ENUMERATION OF RATIONAL CURVES ON MINIMAL RATIONAL SURFACES

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Chapter 1

Introduction

Let S be a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, for example $\mathbb{P}^1 \times \mathbb{P}^1$, and let D be an irreducible curve on S. To the curve D, we associate a projective space $|D|$. We study the geometry of the Severi variety $V(D)$ parametrizing irreducible rational curves on minimal rational ruled surfaces, i.e. $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, in the projective space $|D|$ for a given curve D. In particular, we compute the Severi degree. That is, we compute the number of irreducible rational curves through $\dim V(D)$ general points on $\mathbb{F}_n$, where $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, $(n \geq 0)$.

Consider the question of determining these Severi degrees on the projective plane $\mathbb{P}^2$. Since $3d-1$ will prove to be the $\dim V(D)$ where D is a curve of degree d, this is equivalent to asking how many rational plane curves of degree d pass through $3d-1$ general points. For example, first take a degree one curve, i.e. a line in the plane. There are infinitely many lines though one point, no lines through three general points, but one line through two points. In other words $N(D) = 1$ when D is a line. Continuing with the example in the plane, we ask how many conics $(d = 2)$

pass through five general points? The answer again is one, since five points determine a unique conic. For $d = 1$ and $d = 2$ the Severi variety is in fact equal to the complete linear system of the degree d rational curve. For $d \geq 3$ the results are not so easily anticipated. There are twelve rational cubics passing through nine general points; equivalently a general pencil of elliptic curves has twelve nodal cubics, so $N(D) = 12$ when D is a cubic. In the late 19^{th} century Zeuthen determined that there were 620 rational quartics passing through 11 general points. Only in 1993 was it shown by Kontsevich that there are 87304 rational plane quintics passing through 14 general points.

Below we give some facts about the mininal rational surfaces so that we can precisely state this enumerative problem. We conclude the chapter with a history of the problem.

1.1 Preliminary Facts Needed About $\mathbb{F}_n$

We note here that we work over the complex numbers, so by $\mathbb{P}^n$ we mean $\mathbb{P}^n_{\mathbb{C}}$ and by an irreducible rational curve we mean a curve whose normalization is $\mathbb{P}^1$.

Every minimal ruled surface over $\mathbb{P}^1$ is of the form $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for some $n \geq 0$. We denote this surface by $\mathbb{F}_n$, the n^{th} Hirzebruch surface. The Hirzebruch surfaces are birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ and hence to $\mathbb{P}^2$ so they are all rational.

The Picard group of $\mathbb{F}_n$ is generated by the classes E and F, where

$$E^2 = -n, \quad E \cdot F = 1, \text{ and } \quad F^2 = 0.$$

For $n > 0$, E is the unique irreducible curve on $\mathbb{F}_n$ with negative self-intersection.

The class F is the class of the fiber of the ruling on $\mathbb{F}_n$.

If we denote a section of the $\mathbb{P}^1$-bundle $\mathbb{F}_n \to \mathbb{P}^1$ disjoint from E by C, then $C \sim E + nF$. So the classes C and F also generate the Picard group of $\mathbb{F}_n$, with intersection pairings given by

$$C^2 = n, \qquad C \cdot F = 1, \qquad \text{and} \qquad F^2 = 0.$$

For divisor classes on $\mathbb{F}_n$, consider their possible self-intersections. We have $E^2 = -n$, $F^2 = 0$ and $D^2 \geq n$ corresponding to $D \sim aC + bF$ with $a, b > 0$. The divisor classes D for which $D^2 > n$ are the interesting $N(D)$'s, so for our purposes it is more convenient to write a divisor class D as

$$D \sim aC + bF$$

for $a, b \in \mathbb{N}$.

A complete linear system $|D|$ contains an irreducible curve if and only if $D = E$ or $D \sim aC + bF$ when $a \geq 0$ and $b \geq 0$. The general member of $|D|$ is a reduced connected curve if and only if $D = E$, $D = F$, $D \sim C + bF$ when $b > -an$, or $D \sim aC + bF$ when $a \geq 2$ and $b \geq -n$.

The canonical class of $\mathbb{F}_n$ is

$$K \sim -2E - (2+n)F \sim -2C - (2-n)F.$$

Note that if $D \sim aC + bF$ is effective, then $K - D$ is not effective so $H^2(D) = 0$ by Serre Duality. Therefore Riemann-Roch for the divisor class D on the surface $\mathbb{F}_n$ becomes

$$h^0(D) - h^1(D) = \frac{1}{2}(D^2 - D \cdot K) + \chi(\mathcal{O}_{\mathbb{F}_n}).$$

As a rational surface, $\mathbb{F}_n$ has birational invariants $q = 0$ and $p_g = 0$ so $\chi(\mathcal{O}_{\mathbb{F}_n}) = 1 - q + p_g = 1$. For $D > 0$ we have $D \cdot F \geq 0$ so

$$H^1(\mathcal{O}_{\mathbb{F}_n}(D)) \cong H^1(\mathbb{P}^1, \phi_*\mathcal{O}_{\mathbb{F}_n}(D)),$$

where ϕ is the ruling $\phi : \mathbb{F}_n \to \mathbb{P}^1$. For $D \sim aE + bF$,

$$\phi_*\mathcal{O}_{\mathbb{F}_n}(D) = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))^{\otimes a} \otimes \mathcal{O}_{\mathbb{P}^1}(b).$$

But this is a sum of terms of the form $\mathcal{O}_{\mathbb{P}^1}(c)$ where $c \geq b - an$ which is always positive since D is irreducible, here we follow Hartshorne's line of proof in Lemma 2.4 on p. 379. Therefore $h^1(\mathcal{O}_{\mathbb{F}_n}(D)) = 0$. Thus

$$h^0(D) = \frac{1}{2}(D^2 - D \cdot K) + 1$$

and so we can conclude that

$$\dim |D| = \frac{1}{2}(D^2 - D \cdot K).$$

1.2 Statement of the Problem

Let $S = \mathbb{F}_n$ and let D be an effective divisor on S.

Definition 1.2.1 *Let $V(D)$ be the closure of the locus of all points parametrizing irreducible rational curves in the projective space $|D|$.*

We call this variety the Severi variety. A general point of the Severi variety represents a curve with p_a nodes , where p_a is the arithmetic genus of D.

We denote the dimension of $V(D)$ by $r(D)$. Since a general point of $V(D)$ is known to be a curve with $p_a(D)$ nodes, and no other singularities, then $V(D)$ has dimension

$$\begin{aligned} r(D) &= \dim |D| - p_a(D) \\ &= \frac{1}{2}(D^2 - D \cdot K_S) - \left(1 + \frac{1}{2}(D^2 + D \cdot K_S)\right) \\ &= -K_S \cdot D - 1 \end{aligned}$$

The degree of $V(D)$ we denote by $N(D)$. Called the Severi degree, $N(D)$ represents the number of irreducible rational curves in $|D|$ passing through $r(D)$ general points on S. It is the goal of this work to find an explicit formula for $N(D)$ on $S = \mathbb{F}_n$.

1.3 Further Notation

Definition 1.3.1 *For a positive integer m, let $V_m(D) \subset V(D) \subset |D|$ be the closure of the locus representing irreducible rational curves in $|D|$ meeting E at a smooth point with multiplicity at least m.*

Let $r_m(D) = \dim V_m(D)$ and $N_m(D) = \deg V_m(D)$. Caporaso and Harris in [CH1] (Proposition 2.1 on p.21) show that $r_m(D) = -K_S \cdot D - m$.

1.4 History

This basic enumerative problem of determining Severi degrees has remained unsolved until very recently. Interest in this problem has been revitalized by recent ideas in quantum field theory which lead to the definition of quantum cohomology. As a by-product, formulas enumerating rational curves on certain varieties were proved. For example, in 1993 Kontsevich (in [K]) derived a beautiful formula for rational curves in the plane, assuming associativity of the quantum product (not known at the time). Kontsevich's well known recursive formula for a divisor of degree d on $\mathbb{P}^2$:

$$N(d) = \sum_{d_1+d_2=d} N(d_1)N(d_2)d_1^2 d_2 \left[d_2 \binom{3d-4}{3d_1-2} - d_1 \binom{3d-4}{3d_1-1} \right].$$

In addition, Kontesevich and Manin have a similar recursive formula for $N(D)$ on $\mathbb{F}_0$ and $\mathbb{F}_1$. Their technique was dependent upon the fact that those surfaces are convex and hence the technique will not extend to $\mathbb{F}_n$ for $n \geq 2$.

L. Caporaso and J. Harris in a very long paper, [CH1], were able to find a very nice closed formula for the degree of the Severi variety for the divisor $2C$ on $\mathbb{F}_n$. Recall that C is a section of the $\mathbb{P}^1$-bundle $\mathbb{F}_n \to \mathbb{P}^1$.

Theorem 1.4.1 [CH1] (p.80) *Let $N(2C)$ be the number of irreducible rational curves in the linear series $|2C|$ on $\mathbb{F}_n$ passing through $2n+3$ points, then*

$$N(2C) = \sum_{k=0}^{n-1} (n-k)^2 \binom{2n+2}{k}$$

Others investigated particular divisor classes on $\mathbb{F}_n$. Using excess intersection and the moduli space of stable maps, D. Abramovich and A. Bertram calculated Severi degrees in all classes on $\mathbb{F}_2$, and in certain classes on $\mathbb{F}_n$ [V, p.11]. Notably, they were able to write a formula for $N(2C+bF)$ on $\mathbb{F}_n$. Their formula determines $N(2C+bF)$ on $\mathbb{F}_n$ in terms of $N(2C+(b+1)F)$ on $\mathbb{F}_{n-1}$. So their recursion in a sense is in terms of a worse divisor on a better surface.

In [CH2] L. Caporaso and J. Harris developed the Rational Fibration Method and applied it to the cases $S = \mathbb{P}^2$, $S = \mathbb{F}_2$, and $S = \mathbb{F}_3$. The first case provides a simpler proof of Kontsevich's formula. When applying the method on $\mathbb{F}_3$, we see the occurrence in codimension 1 of degenerate loci that are no longer of type $V(D)$; instead we see loci of curves satisfying tangency conditions with E. So for $\mathbb{F}_3$, Caporaso and Harris get an inductive formula expressing $N(D)$ in terms of degrees of these tangential Severi varieties. The results of [CH2] are as follows:

Theorem 1.4.2 [CH2] (p.15) *For any effective divisor $D \neq E$ on $\mathbb{F}_2$*

$$
\begin{aligned}
N(D) &= \frac{1}{2} \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times \\
&\qquad \left[(C \cdot D_1)(C \cdot D_2)\binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2\binom{r(D)-3}{r(D_1)-2}\right] + \\
&+ \sum_{D_1+D_2=D-E} N(D_1)N(D_2)(D_1 \cdot E)(D_2 \cdot E) \times \\
&\qquad \left[(D_1 \cdot C)(D_2 \cdot C)\binom{r(D)-3}{r(D_1)-1} - (D_2 \cdot C)^2\binom{r(D)-3}{r(D_1)-2}\right].
\end{aligned}
$$

Theorem 1.4.3 [CH2] (p.23) *For any effective divisor $D \neq E$ on $\mathbb{F}_3$*

$$\begin{aligned}
N(D) &= \frac{1}{3} \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times \\
&\qquad \left[(C \cdot D_1)(C \cdot D_2) \binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2 \binom{r(D)-3}{r(D_1)-2} \right] + \\
&+ \sum_{D_1+D_2=D-E} N_2(D_1)N(D_2)(E \cdot D_2) \times \\
&\qquad \left[(D_1 \cdot C)(D_2 \cdot C) \binom{r(D)-3}{r(D_1)-2} - (D_2 \cdot C)^2 \binom{r(D)-3}{r(D_1)-3} \right] + \\
&+ \sum_{D_1+D_2=D-E} N(D_1)N_2(D_2)(E \cdot D_1) \times \\
&\qquad \left[(D_1 \cdot C)(D_2 \cdot C) \binom{r(D)-3}{r(D_1)-1} - (D_2 \cdot C)^2 \binom{r(D)-3}{r(D_1)-2} \right] + \\
&+ \frac{1}{3} \sum_{D_1+D_2+D_3=D-E} \prod_{i=1}^{3} N(D_i)(D_i \cdot E) \times \\
&\qquad [(2(C \cdot D_1)(C \cdot D_2) + (C \cdot D_1)(C \cdot D_3) + \\
&\qquad + (C \cdot D_2)(C \cdot D_3) - (C \cdot D_3)^2) \binom{r(D)-3}{r(D_1)-1, r(D_2)-1} + \\
&\qquad + \left((C \cdot D_2)^2 + (C \cdot D_3)^2 + (C \cdot D_2)(C \cdot D_3) \right) \binom{r(D)-3}{r(D_1)-2, r(D_2)} \Bigg] .
\end{aligned}$$

These formulas are in the spirit of Kontsevich. The recursion involves degrees of Severi varities of smaller divisors, possibly with tangency conditions. All other components of the calculation are very easily calculated.

1.5 Results of Ravi Vakil

R. Vakil recently computed Severi degrees for divisors of *arbitrary* genus on $\mathbb{F}_n$, not just of rational curves. That is, count the number of curves of genus g through $-K_{\mathbb{F}_n} \cdot D + g - 1$ general points. The results for the irreducible case follow.

Definition 1.5.1 *Let $W^{D,g}(\alpha, \beta, \Gamma)$ be the closure (in $|D|$) of the locus of irreducible curves in S in a divisor class D of geometric genus g, not containing E, with (informally) α_k "assigned" points of contact of order k and β_k "unassigned" points of contact of order k with E and let $N_{irr}^{D,g}(\alpha, \beta)$ be its degree.*

Theorem 1.5.2 [V] (p. 7) *If* $\dim W^{D,g}(\alpha, \beta, \Gamma) > 0$ *then*

$$\begin{aligned} N_{irr}^{D,g}(\alpha, \beta) &= \sum_{\beta_k > 0} k N_{irr}^{D,g}(\alpha + e_k, \beta - e_k) \\ &+ \sum \frac{1}{\alpha} \binom{\alpha}{\alpha^1, ..., \alpha^l, \alpha - \sum \alpha^i} \binom{\Upsilon^{D,g}(\beta) - 1}{\Upsilon^{D^1,g^1}(\beta^1), ..., \Upsilon^{D^l,g^l}(\beta^l)} \\ &\cdot \prod_{i=1}^{l} \binom{\beta^i}{\gamma^i} I^{\beta^i - \gamma^i} N_{irr}^{D^i,g^i}(\alpha^i, \beta^i) \end{aligned}$$

where the second sum runs over choices of $D^i, g^i, \alpha^i, \beta^i, \gamma^i (1 \leq i \leq l)$, where D^i is a divisor class, g^i is a non-negative integers, $\alpha^i, \beta^i, \gamma^i$ are sequences of non-negative integers, $\sum D^i = D - E$, $\sum \gamma^i = \beta$, $\beta^i \geq \gamma^i$, and σ is the number of symmetries of the set $\{(D^i, g^i, \alpha^i, \beta^i, \gamma^i)\}_{1 \leq i \leq l}$.

The formula uses the following definitions and notation. For any sequence $\alpha = (\alpha_1, \alpha_2, ...)$ of non-negative integers with all but finitely many α_i zero, set

$$|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + ...$$

$$I\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + ...$$

$$I^\alpha = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} ...$$

$$\alpha! = \alpha_1! \alpha_2! \alpha_3! ...$$

and

$$\binom{\alpha}{\alpha'} = \binom{\alpha_1}{\alpha'_1} \binom{\alpha_2}{\alpha'_2} \binom{\alpha_3}{\alpha'_3} ....$$

Let e_k be the sequence $(0,...,0,1,0,...,0)$ that is zero except for a 1 in the k^{th} term. Let

$$\Upsilon^{D,g}(\beta) := -(K_S + E)\cdot D + |\beta| + g - 1.$$

So with the "seed data" $N_{irr}^{F,0}(e_1, 0) = 1$, this formula inductively counts irreducible curves of any genus in any divisor class of $\mathbb{F}_n$ by allowing the assigned points of contact to get worse and the unassigned points of contact to get better. While it is indeed a very elegant formula, in practice it is very difficult to compute with, even in the simplest of cases.

1.6 Results of This Paper

Theorem 1.6.1 *Let $D \neq E$ be an effective divisor on $\mathbb{F}_n$. Then*

$$\begin{aligned}
nN(D) = &\sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1\cdot D_2)\times \\
&\qquad \left[(C\cdot D_1)(C\cdot D_2)\binom{r(D)-3}{r(D_1)-1} - (C\cdot D_2)^2\binom{r(D)-3}{r(D_1)-2}\right] + \\
&+\sum_{\substack{D_1+\ldots+D_s=D-E\\ \{D_3,\ldots,D_s\}}} \Delta'\left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\left(\frac{\gamma_1+\gamma_2}{2}\right)(C\cdot D)^2 + \\
&-\sum_{\substack{D_1+\ldots+D_s=D-E\\ \{D_2,\ldots,D_s\}}} \Delta\left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\left[\gamma_1(C\cdot D_1 - C\cdot D)^2 + \sum_{i=2}^{s}\gamma_i(C\cdot D_i)^2\right]
\end{aligned} \tag{1.1}$$

This formula calculates the number of rational curves in *any* class on $\mathbb{F}_n$ in the style of Kontsevich and of Caporaso and Harris. The recursion is in terms of the degrees of Severi varieties of smaller divisors with possible tangency conditions and otherwise involves only intersection products of curves on $\mathbb{F}_n$, which are easily calculated. We also note that the main theorem in the case of $\mathbb{P}^2$ gives a proof for Kontsevich's formula, and in the case of $\mathbb{F}_2$ and $\mathbb{F}_3$ agrees with the formulas of Caporaso and Harris above. Unexpectedly, the theorem gives a simpler proof of the closed formula of Caporaso and Harris for $N(2C)$ on $\mathbb{F}_n$. As a result of the simplicity of the objects involved in the calculation, I have written a Maple program implementing the formula.

1.7 Outline of Approach

Chapter Two describes in detail the method used to calculate these Severi degrees. The approach taken is inspired by the Rational Fibration Method of L. Caporaso and J. Harris in [CH2]. This method builds a surface $\mathcal{Y}$ for a divisor D and a generically finite map $\pi : \mathcal{Y} \to \mathbb{F}_n$ whose degree is the Severi degree $N(D)$. In order to calculate the degree of π, we must first determine the Néron-Severi group of $\mathcal{Y}$. To this end, the remaining part of Chapter Two will address issues necessary to fully describe the Néron-Severi group of $\mathcal{Y}$.

The third chapter begins with a statement of the main theorem and then proceeds with its proof. Chapter Four uses this theorem to calculate some Severi degrees in the plane as well as on $\mathbb{F}_2$. Chapter Four also gives a different proof for a closed formula for the degree of the Severi variety for $2C$ on $\mathbb{F}_n$ using the main theorem.

Chapters Five and Six apply the Rational Fibration Method to the tangential Severi varieties $V_m(D)$ with the goal of writing an explicit formula for $N_m(D)$. Specifically, Chapter Five classifies and describes the reducible fibers of $\mathcal{Y} \to B$ so that we might write down the Néron-Severi group of $\mathcal{Y}$. This is the most delicate issue in the construction. Chapter Six states and proves the theorem giving the explicit formula for $N_m(D)$. We finish by giving in Chapter Seven some examples using the formula for $N_m(D)$.

Chapter 2

Methods and Techniques

The Rational Fibration method of [CH2] builds a surface $\mathcal{Y}$ and a map π such that the degree of π is the Severi degree $N(D)$. This chapter will describe this method in detail and explain how the Severi degree appears as a result of its construction. We then use a proposition of Caporaso and Harris in [CH1] to fully describe the Néron-Severi group of $\mathcal{Y}$.

2.1 The Rational Fibration Method

We begin by briefly summarizing the Rational Fibration Method. The construction begins by taking a linear section Γ of $V(D)$, so Γ parametrizes the irreducible rational curves in $|D|$ passing through $r(D)-1$ general points. Let χ be the universal family of curves corresponding to Γ and let $f : \chi \to \Gamma$ be its projection onto Γ. Assume for the moment that χ is a smooth surface and that Γ is a smooth curve. Let π be the inclusion followed by projection, $\pi : \chi \hookrightarrow \Gamma \times \mathbb{F}_n \to \mathbb{F}_n$.

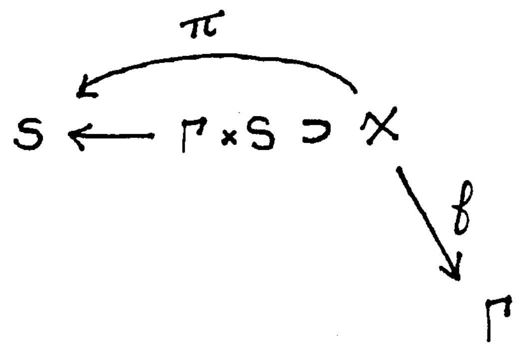

Figure 2.1: General Construction.

By carefully considering the degree of π we will see that $\deg \pi = N(D)$. We now describe this construction and conclusion more precisely.

Let $S = \mathbb{F}_n$ and let D be an effective divisor on S with nonnegative self-intersection. Choose $r(D)-1$ general points $q_1, ..., q_{r(D)-1} \in S$ and let Γ be the closure in $|D|$ of the set of irreducible rational curves passing through these points. If H_{q_i} is the hyperplane in $|D|$ parametrizing curves through q_i then

$$\Gamma = V(D) \cap H_{q_1} \cap ... \cap H_{q_{r(D)-1}},$$

a linear section of $V(D)$. We note that $V = V(D)$ is non-singular off ∂V so by Bertini's Theoreom $V \cap H_{q_1}$ is non-singular off $\partial V \cap H_{q_1}$. Therefore the singular locus of Γ is contained in $\partial\Gamma$, which parametrizes the reducible fibers. Let $\chi \subset \Gamma \times S$ be the universal family over Γ, i.e. the subscheme of $\Gamma \times S$ whose fiber over each $[X] \in \Gamma$

is X (the family of curves corresponding to $\Gamma \subset |D|$). A general fiber of $\chi \to \Gamma$ is an irreducible nodal rational curve, so there are a finite number of special fibers which are reducible, possibly have tangency conditions with E, and at worst have nodes away from E (Prop 2.1 of [CH1]). We would like to build a family from $\chi \to \Gamma$ whose general fiber is the normalization of its corresponding fiber in $\chi \to \Gamma$. So we do a series of normalizations. Normalizing Γ gives $\Gamma^\nu \to \Gamma$. Γ^ν is a smooth curve. Then take the normalization χ^ν of $\chi \times_\Gamma \Gamma^\nu$ to give $\chi^\nu \to \Gamma^\nu$ with general fiber isomorphic to $\mathbb{P}^1$. If we represent χ by

$$\chi = \{(\gamma, D_\gamma) | \gamma \in \Gamma, D_\gamma \text{ curve on } S\}$$

then

$$\begin{aligned} \chi \times_\Gamma \Gamma^\nu &= \{(\gamma, D_\gamma, \tilde{\gamma}) | \gamma \in \Gamma, D_\gamma \text{ curve on } S, \tilde{\gamma} \in \Gamma^\nu \text{ and } \nu(\tilde{\gamma}) = \gamma\} \\ &= \{(D_\gamma, \tilde{\gamma}) | \nu(\tilde{\gamma}) = \gamma\}. \end{aligned}$$

If we let X^ν denote a fiber of $\chi^\nu \to \Gamma^\nu$, then X^ν may differ from the normalization of X. We can think of χ as having a locus of assigned nodes and χ^ν is the normalization of each fiber only at these assigned nodes. Finally we apply a semi-stable reduction by making a base change $B \to \Gamma^\nu$ and blowing up the total space of the pullback family $\chi^\nu \times_{\Gamma^\nu} B$. This gives a family $\mathcal{Y} \to B$ whose total space is smooth, whose general fiber is a smooth rational curve, and whose special fibers are all nodal curves. We will denote the composite map by $\pi : \mathcal{Y} \to S$.

$$\pi : \mathcal{Y} \to \chi^\nu \times_{\Gamma^\nu} B \to \chi^\nu \to \chi \hookrightarrow \Gamma \times S \to S$$

Note that π is a generically finite map. Recalling that Γ parametrizes all curves

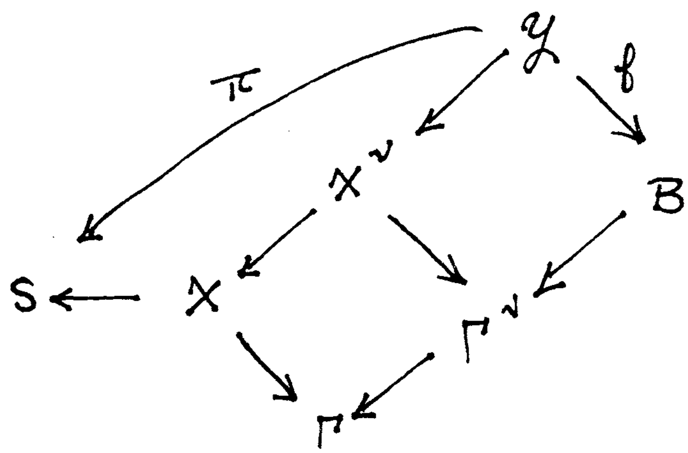

Figure 2.2: Construction of the Surface $\mathcal{Y}$.

passing through the $r(D)-1$ general points, we calculate the degree of π. To calculate the degree of π we consider for arbitrary $s \in S$ the following set:

$$\pi^{-1}(s) = \{([X], s) | [X] \in \Gamma\}.$$

That is, we consider the set of all curves parametrized by Γ passing through s for $s \in S$. Therefore the degree of π is equal to the number of irreducible rational curves in $|D|$ passing through $q_1, ..., q_{r(D)-1}$ and s, i.e. $\deg \pi = N(D)$.

2.2 Classification of Reducible Fibers of $\chi \to \Gamma$

As a ruled surface, the Picard group of $\mathcal{Y}$ is freely generated by the class of a fiber of the ruling, the class of a section of the ruling and the classes of all the irreducible curves contained in fibers of the ruling and disjoint from the section. The Proposition below is a restatement of Proposition 2.5 of [CH1]. It classifies the reducible fibers of $\chi \to \Gamma$. Note: the construction of [CH1] involves the same $\chi \to \Gamma$ as the rational fibration construction. The ideas in this Proposition will be used to describe the reducible fibers of $\mathcal{Y} \to B$ so that we can write down the Néron-Severi group of $\mathcal{Y}$.

Proposition 2.2.1 (Proposition 2.5 of [CH1] on p.28) *Let $X \subset S = \mathbb{F}_n$ be any reducible fiber of the family $\chi \to \Gamma$.*

1. *If X does not contain E, then X has exactly two irreducible components X_1 and X_2, with $[X_i] \in V(D_i)$ and $D_1 + D_2 = D$. Moreover, each $[X_i]$ is a general point in $V(D_i)$.*

2. *If X does contain E, then X has irreducible components $E, X_1, ..., X_s$, with $[X_i] \in V(D_i)$ and $E+D_1+...+D_s = D$. Moreover, each X_i is general in $V_{m_i}(D_i)$ for some collection $m_1, ..., m_s$ of positive integers such that $\sum_{i=1}^{s} m_i = n$.*

Notation. If X is any reducible fiber of the family $\chi \to \Gamma$ not containing E, we call its corresponding fibers of $f : \mathcal{Y} \to B$ type **J fibers**. If X is any reducible fiber of the family $\chi \to \Gamma$ containing E, we call its corresponding fibers of $f : \mathcal{Y} \to B$ type **K fibers**.

2.3 Classification of Reducible Fibers of $f : \mathcal{Y} \to B$

Lemma 2.3.1 Type J *fibers have two smooth irreducible components, J_1 and J_2. J_1 and J_2 meet transversally at one point, such that $\pi(J_i) = D_i$, $D_i > 0$, and $D_i \neq E$.*

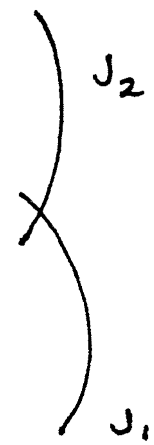

Figure 2.3: Type J Reducible Fibers of $f : \mathcal{Y} \to B$

Notation. By convention we denote the component of a type J fiber containing q_1 by J_1. Let $j(D_1, D_2)$ denote the number of fibers of type J such that $\pi(J_i) = D_i$ and $D_1 + D_2 = D$. And let B_J be the set of points $b \in B$ such that the fiber X_b over b is a fiber of type J.

Note: The type J fibers are derived directly from Propositions 2.6 and 2.7 of [CH1] on pages 33 and 37. We do not reprove the results here.

The type K fibers described in the Lemma below are a generalization of Propositions 2.6 and 2.7 of [CH1]. The results here apply the ideas of the Propositions to a more general object. Caporaso and Harris allow only one component of the decomposition to meet E at a smooth point with multiplicity greater than one. We allow each component D_i to meets E at a smooth point of multiplicity m_i where $m_i \geq 1$.

Lemma 2.3.2 **Type K** *fibers have irreducible components* $K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i-1}, K_i$ *with* $i = 1, ..., s$ *such that* $\pi(K_i) = D_i$, $\pi(K_E) = E$, $D_i > 0$, *and* $D_i \neq E$. $K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i-1}, K_i$ *form a chain in the given order, i.e.*

$$K_E \cdot K_{i,1} = K_{i,1} \cdot K_{i,2} = K_{i,2} \cdot K_{i,3} = ... = K_{i,\gamma_i-2} \cdot K_{i,\gamma_i-1} = K_{i,\gamma_i-1} \cdot K_i = 1,$$

and no other intersections.

Notation. For the type K fibers: let m_i be the multiplicity with which D_i meets E at a smooth point and γ_i be $\frac{k}{m_i}$, where we assume $k = \text{lcm } (m_1, m_2, ..., m_s)$. By convention we denote the component containing q_1 by K_1. Let $k(D_1, D_2, ..., D_s)$ denote the number of fibers of type K such that $\pi(K_i) = D_i$, $\pi(K_E) = E$, and $\sum_{i=1}^{s} D_i = D - E$. And let B_K be the set of points $b \in B$ such that the fiber X_b over b is a fiber of type K.

Definition 2.3.3 *If P is a limit of nodes of fibers of $\chi \to \Gamma$ near X in the chosen branch – that is, if (P, b) is in the closure of the singular locus of the map $\chi \times_\Gamma (\Gamma^\nu - \{b\}) \to \Gamma^\nu$ - we will say that P is an old node of X. If (P, b) is an isolated singular point of the map $\chi \times_\Gamma (\Gamma^\nu - \{b\}) \to \Gamma^\nu$ we will say that P is a new node of X.*

Proof. (of Lemma 2.3.2) We describe the fibers of type K of $\mathcal{Y} \to B$ in two parts. We begin by analyzing the local geometry of Γ around a point corresponding to a fiber of type K. Then we analyze the singularities of the total space of the normalized family $\chi^\nu \to \Gamma^\nu$ along the fiber corresponding to the fiber of type K.

Fix a point $[X] \in \Gamma$ such that X is a reducible fiber of $\chi \to \Gamma$ containing E as a component. By Proposition 2.2.1, X must be of the form $X = E \cup X_1 \cup ... \cup X_s$ where

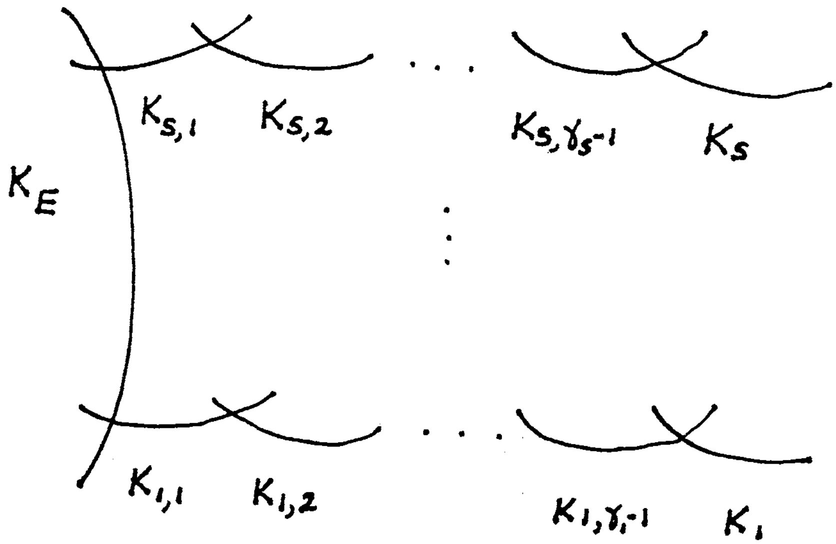

Figure 2.4: Type K Reducible Fibers of $f : \mathcal{Y} \to B$

X_i is a general member of the family $V_{m_i}(D_i)$ for some collection of positive integers $m_1, ..., m_s$ such that $\sum_i m_i = n$. Since X_i is an irreducible rational curve with $p_a(D_i)$ where

$$p_a(D_i) = 1 + \frac{1}{2}(D_i^2 + D_i \cdot K_S)$$

then the total number of nodes on X will be

$$
\begin{aligned}
\sum_{i=1}^{s} p_a(D_i) \quad &+ \quad \sum_{\substack{i,j \\ i \neq j}} D_i \cdot D_j + \sum_{i=1}^{s} (D_i \cdot E - m_i) = \\
&= \sum_{i=1}^{s} \left(1 + \frac{1}{2}(D_i^2 + D_i \cdot K_S)\right) + \sum_{\substack{i,j \\ i \neq j}} D_i \cdot D_j + \sum_{i=1}^{s} (D_i \cdot E - m_i) \\
&= s + \frac{1}{2}\left(\sum_{i=1}^{s} D_i^2 + \sum_{i=1}^{s} D_i \cdot K_S\right) + \sum_{\substack{i,j \\ i \neq j}} D_i \cdot D_j + \sum_{i=1}^{s} (D_i \cdot E - m_i) \\
&= s + 1 + \frac{1}{2}\left(-n + \sum_{i=1}^{s} D_i^2 + (n-2) + \sum_{i=1}^{s} D_i \cdot K_s\right) + \\
&\qquad + \sum_{\substack{i,j \\ i \neq j}} D_i \cdot D_j + \sum_{i=1}^{s} (D_i \cdot E - m_i) \\
&= s + 1 + \frac{1}{2}\left(E^2 + \sum_{i=1}^{s} D_i^2 + E \cdot K_S + \sum_{i=1}^{s} D_i \cdot K_S\right) + \\
&\qquad + \sum_{\substack{i,j \\ i \neq j}} D_i \cdot D_j + \sum_{i=1}^{s} (D_i \cdot E - m_i) \\
&= s + p_a(D) - \sum_{i=1}^{s} m_i.
\end{aligned}
$$

Thus X has $s + p_a(D) - \sum_{i=1}^{s} m_i$ nodes and s tacnodes of order $m_1, ..., m_s$. Then as in [CH1] we see that in the normalization of the total space of the family, the fiber corresponding to X will consist of a curve $\tilde{E}$ mapping to E, plus the normalizations $\tilde{X}_i$ of the curves X_i, each meeting E at one point and disjoint from each other. All the nodes of X arising from points of pairwise intersection of the X_i are old. If X_i has a point of contact order $m_i > 1$ with E, that must be the image of the point $\tilde{X}_i \cap \tilde{E} \in \chi^\nu$; all the other points of $X_i \cap E$ will be old nodes of X. If X_i intersects E transversely, then any one of its points $X_i \cap E$ can be a new node. This completes the first part of our analysis.

Now we analyze the singularities of the total space of the normalized family $\chi^\nu \to \Gamma^\nu$ along the fiber X^ν. First we introduce some notation.

Notation. We will denote by $P_1^i, ..., P_{l_i}^i$, $1 \leq i \leq s$, the new nodes of X along E coming from components of X meeting E only transversely; and by P^i, $1 \leq i \leq s$, the double points of X other than nodes, coming from a point of contact order $m_i \geq 2$ of E with another component of X, if any. We recall that the nearby fibers of our family are smooth near P_j^i, and that there will be one point p_j^i of χ_j^i lying over each P_j^i, which will be a node of X^ν, while the nearby fibers have $m_i - 1$ nodes tending to the point P^i; thus the normalization $X^\nu \to X$ will again have one point p^i lying over P^i, and that point will be a node of X^ν.

Consider the family $\chi^\nu \to \Gamma^\nu$ in a neighborhood of the whole fiber X^ν, the fiber corresponding to X. Recall that $\mathcal{Y}$ is the minimal desingularization of χ^ν. X^ν has only nodes as singularities so χ^ν will have a singularity of type A_n at each node. Suppose that $p^i \in X^\nu$ is an $A_{\gamma_i - 1}$ singularity, for some γ_i. Resolving p^i gives a chain of $\gamma_i - 1$ smooth rational curves with self-intersection -2 in $\mathcal{Y}$. We will denote the component of X meeting E at P^i by K_i. Now consider the pull-back of E from S to $\mathcal{Y}$ by π:

$$\pi^* E = k\tilde{E} + \sum_{i=1}^{s} \left(a_i K_i + \sum_{j=1}^{\gamma_i - 1} a_{i,j} K_{i,j} \right) + E'$$

where $k \in \mathbb{Z}_+$ and $\tilde{E}$ is the proper transform of E and E' is a curve in $\mathcal{Y}$ meeting the fiber only at the K_i, with

$$E' \cdot K_i = E \cdot \pi(K_i) - m_i.$$

Since π maps $K_{i,j}$ to points in S, then $\deg_{K_{i,j}}(\pi^* E) = 0$ and

$$\begin{aligned} \pi^* E \cdot K_{i,j} &= a_{i,j-1} K_{i,j-1} \cdot K_{i,j} + a_{i,j} K_{i,j}^2 + a_{i,j+1} K_{i,j+1} \cdot K_{i,j} \\ 0 &= a_{i,j-1} - 2a_{i,j} + a_{i,j+1} \end{aligned}$$

setting $a_{i,\gamma_i} = 0$ and $a_{i,0} = k$.

On the other hand, π restricted to K_i meets E at $P^i = \pi(p^i)$ with multiplicity m_i so the multiplicity at p^i of the restriction to K_i of the divisor $\pi^*(E) - E'$ is m_i implying that $a_{i,\gamma_i - 1} = m_i$: $m_i = (\pi^* E - E') \cdot K_i = a_{i,\gamma_i-1} K_{i,\gamma_i-1} \cdot K_i$, so $a_{i,\gamma_i-1} = m_i$. Similarly $a_{i,\gamma_i-2} - 2a_{i,\gamma_i-1} + a_{i,\gamma_i} = 0$, so $a_{i,\gamma_i-2} = 2m_i$ and $a_{i,\gamma_i-3} - 2a_{i,\gamma_i-2} + a_{i,\gamma_i-1} = 0$, so $a_{i,\gamma_i-3} = 3m_i$. Continuing $a_{i,0} - 2a_{i,1} + a_{i,2} = 0$, so $a_{i,0} - 2(\gamma_i - 1)m_i + (\gamma_i - 2)m_i = 0$ and therefore $a_{i,0} - \gamma_i m_i = 0$; finally $k = a_{i,0} = \gamma_i m_i$. Therefore $p_i \in \chi^\nu$ is a singularity of type $A_{\gamma_i - 1}$, where $\gamma_i = \dfrac{k}{m_i}$. Clearly $\mathrm{lcm}(m_1, m_2, ..., m_s)|k$. This completes our description of the fibers of type K. □

2.4 Néron-Severi Group of $\mathcal{Y}$

Now we address the main goal of this chapter. Since $\mathcal{Y}$ is a ruled surface, the Néron-Severi group of $\mathcal{Y}$ is freely generated by the class of a fiber of the ruling, the class of a section of the ruling, and the classes of all the irreducible curves contained in fibers of the ruling and disjoint from the section. Let Y be the class of a fiber of $\mathcal{Y}$ and A correspond to a section of $f : \mathcal{Y} \to B$ parametrizing curves through the base point q_1. We choose the following as a basis for the Néron-Severi group of $\mathcal{Y}$:

$$\{A, Y\} \cup \{J_2\}_{b \in B_J} \cup \{K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i - 1}, K_i\}_{b \in B_K, i=1,...,s} - \{K_1\}.$$

Note: One can readily see that $J_1 = Y - J_2$ and likewise for the type K fibers.

The below relations follow easily:

$$A \cdot Y = 1, \quad Y^2 = 0, \quad J_2^2 = -1,$$

$$K_E^2 = -s, \quad K_{i,j}^2 = -2, \quad K_i^2 = -1$$

$$K_E \cdot K_{i,1} = 1, \quad K_{i,j} \cdot K_{i,j+1} = 1, \quad K_{i,\gamma-1} \cdot K_i = 1.$$

Other than these and A^2, there are no additional non-zero intersections. The calculation of A^2 is a delicate one; we compute it in the next chapter.

2.5 Counting Reducible Fibers of $f : \mathcal{Y} \to B$

Taking into account the results of the above classification of fibers of $\mathcal{Y} \to B$, we count the reducible fibers of type J and type K on $\mathcal{Y}$. This count will be used in the calculation of $N(D)$.

Lemma 2.5.1 *1. If X is a reducible fiber of $\mathcal{Y} \to B$ not containing E, then the number of type J fibers for a given decomposition $D = D_1 + D_2$, denoted $j(D_1, D_2)$, is*

$$\binom{r(D) - 2}{r(D_1) - 1} N(D_1)N(D_2)(D_1 \cdot D_2).$$

2. If X is a reducible fiber of $\mathcal{Y} \to B$ containing E, then the number of type K fibers for a given decomposition $D = D_1 + ... + D_s$, denoted $k(D_1, D_2, ..., D_s)$, is

$$\Delta \prod_{i=1}^{s} N(D_i)\Lambda(D_i),$$

where

$$\Delta = \frac{1}{R}\binom{r(D)-2}{r_{m_1}(D_1)-1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})},$$

$$\Lambda(D_i) = \begin{cases} E\cdot D_i & m_i = 1 \\ 1 & m_i \geq 2 \end{cases},$$

and R represents the repetition factor accounting for repetition of the components in the set $\{D_2, ..., D_s\}$.

Proof. (*Part 1.*) If X is a reducible fiber of $\chi \to \Gamma$ not containing E as a component, then X must contain two components X_1 and X_2 meeting transversely at one point such that $\pi(X_i) = D_i$, $D_i > 0$, $D_i \neq E$, and $D = D_1 + D_2$.

Since D must pass through $r(D) - 1$ general points, each X_i can hold at most $r(D_i)$ of these $r(D) - 1$ general points. Since $D = D_1 + D_2$,

$$\begin{aligned} r(D) - 1 &= (-K_S \cdot D - 1) - 1 \\ &= -K_S \cdot (D_1 + D_2) - 2 \\ &= -K_S \cdot D_1 - 1 - K_S \cdot D_2 - 1 \\ &= r(D_1) + r(D_2). \end{aligned}$$

It follows that X_i must contain exactly $r(D_i)$ points. Recall that by convention the point q_1 lies on X_1, so there are $r(D) - 2$ choose $r(D_1) - 1$ ways to distribute the $r(D) - 1$ points on the two curves. For each such distribution of points there exist $N(D_i)$ curves $X_i \in V(D_i)$ containing the $r(D_i)$ points. So there are

$$\binom{r(D)-2}{r(D_1)-1} N(D_1)N(D_2)$$

such $[X] \in \Gamma$. For each $[X] \in \Gamma$, Γ has $D_1 \cdot D_2$ smooth branches (Proposition 2.6 in [CH1]). So there will be $D_1 \cdot D_2$ points of Γ^ν lying over each $[X]$. Finally we note that χ^ν is smooth along such fibers (Proposition 2.7 in [CH1] on p.37). Therefore

$$j(D_1, D_2) = \binom{r(D) - 2}{r(D_1) - 1} N(D_1)N(D_2)(D_1 \cdot D_2).$$

(*Part 2.*) If X is a reducible fiber of $\chi \to \Gamma$ containing E as a component, then X has irreducible components

$$\{K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i - 1}, K_i\}$$

with $i = 1, ..., s$ such that $\pi(K_i) = D_i$, $\pi(K_E) = E$, $D_i > 0$, and $D_i \neq E$. For each i let m_i be the multiplicity with which D_i meets E at a smooth point.

Since D must pass through $r(D) - 1$ general points, each X_i can contain at most $r_{m_i}(D_i)$ of the $r(D) - 1$ general points $q_1, ..., q_{r(D)-1}$. Since $D = D_1 + ... + D_s + E$ and $\sum_{i=1}^{s} m_i = n$,

$$\begin{aligned}
r(D) - 1 &= (-K_S \cdot D - 1) - 1 \\
&= -K_S \cdot (D_1 + ... + D_s + E) - 2 \\
&= -K_S \cdot D_1 - K_S \cdot D_2 - ... - K_S \cdot D_s - K_S \cdot E - 2 \\
&= -K_S \cdot D_1 - ... - K_S \cdot D_s - n + 2 - 2 \\
&= \sum_{i=1}^{s} -K_S \cdot D_i - n \\
&= \sum_{i=1}^{s} -K_S \cdot D_i - \sum_{i=1}^{s} m_i \\
&= \sum_{i=1}^{s} (-K_S \cdot D_i - m_i) \\
&= \sum_{i=1}^{s} r_{m_i}(D_i).
\end{aligned}$$

So it follows that each X_i must contain exactly $r_{m_i}(D_i)$ points. Recalling that the point q_1 lies on X_1, then there are

$$\binom{r(D)-2}{r_{m_1}(D_1)-1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})}$$

ways to distribute the $r(D)-1$ points on the s curves. For each distribution of points there exist $N_{m_i}(D_i)$ curves $X_i \in V_{m_i}(D_i)$ containing the $r_{m_i}(D_i)$ points. Thus there are

$$\binom{r(D)-2}{r_{m_1}(D_1)-1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})} \prod_{i=1}^{s} N_{m_i}(D_i)$$

such $[X] \in \Gamma$.

By Lemma 2.3.1 in a neighborhood of $[X] \in \Gamma$, Γ consists of $\prod_{\{m_i=1\}}(D_i \cdot E)$ smooth branches, Γ_α (where $\alpha = (\alpha_1, ..., \alpha_s)$ with α_i removed if $m_i \neq 1$), and, for all i such that D_i has a point P^i of intersection multiplicity $m_i \geq 2$ with E, exactly $m_i - 1$ nodes of nearby fibers will tend to P^i. Along the smooth branches Γ_α, each point P_{i,α_i} has a single point lying over it which will be a node of the fiber X^ν of $\chi^\nu \to \Gamma^\nu$ corresponding to $[X] \in \Gamma$. The fibers X^ν of $\chi^\nu \to \Gamma^\nu$ corresponding to $[X] \in \Gamma$ are all the curves obtained by normalizing X at all the nodes of the D_i, at all but one of the points of intersection of E with each of the components D_i with $m_i = 1$, and at all the transverse points of intersection of D_i with E for $m_i \geq 2$; finally then taking the partial normalization of X at P_i having an ordinary node over p_i. Therefore we are able to conclude that

$$k(D_1, ..., D_s) = \Delta \prod_{i=1}^{s} N(D_i)\Lambda(D_i),$$

where

$$\Delta = \frac{1}{R}\binom{r(D)-2}{r_{m_1}(D_1)-1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})},$$

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2 \end{cases},$$

and R represents the repetition factor accounting for repetition of the components in the set $\{D_2, ..., D_s\}$. Note: D_1 is distinguished since by convention q_1 lies on D_1.

□

Chapter 3

The general recursion for $\mathbb{F}_n$

We will now prove the main theorem. The proof is motivated by the following fact: given any two line bundles L and M on S, we have

$$\pi^*L \cdot \pi^*M = \deg \pi(L \cdot M) = N(D)(L \cdot M).$$

We begin by proving some useful lemmas. In particular, we write π^*L as a linear combination of the elements of the Néron-Severi group of $\mathcal{Y}$, and we calculate A^2. Once we have done this, we will have all the necessary information to enable us to calculate $\pi^*L \cdot \pi^*M$ for any line bundles L and M on $S = \mathbb{F}_n$.

3.1 Theorem

We recall the necessary facts and definitions related to the type K fibers. A reducible fiber of the family $\chi \to \Gamma$ has irreducible components $E, X_1, X_2, ..., X_s$ with $D = E + D_1 + D_2 + ... + D_s$, X_i is general in $V_{m_i}(D_i)$ for a collection of positive integers $m_1, m_2, ..., m_s$ such that $\sum_{i=1}^{s} m_i = n$.

(Note: $\dim V_{m_i}(D_i) = r_{m_i}(D_i) = -K_S \cdot D_i - m_i$.) The corresponding components X_i on χ^ν have singularities of type $A_{\gamma_i - 1}$ where $\gamma_i = \frac{k}{m_i}$ and we assume for computational purposes that $k = \mathrm{lcm}(m_1, m_2, ..., m_s)$. Related to the number of type K fibers for a particular decomposition we have:

$$\Delta = \frac{1}{R}\binom{r(D)-2}{r_{m_1}(D_1)-1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})}$$

where R represents the repetition factor accounting for repetition of components in the set $\{D_2, D_3, ..., D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2. \end{cases}$$

The calculation of A^2, to be shown later, involves choosing a section A' disjoint from A. As a result we see a corresponding definition for Δ' describing how the remaining $r(D) - 3$ points (not counting q_1 and q_2) can be distributed on the s curves:

$$\Delta' = \frac{1}{R'}\binom{r(D)-3}{r_{m_1}(D_1)-1, r_{m_2}(D_2)-1, r_{m_3}(D_3), r_{m_4}(D_4), ..., r_{m_{s-1}}(D_{s-1})},$$

where R' represents the repetition factor accounting for repetition of components in the set $\{D_3, ..., D_s\}$. These are the ingredients in the following theorem.

Theorem 3.1.1 *Let $D \neq E$ be an effective divisor on $\mathbb{F}_n$. Then*

$$\begin{aligned} nN(D) = &\sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times \\ &\qquad \left[(C \cdot D_1)(C \cdot D_2)\binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2\binom{r(D)-3}{r(D_1)-2}\right] + \\ &+ \sum_{\substack{D_1+...+D_s=D-E \\ \{D_3,...,D_s\}}} \Delta' \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\left(\frac{\gamma_1+\gamma_2}{2}\right)(C \cdot D)^2 + \\ &- \sum_{\substack{D_1+...+D_s=D-E \\ \{D_2,...,D_s\}}} \Delta \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\left[\gamma_1(C \cdot D_1 - C \cdot D)^2 + \sum_{i=2}^{s} \gamma_i(C \cdot D_i)^2\right] \end{aligned} \tag{3.1}$$

We make a few notes regarding the use of the formula. For a given effective divisor D, there are two types of decompositions: type J and type K. For the type J decompositions, D decomposes into 2 components: D_1 and D_2, each effective divisors. These are the decompositions which are allowable in the first sum, clearly a finite sum. In this sum symmetric decompositions are included when $D_1 \neq D_2$.

The remaining two sums determine the contributions coming from the type K fibers. These are the decompositions which contain E as a component as described above such that a corresponding reducible fiber of $\chi \to \Gamma$ has irreducible components $E, X_1, X_2, ..., X_s$ with $D - E = D_1 + D_2 + ... + D_s$, X_i is general in $V_{m_i}(D_i)$ for a collection of positive integers $m_1, m_2, ..., m_s$ such that $\sum_{i=1}^{s} m_i = n$. We note here that in particular we see that $s \leq n$, so again we see that the sums are finite.

The middle summation is related to A^2, which as we will see later, requires that D_1 and D_2 be distinguished. So for a given decomposition of this type, this sum will accept only the permutations in which D_1 and D_2 are distinguished. For example, suppose $D - E = \tilde{D} + F + F + F$ where $\tilde{D} \neq F$. Then allowable permutations for this decomposition would be $\tilde{D}+F+F+F$, $F+\tilde{D}+F+F$ and $F+F+\tilde{D}+F$. Note: $F + F + F + \tilde{D}$ is considered the same as the third permutation since they agree in the first two components.

The last sum, again coming from the type K fibers, requires that only D_1 be distinguished. Using the same example, suppose $D - E = \tilde{D} + F + F + F$ where $\tilde{D} \neq F$. Then allowable permutations for this decomposition would be $\tilde{D}+F+F+F$ and $F + \tilde{D} + F + F$. Note: $F + F + \tilde{D} + F$ and $F + F + F + \tilde{D}$ are considered the same as the second permutation since they agree in the first component.

3.2 Proof of Theorem: Some Useful Lemmas

Let L be any line bundle in Pic $\mathbb{F}_n$. We can write the class of its pullback to $\mathcal{Y}$ as a linear combination of the elements of the Néron-Severi group of $\mathcal{Y}$. Since we know the image in $\mathbb{F}_n$ of the components of the reducible fibers of $f : \mathcal{Y} \to B$, we can calculate the degrees on all such components of π^*L of any line bundle.

Take any effective divisor class D on S with nonnegative self-intersection and $V(D) \neq \emptyset$. Choose $r(D) - 1$ general points $q_1, q_2, ..., q_{r(D)-1}$ on S. Consider the family $\chi \to \Gamma$ of curves $X \in V(D)$ passing through the q_i. Let $\chi^\nu \to \Gamma^\nu$, $\mathcal{Y} \to B$, and

$$\pi : \mathcal{Y} \to \chi^\nu \times_{\Gamma^\nu} B \to \chi^\nu \to \chi \hookrightarrow \Gamma \times S \to S$$

be as described in the set-up of the Rational Fibration method in Section 2.1.

Lemma 3.2.1 *For L any line bundle in Pic($\mathbb{F}_n$),*

$$\begin{aligned} \pi^*L &= (L \cdot D)A - (L \cdot D)A^2Y - \sum_{b \in B_J} (L \cdot D_2)J_2 + \\ &+ \sum_{b \in B_K} \left[\gamma_1(L \cdot D_1 - L \cdot D)K_E + \sum_{j=1}^{\gamma_i - 1} (\gamma_1 - j)(L \cdot D_1 - L \cdot D)K_{1,j} + \right. \\ &\left. + \sum_{i=2}^{s} \sum_{j=1}^{\gamma_i} (\gamma_1(L \cdot D_1 - L \cdot D) - jL \cdot D_i)K_{i,j} \right] \end{aligned} \tag{3.2}$$

where B_J and B_K are the subsets of points of B parametrizing fibers of type J and type K respectively.

Proof. Take L any line bundle in Pic($\mathbb{F}_n$). Recall that Pic $\mathcal{Y}$ is generated by a section of the ruling, A, a fiber of the ruling, F, and all the irreducible curves contained in fibers of the ruling and disjoint from the section. So we can write the class of the pullback of L to $\mathcal{Y}$ as a linear combination of

$$\{A, Y\} \cup \{J_2\}_{b\in B_J} \cup \{K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i-1}, K_i\}_{b\in B_K, i=1,\dots,s} - \{K_1\}.$$

We define the coefficient of $\square$ as $a_\square$ in this linear combination allowing us to write the pullback of L as:

$$\pi^* L = a_A A + a_Y Y + J^L + K^L$$

where

$$J^L = \sum_{b\in B_J} a_{J_2} J_2$$

and

$$K^L = \sum_{b\in B_K} \left(a_E K_E + \sum_{j=1}^{\gamma_1 - 1} a_{1,j} K_{1,j} + \sum_{i=2}^{s} \sum_{j=1}^{\gamma_i} a_{i,j} K_{i,j} \right).$$

Note: here $K_{i,\gamma_i} = K_i$. Now we determine the coefficients $a_\square$ in the above expression for $\pi^* L$ by evaluating the following products: $L \cdot D = \pi^* L \cdot Y = a_A A \cdot Y$, so $a_A = L \cdot D$; since π collapses A to the base point q we have $0 = \pi^* L \cdot A = a_A A^2 + a_Y Y \cdot A$, so $a_Y = -(L \cdot D) A^2$; $L \cdot D_2 = \pi^* L \cdot J_2 = a_{J_2} J_2^2$, and so $a_{J_2} = -(L \cdot D_2)$.

We similarly determine the coefficients of the type K fibers. Now $L \cdot D_1 = \pi^* L \cdot K_1 = a_{1,\gamma_1-1} K_{1,\gamma_1-1} \cdot K_1 + (L \cdot D) A \cdot K_1$, so $a_{1,\gamma_1-1} = L \cdot D_1 - L \cdot D$; similarly $0 = \pi^* L \cdot K_{1,\gamma_1-1} = a_{1,\gamma_1-2} K_{1,\gamma_1-2} \cdot K_{1,\gamma_1-1} + a_{1,\gamma_1-1} K_{1,\gamma_1-1}^2$, so $a_{1,\gamma_1-2} = 2(L \cdot D_1 - L \cdot D)$; and $0 = \pi^* L \cdot K_{1,\gamma_1-2} = a_{1,\gamma_1-3} K_{1,\gamma_1-3} \cdot K_{1,\gamma_1-2} + a_{1,\gamma_1-2} K_{1,\gamma_1-2}^2 + a_{1,\gamma_1-1} K_{1,\gamma_1-1} \cdot K_{1,\gamma_1-2}$, so $a_{1,\gamma_1-3} = 3(L \cdot D_1 - L \cdot D)$. Continuing in this manner we are able to write the coefficient of $K_{1,j}$ in general: $a_{1,\gamma_1-j} = j(L \cdot D_1 - L \cdot D)$ so $a_{1,j} = (\gamma_1 - j)(L \cdot D_1 - L \cdot D)$ for all j. We now have enough information to determine the coefficient of K_E:

$$0 = \pi^* L \cdot K_{1,1} = a_E K_E \cdot K_{1,1} + a_{1,1} K_{1,1}^2 + a_{1,2} K_{1,2} \cdot K_{1,1},$$

so

$$a_E = 2a_{1,1} - a_{1,2} = 2(\gamma_1 - 1)(L \cdot D_1 - L \cdot D) - (\gamma_1 - 2)(L \cdot D_1 - L \cdot D)$$

and thus $a_E = \gamma_1(L \cdot D_1 - L \cdot D)$.

For the remainder of this proof we assume $i \neq 1$. Now $L \cdot D_i = \pi^*L \cdot K_i = a_{i,\gamma_i-1}K_{i,\gamma_i-1} \cdot K_i + a_iK_i^2$, so $a_i = a_{i,\gamma_i-1} - L \cdot D_i$; $0 = \pi^*L \cdot K_{i,\gamma_1-1} = a_{i,\gamma_i-2}K_{i,\gamma_i-2} \cdot K_{i,\gamma_i-1} + a_{i,\gamma_i-1}K_{i,\gamma_i-1}^2 + a_iK_i \cdot K_{i,\gamma_i-1}$, so $a_{i,\gamma_i-1} = a_{i,\gamma_i-2} - L \cdot D_i$; continuing gives $a_{i,j+1} = a_{i,j} - L \cdot D_i$. But recall that we also have $a_{i,j-1} - 2a_{i,j} + a_{i,j+1} = 0$ so $0 = \pi^*L \cdot K_{i,1} = a_EK_E \cdot K_{i,1} + a_{i,1}K_{i,1}^2 + a_{i,2}K_{i,2} \cdot K_{i,1}$, so $2a_{i,1} = a_E - a_{i,2} = \gamma_1(L \cdot D_1 - L \cdot D) + a_{i,1} - L \cdot D_i$ so $a_{i,1} = \gamma_1(L \cdot D_1 - L \cdot D) - L \cdot D_i$; Recall: $a_{i,j+1} = a_{i,j} - L \cdot D_i$ So $a_{i,2} = a_{i,1} - L \cdot D_i$, $a_{i,3} = a_{i,2} - L \cdot D_i = a_{i,1} - 2L \cdot D_i$, $a_{i,4} = a_{i,3} - L \cdot D_i = a_{i,1} - 3L \cdot D_i$. Continue, giving $a_{i,j} = a_{i,j-1} - L \cdot D_i = a_{i,1} - (j-1)L \cdot D_i = \gamma_1(L \cdot D_1 - L \cdot D) - jL \cdot D_i$. $a_i = a_{i,\gamma_i-1} - L \cdot D_i = \gamma_1(L \cdot D_1 - L \cdot D) - \gamma_iL \cdot D_i$

And so all the coefficients are as claimed in the lemma. □

Next we compute A^2. To do this we choose a base point $q_2 \neq q_1$ so that q_2 determines a second section A' of $f : \mathcal{Y} \to B$ disjoint from A. Then, by symmetry,

$$2A^2 = (A - A')^2.$$

By writing A' in terms of the Néron-Severi group of $\mathcal{Y}$ we can calculate $(A - A')^2$, allowing us to solve for A^2.

Lemma 3.2.2 *If A corresponds to a section of $f : \mathcal{Y} \to B$ parametrizing curves through q_1 where $f : \mathcal{Y} \to B$ is as described in Section 1.2 then*

$$A^2 = \frac{1}{2}\Bigg(- \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r(D)-3}{r(D_1)-1} + \\ - \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_3,\ldots,D_s\}}} (\gamma_1+\gamma_2)\Delta' \prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\Bigg)$$

where

$$\Delta' = \frac{1}{R'}\binom{r(D)-3}{r_{m_1}(D_1)-1, r_{m_2}(D_2)-1, r_{m_3}(D_3), r_{m_4}(D_4), \ldots, r_{m_{s-1}}(D_{s-1})},$$

R' represents the repetition factor accounting for repetition of components in the set $\{D_3, \ldots, D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2 \end{cases}.$$

Proof. Choose a base point $q_2 \neq q_1$. The point q_2 determines a section A' of $f : \mathcal{Y} \to B$ parametrizing curves through q_2. A and A' are determined by the distinct base points q_1 and q_2 and as such are disjoint. By symmetry $A^2 = (A')^2$ and $A \cdot A' = 0$ so

$$2A^2 = (A - A')^2.$$

Let $S_J \subset B_J$ be the subset of points on B parametrizing reducible fibers of type J for which q_1 and q_2 lie on distinct components. Let $A_J(D_1, D_2)$ denote the number of such fibers, so

$$A_J(D_1, D_2) = N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r(D)-3}{r(D_1)-1}.$$

This follows from the proof for $j(D_1, D_2)$ noting that q_2 lies on J_2. Define S_K similarly for fibers of type K in which q_1 and q_2 lie on different components. Let

$A_K(D_1, D_2, ..., D_s)$ denote the number of such fibers of type K, so

$$A_K(D_1, D_2, ..., D_s) = \Delta' \prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)$$

where

$$\Delta' = \frac{1}{R'}\binom{r(D)-3}{r_{m_1}(D_1)-1, r_{m_2}(D_2)-1, r_{m_3}(D_3), r_{m_4}(D_4), ..., r_{m_{s-1}}(D_{s-1})}$$

and R' represents the repetition factor accounting for repetition of components in the set $\{D_3, ..., D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2. \end{cases}$$

This follows from the proof for $k(D_1, ..., D_s)$ noting that q_2 lies on D_2.

Now we determine the coefficients of $A' - A$. For the type J fibers, let σ_J be the blowdown of J_2. Let $\bar{A} := \sigma_J(A)$, and $\bar{A}' := \sigma_J(A')$. By standard properties of blowdowns,

$$Y \equiv J_1 + J_2, \quad A = \sigma_J^*(\bar{A}), \quad A' = \sigma_J^*(\bar{A}') - J_2, \quad \text{and} \quad \sigma_J^*(\bar{A}' - \bar{A}) = lY,$$

for some l. It follows that in terms of the type J fibers

$$A' - A = lY - J_2 = (l-1)Y + J_1 + J_2 - J_2 = (l-1)Y + J_1.$$

For the type K fibers, let σ_K be the blowdown of $K_i, K_{i,\gamma_i-1}, ..., K_{i,2}, K_{i,1}$ in the listed order beginning with $i = s$ down to $i = 2$, then blow down $K_E, K_{1,1}, ..., K_{1,\gamma_1-1}$. Let $\bar{A} := \sigma_K(A)$ and $\bar{A}' := \sigma_K(A')$. By standard properties of blowdowns,

$Y \equiv K_E + \sum_{i=1}^{s}\left(K_i + \sum_{j=1}^{\gamma_i-1} K_{i,j}\right),$

$A = \sigma_K^*(\bar{A})$, and

$$\begin{aligned} A' &= \sigma_K^*(\bar{A}') - K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1 - 1)K_{1,1} - \gamma_1 K_E + \\ &\quad -\gamma_1 \sum_{i\geq 3} \left(\sum_{j=1}^{\gamma_i - 1} K_{i,j} + K_i \right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\ &\quad -(\gamma_1+\gamma_2-1)K_{2,\gamma_2-1} - (\gamma_1+\gamma_2)K_2. \end{aligned}$$

Now we know $\sigma_K^*(\bar{A}' - \bar{A}) = lY$. So

$$\begin{aligned} A' - A &= lY - K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1 - 1)K_{1,1} - \gamma_1 K_E + \\ &\quad -\gamma_1 \sum_{i\geq 3} \left(\sum_{j=1}^{\gamma_i - 1} K_{i,j} + K_i \right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\ &\quad -(\gamma_1+\gamma_2-1)K_{2,\gamma_2-1} - (\gamma_1+\gamma_2)K_2 \\ &= (l-\gamma_1)Y + \gamma_1 Y - K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1 - 1)K_{1,1} - \gamma_1 K_E + \\ &\quad -\gamma_1 \sum_{i\geq 3} \left(\sum_{j=1}^{\gamma_i - 1} K_{i,j} + K_i \right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\ &\quad -(\gamma_1+\gamma_2-1)K_{2,\gamma_2-1} - (\gamma_1+\gamma_2)K_2 \\ &= (l-\gamma_1)Y + \gamma_1 \left(K_E + \sum_{i=1}^{s} \left(K_i + \sum_{j=1}^{\gamma_i - 1} K_{i,j} \right) \right) + \\ &\quad -K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1 - 1)K_{1,1} - \gamma_1 K_E + \\ &\quad -\gamma_1 \sum_{i\geq 3} \left(\sum_{j=1}^{\gamma_i - 1} K_{i,j} + K_i \right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\ &= (l-\gamma_1)Y + \Big((\gamma_1)K_1 + K_{1,1} + 2K_{1,2} + \dots + +(\gamma_1 - 1)K_{1,\gamma_1-1} \Big) + \\ &\qquad\qquad -\Big((\gamma_2)K_2 + K_{2,1} + 2K_{2,2} + \dots + +(\gamma_2 - 1)K_{1,\gamma_2-1} \Big) \end{aligned}$$

Let $\kappa_i = K_{i,1} + 2K_{i,2} + \dots + (\gamma_i - 1)K_{i,\gamma_i - 1} + (\gamma_i)K_i$. Let σ blow down all J_2's and all components of the type K fibers except K_1 as above. Then arguing as before we have:

$$A' - A = mY + \sum_{b\in S_J} J_1 + \sum_{b\in S_K} (\kappa_1 - \kappa_2)$$

and so

$$
\begin{aligned}
2A^2 = (A' - A)^2 &= m^2Y^2 + \sum_{b\in S_J} J_1^2 + \sum_{b\in S_K}(\kappa_1 - \kappa_2)^2 \\
&= \sum_{b\in S_J}(-1) - \sum_{b\in S_K}(\gamma_1 + \gamma_2).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
A^2 &= \frac{1}{2}\left(\sum_{b\in S_J}(-1) - \sum_{b\in S_K}(\gamma_1+\gamma_2)\right) \\
&= \frac{1}{2}\left(-\sum_{D_1+D_2=D} A_J(D_1,D_2) - \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_3,\ldots,D_s\}}}(\gamma_1+\gamma_2)A_K(D_1,..,D_s)\right)
\end{aligned}
$$

where in the decompositions of $D - E$ above, the first and second components are distinguished, as claimed in the Lemma. □

3.3 Actual Proof of Theorem

Proof. Let C be a section of the $\mathbb{P}^1$-bundle $\mathbb{F}_n \to \mathbb{P}^1$ disjoint from E, $C \sim E + nF$. Using the relation developed in Section 2.1, $\pi^*C \cdot \pi^*C = (C \cdot C)\deg\pi$. Since $C \cdot C = n$ and $\deg\pi = N(D)$, $\pi^*C \cdot \pi^*C = nN(D)$ and so $N(D)$ can be calculated from evaluating $\pi^*C \cdot \pi^*C$.

By Lemma 3.2.1 on page 32, letting $L = C$,

$$
\begin{aligned}
\pi^*C &= (C\cdot D)A - (C\cdot D)A^2Y - \sum_{b\in B_J}(C\cdot D_2)J_2 + \\
&\quad + \sum_{b\in B_K}\Bigg[\gamma_1(C\cdot D_1 - C\cdot D)K_E + \sum_{j=1}^{\gamma_i-1}(\gamma_1 - j)(C\cdot D_1 - C\cdot D)K_{1,j} + \\
&\qquad + \sum_{i=2}^{s}\sum_{j=1}^{\gamma_i}(\gamma_1(C\cdot D_1 - C\cdot D) - jC\cdot D_i)K_{i,j}\Bigg].
\end{aligned}
$$

Using short-hand notation

$$\pi^*C \;=\; (C\cdot D)A - (C\cdot D)A^2Y + J^C + K^C,$$

we compute the intersection product on $\mathcal{Y}$ of the pull-back of line bundle C on $\mathbb{F}_n$ with itself. We obtain

$$\begin{aligned}
\pi^*C\cdot\pi^*C &= \Big((C\cdot D)A - (C\cdot D)A^2Y + J^C + K^C\Big) \\
&\quad \cdot\Big((C\cdot D)A - (C\cdot D)A^2Y + J^C + K^C\Big) \\
&= (C\cdot D)^2A^2 - 2(C\cdot D)^2A^2A\cdot Y + (C\cdot D)^2(A^2)^2Y^2 + \\
&\quad +J^C\cdot J^C + K^C\cdot K^C \\
&= -(C\cdot D)^2A^2 + J^C\cdot J^C + K^C\cdot K^C.
\end{aligned}$$

A straighforward calculation yields

$$\begin{aligned}
nN(D) &= \pi^* C \cdot \pi^* C \\
&= -(C\cdot D)^2 A^2 - \sum_{D_1+D_2=D} (C\cdot D_2)^2 j(D_1,D_2) + \\
&\quad + \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_2,\ldots,D_s\}}} \Bigg[- s\gamma_1^2 (C\cdot D_1 - C\cdot D)^2 + \\
&\qquad +2\gamma_1(\gamma_1-1)(C\cdot D_1 - C\cdot D)^2 + \\
&\quad + \sum_{j=1}^{\gamma_1-1} -2(\gamma_1-j)^2 (C\cdot D_1 - C\cdot D)^2 + \\
&\quad + \sum_{j=1}^{\gamma_1-2} 2(\gamma_1-j)(\gamma_1-j-1)(C\cdot D_1 - C\cdot D)^2 + \\
&\quad + \sum_{i=2}^{s} \Big[2\gamma_1 (C\cdot D_1 - C\cdot D)(\gamma_1(C\cdot D_1 - C\cdot D) - C\cdot D_i) \\
&\qquad -(\gamma_1(C\cdot D_1 - C\cdot D) - \gamma_i C\cdot D_i)^2 + \\
&\quad + \sum_{j=1}^{\gamma_i-1} (2(\gamma_1(C\cdot D_1 - C\cdot D) - jC\cdot D_i)\times \\
&\qquad \times(\gamma_1(C\cdot D_1 - C\cdot D) - (j+1)C\cdot D_i) + \\
&\qquad -2(\gamma_1(C\cdot D_1 - C\cdot D) - jC\cdot D_i)^2) \Big]\Bigg] k(D_1, D_2, \ldots, D_s).
\end{aligned} \tag{3.3}$$

Some tedious but elementary manipulation of the coefficient of $k(D_1, D_2, \ldots, D_s)$ yields

$$\begin{aligned}
nN(D) &= -(C\cdot D)^2 A^2 - \sum_{D_1+D_2=D} (C\cdot D_2)^2 j(D_1,D_2) + \\
&\quad - \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_2,\ldots,D_s\}}} \left[\gamma_1 (C\cdot D_1 - C\cdot D)^2 + \sum_{i=2}^{s} \gamma_i (C\cdot D_i)^2 \right] k(D_1,\ldots,D_s).
\end{aligned} \tag{3.4}$$

Considering only the type J components of the above expression for $nN(D)$, and using the following facts:

$$D = D_1 + D_2 \Rightarrow (C \cdot D)^2 = (C \cdot D_1)^2 + 2(C \cdot D_1)(C \cdot D_2) + (C \cdot D_2)^2,$$

$$\binom{r(D)-3}{r(D_1)-1} + \binom{r(D)-3}{r(D_1)-2} = \binom{r(D)-2}{r(D_1)-1},$$

$$j(D_1, D_2) = N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r(D)-2}{r(D_1)-1}, \text{ and}$$

$$\text{(J part)} \quad A^2 = -\frac{1}{2} \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r(D)-3}{r(D_1)-1}$$

gives

$$\begin{aligned}
&-(C \cdot D)^2 A^2 - \sum_{D_1+D_2=D} (C \cdot D_2)^2 j(D_1, D_2) \\
= \; & \frac{1}{2}(C \cdot D)^2 \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r(D)-3}{r(D_1)-1} + \\
& \quad - \sum_{D_1+D_2=D} (C \cdot D_2)^2 N(D_1)N(D_2)(D_1 \cdot D_2)\binom{r(D)-2}{r(D_1)-1} \\
= \; & \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times \\
& \quad \left[\frac{1}{2}(C \cdot D)^2 \binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2 \binom{r(D)-2}{r(D_1)-1}\right] \\
= \; & \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times \\
& \left[\frac{1}{2}\left((C \cdot D_1)^2 + 2(C \cdot D_1)(C \cdot D_2) + (C \cdot D_2)^2\right)\binom{r(D)-3}{r(D_1)-1} + \right. \\
& \quad \left. -(C \cdot D_2)^2 \binom{r(D)-2}{r(D_1)-1}\right] \\
= \; & \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times \\
& \left[\frac{1}{2}\left((C \cdot D_1)^2 + 2(C \cdot D_1)(C \cdot D_2) + (C \cdot D_2)^2\right)\binom{r(D)-3}{r(D_1)-1} + \right. \\
& \quad \left. -(C \cdot D_2)^2 \left(\binom{r(D)-3}{r(D_1)-1} + \binom{r(D)-3}{r(D_1)-2}\right)\right]
\end{aligned}$$

$$= \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times$$

$$\left[(C \cdot D_1)(C \cdot D_1)\binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2\binom{r(D)-3}{r(D_1)-2} + \right.$$

$$\left. + \binom{r(D)-3}{r(D_1)-1}\left(\frac{1}{2}(C \cdot D_1)^2 - \frac{1}{2}(C \cdot D_2)^2\right)\right]$$

$$= \sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times$$

$$\left[(C \cdot D_1)(C \cdot D_1)\binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2\binom{r(D)-3}{r(D_1)-2}\right]$$

The last equality was due to cancellation of symmetric divisors.

For the components of type K only:

$$- \quad (C \cdot D)^2 A^2 - \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_2,\ldots,D_s\}}} \left[\gamma_1 (C \cdot D_1 - C \cdot D)^2 + \sum_{i=2}^{s} \gamma_i (C \cdot D_i)^2\right] k(D_1, \ldots, D_s)$$

$$= \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_3,\ldots,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right) \left[\frac{\Delta'}{2}(\gamma_1+\gamma_2)(C \cdot D)^2\right] +$$

$$- \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_2,\ldots,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right) \left[\gamma_1 \Delta (C \cdot D_1 - C \cdot D)^2 + \sum_{i=2}^{s} \gamma_i \Delta (C \cdot D_i)^2\right]$$

Combining these gives this result: $nN(D) =$

$$\sum_{D_1+D_2=D} N(D_1)N(D_2)(D_1 \cdot D_2) \times$$

$$\left[(C \cdot D_1)(C \cdot D_2)\binom{r(D)-3}{r(D_1)-1} - (C \cdot D_2)^2\binom{r(D)-3}{r(D_1)-2}\right] +$$

$$+ \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_3,\ldots,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right) \left[\frac{\Delta'}{2}(\gamma_1+\gamma_2)(C \cdot D)^2\right] +$$

$$- \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_2,\ldots,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right) \left[\gamma_1 \Delta (C \cdot D_1 - C \cdot D)^2 + \sum_{i=2}^{s} \gamma_i \Delta (C \cdot D_i)^2\right]$$

□

Chapter 4

Examples

4.1 The plane.

Recall in the plane a divisor class is determined by its degree. We consider the variety of cubic $V(3) \subset \mathbb{P}^9$, $\dim V = 8$. V is irreducible and smooth at its general points. It contains an irreducible subvariety of codimension one whose general points parametrize reducible cubics given by the union of a line and a conic. V is singular along this subvariety. It is well-known that the degree of the singular locus is 12. We will be calculating this same 12 using the techniques of this paper.

Fix 7 general points $q_1, q_2 ..., q_7$ in the plane and let $\Gamma \subset V$ be the irreducible curve parametrizing all nodal cubics through the points. Let $\chi \to \Gamma$ be the corresponding family. This family has $\binom{7}{2}$ reducible fibers corresponding to reducible cubics of type $C_1 \cup C_2$ where C_1 is a line through two of the points $q_1, q_2, ..., q_7$ and C_2 is a conic through the other five points. If t is a point on Γ such that X_t, the fiber above it in the family $\chi \to \Gamma$, is one of these reducible curves then t is a node of Γ.

Let B be the normalization of Γ. Let $\mathcal{Y}$ be the normalization of the fiber product of χ and B over Γ, i.e.

$$\mathcal{Y} = (\chi \times_\Gamma B)^\nu.$$

This is a smooth surface. $f : \mathcal{Y} \to B$ has general fiber isomorphic to $\mathbb{P}^1$ and special fibers that are at worst nodal. There are $j(C_1, C_2) = 2\binom{7}{2}$ reducible nodal fibers and no other singular fibers. Let $\pi : \mathcal{Y} \to \mathbb{P}^2$.

Let Y represent the class of a fiber of $f : \mathcal{Y} \to B$ so that $Y^2 = 0$. Let A represent the class of a section corresponding to one of the seven base points, denoted by q. Let $B' \subset B$ be the set of points corresponding to reducible fibers. Note: B' consists of exactly $2\binom{7}{2}$ points as calculated earlier. For $b \in B'$ let $J_{1,b}$ and $J_{2,b}$ be the two components of the fiber such that $A \cdot J_{1,b} = 1$ and $A \cdot J_{2,b} = 0$. Then $\{Y, A, \{J_{2,b}\}_{b \in B'}\}$ generate the Néron-Severi group of $\mathcal{Y}$ with

$$A \cdot Y = 1, \quad J_{2,b}^2 = -1, \quad \text{and} \quad A \cdot J_{2,b} = Y \cdot J_{2,b} = 0.$$

Let L be the hyperplane class in $\mathbb{P}^2$. Then

$$\pi^* L \cdot \pi^* L = L \cdot L \deg \pi = N$$

and

$$\pi^* L = a_Y Y + a_A A + \sum_{b \in B'} a_{J_{2,b}} J_{2,b}.$$

Calculating the coefficients: $\pi^* L \cdot Y = 3$ so $a_A = 3$; $\pi^* L \cdot A = 0$ so $a_Y = -3A^2$; and $\pi^* L \cdot J_{2,b} = L \cdot \pi_* J_{2,b} = \deg \pi_* J_{2,b}$, so $a_{J_{2,b}} = -\deg \pi_* J_{2,b}$. This gives $\pi^* L \cdot \pi^* L = -9A^2 - \sum_{b \in B'} (\deg \pi_* J_{2,b})^2$.

Next we compute A^2. Pick any one of the other base points and call it q'. Let A' be its corresponding section. Note that $A^2 = (A')^2$ and $A \cdot A' = 0$ so $2A^2 = (A - A')^2$.

To compute the right-hand side, let

$$S_J = \{b \in B' | q' \in \pi(J_{2,b})\},$$

i.e. the collection of points $b \in B'$ such that the sections A and A' meet different components of the fiber. For every $b \notin S_J$, A and $A' - \sum J_{2,b}$ have the same intersection number with every component of the every fiber of $Y \to B$:

$$A \cdot (J_{1,b} + J_{2,b}) = 1, \quad (A' - \sum J_{2,b}) \cdot (J_{1,b} + J_{2,b}) = 1.$$

For every $b \in S_J$, we have $A \cdot J_{1,b} = 1$ and $A \cdot J_{2,b} = 0$, while $A' \cdot J_{1,b} = 0$ and $A' \cdot J_{2,b} = 1$. Therefore, A and $A' - \sum_{b \in S_J} J_{2,b}$ differ by a multiple of the class of Y, i.e.

$$A - A' = -\sum_{b \in S_J} J_{2,b} + nY.$$

So

$$2A^2 = (A - A')^2 = \sum_{b \in S_J} {J_{2,b}}^2 = \sum_{b \in S_J} -1 = -\#(S_J),$$

that is the number of reducible fibers with q and q' on different components.

Now we count the number of fibers of this type. There are 5 curves with q on the line and q' on the conic and 5 with q' on the line and q on the conic, giving a total of 10. Since Γ has a node at each point corresponding to a curve of this type, then after normalizing Γ we have 20 curves of this type. Therefore $2A^2 = -20$, so $A^2 = -10$.

Finally we compute $\sum(\deg \pi_* J_{2,b})^2$. There are $\binom{6}{1}$ points on Γ corresponding to reducible fibers with q on the line. Since Γ has a node at each of these points, then after normalizing Γ we have 12 curves of this type. The point q being on the line implies that $J_{1,b}$ is the line and $J_{2,b}$ is the conic which has self-intersection 4.

There are $\binom{6}{2}$ points on Γ corresponding to reducible fibers with q on the conic. The curve Γ has a node at each of these points, so after normalizing Γ there are $2\binom{6}{2}$ such points. The point q being on the conic implies that $J_{1,b}$ is the conic and $J_{2,b}$ is the line which has self-intersection 1, so

$$\sum(\deg \pi_* W_b)^2 = 2(6)(4) + 2\binom{6}{2} = 78.$$

Thus N = -9(-10) - 78 =12.

4.2 Proposition for $N(2C)$ on $\mathbb{F}_n$

In this section we show how our formula from Theorem 3.1.1 can be used to give another proof of Caporaso and Harris' formula for $N(2C)$.

Proposition 4.2.1 (Caporaso and Harris, [CH1] Theorem 3.3 on p. 80) *Let $N(2C)$ be the number of irreducible rational curves in the linear series $|2C|$ on $\mathbb{F}_n$ passing through $2n+3$ points, then*

$$N(2C) = \sum_{k=0}^{n-1}(n-k)^2\binom{2n+2}{k}.$$

4.2.1 Proof of Proposition

Proof. Apply the formula in Theorem 3.1.1 to the case of $D = 2C$ on $\mathbb{F}_n$. If $D = 2C$ then we have decompositions $2C = C + C$ (type J) and $2C - E = (C + bF) + F + F + ... + F$ where there are $n - b$ copies of F (type K). Here are the relevant numbers:

Type J Fiber:

$$r(2C) = -K_S \cdot 2C - 1 = (2E + (n+2)F) \cdot 2C - 1 = 2n + 3$$

$$r(C) = -K_S \cdot C - 1 = (2E + (n+2)F) \cdot C - 1 = n + 1$$

For the decomposition $2C = C + C$, the contribution to $nN(2C)$ is

$$\begin{aligned} N(C)N(C)(C \cdot C) &\left[(C \cdot C)(C \cdot C)\binom{r(2C)-3}{r(C)-1} - (C \cdot C)^2\binom{r(2C)-3}{r(C)-2}\right] \\ &= n\left[n^2\binom{2n}{n} - n^2\binom{2n}{n-1}\right] \\ &= n^3\left[\binom{2n}{n} - \binom{2n}{n-1}\right]. \end{aligned}$$

Type K Fibers: For each b, $b = 1, ..., n-1$, there is a decomposition $2C - E = (C + bF) + F + F + ... + F$. In such a case $\sum_{i=1}^{n-b+1} m_i = n$. If $D_i = C + bF$ then $m_i = b$ giving $\gamma_i = 1$ and if $D_j = F$ then $m_j = 1$ giving $\gamma_j = b$ $\left(\text{note: } \sum_{i=1}^{n-b+1} m_i = b + (n-b)(1) = n\right)$. The dimensions are calculated as

$$r_b(C + bF) = -K_S \cdot (C + bF) - b = (2E + (n+2)F) \cdot (C + bF) - b = n + b + 2$$

$$r(F) = -K_S \cdot F - 1 = (2E + (n+2)F) \cdot F - 1 = n + 1$$

The contribution of these fibers in the formula of Theorem 3.1.1 is as follows:

$$\begin{aligned} \sum_{b=1}^{n-1} N_b(C+bF) &\left[\frac{1}{2}\frac{1}{(n-b-1)!}\binom{2n}{n+b+1, 0, 1, 1, ..., 1}(b+1)(C \cdot 2C)^2 + \right. \\ &+ \frac{1}{2}\frac{1}{(n-b-1)!}\binom{2n}{0, n+b+1, 1, 1, 1, ..., 1}(1+b)(C \cdot 2C)^2 + \\ &\left. + \frac{1}{2}\frac{1}{(n-b-2)!}\binom{2n}{0, 0, 1, 1, ..., 1}(b+b)(C \cdot 2C)^2\right] + \\ - \sum_{b=1}^{n-1} N_b(C+bf) &\left[\frac{1}{(n-b)!}\binom{2n+1}{n+b+1, 1, 1, ..., 1} \times \right. \\ &\left(1(C \cdot (C+bF) - C \cdot 2C)^2 + b(n-b)(C \cdot F)^2\right) + \\ &+ \frac{1}{(n-b-1)!}\binom{2n+1}{0, 1, 1, ..., 1} \times \\ &\left.\left(b(C \cdot F - C \cdot 2C)^2 + 1(C \cdot (C+bF))^2 + b(n-b-1)(C \cdot F)^2\right)\right] \end{aligned}$$

Note that in the first sum the first two components are distinguished so we sum over the permutations $(C+bF)+F+...+F$, $F+(C+bF)+F+...+F$, and $F+F+...+F+(C+bF)$. While in the second sum, only the first component is distinguished so we sum only over the permutations $(C+bF)+F+...+F$ and $F+(C+bF)+F+...+F$. We can then simplify to

$$\begin{aligned}
&\sum_{b=1}^{n-1} b\left[\binom{2n}{n+b+1}(b+1)4n^2+\binom{2n}{n+b+2}4bn^2\right]+\\
-&\sum_{b=1}^{n-1} b\left[\binom{2n+1}{n+b+1}\Big((b-n)^2+b(n-b)\Big)+\right.\\
&\qquad\left.+\binom{2n+1}{n+b+2}\Big(b(1-2n)^2+(n+b)^2+b(n-b-1)\Big)\right]
\end{aligned}$$

Combining the above contributions of the type J and type K fibers gives:

$$\begin{aligned}
nN(2C) &= n^3\left[\binom{2n}{n}-\binom{2n}{n-1}\right]+\\
&+\sum_{b=1}^{n-1} b\left[\binom{2n}{n+b+1}(b+1)4n^2+\binom{2n}{n+b+2}4bn^2\right]+\\
&-\sum_{b=1}^{n-1} b\left[\binom{2n+1}{n+b+1}\Big((b-n)^2+b(n-b)\Big)+\right.\\
&\qquad\left.+\binom{2n+1}{n+b+2}\Big(b(1-2n)^2+(n+b)^2+b(n-b-1)\Big)\right]
\end{aligned}$$

We complete the proof by simplifying the above formula to the desired form. This simplification is motivated by the very clever ideas used the proof of this same proposition in [CH1]. We begin by labeling pieces of the formula and simplifying them individually. Let

$$\begin{aligned}
A &= n^3\left[\binom{2n}{n}-\binom{2n}{n-1}\right],\\
A'_b &= \left[\binom{2n}{n+b+1}(b+1)4bn^2+\binom{2n}{n+b+2}4b^2n^2\right], \text{and}
\end{aligned}$$

$$A''_b = \left[\binom{2n+1}{n+b+1} b\Big((b-n)^2 + b(n-b)\Big) + \right.$$
$$\left. + \binom{2n+1}{n+b+2} b\Big(b(1-2n)^2 + (n+b)^2 + b(n-b-1)\Big)\right].$$

In this notation,

$$nN(2C) = A + \sum_{b=1}^{n-1} (A'_b - A''_b).$$

We begin by simplifying A:

$$\begin{aligned}
A &= n^3 \left[\binom{2n}{n} - \binom{2n}{n-1}\right] \\
&= n^3 \left(\frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!}\right) \\
&= n^3 \left(\frac{(2n)!}{n!(n+1)!}((n+1)-n)\right) \\
&= n^3 \left(\frac{(2n)!}{n!(n+1)!}\right) \\
&= n^3 \left[\frac{1}{n}\binom{2n}{n-1}\right] \\
&= n^2 \binom{2n}{n-1}.
\end{aligned}$$

We simplify A'_b using the identity

$$\binom{2n}{n+b+2} + \binom{2n}{n+b+1} = \binom{2n+1}{n+b+2}.$$

$$\begin{aligned}
A'_b &= \binom{2n}{n+b+2} 4b^2n^2 + \binom{2n}{n+b+1} b(b+1)4n^2 \\
&= \binom{2n}{n+b+2} 4b^2n^2 + \binom{2n}{n+b+1} 4b^2n^2 + \binom{2n}{n+b+1} 4bn^2 \\
&= \binom{2n+1}{n+b+2} 4b^2n^2 + \binom{2n}{n+b+1} 4bn^2
\end{aligned}$$

Now we simplify A''_b using a similar identity

$$\binom{2n+1}{n+b+2} + \binom{2n+1}{n+b+1} = \binom{2n+2}{n+b+2}.$$

$$
\begin{aligned}
A_b'' &= \binom{2n+1}{n+b+1} b\Big((b-n)^2 + b(n-b)\Big) + \\
&\qquad + \binom{2n+1}{n+b+2} b\Big(b(1-2n)^2 + (n+b)^2 + b(n-b-1)\Big) \\
&= \binom{2n+1}{n+b+1} b(n^2 - bn) + \binom{2n+1}{n+b+2} b(n^2 - bn + 4bn^2) \\
&= \binom{2n+1}{n+b+2} b(n^2 - bn) + \binom{2n+1}{n+b+2} b(n^2 - bn) + \binom{2n+1}{n+b+2} 4b^2n^2 \\
&= \binom{2n+2}{n+b+2} b(n^2 - bn) + \binom{2n+1}{n+b+2} 4b^2n^2
\end{aligned}
$$

Combining these we get

$$
\begin{aligned}
nN(2C) &= A + \sum_{b=1}^{n-1} (A_b' - A_b'') \\
&= n^2\binom{2n}{n-1} + \sum_{b=1}^{n-1} \Bigg[\binom{2n+1}{n+b+2} 4b^2n^2 + \binom{2n}{n+b+1} 4bn^2 + \\
&\qquad - \binom{2n+2}{n+b+2} b(n^2 - bn) - \binom{2n+1}{n+b+2} 4b^2n^2\Bigg] \\
&= n^2\binom{2n}{n-1} + \sum_{b=1}^{n-1} \Bigg[\binom{2n}{n+b+1} 4bn^2 - \binom{2n+2}{n+b+2} b(n^2 - bn)\Bigg] \\
&= n^2\binom{2n}{n-1} + \sum_{b=1}^{n-1} \binom{2n}{n-b-1} 4bn^2 - \sum_{b=1}^{n-1} \binom{2n+2}{n-b} nb(n-b) \\
&= n^2\binom{2n}{n-1} + \sum_{k=1}^{n-1} \binom{2n}{k-1} 4n^2(n-k) - \sum_{k=1}^{n-1} \binom{2n+2}{k} nk(n-k)
\end{aligned}
$$

On page 78 of [CH1], Caporaso and Harris simplify this expression as follows:

$$
\begin{aligned}
nN(2C) &= n^2\binom{2n}{n-1} + \sum_{k=1}^{n-1} \binom{2n}{k-1} 4n^2(n-k) - \sum_{k=1}^{n-1} \binom{2n+2}{k} nk(n-k) \\
&= n^2 \sum_{k=0}^{n-1} (n-k)\binom{2n+2}{k} - \sum_{k=1}^{n-1} \binom{2n+2}{k} nk(n-k) \\
&= n \sum_{k=0}^{n-1} (n-k)^2 \binom{2n+2}{k}
\end{aligned}
$$

□

4.2.2 Application of Proposition using Maple

Programming this formula in Maple to calculate $N(2C)$ on $\mathbb{F}_n$ yields the following numbers.

with(combinat, numbcomb);

for n from 2 to 10 do

$$N2C[n] \quad := \quad \text{add}\Big((n-k)^2 * \text{numbcomb}(2*n+2, k), k = 0..n-1\Big);$$

od:

print(N2C);

n=2: 10, n=3: 69, n=4: 406, n=5: 2186, n=6: 11124, n=7: 54445, n=8: 259006, n=9: 1205790, n=10: 5519020

4.3 Examples on $\mathbb{F}_2$.

4.3.1 $D = 2C + F$

We show that $N(2C+F) = 93$. If $D = 2C+F$ then we have decomposition $2C+F = C+(C+F)$ (type J), $2C+F = (2C)+F$ (type J), and $2C-E = (C+F)+F$ (type K).

TYPE J

Relevant counts for the decomposition $2C+F = C+(C+F)$:

Dimensions:

$$r(2C+F) = -K_S \cdot (2C+F) - 1 = (2E+4F) \cdot (2C+F) - 1 = 9$$

$$r(C) = -K_S \cdot C - 1 = (2E + 4F) \cdot C - 1 = 3$$

$$r(C + F) = -K_S \cdot (C + F) - 1 = (2E + 4F) \cdot (C + F) - 1 = 5$$

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $2C+F = C+(C+F)$:

(q_1 lies on C)

$$\begin{aligned} j(C, C+F) &= N(C)N(C+F)(C \cdot C+F)\binom{r(2C+F)-2}{r(C)-1} \\ &= 3\binom{7}{2} = 63 \end{aligned}$$

(q_1 lies on C and q_2 lies on $C + F$)

$$\begin{aligned} A_J(C, C+F) &= N(C)N(C+F)(C \cdot C+F)\binom{r(2C+F)-3}{r(C)-1} \\ &= 3\binom{6}{2} = 45 \end{aligned}$$

Contribution of $2C + F = C + (C + F)$ to $2N(2C + F)$:

$$\begin{aligned} &\frac{1}{2} \quad (C \cdot (2C+F))^2 A_J(C, C+F) - (C \cdot (C+F))^2 j(C, C+F) \\ &= \frac{1}{2}(25)(45) - (9)(63) = \frac{-9}{2} \end{aligned}$$

Relevant counts for the symmetric decomposition $2C + F = (C + F) + C$:

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $2C+F = (C+F)+C$:

(q_1 lies on $C + F$)

$$\begin{aligned} j(C+F, C) &= N(C+F)N(C)(C+F \cdot C)\binom{r(2C+F)-2}{r(C+F)-1} \\ &= 3\binom{7}{4} = 105 \end{aligned}$$

(q_1 lies on $C + F$ and q_2 lies on C)

$$\begin{aligned} A_J(C+F, C) &= N(C+F)N(C)(C+F \cdot C)\binom{r(2C+F)-3}{r(C+F)-1} \\ &= 3\binom{6}{4} = 45 \end{aligned}$$

Contribution of $2C + F = (C + F) + C$ to $2N(2C + F)$:

$$\begin{aligned} \frac{1}{2} & (C \cdot (2C+F))^2 A_J(C+F, C) - (C \cdot C)^2 j(C+F, C) \\ = & \frac{1}{2}(25)(45) - (4)(105) = \frac{285}{2} \end{aligned}$$

Total contribution for the decomposition $2C + F = (C + F) + C = C + (C + F)$ is $\frac{-9}{2} + \frac{285}{2} = 138$.

Relevant counts for the decomposition $2C + F = (2C) + F$:

Dimensions:

$$r(2C) = -K_S \cdot 2C - 1 = (2E + 4F) \cdot 2C - 1 = 7$$

$$r(F) = -K_S \cdot F - 1 = (2E + 4F) \cdot F - 1 = 1$$

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $2C + F = (2C) + F$:

(q_1 lies on $2C$)

$$\begin{aligned} j(2C, F) &= N(2C)N(F)(2C \cdot F)\binom{r(2C+F)-2}{r(2C)-1} \\ &= 10(2)\binom{7}{6} = 140 \end{aligned}$$

(q_1 lies on $2C$ and q_2 lies on F)

$$\begin{aligned} A_J(2C, F) &= N(2C)N(F)(2C \cdot F)\binom{r(2C+F)-3}{r(2C)-1} \\ &= 10(2)\binom{6}{6} = 20 \end{aligned}$$

Contribution of $2C + F = (2C) + F$ to $2N(2C + F)$:

$$\begin{aligned} \frac{1}{2} & (C \cdot (2C+F))^2 A_J(2C, F) - (C \cdot F)^2 j(2C, F) \\ = & \frac{1}{2}(25)(20) - (1)(140) = 110 \end{aligned}$$

Relevant counts for the symmetric decomposition $2C + F = F + (2C)$:

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $2C + F = F + (2C)$:

(q_1 lies on F)

$$\begin{aligned} j(F, 2C) &= N(F)N(2C)(F \cdot 2C)\binom{r(2C+F)-2}{r(F)-1} \\ &= 10(2)\binom{7}{0} = 20 \end{aligned}$$

(q_1 lies on F and q_2 lies on $2C$)

$$\begin{aligned} A_J(F, 2C) &= N(F)N(2C)(F \cdot 2C)\binom{r(2C+F)-3}{r(F)-1} \\ &= 10(2)\binom{6}{0} = 20 \end{aligned}$$

Contribution of $2C + F = F + (2C)$ to $2N(2C + F)$:

$$\begin{aligned} \frac{1}{2} &\quad (C \cdot (2C+F))^2 A_J(F, 2C) - (C \cdot 2C)^2 j(F, 2C) \\ &= \frac{1}{2}(25)(20) - (16)(20) = -70 \end{aligned}$$

Total contribution for the decomposition $2C+F = (2C)+F = F+(2C)$ is $110-70 = 40$.

TYPE K

Relevant counts for the decomposition $(2C + F) - E = (C + 2F) + F$:

Dimensions:

$$r(C + 2F) = -K_S \cdot (C + 2F) - 1 = (2E + 4F) \cdot (C + 2F) - 1 = 7$$

$$(C + 2F) \cdot E = 2 \geq m_{C+F} \geq 1$$

$$r(F) = -K_S \cdot F - 1 = (2E + 4F) \cdot F - 1 = 1$$

$$F \cdot E = 1 \geq m_F \geq 1$$

Classification of multiplicities:

$$2 = m_{C+2F} + m_F \Rightarrow m_{C+2F} = m_F = 1, \gamma_{C+F} = \gamma_F = 1$$

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $(2C + F) - E = (C + 2F) + F$:

(q_1 lies on $C + 2F$)

$$\begin{aligned} k(C+2F, F) &= N(C+2F)N(F)(E \cdot (C+2F))(E \cdot F)\binom{r(2C+F)-2}{r(C+2F)-1} \\ &= 2\binom{7}{6} = 14 \end{aligned}$$

(q_1 lies on $C + 2F$ and q_2 lies on F)

$$\begin{aligned} A_K(C+2F, F) &= N(C+2F)N(F)(E \cdot (C+2F))(E \cdot F)\binom{r(2C+F)-3}{r(C+2F)-1} \\ &= 2\binom{6}{6} = 2 \end{aligned}$$

Contribution of $(2C + F) - E = (C + 2F) + F$ to $2N(2C + F)$:

$$\begin{aligned} \frac{1}{2} &\ (C \cdot (2C+F))^2(\gamma_{C+2F} + \gamma_F)A_K(C+2F, F) \\ &\quad - \left(\gamma_{C+2F}(C \cdot (C+2F) - C \cdot (2C+F))^2 + \gamma_F(C \cdot F)^2\right) k(C+2F, F) \\ = &\ \frac{1}{2}(25)(1+1)(2) - (1(4-5)^2 + 1(1)^2)(14) = 22 \end{aligned}$$

Relevant counts for the symmetric decomposition $(2C + F) - E = F + (C + 2F)$:

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $(2C + F) - E = (C + 2F) + F$:

(q_1 lies on F)

$$\begin{aligned} k(F, C+2F) &= N(F)N(C+2F)(E\cdot F)(E\cdot(C+2F))\binom{r(2C+F)-2}{r(F)-1} \\ &= 2\binom{7}{0} = 2 \end{aligned}$$

(q_1 lies on F and q_2 lies on $C+2F$)

$$\begin{aligned} A_K(F, C+2F) &= N(F)N(C+2F)(E\cdot F)(E\cdot(C+2F))\binom{r(2C+F)-3}{r(F)-1} \\ &= 2\binom{6}{0} = 2 \end{aligned}$$

Contribution of $(2C+F)-E = F+(C+2F)$ to $2N(2C+F)$:

$$\begin{aligned} &\frac{1}{2}\ (C\cdot(2C+F))^2(\gamma_F+\gamma_{C+2F})A_K(F, C+2F) \\ &\qquad -\left(\gamma_F(C\cdot F - C\cdot(2C+F))^2+\gamma_{C+2F}(C\cdot(C+2F))^2\right)k(F, C+2F) \\ &= \frac{1}{2}(25)(1+1)(2)-(1(1-5)^2+1(4)^2)(2) = -14 \end{aligned}$$

Total contribution for the decomposition $2C+F-E=(C+2F)+F=F+(C+2F)$ is $22-14=8$. Therefore $2N(2C+F)=138+40+8=186$, and so $N(2C+F)=93$.

4.3.2 $D = 3C$

We show that $N(3C)=2232$. If $D=3C$ then we have decomposition $3C=2C+C$ (type J), $2C-E=(2C+F)+F$ (type K), and $2C-E=(C+F)+(C+F)$ (type K).

TYPE J

Relevant counts for the decompositions $3C=2C+C$:

Dimensions:

$$r(3C) = -K_S\cdot 3C - 1 = (2E+4F)\cdot 3C - 1 = 11$$

$$r(2C) = -K_S \cdot 2C - 1 = (2E+4F)\cdot 2C - 1 = 7$$

$$r(C) = -K_S \cdot C - 1 = (2E+4F)\cdot C - 1 = 3$$

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $3C = 2C + C$:

(q_1 lies on $2C$)

$$\begin{aligned} j(2C, C) &= N(2C)N(C)(2C\cdot C)\binom{r(3C)-2}{r(2C)-1} \\ &= 10(4)\binom{9}{6} = 3360 \end{aligned}$$

(q_1 lies on $2C$ and q_2 lies on C)

$$\begin{aligned} A_J(2C, C) &= N(2C)N(C)(2C\cdot C)\binom{r(3C)-3}{r(2C)-1} \\ &= 10(4)\binom{8}{6} = 1120 \end{aligned}$$

Contribution of $3C = 2C + C$ to $2N(3C)$:

$$\begin{aligned} &\frac{1}{2} \quad (C\cdot 3C)^2 A_J(2C, C) - (C\cdot C)^2 j(2C, C) \\ &= \frac{1}{2}(36)(1120) - (4)(3360) = 6720 \end{aligned}$$

Relevant counts for the symmetric decomposition $3C = C + 2C$:

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $3C = C + 2C$:

(q_1 lies on C)

$$\begin{aligned} j(C, 2C) &= N(C)N(2C)(C\cdot 2C)\binom{r(3C)-2}{r(C)-1} \\ &= 10(4)\binom{9}{2} = 1440 \end{aligned}$$

(q_1 lies on C and q_2 lies on $2C$)

$$\begin{aligned} A_J(C, 2C) &= N(C)N(2C)(C\cdot 2C)\binom{r(3C)-3}{r(2C)-1} \\ &= 10(4)\binom{8}{2} = 1120 \end{aligned}$$

Contribution of $3C = C + 2C$ to $2N(3C)$:

$$\begin{aligned} & \frac{1}{2} \quad (C \cdot 3C)^2 A_J(C, 2C) - (C \cdot 2C)^2 j(C, 2C) \\ = & \frac{1}{2}(36)(1120) - (16)(1440) = -2880 \end{aligned}$$

Total contribution for the decomposition $3C = 2C + C = C + 2C$ is $6720 - 2880 = 3840$.

TYPE K

Relevant counts for the decomposition $3C - E = (2C + F) + F$:

$$r(2C + F) = -K_S \cdot (2C + F) - 1 = (2E + 4F) \cdot (2C + F) - 1 = 9$$

$$(2C + F) \cdot E = 1 \geq m_{2C+F} \geq 1$$

$$r(F) = -K_S \cdot F - 1 = (2E + 4F) \cdot F - 1 = 1$$

$$F \cdot E = 1 \geq m_F \geq 1$$

$$m_{2C+F} = m_F = 1 \Rightarrow \gamma_{2C+F} = \gamma_F = 1$$

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $3C - E = (2C + F) + F$:

(q_1 lies on $2C + F$)

$$\begin{aligned} k(2C + F, F) &= N(2C + F)N(F)(E \cdot (2C + F))(E \cdot F)\binom{r(3C) - 2}{r(2C + F) - 1} \\ &= \binom{9}{8} = 9N(2C + F) \end{aligned}$$

(q_1 lies on $2C + F$ and q_2 lies on F)

$$\begin{aligned} A_K(2C + F, F) &= N(2C + F)N(F)(E \cdot (2C + F))(E \cdot F)\binom{r(3C) - 3}{r(2C + F) - 1} \\ &= \binom{8}{8} = N(2C + F) \end{aligned}$$

Contribution of $3C - E = (2C + F) + F$ to $2N(3C)$:

$$\begin{aligned}
\frac{1}{2} & \quad (C \cdot 3C)^2(\gamma_{2C+F} + \gamma_F)A_K(2C+F, F) \\
& \qquad - \left(\gamma_{2C+F}(C \cdot (2C+F) - C \cdot 3C)^2 + \gamma_F(C \cdot F)^2\right) k(2C+F, F) \\
= & \quad \frac{1}{2}(36)(1+1)N(2C+F) - (1(5-6)^2 + 1(1)^2)9N(2C+F) = 18N(2C+F)
\end{aligned}$$

Relevant counts for the symmetric decomposition $3C - E = F + (2C + F)$:

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $3C-E = F+(2C+F)$:

(q_1 lies on F)

$$\begin{aligned}
k(F, 2C+F) &= N(F)N(2C+F)(E \cdot F)(E \cdot (2C+F))\binom{r(3C)-2}{r(F)-1} \\
&= N(2C+F)\binom{9}{0} = N(2C+F)
\end{aligned}$$

(q_1 lies on F and q_2 lies on $2C + F$)

$$\begin{aligned}
A_K(F, 2C+F) &= N(F)N(2C+F)(E \cdot F)(E \cdot (2C+F))\binom{r(3C)-3}{r(F)-1} \\
&= N(2C+F)\binom{8}{0} = N(2C+F)
\end{aligned}$$

Contribution of $3C - E = F + (2C + F)$ to $2N(3C)$:

$$\begin{aligned}
\frac{1}{2} & \quad (C \cdot 3C)^2(\gamma_F + \gamma_{2C+F})A_K(F, 2C+F) \\
& \qquad - \left(\gamma_F(C \cdot F - C \cdot 3C)^2 + \gamma_{2C+F}(C \cdot (2C+F))^2\right) k(F, 2C+F) \\
= & \quad \frac{1}{2}(36)(1+1)N(2C+F) - (1(1-6)^2 + 1(5)^2)N(2C+F) = -14N(2C+F)
\end{aligned}$$

Total contribution for the decomposition $3C - E = (2C + F) + F = F + (2C + F)$ is $18N(2C + F) - 14N(2C + F) = 4N(2C + F) = 4(93) = 372$.

Relevant counts for the decomposition $3C - E = (C + F) + (C + F)$:

$$r(C+F) = -K_S \cdot (C+F) - 1 = (2E+4F) \cdot (C+F) - 1 = 5$$

$$E \cdot (C+F) = 1 \geq m_{C+F} \geq 1 \Rightarrow m_{C+F} = \gamma_{C+F} = 1$$

Number of fibers of $\mathcal{Y} \to B$ of this type for the decomposition $3C - E = (C+F) + (C+F)$:

(q_1 lies on $C+F$)

$$\begin{aligned} k(C+F, C+F) &= N(C+F)^2(E\cdot(C+F))(E\cdot(C+F))\binom{r(3C)-2}{r(C+F)-1} \\ &= \binom{9}{4} = 126 \end{aligned}$$

(q_1 lies on $C+F$ and q_2 lies on $C+F$)

$$\begin{aligned} A_K(C+F, C+F) &= N(C+F)^2(E\cdot(C+F))(E\cdot(C+F))\binom{r(3C)-3}{r(C+F)-1} \\ &= \binom{8}{4} = 70 \end{aligned}$$

Contribution of $3C - E = (C+F) + (C+F)$ to $2N(3C)$:

$$\begin{aligned} &\frac{1}{2} \quad (C\cdot 3C)^2(\gamma_{C+F}+\gamma_{C+F})A_K(C+F, C+F) + \\ &\qquad - \left(\gamma_{C+F}(C\cdot(C+F) - C\cdot 3C)^2 + \gamma_{C+F}(C\cdot(C+F))^2\right) k(C+F, C+F) \\ = &\ \frac{1}{2}(36)(1+1)(70) - (1(3-6)^2 + 1(3)^2)126 = 252 \end{aligned}$$

Therefore

$$\begin{aligned} 2N(3C) &= 3840 + 372 + 252 \\ &= 4464, \end{aligned}$$

and $N(3C) = 2232$.

Chapter 5

The Geometry of $V_m(D)$

Now we apply the Rational Fibration Method to the tangential Severi varieties $V_m(D)$ with the goal of writing an explicit formula for its degree. We begin by describing the Rational Fibration Method in this context.

5.1 The Rational Fibration Method for $V_m(D)$

Let $S = \mathbb{F}_n$ and let D be an effective divisor on S with nonnegative self-intersection. Let $V_m(D)$ be the closure of the locus of all points parametrizing irreducible rational curves in $|D|$ meeting E at a smooth point with multiplicity m. Note: $V_m(D) \subset V(D) \subset |D|$. The dimension of $V_m(D)$, which we assume to be nonempty, we denote by $r_m(D)$. By proposition 2.1 of [CH1],

$$r_m(D) = -K_S \cdot D - m,$$

where K_S is the canonical class of S. We denote the degree of $V_m(D)$ by $N_m(D)$.

We begin by choosing $r_m(D) - 1$ general points $q_1, q_2, ..., q_{r_m(D)-1} \in S$ and let

$\Gamma_m(D) \subset V_m(D)$ be the closure of the locus of points $[X] \in V_m(D)$ corresponding to the irreducible rational curves X passing through these points. Equivalently, if for any point $p \in S$, H_p is the hyperplane in $|D|$ of points corresponding to curves passing through p, then $\Gamma_m(D)$ will be the one-dimensional linear section of $V_m(D)$:

$$\Gamma_m(D) = V_m(D) \cap_{i=1}^{r_m(D)-1} H_{q_i}.$$

Thus $\Gamma_m(D)$ is the closure in $V_m(D)$ of the set of irreducible rational curves passing through $q_1, ..., q_{r_m(D)-1}$ and meeting E at a smooth point with multiplicity m. Let $\chi_m(D) \subset \Gamma_m \times S$ be the universal family over $\Gamma_m(D)$, i.e. the family of curves corresponding to Γ_m. The fibers of $\chi_m \to \Gamma_m$ correspond to the curves parametrized by Γ_m. We would like to build a family from $\chi_m \to \Gamma_m$ whose general fiber is the normalization of its corresponding fiber in $\chi_m \to \Gamma_m$. So we do a series of normalizations. Normalizing Γ_m gives $\Gamma_m^\nu \to \Gamma_m$. Then take the normalization χ_m^ν of $\chi_m \times_{\Gamma_m} \Gamma_m^\nu$ to give $\chi_m^\nu \to \Gamma_m^\nu$. Finally we apply a semi-stable reduction by making a base change $B \to \Gamma_m^\nu$ and blowing up the total space of the pullback family $\chi_m^\nu \times_{\Gamma_m^\nu} B$. This gives a family $\mathcal{Y}_m \to B$. We will denote the composite map by $\pi : \mathcal{Y}_m \to B$.

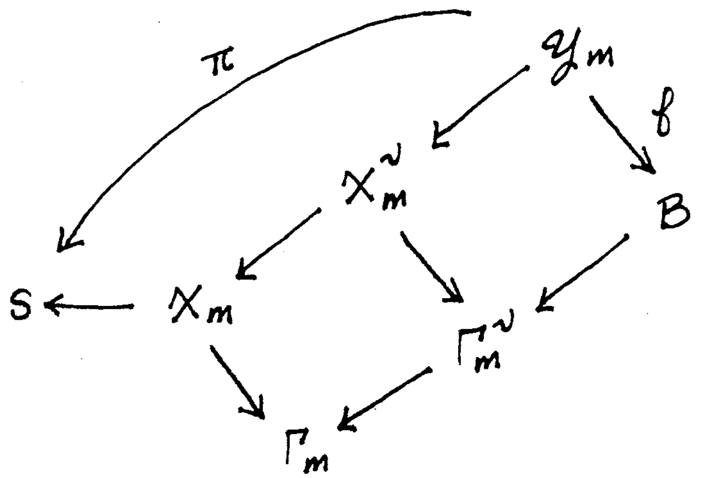

Figure 5.1: Construction of the Surface $\mathcal{Y}_m$.

We study the geometry of the general point of the boundary of $V_m(D)$. To do this we list all types of reducible fibers that occur in the family $\chi_m \to \Gamma_m$. Remark: By [CH1] Proposition 2.1, away from any points of tangency with E, X has only nodes as singularities.

5.2 Classification of Reducible Fibers of $\chi_m \to \Gamma_m$

Here we describe the reducible fibers of the family $\chi_m \to \Gamma_m$ so that we may completely describe the reducible fibers of $\mathcal{Y}_m \to B$ and the Néron-Severi group of $\mathcal{Y}_m$.

Proposition 5.2.1 *Let $X \subset S = \mathbb{F}_n$ be any reducible fiber of the family $\chi_m \to \Gamma_m$. We assume $m \leq n$.*

1. *If X does not contain E, then X has exactly two irreducible components X_1 and X_2, with $[X_j] \in V(D_j)$ and $D_1 + D_2 = D$ and either*

   (a) *Each $[X_j]$ is a general point in $V_{m_j}(D_j)$ for some $m_1, m_2 \in \mathbb{Z}_+$ such that $X_1 \cap X_2 \cap E = \emptyset$ and $m_1 + m_2 = m + 1$.*

   OR

   (b) *Each $[X_j]$ is a general point in $V_{m_j}(D_j)$ for some $m_1, m_2 \in \mathbb{Z}_+$ such that $\#(X_1 \cap X_2 \cap E) = \tau \geq 1$ and $m_1 + m_2 = m + 1 - \tau$.*

2. *If X does contain E, then X has irreducible components $E, X_1, ..., X_s$, with $[X_j] \in V(D_j)$ and $E + D_1 + ... + D_s = D$. Moreover each $[X_j]$ is a general point in $V_{m_j}(D_j)$ for some collection $m_1, ..., m_s$ of positive integers such that $\sum_{i=1}^{s} m_j = n + m - 1$.*

Notation. If X is any reducible fiber in the family $\chi_m \to \Gamma_m$ not containing E such that $X_1 \cap X_2 \cap E = \emptyset$, we call its corresponding fibers of $f : \mathcal{Y}_m \to B$ **type $\mathbf{J_m}$** fibers. And let B_{J_m} be the set of points $b \in B$ such that the fiber X_b over b is a fiber of type J_m. If, on the other hand, X is any reducible fiber in the family $\chi_m \to \Gamma_m$ not containing E such that $X_1 \cap X_2 \cap E \neq \emptyset$, we call its corresponding fibers of $f : \mathcal{Y}_m \to B$ **type $\tilde{\mathbf{J}}_\mathbf{m}$** fibers. For fibers of this type we assume $\tau \leq 2$. And let $B_{\tilde{J}_m}$ be the set of points $b \in B$ such that the fiber X_b over b is a fiber of type $\tilde{J}_m$. If X is any reducible fiber in the family $\chi_m \to \Gamma_m$ containing E, we call its corresponding fibers of $f : \mathcal{Y}_m \to B$ **type $\mathbf{K_m}$** fibers. And let B_{K_m} be the set of points $b \in B$ such that the fiber X_b over b is a fiber of type K_m.

Proof.

(*Part 1.*: X does not contain E) We write the divisor X as a sum $X = \sum_{j=1}^{s} a_j X_j$ where $a_j > 0$ and the X_j are irreducible curves in S. Since $[X] \in V_m(D) \subset V(D)$, X is a (reducible) rational curve and so all the curves X_j must be rational. We can see this by considering any one-parameter family $\chi' \to \Gamma'$ of irreducible rational curves specializing to X. With this family apply the same sequence of normalizations to arrive at a family $\mathcal{Y}' \to B'$ of nodal curves, with general fiber $\mathbb{P}^1$, that admits a regular map $\mathcal{Y}' \to \chi'$. Since the fibers of $\mathcal{Y}' \to B'$ are reduced curves of arithmetic genus zero, every component of X is dominated by a rational curve and so must be rational. Therefore $[X_j] \in V(D_j)$ where D_j are divisor classes such that $\sum a_j D_j = D$.

We begin by showing that X_j belongs to $V_{m_j}(D_j)$ for suitable m_j. We approach this by limiting the number of points of intersection of the curves X_j with E. This gives a better bound on the dimension of the family of such curves X.

Now, say $X_j \in \tilde{V}_{\bar{m}^j}(D_j)$ where $\bar{m}^j = (m_1^j, ..., m_k^j)$ is a sequence of positive integers with $\sum_i m_i^j = X_j \cdot E$. Let $\nu_j : X_j^{\nu} \to X_j$ be the normalization map and Y be a reducible fiber of $\mathcal{Y}$. Choose any irreducible component X_j^0 of Y dominating X_j (hence dominating the normalization X_j^{ν}), and let $\pi_j : X_j^0 \to X_j^{\nu} \to X_j$ be the restriction of $\pi : \mathcal{Y} \to B$ to X_j^0.

By counting points, clearly $\sum_i (m_i^j - 1) \geq X_j \cdot E - \#(X_j \cap E)$ so

$$\begin{aligned}\sum_{i,j}(m_i^j - 1) &\geq \sum_j X_j \cdot E - \sum_j \#(X_j \cap E) \\ &= \sum_j X_j \cdot E - \sum_j \#(a_j X_j \cap E)\end{aligned}$$

By assumption $\tau = \#\{(X_j \cap E) \cap (X_i \cap E), \text{for } i \neq j\}$ so

$$\begin{aligned}\sum_{i,j}(m_i^j - 1) &\geq \sum_j X_j \cdot E - \left(\#\left(\left(\sum_j a_j X_j\right) \cap E\right) + \tau\right) \\ &\geq \sum_j X_j \cdot E - \#(X \cap E) - \tau\end{aligned}$$

$D \equiv X = \sum a_j X_j$ so

$$\begin{aligned}\sum_{i,j}(m_i^j - 1) &\geq \sum_j X_j \cdot E - \#(D \cap E) - \tau \\ &\geq \left(D \cdot E - \sum_j (a_j - 1) D_j \cdot E\right) - (D \cdot E - m + 1) - \tau \\ &= -\sum_j (a_j - 1)(D_j \cdot E) + m - 1 - \tau. \qquad (5.1)\end{aligned}$$

Now

$$\begin{aligned}\sum_j \dim V_{\bar{m}^j}(D_j) &= \sum_j \left(r(D_j) - \sum_i (m_i^j - 1)\right) \\ &= \sum_j (-K_S \cdot D_j - 1) - \sum_{i,j}(m_i^j - 1).\end{aligned}$$

On the other hand,

$$\begin{aligned}\dim V_m(D) - 1 = r_m(D) - 1 &= -K_S \cdot D - m - 1 \\ &= -K_S \cdot (\sum_j a_j D_j) - m - 1 \\ &= \sum_j a_j(-K_S \cdot D_j) - m - 1.\end{aligned}$$

Assume for now that $\tau \leq 2$. Since the components meet along E in τ points, this imposes τ independent conditions on $\sum_j V_{\bar{m}^j}(D_j)$. So $\dim V_m(D) - 1$ is at most $\sum_j V_{\bar{m}^j}(D_j) - \tau$, thus

$$\sum_j a_j(-K_S \cdot D_j) - m - 1 \;\leq\; \sum_j(-K_S \cdot D_j - 1) - \sum_{i,j}(m_i^j - 1) - \tau. \quad (5.2)$$

Then using inequality 5.1, the above becomes

$$\begin{aligned}\sum_j a_j(-K_S \cdot D_j) - m - 1 \;&\leq\; \sum_j -K_S \cdot D_j - s + \\ &+\sum_j (a_j - 1)D_j \cdot E - m + 1 + \tau - \tau,\end{aligned}$$

which simplifies to

$$\sum_j (a_j - 1)(-K_S - E) \cdot D_j + s - 2 \leq 0.$$

On $S = \mathbb{F}_n$, $K_S = -2E - (n+2)F$, so for any divisor $\alpha C + \beta F$ with $\alpha, \beta \geq 0$,

$$(-K_S - E) \cdot (\alpha C + \beta F) = (E + (n+2)F) \cdot (\alpha C + \beta F) = \beta + \alpha(n+2) \geq 0.$$

Thus, since $s \geq 2$, $\sum_j (a_j - 1)(-K_S - E) \cdot D_j + s - 2 \leq 0$ can only happen if $a_j = 1$ for all j and $s = 2$. Since $a_j = 1$ for all j, there is a unique component of Y mapping to each X_j and so each X_j, $j = 1$ or 2, can have at most one point of intersection multiplicity m_j. Therefore each X_j is in $V_{m_j}(D_j)$ for some positive integers m_j, $j = 1$ or 2.

The inequality 5.1 now simplifies to $(m_1 - 1) + (m_2 - 1) \geq m - 1 - \tau \Rightarrow m_1 + m_2 \geq m + 1 - \tau$. Inequality 5.2 becomes $(m_1 - 1) + (m_2 - 1) \leq m + 1 - 2 - \tau \Rightarrow m_1 + m_2 \leq m + 1 - \tau$. Combining these gives $m_1 + m_2 = m + 1 - \tau$.

This completes *part 1.*

(*Part 2.*: X contains E) Now suppose $X = aE + \sum_{j=2}^{s} a_j X_j$. Here we show that X_j belongs to $V_{m_j}(D_j)$ for suitable m_j. We approach this by limiting the number of points of intersection of the curves X_j with E. This gives a better bound on the dimension of the family of such curves X.

Consider the family $\mathcal{Y} \to B$. The total space of $\mathcal{Y}$ is smooth and every fiber is a union of smooth rational curves meeting transversely, and whose dual graph is a tree. Take Y a special fiber of $\mathcal{Y}$ and decompose it into Y_E, the union of the irreducible components mapping to E, and Y_R, the union of the remaining components. Then take Y_R and decompose it into s parts, such that Y_j is the union of the irreducible components mapping to X_j. Let $\{Z_i\}_{i \in I}$ be the irreducible components of Y_E. For each i, let α_i be the degree of the map $\mu|_{Z_i} : Z_i \to E$, so $\sum \alpha_i = a$. Similarly, let $\{Z_{j,i}\}_{i \in I_j}$ denote the connected components of Y_j and $\alpha_{j,i}$ the degree of the restriction map $\mu|_{Z_{j,i}} : Y_j \to X_j$, so that $\sum_i \alpha_{j,i} = a_j$.

Let ε be the number of points of intersection of Y_E with Y_R. Since the dual graph of Y is a tree, then the number of pairwise points of intersection of the connected components $Z_{j,i}$ of Y_j and the connected components Z_i of Y_E is equal to the total number of all such connected components minus one. In other words,

$$\begin{aligned} \varepsilon = \#(Y_E \cap Y_R) &= \#\{\text{connected components of } \; Y_E\} + \\ &\quad + \sum_j \#\{\text{connected components of } \; Y_j\} - 1 \\ &\quad \leq a + \sum_j a_j - 1. \end{aligned}$$

Now suppose $X_j \in \tilde{V}_{\bar{m}^j}$ for each $j = 1, ..., s$. Let $\nu_j : X_j^\nu \to X_j$ be the normalization map. Choose any irreducible component X_j^0 of Y dominating X_j (hence dominating the normalization X_j^ν), and let $\pi_j : X_j^0 \to X_j^\nu \to X_j$ be the restriction of π to X_j^ν.

The total number of points of the pullback $\nu_j^*(E)$ of E to X_j^ν is

$$\begin{aligned} \#\nu_j^*(E) &\leq \#\pi_j^*(E) \\ &= \#(X_j^0 \cap Y_E), \end{aligned}$$

and hence

$$\begin{aligned} \sum_j \#\nu_j^*(E) &\leq \sum_j \#(X_j^0 \cap Y_E) \\ &\leq \#(Y_R \cap Y_E) \\ &= \varepsilon \end{aligned}$$

with strict inequality if any $a_j > 1$. But the sum of degrees of E on the curves X_j satisfies

$$\begin{aligned} \sum_j \deg(\pi_j^* E) &\geq (\sum_j X_j) \cdot E \\ &= ((D - aE - \sum_j (a_j - 1)D_j) \cdot E) \\ &= D \cdot E + an - \sum_j (a_j - 1)D_j \cdot E. \end{aligned}$$

Comparing $\#\nu_j^*(E)$, the number of points of the pullbacks of E to the normalization X_j^ν with the degree of this pullback, $\deg(\pi_j^* E)$, we conclude that there must be multiplicities in these divisors: specifically, the sum $\sum_{i,j}(m_i^j - 1)$ of the multiplicities minus one must be the differences of these numbers, so that

$$
\begin{aligned}
\sum_{i,j}(m_i^j - 1) \;\geq\; & \sum_j \deg \pi_j^*(E) - \varepsilon - (D \cdot E - (m-1)) \\
& \geq D \cdot E + an - \sum_j (a_j - 1) D_j \cdot E + \\
& \qquad -a - \sum_j a_j + 1 - D \cdot E + m - 1 \\
& = a(n-1) + 1 - \sum_j (a_j - 1) D_j \cdot E - \sum_j a_j + m - 1 \qquad (5.3)
\end{aligned}
$$

This allows us to bound the number of degrees of freedom of the curves X_j:

$$
\begin{aligned}
\sum_j \dim V_{\bar{m}^j}(D_j) \;=\; & \sum_j r(D_j) - \sum_{i,j}(m_i^j - 1) \\
= \; & \sum_j (-K_S \cdot D_j - 1) - \sum_{i,j}(m_i^j - 1) \\
\leq & \sum_j -K_S \cdot D_j - s - a(n-1) - 1 + \\
& \qquad + \sum_j (a_j - 1) D_j \cdot E + \sum_j a_j - m + 1
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\dim V_m(D) - 1 \;&=\; r_m(D) - 1 \\
&= -K_S \cdot D - m - 1 \\
&= a(-K_S \cdot E) + \sum_j a_j(-K_S \cdot D_j) - m - 1 \\
&= -a(n-2) - m - 1 + \sum_j a_j(-K_S \cdot D_j).
\end{aligned}
$$

But $\dim V_m(D) - 1$ must be at most $\sum_j \dim V_{\bar{m}^j}(D_j)$. So

$$
-a(n-2) - m - 1 + \sum_j a_j(-K_S \cdot D_j) \;\leq\; \sum_j -K_S \cdot D_j - s - \sum_{i,j}(m_i^j - 1) \qquad (5.4)
$$

$$
\leq \sum_j -K_S \cdot D_j - s - a(n-1) - 1 \;+\; \sum_j (a_j - 1)(D_j \cdot E) + \sum_j a_j - m + 1
$$

and so

$$a - 2 + \sum_j (a_j - 1)[((-K_S - E) \cdot D_j) - 1] \leq 0.$$

Now

$$(-K_S - E) \cdot D_j = (C + 2F) \cdot D_j \geq n + 2$$

for any curve D_j on $S = \mathbb{F}_n$ other than E and F, so for $D_j \neq F$ we have

$$\begin{aligned} 0 \geq\ & a - 2 + (a_j - 1)((-K_S - E) \cdot D_j) \\ & \geq a - 2 + (a_j - 1)(n + 2) \\ & \geq a - 2 + (a_j - 1)(m + 2) \end{aligned}$$

which can be true if and only if $a_j = 1$, since $a \geq 1$, $a_j \geq 1$, and $m \geq 1$. Since $a_j = 1$ for all j, there is a unique component of Y mapping to each X_j, so each X_j can have at most one point of intersecion multiplicity $m_j > 1$ with E. The inequality 5.3 now simplifies to $\sum_j (m_j - 1) \geq a(n-1) + 1 - s + m - 1$ and so $\sum_j m_j \geq a(n-1) + m$. Inequality 5.4 becomes $\sum_j (m_j - 1) \leq a(n-2) + m + 1 - s$ and so $\sum_j m_j \leq a(n-2) + m + 1$. This gives $a(n-1) + m \leq a(n-2) + m + 1$, thus $a \leq 1$. So we have $\sum_{j=1}^s m_j = n + m - 1$. Therefore each X_j is a general member of $V_{m_j}(D_j)$ for some positive integers $m_1, ..., m_s$ with $\sum_{j=1}^s m_j = n + m - 1$. □

Determining the reducible fibers of $\mathcal{Y}_m \to B_m$ from the reducible fibers of $\chi_m \to \Gamma_m$ follows as in the $V(D)$ case with analagous arguments for the $\tilde{J}_m$ type fibers. We assume that the singularity arising in the $\tilde{J}_m$ type fibers is of type $A_{\gamma-1}$. When this singularity is resolved, the resulting fiber is a chain $\tilde{J}_1, \tilde{J}_{0,1}, \tilde{J}_{0,2}, ..., \tilde{J}_{0,\gamma-1}, \tilde{J}_2$.

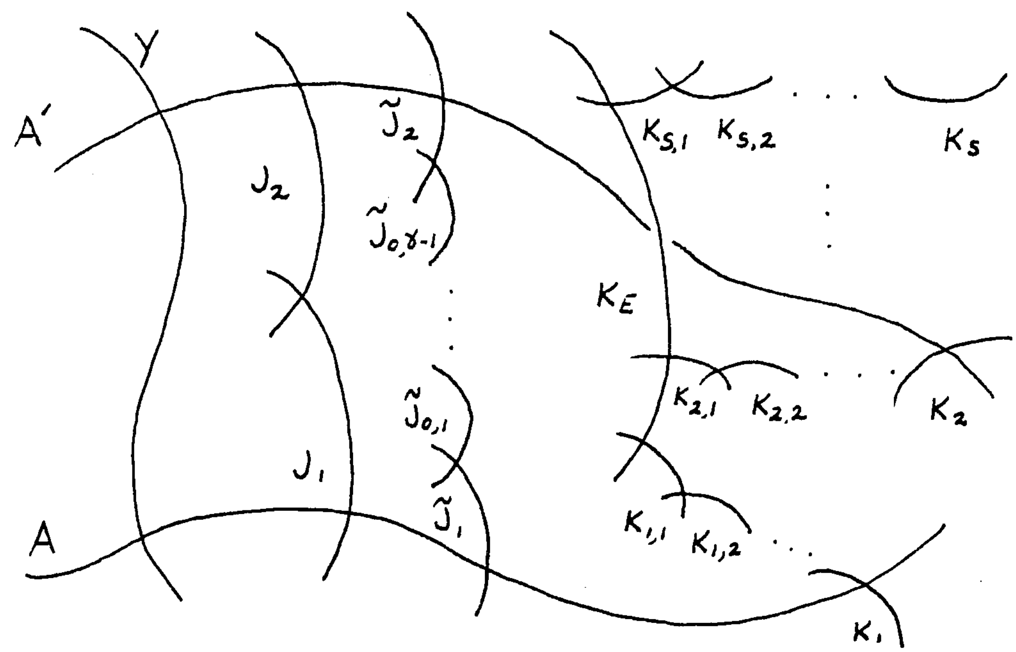

Figure 5.2: The Surface $\mathcal{Y}_m$.

5.3 Néron-Severi Group of $\mathcal{Y}_m$

Since $\mathcal{Y}_m$ is a ruled surface then as before, the Néron-Severi group of $\mathcal{Y}$ is freely generated by the class of a fiber of the ruling, the class of a section of the ruling, and the classes of all the irreducible curves contained in fibers of the ruling and disjoint from the section. Let Y be the class of a fiber of $\mathcal{Y}_m$ and A correspond to a section of $f : \mathcal{Y}_m \to B$ parametrizing curves through the base point q_1. We choose the following set of generators for the Néron-Severi group of $\mathcal{Y}_m$:

$$\{A, Y\} \cup \{J_2\}_{b \in B_{J_m}} \cup \{\tilde{J}_{0,1}, \tilde{J}_{0,2}, ..., \tilde{J}_{0,\gamma-1}, \tilde{J}_2\}_{b \in B_{\tilde{J}_m}} \cup$$

$$\{K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i - 1} K_i\}_{b \in B_{K_m}, i=1,...,s} - \{K_1\}$$

The below relations follow easily:

$$A \cdot Y = 1, \quad Y^2 = 0, \quad J_2^2 = -1,$$

$$\tilde{J}_{0,j}^2 = -2, \quad \tilde{J}_2^2 = -1, \quad \tilde{J}_{0,j} \cdot \tilde{J}_{0,j+1} = 1, \quad \tilde{J}_{0,m-1} \cdot \tilde{J}_2 = 1$$

$$K_E^2 = -s, \quad K_{i,j}^2 = -2, \quad K_i^2 = -1,$$

$$K_E \cdot K_{i,1} = 1, \quad K_{i,j} \cdot K_{i,j+1} = 1, \qquad K_{i,\gamma_i - 1} \cdot K_i = 1$$

Other than these and A^2, there are no additional non-zero intersections. The calculation of A^2 is done in the next chapter.

5.4 Counting Reducible Fibers of $f : \mathcal{Y}_m \to B$

Now we calculate the number of fibers of type J_m, type $\tilde{J}_m$ and type K_m on $\mathcal{Y}_m$. This count will be used in the calculation of $N_m(D)$.

Lemma 5.4.1 *1. If X is a reducible fiber of $\mathcal{Y}_m \to B$ of type J_m then the number of such fibers for a given decomposition $D = D_1 + D_2$, denoted $j_m(D_1, D_2)$, is*

$$\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1} N_{m_1}(D_1) N_{m_2}(D_2)(D_1 \cdot D_2).$$

2. If X is a reducible fiber of $\mathcal{Y}_m \to B_m$ of type $\tilde{J}_m$ with $\tau = 1$, and, assuming that the components meet at the smooth point of multiplicity m_i, then the number of such fibers for a given decomposition $D = D_1 + D_2$, denoted $\tilde{j}_m(D_1, D_2)$, is

$$\left(\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1} \Theta(D_1) + \binom{r_m(D) - 2}{r_{m_1}(D_1) - 2} \Theta(D_2) \right) N_{m_1}(D_1) N_{m_2}(D_2),$$

where

$$\Theta(D_i) = \begin{cases} E \cdot D_i & m_1 = 1 \text{ AND } m_2 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

3. If X is a reducible fiber of $\mathcal{Y}_m \to B_m$ containing E, then the number of type K_m fibers for a given decomposition $D-E = D_1+...+D_s$, denoted $k_m(D_1, D_2, ..., D_s)$, is

$$\Delta \prod_{i=1}^{s} N(D_i)\Lambda(D_i),$$

where

$$\Delta = \frac{1}{R}\binom{r_m(D)-2}{r_{m_1}(D_1)-1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})},$$

and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2. \end{cases}$$

R represents the repetition factor accounting for repetition of the components in the set $\{D_2, ..., D_s\}$.

Proof. (*Part 1.*) If X is a reducible fiber of $\chi_m \to \Gamma_m$ not containing E as a component, then X must contain exactly two components X_1 and X_2 meeting transversely at one point such that $\pi(X_i) = D_i$, $D_i > 0$, $D_i \neq E$, and $D = D_1 + D_2$.

Since D must pass through $r_m(D) - 1$ general points, each X_i can hold at most $r_{m_i}(D_i)$ of these $r_m(D)-1$ general points. Since $D = D_1+D_2$ and $m_1+m_2 = m+1-\tau$,

$$\begin{aligned} r_m(D) - 1 &= (-K_S \cdot D - 1) - m + 1 - 1 \\ &= -K_S \cdot D - (m+1) \\ &= -K_S \cdot (D_1 + D_2) - (m+1) \\ &= -K_S \cdot D_1 - K_S \cdot D_2 - m_1 - m_2 - \tau \\ &= -K_S \cdot D_1 - m_1 - K_S \cdot D_2 - m_2 - \tau \\ &= r_{m_1}(D_1) + r_{m_2}(D_2) - \tau \end{aligned}$$

If $\tau = 0$ (so type J_m fibers) then it follows that X_i must contain exactly $r_{m_i}(D_i)$ points. Recalling that the point q_1 lies on X_1, then there are

$$\binom{r_m(D)-2}{r_{m_1}(D_1)-1}$$

ways to distribute the $r_m(D)-1$ points on the two curves. For each distribution of points there exist $N_{m_i}(D_i)$ curves $X_i \in V_{m_i}(D_i)$ containing $r_{m_i}(D_i)$ points. So there are

$$\binom{r_m(D)-2}{r_{m_1}(D_1)-1} N_{m_1}(D_1)N_{m_2}(D_2)$$

such $[X] \in \Gamma_m$. For each $[X] \in \Gamma_m$, Γ_m has $D_1 \cdot D_2$ smooth branches (Proposition 2.6 in [CH1]). So there will be $D_1 \cdot D_2$ points of Γ_m^ν lying over each $[X]$. Finally we note that χ_m^ν is smooth along such fibers (Proposition 2.7 in [CH1]). Therefore

$$j_m(D_1, D_2) = \binom{r_m(D)-2}{r_{m_1}(D_1)-1} N_{m_1}(D_1)N_{m_2}(D_2)(D_1 \cdot D_2).$$

This completes *part 1.*

(*Part 2.*) If $\tau = 1$ (so type $\tilde{J}_m$ fibers) then

$$r_m(D) - 1 = r_{m_1}(D_1) + r_{m_2}(D_2) - 1.$$

On the other hand $\tau = 1$ implies that one component must pass through a point of intersection of the other component with E. This may be imposed on either of the two curves. If the condition is imposed on the second curve then it follows that X_1 must contain exactly $r_{m_1}(D_1)$ points and X_2 must contain exactly $r_{m_2}(D_2)-1$ points. Recalling that the point q_1 lies on X_1, then there are

$$\binom{r_m(D)-2}{r_{m_1}(D_1)-1}$$

ways to distribute the $r_m(D)-1$ points on the two curves. If the condition is imposed on the first curve then it follows that X_1 must contain exactly $r_{m_1}(D_1)-1$ points and X_2 must contain exactly $r_{m_2}(D_2)$ points . Recalling that the point q_1 lies on X_1, then there are

$$\binom{r_m(D)-2}{r_{m_1}(D_1)-2}$$

ways to distribute the $r_m(D)-1$ points on the two curves. For each distribution of points there exist $N_{m_i}(D_i)$ curves $X_i \in V_{m_i}(D_i)$ containing $r_{m_i}(D_i)$ points, counting the condition where appropriate. So there are

$$\left(\binom{r_m(D)-2}{r_{m_1}(D_1)-1}+\binom{r_m(D)-2}{r_{m_1}(D_1)-2}\right)N_{m_1}(D_1)N_{m_2}(D_2).$$

such $[X] \in \Gamma_m$.

It is also possible that the condition might be imposed at any of the points of $E \cap D_i$. We assume that the condition is imposed at the point of multiplicity m_i. In other words there is only a choice when $m_1 = m_2 = 1$. For each $[X] \in \Gamma_m$, Γ_m has $\Theta(D_i)$ smooth branches where X_i is not the component that the condition is imposed on(Proposition 2.6 in [CH1]), where

$$\Theta(D_i) = \begin{cases} E \cdot D_i & m_1 = 1 \text{ AND } m_2 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

So there will be $\Theta(D_i)$ points of Γ_m^ν lying over each $[X] \in \Gamma_m$. Finally we note that χ_m^ν is smooth along such fibers (Proposition 2.7 in [CH1]). Therefore

$$\tilde{\jmath}_m(D_1,D_2) = \left(\binom{r_m(D)-2}{r_{m_1}(D_1)-1}\Theta(D_1)+\binom{r_m(D)-2}{r_{m_1}(D_1)-2}\Theta(D_2)\right)N_{m_1}(D_1)N_{m_2}(D_2).$$

This completes *part 2.*

(*Part 3.*) If X is a reducible fiber of $\chi_m \to \Gamma_m$ containing E as a component, then X has irreducible components

$$\{K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i-1}, K_i\}$$

with $i = 1, ..., s$ such that $\pi(K_i) = D_i$, $\pi(K_E) = E$, $D_i > 0$, and $D_i \neq E$. For each i let m_i be the multiplicity with which D_i meets E at a smooth point.

Since D must pass through $r_m(D) - 1$ general points, X_i can contain at most $r_{m_i}(D_i)$ of the $r(D) - 1$ general points $q_1, ..., q_{r_m(D)-1}$. Since $D = D_1 + ... + D_s + E$ and $\sum_{i=1}^{s} m_i = n + m - 1$, then

$$\begin{aligned}
r_m(D) - 1 &= (-K_S \cdot D - 1) - m + 1 - 1 \\
&= -K_S \cdot (D_1 + ... + D_s + E) - (m+1) \\
&= -K_S \cdot D_1 - K_S \cdot D_2 - ... - K_S \cdot D_s - K_S \cdot E - (m+1) \\
&= -K_S \cdot D_1 - ... - K_S \cdot D_s - n + 2 - (m+1) \\
&= \sum_{i=1}^{s} -K_S \cdot D_i - n - m + 1 \\
&= \sum_{i=1}^{s} -K_S \cdot D_i - \sum_{i=1}^{s} m_i \\
&= \sum_{i=1}^{s} (-K_S \cdot D_i - m_i) \\
&= \sum_{i=1}^{s} r_{m_i}(D_i).
\end{aligned}$$

It follows that each X_i must contain exactly $r_{m_i}(D_i)$ points. Recalling that the point q_1 lies on X_1, then there are

$$\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})}$$

ways to distribute the $r_m(D) - 1$ points on the s curves. For each distribution of points there exist $N_{m_i}(D_i)$ curves $X_i \in V_{m_i}(D_i)$ containing $r_{m_i}(D_i)$ points. So there are

$$\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_{s-1}}(D_{s-1})} \prod_{i=1}^{s} N_{m_i}(D_i)$$

such $[X] \in \Gamma$.

By [CH1] Proposition 2.6 and 2.7 we have the following: In a neighborhood of $[X] \in \Gamma_m$, Γ consist of $\Pi_{m_i=1} D_i \cdot E$ smooth branches, Γ_α, and for all i such that D_i has a point P_i of intersection multiplicity $m_i \geq 2$ with E, exactly $m_i - 1$ nodes of nearby fibers will tend to P_i. Along the smooth branch γ_α, each point P_{i,α_i} has a single point lying over it which will be a node of the fiber X^ν of $\chi_m^\nu \to \Gamma_m^\nu$ corresponding to $[X] \in \Gamma_m$.

The fibers X^ν of $\chi_m^\nu \to \Gamma_m^\nu$ corresponding to $[X] \in \Gamma_m$ are all curves obtained by normalizing X at all the nodes of the D_i, at all but one of the points of intersection of E with each of the components D_i with $m_i = 1$, at all the transverse points of intersection of D_i with E for $m_i \geq 2$, and finally taking the partial normalization of X at P_i having an ordinary node over P_i. Therefore we are able to conclude that $k_m(D_1, ..., D_s)$ is as stated in the Lemma. $\square$

Chapter 6

The general recursion for $N_m(D)$

We will now prove the theorem below. Just as for the calculation of $N(D)$, the proof is motivated by the following fact: given any two line bundles L and M on S, we have

$$\pi^*L \cdot \pi^*M = \deg \pi(L \cdot M) = N_m(D)(L \cdot M).$$

We begin by proving some useful lemmas. In particular, we write π^*L as a linear combination of the elements of the Néron-Severi group of $\mathcal{Y}_m$, and we calculate A^2.

6.1 Theorem

We recall the necessary facts and definitions needed to use the below theorems. In general we have $r_m(D) = -K_S \cdot D - m$. A reducible fiber of the family $\chi_m \to \Gamma_m$ of type J_m or type $\tilde{J}_m$ has irreducible components X_1, X_2, with $D = D_1 + D_2$, X_i is general in $V_{m_i}(D_i)$ for positive integers m_1, m_2 such that $m_1 + m_2 = m + 1 - \tau$ where $\tau = \#(X_1 \cap X_2 \cap E)$. Related to the type $\tilde{J}_m$ fibers for a particular decomposition we have: $\tilde{\mathcal{J}}(D_1, D_2, \tau, \gamma)$ representing the coefficient of $N_{m_1}(D_1)N_{m_2}(D_2)$ in the formula.

A reducible fiber of the family $\chi_m \to \Gamma_m$ of type K_m has irreducible components $E, X_1, ..., X_s$, with $D = E + D_1 + ... + D_s$, X_i is general in $V_{m_i}(D_i)$ for a collection of positive integers $m_1, ..., m_s$ such that $\sum_{i=1}^{s} m_i = n + m - 1$. The corresponding components X_i on χ_m^ν have singularities of type $A_{\gamma_i - 1}$ where $\gamma_i = \frac{k}{m_i}$ and we assume for computational purposes that $k = \text{lcm}(m_1, ..., m_s)$. Related to the number of type K_m fibers for a particular decomposition we have:

$$\Delta = \frac{1}{R}\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_s - 1}(D_{s-1})}$$

where R represents the repetition factor accounting for repetition of components in the set $\{D_2, ..., D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2. \end{cases}$$

The calculation of A^2, to be shown later, involves choosing a section A' disjoint from A. As a result we see a corresponding definition for Δ' describing how the remaining $r_m(D) - 3$ points (not counting q_1 and q_2) can be distributed on the s curves:

$$\Delta' = \frac{1}{R'}\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1, r_{m_2}(D_2) - 1, r_{m_3}(D_3), r_{m_4}(D_4), ..., r_{m_s - 1}(D_{s-1})},$$

where R' represents the repetition factor accounting for repetition of components in the set $\{D_3, ..., D_s\}$. These are the ingredients in the following theorem.

Theorem 6.1.1 *For any effective divisor $D \neq E$ on $\mathbb{F}_n$,*

$$\begin{aligned} nN_m(D) = & \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m+1}} N_{m_1}(D_1)N_{m_2}(D_2)(D_1 \cdot D_2) \times \\ & \left[(C \cdot D_1)(C \cdot D_2)\binom{r_m(D)-3}{r_{m_1}(D_1)-1} - (C \cdot D_2)^2\binom{r_m(D)-3}{r_{m_1}(D_1)-2}\right] + \\ + & \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m+1-\tau}} N_{m_1}(D_1)N_{m_2}(D_2)\tilde{\mathcal{J}}(D_1, D_2, \tau, \gamma) \\ + & \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_3,\ldots,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\left[\frac{\Delta'}{2}(\gamma_1+\gamma_2)(C \cdot D)^2\right] + \\ - & \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_2,\ldots,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\Delta\left[\gamma_1(C \cdot D_1 - C \cdot D)^2 + \sum_{i=2}^{s}\gamma_i(C \cdot D_i)^2\right]. \end{aligned} \tag{6.1}$$

6.2 Theorem case $\tau = 1$:

We recall the necessary facts and definitions needed to use the below theorems. In general we have $r_m(D) = -K_S \cdot D - m$. A reducible fiber of the family $\chi_m \to \Gamma_m$ of type J_m or type $\tilde{J}_m$ has irreducible components X_1, X_2, with $D = D_1 + D_2$, X_i is general in $V_{m_i}(D_i)$ for positive integers m_1, m_2 such that $m_1 + m_2 = m + 1 - \tau$ where $\tau = \#(X_1 \cap X_2 \cap E)$. Related to the number of type $\tilde{J}_m$ fibers for a particular decomposition we have:

$$\Theta(D_i) = \begin{cases} E \cdot D_i & m_1 = 1 \text{ AND } m_2 = 1 \\ 1 & \textit{otherwise.} \end{cases}$$

A reducible fiber of the family $\chi_m \to \Gamma_m$ of type K_m has irreducible components $E, X_1, ..., X_s$, with $D = E + D_1 + ... + D_s$, X_i is general in $V_{m_i}(D_i)$ for a collection of positive integers $m_1, ..., m_s$ such that $\sum_{i=1}^{s} m_i = n + m - 1$. The corresponding components X_i on χ_m^ν have singularities of type $A_{\gamma_i - 1}$ where $\gamma_i = \frac{k}{m_i}$ and we assume for computational purposes that $k = \text{lcm}(m_1, ..., m_s)$. Related to the number of type K_m fibers for a particular decomposition we have:

$$\Delta = \frac{1}{R}\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1, r_{m_2}(D_2), r_{m_3}(D_3), ..., r_{m_s - 1}(D_{s-1})}$$

where R represents the repetition factor accounting for repetition of components in the set $\{D_2, ..., D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2. \end{cases}$$

The calculation of A^2, to be shown later, involves choosing a section A' disjoint from A. As a result we see a corresponding definition for Δ' describing how the remaining

$r_m(D) - 3$ points (not counting q_1 and q_2) can be distributed on the s curves:

$$\Delta' = \frac{1}{R'}\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1, r_{m_2}(D_2) - 1, r_{m_3}(D_3), r_{m_4}(D_4), ..., r_{m_s-1}(D_{s-1})}$$

, where R' represents the repetition factor accounting for repetition of components in the set $\{D_3, ..., D_s\}$. These are the ingredients in the following theorem.

Theorem 6.2.1 *For any effective divisor $D \neq E$ on $\mathbb{F}_n$, and assuming $\tau = 1$ for all $\tilde{J}_m$ fibers we have*

$$\begin{aligned}
nN_m(D) = & \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m+1}} N_{m_1}(D_1)N_{m_2}(D_2)(D_1 \cdot D_2) \times \\
& \left[(C \cdot D_1)(C \cdot D_2)\binom{r_m(D) - 3}{r_{m_1}(D_1) - 1} - (C \cdot D_2)^2\binom{r_m(D) - 3}{r_{m_1}(D_1) - 2}\right] + \\
+ & \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m}} N_{m_1}(D_1)N_{m_2}(D_2)(C \cdot D)^2 \times \\
& \left[\Theta(D_1)\binom{r_m(D) - 3}{r_{m_1}(D_1) - 1} + \Theta(D_2)\binom{r_m(D) - 3}{r_{m_1}(D_1) - 2}\right] + \\
- & \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m}} \gamma N_{m_1}(D_1)N_{m_2}(D_2)(C \cdot D_2)^2 \times \\
& \left[\Theta(D_1)\binom{r_m(D) - 2}{r_{m_1}(D_1) - 1} + \Theta(D_2)\binom{r_m(D) - 2}{r_{m_1}(D_1) - 2}\right] + \\
+ & \sum_{\substack{D_1+...+D_s=D-E \\ \{D_3,...,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\left[\frac{\Delta'}{2}(\gamma_1 + \gamma_2)(C \cdot D)^2\right] + \\
- & \sum_{\substack{D_1+...+D_s=D-E \\ \{D_2,...,D_s\}}} \left(\prod_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\right)\Delta\left[\gamma_1(C \cdot D_1 - C \cdot D)^2 + \sum_{i=2}^{s}\gamma_i(C \cdot D_i)^2\right]
\end{aligned} \tag{6.2}$$

6.3 Proof of Theorem: Some Useful Lemmas

Let L be any line bundle in Pic $\mathbb{F}_n$. We can write the class of its pullback to $\mathcal{Y}_m$ as a linear combination of the elements of the Néron-Severi group of $\mathcal{Y}_m$. Since we know the image in $\mathbb{F}_n$ of the components of the reducible fibers of $f : \mathcal{Y}_m \to B_m$, we can calculate the degrees on all such components of π^*L of any line bundle.

Take any effective divisor class D on S with nonnegative self-intersection and $V_m(D) \neq \emptyset$. Choose $r_m(D) - 1$ general points $q_1, q_2, ..., q_{r_m(D)-1}$ on S. Consider the family $\chi_m \to \Gamma_m$ of curves $X \in V_m(D)$ passing through the q_i. Let $\chi_m^\nu \to \Gamma_m^\nu$, $\mathcal{Y}_m \to B_m$, and

$$\pi : \mathcal{Y}_m \to \chi_m^\nu \times_{\Gamma_m^\nu} B_m \to \chi_m^\nu \to \chi_m \hookrightarrow \Gamma_m \times S \to S$$

be as described in the set-up of the Rational Fibration method in Section 2.1.

Lemma 6.3.1 *For L any line bundle in Pic($\mathbb{F}_n$),*

$$\begin{aligned}\pi^*L &= (L\cdot D)A - (L\cdot D)A^2Y - \sum_{b\in B_{J_m}} (L\cdot D_2)J_2 + \\ &- \sum_{b\in B_{\tilde{J}_m}} \left[\sum_{j=1}^{\gamma-1} j(L\cdot D_2)\tilde{J}_{0,j} + \gamma(L\cdot D_2)\tilde{J}_2\right] + \\ &+ \sum_{b\in B_{K_m}} \left[\gamma_1(L\cdot D_1 - L\cdot D)K_E + \sum_{j=1}^{\gamma_i-1}(\gamma_1 - j)(L\cdot D_1 - L\cdot D)K_{1,j} + \right.\\ &\left. + \sum_{i=2}^{s}\sum_{j=1}^{\gamma_i}(\gamma_1(L\cdot D_1 - L\cdot D) - jL\cdot D_i)K_{i,j}\right] \qquad (6.3)\end{aligned}$$

where B_{J_m}, $B_{\tilde{J}_m}$, and B_{K_m} are the subsets of points of B parametrizing fibers of type J_m, $\tilde{J}_m$, and K_m respectively.

Proof. Take L any line bundle in Pic($\mathbb{F}_n$). Recall that Pic$\mathcal{Y}_m$ is generated by a section of the ruling, A, a fiber of the ruling, F, and all the irreducible curves contained in fibers of the ruling and disjoint from the section. So we can write the class of its pullback to $\mathcal{Y}_m$ as a linear combination of

$$\{A, Y\} \cup \{J_2\}_{b \in B_{J_m}} \cup \{\tilde{J}_{0,1}, \tilde{J}_{0,2}, ..., \tilde{J}_{\gamma-1}, \tilde{J}_2\}_{b \in B_{\tilde{J}_m}} \cup$$

$$\{K_E, K_{i,1}, K_{i,2}, ..., K_{i,\gamma_i-1}, K_i\}_{b \in B_{K_m}, i=1,...,s} - \{K_1\}.$$

We define the coefficient of $\Box$ as $a_\Box$ in this linear combination allowing us to write the pullback of L as:

$$\pi^* L = a_A A + a_Y Y + J_m^L + \tilde{J}_m^L + K_m^L$$

where

$$J_m^L = \sum_{b \in B_{J_m}} a_{J_2} J_2,$$

$$\tilde{J}_m^L = \sum_{b \in B_{\tilde{J}_m}} \left(\sum_{j=1}^{\gamma} a_{0,j} \tilde{J}_{0,j} \right),$$

and

$$K_m^L = \sum_{b \in B_{K_m}} \left(a_E K_E + \sum_{j=1}^{\gamma_1 - 1} a_{1,j} K_{1,j} + \sum_{i=2}^{s} \sum_{j=1}^{\gamma_i} a_{i,j} K_{i,j} \right).$$

Note: here $\tilde{J}_{0,\gamma} = \tilde{J}_2$ and $K_{i,\gamma_i} = K_i$. Now we determine the coefficients for $\pi^* L$ by evaluating the following products.

Since $L \cdot D = \pi^* L \cdot Y = a_A A \cdot Y$, $a_A = L \cdot D$.

Since π collapses A to the base point q we have $0 = \pi^* L \cdot A = a_A A^2 + a_Y Y \cdot A$, so $a_Y = -(L \cdot D) A^2$.

Type J_m:

Since $L \cdot D_2 = \pi^* L \cdot J_2 = a_{J_2} J_2^2$, $a_{J_2} = -(L \cdot D_2)$.

Type $\tilde{J}_m$:

Since $L \cdot D_1 = \pi^* L \cdot \tilde{J}_1 = (L \cdot D) A \cdot \tilde{J}_1 + a_{0,1} \tilde{J}_{0,1} \cdot \tilde{J}_1$, $a_{0,1} = L \cdot D_1 - L \cdot D = -L \cdot D_2$. Since $0 = \pi^* L \cdot \tilde{J}_{0,1} = -L \cdot D_2 \tilde{J}_{0,1}^2 + a_{0,2} \tilde{J}_{0,2} \cdot \tilde{J}_{0,1}$, $a_{0,2} = -(-L \cdot D_2)(-2) = -2L \cdot D_2$. Since $0 = \pi^* L \cdot \tilde{J}_{0,2} = -L \cdot D_2 \tilde{J}_{0,1} \cdot \tilde{J}_{0,2} - 2L \cdot D_2 \tilde{J}_{0,2}^2 + a_{0,3} \tilde{J}_{0,3} \cdot \tilde{J}_{0,2}$, so $a_{0,3} = -3L \cdot D_2$ Continuing: $a_{0,j} = -jL \cdot D_2$ for all j.

Type K_m:

Since $L \cdot D_1 = \pi^* L \cdot K_1 = a_{1,\gamma_1 - 1} K_{1,\gamma_1 - 1} \cdot K_1 + (L \cdot D) A \cdot K_1$, $a_{1,\gamma_1 - 1} = L \cdot D_1 - L \cdot D$. Since $0 = \pi^* L \cdot K_{1,\gamma_1 - 1} = a_{1,\gamma_1 - 2} K_{1,\gamma_1 - 2} \cdot K_{1,\gamma_1 - 1} + a_{1,\gamma_1 - 1} K_{1,\gamma_1 - 1}^2$, we have $a_{1,\gamma_1 - 2} = 2(L \cdot D_1 - L \cdot D)$. Since $0 = \pi^* L \cdot K_{1,\gamma_1 - 2} = a_{1,\gamma_1 - 3} K_{1,\gamma_1 - 3} \cdot K_{1,\gamma_1 - 2} + a_{1,\gamma_1 - 2} K_{1,\gamma_1 - 2}^2 + a_{1,\gamma_1 - 1} K_{1,\gamma_1 - 1} \cdot K_{1,\gamma_1 - 2}$, we have $a_{1,\gamma_1 - 3} = 3(L \cdot D_1 - L \cdot D)$. Continuing: $a_{1,\gamma_1 - j} = j(L \cdot D_1 - L \cdot D)$. Thus $a_{1,j} = (\gamma_1 - j)(L \cdot D_1 - L \cdot D)$ for all j. Now $0 = \pi^* L \cdot K_{1,1} = a_E K_E \cdot K_{1,1} + a_{1,1} K_{1,1}^2 + a_{1,2} K_{1,2} \cdot K_{1,1}$, so $a_E = 2a_{1,1} - a_{1,2} = 2(\gamma_1 - 1)(L \cdot D_1 - L \cdot D) - (\gamma_1 - 2)(L \cdot D_1 - L \cdot D)$. Thus $a_E = \gamma_1 (L \cdot D_1 - L \cdot D)$. For $i \neq 1$; $L \cdot D_i = \pi^* L \cdot K_i = a_{i,\gamma_i - 1} K_{i,\gamma_i - 1} \cdot K_i + a_i K_i^2$, so $a_i = a_{i,\gamma_i - 1} - L \cdot D_i$. For $i \neq 1$; $0 = \pi^* L \cdot K_{i,\gamma_1 - 1} = a_{i,\gamma_i - 2} K_{i,\gamma_i - 2} \cdot K_{i,\gamma_i - 1} + a_{i,\gamma_i - 1} K_{i,\gamma_i - 1}^2 + a_i K_i \cdot K_{i,\gamma_i - 1}$, so $a_{i,\gamma_i - 1} = a_{i,\gamma_i - 2} - L \cdot D_i$. Continuing: $a_{i,j+1} = a_{i,j} - L \cdot D_i$. We also know that $a_{i,j-1} - 2a_{i,j} + a_{i,j+1} = 0$. Now $0 = \pi^* L \cdot K_{i,1} = a_E K_E \cdot K_{i,1} + a_{i,1} K_{i,1}^2 + a_{i,2} K_{i,2} \cdot K_{i,1}$, so $2a_{i,1} = a_E - a_{i,2} = \gamma_1 (L \cdot D_1 - L \cdot D) + a_{i,1} - L \cdot D_i$ and so $a_{i,1} = \gamma_1 (L \cdot D_1 - L \cdot D) - L \cdot D_i$. Recall that $a_{i,j+1} = a_{i,j} - L \cdot D_i$, so $a_{i,2} = a_{i,1} - L \cdot D_i$, $a_{i,3} = a_{i,2} - L \cdot D_i = a_{i,1} - 2L \cdot D_i$, and $a_{i,4} = a_{i,3} - L \cdot D_i = a_{i,1} - 3L \cdot D_i$. Continuing, we get $a_{i,j} = a_{i,j-1} - L \cdot D_i = a_{i,1} - (j-1) L \cdot D_i = \gamma_1 (L \cdot D_1 - L \cdot D) - jL \cdot D_i$. Finally $a_i = a_{i,\gamma_i - 1} - L \cdot D_i = \gamma_1 (L \cdot D_1 - L \cdot D) - \gamma_i L \cdot D_i$, and so all the coefficients are as claimed in the lemma.

□

Next we compute A^2. We do so using the same techniques as used in the calculation of A^2 for $V(D)$.

Lemma 6.3.2 *If A corresponds to a section of $f : \mathcal{Y}_m \to B_m$ parametrizing curves through q_1 where $f : \mathcal{Y}_m \to B_m$ is as described in Section 5.1 then*

$$A^2 = \frac{1}{2}\Bigg[- \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m+1}} N_{m_1}(D_1)N_{m_2}(D_2)(D_1 \cdot D_2)\binom{r_m(D)-3}{r_{m_1}(D_1)-1} +$$
$$- \sum_{\substack{D_1+D_2=D \\ m_1+m_2=m}} \gamma N_{m_1}(D_1)N_{m_2}(D_2) \times$$
$$\left(\Theta(D_1)\binom{r_m(D)-3}{r_{m_1}(D_1)-1} + \Theta(D_2)\binom{r_m(D)-3}{r_{m_1}(D_1)-2}\right) +$$
$$- \sum_{\substack{D_1+\ldots+D_s=D-E \\ \{D_3,\ldots,D_s\}}} (\gamma_1+\gamma_2)\Delta' \Pi_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i)\Bigg]$$

where

$$\Delta' = \frac{1}{R'}\binom{r_m(D)-3}{r_{m_1}(D_1)-1, r_{m_2}(D_2)-1, r_{m_3}(D_3), r_{m_4}(D_4), \ldots, r_{m_{s-1}}(D_{s-1})},$$

R' represents the repetition factor accounting for repetition of components in the set $\{D_3, \ldots, D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2 \end{cases}.$$

Proof. Choose a base point $q_2 \neq q_1$. q_2 determines a section A' of $f : \mathcal{Y}_m \to B_m$ parametrizing curves through q_2. A and A' are determined by the distinct base points q_1 and q_2 and as such are disjoint. By symmetry $A^2 = (A')^2$ and $A \cdot A' = 0$ so

$$2A^2 = (A - A')^2.$$

Let $S_{J_m} \subset B_{J_m}$ be the subset for which q_1 and q_2 lie on distinct components. Let $A_{J_m}(D_1, D_2)$ denote the number of such fibers of type J_m, so

$$A_{J_m}(D_1, D_2) = N_{m_1}(D_1)N_{m_2}(D_2)(D_1 \cdot D_2)\binom{r_m(D)-3}{r_{m_1}(D_1)-1}.$$

This follows from the proof for $j_m(D_1, D_2)$ noting that q_2 lies on J_2. Define $S_{\tilde{J}_m}$ and S_{K_m} similarly for fibers of type $\tilde{J}_m$ and K_m in which q_1 and q_2 lie on different components. Let $A_{\tilde{J}_m}(D_1, D_2)$ denote the number of such fibers of type $\tilde{J}_m$, so

$$A_{\tilde{J}_m}(D_1, D_2) = N_{m_1}(D_1)N_{m_2}(D_2)\left(\Theta(D_1)\binom{r_m(D)-3}{r_{m_1}(D_1)-1} + \Theta(D_2)\binom{r_m(D)-3}{r_{m_1}(D_1)-2}\right).$$

This follows from the proof for $\tilde{j}_m(D_1, D_2)$ noting that q_2 lies on $\tilde{J}_2$. Similarly let $A_{K_m}(D_1, D_2, ..., D_s)$ denote the number of such fibers of type K_m, so

$$A_{K_m}(D_1, D_2, ..., D_s) = \Delta' \Pi_{i=1}^{s} N_{m_i}(D_i)\Lambda(D_i).$$

where

$$\Delta' = \frac{1}{R'}\binom{r_m(D)-3}{r_{m_1}(D_1)-1, r_{m_2}(D_2)-1, r_{m_3}(D_3), r_{m_4}(D_4), ..., r_{m_{s-1}}(D_{s-1})}$$

and R' represents the repetition factor accounting for repetition of components in the set $\{D_3, ..., D_s\}$, and

$$\Lambda(D_i) = \begin{cases} E \cdot D_i & m_i = 1 \\ 1 & m_i \geq 2. \end{cases}$$

This follows from the proof for $k_m(D_1, ..., D_s)$ noting that q_2 lies on D_2. Now we determine the coefficients of $A' - A$. For the type J_m fibers, let σ_J be the blowdown of J_2. Let $\bar{A} := \sigma_J(A)$, and $\bar{A}' := \sigma_J(A')$. The following is clear:

$$Y \sim J_1 + J_2, \quad A = \sigma_J^*(\bar{A}), \quad A' = \sigma_J^*(\bar{A}') - J_2, \quad \text{and} \quad \sigma_J^*(\bar{A}' - \bar{A}) = lY,$$

for some l. It follows that, in terms of the type J_m fibers,

$$A' - A = lY - J_2 = (l-1)Y + J_1 + J_2 - J_2 = (l-1)Y + J_1.$$

For the type $\tilde{J}_m$ fibers, let $\sigma_{\tilde{J}}$ be the blowdown of $\tilde{J}_2, \tilde{J}_{0,\gamma-1}, ..., \tilde{J}_{0,2}, \tilde{J}_{0,1}$ in the listed order. Let $\bar{A} := \sigma_{\tilde{J}}(A)$ and $\bar{A}' := \sigma_{\tilde{J}}(A')$.

$Y \sim \tilde{J}_1 + \sum_{j=1}^{\gamma-1} \tilde{J}_{0,j} + \tilde{J}_2$

$A = \sigma_{\tilde{J}}^*(\bar{A})$

$A' = \sigma_{\tilde{J}}^*(\bar{A}') - \tilde{J}_{0,1} - 2\tilde{J}_{0,2} - ... - (\gamma - 1)\tilde{J}_{0,\gamma-1} - \gamma_1 \tilde{J}_2$

Now we know $\sigma_{\tilde{J}}^*(\bar{A}' - \bar{A}) = lY$. So

$$\begin{aligned} A' - A &= lY - \tilde{J}_{0,1} - 2\tilde{J}_{0,2} - ... - (\gamma - 1)\tilde{J}_{0,\gamma-1} - \gamma \tilde{J}_2 \\ &= (l - \gamma)Y + \gamma\tilde{J}_1 + \gamma\tilde{J}_{0,1} + ... + \gamma\tilde{J}_{0,\gamma-1} + \gamma\tilde{J}_2 \\ &\qquad -\tilde{J}_{0,1} - 2\tilde{J}_{0,2} - ... - (\gamma - 1)\tilde{J}_{0,\gamma-1} - \gamma\tilde{J}_2 \\ &= (l - \gamma)Y + \gamma\tilde{J}_1 + (\gamma - 1)\tilde{J}_{0,1} + ... + 2\tilde{J}_{0,\gamma-2} + \tilde{J}_{0,\gamma-1} \end{aligned}$$

For the type K_m fibers, let σ_K be the blowdown of $K_i, K_{i,\gamma_i-1}, ..., K_{i,2}, K_{i,1}$ in the listed order beginning with $i = s$ down to $i = 2$, then blow down $K_E, K_{1,1}, ..., K_{1,\gamma_1-1}$. Let $\bar{A} := \sigma_K(A)$ and $\bar{A}' := \sigma_K(A')$.

$Y \sim K_E + \sum_{i=1}^{s} \left(K_i + \sum_{j=1}^{\gamma_i-1} K_{i,j} \right)$

$A = \sigma_K^*(\bar{A})$

$$\begin{aligned} A' &= \sigma_K^*(\bar{A}') - K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - ... - (\gamma_1 - 1)K_{1,1} - \gamma_1 K_E + \\ &\quad -\gamma_1 \sum_{i\geq 3} \left(\sum_{j=1}^{\gamma_i-1} K_{i,j} + K_i \right) - (\gamma_1 + 1)K_{2,1} - (\gamma_1 + 2)K_{2,2} - ... + \\ &\quad -(\gamma_1 + \gamma_2 - 1)K_{2,\gamma_2-1} - (\gamma_1 + \gamma_2)K_2 \end{aligned}$$

Now we know $\sigma_K^*(\bar{A}' - \bar{A}) = lY$. So

$$\begin{aligned}
A'-A &= lY - K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1-1)K_{1,1} - \gamma_1 K_E + \\
&\quad -\gamma_1 \sum_{i\geq 3}\left(\sum_{j=1}^{\gamma_i-1} K_{i,j} + K_i\right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\
&\quad -(\gamma_1+\gamma_2-1)K_{2,\gamma_2-1} - (\gamma_1+\gamma_2)K_2 \\
&= (l-\gamma_1)Y + \gamma_1 Y - K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1-1)K_{1,1} - \gamma_1 K_E + \\
&\quad -\gamma_1 \sum_{i\geq 3}\left(\sum_{j=1}^{\gamma_i-1} K_{i,j} + K_i\right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\
&\quad -(\gamma_1+\gamma_2-1)K_{2,\gamma_2-1} - (\gamma_1+\gamma_2)K_2 \\
&= (l-\gamma_1)Y + \gamma_1\left(K_E + \sum_{i=1}^{s}\left(K_i + \sum_{j=1}^{\gamma_i-1} K_{i,j}\right)\right) + \\
&\quad -K_{1,\gamma_1-1} - 2K_{1,\gamma_1-2} - \dots - (\gamma_1-1)K_{1,1} - \gamma_1 K_E + \\
&\quad -\gamma_1 \sum_{i\geq 3}\left(\sum_{j=1}^{\gamma_i-1} K_{i,j} + K_i\right) - (\gamma_1+1)K_{2,1} - (\gamma_1+2)K_{2,2} - \dots + \\
&= (l-\gamma_1)Y + \Big((\gamma_1)K_1 + K_{1,1} + 2K_{1,2} + \dots + +(\gamma_1-1)K_{1,\gamma_1-1}\Big) + \\
&\qquad -\Big((\gamma_2)K_2 + K_{2,1} + 2K_{2,2} + \dots + +(\gamma_2-1)K_{1,\gamma_2-1}\Big)
\end{aligned}$$

Let $\kappa_i = K_{i,1} + 2K_{i,2} + \dots + (\gamma_i - 1)K_{i,\gamma_i-1} + (\gamma_i)K_i$

Now let σ blowdown all J_2's, all the components of the type $\tilde{J}_m$ fibers except $\tilde{J}_1$, and all components of the type K_m fibers except K_1 as above. Then arguing as before we have:

$$A' - A = mY + \sum_{b\in S_{J_m}} J_1 + \sum_{b\in S_{\tilde{J}_m}}\left(\gamma\tilde{J}_1 + \sum_{j=1}^{\gamma-2}(\gamma-j)\tilde{J}_{0,j}\right) + \sum_{b\in S_{K_m}}(\kappa_1 - \kappa_2)$$

$$\begin{aligned} 2A^2 = (A' - A)^2 &= m^2 Y^2 + \sum_{b \in S_{J_m}} J_1^2 + \\ &+ \sum_{b \in S_{\tilde{J}_m}} \left(\gamma \tilde{J}_1 + \sum_{j=1}^{\gamma-2} (\gamma - j) \tilde{J}_{0,j} \right)^2 + \sum_{b \in S_{K_m}} (\kappa_1 - \kappa_2)^2 \\ &= -\sum_{b \in S_{J_m}} 1 - \sum_{b \in S_{\tilde{J}_m}} \gamma - \sum_{b \in S_{K_m}} (\gamma_1 + \gamma_2) \end{aligned}$$

Therefore

$$\begin{aligned} A^2 &= \frac{1}{2} \left(-\sum_{b \in S_{J_m}} 1 - \sum_{b \in S_{\tilde{J}_m}} \gamma - \sum_{b \in S_{K_m}} (\gamma_1 + \gamma_2) \right) \\ &= \frac{1}{2} \left(- \sum_{\substack{D_1 + D_2 = D \\ m_1 + m_2 = m+1}} A_{J_m}(D_1, D_2) \right. \\ &\quad - \sum_{\substack{D_1 + D_2 = D \\ m_1 + m_2 = m}} A_{\tilde{J}_m}(D_1, D_2) \\ &\quad \left. - \sum_{\substack{D_1 + \ldots + D_s = D - E \\ \{D_3, \ldots, D_s\}}} (\gamma_1 + \gamma_2) A_{K_m}(D_1, .., D_s) \right) \end{aligned}$$

where in the decompositions of $D - E$ above, the first and second components are distinguished. □

6.4 Proof of $N_m(D)$ Recursion Theorem

Proof. of Theorem 6.1 Let C be a section of the $\mathbb{P}^1$-bundle $\mathbb{F}_n \to \mathbb{P}^1$ disjoint from E, $C \sim E + nF$. Now we calculate $\pi^* C \cdot \pi^* C$. As for $N(D)$, since $\pi^* C \cdot \pi^* C = C \cdot C \deg \pi$ then

$$\pi^* C \cdot \pi^* C = n N_m(D).$$

By Lemma 6.3.1 on page 84

$$\begin{aligned}\pi^*C &= (C\cdot D)A-(C\cdot D)A^2Y-\sum_{b\in B_{J_m}}(C\cdot D_2)J_2+\\ &-\sum_{b\in B_{\tilde{J}_m}}\left[\sum_{j=1}^{\gamma-1}j(C\cdot D_2)\tilde{J}_{0,j}+\gamma(C\cdot D_2)\tilde{J}_2\right]+\\ &+\sum_{b\in B_{K_m}}\left[\gamma_1(C\cdot D_1-C\cdot D)K_E+\sum_{j=1}^{\gamma_i-1}(\gamma_1-j)(C\cdot D_1-C\cdot D)K_{1,j}+\right.\\ &\qquad\left.+\sum_{i=2}^{s}\sum_{j=1}^{\gamma_i}(\gamma_1(C\cdot D_1-C\cdot D)-jC\cdot D_i)K_{i,j}\right].\end{aligned}$$

Using short-hand notation

$$\pi^*C = (C\cdot D)A-(C\cdot D)A^2Y+J_m^C+\tilde{J}_m^C+K_m^C$$

we compute the intersection product on $\mathcal{Y}_m$ of the pull-back of line bundle C on $\mathbb{F}_n$ with itself. This gives

$$\pi^*C\cdot\pi^*C=-(C\cdot D)^2A^2+J_m^C\cdot J_m^C+\tilde{J}_m^C\cdot\tilde{J}_m^C+K_m^C\cdot K_m^C.$$

Similar to the $N(D)$ case we obtain:

$$\begin{aligned}nN_m(D) &= -(C\cdot D)^2A^2-\sum_{\substack{D_1+D_2=D\\ m_1+m_2=m+1}}(C\cdot D_2)^2 j_m(D_1,D_2)+\\ &-\sum_{\substack{D_1+D_2=D\\ m_1+m_2=m}}\gamma(C\cdot D_2)^2\tilde{j}_m(D_1,D_2)+\\ &-\sum_{\substack{D_1+\ldots+D_s=D-E\\ \{D_2,\ldots,D_s\}}}\left[\gamma_1(C\cdot D_1-C\cdot D)^2+\sum_{i=2}^{s}\gamma_i(C\cdot D_i)^2\right]k_m(D_1,\ldots,D_s).\end{aligned}$$

Simplifying as in the $N(D)$ case gives the desired result. □

Chapter 7

Example $N_2(C+bF)$ on $\mathbb{F}_n$

Before using the formula to do the calculation, we make a few remarks regarding the geometry of this example to give some insight into the computation. $V_2(C+bF)$ has dimension

$$r_2(C+bF) = (2E+(n+2)F)(C+bF) - 2 = 2b+n,$$

so we choose $2b+n-1$ points $q_1, ..., q_{2b+n-1}$. Consider the family χ_2 of curves in $V_2(C+bF)$ through these points. Let A be the class of q_1 and Y an irreducible fiber in the family. There are two types of reducible fibers of χ_2: type J and type $\tilde{J}$.

The first type, type J, is a decomposition of $X \in \chi$ into $X_1 + X_2$ where $\pi(X_1) = C+(b-1)F$ and $\pi(X_2) = F$ such that X_1 is general in $V_2(C+(b-1)F)$, X_2 is general in $V(F)$ and $q_1 \in X_1$. We also have a decomposition as above but with $\pi(X_1) = F$ and $\pi(X_2) = C+(b-1)F$ such that X_1 is general in $V(F)$, X_2 is general in $V_2(C+(b-1)F)$, and $q_1 \in X_1$.

The second type, type $\tilde{J}$, is a decomposition of $X \in \chi$ into $X_1 + X_2$ where

$\pi(X_1) = C+(b-1)F$ and $\pi(X_2) = F$ such that X_1 is in $V(C+(b-1)F)$, X_2 is $V(F)$, $q_1 \in X_1$, and X_1 and X_2 intersect at a point on E. We also have a decomposition as above but with $\pi(X_1) = F$ and $\pi(X_2) = C + (b-1)F$ such that X_1 is in $V(F)$, X_2 is in $V(C + (b-1)F)$, $q_1 \in X_1$, and X_1 and X_2 intersect at a point on E.

We make a few comments about decompositions of type $\tilde{J}$. We have $r_{m_1}(D_1) + r_{m_2}(D_2) - 1 = r_2(D) - 1$. This says that the component X_1 may contain $r_{m_1}(D_1)$ or $r_{m_1}(D_1) - 1$ of the $r_2(D) - 1 = 2b+n$ points. If X_1 contains $r_{m_1}(D_1)$ of the points then the component X_2 must intersect X_1 at any one of its $X_1 \cdot E$ points of intersection with E. Note: $X_i \in V(D_i) \Rightarrow X_i$ meets E transversely. If X_1 contains $r_{m_1}(D_1) - 1$ of the points then the component X_2 contain exactly $r_{m_2}(D_2)$ of the points and X_1 must intersect X_2 at any one of its $X_2 \cdot E$ points of intersection with E. We also note that since $\dim V(F) = r(F) = 1$, the component X_i such that $\pi(X_i) = F$ can not both contain a point and have the condition imposed on it that it must pass through a point on E. As a result, several pieces of the computation will drop out, i.e. have no contribution to the calculation.

The relevant dimensions are

$$r_2(C + (b-1)F) = 2b + n - 2, \qquad r(C + (b-1)F) = 2b + n - 1, \text{ and } \qquad r(F) = 1.$$

So $nN_2(C+bF)$

$$
\begin{aligned}
&= N_2(C+(b-1)F)N(F)(C+(b-1)F\cdot F)\times \\
&\quad \left[(C\cdot C+(b-1)F)(C\cdot F)\binom{2b+n-3}{2b+n-3}-(C\cdot F)^2\binom{2b+n-3}{2b+n-4}\right] \\
&+ N(F)N_2(C+(b-1)F)(F\cdot C+(b-1)F)\times \\
&\quad \left[(C\cdot F)(C\cdot C+(b-1)F)\binom{2b+n-3}{0}-(C\cdot C+(b-1)F)^2\binom{2b+n-3}{-1}\right] \\
&+ (C\cdot C+bF)^2N(C+(b-1)F)N(F)\times \\
&\quad \left[(C+(b-1)F\cdot E)\binom{2b+n-3}{2b+n-2}+(F\cdot E)\binom{2b+n-3}{2b+n-3}\right] \\
&- 2(C\cdot F)^2N(C+(b-1)F)N(F)\times \\
&\quad \left[(C+(b-1)F\cdot E)\binom{2b+n-2}{2b+n-2}+(F\cdot E)\binom{2b+n-3}{2b+n-3}\right] \\
&+ (C\cdot C+bF)^2N(F)N(C+(b-1)F)\times \\
&\quad \left[(F\cdot E)\binom{2b+n-3}{0}+(C+(b-1)F\cdot E)\binom{2b+n-3}{-1}\right] \\
&- 2(C\cdot C+(b-1)F)^2N(F)N(C+(b-1)F)\times \\
&\quad \left[(F\cdot E)\binom{2b+n-2}{0}+(C+(b-1)F\cdot E)\binom{2b+n-3}{-1}\right] \\
&= 2(b-2)((n+b-1)-(2b+n-3))+2(b-2)(n+b-1)+ \\
&\quad (n+b)^2-2(b-1+2b+n-2)+(n+b)^2-2(n+b-1)^2 \\
&= 2n(b-1).
\end{aligned}
$$

Thus $N_2(C+bF)=2(b-1)$.

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