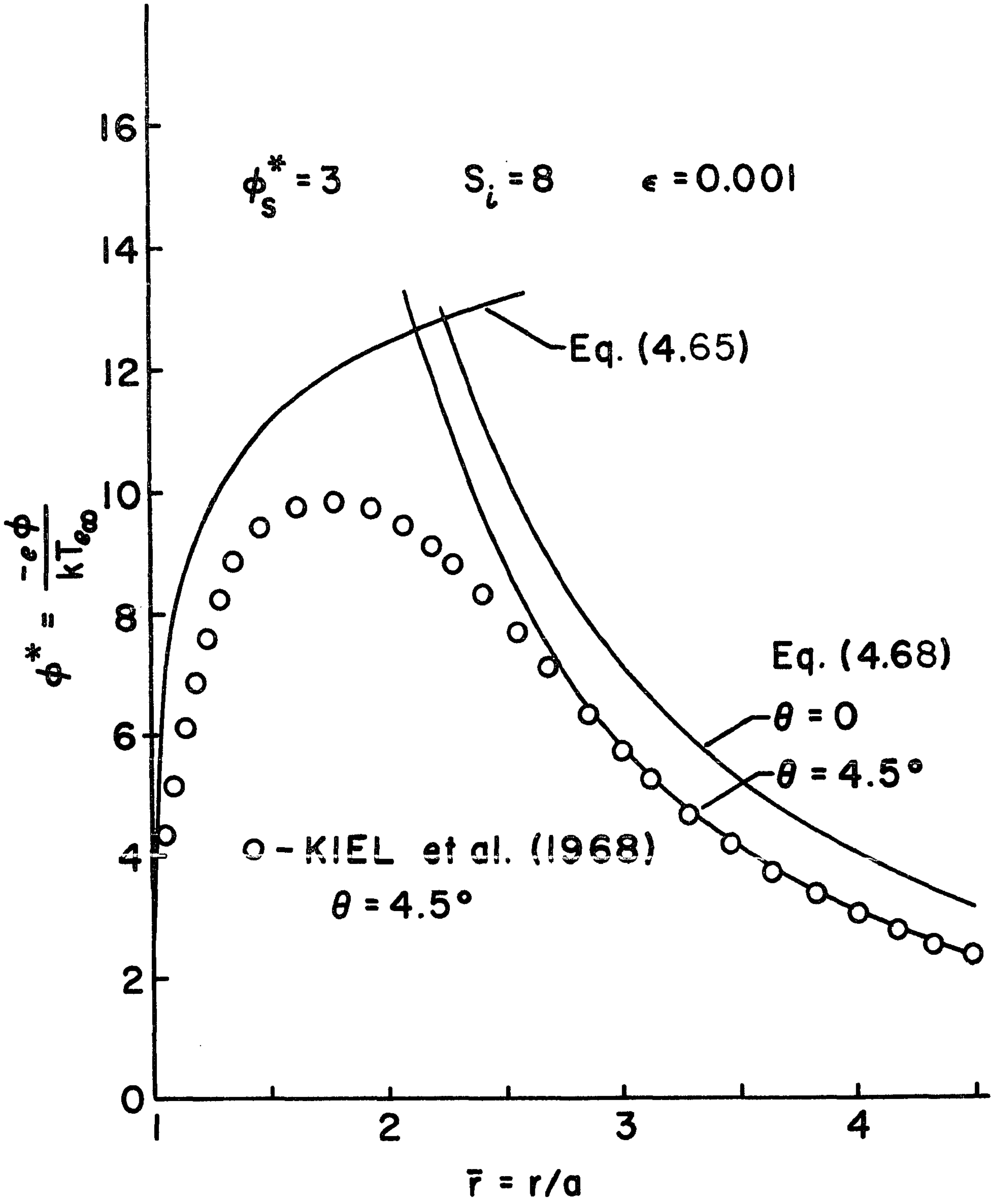


Fig. 7. Electric Potential in the Near Wake, ϵ = 0.001.

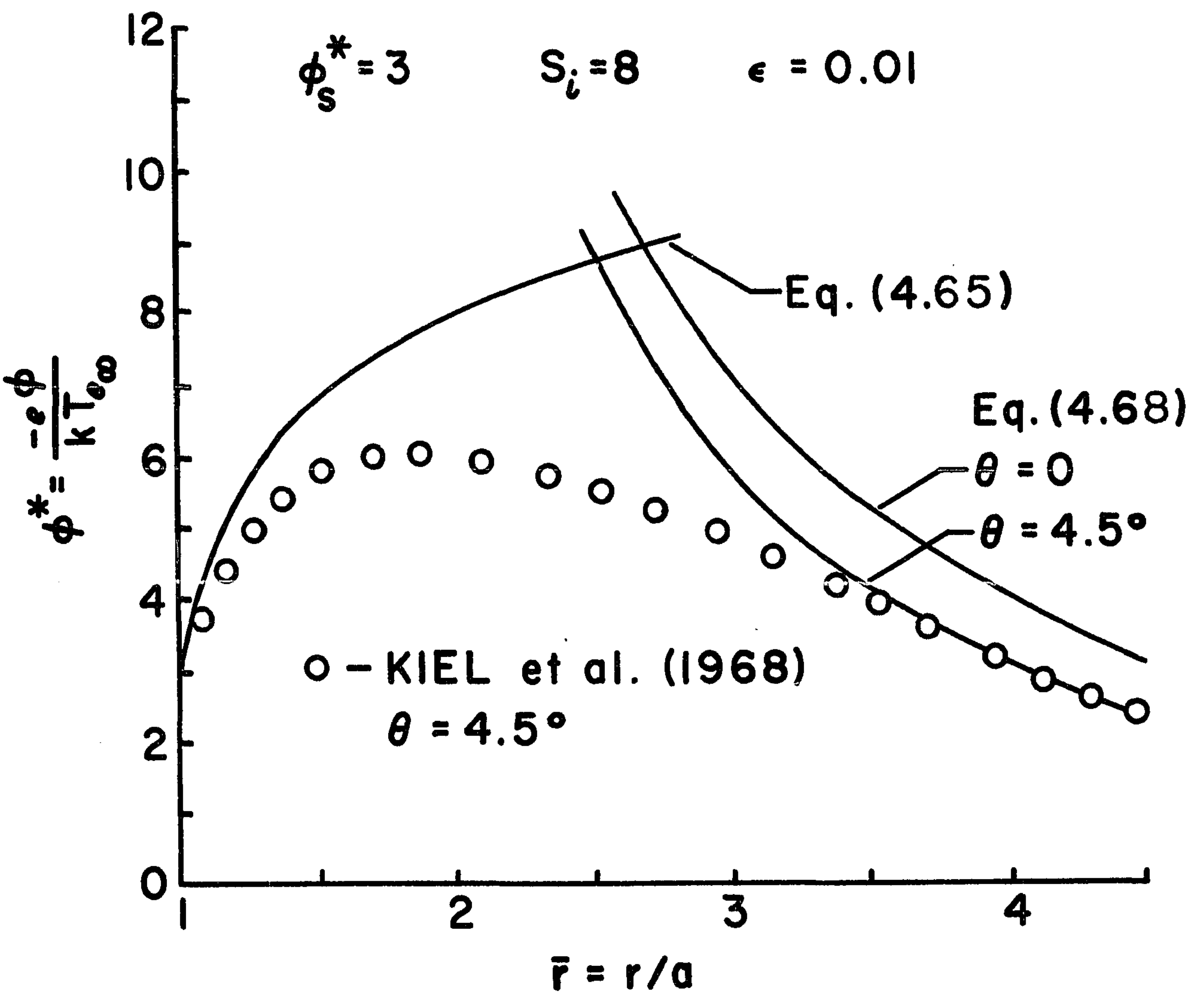


Fig. 8. Electric Potential in the Near Wake, ϵ = 0.01.

CHAPTER V

FIRST-ORDER CORRECTION FOR THE ION DISTRIBUTION FUNCTION

Method of Characteristics

We now return to the first-order distribution function $\bar{f}_1$ given as part of f_{i1} by Eq. (3.8). In dimensional velocity and space variables, Eq. (3.9) for $\bar{f}_1$ reads

$$\vec{\xi}\cdot\frac{\partial \bar{f}_1}{\partial \vec{r}} = - \vec{U}_\infty\cdot\frac{\partial \phi_o'}{\partial \vec{r}} \tag{5.1}$$

where

$$\phi_o' = \frac{\phi_o}{\phi_s}$$

Consider a solution to Eq. (5.1) by means of the theory of characteristics. If the right-hand side is regarded as known, the characteristic equations are

$$\frac{d\bar{f}_1}{dt} = - \vec{U}_\infty\cdot\frac{\partial \phi_o'}{\partial \vec{r}} \tag{5.2a}$$

$$\frac{d\vec{r}}{dt} = \vec{\xi} \tag{5.2b}$$

$$\frac{d\vec{\xi}}{dt} = 0 \tag{5.2c}$$

The characteristics of Eq. (5.1) in physical space are the free-particle trajectories, which are straight lines. The value of the function $\bar{f}_1$ varies along the straight-line trajectories depending on ϕ_o'. By contrast, note that the characteristics for the Vlasov Eq. (2.1) are curved trajectories in physical space, depending on ϕ, and that f_i is a constant on these trajectories.

There are three physical constants of motion of interest along a free-particle trajectory: the energy, E, angular momentum about the origin, L, and the z-component of angular momentum, L_z.

$$E = \tfrac{1}{2}m_i\xi^2 = \frac{m_i}{2}(\xi_r^2+\xi_\theta^2+\xi_\psi^2) \tag{5.3a}$$

$$L^2 = m_i^2 r^2(\xi_\theta^2+\xi_\psi^2) \tag{5.3b}$$

$$L_z = m_i r\xi_\psi \sin\theta \tag{5.3c}$$

In terms of these constants, the variation of velocity components along a trajectory (or characteristic) can be expressed as

$$\xi = \sqrt{2E/m_i} \tag{5.4a}$$

$$\xi_r = \pm\sqrt{2E/m_i - L^2/m_i^2 r^2} \tag{5.4b}$$

$$\xi_\theta = \pm \frac{L}{m_i r\sin\theta}\sqrt{\sin^2\theta - L_z^2/L^2} \tag{5.4c}$$

The orbit equation relates θ as a function of r along a particle trajectory. The orbit equation can be obtained from Eq. (5.4). Since $\xi_r = dr/dt$, $\xi_\theta = r\, d\theta/dt$, we obtain from Eq. (5.4c) by separation of variables

$$\frac{-d(\cos\theta)}{\sqrt{1-\frac{L_z^2}{L^2}-\cos^2\theta}} = \pm\frac{L dr}{m_i r^2 \xi_r} \tag{5.5}$$

The integration of this equation along a particle trajectory gives

$$\cos^{-1}\frac{\cos\theta}{\sqrt{1-\frac{L_z^2}{L^2}}} - \cos^{-1}\frac{\cos\theta_\infty}{\sqrt{1-\frac{L_z^2}{L^2}}} = \pm I_\infty \tag{5.6}$$

where $\cos\theta_\infty$ is a constant, and

$$I_\infty \equiv \frac{L}{m_i}\int_\infty^r \frac{dr}{r^2\xi_r} \tag{5.7}$$

Solving Eq. (5.6) for $\cos\theta_\infty$ and making use of (5.4c) gives

$$\cos\theta_\infty = \cos\theta\cos I_\infty + \frac{m_i r}{L}\xi_\theta \sin\theta\sin I_\infty \tag{5.8}$$

In the same manner the Eq. (5.6) can be solved for $\cos\theta$ to give

$$\cos\theta = \cos\theta_\infty \cos I_\infty \mp \sqrt{\sin^2\theta_\infty - \frac{L_z^2}{L^2}}\sin I_\infty \tag{5.9}$$

It can be shown from expressions (5.4b) and (5.7) that $\cos I_\infty = -\xi_r/\xi$, and $\sin I_\infty = L/m_i r\xi$. Hence the Eqs. (5.8) and (5.9) become

$$\cos\theta_\infty = -\frac{\xi_r}{\xi}\cos\theta + \frac{\xi_\theta}{\xi}\sin\theta \tag{5.10a}$$

$$\cos\theta = -\frac{\xi_r}{\xi}\cos\theta_\infty \mp \frac{L}{m_i r\xi}\sqrt{\sin^2\theta_\infty - \frac{L_z^2}{L^2}} \tag{5.10b}$$

where the minus sign is used when $\xi_\theta>0$ and the plus sign when $\xi_\theta<0$.

Since $\xi_r = dr/dt$, the solution for $\bar{f}_1$ can be expressed from (5.2a) as

$$\bar{f}_1 = -U_\infty \int_\infty^r \left(\frac{\partial\phi_o'}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\phi_o'}{\partial\theta}\sin\theta\right)\frac{dr}{\xi_r} \tag{5.11}$$

This integral can be evaluated when $\phi_o' = \phi_o'(r,\theta)$ is known, since θ is given as a function of r by (5.10).

Expression (5.11) is the formal solution for $\bar{f}_1$, but its use remains difficult because ϕ_o' is a complicated function of r and θ. For practical approximation purposes, it is therefore useful to consider the special case of a central potential $\phi_o' = \phi_o'(r)$. This is not a bad approximation for the region in front of a sphere.

Spherical Potential

For a spherical potential, we have $\partial\phi_o'/\partial\theta = 0$, and expression (5.10b) substituted into Eq. (5.11) yields

$$\bar{f}_1 = -U_\infty[-\frac{\phi_o'}{\xi}\cos\theta_\infty \mp \frac{L}{m_i\xi}\sqrt{\sin^2\theta_\infty - \frac{L_z^2}{L^2}}\int_\infty^r \frac{\frac{d\phi_o'}{dr}dr}{r\,\xi_r}] \tag{5.12}$$

The constant of the motion $\cos\theta_\infty$ is given in terms of the local position and velocity by (5.10a). The other constant of the motion can be evaluated in terms of the local variables by (5.10b):

$$\mp\frac{L}{m_i\xi}\sqrt{\sin^2\theta_\infty - \frac{L_z^2}{L^2}} = r[\cos\theta + \frac{\xi_r}{\xi}\cos\theta_\infty] \tag{5.13}$$

Again using (5.10a), we get

$$\mp\frac{L}{m_i\xi}\sqrt{\sin^2\theta_\infty - \frac{L_z^2}{L^2}} = r[\frac{\xi_\theta^2+\xi_\psi^2}{\xi^2}\cos\theta + \frac{\xi_r\xi_\theta}{\xi^2}\sin\theta] \tag{5.14}$$

Equation (5.12) can now be written

$$\begin{aligned}\bar{f}_1 = &-\frac{U_\infty\phi_o'}{\xi}(\frac{\xi_r}{\xi}\cos\theta - \frac{\xi_\theta}{\xi}\sin\theta)\\ &-\frac{U_\infty}{\xi}(\frac{\xi_\theta^2+\xi_\psi^2}{\xi^2}\cos\theta + \frac{\xi_r\xi_\theta}{\xi^2}\sin\theta)I(\phi_o')\end{aligned} \tag{5.15}$$

where

$$I(\phi_o') \equiv r\xi\int_\infty^r \frac{\frac{d\phi_o'}{dr}dr}{r\,\xi_r} \tag{5.16}$$

Making use of expressions (5.4) for ξ_r/ξ, and substituting the dummy variable u for r, we express the integral $I(\phi_o')$ as

$$I(\phi_o') = -r\int_\infty^r \frac{\frac{d\phi_o'}{du}\,du}{\sqrt{u^2-r_m^2}} \qquad \xi_r \leq 0 \tag{5.17a}$$

$$I(\phi_o') = -r\int_\infty^{r_m} \frac{\frac{d\phi_o'}{du}\,du}{\sqrt{u^2-r_m^2}} + r\int_{r_m}^{r} \frac{\frac{d\phi_o'}{du}\,du}{\sqrt{u^2-r_m^2}} \qquad \xi_r > 0 \tag{5.17b}$$

where

$$r_m^2 \equiv r^2\,\frac{\xi_\theta^2+\xi_\psi^2}{\xi^2} \tag{5.18}$$

The integral $I(\phi_o')$ is the term that contains the Debye length in the first approximation for the ion distribution function.

A compact form for the first-order function $f_i = f_{io}[1+\delta f_{i1}]$ can be obtained by introducing spherical coordinates in velocity space as follows:

$$\begin{aligned} \xi_r &= \xi\cos\gamma \\ \xi_\theta &= \xi\sin\gamma\cos\varepsilon \\ \xi_\psi &= \xi\sin\gamma\sin\varepsilon \end{aligned} \tag{5.19}$$

By means of (3.4a), (3.8) and (5.15), we have for the first-order function

$$f_i = f_{io} + \frac{S_i^2\delta}{B_i}\left[-\frac{\phi_o'}{\xi}\frac{\partial f_{io}}{\partial\xi} + \frac{\sin\gamma}{\xi^2}\frac{\partial f_{io}}{\partial\gamma}I(\phi_o')\right] \tag{5.20a}$$

$$= f_{io} + \frac{1}{\xi}\left(\frac{e\phi_o}{m_i}\right)\frac{\partial f_{io}}{\partial\xi} - \frac{\sin\gamma}{\xi^2}\frac{\partial f_{io}}{\partial\gamma}I\left(\frac{e\phi_o}{m_i}\right) \tag{5.20b}$$

The first and second terms on the right-hand side are equivalent to the ad hoc approximation used by Prager and Rasmussen [1967].

Models for Spherical Potential

It is useful to evaluate the integral $I(\phi_o')$ for various spherical models. A simple model that represents the electric field qualitatively in front of the sphere at least is the linear model shown in Fig. 9. Because the density of the ions in the near wake is practically zero for large S_i and small ε, the nature of ϕ_o' aft of the sphere is inconsequential for computing moments of the distribution function on the sphere surface.

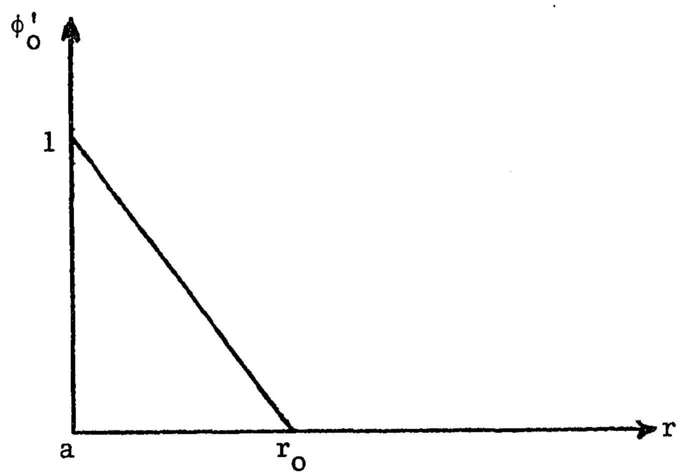


Figure 9. Linear Potential Model

For the above model, we have

$$\begin{aligned} \frac{\partial \phi_o'}{\partial r} &= -\left|\frac{\partial \phi_o'}{\partial r}\right|_s = \text{const.} \qquad && a \le r \le r_o \\ &= 0 && r > r_o \end{aligned} \tag{5.21}$$

where

$$r_o = a + \frac{1}{\left|\frac{\partial \phi_o'}{\partial r}\right|_s} \tag{5.22}$$

If we now make use of the analysis in <u>Inner Expansion or the Boundary-Layer Approximation</u>, we can use the value of $(\partial \phi_o^i/\partial \zeta)_s$ near $\theta = \pi$ and write

$$\frac{d\phi_o'}{dr} = -\frac{1}{a\varepsilon\phi_s^*}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s = \text{const.} \qquad a \le r \le r_o$$

$$= 0 \qquad\qquad r > r_o \tag{5.23}$$

where

$$r_o = a\left[1+\varepsilon\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\right] \tag{5.24}$$

On the surface of the sphere, $r = a$, we now get for $I(\phi_o')$ from Eqs. (5.17a) and (5.23)

$$I(\phi_o') = a\int_a^{r_o}\frac{\frac{d\phi_o'}{du}\,du}{\sqrt{u^2-r_m^2}}$$

$$= -\frac{1}{\varepsilon\phi_s^*}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s \ell n\left[\frac{r_o+\sqrt{r_o^2-r_m^2}}{a+\sqrt{a^2-r_m^2}}\right] \tag{5.25}$$

where
$$r_m^2 = a^2\,\frac{\xi_\theta^2+\xi_\psi^2}{\xi^2} = a^2\sin^2\gamma$$

on the surface of the sphere. Further, expanding the Eq. (5.25) in the series in small ε, we obtain when $\gamma \ne \frac{\pi}{2}$

$$I(\phi_o') = -\frac{1}{|\cos\gamma|} + \tfrac{1}{2}\varepsilon\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\;\frac{1}{|\cos^3\gamma|} + 0(\varepsilon^2) \tag{5.26a}$$

and when $\gamma = \frac{\pi}{2}$

$$I(\phi_o') = -\sqrt{\frac{1}{\varepsilon\phi_s^*}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\left[\sqrt{2} - \frac{\sqrt{2}}{12}\,\varepsilon\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s} + 0(\varepsilon^2)\right] \tag{5.26b}$$

Another simple model that represents the electric field qualitatively in front of the sphere at least, but does not have an abrupt change of the potential gradient at the point r_0 as in the previous linear model and thus is more realistic, is the parabolic model shown in Fig. 10.

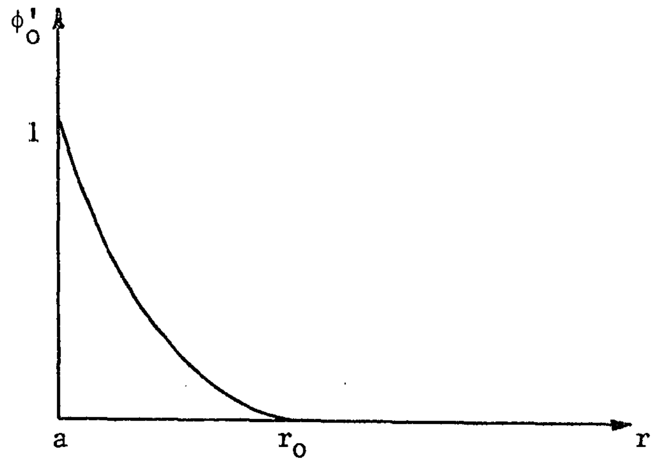


Figure 10. Parabolic Potential Model.

For this model, we have

$$\frac{\partial\phi'_o}{\partial r} = -\left|\frac{\partial\phi'_o}{\partial r}\right|_s + \frac{1}{2}\left|\frac{\partial\phi'_o}{\partial r}\right|_s^2 (r-a) \qquad a\leq r\leq r_o$$

$$= 0 \qquad r>r_o \tag{5.27}$$

where

$$r_o = a + \frac{2}{\left|\frac{\partial\phi'_o}{\partial r}\right|_s} \tag{5.28}$$

As mentioned in the previous model, we can use the value of $(\frac{\partial\phi^i_o}{\partial\zeta})_s$ near $\theta = \pi$ and write

$$\frac{d\phi'_o}{dr} = -\frac{1}{a\epsilon\phi^*_s}\left|\frac{\partial\phi^i_o}{\partial\zeta}\right|_s + \frac{1}{2}\frac{1}{(a\epsilon\phi^*_s)^2}\left|\frac{\partial\phi^i_o}{\partial\zeta}\right|_s^2 (r-a) \qquad a\leq r\leq r_o$$

$$= 0 \qquad r>r_o \tag{5.29}$$

where

$$r_o = a[1+2\varepsilon \frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}] \tag{5.30}$$

On the surface of the sphere, r = a, we now obtain for $I(\phi_o')$ from Eqs. (5.17a) and (5.29)

$$I(\phi_o') = a\int_a^{r_o} \frac{\frac{d\phi_o'}{du}\, du}{\sqrt{u^2-r_m^2}}$$

$$= -\frac{1}{\varepsilon\phi_s^*}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s [1+\tfrac{1}{2}\frac{1}{\varepsilon\phi_s^*}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s]\ln[\frac{r_o+\sqrt{r_o^2-r_m^2}}{a+\sqrt{a^2-r_m^2}}]$$

$$+\frac{1}{2a}\frac{1}{(\varepsilon\phi_s^*)^2}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s^2[\sqrt{r_o^2-r_m^2} - \sqrt{a^2-r_m^2}] \tag{5.31}$$

where

$$r_m^2 = a^2 \frac{\xi_\theta^2+\xi_\psi^2}{\xi^2} = a^2 \sin^2\gamma$$

on the surface of the sphere. Further, expanding Eq. (5.31) in the series in small ε, we obtain when $\gamma \neq \frac{\pi}{2}$

$$I(\phi_o') = -\frac{1}{|\cos\gamma|} + \frac{2}{3}\varepsilon \frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\frac{1}{|\cos^3\gamma|} + O(\varepsilon^2) \tag{5.32a}$$

and when $\gamma = \frac{\pi}{2}$

$$I(\phi_o') = -\sqrt{\frac{1}{\varepsilon\phi_s^*}\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\,[\frac{4}{3} - \frac{2}{15}\varepsilon\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s} + O(\varepsilon^2)] \tag{5.32b}$$

Equations (5.25) and (5.31) evaluated at $\gamma = \pi/2$ have singularities as ε tends to zero, as can be seen from Eqs. (5.26b) and (5.32b).

This singularity as ε tends to zero arises because of the behavior of the integrand $1/\sqrt{r^2-r_m^2}$ in the integral (5.17a). It is not possible to remove the singularity by assuming the similar simple models for the spherical potential such as Eq. (5.21) or (5.27). Nevertheless, the expressions (5.26a) and (5.32a) illustrate how the Debye length, or ε, enters the first-order perturbation away from $\gamma = \pi/2$.

The next step is to compute the induced drag and the energy transfer by evaluating the appropriate moments on the sphere. Except for the term containing $I(\phi_o')$ the results are the same as that of Prager and Rasmussen [1967]. It will be seen later that it is necessary to find a spherical model which will produce $I(\phi_o')$ free from singularity at $\gamma = \pi/2$ as ε tends to zero in order to obtain meaningful results of the moments on the sphere.

CHAPTER VI

MOMENTS ON THE SURFACE

Moments for the Ions

The distribution function for the ions is nonzero on the surface only for $\xi_r<0$. The first-order ion distribution function on the surface of the sphere is, from Eq. (5.20a) and (5.17a)

$$f_i(\vec{r}_s,\vec{\xi}) = f_{io} - \frac{\bar{\phi}_s}{2}\frac{1}{\xi}\frac{\partial f_{io}}{\partial \xi} + \frac{\bar{\phi}_s}{2}\frac{\sin\gamma}{\xi^2}\frac{\partial f_{io}}{\partial \gamma} I(\phi_o') \tag{6.1}$$

where $$\bar{\phi}_s \equiv -\frac{2e\phi_s}{m_i} = 2U_\infty^2\delta$$

$$I(\phi_o') = a\int_a^\infty \frac{\frac{d\phi_o'}{du}\,du}{\sqrt{u^2-a^2\sin^2\gamma}} \qquad \frac{\pi}{2}\leq\gamma\leq\pi \tag{6.2}$$

In rectangular coordinates, f_{io} is given as

$$f_{io} = A_i\exp[-B_i\{\xi^2+U_\infty^2-2U_\infty(\xi_r\cos\theta-\xi_\theta\sin\theta)\}] \tag{6.3a}$$

while in spherical coordinates as

$$f_{io} = A_i\exp[-B_i\{\xi^2+U_\infty^2-2U_\infty\xi(\cos\gamma\ \cos\theta-\sin\gamma\ \cos\varepsilon\ \sin\theta)\}] \tag{6.3b}$$

The negative surface potential causes the potential field to be attractive for the ions. Since the total energy of an ion particle is conserved, there is a restriction $\xi\geq\sqrt{-\frac{2e\phi_s}{m_i}} = \sqrt{\bar{\phi}_s}$ on the surface of the sphere. It is convenient to write the moments with the velocity components expressed in spherical components given by (5.19). In spherical coordinates in velocity space we have

$$d^3\xi \equiv \xi^2 \sin\gamma \, d\xi \, d\gamma \, d\varepsilon \tag{6.4}$$

The general tensor moments on the surface for the ions can now be written as

$$n_i\langle \xi_r^\ell \xi_\theta^m \xi_\psi^n \rangle = \int_{\sqrt{\bar{\phi}_s}}^{\infty} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{2\pi} \xi_r^\ell \xi_\theta^m \xi_\psi^n f_i \; \xi^2 \sin\gamma \, d\xi \, d\gamma \, d\varepsilon \tag{6.5a}$$

$$= \int_{\sqrt{\bar{\phi}_s}}^{\infty} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{2\pi} \xi^{\ell+m+n+2} g(\gamma,\varepsilon) f_i d\xi \, d\gamma \, d\varepsilon \tag{6.5b}$$

where

$$g(\gamma,\varepsilon) = \cos^\ell\gamma \, \sin^{m+n+1}\gamma \, \cos^m\varepsilon \, \sin^n\varepsilon \tag{6.6}$$

Substitution of Eq. (6.1) for f_i gives

$$\begin{aligned} n_i\langle \xi_r^\ell \xi_\theta^m \xi_\psi^n \rangle = & \int_{\sqrt{\bar{\phi}_s}}^{\infty} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{2\pi} \xi^{\ell+m+n+2} \, g(\gamma,\varepsilon) f_{io} d\xi \, d\gamma \, d\varepsilon \\ & - \frac{\bar{\phi}_s}{2} \int_{\sqrt{\bar{\phi}_s}}^{\infty} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{2\pi} \xi^{\ell+m+n+1} \, g(\gamma,\varepsilon) \frac{\partial f_{io}}{\partial \xi} d\xi \, d\gamma \, d\varepsilon \\ & + \frac{\bar{\phi}_s}{2} \int_{\sqrt{\bar{\phi}_s}}^{\infty} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{2\pi} \xi^{\ell+m+n} \, g(\gamma,\varepsilon) \sin\gamma \frac{\partial f_{io}}{\partial \gamma} I(\phi_o') d\xi \, d\gamma \, d\varepsilon \end{aligned} \tag{6.7}$$

Since $\bar{\phi}_s$ is of order δ, it can be shown that the moments for the ions can be approximated to the order $\bar{\phi}_s$ as

$$n_i\langle\xi_r^{\ell}\xi_\theta^{m}\xi_\psi^{n}\rangle = \int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \xi^{\ell+m+n+2} g(\gamma,\varepsilon) f_{io} d\xi\ d\gamma\ d\varepsilon$$

$$- \frac{\bar{\phi}_s}{2}\int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \xi^{\ell+m+n+1} g(\gamma,\varepsilon)\frac{\partial f_{io}}{\partial\xi}\ d\xi\ d\gamma\ d\varepsilon$$

$$+ \frac{\bar{\phi}_s}{2}\int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \xi^{\ell+m+n} g(\gamma,\varepsilon)\sin\gamma\ \frac{\partial f_{io}}{\partial\gamma}\ I(\phi_o')d\xi\ d\gamma\ d\varepsilon \tag{6.8}$$

The second and the third integrals can be integrated by parts with respect to ξ and γ respectively. It is more convenient to carry out the integrations in rectangular coordinates, ξ_r, ξ_θ, ξ_ψ. The first integral can be easily written in rectangular coordinates, but to do so for the second integral after integration by parts is not as easy. It can be done by making use of the rewritten form of f_{io}

$$f_{io} = A_i\,\xi^2 \exp[-B_i(U_\infty^2-2U_\infty\xi_r\cos\theta+2U_\infty\xi_\theta\sin\theta)]\int_{B_i}^\infty e^{-C\xi^2}dC \tag{6.9}$$

which provides the needed factor ξ^2 to write the integral in rectangular coordinates. Now we can rewrite the moments for the ions in the more convenient form

$$n_i\langle\xi_r^{\ell}\xi_\theta^{m}\xi_\psi^{n}\rangle = I_1 + \frac{\bar{\phi}_s}{2}I_2 - \frac{\bar{\phi}_s}{2}I_3 \qquad \text{if} \quad \ell \neq 0 \tag{6.10a}$$

$$= I_1 + \frac{\bar{\phi}_s}{2}I_2 - \frac{\bar{\phi}_s}{2}(I_3+I_4) \qquad \text{if} \quad \ell = 0 \tag{6.10b}$$

where

$$I_1 = \int_{-\infty}^{o}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_r^{\ell}\xi_\theta^m\xi_\psi^n f_{io}\, d\xi_r\, d\xi_\theta\, d\xi_\psi \tag{6.11}$$

$$I_2 = (\ell+m+n+1)A_i\int_{B_i}^{\infty}\int_{-\infty}^{o}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_r^{\ell}\xi_\theta^m\xi_\psi^n \exp[-C\{(\xi_r - \frac{\sqrt{B_i}}{C} S_i\cos\theta)^2$$

$$+(\xi_\theta + \frac{\sqrt{B_i}}{C} S_i\sin\theta)^2 + \xi_\psi^2\} + S_i^2(\frac{B_i}{C} - 1)]\, dC d\xi_r d\xi_\theta d\xi_\psi \tag{6.12}$$

$$I_3 = \int_{o}^{\infty}\int_{\frac{\pi}{2}}^{\pi}\int_{o}^{2\pi} \xi^{\ell+m+n}\cos^m\varepsilon\, \sin^n\varepsilon\, f_{io}\, \frac{d}{d\gamma}[\cos^{\ell}\gamma\, \sin^{m+n+2}\gamma\;\; I(\phi_o')]\, d\xi\, d\gamma\, d\varepsilon \tag{6.13}$$

$$I_4 = A_i\; I(\phi')_{\gamma=\pi/2} \int_{o}^{\infty}\int_{o}^{2\pi} \xi^{m+n}\cos^m\varepsilon\, \sin^n\varepsilon\, \exp[-(\sqrt{B_i}\xi + S_i\cos\varepsilon\, \sin\theta)^2$$

$$- S_i^2(1-\cos^2\varepsilon\, \sin^2\theta)]\, d\xi\, d\varepsilon \tag{6.14}$$

The term I_4 arises because the integrated part does not vanish for $\ell = 0$. Either Eq. (3.6) or (6.3a) for f_{io} in rectangular coordinates can be used to evaluate the integral (6.11). To evaluate the integral (6.13), the Eq. (6.3b) for f_{io} in spherical coordinates should be used.

Moments for the Electrons

The zeroth-order electron distribution function on the surface of the sphere for small electron speed ratio, $S_e \ll 1$, can be written from (3.15b) as

$$f_{e_o}(\vec{r}_s, \vec{\xi}) = A_e\, e^{-\phi_s^*}\, e^{-(\sqrt{B_e}\,\vec{\xi} - \vec{S}_e)^2}$$

$$= A_e\, e^{-\phi_s^*}\, e^{-B_e\xi^2} + O(S_e) \tag{6.15}$$

The tensor moments on the surface for the electrons is obtained by integrating over velocity space for $\xi_r \leq 0$. In rectangular coordinates in velocity space, we have

$$n_e\langle\xi_r^\ell\xi_\theta^m\xi_\psi^n\rangle = \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_r^\ell\xi_\theta^m\xi_\psi^n \, f_{e_o} \, d\xi_r d\xi_\theta d\xi_\psi \tag{6.16}$$

Neglecting terms of order S_e in f_{e_o}, we write the tensor moments for the electrons

$$n_e\langle\xi_r^\ell\xi_\theta^m\xi_\psi^n\rangle = A_e e^{-\phi_s^*}\int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_r^\ell\xi_\theta^m\xi_\psi^n \, e^{-B_e\xi^2} \, d\xi_r d\xi_\theta d\xi_\psi \tag{6.17}$$

The potential enters exponentially for the electrons, whereas it enters algebraically for the ions in these approximations.

Moments for the Neutral Particles

The neutral particles are not affected by the potential. For $\xi_r \leq 0$, the distribution function on the surface of the sphere is

$$f_o(\vec{r}_s,\vec{\xi}) = A_o \, e^{-B_o(\vec{\xi}-\vec{U}_\infty)^2} \tag{6.18}$$

The tensor moments for $\xi_r \leq 0$ is then written as

$$n_o\langle\xi_r^\ell\xi_\theta^m\xi_\psi^n\rangle = \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\xi_r^\ell\xi_\theta^m\xi_\psi^n \, f_o \, d\xi_r d\xi_\theta d\xi_\psi \tag{6.19}$$

For $\xi_r > 0$, the particles are emitted from the surface with a diffuse distribution given by (2.7) as

$$f_d(\vec{r}_s,\vec{\xi}) = n_d(\vec{r}_s)\left(\frac{B_w}{\pi}\right)^{3/2} e^{-B_w\xi^2} \tag{2.7}$$

where

$$B_w = \frac{m_o}{2kT_w}$$

and $n_d(\vec{r}_s)$ is a function determined from the condition of zero mass flux normal to the surface or Eq. (2.8). The tensor moments for the emitted particles at the surface is then given by

$$[n_o<\xi_r^\ell \xi_\theta^m \xi_\psi^n>]_{emit} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \xi_r^\ell \xi_\theta^m \xi_\psi^n \; f_d \; d\xi_r \; d\xi_\theta d\xi_\psi \tag{6.20}$$

The Floating Potential

The floating potential is the value of ϕ_s that satisfies Eq. (2.9), which for J = 0 can be written as

$$\int_0^\pi n_e<\xi_r>\sin\theta \; d\theta = \int_0^\pi n_i<\xi_r>\sin\theta \; d\theta \tag{6.21}$$

The radial electron number flux $n_e<\xi_r>$ can be obtained by setting $\ell = 1$, $m = n = 0$ in Eq. (6.17)

$$n_e<\xi_r> = - \frac{n_{e\infty}}{2\sqrt{\pi B_e}} e^{-\phi_s^*} \tag{6.22}$$

Hence

$$\int_0^\pi n_e<\xi_r>\sin\theta \; d\theta = - \frac{n_{e\infty}}{\sqrt{\pi B_e}} e^{-\phi_s^*} \tag{6.23}$$

Similarly setting $\ell = 1$, $m = n = 0$ in Eq. (6.10a), the radial ion number flux $n_i<\xi_r>$ is obtained as

$$n_i<\xi_r> = I_1 + \frac{\bar{\phi}_s}{2} I_2 - \frac{\bar{\phi}_s}{2} I_{3(\xi_r)} \tag{6.24}$$

where

$$I_1 = - \frac{n_{i\infty}}{2\sqrt{\pi B_i}} [e^{-S_i^2\cos^2\theta} - \sqrt{\pi} S_i \cos\theta \; \mathrm{erfc}(S_i \cos\theta)] \tag{6.25}$$

$$I_2 = -\frac{n_{i\infty}}{S_i^2}\sqrt{\frac{B_i}{\pi}}\,[e^{-S_i^2\cos^2\theta} - e^{-S_i^2} - \sqrt{\pi}S_i\cos\theta\,\mathrm{erfc}(S_i\cos\theta)$$

$$+\sqrt{\pi}\cos\theta\,e^{-S_i^2}\int_0^{S_i} e^{x^2}\,\mathrm{erfc}(x\cos\theta)dx] \tag{6.26}$$

$$I_{3(\xi_r)} = \int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \xi f_{io}\frac{d}{d\gamma}[\cos\gamma\,\sin^2\gamma\,I(\phi_o')]d\xi d\gamma d\varepsilon \tag{6.27a}$$

$$= 2n_{i\infty}\sqrt{\frac{B_i}{\pi}}\,e^{-S_i^2}\int_0^\infty\int_{\frac{\pi}{2}}^{\pi} x\,e^{-(x^2-2xS_i\cos\gamma\,\cos\theta)}I_o(2xS_i\sin\gamma\,\sin\theta)$$

$$\frac{d}{d\gamma}[\cos\gamma\,\sin^2\gamma\,I(\phi_o')]dxd\gamma \tag{6.27b}$$

Hence

$$\int_0^\pi n_i\langle\xi_r\rangle\sin\theta d\theta = -\frac{n_{i\infty}}{2S_i\sqrt{B_i}}[(\tfrac{1}{2}+S_i^2)\mathrm{erf}\,S_i + \frac{S_i}{\sqrt{\pi}}e^{-S_i^2} + \frac{T_{e\infty}}{T_{i\infty}}\phi_s^*\,\mathrm{erf}\,S_i]$$

$$-\frac{\bar{\phi}_s}{2}\int_0^\pi I_{3(\xi_r)}\sin\theta d\theta \tag{6.28}$$

From (6.21), (6.23) and (6.28) we obtain

$$e^{-\phi_s^*} = \frac{n_{i\infty}}{n_{e\infty}}\frac{\sqrt{\pi}}{2S_i}\sqrt{\frac{m_e}{m_i}\frac{T_{i\infty}}{T_{e\infty}}}\,[(S_i^2+\tfrac{1}{2})\mathrm{erf}\,S_i + \frac{S_i}{\sqrt{\pi}}e^{-S_i^2}$$

$$+\frac{T_{e\infty}}{T_{i\infty}}\phi_s^*\{\mathrm{erf}\,S_i + \frac{U_\infty}{n_{i\infty}}\int_0^\pi I_{3(\xi_r)}\sin\theta\,d\theta\}] \tag{6.29}$$

where $I_{3(\xi_r)}$ is given by (6.27). This expression is the same as that obtained by Prager and Rasmussen [1967] except for the last term involving $I_{3(\xi_r)}$. This additional term contains the Debye length, and provides the dependence of the floating potential on the Debye length.

It is not possible to evaluate the extra term exactly, even if the simplest spherical potential model is used. However, for large ion speed ratio, $S_i \gg 1$, an asymptotic expression of this term can be obtained without specifying a certain model for the spherical potential by means of Laplace's method of asymptotic integration (see Appendices A and B).

Substituting Eq. (B.8), we can write Eq. (6.29) for $S_i \gg 1$ as

$$e^{-\phi_s^*} = \frac{n_{i\infty}}{n_{e\infty}} \frac{\sqrt{\pi}S_i}{2} \sqrt{\frac{m_e}{m_i} \frac{T_{i\infty}}{T_{e\infty}}} \Big[1 + \frac{1}{2S_i^2} + \frac{T_{e\infty}}{T_{i\infty}} \frac{\phi_s^*}{S_i^2} \Big\{1 - \tfrac{1}{2} I(\phi_o')_{\gamma=\pi/2} \int_o^{\frac{\pi}{2}} \mathrm{erfc}(S_i \cos\theta) d\theta\Big\}\Big] \tag{6.30}$$

It is interesting to observe that $I_{3(\xi_r)} = 0$ when $S_i = 0$, and the expression (6.29) reduces to

$$e^{-\phi_s^*} = \frac{n_{i\infty}}{n_{e\infty}} \sqrt{\frac{m_e}{m_i} \frac{T_{i\infty}}{T_{e\infty}}} \Big(1 + \frac{T_{e\infty}}{T_{i\infty}} \phi_s^*\Big) \tag{6.31}$$

In this case, the floating potential is independent of the Debye length.

Determination of the Diffuse Function $n_d(\vec{r}_s)$

The condition for zero mass flux normal to the surface is satisfied when the function $n_d(\vec{r}_s)$ in the diffuse distribution function is determined so that Eq. (2.6) is identically satisfied. The function $n_d(\vec{r}_s)$ is determined from Eq. (2.8) which can be written as

$$n_d(\vec{r}_s) = -\sqrt{\frac{2\pi}{m_o k T_w}} \, [m_i n_i \langle\xi_r\rangle + m_e n_e \langle\xi_r\rangle + m_o n_o \langle\xi_r\rangle] \tag{6.32}$$

The radial ion and electron number fluxes, $n_i\langle\xi_r\rangle$ and $n_e\langle\xi_r\rangle$, have been obtained in the previous section. The radial number flux for the incoming neutral particles $n_o\langle\xi_r\rangle$ can be obtained by setting $\ell = 1$, $m = n = 0$ in Eq. (6.19).

$$n_o\langle\xi_r\rangle = -\frac{n_{o\infty}}{2\sqrt{\pi}B_o}\left[e^{-S_o^2\cos^2\theta} - \sqrt{\pi}S_o\cos\theta\ \mathrm{erfc}(S_o\cos\theta)\right] \qquad (6.33)$$

Substitution of (6.22), (6.24) and (6.33) to Eq. (6.32) gives

$$n_d(\vec{r}_s) = \sqrt{\frac{B_w}{B_i}}\left[\frac{m_i}{m_o} n_{i\infty}G_1(S_i,\theta) + n_{o\infty}\sqrt{\frac{B_i}{B_o}}\,G_1(S_o,\theta) + \frac{m_e}{m_o} n_{e\infty}\sqrt{\frac{B_i}{B_e}}\,e^{-\phi_s^*} + \frac{m_i}{m_o} n_{i\infty}\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}\left\{G_o(S_i,\theta) + \frac{S_i^2}{n_{i\infty}}\sqrt{\frac{\pi}{B_i}}\,I_{3(\xi_r)}\right\}\right] \qquad (6.34)$$

where

$$G_1(S,\theta) \equiv e^{-S^2\cos^2\theta} - \sqrt{\pi}S\cos\theta\ \mathrm{erfc}(S\cos\theta) \qquad (6.35)$$

$$G_o(S,\theta) \equiv G_1(S,\theta) - e^{-S^2} + \sqrt{\pi}\cos\theta\, e^{-S^2}\int_o^S e^{x^2}\mathrm{erfc}(x\cos\theta)dx \qquad (6.36)$$

and $I_{3(\xi_r)}$ is given by (6.27). Prager and Rasmussen [1967] obtained the same expression as Eq. (6.34) except for the last term involving $I_{3(\xi_r)}$, which contains the Debye length and provides the dependence of $n_d(\vec{r}_s)$ on the Debye length. Although $I_{3(\xi_r)}$ can not be determined exactly even for the simplest spherical potential model, asymptotic expressions of $I_{3(\xi_r)}$ for large ion speed ratio, $S_i \gg 1$, are obtained without specifying a certain model for the spherical potential in Appendix A. The third term in the brackets in the expression (6.34) is negligible because the ratio m_e/m_o is a very small number. The ratio m_i/m_o can be taken as unity.

The Slip Velocity

The mean velocity at the surface of the sphere will have only a component in the $\hat{e}_\theta$ direction because the $\hat{e}_r$ component is zero due to mass conservation, and $\hat{e}_\psi$ component is zero owing to symmetry. The mean value of velocity at the surface is defined as follows:

$$V_\theta = \frac{\Sigma m_s n_s \langle\xi_\theta\rangle}{\Sigma m_s n_s} \tag{6.37}$$

where the summation is over all the electron, ion, and neutral species.

Setting $m = 1$, $\ell = n = 0$ in Eqs. (6.19), (6.20), (6.17) and (6.10b) gives

$$n_o\langle\xi_\theta\rangle = -\tfrac{1}{2} n_{o\infty} U_\infty \sin\theta \, \mathrm{erfc}(S_o\cos\theta) \tag{6.38a}$$

$$[n_o\langle\xi_\theta\rangle]_{emit} = 0 \tag{6.38b}$$

$$n_e\langle\xi_\theta\rangle = 0(S_e) \tag{6.38c}$$

$$n_i\langle\xi_\theta\rangle = I_1 + \frac{\bar{\phi}_s}{2} I_2 - \frac{\bar{\phi}_s}{2}(I_3+I_4)_{\xi_\theta} \tag{6.38d}$$

where

$$I_1 = -\tfrac{1}{2} n_{i\infty} U_\infty \sin\theta \, \mathrm{erfc}(S_i\cos\theta) \tag{6.39}$$

$$I_2 = -n_{i\infty}\frac{B_i}{S_i^2} U_\infty \sin\theta\left[\mathrm{erfc}(S_i\cos\theta) - \frac{e^{-S_i^2}}{S_i}\left\{\frac{\cos\theta(1-e^{S_i^2\sin^2\theta})}{\sqrt{\pi}\,\sin^2\theta} + \int_o^S e^{x^2}\mathrm{erfc}(x\cos\theta)dx\right\}\right] \tag{6.40}$$

$$I_{3(\xi_\theta)} = \int_o^\infty\int_{\frac{\pi}{2}}^{\pi}\int_o^{2\pi} \xi\cos\varepsilon \, f_{io}\frac{d}{d\gamma}[\sin^3\gamma \, I(\phi_o')]d\xi d\gamma d\varepsilon \tag{6.41a}$$

$$= -2n_{i\infty}\sqrt{\frac{B_i}{\pi}}\,e^{-S_i^2}\int_o^\infty\int_{\frac{\pi}{2}}^{\pi} x\,e^{-(x^2-2xS_i\cos\gamma\cos\theta)} I_1(2xS_i\sin\gamma\sin\theta) \frac{d}{d\gamma}[\sin^3\gamma \, I(\phi_o')]dxd\gamma \tag{6.41b}$$

$$I_{4(\xi_\theta)} = -\frac{n_{i\infty}}{2}\frac{\sqrt{B_i}}{\pi} e^{-S_i^2} I(\phi_o')_{\gamma=\pi/2}\int_0^{2\pi} S_i \cos^2\varepsilon\sin\theta\, e^{S_i^2\cos^2\varepsilon\sin^2\theta}$$
$$\mathrm{erfc}(S_i\cos\varepsilon\sin\theta)d\varepsilon \qquad (6.42)$$

The number densities on the surface can be obtained by setting $\ell = m = n = 0$ in the Eqs. (6.19), (6.20), (6.17) and (6.10b)

$$n_o = \tfrac{1}{2} n_{o\infty}\mathrm{erfc}(S_o\cos\theta) \qquad (6.43a)$$

$$n_{o_{emit}} = \tfrac{1}{2} n_d(\vec{r}_s) \qquad (6.43b)$$

$$n_e = \tfrac{1}{2} n_{e\infty}\, e^{-\phi_s^*} \qquad (6.43c)$$

$$n_i = I_1 + \frac{\bar{\phi}_s}{2} I_2 - \frac{\bar{\phi}_s}{2}(I_3+I_4)_{n_i} \qquad (6.43d)$$

where

$$I_1 = \tfrac{1}{2} n_{i\infty}\,\mathrm{erfc}(S_i\cos\theta) \qquad (6.44)$$

$$I_2 = n_{i\infty}\frac{B_i}{S_i} e^{-S_i^2}\int_0^{S_i} e^{x^2}\,\mathrm{erfc}(x\cos\theta)dx \qquad (6.45)$$

$$I_{3(n_i)} = \int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} f_{io}\frac{d}{d\gamma}[\sin^2\gamma\, I(\phi_o')]d\xi d\gamma d\varepsilon \qquad (6.46a)$$

$$= 2n_{i\infty}\frac{B_i}{\sqrt{\pi}} e^{-S_i^2}\int_0^\infty\int_{\frac{\pi}{2}}^{\pi} e^{-(x^2-2xS_i\cos\gamma\cos\theta)}$$
$$I_o(2xS_i\sin\gamma\sin\theta)\frac{d}{d\gamma}[\sin^2\gamma\, I(\phi_o')]dxd\gamma \qquad (6.46b)$$

$$I_{4(n_i)} = \frac{n_{i\infty}}{2}\frac{B_i}{\pi} e^{-S_i^2} I(\phi_o')_{\gamma=\pi/2}\int_0^{2\pi} e^{S_i^2\cos^2\varepsilon\sin^2\theta}\mathrm{erfc}(S_i\cos\varepsilon\sin\theta)d\varepsilon$$
$$(6.47)$$

Contribution of the electrons is small and can be neglected. Substitution of (6.38) and (6.43) to expression (6.37) gives

$$V_\theta = -U_\infty \sin\theta\left[\frac{m_o n_{o\infty}\mathrm{erfc}(S_o\cos\theta)+m_i n_{i\infty}\left\{\mathrm{erfc}(S_i\cos\theta)+\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}G_2(S_i,\theta)\right\}+K_o}{m_o n_{o\infty}\mathrm{erfc}(S_o\cos\theta)+m_o n_d(\vec{r}_s)+m_i n_{i\infty}\left\{\mathrm{erfc}(S_i\cos\theta)+\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}G_3(S_i,\theta)\right\}+K_1}\right] \tag{6.48}$$

where

$$G_2(S,\theta) = \mathrm{erfc}(S\cos\theta) - \frac{e^{-S^2}}{S}\left[\frac{\cos\theta(1-e^{S^2\sin^2\theta})}{\sqrt{\pi}\ \sin^2\theta} + \int_o^S e^{x^2}\mathrm{erfc}(x\cos\theta)dx\right] \tag{6.49}$$

$$G_3(S,\theta) = S\ e^{-S^2}\int_o^S e^{x^2}\mathrm{erfc}(x\cos\theta)dx \tag{6.50}$$

$$K_o = \frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}\frac{m_i U_\infty}{\sin\theta}(I_3+I_4)_{\xi_\theta} \tag{6.51}$$

$$K_1 = -\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}m_i U_\infty^2(I_3+I_4)_{n_i} \tag{6.52}$$

The integrals $I_{3(\xi_\theta)}$, $I_{4(\xi_\theta)}$, $I_{3(n_i)}$ and $I_{4(n_i)}$ are given by Eqs. (6.41), (6.42), (6.46) and (6.47) respectively. Although $I_{3(\xi_\theta)}$ and $I_{3(n_i)}$ can not be integrated exactly, asymptotic expressions of $I_{3(\xi_\theta)}$ and $I_{3(n_i)}$ for large ion speed ratio, $S_i \gg 1$, are obtained in Appendix A.

Consider a simplified special case of the slip velocity. Let $T_{i\infty} = T_{e\infty} = T_{o\infty} = T_\infty$ and introduce the degree of ionization, α, as

$$\alpha \equiv \frac{m_i n_{i\infty}}{m_i n_{i\infty}+m_o n_{o\infty}} \tag{6.53}$$

Then $S_i \approx S_o$, since $m_i/m_o \approx 1$. The slip velocity then becomes

$$V_\theta = -U_\infty \sin\theta\left[\frac{\operatorname{erfc}(S_i\cos\theta)+\alpha\frac{\phi_s^*}{S_i^2}G_2(S_i,\theta)+\frac{\alpha}{m_i n_{i\infty}}K_o}{\operatorname{erfc}(S_i\cos\theta)+\sqrt{\frac{T_\infty}{T_w}}G_1(S_i,\theta)+\alpha\frac{\phi_s^*}{S_i^2}\left(\sqrt{\frac{T_\infty}{T_w}}G_o+G_3\right)+\frac{\alpha}{m_i n_{i\infty}}(K_1+K_2)}\right]$$

(6.54)

where

$$K_2 = \frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}m_i U_\infty^2 \sqrt{\pi B_w}\, I_{3(\xi_r)} \qquad (6.55)$$

The integral $I_{3(\xi_r)}$ is given by Eq. (6.27) and its asymptotic expressions for $S_i >> 1$ in Appendix A.

The expressions (6.48) and (6.54) are the same as those obtained by Prager and Rasmussen [1967] except the terms K_o, K_1 and K_2 which contain the Deybe length and provide the dependence of the slip velocity on the Debye length. In the expression (6.48), the term K_2 is included in the diffuse function, $n_d(\vec{r}_s)$.

Molecular Drag

The drag induced by rarefied plasma flows past a sphere consists of two parts. One part is due to the contribution from the electric Maxwell stresses, and the other part is due to the contribution from the momentum flux of the ions, electrons and neutral particles. The former is called Maxwell drag which has been obtained in Chapter IV. The Maxwell drag is negative and hence actually is a thrust. The latter is called molecular drag and will be evaluated in this section.

The molecular drag is determined from the momentum flux tensor as given by expression (2.11) which can be written as

$$\overleftrightarrow{\sigma}_f \equiv - \sum_s^{i,e,o} m_s n_s \langle \vec{\xi}\,\vec{\xi} \rangle \tag{6.56}$$

If $\hat{e}_z$ is a unit vector in the direction of the free stream, the molecular drag is

$$D_f = \oiint_{\text{sphere}} \hat{e}_r \cdot \overleftrightarrow{\sigma}_f \cdot \hat{e}_z \, dS$$

$$= 2\pi a^2 \left[- \sum_s^{i,e,o} m_s \int_0^\pi n_s \langle \xi_r \xi_r \rangle \cos\theta \, \sin\theta d\theta + \sum_s^{i,e,o} m_s \int_0^\pi n_s \langle \xi_r \xi_\theta \rangle \sin^2\theta d\theta \right] \tag{6.57}$$

The term involving $\xi_r\xi_r$ is the contribution from the normal stress and $\xi_r\xi_\theta$ from the tangential stress.

Setting $\ell = 2$, $m = n = 0$ in Eqs. (6.19), (6.20), (6.17) and (6.10a) gives

$$n_o \langle \xi_r \xi_r \rangle = \frac{n_{o\infty}}{2B_o} \left[(\tfrac{1}{2} + S_o^2 \cos^2\theta) \operatorname{erfc}(S_o \cos\theta) - \frac{1}{\sqrt{\pi}} S_o \cos\theta \, e^{-S_o^2 \cos^2\theta} \right] \tag{6.58a}$$

$$[n_o \langle \xi_r \xi_r \rangle]_{emit} = \frac{1}{4B_w} n_d(\vec{r}_s) \tag{6.58b}$$

$$n_e \langle \xi_r \xi_r \rangle = \frac{n_{e\infty}}{4B_e} e^{-\phi_s^*} + O(Se) \tag{6.58c}$$

$$n_i \langle \xi_r \xi_r \rangle = I_1 + \frac{\bar{\phi}_s}{2} I_2 - \frac{\bar{\phi}_s}{2} I_{3(\xi_r^2)} \tag{6.58d}$$

where

$$I_1 = \frac{n_{i\infty}}{2B_i} \left[(\tfrac{1}{2} + S_i^2 \cos^2\theta) \operatorname{erfc}(S_i \cos\theta) - \frac{1}{\sqrt{\pi}} S_i \cos\theta \, e^{-S_i^2 \cos^2\theta} \right] \tag{6.59}$$

$$I_2 = 3n_{i\infty} \frac{e^{-S_i^2}}{S_i^3} \int_0^{S_i} x^2 e^{x^2} \left[(\tfrac{1}{2} + x^2 \cos^2\theta) \operatorname{erfc}(x \cos\theta) - \frac{1}{\sqrt{\pi}} x \cos\theta \, e^{-x^2 \cos^2\theta} \right] dx \tag{6.60}$$

$$I_{3(\xi_r^2)} = \int_0^\infty \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\pi} \xi^2 \; f_{io} \frac{d}{d\gamma} \left[\cos^2\gamma \; \sin^2\gamma \; I(\phi_o')\right] d\xi d\gamma d\varepsilon \tag{6.61a}$$

$$= 2n_{i\infty} \frac{e^{-S_i^2}}{\sqrt{\pi}} \int_0^\infty \int_{\frac{\pi}{2}}^{\pi} x^2 e^{-(x^2-2xS_i\cos\gamma\;\cos\theta)} I_o(2xS_i\sin\gamma\;\sin\theta)$$

$$\frac{d}{d\gamma}\left[\cos^2\gamma \; \sin^2\gamma \; I(\phi_o')\right] dx d\gamma \tag{6.61b}$$

Hence

$$m_o\int_0^\pi n_o\langle\xi_r\xi_r\rangle\cos\theta\;\sin\theta d\theta$$

$$= \frac{m_o n_{o\infty}}{2B_o S_o^2}\left[\left(\frac{1}{8}-\frac{1}{2}S_o^2-\frac{1}{2}S_o^4\right)\text{erf}\;S_o - \frac{S_o}{\sqrt{\pi}}e^{-S_o^2}\left(\tfrac{1}{4}+\tfrac{1}{2}S_o^2\right)\right] \tag{6.62a}$$

$$m_o\int_0^\pi [n_o\langle\xi_r\xi_r\rangle]_{emit}\;\cos\theta\;\sin\theta d\theta$$

$$= -\frac{\sqrt{\pi}}{6B_w}\sqrt{\frac{B_w}{B_i}}\left[m_i n_{i\infty} S_i + m_o n_{o\infty}\sqrt{\frac{B_i}{B_o}}\,S_o + m_i n_{i\infty}\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}\{S_i - D(S_i)\}\right]$$

$$+\;\bar{\phi}_s\,\frac{m_i}{4}\sqrt{\frac{\pi}{B_w}}\int_0^\pi I_{3(\xi_r)}\cos\theta\;\sin\theta d\theta \tag{6.62b}$$

where $D(S_i)$ is the Dawson function given by

$$D(S) = e^{-S^2}\int_0^S e^{x^2}dx$$

and $I_{3(\xi_r)}$ is given by (6.27).

$$m_e\int_0^\pi n_e\langle\xi_r\xi_r\rangle\;\cos\theta\;\sin\theta d\theta = m_e O(S_e) \tag{6.62c}$$

$$m_i\int_0^\pi n_i\langle\xi_r\xi_r\rangle \cos\theta \sin\theta d\theta$$

$$= \frac{m_i n_{i\infty}}{2B_i S_i^2}\left[\left(\frac{1}{8}-\frac{1}{2}S_i^2-\frac{1}{2}S_i^4\right)\text{erf } S_i - \frac{S_i}{\sqrt{\pi}}e^{-S_i^2}\left(\tfrac{1}{4}+\tfrac{1}{2}S_i^2\right)\right]$$

$$+ \bar{\phi}_s \frac{3m_i n_{i\infty}}{2S_i^2}\left[\left(\frac{1}{8}-\frac{1}{4}S_i^2\right)\text{erf } S_i - \frac{S_i}{4\sqrt{\pi}}e^{-S_i^2}\right]$$

$$- \bar{\phi}_s \frac{m_i}{2}\int_0^\pi I_{3(\xi_r^2)} \cos\theta \sin\theta d\theta \qquad (6.62d)$$

Setting $\ell = m = 1$, $n = 0$ in Eqs. (6.19), (6.20), (6.17) and (6.10a) gives

$$n_o\langle\xi_r\xi_\theta\rangle = \frac{n_{o\infty}}{2B_o} S_o \sin\theta\left[\frac{1}{\sqrt{\pi}} e^{-S_o^2\cos^2\theta} - S_o \cos\theta \text{ erfc}(S_o \cos\theta)\right] \qquad (6.63a)$$

$$[n_o\langle\xi_r\xi_\theta\rangle]_{emit} = 0 \qquad (6.63b)$$

$$n_e\langle\xi_r\xi_\theta\rangle = 0(S_e) \qquad (6.63c)$$

$$n_i\langle\xi_r\xi_\theta\rangle = I_1 + \frac{\bar{\phi}_s}{2} I_2 - \frac{\bar{\phi}_s}{2} I_{3(\xi_r\xi_\theta)} \qquad (6.63d)$$

where

$$I_1 = \frac{n_{i\infty}}{2B_i} S_i \sin\theta\left[\frac{1}{\sqrt{\pi}} e^{-S_i^2\cos^2\theta} - S_i \cos\theta \text{ erfc}(S_i\cos\theta)\right] \qquad (6.64)$$

$$I_2 = 3n_{i\infty}\frac{e^{-S_i^2}}{S_i^3}\int_0^{S_i} x^3 e^{x^2}\left[x\cos\theta \text{ erfc}(x\cos\theta) - \frac{1}{\sqrt{\pi}}e^{-x^2\cos^2\theta}\right](-\sin\theta)dx \qquad (6.65)$$

$$I_{3(\xi_r\xi_\theta)} = \int_0^\infty\int_{\frac{\pi}{2}}^\pi\int_0^{2\pi} \xi^2 \cos\varepsilon \, f_{io} \frac{d}{d\gamma}[\cos\gamma \sin^3\gamma \, I(\phi_o')]d\xi d\gamma d\varepsilon \qquad (6.66a)$$

$$= -2n_{i\infty}\frac{e^{-S_i^2}}{\sqrt{\pi}}\int_0^\infty\int_{\frac{\pi}{2}}^\pi x^2 e^{-(x^2-2xS_i\cos\gamma\cos\theta)} I_1(2xS_i\sin\gamma\sin\theta) \frac{d}{d\gamma}[\cos\gamma \sin^3\gamma \, I(\phi_o')]dxd\gamma \qquad (6.66b)$$

Hence

$$m_o\int_0^\pi n_o<\xi_r\xi_\theta>\sin^2\theta\ d\theta$$
$$= \frac{m_o n_{o\infty}}{2B_o S_o^2}\left[\frac{S_o}{\sqrt{\pi}}\ e^{-S_o^2}(\tfrac{1}{4}+\tfrac{1}{2}S_o^2)-(\frac{1}{8}-\frac{1}{2}\ S_o^2-\frac{1}{2}S_o^4)\text{erf}\ S_o\right] \tag{6.67a}$$

$$m_o\int_0^\pi [n_o<\xi_r\xi_\theta>]_{emit}\ \sin^2\theta\ d\theta = 0 \tag{6.67b}$$

$$m_e\int_0^\pi n_e<\xi_r\xi_\theta>\sin^2\theta\ d\theta = m_e\ 0(S_e) \tag{6.67c}$$

$$m_i\int_0^\pi n_i<\xi_r\xi_\theta>\sin^2\theta\ d\theta$$
$$= -\frac{m_i n_{i\infty}}{2B_i S_i^2}\left[(\frac{1}{8}-\frac{1}{2}\ S_i^2-\frac{1}{2}\ S_i^4)\text{erf}\ S_i\ -\frac{S_i}{\sqrt{\pi}}\ e^{-S_i^2}(\tfrac{1}{4}+\tfrac{1}{2}S_i^2)\right]$$
$$-\ \bar{\phi}_s\ \frac{3m_i n_{i\infty}}{2S_i^2}\left[(\frac{1}{8}-\frac{1}{4}\ S_i^2)\text{erf}\ S_i\ -\frac{1}{4\sqrt{\pi}}\ S_i e^{-S_i^2}\right]$$
$$-\ \bar{\phi}_s\ \frac{m_i}{2}\int_0^\pi I_{3(\xi_r\xi_\theta)}\sin^2\theta\ d\theta \tag{6.67d}$$

The contribution of the electrons to the drag is small and can be neglected. Substitution of (6.62) and (6.67) to Eq. (6.57) gives the desired molecular drag. In terms of a drag coefficient, we can write the molecular drag as

$$C_{D_f} = \frac{D_f}{\tfrac{1}{2}(m_i n_{i\infty}+m_o n_{o\infty})U_\infty^2\pi a^2}$$
$$= (C_{D_f})_o + (C_{D_f})_\phi \tag{6.68}$$

where $(C_{D_f})_o$ is independent of the floating potential, ϕ_s^*, and the Debye length, and $(C_{D_f})_\phi$ contains the floating potential and the Debye length. In terms of the degree of ionization, α, we have

$$(C_{D_f})_o = (1-\alpha)\bar{C}_D(S_o, \frac{T_w}{T_{o\infty}}) + \alpha\bar{C}_D(S_i, \frac{T_w}{T_{i\infty}}) \tag{6.69}$$

$$(C_{D_f})_\phi = \alpha\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}[F(S_i, \frac{T_w}{T_{i\infty}}) + H(\phi_s^*,\varepsilon)] \tag{6.70}$$

where

$$\bar{C}_D(S_i, \frac{T_w}{T_{i\infty}}) = 2[(1+\frac{1}{S_i^2}-\frac{1}{4S_i^4})\mathrm{erf}\, S_i + \frac{e^{-S_i^2}}{\sqrt{\pi}}(\frac{1}{S_i}+\frac{1}{2S_i^3})+\frac{\sqrt{\pi}}{3S_i}\sqrt{\frac{T_w}{T_{i\infty}}}] \tag{6.71}$$

$$F(S_i, \frac{T_w}{T_{i\infty}}) = 3[(1-\frac{1}{2S_i^2})\mathrm{erf}\, S_i + \frac{e^{-S_i^2}}{\sqrt{\pi}S_i}]+\frac{2\sqrt{\pi}}{3S_i}\sqrt{\frac{T_w}{T_{i\infty}}}[1-\frac{D(S_i)}{S_i}] \tag{6.72}$$

$$H(\phi_s^*,\varepsilon) = -\frac{1}{n_{i\infty}}[\sqrt{\frac{\pi}{B_w}}\int_0^\pi I_{3(\xi_r)}\cos\theta\,\sin\theta d\theta + 2\{\int_0^\pi I_{3(\xi_r\xi_\theta)}\sin^2\theta\, d\theta - \int_0^\pi I_{3(\xi_r^2)}\cos\theta\,\sin\theta\, d\theta\}] \tag{6.73}$$

Here $D(S_i)$ is the Dawson function as given after Eq. (6.62b), and $I_{3(\xi_r)}$, $I_{3(\xi_r^2)}$, $I_{3(\xi_r\xi_\theta)}$ are given by Eqs. (6.27), (6.61) and (6.66) respectively. For $\alpha = 0$, the expression (6.68) yields the usual formula for free-molecule drag with diffuse reflection. The expression (6.68) for the molecular drag is the same as that obtained by Prager and Rasmussen [1967] except the term $H(\phi_s^*,\varepsilon)$ in $(C_{D_f})_\phi$. This additional term contains the Debye length, and provides the dependence of the molecular drag on the Debye length.

It is not possible to evaluate the extra term exactly, even if the simplest spherical potential model is used. However, for large ion speed ratio, $S_i >> 1$, an asymptotic expression of this term is obtained without specifying a certain model for the spherical potential in Appendix B. Since the Maxwell drag is negative as has been shown in Chapter IV, the molecular drag given by Eq. (6.68) yields the maximum value of drag.

Energy Flux to the Sphere

The energy gained or lost by the sphere is equal to the total kinetic energy flux to the sphere due to the incident and emitted particles plus the energy gained because of recombination of the ions at the surface. Other changes in internal energy of the molecules shall be neglected. Assuming that recombination occurs immediately with the impact of an ion with the sphere, we express the local energy flux to the sphere surface as

$$q = - \sum_{s}^{i,e,o} \frac{m_s}{2} \int \xi_r \xi^2 f_s d^3\xi - \mathcal{E} \int \xi_r f_i \; d^3\xi \tag{6.74}$$

where the summation is over the ions, electrons, and neutral particles incident on the sphere and the neutral particles emitted diffusely from the sphere. The symbol $\mathcal{E}$ denotes the ionization potential per molecule.

The Eq. (6.74) can be written as

$$q = - \sum_{s}^{i,e,o} \frac{m_s}{2} n_s \langle \xi_r \xi^2 \rangle - \mathcal{E} n_i \langle \xi_r \rangle \tag{6.75a}$$

$$= - \sum_{s}^{i,e,o} \frac{m_s}{2} [n_s \langle \xi_r^3 \rangle + n_s \langle \xi_r \xi_\theta^2 \rangle + n_s \langle \xi_r \xi_\psi^2 \rangle] - \mathcal{E} n_i \langle \xi_r \rangle \tag{6.75b}$$

Evaluating $n_s \langle \xi_r^3 \rangle$, $n_s \langle \xi_r \xi_\theta^2 \rangle$ and $n_s \langle \xi_r \xi_\psi^2 \rangle$ from Eqs. (6.19), (6.20), (6.17) and (6.10a), and adding together, we have

$$n_o\langle\xi_r\xi^2\rangle = -\frac{n_{o\infty}}{2\sqrt{\pi}B_o^{3/2}}\left[(S_o^2+\tfrac{5}{2})G_1(S_o,\theta)-\tfrac{1}{2}e^{-S_o^2\cos^2\theta}\right] \tag{6.76a}$$

$$[n_o\langle\xi_r\xi^2\rangle]_{emit} = \frac{1}{\sqrt{\pi}\,B_w^{3/2}}\,n_d(\vec{r}_s) \tag{6.76b}$$

$$n_e\langle\xi_r\xi^2\rangle = -\frac{n_{e\infty}}{\sqrt{\pi}\,B_e^{3/2}}\,e^{-\phi_s^*} \tag{6.76c}$$

$$n_i\langle\xi_r\xi^2\rangle = -\frac{n_{i\infty}}{2\sqrt{\pi}\,B_i^{3/2}}\left[(S_i^2+\tfrac{5}{2})G_1(S_i,\theta)-\tfrac{1}{2}e^{-S_i^2\cos^2\theta}\right]$$

$$-\bar{\phi}_s\frac{n_{i\infty}}{\sqrt{\pi}\,B_i}G_1(S_i,\theta)-\frac{\bar{\phi}_s}{2}I_{3(\xi_r\xi^2)} \tag{6.76d}$$

where $G_1(S,\theta)$ is given by (6.35), and

$$I_{3(\xi_r\xi^2)} = \int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\xi^3\,f_{io}\,\frac{d}{d\gamma}\,[\cos\gamma\,\sin^2\gamma\,I(\phi_o')]d\xi\,d\gamma\,d\varepsilon \tag{6.77a}$$

$$= 2n_{i\infty}\frac{e^{-S_i^2}}{\sqrt{\pi}\,B_i}\int_0^\infty\int_{\frac{\pi}{2}}^{\pi}x^3\,e^{-(x^2-2xS_i\cos\gamma\,\cos\theta)}I_o(2xS_i\sin\gamma\,\sin\theta)$$

$$\frac{d}{d\gamma}\,[\cos\gamma\,\sin^2\gamma\,I(\phi_o')]dxd\gamma \tag{6.77b}$$

The radial ion number flux $n_i\langle\xi_r\rangle$ is given by (6.24) which can be written as

$$n_i\langle\xi_r\rangle = -\frac{n_{i\infty}}{2\sqrt{\pi}\,B_i}\left[G_1(S_i,\theta)+\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}G_o(S_i,\theta)\right]-\frac{\bar{\phi}_s}{2}I_{3(\xi_r)} \tag{6.78}$$

where $G_o(S,\theta)$ is given by (6.36) and $I_{3(\xi_r)}$ by (6.27). Substitution of (6.76) and (6.78) to the expression (6.75a) gives the desired energy flux to the surface of the sphere. The contributions to the heat transfer can be divided into a part corresponding to the usual free-molecule heat transfer, a part due to electron heating, a part due to the excess over the free-molecule heat transfer caused by the floating potential, and a part due to recombination of the ions at the surface. It is convenient to represent the heat transfer by the nondimensional form

$$C_q \equiv \frac{q}{\frac{1}{2}(m_i n_{i\infty}+m_o n_{o\infty})U_\infty^3} \tag{6.79}$$

We then have

$$C_q = (C_q)_o + (C_q)_e + (C_q)_\phi + (C_q)_\mathcal{E} \tag{6.80}$$

The term $(C_q)_o$ does not depend on the floating potential ϕ_s^* and comes from the ions and neutral particles. It is given by

$$(C_q)_o = (1-\alpha)G_4(S_o,\theta) + \alpha G_4(S_i,\theta) - \frac{1}{S_i^2}\frac{m_i}{m_o}\frac{T_w}{T_{i\infty}}\left[(1-\alpha)\frac{G_1(S_o,\theta)}{\sqrt{\pi}\,S_o} + \alpha\frac{G_1(S_i,\theta)}{\sqrt{\pi}\,S_i}\right] \tag{6.81}$$

where

$$G_4(S,\theta) = \frac{1}{2\sqrt{\pi}\,S^3}\left[(S^2+\frac{5}{2})G_1(S,\theta) - \frac{1}{2}e^{-S^2\cos^2\theta}\right] \tag{6.82a}$$

$$= (\frac{1}{S^3}+\frac{1}{2S})\frac{e^{-S^2\cos^2\theta}}{\sqrt{\pi}} - \frac{\cos\theta}{4}(\frac{5}{S^2}+2)\,\mathrm{erfc}(S\cos\theta) \tag{6.82b}$$

For $\alpha = 0$, the Eq. (6.81) which is the same as that obtained by Prager and Rasmussen [1967] yields the free-molecule energy transfer for diffuse reflection.

The heating due to the transport of kinetic energy by incoming electrons is (neglecting the term containing m_e/m_o which is very small)

$$(C_q)_e = \frac{\alpha}{S_i^3} \frac{T_{e\infty}}{T_{i\infty}} \left[\frac{n_{e\infty}}{n_{i\infty}} \sqrt{\frac{m_i}{m_e} \frac{T_{e\infty}}{T_{i\infty}}} \frac{e^{-\phi_s^*}}{\sqrt{\pi}}\right] \tag{6.83}$$

The electron heating is symmetric because terms of order S_e were neglected in the electron distribution function. The expression (6.83) is the same as that obtained by Prager and Rasmussen [1967]. If the floating potential is determined by Eq. (6.29) then, $(C_q)_e$ becomes

$$(C_q)_e = \frac{\alpha}{2S_i^2} \frac{T_{e\infty}}{T_{i\infty}} \Big[(1+ \frac{1}{2S_i^2})\mathrm{erf}\, S_i + \frac{e^{-S_i^2}}{\sqrt{\pi}\, S_i}$$

$$+ \frac{T_{e\infty}}{T_{i\infty}} \frac{\phi_s^*}{S_i^2} \left\{\mathrm{erf}\, S_i + \frac{U_\infty}{n_{i\infty}} \int_0^\pi I_{3(\xi_r)} \sin\theta d\theta\right\}\Big] \tag{6.84}$$

This expression is the same as that obtained by Prager and Rasmussen [1967] except for the last term involving $I_{3(\xi_r)}$, which contains the Debye length, and provides the dependence of $(C_q)_e$ on the Debye length. For large ion speed ratio, S_i>>1, an asymptotic expression for the extra term is obtained without specifying a certain model for the spherical potential (see Appendices A and B). Substituting Eq. (B.8), We can write Eq. (6.84) for S_i>>1 as

$$(C_q)_e = \frac{\alpha}{2S_i^2} \frac{T_{e\infty}}{T_{i\infty}} \Big[1+ \frac{1}{2S_i^2}$$

$$+ \frac{T_{e\infty}}{T_{i\infty}} \frac{\phi_s^*}{S_i^2} \left\{1- \tfrac{1}{2} I(\phi_o')_{\gamma=\pi/2} \int_0^{\frac{\pi}{2}} \mathrm{erfc}(S_i \cos\theta) d\theta\right\}\Big] \tag{6.85}$$

The contribution $(C_q)_\phi$ comes from the additional ions that strike the sphere owing to the floating potential. We have

$$(C_q)_\phi = \alpha\left(\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}\right)\left[\frac{G_1(S_i,\theta)}{\sqrt{\pi}\,S_i} - \frac{1}{S_i^2}\left(\frac{m_i}{m_o}\frac{T_w}{T_{i\infty}}\right)\frac{G_o(S_i,\theta)}{\sqrt{\pi}\,S_i}\right.$$

$$\left. + \frac{1}{2n_{i\infty}U_\infty}\left\{I_{3(\xi_r\xi^2)} - \frac{2}{B_w}I_{3(\xi_r)}\right\}\right] \qquad (6.86)$$

where $G_1(S,\theta)$, $G_o(S,\theta)$, $I_{3(\xi_r\xi^2)}$ and $I_{3(\xi_r)}$ are given by Eqs. (6.35), (6.36), (6.77) and (6.27) respectively. The contribution of electrons that enters through the diffuse function $n_d(\vec{r}_s)$ given by (6.34) was neglected. The expression (6.86) is the same as that obtained by Prager and Rasmussen [1967] except that the expression (6.86) has an extra term, i.e., the last term involving $I_{3(\xi_r\xi^2)}$ and $I_{3(\xi_r)}$, which contains the Debye length and provides the dependence of $(C_q)_\phi$ on the Debye length, and that it has not a quadratic term,

$$\left(\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}\right)^2 ,$$

as their result does. This is because the present analysis is confined to the first order of δ, and it can be shown that a linear term

$$\left(\frac{T_{e\infty}}{T_{i\infty}}\frac{\phi_s^*}{S_i^2}\right)$$

is of order δ. Although the exact evaluation of the extra term is not possible even for the simplest spherical potential model, asymptotic expressions of $I_{3(\xi_r)}$ and $I_{3(\xi_r\xi^2)}$ for large ion speed ratio, $S_i \gg 1$, are obtained without specifying a certain model for the spherical potential in Appendix A.

The heating of the sphere because of the recombination of the ions is given by

$$(C_q)_{\mathcal{E}} = \frac{\alpha}{2}\,(\frac{\mathcal{E}^*}{S_i^2})\,[\frac{G_1(S_i,\theta)}{\sqrt{\pi}\,S_i} + (\frac{T_{e\infty}}{T_{i\infty}}\,\frac{\phi_s^*}{S_i^2})\{\frac{G_o(S_i,0)}{\sqrt{\pi}\,S_i} + \frac{U_\infty}{n_{i\infty}}\,I_{3(\xi_r)}\}] \tag{6.87}$$

where

$$\mathcal{E}^* \equiv \frac{\mathcal{E}}{kT_{i\infty}} \tag{6.88}$$

The expression (6.87) is the same as that obtained by Prager and Rasmussen [1967] except for the last term involving $I_{3(\xi_r)}$, which contains the Debye length, and provides the dependence of $(C_q)_{\mathcal{E}}$ on the Debye length.

Discussion

General expressions of floating potential, slip velocity, molecular drag, and energy transfer are obtained. The integral I_3 in these expressions can not be evaluated exactly even for the simplest spherical potential model. However, asymptotic expressions of the integral I_3 for large ion speed ratio, $S_i \gg 1$, are obtained in Appendix A without specifying a particular model for the spherical potential by means of Laplace's method of asymptotic integration. The results of the computation of the moments are the same as those obtained by Prager and Rasmussen [1967] except the term containing the integral I_3 which in turn contains the integral $I(\phi_o')$. The integral $I(\phi_o')$ contains the Debye length and provides the required dependence of the moments on the Debye length.

The linear and parabolic models are used to evaluate the integral $I(\phi_o')$ in Chapter V. The results, Eqs. (5.25) and (5.31), evaluated at $\gamma = \pi/2$ have singularities as ε tends to zero, as can be seen from Eqs. (5.26b) and (5.32b). Nevertheless, Eqs. (5.26a) and (5.32a) serve to illustrate how the Debye length, or ε enters the first-order approximation of the moments on the sphere away from $\gamma = \pi/2$. The asymptotic expressions of the integral I_3 for large ion speed ratio, $S_i >> 1$, obtained in Appendix A indicate that the integral $I(\phi_o')$ and its derivative $I'(\phi_o')$ must not have singularities at $\gamma = \pi/2$ as ε tends to zero, otherwise the results of the moments on the sphere would be meaningless. Thus it is necessary to find a spherical model which will produce $I(\phi_o')$ and $I'(\phi_o')$ that are free from singularities at $\gamma = \pi/2$ as ε tends to zero in order to obtain meaningful results of the moments on the sphere.

It was pointed out in Chapter V that the singularity arises because of the behavior of the factor $1/\sqrt{r^2 - r_m^2}$ in the integrand of the integral $I(\phi_o')$, Eq. (5.17a), and that it is not possible to remove the singularity by assuming the similar simple models for the spherical potential such as Eq. (5.21) or (5.27). The desired spherical potential model must satisfy the boundary conditions

$$\phi_o' = 1, \qquad \frac{\partial \phi_o'}{\partial r} = - \left| \frac{\partial \phi_o'}{\partial r} \right|_s \qquad \text{at} \quad r = a \tag{6.89a}$$

$$\phi_o' = 0, \qquad \frac{\partial \phi_o'}{\partial r} = 0 \qquad \text{at} \quad r = r_o \tag{6.89b}$$

At the same time the model should be able to eliminate the cause of the singularity, the factor $1/\sqrt{r^2 - r_m^2}$ in the integrand of the integral $I(\phi_o')$. Also the model should be simple enough so that the integration, Eq. (B.7), of the additional term $H(\phi_s^*, \varepsilon)$ of the molecular drag can be carried out. This difficult task has to be left for the future investigation.

For the present, however, we can observe the general trend of the effect of the Debye length or ε on the moments from the simple model Eq. (5.21) or (5.27), although the integral $I(\phi_o')$ has singularity at $\gamma=\pi/2$ as ε tends to zero. From Appendix A we can see that the term containing $I(\phi_o')_{\gamma=\pi/2}$ enters the integral I_3 only on the downstream face of the sphere. Since the density of the ions in the near wake is practically zero for large S_i and small ε, the term containing $I(\phi_o')_{\gamma=\pi/2}$ is negligible compared to its counterpart on the upstream face of the sphere.

For example, from Eq. (B.7), the additional term $H(\phi_s^*,\varepsilon)$ of the molecular drag using the linear model Eq. (5.21) and neglecting the term containing $I(\phi_o')_{\gamma=\pi/2}$ compared to its counterpart, the first term, is

$$H(\phi_s^*,\varepsilon) = \frac{\sqrt{\pi}}{S_i}\sqrt{\frac{B_i}{B_w}}\left[-\frac{2}{3}+\frac{2\sqrt{2}}{3}\sqrt{\varepsilon}\sqrt{\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}} + O(\varepsilon)\right] \qquad (6.90)$$

When ε goes to zero, the term $H(\phi_s^*,\varepsilon)$ goes to a finite negative value. For the typical values of $\phi_s^* = 3$ and $S_i = 5$, $\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s = 2.03$ near $\theta = \pi$ for Maxwellian electron-density model. When $\varepsilon = 0.001$, 0.01, and 0.1, the second term in the brackets is 0.0364, 0.115, and 0.364 respectively. We can see that $H(\phi_s^*,\varepsilon)$ is always negative when $\varepsilon \ll 1$, and becomes less negative when ε increases. Thus the term $H(\phi_s^*,\varepsilon)$ reduces the molecular drag, and the molecular drag increases with increasing ε. Therefore the result of Prager and Rasmussen [1967] gives larger value of drag than that of the present more accurate drag.

For the linear model, Eq. (5.21), the integral $I(\phi_o')$ and its derivative $I'(\phi_o')$, when expanded in the series in small ε, are

$$I(\phi_o') = -\frac{1}{|\cos\gamma|}\left[1-\tfrac{1}{2}\varepsilon\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\frac{1}{\cos^2\gamma}+O(\varepsilon^2)\right] \qquad \gamma \neq \frac{\pi}{2} \qquad (6.91)$$

$$I'(\phi_o') = \frac{\sin\gamma}{\cos^2\gamma}\left[1-\frac{3}{2}\varepsilon\frac{\phi_s^*}{\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s}\frac{1}{\cos^2\gamma}+O(\varepsilon^2)\right] \qquad \gamma \neq \frac{\pi}{2} \qquad (6.92)$$

When ε goes to zero, $I(\phi_o')$ goes to a finite negative value and $I'(\phi_o')$ to a finite positive value. For the typical values of $\phi_s^* = 3$ and $S_i = 5$, $\left|\frac{\partial\phi_o^i}{\partial\zeta}\right|_s = 2.03$ near $\theta = \pi$ for Maxwellian electron-density model as mentioned before. When $\varepsilon = 0.001$, 0.01, and 0.1, the second term in the brackets of $I(\phi_o')$ is $0.00074/\cos^2\gamma$, $0.0074/\cos^2\gamma$, and $0.074/\cos^2\gamma$ respectively, and that of $I'(\phi_o')$ is $0.00222/\cos^2\gamma$, $0.0222/\cos^2\gamma$, and $0.222/\cos^2\gamma$ respectively. We can see that $I(\phi_o')$ is always negative and becomes less negative when ε increases and that $I'(\phi_o')$ is always positive and becomes less positive when ε increases, if $\varepsilon \ll 1$ and γ is not too close to $\pi/2$. The trend of the effect of the Debye length or ε on the integrals I_3 can be observed by substituting Eqs. (6.91) and (6.92) into the expressions of I_3 in Appendix A.

CHAPTER VII

CONCLUDING REMARKS

An analytic investigation has been made of the problem of a steady rarefied plasma flow past a sphere by means of the Vlasov-Poisson set of equations. The equations are attacked by means of a perturbation method exploiting the features of an ionospheric satellite, i.e., the velocity of the satellite is much larger than the thermal velocity of ions, but much smaller than that of electrons, and the Debye length is much smaller than the characteristic length of the satellite. The effects of photoemission, the secondary emission of electrons, and the earth's magnetic field are neglected.

The ion distribution function is found to be independent of the electric field to a zeroth-order approximation. Thus the zeroth-order ion number density is the same as that of neutral particles. The zeroth-order electron number density is obtained by assuming three different models for the structure of the electron velocity space. The zeroth-order ion and electron number densities are used in the Poisson's equation, which is a singular perturbation problem, to obtain the zeroth-order solutions for the potential field and the potential gradients at the sphere surface, which in turn are used to obtain the Maxwell drag. A simple formula of the Maxwell drag for the large ion speed ratio shows that it is negative and actually a thrust when the surface potential is large. Simple formulas are also obtained for the potential near the wake axis, which allow rapid computations to be made. They agree qualitatively with more sophisticated (and more complicated) results of numerical integration.

The first-order approximation to the ion distribution function is obtained for the special case of a spherical potential, which is suitable for the sphere problem. The first-order approximate expressions of floating potential, slip velocity, molecular drag, and energy transfer are obtained. The results are the same as those obtained by Prager and Rasmussen [1967] except for the term involving integral I_3 which contains the integral $I(\phi_o')$. The integral $I(\phi_o')$ contains the Debye length and provides the required dependence of the moments on the Debye length.

The asymptotic expressions of the integral I_3 for large ion speed ratio are obtained without specifying a particular model for the spherical potential. It is found that it is necessary to find a spherical potential model which will produce the integral $I(\phi_o')$ and its derivative $I'(\phi_o')$ that are free from singularities at $\gamma = \pi/2$ as ε tends to zero in order to obtain meaningful results of the moments on the sphere. Although the desired spherical potential model is not obtained, the requirements for obtaining it are pointed out.

The simple linear potential model is used to observe the general trend of the effect of the Debye length or ε on the moments, although it has a singularity at $\gamma = \pi/2$ as ε tends to zero. It is found that the molecular drag increases with increasing ε, and that the results of Prager and Rasmussen [1967] gives a larger value of drag than that of the present more accurate drag.

Prager and Rasmussen [1967] obtained the moments by an ad hoc approximation of ion distribution function. Present systematic analysis shows that although they obtained a part of the first-order approximation, they missed the other part, i.e., the part involving the integral I_3 which contains the Debye length and provides the dependence of the moments on the Debye length. As a result, their results are independent of the Debye length and are not realistic.

It is hoped that the present investigation will serve as a basis for further research. In particular, it is desirable to obtain higher order approximations of the potential near the wake axis. It is also desirable to obtain a spherical potential model which will produce the integral $I(\phi_o')$ and its derivative $I'(\phi_o')$ free from singularities at $\gamma = \pi/2$ as ε tends to zero, so that the explicit expressions of the moments can be obtained. In this way, the expressions for the drag and energy transfer can serve as a useful guide for the conducting and interpretation of experiments. Systematic experimental investigations indicating the role of the pertinent parameters are especially needed since data are very scarce.

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APPENDIX A

EVALUATION OF INTEGRALS $I_{3(\xi_r)}$, $I_{3(n_i)}$, $I_{3(\xi_\theta)}$, $I_{3(\xi_r^2)}$, $I_{3(\xi_r\xi_\theta)}$, $I_{3(\xi_r\xi^2)}$

The integral, I_3, appearing in the moments for the ions can not be integrated exactly, even if the simplest spherical potential model is used. However, for large ion speed ratio, $S_i >> 1$, asymptotic expressions for I_3 may be obtained without specifying a certain model for the spherical potential by means of Laplace's method of asymptotic integration outlined in the book by Erdelyi [1956]. As an example $I_{3(\xi_r)}$ will be evaluated in detail.

From Eq. (6.2), we have

$$I(\phi_o') = a\int_a^\infty \frac{\frac{d\phi_o'}{du}\, du}{\sqrt{u^2-a^2\sin^2\gamma}} \qquad \frac{\pi}{2} \le \gamma \le \pi \tag{A.1}$$

$$I'(\phi_o') = \frac{d\ I(\phi_o')}{d\gamma} = a^3\sin\gamma\ \cos\gamma\int_a^\infty \frac{\frac{d\phi_o'}{du}\, du}{(u^2-a^2\sin^2\gamma)^{3/2}} \qquad \frac{\pi}{2} \le \gamma \le \pi \tag{A.2}$$

(1) From Eq. (6.27a)

$$I_{3(\xi_r)} = \int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \xi f_{io} \frac{d}{d\gamma}\left[\cos\gamma\ \sin^2\gamma\ I(\phi_o')\right] d\xi\ d\gamma\ d\varepsilon \tag{A.3}$$

where

$$f_{io} = n_{i\infty}(\frac{B_i}{\pi})^{3/2} \exp[-B_i\{\xi^2+U_\infty^2-2U_\infty\xi(\cos\gamma \cos\theta-\sin\gamma \cos\varepsilon \sin\theta)\}] \tag{A.4a}$$

$$= n_{i\infty}(\frac{B_i}{\pi})^{3/2} e^{-S_i^2} e^{-(B_i\xi^2-2\sqrt{B_i}\,\xi\, S_i\cos\gamma \cos\theta)} e^{-2\sqrt{B_i}\,\xi\, S_i\sin\gamma \sin\theta \cos\varepsilon} \tag{A.4b}$$

For large ion speed ratio, $S_i >> 1$, we can see that the exponent in the integrand has a sharp maximum at $\varepsilon = \pi$. According to Laplace, the major contribution to the value of the integral with respect to ε arises from the immediate vicinity of $\varepsilon = \pi$. Carrying out an expansion in powers of $x = \varepsilon - \pi$ and replacing the limits of the x integration by ∞ and $-\infty$, we obtain

$$I_{3(\xi_r)} \sim n_{i\infty}(\frac{B_i}{\pi})^{3/2} e^{-S_i^2}\int_0^\infty \xi\, e^{-B_i\xi^2} d\xi \int_{\frac{\pi}{2}}^{\pi} e^{2\sqrt{B_i}\,\xi\, S_i(\cos\gamma\cos\theta+\sin\gamma\sin\theta)}$$

$$\frac{d}{d\gamma}[\cos\gamma \sin^2\gamma\; I(\phi_o')]\int_{-\infty}^{\infty} e^{-\sqrt{B_i}\,\xi\, S_i\sin\gamma \sin\theta\, x^2} dx$$

$$= n_{i\infty}\frac{B_i}{\pi}\frac{e^{-S_i^2}}{\sqrt{U_\infty \sin\theta}}\int_0^\infty \sqrt{\xi}\, e^{-B_i\xi^2} d\xi \int_{\frac{\pi}{2}}^{\pi} \sqrt{\sin\gamma}\,[\cos\gamma \sin\gamma\; I'(\phi_o')$$

$$+ (2\cos^2\gamma-\sin^2\gamma) I(\phi_o')] e^{2\sqrt{B_i}\,\xi S_i \cos(\gamma-\theta)} d\gamma \tag{A.5}$$

(i) When $0<\theta<\frac{\pi}{2}$, $\quad S_i >> 1$

The exponent in the integrand has a maximum at $\gamma = \frac{\pi}{2}$ in the range $\pi/2 \leq \gamma \leq \pi$. Expanding in powers of $y = \gamma - \pi/2$ and replacing the upper limit on the integral with respect to y by ∞, we obtain

$$I_{3(\xi_r)} \sim - n_{i\infty} \frac{B_i}{\pi} \frac{e^{-S_i^2}}{\sqrt{U_\infty \sin\theta}} I(\phi_o')_{\gamma=\pi/2} \int_0^\infty \sqrt{\xi}\, e^{-B_i(\xi^2 - 2\xi U_\infty \sin\theta)} d\xi$$

$$\int_0^\infty e^{-2\sqrt{B_i}\,\xi S_i \cos\theta\, y} dy$$

$$= - \frac{n_{i\infty}}{2\pi} \frac{e^{-S_i^2 \cos^2\theta}}{U_\infty \cos\theta \sqrt{U_\infty \sin\theta}} I(\phi_o')_{\gamma=\pi/2} \int_0^\infty \frac{1}{\sqrt{\xi}} e^{-B_i(\xi - U_\infty \sin\theta)^2} d\xi \quad (A.6)$$

The exponent has a sharp maximum at $\xi = U_\infty \sin\theta$. Expanding in powers of $z = \xi - U_\infty \sin\theta$ and replacing the lower limit on the integral with respect to z by $-\infty$, we finally obtain

$$I_{3(\xi_r)} \sim - \frac{n_{i\infty}}{2\pi} \frac{e^{-S_i^2 \cos^2\theta}}{U_\infty \cos\theta U_\infty \sin\theta} I(\phi_o')_{\gamma=\pi/2} \int_{-\infty}^\infty e^{-B_i z^2} dz$$

$$= - \frac{n_{i\infty}}{2} \frac{e^{-S_i^2 \cos^2\theta}}{\sqrt{\pi}\, S_i \cos\theta} \frac{1}{U_\infty \sin\theta} I(\phi_o')_{\gamma=\pi/2} \quad (A.7a)$$

$$\simeq - \frac{n_{i\infty}}{2} \frac{\mathrm{erfc}(S_i \cos\theta)}{U_\infty \sin\theta} I(\phi_o')_{\gamma=\pi/2} \quad (A.7b)$$

(ii) When $\frac{\pi}{2} < \theta < \pi$, $\quad S_i \gg 1$

In this case the exponent in the integrand of Eq. (A.5) has a sharp maximum at $\gamma=\theta$. Expanding in powers of $y = \gamma - \theta$ and replacing the limits of the y integration by ∞ and $-\infty$, we obtain

$$I_{3(\xi_r)} \sim n_{i\infty} \frac{B_i}{\pi} \frac{e^{-S_i^2}}{\sqrt{U_\infty}} [\cos\theta \sin\theta\, I'(\phi_o')_{\gamma=0} + (2\cos^2\theta - \sin^2\theta) I(\phi_o')_{\gamma=0}]$$

$$\int_0^\infty \sqrt{\xi}\, e^{-B_i(\xi^2 - 2\xi U_\infty)} d\xi \int_{-\infty}^\infty e^{-\sqrt{B_i}\,\xi S_i y^2} dy$$

$$= \frac{n_{i\infty}}{U_\infty} \sqrt{\frac{B_i}{\pi}} [\cos\theta \sin\theta\, I'(\phi_o')_{\gamma=\theta} + (2\cos^2\theta - \sin^2\theta) I(\phi_o')_{\gamma=\theta}] \int_0^\infty e^{-B_i(\xi - U_\infty)^2} d\xi$$

(A.8)

The exponent has a sharp maximum at $\xi = U_\infty$. Changing the variable to $z = \xi - U_\infty$ and replacing the lower limit on the integral with respect to z by $-\infty$, we finally obtain

$$I_{3(\xi_r)} \sim \frac{n_{i\infty}}{U_\infty}\sqrt{\frac{B_i}{\pi}}\,[\cos\theta\,\sin\theta\,I'(\phi_o')_{\gamma=\theta} + (2\cos^2\theta - \sin^2\theta)I(\phi_o')_{\gamma=\theta}]\int_{-\infty}^{\infty} e^{-B_i z^2}\,dz$$

$$= \frac{n_{i\infty}}{U_\infty}\,[\cos\theta\,\sin\theta\,I'(\phi_o')_{\gamma=\theta} + (3\cos^2\theta - 1)I(\phi_o')_{\gamma=\theta}] \qquad \text{(A.9)}$$

(iii) When $\theta = \frac{\pi}{2}$, $\qquad S_i >> 1$

From Eq. (A.5)

$$I_{3(\xi_r)} \sim n_{i\infty}\frac{B_i}{\pi}\frac{e^{-S_i^2}}{\sqrt{U_\infty}}\int_0^\infty \sqrt{\xi}\,e^{-B_i\xi^2}\,d\xi\int_{\frac{\pi}{2}}^{\pi}\sqrt{\sin\gamma}\,[\cos\gamma\,\sin\gamma\,I'(\phi_o')$$

$$+ (2\cos^2\gamma - \sin^2\gamma)I(\phi_o')]\,e^{2\sqrt{B_i}\,\xi S_i \sin\gamma}\,d\gamma \qquad \text{(A.10)}$$

The exponent in the integrand has a sharp maximum at $\gamma = \frac{\pi}{2}$. Expanding in powers of $y = \gamma - \pi/2$ and replacing the upper limit on the integral with respect to y by ∞, we obtain

$$I_{3(\xi_r)} \sim -n_{i\infty}\frac{B_i}{\pi}\frac{e^{-S_i^2}}{\sqrt{U_\infty}}\,I(\phi_o')_{\gamma=\pi/2}\int_0^\infty \sqrt{\xi}\,e^{-B_i(\xi^2 - 2\xi U_\infty)}\,d\xi\int_0^\infty e^{-\sqrt{B_i}\,\xi S_i y^2}\,dy$$

$$= -\frac{n_{i\infty}}{2U_\infty}\sqrt{\frac{B_i}{\pi}}\,I(\phi_o')_{\gamma=\pi/2}\int_0^\infty e^{-B_i(\xi - U_\infty)^2}\,d\xi \qquad \text{(A.11)}$$

The exponent has a sharp maximum at $\xi = U_\infty$. Applying the same method used in Eq. (A.8), we finally obtain

$$I_{3(\xi_r)} \sim -\frac{n_{i\infty}}{2U_\infty}\sqrt{\frac{B_i}{\pi}}\, I(\phi_o')_{\gamma=\pi/2}\int_{-\infty}^{\infty} e^{-B_i z^2}\,dz$$

$$= -\frac{n_{i\infty}}{2U_\infty}\, I(\phi_o')_{\gamma=\pi/2} \qquad \text{(A.12)}$$

It is interesting to observe that this result can be obtained from the approximate equation (A.7b) by setting $\theta = \pi/2$.

(iv) When $\theta = 0, \pi,\quad S_i >> 1$

From Eqs. (A.3) and (A.4b)

$$I_{3(\xi_r)} = 2\pi\, n_{i\infty}\left(\frac{B_i}{\pi}\right)^{3/2} e^{-S_i^2}\int_0^\infty \xi\, e^{-B_i\xi^2}\,d\theta\int_{\frac{\pi}{2}}^{\pi}[\cos\gamma\,\sin^2\gamma\, I'(\phi_o')$$

$$+(2\sin\gamma\,\cos^2\gamma-\sin^3\gamma)I(\phi_o')]e^{\pm 2\sqrt{B_i}\,\xi\, S_i\cos\gamma}d\gamma \qquad \text{(A.13)}$$

where the plus sign is used when $\theta = 0$ and the minus sign when $\theta = \pi$.

For the case $\theta = 0$, the exponent has a maximum at $\gamma = \pi/2$ in the range $\pi/2 \leq \gamma \leq \pi$. Expanding in powers of $y = \gamma - \pi/2$ and replacing the upper limit on the integral with respect to y by ∞, we obtain

$$I_{3(\xi_r)} \sim -2\pi\, n_{i\infty}\left(\frac{B_i}{\pi}\right)^{3/2} e^{-S_i^2}\, I(\phi_o')_{\gamma=\pi/2}\int_0^\infty \xi\, e^{-B_i\xi^2}\,d\xi\int_0^\infty e^{-2\sqrt{B_i}\,\xi S_i x}dx$$

$$= -\frac{n_{i\infty}}{2}\frac{\sqrt{B_i}\, e^{-S_i^2}}{S_i}\, I(\phi_o')_{\gamma=\pi/2} \qquad \text{(A.14)}$$

For the case $\theta = \pi$, the exponent has a sharp maximum at $\gamma = \pi$. Expanding in powers of $y = \gamma - \pi$, we find that $I_{3(\xi_r)}$ is asymptotically equal to zero

$$I_{3(\xi_r)} \sim 0 \qquad \text{(A.15)}$$

The other I_3's can be evaluated in the same manner. The results are given as follows.

(2) From Eq. (6.46a)

$$I_{3(n_i)} = \int_0^\infty \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\pi} f_{io} \frac{d}{d\gamma} [\sin^2\gamma \; I(\phi_o')] d\xi \; d\gamma \; d\varepsilon \qquad (A.16)$$

(i) When $0 < \theta < \frac{\pi}{2}$, $S_i >> 1$

$$I_{3(n_i)} \sim \frac{n_{i\infty}}{2} \frac{e^{-S_i^2 \cos^2\theta}}{\sqrt{\pi}\; S_i \cos\theta} \frac{1}{U_\infty^2 \; \sin^2\theta} I'(\phi_o')_{\gamma=\pi/2} \qquad (A.17a)$$

$$\simeq \frac{n_{i\infty}}{2} \frac{\mathrm{erfc}\;(S_i\cos\theta)}{U_\infty^2 \; \sin^2\theta} I'(\phi_o')_{\gamma=\pi/2} \qquad (A.17b)$$

(ii) When $\pi/2 < \theta < \pi$, $S_i >> 1$

$$I_{3(n_i)} \sim \frac{n_{i\infty}}{U_\infty^2} [\sin\theta \; I'(\phi_o')_{\gamma=\theta} + 2\cos\theta \; I(\phi_o')_{\gamma=\theta}] \qquad (A.18)$$

(iii) When $\theta = \pi/2$, $S_i >> 1$

$$I_{3(n_i)} \sim \frac{n_{i\infty}}{2U_\infty^2} I'(\phi_o')_{\gamma=\pi/2} \qquad (A.19)$$

(iv) When $\theta = 0, \pi$, $S_i >> 1$

$$I_{3(n_i)} \sim 0 \qquad (A.20)$$

(3) From Eq. (6.41a)

$$I_{3(\xi_\theta)} = \int_0^\infty \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\pi} \xi \cos\varepsilon \, f_{io} \frac{d}{d\gamma} [\sin^3\gamma \; I(\phi_o')] d\xi \; d\gamma \; d\varepsilon \qquad (A.21)$$

(i) When $0 < \theta < \pi/2$, $S_i >> 1$

$$I_{3(\xi_\theta)} \sim -\frac{n_{i\infty}}{2}\frac{e^{-S_i^2\cos^2\theta}}{\sqrt{\pi}\,S_i\,\cos\theta}\frac{1}{U_\infty\,\sin\theta}\,I'(\phi_o')_{\gamma=\pi/2} \tag{A.22a}$$

$$\simeq -\frac{n_{i\infty}}{2}\frac{\mathrm{erfc}(S_i\cos\theta)}{U_\infty\,\sin\theta}\,I'(\phi_o')_{\gamma=\pi/2} \tag{A.22b}$$

(ii) When $\pi/2 < \theta < \pi$, $S_i >> 1$

$$I_{3(\xi_\theta)} \sim -\frac{n_{i\infty}}{U_\infty}\sin\theta[\sin\theta\; I'(\phi_o')_{\gamma=\theta} + 3\cos\theta\; I(\phi_o')_{\gamma=\theta}] \tag{A.23}$$

(iii) When $\theta = \pi/2$, $S_i >> 1$

$$I_{3(\xi_\theta)} \sim -\frac{n_{i\infty}}{2U_\infty}\,I'(\phi_o')_{\gamma=\pi/2} \tag{A.24}$$

(iv) When $\theta = 0, \pi$

$$I_{3(\xi_\theta)} = 0 \tag{A.25}$$

(4) From Eq. (6.61a)

$$I_{3(\xi_r^2)} = \int_o^\infty\int_{\frac{\pi}{2}}^{\pi}\int_o^{2\pi} \xi^2\; f_{io}\,\frac{d}{d\gamma}\,[\cos^2\gamma\;\sin^2\gamma\; I(\phi_o')]d\xi\; d\gamma\; d\varepsilon \tag{A.26}$$

(i) When $0 < \theta < \pi/2$, $S_i >> 1$

$$I_{3(\xi_r^2)} \sim 0 \tag{A.27}$$

(ii) When $\pi/2 < \theta < \pi$, $S_i >> 1$

$$I_{3(\xi_r^2)} \sim n_{i\infty}\,\cos\theta[\cos\theta\;\sin\theta\; I'(\phi_o')_{\gamma=\theta} + 2(2\cos^2\theta - 1)I(\phi_o')_{\gamma=\theta}] \tag{A.28}$$

(iii) When $\theta = 0,\ \pi/2,\ \pi,\qquad S_i >> 1$

$$I_{3(\xi_r^2)} \sim 0 \tag{A.29}$$

(5) From Eq. (6.66a)

$$I_{3(\xi_r\xi_\theta)} = \int_0^\infty \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\pi} \xi^2 \cos\varepsilon\, f_{io} \frac{d}{d\gamma} [\cos\gamma \sin^3\gamma\ I(\phi_o')] d\xi\, d\gamma\, d\varepsilon \tag{A.30}$$

(i) When $0 < \theta < \pi/2,\qquad S_i >> 1$

$$I_{3(\xi_r\xi_\theta)} \sim \frac{n_{i\infty}}{2} \frac{e^{-S_i^2\cos^2\theta}}{\sqrt{\pi}\, S_i\cos\theta} I(\phi_o')_{\gamma=\pi/2} \tag{A.31a}$$

$$\simeq \frac{n_{i\infty}}{2} \operatorname{erfc}(S_i\cos\theta) I(\phi_o')_{\gamma=\pi/2} \tag{A.31b}$$

(ii) When $\pi/2 < \theta < \pi,\qquad S_i >> 1$

$$I_{3(\xi_r\xi_\theta)} \sim -n_{i\infty} \sin\theta[\cos\theta \sin\theta\ I'(\phi_o')_{\gamma=\theta} + (4\cos^2\theta - 1) I(\phi_o')_{\gamma=\theta}] \tag{A.32}$$

(iii) When $\theta = \pi/2,\qquad S_i >> 1$

$$I_{3(\xi_r\xi_\theta)} \sim \frac{n_{i\infty}}{2} I(\phi_o')_{\gamma=\pi/2} \tag{A.33}$$

(iv) When $\theta = 0,\ \pi$

$$I_{3(\xi_r\xi_\theta)} = 0 \tag{A.34}$$

(6) From Eq. (6.77a)

$$I_{3(\xi_r\xi^2)} = \int_0^\infty \int_{\frac{\pi}{2}}^{\pi} \int_0^{2\pi} \xi^3\ f_{io} \frac{d}{d\gamma} [\cos\gamma \sin^2\gamma\ I(\phi_o')] d\xi\, d\gamma\, d\varepsilon \tag{A.35}$$

(i) When $0 < \theta < \pi/2$, $S_i >> 1$

$$I_{3(\xi_r \xi^2)} \sim -\frac{n_{i\infty}}{2} \frac{e^{-S_i^2 \cos^2\theta}}{\sqrt{\pi}\, S_i \cos\theta} U_\infty \sin\theta \; I(\phi_o')_{\gamma=\pi/2} \tag{A.36a}$$

$$\approx -\frac{n_{i\infty}}{2} \operatorname{erfc}(S_i \cos\theta) U_\infty \sin\theta \; I(\phi_o')_{\gamma=\pi/2} \tag{A.36b}$$

(ii) When $\pi/2 < \theta < \pi$, $S_i >> 1$

$$I_{3(\xi_r \xi^2)} \sim n_{i\infty} U_\infty [\cos\theta \sin\theta \; I'(\phi_o')_{\gamma=\theta} + (3\cos^2\theta - 1) I(\phi_o')_{\gamma=\theta}] \tag{A.37}$$

(iii) When $\theta = \pi/2$, $S_i >> 1$

$$I_{3(\xi_r \xi^2)} \sim -\frac{n_{i\infty}}{2} U_\infty \; I(\phi_o')_{\gamma=\pi/2} \tag{A.38}$$

(iv) When $\theta = 0$, $S_i >> 1$

$$I_{3(\xi_r \xi^2)} \sim -\frac{n_{i\infty}}{4} \frac{e^{-S_i^2}}{\sqrt{B_i} S_i} I(\phi_o')_{\gamma=\pi/2} \tag{A.39}$$

(v) When $\theta = \pi$, $S_i >> 1$

$$I_{3(\xi_r \xi^2)} \sim 0 \tag{A.40}$$

APPENDIX B

EVALUATION OF $H(\phi_s^*,\varepsilon)$

The term $H(\phi_s^*,\varepsilon)$ in the molecular drag coefficient can not be evaluated exactly even for the simplest spherical potential model. However, for large ion speed ratio, $S_i >> 1$, an asymptotic expression for $H(\phi_s^*,\varepsilon)$ may be obtained without specifying a certain model for the spherical potential by utilizing the results in Appendix A.

From Eq. (6.73)

$$H(\phi_s^*,\varepsilon) = -\frac{1}{n_{i\infty}}\left[\sqrt{\frac{\pi}{B_w}}\int_o^\pi I_{3(\xi_r)}\cos\theta\,\sin\theta\,d\theta\right.$$

$$\left.+\,2\left\{\int_o^\pi I_{3(\xi_r\xi_\theta)}\sin^2\theta\,d\theta - \int_o^\pi I_{3(\xi_r^2)}\cos\theta\,\sin\theta\,d\theta\right\}\right] \qquad (B.1)$$

From Eq. (A.7a)

$$\int_o^{\frac{\pi}{2}} I_{3(\xi_r)}\cos\theta\,\sin\theta\,d\theta \sim -\frac{n_{i\infty}}{2S_i^2}\sqrt{\frac{B_i}{\pi}}\,I(\phi_o')_{\gamma=\pi/2}\int_o^{\frac{\pi}{2}} e^{-S_i^2\cos^2\theta}\,d\theta$$

The exponent has a sharp maximum at $\theta = \pi/2$ for $S_i >> 1$. Expanding in powers of $x = \theta-\pi/2$ and replacing the lower limit by $-\infty$, we obtain

$$\int_o^{\frac{\pi}{2}} I_{3(\xi_r)}\cos\theta\,\sin\theta\,d\theta \sim -\frac{n_{i\infty}}{2S_i^2}\sqrt{\frac{B_i}{\pi}}\,I(\phi_o')_{\gamma=\pi/2}\int_{-\infty}^{o} e^{-S_i^2x^2}\,dx$$

$$= -\frac{n_{i\infty}\sqrt{B_i}}{4S_i^3}\,I(\phi_o')_{\gamma=\pi/2} \qquad (B.2)$$

From Eq. (A.9)

$$\int_{\frac{\pi}{2}}^{\pi} I_{3(\xi_r)} \cos\theta \ \sin\theta \ d\theta$$

$$\sim \frac{n_{i\infty}}{U_\infty}\Big[\int_{\frac{\pi}{2}}^{\pi} \cos^2\theta \ \sin^2\theta \ I'(\phi_o')_{\gamma=\theta} \ d\theta + \int_{\frac{\pi}{2}}^{\pi} (3\cos^2\theta - 1)\cos\theta \ \sin\theta \ I(\phi_o')_{\gamma=\theta} d\theta\Big]$$

Substituting Eqs. (A.1) and (A.2) for $I(\phi_o')$ and $I'(\phi_o')$, and carrying out the integration, we obtain

$$\int_{\frac{\pi}{2}}^{\pi} I_{3(\xi_r)} \cos\theta \ \sin\theta \ d\theta$$

$$\sim \frac{n_{i\infty}}{U_\infty}\int_a^\infty \Big[-\frac{2}{3}\frac{u^3}{a^3} + \frac{1}{a}\sqrt{u^2-a^2} + \frac{2}{3}\frac{1}{a^3}(u^2-a^2)^{3/2}\Big]\frac{d\phi_o'}{du} \ du \tag{B.3}$$

From Eqs. (A.31b) and (A.27)

$$\int_o^{\frac{\pi}{2}} I_{3(\xi_r\xi_\theta)} \sin^2\theta \ d\theta \sim \frac{n_{i\infty}}{2} I(\phi_o')_{\gamma=\pi/2}\int_o^{\frac{\pi}{2}} \mathrm{erfc}(S_i\cos\theta)\sin^2\theta \ d\theta \tag{B.4}$$

$$\int_o^{\frac{\pi}{2}} I_{3(\xi_r^2)} \cos\theta \ \sin\theta \ d\theta \sim 0 \tag{B.5}$$

From Eqs. (A.32) and (A.28)

$$\int_{\frac{\pi}{2}}^{\pi} I_{3(\xi_r\xi_\theta)} \sin^2\theta \ d\theta - \int_{\frac{\pi}{2}}^{\pi} I_{3(\xi_r^2)} \cos\theta \ \sin\theta \ d\theta$$

$$\sim - n_{i\infty}\Big[\int_{\frac{\pi}{2}}^{\pi} (\cos\theta - \cos^3\theta) I'(\phi_o')_{\gamma=\theta} d\theta + \int_{\frac{\pi}{2}}^{\pi} (3\cos^2\theta - 1)\sin\theta \ I(\phi_o')_{\gamma=\theta} d\theta\Big] = 0$$

(B.6)

Substituting Eqs. (B.2) $\sim$ (B.6) into Eq. (B.1), we obtain

$$H(\phi_s^*,\varepsilon) \sim -\frac{\sqrt{\pi}}{S_i}\sqrt{\frac{B_i}{B_w}}\int_a^\infty [-\frac{2}{3}\frac{u^3}{a^3}+\frac{1}{a}\sqrt{u^2-a^2}+\frac{2}{3}\frac{1}{a^3}(u^2-a^2)^{3/2}]\frac{d\phi_o'}{du}du$$

$$+ I(\phi_o')_{\gamma=\pi/2}[\frac{\sqrt{\pi}}{4S_i^3}\sqrt{\frac{B_i}{B_w}} - \int_o^{\frac{\pi}{2}} \mathrm{erfc}(S_i\cos\theta)\sin^2\theta\, d\theta] \qquad \text{(B.7)}$$

We can also obtain from Eqs. (A.7b) and (A.9)

$$\int_o^\pi I_{3(\xi_r)}\sin\theta\, d\theta \sim -\frac{n_{i\infty}}{2U_\infty} I(\phi_o')_{\gamma=\pi/2}\int_o^{\frac{\pi}{2}}\mathrm{erfc}(S_i\cos\theta)d\theta$$

$$+\frac{n_{i\infty}}{U_\infty}[\int_{\frac{\pi}{2}}^{\pi}(\cos\theta-\cos^3\theta)I'(\phi_o')_{\gamma=\theta}d\theta + \int_{\frac{\pi}{2}}^{\pi}(3\cos^2\theta-1)\sin\theta\, I(\phi_o')_{\gamma=\theta}d\theta]$$

$$= -\frac{n_{i\infty}}{2U_\infty} I(\phi_o')_{\gamma=\pi/2}\int_o^{\frac{\pi}{2}}\mathrm{erfc}(S_i\cos\theta)d\theta \qquad \text{(B.8)}$$