Graphic Matroid Embeddability and Sarkaria's Theorem

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1. Abstract

In this paper, we intend to explore consequences of Sarkaria's Theorem in the context of Graphic matroids. Sarkaria's Theorem states that if χ is the chromatic number of the Kneser graph of a simplicial complex Δ and Δ has m vertices, then Δ cannot be embedded in $\mathbb{R}^d$ for $d \leq m-\chi-2$. This theorem provides a lower bound for embedding simplicial complexes. For example, it can be used to prove that K_5 is non-planar. We start by covering important mathematical definitions related to the topic at hand, including matroids, graphic matroids, and the Kneser graph of a matroid. Then, we cover some examples to provide understanding of Sarkaria's theorem and motivations for pursuing this topic. We show how Sarkaria's theorem is implemented and how it can tell us a matroid is non-embeddable in dimensions higher than the dimension of the matroid. After proving a special case with the embeddability of the graphic matroid of K_5, we generalize the proof to include all complete graphs. This subsequently, can tell us information regarding the embeddability of any graphic matroid.

2. Definitions

Definition 1. A *graph* is a pair of two sets (V, E). They are a vertex set V and an edge set E, where the elements of E are unordered pairs of vertices from V representing a connection between those two vertices. A *complete graph* is a graph such that if $x, y \in V$ then $(x, y) \in E$. In other words, all vertices share an edge between them. We denote the complete graph as K_n, where n is the number of vertices of the graph.

Definition 2. The *chromatic number* of a graph is the least number of colors needed to color the vertices a graph such that no two vertices that share an edge have the same color.

Definition 3. An *(abstract) simplicial complex* Δ on a vertex set X is a collection of subsets of X such that if $A \subseteq B \in \Delta$ then $A \in \Delta$. The members of Δ are called *simplices* or *faces*. If an element of Δ is not properly contained in any other element of Δ then it is called a *facet*.

The *dimension* of a simplicial complex is the maximum dimension of all of its simplices, or, in other words, the dimension of the largest facets in the complex.

Definition 4. A *matroid* M is a pair (E, I), where E is a finite set called the ground set and I is a family of subsets of E with the following properties:

(1) $\emptyset \in I$
(2) Every subset of an independent set is independent, i.e, if $A \subseteq B \in I$ then $A \in I$.
(3) If $A, B \in I$, and $|A| > |B|$, then there exists some $x \in A \setminus B$ such that $B \cup \{x\} \in I$.

For example:

Let $E = \{1, 2, 3, 4, 5, 6\}$ and set $S \subset E$ is independent if $|S| < 6$

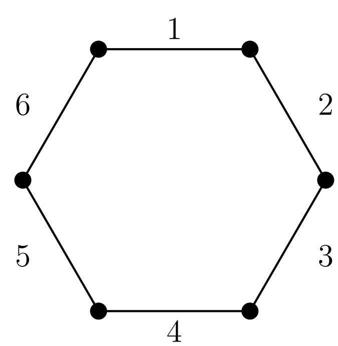
- $\emptyset \in I$, since $0 < 6$
- If $B \in I$, and $A \subseteq B$, then $|A| \leq |B| < 6$, so $A \in I$
- Let $A, B \in I$ and $|A| > |B|$. So there exists an $x \in A \setminus B$ such that $|B \cup \{x\}| \leq |A| < 6$.

Note that (1) and (2) are sufficient to fit the definition of a simplicial complex. We call the simplicial complex of the matroid the *matroid independence complex*.

Definition 5. A *graphic matroid* is a matroid whose ground set is the edge set of a graph G. A set of edges is independent if and only if it does not contain a cycle in G. Any finite graph G can give rise to a graphic matroid $M(G)$ by this definition.

Note that the facets of a graphic matroid are the spanning forests since adding any edge not already in a spanning tree will complete a cycle.

For example, our previous matroid from Definition 4 is graphic:
E = {1,2,3,4,5,6} and set $S \subset E$ is independent if $|S| < 6$



Any subset of edges of size less than 6 is independent since only all 6 edges will make a cycle.

Definition 6. A *minimal non-face* of a matroid M is a set $X \in E$ such that $X \notin I$ but $X \setminus \{a\} \in I$ for all $a \in X$.

Definition 7. The *Kneser graph* of a simplicial complex S, denoted as $KG(S)$, is a graph whose vertices are the minimal non-faces of S. Two vertices are adjacent whenever the corresponding two minimal non-faces are disjoint.

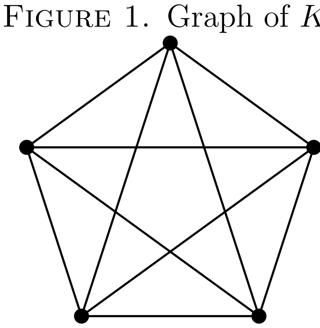
Note that the minimal non-faces of a graphic matroid are precisely all circuits of the graph, since removing any edge of the circuit will result in a forest.

Definition 8. An *Embedding* of a simplicial complex Δ in a space R is an injective and structure-preserving map from Δ to R. In other words, a representation of Δ in R where no there is no self intersection.
For example, a planar graph is an embedding of the graph as seen as a simplicial complex.

3. Introduction to Sarkaria's Theorem

To give a good idea of Sarkaria's Theorem, we will use it to show that K_5 is non-planar. K_5 can be seen as a simplicial complex where the vertex set is precisely the vertex set of the graph and the edges as 1-dimensional simplices.

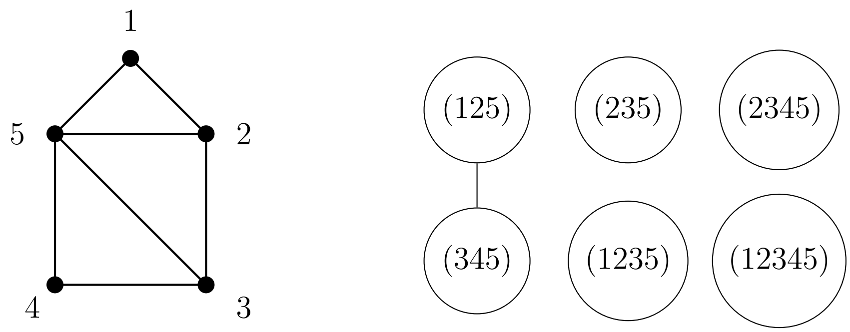
Figure 1. Graph of K_5



Sarkaria's Theorem states that if χ is the chromatic number of the Kneser graph of a simplicial complex Δ and Δ has m vertices, then Δ cannot be embedded in $\mathbb{R}^d$ for $d \leq m - \chi - 2$. So, in this case, $m = 5$. Now, to find χ, we first need to know what the minimal non-faces are for K_5. If we have a subset of any three vertices, then it is not face of K_5. However, remove any one of those three vertices then we have an edge of K_5 which *is* a face. Thus, our minimal non-faces are any subset of 3 vertices in K_5. Since there are only 5 vertices, every minimal non-face must share at least one vertex. Hence, in the Keneser graph, there are no adjacent vertices. Therefore, $\chi = 1$ and Sarkaria's Theorem says that K_5 is non-embeddable in $\mathbb{R}^{5-1-2} = \mathbb{R}^2$.

4. Sarkaria's Theorem and Graphic Matroids

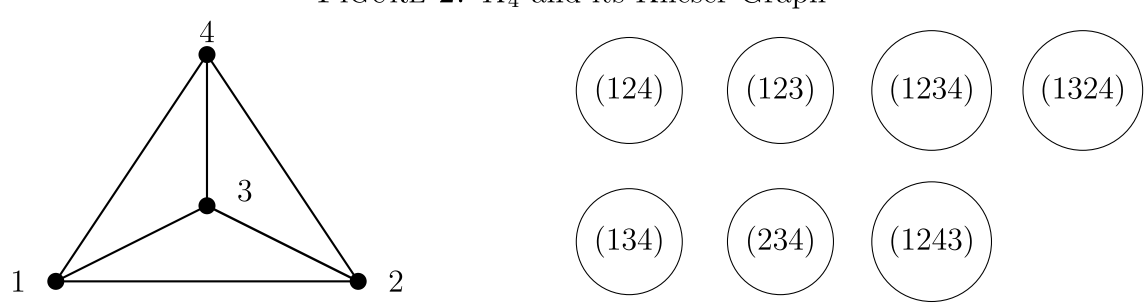
Consider the following graph H (left). It only contains two cycles with disjoint edges, $(1;2;5)$ and $(3;4;5)$. Thus its Kneser graph (right) has one edge.



The graphic matroid of H, $M(H)$, has 7 vertices since H has 7 edges. The chromatic number of the Kneser graph of $M(H)$ is 2. So, by Sarkaria's Theorem, $M(H)$ is not embeddable in $\mathbb{R}^d$ where $d = 7 - 2 - 2 = 3$. Since the spanning tree of H needs only 4 edges, the largest facet of $M(H)$ has 4 vertices. Thus $M(H)$ is 3-dimensional. So Sarkaria's Theorem tell us this matroid is not embeddable in $\mathbb{R}^3$ even though it is 3-dimensional.

The Kneser graph may have no edges and still have the possibility for an interesting result. Let us consider the graphic matroid of K_4, $M(K_4)$. Any two cycles of K_4 intersect. Thus, the Kneser graph of $M(K_4)$ has no edges. Hence, the Kneser graph can be colored with 1 color, so $\chi = 1$. Since the vertices of a graphic matroid are the edges of the graph, $m = 6$. Thus, $M(K_4)$ is not embeddable in $\mathbb{R}^d$ for $d \leq 6-1-2 = 3$.

Figure 2. K_4 and its Kneser Graph



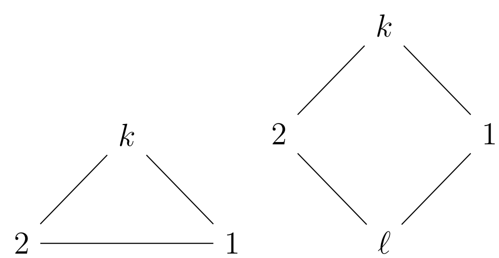
Since a spanning tree of K_4 has 3 edges, the graphic matroid is 2-dimensional. This gives us an interesting result that our 2-dimensional matroid is non-embeddable in a higher dimension, namely $\mathbb{R}^3$.

We can also see a similar example with the graphic matroid of K_5. The Kneser graph for $M(K_5)$ is much more complicated, having over 30 circuts. Thus, we need a new approach to finding the chromatic number of the Kneser graph. As a warm-up to our general result for coloring the Kneser graphs of graphic matroids, let us prove the special case of $M(K_5)$.

Proposition 1. The chromatic number of $KG(M(K_5))$ is 3.

Proof. Note that there are odd cycles in $KG(M(K_5))$, thus the chromatic number is strictly larger than 2. Let $M(K_5)$ be the graphic matroid of K_5. Begin by labeling each vertex of the Kneser graph of $M(K_5)$ by its lexicographically earliest edge not including 1 or 2. For example, the cycle (1 3 4) has the label $\{3, 4\}$, and the cycle (1 5 2 4 3) also has the label $\{3, 4\}$. Name T as the set of cycles labeled in this way. The cycles that are not labeled are 3-cycles of the form (1 2 k), and 4-cycles of the form (1 ℓ 2 k) where $k, \ell \in \{3, 4, 5\}$. Name the sets of these cycles U and V respectively. For these cycles, label the 4 cycles by the edge $\{\min\{k, \ell\}, 5\}$. When $k \neq 5$, label the 3-cycles $\{k, 5\}$, and when $k = 5$, label that 3-cycle $\{4, 5\}$.

FIGURE 3. The special 3-cycles and 4-cycles



We want to show that the labels $\{3, 4\}$, $\{3, 5\}$, and $\{4, 5\}$ are a valid coloring of the graph $KG(M(K_5))$. Since two cycles are connected only if they are disjoint, it is sufficient to show that if two cycles have the same labeling, then there exists an edge $\{i, j\}$ contained in both cycles.

First, let's start with the bulk of our graph, the cycles in T. If $A, B \in T$ have the same label $\{i, j\}$, then $\{i, j\} \in A$ and $\{i, j\} \in B$.

Now, let us consider the cycles in V. If $A, B \in V$ have the same label $\{i, 5\}$ then $i = \min\{\ell, k\}$, and thus $\{i, 2\} \in A$ and $\{i, 2\} \in B$. Let $A \in T$ and $B \in V$ have the same label $\{i, 5\}$. Since A is a cycle, the vertex i has two points it is connected to, n and some j. If $j \neq 1$ and $j \neq 2$, then $\{i, j\}$ is lexicographically earlier than $\{i, 5\}$ contradicting that $\{i, 5\}$ is its label. Thus, $\{i, 1\} \in A$ or $\{i, 2\} \in A$. Since $\{i, 1\} \in B$ and $\{i, 2\} \in B$, in both cases, $A \cap B \neq \emptyset$.

Finally, let us consider the set of cycles U. Since the edge $\{1, 2\} \in A$ if $A \in U$, for all $A, B \in U$, $A \cap B \neq \emptyset$. Let $k \neq n$. If $A \in U$ and $B \in V$ share the same label $\{i, 5\}$, then $\{1, i\} \in A$ and $\{1, i\} \in B$. If $A \in U$ and $B \in T$ have the same label $\{i, 5\}$, then similar to the argument for V and T, $\{i, 1\} \in B$ or $\{i, 2\} \in B$.

If $k = n$, then we have the cycle $C = (1\ 2\ 5)$ and we labeled it $\{4,5\}$. If $A \in T$ shares this label, then $\{4,5\} \in A$. If $\{1,5\} \in A$ or $\{2,5\} \in A$, $C \cap A \neq \emptyset$. So we just need to consider if $\{3,5\} \in A$. If this is the case, then it contradicts that $\{4,5\}$ is A's label since $\{3,5\}$ is lexicographically earlier. Thus, $C \cap A \neq \emptyset$. If $B \in V$ has the label $\{4,5\}$, then $4 = \min(\ell, k)$. Therefore $\ell = 5$ or $k = 5$. Hence, $\{1,5\} \in B$, so $B \cap C \neq \emptyset$.

$\square$

Since K_5 has 10 edges and the chromatic number of $KG(M(K_5))$ is 3, by Sarkaria's Theorem we have that $M(K_5)$ is not embeddable in $\mathbb{R}^d$ where $d \leq 10 - 3 - 2 = 5$. Since a spanning tree in K_5 has 4 edges, $M(K_5)$ is 3 dimensional. Thus, once again we have a matroid non-embeddable in a higher dimension, but with a gap of 2 dimensions this time. This result raises the question if this is true for all graphical matroids of complete graphs, and it the gap continues to grow.

5. Results

Denote chromatic number of $KG(M(K_n)$ as χ. The number of edges in a complete graph on n vertices is given by $\binom{n}{2}$, so our d from Sarkaria's Theorem is $\binom{n}{2} - \chi - 2$. The dimension of $M(K_n)$ is $n-2$ since it takes $n-1$ edges to form a spanning tree of the graph. Therefore, if the gap were to continue increasing with n, it would follow the inequality

$$\binom{n}{2} - \chi - 2 \geq (n-2) + (n-3)$$

With $(n-2) + (n-3)$ representing the forbidden Sarkaria Embedding dimension if the gap between the graphic matroid's dimension and the forbidden embedding dimension were to increase with at least slope 1.

Solving for χ we have

$$\begin{aligned}
\chi &\leq \binom{n}{2} - (n-2) - (n-3) - 2 \\
&= \frac{n^2 - n}{2} - 2n + 3 \\
&= \frac{n^2 - n - 4n + 6}{2} \\
&= \frac{(n-3)(n-2)}{2} \\
&= \binom{n-2}{2}
\end{aligned}$$

Since the sign of χ is negative in Sarkaria's Theorem, if this is an upper bound, it gives us a lower bound for Sarkaria's forbidden embedding dimension.

Proposition 2. The chromatic number χ of $KG(M(K_n))$ is bounded above by $\binom{n-2}{2}$

Proof. Let $M(K_n)$ be the graphic matroid of a complete graph on n vertices. We will be using a similar technique as we did for Proposition 1 to color $KG(M(K_n))$. By showing we can label each cycle by an edge within the cycle, not including 1 or 2, we will only use $\binom{n-2}{2}$ labels. Thus showing that the chromatic number is at most $\binom{n-2}{2}$. For simplicity, we will write cycles using cycle notation, but as opposed to permutations, there is no direction to the cycle. All that matters are adjacent vertices. For example, this means $(1\ 2\ 3\ 4\ 5) = (5\ 4\ 3\ 2\ 1)$

Let M denote the set of cycles of K_n for $n \geq 4$. If $A \in M$ such that there exists some $\{a, b\} \in A$ such that $a > 2$ and $b > 2$, then label A by the lexicographically least (numerically least) edge of A, not including 1 or 2. Let T be the set of all cycles labeled this way.

This labels most elements in M, but leaves out some cycles, namely, any 3-cycle of the form $(1\ 2\ k)$ and 4 cycle of the form $(1\ \ell\ 2\ k)$ with $\ell, k \in [3, n]$. See Figure 3. Name the sets of these cycles U and V respectively. For these cycles, label the 4 cycles by the edge $\{\min\{k, l\}, n\}$. When $k \neq n$, label the 3-cycles $\{k, n\}$, and when $k = n$, label that 3-cycle $\{n-1, n\}$.

We want to show that these labels are a valid coloring of the graph $KG(K_n)$ for all $n \geq 4$. Since two cycles are connected only if they are disjoint, it is sufficient to show that if two cycles have the same labeling, then there exists an edge $\{i, j\}$ contained in both cycles.

Note if $A, B \in T$ have the same label $\{i, j\}$, then $\{i, j\} \in A$ and $\{i, j\} \in B$.

Now, let us consider the cycles in V. If $A, B \in V$ have the same label $\{i, n\}$ then $i = \min\{\ell, k\}$, and thus $\{i, 2\} \in A$ and $\{i, 2\} \in B$. Let $A \in T$ and $B \in V$ have the same label $\{i, n\}$. Since A is a cycle, the vertex i has two points it is connected to, n and some j. If $j \neq 1$ and $j \neq 2$, then $\{i, j\}$ is lexicographically earlier than $\{i, n\}$ contradicting that $\{i, n\}$ is its label. Thus, $\{i, 1\} \in A$ or $\{i, 2\} \in A$. Since $\{i, 1\} \in B$ and $\{i, 2\} \in B$, in both cases, $A \cap B \neq \emptyset$.

Finally, let us consider the set of cycles U. Since the edge $\{1, 2\} \in A$ if $A \in U$, for all $A, B \in U$, $A \cap B \neq \emptyset$. Let $k \neq n$. If $A \in U$ and $B \in V$ share the same label $\{i, n\}$, then $\{1, i\} \in A$ and $\{1, i\} \in B$. If $A \in U$ and $B \in T$ have the same label $\{i, n\}$, then similar to the argument for V and T, $\{i, 1\} \in B$ or $\{i, 2\} \in B$.

If $k = n$, then we have the cycle $C = (1\ 2\ n)$ and we labeled it $\{n-1, n\}$. If $A \in T$ shares this label, then $\{n-1, n\} \in T$. Since A is a cycle, only two edges contain n, $\{n-1, n\}$ and $\{n, j\}$. If $j \neq 1$ and $j \neq 2$, then $\{n, j\}$ is lexicographically earlier than $\{n-1, n\}$ contradicting that A has the label $\{n-1, n\}$. Thus, $j = 1$ or $j = 2$. In either case, since $\{1, n\} \in C$ and $\{2, n\} \in C$, $C \cap B \neq \emptyset$. If $A \in V$ has the label $\{n-1, n\}$, then $n - 1 = \min(\ell, k)$. Therefore $\ell = n$ or $k = n$. Hence, $\{1, n\} \in A$, so $A \cap C \neq \emptyset$. $\square$

We have shown that the graphic matroids of K_n for $n \geq 4$ cannot be embedded in dimension $n-2$, and the gap between the embeddable dimension and the dimension of $M(K_n)$ widens with n with at least slope 1. Specifically, the graphic matroid of K_n cannot be embedded in dimension $2n-5$. Consequently, Proposition 2 also implies the following corollary.

Corollary 1. All graphic matroids $M(G)$ can not be embedded in dimension $m - \binom{n-2}{2} - 2$ where m is the number of edges in G and n the number of vertices.

Proof. Let $M(G)$ be a graphic matorid on n vertices. Note that every circuit of G is a circuit of K_n since every possible circuit is present in K_n. Therefore, $KN(M(G)) \subset KN(M(K_n))$. Hence, a coloring of $KN(M(K_n))$ will work for $KN(M(G))$. So, by proposition 2, the chromatic number is at most $\binom{n-2}{2}$. By Sarkaria's Theorem, we have our result. $\square$

While this result is far reaching, it doesn't always tell us something interesting. For example, let G be an acyclic graph on 5 verticies. Since there are no circuits, our chromatic number of the Kneser graph of $M(G)$ is 0. So, by our corollary, it is not embeddable in $\mathbb{R}^d$ where $d = 4-3-2 = -1$. There is also a related result which says that any embedding of a graphic matroid Δ in $\mathbb{R}^{(m-\chi-1)}$ is chiral. See [1] for more detail. So applying Proposition 2 we have

Corollary 2. An embedding of a graphic matroid Δ in $\mathbb{R}^{(m-\binom{n-2}{2}-1)}$ is chiral.

There are some further questions to explore as well.

- Is this bound sharp? If we go one dimension higher than the forbidden Sarkaria dimension, will $M(K_n)$ be embeddable?
- Can we find a lower bound for when it is guaranteed that we can embed $M(K_n)$
- Is $\chi = \binom{n-2}{2}$?

References

[1] Florian Frick and Michael C. Harrison, "Spaces of embeddings: Nonsingular bilinear maps, chirality, and their generalizations", Proc. Amer. Math. Soc. 150 (2022), 423-437, 2020

[2] Matoušek, "Using the Borsuk-Ulam Theorem", Springer-Verlag, 2003