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ABSTRACT

We introduce a generalized notion of homogeneous strict polynomial functors defined over a superalgebra, A. In particular, we define two closely related families of categories P_A^d and $\mathsf{P}_{(A,\mathfrak{a})}^d$ which generalize the categories P_d of classical homogeneous strict polynomial functors studied by Friedlander and Suslin and the categories $\mathsf{Pol}_{d,\Bbbk}^{(\mathrm{I})}$ and $\mathsf{Pol}_{d,\Bbbk}^{(\mathrm{II})}$ of homogeneous strict polynomial superfunctors defined by Axtell. In particular, we exhibit equivalences between the categories P_A^d, $\mathsf{P}_{(A,\mathfrak{a})}^d$ and the categories of left supermodules for generalized Schur algebras $S^A(m|n,d)$, $T_{\mathfrak{a}}^A(n,d)$, respectively (the latter of which were introduced by Kleshchev and Muth). Moreover, we establish a relationship between webs for $\mathfrak{gl}_n(A)$ and these generalized strict polynomial functors in the form of a faithful (and full under certain assumptions on $\Bbbk$) functor from the category of $\mathfrak{gl}_n(A)$-webs to $\mathsf{P}_{(A,\mathfrak{a})}$.

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Contents

0. Introduction

Schur-Weyl duality (named for Issai Schur and Herman Weyl) is a classic and famous example of a phenomenon in representation theory in which the irreducible representations of two different algebraic objects are related to one another via an explicit construction. Specifically, Schur-Weyl duality relates the representation theory of the general linear group and the symmetric group. There are several formulations, and we provide one such formulation which uses the general linear Lie algebra.

Let $\Bbbk$ be an infinite field. Let $V = \Bbbk^n$ denote column vectors of height n. Then $\mathfrak{gl}(V) = \mathfrak{gl}_n(\Bbbk)$ acts on the left of V via matrix multiplication, and therefore by using the coproduct for $\mathfrak{gl}_n(\Bbbk)$, on the left of $V^{\otimes d}$. The symmetric group $\mathfrak{S}_d$ acts on the right of $V^{\otimes d}$ via place permutation, and it is easy to see that these two actions commute. In particular, these commuting actions give rise to representations from the universal enveloping algebra of $\mathfrak{gl}(V)$ and from the group algebra of $\mathfrak{S}_d$

$$\mathcal{U}\left(\mathfrak{gl}(V)\right) \xrightarrow{\rho} \operatorname{End}_{\Bbbk}\left(V^{\otimes d}\right) \xleftarrow{\sigma} \Bbbk\mathfrak{S}_d$$

such that

$$\rho\left(\mathcal{U}\left(\mathfrak{gl}(V)\right)\right) \subset \operatorname{End}_{\Bbbk\mathfrak{S}_d}\left(V^{\otimes d}\right)$$

$$\sigma\left(\Bbbk\mathfrak{S}_d\right) \subset \operatorname{End}_{\mathcal{U}(\mathfrak{gl}(V))}\left(V^{\otimes d}\right).$$

Schur-Weyl duality tells us that we can say something stronger. We really have

$$\rho\left(\mathcal{U}\left(\mathfrak{gl}(V)\right)\right) = \operatorname{End}_{\Bbbk\mathfrak{S}_d}\left(V^{\otimes d}\right)$$

$$\sigma\left(\Bbbk\mathfrak{S}_d\right) = \operatorname{End}_{\mathcal{U}(\mathfrak{gl}(V))}\left(V^{\otimes d}\right),$$

so the image of each algebra under its representation equals the full centralizer algebra for the other action. See [Gre07] for a classical discussion. This gives a strong correspondence between representations of these objects. If $\Bbbk$ has characteristic 0, we can formulate this correspondence as follows: There is a decomposition of $V^{\otimes d}$ (as $(\mathcal{U}(\mathfrak{gl}_n(\Bbbk)), \Bbbk\mathfrak{S}_d)$-bimodules)

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda V \otimes V_\lambda$$

where the sum ranges over all partitions λ of d, the V_λ are *all* irreducible representations of $\mathfrak{S}_d$ (the Specht modules), and $\mathbb{S}_\lambda V \coloneqq \operatorname{Hom}_{\Bbbk\mathfrak{S}_d}\left(V_\lambda, V^{\otimes d}\right)$ is an irreducible module for $\mathrm{GL}_n(\Bbbk)$ or is zero (zero whenever the number of parts of lambda is greater than n).

Now one may ask themselves what class of irreducible modules the $\mathbb{S}_\lambda V$ account for. These are precisely the irreducible *polynomial representations of degree* d of $\mathrm{GL}_n(\Bbbk)$. A concrete definition of a finite-dimensional degree d polynomial representation of $\mathrm{GL}_n(\Bbbk)$ is as follows. Let V be a finite-dimensional $\Bbbk$-representation of $\mathrm{GL}_n(\Bbbk)$. If one chooses a $\Bbbk$-basis for V, the action of any $g \in \mathrm{GL}_n(\Bbbk)$ on V can be written as a matrix. If there exists a $\Bbbk$-basis of V such that this matrix has entries that are homogeneous degree d polynomials in the original entries of g, then V is said to be a homogeneous degree d polynomial representation.

For example, Consider $\mathrm{GL}_2(\Bbbk)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we let $\{e_1, e_2\}$ be a basis for $\Bbbk^2$, then the symmetric square $S^2(\Bbbk^2)$ has $\{e_1e_1, e_1e_2, e_2e_2\}$ as a basis (where $e_1 \otimes e_2$ and $e_2 \otimes e_1$ both correspond to e_1e_2). We leave it as an exercise to the reader to work out that g acts on $S^2(\Bbbk^2)$ via $\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$. Thus, $S^2(\Bbbk^2)$ is a homogeneous degree 2 polynomial representation of $\mathrm{GL}_2(\Bbbk)$.

These homogeneous polynomial representations can be looked at in another way. Notice that Schur-Weyl duality gives us a surjection

$$\mathcal{U}(\mathfrak{gl}(V)) \twoheadrightarrow \mathrm{End}_{\Bbbk\mathfrak{S}_d}\left(V^{\otimes d}\right) =: S(n,d)$$

where $S(n,d)$ is the *Schur algebra* which was named and whose use was emphasized in Green's monograph [Gre07]. Let $\mathsf{Pol}(n,d)$ denote the category of finite-dimensional homogeneous degree d polynomial representations of $\mathrm{GL}_n(\Bbbk)$. Let ${}_{S(n,d)}\mathsf{mod}$ denote the category of left modules for $S(n,d)$. There is an equivalence of categories

$$\mathsf{Pol}(n,d) \cong {}_{S(n,d)}\mathsf{mod}$$

[Gre07]. So studying modules for the Schur algebra is equivalent to studying polynomial representations of degree d.

Now for any n and d, certain constructions always yield polynomial representations. For example, $(\Bbbk^n)^{\otimes d}$, $S^d(\Bbbk^n)$, and $\bigwedge^d(\Bbbk^n)$ all give (homogeneous degree d) polynomial representations of $\mathrm{GL}_n(\Bbbk)$. We can view $\otimes^d$, S^d, and $\bigwedge^d$ as functors between, say, the category of finite-dimensional $\Bbbk$-vector spaces and itself. Moreover, notice that such a functor $F \in \{\otimes^d, S^d, \bigwedge^d\}$ has the property that for any objects V, W the induced map

$$F_{V,W} : \mathrm{Hom}_{\Bbbk}(V,W) \to \mathrm{Hom}_{\Bbbk}(FV, FW)$$

is a homogeneous polynomial mapping (in the sense that one can choose bases of V and W and of FV and FW such that a matrix $m \in \mathrm{Hom}_{\Bbbk}(V,W)$ is sent to a matrix $Fm \in \mathrm{Hom}_{\Bbbk}(FV,FW)$ which has entries that are homogeneous degree d polynomials in the entries of the original matrix m). So maybe studying functors with such a property is a good way to study homogeneous polynomial representations.

Call such a functor (one from the category of finite vector spaces to itself such that all induced maps on morphism spaces are homogeneous degree d polynomial mappings) a *homogeneous degree d polynomial functor*[1]. Denote the category of homogeneous degree d polynomial functors by F_d and the category of finite-dimensional $\Bbbk\mathfrak{S}_d$-modules by ${}_{\Bbbk\mathfrak{S}_d}\mathsf{mod}$. Then

$$\mathsf{F}_d \cong {}_{\Bbbk\mathfrak{S}_d}\mathsf{mod}$$

when $\Bbbk$ is a field of characteristic 0 (see, for example, Appendix A of Chapter I in [Mac95]). For $n \geqslant d$, we have ${}_{\Bbbk\mathfrak{S}_d}\mathsf{mod} \cong {}_{S(n,d)}\mathsf{mod}$ (see Section 6 of [Gre07]) so that we can write

$$\mathsf{F}_d \cong {}_{S(n,d)}\mathsf{mod},$$

and we see that studying the category of homogeneous degree d polynomial functors is equivalent to studying modules for $S(n,d)$, *for* $\Bbbk$ *a field of characteristic 0*. What about for more general fields?

[1] This notion of polynomial functor is in line with the notion of polynomality that Eilenberg and MacLane introduced in [EM54]. There are other constructions in the literature that carry the name 'polynomial functor' which are not related to this notion. See [GK13] and references therein for a taste of this.

In [FS97], Friedlander and Suslin introduced the category of *strict homogeneous degree d polynomial functors*, P_d, and showed that for $n \geqslant d$, evaluation at $\Bbbk^n$ gives an equivalence of categories

$$\mathsf{P}_d \cong {}_{S(n,d)}\mathsf{mod}.$$

In this sense, $\mathsf{P} := \bigoplus_{d \geqslant 0} \mathsf{P}_d$ encompasses all constructions which yield modules for $S(n,d)$, and hence polynomial representations. It is important to emphasize that they work over $\Bbbk$ an arbitrary field (in [SFB97], the results are extended to unital commutative rings). When $\Bbbk$ is an infinite characteristic 0 field, this recovers the result above from [Mac95]. However, there is a difference between the category F_d and P_d when $\Bbbk$ is a finite field. In general, P_d will not be equivalent to ${}_{\Bbbk\mathfrak{S}_d}\mathsf{mod}$. See [AR17] for a nice exposition on the relationship between P^d, ${}_{S(n,d)}\mathsf{mod}$, ${}_{\Bbbk\mathfrak{S}_d}\mathsf{mod}$, and the so-called *Schur functor*. The difference between F_d and P_d boils down to the fact that strict polynomial functors require the induced linear maps to be *strict* polynomial instead of just polynomial - See Appendix A for a more detailed discussion.

What makes the category P_d, and hence P, desireable is that cohomology calculations are easier than in F_d (see [Pir00] for a readable technical discussion). Morally, when studying modules for the Schur algebra, there is an explicit dependence on n and d. But a homogeneous polynomial functor only depends on d, and the category P gathers all homogeneous degree d polynomial functors together which allows one to take advantage of independence from n to perform cohomological calculations. Friedlander and Suslin use P in this way to show in [FS97] that the cohomology ring of any finite $\Bbbk$-group scheme is a finitely generated $\Bbbk$-algebra.

So from our point of view, Schur-Weyl duality and [Mac95, FS97] provide good reasons to study modules for the Schur algebra and that a good way to do so is via strict polynomial functors. Now Schur-Weyl duality has many generalizations. For some examples, see [DDH05, Jim86, BR87, Ser85] and the references therein. We will focus on 'super' generalizations.

Specifically, for this thesis, we want to handle two extra features. One is to work in the 'super' setting, and the other is to replace $\mathfrak{S}_d$ with a certain (super) algebra $A \wr \mathfrak{S}_d$ which we will define later in the thesis. By the 'super' setting, we mean that we work with $\mathbb{Z}_2$-graded $\Bbbk$-modules (which we call $\Bbbk$-supermodules). So each module V can be written in homogenous components $V = V_0 \oplus V_1$ where we say elements in V_0 are *even* (or have parity 0) and elements in V_1 are *odd* (or have parity 1). The parity of an element is denoted $\overline{x} \in \{0, 1\}$. Morphisms between $\mathbb{Z}_2$-graded $\Bbbk$-modules are just $\Bbbk$-linear maps, but we also have $\mathrm{Hom}_\Bbbk(V, W) = \mathrm{Hom}_\Bbbk(V, W)_0 \oplus \mathrm{Hom}_\Bbbk(V, W)_1$ where $\mathrm{Hom}_\Bbbk(V, W)_0$ consists of maps which preserve grading (so the parity of $f(x)$ is equal to the parity of x for $f \in \mathrm{Hom}_\Bbbk(V, W)_0$ and x a homogenous element of V) and $\mathrm{Hom}_\Bbbk(V, W)_1$ consists of maps which reverse grading (so the parity of $f(x)$ is equal to the parity of x plus 1 for $f \in \mathrm{Hom}_\Bbbk(V, W)_1$ and x a homogenous element of V).

Let $\Bbbk^{m|n}$ denote the $\Bbbk$-supermodule which has a homogeneous basis of m-many even elements and n-many odd elements. One can think of this module as column vectors of height $m+n$ with a basis given by $(0 \cdots 0\,1\,0 \cdots 0)^\top$ where the even basis elements are those column vectors with 1 in the the first m-many slots, and the odd basis elements are those column vectors with 1 in the last n-many slots.

A superalgebra A is a $\Bbbk$-supermodule, $A = A_0 \oplus A_1$, along with a bilinear multiplication $A \times A \to A$ such that $A_i A_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$. There is a super version of Lie algebras which uses analogous axioms to the usual definition, but with an extra sign that appears which is dependent on the parity of elements involved (we discuss this in detail later in the thesis). We have $\mathfrak{gl}(\Bbbk^{m|n}) = \mathrm{End}_{\Bbbk}(\Bbbk^{m|n})$ being a Lie superalgebra under the supercommutator bracket for $\mathrm{End}_{\Bbbk}(\Bbbk^{m|n})$ which is defined as $[x, y] \coloneqq xy - (-1)^{\overline{x}\cdot\overline{y}} yx$ for homogeneous elements x, y (and extended $\Bbbk$-linearly to non-homogeneous elements). We can also take the universal enveloping algebra of this Lie superalgebra, $\mathcal{U}(\mathfrak{gl}(V))$.

We have the following super versions of Schur-Weyl duality [BR87, Ser85]: Let $V = \Bbbk^{m|n}$. There are commuting actions giving rise to representations

$$\mathcal{U}(\mathfrak{gl}(V)) \xrightarrow{\rho} \mathrm{End}_{\Bbbk}\left(V^{\otimes d}\right) \xleftarrow{\sigma} \Bbbk\mathfrak{S}_d$$

such that

$$\mathcal{U}(\mathfrak{gl}(V)) \twoheadrightarrow \mathrm{End}_{\Bbbk\mathfrak{S}_d}\left(V^{\otimes d}\right),$$

where now $\mathfrak{S}_d$ acts on $V^{\otimes d}$ via *signed* place permutation which we define in detail later in the thesis.

When $m = n$, there exists an odd involution $c \in \mathrm{End}_{\Bbbk}(V)$, and one can define the type Q Lie superalgebra

$$\mathfrak{q}(n) \coloneqq \{x \in \mathfrak{gl}(V) \mid [x, c] = 0\}.$$

Restricting the representations from above gives Sergeev Duality: Let $V = \Bbbk^{n|n}$. The representations

$$\mathcal{U}(\mathfrak{q}(n)) \xrightarrow{\rho} \mathrm{End}_{\Bbbk}\left(V^{\otimes d}\right) \xleftarrow{\sigma} \mathcal{C}(1) \wr \mathfrak{S}_d$$

are such that

$$\mathcal{U}(\mathfrak{q}(n)) \twoheadrightarrow \mathrm{End}_{\mathcal{C}(1)\wr\mathfrak{S}_d}\left(V^{\otimes d}\right),$$

where $\mathcal{C}(1) = \Bbbk[c]/\langle c^2 - 1\rangle$ is the Clifford (super)algebra on one odd generator, and $\mathcal{C}(1) \wr \mathfrak{S}_d$ is the wreath product (super)algebra (which we define in detail later in the thesis) called the Sergeev superalgebra.

There are analogs of the Schur algebra appearing in both dualities. For $V = \Bbbk^{m|n}$, $\mathrm{End}_{\Bbbk\mathfrak{S}_d}\left(V^{\otimes d}\right)$ is the Schur superalgebra $S(m|n, d)$ studied in [BR87, Ser85, Don01], etc. And $\mathrm{End}_{\mathcal{C}(1)\wr\mathfrak{S}_d}\left(V^{\otimes d}\right)$ is the Schur superalgebra of type Q, $\mathcal{Q}(n, d)$, studied in [Ser85, BK02] etc.

In [Axt13], Axtell considered two super variations of the classical strict polynomial functors: $\mathsf{Pol}_d^{(\mathrm{I})}$ and $\mathsf{Pol}_d^{(\mathrm{II})}$. He proved that for $m, n \geqslant d$, evaluation at $\Bbbk^{m|n}$ and $\mathcal{C}(1)^n$ gives equivalences

$$\mathsf{Pol}_d^{(\mathrm{I})} \cong {}_{S(m|n,d)}\mathsf{smod} \quad \text{and} \quad \mathsf{Pol}_d^{(\mathrm{II})} \cong {}_{\mathcal{Q}(n,d)}\mathsf{smod},$$

respectively (where ${}_A\mathsf{smod}$ denotes the category of left supermodules for a superalgebra A). Using $\mathsf{Pol}_d^{(\mathrm{I})}$, Drupieski proved in [Dru16] that the cohomology ring of any finite $\Bbbk$-supergroup scheme is a finitely generated $\Bbbk$-algebra. Thus, we have generalizations of two of the main theorems from [FS97] to the super setting.

Axtell's constructions take advantage of the similar structure of these two generalizations of the Schur algebra. Notice that we can view super Schur-Weyl duality as having $\mathfrak{gl}_{m|n}(\Bbbk)$ and $\Bbbk \wr \mathfrak{S}_d$ acting on $(\Bbbk^{m|n})^{\otimes d}$, these actions commute, and

$$S(m|n, d) = \mathrm{End}_{\Bbbk\wr\mathfrak{S}_d}\left(\left(\Bbbk^{m|n}\right)^{\otimes d}\right) \cong \left(\mathrm{End}_{\Bbbk}(\Bbbk^{m|n})^{\otimes d}\right)^{\mathfrak{S}_d}.$$

Moreover, we have $\mathfrak{q}(n) \cong \mathfrak{gl}_n(\mathcal{C}(1))$ as Lie superalgebras, and $(\Bbbk^{n|n})^{\otimes d} \cong (\mathcal{C}(1)^n)^{\otimes d}$ as supermodules. So $\mathfrak{gl}_n(\mathcal{C}(1))$ and $\mathcal{C}(1) \wr \mathfrak{S}_d$ both act on $(\mathcal{C}(1)^n)^{\otimes d}$, these actions commute, and by Sergeev duality,

$$\mathcal{Q}(n, d) \cong \mathrm{End}_{\mathcal{C}(1)\wr\mathfrak{S}_d}\left((\mathcal{C}(1)^n)^{\otimes d}\right) \cong \left(\mathrm{End}_{\mathcal{C}(1)}(\mathcal{C}(1)^n)^{\otimes d}\right)^{\mathfrak{S}_d}.$$

So Axtell focused on taking certain superalgebras A and taking the corresponding super Schur algebra for A as being wreath product endomorphisms of the d-fold tensor product of some number of copies of the algebra. $\mathsf{Pol}_d^{(\mathrm{I})}$ uses $A = \Bbbk$ and $\mathsf{Pol}_d^{(\mathrm{II})}$ uses $A = \mathcal{C}(1)$.

Now taking this further, if we let A be any superalgebra and let $V = A^n$, then $\mathfrak{gl}(V) = \mathfrak{gl}_n(A)$ acts on the left of $V^{\otimes d}$, $A \wr \mathfrak{S}_d$ acts on the right of $V^{\otimes d}$, and these actions commute. So there are (at least) two natural candidates for a generalized Schur algebra corresponding to A. One could either take the image of $\mathcal{U}(\mathfrak{gl}(V))$ inside $\mathrm{End}_{\Bbbk}(V^{\otimes d})$ or can take $\mathrm{End}_{A\wr\mathfrak{S}_d}\left((A^n)^{\otimes d}\right)$. A Schur-Weyl type result would ensure that these two choices agreed, but they don't have to, in general. We opt for the latter choice and define

$$S^A(n,d) \coloneqq \mathrm{End}_{A\wr\mathfrak{S}_d}\left((A^n)^{\otimes d}\right) \cong \left(\mathrm{End}_A(A^n)^{\otimes d}\right)^{\mathfrak{S}_d}$$

which is the generalized Schur algebra studied by Evseev, Kleshchev, Muth, etc. [EK17, EK18, KM20]. In fact, we can extend this definition slightly:

$$S^A(m|n,d) \coloneqq \mathrm{End}_{A\wr\mathfrak{S}_d}\left((A^{m|n})^{\otimes d}\right) \cong \left(\mathrm{End}_A(A^{m|n})^{\otimes d}\right)^{\mathfrak{S}_d}$$

where $A^{m|n} \coloneqq A^m \oplus (\Pi A)^n$, and ΠA is the $\Bbbk$-supermodule which has the same underlying $\Bbbk$-module structure as A but with all parities reversed (called the parity shift of A).

It is natural to ask at this point if there are generalized polynomial functors which correspond to these generalized Schur algebras. Answering this question affirmatively is the purpose of this thesis. We will introduce two flavors of generalized polynomial functors. The first we will call *strict polynomial superfunctors* which will correspond to $S^A(m|n,d)$. We include 'super' in the name since, even if we take A to be completely even, the parity shifts present ensure a non-trivial $\mathbb{Z}_2$-grading, so this really is unavoidably super.

The second flavor we call *generalized strict polynomial functors* because they correspond to the generalized Schur algebra $T^A_{\mathfrak{a}}(n,d) \subset S^A(n,d)$ from [KM20]. We exclude 'super' from the name since there will be no parity shifts and so, for example, choosing $A = \Bbbk$ to be completely even reduces to the classical non-super case of $S(n,d)$. Note that we work with the more subtly defined $T^A_{\mathfrak{a}}(n,d)$ instead of $S^A(n,d)$. See [KM20] for a discussion of why this subalgebra is, from various viewpoints, the 'correct' generalization of the Schur algebra. What's more, taking $\mathfrak{a} = A_0$ recovers $S^A(n,d)$, so we lose no information by choosing this algebra.

This is the reason, however, we must introduce two separate (yet closely related) versions of polynomial functors. It is our opinion that when studying superalgebras and their modules, it is most natural to include parity shifts of the algebra. There is a clear way to include parity shifts of A in the definition of $S^A(n,d)$ which gives us $S^A(m|n,d)$. However, $T^A_{\mathfrak{a}}(n,d)$ doesn't allow parity shifts of A, and it is an open question to appropriately define $T^A_{\mathfrak{a}}(m|n,d)$ so that it does. Hence our two separate treatments. The reader will only notice a difference in proof technique in lemma 6.7.

It is also worth noting that in [Mac80], MacDonald gives a more general treatment of the result mentioned above in [Mac95]. He defines polynomial functors over a (non-super) $\Bbbk$-algebra A and obtains an equivalence with wreath product modules for $\Bbbk$ an infinite characteristic 0 field. See Appendix C for a discussion on how MacDonald's construction relates to our generalized strict polynomial functors.

The structure of the thesis is as follows: Sections 1 through 5 give constructions needed for defining our polynomial functor categories and proving our main results. Section 6 and its subsections introduce our categories, prove equivalence statements that relate these categories to the corresponding generalized Schur algebra, and explore some of the structure of these categories. Finally, section 7 and its subsections explore the relationship between our generalized polynomial functors and the category of webs for $\mathfrak{gl}_n(A)$. Appendix A explores the difference between strict and non-strict polynomial functors, Appendix B discusses the use of free vs projective A-supermodules in the definitions of our generalized functors, and Appendix C explores the relationship between [Mac80] and our generalized strict polynomial functors.

1. Superalgebras and Supermodules

Here we discuss the required preliminaries regarding superalgebras and their supermodules.

1.1. Basics of Superalgebras and Their Supermodules

We are interested in unital superalgebras, A, over a commutative domain, $\Bbbk$, which is not characteristic 2. By this, we mean that A is a $\Bbbk$-module (both left and right since $\Bbbk$ is commutative), has a direct sum decomposition $A = A_0 \oplus A_1$ where A_0, A_1 are free $\Bbbk$-modules, and comes equipped with a bilinear multiplication $A \times A \to A$ such that $A_i A_j \subset A_{i+j}$ for $i, j \in \mathbb{Z}_2$ (i.e. A is $\mathbb{Z}_2$-graded). Here, the parity of an element $x \in A_i$ is denoted $\overline{x} := i$. Anything with parity 0 is called even, and anything with parity 1 is called odd.

A right A-supermodule, M, is a right module in the usual sense, along with a direct sum decomposition (as free $\Bbbk$-modules) $M = M_0 \oplus M_1$ such that $M_i A_j \subset M_{i+j}$ for $i, j \in \mathbb{Z}_2$. For every supermodule M, we have a parity operator Π which is the identity on the underlying space, but which reverses the parity of all elements of M. We denote this new module ΠM. The action of A on ΠM is the same as on M. So in particular, ΠA is a right A-supermodule via right multiplication.

Given any $\Bbbk$-supermodules V and W, we have $V \oplus W$ is a $\Bbbk$-supermodule where $(V \oplus W)_0 = V_0 \oplus W_0$ and $(V \oplus W)_1 = V_1 \oplus W_1$. Moreover, $V \otimes W = V \otimes_{\Bbbk} W$ is a $\Bbbk$-supermodule where $(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)$ and $(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$. This generalizes to any finite tensor product of $\Bbbk$-supermodules. Unless otherwise stated, all tensor products are over $\Bbbk$.

Given two superalgebras A, B, we view $A \otimes B$ as a superalgebra where multiplication is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{\overline{b_1} \cdot \overline{a_2}} a_1 a_2 \otimes b_1 b_2,$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$ are homogeneous. The definition is then extended $\Bbbk$-linearly to non-homogeneous elements. We employ this technique of defining things on homogenous elements and extending linearly many times in this thesis and won't mention it explicitly again.

A homomorphism $\varphi : V \to W$ between *right* A-supermodules is a $\Bbbk$-linear map such that

$$\varphi(va) = \varphi(v)a$$

for $v \in V$ and $a \in A$. The space of right A-supermodule homomorphisms $\mathrm{Hom}_A(V, W)$ has a $\mathbb{Z}_2$-grading $\mathrm{Hom}_A(V, W) = \mathrm{Hom}_A(V, W)_0 \oplus \mathrm{Hom}_A(V, W)_1$ where $\mathrm{Hom}_A(V, W)_0$ consists of maps which preserve grading (so $\overline{\varphi(v)} = \overline{v}$ for $\varphi \in \mathrm{Hom}_A(V, W)_0$ and v a homogenous element of V) and $\mathrm{Hom}_A(V, W)_1$ consists of maps which reverse grading (so $\overline{\varphi(v)} = \overline{v} + 1$ for $\varphi \in \mathrm{Hom}_A(V, W)_1$ and v a homogenous element of V). Then smod_A denotes the category of *right* A-supermodules.

We can similarly define the category ${}_A\mathsf{smod}$ of *left* A-supermodules where a homogeneous homomorphism $\varphi : V \to W$ of left A-supermodules is a $\Bbbk$-linear map which either preserves parity (even) or reverses parity (odd) and is such that

$$\varphi(av) = (-1)^{\overline{\varphi}\cdot\overline{a}} a\varphi(v)$$

for $a \in A$ homogeneous and $v \in V$. An arbitrary homomorphism of left A-supermodules is a sum of homogeneous ones.

By a free finitely-generated (or 'finite' for brevity) right A-supermodule, M, we mean that there is an even isomorphism (as A-supermodules, where the action on the latter space is by right multiplication in A)

$$M \cong \left(\bigoplus_i^m A\right) \oplus \left(\bigoplus_j^n \Pi A\right).$$

We denote this latter space by $A^{m|n}$. In practice, this just means that having a free finite right A-supermodule means we can find a homogeneous finite rank A-basis for our module M, where we have m-many even basis elements and n-many odd basis elements. Note that $A^{\oplus m} = A^m = A^{m|0}$.

Example 1.1. Consider a free A-supermodule V which has an A-basis $\{v_1, v_2, v_3\}$ where v_1 and v_2 are both even, and v_3 is odd. This means that any element $x \in V$ can be uniquely expressed as an A-linear combination of elements from $\{v_1, v_2, v_3\}$.

On the other hand, consider $A \oplus A \oplus \Pi A$ as an A-supermodule under right multiplication. That is, $(a, b, c)d := (ad, bd, cd)$ for $a, b, d \in A$ and $c \in \Pi A$. Notice that every (a, b, c) can be written uniquely as $(1_A, 0, 0)a + (0, 1_A, 0)b + (0, 0, 1_{\Pi A})c$.

So, identifying $v_1 \mapsto (1_A, 0, 0)$, $v_2 \mapsto (0, 1_A, 0)$, and $v_3 \mapsto (0, 0, 1_{\Pi A})$ gives an isomorphism $V \cong A \oplus A \oplus \Pi A$ as A-supermodules. Note that, for example, the parity of $(0, 0, 1_{\Pi A})$ is odd since the parity of 1_A is even, and this matches the parity of v_3.

Remark 1.2. We have a natural isomorphism $M_{m_2+n_2,m_1+n_1}(A) \cong \mathrm{Hom}_A(A^{m_1|n_1}, A^{m_2|n_2})$ where the former space denotes $(m_2 + n_2) \times (m_1 + n_1)$ matrices with entries in A. Here, we consider $A^{m_1|n_1}$ as column vectors of height $m_1 + n_1$ (similarly for $A^{m_2|n_2}$), and a matrix M is associated to the map given by multiplication on the left by M.

Explicitly, it is easy to see that a matrix M determines a right A-homomorphism $A^{m_1|n_1} \to A^{m_2|n_2}$ since acting on an input corresponds to scaling each entry of the input column vector on the right, and acting on an output also corresponds to scaling entries on the right.

Conversely, given $f \in \mathrm{Hom}_A(A^{m_1|n_1}, A^{m_2|n_2})$, let f_j denote the restriction of f to the j^{th} summand of $A^{m_1|n_1}$. Then $f = \sum_{j=1}^{m_1+n_1} f_j$. Let π_i denote projection onto the i^{th} summand of $A^{m_2|n_2}$. Let f_{ij} denote the map $\pi_i \circ f_j$. Note that f_{ij} is an A-map of one of the following forms: $A \to A$, $A \to \Pi A$, $\Pi A \to A$, or $\Pi A \to \Pi A$. This depends on the indices i and j, for example, if $0 \leqslant j \leqslant m_1$ and $m_2 + 1 \leqslant i \leqslant m_2 + n_2$, then $f_{ij} : A \to \Pi A$. Regardless of whether f_{ij} lands in A or ΠA, f_{ij} is determined by its value on $1_A \in A = \Pi A$. So, forgetting parity for a moment, $f_{ij}(1_A)$ is an element in A. Notice that the i^{th} coordinate of $f(x)$ is given by $\sum_j f_{ij}(x)$. It then follows that f determines a matrix in $M_{m_2+n_2,m_1+n_1}(A)$ whose ij entry is given by $f_{ij}(1_A)$.

Denote the A-basis elements of $A^{m_1|n_1}$ and $A^{m_2|n_2}$ by x_j and y_i, respectively. Let φ^a_{ij} in $\mathrm{Hom}_A(A^{m_1|n_1}, A^{m_2|n_2})$ denote the map $x_k \mapsto \delta_{jk}(y_i)a$. Then x_k and y_i are column vectors with a 1_A in the k^{th} and i^{th} positions, respectively, and zeros elsewhere, and φ_{ij}^a is the matrix with zeros everywhere except for an a in the ij position (like a matrix unit but with an element a instead of 1_A). It follows that, for the f mentioned in the previous paragraph, $f = \sum_{ij} \varphi_{ij}^{f_{ij}(1_A)}$.

In this thesis, we will constantly make use of the fact that any A-map, f, between free finite right A-supermodules can be identified with a matrix in this way (after fixing A-bases for the source and target modules), that is, $f = \sum_{ij} \varphi_{ij}^{a_{ij}}$ for $a_{ij} \in A$. Notice that for a homogeneous, the parity of φ_{ij}^a is $\overline{x_j} + \overline{y_i} + \overline{a}$. This means that a homogeneous map f can be written as a matrix where each entry is homogeneous (the parity of which depends on its position in the matrix as well as the parity of the map f).

For square matrices, we denote $M_{m|n}(A) \coloneqq M_{m+n,m+n}(A) \cong \mathrm{End}_A(A^{m|n})$.

Next, we recall that for V, W right supermodules for superalgebras A, B, respectively, $V \otimes W$ is a right $A \otimes B$-supermodule with action on homogeneous elements given by

$$(v \otimes w).(a \otimes b) \coloneqq (-1)^{\overline{w}\cdot\overline{a}} va \otimes wb. \tag{1}$$

The analogous left action works for V, W left supermodules, as well.

Moreover, if $\varphi \in \mathrm{Hom}_A(V_1, W_1)$ and $\psi \in \mathrm{Hom}_B(V_2, W_2)$, then we may define $\varphi \boxtimes \psi \in \mathrm{Hom}_{A\otimes B}(V_1 \otimes V_2, W_1 \otimes W_2)$ via

$$\varphi \boxtimes \psi(v_1 \otimes v_2) \coloneqq (-1)^{\overline{\psi}\cdot\overline{v_1}} \varphi(v_1) \otimes \psi(v_2). \tag{2}$$

Note that this *same* formula holds for left supermodules, as well.

As in (1), for finite free right A-supermodules V, W we have a right $A \otimes A$ action on $V \otimes W$ given by

$$(x \otimes y).(a_1 \otimes a_2) \coloneqq (-1)^{\overline{y}\cdot\overline{a_1}} x.a_1 \otimes y.a_2.$$

This extends in the obvious way for finite tensor products of finite free right A-supermodules. For example, $V \otimes W \otimes X$ is a right $A \otimes A \otimes A$-supermodule with action

$$(v \otimes w \otimes x).(a_1 \otimes a_2 \otimes a_3) \coloneqq (-1)^{\overline{a_1}(\overline{w}+\overline{x})+\overline{a_2}\cdot\overline{x}} v.a_1 \otimes w.a_2 \otimes x.a_3.$$

Let us set some notation that will aid us in keeping track of signs. Let H_A denote the set of homogeneous elements of A and H_V denote the homogeneous elements of a free finite right A-supermodule, V. Let $\vec{v} \in H_V^d$ and $\vec{x} \in H_A^d$. Define

$$s(\vec{v} \to \vec{x}) \coloneqq \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \overline{x_i} \cdot \overline{v_j}. \tag{3}$$

Remark 1.3. This sign is constructed so that we have

$$(v_1 \otimes \cdots \otimes v_d).(x_1 \otimes \cdots \otimes x_d) = (-1)^{s(\vec{v}\to\vec{x})} v_1x_1 \otimes \cdots \otimes v_dx_d.$$

From now on, when it is clear from context, we will abuse notation and not explicitly make a distinction between $v_1 \otimes \cdots \otimes v_d$ and the corresponding tuple $\vec{v}$.

Moreover, we can extend this definition to other situations, as well. For example, if $\vec{\varphi}$ is a tuple of homogeneous A-maps from V to a free finite right A-supermodule W, we have

$$(\varphi_1 \boxtimes \cdots \boxtimes \varphi_d)(v_1 \otimes \cdots \otimes v_d) = (-1)^{s(\vec{\varphi}\to\vec{v})} \varphi_1(v_1) \otimes \cdots \otimes \varphi_d(v_d).$$

Next, note that for any $\Bbbk$-supermodule V, $\mathfrak{S}_d$ acts on $V^{\otimes d}$ on the right via

$$(v_1 \otimes \cdots \otimes v_d).t_i := (-1)^{\overline{v_i}\cdot\overline{v_{i+1}}} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d, \tag{4}$$

where t_i is the simple transposition swapping i and $i+1$.

From this, one can deduce that for a general element $\sigma \in \mathfrak{S}_d$, we have

$$(v_1 \otimes \cdots \otimes v_d).\sigma = (-1)^{s(\vec{v},\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},$$

where

$$s(\vec{v},\sigma) := \#\{(k,\ell) \in [1,d]^2 \mid k < \ell,\ \sigma^{-1}(k) > \sigma^{-1}(\ell),\ \overline{v_k} = 1 = \overline{v_\ell}\}, \tag{5}$$

and $\vec{v} = (v_1, \ldots, v_d)$.

Remark 1.4. It's easy to see that $s(\vec{v},\sigma\tau) = s(\vec{v},\sigma) + s(\vec{v}\sigma,\tau)$ for all $\vec{v} \in H_V^d$ and all $\sigma,\tau \in \mathfrak{S}_d$.

Now we should analyze how the parity shift of supermodules plays with tensor product. Before summarizing this in the next lemma, we set some notation. First of all, we will denote by $\pi x \in \Pi A$ the element $x \in A$ but thought of with the opposite parity. So $\overline{\pi x} = \overline{x} + 1$. Similarly, denote by $\pi(x \otimes y) \in \Pi(A \otimes A)$ the element $x \otimes y$ in $A \otimes A$ but thought of with the opposite parity. Then for example, the element $x \otimes \pi y \in A \otimes \Pi A$ denotes that $x \in A$ has the usual parity, and πy has the opposite parity of $y \in A$. So $\overline{\pi y} = \overline{y} + 1$, $\overline{x \otimes \pi y} = \overline{x} + \overline{\pi y} = \overline{x} + \overline{y} + 1$, and $\overline{\pi(x \otimes y)} = \overline{x \otimes y} + 1 = \overline{x} + \overline{y} + 1$.

Now note that for $a,b,x,y \in A$ homogeneous, we have $A \otimes A$ acting on $\Pi(A \otimes A)$ via $\pi(x \otimes y).(a \otimes b) = (-1)^{\overline{a}\cdot\overline{y}}\pi(xa \otimes yb)$ since the action on the parity shift of a module is the same as that on the original (so we just use the usual multiplication rule on $A \otimes A$ except we view the resulting element in $\Pi(A \otimes A)$). Moreover, in $A \otimes \Pi A$, we have $(x \otimes \pi y).(a \otimes b) = (-1)^{\overline{a}\cdot\overline{\pi y}} xa \otimes (\pi y)b = (-1)^{\overline{a}(\overline{y}+1)} xa \otimes \pi(yb)$. Notice that the partiy of the resulting elements in each space are the same, but the elements themselves differ potentially by a sign.

Lemma 1.5. *Let $\{A_i \mid 1 \leqslant i \leqslant n\}$ be an ordered collection of free right A-supermodules such that each $A_i = \Pi^{p_i} A$ where $p_i \in \{0,1\}$, $\Pi^0 A := A$, and $\Pi^1 A := \Pi A$. Suppose the action on each A_i is given by right multiplication and that there are an even number of $A_i = \Pi^1 A$.*

Define a map $\Phi_{\vec{p}} : \bigotimes_{i=1}^{n} A \to \bigotimes_{i=1}^{n} A_i$ on homogeneous pure tensors (which implies that each component is a homogeneous element of the corresponding A) via

$$v_1 \otimes \cdots \otimes v_n \mapsto (-1)^{\sum_{i=1}^{n-1} \overline{v_i}\cdot\alpha_i} v_1' \otimes \cdots \otimes v_n',$$

where

$$\alpha_i := \begin{cases} 0 & \text{if there are an even number of } A_j = \Pi^1 A \text{ for } j \in \{i+1,\ldots,n\} \\ 1 & \text{if there are an odd number of } A_j = \Pi^1 A \text{ for } j \in \{i+1,\ldots,n\} \end{cases},$$

$\vec{p} = (p_1,\ldots,p_n) \in \{0,1\}^n$, and v_i' is either v_i or πv_i depending on whether $A_i = \Pi^0 A$ or $A_i = \Pi^1 A$, respectively.

Then $\Phi_{\vec{p}}$ is an even isomorphism of right $A^{\otimes n}$-supermodules.

Proof. First, we have $\overline{v_i'} = \overline{v_i} + p_i$. From this, it is obvious that $\Phi_{\vec{p}}$ is an even map (preserves grading) since there are only an even number of $A_i = \Pi^1 A$, and so it follows that $\overline{v_1 \otimes \cdots \otimes v_n} = \overline{v_1' \otimes \cdots \otimes v_n'}$. Moreover, it is obvious that $\Phi_{\vec{p}}$ is a bijection since the underlying set of elements of each module are the same set, and $\Phi_{\vec{p}}$ takes a homogeneous pure tensor to $\pm$ itself.

All that remains to be checked is that $\Phi_{\vec{p}}$ is an $A^{\otimes n}$-map. It suffices to check this for homogeneous pure tensors since $\Phi_{\vec{p}}$ is defined to be $\Bbbk$-linear.

For each i, let $v_i \in A$ be homogeneous, and let $a_1 \otimes \cdots \otimes a_d \in A^{\otimes n}$ be a homogeneous pure tensor (so each $a_j \in A$ is homogeneous). Notice that each $v_i' \in A_i$ is also homogeneous. Then on one hand, we have

$$\begin{aligned}\Phi_{\vec{p}}((v_1 \otimes \cdots \otimes v_n)(a_1 \otimes \cdots \otimes a_n)) &= \Phi_{\vec{p}}\left((-1)^{s(\vec{v}\to\vec{a})} v_1a_1 \otimes \cdots \otimes v_na_n\right) \\ &= (-1)^{s(\vec{v}\to\vec{a})}\Phi_{\vec{p}}\left(v_1a_1 \otimes \cdots \otimes v_na_n\right) \\ &= (-1)^{s(\vec{v}\to\vec{a}) + \sum\limits_{i=1}^{n-1} \overline{v_ia_i}\cdot\alpha_i}(v_1a_1)' \otimes \cdots \otimes (v_na_n)'. \qquad (6)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Phi_{\vec{p}}(v_1 \otimes \cdots \otimes v_n)(a_1 \otimes \cdots \otimes a_n) &= (-1)^{\sum\limits_{i=1}^{n-1} \overline{v_i}\cdot\alpha_i}(v_1' \otimes \cdots \otimes v_n')(a_1 \otimes \cdots \otimes a_n) \\ &= (-1)^{s(\vec{v}'\to\vec{a}) + \sum\limits_{i=1}^{n-1} \overline{v_i}\cdot\alpha_i}(v_1'a_1) \otimes \cdots \otimes (v_n'a_n). \qquad (7)\end{aligned}$$

Once we show that (6) = (7), we will have our result. To this end, first notice that $(v_ia_i)'$ is equal to v_ia_i if $A_i = \Pi^0 A$ or is equal to $\pi(v_ia_i)$ if $A_i = \Pi^1 A$. On the other hand, $v_i'a_i$ is equal to v_ia_i if $A_i = \Pi^0 A$ or is equal to $(\pi v_i)a_i$ if $A_i = \Pi^1 A$. Since $\pi(v_ia_i) = (\pi v_i)a_i$, we see that the pure tensors in (6) and (7) are the same. This means we just need to check that the signs are the same.

Notice (by our definition of α_j) that

$$\begin{aligned}\overline{a_j}(\overline{v_{j+1}' + \cdots + v_n'}) &= \overline{a_j}((\overline{v_{j+1}} + p_{j+1}) + \cdots + (\overline{v_n} + p_n)) \\ &= \overline{a_j}(\overline{v_{j+1}} + \cdots + \overline{v_n}) + \overline{a_j}(p_{j+1} + \cdots + p_n) \\ &= \overline{a_j}(\overline{v_{j+1}} + \cdots + \overline{v_n}) + \overline{a_j}\cdot\alpha_j.\end{aligned}$$

It follows that $s(\vec{v}' \to \vec{a}) = s(\vec{v} \to \vec{a}) + \sum\limits_{i=1}^{n-1} \overline{a_i}\cdot\alpha_i$. Since $\overline{v_ja_j} = \overline{v_j} + \overline{a_j}$, it follows that (6) = (7), as desired. ◻

Lemma 1.6. *Let $\{A_i \mid 1 \leqslant i \leqslant n\}$ be an ordered collection of free right A-supermodules such that each $A_i = \Pi^{p_i}A$ where $p_i \in \{0,1\}$, $\Pi^0 A := A$, and $\Pi^1 A := \Pi A$. Suppose the action on each A_i is given by right multiplication and that there are an odd number of $A_i = \Pi^1 A$.*

Define a map $\Phi_{\vec{p}}^{\pi} : \Pi\left(\bigotimes\limits_{i=1}^{n} A\right) \to \bigotimes\limits_{i=1}^{n} A_i$ on homogeneous pure tensors (which implies that each component is a homogeneous element of the corresponding A) via

$$\pi(v_1 \otimes \cdots \otimes v_n) \mapsto (-1)^{\sum\limits_{i=1}^{n-1} \overline{v_i}\cdot\alpha_i} v_1' \otimes \cdots \otimes v_n',$$

where

$$\alpha_i := \begin{cases} 0 & \text{if there are an even number of } A_j = \Pi^1 A \text{ for } j \in \{i+1,\ldots,n\} \\ 1 & \text{if there are an odd number of } A_j = \Pi^1 A \text{ for } j \in \{i+1,\ldots,n\} \end{cases},$$

$\vec{p} = (p_1,\ldots,p_n) \in \{0,1\}^n$, and v_i' is either v_i or πv_i depending on whether $A_i = \Pi^0 A$ or $A_i = \Pi^1 A$, respectively.

Then $\Phi_{\vec{p}}^{\pi}$ is an even isomorphism of right $A^{\otimes n}$-supermodules.

Proof. The proof is almost identical to that of lemma 1.5, except that here, $\Phi^{\pi}_{\vec{n}}$ is grading-preserving since there are an odd number of $A_i = \Pi^1 A$. ∎

Example 1.7. Consider $\Phi^{\pi}_{(1,0,1,1)} : \Pi(A \otimes A \otimes A \otimes A) \to \Pi A \otimes A \otimes \Pi A \otimes \Pi A$. Then (suppressing $\otimes$ notation):

$$\begin{aligned}
\Phi^{\pi}_{(1,0,1,1)}(\pi(v_1v_2v_3v_4)(a_1a_2a_3a_4)) &= \Phi^{\pi}_{(1,0,1,1)}\left((-1)^{s(\vec{v}\to\vec{a})}(v_1a_1)(v_2a_2)(v_3a_3)(v_4a_4)\right) \\
&= (-1)^{s(\vec{v}\to\vec{a})}\Phi^{\pi}_{(1,0,1,1)}\left((v_1a_1)(v_2a_2)(v_3a_3)(v_4a_4)\right) \\
&= (-1)^{s(\vec{v}\to\vec{a})+\overline{v_3a_3}}\pi(v_1a_1)(v_2a_2)\pi(v_3a_3)\pi(v_4a_4)
\end{aligned}$$

versus

$$\begin{aligned}
\Phi^{\pi}_{(1,0,1,1)}(\pi(v_1v_2v_3v_4))(a_1a_2a_3a_4) &= (-1)^{\overline{v_3}}((\pi v_1)(v_2)(\pi v_3)(\pi v_4))(a_1a_2a_3a_4) \\
&= (-1)^{\gamma+\overline{v_3}}([\pi v_1]a_1)(v_2a_2)([\pi v_3]a_3)([\pi v_4]a_4)
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \overline{a_1}(\overline{v_2} + \overline{\pi v_3} + \overline{\pi v_4}) + \overline{a_2}(\overline{\pi v_3} + \overline{\pi v_4}) + \overline{a_3} \cdot \overline{\pi v_4} \\
&= \overline{a_1}(\overline{v_2} + (\overline{v_3} + 1) + (\overline{v_4} + 1)) + \overline{a_2}((\overline{v_3} + 1) + (\overline{v_4} + 1)) + \overline{a_3}(\overline{v_4} + 1) \\
&= \overline{a_1}(\overline{v_2} + \overline{v_3} + \overline{v_4}) + \overline{a_2}(\overline{v_3} + \overline{v_4}) + \overline{a_3} \cdot \overline{v_4} + \overline{a_3} \\
&= s(\vec{v} \to \vec{a}) + \overline{a_3}.
\end{aligned}$$

So we have $s(\vec{v} \to \vec{a}) + \overline{v_3a_3} = s(\vec{v} \to \vec{a}) + \overline{v_3} + \overline{a_3} = \gamma + \overline{v_3}$, and we see that $\Phi^{\pi}_{(1,0,1,1)}$ is an $A^{\otimes 4}$-map.

Proposition 1.8. *Let $V_1, \ldots, V_d$ be finite free right A-supermodules. Identify each $V_k \cong A^{m_k|n_k}$ for some nonnegative integers m_k, n_k. This amounts to choosing for each V_k a corresponding homogeneous A-basis $\{v_{k_t} \mid 1 \leqslant t \leqslant m_k + n_k\}$. Then $V_1 \otimes \cdots \otimes V_d$ is a free right $A^{\otimes d}$-supermodule with basis $\{v_{\vec{t}}\}$ where $v_{\vec{t}} = v_{1_{t_1}} \otimes v_{2_{t_2}} \otimes \cdots \otimes v_{d_{t_d}}$. Here, $\overline{v_{\vec{t}}} = \overline{v_{1_{t_1}}} + \cdots + \overline{v_{d_{t_d}}}$.*

Proof. Let's begin by setting some notation. Let $A^0 = A$ and $A^1 = \Pi A$. For $\vec{i} \in \{0,1\}^d$, let $A^{\vec{i}} = A^{i_1} \otimes \cdots \otimes A^{i_d}$ (so that each $A^{\vec{i}}$ is a length d tensor product of modules with each component being either A or ΠA). For $\vec{i} \in \{0,1\}^d$, let $\alpha_{\vec{i}} = (\alpha_{i_1})(\alpha_{i_2}) \cdots (\alpha_{i_d})$ be the product of integers where

$$\alpha_{i_j} = \begin{cases} m_j & \text{if } A^{i_j} = A \\ n_j & \text{if } A^{i_j} = \Pi A \end{cases} = \begin{cases} m_j & \text{if } i_j = 0 \\ n_j & \text{if } i_j = 1 \end{cases}.$$

Let $\{0,1\}^d_E$ denote the subset of $\{0,1\}^d$ consisting of tuples that only have an even number of components containing a 1. Let $\{0,1\}^d_O$ denote the subset of $\{0,1\}^d$ consisting of tuples that only have an odd number of components containing a 1 (so $\{0,1\}^d = \{0,1\}^d_O \sqcup \{0,1\}^d_E$).

Then (by distributivity of tensor and direct sum) we have

$$\begin{aligned}
V_1 \otimes \cdots \otimes V_d &\cong A^{m_1|n_1} \otimes \cdots \otimes A^{m_d|n_d} \\
&= \bigoplus_{\vec{i} \in \{0,1\}^d} \left(\bigoplus_{k=1}^{\alpha_{\vec{i}}} A^{\vec{i}} \right) \\
&\cong \bigoplus_{\vec{i} \in \{0,1\}^d_E} \left(\bigoplus_{k=1}^{\alpha_{\vec{i}}} A^{\otimes d} \right) \oplus \bigoplus_{\vec{j} \in \{0,1\}^d_O} \left(\bigoplus_{k=1}^{\alpha_{\vec{j}}} \Pi\left(A^{\otimes d}\right) \right) \\
&= \bigoplus_{i=1}^{M} A^{\otimes d} \oplus \bigoplus_{j=1}^{N} \Pi\left(A^{\otimes d}\right), \qquad (8)
\end{aligned}$$

where M, N are (probably large) integers which are easy to figure out, but unnecessary to write down. Moreover, the isomorphism in the third line above comes from applying the appropriate inverses of module isomorphisms from lemmas 1.5 and 1.6 to each summand. In particular, for a given $\vec{i}$, you must apply $(\Phi_{\vec{i}})^{-1}$ if $\vec{i} \in \{0,1\}^d_E$ and $\left(\Phi^{\pi}_{\vec{i}}\right)^{-1}$ if $\vec{i} \in \{0,1\}^d_O$.

Now, by definition, we see that $V_1 \otimes \cdots \otimes V_d$ is a finite free right $A^{\otimes d}$-supermodule since it is isomorphic to a finite direct sum of copies of $A^{\otimes d}$ and $\Pi\left(A^{\otimes d}\right)$. Furthermore, if one chases the maps involved, it is easy to see that the image of our claimed basis is (up to signs) the canonical basis for $\bigoplus_{i=1}^{M} A^{\otimes d} \oplus \bigoplus_{j=1}^{N} \Pi\left(A^{\otimes d}\right)$. ◻

Example 1.9. Let's trace through the calculations in proposition 1.8 in the case where we are tensoring together two free modules:

$$\begin{aligned}
V \otimes W &\cong A^{m_1|n_1} \otimes A^{m_2|n_2} \\
&= \left(\bigoplus_{i=1}^{m_1} A \oplus \bigoplus_{j=1}^{n_1} \Pi A \right) \otimes \left(\bigoplus_{k=1}^{m_2} A \oplus \bigoplus_{\ell=1}^{n_2} \Pi A \right) \\
&= \left(\bigoplus_{a=1}^{m_1 \cdot m_2} A \otimes A \right) \oplus \left(\bigoplus_{b=1}^{m_1 \cdot n_2} A \otimes \Pi A \right) \oplus \left(\bigoplus_{c=1}^{n_1 \cdot m_2} \Pi A \otimes A \right) \oplus \left(\bigoplus_{d=1}^{n_1 \cdot n_2} \Pi A \otimes \Pi A \right) \\
&= \left(\bigoplus_{a=1}^{\alpha_{(0,0)}} A^{(0,0)} \right) \oplus \left(\bigoplus_{b=1}^{\alpha_{(0,1)}} A^{(0,1)} \right) \oplus \left(\bigoplus_{c=1}^{\alpha_{(1,0)}} A^{(1,0)} \right) \oplus \left(\bigoplus_{d=1}^{\alpha_{(1,1)}} A^{(1,1)} \right) \\
&= \bigoplus_{\vec{i} \in \{0,1\}^2} \left(\bigoplus_{k=1}^{\alpha_{\vec{i}}} A^{\vec{i}} \right) \\
&\cong \left(\bigoplus_{a=1}^{m_1 \cdot m_2} A \otimes A \right) \oplus \left(\bigoplus_{b=1}^{m_1 \cdot n_2} \Pi(A \otimes A) \right) \oplus \left(\bigoplus_{c=1}^{n_1 \cdot m_2} \Pi(A \otimes A) \right) \oplus \left(\bigoplus_{d=1}^{n_1 \cdot n_2} A \otimes A \right)
\end{aligned}$$

In practice, you wouldn't need to write the red lines in the calculation above, but we've included them for comparison to the second line of equation (8). The isomorphism in the third line of (8) corresponds to going from the third line above to the sixth line above. Here, it is easy to see how to apply the inverses of isomorphisms from lemmas 1.5 and 1.6. In our example, you apply the identity map to the terms from the first chunk (corresponding to $\vec{i} =$

$(0,0)$), you apply the map $\left(\Phi^{\pi}_{(0,1)}\right)^{-1}$ to the terms from the second chunk (corresponding to $\vec{i} = (0,1)$), you apply the map $\left(\Phi^{\pi}_{(1,0)}\right)^{-1}$ to the terms from the third chunk (corresponding to $\vec{i} = (0,1)$), and you apply the map $\left(\Phi_{(1,1)}\right)^{-1}$ to the terms from the final chunk (corresponding to $\vec{i} = (1,1)$). Moreover, in this example, $M = m_1m_2 + n_1n_2$ and $N = m_1n_2 + n_1m_2$.

Lemma 1.10. *Let A be a superalgebra and V, W be finite free right A-supermodules. Then*

$$\Phi : \mathrm{Hom}_A(V,W)^{\otimes d} \to \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$$

given by

$$f_1 \otimes \cdots \otimes f_d \mapsto f_1 \boxtimes \cdots \boxtimes f_d$$

is an even isomorphism of $\Bbbk$-supermodules.

Proof. First of all, each space is clearly a $\Bbbk$-module in the obvious way. To see that Φ is even, suppose $f_1 \otimes \cdots \otimes f_d \in \mathrm{Hom}_A(V,W)^{\otimes d}$ is homogeneous. This means that each component $f_i \in \mathrm{Hom}_A(V,W)$ is a homogeneous map, and $\overline{f_1 \otimes \cdots \otimes f_d} = \overline{f_1} + \cdots + \overline{f_d}$. Now consider a homogeneous pure tensor $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$. Again, each $v_i \in V$ is homogeneous, and we have $(f_1 \boxtimes \cdots \boxtimes f_d)(v_1 \otimes \cdots \otimes v_d) = (-1)^{s(\vec{f} \to \vec{v})} f_1(v_1) \otimes \cdots \otimes f_d(v_d)$. We see that

$$\begin{aligned}\overline{f_1(v_1) \otimes \cdots \otimes f_d(v_d)} &= \overline{f_1(v_1)} + \cdots + \overline{f_d(v_d)} \\ &= \overline{f_1} + \overline{v_1} + \cdots + \overline{f_d} + \overline{v_d} \\ &= (\overline{f_1} + \cdots + \overline{f_d}) + (\overline{v_1} + \cdots + \overline{v_d}),\end{aligned}$$

and it follows that $\overline{\Phi(f_1 \otimes \cdots \otimes f_d)} = \overline{f_1 \otimes \cdots \otimes f_d}$.

Now, to show that Φ is an isomorphism, we will construct a ($\Bbbk$-linear) inverse map Ψ. Note that since Φ is defined to be $\Bbbk$-linear, once we have our inverse, we have the isomorphism as $\Bbbk$-modules. We wish to define Ψ on some special elements of $\mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$. In order to do this, choose isomorphisms $V \cong A^{m_1|n_1}$ and $W \cong A^{m_2|n_2}$ which are compatible with isomorphisms $V^{\otimes d} \cong \left(A^{\otimes d}\right)^{M_1|N_1}$ and $W^{\otimes d} \cong \left(A^{\otimes d}\right)^{M_2|N_2}$, which is possible by proposition 1.8. This is equivalent to choosing homogeneous A-bases $\{v_j\}$ for V and $\{w_i\}$ for W and then getting pure tensors of these basis elements as homogeneous $A^{\otimes d}$-bases for $V^{\otimes d}$ and $W^{\otimes d}$. In particular, we denote by $v_{\vec{j}} = v_{j_1} \otimes \cdots \otimes v_{j_d}$ a $A^{\otimes d}$-basis element of $V^{\otimes d}$ (similarly for $W^{\otimes d}$).

Then by remark 1.2, we can view both $\mathrm{Hom}_A(V,W)$ and $\mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ as matrices indexed by the appropriate bases just described. In particular, every $f \in \mathrm{Hom}_A(V,W)$ can be written as $f = \sum_{i,j} \varphi_{ij}^{a_{ij}}$ for some $a_{ij} \in A$ where the φ_{ij}^{a} are as in remark 1.2. Extending this notation, we let $\varphi_{\vec{i}\vec{j}}^{a_{\vec{i}\vec{j}}} \in \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ for $a_{\vec{i}\vec{j}} \in A^{\otimes d}$ denote the map $v_{\vec{k}} \mapsto \delta_{\vec{j}\vec{k}} w_{\vec{i}} a_{\vec{i}\vec{j}}$.

Now, note that any $g \in \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ can first be written as a sum of homogeneous maps $g = g_0 + g_1$ (where we've absorbed the $\Bbbk$-coefficients into the maps g_1 and g_2). Again by remark 1.2 and using our notation from above, for $k \in \{0,1\}$, $g_k = \sum_{\vec{i},\vec{j}} \varphi_{\vec{i}\vec{j}}^{a^k_{\vec{i}\vec{j}}}$ where each $a^k_{\vec{i}\vec{j}}$ is a homogeneous element of $A^{\otimes d}$. Now it is true that $a^k_{\vec{i}\vec{j}}$ need not be a pure tensor. However, each of these elements can be written as a sum of homogeneous pure tensors. Since there are finitely many $a^k_{\vec{i}\vec{j}}$ and each can be written as a finite sum of homogeneous pure tensors, there is a largest number of pure tensors that appears among all expansions. Let this number be

ζ. Then each expansion into sums of pure tensors can be writte as $a^k_{\vec{i}\vec{j}} = \sum_{z=1}^{\zeta} \left(a^k_{\vec{i}\vec{j}}\right)_z$ where possibly some of the $\left(a^k_{\vec{i}\vec{j}}\right)_z$ are zero. So we have $g_k = \sum_{z=1}^{\zeta} \left(\sum_{\vec{i},\vec{j}} \varphi_{\vec{i}\vec{j}}^{\left(a^k_{\vec{i}\vec{j}}\right)_z} \right)$.

Thinking of matrices, this procedure is equivalent to taking the matrix for g and first writing it as a sum of two homogeneous matrices $g = g_0 + g_1$. Then we take each homogeneous matrix and expand each of its homogeneous entries into a sum of homogeneous pure tensors. There is an entry which has the largest number of terms (which is ζ). Then each entry can be expanded into ζ-many terms (with some being zero). Then we write this matrix as a sum of ζ-many matrices in the obvious way, where now each matrix has entries consisting solely of homogeneous pure tensors from $A^{\otimes d}$.

We will define Ψ on these special elements and then extend linearly. Specifically, let's define Ψ on elements of the form $\varphi^a_{\vec{i}\vec{j}}$ where $a = a_1 \otimes \cdots \otimes a_d$ is a homogeneous pure tensor in $A^{\otimes d}$ via

$$\Psi(\varphi^a_{\vec{i}\vec{j}}) = (-1)^{s\left(\varphi^a_{\vec{i}\vec{j}} \to v_{\vec{j}}\right) + s(w_{\vec{i}} \to a)} \varphi^{a_1}_{i_1 j_1} \otimes \cdots \otimes \varphi^{a_d}_{i_d j_d}, \tag{9}$$

where we've abused notation and think of $\varphi^a_{\vec{i}\vec{j}}$ as the tuple $(\varphi^{a_1}_{i_1 j_1}, \dots, \varphi^{a_d}_{i_d j_d})$, $v_{\vec{j}}$ as the tuple $(v_{j_1}, \dots, v_{j_d})$, $w_{\vec{i}}$ as the tuple $(w_{i_1}, \dots, w_{i_d})$, and a as the tuple $(a_1, \dots, a_d)$. Explicitly, we have

$$s\left(\varphi^a_{\vec{i}\vec{j}} \to v_{\vec{j}}\right) = \sum_{r=1}^{d-1} \overline{v_{j_r}} \left(\sum_{t=r+1}^{d} \overline{\varphi^{a_t}_{i_t j_t}} \right) = \sum_{r=1}^{d-1} \overline{v_{j_r}} \left(\sum_{t=r+1}^{d} \overline{w_{i_t}} + \overline{v_{j_t}} + \overline{a_t} \right),$$

and

$$s(w_{\vec{i}} \to a) = \sum_{r=1}^{d-1} \overline{a_r} \left(\sum_{t=r+1}^{d} \overline{w_{i_t}} \right).$$

But, there is a well-definedness issue coming from the fact that there may be multiple ways to write a homogeneous $a \in A^{\otimes d}$ as a sum of (homogeneous) pure tensors.

To ensure that we really can define Ψ just on these special elements as in (9), we just need to check the following middle-linearity conditions:

- $\Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k+a'_k,\dots,a_d)}\right) = \Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k,\dots,a_d)}\right) + \Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a'_k,\dots,a_d)}\right)$ for each slot, and
- $\Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k\alpha,a_{k+1},\dots,a_d)}\right) = \Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k,\alpha a_{k+1},\dots,a_d}\right)$ for each appropriate slot for $\alpha \in \Bbbk$.

For the first bullet point, we have $\Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k+a'_k,\dots,a_d)}\right)$

$$
\begin{aligned}
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k+a'_k,\dots,a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k+a'_k,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k+a'_k}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k+a'_k,\dots a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k+a'_k,\dots a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&\quad+(-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k+a'_k,\dots a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k+a'_k,\dots a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a'_k}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k,\dots a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k,\dots a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&\quad+(-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a'_k,\dots a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a'_k,\dots a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a'_k}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= \Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k,\dots,a_d)}\right)+\Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a'_k,\dots,a_d)}\right),
\end{aligned}
$$

where the second equality follows from the obvious fact that $\varphi_{ij}^{b+c}=\varphi_{ij}^{b}+\varphi_{ij}^{c}$ for $b,c\in A$, and the third equality comes from the fact that $\overline{a_k+a'_k}=\overline{a_k}=\overline{a'_k}$ since the element $a_k+a'_k$ is homogeneous.

For the second bullet point, we have $\Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k\alpha,a_{k+1},\dots a_d)}\right)$

$$
\begin{aligned}
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k\alpha,\dots,a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k\alpha,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k\alpha}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k\alpha,\dots,a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k\alpha,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k}\alpha\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k\alpha,\dots,a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k\alpha,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k}\otimes\alpha\varphi_{i_{k+1}j_{k+1}}^{a_{k+1}}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k\alpha,\dots,a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_k\alpha,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_kj_k}^{a_k}\otimes\varphi_{i_{k+1}j_{k+1}}^{\alpha a_{k+1}}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,\alpha a_{k+1},\dots,a_d)}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,\alpha a_{k+1},\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_{k+1}j_{k+1}}^{\alpha a_{k+1}}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\\
&= \Psi\left(\varphi_{\vec{i}\vec{j}}^{(a_1,\dots,a_k,\alpha a_{k+1},\dots a_d)}\right),
\end{aligned}
$$

where the second and fourth equalities come from the easy fact that $\alpha\varphi_{ij}^{a}=\varphi_{ij}^{\alpha a}=\varphi_{ij}^{a\alpha}=\varphi_{ij}^{a}\alpha$, the third equality comes from the fact that our tensors are over $\Bbbk$ so the α can go between slots, and the fifth equality comes from the fact that $\overline{\alpha a}=\overline{a}=\overline{a\alpha}$ for all $a\in A$ and $\alpha\in\Bbbk$. Thus, we see that Ψ is well-defined. Now we just need to check that Ψ is actually the inverse to Φ.

So first, consider $\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d}\in\operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d},W^{\otimes d})$ for each $a_k\in A$ homogeneous (by our above discussion, it suffices to consider such an element). Then

$$
\begin{aligned}
(\Phi\circ\Psi)\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d}\right) &= \Phi\left((-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\otimes\cdots\otimes\varphi_{i_dj_d}^{a_d}\right)\\
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d}\to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1,\dots,a_d))}\varphi_{i_1j_1}^{a_1}\boxtimes\cdots\boxtimes\varphi_{i_dj_d}^{a_d}. \qquad (10)
\end{aligned}
$$

Now to analyze the map in (10), we can just feed in $A^{\otimes d}$-basis elements of $V^{\otimes d}$. In particular, we have $\left(\varphi_{i_1j_1}^{a_1} \boxtimes \cdots \boxtimes \varphi_{i_dj_d}^{a_d}\right)\left(v_{\vec{k}}\right)$

$$\begin{aligned}
&= (-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d} \to v_{\vec{k}}\right)} \left(\varphi_{i_1j_1}^{a_1}(v_{k_1}) \otimes \cdots \otimes \varphi_{i_dj_d}^{a_d}(v_{k_d})\right) \\
&= \delta_{\vec{j}\vec{k}}(-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d} \to v_{\vec{k}}\right)} w_{i_1}a_1 \otimes \cdots \otimes w_{i_d}a_d \\
&= \delta_{\vec{j}\vec{k}}(-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d} \to v_{\vec{k}}\right)+s(w_{\vec{i}}\to(a_1,\ldots,a_d))} (w_{i_1} \otimes \cdots \otimes w_{i_d})(a_1 \otimes \cdots \otimes a_d),
\end{aligned} \tag{11}$$

which gives zero unless $\vec{j} = \vec{k}$. Thus, it follows that $(\Phi \circ \Psi)\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d}\right) = \varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d}$ (notice that the signs cancel since we work mod 2). Thus, $\Phi\circ\Psi$ is the identity on $\mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ as desired.

To see the other direction, we first note that it suffices to consider elements of the form $\varphi_{i_1j_1}^{a_1} \otimes \cdots \otimes \varphi_{i_dj_d}^{a_d} \in \mathrm{Hom}_A(V,W)^{\otimes d}$ where the $a_k \in A$ are homogeneous. This is because we can consider homogeneous elements (all of whose component maps are homogeneous) and then given any such $f_1 \otimes \cdots \otimes f_d \in \mathrm{Hom}_A(V,W)^{\otimes d}$, each f_k can be uniquely written (as a matrix) in the form $\sum\limits_{ij} \varphi_{ij}^{a_{ij}}$ for homogeneous $a_{ij} \in A$. Then we expand and get a large sum of elements of the form $\varphi_{i_1j_1}^{a_1} \otimes \cdots \otimes \varphi_{i_dj_d}^{a_d}$. Since this process is unique, it suffice to just check on these nice elements.

So, we have

$$\begin{aligned}
(\Psi \circ \Phi)\left(\varphi_{i_1j_1}^{a_1} \otimes \cdots \otimes \varphi_{i_dj_d}^{a_d}\right) &= \Psi\left(\varphi_{i_1j_1}^{a_1} \boxtimes \cdots \boxtimes \varphi_{i_dj_d}^{a_d}\right) \\
&= \Psi\left((-1)^{s\left(\varphi_{\vec{i}\vec{j}}^{a_1\otimes\cdots\otimes a_d} \to v_{\vec{j}}\right)+s(w_{\vec{i}}\to(a_1\cdots a_d))} \varphi_{\vec{i}\vec{j}}^{(a_1\otimes\cdots\otimes a_d)}\right) \\
&= \varphi_{i_1j_1}^{a_1} \otimes \cdots \otimes \varphi_{i_dj_d}^{a_d},
\end{aligned}$$

where the second equality follows from (11), and the third equality comes from the fact that the signs cancel since we work mod 2. Thus, $\Psi \circ \Phi$ is the identity on $\mathrm{Hom}_A(V,W)^{\otimes d}$, and we see that Ψ really is an inverse to Φ. Thus, we have our desired isomorphism. ◻

1.2. A $\mathfrak{gl}_n(A)$ Action

Let $V = A^n$. Recall that $\mathfrak{gl}(V) = \mathfrak{gl}_n(A)$ is the Lie superalgebra of $n \times n$ matrices with entries in A whose Lie bracket is given by

$$[x,y] \coloneqq xy - (-1)^{\overline{x}\cdot\overline{y}}yx$$

for x, y homogeneous elements of $M_n(A)$. For any $a \in A$, let $E_{ij}^a \in \mathfrak{gl}_n(A)$ denote the usual elementary matrix but with a a instead of a 1 in the ij entry (as a map $A^n \to A^n$, $E_{ij}^a = \varphi_{ij}^a$, but we've used a different name to emphasize that we think of this as a matrix). This element has parity $\overline{a}$. For any $x \in A$, let $v_k^x \in V$ denote the column vector of height n which has x in the k^{th} position and zeros elsewhere. If x is homogeneous, then we have

$$E_{ij}^a v_k^x = \delta_{jk} v_i^{ax}, \tag{12}$$

which has parity $\overline{a} + \overline{x}$.

This yields a natural left action of $\mathfrak{gl}_n(A)$ on A^n given by usual matrix multiplication that one can easily check makes A^n into a left $\mathfrak{gl}_n(A)$-supermodule.

Using the coproduct for $\mathfrak{gl}_n(A)$, we get an action on $(A^n)^{\otimes d}$. This plays well with the right action of $\mathfrak{S}_d$ on $(A^n)^{\otimes d}$ given by signed place permutation as in (4), as we will see below.

Let $\{v_i\}$ denote the A-basis for A^n and B denote the $\Bbbk$-basis for A. For $x \in B$, let v_i^x denote the column vector with x in the i^{th} position and zeros elsewhere. Then $\{v_i^x \mid x \in B\}$ is a $\Bbbk$-basis for A^n, and pure tensors of these form a $\Bbbk$-basis for $(A^n)^{\otimes d}$.

Proposition 1.11. *$(A^n)^{\otimes d}$ is a $(\mathfrak{gl}_n(A), \mathfrak{S}_d)$-bisupermodule, and these actions commute.*

Proof. It suffices to check this on a $\Bbbk$-basis element $v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_d}^{x_d}$ of $(A^n)^{\otimes d}$ where each $x_i \in B$. Similarly, since $\{E_{ij}^b \mid b \in B\}$ spans $\mathfrak{gl}_n(A)$, and the simple transpositions t_k span $\mathfrak{S}_d$, it suffices to consider these elements, too.

A general term from $\left(E_{ij}^b.(v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_d}^{x_d})\right).t_k$ looks like

$$(-1)^{\bar{b}(\overline{x_1}+\cdots+\overline{x_{s-1}})}\left(v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_{s-1}}^{x_{s-1}} \otimes E_{ij}^b v_{r_s}^{x_s} \otimes v_{r_{s+1}}^{x_{s+1}} \otimes \cdots \otimes v_{r_d}^{x_d}\right).t_k \tag{13}$$

for some $1 \leqslant s \leqslant d$. We will focus on the interesting case when $s = k$ and leave the other calculations to the reader.

In this case, notice that $\left(v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_{s-1}}^{x_{s-1}} \otimes E_{ij}^b v_{r_s}^{x_s} \otimes v_{r_{s+1}}^{x_{s+1}} \otimes \cdots \otimes v_{r_d}^{x_d}\right).t_k$ equals

$$\delta_{j,r_k}\left(v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_{k-1}}^{x_{k-1}} \otimes v_i^{bx_k} \otimes v_{r_{k+1}}^{x_{k+1}} \otimes \cdots \otimes v_{r_d}^{x_d}\right).t_k,$$

which equals

$$\delta_{j,r_k}(-1)^{(\bar{b}+\overline{x_k})\overline{x_{k+1}}}\left(v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_{k-1}}^{x_{k-1}} \otimes v_{r_{k+1}}^{x_{k+1}} \otimes v_i^{bx_k} \otimes \cdots \otimes v_{r_d}^{x_d}\right).$$

So in this setting, (13) becomes

$$\delta_{j,r_k}(-1)^{\bar{b}(\overline{x_1}+\cdots+\overline{x_{k-1}})+(\bar{b}+\overline{x_k})\overline{x_{k+1}}}\left(v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_{k-1}}^{x_{k-1}} \otimes v_{r_{k+1}}^{x_{k+1}} \otimes v_i^{bx_k} \otimes \cdots \otimes v_{r_d}^{x_d}\right). \tag{14}$$

The relevant corresponding term from $E_{ij}^b.\left((v_{r_1}^{x_1} \cdots v_{r_d}^{x_d}).t_k\right)$ is the following:

$$\delta_{j,r_k}\left((-1)^{\overline{x_k}\cdot\overline{x_{k+1}}+\bar{b}(\overline{x_1}+\cdots+\overline{x_{k-1}}+\overline{x_{k+1}})} v_{r_1}^{x_1} \otimes \cdots \otimes v_{r_{k-1}}^{x_{k-1}} \otimes v_{r_{k+1}}^{x_{k+1}} \otimes v_i^{bx_k} \otimes \cdots \otimes v_{r_d}^{x_d}\right). \tag{15}$$

It is clear that (14) = (15), and our result follows. ∎

1.3. The Opposite Superalgebra

Given any superalgebra A, one may form its opposite superalgebra A^{sop} as follows. As a $\mathbb{Z}_2$-graded free $\Bbbk$-supermodule, $A^{\mathsf{sop}} = A$ so that $A_0^{\mathsf{sop}} = A_0$ and $A_1^{\mathsf{sop}} = A_1$. The only difference is in the multiplication. Let $\bullet$ denote the multiplication in A^{sop}. Then we have

$$x \bullet y \coloneqq (-1)^{\overline{x}\cdot\overline{y}} yx$$

where yx is the product in A.

Proposition 1.12. *There exist well-defined covariant functors $(\_\_)^{\mathsf{sop}} : \mathsf{smod}_A \to {}_{A^{\mathsf{sop}}}\mathsf{smod}$ and $(\_\_)^{\mathsf{sop}} : {}_A\mathsf{smod} \to \mathsf{smod}_{A^{\mathsf{sop}}}$.*

Proof. First of all, if M is any right A-supermodule, we can define a left A^{sop}-supermodule structure on M via

$$a.m \coloneqq (-1)^{\overline{a}\cdot\overline{m}} ma \tag{16}$$

where ma denotes the right action of a on m. The only axiom which may not be immediately obvious is associativity, so we check that here. Let a, b be homogeneous elements of A and

m a homogeneous element of M. Then

$$\begin{aligned}(a \bullet b).m = (-1)^{\overline{a}\cdot\overline{b}}(ba).m &= (-1)^{\overline{a}\cdot\overline{b}+\overline{ba}\cdot\overline{m}}m(ba)\\ &= (-1)^{\overline{a}\cdot\overline{b}+\overline{b}\cdot\overline{m}+\overline{a}\cdot\overline{m}}(mb)a,\end{aligned} \tag{17}$$

whereas

$$\begin{aligned}a.(b.m) &= a.\left((-1)^{\overline{b}\cdot\overline{m}}mb\right)\\ &= (-1)^{\overline{b}\cdot\overline{m}+\overline{a}\cdot\overline{mb}}(mb)a\\ &= (-1)^{\overline{b}\cdot\overline{m}+\overline{a}\cdot\overline{m}+\overline{a}\cdot\overline{b}}(mb)a.\end{aligned} \tag{18}$$

Since (17) = (18), we have what we wanetd.

So the functor $(\_\_)^{\mathsf{sop}}$ sends an object M to the left supermodule defined in (16) which we will denote M^{sop}. The functor is the identity on morphisms. To check that this makes sense, suppose f is a right A-supermodule homomorphism from $M \to N$. Then

$$\begin{aligned}f(a.m) &= f\left((-1)^{\overline{a}\cdot\overline{m}}ma\right)\\ &= (-1)^{\overline{a}\cdot\overline{m}}f(m)a\\ &= (-1)^{\overline{a}\cdot\overline{m}+\overline{a}\cdot\overline{f(m)}}a.f(m)\\ &= (-1)^{\overline{a}\cdot\overline{f}}a.f(m),\end{aligned} \tag{19}$$

where the second line comes from f being a right A-map, and the last line comes from $\overline{f(m)} = \overline{f} + \overline{m}$. Hence, f really is a left A^{sop}-map. Composition and identites obviously work out, so we have $(\_\_)^{\mathsf{sop}} : \mathsf{smod}_A \to {}_{A^{\mathsf{sop}}}\mathsf{smod}$.

Now if M is a left A-supermodule, then M^{sop} is the right A^{sop}-supermodule with action given by

$$m.a \coloneqq (-1)^{\overline{m}\cdot\overline{a}}am, \tag{20}$$

where am denotes the left action of a on m. Analogous computations to those above show that $(\_\_)^{\mathsf{sop}} : {}_A\mathsf{smod} \to \mathsf{smod}_{A^{\mathsf{sop}}}$ is also well-defined. ∎

Proposition 1.13. $(\_\_)^{\mathsf{sop}} : \mathsf{smod}_A \to {}_{A^{\mathsf{sop}}}\mathsf{smod}$ *and* $(\_\_)^{\mathsf{sop}} : {}_A\mathsf{smod} \to \mathsf{smod}_{A^{\mathsf{sop}}}$ *are equivalences of categories.*

Proof. It is obvious that $(A^{\mathsf{sop}})^{\mathsf{sop}} = A$, and it is immediate that the following diagram commutes:

$$\begin{tikzcd} \mathsf{smod}_A \arrow[r, "(\_\_)^{\mathsf{sop}}"] \arrow[rr, bend right, "\mathrm{id}"'] & {}_{A^{\mathsf{sop}}}\mathsf{smod} \arrow[r, "(\_\_)^{\mathsf{sop}}"] \arrow[rr, bend left, "\mathrm{id}"] & \mathsf{smod}_A \arrow[r, "(\_\_)^{\mathsf{sop}}"] & {}_{A^{\mathsf{sop}}}\mathsf{smod} \end{tikzcd}$$

Hence $\mathsf{smod}_A \cong {}_{A^{\mathsf{sop}}}\mathsf{smod}$. Similarly, ${}_A\mathsf{smod} \cong \mathsf{smod}_{A^{\mathsf{sop}}}$. ∎

Next, we want to work with square matrices with entries in A and A^{sop}. Let $X * Y$ denote the product in $M_n(A^{\mathsf{sop}})$ (which is usual matrix multiplication, but the multiplication of entries happens in A^{sop}). Precisely, if $X = (x_{ij})$ and $Y = (y_{ij})$, we have

$$(X * Y)_{rs} = \sum_{t=1}^{n} x_{rt} \bullet y_{ts}.$$

Let X^{T} denote the transpose of a matrix X.

Lemma 1.14. *For $X, Y \in M_n(A)$, we have*

$$(XY)^{\mathsf{T}} = (-1)^{\overline{X}\cdot\overline{Y}} Y^{\mathsf{T}} * X^{\mathsf{T}}$$

and

$$(X * Y)^{\mathsf{T}} = (-1)^{\overline{X}\cdot\overline{Y}} Y^{\mathsf{T}} X^{\mathsf{T}}.$$

Proof. Let $X = (x_{ij})$ and $Y = (y_{ij})$. We have

$$(XY)_{rs} = \sum_{t=1}^{n} x_{rt} y_{ts}$$

so that

$$(XY)^{\mathsf{T}}_{rs} = \sum_{t=1}^{n} x_{st} y_{tr}. \tag{21}$$

On the other hand, we have

$$\begin{aligned}
(Y^{\mathsf{T}} * X^{\mathsf{T}})_{rs} &= \sum_{t=1}^{n} y^{\mathsf{T}}_{rt} \bullet x^{\mathsf{T}}_{ts} \\
&= \sum_{t=1}^{n} y_{tr} \bullet x_{st} \\
&= \sum_{t=1}^{n} (-1)^{\overline{y_{tr}}\cdot\overline{x_{st}}} x_{st} y_{tr} \\
&= (-1)^{\overline{Y}\cdot\overline{X}} \sum_{t=1}^{n} x_{st} y_{tr}, \qquad (22)
\end{aligned}$$

where the last line follows from the fact that a homogeneous matrix has all of its entries being homogeneous of the same parity. Comparing (21) and (22), we have our first identity. The second identity is proved with similar calculations. $\square$

Next, we'll consider $\mathfrak{gl}_n(A^{\mathsf{sop}})$ which is a Lie superalgebra with the usual supercommutator bracket but where the individual matrix multiplications are taking place in $M_n(A^{\mathsf{sop}})$ so that we have

$$[X, Y] := X * Y - (-1)^{\overline{X}\cdot\overline{Y}} Y * X.$$

Proposition 1.15. *There exist well-defined functors $\mathcal{T} : \mathsf{smod}_{\mathfrak{gl}_n(A)} \to {}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}\mathsf{smod}$ and $\mathcal{T} : {}_{\mathfrak{gl}_n(A)}\mathsf{smod} \to \mathsf{smod}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}$.*

Proof. Given M a right $\mathfrak{gl}_n(A)$ module, M can be equipped with the following left $\mathfrak{gl}_n(A^{\mathsf{sop}})$ action:

$$X.m := (-1)^{\overline{m}\cdot\overline{X}} m X^{\mathsf{T}}, \tag{23}$$

where mX^T denotes the right action of the element X^T on m. To see that this is actually a Lie superalgebra action, note that

$$\begin{aligned}
[X,Y].m &= (X * Y - (-1)^{\overline{X}\cdot\overline{Y}} Y * X).m \\
&= (-1)^{(\overline{X}+\overline{Y})\cdot\overline{m}} m(X * Y - (-1)^{\overline{X}\cdot\overline{Y}} Y * X)^\mathsf{T} \\
&= (-1)^{(\overline{X}+\overline{Y})\cdot\overline{m}} m((-1)^{\overline{X}\cdot\overline{Y}} Y^\mathsf{T} X^\mathsf{T} - X^\mathsf{T} Y^\mathsf{T}) \\
&= (-1)^{(\overline{X}+\overline{Y})\cdot\overline{m}+\overline{X}\cdot\overline{Y}} (mY^\mathsf{T})X^\mathsf{T} - (-1)^{(\overline{X}+\overline{Y})\cdot\overline{m}} (mX^\mathsf{T})Y^\mathsf{T}, \qquad (24)
\end{aligned}$$

where the second line comes from the fact that $\overline{[X,Y]} = \overline{X} + \overline{Y}$ whether viewed over A or A^{sop}, and the third line follows from lemma 1.14.

On the other hand, we have

$$\begin{aligned}
X.(Y.m) - (-1)^{\overline{X}\cdot\overline{Y}} Y.(X.m) &= X.((-1)^{\overline{Y}\cdot\overline{m}} mY^\mathsf{T}) - (-1)^{\overline{X}\cdot\overline{Y}} Y.((-1)^{\overline{X}\cdot\overline{m}} mX^\mathsf{T}) \\
&= (-1)^{\overline{Y}\cdot\overline{m}+\overline{X}(\overline{m}+\overline{Y})} (mY^\mathsf{T})X^\mathsf{T} - (-1)^{\overline{X}\cdot\overline{Y}+\overline{X}\cdot\overline{m}+\overline{Y}(\overline{m}+\overline{X})} (mX^\mathsf{T})Y^\mathsf{T} \qquad (25)
\end{aligned}$$

where the second line comes from the fact that the parity of the transpose is clearly the parity of the original matrix. Since (24) = (25), we have our result. Then $\mathcal{T}M$ is the left $\mathfrak{gl}_n(A^{\mathsf{sop}})$ module with action as above.

The same calculation as (19) shows that $\mathcal{T}$ is the identity on homs and that this makes sense.

Now for any left $\mathfrak{gl}_n(A)$ module M, define a right $\mathfrak{gl}_n(A^{\mathsf{sop}})$ action via

$$m.X \coloneqq (-1)^{\overline{m}\cdot\overline{X}} X^\mathsf{T} m. \qquad (26)$$

Analogous calculations as above show that this action gives a right $\mathfrak{gl}_n(A^{\mathsf{sop}})$ module $\mathcal{T}M$. ∎

Proposition 1.16. $\mathcal{T} : \mathsf{smod}_{\mathfrak{gl}_n(A)} \to {}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}\mathsf{smod}$ *and* $\mathcal{T} : {}_{\mathfrak{gl}_n(A)}\mathsf{smod} \to \mathsf{smod}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}$ *are equivalences of categories.*

Proof. Since $(A^{\mathsf{sop}})^{\mathsf{sop}} = A$, it is immediate that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{smod}_{\mathfrak{gl}_n(A)} \arrow[r, "\mathcal{T}"] \arrow[rr, bend right, "\mathrm{id}"'] & {}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}\mathsf{smod} \arrow[r, "\mathcal{T}"] \arrow[rr, bend left, "\mathrm{id}"] & \mathsf{smod}_{\mathfrak{gl}_n(A)} \arrow[r, "\mathcal{T}"] & {}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}\mathsf{smod}
\end{tikzcd}$$

Hence $\mathsf{smod}_{\mathfrak{gl}_n(A)} \cong {}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}\mathsf{smod}$. Similarly, ${}_{\mathfrak{gl}_n(A)}\mathsf{smod} \cong \mathsf{smod}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}$. ∎

Remark 1.17. Let V_n denote the right $\mathfrak{gl}_n(A^{\mathsf{sop}})$ module which is row vectors whose entries are in A^{sop} with the action given by matrix multiplication on the right. Let A^n denote the left $\mathfrak{gl}_n(A)$ module which is column vectors with entries in A and action given by left matrix multiplication.

There is a canonical $\Bbbk$-module isomorphism $\mathsf{t} : V_n \to A^n$. We claim that $\mathsf{t}(\mathcal{T}V_n) = A_n$ as left $\mathfrak{gl}_n(A)$ modules, that is, V_n can be identified with A_n under $\mathcal{T}$.

To see this claim, first consider the right action on V_n. Let X be a homogeneous element of $\mathfrak{gl}_n(A^{\mathsf{sop}})$ and $\vec{v} = (v_1, \dots, v_n)$ be a homogeneous element of V_n. Note that the r^{th} component of $\vec{v}X$ is

$$\sum_{t=1}^{n} v_t \bullet x_{tr}.$$

Then on $\mathcal{T}V_n$, X acts as $X.\vec{v} = (-1)^{\overline{\vec{v}}\cdot\overline{X}}\vec{v}X^{\mathsf{T}}$ whose r^{th} component is

$$\begin{aligned}(-1)^{\overline{\vec{v}}\cdot\overline{X}}\sum_{t=1}^{n} v_t \bullet x_{rt} &= (-1)^{\overline{\vec{v}}\cdot\overline{X}}\sum_{t=1}^{n}(-1)^{\overline{v_t}\cdot\overline{x_{rt}}}x_{rt}v_t\\ &= (-1)^{\overline{\vec{v}}\cdot\overline{X}}\sum_{t=1}^{n}(-1)^{\overline{\vec{v}}\cdot\overline{X}}x_{rt}v_t\\ &= \sum_{t=1}^{n} x_{rt}v_t,\end{aligned}$$

which is clearly the r^{th} component of $X\begin{pmatrix} v_1\\ \vdots \\ v_n\end{pmatrix}$, so we have our claim. Note that the second line above follows from the fact that $\vec{v}$ being homogeneous means each of its entries are homogeneous of the same degree.

Moreover, say $f : V_n \to M$ is a map of right $\mathfrak{gl}_n(A^{\mathsf{sop}})$ modules. Then as a (free) $\Bbbk$-module map, f is determined by its values on a $\Bbbk$-basis for V_n. Letting B be a $\Bbbk$-basis for A, we have $\{v_i^b \mid b \in B\}$ being a $\Bbbk$-basis for V_n. Since $\mathcal{T}$ is identity on morphisms, f is unchanged as a map $\mathcal{T}V_n \to \mathcal{T}M$. Let $\{w_i^b\}$ be a $\Bbbk$-basis for A^n (so the map $\mathtt{t}$ above identifies v_i^b with w_i^b). Let $f' : A^n \to \mathcal{T}M$ be given by $f'(w_i^b) = f(v_i^b)$. Then under our identification, the functor $\mathcal{T}$ sends V_n to A^n and maps $f : V_n \to M$ to maps $f' : A^n \to \mathcal{T}M$. From this point on, we will not make a distinction between the map f and f'.

2. More $\mathfrak{S}_d$-actions

Lemma 2.1. *Let V be a right finite free A-supermodule so that $V^{\otimes d}$ is a right $A^{\otimes d}$-supermodule as in* (1)*. For any $v \in V^{\otimes d}$, $a \in A^{\otimes d}$, and $\sigma \in \mathfrak{S}_d$, we have*

$$(v.a).\sigma = (v.\sigma).(a.\sigma).$$

Proof. Without loss of generality, we show this for $\sigma = t_i$ a simple transposition and homogeneous elements. So $[(v_1 \otimes \cdots \otimes v_d).(a_1 \otimes \cdots \otimes a_d)].t_i$

$$\begin{aligned}&= (-1)^{s(\vec{v}\to\vec{a})}(v_1.a_1 \otimes \cdots \otimes v_d.a_d).t_i\\ &= (-1)^{s(\vec{v}\to\vec{a})+\overline{v_i.a_i}\cdot\overline{v_{i+1}.a_{i+1}}}v_1.a_1 \otimes \cdots \otimes v_{i+1}.a_{i+1} \otimes v_i.a_i \otimes \cdots \otimes v_d.a_d. \qquad (27)\end{aligned}$$

On the other hand, we have $[(v_1 \otimes \cdots \otimes v_d).t_i].[(a_1 \otimes \cdots \otimes a_d).t_i]$

$$\begin{aligned}&= \left[(-1)^{\overline{v_i}\cdot\overline{v_{i+1}}}v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d\right].\left[(-1)^{\overline{a_i}\cdot\overline{a_{i+1}}}a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_d\right]\\ &= (-1)^{\overline{v_i}\cdot\overline{v_{i+1}}+\overline{a_i}\cdot\overline{a_{i+1}}+s(\vec{v}t_i\to\vec{a}t_i)}v_1.a_1 \otimes \cdots \otimes v_{i+1}.a_{i+1} \otimes v_i.a_i \otimes \cdots \otimes v_d.a_d. \qquad (28)\end{aligned}$$

Notice that $s(\vec{v}t_i \to \vec{a}t_i)$ equals

$$\overline{a_1}(\overline{v_2} + \cdots + \overline{v_d}) + \cdots + \overline{a_{i+1}}(\overline{v_i} + \overline{v_{i+2}} + \cdots + \overline{v_d}) + \overline{a_i}(\overline{v_{i+2}} + \cdots + \overline{v_d}) + \cdots + \overline{a_{d-1}}\overline{v_d}.$$

It follows that $s(\vec{v}t_i \to \vec{a}t_i) = s(\vec{v} \to \vec{a}) - \overline{a_i} \cdot \overline{v_{i+1}} + \overline{a_{i+1}} \cdot \overline{v_i}$. Moreover, we have

$$\begin{aligned}\overline{v_i.a_i} \cdot \overline{v_{i+1}.a_{i+1}} &= (\overline{v_i} + \overline{a_i})(\overline{v_{i+1}} + \overline{a_{i+1}}) \\ &= \overline{v_i} \cdot \overline{v_{i+1}} + \overline{v_i} \cdot \overline{a_{i+1}} + \overline{a_i} \cdot \overline{v_{i+1}} + \overline{a_i} \cdot \overline{a_{i+1}},\end{aligned}$$

and it follows that (27)=(28), as required. ∎

Definition 2.2. The **wreath product** $A \wr \mathfrak{S}_d$ is the superalgebra which as a $\Bbbk$-supermodule is $\Bbbk\mathfrak{S}_d \otimes A^{\otimes d}$ (where $\Bbbk\mathfrak{S}_d$ is concentrated in degree 0) with multiplication given by

$$(\sigma \otimes a)(\sigma' \otimes a') = \sigma\sigma' \otimes (a\sigma')a'$$

for any $\sigma \in \mathfrak{S}_d$ and $a \in A^{\otimes d}$. We identify $\Bbbk\mathfrak{S}_d$ with $\Bbbk\mathfrak{S}_d \otimes 1$ and $A^{\otimes d}$ with $1 \otimes A^{\otimes d}$.

Then for any right A-supermodule V, $V^{\otimes d}$ is a right $A \wr \mathfrak{S}_d$-supermodule via

$$v.(\sigma \otimes a) := (v.\sigma).a \tag{29}$$

for $v \in V^{\otimes d}$, $a \in A^{\otimes d}$, $\sigma \in \mathfrak{S}_d$. To see this is well-defined, it suffices to check the multiplication relation for $A \wr \mathfrak{S}_d$ since the $\mathfrak{S}_d$ and $A^{\otimes d}$ actions are already well-defined. So note that

$$\begin{aligned}(v.(\sigma_1 \otimes a_1)).(\sigma_2 \otimes a_2) &= ((v.\sigma_1).a_1).(\sigma_2 \otimes a_2) \\ &= [((v.\sigma_1).a_1).\sigma_2].a_2 \\ &= [((v.\sigma_1).\sigma_2).(a_1.\sigma_2)].a_2,\end{aligned} \tag{30}$$

where this last equality follows from lemma 2.1 (since $v.\sigma_1 \in V^{\otimes d}$). On the other hand,

$$\begin{aligned}v.(\sigma_1\sigma_2 \otimes (a_1.\sigma_2)a_2) &= (v.\sigma_1\sigma_2).[(a_1.\sigma_2)a_2] \\ &= ((v.\sigma_1).\sigma_2).[(a_1.\sigma_2)a_2] \\ &= [((v.\sigma_1).\sigma_2).(a_1.\sigma_2)].a_2,\end{aligned} \tag{31}$$

where the last two lines hold because of the well-definedness of the $\mathfrak{S}_d$ and $A^{\otimes d}$ actions, respectively. So (30)=(31), as desired.

Now we define a right action of $\mathfrak{S}_d$ on $\mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ as follows: For $\sigma \in \mathfrak{S}_d$ and $\varphi \in \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$, define

$$\varphi.\sigma := \varphi^\sigma \qquad \text{where} \qquad \varphi^\sigma(v) := (\varphi(v.\sigma^{-1})).\sigma \quad \forall v \in V^{\otimes d} \tag{32}$$

Clearly, $1 \in \mathfrak{S}_d$ acts as identity, and for $\sigma, \alpha \in \mathfrak{S}_d$, we have

$$\begin{aligned}(\varphi^\sigma)^\alpha(v) &= \left(\varphi^\sigma(v.\alpha^{-1})\right).\alpha \\ &= \left(\varphi\left((v.\alpha^{-1}).\sigma^{-1}\right).\sigma\right).\alpha \\ &= \left(\varphi\left((v.(\alpha^{-1}\sigma^{-1})\right).\sigma\right).\alpha \\ &= \left(\varphi\left((v.(\sigma\alpha)^{-1})\right).\sigma\right).\alpha \\ &= \varphi\left((v.(\sigma\alpha)^{-1})\right).(\sigma\alpha) \\ &= \varphi^{\sigma\alpha}(v).\end{aligned}$$

This shows our action is well-defined.

Remark 2.3. Notice that a map $f \in \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ is invariant under the $\mathfrak{S}_d$ action from (32) if and only if $f(v.\sigma) = f(v).\sigma$ for all $\sigma \in \mathfrak{S}_d$ and $v \in V^{\otimes d}$.

Lemma 2.4. *For V, W finite free right A-supermodules, we have*

$$\mathrm{Hom}_A(V,W)^{\otimes d} \cong \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$$

as $\mathfrak{S}_d$-supermodules.

Proof. By lemma 1.10, we have the isomorphism as $\Bbbk$-supermodules. Denote this isomorphism Φ. Then we wish to show Φ is also an $\mathfrak{S}_d$-map. We will show this on homogeneous pure tensors and generators (simple transpositions) of $\mathfrak{S}_d$. Let $f_j \in \mathrm{Hom}_A(V,W)$ and $v_j \in V$ for $1 \leqslant j \leqslant d$. We wish to show that acting by t_i then applying Φ is the same as applying Φ then acting by t_i. The former is as follows (we omit the $\otimes$ and $\boxtimes$ symbols for brevity):

$$\begin{aligned}\left[(f_1 \cdots f_d).t_i\right](v_1 \cdots v_d) &= (-1)^{\overline{f_i}\cdot\overline{f_{i+1}}}(f_1 \cdots f_{i+1} f_i \cdots f_d)(v_1 \cdots v_d) \\ &= (-1)^{\overline{f_i}\cdot\overline{f_{i+1}} + s(\vec{f}t_i \to \vec{v})} f_1(v_1) \cdots f_{i+1}(v_i) f_i(v_{i+1}) \cdots f_d(v_d). \end{aligned} \tag{33}$$

The latter is given by

$$\begin{aligned}(f_1 \cdots f_d)^{t_i}(v_1 \cdots v_d) &= \left[(f_1 \cdots f_d)\left((v_1 \cdots v_d).t_i^{-1}\right)\right].t_i \\ &= \left[(f_1 \cdots f_d)\left((-1)^{\overline{v_i}\cdot\overline{v_{i+1}}} v_1 \cdots v_{i+1} v_i \cdots v_d\right)\right].t_i \\ &= \left[(-1)^{\overline{v_i}\cdot\overline{v_{i+1}} + s(\vec{f} \to \vec{v}t_i)} f_1(v_1) \cdots f_i(v_{i+1}) f_{i+1}(v_i) \cdots f_d(v_d)\right].t_i \\ &= (-1)^{\overline{v_i}\cdot\overline{v_{i+1}} + s(\vec{f} \to \vec{v}t_i) + \overline{f_{i+1}(v_i)}\cdot\overline{f_i(v_{i+1})}} f_1(v_1) \cdots f_{i+1}(v_i) f_i(v_{i+1}) \cdots f_d(v_d). \end{aligned} \tag{34}$$

Notice that

$$s(\vec{f}t_i \to \vec{v}) = \overline{f_d}(\overline{v_1} + \cdots + \overline{v_{d-1}}) + \cdots + \overline{f_i}(\overline{v_1} + \cdots + \overline{v_{i-1}} + \overline{v_i}) + \overline{f_{i+1}}(\overline{v_1} + \cdots + \overline{v_{i-1}}) + \cdots + \overline{f_2}\cdot\overline{v_1}$$

and

$$s(\vec{f} \to \vec{v}t_i) = \overline{f_d}(\overline{v_1} + \cdots + \overline{v_{d-1}}) + \cdots + \overline{f_{i+1}}(\overline{v_1} + \cdots + \overline{v_{i-1}} + \overline{v_{i+1}}) + \overline{f_i}(\overline{v_1} + \cdots + \overline{v_{i-1}}) + \cdots + \overline{f_2}\cdot\overline{v_1},$$

so we see that $s(\vec{f} \to \vec{v}t_i) = s(\vec{f}t_i \to \vec{v}) - \overline{f_i}\cdot\overline{v_i} + \overline{f_{i+1}}\cdot\overline{v_{i+1}}$. Hence (working mod 2),

$$\begin{aligned}\overline{v_i}\cdot\overline{v_{i+1}} + s(\vec{f} \to \vec{v}t_i) + \overline{f_{i+1}(v_i)}\cdot\overline{f_i(v_{i+1})} &= \overline{v_i}\cdot\overline{v_{i+1}} + s(\vec{f} \to \vec{v}t_i) + (\overline{f_{i+1}} + \overline{v_i})(\overline{f_i} + \overline{v_{i+1}}) \\ &= s(\vec{f} \to \vec{v}t_i) + \overline{f_{i+1}}\cdot\overline{f_i} + \overline{f_{i+1}}\cdot\overline{v_{i+1}} + \overline{v_i}\cdot\overline{f_i} \\ &= s(\vec{f}t_i \to \vec{v}) + \overline{f_{i+1}}\cdot\overline{f_i}.\end{aligned}$$

Thus, (33)=(34), as required. ∎

3. Category of Divided Powers

For a superalgebra A, let V denote the category of finite free right A-supermodules (including parity shifts). Recall that for a $\Bbbk$-supermodule V, the d^{th} divided powers of V is the $\Bbbk$-supermodule $\Gamma^d(V) \coloneqq \left(V^{\otimes d}\right)^{\mathfrak{S}_d}$, the space of $\mathfrak{S}_d$-invariants, where $\mathfrak{S}_d$ acts on $V^{\otimes d}$ as in (4).

Definition 3.1. Let $\Gamma^d\mathsf{V}$ denote the **category of d^{th} divided powers** whose objects are the same as those for V and whose morphism spaces are defined to be

$$\mathrm{Hom}_{\Gamma^d\mathsf{V}}(V,W) \coloneqq \Gamma^d\,\mathrm{Hom}_A(V,W).$$

In order to define composition, we make use of the following:

Lemma 3.2. *For V, W in V, there is an isomorphism*

$$\Gamma^d \operatorname{Hom}_A(V, W) \cong \operatorname{Hom}_{A \wr \mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d})$$

Proof. Recall that $A \wr \mathfrak{S}_d$ acts on $V^{\otimes d}$ and $W^{\otimes d}$ by (29). By definition, $\Gamma^d \operatorname{Hom}_A(V, W) = \left(\operatorname{Hom}_A(V, W)^{\otimes d}\right)^{\mathfrak{S}_d}$. By lemma 2.4, $\operatorname{Hom}_A(V, W)^{\otimes d} \cong \operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})$ as $\mathfrak{S}_d$-supermodules. Therefore, $\Gamma^d \operatorname{Hom}_A(V, W) \cong \left(\operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})\right)^{\mathfrak{S}_d}$. We claim that

$$\left(\operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})\right)^{\mathfrak{S}_d} = \operatorname{Hom}_{A \wr \mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}).$$

To see this, first suppose $\varphi \in \left(\operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})\right)^{\mathfrak{S}_d}$. Then for $v \in V^{\otimes d}$ and $\sigma \otimes a \in A \wr \mathfrak{S}_d$,

$$\begin{aligned}
\varphi(v.(\sigma \otimes a)) &= \varphi((v.\sigma).a) \\
&= \varphi(v.\sigma).a \\
&= \left[\left(\varphi^{\sigma^{-1}}(v)\right).\sigma\right].a \\
&= [\varphi(v).\sigma].a \\
&= \varphi(v).(\sigma \otimes a).
\end{aligned}$$

Hence $\varphi \in \operatorname{Hom}_{A \wr \mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d})$. The second equality follows since φ is a $A^{\otimes d}$-map. The third equality holds since $\varphi^{\sigma^{-1}}(v) = \varphi(v.\sigma).\sigma^{-1}$ which implies $\left(\varphi^{\sigma^{-1}}(v)\right).\sigma = \varphi(v.\sigma)$ (recall this action is given by (32)). The fourth equality follows since φ is invariant under the $\mathfrak{S}_d$-action, so that $\varphi^{\sigma^{-1}} = \varphi$.

Next, suppose $\varphi \in \operatorname{Hom}_{A \wr \mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d})$. Then for $\sigma \in \mathfrak{S}_d$ and $v \in V^{\otimes d}$, we have $\sigma \otimes 1 \in A \wr \mathfrak{S}_d$, and so

$$\begin{aligned}
\varphi^{\sigma}(v) &= \varphi(v.\sigma^{-1}).\sigma \\
&= \varphi(v.(\sigma^{-1} \otimes 1)).\sigma \\
&= \left(\varphi(v).(\sigma^{-1} \otimes 1)\right).\sigma \\
&= \left(\varphi(v).(\sigma^{-1} \otimes 1)\right).(\sigma \otimes 1) \\
&= \varphi(v).\left((\sigma^{-1} \otimes 1)(\sigma \otimes 1)\right) \\
&= \varphi(v).\left(\sigma^{-1}\sigma \otimes (1.\sigma)1\right) \\
&= \varphi(v).(1 \otimes 1) \\
&= \varphi(v).
\end{aligned}$$

Thus, $\varphi \in \left(\operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})\right)^{\mathfrak{S}_d}$. The third equality follows from φ being a $A \wr \mathfrak{S}_d$-map. The fifth equality comes from the well-definedness of the $A \wr \mathfrak{S}_d$-action (see section 2). We've shown our claim, and hence the desired result. ∎

Using this isomorphism, composition in $\mathsf{smod}_{A \wr \mathfrak{S}_d}$ induces the composition in $\Gamma^d \mathsf{V}$.

Definition 3.3. Define the **Schur superalgebra over *A*** to be

$$S^A(m|n, d) := \left(M_{m|n}(A)^{\otimes d}\right)^{\mathfrak{S}_d} \cong \left(\operatorname{End}_A(A^{m|n})^{\otimes d}\right)^{\mathfrak{S}_d}.$$

Remark 3.4. It follows from the definition and lemma 3.2 that $S^A(m|n, d) \cong \operatorname{End}_{\Gamma^d \mathsf{V}}(A^{m|n}) \cong \operatorname{End}_{A \wr \mathfrak{S}_d}\left((A^{m|n})^{\otimes d}\right)$.

4. Generalized Schur Algebras & Schurified Categories

For this section, we follow the work of [KM20] but extend definitions and results for non-square matrices. We'll want to consider $\Bbbk$-bases for the superalgebra A and its supermodules.

Suppose V_i for $1 \leqslant i \leqslant d$ are free $\Bbbk$-supermodules with homogeneous $\Bbbk$-bases $\{x^i_j \mid j \in J_i\}$ for some index sets J_i. Then $\{x^1_{j_1} \otimes x^2_{j_2} \otimes \cdots \otimes x^d_{j_d} \mid \vec{j} \in \bigoplus_{i=1}^d J_i\}$ is a homogeneous $\Bbbk$-basis for $V_1 \otimes \cdots \otimes V_d$.

In the special case where each $V_i = V$ with homogeneous $\Bbbk$-basis $\{x_j \mid j \in J\}$, we have $\{x_{\vec{j}} \coloneqq x_{j_1} \otimes \cdots \otimes x_{j_d} \mid \vec{j} \in J^d\}$ as a homogeneous $\Bbbk$-basis for $V^{\otimes d}$.

Now consider V a free $\Bbbk$-supermodule with homogeneous basis B. For $\vec{b} \in B^d$, let $v_{\vec{b}} \coloneqq b_1 \otimes \cdots \otimes b_d \in V^{\otimes d}$ (which is a homogeneous $\Bbbk$-basis element). Note that $\mathfrak{S}_d$ acts on the right of B^d via (unsigned) place permutation, so for $\sigma \in \mathfrak{S}_d$, we have $\vec{b}\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(d)})$ and so $v_{\vec{b}}.\sigma = (-1)^{s(\vec{b},\sigma)} v_{\vec{b}\sigma}$.

Moreover, let

$$R(B,d) \coloneqq \{\vec{b} \in B^d \mid \text{ for } k < \ell,\ b_k = b_\ell \text{ only if } \overline{b_k} = 0 = \overline{b_\ell}\}. \tag{35}$$

So the set $R(B,d)$ indexes the $\Bbbk$-basis elements of $V^{\otimes d}$ which do not have any repeated odd components. Note that $R(B,d) \subset B^d$ is $\mathfrak{S}_d$-invariant, so it makes sense to talk about the orbits $R(B,d)/\mathfrak{S}_d$.

Let $\mathcal{R}_{\vec{b}}$ denote the set of shortest coset representatives of $\mathrm{Stab}(\vec{b})\backslash\mathfrak{S}_d$. Then $\{\vec{b}\sigma \mid \sigma \in \mathcal{R}_{\vec{b}}\}$ is the set of distinct elements in the orbit of $\vec{b}$. Define

$$\widetilde{v_{\vec{b}}} \coloneqq \sum_{\sigma \in \mathcal{R}_{\vec{b}}} v_{\vec{b}}.\sigma = \sum_{\sigma \in \mathcal{R}_{\vec{b}}} (-1)^{s(\vec{b},\sigma)} v_{\vec{b}\sigma}. \tag{36}$$

In other terms, we have $\widetilde{v_{\vec{b}}} = (b_1 \widetilde{\otimes \cdots \otimes} b_d) = \sum_{\sigma \in \mathcal{R}_{\vec{b}}} (-1)^{s(\vec{b},\sigma)} b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(d)}$ is a certain signed sum of all distinct permutations of the basis element $v_{\vec{b}}$.

Lemma 4.1. *Suppose V is a free $\Bbbk$-supermodule with homogeneous $\Bbbk$-basis B. Let $\Omega(B,d)$ denote a set of orbit representatives for $R(B,d)/\mathfrak{S}_d$. Then*

$$\left\{ \widetilde{v_{\vec{b}}} \,\middle|\, \vec{b} \in \Omega(B,d) \right\}$$

is a homogeneous $\Bbbk$-basis for $\left(V^{\otimes d}\right)^{\mathfrak{S}_d}$.

Proof. There are two obvious steps to the proof. First, we must show that these elements span. Second, we must show that they are linearly independent. The second point is easily seen. For a given $\vec{b}$, the sum $\widetilde{v_{\vec{b}}}$ ranges over distinct elements of the orbit of $\vec{b}$. This means that each term in the sum $\widetilde{v_{\vec{b}}}$ is a distinct $\Bbbk$-basis element of $V^{\otimes d}$. Moreover, since our proposed basis set is indexed by orbits of $R(B,d)$, two distinct elements $\widetilde{v_{\vec{b}}} \neq \widetilde{v_{\vec{c}}}$ must share no common summands since $\vec{b}$ and $\vec{c}$ are in separate disjoint orbits. Therefore any finite sum of the $\widetilde{v_{\vec{b}}}$ is just a larger finite sum of distinct $\Bbbk$-basis elements of $V^{\otimes d}$, and so it follows that any finite collection of the $\widetilde{v_{\vec{b}}}$ is linearly independent. So we have our second step.

Now we just need to tackle spanning. Choose $f \in \left(V^{\otimes d}\right)^{\mathfrak{S}_d}$. In particular, $f \in V^{\otimes d}$ so it can be written as a linear combination of $\Bbbk$-basis elements $f = \sum_{\vec{b} \in B^d} \alpha_{\vec{b}} v_{\vec{b}}$ where $\alpha_{\vec{b}} \in \Bbbk$ (and all but finitely many are nonzero). Since $f \in \left(V^{\otimes d}\right)^{\mathfrak{S}_d}$, this means for all $\sigma \in \mathfrak{S}_d$, $f.\sigma = f$. In particular, we have

$$\begin{aligned} f.\sigma &= \left(\sum_{\vec{b}\in B^d} \alpha_{\vec{b}} v_{\vec{b}}\right)\sigma \\ &= \sum_{\vec{b}\in B^d} \alpha_{\vec{b}}\left(v_{\vec{b}}\right)\sigma \\ &= \sum_{\vec{b}\in B^d} (-1)^{s(\vec{b},\sigma)} \alpha_{\vec{b}} v_{\vec{b}\sigma}. \end{aligned} \tag{37}$$

Now since σ is a bijection, we can reindex the sum for f:

$$\begin{aligned} f &= \sum_{\vec{b}\in B^d} \alpha_{\vec{b}} v_{\vec{b}} \\ &= \sum_{\vec{b}\in B^d} \alpha_{\vec{b}\sigma} v_{\vec{b}\sigma}. \end{aligned} \tag{38}$$

Since (37) = (38), the fact that these sums range over basis elements implies that

$$\alpha_{\vec{b}\sigma} = (-1)^{s(\vec{b},\sigma)} \alpha_{\vec{b}} \tag{39}$$

for all $\vec{b} \in B^d$ and $\sigma \in \mathfrak{S}_d$.

So (39) relates the coefficient corresponding to a given $\vec{b} \in B^d$ with the coefficients corresponding to tuples in the orbit of $\vec{b}$. In particular, $\alpha_{\vec{b}\sigma} = \pm\alpha_{\vec{b}}$ for all σ. The only interesting case is when $\vec{b}\tau = \vec{b}$ and $s(\vec{b},\tau) = 1 \bmod 2$ for some τ. For in this case, we would have $\alpha_{\vec{b}} = \alpha_{\vec{b}\tau} = -\alpha_{\vec{b}}$. Since $\operatorname{char}(\Bbbk) \neq 2$ and $\Bbbk$ is a domain, this would imply that $\alpha_{\vec{b}} = 0$. We would like to characterize when this happens.

Given $\vec{b} \in B^d$, two things can happen. Case 1 is that $\operatorname{Stab}(\vec{b}) = \{\mathrm{id}\}$. Then there are no elements of $\mathfrak{S}_d$ which fix $\vec{b}$, so $\alpha_{\vec{b}}$ can be anything in $\Bbbk$. Notice that this also means $\vec{b}$ has no repeated entries (odd or even), for otherwise there would be an appropriate transposition fixing $\vec{b}$.

Case 2 is that there exists some non-identity element $\tau \in \operatorname{Stab}(\vec{b})$. If $s(\vec{b},\tau) = 0 \bmod 2$, this does not imply $\alpha_{\vec{b}} = 0$. So in particular, if every element which fixes $\vec{b}$ has this property, then we still have that $\alpha_{\vec{b}}$ can be anything in $\Bbbk$. So the only issue is when $s(\vec{b},\tau) = 1 \bmod 2$. We claim that the existance of a non-identity element $\tau \in \operatorname{Stab}(\vec{b})$ such that $s(\vec{b},\tau) = 1 \bmod 2$ is equivalent to the condition that $\vec{b}$ has at least one repeated odd entry.

To see this claim, first suppose that $\vec{b}$ has at least one repeated odd entry. That is, for some $1 \leqslant k < \ell \leqslant d$, we have $b_k = b_\ell$. Let $\tau = (k\ \ell)$ be the transposition which only swaps the k and ℓ positions. Then clearly $\vec{b}\tau = \vec{b}$. Moreover, one can easily check that (working modulo 2)

$$\begin{aligned}
s(\vec{b},\tau) &= \overline{b_\ell}(\overline{b_{k+1}} + \cdots + \overline{b_{\ell-1}}) + \overline{b_\ell}\cdot\overline{b_k} + \overline{b_{k+1}}\cdot\overline{b_k} + \cdots + \overline{b_{\ell-1}}\cdot\overline{b_k} \\
&= \overline{b_\ell}(\overline{b_{k+1}} + \cdots + \overline{b_{\ell-1}}) + \overline{b_k}(\overline{b_{k+1}} + \cdots + \overline{b_{\ell-1}}) + \overline{b_\ell}\cdot\overline{b_k} \\
&= (\overline{b_\ell} + \overline{b_k})(\overline{b_{k+1}} + \cdots + \overline{b_{\ell-1}}) + \overline{b_\ell}\cdot\overline{b_k} \\
&= (1+1)(\overline{b_{k+1}} + \cdots + \overline{b_{\ell-1}}) + 1\cdot 1 \\
&= 1 \mod 2.
\end{aligned} \tag{40}$$

So we have one implication.

To see the reverse implication, suppose that $\vec{b}$ has no repeated odd entries. If there are no non-identity elements in $\mathrm{Stab}(\vec{b})$, then the condition we want holds. If there is some non-identity $\tau \in \mathrm{Stab}(\vec{b})$, then we just need to show that $s(\vec{b},\tau) = 0 \mod 2$. So say we have $\tau \neq \mathrm{id}$ such that $\vec{b}\tau = \vec{b}$. Since $\vec{b}$ has no repeated odd entries, the only way this can hold is if τ fixes k for all odd b_k. So the only entries of $\vec{b}$ which get shuffled by τ are even. Since $s(\vec{b},\tau)$ counts the number of odd entries which have passed over other odd entries, it must be that $s(\vec{b},\tau) = 0$ as desired.

To summarize, we've shown that for $f = \sum_{\vec{b}\in B^d} \alpha_{\vec{b}} v_{\vec{b}} \in \left(V^{\otimes d}\right)^{\mathfrak{S}_d}$, the coefficient $\alpha_{\vec{b}} = 0$ whenever $\vec{b}$ has repeated odd entries. Moreover, given $\vec{b}$, the coefficients $\alpha_{\vec{b}\sigma}$ are all determined, so the only possible nonzero coefficients correspond to $\vec{b} \in R(B,d)$. Hence we can group our sum as

$$\begin{aligned}
f = \sum_{\vec{b}\in B^d} \alpha_{\vec{b}} v_{\vec{b}} &= \sum_{\vec{b}\in\Omega(B,d)} \left(\sum_{\sigma\in\mathcal{R}_{\vec{b}}} \alpha_{\vec{b}\sigma} v_{\vec{b}\sigma} \right) \\
&= \sum_{\vec{b}\in\Omega(B,d)} \left(\sum_{\sigma\in\mathcal{R}_{\vec{b}}} (-1)^{s(\vec{b},\sigma)} \alpha_{\vec{b}} v_{\vec{b}\sigma} \right) \\
&= \sum_{\vec{b}\in\Omega(B,d)} \alpha_{\vec{b}} \left(\sum_{\sigma\in\mathcal{R}_{\vec{b}}} (-1)^{s(\vec{b},\sigma)} v_{\vec{b}\sigma} \right) \\
&= \sum_{\vec{b}\in\Omega(B,d)} \alpha_{\vec{b}} \widetilde{v_{\vec{b}}}.
\end{aligned}$$

In particular, we see that $\left\{ \widetilde{v_{\vec{b}}} \,\middle|\, \vec{b} \in \Omega(B,d) \right\}$ spans $\left(V^{\otimes d}\right)^{\mathfrak{S}_d}$ as desired. ◻

Remark 4.2. Note that for two different elements $\vec{b}, \vec{c} \in B^d$ in the same $\mathfrak{S}_d$-orbit, we have

$$\widetilde{v_{\vec{b}}} = \pm \widetilde{v_{\vec{c}}}.$$

So a different choice of orbit representatives $\Omega(B,d)$ yields $\pm$ the $\Bbbk$-basis from lemma 4.1.

Next, we consider a free $\Bbbk$-superalgebra A as before. If we assume our supermodules are of the form $A^{m|0} = A^m$ for some m, remark 1.2 says we have a natural isomorphism $M_{m,n}(A) \cong \mathrm{Hom}_A(A^n, A^m)$, and for any $a \in A$, we let φ^a_{rs} denote the matrix with a in the r,s entry and zeros elsewhere (In [KM20], this was denoted $\xi^a_{r,s}$).

Definition 4.3. Fix a (unital) $\Bbbk$-subalgebra $\mathfrak{a} \subset A_0$ such that $\mathfrak{a}$ and $A/\mathfrak{a}$ are both free as $\Bbbk$-modules. Such a pair $(A, \mathfrak{a})$ is called a **(unital) good pair**.

Now choose a unital good pair and fix a $\Bbbk$-module complement, $\mathfrak{c}$, of $\mathfrak{a}$ in A_0 along with $\Bbbk$-bases $B_{\mathfrak{a}}$, $B_{\mathfrak{c}}$, and B_1 for $\mathfrak{a}$, $\mathfrak{c}$, and A_1, respectively. So $B_0 = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}}$ is a $\Bbbk$-basis for A_0 and $B = B_0 \sqcup B_1$ is a homogeneous $\Bbbk$-basis for A.

Since we aren't considering parity shifts, $\overline{\varphi^a_{rs}} = \overline{a}$, and $M_{m,n}(A)$ inherits a (free) $\Bbbk$-supermodule structure, and moreover, we have that

$$C_{m,n} := \{\varphi^b_{rs} \mid 1 \leqslant s \leqslant n, 1 \leqslant r \leqslant m, b \in B\} \tag{41}$$

is a homogeneous $\Bbbk$-basis for $M_{m,n}(A)$.

We will now introduce some notation which generalizes important objects defined in [KM20]. For any set P consisting of homogeneous elements of A, and letting $[s,t] := \{s, s+1, \ldots, t\}$, define $\mathrm{Tri}^P(m,n;d)$ to be the set

$$\{(\vec{p},\vec{r},\vec{s}) \in P^d \times [1,m]^d \times [1,n]^d \mid \text{for } k \neq \ell,\ (p_k,r_k,s_k) = (p_\ell,r_\ell,s_\ell) \text{ only if } \overline{p_k} = 0 = \overline{p_\ell}\}. \tag{42}$$

Notice that any element of $(C_{m,n})^{\otimes d}$ uniquely determines a triple $(\vec{b},\vec{r},\vec{s}) \in B^d \times [1,m]^d \times [1,n]^d$. Then a point in $R(C_{m,n},d)$ also determines a triple $(\vec{b},\vec{r},\vec{s})$. This triple corresponds to an element of $\mathrm{Tri}^B(m,n;d)$. Then recalling notation (35), we see that $\mathrm{Tri}^B(m,n;d)$ indexes the $\Bbbk$-basis elements of $M_{m,n}(A)^{\otimes d}$ which do not have any repeated odd components.

Now $\mathrm{Tri}^P(m,n;d)$ is $\mathfrak{S}_d$-invariant (under the diagonal action) so it makes sense to talk about orbits. Moreover, for $(\vec{p},\vec{r},\vec{s}) \in \mathrm{Tri}^P(m,n;d)$, we denote the stabilizer of this element by

$$\mathfrak{S}_{\vec{p},\vec{r},\vec{s}} := \{\sigma \in \mathfrak{S}_d \mid (\vec{p},\vec{r},\vec{s})\sigma = (\vec{p},\vec{r},\vec{s})\}. \tag{43}$$

Let ${}^{\vec{p},\vec{r},\vec{s}}\mathscr{D}$ denote a set of shortest coset representatives for $\mathfrak{S}_{\vec{p},\vec{r},\vec{s}}\backslash\mathfrak{S}_d$. Then $\{(\vec{p},\vec{r},\vec{s})\sigma \mid \sigma \in {}^{\vec{p},\vec{r},\vec{s}}\mathscr{D}\}$ is the set of distinct elements in the orbit $[(\vec{p},\vec{r},\vec{s})]$.

Considering the right $\mathfrak{S}_d$-action from (4) on $M_{m,n}(A)^{\otimes d}$, we see that for $a_1, \ldots, a_d \in A$ homogeneous, $1 \leqslant s_1, \ldots, s_d \leqslant n$, $1 \leqslant r_1, \ldots, r_d \leqslant m$, and $\sigma \in \mathfrak{S}_d$, that

$$\left(\varphi^{a_1}_{r_1 s_1} \otimes \cdots \otimes \varphi^{a_d}_{r_d s_d}\right)^\sigma = (-1)^{s(\vec{a},\sigma)} \varphi^{a_{\sigma 1}}_{r_{\sigma 1} s_{\sigma 1}} \otimes \cdots \otimes \varphi^{a_{\sigma d}}_{r_{\sigma d} s_{\sigma d}}.$$

Definition 4.4. We define the ***A*-Schur space** $S^A(m,n;d)$ to be the space of invariants

$$\begin{aligned} S^A(m,n;d) &:= \left(M_{m,n}(A)^{\otimes d}\right)^{\mathfrak{S}_d} \\ &\cong \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes d}\right)^{\mathfrak{S}_d}, \end{aligned}$$

which is no more than $\Gamma^d(M_{m,n}(A)) \cong \Gamma^d(\mathrm{Hom}_A(A^n, A^m))$.

Remark 4.5. Notice that whenever $m = n$, $M_n(A) := M_{n,n}(A)$ is naturally a $\Bbbk$-superalgebra with product given by matrix multiplication (composition if working with $\mathrm{End}_A(A^n)$). In this case, we have $S^A(n,d) := S^A(n,n;d)$ is a $\Bbbk$-superalgebra which we call the ***A*-Schur algebra**. This is the algebra considered in [KM20]. Moreover, notice that $S^A(m|0,d)$ from definition 3.3 equals $S^A(m,d)$.

For free finite right A-supermodules $V \cong A^n$ and $W \cong A^m$, it makes sense to denote $S^A(V,W;d) := \Gamma^d(\mathrm{Hom}_A(V,W))$. In particular, if $V = A^n$ and $W = A^m$, we have $S^A(V,W;d) = S^A(A^n,A^m;d) = S^A(m,n;d)$.

For $(\vec{a}, \vec{r}, \vec{s}) \in A^d \times [1,m]^d \times [1,n]^d$, define

$$\varphi_{(\vec{a},\vec{r},\vec{s})} \coloneqq \varphi^{a_1}_{r_1 s_1} \otimes \cdots \otimes \varphi^{a_d}_{r_d s_d} \in M_{m,n}(A)^{\otimes d}.$$

Let H be the set of all nonzero homogeneous elements of A. For $(\vec{a}, \vec{r}, \vec{s}) \in \mathrm{Tri}^H(m,n;d)$, define

$$\begin{aligned}\widetilde{\varphi}_{(\vec{a},\vec{r},\vec{s})} &\coloneqq \sum_{\sigma \in {}^{\vec{a},\vec{r},\vec{s}}\mathscr{D}} \varphi_{(\vec{a},\vec{r},\vec{s})} \cdot \sigma \\ &= \sum_{\sigma \in {}^{\vec{a},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{a},\sigma)} \varphi^{a_{\sigma 1}}_{r_{\sigma 1} s_{\sigma 1}} \otimes \cdots \otimes \varphi^{a_{\sigma d}}_{r_{\sigma d} s_{\sigma d}}\end{aligned}$$

in the same vein as (36). Note that $\widetilde{\varphi}_{(\vec{a},\vec{r},\vec{s})}$ is analagous to the element $\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}$ from [KM20] defined in their equation (3.2). In fact, they are equal when we have $m = n$.

Then it follows immediately from lemma 4.1 that we have

Corollary 4.6. *Suppose A is a free $\Bbbk$-superalgebra with homogeneous $\Bbbk$-basis B. Choose a set of orbit representatives $\Omega(B,m,n;d)$ for $\mathrm{Tri}^B(m,n;d)/\mathfrak{S}_d$. Then*

$$\left\{ \widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} \;\middle|\; (\vec{b},\vec{r},\vec{s}) \in \Omega(B,m,n;d) \right\}$$

is a homogeneous $\Bbbk$-basis for $S^A(m,n;d) = \left(M_{m,n}(A)^{\otimes d}\right)^{\mathfrak{S}_d}$.

We need some more notation. Let $(\vec{b},\vec{r},\vec{s}) \in \mathrm{Tri}^B(m,n;d)$. For $b \in B$, $r \in [1,m]$, and $s \in [1,n]$, let

$$[\vec{b},\vec{r},\vec{s}]^b_{r,s} \coloneqq \#\{k \in [1,d] \mid (b_k, r_k, s_k) = (b,r,s)\}. \tag{44}$$

So $[\vec{b},\vec{r},\vec{s}]^b_{r,s}$ counts the number of components of $\varphi_{(\vec{b},\vec{r},\vec{s})}$ which equal a given element φ^b_{rs}.

Define

$$[\vec{b},\vec{r},\vec{s}]^! \coloneqq \prod_{\substack{b \in B \\ r \in [1,m] \\ s \in [1,n]}} [\vec{b},\vec{r},\vec{s}]^b_{r,s}! = \prod_{\substack{b \in B_0 \\ r \in [1,m] \\ s \in [1,n]}} [\vec{b},\vec{r},\vec{s}]^b_{r,s}!, \tag{45}$$

where the equality comes from the fact that $[\vec{b},\vec{r},\vec{s}]^b_{r,s} \not> 1$ whenever $b \in B_1$ since $(\vec{b},\vec{r},\vec{s})$ contains no repeated odd entries. Moreover, this product is always finite since only finitely many $[\vec{b},\vec{r},\vec{s}]^b_{r,s}$ can ever be greater than 1. One last observation is that $|\mathfrak{S}_{\vec{b},\vec{r},\vec{s}}| = [\vec{b},\vec{r},\vec{s}]^!$.

Define

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{a}} \coloneqq \prod_{\substack{b \in B_{\mathfrak{a}} \\ r \in [1,m] \\ s \in [1,n]}} [\vec{b},\vec{r},\vec{s}]^b_{r,s}!, \tag{46}$$

and

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \coloneqq \prod_{\substack{b \in B_{\mathfrak{c}} \\ r \in [1,m] \\ s \in [1,n]}} [\vec{b},\vec{r},\vec{s}]^b_{r,s}!, \tag{47}$$

and

$$\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \coloneqq [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}. \tag{48}$$

This element is analogous to the $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$ defined in [KM20].

Definition 4.7. We define the **generalized A-Schur space** $T^A_{\mathfrak{a}}(m,n;d)$ to be the $\Bbbk$-submodule of $S^A(m,n;d)$ given by

$$T^A_{\mathfrak{a}}(m,n;d) \coloneqq \operatorname{span}_{\Bbbk}\left(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \mid (\vec{b},\vec{r},\vec{s}) \in \operatorname{Tri}^B(m,n;d)\right).$$

Remark 4.8. As in remark 4.5 with the A-Schur space, if $V = A^n$ and $W = A^m$ are free finite right A-supermodules, then we denote $T^A_{\mathfrak{a}}(V,W;d) \coloneqq T^A_{\mathfrak{a}}(m,n;d)$.

It is clear that we have the following:

Corollary 4.9. *Suppose A, B, and $\Omega(B,m,n;d)$ are as in corollary 4.6. Then*

$$\left\{\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \,\middle|\, (\vec{b},\vec{r},\vec{s}) \in \Omega(B,m,n;d)\right\}$$

is a homogeneous $\Bbbk$-basis for $T^A_{\mathfrak{a}}(m,n;d)$.

Remark 4.10. First of all, notice that when $\mathfrak{a} = A_0$ we have $\mathfrak{c} = \varnothing$ and so $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} = 1$. It follows that $T^A_{A_0}(m,n;d) = S^A(m,n;d)$. Second, for any allowable choice of $\mathfrak{a}$, we have $S^{\mathfrak{a}}(m,n;d) \subset T^A_{\mathfrak{a}}(m,n;d)$ since the tuples $(\vec{b},\vec{r},\vec{s})$ which index elements of $S^{\mathfrak{a}}(m,n;d)$ have no $b_k \in \mathfrak{c}$ so that $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} = 1$ for all of these elements. Similarly, we have $S^{A_1}(m,n;d) \subset T^A_{\mathfrak{a}}(m,n;d)$.

Note that when $m = n$, our $T^A_{\mathfrak{a}}(m,n;d)$ is equal to the $T^A_{\mathfrak{a}}(n;d)$ in [KM20], which is a $\Bbbk$-subsuperalgebra of $S^A(n,d)$ which is *independent of basis*. This means (among other things) that $T^A_{\mathfrak{a}}(n;d)$ is closed under multiplication (composition when thinking of morphisms $A^n \to A^n$). We'd like similar results in our non-square setting. Specifically, the first thing we want is for the composition map

$$T^A_{\mathfrak{a}}(k,m;d) \otimes T^A_{\mathfrak{a}}(m,n;d) \to T^A_{\mathfrak{a}}(k,n;d)$$

to be well-defined.

Remark 4.11. This can be shown by essentially the same argument used in Proposition 3.12 from [KM20] which shows that $T^A_{\mathfrak{a}}(n,n;d)$ is a $\Bbbk$-subalgebra of $S^A(n,n;d)$. This argument boils down to showing that a certain coefficient divides a product of certain integers. The difference is that all values considered in [KM20] correspond to square matrices - in our setting, each of these values can be generalized by allowing for non-square matrices, but all of the arguments involved still go through without issue. The relevant result from [KM20] is Corollary 3.7 which relies on their Proposition 3.6 and Lemma 2.11. However, since these computations are rather involved, we will choose to argue by embedding the non-square case into the square case:

First, fix m,n,d. Let $N \coloneqq \max(m,n)$. Then we have an embedding

$$\iota : M_{m,n}(A) \hookrightarrow M_N(A)$$

given by

$$\varphi^a_{ij} \mapsto \varphi^a_{ij}, \tag{49}$$

where $1 \leqslant i \leqslant m \leqslant N$ and $1 \leqslant j \leqslant n \leqslant N$. So this embedding sticks an $m \times n$ matrix into the upper left corner of an $N \times N$ matrix with zeros outside of this corner. Note that this embedding works for any $N' \geqslant \max(m,n)$.

Moreover, we know that matrix multiplication is bilinear, where for $\varphi \in M_{m,n}(A)$ and $\psi \in M_{k,m}(A)$, we have $\psi\varphi \in M_{k,m}(A)$. It is easy to see that $\iota(\psi\varphi) = \iota(\psi)\iota(\varphi)$ where we embed into $M_{\max(m,n,k)}(A)$. So under this embedding, we can keep track of the composition of nonsquare matrices in terms of the composition of square matrices.

This induces an embedding

$$\iota^d : M_{m,n}(A)^{\otimes d} \hookrightarrow M_N(A)^{\otimes d}$$

given by

$$\varphi_{(\vec{a},\vec{i},\vec{j})} \mapsto \varphi_{(\vec{a},\vec{i},\vec{j})}.$$

It is almost immediate that ι^d is not just a map as supermodules, but also as right $\mathfrak{S}_d$-modules. Therefore, it induces a map

$$S^A(m,n;d) = \left(M_{m,n}(d)^{\otimes d}\right)^{\mathfrak{S}_d} \hookrightarrow \left(M_N(d)^{\otimes d}\right)^{\mathfrak{S}_d} = S^A(N,N;d).$$

Notice that each $\Bbbk$-basis element $\widetilde{\varphi}_{(\vec{a},\vec{r},\vec{s})} \in S^A(m,n;d)$ (where $\vec{r} \in [1,m]^d$ and $\vec{s} \in [1,n]^d$) is sent to a $\Bbbk$-basis element, $\widetilde{\varphi}_{(\vec{a},\vec{r},\vec{s})}$ of $S^A(N,N;d)$ (where now we view $\vec{r} \in [1,N]^d$ and $\vec{s} \in [1,N]^d$ by appending zeros where necessary). Since ι^d is $\Bbbk$-linear, it follows that ι^d actually induces a map

$$T^A_{\mathfrak{a}}(m,n;d) \hookrightarrow T^A_{\mathfrak{a}}(N,N;d.)$$

In particular, given some $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T^A_{\mathfrak{a}}(m,n;d)$, we know that $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} = [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$ where $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} = \prod_{\substack{b\in B_{\mathfrak{c}}\\ r\in[1,m]\\ s\in[1,n]}} [\vec{b},\vec{r},\vec{s}]^b_{r,s}!$. Now for some $\widetilde{\eta}_{(\vec{a},\vec{p},\vec{q})} \in T^A_{\mathfrak{a}}(N,N;d)$, we have $\widetilde{\eta}_{(\vec{a},\vec{p},\vec{q})} = [\vec{a},\vec{p},\vec{q}]^!_{\mathfrak{c}}\widetilde{\varphi}_{(\vec{a},\vec{p},\vec{q})}$ where $[\vec{a},\vec{p},\vec{q}]^!_{\mathfrak{c}} = \prod_{\substack{a\in B_{\mathfrak{c}}\\ p,q\in[1,N]}} [\vec{a},\vec{p},\vec{q}]^a_{p,q}!$. It is clear that if we had $\vec{p} \in [1,m]^d$ and $\vec{q} \in [1,n]^d$, that $[\vec{a},\vec{p},\vec{q}]^!_{\mathfrak{c}} = \prod_{\substack{a\in B_{\mathfrak{c}}\\ p\in[1,m]\\ q\in[1,n]}} [\vec{a},\vec{p},\vec{q}]^a_{p,q}!$. So ι^d really does send things to the correct elements.

To summarize, we have the following commuting square:

$$\begin{tikzcd}
T^A_{\mathfrak{a}}(m,n;d) \arrow[r, hook] \arrow[d, hook, "\iota^d"'] & S^A(m,n;d) \arrow[d, hook, "\iota^d"] \\
T^A_{\mathfrak{a}}(N,N;d) \arrow[r, hook] & S^A(N,N;d)
\end{tikzcd}.$$

Proposition 4.12. *The composition map*

$$T^A_{\mathfrak{a}}(k,m;d) \otimes T^A_{\mathfrak{a}}(m,n;d) \to T^A_{\mathfrak{a}}(k,n;d)$$

is a well-defined even map of $\Bbbk$*-supermodules.*

Proof. Let $\widetilde{\eta}_{(\vec{a},\vec{p},\vec{q})} \in T^A_{\mathfrak{a}}(k,m;d)$ and $\widetilde{\eta}_{(\vec{c},\vec{u},\vec{v})} \in T^A_{\mathfrak{a}}(m,n;d)$. Let $M = \max(k,m,n)$. Then under the ι^d embedding, we view $\widetilde{\eta}_{(\vec{a},\vec{p},\vec{q})}$ and $\widetilde{\eta}_{(\vec{c},\vec{u},\vec{v})}$ as both sitting in $T^A_{\mathfrak{a}}(M,M;d)$.

Then this composition map is really multiplication within $S^A(M,M;d)$ so is clearly even and $\Bbbk$-bilinear. So the only thing to check is that the target of this map really is $T^A_{\mathfrak{a}}(k,n;d)$.

It is shown in Proposition 3.12 of [KM20] that $\widetilde{\eta}_{(\vec{a},\vec{p},\vec{q})}\widetilde{\eta}_{(\vec{c},\vec{u},\vec{v})}$ is some linear combination of elements of the form $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}$ where, in particular, $\vec{r} \sim \vec{p}$ and $\vec{s} \sim \vec{v}$ (with respect to the $\mathfrak{S}_d$-action). This means that the only terms which contribute to this linear combination are those with $\vec{r} \in [1,k]^d$ and $\vec{s} \in [1,n]^d$. It follows that this linear combination (which we view as living in $T^A_{\mathfrak{a}}(M,M;d)$) really sits in $T^A_{\mathfrak{a}}(k,n;d)$. ∎

The second thing we want is that $T_{\mathfrak{a}}^A(m,n;d)$ is independent of the choice of basis $B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_1$ for A and that it only depends on the choice of subalgebra $\mathfrak{a}$. Before showing this, we need some more setup (we folllow [KM20] section 4 for this).

Definition 4.13. For $d, e \in \mathbb{N}$, let ${}^{(d,e)}\mathscr{D}$ be the set of shortest coset representatives for $(\mathfrak{S}_d \times \mathfrak{S}_e)\backslash\mathfrak{S}_{d+e}$. Given $\varphi_1 \in M_n(A)^{\otimes d}$ and $\varphi_2 \in M_n(A)^{\otimes e}$, define

$$\varphi_1 * \varphi_2 \coloneqq \sum_{\sigma \in {}^{(d,e)}\mathscr{D}} (\varphi_1 \otimes \varphi_2)\,.\sigma.$$

Note that $\bigoplus\limits_{d\geqslant 0} M_n(A)^{\otimes d}$ is an associative supercommutative superalgebra under this $*$-product. Let

$$Y_n \coloneqq \operatorname{span}_{\Bbbk}(\varphi_{rs}^b \mid r,s \in [1,n],\, b \in B_{\mathfrak{c}} \sqcup B_1) \subset M_n(A)$$

and

$$Y_{m,n} \coloneqq \operatorname{span}_{\Bbbk}(\varphi_{rs}^b \mid r \in [1,m],\, s \in [1,n],\, b \in B_{\mathfrak{c}} \sqcup B_1) \subset M_{m,n}(A).$$

Note that for $N = \max(m,n)$, we can view $Y_{m,n} \subset Y_N$ in light of the embedding ι in (49). For any set $M \subset M_n(A)$, define

$$\operatorname{Star}^d M \coloneqq \underbrace{M * \cdots * M}_{d \text{ terms}}.$$

Lemma 4.2 from [KM20] shows that $S^A(n,n;d)*S^A(n,n;e) \subset S^A(n,n;d+e)$ and $T_{\mathfrak{a}}^A(n,n;d)* T_{\mathfrak{a}}^A(n,n;e) \subset T_{\mathfrak{a}}^A(n,n;d+e)$. Furthermore, if one analyzes the tuples and coefficients involved in this lemma, it is easy to see that $S^A(m,n;d) * S^A(m,n;e) \subset S^A(m,n;d+e)$ and $T_{\mathfrak{a}}^A(m,n;d)*T_{\mathfrak{a}}^A(m,n;e) \subset T_{\mathfrak{a}}^A(m,n;d+e)$ (where in order to make sense of the $*$-product, we are utilizing the ι embedding from (49)). It follows that $\operatorname{Star}^d Y_n \subset T_{\mathfrak{a}}^A(n,n;d)$. Then viewing $Y_{m,n} \subset Y_N$, we can make sense out of $\operatorname{Star}^d Y_{m,n}$ and say that $\operatorname{Star}^d Y_{m,n} \subset T_{\mathfrak{a}}^A(m,n;d) \subset T_{\mathfrak{a}}^A(N,N;d)$.

The following generalizes lemma 4.9 from [KM20]:

Lemma 4.14. *We have*

$$T_{\mathfrak{a}}^A(m,n;d) = \bigoplus_{e=0}^{d} S^{\mathfrak{a}}(m,n;d-e) * \operatorname{Star}^e Y_{m,n}.$$

Proof. From remark 4.10, we know that $S^{\mathfrak{a}}(m,n;d-e) \subset T_{\mathfrak{a}}^A(m,n;d-e)$. We've also already observed that $\operatorname{Star}^e Y_{m,n} \subset T_{\mathfrak{a}}^A(m,n;e)$. So it follows that the right hand side is included in the left hand side.

For the reverse containment, note that it is shown in the proof of lemma 4.9 of [KM20] that for $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T_{\mathfrak{a}}^A(N,N;e)$, we have

$$\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} = \pm \left(\mathop{\Large\ast}_{\substack{b \in B_{\mathfrak{a}} \\ r \in [1,N] \\ s \in [1,N]}} \left(\varphi_{rs}^b\right)^{\otimes [\vec{b},\vec{r},\vec{s}]_{r,s}^b} \right) * \left(\mathop{\Large\ast}_{\substack{b \in B_{\mathfrak{a}} \sqcup B_1 \\ r \in [1,N] \\ s \in [1,N]}} \left(\varphi_{rs}^b\right)^{\otimes [\vec{b},\vec{r},\vec{s}]_{r,s}^b} \right), \tag{50}$$

where this first term is in $S^{\mathfrak{a}}(N,N;d-e)$, and the second term is in $\operatorname{Star}^e Y_N$ (these containments follow from lemma 4.6 in [KM20]).

Now observe that if $(\vec{b},\vec{r},\vec{s})$ is such that $\vec{r}\in[1,m]^d\subset[1,N]^d$ and $\vec{s}\in[1,n]^d\subset[1,N]^d$, then $[\vec{b},\vec{r},\vec{s}]^b_{r,s}=0$ whenever $r\notin[1,m]$ and $s\notin[1,n]$. This type of tuple corresponds to an element $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\in T^A_{\mathfrak{a}}(m,n;e)\subset T^A_{\mathfrak{a}}(N,N;e)$, and for such an element, (50) becomes

$$\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}=\pm\left(\mathop{\Large\ast}_{\substack{b\in B_{\mathfrak{a}}\\ r\in[1,m]\\ s\in[1,n]}}\left(\varphi^b_{rs}\right)^{\otimes[\vec{b},\vec{r},\vec{s}]^b_{r,s}}\right)*\left(\mathop{\Large\ast}_{\substack{b\in B_{\mathfrak{a}}\sqcup B_1\\ r\in[1,m]\\ s\in[1,n]}}\left(\varphi^b_{rs}\right)^{\otimes[\vec{b},\vec{r},\vec{s}]^b_{r,s}}\right)$$

with the first term in $S^{\mathfrak{a}}(m,n;d-e)$ and the second term in $\operatorname{Star}^e Y_{m,n}$. ◻

Proposition 4.15. *The space $T^A_{\mathfrak{a}}(m,n;d)$ is independent of the choice of basis $B=B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}}\sqcup B_1$ for $(A,\mathfrak{a})$ and only depends on the choice of subalgebra $\mathfrak{a}$.*

Proof. The proof is almost the same as that of Proposition 4.11 from [KM20]. Let $B=B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}}\sqcup B_1$ and $B'=B'_{\mathfrak{a}}\sqcup B'_{\mathfrak{c}}\sqcup B'_1$ be two distinct choices of $(A,\mathfrak{a})$-bases. Let $Y_{m,n}=\operatorname{span}_{\Bbbk}(\varphi^b_{rs}\mid r\in[1,m],\ s\in[1,n],\ b\in B_{\mathfrak{c}}\sqcup B_1)$ and $Y'_{m,n}=\operatorname{span}_{\Bbbk}(\varphi^b_{rs}\mid r\in[1,m],\ s\in[1,n],\ b\in B'_{\mathfrak{c}}\sqcup B'_1)$.

Notice that $B'_{\mathfrak{c}}\subset\operatorname{span}_{\Bbbk}(B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}})$ and $B'_1\subset\operatorname{span}_{\Bbbk}(B_1)$, so any element in $\operatorname{Star}^e Y'_{m,n}$ can be expanded in the basis B into a sum of elements of the form $\varphi^{a_1}_{r_1s_1}*\cdots*\varphi^{a_e}_{r_es_e}$ with $a_i\in B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}}\sqcup B_1$. That is,

$$\operatorname{Star}^e Y'_{m,n}\subset\bigoplus_{f=0}^{e}\operatorname{Star}^{e-f}Y_{m,n}*\operatorname{Star}^f Y^{\mathfrak{a}}_{m,n},\tag{51}$$

where $Y^{\mathfrak{a}}_{m,n}=\operatorname{span}_{\Bbbk}(\varphi^b_{rs}\mid r\in[1,m],\ s\in[1,n],\ b\in B_{\mathfrak{a}})$.

Letting $'T^A_{\mathfrak{a}}(m,n;d)$ be defined using basis B', we know from lemma 4.14 that

$$'T^A_{\mathfrak{a}}(m,n;d)=\bigoplus_{e=0}^{d}S^{\mathfrak{a}}(m,n;d-e)*\operatorname{Star}^e Y'_{m,n}.$$

Then using (51), we have

$$'T^A_{\mathfrak{a}}(m,n;d)=\bigoplus_{e=0}^{d}S^{\mathfrak{a}}(m,n;d-e)*\left(\bigoplus_{f=0}^{e}\operatorname{Star}^{e-f}Y_{m,n}*\operatorname{Star}^f Y^{\mathfrak{a}}_{m,n}\right).$$

Since $S^{\mathfrak{a}}(m,n;d-e)\subset T^A_{\mathfrak{a}}(m,n;d-e)$, $\operatorname{Star}^{e-f}Y_{m,n}\subset T^A_{\mathfrak{a}}(m,n;e-f)$, and $\operatorname{Star}^f Y^{\mathfrak{a}}_{m,n}\subset T^A_{\mathfrak{a}}(m,n;f)$, it follows that $'T^A_{\mathfrak{a}}(m,n;d)\subset T^A_{\mathfrak{a}}(m,n;d)$. Similarly, $T^A_{\mathfrak{a}}(m,n;d)\subset{}'T^A_{\mathfrak{a}}(m,n;d)$, completing the proof. ◻

Let A be a superalgebra over $\Bbbk$ and choose $\mathfrak{a}\subset A_0$ so that $(A,\mathfrak{a})$ forms a unital good pair.

Definition 4.16. Let $\mathsf{T}^A_{\mathfrak{a}}(d)$ denote the **d^{th} Schurified category of free finite unshifted right A-supermodules** whose objects are finite free *unshifted* right A-supermodules, that is, are of the form $V=A^n$, and whose morphism sets are defined to be

$$\operatorname{Hom}_{\mathsf{T}^A_{\mathfrak{a}}(d)}(A^n,A^m):=T^A_{\mathfrak{a}}(m,n;d).$$

Remark 4.17. Only allowing objects $V=A^n$ means we are working with free unshifted A-supermodules with a distinguished choice of even A-basis. We've defined $T^A_{\mathfrak{a}}(m,n;d)$ in terms of matrices with entries in A, so it's necessary to have these bases specified. It may be interesting to define $T^A_{\mathfrak{a}}(m,n;d)$ independently of the A-bases so that one can make sense of the space for free supermodules of the form $V\cong A^n$, however, we won't do so in this thesis.

Remark 4.18. If we have $\mathfrak{a} = A_0$, then $T_{\mathfrak{a}}^A(m,n;d) = S^A(m,n;d) = \Gamma^d(M_{m,n}(A)) \cong \Gamma^d \operatorname{Hom}_A(A^n, A^m)$. So in this case, $\mathsf{T}_{\mathfrak{a}}^A(d)$ is just the d^{th} divided powers category $\Gamma^d\mathcal{V}$ where $\mathcal{V}$ is the category of finite free unshifted right A-supermodules of the form $V = A^n$. In certain contexts, we may also denote the category by $\mathsf{S}^A(d)$.

Remark 4.19. For $d = 0$, note that $M_{m,n}(A)^{\otimes 0} := \Bbbk$ so that $S^A(m,n;0) = \Bbbk$. It follows that $T_{\mathfrak{a}}^A(m,n;0) = \Bbbk$, as well. Thus $\mathsf{T}_{\mathfrak{a}}^A(0) = \mathsf{S}^A(0)$ is the category whose objects are finite free unshifted right A-supermodules and whose morphism spaces always equal $\Bbbk$.

5. Categories Enriched over $\mathsf{smod}_{\Bbbk}$

The definitions in this section are inspired by those in [BE17] (which deals with $\Bbbk$ a field of characteristic not 2).

Let $\mathsf{smod}_{\Bbbk}$ denote the category of all (not necessarily finite-dimensional) $\Bbbk$-supermodules. A **supercategory** C is an $\mathsf{smod}_{\Bbbk}$-*enriched* category meaning the morphism spaces in C are $\Bbbk$-supermodules and composition in C is bilinear and even - that is, composition induces a map

$$\operatorname{Hom}_{\mathsf{C}}(V,W) \otimes \operatorname{Hom}_{\mathsf{C}}(U,V) \to \operatorname{Hom}_{\mathsf{C}}(U,W)$$

which is an even linear map. Sometimes we use the following notation for morphism spaces:

$$\mathsf{C}(V,W) := \operatorname{Hom}_{\mathsf{C}}(V,W).$$

For a supercategory C, we let C_{ev} denote the *even underlying category* which has the same objects as C but only the even morphisms. Since C_{ev} only involves even morphisms, it makes sense to ask whether or not this category is abelian. We say that a supercategory C is **abelian** if C_{ev} is abelian in the usual sense.

A **superfunctor** $F : \mathsf{C} \to \mathsf{D}$ between supercategories is a covariant functor such that for every $V, W \in \mathsf{C}$, the induced map

$$F_{V,W} : \operatorname{Hom}_{\mathsf{C}}(V,W) \to \operatorname{Hom}_{\mathsf{D}}(F(V), F(W))$$

is an *even* k-linear map. F is **full** if $F_{V,W}$ is surjective for every pair V, W, and F is **faithful** if $F_{V,W}$ is injective for every pair V, W. F is called **fully faithful** if $F_{V,W}$ is both injective and surjective. F is called **essentially surjective** if for every object Y in D, there exists an object X in C such that D is isomorphic to FX via an even morphism.

A **supernatural transformation** $\eta : F \to G$ between two superfunctors $F, G : \mathsf{C} \to \mathsf{D}$ is a family of morphisms $\eta_X = \eta_X^0 + \eta_X^1 \in \operatorname{Hom}_{\mathsf{D}}(F(X), G(X))$ for X an object in C such that $\overline{\eta_X^p} = p$ and

$$\eta_Y^p \circ F(f) = (-1)^{\overline{f} \cdot p} G(f) \circ \eta_X^p$$

for $p \in \mathbb{Z}_2$ and $f \in \operatorname{Hom}_{\mathsf{C}}(X,Y)$.

Note that the supernatural transformation η decomposes as a sum of homogeneous supernatural transformations $\eta_0 + \eta_1$ where $(\eta_p)_X = \eta_X^p$ making the space $\operatorname{SNat}(F,G)$ of all supernatural transformations from F to G into a $\Bbbk$-supermodule.

A superfunctor $F : \mathsf{C} \to \mathsf{D}$ between supercategories is an **superequivalence** (or equivalence) if there is a superfunctor $G : \mathsf{D} \to \mathsf{C}$ such that $F \circ G$ and $G \circ F$ are isomorphic to identities via even supernatural transformations.

Remark 5.1. Having a fully faithful essentially surjective functor between non-super categories is equivalent to having an equivalence between those categories. See, for example, Theorem 1.59 of [Rie16]. It's easy to see that this extends to our supercategory setting since all of the relevent maps in our definitions are even.

Fix a superalgebra A. Define the **parity shift functor** $\Pi : \mathsf{smod}_A \to \mathsf{smod}_A$ by sending a supermodule V to ΠV which has the same underlying $\Bbbk$-supermodule structure but with opposite $\mathbb{Z}_2$-grading. The right A-action is the same as that in V. On morphisms $f : V \to W$, we have $\Pi(f) = (-1)^{\overline{f}} f$ (so for even maps, Πf acts the same as f did, and on odd maps, it acts by the negative).

Note that we may apply the parity shift to $\mathsf{smod}_{\Bbbk}$, and we denote

$$\Bbbk^{m|n} := \Bbbk^m \oplus (\Pi \Bbbk)^n.$$

We remark also that we are most interested in the case of right supermodules, but for left supermodules, $\Pi : {}_A\mathsf{smod} \to {}_A\mathsf{smod}$ sends V to ΠV but with a new action defined (in terms of the old action) by $a.v := (-1)^{\overline{a}} av$.

Proposition 5.2. *Let A be a superalgebra and consider ${}_A\mathsf{smod}$. Then for every $M \in {}_A\mathsf{smod}$, we have a natural (odd) isomorphism*

$$\mathrm{Hom}_{{}_A\mathsf{smod}}(A, M) \cong \mathrm{Hom}_{{}_A\mathsf{smod}}(\Pi A, M).$$

Proof. Let $e : A \to \Pi A$ be the odd linear map which is the identity on the underlying $\Bbbk$-supermodule. Then $e^{-1} : \Pi A \to A$ is also the odd linear map which is identity on the underlying $\Bbbk$-supermodule.

Define $\varphi : \mathrm{Hom}_{{}_A\mathsf{smod}}(A, M) \to \mathrm{Hom}_{{}_A\mathsf{smod}}(\Pi A, M)$ via $f \mapsto f \circ e^{-1}$. To see that $\varphi(f)$ is actually an A-map, let $a, x \in A$ and $f \in \mathrm{Hom}_{{}_A\mathsf{smod}}(A, M)$ be homogeneous. We have

$$\begin{aligned}
\varphi(f)(a.x) &= \varphi(f)\left((-1)^{\overline{a}} ax\right) \\
&= (f \circ e^{-1})\left((-1)^{\overline{a}} ax\right) \\
&= (-1)^{\overline{a}} f(e^{-1}(ax)) \\
&= (-1)^{\overline{a}} f(ax) \\
&= (-1)^{\overline{a} + \overline{a} \cdot \overline{f}} a f(x) \\
&= (-1)^{\overline{a}(1+\overline{f})} a f(x) \\
&= (-1)^{\overline{a} \cdot \overline{\varphi(f)}} a f(e^{-1}(x)) \\
&= (-1)^{\overline{a} \cdot \overline{\varphi(f)}} a \varphi(f)(x),
\end{aligned}$$

where the fifth equality comes from the fact that f is a A-map, and the second-to-last equality comes from the fact that $\overline{\varphi(f)} = \overline{f \circ e^{-1}} = \overline{f} + 1$.

Let $\psi : \mathrm{Hom}_{{}_A\mathsf{smod}}(\Pi A, M) \to \mathrm{Hom}_{{}_A\mathsf{smod}}(A, M)$ be given by $g \mapsto g \circ e$. To see that ψ is well-defined, let $a, x \in A$ and $g \in \mathrm{Hom}_{{}_A\mathsf{smod}}(\Pi A, M)$ be homogeneous. We have

$$\begin{aligned} a\psi(g)(x) &= a(g \circ e)(x) \\ &= ag(x) \\ &= (-1)^{\overline{a}\cdot\overline{g}} g(a.x) \\ &= (-1)^{\overline{a}\cdot\overline{g}+\overline{a}} g(ax) \\ &= (-1)^{\overline{a}\cdot\overline{\psi(g)}} (g \circ e)(ax) \\ &= (-1)^{\overline{a}\cdot\overline{\psi(g)}} \psi(g)(ax), \end{aligned}$$

where the third equality follows from g being an A-map, and the fifth equality follows from the fact that $\overline{\psi(g)} = \overline{g \circ e} = \overline{g} + 1$. It is obvious that φ and ψ are mutually inverse, and hence we have our isomorphism. ∎

Now given a supercategory C, we consider the category

$$\mathsf{C}\text{-}\mathsf{smod} := \mathsf{sFun}_{\Bbbk}(\mathsf{C}, \mathsf{smod}_{\Bbbk})$$

of superfunctors between C and $\mathsf{smod}_{\Bbbk}$. It consists of superfunctors as objects and supernatural transformations as morphisms. By our previous definitions, we observe that $\mathsf{C}\text{-}\mathsf{smod}$ is a supercategory (with $\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(F, G) = \mathrm{SNat}(F, G)$).

Proposition 5.3. *For each object X in C there is a superfunctor*

$$\mathrm{ev}_X^{\mathsf{C}} : \mathsf{C}\text{-}\mathsf{smod} \to {}_{\mathrm{End}_{\mathsf{C}}(X)}\mathsf{smod}$$

given by evaluation at X.

Proof. Given F in $\mathsf{C}\text{-}\mathsf{smod}$, why is it that $F(X)$ is a left $\mathrm{End}_{\mathsf{C}}(X)$-supermodule? Well, first note that F induces an even linear map $F_{X,X} : \mathrm{Hom}_{\mathsf{C}}(X, X) \to \mathrm{Hom}_{\Bbbk}(F(X), F(X))$, that is, $\mathrm{End}_{\mathsf{C}}(X) \to \mathrm{End}_{\Bbbk}(F(X))$.

So for $f \in \mathrm{End}_{\mathsf{C}}(X)$ and $y \in F(X)$, we define $f.y := F(f)(y)$. Moreover, for $f, g \in \mathrm{End}_{\mathsf{C}}(X)$ and $y \in F(X)$, we have $f.(g.y) = f.(Fg(y)) = Ff(Fg(y))$. On the other hand, since F respects composition, we have $(fg).y = F(fg)(y) = (FfFg)(y) = Ff(Fg(y))$. Finally, F sends the identity to the identity which means $1.y = y$, and we see that we really do have a $\mathrm{End}_{\mathsf{C}}(X)$-action on $F(X)$.

What about morphisms? Suppose $\eta \in \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(F, G)$. Then $\mathrm{ev}_X^{\mathsf{C}}$ sends η to its section at X, $\eta_X : F(X) \to G(X)$. We wish to see that η_X is a $\mathrm{End}_{\mathsf{C}}(X)$-map. That is, we wish to show that for every $y \in F(X)$ and every $f \in \mathrm{End}_{\mathsf{C}}(X)$,

$$\eta_X(f.y) = (-1)^{\overline{\eta_X}\cdot\overline{f}} f.(\eta_X(y)).$$

Note that $\eta_X(f.y) = \eta_X(Ff(y))$ and $f.(\eta_X(y)) = Gf(\eta_X(y))$. Since η is a supernatural transformation, we know $\eta_X \circ Ff = (-1)^{\overline{\eta}\cdot\overline{f}} Gf \circ \eta_X$ which exactly gives $\eta_X(Ff(y)) = (-1)^{\overline{\eta_X}\cdot\overline{f}} Gf(\eta_X(y))$, as desired. That composition and identity are respected is obvious. ∎

Next, notice that $\Pi : \mathsf{smod}_{\Bbbk} \to \mathsf{smod}_{\Bbbk}$ is superfunctor. Then $\Pi \circ \_ : \mathsf{C}\text{-}\mathsf{smod} \to \mathsf{C}\text{-}\mathsf{smod}$ is a superfunctor. Specifically, given $F \in \mathsf{C}\text{-}\mathsf{smod}$, denote $(\Pi \circ \_)(F)$ by ΠF the functor which sends an object X in C to $\Pi(F(X))$ and sends a morphism $f : X \to Y$ in C to the linear map $(-1)^{\overline{F(f)}} F(f)$ (which equals $(-1)^{\overline{f}} F(f)$ since F is even). Also, a morphism $\eta \in$

$\mathrm{Hom}_{\textsf{C-smod}}(F,G)$ gets sent to the supernatural transformation $\Pi\eta$ which has $\Pi\eta_X^p = \Pi(\eta_X^p)$ as its homogeneous components.

Furthermore, for V in C, if we let $\mathrm{id}_{\Pi V\to V} : \Pi V \to V$ be the odd linear map which is the identity on the underlying $\Bbbk$-supermodule, we get an induced odd supernatural transformation $\mathrm{id}_{\Pi F\to F} : \Pi F \to F$ for any F in $\textsf{C-smod}$ (which has $\mathrm{id}_{\Pi F(V)\to F(V)}$ as its sections). Similarly, we have an odd supernatural transformation $\mathrm{id}_{F\to\Pi F} : F \to \Pi F$. These are clearly isomorphisms.

Proposition 5.4. *Let* C *be a supercategory and consider* $\textsf{C-smod}$*. Then for every* F, G *in* $\textsf{C-smod}$*, we have an odd natural isomorphism*

$$\mathrm{Hom}_{\textsf{C-smod}}(F,G) \cong \mathrm{Hom}_{\textsf{C-smod}}(\Pi F,G) \qquad \textit{via} \qquad \eta \mapsto \eta \circ \mathrm{id}_{\Pi F\to F}$$

Proof. The discussion immediately preceeding this proposition shows that this map is well-defined. An inverse to this map is given by $\rho \mapsto \rho \circ \mathrm{id}_{F\to\Pi F} : F \to \Pi F$. ∎

Denote by $h^X := \mathrm{Hom}_{\mathsf{C}}(X, \_)$ the obvious functor in $\textsf{C-smod}$.

Proposition 5.5. *Let* C *be a supercategory and consider* $\textsf{C-smod}$*. Then for every object* X *in* C *and every* F *in* $\textsf{C-smod}$*, we have a natural isomorphism*

$$\mathrm{Hom}_{\textsf{C-smod}}(h^X, F)_{\bar{1}} \cong \mathrm{Hom}_{\textsf{C-smod}}(h^X, \Pi F)_{\bar{0}}.$$

Proof. We will construct mutually inverse maps φ and ψ as follows. First, for any object Y in C, and for any object F in $\textsf{C-smod}$, let $e_Y : F(Y) \to \Pi F(Y)$ denote the linear map which is just the identity on the underlying $\Bbbk$-supermodule. Note that we use the same notation regardless of which functor F we are talking about, and also that e_Y is always an odd map. It follows that the e_Y define an odd homogeneous supernatural transformation $e \in \mathrm{Hom}_{\textsf{C-smod}}(F, \Pi F)_{\bar{1}}$ for any object F in $\textsf{C-smod}$.

To verify this claim, note that given any $y \in F(Y)$ and $f : Y \to Z$, we have

$$\begin{aligned}
[(\Pi F)(f)] \circ [e_Y(y)] &= [(-1)^{\bar{f}} F(f)] \circ [e_Y(y)] \\
&= [(-1)^{\bar{f}} F(f)](y) \\
&= (-1)^{\bar{f}} F(f)(y).
\end{aligned}$$

On the other hand,

$$e_Z \circ [F(f)(y)] = F(f)(y),$$

and we have our claim.

Now define $\varphi : \mathrm{Hom}_{\textsf{C-smod}}(h^X, F)_{\bar{1}} \to \mathrm{Hom}_{\textsf{C-smod}}(h^X, \Pi F)_{\bar{0}}$ via $\eta \mapsto e \circ \eta$. Define $\psi : \mathrm{Hom}_{\textsf{C-smod}}(h^X, \Pi F)_{\bar{0}} \to \mathrm{Hom}_{\textsf{C-smod}}(h^X, F)_{\bar{1}}$ via $\nu \mapsto e \circ \nu$.

To see that φ is well-defined, note that

$$\begin{aligned}[(\Pi F)(f)] \circ [e_Y \circ \eta_Y] &= [(\Pi F)(f) \circ e_Y] \circ \eta_Y \\ &= [(-1)^{\overline{f}} e_Z \circ F(f)] \circ \eta_Y \\ &= (-1)^{\overline{f}} e_Z \circ [F(f) \circ \eta_Y] \\ &= (-1)^{\overline{f}} e_Z \circ [(-1)^{\overline{f} \cdot \overline{\eta}} \eta_Z \circ h^X(f)] \\ &= (-1)^{\overline{f}} e_Z \circ [(-1)^{\overline{f} \cdot 1} \eta_Z \circ h^X(f)] \\ &= (-1)^{\overline{f} + \overline{f}} [e_Z \circ \eta_Z] \circ h^X(f) \\ &= [e_Z \circ \eta_Z] \circ h^X(f),\end{aligned}$$

so indeed we see that $e \circ \eta$ is an even supernatural transformation from h^X to ΠF.

Similarly, to see that ψ is well-defined, note that $\Pi \circ \Pi$ is the identity, so

$$\begin{aligned}[F(f)] \circ [e_Y \circ \psi_Y] &= [(-1)^{\overline{f}} e_Z \circ \Pi F(f)] \circ \psi_Y \\ &= (-1)^{\overline{f}} e_Z \circ [\Pi F(f) \circ \psi_Y] \\ &= (-1)^{\overline{f}} e_Z \circ [\psi_Z \circ h^X(f)] \\ &= (-1)^{\overline{f}} [e_Z \circ \psi_Z] \circ h^X(f),\end{aligned}$$

and hence $e \circ \psi$ is an odd supernatural transformation from h^X to F.

That ϕ and ψ are mutually inverse follows from the fact that $\Pi \circ \Pi$ is the identity. ◻

Lemma 5.6 (Yoneda). *Let C be a supercategory. For every object X in C and every F in $\mathsf{C}\text{-}\mathsf{smod}$, we have an isomorphism*

$$\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^X, F) \cong F(X)$$

which is natural (with respect to even morphisms) in both X and F.

Proof. Define $\Phi : \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^X, F)_{\overline{0}} \to F(X)_{\overline{0}}$ via $\eta \mapsto \eta_X(\mathrm{id}_X)$. Define $\Psi : F(X)_{\overline{0}} \to \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^X, F)_{\overline{0}}$ via $v \mapsto \psi^v$ the supernatural transformation whose section is defined by $\psi^v_Y(f) := F(f)(v)$ for $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$.

Note that id_X is an even element of $\mathrm{Hom}_{\mathsf{C}}(X, X)$ and since $\eta \in \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^X, F)_{\overline{0}}$, it follows that $\eta \mapsto \eta_X(\mathrm{id}_X) \in F(X)_{\overline{0}}$ under Φ, hence, it is well-defined. Next, note that for $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$,

$$\begin{aligned}[F(g) \circ \psi^v_Y](f) &= F(g) \circ [F(f)(v)] \\ &= F(gf)(v).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[\psi^v_Y \circ h^X(g)](f) &= \psi^v_Y \circ [g \circ f] \\ &= F(gf)(v).\end{aligned}$$

Hence ψ^v is an even supernatural transformation, and we see that Ψ is well-defined. Now we check that these maps are mutually inverse:

$$\begin{aligned}[\Phi \circ \Psi](v) &= \Phi(\psi^v) \\ &= \psi^v_X(\mathrm{id}_X) \\ &= F(\mathrm{id}_X)(v) \\ &= \mathrm{id}_{F(X)}(v) \\ &= v\end{aligned}$$

and

$$\begin{aligned}[\Psi \circ \Phi](\eta) &= \Psi \circ [\eta_X(\mathrm{id}_X)] \\ &= \psi^{\eta_X(\mathrm{id}_X)},\end{aligned}$$

where $\psi^{\eta_X(\mathrm{id}_X)}$ has sections given by:

$$\begin{aligned}\psi^{\eta_X(\mathrm{id}_X)}_Y(f) &= F(f) \circ [\eta_X(\mathrm{id}_X)] \\ &= [F(f) \circ \eta_X](\mathrm{id}_X) \\ &= [\eta_Y \circ h^X(f)](\mathrm{id}_X) \\ &= \eta_Y \circ [f \circ \mathrm{id}_X] \\ &= \eta_Y(f).\end{aligned}$$

Therefore, $\psi^{\eta_X(\mathrm{id}_X)}$ is the same supernatural transformation as η. We've just established $\mathrm{Hom}_{\textsf{C-smod}}(h^X, F)_{\bar{0}} \cong F(X)_{\bar{0}}$.

Now by Proposition 5.5, we know that $\mathrm{Hom}_{\textsf{C-smod}}(h^X, F)_{\bar{1}} \cong \mathrm{Hom}_{\textsf{C-smod}}(h^X, \Pi F)_{\bar{0}}$. By the result just proven, $\mathrm{Hom}_{\textsf{C-smod}}(h^X, \Pi F)_{\bar{0}} \cong (\Pi F)(X)_{\bar{0}}$. But $(\Pi F)(X)_{\bar{0}} = (\Pi \circ \_)(F(X)_{\bar{0}}) \cong F(X)_{\bar{1}}$. It follows that we have our full isomorphism $\mathrm{Hom}_{\textsf{C-smod}}(h^X, F) \cong F(X)$.

To see that Φ is natural in F, suppose $\nu : F \to G$ is an (even) supernatural transformation. Then $\nu \circ \_ : \mathrm{Hom}_{\textsf{C-smod}}(h^X, F) \to \mathrm{Hom}_{\textsf{C-smod}}(h^X, G)$ since the composition of (even) supernatural transformations is again a (even) supernatural transformation. Then $(\nu_P \circ \Phi)(\eta) = \nu_P(\eta_P(\mathrm{id}_P))$. On the other hand, $(\Phi \circ (\nu \circ \_))(\eta) = \Phi(\nu \circ \eta) = (\nu \circ \eta)_P(\mathrm{id}_P) = \nu_P(\eta_P(\mathrm{id}_P))$. Hence, the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\textsf{C-smod}}(h^X, F) \arrow[r, "\Phi"] \arrow[d, "\nu\circ\_"'] & F(X) \arrow[d, "\nu_X"] \\ \mathrm{Hom}_{\textsf{C-smod}}(h^X, G) \arrow[r, "\Phi"'] & G(X) \end{tikzcd} .$$

To see that Φ is functorial in X, let $f \in \mathrm{Hom}_{\textsf{C}}(X_1, X_2)$ and $g \in \mathrm{Hom}_{\textsf{C}}(Y_1, Y_2)$ be even. Then the associativity of composition implies that the following square commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\textsf{C}}(X_2, Y_1) \arrow[r, "g\circ\_"] \arrow[d, "\_\circ f"'] & \mathrm{Hom}_{\textsf{C}}(X_2, Y_2) \arrow[d, "\_\circ f"] \\ \mathrm{Hom}_{\textsf{C}}(X_1, Y_1) \arrow[r, "g\circ\_"'] & \mathrm{Hom}_{\textsf{C}}(X_1, Y_2) \end{tikzcd} .$$

From this we see easily that for an even $\eta \in \mathrm{Hom}_{\textsf{C-smod}}(h^{X_1}, F)$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X_2,Y_1) \arrow[r, "g\circ_"] \arrow[d, "_\circ f"'] & \mathrm{Hom}_{\mathsf{C}}(X_2,Y_2) \arrow[d, "_\circ f"] \\
\mathrm{Hom}_{\mathsf{C}}(X_1,Y_1) \arrow[r, "g\circ_"'] \arrow[d, "\eta_{Y_1}"'] & \mathrm{Hom}_{\mathsf{C}}(X_1,Y_2) \arrow[d, "\eta_{Y_2}"] \\
F(Y_1) \arrow[r, "Fg"'] & F(Y_2)
\end{tikzcd} .$$

Moreover, this means $\eta\circ(\_\circ f)\in \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^{X_2},F)$. So $f: X_1\to X_2$ induces a map $f^*:\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^{X_1},F)\to \mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^{X_2},F)$.

Note that $(\Phi\circ f^*)(\eta)=[\eta\circ(\_\circ f)]_{X_2}(\mathrm{id}_{X_2})=\eta_{X_2}(f)$. On the other hand, using the above diagram, we see that $(Ff\circ\Phi)(\eta)=Ff(\eta_{X_1}(\mathrm{id}_{X_1}))=[\eta_{X_2}\circ(f\circ\_)](\mathrm{id}_{X_1})=\eta_{X_2}(f)$. Hence the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^{X_1},F) \arrow[r, "\Phi"] \arrow[d, "f^*"'] & F(X_1) \arrow[d, "Ff"] \\
\mathrm{Hom}_{\mathsf{C}\text{-}\mathsf{smod}}(h^{X_2},F) \arrow[r, "\Phi"'] & F(X_2)
\end{tikzcd} ,$$

and we have our desired result. ◻

Remark 5.7. Note that $\mathsf{smod}_{\Bbbk}$ is a symmetric monoidal supercategory under the usual tensor product of $\Bbbk$-modules where the braiding is the super flip map

$$x\otimes y\mapsto (-1)^{\overline{x}\cdot\overline{y}}y\otimes x,$$

and where composition of morphisms is defined as

$$(f\otimes g)\circ(h\otimes k)=(-1)^{\overline{g}\cdot\overline{h}}(f\circ h)\otimes(g\circ k).$$

Then using this monoidal structure, one can introduce an operation $\_\boxtimes\_$ which makes the category sCat of all supercategories monoidal. For any supercategories $\mathcal{C}$ and $\mathcal{D}$, we can form a supercategory $\mathcal{C}\boxtimes\mathcal{D}$ whose objects are ordered pairs of objects (X,Y) where X and Y are objects of $\mathcal{C}$ and $\mathcal{D}$, respectively, and whose morphism spaces are defined as

$$\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}((W,X),(Y,Z)):=\mathrm{Hom}_{\mathcal{C}}(W,Y)\otimes\mathrm{Hom}_{\mathcal{D}}(X,Z).$$

Composition is given by the formula above. The unit object $\boldsymbol{I}$ is the supercategory with one object whose morphism space is $\Bbbk$ concentrated in degree 0.

6. Generalized Strict Polynomial Functors & Strict Polynomial Superfunctors

Let $\Bbbk$ be a commutative domain which is not characteristic 2. Let A be a unital (free) $\Bbbk$-superalgebra. Let V denote the category of free finite right A-supermodules including parity shifts, so that an object looks like $V\cong A^{m|n}$. Let $\mathcal{V}$ denote the category of free finite right A-supermodules without parity shifts (unshifted), so an object looks like $V=A^n$. Then we have V, $\mathcal{V}$, $\Gamma^d\mathsf{V}$, and $\Gamma^d\mathcal{V}$ all being supercategories. Moreover, it follows from definitions and proposition 4.12 that for any unital good pair $(A,\mathfrak{a})$, $\mathsf{T}^A_{\mathfrak{a}}(d)$ is a supercategory.

There are two flavors of generalized strict polynomial functors we wish to introduce. The differences are whether or not we consider parity shifts of supermodules and which morphisms we allow. In what follows, these differences are only apparent in the proof of lemma 6.7.

Definition 6.1. A **homogeneous generalized strict polynomial functor over $(A, \mathfrak{a})$ of degree d** is a superfunctor

$$F : \mathsf{T}^A_{\mathfrak{a}}(d) \to \mathsf{smod}_{\Bbbk}.$$

Such a functor is called a *generalized strict polynomial functor* because of its connection with generalized Schur (super)algebras. We emphasize 'general' in the name and not 'super' since taking A to be concentrated in even degree still yields an interesting generalized Schur algebra which is not $\mathbb{Z}_2$-graded.

Definition 6.2. A **homogeneous strict polynomial superfunctor over A of degree d** is a superfunctor

$$F : \Gamma^d \mathsf{V} \to \mathsf{smod}_{\Bbbk}.$$

Such a functor is called a *strict polynomial superfunctor* since the main emphasis here is to consider free A-supermodules with even and odd A-basis elements. In particular, even if A is concentrated in degree 0, the corresponding Schur superalgebra (see definition 3.3) will usually be a superalgebra.

Definition 6.3. Let $\mathsf{P}^d_{(A,\mathfrak{a})}$ denote the (super)category of degree d homogeneous generalized strict polynomial functors over $(A, \mathfrak{a})$ whose morphisms are supernatural transformations. That is,

$$\mathsf{P}^d_{(A,\mathfrak{a})} := \mathsf{T}^A_{\mathfrak{a}}(d)\text{-}\mathsf{smod}.$$

Then the category $\mathsf{P}_{(A,\mathfrak{a})}$ of arbitrary generalized strict polynomial functors over $(A, \mathfrak{a})$ is defined to be

$$\mathsf{P}_{(A,\mathfrak{a})} := \bigoplus_{d \in \mathbb{N}} \mathsf{P}^d_{(A,\mathfrak{a})}.$$

Definition 6.4. Let P^d_A denote the (super)category of degree d homogeneous strict polynomial superfunctors over A whose morphisms are supernatural transformations. That is,

$$\mathsf{P}^d_A := \Gamma^d \mathsf{V}\text{-}\mathsf{smod}.$$

Then the category P_A of arbitrary strict polynomial superfunctors over A is defined to be

$$\mathsf{P}_A := \bigoplus_{d \in \mathbb{N}} \mathsf{P}^d_A.$$

It follows from remark 4.18 that

$$\mathsf{P}^d_{(A,A_0)} = \Gamma^d \mathcal{V}\text{-}\mathsf{smod}.$$

In this way, one can recover $\mathsf{P}_{(A,A_0)}$ from P_A by restricting to $\mathcal{V}$. However, for $\mathfrak{a} \neq A_0$, there is no way to do this.

Note that $(\mathsf{smod}_{\Bbbk})_{\mathsf{ev}}$, the category with the same objects as $\mathsf{smod}_{\Bbbk}$ but only even morphisms, is an abelian category. Since kernels, cokernels, products and coproducts can be computed pointwise in $(\mathsf{smod}_{\Bbbk})_{\mathsf{ev}}$, it follows that $(\mathsf{P}_{(A,\mathfrak{a})})_{\mathsf{ev}}$ and $(\mathsf{P}_A)_{\mathsf{ev}}$ are also abelian. Thus the supercategories $\mathsf{P}_{(A,\mathfrak{a})}$ and P_A are abelian.

Remark 6.5. Define

$$\mathrm{ev}^d_V : \mathsf{P}^d_{(A,\mathfrak{a})} \to {}_{\mathrm{End}_{\mathsf{T}^A_{\mathfrak{a}}(d)}(V)}\mathsf{smod}$$

to be evaluation at V. Notice that $\mathrm{ev}_V^d = \mathrm{ev}_V^{\mathsf{C}}$ from proposition 5.3 for $\mathsf{C} = \mathsf{T}_{\mathfrak{a}}^A(d)$. Then proposition 5.3 implies that ev_V^d is well-defined. So in particular, for $V = A^n$,

$$\mathrm{ev}_{A^n}^d : \mathsf{P}_{(A,\mathfrak{a})}^d \to {}_{\mathrm{End}_{\mathsf{T}_{\mathfrak{a}}^A(d)}(A^n)}\mathsf{smod} = {}_{T_{\mathfrak{a}}^A(n,n;d)}\mathsf{smod}.$$

Similarly, evaluation at V,

$$\mathrm{ev}_V^d : \mathsf{P}_A^d \to {}_{\mathrm{End}_{\Gamma^d\mathsf{V}}(V)}\mathsf{smod},$$

is well-defined. In particular, by remark 3.4,

$$\mathrm{ev}_{A^{m|n}}^d : \mathsf{P}_A^d \to {}_{\mathrm{End}_{\Gamma^d\mathsf{V}}(A^{m|n})}\mathsf{smod} = {}_{S^A(m|n,d)}\mathsf{smod}.$$

6.1. Equivalences with ${}_{S^A(m|n,d)}\mathsf{smod}$ and ${}_{T_{\mathfrak{a}}^A(n,n;d)}\mathsf{smod}$

The proof technique used in this section follows a general framework that shows when the functor from proposition 5.3 induces an equivalence of (super)categories. We follow and expand upon this framework found in Appendix A of [Axt13] which in turn is a super-generalization of the non-super result from Appendix A of [Tou13]. To be explicit, we go through the proofs in detail for our specific supercategories $\mathsf{P}_{(A,\mathfrak{a})}, \mathsf{P}_A$.

Let $\mathcal{P} \in \{\mathsf{P}_{(A,\mathfrak{a})}, \mathsf{P}_A\}$. An object being projective is a property concerning even morphisms, so we say that F in $\mathcal{P}$ is projective if F is a projective object in $\mathcal{P}_{\mathsf{ev}}$. Since $\mathcal{P}_{\mathsf{ev}}$ is abelian, an object F in $\mathcal{P}_{\mathsf{ev}}$ being projective is equivalent to $\mathrm{Hom}_{\mathcal{P}}(F,\_) : \mathcal{P}_{\mathsf{ev}} \to (\mathsf{smod}_{\Bbbk})_{\mathsf{ev}}$ being exact. Note that for $\eta : Q \to R$ a supernatural transformation, $\mathrm{Hom}_{\mathcal{P}}(F,\_)(\eta) = \eta \circ \_$. It follows that if η is an even morphsim, then $\eta \circ \_$ is an even linear map, so that $\mathrm{Hom}_{\mathcal{P}}(F,\_) : \mathcal{P}_{\mathsf{ev}} \to (\mathsf{smod}_{\Bbbk})_{\mathsf{ev}}$ makes sense for any F in $\mathcal{P}$.

We also make use of the following equivalent definition of F being a projective object in $\mathcal{P}_{\mathsf{ev}}$: An epimorphism $\rho : H \to G$ in $\mathcal{P}_{\mathsf{ev}}$ is an (even) epi supernatural transformation $\rho : H \to G$, by which we mean that every section $\rho_X : H(X) \to G(X)$ is an (even) surjection in $\mathsf{smod}_{\Bbbk}$. Then given objects G, H in $\mathcal{P}_{\mathsf{ev}}$ and an (even) morphism $\eta : F \to G$ and (even) epimorphism $\rho : H \to G$, F is projective iff there exists a lift $\tilde{\eta} : F \to H$ such that $\rho\tilde{\eta} = \eta$.

Lemma 6.6. *Let $(\mathcal{P}^d, \mathcal{C}^d) \in \{(\mathsf{P}_{(A,\mathfrak{a})}^d, \mathsf{T}_{\mathfrak{a}}^A(d)), (\mathsf{P}_A^d, \Gamma^d\mathsf{V})\}$. For any d and any object P in $\mathcal{C}^d$, $h^{d,P} := \mathrm{Hom}_{\mathcal{C}^d}(P,\_)$ is a projective object in $\mathcal{P}^d$.*

Proof. Let F, G in $\mathcal{P}^d$. Let $\eta : F \to G$ be an even epi supernatural transformation. This means for every $X \in \mathcal{C}^d$, the section $\eta_X : F(X) \to G(X)$ is an (even) surjection. Then the naturality of the Yoneda isomorphism (lemma 5.6) gives a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{P}^d}(h^{d,P}, F) \arrow[r, "\cong"] \arrow[d, "\eta\circ\_"'] & F(P) \arrow[d, two heads, "\eta_P"] \\ \mathrm{Hom}_{\mathcal{P}^d}(h^{d,P}, G) \arrow[r, "\cong"'] & G(P) \end{tikzcd},$$

and since η_P is surjective, we see that $\eta \circ \_$ must also be surjective. This means that given a morphism $\psi : h^{d,P} \to G$ and an epimorphism $\eta : F \to G$, that there exists some $\nu : h^{d,P} \to F$ such that $\psi = \eta \circ \nu$. This is the definition of $h^{d,P}$ being a projective object. Since we are working in an abelian category, this is equivalent to $\mathrm{Hom}_{\mathcal{P}^d}(h^{d,P},\_)$ being right exact (and hence exact). ∎

Lemma 6.7. *Let $m, n \geqslant d$. Let $(\mathcal{C}, X) \in \{(\mathsf{T}_{\mathfrak{a}}^A(d), A^n), (\Gamma^d\mathsf{V}, A^{m|n})\}$. For all V, W in $\mathcal{C}$, the map induced by composition*

$$\mathrm{Hom}_{\mathcal{C}}(X, W) \otimes \mathrm{Hom}_{\mathcal{C}}(V, X) \to \mathrm{Hom}_{\mathcal{C}}(V, W)$$

is surjective.

Proof. We will provide two slightly different proofs for $\mathsf{T}_{\mathfrak{a}}^A(d)$ and $\Gamma^d\mathsf{V}$. For $\mathsf{T}_{\mathfrak{a}}^A(d)$, we work with $\Bbbk$-bases. This technique is possible for $\Gamma^d\mathsf{V}$, as well, but the indexing becomes much more cumbersome. Hence, we choose to work with A-bases for this case which we believe is much more readable. It also better highlights the role that parity shifts of A play.

We'll start with $\mathsf{T}_{\mathfrak{a}}^A(d)$. Let V and W be objects of $\mathsf{T}_{\mathfrak{a}}^A(d)$ so that $V = A^\ell$ and $W = A^k$ for some $\ell, k \in \mathbb{Z}_{>0}$. So then we see the map in question is really

$$T_{\mathfrak{a}}^A(k,n;d) \otimes T_{\mathfrak{a}}^A(n,\ell;d) \to T_{\mathfrak{a}}^A(k,\ell;d).$$

First, we choose a $\Bbbk$-basis $B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_1$ for A such that $B_{\mathfrak{a}}$ contains 1_A. Let $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}$ be a $\Bbbk$-basis element of $T_{\mathfrak{a}}^A(k,\ell;d)$ as in corollary 4.9. Then to show this map is surjective, we need to find elements of $T_{\mathfrak{a}}^A(k,n;d)$ and $T_{\mathfrak{a}}^A(n,\ell;d)$ which compose to give $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}$.

Recall that

$$\begin{aligned}\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} &\coloneqq \sum_{\sigma\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} \varphi_{(\vec{b},\vec{r},\vec{s})}.\sigma \\ &= \sum_{\sigma\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)}\varphi^{b_{\sigma 1}}_{r_{\sigma 1}s_{\sigma 1}} \otimes \cdots \otimes \varphi^{b_{\sigma d}}_{r_{\sigma d}s_{\sigma d}}.\end{aligned}$$

Note that for $\vec{1}_A = (1_A, \ldots, 1_A) \in B^d$ and for any $\vec{p} \in [1,k]^d$ and $\vec{q} \in [1,n]^d$ we have $(\vec{1}_A, \vec{p}, \vec{q}) \in \mathrm{Tri}^B(k,n;d)$. Therefore, $\widetilde{\eta}_{(\vec{1}_A,\vec{r},\vec{q})}$ is a $\Bbbk$-basis element of $T_{\mathfrak{a}}^A(k,n;d)$ for any $\vec{q} \in [1,n]^d$.

Now we claim that it is possible to choose $\vec{i} \in [1,n]^d$ such that $\mathfrak{S}_{\vec{b},\vec{r},\vec{s}} = \mathfrak{S}_{\vec{1}_A,\vec{r},\vec{i}} = \mathfrak{S}_{\vec{b},\vec{i},\vec{s}}$. To see this, first notice that for any tuple $(\vec{h},\vec{u},\vec{v}) \in B^d \times [1,t_1]^d \times [1,t_2]^d$ (for any $t_1, t_2 \geqslant 1$), we have $\mathfrak{S}_{\vec{h},\vec{u},\vec{v}} = \mathfrak{S}_{\vec{h}} \cap \mathfrak{S}_{\vec{u}} \cap \mathfrak{S}_{\vec{v}}$. So we need to be able to choose $\vec{i} \in [1,n]^d$ such that its stabilizer under the $\mathfrak{S}_d$ action is equal to the smallest of the stabilizers of $\vec{b}, \vec{r}, \vec{s}$. The worst case scenario is when $\mathfrak{S}_{\vec{b},\vec{r},\vec{s}} = \{1\}$. Therefore, we just need to be able to choose distinct entries for $\vec{i}$ which is possible since we've assumed that $n \geqslant d$.

So for this chosen $\vec{i}$, we have $\widetilde{\eta}_{(\vec{1}_A,\vec{r},\vec{i})}$ is a $\Bbbk$-basis element of $T_{\mathfrak{a}}^A(k,n;d)$. Moreover, it is easy to see that $\widetilde{\eta}_{(\vec{b},\vec{i},\vec{s})}$ is a $\Bbbk$-basis element of $T_{\mathfrak{a}}^A(n,\ell;d)$. Notice that $[\vec{1}_A,\vec{r},\vec{i}]^!_{\mathfrak{c}} = 1$ so that $\widetilde{\eta}_{(\vec{1}_A,\vec{r},\vec{i})} = \widetilde{\varphi}_{(\vec{1}_A,\vec{r},\vec{i})}$. Also, note that since we've chosen $\vec{i}$ to have the appropriate stabilizer, we have $[\vec{b},\vec{i},\vec{s}]^!_{\mathfrak{c}} = [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}$. Thus, we have

$$\begin{aligned}\widetilde{\eta}_{(\vec{1}_A,\vec{r},\vec{i})}\widetilde{\eta}_{(\vec{b},\vec{i},\vec{s})} &= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\widetilde{\varphi}_{(\vec{1}_A,\vec{r},\vec{i})}\widetilde{\varphi}_{(\vec{b},\vec{i},\vec{s})} \\ &= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\left(\sum_{\sigma\in{}^{\vec{1}_A,\vec{r},\vec{i}}\mathscr{D}} \varphi^{1_A}_{r_{\sigma 1}i_{\sigma 1}}\otimes\cdots\otimes\varphi^{1_A}_{r_{\sigma d}i_{\sigma d}}\right)\left(\sum_{\tau\in{}^{\vec{b},\vec{i},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\tau)}\varphi^{b_{\tau 1}}_{i_{\tau 1}s_{\tau 1}}\otimes\cdots\otimes\varphi^{b_{\tau d}}_{i_{\tau d}s_{\tau d}}\right) \\ &= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\sigma\in{}^{\vec{1}_A,\vec{r},\vec{i}}\mathscr{D}}\ \sum_{\tau\in{}^{\vec{b},\vec{i},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\tau)}\varphi^{1_A}_{r_{\sigma 1}i_{\sigma 1}}\varphi^{b_{\tau 1}}_{i_{\tau 1}s_{\tau 1}} \otimes \cdots \otimes \varphi^{1_A}_{r_{\sigma d}i_{\sigma d}}\varphi^{b_{\tau d}}_{i_{\tau d}s_{\tau d}},\end{aligned} \tag{52}$$

where the second line follows since $s(\vec{1}_A,\sigma) = 0$ for all σ, and there is no extra sign appearing in the third line since 1_A is even.

Now since $\mathfrak{S}_{\vec{b},\vec{r},\vec{s}} = \mathfrak{S}_{\vec{1}_A,\vec{r},\vec{i}} = \mathfrak{S}_{\vec{b},\vec{i},\vec{s}}$, each sum above ranges over the same set of elements of $\mathfrak{S}_d$. Moreover, since these elements give distinct elements of the orbit, we see that $i_{\sigma j} = i_{\tau j}$

for each $j \in [1,d]$ only when $\sigma = \tau$. Then it follows that we have

$$\begin{aligned}
(52) &= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\tau)} \varphi^{1_A b_{\sigma 1}}_{r_{\sigma 1} s_{\sigma 1}} \otimes \cdots \otimes \varphi^{1_A b_{\sigma d}}_{r_{\sigma d} s_{\sigma d}} \\
&= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\tau)} \varphi^{b_{\sigma 1}}_{r_{\sigma 1} s_{\sigma 1}} \otimes \cdots \otimes \varphi^{b_{\sigma d}}_{r_{\sigma d} s_{\sigma d}} \\
&= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} \\
&= \widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})},
\end{aligned}$$

as desired. Note that being able to choose the appropriately sized stabilizer allows us to avoid getting too few or too many terms in our morphisms.

Now we handle $\Gamma^d\mathsf{V}$. For $V, W, A^{m|n}$ in $\Gamma^d\mathsf{V}$, Let $\{v_k\}$, $\{w_\ell\}$, $\{e_i\}$ be A-bases for $V, W, A^{m|n}$, respectively. We need to consider the $\mathfrak{S}_d$-action on the morphism spaces in question. We view our morphism spaces $\mathrm{Hom}_{\Gamma^d\mathsf{V}}(V,W)$, for example, as $\mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ instead of $\left(\mathrm{Hom}_A(V,W)^{\otimes d}\right)^{\mathfrak{S}_d}$ which is possible as observed in the proof of lemma 3.2.

First, we must introduce some notation. Let $v_{\vec{r}} = v_{r_1} \otimes \cdots \otimes v_{r_d} \in V^{\otimes d}$. Then $v_{\vec{r}}\sigma = (-1)^{s(v_{\vec{r}},\sigma)} v_{\vec{r}\sigma}$. Similarly for the other modules.

Now for $f \in \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ to be invariant under the $\mathfrak{S}_d$-action, we need $f(v_{\vec{r}}.\sigma) = f(v_{\vec{r}})\sigma$ for each basis element $v_{\vec{r}}$ and each σ. So, as in remark 1.2 and lemma 1.10, write f as a matrix so that $f = \sum_{\vec{\ell},\vec{k}} \gamma^{a_{\vec{\ell}\vec{k}}}_{\vec{\ell}\vec{k}}$ (where $\gamma^{a_{\vec{\ell}\vec{k}}}_{\vec{\ell}\vec{k}}$ is the map $v_{\vec{j}} \mapsto \delta_{\vec{k}\vec{j}} w_{\vec{\ell}} a_{\vec{\ell}\vec{k}}$). On one hand, we have

$$\begin{aligned}
\left(\sum_{\vec{\ell},\vec{k}} \gamma^{a_{\vec{\ell}\vec{k}}}_{\vec{\ell}\vec{k}}\right)(v_{\vec{r}}.\sigma) &= \sum_{\vec{\ell},\vec{k}} \left(\gamma^{a_{\vec{\ell}\vec{k}}}_{\vec{\ell}\vec{k}}\right)(v_{\vec{r}}.\sigma) \\
&= \sum_{\vec{\ell},\vec{k}} \left(\gamma^{a_{\vec{\ell}\vec{k}}}_{\vec{\ell}\vec{k}}\right)\left((-1)^{s(v_{\vec{r}},\sigma)} v_{\vec{r}\sigma}\right) \\
&= \sum_{\vec{\ell}} (-1)^{s(v_{\vec{r}},\sigma)} w_{\vec{\ell}} a_{\vec{\ell},\vec{r}\sigma} \\
&= \sum_{\vec{\ell}} (-1)^{s(v_{\vec{r}},\sigma)} w_{\vec{\ell}\sigma} a_{\vec{\ell}\sigma,\vec{r}\sigma} \qquad (53)
\end{aligned}$$

with the last line coming from reindexing the sum, which is possible since σ is a bijection.

On the other hand,

$$\begin{aligned}
\left(\left(\sum_{\vec{\ell},\vec{k}} \gamma^{a_{\vec{\ell}\vec{k}}}_{\vec{\ell}\vec{k}}\right)(v_{\vec{r}})\right)\sigma &= \left(\sum_{\vec{\ell}} w_{\vec{\ell}} a_{\vec{\ell}\vec{r}}\right)\sigma \\
&= \sum_{\vec{\ell}} (w_{\vec{\ell}}.\sigma)(a_{\vec{\ell}\vec{r}}.\sigma) \\
&= \sum_{\vec{\ell}} (-1)^{s(w_{\vec{\ell}},\sigma)} w_{\vec{\ell}\sigma}(a_{\vec{\ell}\vec{r}}.\sigma). \qquad (54)
\end{aligned}$$

Since we must have (53)=(54), and since the $w_{\vec{\ell}\sigma}$ are A-basis elements, it must be that

$$a_{\vec{\ell}\sigma,\vec{r}\sigma} = (-1)^{s(w_{\vec{\ell}},\sigma)+s(v_{\vec{r}},\sigma)}(a_{\vec{\ell},\vec{r}}).\sigma, \qquad (55)$$

which holds for all admittable tuples $\vec{\ell}$ and $\vec{r}$ and all $\sigma \in \mathfrak{S}_d$. Notice that this can impose restrictions on what $a_{\vec{\ell},\vec{r}}$ can be.

Let $\{\vec{s},\vec{r}\}$ denote the orbit of the pair of tuples $\vec{s},\vec{r}$ under the $\mathfrak{S}_d$ action given by $(\vec{s},\vec{r})\sigma := (\vec{s}\sigma,\vec{r}\sigma)$. Let $\mathcal{O}$ denote the set of such orbits. Then relation (55) tells us that we may write f as

$$f = \sum_{\{\vec{\ell},\vec{k}\}\in\mathcal{O}} \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{k})}} \gamma^{(b_{\vec{\ell}\vec{k}}.\sigma)}_{\vec{\ell}\sigma,\vec{k}\sigma}(-1)^{s(w_{\vec{\ell}},\sigma)+s(v_{\vec{k}},\sigma)} \tag{56}$$

where we group the terms by $\mathfrak{S}_d$-orbit so that the inner sum runs over all permutations which give distinct elements of the given orbit. We have similar equations for the other morphism spaces, as well.

In particular, for $g \in \operatorname{Hom}_{A^{\otimes d}}\left(V^{\otimes d}, \left(A^{m|n}\right)^{\otimes d}\right)^{\mathfrak{S}_d}$, the relation from (55) becomes

$$b_{\vec{i}\sigma,\vec{k}\sigma} = (-1)^{s(e_{\vec{i}},\sigma)+s(v_{\vec{k}},\sigma)}(b_{\vec{i},\vec{k}}).\sigma \tag{57}$$

so that we can write

$$g = \sum_{\{\vec{i},\vec{k}\}\in\mathcal{O}} \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{i},\vec{k})}} \psi^{(b_{\vec{i}\vec{k}}.\sigma)}_{\vec{i}\sigma,\vec{k}\sigma}(-1)^{s(e_{\vec{i}},\sigma)+s(v_{\vec{k}},\sigma)}.$$

For $h \in \operatorname{Hom}_{A^{\otimes d}}\left(\left(A^{m|n}\right)^{\otimes d}, W^{\otimes d}\right)^{\mathfrak{S}_d}$, the relation from (55) becomes

$$c_{\vec{\ell}\sigma,\vec{i}\sigma} = (-1)^{s(w_{\vec{\ell}},\sigma)+s(e_{\vec{i}},\sigma)}(c_{\vec{\ell},\vec{i}}).\sigma \tag{58}$$

so that we can write

$$h = \sum_{\{\vec{\ell},\vec{i}\}\in\mathcal{O}} \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{i})}} \varphi^{(c_{\vec{\ell}\vec{i}}.\sigma)}_{\vec{\ell}\sigma,\vec{i}\sigma}(-1)^{s(w_{\vec{\ell}},\sigma)+s(e_{\vec{i}},\sigma)}.$$

Notice from (56) that each element of $\operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ is determined by $|\mathcal{O}|$-many elements of $A^{\otimes d}$. Therefore, to surject onto a given $f \in \operatorname{Hom}_{A^{\otimes d}}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$ with our composition map, it is enough to hit every

$$\sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{k})}} \gamma^{(a_{\vec{\ell}\vec{k}}.\sigma)}_{\vec{\ell}\sigma,\vec{k}\sigma}(-1)^{s(w_{\vec{\ell}},\sigma)+s(v_{\vec{k}},\sigma)}$$

for a given initial seed $\{\vec{\ell},\vec{k}\} \in \mathcal{O}$ and corresponding $a_{\vec{\ell},\vec{k}} \in A^{\otimes d}$ (for we can then take a linear combination of compositions to surject onto a general element).

In order to do this, there are two main steps. First, we need to ensure that we can cover each term in the sum regardless of how many terms there are (i.e. regardless of how small the stabilizer of $\{\vec{\ell},\vec{k}\}$ is). This means we need enough freedom to choose a tuple $\vec{i}$ corresponding to a basis element of $A^{m|n}$ with the appropriate sized stabilizer. Specifically, given a $\{\vec{\ell},\vec{k}\} \in \mathcal{O}$ and corresponding $a_{\vec{\ell},\vec{k}} \in B^{\otimes d}$, choose a $\vec{i}$ (corresponding to $e_{\vec{i}}$ of $(A^{m|n})^{\otimes d}$) such that $\operatorname{Stab}_{\mathfrak{S}_d}\left(\vec{i}\right) = \operatorname{Stab}_{\mathfrak{S}_d}\left((\vec{\ell},\vec{k})\right)$ (which is possible since $m+n \geqslant d$ - this ensures that there are enough distinct basis elements of $A^{m|n}$ so that we can find a tuple $\vec{i}$ which has trivial stabilizer, if necessary). Also notice then that $\operatorname{Stab}_{\mathfrak{S}_d}\left((\vec{\ell},\vec{i})\right) = \operatorname{Stab}_{\mathfrak{S}_d}\left((\vec{\ell},\vec{k})\right) = \operatorname{Stab}_{\mathfrak{S}_d}\left((\vec{i},\vec{k})\right)$.

The next step requires us to choose the entries of our matrices in a nice way to ensure we get the correct final entries. Basically, we need enough even and odd basis elements of $A^{m|n}$ to do so. In particular, we would like to let $c_{\vec{\ell}\vec{i}} = 1$ in $A^{\otimes d}$. Now looking at (58), this is only possible if we can also choose $\vec{i}$ from above so that $s(w_{\vec{\ell}}, \sigma) = s(e_{\vec{i}}, \sigma)$ for all σ. To see this claim, notice that if there are any τ such that $\vec{\ell}\tau = \vec{\ell}$, then since $\vec{i}$ has the same stabilizer, we have $\vec{i}\tau = \vec{i}$. But then $c_{\vec{\ell}\tau,\vec{i}\tau} = c_{\vec{\ell},\vec{i}}$, and so we must have $c_{\vec{\ell},\vec{i}} = (-1)^{s(w_{\vec{\ell}},\tau)+s(e_{\vec{i}},\tau)}(c_{\vec{\ell},\vec{i}})\tau$. If we want $c_{\vec{\ell},\vec{i}} = 1$, then $(c_{\vec{\ell},\vec{i}})\tau = c_{\vec{\ell},\vec{i}} = 1$, and we must have $1 = (-1)^{s(w_{\vec{\ell}},\tau)+s(e_{\vec{i}},\tau)}1$. Note that if the stabilizer of $\vec{\ell}$ is trivial, then this is a non-issue, but it could be that $\vec{\ell}$ is fixed by the entire symmetric group.

The only way to ensure $s(w_{\vec{\ell}}, \sigma) = s(e_{\vec{i}}, \sigma)$ for all σ is to choose $\vec{i}$ so that it corresponds to a basis element of $\left(A^{m|n}\right)^{\otimes d}$ which has the same number of even (and odd) components as the element in $W^{\otimes d}$ corresponding to $\vec{\ell}$. This is always possible so long as $m, n \geqslant d$ (just having $m, n \geqslant 1$ ensures you can get the correct number of even and odd components, but we also need the option to make all of these distinct to ensure we get the correct number of terms).

So, choosing the appropriate $\vec{i}$ so that we can set $c_{\vec{\ell}\vec{i}} = 1$, we have $c_{\vec{\ell}\sigma,\vec{i}\sigma} = (-1)^{s(w_{\vec{\ell}},\sigma)+s(e_{\vec{i}},\sigma)} = (-1)^{s(w_{\vec{\ell}},\sigma)+s(w_{\vec{\ell}},\sigma)} = 1$ so that

$$H = \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{i})}} \varphi^1_{\vec{\ell}\sigma,\vec{i}\sigma}$$

is an element of $\mathrm{Hom}_{A^{\otimes d}}((A^{m|n})^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d}$. In terms of matrices, this is like choosing a sparse matrix which has only the identity of $A^{\otimes d}$ as entries (which correspond to the orbit of initial seed $(\vec{\ell}, \vec{i})$).

For the other matrix, we want to be able to use entries that align exactly with the entries of the target matrix. In particular, for the same fixed $\vec{i}$, we want $b_{\vec{i}\sigma,\vec{k}\sigma} = a_{\vec{\ell}\sigma,\vec{k}\sigma}$ for all σ. Looking at relations (55) and (57), we see that this is possible since we've already ensured that $s(e_{\vec{i}}, \sigma) = s(w_{\vec{\ell}}, \sigma)$ for all σ.

So, simply letting $b_{\vec{i}\vec{k}} = a_{\vec{\ell}\vec{k}}$, we see that

$$G = \sum_{\substack{\tau\in\mathfrak{S}_d \\ (\vec{x}\tau,\vec{y}\tau)=(\vec{i},\vec{k})}} \psi^{(a_{\vec{\ell}\vec{k}}\cdot\tau)}_{\vec{i}\tau,\vec{k}\tau}(-1)^{s(w_{\vec{\ell}},\tau)+s(v_{\vec{k}},\tau)}$$

is an element of $\mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d}, (A^{m|n})^{\otimes d})^{\mathfrak{S}_d}$. Then

$$\begin{aligned} H \circ G &= \left(\sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{i})}} \varphi^1_{\vec{\ell}\sigma,\vec{i}\sigma}\right)\left(\sum_{\substack{\tau\in\mathfrak{S}_d \\ (\vec{x}\tau,\vec{y}\tau)=(\vec{i},\vec{k})}} \psi^{(a_{\vec{\ell}\vec{k}}\cdot\tau)}_{\vec{i}\tau,\vec{k}\tau}(-1)^{s(w_{\vec{\ell}},\tau)+s(v_{\vec{k}},\tau)}\right) \\ &= \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{i})}} \sum_{\substack{\tau\in\mathfrak{S}_d \\ (\vec{x}\tau,\vec{y}\tau)=(\vec{i},\vec{k})}} \varphi^1_{\vec{\ell}\sigma,\vec{i}\sigma}\psi^{(a_{\vec{\ell}\vec{k}}\cdot\tau)}_{\vec{i}\tau,\vec{k}\tau}(-1)^{s(w_{\vec{\ell}},\tau)+s(v_{\vec{k}},\tau)}. \end{aligned} \tag{59}$$

Since $\varphi^1_{\vec{\ell}\sigma,\vec{i}\sigma}\psi^{(a_{\vec{\ell}\vec{k}}\cdot\tau)}_{\vec{i}\tau,\vec{k}\tau} = \delta_{\vec{i}\sigma,\vec{i}\tau}\gamma^{(a_{\vec{\ell}\vec{k}}\cdot\tau)}_{\vec{\ell}\sigma,\vec{k}\tau}$, the only way for this term in (59) to give something nonzero is if $\vec{i}\sigma = \vec{i}\tau$ for some σ, τ. But each sum ranges over only those elements of $\mathfrak{S}_d$ which yield *distinct* elements of the orbit of $\vec{i}$. So $\vec{i}\sigma = \vec{i}\tau$ only when $\sigma = \tau$. Thus, we have

$$\begin{aligned}(59) &= \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{i})}} \gamma_{\vec{\ell}\sigma,\vec{k}\sigma}^{(a_{\vec{\ell}\vec{k}}.\sigma)}(-1)^{s(w_{\vec{\ell}},\sigma)+s(v_{\vec{k}},\sigma)} \\ &= \sum_{\substack{\sigma\in\mathfrak{S}_d \\ (\vec{p}\sigma,\vec{q}\sigma)=(\vec{\ell},\vec{k})}} \gamma_{\vec{\ell}\sigma,\vec{k}\sigma}^{(a_{\vec{\ell}\vec{k}}.\sigma)}(-1)^{s(w_{\vec{\ell}},\sigma)+s(v_{\vec{k}},\sigma)}, \end{aligned} \tag{60}$$

where the last equality follows from the fact that $\mathrm{Stab}_{\mathfrak{S}_d}\left((\vec{\ell},\vec{i})\right) = \mathrm{Stab}_{\mathfrak{S}_d}\left((\vec{\ell},\vec{k})\right)$. Therefore, we see that our composition map is surjective, as desired. ∎

Lemma 6.8. *Let $m,n \geqslant d$. Let $(\mathcal{P},\mathcal{C},Z) \in \{(\mathsf{P}^d_{(A,\mathfrak{a})}, \mathsf{T}^A_{\mathfrak{a}}(d), A^n), (\mathsf{P}^d_A, \Gamma^d\mathsf{V}, A^{m|n})\}$. For all F in $\mathcal{P}$ and V in $\mathcal{C}$, the canonical map*

$$\mathrm{Hom}_{\mathcal{C}}(Z,V)\otimes F(Z) \to F(V), \quad f\otimes x \mapsto Ff(x)$$

is surjective.

Proof. Let $\mathrm{Id}_{V^d} \in \mathrm{Hom}_{\mathcal{C}}(V,V)$ denote the identity map (note that for $\mathcal{C} = \mathsf{T}^A_{\mathfrak{a}}(d)$ we have $\mathrm{Id}_{V^d} = \sum_{i=1}^{r} \widetilde{\eta}_{(\vec{1_A},(i,\dots,i),(i,\dots,i))}$ where $V = A^r$).

By lemma 6.7, we may write $\mathrm{Id}_{V^d} = \sum_{i\in I} \alpha_i\otimes\beta_i$ for some index set I with $\alpha_i \in \mathrm{Hom}_{\mathcal{C}}(Z,V)$ and $\beta_i \in \mathrm{Hom}_{\mathcal{C}}(V,Z)$. Let $y \in F(V)$. Then $F\beta_i(y) \in F(Z)$, and under the canonical map, we have

$$\begin{aligned}\sum_{i\in I}\alpha_i\otimes F\beta_i(y) &\mapsto \sum_{i\in I} F\alpha_i\left(F\beta_i(y)\right) \\ &= \sum_{i\in I} F(\alpha_i\circ\beta_i)(y) \\ &= F\left(\sum_{i\in I}\alpha_i\circ\beta_i\right)(y) \\ &= F\left(\mathrm{Id}_{V^d}\right)(y) \\ &= \mathrm{Id}_{FV}(y) \\ &= y,\end{aligned}$$

where the first equality follows from the fact that F respects compositions, the second equality follows from the fact that F is a $\Bbbk$-linear functor, and the fourth equality follows from the fact that F sends the identity to the identity. ∎

Lemma 6.9. *Let $m,n \geqslant d$. Let $(\mathcal{P},\mathcal{C},X) \in \{(\mathsf{P}^d_{(A,\mathfrak{a})}, \mathsf{T}^A_{\mathfrak{a}}(d), A^n), (\mathsf{P}^d_A, \Gamma^d\mathsf{V}, A^{m|n})\}$. Then $\{h^{d,X}, \Pi h^{d,X}\}$ is a projective generator of $(\mathcal{P})_{\mathsf{ev}}$ where $h^{d,X} = \mathrm{Hom}_{\mathcal{C}}(X,\_)$.*

We denote h^{d,A^n} by $h^{d,n}$ and denote $h^{d,A^{m|n}}$ by $h^{d,m|n}$.

Proof. By lemma 6.6, we know that $h^{d,X}$ is a projective object in $\mathcal{P}$. To see that $\Pi h^{d,X}$ is also projective, suppose we have a supernatural transformation $\nu : \Pi h^{d,X} \to G$ and an epi supernatural transformation $\eta : F \to G$ for F,G in $\mathcal{P}$ (which we take to mean each section is an epimorphism in $\mathsf{smod}_{\Bbbk}$). Recall the supernatural transformation $e : h^{d,X} \to \Pi h^{d,X}$

from proposition 5.5. Then $\nu \circ e$ is a supernatural transformation from $h^{d,X}$ to G. Since $h^{d,X}$ is projective, there exists a supernatural transformation $\overline{\nu \circ e}$ from $h^{d,X}$ to F such that $\eta \circ \overline{\nu \circ e} = \nu \circ e$. Notice that $\overline{\nu \circ e} \circ e^{-1}$ is a supernatural transformation from $\Pi h^{d,X} \to F$ (where e^{-1} has sections given by e_Y^{-1}) such that $\nu = \eta \circ (\overline{\nu \circ e} \circ e^{-1})$. It folllows that $\Pi h^{d,X}$ is projective.

$$\begin{tikzcd} & & F \arrow[d, two heads, "\eta"] \\ h^{d,X} \arrow[r, "e"'] \arrow[urr, dashed, bend left, "\overline{\nu\circ e}"] & \Pi h^{d,X} \arrow[r, "\nu"'] \arrow[ur, dotted, "\overline{\nu\circ e}\circ e^{-1}"] & G \end{tikzcd}$$

Now for any F in $\mathcal{P}$, $h^{d,X} \otimes F(X)$ is in $\mathcal{P}$ where V in $\mathcal{C}$ is mapped to the $\Bbbk$-supermodule $h^{d,X}(V) \otimes F(X)$, and a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ is sent to the linear map $h^{d,X}(f) \otimes \mathrm{id}_{F(X)}$.

Next, note that the canonical map from lemma 6.8 gives rise to an *even* supernatural transformation

$$\mu : h^{d,X} \otimes F(X) \to F$$

whose sections are given by the canonical map: $\mu_V : h^{d,X}(V) \otimes F(X) \to F(V)$. To see that this is the case, consider $g \otimes x \in h^{d,X}(V) \otimes F(X)$ and $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$. Then recall that $\mu_V(g \otimes x) = Fg(x)$ and $\overline{Fg(x)} = \overline{Fg} + \overline{x} = \overline{g} + \overline{x}$ since F is even on morphisms. But $\overline{g} + \overline{x} = \overline{g \otimes x}$ and we see that μ_V preserves parity hence is even. Furthermore, we have

$$\begin{aligned} \left[\mu_W \circ \left(h^{d,X}(f) \otimes \mathrm{id}_{F(X)}\right)\right](g \otimes x) &= \mu_W\left((f \circ g) \otimes x\right) \\ &= F(f \circ g)(x) \end{aligned}$$

and

$$\begin{aligned} \left[Ff \circ \mu_V\right](g \otimes x) &= Ff\left(Fg(x)\right) \\ &= F(f \circ g)(x). \end{aligned}$$

Hence the following diagram commutes

$$\begin{tikzcd} h^{d,X}(V) \otimes F(X) \arrow[r, "h^{d,X}(f)\otimes \mathrm{id}_{F(X)}"] \arrow[d, "\mu_V"'] & h^{d,X}(W) \otimes F(X) \arrow[d, "\mu_W"] \\ F(V) \arrow[r, "Ff"'] & F(W) \end{tikzcd},$$

and we see that μ really is a (even) supernatural transformation. Moreover, since each μ_V is surjective by lemma 6.8, we have that μ is an epi supernatural transformation.

Now since $F(X)$ is some (possibly infinitely generated) $\Bbbk$-supermodule, there exists an even $\Bbbk$-linear surjection

$$\phi_F : \Bbbk^{I|J} \twoheadrightarrow F(X)$$

for the free $\Bbbk$-supermodule $\Bbbk^{I|J} := \left(\bigoplus_I \Bbbk\right) \oplus \left(\bigoplus_J \Pi\Bbbk\right)$ for some index sets I and J.

We claim that ϕ_F induces an even epi supernatural transformation

$$\Phi_F : h^{d,X} \otimes \Bbbk^{I|J} \to h^{d,X} \otimes F(X) \quad \text{with sections} \quad (\Phi_F)_V = \mathrm{Id}_{h^{d,X}(V)} \otimes \phi_F.$$

Indeed, it's easy to verify that for every $f \in \mathrm{Hom}_{\mathsf{C}}(V, W)$, the following diagram commutes

$$\begin{tikzcd}
h^{d,X}(V) \otimes \Bbbk^{I|J} \arrow[r, "h^{d,X}(f)\otimes \mathrm{id}_{\Bbbk^{I|J}} = (f\circ_)\otimes \mathrm{id}_{\Bbbk^{I|J}}"] \arrow[d, "(\Phi_F)_V"'] & h^{d,X}(W) \otimes \Bbbk^{I|J} \arrow[d, "(\Phi_F)_W"] \\
h^{d,X}(V) \otimes F(X) \arrow[r, "h^{d,X}(f)\otimes \mathrm{id}_{F(X)} = (f\circ_)\otimes \mathrm{id}_{F(X)}"'] & h^{d,X}(W) \otimes F(X)
\end{tikzcd}$$

and that each $\mathrm{Id}_{h^{d,X}(V)} \otimes \phi_F$ is surjective since ϕ_F is.

Next, we mention that for F, G in $\mathcal{P}$, we may form $F \oplus G, F \otimes G$ in $\mathcal{P}$ as expected. For V, W in $\mathcal{C}$, V maps to $F(V) \oplus G(V)$ and $F(V) \otimes G(V)$, respectively. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ is sent the linear maps $Ff \oplus Gf$ and $Ff \otimes Gf$, respectively, where each map acts componentwise, as expected. These notions extend to arbitrary index sets. Now for any V in $\mathcal{C}$, $h^{d,X}(V) \otimes \Bbbk^{I|J} = h^{d,X}(V) \otimes \left(\bigoplus_I \Bbbk \oplus \bigoplus_J \Pi\Bbbk \right)$ which is canonically isomorphic to the $\Bbbk$-supermodule $\left(\bigoplus_I h^{d,X}(V) \otimes \Bbbk \right) \oplus \left(\bigoplus_J h^{d,X}(V) \otimes \Pi\Bbbk \right)$. Similar calculations as above show that this isomorphism induces the sections of an (even) supernatural isomorphism

$$h^{d,X} \otimes \Bbbk^{I|J} \cong \left(\bigoplus_I h^{d,X} \otimes \Bbbk \right) \oplus \left(\bigoplus_J h^{d,X} \otimes \Pi\Bbbk \right).$$

Furthermore, $h^{d,X}(V) \otimes \Bbbk$ is canonically isomorphic to $h^{d,X}(V)$ and $h^{d,X}(V) \otimes \Pi\Bbbk$ is canonically isomorphic to $\Pi h^{d,X}(V)$, and therefore $h^{d,X} \otimes \Bbbk$ is supernaturally isomorphic to $h^{d,X}$, and $h^{d,X} \otimes \Pi\Bbbk$ is supernaturally isomorphic to $\Pi h^{d,X}$, all via even morphisms. Therefore, we have

$$h^{d,X} \otimes \Bbbk^{I|J} \cong \left(\bigoplus_I h^{d,X} \right) \oplus \left(\bigoplus_J \Pi h^{d,X} \right).$$

Composing this supernatural isomorphism with Φ_F yields an even epi supernatural transformation

$$\left(\bigoplus_I h^{d,X} \right) \oplus \left(\bigoplus_J \Pi h^{d,X} \right) \twoheadrightarrow h^{d,X} \otimes F(X). \tag{61}$$

Thus, given any F in $\mathcal{P}$, by our observations above, composing the epi supernatural transformation from (61) with the epi supernatural transformation μ yields an even epi supernatural transformation

$$\left(\bigoplus_I h^{d,X} \right) \oplus \left(\bigoplus_J \Pi h^{d,X} \right) \twoheadrightarrow F.$$

That is, $\{h^{d,X}, \Pi h^{d,X}\}$ is a generator of $(\mathcal{P})_{\mathsf{ev}}$. □

Theorem 6.10. *Let $m, n \geqslant d$. Let $(\mathcal{P}, \mathcal{C}, X) \in \{(\mathsf{P}^d_{(A,\mathfrak{a})}, \mathsf{T}^A_{\mathfrak{a}}(d), A^n), (\mathsf{P}^d_A, \Gamma^d\mathsf{V}, A^{m|n})\}$. For brevity, we let $\mathcal{E} = \mathrm{End}_{\mathcal{C}}(X)$ (So for $\mathcal{C} = \mathsf{T}^A_{\mathfrak{a}}(d)$, we have $\mathcal{E} = T^A_{\mathfrak{a}}(n, n; d)$ and for $\mathcal{C} = \Gamma^d\mathsf{V}$, we have $\mathcal{E} = S^A(m|n, d)$). Then evaluation at X gives an equivalence of categories*

$$\mathcal{P} \cong {}_{\mathcal{E}}\mathsf{smod}.$$

Proof. Remark 6.5 explains why $\mathsf{ev}^d_X : \mathcal{P} \to {}_{\mathcal{E}}\mathsf{smod}$ makes sense. By remark 5.1, it suffices to show that this functor is fully faithful and essentially surjective. To show fully faithful, it suffices to show $\mathrm{Hom}_{\mathcal{P}}(G, F) \cong \mathrm{Hom}_{\mathcal{E}}(G(X), F(X))$ as $\Bbbk$-supermodules for any F, G in $\mathcal{P}$. To this end, Notice that the diagram below left commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}}(h^{d,X},F) \arrow[r, "\cong"] \arrow[d, "\mathsf{ev}_X^d"'] & F(X) \\
\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},F(X)) \arrow[ur, "\cong"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\eta \arrow[r, mapsto] \arrow[d, mapsto] & \eta_X\,(\mathrm{Id}_X) \\
\eta_X \arrow[ur, mapsto] &
\end{tikzcd}$$

where the top arrow corresponds to the Yoneda isomorphism, the vertical arrow corresponds to the map induced by evaluation, and the diagonal arrow is the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},F(X)) \cong F(X) \quad \text{via} \quad f \mapsto f(1_{\mathcal{E}}) = f(\mathrm{Id}_X).$$

The diagram above right shows what happens on a given element.

Hence the map induced by evaluation actually gives an isomorphism $\mathrm{Hom}_{\mathcal{P}}(h^{d,X},F) \cong \mathrm{Hom}_{\mathcal{E}}(\mathcal{E},F(X))$. Moreover, proposition 5.4 gives $\mathrm{Hom}_{\mathcal{P}}(h^{d,X},F) \cong \mathrm{Hom}_{\mathcal{P}}(\Pi h^{d,X},F)$, and proposition 5.2 gives $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},F(X)) \cong \mathrm{Hom}_{\mathcal{E}}(\Pi\mathcal{E},F(X))$. Therefore, $\mathrm{Hom}_{\mathcal{P}}(\Pi h^{d,X},F) \cong \mathrm{Hom}_{\mathcal{E}}(\Pi\mathcal{E},F(X))$. This, along with the proof of lemma 6.9 and with general facts about how Hom interacts with direct sums and products, gives the following for any index sets I, J:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{P}}(h^{d,X}\otimes \Bbbk^{I|J},F) &\cong \mathrm{Hom}_{\mathcal{P}}\left(h^{d,X}\otimes\left[\left(\bigoplus_I \Bbbk\right)\oplus\left(\bigoplus_J \Pi\Bbbk\right)\right],F\right)\\
&\cong \mathrm{Hom}_{\mathcal{P}}\left(\left(\bigoplus_I h^{d,X}\otimes\Bbbk\right)\oplus\left(\bigoplus_J h^{d,X}\otimes\Pi\Bbbk\right),F\right)\\
&\cong \mathrm{Hom}_{\mathcal{P}}\left(\left(\bigoplus_I h^{d,X}\right)\oplus\left(\bigoplus_J \Pi h^{d,X}\right),F\right)\\
&\cong \left(\prod_I \mathrm{Hom}_{\mathcal{P}}(h^{d,X},F)\right)\times\left(\prod_J \mathrm{Hom}_{\mathcal{P}}(\Pi h^{d,X},F)\right)\\
&\cong \left(\prod_I \mathrm{Hom}_{\mathcal{E}}(\mathcal{E},F(X))\right)\times\left(\prod_J \mathrm{Hom}_{\mathcal{E}}(\Pi\mathcal{E},F(X))\right)\\
&\cong \mathrm{Hom}_{\mathcal{E}}\left(\left(\bigoplus_I \mathcal{E}\right)\oplus\left(\bigoplus_J \Pi\mathcal{E}\right),F(X)\right)\\
&\cong \mathrm{Hom}_{\mathcal{E}}\left(\left(\bigoplus_I \mathcal{E}\otimes\Bbbk\right)\oplus\left(\bigoplus_J \mathcal{E}\otimes\Pi\Bbbk\right),F(X)\right)\\
&\cong \mathrm{Hom}_{\mathcal{E}}\left(\mathcal{E}\otimes\Bbbk^{I|J},F(X)\right). \qquad (62)
\end{aligned}$$

By lemma 6.9, there is an even epi supernatural transformation $h^{d,X}\otimes\Bbbk^{I_1|J_1}\to G$ for some index sets I_1, J_1. Since this morphism is even, it makes sense to talk about its kernel. More precisely, for each V in $\mathcal{C}$, the section of this supernatural transformation is an even surjection $h^{d,X}(V)\otimes\Bbbk^{I_1|J_1}\to G(V)$. Therefore, we may compute its kernel. The kernels of these sections lift to a kernel of the supernatural transformation, say K. Then again by lemma 6.9, there is an even epi supernatural transformation $h^{d,X}\otimes\Bbbk^{I_2|J_2}\to K$ for some index sets I_2, J_2. This gives rise to the exact sequence

$$h^{d,X}\otimes\Bbbk^{I_2|J_2} \longrightarrow h^{d,X}\otimes\Bbbk^{I_1|J_1} \longrightarrow G \longrightarrow 0\ . \qquad (63)$$

Applying the left-exact contravariant functor $\mathrm{Hom}_{\mathcal{P}}(\_\,, F)$ yields the following exact sequence:

$$\mathrm{Hom}_{\mathcal{P}}(h^{d,X} \otimes \Bbbk^{I_2|J_2}, F) \longleftarrow \mathrm{Hom}_{\mathcal{P}}(h^{d,X} \otimes \Bbbk^{I_1|J_1}, F) \longleftarrow \mathrm{Hom}_{\mathcal{P}}(G, F) \longleftarrow 0 \ . \tag{64}$$

Evaluating at X turns sequence (63) into the exact sequence

$$h^{d,X}(X) \otimes \Bbbk^{I_2|J_2} \longrightarrow h^{d,X}(X) \otimes \Bbbk^{I_1|J_1} \longrightarrow G(X) \longrightarrow 0 \ ,$$

which may be written as

$$\mathcal{E} \otimes \Bbbk^{I_2|J_2} \longrightarrow \mathcal{E} \otimes \Bbbk^{I_1|J_1} \longrightarrow G(X) \longrightarrow 0 \ . \tag{65}$$

Applying the left-exact contravariant functor $\mathrm{Hom}_{\mathcal{E}}(\_\,, F(X))$ to (65) yields the following exact sequence:

$$\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes \Bbbk^{I_2|J_2}, F(X)) \leftarrow \mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes \Bbbk^{I_1|J_1}, F(X)) \leftarrow \mathrm{Hom}_{\mathcal{E}}(G(X), F(X)) \leftarrow 0 \ .$$

Finally, evaluation at X gives rise to the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}}(h^{d,X} \otimes \Bbbk^{I_2|J_2}, F) \arrow[d] & \mathrm{Hom}_{\mathcal{P}}(h^{d,X} \otimes \Bbbk^{I_1|J_1}, F) \arrow[l] \arrow[d] & \mathrm{Hom}_{\mathcal{P}}(G, F) \arrow[l] \arrow[d] & 0 \arrow[l] \\
\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes \Bbbk^{I_2|J_2}, F(X)) & \mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes \Bbbk^{I_1|J_1}, F(X)) \arrow[l] & \mathrm{Hom}_{\mathcal{E}}(G(X), F(X)) \arrow[l] & 0 \arrow[l]
\end{tikzcd} .$$

Here, the leftmost vertical arrow and the middle vertical arrow are isomorphisms by (62). We wish to show that the rightmost vertical arrow is an isomorphism. To this end, first suppose that $\eta \in \mathrm{Hom}_{\mathcal{P}}(G, F)$ is mapped to zero under the evaluation map. Then we have the following section of our diagram:

$$\begin{tikzcd}
 & \eta \arrow[d, mapsto] \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd} .$$

Since the middle vertical arrow is an isomorphism, we have

$$\begin{tikzcd}
0 \arrow[d, leftrightarrow] & \eta \arrow[d, mapsto] \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd} .$$

Since the diagram commutes, we see that $\eta \mapsto 0$ under the middle horizontal arrow in the top row. But this map is injective, and hence it must be that $\eta = 0$. Thus, we have that the rightmost vertical arrow is injective.

Next, suppose $f \in \mathrm{Hom}_{\mathcal{E}}(G(X), F(X))$. Then we have the following section of the diagram:

$$0 \longmapsfrom f' \longmapsfrom f \ .$$

Since the leftmost and middle vertical arrows are isomorphisms, and since the leftmost square commutes, we have

$$\begin{tikzcd}
0 & \eta' \arrow[l, mapsto] & \\
0 \arrow[u, leftrightarrow] & f' \arrow[l, mapsto] \arrow[u, leftrightarrow] & f \arrow[l, mapsto]
\end{tikzcd}.$$

So we see that $\eta' \in \mathrm{Hom}_{\mathcal{P}}(h^{d,X} \otimes \Bbbk^{I_1|J_1}, F)$ is in the kernel of the leftmost horizontal map in the top row. Since that row is exact in this place, there must be an element $\eta \in \mathrm{Hom}_{\mathcal{P}}(G, F)$ such that

$$\begin{tikzcd}
0 & \eta' \arrow[l, mapsto] & \eta \arrow[l, mapsto] \\
0 \arrow[u, leftrightarrow] & f' \arrow[l, mapsto] \arrow[u, leftrightarrow] & f \arrow[l, mapsto]
\end{tikzcd}.$$

Since the middle square in our diagram commutes, we have

$$\begin{tikzcd}
\eta' & \eta \arrow[l, mapsto] \arrow[d, mapsto] \\
f' \arrow[u, leftrightarrow] & g \arrow[l, mapsto]
\end{tikzcd}$$

for some $g \in \mathrm{Hom}_{\mathcal{E}}(G(X), F(X))$. But the middle horizontal arrow in the bottom row of the diagram is injective, and hence since f and g both map to f' under this map, $f = g$. Thus, we've found an element $\eta \in \mathrm{Hom}_{\mathcal{P}}(G, F)$ which maps to f, so the rightmost vertical arrow is surjective. Thus, we have the isomorphism we were looking for, and we see that ev_X^d is fully-faithful.

Next, we wish to see that ev_X^d is essentially surjective. In order to do so, we first observe that for any V in $\mathcal{C}$, $h^{d,X}(V) = \mathrm{Hom}_{\mathcal{C}}(X, V)$ is a right $\mathcal{E}$-supermodule. For $f \in h^{d,X}(V)$ and $x \in \mathcal{E} = \mathrm{End}_{\mathcal{C}}(X)$, we define $f.x \coloneqq f \circ x$. Since composition in $\mathcal{C}$ is well-defined, so is the action. Moreover, for $y \in \mathcal{E}$, note that

$$\begin{aligned}
f.(xy) &= f \circ (x \circ y) \\
&= (f \circ x) \circ y \\
&= (f.x).y
\end{aligned}$$

so we really do have an action (clearly $1 \in \mathcal{E}$ acts as the identity). Therefore, it makes sense to write $h^{d,X}(V) \otimes_{\mathcal{E}} M$ for any V in $\mathcal{C}$ and M in ${}_{\mathcal{E}}\mathsf{smod}$.

Consider the functor $h^{d,X} \otimes_{\mathcal{E}} M \in \mathcal{P}$. Then evaluation at X gives $h^{d,X}(X) \otimes_{\mathcal{E}} M = \mathcal{E} \otimes_{\mathcal{E}} M \cong M$. Hence, ev_X^d is essentially surjective. ◻

6.2. Tensor Product of Generalized Strict Polynomial Functors

In this section, we work out in detail what it means to take the tensor product of two generalized strict polynomial functors. One should hopefully see how the argument goes for strict polynomial superfunctors, as well. The first step in understanding this is to understand the space $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d \times \mathfrak{S}_e}$ where we've canonically identified $\mathfrak{S}_d \times \mathfrak{S}_e$ with a subgroup of $\mathfrak{S}_{d+e}$. We know that

$$\left\{ \widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} \;\middle|\; (\vec{b}, \vec{r}, \vec{s}) \in \Omega(B, m, n; d+e) \right\}$$

gives a homogeneous $\Bbbk$-basis for $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_{d+e}}$ where $\Omega(B, m, n; d+e)$ is a set of orbit representatives for $\mathrm{Tri}^B(m, n; d+e)/\mathfrak{S}_{d+e}$. It's easy to see that a simple modification of the proof of this fact yields

$$\left\{ \sum_{\rho \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}} (-1)^{s(\vec{b},\rho)} \varphi^{b_{\rho 1}}_{r_{\rho 1} s_{\rho 1}} \otimes \cdots \otimes \varphi^{b_{\rho(d+e)}}_{r_{\rho(d+e)} s_{\rho(d+e)}} \;\middle|\; (\vec{b}, \vec{r}, \vec{s}) \in \Omega(B, m, n; d \times e) \right\}$$

as a homogeneous $\Bbbk$-basis for $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d \times \mathfrak{S}_e}$ where ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}$ is the set of shortest coset representatives for $\mathrm{Stab}_{\mathfrak{S}_d \times \mathfrak{S}_e}(\vec{b}, \vec{r}, \vec{s}) \backslash \mathfrak{S}_d \times \mathfrak{S}_e$ and where $\Omega(B, m, n; d \times e)$ is a set of orbit representatives for $\mathrm{Tri}^B(m, n; d+e)/\mathfrak{S}_d \times \mathfrak{S}_e$.

Moreover, one can place a total order on B which will induce a total order (lexicographic order) on $\mathrm{Tri}^B(m, n; d+e)$. It follows that

$$\{(\vec{b}, \vec{r}, \vec{s}) \in \mathrm{Tri}^B(m, n; d+e) \mid (\vec{b}, \vec{r}, \vec{s}) \leqslant (\vec{b}, \vec{r}, \vec{s})\sigma \ \forall \sigma \in \mathfrak{S}_{d+e}\}$$

is a set of representatives for $\mathrm{Tri}^B(m, n; d+e)/\mathfrak{S}_{d+e}$. Similarly,

$$\{(\vec{b}, \vec{r}, \vec{s}) \in \mathrm{Tri}^B(m, n; d+e) \mid (\vec{b}, \vec{r}, \vec{s}) \leqslant (\vec{b}, \vec{r}, \vec{s})\sigma \ \forall \sigma \in \mathfrak{S}_d \times \mathfrak{S}_e\}$$

is a set of representatives for $\mathrm{Tri}^B(m, n; d+e)/\mathfrak{S}_d \times \mathfrak{S}_e$. So from now on, we will consider our initial seeds to be from these sets of representatives. In particular, this means that any repeated componets in a seed must appear sequentially.

It will be helpful to understand the natural inclusion

$$\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_{d+e}} \subset \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d \times \mathfrak{S}_e}.$$

The next lemma demonstrates that a $\Bbbk$-basis element of $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_{d+e}}$ is sent to a sum of $\Bbbk$-basis elements of $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d \times \mathfrak{S}_e}$ where the sum is indexed by permutations of our initial seed $(\vec{b}, \vec{r}, \vec{s})$ which respect the lexicographic order in the first d slots and the last e slots. For example, if $d = 3$ and $e = 2$ and the initial word looked like $CCDEE$, then $CEECD$ is such a permutation, but $ECCDE$ is not.

Define ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e} \subset {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}$ to be the set

$${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e} := \left\{ \tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \;\middle|\; \begin{array}{l} (b_{\tau i}, r_{\tau i}, s_{\tau i}) \leqslant (b_{\tau(i+1)}, r_{\tau(i+1)}, s_{\tau(i+1)}) \text{ for } 1 \leqslant i \leqslant d-1, \\ (b_{\tau j}, r_{\tau j}, s_{\tau j}) \leqslant (b_{\tau(j+1)}, r_{\tau(j+1)}, s_{\tau(j+1)}) \text{ for } d+1 \leqslant j \leqslant d+e-1 \end{array} \right\}.$$

Remark 6.11. Later, we'll need to understand the relationship between ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}$ and ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{e,d}$. Let $\zeta \in \mathfrak{S}_{d+e}$ be the permutation which swaps the first d-many and last e-many entries, keeping the same relative order (so ζ takes $d+1$ to 1, $d+2$ to 2, etc. and takes 1 to $e+1$, …, and d to $d+e$). Then it is obvious that ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{e,d} = {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}\zeta$. That is, for every $\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}$, it is clear that $\tau\zeta \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{e,d}$ and that this gives a bijection.

Furthermore,

$$(b_{\tau 1}, b_{\tau 2}, \ldots, b_{\tau(d)}) = (b_{\zeta\tau(e+1)}, \ldots, b_{\zeta\tau(e+d)})$$

and

$$(b_{\tau(d+1)}, \ldots, b_{\tau(d+e)}) = (b_{\zeta\tau 1}, \ldots, b_{\zeta\tau(e)}),$$

and similarly for $\vec{r}$ and $\vec{s}$.

Lemma 6.12. *Let $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} \in \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_{d+e}}$ be such that $(\vec{b},\vec{r},\vec{s})$ is in lexicographic order as discussed above. Then the inclusion*

$$\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_{d+e}} \subset \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d\times\mathfrak{S}_e}$$

sends

$$\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} \mapsto \sum_{\tau\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \sum_{\rho\in {}^{\vec{b}\tau,\vec{r}\tau,\vec{s}\tau}\mathscr{E}} (-1)^{s(\vec{b}\tau,\rho)} \varphi^{b_{\tau\rho1}}_{r_{\tau\rho1}s_{\tau\rho1}} \otimes\cdots\otimes \varphi^{b_{\tau\rho(d+e)}}_{r_{\tau\rho(d+e)}s_{\tau\rho(d+e)}},$$

which by remark 1.4 is equal to

$$\sum_{\tau\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} \sum_{\rho\in {}^{\vec{b}\tau,\vec{r}\tau,\vec{s}\tau}\mathscr{E}} (-1)^{s(\vec{b},\tau\rho)} \varphi^{b_{\tau\rho1}}_{r_{\tau\rho1}s_{\tau\rho1}} \otimes\cdots\otimes \varphi^{b_{\tau\rho(d+e)}}_{r_{\tau\rho(d+e)}s_{\tau\rho(d+e)}}.$$

Proof. First of all, given a τ and ρ from the double summation, $\tau\rho$ clearly corresponds to some term in $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$.

On the other hand, given σ from $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$, there exists some $\gamma\in\mathfrak{S}_{d+e}$ such that

$$(b_{\sigma\gamma i}, r_{\sigma\gamma i}, s_{\sigma\gamma i}) \leqslant (b_{\sigma\gamma(i+1)}, r_{\sigma\gamma(i+1)}, s_{\sigma\gamma(i+1)}) \text{ for } 1\leqslant i\leqslant d-1$$

and

$$(b_{\sigma\gamma j}, r_{\sigma\gamma j}, s_{\sigma\gamma j}) \leqslant (b_{\sigma\gamma(j+1)}, r_{\sigma\gamma(j+1)}, s_{\sigma\gamma(j+1)}) \text{ for } d+1\leqslant j\leqslant d+e-1.$$

Then letting $\tau=\sigma\gamma$ and $\rho=\gamma^{-1}$, we see that the σ term of $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$ corresponds to the τ,ρ term of the double sum. Clearly, this gives a one-to-one correspondence between the terms of the two expressions. ∎

Remark 6.13. There is a canonical isomorphism of $\Bbbk$-supermodules

$$\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d\times\mathfrak{S}_e} \cong \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes d}\right)^{\mathfrak{S}_d} \otimes \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes e}\right)^{\mathfrak{S}_e}$$

as follows: Consider a $\Bbbk$-basis element

$$\sum_{\rho\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}} (-1)^{s(\vec{b},\rho)} \varphi^{b_{\rho1}}_{r_{\rho1}s_{\rho1}} \otimes\cdots\otimes \varphi^{b_{\rho(d+e)}}_{r_{\rho(d+e)}s_{\rho(d+e)}}$$

of $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d\times\mathfrak{S}_e}$. For each $\rho\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}$, we have that ρ is identified with some $(\alpha,\beta)\in\mathfrak{S}_d\times\mathfrak{S}_e$. Let $\vec{b}'\in B^d$ be such that $b'_i=b_i$ for $1\leqslant i\leqslant d$, and let $\vec{b}''\in B^e$ be such that $b''_j=b_{d+j}$ for $1\leqslant j\leqslant e$. Similarly for $\vec{r}', \vec{r}'', \vec{s}', \vec{s}''$. Then we have $(\vec{b},\vec{r},\vec{s}) = (\vec{b}'\vec{b}'', \vec{r}'\vec{r}'', \vec{s}'\vec{s}'')$. Moreover, we have $\alpha\in {}^{\vec{b}',\vec{r}',\vec{s}'}\mathscr{D}$ and $\beta\in {}^{\vec{b}'',\vec{r}'',\vec{s}''}\mathscr{D}$. Conversely, any $(\alpha,\beta)\in {}^{\vec{b}',\vec{r}',\vec{s}'}\mathscr{D}\times {}^{\vec{b}'',\vec{r}'',\vec{s}''}\mathscr{D}$ determines some $\rho\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}$.

So for a given $\rho\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}$, we can write $(-1)^{s(\vec{b},\rho)} \varphi^{b_{\rho1}}_{r_{\rho1}s_{\rho1}} \otimes\cdots\otimes \varphi^{b_{\rho(d+e)}}_{r_{\rho(d+e)}s_{\rho(d+e)}}$ as

$$(-1)^{s(\vec{b}',\alpha)+s(\vec{b}'',\beta)} \varphi^{b'_{\alpha1}}_{r'_{\alpha1}s'_{\alpha1}} \otimes\cdots\otimes \varphi^{b'_{\alpha d}}_{r'_{\alpha d}s'_{\alpha d}} \otimes \varphi^{b''_{\beta1}}_{r''_{\beta1}s''_{\beta1}} \otimes\cdots\otimes \varphi^{b''_{\beta e}}_{r''_{\beta e}s''_{\beta e}},$$

and it follows that

$$\sum_{\rho\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{E}} (-1)^{s(\vec{b},\rho)} \varphi^{b_{\rho1}}_{r_{\rho1}s_{\rho1}} \otimes\cdots\otimes \varphi^{b_{\rho(d+e)}}_{r_{\rho(d+e)}s_{\rho(d+e)}}$$

is equal to

$$\left(\sum_{\alpha\in {}^{\vec{b}',\vec{r}',\vec{s}'}\mathscr{D}} (-1)^{s(\vec{b}',\alpha)} \varphi^{b'_{\alpha1}}_{r'_{\alpha1}s'_{\alpha1}} \otimes\cdots\otimes \varphi^{b'_{\alpha d}}_{r'_{\alpha d}s'_{\alpha d}}\right) \otimes \left(\sum_{\beta\in {}^{\vec{b}'',\vec{r}'',\vec{s}''}\mathscr{D}} (-1)^{s(\vec{b}'',\beta)} \varphi^{b''_{\beta1}}_{r''_{\beta1}s''_{\beta1}} \otimes\cdots\otimes \varphi^{b''_{\beta e}}_{r''_{\beta e}s''_{\beta e}}\right),$$

which is equal to the $\Bbbk$ basis element

$$\widetilde{\varphi}_{(\vec{b}',\vec{r}',\vec{s}')} \otimes \widetilde{\varphi}_{(\vec{b}'',\vec{r}'',\vec{s}'')}$$

of $\left(\mathrm{Hom}_A(A^n, A^m)^{\otimes d}\right)^{\mathfrak{S}_d} \otimes \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes e}\right)^{\mathfrak{S}_e}$.

This remark along with lemma 6.12 imply the following (where we use the notation from remark 4.5 for supermodules A^n, A^m):

Corollary 6.14. *We have an inclusion of categories $\mathsf{S}^A(d+e) \hookrightarrow \mathsf{S}^A(d) \otimes \mathsf{S}^A(e)$ induced by the sequence of maps*

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{S}^A(d+e)}(A^n, A^m) &= S^A(m, n; d+e) \\
&= \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_{(d+e)}} \\
&\subset \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes(d+e)}\right)^{\mathfrak{S}_d \times \mathfrak{S}_e} \\
&\cong \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes d}\right)^{\mathfrak{S}_d} \otimes \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes e}\right)^{\mathfrak{S}_e} \\
&= S^A(A^n, A^m; d) \otimes S^A(A^n, A^m; e) \\
&= \mathrm{Hom}_{\mathsf{S}^A(d)}(A^n, A^m) \otimes \mathrm{Hom}_{\mathsf{S}^A(e)}(A^n, A^m),
\end{aligned} \tag{66}$$

which send a $\Bbbk$ basis element $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$ to the sum of $\Bbbk$-basis elments

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \widetilde{\varphi}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}.$$

We are actually interested in $\mathsf{T}^A_{\mathfrak{a}}(d+e)$. One can still apply this chain of maps to an element $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T^A_{\mathfrak{a}}(m, n; d+e)$ and end up in $S^A(m, n; d) \otimes S^A(m, n; e)$. But we would like to say that we actually land in $T^A_{\mathfrak{a}}(m, n; d) \otimes T^A_{\mathfrak{a}}(m, n; e)$.

Proposition 6.15. *The chain of maps from* (66) *induces*

$$\mathsf{T}^A_{\mathfrak{a}}(d+e) \hookrightarrow \mathsf{T}^A_{\mathfrak{a}}(d) \otimes \mathsf{T}^A_{\mathfrak{a}}(e).$$

Proof. We have $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} = [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$ which the chain of maps in (66) identifies with

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \widetilde{\varphi}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}.$$

Now consider all (x, y, z) that contribute to $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}$, that is, such that $x \in \mathfrak{c}$, $y \in [1, m]$, $z \in [1, n]$, and $[\vec{b},\vec{r},\vec{s}]^x_{y,z}! \neq 0!$. Suppose there are p-many tuples. Enumerate them (x_1, y_1, z_1), $\dots$, (x_p, y_p, z_p). Suppose $[\vec{b},\vec{r},\vec{s}]^{x_i}_{y_i,z_i}! = q_i!$ (so $\varphi^{x_i}_{y_i z_i}$ appears q_i-many times in $\varphi_{(\vec{b},\vec{r},\vec{s})}$). Then

$$\begin{aligned}
[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} &= \prod_{i=1}^{p} [\vec{b},\vec{r},\vec{s}]^{x_i}_{y_i,z_i}! \\
&= \prod_{i=1}^{p} q_i!.
\end{aligned}$$

Now for a given $\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}$, say (x_i, y_i, s_i) appears $k_{\tau,i}$-many times in $(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')$ and $(q_i - k_{\tau,i})$-many times in $(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')$ (for appropriate values of $k_{\tau,i}$). That means

$[\vec{b}\tau', \vec{r}\tau', \vec{s}\tau']^{x_i}_{y_i,z_i}! = k_{\tau,i}!$ and $[\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'']^{x_i}_{y_i,z_i}! = (q_i - k_{\tau,i})!$. It follows that $[\vec{b}\tau', \vec{r}\tau', \vec{s}\tau']^!_{\mathfrak{c}} = \prod_{i=1}^{p} k_{\tau,i}!$ and $[\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'']^!_{\mathfrak{c}} = \prod_{i=1}^{p}(q_i - k_{\tau,i})!$. Finally, it follows that

$$[\vec{b}, \vec{r}, \vec{s}]^!_{\mathfrak{c}} \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \widetilde{\varphi}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}$$

is equal to

$$\prod_{i=1}^{p} q_i! \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \widetilde{\varphi}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}$$

which equals

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} q_i! \widetilde{\varphi}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}$$

which equals

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} \prod_{i=1}^{p} k_{\tau,i}! \prod_{i=1}^{p} (q_i - k_{\tau,i})! \widetilde{\varphi}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}$$

which equals

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} [\vec{b}\tau', \vec{r}\tau', \vec{s}\tau']^!_{\mathfrak{c}} [\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'']^!_{\mathfrak{c}} \widetilde{\varphi}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')} \otimes \widetilde{\varphi}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}$$

which equals

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} \widetilde{\eta}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')} \otimes \widetilde{\eta}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}$$

which lives in $\mathsf{T}^A_{\mathfrak{a}}(m, n; d) \otimes \mathsf{T}^A_{\mathfrak{a}}(m, n; e)$, as desired. ∎

Remark 6.16. For generalized polynomial functors F in $\mathsf{P}^d_{(A,\mathfrak{a})}$ and G in $\mathsf{P}^e_{(A,\mathfrak{a})}$, $F \otimes G$ is a generalized polynomial functor in $\mathsf{P}^{d+e}_{(A,\mathfrak{a})}$ which sends an object V to $F(V) \otimes G(V)$. Now $F \otimes G$ must take a morphism from $T^A_{\mathfrak{a}}(d+e)$ to a $\Bbbk$-linear map. F and G only know how to handle morphisms from the d and e spaces, respectively. But proposition 6.15 allows us to handle this issue in the following way:

$$(F \otimes G)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}) := \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} F(\widetilde{\eta}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')}) \boxtimes G(\widetilde{\eta}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}).$$

It is often easier in practice to work with this equivalent formulation:

$$[\vec{b}, \vec{r}, \vec{s}]^!_{\mathfrak{c}} \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} F(\widetilde{\varphi}_{(\vec{b}\tau', \vec{r}\tau', \vec{s}\tau')}) \boxtimes G(\widetilde{\varphi}_{(\vec{b}\tau'', \vec{r}\tau'', \vec{s}\tau'')}).$$

We will make use of this throughout the rest of the thesis.

Proposition 6.17. *The tensor product of generalized polynomial functors described above makes $\mathsf{P}_{(A,\mathfrak{a})}$ into a monoidal supercategory.*

Proof. We will show the result working with generalized polynomial functors of homogeneous degree. It is clear how to extend if one has direct sums of such functors using the fact that, by definition, our tensor product of polynomial functors distributes over our formal direct sum:

$$(F \oplus G) \otimes H = (F \otimes H) \oplus (G \otimes H).$$

First, consider the supercategory $\mathsf{P}_{(A,\mathfrak{a})} \boxtimes \mathsf{P}_{(A,\mathfrak{a})}$ (as in remark 5.7). Then define the superfunctor $\_ \otimes \_ : \mathsf{P}_{(A,\mathfrak{a})} \boxtimes \mathsf{P}_{(A,\mathfrak{a})} \to \mathsf{P}_{(A,\mathfrak{a})}$ in the following way: On objects, we have

$$(F, G) \mapsto F \otimes G,$$

and for a morphism $\alpha \otimes \beta \in \mathrm{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(E, G) \otimes \mathrm{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(F, H)$, we have that $(\_ \otimes \_)(\alpha \otimes \beta)$ is the supernatural transformation $E \otimes F \to G \otimes H$ whose sections at V are given by

$$\alpha_V \boxtimes \beta_V.$$

In order to check that these sections actually define a supernatural transformation, we will suppose E, F, G, H are homogeneous. In particular, we can assume E and G are of degrees d and that F and H are of degree e. Now for any free finite right A-supermodules V, W, we must check the following:

$$(\alpha_W \boxtimes \beta_W) \circ (E \otimes F)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}) = (-1)^{\overline{\vec{b}}\cdot\overline{\alpha\otimes\beta}} (G \otimes H)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}) \circ (\alpha_V \boxtimes \beta_V).$$

Well, using remark 6.16, we have $(\alpha_W \boxtimes \beta_W) \circ (E \otimes F)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$

$$\begin{aligned} &= (\alpha_W \boxtimes \beta_W) \circ \left(\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} E(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}) \boxtimes F(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}) \right) \\ &= \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau) + \overline{\beta_W}\cdot\overline{\vec{b}\tau'}} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} \left(\alpha_W \circ E(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}) \right) \boxtimes \left(\beta_W \circ F(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}) \right), \end{aligned} \tag{67}$$

where the extra sign in the second line follows from the fact that

$$\overline{E(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')})} = \overline{\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}} = \overline{\vec{b}\tau'}$$

(and by $\overline{\vec{b}}$, for example, we mean $\overline{b_1 \otimes \cdots \otimes b_d} = \overline{b_1} + \cdots + \overline{b_d}$).

Now since $\alpha \in \mathrm{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(E, G)$ and $\beta \in \mathrm{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(F, H)$, we know that

$$\alpha_W \circ E(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}) = (-1)^{\overline{\alpha}\cdot\overline{\vec{b}\tau'}} G(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}) \circ \alpha_V$$

and

$$\beta_W \circ F(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}) = (-1)^{\overline{\beta}\cdot\overline{\vec{b}\tau''}} H(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}) \circ \beta_V.$$

So we can write (67) as

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau) + \overline{\beta_W}\cdot\overline{\vec{b}\tau'} + \overline{\alpha}\cdot\overline{\vec{b}\tau'} + \overline{\beta}\cdot\overline{\vec{b}\tau''}} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} \left(G(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}) \circ \alpha_V \right) \boxtimes \left(H(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}) \circ \beta_V \right). \tag{68}$$

Now on the other hand, we have $(-1)^{\bar{\vec{b}}\cdot\overline{\alpha\otimes\beta}}(G\otimes H)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})\circ(\alpha_V\boxtimes\beta_V)$

$$= \left(\sum_{\tau\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}}(-1)^{s(\vec{b},\tau)+\bar{\vec{b}}\cdot\overline{\alpha\otimes\beta}}\prod_{i=1}^{p}\binom{q_i}{k_{\tau,i}}G(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')})\boxtimes H(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')})\right)\circ(\alpha_V\boxtimes\beta_V)$$

$$= \sum_{\tau\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}}(-1)^{s(\vec{b},\tau)+\bar{\vec{b}}\cdot\overline{\alpha\otimes\beta}+\overline{\vec{b}\tau''}\cdot\overline{\alpha_V}}\prod_{i=1}^{p}\binom{q_i}{k_{\tau,i}}\left(G(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')})\circ\alpha_V\right)\boxtimes\left(H(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')})\circ\beta_V\right). \tag{69}$$

So we see that (68) and (69) agree up to signs. To check that the signs also match, note that

$$\begin{aligned}(-1)^{s(\vec{b},\tau)+\bar{\vec{b}}\cdot\overline{\alpha\otimes\beta}+\overline{\vec{b}\tau''}\cdot\overline{\alpha_V}} &= (-1)^{s(\vec{b},\tau)+\bar{\vec{b}}\cdot\overline{\alpha}+\bar{\vec{b}}\cdot\overline{\beta}+\overline{\vec{b}\tau''}\cdot\overline{\alpha}}\\ &= (-1)^{s(\vec{b},\tau)+\bar{\vec{b}}\cdot\overline{\beta}+\overline{\vec{b}\tau'}\cdot\overline{\alpha}}\\ &= (-1)^{s(\vec{b},\tau)+\overline{\beta_W}\cdot\overline{\vec{b}\tau'}+\overline{\alpha}\cdot\overline{\vec{b}\tau'}+\overline{\beta}\cdot\overline{\vec{b}\tau''}},\end{aligned}$$

where the second line follows from the fact that $(-1)^{\bar{\vec{b}}\cdot\overline{\alpha}} = (-1)^{\overline{\vec{b}\tau}\cdot\overline{\alpha}} = (-1)^{\overline{\vec{b}\tau'}\cdot\overline{\alpha}+\overline{\vec{b}\tau''}\cdot\overline{\alpha}}$, and the third line follows from the similar fact that $(-1)^{\bar{\vec{b}}\cdot\overline{\beta}} = (-1)^{\overline{\vec{b}\tau}\cdot\overline{\beta}} = (-1)^{\overline{\vec{b}\tau'}\cdot\overline{\beta}+\overline{\vec{b}\tau''}\cdot\overline{\beta}}$. Thus, (68) = (69) as desired.

Next, we will define our identity object $\boldsymbol{I}$ in $\mathsf{P}^0_{(A,\mathfrak{a})}\subset\mathsf{P}_{(A,\mathfrak{a})}$. By remark 4.19, we need $\boldsymbol{I}$ to be a functor which takes a free finite right A-supermodule V to some $\Bbbk$-supermodule $\boldsymbol{I}(V)$ and takes a morphism $\beta\in\mathrm{Hom}_{\mathsf{T}^A_{\mathfrak{a}}(0)}(V,W)=\Bbbk$ to a $\Bbbk$-map $\boldsymbol{I}(\beta)\in\mathrm{Hom}_{\Bbbk}(\boldsymbol{I}(V),\boldsymbol{I}(W)$. So let

$$\boldsymbol{I}(V) := \Bbbk \qquad \text{and} \qquad \begin{aligned}\boldsymbol{I}(\beta):\Bbbk&\to\Bbbk\\ x&\mapsto\beta x\end{aligned}\ . \tag{70}$$

Next, we need to define an even supernatural isomorphism $\alpha : (\_\otimes\_)\otimes\_\to\_\otimes(\_\otimes\_)$ where we view each of $(\_\otimes\_)\otimes\_$ and $\_\otimes(\_\otimes\_)$ as superfunctors $\mathsf{P}_{(A,\mathfrak{a})}\boxtimes\mathsf{P}_{(A,\mathfrak{a})}\boxtimes\mathsf{P}_{(A,\mathfrak{a})}\to\mathsf{P}_{(A,\mathfrak{a})}$. So α should have sections $\alpha_{E,F,G}$ which are themselves even supernatural transformations $(E\otimes F)\otimes G\to E\otimes(F\otimes G)$. So define the section at V of a given $\alpha_{E,F,G}$ to be the (even) associator for $\mathsf{smod}_{\Bbbk}$:

$$(x\otimes y)\otimes z\mapsto x\otimes(y\otimes z).$$

We should check that these make $\alpha_{E,F,G}$ into an even supernatural transformation. Suppose E,F,G are homogeneous of degrees a,b,c, respectively. Then for $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\in T^A_{\mathfrak{a}}(V,W;a+b+c)$, we must check that

$$\alpha_{E,F,G_W}\circ[(E\otimes F)\otimes G](\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}) = [E\otimes(F\otimes G)](\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})\circ\alpha_{E,F,G_V}. \tag{71}$$

Using the formula from remark 6.16, we see that $[(E\otimes F)\otimes G](\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$ is $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}$ times the following:

$$\begin{aligned}&\sum_{\tau\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a+b,c}}(-1)^{s(\vec{b},\tau)}(E\otimes F)(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')})\boxtimes G(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')})\\ &= \sum_{\substack{\tau\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a+b,c}\\ \rho\in{}^{\vec{b}\tau',\vec{r}\tau',\vec{s}\tau'}\mathscr{D}_{a,b}}}(-1)^{s(\vec{b},\tau)+s(\vec{b}\tau',\rho)}\Big(E(\widetilde{\eta}_{((\vec{b}\tau')\rho',(\vec{r}\tau')\rho',(\vec{s}\tau')\rho')})\boxtimes F(\widetilde{\eta}_{((\vec{b}\tau')\rho'',(\vec{r}\tau')\rho'',(\vec{s}\tau')\rho'')})\Big)\boxtimes G(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}).\end{aligned}$$

On the other hand, we have $[E \otimes (F \otimes G)](\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$ is $[\vec{b},\vec{r},\vec{s}]^{!}_{\mathfrak{c}}$ times the following:

$$\sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b+c}} (-1)^{s(\vec{b},\sigma)} E(\widetilde{\eta}_{(\vec{b}\sigma',\vec{r}\sigma',\vec{s}\sigma')}) \boxtimes (F \otimes G)(\widetilde{\eta}_{(\vec{b}\sigma'',\vec{r}\sigma'',\vec{s}\sigma'')})$$

$$= \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b+c} \\ \kappa \in {}^{\vec{b}\sigma'',\vec{r}\sigma'',\vec{s}\sigma''}\mathscr{D}_{b,c}}} (-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma'',\kappa)} E(\widetilde{\eta}_{(\vec{b}\sigma',\vec{r}\sigma',\vec{s}\sigma')}) \boxtimes \Big(F(\widetilde{\eta}_{((\vec{b}\sigma'')\kappa',(\vec{r}\sigma'')\kappa',(\vec{s}\sigma'')\kappa')}) \boxtimes G(\widetilde{\eta}_{((\vec{b}\sigma'')\kappa'',(\vec{r}\sigma'')\kappa'',(\vec{s}\sigma'')\kappa'')}) \Big)$$

Now because the sections of $\alpha_{E,F,G}$ are the associator for $\mathsf{smod}_{\Bbbk}$, checking (71) is equivalent to showing that

$$\sum_{\substack{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a+b,c} \\ \rho \in {}^{\vec{b}\tau',\vec{r}\tau',\vec{s}\tau'}\mathscr{D}_{a,b}}} (-1)^{s(\vec{b},\tau)+s(\vec{b}\tau',\rho)} E(\widetilde{\eta}_{((\vec{b}\tau')\rho',(\vec{r}\tau')\rho',(\vec{s}\tau')\rho')}) \boxtimes F(\widetilde{\eta}_{((\vec{b}\tau')\rho'',(\vec{r}\tau')\rho'',(\vec{s}\tau')\rho'')}) \boxtimes G(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}) \tag{72}$$

is equal to

$$\sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b+c} \\ \kappa \in {}^{\vec{b}\sigma'',\vec{r}\sigma'',\vec{s}\sigma''}\mathscr{D}_{b,c}}} (-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma'',\kappa)} E(\widetilde{\eta}_{(\vec{b}\sigma',\vec{r}\sigma',\vec{s}\sigma')}) \boxtimes F(\widetilde{\eta}_{((\vec{b}\sigma'')\kappa',(\vec{r}\sigma'')\kappa',(\vec{s}\sigma'')\kappa')}) \boxtimes G(\widetilde{\eta}_{((\vec{b}\sigma'')\kappa'',(\vec{r}\sigma'')\kappa'',(\vec{s}\sigma'')\kappa'')}). \tag{73}$$

In order to show this, we will consider the set ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c} \subset {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}$ defined to be

$${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c} := \left\{ \tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \;\middle|\; \begin{array}{l} (b_{\tau i}, r_{\tau i}, s_{\tau i}) \leqslant (b_{\tau(i+1)}, r_{\tau(i+1)}, s_{\tau(i+1)}) \text{ for } 1 \leqslant i \leqslant a-1, \\ (b_{\tau j}, r_{\tau j}, s_{\tau j}) \leqslant (b_{\tau(j+1)}, r_{\tau(j+1)}, s_{\tau(j+1)}) \text{ for } a+1 \leqslant j \leqslant a+b-1, \\ (b_{\tau k}, r_{\tau k}, s_{\tau k}) \leqslant (b_{\tau(k+1)}, r_{\tau(k+1)}, s_{\tau(k+1)}) \text{ for } a+b+1 \leqslant k \leqslant a+b+c-1 \end{array} \right\}$$

Each summand from (72) is indexed by a choice of $\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a+b,c}$ and $\rho \in {}^{\vec{b}\tau',\vec{r}\tau',\vec{s}\tau'}\mathscr{D}_{a,b}$. For such a choice, first note that we can view ρ as actually acting on $(\vec{b}\tau, \vec{r}\tau, \vec{s}\tau)$ by having ρ fix the last c-many components (se we view $\rho \in \mathfrak{S}_{a+b+c}$ instead of as being in $\mathfrak{S}_{a+b}$). Then the sign $(-1)^{s(\vec{b}\tau',\rho)}$ from (72) can be written as $(-1)^{s(\vec{b}\tau,\rho)}$. Then it's clear that $\tau\rho \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}$ and that $(-1)^{s(\vec{b},\tau)+s(\vec{b}\tau,\rho)} = (-1)^{s(\vec{b},\tau\rho)}$. On the other hand, given some $\nu \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}$, we can associate a *unique* element $\nu^* \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a+b,c}$ such that the last c-many components of $(\vec{b},\vec{r},\vec{s})\nu^*$ are the same as the last c-many of $(\vec{b},\vec{r},\vec{s})\nu$. Then there is a *unique* element $\nu^{**} \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}$ such that ν^{**} fixes the last c-many components of $(\vec{b},\vec{r},\vec{s})\nu^*$ and such that the first a-many components of $((\vec{b},\vec{r},\vec{s})\nu^*)\nu^{**}$ are the same as the first a-many components of $(\vec{b},\vec{r},\vec{s})\nu$ (take $\nu^{**} = (\nu^*)^{-1}\nu$). So we have $\nu = \nu^*\nu^{**}$. But ν^{**} can be viewed as an element of ${}^{(\vec{b}\nu^*)',(\vec{r}\nu^*)',(\vec{s}\nu^*)'}\mathscr{D}_{a,b}$, and we have that (ν^*, ν^{**}) corresponds to a summand of (72).

Moreover, it is clear that this process determines a one-to-one correspondence between pairs which index the sum (72) and elements of ${}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}$ (where (τ,ρ) is associated to $\tau\rho$). Thus, we may rewrite (72) as

$$\sum_{\nu \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}} (-1)^{s(\vec{b},\nu)} E(\widetilde{\eta}_{\vec{b}\nu',\vec{r}\nu',\vec{s}\nu'}) \boxtimes F(\widetilde{\eta}_{\vec{b}\nu'',\vec{r}\nu'',\vec{s}\nu''}) \boxtimes G(\widetilde{\eta}_{\vec{b}\nu''',\vec{r}\nu''',\vec{s}\nu'''}) \tag{74}$$

where by, $\vec{b\nu}'''$ for example, we mean $\vec{b\nu}'''$ is a length c tuple whose entries are the same as the last c-many entries of $\vec{b\nu}$.

Similarly for (73), we can associate a pair (σ, κ) which indexes a summand to an element $\sigma\kappa \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}$ which gives a one-to-one correspondence (to go the other way, first collect the appropriate first a-many components, then shuffle the remaining components). So we may rewrite (73) as

$$\sum_{\nu \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{a,b,c}} (-1)^{s(\vec{b},\nu)} E(\widetilde{\eta}_{\vec{b\nu}',\vec{r\nu}',\vec{s\nu}'}) \boxtimes F(\widetilde{\eta}_{\vec{b\nu}'',\vec{r\nu}'',\vec{s\nu}''}) \boxtimes G(\widetilde{\eta}_{\vec{b\nu}''',\vec{r\nu}''',\vec{s\nu}'''}),$$

which is exactly (74). Thus (72) = (73) as desired.

So now we appropriately defined sections of α, and we need to check that these make α into an even supernatural transformation. This boils down to checking the following:

$$\alpha_{F,G,H} \circ (\eta_1 \boxtimes \eta_2) \boxtimes \eta_3 = \eta_1 \boxtimes (\eta_2 \boxtimes \eta_3) \circ \alpha_{C,D,E} \tag{75}$$

where it suffices to take C, F in $\mathsf{P}^x_{(A,\mathfrak{a})}$, D, G in $\mathsf{P}^y_{(A,\mathfrak{a})}$, E, H in $\mathsf{P}^z_{(A,\mathfrak{a})}$, and $\eta_1 \otimes \eta_2 \otimes \eta_3 \in \operatorname{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(C, F) \otimes \operatorname{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(D, G) \otimes \operatorname{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(E, H)$.

This means we must check that for any V, the sections at V of both sides of (75) are equal. This is a calculation in $\mathsf{smod}_{\Bbbk}$ which holds there since $\mathsf{smod}_{\Bbbk}$ is monoidal (and our monoidal structure is defined pointwise in terms of the monoidal structure of $\mathsf{smod}_{\Bbbk}$). It also follows from the $\mathsf{smod}_{\Bbbk}$ structure that the sections of α are isomorphisms, so α is a supernatural isomorphism.

Next we need even supernatural isomorphisms $\ell : \boldsymbol{I} \otimes \_ \to \_$ and $r : \_ \otimes \boldsymbol{I} \to \_$. First, we will define the sections of ℓ. Given any F in $\mathsf{P}_{(A,\mathfrak{a})}$, define $\ell_F : \boldsymbol{I} \otimes F \to F$ to be the supernatural transformation whose sections at V are $\ell_{FV} : \Bbbk \otimes F(V) \to F(V)$ given by

$$\beta \otimes x \mapsto \beta x.$$

This is easily seen to define an even supernatural transformation. Now to check that the sections ℓ_F make ℓ a supernatural transformation, we must check

$$\ell_G \circ \operatorname{id} \otimes \gamma = \gamma \circ \ell_F \tag{76}$$

for F, G in $\mathsf{P}_{(A,\mathfrak{a})}$ and $\gamma \in \operatorname{Hom}_{\mathsf{P}_{(A,\mathfrak{a})}}(F, G)$. But again, this means we must check that for any V, the sections at V of both sides of (76) are equal. This is easy to see (and again follows from the monoidal structure of $\mathsf{smod}_{\Bbbk}$). It also follows that the sections of ℓ are isomorphisms, so ℓ is a supernatural isomorphism, as desired.

The situation for $r : \_ \otimes \boldsymbol{I} \to \_$ is similar. We have $r_F : F \otimes \boldsymbol{I} \to F$ being the supernatural transformation whose sections at V are $r_{FV} : F(V) \otimes \Bbbk \to F(V)$ given by

$$x \otimes \beta \mapsto x\beta = \beta x.$$

The arguments that show r is a supernatural isomorphism are analagous to those used for ℓ.

Now we just need to show that these definitions adhere to the coherence conditions analagous to those for monoidal (non-super)categories. But these coherence conditions amount to showing that some compositions of supernatural transformations are equal, which is all computed pointwise at each section of these morphisms. These computations all take place in $\mathsf{smod}_{\Bbbk}$, and we know the relations hold there. Thus, the coherence conditions are satisfied, and we are done.

We claim that $\mathsf{P}_{(A,\mathfrak{a})}$ is also symmetric monoidal, induced by the super flip map

$$m \otimes n \mapsto (-1)^{\overline{m}\cdot\overline{n}} n \otimes m.$$

Proposition 6.18. $\mathsf{P}_{(A,\mathfrak{a})}$ *is a symmetric monoidal supercategory.*

Proof. As for proposition 6.17, it suffices to check this for homogeneous generalized polynomial functors. So we need to check that for any F in $\mathsf{P}^d_{(A,\mathfrak{a})}$ and G in $\mathsf{P}^e_{(A,\mathfrak{a})}$, $F \otimes G$ in $\mathsf{P}^{d+e}_{(A,\mathfrak{a})}$ is supernaturally isomorphic to $G \otimes F$ in both F and G. In particular, we need a supernatural isomorphism, which we will call $\mathsf{flip}_{F,G} : F \otimes G \to G \otimes F$. So, for any $V = A^n$, define the section of $\mathsf{flip}_{F,G}$ at V to be $\mathsf{flip}_{F,G_V} : (F \otimes G)(V) \to (G \otimes F)(V)$ via

$$u_f \otimes u_g \mapsto (-1)^{\overline{u_f}\cdot\overline{u_g}} u_g \otimes u_f,$$

where u_f, u_g are homogeneous elements of $F(V)$ and $G(V)$, respectively.

The flip is obviously even and whose sections yield isomorphisms of $\Bbbk$-supermodules. To check that it is a supernatural transformation, recall that $\left\{\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \,\middle|\, (\vec{b},\vec{r},\vec{s}) \in \Omega(B,m,n;d)\right\}$ gives a $\Bbbk$-basis for $T^A_{\mathfrak{a}}(m,n;d)$ ($= T^A_{\mathfrak{a}}(V,W;d)$ for $V = A^n$ and $W = A^m$). Since the naturality condition in this case is a matter of $\Bbbk$-linear maps commuting, it suffices to compute on homogeneous elements $u_f \in F(V)$ and $u_g \in G(V)$ and $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T^A_{\mathfrak{a}}(m,n;d+e)$.

First, applying $(F \otimes G)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$ to $u_f \otimes u_g \in F(V) \otimes G(V)$ gives (using the formulation as in remark 6.16)

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} F(\widetilde{\varphi}_{((\vec{b}\tau)',(\vec{r}\tau)',(\vec{s}\tau)')}) \boxtimes G(\widetilde{\varphi}_{((\vec{b}\tau)'',(\vec{r}\tau)'',(\vec{s}\tau)'')})(u_f \otimes u_g), \tag{77}$$

which equals

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)+\overline{(\vec{b}\tau)''}\cdot\overline{u_f}} F(\widetilde{\varphi}_{((\vec{b}\tau)',(\vec{r}\tau)',(\vec{s}\tau)')})(u_f) \otimes G(\widetilde{\varphi}_{((\vec{b}\tau)'',(\vec{r}\tau)'',(\vec{s}\tau)'')})(u_g) \tag{78}$$

since G is even and since $\overline{\widetilde{\varphi}_{((\vec{b}\tau)'',(\vec{r}\tau)'',(\vec{s}\tau)'')}} = \overline{((\vec{b}\tau)'',(\vec{r}\tau)'',(\vec{s}\tau)'')} = \overline{(\vec{b}\tau)''}$. Then applying the flip to equation (78) gives $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}$ times the following:

$$\sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)+\overline{(\vec{b}\tau)''}\cdot\overline{u_f}+\left(\overline{(\vec{b}\tau)'}+\overline{u_f}\right)\left(\overline{(\vec{b}\tau)''}+\overline{u_g}\right)} G(\widetilde{\varphi}_{((\vec{b}\tau)'',(\vec{r}\tau)'',(\vec{s}\tau)'')})(u_g) \otimes F(\widetilde{\varphi}_{((\vec{b}\tau)',(\vec{r}\tau)',(\vec{s}\tau)')})(u_f), \tag{79}$$

where the extra signs appear as written because of similar reasoning as in the previous paragraph.

On the other hand, if we first apply the flip to $u_f \otimes u_g$ to get $(-1)^{\overline{u_f}\cdot\overline{u_g}} u_g \otimes u_f$ and then apply $(G \otimes F)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$, we get

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\rho \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{e,d}} (-1)^{s(\vec{b},\rho)+\overline{u_f}\cdot\overline{u_g}} G(\widetilde{\varphi}_{((\vec{b}\rho)',(\vec{r}\rho)',(\vec{s}\rho)')}) \boxtimes F(\widetilde{\varphi}_{((\vec{b}\rho)'',(\vec{r}\rho)'',(\vec{s}\rho)'')})(u_g \otimes u_f), \tag{80}$$

where note that here, our prime and double-prime notation on the vectors refer to taking the first e-many and last d-many entries (which is reversed from above). Then (80) is equal to

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\rho\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{e,d}} (-1)^{s(\vec{b},\rho)+\overline{u_f}\cdot\overline{u_g}+\overline{(\vec{b}\rho)''}\cdot\overline{u_g}} G(\widetilde{\varphi}_{((\vec{b}\rho)',(\vec{r}\rho)',(\vec{s}\rho)')})(u_g)\otimes F(\widetilde{\varphi}_{((\vec{b}\rho)'',(\vec{r}\rho)'',(\vec{s}\rho)'')})(u_f). \tag{81}$$

Now by remark 6.11 (and using ζ as in the remark), we can reindex (81) to get

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\tau\in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau\zeta)+\overline{u_f}\cdot\overline{u_g}+\overline{(\vec{b}\tau\zeta)''}\cdot\overline{u_g}} G(\widetilde{\varphi}_{((\vec{b}\tau\zeta)',(\vec{r}\tau\zeta)',(\vec{s}\tau\zeta)')})(u_g)\otimes F(\widetilde{\varphi}_{((\vec{b}\tau\zeta)'',(\vec{r}\tau\zeta)'',(\vec{s}\tau\zeta)'')})(u_f). \tag{82}$$

Again by remark 6.11, we know that

$$G(\widetilde{\varphi}_{((\vec{b}\tau\zeta)',(\vec{r}\tau\zeta)',(\vec{s}\tau\zeta)')})(u_g)\otimes F(\widetilde{\varphi}_{((\vec{b}\tau\zeta)'',(\vec{r}\tau\zeta)'',(\vec{s}\tau\zeta)'')})(u_f)$$

equals

$$G(\widetilde{\varphi}_{((\vec{b}\tau)'',(\vec{r}\tau)'',(\vec{s}\tau)'')})(u_g)\otimes F(\widetilde{\varphi}_{((\vec{b}\tau)',(\vec{r}\tau)',(\vec{s}\tau)')})(u_f),$$

so the terms in equations (79) and (82) match, up to signs. So, let's match the signs. From (79), we have

$$\begin{aligned}
(-1)^{s(\vec{b},\tau)+\overline{(\vec{b}\tau)''}\cdot\overline{u_f}+\left(\overline{(\vec{b}\tau)'}+\overline{u_f}\right)\left(\overline{(\vec{b}\tau)''}+\overline{u_g}\right)} &= (-1)^{s(\vec{b},\tau)+\overline{(\vec{b}\tau)''}\cdot\overline{u_f}+\overline{(\vec{b}\tau)'}\cdot\overline{(\vec{b}\tau)''}+\overline{(\vec{b}\tau)'}\cdot\overline{u_g}+\overline{u_f}\cdot\overline{(\vec{b}\tau)''}+\overline{u_f}\cdot\overline{u_g}}\\
&= (-1)^{s(\vec{b},\tau)+\overline{(\vec{b}\tau)'}\cdot\overline{(\vec{b}\tau)''}+\overline{(\vec{b}\tau)'}\cdot\overline{u_g}+\overline{u_f}\cdot\overline{u_g}}\\
&= (-1)^{s(\vec{b},\tau)+s(\vec{b}\tau,\zeta)+\overline{(\vec{b}\tau)'}\cdot\overline{u_g}+\overline{u_f}\cdot\overline{u_g}}\\
&= (-1)^{s(\vec{b},\tau)+s(\vec{b}\tau,\zeta)+\overline{(\vec{b}\tau\zeta)''}\cdot\overline{u_g}+\overline{u_f}\cdot\overline{u_g}}\\
&= (-1)^{s(\vec{b},\tau\zeta)+\overline{(\vec{b}\tau\zeta)''}\cdot\overline{u_g}+\overline{u_f}\cdot\overline{u_g}},
\end{aligned}$$

which is precisely the sign from the corresponding term in (82). Thus, we have (79) = (82) as desired.

It just remains to show that flip is natural in both arguments. This amounts to showing

$$\mathsf{flip}_{E,G}\circ \mathrm{id}\otimes\eta = \eta\otimes\mathrm{id}\circ\mathsf{flip}_{E,F}$$

for $\eta\in \mathrm{Hom}_{\mathsf{P}^e_{(A,\mathfrak{a})}}(F,G)$ and

$$\mathsf{flip}_{F,G}\circ\beta\otimes\mathrm{id} = \mathrm{id}\otimes\beta\circ\mathsf{flip}_{E,G}$$

for $\beta\in\mathrm{Hom}_{\mathsf{P}^d_{(A,\mathfrak{a})}}(E,F)$. These both need to hold at every section V which is a calculation taking place in $\mathsf{smod}_{\Bbbk}$ which we know holds since $\mathsf{smod}_{\Bbbk}$ is symmetric monoidal under the flip map. ◻

6.3. The Superfunctor S^d

We claim that the prototypical examples of classical strict polynomial functors such as $\otimes^d$, $\bigwedge^d$, S^d, and Γ^d all have analogs for the categories $\mathsf{P}^d_{(A,\mathfrak{a})}$ and P^d_A. In this subsection, we will analyze S^d in full detail since it will play such an important role in what follows. In particular, we work with $\mathsf{P}^d_{(A,\mathfrak{a})}$, but one should be able to see how the argument goes for P^d_A, as well.

First, we will define the generalized strict polynomial functor S^d in $\mathsf{P}^d_{(A,\mathfrak{a})}$ for any unital superalgebra good pair $(A,\mathfrak{a})$. Let U be a free finite right A-supermodule. Then in particular,

U is a $\Bbbk$-supermodule, and the tensor algebra makes sense, $T(U) := \bigoplus_{i=0} U^{\otimes i}$. This object is a $\Bbbk$-supermodule inheriting it's $\mathbb{Z}_2$-grading from that of U. Moreover, it is a $\Bbbk$-superalgebra under the product $\otimes$. It also has an obvious $\mathbb{Z}$-grading by degree (the number of tensors in a given summand), and we denote $T^d(U) := U^{\otimes d}$ its ($\mathbb{Z}$-)degree d part.

Consider the ideal $J := \langle x \otimes y - (-1)^{\overline{x} \cdot \overline{y}} y \otimes x \rangle \subset T(U)$ where x, y are homogeneous elements of U. We define $S(U) := T(U)/J$ as a quotient $\Bbbk$-superalgebra. Since the ideal J is homogeneous (with respect to the $\mathbb{Z}$-grading), we get $S^d(U) = T^d(U)/J$ and $S(U) = \bigoplus_{i=0} S^i(U)$. Note that it is also homogeneous with respect to our $\mathbb{Z}_2$-grading so that $S(U)$ (and each $S^d(U)$) is a $\Bbbk$-supermodule. The parity of a coset in $S(U)$ (or $S^d(U)$) is the parity of the representative.

Abusing notation, we have S^d in $\mathsf{P}^d_{(A,\mathfrak{a})}$ send U to the $\Bbbk$-supermodule $S^d(U)$ just defined. Now we need to see what S^d does to a morphism.

Recall that our morphisms live in $T^A_{\mathfrak{a}}(m,n;d) \subset S^A(m,n;d) \cong \left(\mathrm{Hom}_A(A^n, A^m)^{\otimes d}\right)^{\mathfrak{S}_d}$. We need some lemmas before we proceed.

Let H denote the set of homogeneous elements of A. Let $\vec{b}, \vec{x} \in H^d$. Then we say

$$\vec{bx} := (b_1x_1, b_2x_2, \ldots, b_dx_d).$$

For any $\sigma \in \mathfrak{S}_d$, we have $\vec{b}\sigma = (b_{\sigma 1}, \ldots, b_{\sigma d})$ so that

$$\vec{b\sigma x} := (b_{\sigma 1}x_1, \ldots, b_{\sigma d}x_d).$$

Lemma 6.19. *For $\vec{b}, \vec{x} \in H^{d+e}$, $\vec{r} \in [1,m]^{d+e}$, and $\vec{s} \in [1,n]^{d+e}$, let $\vec{b}' = (b_1, \ldots, b_d)$, $\vec{b}'' = (b_{d+1}, \ldots, b_{d+e})$, $\vec{x}' = (x_1, \ldots, x_d)$, $\vec{x}'' = (x_{d+1}, \ldots, x_{d+e})$, $\vec{r}' = (r_1, \ldots, r_d)$, $\vec{r}'' = (r_{d+1}, \ldots, r_{d+e})$, $\vec{s}' = (s_1, \ldots, s_d)$, and $\vec{s}'' = (s_{d+1}, \ldots, s_{d+e})$. Then*

$$(-1)^{\overline{\varphi_{(\vec{b}'', \vec{r}'', \vec{s}'')}} \cdot \overline{\vec{x}'} + s(\vec{b}' \to \vec{x}') + s(\vec{b}'' \to \vec{x}'')} = (-1)^{s(\vec{b} \to \vec{x})}.$$

Proof. Well

$$\begin{aligned}
\overline{\varphi_{(\vec{b}'', \vec{r}'', \vec{s}'')}} \cdot \overline{\vec{x}'} &= \left(\overline{\varphi^{b_{d+1}}_{r_{d+1}s_{d+1}}} + \cdots + \overline{\varphi^{b_{d+e}}_{r_{d+e}s_{d+e}}}\right)\left(\overline{x_1} + \cdots + \overline{x_d}\right) \\
&= \left(\overline{b_{d+1}} + \cdots + \overline{b_{d+e}}\right)\left(\overline{x_1} + \cdots + \overline{x_d}\right) \\
&= \sum_{i=1}^{d} \sum_{j=1}^{e} \overline{x_i} \cdot \overline{b_{d+j}},
\end{aligned}$$

which implies

$$\begin{aligned}
(-1)^{\overline{\varphi_{(\vec{b}'', \vec{r}'', \vec{s}'')}} \cdot \overline{\vec{x}'} + s(\vec{b}' \to \vec{x}') + s(\vec{b}'' \to \vec{x}'')} &= (-1)^{\left(\sum_{i=1}^{d} \sum_{j=1}^{e} \overline{x_i} \cdot \overline{b_{d+j}}\right) + \left(\sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \overline{x_i} \cdot \overline{b_j}\right) + \left(\sum_{i=1}^{e-1} \sum_{j=i+1}^{e} \overline{x_{d+i}} \cdot \overline{b_{d+j}}\right)} \\
&= (-1)^{\sum_{i=1}^{d+e-1} \sum_{j=i+1}^{d+e} \overline{x_i} \cdot \overline{b_j}} \\
&= (-1)^{s(\vec{b} \to \vec{x})},
\end{aligned}$$

as desired. ∎

From this, it is easy to see the following:

Corollary 6.20. *For $V = A^n$, we have an A-basis for V, $\{v_1, \ldots, v_n\}$. Now for $b \in B$, if we let v_i^b denote the element $v_i.b$ (so v_i^b corresponds to a column vector of height n with zeros everywhere except for a b in the i^{th} slot), we see that $\{v_i^b \mid b \in B,\ i \in [1,n]\}$ is a $\Bbbk$-basis for V. For $\vec{b}, \vec{x} \in B^{d+e}$, $\vec{r} \in [1,m]^{d+e}$, and $\vec{s} \in [1,n]^{d+e}$, we have that*

$$\left(\varphi_{r_1 s_1}^{b_1} \boxtimes \cdots \boxtimes \varphi_{r_d s_d}^{b_d}\right) \boxtimes \left(\varphi_{r_{d+1} s_{d+1}}^{b_{d+1}} \boxtimes \cdots \boxtimes \varphi_{r_{d+e} s_{d+e}}^{b_{d+e}}\right) \left(v_{s_1}^{x_1} \otimes \cdots \otimes v_{s_d}^{x_d}\right) \otimes \left(v_{s_{d+1}}^{x_{d+1}} \otimes \cdots \otimes v_{s_{d+e}}^{x_{d+e}}\right)$$

is equal to

$$(-1)^{s(\vec{b} \to \vec{x})} v_{r_1}^{b_1 x_1} \otimes \cdots \otimes v_{r_{d+e}}^{b_{d+e} x_{d+e}}.$$

Lemma 6.21. *For any $\vec{b}, \vec{x} \in H^d$ and for any simple transposition $t_i \in \mathfrak{S}_d$,*

$$(-1)^{s(\vec{b} \to \vec{x}) + s(\vec{bx}, t_i)} = (-1)^{s(\vec{b}, t_i) + s(\vec{x}, t_i) + s(\vec{b}t_i \to \vec{x}t_i)}.$$

Proof. In this case, it is easy to see that

$$\begin{aligned}(-1)^{s(\vec{bx}, t_i)} &= (-1)^{\overline{b_i x_i} \cdot \overline{b_{i+1} x_{i+1}}} \\ &= (-1)^{(\overline{b_i} + \overline{x_i}) \cdot (\overline{b_{i+1}} + \overline{x_{i+1}})} \\ &= (-1)^{\overline{b_i} \cdot \overline{b_{i+1}} + \overline{b_i} \cdot \overline{x_{i+1}} + \overline{x_i} \cdot \overline{b_{i+1}} + \overline{x_i} \cdot \overline{x_{i+1}}}\end{aligned}$$

so that the left hand side of our desired equality is

$$(-1)^{s(\vec{b} \to \vec{x}) + s(\vec{bx}, t_i)} = (-1)^{\left(\sum_{r=1}^{d-1} \sum_{s=r+1}^{d} \overline{x_r} \cdot \overline{b_s}\right) + \overline{b_i} \cdot \overline{b_{i+1}} + \overline{b_i} \cdot \overline{x_{i+1}} + \overline{x_i} \cdot \overline{b_{i+1}} + \overline{x_i} \cdot \overline{x_{i+1}}}. \tag{83}$$

It is similarly easy to see that

$$(-1)^{s(\vec{b}, t_i) + s(\vec{x}, t_i)} = (-1)^{\overline{b_i} \cdot \overline{b_{i+1}} + \overline{x_i} \cdot \overline{x_{i+1}}}.$$

Notice further that

$$\begin{aligned}(-1)^{s(\vec{b}t_i \to \vec{x}t_i)} &= (-1)^{\left(\sum_{r=1}^{d-1} \sum_{s=r+1}^{d} \overline{x_r} \cdot \overline{b_s}\right) - \overline{x_i} \cdot \overline{b_{i+1}} + \overline{x_{i+1}} \cdot \overline{b_i}} \\ &= (-1)^{\left(\sum_{r=1}^{d-1} \sum_{s=r+1}^{d} \overline{x_r} \cdot \overline{b_s}\right) + \overline{x_i} \cdot \overline{b_{i+1}} + \overline{x_{i+1}} \cdot \overline{b_i}},\end{aligned}$$

where this last equality follows since the calculation essentially takes place modulo 2, so adding 0 or 1 is equivalent to subtracting 0 or 1, respectively.

So our desired right hand side is

$$(-1)^{s(\vec{b}, t_i) + s(\vec{x}, t_i) + s(\vec{b}t_i \to \vec{x}t_i)} = (-1)^{\overline{b_i} \cdot \overline{b_{i+1}} + \overline{x_i} \cdot \overline{x_{i+1}} + \left(\sum_{r=1}^{d-1} \sum_{s=r+1}^{d} \overline{x_r} \cdot \overline{b_s}\right) + \overline{x_i} \cdot \overline{b_{i+1}} + \overline{x_{i+1}} \cdot \overline{b_i}}. \tag{84}$$

Now it is clear that (83) = (84), and we are done. ∎

Lemma 6.22. *Suppose that for every $\vec{b}, \vec{x} \in H^d$, $\sigma, \tau \in \mathfrak{S}_d$ are such that*

$$(-1)^{s(\vec{b} \to \vec{x}) + s(\vec{bx}, \sigma)} = (-1)^{s(\vec{b}, \sigma) + s(\vec{x}, \sigma) + s(\vec{b}\sigma \to \vec{x}\sigma)}$$

and

$$(-1)^{s(\vec{b} \to \vec{x}) + s(\vec{bx}, \tau)} = (-1)^{s(\vec{b}, \tau) + s(\vec{x}, \tau) + s(\vec{b}\tau \to \vec{x}\tau)}.$$

Then for every $\vec{b}$ and $\vec{x}$,

$$(-1)^{s(\vec{b} \to \vec{x}) + s(\vec{bx}, \sigma\tau)} = (-1)^{s(\vec{b}, \sigma\tau) + s(\vec{x}, \sigma\tau) + s(\vec{b}(\sigma\tau) \to \vec{x}(\sigma\tau))}.$$

Proof. We have

$$\begin{aligned}
(-1)^{s(\vec{b}\to\vec{x})+s(\vec{bx},\sigma\tau)} &= (-1)^{s(\vec{b}\to\vec{x})+s(\vec{bx},\sigma)+s(\vec{bx}\sigma,\tau)} \\
&= (-1)^{s(\vec{b},\sigma)+s(\vec{x},\sigma)+s(\vec{b}\sigma\to\vec{x}\sigma)+s(\vec{bx}\sigma,\tau)} \\
&= (-1)^{s(\vec{b},\sigma)+s(\vec{x},\sigma)+s(\vec{b}\sigma\to\vec{x}\sigma)+s(\overrightarrow{(b\sigma)(x\sigma)},\tau)} \\
&= (-1)^{s(\vec{b},\sigma)+s(\vec{x},\sigma)+s(\vec{b}\sigma,\tau)+s(\vec{x}\sigma,\tau)+s((\vec{b}\sigma)\tau\to(\vec{x}\sigma)\tau)} \\
&= (-1)^{s(\vec{b},\sigma\tau)+s(\vec{x},\sigma\tau)+s(\vec{b}(\sigma\tau)\to\vec{x}(\sigma\tau))},
\end{aligned}$$

where the first equality follows from expanding the second term using remark 1.4, the second equality follows from our hypothesis applied to the first two terms, the fourth equality follows from applying our hypothesis to the last two terms, and the final equality follows from using remark 1.4 to collapse terms. ∎

Corollary 6.23. *For every $\vec{b},\vec{x}\in H^d$ and every $\sigma\in\mathfrak{S}_d$, we have*

$$(-1)^{s(\vec{b}\to\vec{x})+s(\vec{bx},\sigma)} = (-1)^{s(\vec{b},\sigma)+s(\vec{x},\sigma)+s(\vec{b}\sigma\to\vec{x}\sigma)}.$$

Proof. By lemma 6.21, we know this equality holds for all simple transpositions. For a given $\sigma\in\mathfrak{S}_d$, σ can be written as a product of simple transpositions. Then one may iteratively use lemma 6.22 to obtain the desired result. ∎

For a chosen basis $B = B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}}\sqcup B_{A_1}$ of our good pair, and for $V = A^n$ and $W = A^m$, recall that $\left\{\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}\,\middle|\,(\vec{b},\vec{r},\vec{s})\in\Omega(B,m,n;d)\right\}$ gives a homogeneous $\Bbbk$-basis for $S^A(V,W;d) = S^A(m,n;d)$ where

$$\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} = \sum_{\sigma\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)}\varphi^{b_{\sigma1}}_{r_{\sigma1}s_{\sigma1}}\otimes\cdots\otimes\varphi^{b_{\sigma d}}_{r_{\sigma d}s_{\sigma d}}.$$

Then $\left\{\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\,\middle|\,(\vec{b},\vec{r},\vec{s})\in\Omega(B,m,n;d)\right\}$ gives us a homogeneous $\Bbbk$-basis for $T^A_{\mathfrak{a}}(V,W;d)= T^A_{\mathfrak{a}}(m,n;d)$ where

$$\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} = [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}.$$

So, a general element in $T^A_{\mathfrak{a}}(V,W;d)$ can be written as a finite $\Bbbk$-linear combination of the $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}$. Since generalized polynomial functors induce even $\Bbbk$-linear maps between morphism spaces, it suffices to consider only the $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}$.

It follows from lemma 2.4 that we have $S^A(V,W;d)\cong \mathrm{Hom}_{A^{\otimes d}}(V^{\otimes d},W^{\otimes d})^{\mathfrak{S}_d}$ via

$$\sum\alpha_{\vec{i}}f_{i_1}\otimes\cdots\otimes f_{i_d}\mapsto\sum\alpha_{\vec{i}}f_{i_1}\boxtimes\cdots\boxtimes f_{i_d}.$$

Since $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\in T^A_{\mathfrak{a}}(V,W;d)\subset S^A(V,W;d)$, we can consider the image of this element under the aforementioned map. Call this $\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}$. Explicitly,

$$\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})} = [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\sum_{\sigma\in{}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)}\varphi^{b_{\sigma1}}_{r_{\sigma1}s_{\sigma1}}\boxtimes\cdots\boxtimes\varphi^{b_{\sigma d}}_{r_{\sigma d}s_{\sigma d}}.$$

So $\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}$ is, in particular, a $\Bbbk$-linear map from $T^d(V)\to T^d(W)$. What we need is a map from $S^d(V)\to S^d(W)$. Well first, note that the degree d part of the ideal $\langle x\otimes y-(-1)^{\overline{x}\cdot\overline{y}}y\otimes x\rangle\subset$

$T(V)$ can be written as

$$\mathrm{span}_{\Bbbk}\{u_{i_1}\otimes u_{i_2}\otimes\cdots\otimes u_{i_d}-(-1)^{\overline{u_{i_j}}.\overline{u_{i_{j+1}}}}u_{t_j(i_1)}\otimes u_{t_j(i_2)}\otimes\cdots\otimes u_{t_j(i_d)}\mid 1\leqslant j\leqslant d-1,\ u_{i_k}\in V\}$$

which is equal to

$$I^d(V)\coloneqq \mathrm{span}_{\Bbbk}\{u_{i_1}\otimes\cdots\otimes u_{i_d}-(u_{i_1}\otimes\cdots\otimes u_{i_d}).t_j\mid 1\leqslant j\leqslant d-1,\ u_{i_k}\in V\},$$

where $t_j\in\mathfrak{S}_d$ is a simple transposition. Similarly for $T(W)$. By the universal property of quotients, we get the induced map below (which by an abuse of notation we again refer to as $\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}$)

$$\begin{tikzcd} T^d(V) \arrow[r, "\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}"] \arrow[d, two heads, "p"'] & T^d(W) \arrow[d, two heads, "p"] \\ S^d(V) \arrow[r, dashed, "\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}"'] & S^d(W) \end{tikzcd} \tag{85}$$

so long as $I^d(V)$ is in the kernel of $p\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}$. To see that this is the case, first note that

$$\begin{aligned}\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}(u_{i_1}\otimes\cdots\otimes u_{i_d}-(u_{i_1}\otimes\cdots\otimes u_{i_d}).t_j) &= \widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}(u_{i_1}\otimes\cdots\otimes u_{i_d})-\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}((u_{i_1}\otimes\cdots\otimes u_{i_d}).t_j)\\ &= \widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}(u_{i_1}\otimes\cdots\otimes u_{i_d})-\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}(u_{i_1}\otimes\cdots\otimes u_{i_d}).t_j.\end{aligned} \tag{86}$$

Now $\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}(u_{i_1}\otimes\cdots\otimes u_{i_d})$ is some linear combination of pure tensors in $T^d(W)$, and since the t_j acts linearly, it follows that (86) is in $I^d(W)$. Thus, p maps this element to 0 in $S^d(W)$, and our desired result follows. This induced map is where the functor S^d sends the morphism $\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}$. In other words,

$$S^d(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})[u_1\cdots u_d]=\left[\widetilde{\eta}'_{(\vec{b},\vec{r},\vec{s})}(u_1\cdots u_d)\right], \tag{87}$$

and we have the functor S^d defined.

Let's take a short digression.

Remark 6.24. It follows from proposition 1.11 that the $\mathfrak{gl}_n(A)$ action on $T^d(A^n)$ descends to an action on $S^d(A^n)$. To see this, note that acting by $x\in\mathfrak{gl}_n(A)$ gives a linear map from $T^d(A^n)\to T^d(A^n)$. The fact that this action commutes with the $\mathfrak{S}_d$ action implies that a similar calculation as in (86) holds, and we have our claim.

Speaking of $\mathfrak{gl}_n(A)$, we note that remark 1.17 extends to symmetric powers of those modules in question. Let $\mathsf{smod}^{\mathcal{S}}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}$ denote the full monoidal subcategory of right $\mathfrak{gl}_n(A^{\mathsf{sop}})$ supermodules tensor-generated by the objects $S^d(V_n)$. Let ${}_{\mathfrak{gl}_n(A)}\mathsf{smod}^{\mathcal{S}}$ denote the full monoidal subcategory of left $\mathfrak{gl}_n(A)$ supermodules tensor-generated by the objects $S^d(A_n)$. It follows from proposition 1.16 and remark 1.17 that we have:

Proposition 6.25. *$\mathcal{T}$ restricts to an equivalence of categories* $\mathsf{smod}^{\mathcal{S}}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}\cong {}_{\mathfrak{gl}_n(A)}\mathsf{smod}^{\mathcal{S}}$.

Returning from our digression, we will be interested later in the functor $S^d\otimes S^e$ for some d,e. From remark 6.16, we can identify $S^d\otimes S^e$ as a functor sitting in $\mathsf{P}^{d+e}_{(A,\mathfrak{a})}$ which sends V

to $(S^d \otimes S^e)(V) = S^d(V) \otimes S^e(V)$ and on morphisms

$$(S^d \otimes S^e)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}) \coloneqq \sum_{\tau \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}_{d,e}} (-1)^{s(\vec{b},\tau)} \prod_{i=1}^{p} \binom{q_i}{k_{\tau,i}} S^d(\widetilde{\eta}_{(\vec{b}\tau',\vec{r}\tau',\vec{s}\tau')}) \boxtimes S^e(\widetilde{\eta}_{(\vec{b}\tau'',\vec{r}'',\vec{s}\tau'')}).$$

When actually doing calculations, it is often much easier to view $(S^d \otimes S^e)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$ as

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)} (\varphi^{b_{\sigma 1}}_{r_{\sigma 1} s_{\sigma 1}} \boxtimes \cdots \boxtimes \varphi^{b_{\sigma d}}_{r_{\sigma d} s_{\sigma d}}) \boxtimes (\varphi^{b_{\sigma(d+1)}}_{r_{\sigma(d+1)} s_{\sigma(d+1)}} \boxtimes \cdots \boxtimes \varphi^{b_{\sigma(d+e)}}_{r_{\sigma(d+e)} s_{\sigma(d+e)}}).$$

We will make use of this throughout the rest of the thesis.

7. Connection Between $\mathsf{P}_{(A,\mathfrak{a})}$ and $\mathfrak{gl}_n(A)$-Webs

In this section, we will establish a connection between generalized strict polynomial functors and webs for $\mathfrak{gl}_n(A)$ as defined in [DKMZ22] (which we will denote $\mathsf{Web}_{(A,\mathfrak{a})}$).

7.1. The category $\mathsf{Web}_{(A,\mathfrak{a})}$

In this subsection, we will briefly summarize enough of [DKMZ22] for the reader to get a feel for the category $\mathsf{Web}_{(A,\mathfrak{a})}$.

Remark 7.1. The web category $\mathsf{Web}_{\mathcal{A},\boldsymbol{a}}$ is defined in section 3.1 of [DKMZ22] in terms of small $\Bbbk$-linear supercatgories $\mathcal{A}$ and subcategories $\boldsymbol{a}$ which form a 'good pair'. As described in section 2.2 of [DKMZ22], small $\Bbbk$-linear supercategories $\mathcal{A}$ are in one-to-one correspondence with locally unital $\Bbbk$-superalgebras A. So $\mathsf{Web}_{\mathcal{A},\boldsymbol{a}}$ describes webs for locally unital superalgebras which come along with a distinguished system of orthogonal idempotents I. In particular, $\mathrm{Ob}(\mathcal{A}) = I$, and for our thesis, we're considering unital superalgebras, so we can take $I = \{1_A\}$ which corresponds to a small $\Bbbk$-linear supercategory $\mathcal{A}$ with a single object. Then a good pair $(\mathcal{A}, \boldsymbol{a})$ corresponds to a unital good pair $(A, \mathfrak{a})$ as we defined in definition 4.3, and as described in section 1.2 of [DKMZ22], $\mathsf{Web}_{\mathcal{A},\boldsymbol{a}}$ corresponds to webs defined for the unital good pair $(A, \mathfrak{a})$. The effect this has is that we can ignore the idempotents in the diagrams of definition 3.1.1 of [DKMZ22]. This is the category we will call $\mathsf{Web}_{(A,\mathfrak{a})}$ from now on.

That said, our theory of generalized strict polynomial functors can work with locally unital superalgebras, as well. The only place where the unit of A is essential is in the proof of lemma 6.7 where we force the entries of certain matrices to be 1_A. Since the matrices involved have finite size, there are only finitely many relevant idempotents from I. Then one can replace 1_A with the sum of these relevant idempotents, and the result still goes through.

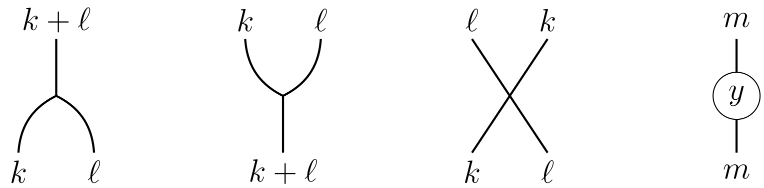
Definition 7.2. Let A be a $\Bbbk$-superalgebra and $(A, \mathfrak{a})$ be a unital good pair. Then $\mathsf{Web}_{(A,\mathfrak{a})}$ is the monoidal supercategory generated by the following objects subject to the following relations:

The objects will be tuples of nonnegative integers with concatenation of words providing the monoidal product. In paticular, the objects of $\mathsf{Web}_{(A,\mathfrak{a})}$ are given by

$$\mathrm{Ob}(\mathsf{Web}_{(A,\mathfrak{a})}) = \{\vec{x} \coloneqq (x_1, \dots, x_t) \in \mathbb{Z}^t_{\geqslant 0} \mid t \in \mathbb{Z}_{\geqslant 0}\}$$

with 0 indicating the unit object.

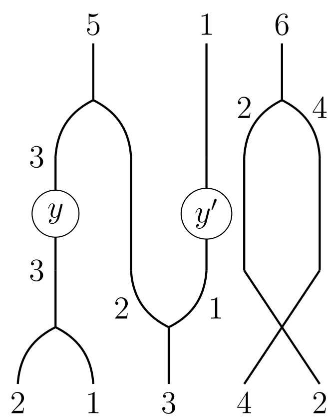
The generating morphisms of $\mathsf{Web}_{(A,\mathfrak{a})}$ are given by the diagrams[2]:



(88)

for $k, \ell \in \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{Z}_{>0}$, $\begin{cases} y \in A & \text{if} \quad m = 1 \\ y \in \mathfrak{a} & \text{if} \quad m > 1 \end{cases}$, where diagrams are to be read from bottom to top. We call these morphisms **split**, **merge**, **crossing**, and **coupon** respectively. Morphisms in $\mathsf{Web}_{(A,\mathfrak{a})}$ are then $\Bbbk$-linear combinations of diagrams built by repeated concatenation of generating diagrams, subject to a fairly simple set of local relations found in definition 3.1.1 of [DKMZ22].

For example, a typical morphism in $\mathsf{Web}_{(A,\mathfrak{a})}((2,1,3,4,2),(5,1,6))$ looks like:



(89)

where $y \in \mathfrak{a}$ and $y' \in A$.

Splits, merges and crossings have parity 0, and the parity of the y coupon is $\overline{y}$. We say a strand labeled by k has *thickness* k. We refer to strands of thickness 1 as *thin* strands, otherwise we call them *thick*. We emphasize that thick strands may only be decorated by coupons in the (even) subalgebra $\mathfrak{a}$, whereas thin strands may be decorated with arbitrary coupons in A. Going forward, we will use the following conventions:

- Strands of thickness 0 (and any coupons thereon) are to be deleted;
- Diagrams containing a strand of negative thickness are to be read as zero.

The language we've used thus far implies that the category $\mathsf{Web}_{(A,\mathfrak{a})}$ is related to $\mathfrak{gl}_n(A)$ in some way. We explain this next. Let $\mathsf{smod}^{\mathcal{S}}_{\mathfrak{gl}_n(A)}$ denote the full subsupercategory of *right* $\mathfrak{gl}_n(A)$-supermodules tensor-generated by $S^d(V_n)$, for various d, where V_n is the free *left* A-supermodule of rank n viewed as row vectors. Theorem 5.5.1 of [DKMZ22] establishes a family of monoidal superfunctors $G_n : \mathsf{Web}_{(A,\mathfrak{a})} \to \mathsf{smod}^{\mathcal{S}}_{\mathfrak{gl}_n(A)}$. These functors are asymptotically locally faithful, which means the following: For any objects $\vec{x}, \vec{y}$, there exists $N > 0$

[2]Since we're working with unital superalgebras, it's actually redundant to include the crossing diagram and the related relations. We include it here, however, because it corresponds to an important morphism we will describe below and is necessary for the locally-unital case.

such that the map between morphism spaces induced by G_n is injective (faithful) for all $n \geqslant N$.

Moreover, when $\Bbbk$ is a characteristic 0 field, the G_n are also asymptotically locally full. In fact, under certain additional assumptions on A, the G_n are full by corollary 6.6.3 and proposition 9.3.2 of [DKMZ22]. So in this way, $(A, \mathfrak{a})$-webs describe morphisms between $\mathfrak{gl}_n(A)$-modules of the form $S^{d_1}(V_n) \otimes \cdots \otimes S^{d_m}(V_n)$.

We can describe the functor G_n explicitly. First, as described in [DKMZ22], note that for any objects M, N in $\mathsf{smod}_\Bbbk$, the superalgebra $S(M)$ has the obvious associative product $\nabla : S(M) \otimes S(M) \to S(M)$, coassociative coproduct $\Delta : S(M) \to S(M) \otimes S(M)$, and projections $p_d : S(M) \to S^d(M)$ and inclusions $\iota_d : S^d(M) \to S(M)$ for all $d \in \mathbb{Z}_{\geqslant 0}$.

Consider the maps

$$ {}_M\operatorname{spl}^{d,e}_{d+e} : S^{d+e}(M) \to S^d(M) \otimes S^e(M) \quad \text{via} \quad {}_M\operatorname{spl}^{d,e}_{d+e} := (p_d \otimes p_e) \circ \Delta \circ \iota_{d+e}, \tag{90} $$

$$ {}_M\operatorname{mer}^{d+e}_{d,e} : S^d(M) \otimes S^e(M) \to S^{d+e}(M) \quad \text{via} \quad {}_M\operatorname{mer}^{d+e}_{d,e} := p_{d+e} \circ \nabla \circ (\iota_d \otimes \iota_e), \tag{91} $$

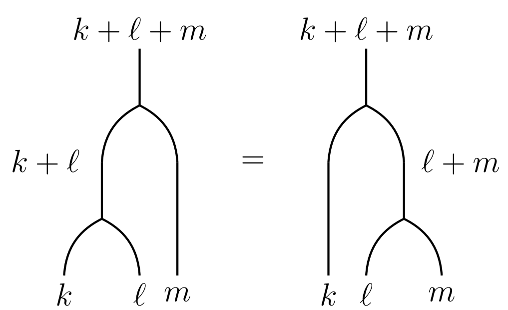
$$ \tau_{M,N} : M \otimes N \to N \otimes M \quad \text{via} \quad m \otimes n \mapsto (-1)^{\overline{m} \cdot \overline{n}} n \otimes m, \tag{92} $$

and for $y \in A$,

$$ L^y_d : S^d(V_n) \to S^d(V_n) \quad \text{via} \quad \begin{cases} v^x_r \mapsto v^{yx}_r & \text{if} \quad d = 1 \\ v^{x_1}_{r_1} \cdots v^{x_d}_{r_d} \mapsto v^{yx_1}_{r_1} \cdots v^{yx_d}_{r_d} & \text{if} \quad d > 1 \text{ and } y \in \mathfrak{a} \end{cases}. \tag{93} $$

These are all even homomorphisms of right $\mathfrak{gl}_n(A)$-supermodules. Then the monoidal functor G_n sends an object d of $\mathsf{Web}_{(A,\mathfrak{a})}$ to $S^d(V_n)$ and sends the generating diagrams split, merge, crossing, and y-coupon, to the maps ${}_{V_n}\operatorname{spl}^{d,e}_{d+e}$, ${}_{V_n}\operatorname{mer}^{d+e}_{d,e}$, $\tau_{S^d(V^n), S^e(V^n)}$, and L^y_d, respectively.

Theorem 5.5.1 of [DKMZ22] shows that G_n preserves the defining relations of $\mathsf{Web}_{(A,\mathfrak{a})}$, so is well-defined. The relations in $\mathsf{Web}_{(A,\mathfrak{a})}$ are essentially constructed so that this is the case. For example, one defining relation of $\mathsf{Web}_{(A,\mathfrak{a})}$ is the following *merge associativity*:



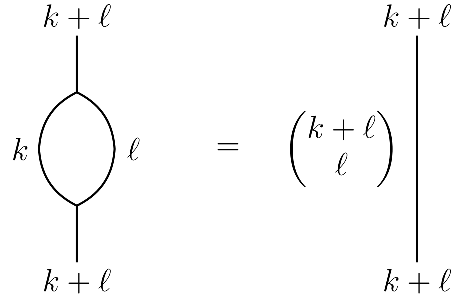
(94)

One wants to check, then, that

$$ {}_{V_n}\operatorname{mer}^{k+\ell+m}_{k+\ell,m} \circ ({}_{V_n}\operatorname{mer}^{k+\ell}_{k,\ell} \otimes \mathrm{id}) = {}_{V_n}\operatorname{mer}^{k+\ell+m}_{k,\ell+m} \circ (\mathrm{id} \otimes {}_{V_n}\operatorname{mer}^{\ell+m}_{\ell,m}). $$

This follows almost immediately from the fact that the multiplication in $S(V_n)$ is associative (and note that because this multiplication is associative, you'd want this to be reflected in the relations for $\mathsf{Web}_{(A,\mathfrak{a})}$ if G_n is to be (fully) faithful).

Another defining relation of $\mathsf{Web}_{(A,\mathfrak{a})}$ is the so-called *knothole relation*:



(95)

Then one wants to check that

$$ {}_{V_n}\operatorname{mer}_{k,\ell}^{k+\ell} \circ {}_{V_n}\operatorname{spl}_{k+\ell}^{k,\ell} = \binom{k+\ell}{\ell} \operatorname{id}. $$

We'll check this for a small example ($k = 2 = \ell$) to give the reader a feel for how it goes. We'll simplify things and choose a homogeneous element $wxyz \in S^4(V_n)$ such that each w, x, y, z are even elements of V_n. Then ${}_{V_n}\operatorname{mer}_{2,2}^{4} \circ {}_{V_n}\operatorname{spl}_{4}^{2,2}(wxyz)$ is equal to[3]

$$ {}_{V_n}\operatorname{mer}_{2,2}^{4}\left(wx \otimes yz + wy \otimes xz + wz \otimes xy + xy \otimes wz + xz \otimes wy + yz \otimes wx\right) $$

which equals

$$ 6(wxyz) = \binom{4}{2} wxyz $$

in $S^4(V_n)$, as desired.

The core idea of the remainder of this section is that under appropriate assumptions, $\mathsf{Web}_{(A,\mathfrak{a})}$ is equivalent to a subcategory of $\mathsf{P}_{(A,\mathfrak{a})}$. In order to establish this claim, we will make use of the functors $G_n : \mathsf{Web}_{(A,\mathfrak{a})} \to \mathsf{smod}_{\mathfrak{gl}_n(A)}$. So we first want to relate $\mathsf{P}_{(A,\mathfrak{a})}$ with $\mathfrak{gl}_n(A)$-supermodules.

7.2. $\mathsf{P}_{(A,\mathfrak{a})}$ and $\mathfrak{gl}_n(A)$-Supermodules

By theorem 6.10, we know that evaluation at A^n gives an equivalence of categories $\mathsf{P}^d_{(A,\mathfrak{a})} \cong {}_{T^A_{\mathfrak{a}}(n,n;d)}\mathsf{smod}$ whenever $n \geqslant d$. We wish to establish here that we can view evaluation as instead landing us in the category ${}_{\mathfrak{gl}_n(A)}\mathsf{smod}$.

In order to make this precise, fix some F in $\mathsf{P}^d_{(A,\mathfrak{a})}$ and a homogeneous $x \in \mathfrak{gl}_n(A)$. It is essentially automatic that $\mathfrak{gl}_n(A) \cong \operatorname{End}_A(A^n)$ as superspaces (where this later space is the space of all *right* A-supermodule endomorphisms of A^n). Then we can associate to x the element $x' \in \operatorname{End}_{\mathsf{S}^A(d)}(A^n) = \left(\operatorname{End}_A(A^n)^{\otimes d}\right)^{\mathfrak{S}_d}$ given by

$$ x' := \sum_{i=1}^{d} \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes x \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \tag{96} $$

where in each term of this summand, x is in the i^{th} slot, and we've denoted by $\mathbb{1}$ the map $\operatorname{id}_{A^n} \in \operatorname{End}_A(A^n)$. Notice that for x homogeneous, each summand in x' has parity equal to $\overline{x}$ since $\overline{\mathbb{1}} = 0$, and there are no sign issues. Thus, $\overline{x'} = \overline{x}$.

[3]Note that ${}_{V_n}\operatorname{spl}_{k+\ell}^{k,\ell}$ is, after identifying $A^n \cong V_n$, the $\Bbbk$-map defined by the section of $\varsigma_{k+\ell}^{k,\ell}$ defined in proposition 7.7 below.

Of course, what we actually want is for x' to be in $\mathrm{End}_{\mathsf{T}^A_{\mathfrak{a}}(d)}(A^n) = T^A_{\mathfrak{a}}(n,n;d)$. To see that this is the case, first note that x' can be written as a $\Bbbk$-linear combination of the $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$ which comprise the $\Bbbk$-basis for $S^A(n,n;d)$. To actually achieve this, one would expand each summand of x'.

Precisely, first choose a basis $B_{\mathfrak{a}}$ of $\mathfrak{a}$ which contains 1_A and so that $(A,\mathfrak{a})$ is a unital good pair. Moreover, we will assume that the entries x_{rs} of x are in the $\Bbbk$-basis B of A.

Then note that

$$\mathbb{1} = \sum_{i=1}^{n} \varphi_{ii}^{1_A}$$

and

$$x = \sum_{r,s\in[1,n]} \varphi_{rs}^{x_{rs}}.$$

So expanding a given term $\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes x\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$ of x' yields sums of elements of the form

$$\varphi_{i_1i_1}^{1_A}\otimes\cdots\otimes\varphi_{i_{k-1}i_{k-1}}^{1_A}\otimes\varphi_{i_kj_k}^{x_{i_kj_k}}\otimes\varphi_{i_{k+1}i_{k+1}}^{1_A}\otimes\cdots\otimes\varphi_{i_di_d}^{1_A}.$$

Such a term will contribute to the basis element $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})}$ where $\vec{b} = (1_A,\dots,x_{i_kj_k},\dots,1_A)$, $\vec{r} = (i_1,\dots,i_d)$, and $\vec{s} = (i_1,\dots,i_{k-1},j_k,i_{k+1},\dots,i_d)$. Since $x_{i_kj_k}$ is the only element with the potential to be in $\mathfrak{c}$, it follows that $[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} = 1$. Therefore, $\widetilde{\varphi}_{(\vec{b},\vec{r},\vec{s})} = \widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}$ in this case. It follows that the expansion of x' into the $\Bbbk$-basis of $S^A(n,n;d)$ is actually also an expansion into the $\Bbbk$-basis of $T^A_{\mathfrak{a}}(n,n;d)$, and hence $x'\in T^A_{\mathfrak{a}}(n,n;d)$, as desired.

Proposition 7.3. *For $n\geqslant d$ and every F in $\mathsf{P}^d_{(A,\mathfrak{a})}$, the map $\rho_F : \mathfrak{gl}_n(A)\to \mathrm{End}_{\Bbbk}(F(A^n))$ given by $\rho_F(x) = F(x')$ is a homomorphism of Lie superalgebras. Therefore, $F(A^n)$ is a left $\mathfrak{gl}_n(A)$-supermodule.*

Proof. First of all, the target of ρ_F makes sense since, by definition, F knows how to take morphisms $x'\in\mathrm{End}_{\mathsf{T}^A_{\mathfrak{a}}(d)}(A^n)$ to morphisms $F(x')\in\mathrm{End}_{\Bbbk}(F(A^n))$. To see that ρ_F is a Lie superalgebra homomorphism, we must check that it respects the supercommutator. So, choose any homogeneous $x,y\in\mathfrak{gl}_n(A)$. Then on one hand, (suppressing the $\otimes$ symbol) we have

$$\begin{aligned}[x,y]' &= \big(xy-(-1)^{\overline{x}\cdot\overline{y}}yx\big)' \\ &= \sum_{i=1}^{d}\mathbb{1}\cdots\big(xy-(-1)^{\overline{x}\cdot\overline{y}}yx\big)\cdots\mathbb{1} \\ &= \left(\sum_{i=1}^{d}\mathbb{1}\cdots xy\cdots\mathbb{1}\right)-(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{j=1}^{d}\mathbb{1}\cdots yx\cdots\mathbb{1}\right). \end{aligned} \tag{97}$$

On the other hand, we have

$$\begin{aligned}[x',y'] &= x'y'-(-1)^{\overline{x'}\cdot\overline{y'}}y'x' \\ &= x'y'-(-1)^{\overline{x}\cdot\overline{y}}y'x' \\ &= \left(\sum_{p=1}^{d}\mathbb{1}\cdots x\cdots\mathbb{1}\right)\left(\sum_{q=1}^{d}\mathbb{1}\cdots y\cdots\mathbb{1}\right)-(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{r=1}^{d}\mathbb{1}\cdots y\cdots\mathbb{1}\right)\left(\sum_{s=1}^{d}\mathbb{1}\cdots x\cdots\mathbb{1}\right) \end{aligned} \tag{98}$$

Now consider the product $(\mathbb{1}\cdots x\cdots\mathbb{1})(\mathbb{1}\cdots y\cdots\mathbb{1})$ where x is in the i^{th} slot and y is in the j^{th} slot. Since $\overline{\mathbb{1}}=0$, we pick up a sign of $(-1)^{\overline{x}\cdot\overline{y}}$ only when x passes over y, which happens for $1\leqslant j\leqslant i-1$. In general, we have

$$(\mathbb{1}\cdots x\cdots\mathbb{1})(\mathbb{1}\cdots y\cdots\mathbb{1})\begin{cases}(-1)^{\overline{x}\cdot\overline{y}}(\mathbb{1}\cdots y\cdots x\cdots\mathbb{1}) & \text{if}\quad 1\leqslant j\leqslant i-1\\ \mathbb{1}\cdots xy\cdots\mathbb{1} & \text{if}\quad j=i\\ \mathbb{1}\cdots x\cdots y\cdots\mathbb{1} & \text{if}\quad i+1\leqslant j\leqslant d\end{cases}. \tag{99}$$

So

$$\begin{aligned}x'y'&=\left(\sum_{p=1}^{d}\mathbb{1}\cdots x\cdots\mathbb{1}\right)\left(\sum_{q=1}^{d}\mathbb{1}\cdots y\cdots\mathbb{1}\right)\\&=\sum_{p=1}^{d}\left[(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{q=1}^{p-1}\mathbb{1}\cdots y\cdots x\cdots\mathbb{1}\right)+(\mathbb{1}\cdots xy\cdots\mathbb{1})+\left(\sum_{q=p+1}^{d}\mathbb{1}\cdots x\cdots y\cdots\mathbb{1}\right)\right],\end{aligned}$$

where x is in the p^{th} slot, and y is in the q^{th} slot (so in particular, the middle term above has xy in the $p=q$ slot). So we can write

$$\begin{aligned}x'y'&=\sum_{p=1}^{d}(\mathbb{1}\cdots xy\cdots\mathbb{1})\\&\quad+\sum_{p=1}^{d}\left[(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{q=1}^{p-1}\mathbb{1}\cdots y\cdots x\cdots\mathbb{1}\right)+\left(\sum_{q=p+1}^{d}\mathbb{1}\cdots x\cdots y\cdots\mathbb{1}\right)\right],\end{aligned}\tag{100}$$

and then

$$\begin{aligned}y'x'&=\left(\sum_{r=1}^{d}\mathbb{1}\cdots y\cdots\mathbb{1}\right)\left(\sum_{s=1}^{d}\mathbb{1}\cdots x\cdots\mathbb{1}\right)\\&=\sum_{r=1}^{d}\left[(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{s=1}^{r-1}\mathbb{1}\cdots x\cdots y\cdots\mathbb{1}\right)+(\mathbb{1}\cdots yx\cdots\mathbb{1})+\left(\sum_{s=r+1}^{d}\mathbb{1}\cdots y\cdots x\cdots\mathbb{1}\right)\right],\end{aligned}\tag{101}$$

where y is in the r^{th} slot, and x is in the s^{th} slot (so in particular, the middle term above has yx in the $r=s$ slot).

Note that (101) is indexed such that the position of y is chosen first, then the position of x. To better match (100), let us switch the order of the sums in (101) so that we first choose the position of x (which will correspond to s) and then the position of y (which will correspond to r). We have

$$\begin{aligned}y'x'&=\sum_{s=1}^{d}\left[\sum_{r=1}^{s-1}(\mathbb{1}\cdots y\cdots x\cdots\mathbb{1})+(\mathbb{1}\cdots yx\cdots\mathbb{1})+(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{r=s+1}^{d}\mathbb{1}\cdots x\cdots y\cdots\mathbb{1}\right)\right]\\&=\sum_{s=1}^{d}(\mathbb{1}\cdots yx\cdots\mathbb{1})\\&\quad+\sum_{s=1}^{d}\left[\sum_{r=1}^{s-1}(\mathbb{1}\cdots y\cdots x\cdots\mathbb{1})+(-1)^{\overline{x}\cdot\overline{y}}\left(\sum_{r=s+1}^{d}\mathbb{1}\cdots x\cdots y\cdots\mathbb{1}\right)\right].\end{aligned}\tag{102}$$

Since we work modulo 2 when computing parity, we have $-(-1)^{\overline{x}\cdot\overline{y}}(-1)^{\overline{x}\cdot\overline{y}}=-1$, so it follows that

$$\begin{aligned}-(-1)^{\overline{x}\cdot\overline{y}}y'x' = -(-1)^{\overline{x}\cdot\overline{y}}\sum_{s=1}^{d}(\mathbb{1}\cdots yx\cdots\mathbb{1})\\ +\sum_{s=1}^{d}\left[-(-1)^{\overline{x}\cdot\overline{y}}\sum_{r=1}^{s-1}(\mathbb{1}\cdots y\cdots x\cdots\mathbb{1})-\left(\sum_{r=s+1}^{d}\mathbb{1}\cdots x\cdots y\cdots\mathbb{1}\right)\right]. \quad (103)\end{aligned}$$

So finally, lining up our terms from (100) and (103) (and remembering that p and s correspond to the x position, and q and r correspond to the y position), we see that (98) = (97). That is, $[x,y]'=[x',y']$.

It follows then, that

$$\begin{aligned}\rho_F([x,y]) &= F([x,y]')\\ &= F([x',y'])\\ &= F(x'y'-(-1)^{\overline{x}\cdot\overline{y}}y'x')\\ &= F(x')F(y')-(-1)^{\overline{x}\cdot\overline{y}}F(y')F(x')\\ &= \rho_F(x)\rho_F(y)-(-1)^{\overline{x}\cdot\overline{y}}\rho_F(y)\rho_F(x)\\ &= \rho_F(x)\rho_F(y)-(-1)^{\overline{\rho_F(x)}\cdot\overline{\rho_F(y)}}\rho_F(y)\rho_F(x)\\ &= [\rho_F(x),\rho_F(y)],\end{aligned}$$

where the second to last equality follows from the fact that $\rho_F(x)=F(x')$ which has parity equal to $\overline{x}$ since F is even and $\overline{x'}=\overline{x}$, as mentioned above. Then ρ_F is even, so $\overline{\rho_F(x)}=\overline{x}$. Similarly for y. So we have our desired result. ◻

Now suppose for F,G in $\mathsf{P}^d_{(A,\mathfrak{a})}$ we have a homogeneous supernatural transformation $\eta: F\to G$. In particular, if we have homogeneous $x\in\mathfrak{gl}_n(A)$, this means $\eta_{A^n}\circ F(x')=(-1)^{\overline{\eta}\cdot\overline{x'}}G(x')\circ\eta_{A^n}$. But this is equivalent to having $\eta_{A^n}\circ\rho_F(x)=(-1)^{\overline{\eta}\cdot\overline{\rho_F(x)}}\rho_G(x)\circ\eta_{A^n}$ which exactly means η_{A^n} is a homomorphism of $\mathfrak{gl}_n(A)$-supermodules. Thus, we have the following result (that composition and identity are respected follows from η being a supernatural transformation):

Proposition 7.4. *For $n\geqslant d$, evaluation at A^n gives a functor $\mathsf{ev}_{A^n}:\mathsf{P}^d_{(A,\mathfrak{a})}\to{}_{\mathfrak{gl}_n(A)}\mathsf{smod}$ where ${}_{\mathfrak{gl}_n(A)}\mathsf{smod}$ denotes the category of left supermodules for $\mathfrak{gl}_n(A)$.*

We can now think about sections of natural transformations not just as $\Bbbk$-maps (or even $T^A_{\mathfrak{a}}(n,n;d)$-maps), but under the appropriate conditions, as $\mathfrak{gl}_n(A)$-maps. This will be helpful in what follows.

7.3. **Merge Morphism**

We know from remark 6.24 that $S^d(A^n)$ is a left $\mathfrak{gl}_n(A)$-supermodule. Our aim is to define a certain morphism in $\mathsf{P}^d_{(A,\mathfrak{a})}$ whose sections are $\mathfrak{gl}_n(A)$-supermodule maps.

Proposition 7.5. *For $S^d\otimes S^e, S^{d+e}$ in $\mathsf{P}^{d+e}_{(A,\mathfrak{a})}$, the map $\boldsymbol{\mu}^{d+e}_{d,e}:S^d\otimes S^e\to S^{d+e}$ is an even supernatural transformation whose sections $(\boldsymbol{\mu}^{d+e}_{d,e})_V:(S^d\otimes S^e)(V)\to S^{d+e}(V)$ are given by multiplication inside $S(V)$. We call this the **merge** morphism.*

Proof. We should check that this definition makes sense. First of all, for $x \in S^d(V)$ and $y \in S^e(V)$, $(\boldsymbol{\mu}_{d,e}^{d+e})_V(x \otimes y)$ just multiplies these elements inside $S(V)$ (by concatenating the representatives), resulting in an element which lives in $S^{d+e}(V)$.

Next, consider an element $[u_{i_1} \cdots u_{i_a}] \otimes [u_{j_1} \cdots u_{j_b}] \in S^a(V) \otimes S^b(V)$. By this notation, we mean, for example, that $u_{i_1} \cdots u_{i_a}$ is a pure tensor in $T^a(V)$ (where we've supressed the $\otimes$ notation) with each u_{i_k} being a homogeneous element of V, and $[u_{i_1} \cdots u_{i_a}]$ denotes the corresponding coset in $S^d(V)$. The parity of $[u_{i_1} \cdots u_{i_a}]$ is $\overline{u_{i_1}} + \cdots + \overline{u_{i_a}}$, and similarly for $[u_{j_1} \cdots u_{j_b}]$. Then the parity of $[u_{i_1} \cdots u_{i_a}] \otimes [u_{j_1} \cdots u_{j_b}]$ is $(\overline{u_{i_1}} + \cdots + \overline{u_{i_a}}) + (\overline{u_{j_1}} + \cdots + \overline{u_{j_b}})$ which is precisely the parity of $[u_{i_1} \cdots u_{i_a} u_{j_1} \cdots u_{j_b}] \in S^{d+e}(V)$. Hence each section of $\boldsymbol{\mu}_{d,e}^{d+e}$ is an even map, so $\boldsymbol{\mu}_{d,e}^{d+e}$ is even.

What remains to be seen is that $\boldsymbol{\mu}_{d,e}^{d+e}$ is actually a supernatural transformation. To check this, first note that since $V = A^n$, we have an A-basis for V, $\{v_1, \ldots, v_n\}$. Now for $b \in B$, if we let v_i^b denote the element $v_i.b$ (so v_i^b corresponds to a column vector of height n with zeros everywhere except for a b in the i^{th} slot), we see that $\{v_i^b \mid b \in B,\ i \in [1,n]\}$ is a $\Bbbk$-basis for V. Moreover, recall that $\left\{ \widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \,\middle|\, (\vec{b},\vec{r},\vec{s}) \in \Omega(B,m,n;d) \right\}$ gives a $\Bbbk$-basis for $T_{\mathfrak{a}}^A(m,n;d)$ $(= T_{\mathfrak{a}}^A(V,W;d)$ for $W = A^m)$. Since the naturality condition in this case is a matter of $\Bbbk$-linear maps commuting (remember our sections are even, so we really do get a commuting square), the previous observations imply that it suffices to compute on $[v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \otimes [v_{t_{d+1}}^{x_{d+1}} \cdots v_{t_{d+e}}^{x_{d+e}}] \in S^d(V) \otimes S^e(V)$ (where $\vec{x} \in B^{d+e}$ and $\vec{t} \in [1,n]^{d+e}$) and $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T_{\mathfrak{a}}^A(m,n;d+e)$.

First of all, it is clear that either path of our desired square yields 0 if $\vec{t} \not\sim \vec{s}$. So, we will further assume that $\vec{t} = \vec{s}\gamma$ for some $\gamma \in \mathfrak{S}_{d+e}$. Then applying $(S^d \otimes S^e)(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$ to $[v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \otimes [v_{t_{d+1}}^{x_{d+1}} \cdots v_{t_{d+e}}^{x_{d+e}}]$ is equivalent to applying the morphism

$$[\vec{b},\vec{r},\vec{s}]_{\mathfrak{c}}^{!} \sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)} \varphi_{r_{\sigma 1} s_{\sigma 1}}^{b_{\sigma 1}} \cdots \varphi_{r_{\sigma d} s_{\sigma d}}^{b_{\sigma d}} \boxtimes \varphi_{r_{\sigma(d+1)} s_{\sigma(d+1)}}^{b_{\sigma(d+1)}} \cdots \varphi_{r_{\sigma(d+e)} s_{\sigma(d+e)}}^{b_{\sigma(d+e)}}$$

to $[v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \otimes [v_{t_{d+1}}^{x_{d+1}} \cdots v_{t_{d+e}}^{x_{d+e}}]$ which equals

$$[\vec{b},\vec{r},\vec{s}]_{\mathfrak{c}}^{!} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x})} \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1} \cdots v_{r_{\sigma d}}^{b_{\sigma d} x_d} \right] \otimes \left[v_{r_{\sigma(d+1)}}^{b_{\sigma(d+1)} x_{d+1}} \cdots v_{r_{\sigma(d+e)}}^{b_{\sigma(d+e)} x_{d+e}} \right] \tag{104}$$

by corollary 6.20. Now applying the merge to (104) gives

$$[\vec{b},\vec{r},\vec{s}]_{\mathfrak{c}}^{!} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x})} \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1} \cdots v_{r_{\sigma(d+e)}}^{b_{\sigma(d+e)} x_{d+e}} \right]. \tag{105}$$

On the other hand, first applying merge to $[v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \otimes [v_{t_{d+1}}^{x_{d+1}} \cdots v_{t_{d+e}}^{x_{d+e}}]$ gives $[v_{t_1}^{x_1} \cdots v_{t_{d+e}}^{x_{d+e}}]$. Then applying $S^{d+e}(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})$ gives

$$[\vec{b},\vec{r},\vec{s}]_{\mathfrak{c}}^{!} \left[\sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)} \left(\varphi_{r_{\sigma 1} s_{\sigma 1}}^{b_{\sigma 1}} \cdots \varphi_{r_{\sigma(d+e)} s_{\sigma(d+e)}}^{b_{\sigma(d+e)}} \right) \left(v_{s_{\gamma 1}}^{x_1} \cdots v_{s_{\gamma(d+e)}}^{x_{d+e}} \right) \right]$$

which equals

$$[\vec{b}, \vec{r}, \vec{s}]^{!}_{\mathfrak{c}} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma\to\vec{x})} \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1} \cdots v_{r_{\sigma(d+e)}}^{b_{\sigma(d+e)} x_{d+e}} \right]. \tag{106}$$

Since (105) = (106), we have our result. ◻

Proposition 7.6. *For any d, e, the section $(\boldsymbol{\mu}_{d,e}^{d+e})_{A^n} : (S^d \otimes S^e)(A^n) \to S^{d+e}(A^n)$ is a map of $\mathfrak{gl}_n(A)$-supermodules.*

Proof. First, of all, by remark 6.24, the objects in question are actually $\mathfrak{gl}_n(A)$-supermodules. We consider a homogeneous element $[u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}] \in S^d(A^n) \otimes S^e(A^n)$. Now let $x \in \mathfrak{gl}_n(A)$ be homogeneous. Then

$$\begin{aligned} x([u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}]) &= x[u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}] + (-1)^{\alpha}[u_{i_1} \cdots u_{i_d}] \otimes x[u_{j_1} \cdots u_{j_e}] \\ &= \sum_{r=1}^{d} (-1)^{\theta_r} [u_{i_1} \cdots x u_{i_r} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}] \\ &\quad + (-1)^{\alpha}[u_{i_1} \cdots u_{i_d}] \otimes \sum_{s=1}^{e} (-1)^{\eta_s} [u_{j_1} \cdots x u_{j_s} \cdots u_{j_e}], \end{aligned} \tag{107}$$

where $\alpha = \overline{x}(\overline{u_{i_1}} + \cdots + \overline{u_{i_d}})$, $\theta_r = \overline{x}(\overline{u_{i_1}} + \cdots + \overline{u_{i_{r-1}}})$, and $\eta_s = \overline{x}(\overline{u_{j_1}} + \cdots + \overline{u_{j_{s-1}}})$.

So if we take the merge of (107), we get

$$\begin{aligned} (\boldsymbol{\mu}_{d,e}^{d+e})_{A^n} \left(x([u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}]) \right) &= \sum_{r=1}^{d} (-1)^{\theta_r} [u_{i_1} \cdots x u_{i_r} \cdots u_{i_d} u_{j_1} \cdots u_{j_e}] \\ &\quad + \sum_{s=1}^{e} (-1)^{\eta_s + \alpha} [u_{i_1} \cdots u_{i_d} u_{j_1} \cdots x u_{j_s} \cdots u_{j_e}]. \end{aligned} \tag{108}$$

On the other hand,

$$\begin{aligned} x\left((\boldsymbol{\mu}_{d,e}^{d+e})_{A^n}([u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}]) \right) &= x\left([u_{i_1} \cdots u_{i_d} u_{j_1} \cdots u_{j_e}] \right) \\ &= \sum_{t=1}^{d} (-1)^{\kappa_t} [u_{i_1} \cdots x u_{i_t} \cdots u_{i_d} u_{j_1} \cdots u_{j_e}] \\ &\quad + \sum_{t'=1}^{e} (-1)^{\kappa'_{t'}} [u_{i_1} \cdots u_{i_d} u_{j_1} \cdots x u_{j_{t'}} \cdots u_{j_e}], \end{aligned} \tag{109}$$

where the t ranges through i_1 to i_d and t' ranges through j_1 to j_e. We have that $\kappa_t = \overline{x}(\overline{u_{i_1}} + \cdots + \overline{u_{i_{t-1}}})$ and $\kappa'_{t'} = \overline{x}(\overline{u_{i_1}} + \cdots + \overline{u_{i_d}} + \overline{u_{j_1}} + \cdots + \overline{u_{j_{t'-1}}})$.

We want (108) = (109) which will follow so long as $\theta_r = \kappa_r$ for all $1 \leqslant r \leqslant d$ and $\eta_s + \alpha = \kappa'_s$ for all $1 \leqslant s \leqslant e$. Luckily, this is immediate. ◻

7.4. **Split Morphism**

Let's keep the train rolling.

Proposition 7.7. *For $S^d \otimes S^a, S^{d+e}$ in $\mathsf{P}^{d+e}_{(A,\mathfrak{a})}$, the map $\varsigma^{d,e}_{d+e} : S^{d+e} \to S^d \otimes S^e$ is an even supernatural transformation whose sections $(\varsigma^{d,e}_{d+e})_V : S^{d+e}(V) \to (S^d \otimes S^e)(V)$ are given by*

$$[u_1 \cdots u_{d+e}] \mapsto \sum_{P \sqcup Q = [1,d+e]} (-1)^{\varepsilon(P,Q)} [u_{p_1} \cdots u_{p_d}] \otimes [u_{q_1} \cdots u_{q_e}].$$

Here, $[u_1 \cdots u_{d+e}]$ is as in the proof of proposition 7.5 where each u_i is a homogeneous element of V, $P = \{p_1 < \cdots < p_d\}$, $Q = \{q_1 < \cdots < q_e\}$, and

$$\varepsilon(P,Q) = \#\{(p,q) \in P \times Q \mid p > q,\ \overline{u_p} = \overline{u_q} = 1\}.$$

*We call this the **split** morphism.*

Proof. First, we check that $\varsigma^{d,e}_{d+e}$ is even. To so, we need to know that each section $(\varsigma^{d,e}_{d+e})_V$ is an even map. So, consider $[u_1 \cdots u_{d+e}] \in S^{d+e}(V)$ where each u_i is a homogeneous element in V. Now given any P, Q as in the formula for $(\varsigma^{d,e}_{d+e})_V$, the parity of $[u_{p_1} \cdots u_{p_d}] \otimes [u_{q_1} \cdots u_{q_e}]$ is $(\overline{u_{p_1}} + \cdots + \overline{u_{p_d}}) + (\overline{u_{q_1}} + \cdots + \overline{u_{q_e}})$ which equals $\overline{u_1} + \cdots + \overline{u_{d+e}}$ since $P \sqcup Q = [1, d+e]$. Therefore, each summand in $(\varsigma^{d,e}_{d+e})_V([u_1 \cdots u_{d+e}])$ has the same parity as $[u_1 \cdots u_{d+e}]$. Hence $\overline{[u_1 \cdots u_{d+e}]} = \overline{(\varsigma^{d,e}_{d+e})_V([u_1 \cdots u_{d+e}])}$, and we indeed see that our map is even.

What remains to be seen is that $\varsigma^{d,e}_{d+e}$ is actually a supernatural transformation. To check this, let $\{v_1, \ldots, v_n\}$ be an A-basis for $V = A^n$ so that $\{v_i^b \mid b \in B,\ i \in [1,n]\}$ is a $\Bbbk$-basis for V. Since the naturality condition in this case is a matter of $\Bbbk$-linear maps commuting (remember our sections are even, so we really do get a commuting square), the previous observations imply that it suffices to compute on $[v_{t_1}^{x_1} \cdots v_{t_{d+e}}^{x_{d+e}}] \in S^{d+e}(V)$ (where $\vec{x} \in B^{d+e}$ and $\vec{t} \in [1,n]^{d+e}$) and $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T^A_{\mathfrak{a}}(m,n;d+e)$.

First of all, it is clear that either path of our desired square yields 0 if $\vec{t} \not\sim \vec{s}$. So, we will further assume that $\vec{t} = \vec{s}\gamma$ for some $\gamma \in \mathfrak{S}_{d+e}$. Then

$$\begin{aligned}
S^{d+e}\Big(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\Big)\big([v_{t_1}^{x_1} \cdots v_{t_{d+e}}^{x_{d+e}}]\big) &= S^{d+e}\Big(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\Big)\Big([v_{s_{\gamma 1}}^{x_1} \cdots v_{s_{\gamma(d+e)}}^{x_{d+e}}]\Big) \\
&= \left[\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\left(v_{s_{\gamma 1}}^{x_1} \cdots v_{s_{\gamma(d+e)}}^{x_{d+e}}\right)\right] \\
&= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \left[\sum_{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D}} (-1)^{s(\vec{b},\sigma)} \varphi^{b_{\sigma 1}}_{r_{\sigma 1} s_{\sigma 1}} \cdots \varphi^{b_{\sigma(d+e)}}_{r_{\sigma(d+e)} s_{\sigma(d+e)}} \Big(v_{s_{\gamma 1}}^{x_1} \cdots v_{s_{\gamma(d+e)}}^{x_{d+e}}\Big)\right] \\
&= [\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x})} \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1} \cdots v_{r_{\sigma(d+e)}}^{b_{\sigma(d+e)} x_{d+e}}\right]. \quad (110)
\end{aligned}$$

Then applying the split gives

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x})} \left(\sum_{P,Q} (-1)^{\varepsilon(P,Q)} \left[v_{r_{\sigma(p_1)}}^{b_{\sigma(p_1)} x_{p_1}} \cdots v_{r_{\sigma(p_d)}}^{b_{\sigma(p_d)} x_{p_d}}\right] \otimes \left[v_{r_{\sigma(q_1)}}^{b_{\sigma(q_1)} x_{q_1}} \cdots v_{r_{\sigma(q_e)}}^{b_{\sigma(q_e)} x_{q_e}}\right]\right) \quad (111)$$

Now notice that we can associate to a choice of P and Q a unique permutation $\tau_{P,Q} \in \mathfrak{S}_{d+e}$. Explicity, we have $\tau_{P,Q}(i) = p_i$ for $i \in [1,d]$ and $\tau_{P,Q}(d+j) = q_j$ for $j \in [1,e]$. To shorten notation, we will denote by $\mathfrak{S}_{P,Q}$ the set of all such $\tau_{P,Q}$. Then we may write the inner sum in (111), $\sum\limits_{P,Q}(-1)^{\varepsilon(P,Q)}\Big[v_{r_{\sigma(p_1)}}^{b_{\sigma(p_1)}x_{p_1}}\cdots v_{r_{\sigma(p_d)}}^{b_{\sigma(p_d)}x_{p_d}}\Big]\otimes\Big[v_{r_{\sigma(q_1)}}^{b_{\sigma(q_1)}x_{q_1}}\cdots v_{r_{\sigma(q_e)}}^{b_{\sigma(q_e)}x_{q_e}}\Big]$, as

$$\sum_{\tau\in\mathfrak{S}_{P,Q}}(-1)^{s(\overrightarrow{b\sigma x},\tau)}\Big[v_{r_{\sigma(\tau 1)}}^{b_{\sigma(\tau 1)}x_{\tau 1}}\cdots v_{r_{\sigma(\tau d)}}^{b_{\sigma(\tau d)}x_{\tau d}}\Big]\otimes\Big[v_{r_{\sigma(\tau(d+1))}}^{b_{\sigma(\tau(d+1))}x_{\tau(d+1)}}\cdots v_{r_{\sigma(\tau(d+e))}}^{b_{\sigma(\tau(d+e))}x_{\tau(d+e)}}\Big] \tag{112}$$

so that

$$(111)=[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\sum_{\substack{\sigma\in^{\vec{b},\vec{r},\vec{s}}\mathscr{D}\\ \vec{s}\sigma=\vec{s}\gamma}}(-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma\to\vec{x})}(112).$$

On the other hand, we have

$$\begin{aligned}(\varsigma^{d,e}_{d+e})_V\left(\left[v_{t_1}^{x_1}\cdots v_{t_{d+e}}^{x_{d+e}}\right]\right)&=(\varsigma^{d,e}_{d+e})_V\left(\left[v_{s_{\gamma 1}}^{x_1}\cdots v_{s_{\gamma(d+e)}}^{x_{d+e}}\right]\right)\\ &=\sum_{\tau\in\mathfrak{S}_{P,Q}}(-1)^{s(\vec{x},\tau)}\left[v_{s_{\gamma(\tau 1)}}^{x_{\tau 1}}\cdots v_{s_{\gamma(\tau d)}}^{x_{\tau d}}\right]\otimes\left[v_{s_{\gamma(\tau(d+1))}}^{x_{\tau(d+1)}}\cdots v_{s_{\gamma(\tau(d+e))}}^{x_{\tau(d+e)}}\right].\end{aligned} \tag{113}$$

Now applying $S^d\otimes S^e\left(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\right)$ to a summand $\left[v_{s_{\gamma(\tau 1)}}^{x_{\tau 1}}\cdots v_{s_{\gamma(\tau d)}}^{x_{\tau d}}\right]\otimes\left[v_{s_{\gamma(\tau(d+1))}}^{x_{\tau(d+1)}}\cdots v_{s_{\gamma(\tau(d+e))}}^{x_{\tau(d+e)}}\right]$ in (113) gives the following:

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\sum_{\substack{\alpha\in^{\vec{b},\vec{r},\vec{s}}\mathscr{D}\\ \vec{s}\alpha=\vec{s}(\gamma\tau)}}(-1)^{s(\vec{b},\alpha)+s(\vec{b}\alpha\to\vec{x}\tau)}\left[v_{r_{\alpha 1}}^{b_{\alpha 1}x_{\tau 1}}\cdots v_{r_{\alpha d}}^{b_{\alpha d}x_{\tau d}}\right]\otimes\Big[v_{r_{\alpha(d+1)}}^{b_{\alpha(d+1)}x_{\tau(d+1)}}\cdots v_{r_{\alpha(d+e)}}^{b_{\alpha(d+e)}x_{\tau(d+e)}}\Big] \tag{114}$$

by corollary 6.20. Therefore,

$$S^d\otimes S^e\left(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}\right)(113)=\sum_{\tau\in\mathfrak{S}_{P,Q}}(-1)^{s(\vec{x},\tau)}(114).$$

So we just need to show that

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\sum_{\substack{\sigma\in^{\vec{b},\vec{r},\vec{s}}\mathscr{D}\\ \vec{s}\sigma=\vec{s}\gamma}}(-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma\to\vec{x})}(112)=\sum_{\tau\in\mathfrak{S}_{P,Q}}(-1)^{s(\vec{x},\tau)}(114).$$

First, we claim that each side of this equation consists of the same number of terms. This amounts to showing $\#\{\sigma\mid\vec{s}\sigma=\vec{s}\gamma\}=\#\{\alpha\mid\vec{s}\alpha=\vec{s}(\gamma\tau)\}$. Consider the set map from the first set to the second given by $\sigma\mapsto\sigma\tau$. This is easily seen to be a bijection with inverse given by $\alpha\mapsto\alpha\tau^{-1}$.

This actually also tells us how the terms from each side match up - the σ,τ term from the left matches the $\tau,\sigma\tau$ term from the right. Explicitly, since each $\alpha\in\{\alpha\mid\vec{s}\alpha=\vec{s}(\gamma\tau)\}$ can be written as $\alpha=\sigma\tau$ for some $\sigma\in\{\sigma\mid\vec{s}\sigma=\vec{s}\gamma\}$, we can reindex the summation in (114) so that (114) equals

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}}\sum_{\substack{\sigma\in^{\vec{b},\vec{r},\vec{s}}\mathscr{D}\\ \vec{s}\sigma=\vec{s}\gamma}}(-1)^{s(\vec{b},\sigma\tau)+s(\vec{b}(\sigma\tau)\to\vec{x}\tau)}\left[v_{r_{(\sigma\tau)1}}^{b_{(\sigma\tau)1}x_{\tau 1}}\cdots v_{r_{(\sigma\tau)d}}^{b_{(\sigma\tau)d}x_{\tau d}}\right]\otimes\Big[v_{r_{(\sigma\tau)(d+1)}}^{b_{(\sigma\tau)(d+1)}x_{\tau(d+1)}}\cdots v_{r_{(\sigma\tau)(d+e)}}^{b_{(\sigma\tau)(d+e)}x_{\tau(d+e)}}\Big] \tag{115}$$

So now we want

$$[\vec{b},\vec{r},\vec{s}]^!_{\mathfrak{c}} \sum_{\substack{\sigma\in^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma=\vec{s}\gamma}} (-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma\to\vec{x})}(112) = \sum_{\tau\in\mathfrak{S}_{P,Q}} (-1)^{s(\vec{x},\tau)}(115).$$

It is clear that we have this equality *up to signs*. So to finish our claim, we still need to show that the signs match. Explicitly, we need

$$(-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma\to\vec{x})+s(\overrightarrow{b\sigma x},\tau)} = (-1)^{s(\vec{x},\tau)+s(\vec{b},\sigma\tau)+s(\vec{b}(\sigma\tau)\to\vec{x}\tau)}$$

for each appropriate choice of σ and τ. Well we have

$$\begin{aligned}(-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma\to\vec{x})+s(\overrightarrow{b\sigma x},\tau)} &= (-1)^{s(\vec{b},\sigma)+s(\vec{b}\sigma,\tau)+s(\vec{x},\tau)+s((\vec{b}\sigma)\tau\to\vec{x}\tau)} \\ &= (-1)^{s(\vec{b},\sigma\tau)+s(\vec{x},\tau)+s((\vec{b}\sigma)\tau\to\vec{x}\tau)},\end{aligned}$$

where the first equality comes from applying corollary 6.23 to the last two terms, and the second equality follows from collapsing the first two terms using remark 1.4. ☐

Proposition 7.8. *For any d, e, the section $(\varsigma^{d,e}_{d+e})_{A^n} : S^{d+e}(A^n) \to (S^d \otimes S^e)(A^n)$ is a map of $\mathfrak{gl}_n(A)$-supermodules.*

Proof. Let's consider a homogeneous element $[u_1 \cdots u_{d+e}] \in S^{d+e}(A^n)$ and a homogeneous $x \in \mathfrak{gl}_n(A)$. Then

$$x.[u_1 \cdots u_{d+e}] = \sum_{r=1}^{d+e} (-1)^{\alpha_r}[u_1 \cdots xu_r \cdots u_{d+e}] \tag{116}$$

where $\alpha_r = \overline{x}(\overline{u_1} + \cdots + \overline{u_{r-1}})$.

Now we will introduce new notation to simplify our calculation. Let

$${}_r\vec{v} \coloneqq (u_1, \ldots, xu_r, \ldots u_{d+e})$$

so that ${}_rv_i = u_i$ when $i \neq r$ and ${}_rv_r = xu_r$. Then Applying $(\varsigma^{d,e}_{d+e})_{A^n}$ to (116) gives

$$\sum_{r=1}^{d+e}(-1)^{\alpha_r}\left(\sum_{P\sqcup Q=[1,d+e]} (-1)^{\varepsilon'(P,Q)}[{}_rv_{p_1}\cdots {}_rv_{p_d}] \otimes [{}_rv_{q_1}\cdots {}_rv_{q_e}]\right). \tag{117}$$

On the other hand, we have $x.(\varsigma^{d,e}_{d+e})_{A^n}([u_1 \cdots u_{d+e}])$ is equal to

$$\sum_{P\sqcup Q=[1,d+e]} (-1)^{\varepsilon(P,Q)} \left(x[u_{p_1}\cdots u_{p_d}] \otimes [u_{q_1}\cdots u_{q_e}] + (-1)^{\beta(P,Q)}[u_{p_1}\cdots u_{p_d}] \otimes x[u_{q_1}\cdots u_{q_e}]\right)$$

which equals

$$\begin{aligned}&\sum_{P\sqcup Q=[1,d+e]} (-1)^{\varepsilon(P,Q)} \left(x[u_{p_1}\cdots u_{p_d}] \otimes [u_{q_1}\cdots u_{q_e}]\right) \\ &\quad + \sum_{P\sqcup Q=[1,d+e]} (-1)^{\varepsilon(P,Q)+\beta(P,Q)} \left([u_{p_1}\cdots u_{p_d}] \otimes x[u_{q_1}\cdots u_{q_e}]\right).\end{aligned} \tag{118}$$

Now the first chunk of (118) is equal to

$$\sum_{P\sqcup Q=[1,d+e]} (-1)^{\varepsilon(P,Q)} \left(\sum_{s=1}^{d}(-1)^{\overline{x}(\overline{u_{p_1}}+\cdots+\overline{u_{p_{s-1}}})}[u_{p_1}\cdots xu_{p_s}\cdots u_{p_d}] \otimes [u_{q_1}\cdots u_{q_e}]\right), \tag{119}$$

and the second chunk of (118) is equal to

$$\sum_{P\sqcup Q=[1,d+e]}(-1)^{\varepsilon(P,Q)+\beta(P,Q)}\left(\sum_{t=1}^{e}(-1)^{\overline{x}(\overline{u_{q_1}}+\cdots+\overline{u_{q_{t-1}}})}[u_{p_1}\cdots u_{p_d}]\otimes[u_{q_1}\cdots xu_{q_t}\cdots u_{q_e}]\right). \tag{120}$$

This expansion makes it clear that (117) and (118) have the same number of terms. Now we will match these terms, proving the desired result.

A term from (119) is a choice of sets P, Q and a value of s giving

$$(-1)^{\varepsilon(P,Q)}(-1)^{\overline{x}(\overline{u_{p_1}}+\cdots+\overline{u_{p_{s-1}}})}[u_{p_1}\cdots xu_{p_s}\cdots u_{p_d}]\otimes[u_{q_1}\cdots u_{q_e}],$$

which in our notation from above is equal to

$$(-1)^{\varepsilon(P,Q)}(-1)^{\overline{x}(\overline{{}_{p_s}v_{p_1}}+\cdots+\overline{{}_{p_s}v_{p_{s-1}}})}[{}_{p_s}v_{p_1}\cdots{}_{p_s}v_{p_s}\cdots{}_{p_s}v_{p_d}]\otimes[{}_{p_s}v_{q_1}\cdots{}_{p_s}v_{q_e}]. \tag{121}$$

This corresponds to the $r=p_s$, P,Q term from (117):

$$(-1)^{\alpha_{p_s}}(-1)^{\varepsilon'(P,Q)}[{}_{p_s}v_{p_1}\cdots{}_{p_s}v_{p_d}]\otimes[{}_{p_s}v_{q_1}\cdots{}_{p_s}v_{q_e}]. \tag{122}$$

We just need to make sure that the signs from (121) and (122) match. To do so, we first note that for the element $[u_1\cdots u_{d+e}]$, we have

$$\begin{aligned}(-1)^{\varepsilon(P,Q)}&=(-1)^{\#\{(p,q)\in P\times Q\,|\,p>q,\,\overline{u_p}=\overline{u_q}=1\}}\\&=(-1)^{\sum\limits_{p\in P}\left(\overline{u_p}\cdot\left(\sum\limits_{\substack{q\in Q\\q<p}}\overline{u_q}\right)\right)}.\end{aligned}$$

Now for the term in (122), $\varepsilon'(P,Q)$ comes from the term $[u_1\cdots xu_r\cdots u_{d+e}]$ where $r=p_s$ belongs to P. So it only differs from $\varepsilon(P,Q)$ at the xu_r term. In particular, we have

$$\begin{aligned}(-1)^{\varepsilon'(P,Q)}&=(-1)^{\sum\limits_{\substack{p\in P\\p\neq p_s}}\left(\overline{u_p}\cdot\left(\sum\limits_{\substack{q\in Q\\q<p}}\overline{u_q}\right)\right)+\overline{xu_{p_s}}\left(\sum\limits_{\substack{q\in Q\\q<p_s}}\overline{u_q}\right)}\\&=(-1)^{\sum\limits_{\substack{p\in P\\p\neq p_s}}\left(\overline{u_p}\cdot\left(\sum\limits_{\substack{q\in Q\\q<p}}\overline{u_q}\right)\right)+(\overline{x}+\overline{u_{p_s}})\left(\sum\limits_{\substack{q\in Q\\q<p_s}}\overline{u_q}\right)}\\&=(-1)^{\varepsilon(P,Q)+\overline{x}\left(\sum\limits_{\substack{q\in Q\\q<p_s}}\overline{u_q}\right)}.\end{aligned}$$

Now notice that the sign from (121) is equal to

$$(-1)^{\varepsilon(P,Q)+\overline{x}(\overline{u_{p_1}}+\cdots+\overline{u_{p_{s-1}}})}, \tag{123}$$

and the sign from (122) is equal to

$$(-1)^{\varepsilon'(P,Q)+\alpha_{p_s}}=(-1)^{\varepsilon(P,Q)+\overline{x}\left(\sum\limits_{q<p_s}\overline{u_q}\right)+\overline{x}(\overline{u_1}+\cdots+\overline{u_{p_s-1}})}. \tag{124}$$

Since P is ordered, we know that $\{p_1, \ldots, p_{s-1}\} \subset \{1, \ldots, p_s - 1\}$. Moreover, $\{1, \ldots, p_s - 1\} \setminus \{p_1, \ldots, p_{s-1}\} \subset Q$, and these elements in Q are the only elements which are less than p_s. So they solely contribute to $\sum\limits_{q<p_s} \overline{u_q}$. To summarize, we have (since we work mod 2) that

$$(-1)^{\varepsilon(P,Q)+\overline{x}\left(\sum\limits_{q<p_s} \overline{u_q}\right)+\overline{x}(\overline{u_1}+\cdots+\overline{u_{p_s-1}})} = (-1)^{\varepsilon(P,Q)+\overline{x}(\overline{u_{p_1}}+\cdots+\overline{u_{p_{s-1}}})},$$

and thus (123) = (124), as desired.

Now a term from (120) is a choice of sets P, Q and a value of t giving

$$(-1)^{\varepsilon(P,Q)+\beta(P,Q)}(-1)^{\overline{x}(\overline{u_{q_1}}+\cdots+\overline{u_{q_{t-1}}})}[u_{p_1}\cdots u_{p_d}] \otimes [u_{q_1}\cdots xu_{q_t}\cdots u_{q_e}],$$

which in our notation from above is equal to

$$(-1)^{\varepsilon(P,Q)+\beta(P,Q)}(-1)^{\overline{x}(\overline{{}_{q_t}v_{q_1}}+\cdots+\overline{{}_{q_t}v_{q_{t-1}}})}[{}_{q_t}v_{p_1}\cdots {}_{q_t}v_{p_d}] \otimes [{}_{q_t}v_{q_1}\cdots {}_{q_t}v_{q_t}\cdots {}_{q_t}v_{q_e}]. \tag{125}$$

This corresponds to the $r = q_t$, P, Q term from (117):

$$(-1)^{\alpha_{q_t}}(-1)^{\varepsilon'(P,Q)}[{}_{q_t}v_{p_1}\cdots {}_{q_t}v_{p_d}] \otimes [{}_{q_t}v_{q_1}\cdots {}_{q_t}v_{q_e}]. \tag{126}$$

We just need to make sure that the signs from (125) and (126) match.

Now for the term in (126), $\varepsilon'(P,Q)$ comes from the term $[u_1\cdots xu_r\cdots u_{d+e}]$ where $r = q_t$ belongs to Q. So it only differs from $\varepsilon P, Q$ at the xu_r term. In particular, we have

$$\begin{aligned}
(-1)^{\varepsilon'(P,Q)} &= (-1)^{\sum\limits_{p\in P}\left(\overline{u_p}\cdot\left(\sum\limits_{\substack{q\in Q\setminus\{q_t\}\\ q<p}} \overline{u_q}\right)\right)+\sum\limits_{\substack{p\in P\\ p>q_t}} \overline{u_p}\cdot\overline{xu_{q_t}}} \\
&= (-1)^{\sum\limits_{p\in P}\left(\overline{u_p}\cdot\left(\sum\limits_{\substack{q\in Q\setminus\{q_t\}\\ q<p}} \overline{u_q}\right)\right)+\sum\limits_{\substack{p\in P\\ p>q_t}} \overline{u_p}(\overline{x}+\overline{u_{q_t}})} \\
&= (-1)^{\varepsilon(P,Q)+\sum\limits_{\substack{p\in P\\ p>q_t}} \overline{u_p}\cdot\overline{x}}.
\end{aligned}$$

Now notice that the sign from (125) is equal to

$$(-1)^{\varepsilon(P,Q)+\overline{x}(\overline{u_{p_1}}+\cdots+\overline{u_{p_d}})+\overline{x}(\overline{u_{q_1}}+\cdots+\overline{u_{q_{t-1}}})}, \tag{127}$$

and the sign from (126) is equal to

$$(-1)^{\varepsilon'(P,Q)+\alpha_{q_t}} = (-1)^{\varepsilon(P,Q)+\overline{x}\left(\sum\limits_{p>q_t} \overline{u_p}\right)+\overline{x}(\overline{u_1}+\cdots+\overline{u_{q_t-1}})}. \tag{128}$$

Since Q is ordered, $\{q_1, \ldots, q_{t-1}\} \subset \{1, \ldots, q_t - 1\}$. Moreover, $\{1, \ldots, q_t - 1\} \setminus \{q_1, \ldots, q_{t-1}\} \subset P$, and these elements in P are the only such elements which are less than q_t. So these, along with the elements of p which are larger than q_t make up all of P. That is, $P = (\{1, \ldots, q_t - 1\} \setminus \{q_1, \ldots, q_{t-1}\}) \cup \{p \in P \mid p > q_t\}$. To summarize, we have (since we work mod 2) that

$$(-1)^{\varepsilon(P,Q)+\overline{x}\left(\sum\limits_{p>q_t} \overline{u_p}\right)+\overline{x}(\overline{u_1}+\cdots+\overline{u_{q_t-1}})} = (-1)^{\varepsilon(P,Q)+\overline{x}(\overline{u_{p_1}}+\cdots+\overline{u_{p_d}})+\overline{x}(\overline{u_{q_1}}+\cdots+\overline{u_{q_{t-1}}})},$$

and thus (127) = (128), as desired. $\square$

7.5. **Crossing Morphism**

Recall from propositions 6.17 and 6.18 that $\mathsf{P}_{(A,\mathfrak{a})}$ is symmetric monoidal. For the functors S^d, S^e, we will give the flip map a special notation and call it the **crossing morphism**. Let $\boldsymbol{\tau}_{d,e} : S^d \otimes S^e \to S^e \otimes S^d$ be the supernatural transformation flip_{S^d,S^e} from propositon 6.18.

Proposition 7.9. *For any d, e, the section $(\boldsymbol{\tau}_{d,e})_{A^n} : (S^d \otimes S^e)(A^n) \to (S^e \otimes S^d)(A^n)$ is a map of $\mathfrak{gl}_n(A)$-supermodules.*

Proof. First, of all, by remark 6.24, the objects in question are actually $\mathfrak{gl}_n(A)$-supermodules. We consider a homogeneous element $[u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}] \in S^d(A^n) \otimes S^e(A^n)$. Now let $x \in \mathfrak{gl}_n(A)$ be homogeneous. Then

$$\begin{aligned} x([u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}]) &= x[u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}] + (-1)^{\alpha}[u_{i_1} \cdots u_{i_d}] \otimes x[u_{j_1} \cdots u_{j_e}] \\ &= \sum_{r=1}^{d} (-1)^{\theta_r}[u_{i_1} \cdots xu_{i_r} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}] \\ &\quad + \sum_{s=1}^{e} (-1)^{\eta_s+\alpha}[u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots xu_{j_s} \cdots u_{j_e}], \end{aligned} \tag{129}$$

where $\alpha = \overline{x}(\overline{u_{i_1}} + \cdots + \overline{u_{i_d}})$, $\theta_r = \overline{x}(\overline{u_{i_1}} + \cdots + \overline{u_{i_{r-1}}})$, and $\eta_s = \overline{x}(\overline{u_{j_1}} + \cdots + \overline{u_{j_{s-1}}})$.

So if we apply the crossing to (129), we get

$$\begin{aligned} &\sum_{r=1}^{d} (-1)^{\theta_r + (\overline{u_{j_1}}+\cdots+\overline{u_{j_e}})(\overline{u_{i_1}}+\cdots+\overline{xu_{i_r}}+\cdots+\overline{u_{i_d}})}[u_{j_1} \cdots u_{j_e}] \otimes [u_{i_1} \cdots xu_{i_r} \cdots u_{i_d}] \\ &\quad + \sum_{s=1}^{e} (-1)^{\eta_s+\alpha+(\overline{u_{j_1}}+\cdots+\overline{xu_{j_s}}+\cdots+\overline{u_{j_e}})(\overline{u_{i_1}}+\cdots+\overline{u_{i_d}})}[u_{j_1} \cdots xu_{j_s} \cdots u_{j_e}] \otimes [u_{i_1} \cdots u_{i_d}] \\ &= \sum_{r=1}^{d} (-1)^{\theta_r + (\overline{u_{j_1}}+\cdots+\overline{u_{j_e}})(\overline{u_{i_1}}+\cdots+\overline{xu_{i_r}}+\cdots+\overline{u_{i_d}})}[u_{j_1} \cdots u_{j_e}] \otimes [u_{i_1} \cdots xu_{i_r} \cdots u_{i_d}] \\ &\quad + \sum_{s=1}^{e} (-1)^{\eta_s+(\overline{u_{j_1}}+\cdots+\overline{u_{j_s}}+\cdots+\overline{u_{j_e}})(\overline{u_{i_1}}+\cdots+\overline{u_{i_d}})}[u_{j_1} \cdots xu_{j_s} \cdots u_{j_e}] \otimes [u_{i_1} \cdots u_{i_d}], \end{aligned} \tag{130}$$

where the sign simplifies in the last term since $\overline{xu_{j_s}} = \overline{x} + \overline{u_{j_s}}$, and we can work mod 2 for signs.

On the other hand ,

$$x((\boldsymbol{\tau}_{d,e})_{A^n}([u_{i_1} \cdots u_{i_d}] \otimes [u_{j_1} \cdots u_{j_e}])) = (-1)^{(\overline{u_{i_1}}+\cdots+\overline{u_{i_d}})(\overline{u_{j_1}}+\cdots+\overline{u_{j_e}})} x\left([u_{j_1} \cdots u_{j_e}] \otimes [u_{i_1} \cdots u_{i_d}]\right)$$

which is equal to

$$\begin{aligned} &\sum_{s=1}^{e} (-1)^{\eta_s + (\overline{u_{i_1}}+\cdots+\overline{u_{i_d}})(\overline{u_{j_1}}+\cdots+\overline{u_{j_e}})}[u_{j_1} \cdots xu_{j_s} \cdots u_{j_e}] \otimes [u_{i_1} \cdots u_{i_d}] \\ &\quad + \sum_{r=1}^{d} (-1)^{\theta_r + \overline{x}(\overline{u_{j_1}}+\cdots+\overline{u_{j_e}}) + (\overline{u_{i_1}}+\cdots+\overline{u_{i_d}})(\overline{u_{j_1}}+\cdots+\overline{u_{j_e}})}[u_{j_1} \cdots u_{j_e}] \otimes [u_{i_1} \cdots xu_{i_r} \cdots u_{i_d}]. \end{aligned} \tag{131}$$

Since (130) = (131), we are done. ∎

7.6. **Coupon Morphism**

Let $V = A^n$ have A-basis $\{v_1, \ldots, v_n\}$ so that $\{v_i^b \mid b \in B\}$ is a $\Bbbk$-basis for V. For S^1 in $\mathsf{P}^1_{(A,\mathfrak{a})}$, any $y, x \in B$, and any $n > 0$, define the $\Bbbk$-linear map

$$(\boldsymbol{\kappa}_1^y)_V : S^1(V) \to S^1(V) \qquad \text{via} \qquad v_r^x \mapsto (-1)^{\overline{x}\cdot\overline{y}} v_r^{xy}$$

whose parity is that of y.

For S^d in $\mathsf{P}^d_{(A,\mathfrak{a})}$ (with $d > 1$), any $y \in \mathfrak{a}$, any $\vec{x} \in B^d$, and any $n > 0$, define the (even) linear map

$$(\boldsymbol{\kappa}_d^y)_V : S^d(V) \to S^d(V) \qquad \text{via} \qquad [v_{r_1}^{x_1} \cdots v_{r_d}^{x_d}] \mapsto [v_{r_1}^{x_1 y} \cdots v_{r_d}^{x_d y}].$$

Note that there are no sign issues with $d > 1$ since y must be even in this case. So $(\boldsymbol{\kappa}_d^y)_V$ is an even linear endomorphism of $S^d(V)$ for any finite free right A supermodule V and any $d > 1$. For $d = 1$ it has parity $\overline{y}$.

Proposition 7.10. *The **coupon** morphism $\boldsymbol{\kappa}_d^y : S^d \to S^d$ is a supernatural transformation whose sections are given by the $(\boldsymbol{\kappa}_d^y)_V$ maps above. The parity of $\boldsymbol{\kappa}_d^y$ is the parity of the element $y \in A$.*

Proof. It suffices to check naturality on an element $[v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \in S^d(V)$ for $V = A^n$, $x_i \in B$, $t_i \in [1, n]$ and for the coupon element y to be homogeneous. Consider $\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})} \in T^A_{\mathfrak{a}}(V, W; d)$ for $W = A^m$. Then either path in the naturality square yields 0 whenever $\vec{t} \not\sim \vec{s}$. So, we may assume $\vec{t} = \vec{s}\gamma$ for some $\gamma \in \mathfrak{S}_d$. In this case, for $d \geqslant 2$, we have

$$\begin{aligned}(\boldsymbol{\kappa}_d^y)_W \left(S^d(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})[v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \right) &= [\vec{b}, \vec{r}, \vec{s}]^!_{\mathfrak{c}} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x})} (\boldsymbol{\kappa}_d^y)_W \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1} \cdots v_{r_{\sigma d}}^{b_{\sigma d} x_d} \right] \\ &= [\vec{b}, \vec{r}, \vec{s}]^!_{\mathfrak{c}} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x})} \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1 y} \cdots v_{r_{\sigma d}}^{b_{\sigma d} x_d y} \right]. \end{aligned} \tag{132}$$

On the other hand, we have

$$\begin{aligned} S^d(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})}) \left((\boldsymbol{\kappa}_d^y)_V [v_{t_1}^{x_1} \cdots v_{t_d}^{x_d}] \right) &= S^d(\widetilde{\eta}_{(\vec{b},\vec{r},\vec{s})})[v_{t_1}^{x_1 y} \cdots v_{t_d}^{x_d y}] \\ &= [\vec{b}, \vec{r}, \vec{s}]^!_{\mathfrak{c}} \sum_{\substack{\sigma \in {}^{\vec{b},\vec{r},\vec{s}}\mathscr{D} \\ \vec{s}\sigma = \vec{s}\gamma}} (-1)^{s(\vec{b},\sigma) + s(\vec{b}\sigma \to \vec{x}y)} \left[v_{r_{\sigma 1}}^{b_{\sigma 1} x_1 y} \cdots v_{r_{\sigma d}}^{b_{\sigma d} x_d y} \right]. \end{aligned} \tag{133}$$

Now notice that for $d \geqslant 2$, we must have $y \in \mathfrak{a}$ so that $\overline{y} = 0$. This means $\overline{x_i y} = \overline{x_i}$ for all $i \in [1, d]$. Therefore $s(\vec{b}\sigma \to \vec{x}y) = s(\vec{b}\sigma \to \vec{x})$, and it is obvious that (132) = (133) showing that $\boldsymbol{\kappa}_d^y$ is an even supernatural transformation.

For $d = 1$, (132) degenerates to

$$(\boldsymbol{\kappa}_d^y)_W \left(\varphi_{rs}^b (v_s^x) \right) = (\boldsymbol{\kappa}_d^y)_W (v_r^{bx}) = (-1)^{\overline{y}(\overline{b} + \overline{x})} v_r^{bxy},$$

and (133) degenerates to

$$\varphi_{rs}^b \left((\boldsymbol{\kappa}_d^y)_V (v_s^x) \right) = (-1)^{\overline{y}\cdot\overline{x}} \varphi_{rs}^b \left(v_s^{xy} \right) = (-1)^{\overline{y}\cdot\overline{x}} v_r^{bxy}.$$

These two differ by a sign of $(-1)^{\overline{y}\cdot\overline{b}} = (-1)^{\overline{\boldsymbol{\kappa}_d^y}\cdot\overline{\eta_{(b,r,s)}}}$ which exactly means $\boldsymbol{\kappa}_d^y$ is a supernatural transformation of parity $\overline{y}$. $\square$

Proposition 7.11. *For any d, the section $(\boldsymbol{\kappa}_d^y)_{A^n} : S^d(A^n) \to S^d(A^n)$ is a map of $\mathfrak{gl}_n(A)$-supermodules.*

Proof. Let's consider a $\Bbbk$-basis element $[v_{r_1}^{x_1} \cdots v_{r_d}^{x_d}] \in S^d(A^n)$ and some $E_{ij}^h \in \mathfrak{gl}_n(A)$ with $h \in A$ homogeneous. Then for $d > 1$, we have (suppressing the $\otimes$ symbol)

$$\begin{aligned}
(\boldsymbol{\kappa}_d^y)_{A^n}\left(E_{ij}^h[v_{r_1}^{x_1} \cdots v_{r_d}^{x_d}]\right) &= (\boldsymbol{\kappa}_d^y)_{A^n}\left(\sum_{s=1}^d (-1)^{\overline{h}(\overline{x_1}+\cdots+\overline{x_{s-1}})}[v_{r_1}^{x_1} \cdots E_{ij}^h v_{r_s}^{x_s} \cdots v_{r_d}^{x_d}]\right) \\
&= (\boldsymbol{\kappa}_d^y)_{A^n}\left(\sum_{s=1}^d (-1)^{\overline{h}(\overline{x_1}+\cdots+\overline{x_{s-1}})}\delta_{jr_s}[v_{r_1}^{x_1} \cdots v_i^{hx_s} \cdots v_{r_d}^{x_d}]\right) \\
&= \sum_{s=1}^d (-1)^{\overline{h}(\overline{x_1}+\cdots+\overline{x_{s-1}})}\delta_{jr_s}[v_{r_1}^{x_1 y} \cdots v_i^{hx_s y} \cdots v_{r_d}^{x_d y}] \qquad (134)
\end{aligned}$$

and

$$\begin{aligned}
E_{ij}^h(\boldsymbol{\kappa}_d^y)_{A^n}[v_{r_1}^{x_1} \cdots v_{r_d}^{x_d}] &= E_{ij}^h[v_{r_1}^{x_1 y} \cdots v_{r_d}^{x_d y}] \\
&= \sum_{s=1}^d (-1)^{\overline{h}(\overline{x_1 y}+\cdots+\overline{x_{s-1} y})}[v_{r_1}^{x_1 y} \cdots E_{ij}^h v_{r_s}^{x_s y} \cdots v_{r_d}^{x_d y}] \\
&= \sum_{s=1}^d (-1)^{\overline{h}(\overline{x_1}+\cdots+\overline{x_{s-1}})}\delta_{jr_s}[v_{r_1}^{x_1 y} \cdots v_i^{hx_s y} \cdots v_{r_d}^{x_d y}], \qquad (135)
\end{aligned}$$

where the sign simplifies in this last equality since $\overline{y} = 0$ whenever $d > 1$. Since (134) = (135), we have our result when $d > 1$. When $d = 1$, we have

$$\begin{aligned}
(\boldsymbol{\kappa}_d^y)_{A^n}\left(E_{ij}^h v_r^x\right) &= (\boldsymbol{\kappa}_d^y)_{A^n}\left(\delta_{jr} v_i^{hx}\right) \\
&= \delta_{jr}(-1)^{\overline{y}\cdot\overline{hx}} v_i^{hxy} \\
&= \delta_{jr}(-1)^{\overline{y}(\overline{h}+\overline{x})} v_i^{hxy} \\
&= (-1)^{\overline{y}(\overline{h}+\overline{x})} E_{ij}^h v_r^{xy} \\
&= (-1)^{\overline{y}\cdot\overline{h}} E_{ij}^h (\boldsymbol{\kappa}_d^y)_{A^n}(v_r^x)
\end{aligned}$$

which gives the desired result. ∎

7.7. The Functor $\mathscr{F} : \mathsf{Web}_{(A^{\mathrm{sop}}, \mathfrak{a}^{\mathrm{sop}})} \to \mathsf{P}_{(A,\mathfrak{a})}$

We'd like to establish the relationship between the category $\mathsf{Web}_{(A,\mathfrak{a})}$ from [DKMZ22] and $\mathsf{P}_{(A,\mathfrak{a})}$ in the form of a functor. But there is a subtle discrepency between right and left. Recall that the category $\mathsf{Web}_{(A,\mathfrak{a})}$ describes morphisms between the *right* $\mathfrak{gl}_n(A)$ modules tensor-generated by $S^d(V_n)$, for various n and d, where V_n is the free *left* A-supermodule of rank n viewed as row vectors. In our setting, we have *right* A-supermodules with a *left* $\mathfrak{gl}_n(A)$ action where we view A^n as column vectors. Our claim is that $\mathsf{P}_{(A,\mathfrak{a})}$ corresponds to $\mathsf{Web}_{(A^{\mathrm{sop}}, \mathfrak{a}^{\mathrm{sop}})}$.

Proposition 7.12. *There is a well-defined monoidal functor $\mathscr{F} : \mathsf{Web}_{(A^{\mathrm{sop}}, \mathfrak{a}^{\mathrm{sop}})} \to \mathsf{P}_{(A,\mathfrak{a})}$ which sends an object d of $\mathsf{Web}_{(A^{\mathrm{sop}}, \mathfrak{a}^{\mathrm{sop}})}$ to the polynomial superfunctor S^d in $\mathsf{P}^d_{(A,\mathfrak{a})} \subset \mathsf{P}_{(A,\mathfrak{a})}$ and does the obvious on morphisms: it sends the merge diagram to our merge morphism from*

7.3, the split diagram to our split morphism from 7.4, the crossing diagram to our crossing morphism from 7.5, and the coupon diagram to our coupon morphism from 7.6.

Proof. Consider the maps ${}_{V_n}\mathrm{spl}^{d,e}_{d+e}$, ${}_{V_n}\mathrm{mer}^{d+e}_{d,e}$, and $\tau_{S^dV_n,S^eV_n}$ from [DKMZ22] as we defined above in section 7.1. Remember that here, V_n is row vectors with entries in A^{sop}. But we can identify this as a $\Bbbk$-supermodule with A^n as discussed in remark 1.17. And so as $\Bbbk$-maps, they are exactly the same as the sections at A^n of $\boldsymbol{\varsigma}^{d,e}_{d+e}$, $\boldsymbol{\mu}^{d+e}_{d,e}$, and $\boldsymbol{\tau}_{d,e}$, respectively. It is shown in [DKMZ22] that ${}_{V_n}\mathrm{spl}^{d,e}_{d+e}$, ${}_{V_n}\mathrm{mer}^{d+e}_{d,e}$, and $\tau_{S^dV_n,S^eV_n}$ correspond to the split, merge, and crossing diagrams, respectively, and satisfy the appropriate defining relations of $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}$ for any n. Hence $\boldsymbol{\varsigma}^{d,e}_{d+e}$, $\boldsymbol{\mu}^{d+e}_{d,e}$, and $\boldsymbol{\tau}_{d,e}$ satisfy the same relations.

Now recall the linear map $L^x_d : S^d(V_n) \to S^d(V_n)$ from [DKMZ22]. It is such that

$$v^{z_1}_{r_1}\cdots v^{z_d}_{r_d} \mapsto v^{x\bullet z_1}_{r_1}\cdots v^{x\bullet z_d}_{r_d}.$$

When $d=1$, we have

$$v^z_i \mapsto v^{x\bullet z}_i = (-1)^{\overline{x}\cdot\overline{z}} v^{zx}_i,$$

and when $d>1$, we have

$$v^{z_1}_{r_1}\cdots v^{z_d}_{r_d} \mapsto v^{x\bullet z_1}_{r_1}\cdots v^{x\bullet z_d}_{r_d} = v^{z_1x}_{r_1}\cdots v^{z_dx}_{r_d},$$

where there is no extra sign since x must be even. But after our identification $V_n \cong A^n$ as $\Bbbk$-supermodules, we see that L^x_d is equivalent to the section at A^n of our coupon morphism, $\boldsymbol{\kappa}^x_d$.

Since it is shown in [DKMZ22] that L^x_d satisfies all the relations involving the coupon diagram for all n, it follows that our coupon morphism satisfies the same relations, and we are done. ∎

Remark 7.13. Consider the monoidal superfunctors $G_n : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})} \to \mathsf{smod}^{\mathcal{S}}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}$ from [DKMZ22]. In light of proposition 1.16 and remark 1.17, and viewing the target of G_n as being $\mathsf{smod}_{\mathfrak{gl}_n(A^{\mathsf{sop}})}$, we have the following commuting square for all n:

$$\begin{tikzcd} \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})} \arrow[r, "G_n"] \arrow[d, "\mathscr{F}"'] & \mathsf{smod}_{\mathfrak{gl}_n(A^{\mathsf{sop}})} \arrow[d, "\mathcal{T}"] \\ \mathsf{P}_{(A,\mathfrak{a})} \arrow[r, "\mathsf{ev}_{A^n}"'] & {}_{\mathfrak{gl}_n(A)}\mathsf{smod} \end{tikzcd} . \tag{136}$$

Proposition 7.14. *The functor $\mathscr{F} : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})} \to \mathsf{P}_{(A,\mathfrak{a})}$ is faithful.*

Proof. We wish to show that for any objects $\vec{x},\vec{y}$ in the web category, the induced map $\mathscr{F} : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \to \mathsf{P}_{(A,\mathfrak{a})}(S^{\vec{x}},S^{\vec{y}})$ is injective. Note that any web from $\vec{x}$ to $\vec{y}$ has the property that $\sum x_i = \sum y_j$. Call this value N. Proposition 6.6.1 from [DKMZ22] shows that G_N induces an injection $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \hookrightarrow \mathsf{smod}_{\mathfrak{gl}_N(A^{\mathsf{sop}})}(S^{\vec{x}}(V_N),S^{\vec{y}}(V_N))$. Since $\mathcal{T}$ is an equivalence, the square in (136) yields

$$\begin{tikzcd} \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \arrow[r, hook, "G_N"] \arrow[d, "\mathscr{F}"'] & \mathsf{smod}_{\mathfrak{gl}_N(A^{\mathsf{sop}})}(S^{\vec{x}}(V_N),S^{\vec{y}}(V_N)) \arrow[d, hook, "\mathcal{T}"] \\ \mathsf{P}_{(A,\mathfrak{a})}(S^{\vec{x}},S^{\vec{y}}) \arrow[r, "\mathsf{ev}_{A^N}"'] & {}_{\mathfrak{gl}_N(A)}\mathsf{smod}(S^{\vec{x}}(A^N),S^{\vec{y}}(A^N)) \end{tikzcd} .$$

It follows that $\mathscr{F}$ induces an injection $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \hookrightarrow \mathsf{P}_{(A,\mathfrak{a})}(S^{\vec{x}},S^{\vec{y}})$, as desired. ∎

It's natural to ask when $\mathscr{F}$ is full. Our only restriction is on $\Bbbk$:

Proposition 7.15. *The functor $\mathscr{F} : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})} \to \mathsf{P}_{(A,\mathfrak{a})}$ is full whenever $\Bbbk$ is an algebraically closed field of characteristic 0.*

Proof. We wish to show that for any objects $\vec{x},\vec{y}$ in the web category, the induced map $\mathscr{F} : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \to \mathsf{P}_{(A,\mathfrak{a})}(S^{\vec{x}}, S^{\vec{y}})$ is surjective. Well any web from $\vec{x}$ to $\vec{y}$ has the property that $\sum x_i = \sum y_j$. Call this value N. It follows that $\mathscr{F}$ actually lands in the degree N part of $\mathsf{P}_{(A,\mathfrak{a})}$, that is, we have

$$\mathscr{F} : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \to \mathsf{P}^N_{(A,\mathfrak{a})}(S^{\vec{x}}, S^{\vec{y}}).$$

The functor $G_N : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})} \to \mathsf{smod}^{\mathcal{S}}_{\mathfrak{gl}_N(A^{\mathsf{sop}})}$ is not full, in general. However, by section 5.3 of [EK17], the map $\rho^A_{N,N}$ from proposition 9.3.1 of [DKMZ22] is surjective[4]. Thus, by proposition 9.2.1 of [DKMZ22], the induced map

$$G_N : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \to \mathsf{smod}_{\mathfrak{gl}_N(A^{\mathsf{sop}})}(S^{\vec{x}}(V_N), S^{\vec{y}}(V_N))$$

is surjective. Since G_N is also faithful, this map is an isomorphism on morphism spaces. Similarly, $\mathcal{T}$ induces an isomorphism, and since $\mathscr{F}$ is faithful by proposition 7.14, we have the commuting square:

$$\begin{tikzcd}
\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y}) \arrow[r, hook, two heads, "G_N"] \arrow[d, hook, "\mathscr{F}"'] & \mathsf{smod}_{\mathfrak{gl}_N(A^{\mathsf{sop}})}(S^{\vec{x}}(V_N), S^{\vec{y}}(V_N)) \arrow[d, hook, "\mathcal{T}"] \\
\mathsf{P}^N_{(A,\mathfrak{a})}(S^{\vec{x}}, S^{\vec{y}}) \arrow[r, "\mathsf{ev}_{A^N}"'] & {}_{\mathfrak{gl}_N(A)}\mathsf{smod}(S^{\vec{x}}(A^N), S^{\vec{y}}(A^N)).
\end{tikzcd}$$

Now suppose $\eta \in \mathsf{P}^N_{(A,\mathfrak{a})}(S^{\vec{x}}, S^{\vec{y}})$. Then $\mathsf{ev}_{A^N}(\eta) = \eta_{A^N} \in {}_{\mathfrak{gl}_N(A)}\mathsf{smod}(S^{\vec{x}}(A^N), S^{\vec{y}}(A^N))$. Since $\mathcal{T}$ is an isomorphism, there is a (unique) element $y \in \mathsf{smod}_{\mathfrak{gl}_N(A^{\mathsf{sop}})}(S^{\vec{x}}(V_N), S^{\vec{y}}(V_N))$ such that $\mathcal{T}(y) = \eta_{A^N}$. Similarly, there is a unique element $y' \in \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}(\vec{x},\vec{y})$ such that $G_N(y') = y$, and hence $\mathcal{T}G_N(y') = \eta_{A^N}$. But this means we also have $\mathsf{ev}_{A^N}\mathscr{F}(y') = \eta_{A^N}$. If we can show that ev_{A^N} is injective, it will follow that $\mathscr{F}(y') = \eta$ and hence $\mathscr{F}$ is surjective.

To this end, note that ev_{A^N} can also be thought of as a functor $\mathsf{P}^N_{(A,\mathfrak{a})} \to {}_{T^A_{\mathfrak{a}}(N,N;N)}\mathsf{smod}$ which is an equivalence by theorem 6.10. So, in particular, ev_{A^N} induces an isomorphism

$$\mathsf{ev}_{A^N} : \mathsf{P}^N_{(A,\mathfrak{a})}(S^{\vec{x}}, S^{\vec{y}}) \to {}_{T^A_{\mathfrak{a}}(N,N;N)}\mathsf{smod}(S^{\vec{x}}(A^N), S^{\vec{y}}(A^N))$$

via $\eta \mapsto \eta_{A^N}$. We emphasize that the image of η under this isomorphism is the same exact $\Bbbk$-map as the η_{A^N} of interest above. So if η is nonzero, the map

$$\mathsf{ev}_{A^N} : \mathsf{P}^N_{(A,\mathfrak{a})}(S^{\vec{x}}, S^{\vec{y}}) \to {}_{\mathfrak{gl}_N(A)}\mathsf{smod}(S^{\vec{x}}(A^N), S^{\vec{y}}(A^N))$$

sends η to a nonzero map η_{A^N}, so it is injective, as desired. ∎

Remark 7.16. We've established a faithful (and sometimes full) functor $\mathscr{F} : \mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})} \to \mathsf{P}_{(A,\mathfrak{a})}$. Now $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}$ is not an abelian category, but $\mathsf{P}_{(A,\mathfrak{a})}$ is. So it is a natural question to ask whether $\mathsf{P}_{(A,\mathfrak{a})}$ can be thought of as an 'abelianization' or 'abelian envelope' of $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}$.

There are different notions of what this term should mean depending on the setting and application in mind. See [BEAEO20, BVHP20, Cou21, Fre66, Pre11] and references therein.

[4]See also section 11.2 of [DKMZ22], and keep in mind that for $\Bbbk$ a field of characteristic 0, $T^A_{\mathfrak{a}}(n,n;d) = S^A(n,n;d)$.

But as a starting point, anything called an abelian envelope should satisfy an appropriate universal property of the following basic form: Let C be a (super) category. The pair

$$(\mathsf{C}', F : \mathsf{C} \to \mathsf{C}'),$$

where C' is abelian and F is a (super) functor, is an abelian envelope of C if for any abelian (super) category D and (super) functor $G : \mathsf{C} \to \mathsf{D}$, there exists a (unique up to (super)natural equivalence) (super)functor $G' : \mathsf{C}' \to \mathsf{D}$ such that $G'F = G$.

There should also be some adjectives describing the functors involved. For example, one could require F to be faithful or full (or both). Often, one would want G' to be right-exact or exact. Conditions can be put on G, as well, and one may also want these functors to respect extra structure such as a monoidal product. All of these choices would potentially change the (super)category C'.

Question zero should be when does an abelian envelope exist? Freyd studied this problem in [Fre66] and constructed a category which is now referred to as the *free abelian category* or *Freyd's free abelain category*. See [Pre11] for a modern treatment. In particular, the following is shown in [Pre11]:

Let C be a (non-super) skeletally small preadditive category. A free abelian category on C is a functor $\mathsf{C} \to \mathsf{Ab}(\mathsf{C})$ where $\mathsf{Ab}(\mathsf{C})$ is abelian and has the universal property that for every additive functor $\mathsf{C} \to \mathsf{D}$ where D is abelian, there is an extension to an exact functor from $\mathsf{Ab}(\mathsf{C}) \to \mathsf{D}$, and there is, up to natural equivalence, just one such exact functor. Moreover, if C is a small preadditive category, then $\mathsf{Ab}(\mathsf{C})$ exists and can be realized as (C-mod)-mod.

So (ignoring super vs non-super) $\mathsf{Ab}(\mathsf{Web}_{(\mathsf{A}^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})})$ should exist (after taking the additive closure of $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}$).

Moreover, in [BEAEO20], remark 4.4 and theorem 4.10 (see also example 13.1 of [DKMZ22]) assert that the category of classical strict polynomial functors P is the abelian envelope of $\mathsf{Web}_{(\Bbbk,\Bbbk)}$ in the following sense: any functor $F : \mathsf{Web}_{(\Bbbk,\Bbbk)} \to \mathsf{D}$ to an abelian category D factors through the embedding $\mathsf{Web}_{(\Bbbk,\Bbbk)} \to \mathsf{P}$ to induce a right-exact functor $\mathsf{P} \to \mathsf{D}$, which is monoidal in case F is monoidal.

An interesting question is whether this generalizes to our super setting. What is the appropriate setting in which $\mathsf{P}_{(A,\mathfrak{a})}$ can be thought of as an abelian envelope of $\mathsf{Web}_{(A^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})}$, or at least how does it relate to $\mathsf{Ab}(\mathsf{Web}_{(\mathsf{A}^{\mathsf{sop}},\mathfrak{a}^{\mathsf{sop}})})$ defined above?

Appendix A. Strict vs Non-Strict Polynomial Functors

This section deals with the classical non-super setting. First, we'd like to mention that [FFPS03] has great exposition on the importance and uses of polynomial functors and on the differences between what we call polynomial functors and strict polynomial functors. [Pir00] also contains a nice discussion on this last point as well as a comparison between the cohomology theory involving the category of functors $F : \mathsf{vec}_\Bbbk \to \mathsf{Vec}_\Bbbk$ and the category of strict polynomial functors.

In this appendix, we'll mostly follow [Kra13] (at times verbatim) to highlight the difference between polynomial functors and strict polynomial functors. We reproduce and expand on details here instead of directing the reader to [Kra13] in order to keep this thesis more self-contained. We will use definitions from [Bou03].

For now, let $\Bbbk$ be a commutative ring. We will begin by recalling some standard isomorphisms in the category of finitely generated projective $\Bbbk$-modules, $\mathsf{mod}_\Bbbk^{\mathsf{fgp}}$. For V in $\mathsf{mod}_\Bbbk^{\mathsf{fgp}}$, let $V^* := \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$. Then for V, W in $\mathsf{mod}_{\Bbbk}^{\mathsf{fgp}}$, we have

$$V^* \otimes W \cong \operatorname{Hom}_{\Bbbk}(V, W) \quad \text{via} \quad f \otimes w \mapsto (v \mapsto f(v)w)\,. \tag{137}$$

For V in $\mathsf{mod}_{\Bbbk}^{\mathsf{fgp}}$, the symmetric group on d letters, $\mathfrak{S}_d$, acts on $V^{\otimes d}$ via place permutation. Let $\Gamma^d(V) := \left(V^{\otimes d}\right)^{\mathfrak{S}_d}$ be the $\Bbbk$-submodule of $V^{\otimes d}$ consisting of elements invariant under the $\mathfrak{S}_d$ action. The maximal quotient of $V^{\otimes d}$ on which $\mathfrak{S}_d$ acts trivially is denoted $S^d(V)$. We have

$$\left(\Gamma^d(V)\right)^* \cong S^d\left(V^*\right). \tag{138}$$

Moreover, $S^d(V)$ is in $\mathsf{mod}_{\Bbbk}^{\mathsf{fgp}}$ whenever V is (and hence $\Gamma^d(V)$ is).

Now let $\Bbbk$ be a field. As defined in Chapter 4 section 5.9 of [Bou03] we have the following[5]:

Definition A.1. A *homogeneous polynomial mapping of degree* d between $\Bbbk$-vector spaces V and W is a set map $f : V \to W$ with the following property: there exists a basis $\{v_i\}_{i \in I}$ of V and a family of elements $\{w_\nu \mid \nu \in \mathbb{N}^I,\ |\nu| := \sum_i \nu_i = d\}$ in W such that for all $(\lambda_i) \in \Bbbk^I$,

$$f\left(\sum_{i \in I} \lambda_i v_i\right) = \sum_{\substack{\nu \in \mathbb{N}^I \\ |\nu| = d}} \lambda^\nu w_\nu,$$

where $\lambda^\nu := \lambda_1^{\nu_1}\lambda_2^{\nu_2}\cdots\lambda_{|I|}^{\nu_{|I|}}$. So under f, coefficients transform as homogeneous degree d polynomials. This agrees with the notion of polynomial mapping used in [Mac80].

Let $\operatorname{Pol}^d(V, W)$ denote the space of homogeneous polynomial mappings of degree d from V to W. By proposition 13 in Chapter 4 section 5.9 of [Bou03], $f \in \operatorname{Pol}^d(V, W)$ is equivalent to the existence of a $\Bbbk$-linear map $h : \Gamma^d(V) \to W$ such that

$$h(x \otimes \cdots \otimes x) = f(x)$$

for all $x \in V$ (clearly $x \otimes \cdots \otimes x$ is invariant under the $\mathfrak{S}_d$ action so is an element of $\Gamma^d(V)$). This determines a surjective $\Bbbk$-linear map

$$\operatorname{Hom}_{\Bbbk}(\Gamma^d(V), W) \twoheadrightarrow \operatorname{Pol}^d(V, W). \tag{139}$$

By proposition 16 in Chapter 4 section 5.9 of [Bou03], when $\Bbbk$ is infinite, the above map defines an isomorphism. However, when $\Bbbk$ is finite, this need not be true. In fact, if $\Bbbk$ has less than d elements, the map will have nontrivial kernel. This means that an element of $\operatorname{Hom}_{\Bbbk}(\Gamma^d(V), W)$ need not be determined by its corresponding set map $V \to W$. We'll demonstrate this (and another issue) with an example.

Example A.2. For now, let $\Bbbk$ be some field - we will consider a specific field in a moment. Let $V = \Bbbk^2$ have basis $\{v_1, v_2\}$ and $W = \Bbbk$ have basis $\{w_1\}$. Define

$$\begin{aligned}
\boldsymbol{v}_0 &:= v_1 \otimes v_1 \otimes v_1 \\
\boldsymbol{v}_1 &:= v_1 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_1 \\
\boldsymbol{v}_2 &:= v_1 \otimes v_2 \otimes v_2 + v_2 \otimes v_1 \otimes v_2 + v_2 \otimes v_2 \otimes v_1 \\
\boldsymbol{v}_3 &:= v_2 \otimes v_2 \otimes v_2
\end{aligned}$$

[5] The definition is actually given for $\Bbbk$-modules for $\Bbbk$ any commutative ring, but we give an equivalent formulation in terms of vector space bases for $\Bbbk$ a field.

so that $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ forms a $\Bbbk$-basis for $\Gamma^3 V$. Then for any $a, b \in \Bbbk$, we have that a given $h \in \mathrm{Hom}_{\Bbbk}(\Gamma^3(V), W)$ is such that

$$h((av_1 + bv_2) \otimes (av_1 + bv_2) \otimes (av_1 + bv_2)) = h(a^3\boldsymbol{v}_0 + a^2b\boldsymbol{v}_1 + ab^2\boldsymbol{v}_2 + b^3\boldsymbol{v}_3).$$

Now if we let h be the map defined by

$$\begin{cases} \boldsymbol{v}_0 \mapsto 0 \\ \boldsymbol{v}_1 \mapsto w_1 \\ \boldsymbol{v}_2 \mapsto w_1 \\ \boldsymbol{v}_3 \mapsto 0 \end{cases}$$

then

$$h((av_1 + bv_2) \otimes (av_1 + bv_2) \otimes (av_1 + bv_2)) = (a^2b + ab^2)w_1,$$

so under (139), the corresponding element f in $\mathrm{Pol}^3(V, W)$ is the map

$$av_1 + bv_2 \mapsto (a^2b + ab^2)w_1.$$

Now if $\Bbbk$ contains at least 3 elements, this map is not the zero map. However, if $\Bbbk = \mathbb{Z}/2\mathbb{Z}$, then $a^2 = a$ and $b^2 = b$ so that $a^2b + ab^2 = 2ab = 0$. So in this case, the nontrivial element $h \in \mathrm{Hom}_{\Bbbk}(\Gamma^3(V), W)$ is sent to the zero map, and we see that (139) has nontrivial kernel.

We still have an issue even when $\Bbbk$ has at least d elements.

Example A.3. For now, let $\Bbbk$ be some field. Let $V = \Bbbk^2$ have basis $\{v_1, v_2\}$ and $W = \Bbbk$ have basis $\{w_1\}$. Then we can define a map $f : V \to W$ via

$$av_1 + bv_2 \mapsto (a^2 + b^2)w_1,$$

which seems like it should be a homogeneous degree 2 polynomial mapping. And indeed, if we let $h \in \mathrm{Hom}_{\Bbbk}(\Gamma^2(V), W)$ be defined by

$$\begin{cases} v_1 \otimes v_1 \mapsto w_1 \\ v_1 \otimes v_2 + v_2 \otimes v_1 \mapsto 0 \\ v_2 \otimes v_2 \mapsto 0 \end{cases},$$

then h is the corresponding map under (139) making f a homogeneous degree 2 polynomial mapping.

However, if we focus on $\Bbbk = \mathbb{Z}/2\mathbb{Z}$, then $a^2 + b^2 = a + b$ so that f can also be written as

$$av_1 + bv_2 \mapsto (a + b)w_1,$$

which feels like it should be a degree 1 homogeneous polynomial mapping. And indeed there is a corresponding element of $\mathrm{Hom}_{\Bbbk}(\Gamma^1(V), W) = \mathrm{Hom}_{\Bbbk}(V, W)$ determined by

$$\begin{cases} v_1 \mapsto w_1 \\ v_2 \mapsto w_1 \end{cases}.$$

So f is both a degree 1 and degree 2 homogeneous polynomial mapping, and there is no canonical choice for which is the 'correct' degree.

These two examples demonstrate that the notion of a homogeneous polynomial mapping may not be strong enough. This leads us to the definition of a *homogeneous degree d strict polynomial mapping* (or a *homogeneous polynomial law of degree d* as it's called in [Bou03] - See exercises 9 and 10 of Chapter 4 section 5). Such a mapping is defined in terms of natural transformations between certain functors, but by exercise 10 of Chapter 4 section 5 in [Bou03], the space of homogeneous degree d strict polynomial mappings from V to W is isomorphic to $\mathrm{Hom}_{\Bbbk}(\Gamma^d V, W)$.

By isomorphisms (137) and (138), we have

$$\mathrm{Hom}_{\Bbbk}(\Gamma^d(V), W) \cong \left(\Gamma^d(V)\right)^* \otimes W \cong S^d(V^*) \otimes W$$

so that we can equivalently characterize a homogeneous degree d strict polynomial map as an element of $S^d(V^*) \otimes W$.

Next we analyze the original definition of a homogeneous degree d strict polynomial functor given in [FS97]. First of all, they define a homogeneous degree d strict[6] polynomial map between finite $\Bbbk$-vector spaces V and W as an element of the space $S^d(V^*) \otimes W$. Then we have:

Definition A.4. A homogeneous degree d strict polynomial functor $T : \mathsf{vec}_{\Bbbk} \to \mathsf{vec}_{\Bbbk}$ consists of the following data:

- A function which assigns to each V in $\mathsf{vec}_{\Bbbk}$ a new vector space TV in $\mathsf{vec}_{\Bbbk}$.
- A function which assigns a homogeneous degree d strict polynomial map $T_{V,W}$ from $\mathrm{Hom}_{\Bbbk}(V, W)$ to $\mathrm{Hom}_{\Bbbk}(TV, TW)$ (that is, $T_{V,W} \in S^d(\mathrm{Hom}_{\Bbbk}(V, W)^*) \otimes \mathrm{Hom}_{\Bbbk}(TV, TW)$).

which are subject to the following conditions (the usual conditions that make T a functor):

- For each V in $\mathsf{vec}_{\Bbbk}$, $T_{V,V}(1_V) = 1_{TV}$.
- For U, V, W in $\mathsf{vec}_{\Bbbk}$, the following diagram of (homogeneous) strict polynomial maps commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\Bbbk}(V, W) \times \mathrm{Hom}_{\Bbbk}(U, V) \arrow[r] \arrow[d, "T_{V,W} \times T_{U,V}"'] & \mathrm{Hom}_{\Bbbk}(U, W) \arrow[d, "T_{U,W}"] \\
\mathrm{Hom}_{\Bbbk}(TV, TW) \times \mathrm{Hom}_{\Bbbk}(TU, TV) \arrow[r] & \mathrm{Hom}_{\Bbbk}(TU, TW)
\end{tikzcd} .$$

Now let $\Gamma^d\mathsf{vec}_{\Bbbk}$ be the category whose objects are the same as those of $\mathsf{vec}_{\Bbbk}$ but whose morphism spaces are defined as

$$\mathrm{Hom}_{\Gamma^d\mathsf{vec}_{\Bbbk}}(V, W) := \Gamma^d \,\mathrm{Hom}_{\Bbbk}(V, W).$$

Then the data from definition A.4 is equivalent to having a $\Bbbk$-linear functor $T : \Gamma^d\mathsf{vec}_{\Bbbk} \to \mathsf{vec}_{\Bbbk}$. This is because T obviously sends a finite $\Bbbk$-vector space, V, to a new finite $\Bbbk$-vector space, TV, and the induced map on morphisms is a $\Bbbk$-linear map

$$T_{V,W} : \mathrm{Hom}_{\Gamma^d\mathsf{vec}_{\Bbbk}}(V, W) = \Gamma^d \,\mathrm{Hom}_{\Bbbk}(V, W) \to \mathrm{Hom}_{\Bbbk}(TV, TW)$$

so that $T_{V,W}$ is homogeneous strict polynomial of degree d, which means $T_{V,W}$ is an element of $S^d(\mathrm{Hom}_{\Bbbk}(V, W)^*) \otimes \mathrm{Hom}_{\Bbbk}(TV, TW)$. This illustrates why our definitions 6.1 and 6.2 are generalizations of strict polynomial functors.

It follows from our discussion above that every homogeneous strict polynomial functor T determines a homogeneous polynomial mapping $\mathrm{Hom}_{\Bbbk}(V, W) \to \mathrm{Hom}_{\Bbbk}(TV, TW)$ for every pair of vector spaces V, W. Moreover, when $\Bbbk$ is infinite, such a functor is determined by this data. But when $\Bbbk$ is finite, it is not in general. This illustrates the difference between the strict polynomial functors from [FS97] and the polynomial functors from [Mac80] and [Mac95].

[6] They actually don't use the term 'strict' in this part of the definition, but we will do so here to avoid confusion.

Appendix B. Projective vs Free A-supermodules

In this thesis, we've chosen to work with finitely generated free right A-supermodules of the form $V \cong A^{m|n}$ or $W = A^r$. A large reason for this, other than the ease of computation, is that for general unital good pairs $(A, \mathfrak{a})$, the category $\mathsf{T}^A_{\mathfrak{a}}(d)$ is essentially built from the viewpoint that objects are of the form $V = A^{m|0}$, and there is no obvious way to circumvent this. However, for the case when $\mathfrak{a} = A_0$, we can alter our viewpoint.

Recall that in this situation, we have category equivalences $\mathsf{T}^A_{\mathfrak{a}}(d) \cong \mathsf{S}^A(d) \cong \Gamma^d \mathcal{V}$ where $\mathcal{V}$ denotes the supercategory of free finitely generated unshifted right A-supermodules of the form $V = A^{m|0}$. Then generalized polynomial functors are of the form $\Gamma^d \mathcal{V} \to \mathsf{smod}_{\Bbbk}$. One could feasably replace $\mathcal{V}$ with the category $\mathcal{V}'$ of finite free unshifted right A-supermodules of the form $V \cong A^n$ since $S^A(V, W; d) = \Gamma^d \operatorname{Hom}_A(V, W)$ makes sense.

Strict polynomial superfunctors are of the form $\Gamma^d \mathsf{V} \to \mathsf{smod}_{\Bbbk}$ where V denotes the supercategory of finite free right A-supermodules of the form $A \cong A^{m|n}$.

For various reasons, one may be interested in working with finitely generated projective A-supermodules instead of free supermodules. In this section, we explore when the two starting points yield equivalent formulations of our generalized functors.

Our goal is to show that the supercategory of superfunctors $\Gamma^d \mathsf{V} \to \mathsf{smod}_{\Bbbk}$ is equivalent to the supercategory of superfunctors $\Gamma^d \widehat{\mathsf{V}} \to \mathsf{smod}_{\Bbbk}$ where $\widehat{\mathsf{V}}$ denotes the supercategory of all finitely generated projective right A-supermodules. This will then obviously translate for $\mathcal{V}'$.

We begin with some definitions. Let C be a category.

Definition B.1. An **idempotent** e in C is an element of $\operatorname{Hom}_{\mathsf{C}}(X, X)$ for some object X such that

$$e^2 = e.$$

The idempotent e is a **split idempotent** (or e 'splits') if there exists an object Y (which we call the *underlying object of the splitting*) in C and morphisms $\rho : X \to Y$, $\iota : Y \to X$ such that

$$\rho\iota = \mathrm{id}_Y \quad \text{and} \quad \iota\rho = e.$$

Lemma B.2. *Any two splittings of an idempotent $e : X \to X$ in C are isomorphic. That is, if $\rho : X \to Y$, $\iota : Y \to X$ are such that*

$$\rho\iota = \mathrm{id}_Y \quad \textit{and} \quad \iota\rho = e,$$

and $\rho' : X \to Y'$, $\iota' : Y' \to X$ are such that

$$\rho'\iota' = \mathrm{id}_{Y'} \quad \textit{and} \quad \iota'\rho' = e,$$

then Y is isomorphic to Y'.

Proof. First of all, note that $\rho\iota' : Y' \to Y$ and $\rho'\iota : Y \to Y'$. Then by assumption, we have

$$\begin{aligned}
(\rho\iota')(\rho'\iota) &= \rho(\iota'\rho')\iota \\
&= \rho(e)\iota \\
&= \rho(\iota\rho)\iota \\
&= (\rho\iota)(\rho\iota) \\
&= (\mathrm{id}_Y)(\mathrm{id}_Y) \\
&= \mathrm{id}_Y
\end{aligned}$$

and

$$\begin{aligned}(\rho'\iota)(\rho\iota') &= \rho'(\iota\rho)\iota' \\ &= \rho'(e)\iota' \\ &= \rho'(\iota'\rho')\iota' \\ &= (\rho'\iota')(\rho'\iota') \\ &= (\mathrm{id}_{Y'})(\mathrm{id}_{Y'}) \\ &= \mathrm{id}_{Y'}\,.\end{aligned}$$

Therefore, $Y \cong Y'$. ∎

Definition B.3. The **Karoubi envelope** of a category C, $\mathsf{Kar}(\mathsf{C})$ (also called the idempotent completion of C), is defined as follows:

The objects of $\mathsf{Kar}(\mathsf{C})$ are pairs (X, e) where $e : X \to X$ is an idempotent in C. A morphism $\varphi : (X, e) \to (Y, f)$ is a morphism $\varphi : X \to Y$ in C such that the following diagram commutes in C:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\varphi"'] \arrow[dr, "\varphi"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "f"'] & Y \end{tikzcd} . \tag{140}$$

The identity morphism on (X, e) is

$$\mathrm{id}_{(X,e)} = e : (X, e) \to (X, e).$$

Now we'll justify the term 'idempotent completion'.

Lemma B.4. $\mathsf{Kar}(\mathsf{C})$ *is the universal enlargement of* C *such that every idempotent splits. That is, there exists a fully faithful embedding*

$$E : \mathsf{C} \to \mathsf{Kar}(\mathsf{C}),$$

every idempotent in $\mathsf{Kar}(\mathsf{C})$ *splits, and* $\mathsf{Kar}(\mathsf{C})$ *satisfies the following universal property: For any functor* $F : \mathsf{C} \to \mathsf{D}$ *where every idempotent in* D *splits, there exists a (unique up to natural isomorphism) functor* $F' : \mathsf{Kar}(\mathsf{C}) \to \mathsf{D}$ *such that* $F' \circ E = F$.

Proof. First, we'll show every idempotent $\varphi : (X, e) \to (X, e)$ splits in $\mathsf{Kar}(\mathsf{C})$: By definition, $\varphi : X \to X$ is a morphism in C such that $e\varphi = \varphi = \varphi e$. From this, it is easy to see that we can define a map $\rho : (X, e) \to (X, \varphi)$ given by $\varphi : X \to X$ and a map $\iota : (X, \varphi) \to (X, e)$ given by $\varphi : X \to X$. Since φ is also an idempotent in C, it is easy to check that $\rho\iota = \mathrm{id}_{(X,\varphi)}$ and $\iota\rho = \varphi$.

Next, the embedding

$$E : \mathsf{C} \to \mathsf{Kar}(\mathsf{C}) \quad \text{via} \quad X \mapsto (X, \mathrm{id}_X),$$

where morphisms are mapped to themselves (they trivially satisfy (140)), is clearly fully faithful.

Given any idempotent $e : X \to X$ in C, we have $E(e) : E(X) \to E(X)$ being an idempotent in $\mathsf{Kar}(\mathsf{C})$. By the above observation, $E(e)$ splits. Specifically, we have $\rho : (X, \mathrm{id}_X) \to (X, e)$ given by $e : X \to X$ and a map $\iota : (X, e) \to (X, \mathrm{id}_X)$ given by $e : X \to X$ such that $\rho\iota = \mathrm{id}_{(X,e)}$ and $\iota\rho = E(e)$. In this way, the idempotent $e : X \to X$ is formally split in $\mathsf{Kar}(\mathsf{C})$.

Next, suppose D is a category in which every idempotent splits. Choose a splitting for every idempotent $d : D \to D$ in D. That is, choose an underlying object D_d with maps $\rho_d : D \to D_d$, $\iota_d : D_d \to D$ such that $\rho_d \iota_d = \mathrm{id}_{D_d}$ and $\iota_d \rho_d = d$ (we always split the identity by D and the identity maps).

Then define $F' : \mathsf{Kar}(\mathsf{C}) \to \mathsf{D}$ as follows: On objects

$$F'(X, e) := F(X)_{F(e)},$$

and on morphisms $\phi : (X, e) \to (Y, f)$,

$$F'(\phi) := \rho_{F(f)} F(\phi) \iota_{F(e)}.$$

To see that F' respects the identity, first recall that $\mathrm{id}_{(X,e)} = e$ in $\mathsf{Kar}(\mathsf{C})$. Then we have

$$\begin{aligned}
F'(\mathrm{id}_{(X,e)}) &= F'(e) \\
&= \rho_{F(e)} F(e) \iota_{F(e)} \\
&= \rho_{F(e)} (\iota_{F(e)} \rho_{F(e)}) \iota_{F(e)} \\
&= (\rho_{F(e)} \iota_{F(e)})(\rho_{F(e)} \iota_{F(e)}) \\
&= (\mathrm{id}_{F(X)_{F(e)}})(\mathrm{id}_{F(X)_{F(e)}}) \\
&= \mathrm{id}_{F(X)_{F(e)}} \\
&= \mathrm{id}_{F'(X,e)},
\end{aligned}$$

as desired.

To check that it respects composition, say $\psi : (Y, f) \to (Z, g)$, and note that

$$\begin{aligned}
F'(\psi\phi) &= \rho_{F(g)} F(\psi\phi) \iota_{F(e)} \\
&= \rho_{F(g)} F(\psi f \phi) \iota_{F(e)} \\
&= \rho_{F(g)} F(\psi) F(f) F(\phi) \iota_{F(e)} \\
&= \rho_{F(g)} F(\psi) \iota_{F(f)} \rho_{F(f)} F(\phi) \iota_{F(e)} \\
&= F'(\psi) F'(\phi),
\end{aligned}$$

where the second equality follows from the fact that $\psi f = \psi$ and $f \phi = \phi$ by definition. Thus, F' is well-defined and, moreover, it is easy to see that $F' \circ E = F$.

Now, F' depends on our choices of splittings in D. For example, $F'(X, e)$ is mapped to our choice of underlying object of the splitting of $F(e)$ (which we denoted $F(X)_{F(e)}$). Suppose we chose different splittings in D and defined an analogous functor F^* using these. By lemma B.2, a different choice of splitting of $F(e)$ would yield an isomorphic underlying object. It's easy to check that these isomorphisms define the sections of a natural isomorphism $F' \to F^*$, so that F' is indeed unique up to natural isomorphism. ◻

It is well known that if C is a category of finitely generated free modules over a ring or algebra, then $\mathsf{Kar}(\mathsf{C})$ is equivalent to the category of finitely generated projective modules for that ring or algebra.

We would like to define what it means to take the Karoubi envelope of a supercategory so that we have an analogous equivalence between the envelope of free supermodules and the category of projective supermodules. Let's first define what it means to be projective in a supercategory.

Definition B.5. Let C be a supercategory. An object P in C is **projective** if it is a projective object in C_{ev}. Precisely, this means that given even morphisms $f : P \to X$ and $g : Y \to X$ such that g is an epimorphism, there exists an even lift of f, $\tilde{f} : P \to Y$, such that $g\tilde{f} = f$.

We are most interested in the case when $\mathsf{C} = \mathsf{smod}_A$ for a superalgebra A over a commutative domain $\Bbbk$ which is not characteristic 2. In this setting, note that smod_A is abelian (in the sense that $(\mathsf{smod}_A)_{\mathsf{ev}}$ is abelian), and we have the following:

Lemma B.6. *The following are equivalent for an object P in the category $\mathsf{C} = \mathsf{smod}_A$:*

(1) *P is projective.*
(2) *The superfunctor* $\mathrm{Hom}_{\mathsf{C}}(P, \_) : \mathsf{C}_{\mathsf{ev}} \to (\mathsf{smod}_{\Bbbk})_{\mathsf{ev}}$ *is exact.*

Proof. Let's begin by analyzing condition (2). This means that given any short exact sequence

$$0 \longrightarrow X \xrightarrow{\;a\;} Y \xrightarrow{\;b\;} Z \longrightarrow 0$$

in C_{ev}, applying $\mathrm{Hom}_{\mathsf{C}}(P, \_)$ yields a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{C}}(P, X) \xrightarrow{a\circ\_} \mathrm{Hom}_{\mathsf{C}}(P, Y) \xrightarrow{b\circ\_} \mathrm{Hom}_{\mathsf{C}}(P, Z) \longrightarrow 0$$

in $(\mathsf{smod}_{\Bbbk})_{\mathsf{ev}}$. Well, suppose $\sigma \in \mathrm{Hom}_{\mathsf{C}}(P, X)$ is nonzero. Then there exists an element $t \in P$ such that $\sigma(t) \neq 0$. Since a is injective, it follows that $a(\sigma(t)) \neq 0$, and therefore $a \circ \_$ is injective. This gives exactness in the first place. Exactness in the middle place means the image of $a\circ\_$ is equal to the kernel of $b\circ\_$, or equivalently that $(b\circ\_)\circ(a\circ\_) = 0$. This follows from the fact that $b \circ a = 0$. Thus far, we've used no assumptions about P - we've shown that the functor $\mathrm{Hom}_{\mathsf{C}}(P, \_)$ is *always* left exact. Therefore, $\mathrm{Hom}_{\mathsf{C}}(P, \_)$ is exact precisely when it is right exact, or when our above sequence is exact in the third place.

Now we show that (1) is equivalent to (2). We've just shown above that (2) is equivalent to the above diagram being exact in the third place. This is equivalent to $b\circ\_$ being surjective. This means that given $t \in \mathrm{Hom}_{\mathsf{C}}(P, Z)$, there exists a morphism $\tilde{t} : P \to Y$ such that $t = b\tilde{t}$. Since we already know that $b : Y \to Z$ is surjective, this condition is equivalent to (1). ∎

Lemma B.7. *If P is a projective object in the category $\mathsf{C} = \mathsf{smod}_A$, then every short exact sequence (of even morphisms)*

$$0 \longrightarrow X \xrightarrow{\;a\;} Y \xrightarrow{\;b\;} P \longrightarrow 0$$

splits. That is, there exists a section *of b, an even morphism $c : P \to Y$ such that $bc = \mathrm{id}_P$.*

Moreover, if we let $e := cb$, we have

$$Y = \ker(b) \oplus \mathrm{im}(e) \cong X \oplus P.$$

Proof. Consider the short exact sequence in the lemma statement. The map $b : Y \to P$ is surjective. Therefore, if P is projective, there is a lift $c : P \to Y$ of the identity on P so that $bc = \mathrm{id}_P$.

Next, note that since b and e are even, $\ker(b), \mathrm{im}(e)$ are both subobjects of Y. Now let $y \in Y$. Note that

$$b(y - e(y)) = b(y) - bcb(y) = b(y) - b(y) = 0$$

so $y - e(y) \in \ker(b)$. By exactness, there exists a unique (since a is injective) element $x \in X$ such that

$$a(x) = y - e(y),$$

which means

$$y = a(x) + e(y).$$

The uniqueness of this decomposition shows that $Y = \mathrm{im}(a) \oplus \mathrm{im}(e) = \ker(b) \oplus \mathrm{im}(e)$. Now note that since $bc = \mathrm{id}_P$, it must be that c is injective, hence is an isomorphism onto its image. That is, $P \cong e(Y)$. Similarly, $X \cong a(X)$, and we have our result. ∎

Now we can define what it means to take the Karoubi envelope of a supercategory.

Definition B.8. Let C be a supercategory. The **super Karoubi envelope**, $\mathsf{sKar}(\mathsf{C})$, of C is the supercategory whose objects are pairs (X, e) where X is an object of C and $e : C \to C$ is an *even* idempotent in C. A morphism $\varphi : (X, e) \to (Y, f)$ is a morphism $\varphi : X \to Y$ in C such that the following diagram commutes in C:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "\varphi"'] \arrow[dr, "\varphi"] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "f"'] & Y \end{tikzcd} . \tag{141}$$

The identity morphism on (X, e) is

$$\mathrm{id}_{(X,e)} = e : (X, e) \to (X, e).$$

Remark B.9. Note that since we only consider even idempotents, no signs are introduced in condition (141) regardless of the parity of φ. A good reason to only consider even idempotents is that in our motivating example of $\mathsf{C} = \mathsf{smod}_A$, summands of objects only correspond to such morphisms. The image of a mixed degree idempotent, for example, will not be a subsupermodule of the target object, and hence cannot be a summand. We are particularly interested in summands of free objects which will correspond to projective objects.

Finally, the construction of $\mathsf{sKar}(\mathsf{C})$ differs from that of $\mathsf{Kar}(\mathsf{C}_{\mathsf{ev}})$ only in that $\mathsf{sKar}(\mathsf{C})$ allows for odd (and non-homogeneous) morphisms where $\mathsf{Kar}(\mathsf{C}_{\mathsf{ev}})$ allows for only even morphisms.

Lemma B.10. $\mathsf{sKar}(\mathsf{C})$ *is the universal enlargement of* C *such that every idempotent splits via even morphisms. That is, there exists a fully faithful embedding*

$$E : \mathsf{C} \to \mathsf{sKar}(\mathsf{C}),$$

every idempotent in $\mathsf{sKar}(\mathsf{C})$ *splits via even morphisms, and* $\mathsf{sKar}(\mathsf{C})$ *satisfies the following universal property: For any superfunctor* $F : \mathsf{C} \to \mathsf{D}$ *where every idempotent in* D *splits via even morphisms, there exists a (unique up to supernatural isomorphism) superfunctor* $F' : \mathsf{sKar}(\mathsf{C}) \to \mathsf{D}$ *such that* $F' \circ E = F$.

Proof. First, we note that since we're only concerned with idempotents splitting via even morphisms, lemma B.2 holds in this super setting. Moreover, since F and E induce even maps on morphism spaces, and since our idempotents are even, the same proof of lemma B.4 applies here. In particular, lemma B.2 (applied to the super setting) will induce the appropriate even supernatural isomorphism needed for the 'unique up to supernatural isomorphism' statement. ∎

Let A be a superalgebra over a commutative domain $\Bbbk$ which is not characteristic 2. As in the beginning of this section, let V denote the supercategory of finitely generated free right A-supermodules of the form $A \cong A^{m|n}$, and let $\widehat{\mathsf{V}}$ denote the supercategory of finitely generated projective right A-supermodules.

Lemma B.11. *The identity functor is a fully faithful embedding of supercategories*

$$\mathsf{V} \hookrightarrow \widehat{\mathsf{V}}.$$

Proof. Since both categories are full subcategories of $\mathsf{smod_A}$, we just need to show that any finitely generated free A-supermodule, X, is also projective. Suppose we have (in $\mathsf{smod_A}$) even A-maps $\varphi : X \to P$ and $\psi : P' \to P$ where ψ is surjective. Well X has a finite A-basis B, and φ is determined by its values on B. Since ψ is surjective, each element $\varphi(b) \in P$ (for $b \in B$) has a preimage in P' under ψ, $\varphi(b)'$. Then we can define an even A-map $\tilde{\varphi} : X \to P'$ via $b \mapsto \varphi(b)'$. It's easy to see that $\psi\tilde{\varphi} = \varphi$. So X is projective. ∎

Lemma B.12. *The superfunctor*

$$\mathscr{E} : \mathsf{sKar}(\mathsf{V}) \to \widehat{\mathsf{V}} \quad \textit{via} \quad \begin{cases} (X, e) \mapsto eX \\ (\varphi : (X, e) \to (Y, f)) \mapsto (\varphi|_{eX} : eX \to fY) \end{cases}$$

gives an equivalence of categories (where $\varphi|_{eX}$ denotes the restriction of the map $\varphi : X \to Y$ to $eX \subset X$).

Proof. First of all, let's check that $\mathscr{E}(X, e) = eX$ really is a finitely generated projective A-supermodule. Well, it is clearly a finitely generated right A-supermodule since X is, and those properties are respected by even A-maps. Moreover, since e and $\mathrm{id}_X - e$ are both even, $\mathrm{im}(\mathrm{id}_X - e), \mathrm{im}(e)$ are both sub-supermodules of X.

To see that eX is projective, let's first see that eX is a direct summand of X (viewed in smod_A). Given $x \in X$, we have

$$x = (\mathrm{id}_X - e)(x) + e(x)$$

so that $x \in \mathrm{im}(\mathrm{id}_X - e) \cup \mathrm{im}(e)$. Now if $x' \in \mathrm{im}(e)$, there exists some $x'' \in X$ such that $x' = e(x'')$. But if $x' \in \mathrm{im}(\mathrm{id}_X - e)$, then there is some $x''' \in X$ such that $x' = x''' - e(x''')$. Then we have

$$e(x'') = x''' - e(x'''),$$

and applying e to both sides gives

$$e(x'') = e(x''') - e(x''') = 0$$

since $e^2 = e$. Thus, $x' = e(x'') = 0$ so $\mathrm{im}(\mathrm{id}_X - e) \cap \mathrm{im}(e) = \{0\}$. Therefore, $X = \mathrm{im}(\mathrm{id}_X - e) \oplus \mathrm{im}(e)$, so we have that eX is a summand of X. Notice that e acts as the projection map from X onto the summand eX. Denote the associated inclusion map by i_X.

Now we show that eX is projective. Suppose we have even A-maps $\varphi : eX \to P$ and $\psi : P' \to P$ where ψ is surjective. Well then $\varphi \circ e : X \to P$ is an A-map, and since free objects are projective by lemma B.11, there is a lift $\tilde{\varphi e} : X \to P'$ such that $\psi \circ \tilde{\varphi e} = \varphi e$. If $i_X : eX \to X$ is the inclusion map, we have $\psi(\tilde{\varphi e})i = (\varphi e)i = \varphi$ since $ei = \mathrm{id}_{eX}$. Thus $(\tilde{\varphi e})i : eX \to P'$ is a lift of φ, and we see that eX is projective.

Next, we should show that $\mathscr{E}(\varphi : (X, e) \to (Y, f))$ is actually an A-map between $eX \to fY$. Well, the fact that φ satisfies condition (141) implies that $\varphi|_{eX} : eX \to fY$. Moreover, recall that $\mathrm{id}_{(X,e)} = e$. So $e|_{eX} : eX \to eX$, and since $e^2 = e$, it acts as the identity on eX. Finally, it is easy to see that these restrictions respect composition so that $\mathscr{E}(\psi\varphi) = \mathscr{E}(\psi)\mathscr{E}(\varphi)$. So $\mathscr{E}$ really is a superfunctor (the restriction preserves parity).

Now we'll check that $\mathscr{E}$ is fully faithful. So we need to show that, for any objects $(X, e), (Y, f)$ in $\mathsf{sKar}(\mathsf{V})$, the induced map on morphism spaces

$$\mathscr{E}_{(X,e),(Y,f)} : \mathrm{Hom}_{\mathsf{sKar}(\mathsf{V})}((X, e), (Y, f)) \to \mathrm{Hom}_{\widehat{\mathsf{V}}}(eX, fX)$$

is an isomorphism of $\Bbbk$-supermodules. First of all, let $\varphi \in \mathrm{Hom}_{\mathsf{sKar(V)}}((X,e),(Y,f))$. Since $\mathscr{E}(\varphi)$ is just the restriction of φ to eX, it is obvious that $\mathscr{E}(\varphi) = 0$ only when $\varphi = 0$. So $\mathscr{E}_{(X,e),(Y,f)}$ is injective.

To see that $\mathscr{E}_{(X,e),(Y,f)}$ is surjective, let $\psi \in \mathrm{Hom}_{\widehat{\mathsf{V}}}(eX, fX)$. Then $(i_Y\psi e) : X \to Y$ is an A-map that satisfies condition (141). To see this, note that $(i_Y\psi e)e = (i_Y\psi e)$ since e is an idempotent. Next, we know from above that any element in X may be written as $e(x) + (\mathrm{id}_X - e)(x)$ for some $x \in X$. Then

$$\begin{aligned}
[f(i_Y\psi e)](e(x) + (\mathrm{id}_X - e)(x)) &= [fi_Y\psi](e^2(x) + (e - e^2)(x)) \\
&= [fi_Y\psi](e(x)) \\
&= \psi(e(x)) \\
&= i_Y(\psi(e(x))) \\
&= (i_Y\psi e)(e(x)) \\
&= (i_Y\psi e)(e(x) + (\mathrm{id}_X - e)(x)),
\end{aligned}$$

so we have our condition met. By construction, we have $\mathscr{E}(i_Y\psi e) = \psi$, so $\mathscr{E}_{(X,e),(Y,f)}$ is surjective.

Since we've seen that $\mathscr{E}$ is fully faithful, once we show that it is essentially surjective, we will have our equivalence (by remark 5.1). To this end, choose some object $P \in \widehat{\mathsf{V}}$. Since P is finitely generated, it has some finite generating set G of cardinality $m|n$. Then there's a canonical even surjective A-map $\rho : A^{m|n} \to P$. Let $K = \ker(\rho)$. Then the inclusion of K into $A^{m|n}$ yields the following short exact sequence:

$$0 \longrightarrow K \longrightarrow A^{m|n} \xrightarrow{\ \rho\ } P \longrightarrow 0 \ .$$

By lemma B.7, this sequence splits via a section $c : P \to A^{m|n}$, and we have $P \cong (c\rho)A^{m|n}$ for $c\rho$ an idempotent. Thus, $\mathscr{E}(A^{m|n}, c\rho) \cong P$, and we have our claim. ∎

Now we're ready to show the main result of this section.

Proposition B.13. *There is an equivalence of supercategories*

$$\Gamma^d\mathsf{V}\text{-}\mathsf{smod} \cong \Gamma^d\mathsf{sKar(V)}\text{-}\mathsf{smod}.$$

Proof. First, note that $\Gamma^d\mathsf{sKar(V)}$ has objects (X,e) where X is an object in V and $e : X \to X$ is an idempotent in V. A morphism $\phi : (X,e) \to (Y,f)$ is an element of

$$\Gamma^d\,\mathrm{Hom}_{\mathsf{sKar(V)}}((X,e),(Y,f)) = (\mathrm{Hom}_{\mathsf{sKar(V)}}((X,e),(Y,f))^{\otimes d})^{\mathfrak{S}_d}.$$

So in particular, ϕ is an element of $\mathrm{Hom}_{\mathsf{sKar(V)}}((X,e),(Y,f))^{\otimes d}$ and can be written as

$$\sum_{i \in I} \alpha_i \phi_1^i \otimes \cdots \otimes \phi_d^i \tag{142}$$

where I is some finite index set, each $\alpha_i \in \Bbbk$, and each φ_j^i satisfies condition (141).

Similarly, a morphism $\gamma : X \to Y$ in $\Gamma^d\mathsf{V}$ can be written as

$$\sum_{k \in K} \beta_k \gamma_1^k \otimes \cdots \otimes \gamma_d^k \tag{143}$$

where the only condition on the γ_ℓ^k is that they are A-maps.

For any object X in V, we have id_X being an idempotent so that (X, id_X) is an object of $\Gamma^d\mathsf{sKar}(\mathsf{V})$. Moreover, any morphism $\gamma : X \to Y$ in $\Gamma^d\mathsf{V}$ looks like (143) where each γ_ℓ^k obviously satisfies $\mathrm{id}_Y \gamma_\ell^k = \gamma_\ell^k = \gamma_\ell^k \mathrm{id}_X$. Therefore, γ is actually a morphism $\gamma : (X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y)$. This defines a superfunctor

$$\mathscr{I} : \Gamma^d\mathsf{V} \to \Gamma^d\mathsf{sKar}(\mathsf{V}) \quad \text{via} \quad \begin{cases} X \mapsto (X, \mathrm{id}_X) \\ (\gamma : X \to Y) \mapsto (\gamma : (X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y)) \end{cases}.$$

That $\mathscr{I}$ respects the identity and composition is clear. It's also obvious that this is a fully faithful functor.

Next, consider an object F in $\Gamma^d\mathsf{V}\text{-}\mathsf{smod}$. So F is a superfunctor $F : \Gamma^d\mathsf{V} \to \mathsf{smod}_{\Bbbk}$. We'd like to define a superfunctor

$$\tilde{F} : \Gamma^d\mathsf{sKar}(\mathsf{V}) \to \mathsf{smod}_{\Bbbk}$$

such that $\tilde{F} \circ \mathscr{I} = F$.

To this end, let X be an object of V and $e : X \to X$ be an idempotent. Then since e is an idempotent, it follows that $e^{\otimes d}$ is an element of both $\Gamma^d \mathrm{Hom}_{\mathsf{sKar}(\mathsf{V})}((X, e), (X, e))$ and $\Gamma^d \mathrm{Hom}_{\mathsf{V}}(X, X)$. Therefore, it makes sense to define

$$\tilde{F}(X, e) \coloneqq \mathrm{im}(Fe^{\otimes d}).$$

Also, given any morphism $\phi : (X, e) \to (Y, f)$ in $\Gamma^d\mathsf{sKar}(\mathsf{V})$, it is obvious that ϕ also defines a morphism $\phi : X \to Y$ in $\Gamma^d\mathsf{V}$. So on morphisms, we have

$$\tilde{F}\phi \coloneqq F\phi.$$

Now we'll check that this is a well-defined superfunctor. First, let's see that $\tilde{F}\phi : \mathrm{im}(Fe^{\otimes d}) \to \mathrm{im}(Ff^{\otimes d})$. Let $v \in \mathrm{im}(Fe^{\otimes d})$ so that there is some $w \in FX$ such that $Fe^{\otimes d}(w) = v$. Then for φ as in (142), we have

$$\begin{aligned}
\tilde{F}\phi(v) = F\phi(v) &= F\phi(Fe^{\otimes d}(w)) \\
&= F(\phi e^{\otimes d})(w) \\
&= F\left(\left(\sum_{i \in I} \alpha_i \phi_1^i \otimes \cdots \otimes \phi_d^i\right)\left(e^{\otimes d}\right)\right)(w) \\
&= F\left(\sum_{i \in I} \alpha_i (\phi_1^i e) \otimes \cdots \otimes (\phi_d^i e)\right)(w) \\
&= F\left(\sum_{i \in I} \alpha_i (f\phi_1^i) \otimes \cdots \otimes (f\phi_d^i)\right)(w) \\
&= F\left(\left(f^{\otimes d}\right)\left(\sum_{i \in I} \alpha_i \phi_1^i \otimes \cdots \otimes \phi_d^i\right)\right)(w) \\
&= Ff^{\otimes d}\left(F\phi(w)\right).
\end{aligned}$$

In particular, $\tilde{F}\phi(v) \in \mathrm{im}(Ff^{\otimes d})$.

Next, to see that $\tilde{F}\,\mathrm{id}_{(X,e)} = \mathrm{id}_{\tilde{F}(X,e)}$, first note that $\mathrm{id}_{(X,e)} = e^{\otimes d}$ in $\Gamma^d\mathsf{sKar}(\mathsf{V})$. Moreover, we know from above that $\tilde{F}\,\mathrm{id}_{(X,e)} : \mathrm{im}(Fe^{\otimes d}) \to \mathrm{im}(Fe^{\otimes d})$. So let $v \in \mathrm{im}(Fe^{\otimes d})$ so that there is some $w \in FX$ such that $Fe^{\otimes d}(w) = v$. Then

$$\begin{aligned}\tilde{F}\,\mathrm{id}_{(X,e)}(v) = \tilde{F}e^{\otimes d}(v) &= Fe^{\otimes d}(v) \\ &= Fe^{\otimes d}(Fe^{\otimes d}(w)) \\ &= F(e^{\otimes d}e^{\otimes d})(w) \\ &= F(e^{\otimes d})(w) \\ &= v,\end{aligned}$$

where the fourth line follows from e being an even idempotent. Therefore, $\tilde{F}\,\mathrm{id}_{(X,e)} = \mathrm{id}_{\mathrm{im}(Fe^{\otimes d})} = \mathrm{id}_{\tilde{F}(X,e)}$ as desired. That $\tilde{F}$ respects composition follows from the fact that F does. It should be obvious that we have $\tilde{F} \circ \mathscr{I} = F$ for any F in $\Gamma^d\mathsf{V}\text{-}\mathsf{smod}$.

Now we give a superfunctor

$$\Phi : \Gamma^d\mathsf{V}\text{-}\mathsf{smod} \to \Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod} \quad \text{via} \quad F \mapsto \tilde{F}.$$

On morphisms $\eta : F \to G$, $\Phi\eta$ is the supernatural transformation $\tilde{F} \to \tilde{G}$ whose section at (X, e) is η_X. We should check that this makes sense. It will boil down to knowing that for any idempotent $e : X \to X$, a given section $\eta_X : \mathrm{im}(Fe^{\otimes d}) \to \mathrm{im}(Ge^{\otimes d})$. To see this, we know that since η is a supernatural transformation, for $e^{\otimes d} : X \to X$ in $\Gamma^d\mathsf{V}$, we have

$$\eta_X \circ Fe^{\otimes d} = Ge^{\otimes d} \circ \eta_X,$$

where no signs are present since $e^{\otimes d}$ is even. Now suppose that $v \in \mathrm{im}(Fe^{\otimes d})$ so that there is some $w \in FX$ such that $Fe^{\otimes d}(w) = v$. Then it follows that

$$\begin{aligned}\eta_X(v) &= \eta_X\left(Fe^{\otimes d}(w)\right) \\ &= Ge^{\otimes d}(\eta_X(w)).\end{aligned}$$

Now for a given morphism $\varphi : X \to Y$ in $\Gamma^d\mathsf{V}$, we have

$$\begin{aligned}(\Phi\eta)_{(Y,f)} \circ \Phi F(\phi) &= \eta_Y \circ \tilde{F}\phi \\ &= \eta_Y \circ F\phi \\ &= (-1)^{\overline{\phi}\cdot\overline{\eta}} G\phi \circ \eta_X \\ &= (-1)^{\overline{\phi}\cdot\overline{\eta}} \tilde{G}\phi \circ \eta_X \\ &= (-1)^{\overline{\phi}\cdot\overline{\eta}} \Phi G(\phi) \circ (\Phi\eta)_{(X,e)},\end{aligned}$$

so $\Phi\eta$ really is a supernatural transformation. Since Φ on a supernatural transformation is just the restriction of that transformation, Φ respects the identity and composition, so is well-defined.

Now we'll define a superfunctor

$$\Psi : \Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod} \to \Gamma^d\mathsf{V}\text{-}\mathsf{smod} \quad \text{via} \quad \Psi := \_ \circ \mathscr{I}.$$

In this way, we view Ψ as restriction to the full subcategory $\mathscr{I}\left(\Gamma^d\mathsf{V}\right)$. So given T in $\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}$, ΨT is the functor which sends an X in $\Gamma^d\mathsf{V}$ to $T\mathscr{I}(X) = T(X, \mathrm{id}_X)$ and sends a morphism $\phi : X \to Y$ in $\Gamma^d\mathsf{V}$ to the morphism $T\phi : T(X, \mathrm{id}_X) \to T(Y, \mathrm{id}_Y)$ (which makes sense since the same map ϕ determines a morphism $\phi : (X, \mathrm{id}_X) \to (Y, \mathrm{id}_Y)$). For a morphism $\eta : T \to T'$ in $\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}$, $\Psi\eta$ is the supernatural transformation in $\Gamma^d\mathsf{V}\text{-}\mathsf{smod}$

whose section at X is given by $\eta_{(X,\mathrm{id}_X)}$. Ψ obviously respects identity and composition, so it's a well-defined superfunctor.

Now given F in $\Gamma^d\mathsf{V}\text{-}\mathsf{smod}$, we have $\Psi\Phi(F) = \Psi(\tilde{F}) = \tilde{F} \circ \mathscr{I} = F$. Given a supernatural transformation $\eta : F \to G$ in $\Gamma^d\mathsf{V}\text{-}\mathsf{smod}$, $\Phi(\eta)$ is the supernatural transformation whose section at (X, e) is η_X for any idempotent e. In particular, the section of $\Phi(\eta)$ at (X, id_X) is η_X. Then $\Psi(\Phi(\eta))$ is the supernatural transformation whose section at X is $\Phi(\eta)_{(X,\mathrm{id}_X)} = \eta_X$. Thus, $\Psi\Phi = \mathrm{id}_{\Gamma^d\mathsf{V}\text{-}\mathsf{smod}}$.

Now we claim $\Phi\Psi \cong \mathrm{id}_{\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}}$. Once we show this, we are finished. To this end, we will construct an even supernatural transformation

$$\eta : \mathrm{id}_{\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}} \to \Phi\Psi$$

whose section at T is the supernatural transformation $\eta_T : \mathrm{id}_{\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}}(T) \to \Phi\Psi(T)$ whose section at (X, e) is the morphism $Te^{\otimes d}$. Let's unpack this.

First, we have $\Phi\Psi(T) = \widetilde{(T \circ \mathscr{I})}$ and

$$\widetilde{(T \circ \mathscr{I})}(X, e) = \mathrm{im}((T \circ \mathscr{I})e^{\otimes d}) = \mathrm{im}(Te^{\otimes d}).$$

Moreover, for a supernatural transformation $\sigma : T \to U$ in $\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}$, chasing through the definitions, one sees that $\Phi\Psi(\sigma)$ is the supernatural transformation whose section at (X, e) is $\sigma_{(X,\mathrm{id}_X)} : \mathrm{im}(Te^{\otimes d}) \to \mathrm{im}(Ue^{\otimes d})$.

Now given a morphism $\sigma : T \to U$ in $\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}$, we want to show that

$$\eta_U \circ \mathrm{id}_{\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}}(\sigma) = \Phi\Psi(\sigma) \circ \eta_T.$$

This equation holds precisely if it holds for each section at a given (X, e), that is, if

$$(\eta_U)_{(X,e)} \circ (\mathrm{id}_{\Gamma^d\mathsf{sKar}(\mathsf{V})\text{-}\mathsf{smod}}(\sigma))_{(X,e)} = (\Phi\Psi(\sigma))_{(X,e)} \circ (\eta_T)_{(X,e)}. \tag{144}$$

Well by our above observations, (144) is equivalent to

$$Ue^{\otimes d} \circ \sigma_{(X,e)} = \sigma_{(X,\mathrm{id}_\mathrm{X})} \circ Te^{\otimes d}. \tag{145}$$

Equation (145) holds since $\sigma : T \to U$ is a supernatural transformation, so η does define a supernatural transformation. Moreover, note that since $e^{\otimes d} : (X, e) \to (X, e)$ is the identity morphism in $\Gamma^d\mathsf{sKar}(\mathsf{V})$, $Te^{\otimes d} = \mathrm{id}_{T(X,e)}$ for any T in $\Gamma^d\mathsf{sKar}(\mathsf{V})$. Then it follows that each section of η_T at (X, e) is an isomorphism (the identity), and hence each section η_T is a supernatural isomorphism. Thus, η is a supernatural isomorphism, and we are done. ◻

Remark B.14. Lemma B.12 and proposition B.13 show that $\Gamma^d\mathsf{V}\text{-}\mathsf{smod} \cong \Gamma^d\widehat{\mathsf{V}}\text{-}\mathsf{smod}$. This means that defining the category of strict polynomial superfunctors over finitely generated free right A-supermodules is equivalent to defining them instead over finitely generated projective right A-supermodules.

Appendix C. MacDonald's Polynomial Functors vs Generalized Strict Polynomial Functors

In this section, we sketch the results in [Mac80] and discuss how they relate to our generalized strict polynomial functors. In the introduction, for $\Bbbk$ an infinite field, we defined a homogeneous degree d polynomial functor T to be a functor $T : \mathsf{vec}_\Bbbk \to \mathsf{vec}_\Bbbk$ such that for all objects V, W the induced $\Bbbk$-linear map

$$T_{V,W} : \mathrm{Hom}_\Bbbk(V, W) \to \mathrm{Hom}_\Bbbk(TV, TW)$$

is a homogeneous degree d polynomial mapping (as defined in definition A.1). The goal of [Mac80] is to consider a polynomial functor between more general $\Bbbk$-linear categories than just $\mathsf{vec}_{\Bbbk}$.

Let A, B be $\Bbbk$-linear categories. A homogeneous degree d polynomial functor $T : \mathsf{A} \to \mathsf{B}$ is a covariant functor such that for all objects V, W in A, the induced $\Bbbk$-linear map

$$T_{V,W} : \mathrm{Hom}_{\mathsf{A}}(V, W) \to \mathrm{Hom}_{\mathsf{B}}(TV, TW)$$

is a homogeneous degree d polynomial mapping.

The result in [Mac80] that is relevant for us considers the following conditions: $\Bbbk$ is an infinite field of characteristic 0, A is a $\Bbbk$-algebra, A is taken to be the category ${}_A\mathsf{mod}^{\mathsf{fgp}}$ of finitely generated projective *left* A-modules, and B is taken to be $\mathsf{vec}_{\Bbbk}$. Let F_A^d denote the category of homogeneous degree d polynomial functors $T : {}_A\mathsf{mod}^{\mathsf{fgp}} \to \mathsf{vec}_{\Bbbk}$. Then MacDonald shows that

$$\mathsf{F}_A^d \cong \mathsf{mod}_{A \wr \mathfrak{S}_d}. \tag{146}$$

First, let's see how this generalizes the result mentioned in the introduction from [Mac95]. Taking $A = \Bbbk$, we have $A \wr \mathfrak{S}_d \cong \Bbbk\mathfrak{S}_d$. Moreover, in this setting, $\Bbbk\mathfrak{S}_d$ is a semisimple algebra, and it follows that $\Bbbk\mathfrak{S}_d \cong (\Bbbk\mathfrak{S}_d)^{\mathsf{op}}$. Therefore,

$$\mathsf{mod}_{\Bbbk\mathfrak{S}_d} \cong \mathsf{mod}_{(\Bbbk\mathfrak{S}_d)^{\mathsf{op}}} \cong {}_{\Bbbk\mathfrak{S}_d}\mathsf{mod}$$

so that $\mathsf{mod}_{A\wr\mathfrak{S}_d} \cong {}_{\Bbbk\mathfrak{S}_d}\mathsf{mod}$, and (146) can be written as

$$\mathsf{F}_d := \mathsf{F}_{\Bbbk}^d \cong {}_{\Bbbk\mathfrak{S}_d}\mathsf{mod},$$

which is how we stated the result from [Mac95]. We also explained in the introduction how this relates to the [FS97] result concerning strict polynomial functors. So one can view the [Mac80] result as a generalization of the [FS97] result (when $\Bbbk$ is an infinite field of characteristic 0) by defining the functors over an algebra A.

Our generalized strict polynomial functors $\mathsf{P}^d_{(A,\mathfrak{a})}$ do something similar by defining strict polynomial functors over a superalgebra A for $\Bbbk$ a commutative unital domain of characteristic not equal to 2. So how does $\mathsf{P}^d_{(A,\mathfrak{a})}$ relate to F^d_A?

First of all, take $\Bbbk$ to be an infinite field of characteristic 0 and our unital good pair $(A, \mathfrak{a})$ such that $A = A_0$ is completely even and $\mathfrak{a} = A$. It follows from definitions and theorem 6.10 that

$$\mathsf{P}^d_{(A,\mathfrak{a})} = \mathsf{P}^d_{(A,A)} \cong {}_{S^A(n,d)}\mathsf{smod}$$

whenever $n \geqslant d$. Since A is even, all objects in play are inherently even (as are morphisms), so we really have

$$\mathsf{P}^d_{(A,A)} \cong {}_{S^A(n,d)}\mathsf{mod}.$$

Now when $n \geqslant d$, it's proven in lemma 5.15 of [EK17] that there is a (super)algebra isomorphism

$$\xi_\omega S^A(n, d)\xi_\omega \cong A \wr \mathfrak{S}_d,$$

where ξ_ω is a specific idempotent in $S^A(n, d)$. Now for the case of A being even, the general setting of section 6.2 of [Gre07] applies, and in particular, one can deduce that for $n \geqslant d$,

$$_{\xi_\omega S^A(n,d)\xi_\omega}\mathsf{mod} \cong {}_{S^A(n,d)}\mathsf{mod}$$

so that

$$_{A\wr\mathfrak{S}_d}\mathsf{mod} \cong {}_{S^A(n,d)}\mathsf{mod}.$$

Finally, lemma 10.7 of [DKMZ22] shows that[7]

$$(A^{\mathsf{sop}}) \wr \mathfrak{S}_d \cong (A \wr \mathfrak{S}_d)^{\mathsf{sop}},$$

which in particular implies that (taking $A = A_0$)

$$(A^{\mathsf{op}}) \wr \mathfrak{S}_d \cong (A \wr \mathfrak{S}_d)^{\mathsf{op}}.$$

Putting all of this together, for $A = A_0 = \mathfrak{a}$ and $n \geqslant d$, we have

$$\begin{aligned} \mathsf{P}^d_{(A,A)} &\cong {}_{S^A(n,d)}\mathsf{mod} \\ &\cong {}_{A \wr \mathfrak{S}_d}\mathsf{mod} \\ &\cong \mathsf{mod}_{(A \wr \mathfrak{S}_d)^{\mathsf{op}}} \\ &\cong \mathsf{mod}_{(A^{\mathsf{op}}) \wr \mathfrak{S}_d} \\ &\cong \mathsf{F}^d_{A^{\mathsf{op}}}. \end{aligned}$$

In this way, our generalized strict polynomial functors encompass the [Mac80] result.

[7]Their convention for defining the wreath product $A \wr \mathfrak{S}_d$ is that $\mathfrak{S}_d$ acts on the left of $A^{\otimes d}$ so as a $\Bbbk$-supermodule, $A \wr \mathfrak{S}_d = A^{\otimes d} \otimes \mathfrak{S}_d$ with product $(\vec{a} \otimes \sigma)(\vec{b} \otimes \tau) = \vec{a}\sigma(\vec{b}) \otimes \sigma\tau$. However, it is obvious how to adapt their proof for our 'right-handed' version of the wreath product.

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