TRAJECTORIES OF THE CONTINUOUS NEWTON METHOD APPLIED TO THE PRIMAL-DUAL BARRIER EQUATIONS OF LINEAR PROGRAMMING

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CHAPTER 1

INTRODUCTION AND STATEMENT OF RESULTS

Primal-dual interior point methods were developed in the late 1980's following a history of work primarily by Soviet authors including Shor [Sh],and Nemirovsky and Yudin [NY], leading up to the celebrated paper by Khachian [Kh] in which he establishes the polynomial complexity of linear programming. Next was work by Karmarkar who for the first time developed a polynomial time algorithm [K] for linear programming, the so-called projective scaling algorithm, for which he claimed timing results of practical significance. However, given the proprietary nature of his work (Bell Labs), its impact remained limited. The claims he made immediately stimulated tremendous interest and brought about those methods which are at present considered to be the most powerful and efficient procedures for solving linear programming problems. These are known as primal-dual log-barrier interior-point methods. These methods are not directly based on Karmarkar's work but instead are related to so-called affine scaling methods. The affine scaling algorithm was originally proposed by Dikin [D] in 1967 and later updated by Barnes [Ba] and Vanderbei, Meketon, and Freedman [VMF].

Given $c \in \mathbf{R^n}, b \in \mathbf{R^m}, A \in \mathbf{R^{m\times n}}$, the **symmetric-dual form** of linear programming may be described, conveniently, by

$$\min c^t x \qquad \text{subject to } Ax - s = b$$

$$(LP) \qquad A \in \mathbf{R^{m\times n}},\ \ x \in \mathbf{R^n},\ x_i \geq 0\ \forall i,\ \ s \in \mathbf{R^m},\ s_j \geq 0\ \forall j.$$

Associated to (LP) is the **dual** linear program given by

$$\max b^t y \qquad \text{subject to } A^t y + r = c$$

$$(DP) \qquad A^t \in \mathbf{R^{n\times m}},\ \ y \in \mathbf{R^m},\ y_j \geq 0\ \forall j,\ \ \ r \in \mathbf{R^n},\ r_i \geq 0\ \forall i.$$

Points $(s^*, x^*), (r^*, y^*)$ are solutions of (LP) and (DP) if and only if the **Karush-Kuhn-Tucker (KKT) conditions,**

$$Ax^* - s^* = b,$$

$$A^t y^* + r^* = c,$$

$$X^* r^* = 0,$$

$$Y^* s^* = 0,$$

$$x^*, r^* \in \mathbf{R^n},\ x_i^* \geq 0,\ r_i^* \geq 0,\ \forall i,\ y^*, s^* \in \mathbf{R^m}, y_j^* \geq 0,\ s_j^* \geq 0\ \forall j$$

are satisfied, where X^*, Y^* are the diagonal matrices with diagonal entries x_i^*, y_j^*, respectively.

In 1986, Gill et al. [GMSTW] established a connection between Karmarkar's method and the logarithmic barrier method [Fr], [FM]. The **logarithmic barrier function** [Fr2] for (LP) is

$$B(x, s, \mu) = c^t x - \mu\Big(\sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln s_j\Big)$$

where $\mu > 0$. The approach is to minimize $B(x, s, \mu)$ for a given μ, decrease μ, and then minimize the new $B(x, s, \mu)$. For (LP), the **logarithmic barrier subproblem** becomes

$$\min c^t x - \mu\Big(\sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln s_j\Big)$$

$$\text{subject to } \ Ax - s = b, x \in \mathbf{R^n},\ x_i > 0\ \forall i, s \in \mathbf{R^m}, s_j > 0\ \forall j. \tag{1.1}$$

Suppose that for a given $\mu > 0, (x^*, s^*)$ is a local (hence global) minimum of $B(x, s, \mu)$ such that $Ax^* - s^* = b, x^* \in \mathbf{R^n},\ x_i^* > 0\ \forall i,\ s^* \in \mathbf{R^m}, s_j^* > 0\ \forall j$. Then there exists a KKT vector $\lambda \in \mathbf{R^m_{\geq 0}}$ such that

$$Ax^* - s^* = b,$$

$$A^t\lambda + \mu {X^*}^{-1} e^n = c,$$

$$\mu {S^*}^{-1} e^m = \lambda,$$

where e^k denotes the vector $(1, \ldots, 1) \in \mathbf{R^k}$. Meggido [M] proposed using the logarithmic barrier method simultaneously on the primal and dual problems. Algorithms based on this method were quickly developed, [KMY] and [MA]. Many others joined in and a large body of work appeared in a very short time, [Re],[Me],[LMS], [Go1],[V], [MAR], etc. Given the notation above, the first-order necessary (KKT) conditions for (1.1) are

$$Ax^* - s^* = b, \tag{1.2}$$

$$A^t y^* + r^* = c, \tag{1.3}$$

$$X^* r^* = \mu e^n, \tag{1.4}$$

$$Y^* s^* = \mu e^m, \tag{1.5}$$

$$r^*, x^* \in \mathbf{R^n}, r_i^* > 0, x_i^* > 0\ \forall i,\ s^*, y^* \in \mathbf{R^m}, s_j^* > 0, y_j^* > 0\ \forall j. \tag{1.6}$$

Equations (1.2)-(1.5) are the **primal-dual barrier equations.** Note, the primal-dual barrier equations differ from the KKT conditions only by the presence of $\mu > 0$. We adopt the following notation through out this work.

Definition 1.7. *Denote*

$$\mathbf{R}_+^{\mathbf{2n+2m}} = \{(s,r,x,y) \in \mathbf{R^m} \oplus \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m} : s_j > 0, y_j > 0 \ \forall j, r_i > 0, x_i > 0 \ \forall i\}$$

and

$$\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} = \{(s,r,x,y) \in \mathbf{R^m} \oplus \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m} : s_j \geq 0, y_j \geq 0 \ \forall j, r_i \geq 0, x_i \geq 0 \ \forall i\}.$$

Note that $\mathbf{R}_+^{\mathbf{2n+2m}}, \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ are the convex cones which are the positive, non-negative orthant of $\mathbf{R^m} \oplus \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m}$, respectively.

Given the KKT conditions for (LP) and (DP) and the primal-dual barrier equations, define the following function.

Definition 1.8. *Given $\mu \geq 0$, define $F_\mu : \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \to \mathbf{R^m} \oplus \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m}$ by*

$$F_\mu(s,r,x,y) = \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr - \mu e^n \\ Ys - \mu e^m \end{pmatrix}$$

where X, Y are diagonal matrices with diagonal entries x_i, y_i, respectively.

Remark 1.9. *Given the (LP) setting, $F_\mu(s,r,x,y)$ was defined only on $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. However, we note that $F_\mu(s,r,x,y)$ has components that are polynomial in the components of (s,r,x,y) and therefore $F_\mu(s,r,x,y)$ can, in fact, be defined on all of $\mathbf{R^m} \oplus \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m}$. Given this, we will differentiate F_μ on $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ without regard to the boundary.*

Given Definition 1.8, the Jacobian matrix, $DF_\mu(s,r,x,y)$, of $F_\mu(s,r,x,y)$ is given by

$$\text{(1.10)} \qquad DF_\mu(s,r,x,y) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}$$

where I_{kk} denotes the identity matrix in $\mathbf{R}^{\mathbf{k}\times\mathbf{k}}$.

Definition 1.11. *Given (LP) and (DP), the* **Central Path** *for (LP) and (DP) is the set of all points $(s,r,x,y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ parameterized by $\mu > 0$ and satisfying the Central Path equations given by $F_\mu(s,r,x,y) = 0$.*

Note, from above, if a point $(s^*,r^*,x^*,y^*) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ were on the Central Path corresponding to $\mu = 0$ then the points $(s^*,x^*),(r^*,y^*)$ are solutions for (LP) and (DP). Given (LP), (DP), points $(x,s) \in \mathbf{R}^\mathbf{n} \oplus \mathbf{R}^\mathbf{m}$ such that $x_i \geq 0\ \forall i, s_j \geq 0\ \forall j$ and $(y,r) \in \mathbf{R}^\mathbf{m} \oplus \mathbf{R}^\mathbf{n}$ such that $y_j \geq 0\ \forall j, r_i \geq 0\ \forall i$ are called **feasible** if $Ax - s = b, A^t y + r = c$; otherwise they are called **infeasible**. If feasible points $(x,s),(y,r)$ additionally have strictly positive components they are called **strictly feasible.**

Primal-dual interior-point methods use a Newton-type approach to generate iterates that approach and follow, approximately, the Central Path, by requiring iterates to satisfy a suitable neighborhood condition. In particular, at a given iterate (s^k,r^k,x^k,y^k) and for a prescribed value of μ, the **Newton Vector**

$$\text{(1.12)} \qquad \Phi_{N,\mu}(s,r,x,y) = (-1)DF_\mu(s,r,x,y)^{-1}F_\mu(s,r,x,y)$$

is obtained and then, after a judicious choice of step length σ, the next iterate is calculated as

$$(s^{k+1},r^{k+1},x^{k+1},y^{k+1}) = (s^k,r^k,x^k,y^k) + \sigma\Phi_{N,\mu}(s^k,r^k,x^k,y^k).$$

Properties of the Central Path were given by Meggido [M], Bayer and Lagarias [BL1], [BL2], and Fiacco and McCormick [FM].

Primal-dual interior-point numerical methods are based on the following algorithmic framework where $H(s^k, r^k, x^k, y^k, \mu), N$ are dependent on the specific algorithm and (s^k, r^k, x^k, y^k) denotes the kth iterate of the algorithm.

Given $(s^k, r^k, x^k, y^k) \in \mathbf{R}_+^{\mathbf{2n+2m}}$,

Solve for (p_s, p_r, p_x, p_y) in

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X^k & R^k & 0 \\ Y^k & 0 & 0 & S^k \end{pmatrix} \begin{pmatrix} p_s \\ p_r \\ p_x \\ p_y \end{pmatrix} = -H(s^k, r^k, x^k, y^k, \mu)$$

Set $(s^{k+1}, r^{k+1}, x^{k+1}, y^{k+1}) = (s^k, r^k, x^k, y^k) + \sigma_k(p_s, p_r, p_x, p_y)$ for some $\sigma_k \in (0,1)$ for which $(s^{k+1}, r^{k+1}, x^{k+1}, y^{k+1}) \in N$.

A typical choice for a feasible interior-point methods, [MTY], is

$$H(s^k, r^k, x^k, y^k) = \begin{pmatrix} 0 \\ 0 \\ X^k r^k - \gamma\mu e^n \\ Y^k s^k - \gamma\mu e^m \end{pmatrix}, \quad \mu = \frac{(r^k)^t x^k + (s^k)^t y^k}{n+m}, \quad \gamma \in (0,1)$$

and $N = N_{-\infty}(\beta), \beta \in (0,1)$, where the residual vectors $r_b \in \mathbf{R^m}, r_c \in \mathbf{R^n}$ are given by

$$r_b = Ax - s - b, r_c = A^t y + r - c,$$

and

$$N_{-\infty}(\beta) = \{(s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} : r_b = r_c = 0, x_i r_i, y_j s_j \geq \beta\mu\}.$$

For the primal linear problem, the relationship between algorithmic methods and the continuous trajectories related to the methods has been addressed by Karmarkar [K] and Bayer and Lagarias [BL1], [BL2], [La]. The work by Meggido [M], and Bayer and Lagarias [BL1], [BL2] involved the continuous Central Path trajectory for the primal-dual problem.

Prior to 1993, primal-dual interior-point algorithms assumed a starting point that was strictly feasible, in a given neighborhood of the Central Path, and produced iterates that were strictly feasible. [BL1], [BL2], and [La] studied the geometry of trajectories based on a vector field corresponding to Karmarkar's algorithm [K] and trajectories based on an affine vector field.

Work by [BL1], [BL2], and [La] involved trajectories to solutions of (LP), hence was based on the primal problem with $\mu = 0$, required an initial point (s^0, r^0, x^0, y^0) that was strictly feasible, and required the additional assumption that the feasible set be bounded.

Bayer and Lagarias [BL1], [BL2], [La] studied the geometry of trajectories that are integrals of an affine vector field for the primal LP. Their approach was to use a nonlinear change of variables, a Legendre transform, based upon a projection of the gradient of a logarithmic barrier function applied to the constraints associated to the linear problem. In their work the primal trajectories associated with strictly feasible points in $\mathbf{R}^{\mathbf{n}}_{+}$ were analyzed and it was assumed that the feasible set for the problem was bounded. It was shown that the affine scaling vector field could be realized as a steepest descent vector field of the associated objective function with respect to a certain Riemannian metric defined on the relative interior of the feasible set. Given the Legendre transform coordinate mapping, there exists a global metric such that every geodesic with respect to the metric is an affine primal trajectory associated to a given objective function. Also, it was shown that every affine primal trajectory for a non-constant objective on the feasible set is in fact a geodesic of the global metric. Since the metric geometry is isometric to Euclidean geometry on $\mathbf{R^n}$, the metric geometry is geodesically complete. In studying the central path under the Legendre transform, it was shown that for the primal-dual problem the central path projects onto the central path for the primal and dual problems via orthogonal projections.

While the work of Bayer and Lagarias in the *Transactions of the AMS* [BL1], [BL2], [La], provided much insight about primal methods and trajectories through strictly feasible points, there are questions remaining for primal-dual methods. First, at the time, the state-of-the-art algorithms were primal interior-point methods that were greatly influenced by [K]. Given the prominence of interior-point methods, they restricted their analysis to trajectories through strictly feasible points. For primal-dual methods, information is needed regarding trajectories through arbitrary points $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. That is, what is the behavior of infeasible trajectories and trajectories through points on the boundary of the feasible set? Also, they required that the primal feasible set be bounded, which was motivated by Karmarkar's algorithm which has the same restriction. If (LP) and (DP) have strictly feasible points then at least one of the feasible sets is unbounded. Finally, the use of the Legendre transform to obtain results does not provide much information about the geometry of the trajectories. Hence, their methods differ from this thesis and the exact relationship of our work to theirs must still be clarified.

In recent years, attention has turned from feasible primal-dual methods for linear programming to infeasible primal-dual methods. In particular, it is desirable to start from an arbitrary positive point and produce a sequence of iterates that converges to the solution of the LP. In this setting the restriction of positive iterates and the use of a prescribed neighborhood condition prevents the use of full Newton steps. The first theoretical result on infeasible-type interior-point algorithms was by Kojima, Meggido, and Mizuno [KMM]. Today, most primal-dual methods are based on Mehrotra's [Me] infeasible predictor-corrector method. Other results have come from Zhang [Z], Potra [P1],[P2], Billups and Ferris [BF], and Miao [Mi]. Convergence results to date deal with the convergence of a sequence of iterates $\{(s^k, r^k, x^k, y^k)\}$ and are based on starting points $(s^0, r^0, x^0, y^0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$. Polynomial complexity results to date are based on using the neighborhood condition the iterates satisfy in an appropriate way. A recent comprehensive account of infeasible methods is in Wright [WS].

All primal-dual interior-point methods assume existence of a strictly feasible point. A reformulation of the programming problem can "convert" an infeasible point for the original problem to a feasible point for the new problem. This approach however has some undesirable consequences. The reformulation adds rows and columns to the constraint matrix A from the addition of new primal and dual variables. Therefore the question of the coefficients ("weights") of these new variables needs to be addressed. In fact, the size of the coefficients is hard to determine prior to running the algorithm. Also, the additional columns are typically dense ("large" number of nonzero entries). This together with the required size of the coefficients causes numerical instability and computational inefficiency. Eventually it was shown by Lustig, Marsten, and Shanno [LMS] that as the size of the coefficients approached infinity, the (limiting) directions of the primal-dual equations for the feasible problem coincided with the directions generated by the primal-dual infeasible equations. For infeasible methods, a typical choice for $H(s^k, r^k, x^k, y^k)$, N

are

$$H(s^k, r^k, x^k, y^k) = \begin{pmatrix} Ax^k - s^k - b \\ A^t y^k + r^k - c \\ X^k r^k - \sigma\mu e^n \\ Y^k s^k - \gamma\mu e^m \end{pmatrix}, \quad \mu = \frac{(r^k)^t x^k + (s^k)^t y^k}{n+m}, \quad \sigma \in (0, \frac{1}{2})$$

and

$$N = N_{-\infty}(\gamma, \beta), \mu^0 = \frac{(r^0)^t x^0 + (s^0)^t y^0}{n+m}, \gamma \in (0,1), \beta \geq 1,$$

where the residual vectors $r_b \in \mathbf{R^m}$, $r_c \in \mathbf{R^m}$ are given by

$$r_b^0 = Ax^0 - s^0 - b, r_c^0 = A^t y^0 + r^0 - c$$

and

$$N_{-\infty}(\gamma, \beta) = \{(s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} : \|(r_b, r_c\| \leq \frac{\beta\mu}{\mu^0} \|(r_b^0, r_c^0\|, x_i r_i, y_j s_j \geq \gamma\mu\}$$

Unlike the Simplex Method, primal-dual numerical methods do not have a finite termination property. A stopping criteria is required for these methods. A standard criteria for a given tolerance ϵ, is

$$(x^k)^t r^k + (s^k)^t y^k + \|Ax^k - s^k - b\| + \|A^t y^k + r^k - c\| < \epsilon.$$

It follows that for $\mu = \frac{\epsilon}{n+m}$, if $z^* \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is such that $F_\mu(z^*) = 0$, then z^* satisfies the stopping criteria for numerical methods. Therefore, finding the point on the Central Path corresponding the $\mu = \frac{\epsilon}{n+m}$ provides as good an approximation to a solution of (LP) and (DP) as current numerical methods.

Given the KKT conditions for (LP) and (DP) and the Definition (1.7) of $F_\mu(z)$, it follows that to find solutions to (LP) and (DP) we wish to solve the system $F_0(z) = 0$. Given the problem of solving a system of nonlinear equations $F(v) = 0$, Branin [B] proposed following trajectories $v(t)$ which satisfied a related system of differential equations

$$\mathrm{D}F(v)\frac{dv}{dt} \pm F(v) = 0.$$

This work was related to work done earlier by Davidenko [Da]. The sign of the coefficient of $F(v)$ was changed whenever $det(\mathrm{D}F(v))$ changed sign or a solution of $F(v) = 0$ was approached. The goal was to expand the region of convergence of other methods and to design an algorithm that could find multiple solutions of $F(v) = 0$. Later Smale [S] studied the **continuous (global) Newton** equation

$$\mathrm{D}F(v)\frac{dv}{dt} = -\lambda(v)F(v) \qquad \lambda(v) \in \mathbf{R} \tag{1.13}$$

for a function $F : M \to \mathbf{R^N}$ where $M \subset \mathbf{R^N}$ was a compact domain with a smooth boundary (that is, ∂M was a submanifold of dimension $N-1$).

Definition 1.14. *Given $F : \Omega \subset \mathbf{R^M} \to \mathcal{M}^{\mathbf{M}}$, where $\mathcal{M}^{\mathbf{M}}$ is a manifold of dimension $\mathbf{M}$, denote $E_F = F^{-1}(0)$.*

Under a somewhat restrictive transversality condition on the boundary, Smale proved that the solution, $v(t)$, of the continuous Newton equation had the property that $v(t) \to E$ as $t \to \infty$. His approach was geometric in nature.

Definition 1.15. *Given $F : \Omega \subset \mathbf{R^N} \to \mathcal{M}^{\mathbf{M}}$ as above, define $g_F : \Omega \setminus E_F \subset \mathbf{R^N} \to S^{\mathbf{M-1}}$ by $g_F(z) = \frac{F(z)}{\|F(z)\|}$ where $\|\cdot\|$ denotes the usual Euclidean norm. We call $z \in E_F$ a* **singular point** *of g_F.*

Given $v^0 \in \partial M$, Smale obtained the trajectories, $v(t)$, directly as the inverse images of $g_F^{-1}(g_F(v^0))$. That is, given $F(v^0) = w^0 \neq 0$, consider the ray through w^0,

$$L_+(w^0) = \{\alpha w^0 \in \mathbf{R^N} : \alpha > 0\}.$$

Define

$$A(v^0) = g_F^{-1}(g_F(v^0))$$

and let $C(v^0)$ denote the connected component of v^0 in $A(v^0)$. It follows that F maps $A(v^0)$ into $L_+(w^0)$. Smale established that tangent vectors at $v \in g_F^{-1}(g_F(v^0))$ to the curve $g_F^{-1}(g_F(v^0))$ satisfy (1.13). Smale considered the initial value problem

$$\frac{d\xi}{dt} = \Phi_S(\xi(t)), \qquad \xi(t_0) = v^0, \qquad g_F(\xi(t)) = g_F(v^0),$$

where $v^0 \in Reg(g_F)$ and $\Phi_S(v)$ is constructed so that $\Phi_S(v) \in ker(\mathrm{D}g_F(v))$, $||\Phi_S(v)|| = 1$, and $\Phi_S(v)$ is tangent to $C(v)$. This is done using the fact that $g_F(\xi(t))$ is constant for all t. Hirsch and Smale [HSm] followed this work by using the same approach for the problem of finding solutions for the equation $F(v) = 0$ for a C^2 proper function $F : \mathbf{R^N} \to \mathbf{R^N}$. The **continuous Newton method** approach to solving the equation $F(v) = 0$, is to follow trajectories which are integrals of a vector field $\frac{dv}{dt} = \Phi(v)$ such that the (1.13) holds.

It is possible for points v to exist such that $F(v) \neq 0$ and either $\Phi(v) = 0$ or is undefined (as in the work of Smale and Hirsch). In the work of [HSm], [Sm] these points were excluded by removing a larger related set of points from consideration. Note, as mentioned above, [HSm] established results for functions F which were C^2 and proper. However, in our setting, $F_\mu(z)$ need not be proper. Also, [HSm] had no restriction on where solutions to $F(z) = 0$ were to be found.

It should be noted that the above continuous Newton method is related to **homotopy** [OR] methods in the following way. Given a generic local diffeomorphism $F(v)$ on a domain in $\mathbf{R^N}$, one approach to solving the equation $F(v) = 0$ is to use a homotopy $h(t,v)$. Let $v^0 \in \mathbf{R^N}$, and define $h(t,v)$ as

$$h(t,v) = F(v) - e^{-t}F(v^0).$$

Clearly $h(0,v^0) = 0$ and $h(t,v) \to F(v)$ as $t \to \infty$. If, in fact, there exists a curve $v(t)$ such that

$$0 = h(t,v(t)) = F(v(t)) - e^{-t}F(v^0)$$

then

$$\frac{dv}{dt} = -\mathrm{D}F(v(t))^{-1}F(v(t)).$$

The preceding equation again defines a vector field $\frac{dv}{dt} = \Phi(v)$ such that (1.13) holds for some $\lambda(v) \in \mathbf{R}$.

Given that the function of interest is $F_\mu(z)$, let $A_\mu(z) \subset \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ denote the trajectory given by $g_{F_\mu}^{-1}(g_{F_\mu}(z))$ and let $C_\mu(z)$ denote the connected component of z in $A_\mu(z)$. Note, in fact, $C_\mu(z) \subset \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, which is the domain of $F_\mu(z)$. Given Definition (1.8), a natural choice for $\Phi(z)$ would be the Newton Vector Field, $\Phi_{N,\mu}(z)$ given in (1.12). However, given that

$$DF_\mu(s,r,x,y) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix},$$

it is clear that points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ exist for which $\det(\mathrm{D}F_\mu(z)) = 0$ and hence for which $\Phi_{N,\mu}(z)$ is not defined. Another possibility is the unit vector field, denoted $\Phi_{S,\mu}$, in the direction of

$$\Phi_{A,\mu}(s,r,x,y) = (-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(s,r,x,y))F_\mu(s,r,x,y) \tag{1.16}$$

where $\mathrm{adj}(\mathrm{D}F_\mu(s,r,x,y))$ denotes the classical adjoint of $\mathrm{D}F_\mu(s,r,x,y)$ and the factor $(-1)^{m+1}$ ensures the correct orientation [See Proposition 2.1.2]. The vector field $\Phi_{S,\mu}(z)$ would be a vector field that is suggested by the work of Smale and Hirsch [Sm], [HSm] for the function $F_\mu(z)$. Note that $\Phi_{S,\mu}(z)$ is an extension of $\Phi_{N,\mu}(z)$. Once again, points exist for which $\Phi_{S,\mu}(z)$ is not defined.

In this work, the vector field that is of main interest is the **Adjoint Vector Field** given by (1.16) directly. This is the global extension of $\Phi_{S,\mu}(z)$ (and $\Phi_{N,\mu}(z)$ with respect to $\Phi_{S,\mu}(z)$). That is, under the appropriate reparametrization,

$$\Phi_{N,\mu}(z) \subset \Phi_{S,\mu}(z) \subset \Phi_{A,\mu}(z).$$

Given $\Phi_{A,\mu}(s,r,x,y)$ and $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, the initial value problem associated to (1.13) is

$$\frac{dz}{dt} = \Phi_{A,\mu}(z(t)), \qquad z(0) = z^0. \tag{1.17}$$

The vector fields $\Phi_{A,\mu}(s,r,x,y), \Phi_{S,\mu}(s,r,x,y)$ are superior to $\Phi_{N,\mu}(s,r,x,y)$ as it will be shown that there exists $z_0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $\mathrm{D}F_\mu(z)$ is rank deficient and such that $\Phi_{A,\mu}(z), \Phi_{S,\mu}(z) \neq 0$ [See Theorem 2.1.1 and Corollary 3.7].

Currently, complexity results for infeasible interior-point algorithms are based on algorithms which involve discrete sequences and require a starting point $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$, in order to investigate convergence of approximations limited to a suitable neighborhood of the Central Path. By contrast, the plan of this thesis is to study the global structure of all continuous trajectories of (1.17) as well as certain non-smooth critical trajectories (see Chapter 7 below). It is hoped that this will ultimately clarify the question concerning all starting points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. The results of this thesis imply the existence of infeasible interior-point algorithms of polynomial complexity through arbitrary points $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$ converging to E_{F_0}. In fact, this work establishes the existence of C^0 paths from points $z \in \partial\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}} \setminus \Sigma_{\mu}^{p}$ to E_{F_0}. This conclusion is based on our two main results. First, it is established that through arbitrary points $z(0) \in \mathbf{R}_{+}^{\mathbf{2n+2m}}$ there exists C^1 trajectories, $z(t)$, for which $z(t) \to z^*$ as $t \to \infty$ where z^* is a solution to the primal-dual linear problem $(F_0(z^*) = 0)$ This is done under conditions less restrictive than those for primal-dual interior- point methods. Given that the trajectories are integrals of the vector field $\Phi_{A,0}(z)$, it is established that $\mathbf{R}_{+}^{\mathbf{2n+2m}}$ is contained in the basin of attraction for E_{F_0} with respect to $\Phi_{A,0}(z)$. Second, with the exception of certain critical points, we establish the existence of C^1 trajectories through points in $\partial\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}} \setminus \Sigma_{\mu}^{p}$ as well as in $\mathbf{R}_{+}^{\mathbf{2n+2m}}$ to the Central Path. In particular, for $\mu > 0, z(0) \in \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}} \setminus \Sigma_{\mu}^{p}$, it is shown that the integral trajectories of $\Phi_{A,\mu}(z)$ have the property that $z(t) \to z^* = E_{F_\mu}$ as $t \to \infty$. Given the results established for arbitrary points $z \in \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$, this work establishes the basis for a global convergence theory for linear programming. Note, at this time, certain critical trajectories are still under investigation.

In this work we study the trajectories $F_\mu(z(t)) = e^{-t}F_\mu(z^0)$, which are based on integrals of $\Phi_{A,\mu}(z)$, a vector field related to infeasible primal-dual methods. In particular, we apply the continuous Newton method to the primal-dual barrier equations.

There are other questions answered herein that previous work done does not address. Under the weakened requirement that initial points lie in $\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$, it is shown that fixed points for $\frac{dz}{dt} = \Phi_{A,\mu}(z)$ do exist for which $F_\mu(z) \neq 0$ and their structure is identified. In the work of [HSm], these points were excluded by removing a related set of points from consideration. It is shown here that there exist trajectories through points that [HSm] would exclude which tend to solutions z^* of

$F_\mu(z) = 0$. This is done by making full use of the geometry of linear programming. Given that we seek solutions to $F_0(z) = 0$ under the added restriction that $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, another concern that must be addressed is that of exit points for the trajectory. That is, do points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ exist through which a trajectory exits the set $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$? It is established that in our setting no exit points exist.

The approach taken in the following is largely constructive with emphasis on understanding the geometry associated with the trajectories.

Given (LP) and (DP), the following general assumptions are standard for interior point methods.

$$\{(x,s) : Ax - s = b, x \in \mathbf{R^n}, x_i > 0\ \forall i, s \in \mathbf{R^m}, s_j > 0\ \forall j\} \neq \emptyset. \tag{1.18}$$

$$\{(y,r) : A^t y + r = c, r \in \mathbf{R^n}, r_i > 0\ \forall i, y \in \mathbf{R^m}, y_j > 0\ \forall j\} \neq \emptyset. \tag{1.19}$$

$$\mathrm{Rank}(A) = \min\{m, n\}. \tag{1.20}$$

As a practical assumption, the columns of A are nonzero. It is assumed throughout this work that there exists a unique solution z^* for (LP) and (DP). That is,

$$E_{F_0} = \{z^*\}. \tag{1.21}$$

This is typical in linear programming and is normally based on the assumption that the primal-dual problem is non-degenerative.

Remark. *References herein to vectors $c \in \mathbf{R^n}, b \in \mathbf{R^m}$, and matrix $A \in \mathbf{R^{m\times n}}$ refer to those given in (LP). Given Appendix B, (WLOG) we shall assume that in (LP) $m \leq n$. It is also assumed throughout this work that (1.20) holds and hence it is assumed that* $\mathrm{Rank}(A) = m$. *Finally, references to the measure of a set refer to Lebesgue measure of dimension appropriate to the setting.*

In the first convergence theorem, it is shown that under conditions less restrictive than conditions typical to primal-dual interior-point methods, the integral trajectories of $\Phi_{A,0}(z)$ converge to a solution for (LP) and (DP). In this setting, the restriction of [BL1], [BL2] on the boundedness of the feasible set is not necessary. In fact, we simply require the minimal condition that points $(x, s), (y, r)$ exist which are solutions for (LP) and (DP). From the Duality Theorem for Linear Programming this requirement is equivalent to the condition that

$$\{(s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} : Ax - s = b, A^t y + r = c\} \neq \emptyset. \tag{1.22}$$

Our main results follow.

Theorem 1.23. *Suppose that $\mu = 0, F_\mu(z) = F_\mu(s, r, x, y)$ is given by (1.8) and $\Phi_{A,\mu}(z)$ is given by (1.16). If $z^0 = (s^0, r^0, x^0, y^0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and (1.20)-(1.22) hold, then there exists a unique C^1 solution $z(t) : [0, \infty) \to \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ of (1.17) with the property that $z(t) \to z^* \in E_{F_0}$ as $t \to \infty$.*

Even without hypothesis (1.21), Proposition 6.2.9 establishes $z^* \in E_{F_0}$, for all ω-limit points z^* of $z(t)$.

The next theorem pertains to the question of following a trajectory based on $\Phi_{A,\mu}(z)$ from a given point to the Central Path. In this setting we establish results for points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.

Theorem 1.24. *Let $\mu > 0, F_\mu(z) = F_\mu(s, r, x, y)$ be given by (1.8) and $\Phi_{A,\mu}(z)$ be given by (1.16). Given (1.18)-(1.20), there exists a nowhere dense set $\Sigma_\mu^p \subset \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, of measure zero , such that if $z^0 = (s^0, r^0, x^0, y^0) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$ then there exists a unique C^1 solution $z(t) : [0, \infty) \to \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ of (1.17) with the property that $z(t) \to z^* = E_{F_\mu}$ as $t \to \infty$.*

Chapter 2 identifies the general properties of the vector field $\Phi_{A,\mu}(z)$. It is shown that $\Phi_{A,\mu}(z)$ is C^1 on $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and establishes an important relationship between points z for which $\Phi_{A,\mu}(z) = 0$ and the critical points of the mapping, $g_{F_\mu} : \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F\mu} \to S^{2n+2m-1}$. As per Remark 1.9, we differentiate g_{F_μ} on $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ without regard to the boundary. An overview of the theorems of Smale [S] is also given.

Chapter 3 provides analysis of regular points of g_{F_μ}. It is shown that the set of critical points of g_{F_μ} is of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Also, the structure of the regular values of g_{F_μ} is discussed and it is established that the set of points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $g_{F_\mu}(z)$ is a critical value is of measure zero for $\mu > 0$.

Chapter 4 establishes whether the solution trajectories of (1.17) are transverse to $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. In it, the terminology of transversality is defined based on considering how vectors are transverse to closed half-spaces. It is shown that for $\mu \geq 0$, $\Phi_{A,\mu}(z)$ is not outward transversal to $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ at any point $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. The work done here is based on various matrix manipulations. The definition of Σ_μ^p is given in this chapter.

Chapter 5 provides constructions of other sets, M, for which $\Phi_{A,\mu}(z)$ is inward transversal at $z \in \partial M$. Some of the sets are "hyperbolic" sets and transversality of $\Phi_{A,\mu}(z)$ to ∂M at $z \in \partial M$ is verified by showing that $\Phi_{A,\mu}(z)$ is transversal to the supporting closed half-spaces to the M at $z \in \partial M$. Conditions are established for which $\Phi_{A,\mu}(z)$ is not outward transversal to a collection of the various sets and $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Based on the transversality results in this chapter, an enclosing neighborhood can be constructed for the trajectories $z(t)$. The results here are use in Chapters 6 and 7 to bound the trajectories away from $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ or to bound an individual component of the trajectory $z(t)$.

Chapter 6 identifies the various properties of the trajectories that are solutions to (1.17). The proof of the Theorems 1.23 and 1.24 are done in this chapter and are based heavily on the properties identified earlier in Chapter 6. The work done in Chapter 6 provides complete proofs for the theorems stated by Smale [Sm].

Chapter 7 provides partial results for trajectories corresponding to $\mu = 0$ and $z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. While Theorems (1.23),(1.24) established the existence of C^0 paths from points in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ to points $z^* \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $F_0(z^*) = 0$, these paths, $z(t)$, have the property that $\{z(t) : t \in (0, \infty)\} \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. An unresolved question is the existence of C^0 paths, $z(t)$, such that $\{z(t) : t \in [0, \infty)\} \subset \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and for which $z(t) \to z^*$ such that $F_0(z^*) = 0$. Given in this chapter is a general formulation of the Jacobian matrix $\mathrm{D}\Phi_{A,\mu}(z)$. Based on this formulation, a specific type of critical point of g_{F_μ} is considered for which the various eigenvalues and eigenvectors of the fixed point are identified completely. The trajectories for $\mu = 0$ and through points $z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ are a special type of trajectories in a larger class of trajectories which we call **critical trajectories.** The definition of a critical trajectory is also given in Chapter 7. In the case of $\mu = 0, z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that exactly one component of z^0 is equal to 0, it is shown in Chapter 3 that $\mathrm{D}F_0(z^0)$ is of full rank. Hence, for every such z^0 there exists a C^1 trajectory, $z(t)$, in a neighborhood of z^0 that is a solution to (1.17). A remaining question is whether $g_{F_0}^{-1}(g_{F_0}(z^0))$ is a 1-dimensional manifold (globally) such the trajectory, under the correct orientation and parametrized by $z(t)$, has the property that $z(t) \to z^* \in E_{F_0}$ as $t \to \infty$. Future work to address the above question will involve the study of the associated stable and unstable manifolds. This chapter provides a basis for this future work.

Chapter 8 contains the basic results needed to establish analogous theorems for the **standard form** of linear programming. The standard form is given by

$$\min c^t x \qquad \text{subject to } Ax = b$$

$$A \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}, m \leq n, x \in \mathbf{R}^{\mathbf{n}}, x_i \geq 0\ \forall i. \tag{1.25}$$

The associated dual problem for (1.25) is

$$\max b^t y \qquad \text{subject to } A^t y + r = c$$

$$A \in \mathbf{R}^{\mathbf{n}\times\mathbf{m}},\ n \geq m,\ r \in \mathbf{R}^{\mathbf{n}}, r_i \geq 0\ \forall i\ y \in \mathbf{R}^{\mathbf{m}}. \tag{1.26}$$

Points $(x^*), (y^*, r^*)$ are solutions of (1.25) and (1.26) if and only if the KKT conditions

$$A^t y^* + r^* = c,$$

$$X^* r^* = 0,$$

$$Ax^* = b,$$

$$x^*, r^* \in \mathbf{R}^{\mathbf{n}}, x_i^* \geq 0, r_i^* \geq 0\ \forall i, y^* \in \mathbf{R}^{\mathbf{m}}$$

are satisfied. The optimality conditions for the logarithmic barrier subproblem associated to (1.25) and (1.26) are

$$Ax = b,$$

$$A^t y + r = c,$$

$$Xr = \mu e^n,$$

$$r, x \in \mathbf{R}^{\mathbf{n}}, r_i > 0, x_i > 0\ \forall i\quad y \in \mathbf{R}^{\mathbf{m}}.$$

We adopt the notation

$$\mathcal{R}_+^{2n+m} = \{(r, x, y) \in \mathbf{R}^{\mathbf{n}} \oplus \mathbf{R}^{\mathbf{n}} \oplus \mathbf{R}^{\mathbf{m}} : r_i > 0, x_i > 0\ \forall i\}$$

and

$$\bar{\mathcal{R}}_{+}^{2n+m} = \{(r,x,y) \in \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m} : r_i \geq 0, x_i \geq 0 \ \forall i\}$$

The function based on the optimality conditions is $F_\mu^s(r,x,y) : \bar{\mathcal{R}}_{+}^{2n+m} \to \mathbf{R^n} \oplus \mathbf{R^n} \oplus \mathbf{R^m}$ given by

$$F_\mu^s(r,x,y) = \begin{pmatrix} A^t y + r - c \\ Xr - \mu e^n \\ Ax - b \end{pmatrix} \tag{1.27}$$

and the associated adjoint vector field is

$$\Phi_{A,\mu}^s(r,x,y) = (-1)\mathrm{adj}(DF_\mu(r,x,y))F_\mu(r,x,y). \tag{1.28}$$

The general assumptions for the standard form, (1.25) and (1.26), are

$$\{x : x \in \mathbf{R^n}, x_i > 0 \ \forall i, Ax = b\} \neq \emptyset. \tag{1.29}$$

$$\{(y,r) : r \in \mathbf{R^n}, r_i > 0 \ \forall i, A^t y + r = c\} \neq \emptyset. \tag{1.30}$$

$$\mathrm{rank}(A) = m. \tag{1.31}$$

Once again, we only require the existence of points (x), (y,r) that are solutions of (1.25) and (1.26). As before this is equivalent to the condition that

$$\{(x,r,y) : x, r \in \mathbf{R^n}, x_i \geq 0, r_i \geq 0 \ \forall i, Ax = b, A^t y + r = c\} \neq \emptyset. \tag{1.32}$$

Again we shall assume that

$$E_{F_0^s} = \{z^*\}, \tag{1.33}$$

that is, that there exists a unique solution to (1.25), (1.26).

Theorem 1.34. *Suppose that* $\mu = 0, F^s_\mu(z) = F^s_\mu(r,x,y)$ *is given by (1.27) and* $\Phi^s_{A,\mu}(z)$ *by (1.28). Given (1.31)-(1.33), if* $z^0 = (r^0, x^0, y^0) \in \mathcal{R}^{2n+m}_+$ *then there exists a unique* C^1 *solution* $z(t) : [0,\infty) \to \bar{\mathcal{R}}^{2n+m}_+$ *to* $\frac{dz}{dt} = \Phi^s_{A,\mu}(z(t)),\ z(0) = z^0$ *such that* $z(t) \to z^* \in E_{F^s_0}$ *as* $t \to \infty$.

For the case of $\mu > 0$, to obtain results similar to theorem 1.24, additional restrictions on A are required. The additional restrictions are due to the presence of the non-sign constrained y and the lack of a complementarity condition for y. The added assumption is that

(1.35) $\quad m < n,$ and any set of m columns of A is linearly independent.

Theorem 1.36. *Suppose that* $\mu > 0, F^s_\mu(z) = F^s_\mu(r,x,y)$ *is given by (1.27) and* $\Phi_{A,\mu}(z)$ *is given by (1.28). Given (1.29)-(1.31), (1.35), there exists a nowhere dense set* $\Sigma^p_\mu \subset \partial\bar{\mathcal{R}}^{2n+m}_+$ *of measure zero such that if* $z^0 = (r^0, x^0, y^0) \in \bar{\mathcal{R}}^{2n+m}_+ \setminus \Sigma^p_\mu$ *then there exists a unique* C^1 *solution* $z(t) : [0,\infty) \to \bar{\mathcal{R}}^{2n+m}_+$ *to* $\frac{dz}{dt} = \Phi^s_{A,\mu}(z(t)),\ z(0) = z^0$ *such that* $z(t) \to z^* = E_{F^s_\mu}$ *as* $t \to \infty$.

Appendix A contains includes the various terminology and theorems from differential equations that are used in this work. Appendix B establishes that all linear programming problems in the symmetric-dual form may be given in the form (LP), (DP) with the added condition that $m \leq n$ in (LP).

CHAPTER 2

CONTINUOUS NEWTON VECTOR FIELDS

This chapter identifies the general properties of the vector field $\Phi_{A,\mu}(z)$. It is shown that $\Phi_{A,\mu}(z)$ is C^1 on $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and establishes an important relationship between points z for which $\Phi_{A,\mu}(z) = 0$ and the critical points of the mapping, $g_{F_\mu} : \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu} \to S^{2n+2m-1}$. An overview of the theorems of Smale [S] is also given.

Recall that the initial value problem that is under consideration is

$$\frac{dz}{dt} = \Phi_{A,\mu}(z(t)) \qquad z(0) = z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$$

which is motivated by the trajectories

$$F_\mu(z(t)) = e^{-t} F_\mu(z^0).$$

Appendix A contains the general terminology and theorems on differential equations that will be used in this work. Given $F : \mathbf{R}^{\mathbf{M}} \to \mathbf{R}^{\mathbf{N}}$ such that $F(z)^t = [F_1(z), \ldots, F_N(z)]$,

$$\mathrm{D}F(z) = \left(\tfrac{\partial F_i}{\partial z_j}(z) \right) \in \mathbf{R}^{\mathbf{M \times N}}$$

denotes the Jacobian matrix of $F(z)$ and $\nabla F_i(z) \in \mathbf{R}^{\mathbf{M \times 1}}$ denotes the gradient vector of $F_i(z)$.

We will need the following lemma in the work that is to follow.

Lemma 2.0.1. *Let $g : \mathbf{R}^{\mathbf{M}} \to \mathbf{R}, F : \mathbf{R}^{\mathbf{M}} \to \mathbf{R}^{\mathbf{K}}$ where $F(z)^t = [F_1(z), \ldots, F_K(z)]$ and $g(z), F_i(z)$ are C^1 for every i. Then for $\bar{F}(z) = g(z)F(z), \mathrm{D}\bar{F}(z) = F(z)\nabla g(z)^t + g(z)\mathrm{D}F(z)$.*

Proof: Consider $\bar{F}_i(z) = g(z)F_i(z)$. Then $\frac{\partial \bar{F}_i(z)}{\partial z_j} = \frac{\partial g(z)}{\partial z_j}F_i(z) + g(z)\frac{\partial F_i(z)}{\partial z_j}$. Hence,

$$\begin{aligned}
\mathrm{D}\bar{F}(z) &= \begin{pmatrix} \frac{\partial g(z)}{\partial z_1}F_1(z) + g(z)\frac{\partial F_1(z)}{\partial z_1} & \ldots & \frac{\partial g(z)}{\partial z_M}F_1(z) + g(z)\frac{\partial F_1(z)}{\partial z_M} \\ \vdots & \vdots & \vdots \\ \frac{\partial g(z)}{\partial z_1}F_K(z) + g(z)\frac{\partial F_K(z)}{\partial z_1} & \ldots & \frac{\partial g(z)}{\partial z_M}F_K(z) + g(z)\frac{\partial F_K(z)}{\partial z_M} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial g(z)}{\partial z_1}F_1(z) & \ldots & \frac{\partial g(z)}{\partial z_M}F_1(z) \\ \vdots & \vdots & \vdots \\ \frac{\partial g(z)}{\partial z_1}F_K(z) & \ldots & \frac{\partial g(z)}{\partial z_M}F_K(z) \end{pmatrix} + g(z)\mathrm{D}F(z) \\
&= F(z)\nabla g(z)^t + g(z)\mathrm{D}F(z).
\end{aligned}$$

Since we will be using differential equations to solve for the zero of a function, it is important that we classify the fixed points of the vector fields involved. In particular, we wish to classify the structure of fixed points in the (LP) setting.

Definition 2.0.2. *Let $F : \Omega \subset \mathbf{R}^{\mathbf{N}} \to \mathcal{M}^{\mathbf{M}}$ be a C^r map with $r \geq 1$ where $\mathcal{M}^{\mathbf{M}}$ is a manifold of dimension $\mathbf{M}$. We call $z \in \Omega$ a* **regular point** *of F if $\mathrm{D}F(z)$ is of rank $\min\{\mathbf{M}, \mathbf{N}\}$. $Reg(F) \subset \Omega$ denotes the set of all regular points of F. If $z \in \Omega$ is not a regular point then z is a* **critical point** *of F. $Crit(F) \subset \Omega$ denotes the set of all critical points of F.*

Definition 2.0.3. *Let $F : \Omega \subset \mathbf{R}^{\mathbf{N}} \to \mathcal{M}^{\mathbf{M}}$ be a C^r map with $r \geq 1$ where $\mathcal{M}^{\mathbf{M}}$ is a manifold of dimension $\mathbf{M}$. A point c is a* **regular value** *of F provided $c \in \mathcal{M}^{\mathbf{M}}$ and $F^{-1}(c) \subset Reg(F)$; otherwise c is called a* **critical value** *of F.*

From Smale [Sm] we have the following proposition for which we will include the proof for completeness.

Proposition 2.0.4. *Let $F : \Omega \subset \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ be C^2 and g_F be defined as in (1.15). Then, $v \in ker(\mathrm{D}g_F(z))$ iff $\mathrm{D}F(z)v = \lambda(z)F(z)$ for $\lambda(z) \in \mathbf{R}$.*

Proof: If $z \in \Omega$ such that $F(z) \neq 0$, $\|F(z)\|g_F(z) = F(z)$. It follows that

$$\mathrm{D}_z(\|F(z)\|)g_F(z) + \|F(z)\|\mathrm{D}g_F(z) = \mathrm{D}F(z).$$

Also, $\mathrm{D}_z(\|F(z)\|) = \frac{F(z)^t}{\|F(z)\|}\mathrm{D}F(z)$. Therefore,

$$\|F(z)\|\mathrm{D}g_F(z) = \mathrm{D}F(z) - g_F(z)\frac{F(z)^t}{\|F(z)\|}\mathrm{D}F(z) = \mathrm{D}F(z) - \frac{F(z)F(z)^t\mathrm{D}F(z)}{\|F(z)\|^2}.$$

Hence,

$$\mathrm{D}g_F(z) = \frac{1}{\|F(z)\|}\left[\mathrm{D}F(z) - \frac{F(z)F(z)^t\mathrm{D}F(z)}{\|F(z)\|^2}\right] = \frac{1}{\|F(z)\|}\left[I - \frac{F(z)F(z)^t}{\|F(z)\|^2}\right]\mathrm{D}F(z).$$

Now

$$v \in ker(\mathrm{D}g_F(z)) \Rightarrow \mathrm{D}g_F(z)v = 0 \Rightarrow \left[I - \frac{F(z)F(z)^t}{\|F(z)\|^2}\right]\mathrm{D}F(z)v = 0.$$

It follows that

$$\mathrm{D}F(z)v = \frac{F(z)F(z)^t\mathrm{D}F(z)v}{\|F(z)\|^2}$$

and hence, $\mathrm{D}F(z)v = \lambda(z)F(z)$ where $\lambda(z) = \left[\frac{F(z)^t\mathrm{D}F(z)v}{\|F(z)\|^2}\right] \in \mathbf{R}$. If $\mathrm{D}F(z)v = \lambda(z)F(z)$ for some $\lambda(z) \in \mathbf{R}$ then

$$\begin{aligned}
\mathrm{D}g_F(z)v &= \frac{1}{\|F(z)\|}\left[\mathrm{D}F(z)v - \frac{F(z)F(z)^t\mathrm{D}F(z)v}{\|F(z)\|^2}\right] \\
&= \frac{1}{\|F(z)\|}\left[\lambda(z)F(z) - \frac{F(z)F(z)^t\lambda(z)F(z)}{\|F(z)\|^2}\right] \\
&= \frac{1}{\|F(z)\|}\left[\lambda(z)F(z) - \lambda(z)F(z)\right] \\
&= 0.
\end{aligned}$$

Hence, $v \in ker(\mathrm{D}g_{F_\mu}(z))$.

It should be noted that the above proposition considers g_F as a map into $\mathbf{R}^{\mathbf{N-1}}$. In fact, g_F is a map with range in $S^{\mathbf{N-1}} \subset \mathbf{R}^{\mathbf{N-1}}$. Note, from the proof of Proposition 2.0.4,

$$\mathrm{D}g_F(z) = \frac{1}{\|F(z)\|}\left[I - \frac{F(z)F(z)^t}{\|F(z)\|^2}\right]\mathrm{D}F(z) = \frac{1}{\|F(z)\|}\left[I - g_F(z)g_F^t(z)\right]\mathrm{D}F(z).$$

Recall that given a unit vector $v \in \mathbf{R}^{\mathbf{N-1}}$, $(I - vv^t)$ is the orthogonal projection that is perpendicular to v. In the current setting, it follows that $I - g_F g_F^t$ is the orthogonal projection perpendicular to g_F. Now, from above,

$$v \in ker(\mathrm{D}g_F(z)) \Leftrightarrow \mathrm{D}F(z)v \in ker(I - g_F g_F^t) = \text{span of } g_F(z).$$

Now, if $0 \neq y \in \mathbf{R}^{\mathbf{N-1}}$ is such that there exists $v \in \mathbf{R}^{\mathbf{N}}$ for which $\mathrm{D}g_F(z)v = y$ then $y = \frac{1}{\|F(z)\|}(I - g_F(z)g_F^t(z))\mathrm{D}F(z)v$. It follows that $range(\mathrm{D}g_F(z)) \subset range(I - g_F(z)g_F^t(z))$. Hence, $range(\mathrm{D}g_F(z)) \subset$ span of $g_F^{\perp}(z)$ where $g_F^{\perp}(z)$ denotes the orthogonal complement of $g_F(z)$.

Suppose that $F : \Omega \subset \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{\mathbf{N}}$ is a C^2 map. From Definitions 1.15, 2.0.2, $z \in Reg(g_F)$ if and only if $\mathrm{D}g_F(z)$ is of rank $N-1$. It follows from Proposition 2.0.4 that we have the following corollary.

Corollary 2.0.5 [Sm]. *Let $F : \Omega \subset \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{\mathbf{N}}$ be C^2 and g_F defined as above, then $z \in Reg(g_F)$ if and only if one of the following is true.*

i. $Rank(\mathrm{D}F(z)) = N$.

ii. $Rank(\mathrm{D}F(z)) = N - 1$ and $range(\mathrm{D}F(z)) \cap L = \{0\}$ where $w = F(z)$ and

$$L = L(w) = \{\alpha \cdot w : \alpha \in \mathbf{R}\}.$$

Now, given $z \in \Omega$ such that $0 \neq w = F(z)$,

$$(I - g_F(z)g_F^t(z))\alpha F(z) = \alpha F(z) - g_F(z)g_F(z)^t \alpha F(z) = \alpha F(z) - \alpha F(z) = 0.$$

Hence, $L(w) = ker(I - g_F(z)g_F^t(z))$ and therefore $L^{\perp}(w) = range(I - g_F(z)g_F^t(z))$. This establishes the relationship between Proposition 2.0.4 with g_F viewed as a mapping into $\mathbf{R^{N-1}}$ and the form of $\mathrm{D}g_F(z)$ with g_F viewed as a mapping into S^{N-1}.

We will now identify the important characteristics of the vector fields $\Phi_{A,\mu}(z)$, $\Phi_{S,\mu}(z)$, and, $\Phi_{N,\mu}(z)$.

§2.1 Adjoint Vector Field

Theorem 2.1.1. *Given $\mu \geq 0$, a point $p \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is a fixed point of $\frac{dz}{dt} = \Phi_{A,\mu}(z)$ if and only if $p \in Crit(g_{F_\mu}) \cup E_{F_\mu}$.*

Proof: There are 3 cases, based on $Rank(\mathrm{D}F_\mu(p))$, to consider. If $Rank(\mathrm{D}F_\mu(p)) = 2n + 2m$, $Rank(\mathrm{adj}(\mathrm{D}F_\mu(p)) = 2n + 2m$. It follows from (1.16) that $\Phi_{A,\mu}(p) = 0$ if and only if $F_\mu(p) = 0$, hence $\Phi_{A,\mu}(p) = 0$ if and only if $p \in E_{F_\mu}$. Note, if $p \in E_{F_\mu}$ then $\Phi_{A,\mu}(p) = 0$ by (1.16), regardless of $Rank(\mathrm{D}F_\mu(p))$.

Now suppose that $Rank(\mathrm{D}F_\mu(p)) = 2n + 2m - 1$. Recall that

$$\mathrm{D}F_\mu(p)\mathrm{adj}(\mathrm{D}F_\mu(p)) = \mathrm{adj}(\mathrm{D}F_\mu(p))\mathrm{D}F_\mu(p) = det(\mathrm{D}F_\mu(p))I.$$

It follows that $Rank(\mathrm{adj}(\mathrm{D}F_\mu(p))) = 1$. Since $det(\mathrm{D}F_\mu(p)) = 0$, $range(\mathrm{D}F_\mu(p)) \subset ker(\mathrm{adj}(\mathrm{D}F_\mu(p)))$. Given that $Rank(\mathrm{D}F_\mu(p)) = 2n+2m-1$, $dim(ker(\mathrm{adj}(\mathrm{D}F_\mu(p))))$ $= 2n + 2m - 1$ and therefore $range(\mathrm{D}F_\mu(p)) = ker(\mathrm{adj}(\mathrm{D}F_\mu(p)))$. If $p \in Crit(g_{F_\mu})$, from Corollary 2.0.5, $F_\mu(p) \in range(\mathrm{D}F_\mu(p))$. It follows that $\Phi_{A,\mu}(p) = 0$ as $ker(\mathrm{adj}(\mathrm{D}F_\mu(p))) = range(\mathrm{D}F_\mu(p))$. If p is a fixed point of $\frac{dz}{dt} = \Phi_{A,\mu}(z)$, from (1.16), $F_\mu(p) \in ker(\mathrm{adj}(\mathrm{D}F_\mu(p)))$. If $F_\mu(p) = 0$ then $p \in E_{F_\mu}$, else, $F_\mu(p) \in range(\mathrm{D}F_\mu(p))$ and by Corollary 2.0.5, $p \in Crit(g_{F_\mu})$.

Finally, if $Rank(\mathrm{D}F_\mu(p)) < 2n + 2m - 1, \mathrm{adj}(\mathrm{D}F_\mu(p)) = 0$. If $F_\mu(p) = 0$ then $p \in E_{F_\mu}$ else it follows from (1.16) that $\Phi_{A,\mu}(p) = 0$ and from Corollary 2.0.5 that $p \in Crit(g_{F_\mu})$.

It will be shown (Chapter 3) that almost all points that we will be considering are such that $\mathrm{D}F_\mu(z)$ is of full rank. In fact, it is shown that for all $z \in \mathbf{R}_+^{\mathbf{2n+2m}}, \mathrm{D}F_\mu(z)$ is of full rank. Recall that the Remark following (1.21) holds throughout this work.

Proposition 2.1.2. *Suppose that* $(s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$, *then* $\mathrm{D}F_\mu(s, r, x, y)$ *is of full rank. If* $(s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ *is such that* $\mathrm{D}F_\mu(s, r, x, y)$ *is of full rank,* $sgn(det(\mathrm{D}F_\mu(z))) = (-1)^m$. *It follows that for all* $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ *such that* $\mathrm{D}F_\mu(z)$ *is of full rank,* $sgn(det(\mathrm{D}F_\mu(z))) = (-1)^m$.

Proof: Let $(s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$. By doing elementary operations $\mathrm{D}F_\mu$ can be transformed into

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & 0 & R & -XA^t \\ 0 & 0 & YA & S \end{pmatrix}.$$

Hence the sign of the determinant is completely determined by the sign of

$$(-1)^m \begin{vmatrix} R & -XA^t \\ YA & S \end{vmatrix}.$$

Multiply the first n rows by R^{-1} to form the matrix

$$\begin{pmatrix} I_{nn} & -R^{-1}XA^t \\ YA & S \end{pmatrix}.$$

By elementary row operations we can form the matrix

$$\begin{pmatrix} I_{nn} & -R^{-1}XA^t \\ 0 & S + YAR^{-1}XA^t \end{pmatrix}$$

which leaves the sign of the determinant unchanged. Finally, multiply the last m rows by Y^{-1} to form the matrix

$$\begin{pmatrix} I_{nn} & -R^{-1}XA^t \\ 0 & SY^{-1}+AR^{-1}XA^t \end{pmatrix}.$$

Now the symmetric matrix $SY^{-1}+AR^{-1}XA^t$ is positive definite from the Remark following (1.21) and hence has a positive determinant. So for $(s,r,x,y)\in \mathbf{R}_+^{2n+2m}$, $DF_\mu(s,r,x,y)$ is of full rank. Also the sign of the determinant of $DF_\mu(s,r,x,y)$ is given by $(-1)^m$ for $(s,r,x,y)\in \mathbf{R}_+^{2n+2m}$.

Hence, by the continuity of the determinant, it follows that for any point $(s,r,x,y)\in \bar{\mathbf{R}}_+^{2n+2m}$ for which $DF_\mu(s,r,x,y)$ is of full rank, the sign of the determinant is equal to $(-1)^m$.

In the case that $\mathrm{D}F_\mu(z)$ is of full rank, we have the following formulation of $\mathrm{D}\Phi_{A,\mu}(z)$.

Theorem 2.1.3. *Suppose that* $\mathrm{D}F_\mu(z)$ *is of full rank in a neighborhood about* $z\in\bar{\mathbf{R}}_+^{2n+2m}$. *Then*

$$\begin{aligned}\mathrm{D}\Phi_{A,\mu}(z) =&(-1)^{m+1}(\mathrm{D}F_\mu(z))^{-1}F_\mu(z)\nabla det(\mathrm{D}F_\mu(z))^t - |det(\mathrm{D}F_\mu(z))|I\\ &+(-1)^m\mathrm{D}F_\mu(z)^{-1}\mathrm{D}_z(\mathrm{D}F_\mu(z))adj(\mathrm{D}F_\mu(z))F_\mu(z).\end{aligned}$$

Proof: We consider

$$\mathrm{D}F_\mu(z)\Phi_{A,\mu}(z) = (-1)^{m+1}det(\mathrm{D}F_\mu(z))F_\mu(z).$$

Differentiating and applying a Liebniz rule we get

$$\mathrm{D}_z(\mathrm{D}F_\mu(z))\Phi_{A,\mu}(z)+\mathrm{D}F_\mu(z)\mathrm{D}\Phi_{A,\mu}(z) = (-1)^{m+1}\mathrm{D}_z(det(\mathrm{D}F_\mu(z))F_\mu(z)).$$

From Lemma 2.0.1,

$$\mathrm{D}_z(det(\mathrm{D}F_\mu(z))F_\mu(z)) = F_\mu(z)\nabla det(\mathrm{D}F_\mu(z))^t + det(\mathrm{D}F_\mu(z))\mathrm{D}F_\mu(z)).$$

Note also, from Proposition 2.1.2,

$$(-1)^{m+1}det(\mathrm{D}F_\mu(z)) = -|det(\mathrm{D}F_\mu(z))|.$$

It follows that, since $\mathrm{D}F_\mu(z)$ is of full rank,

$$\begin{aligned}\mathrm{D}\Phi_{A,\mu}(z) &= (-1)^{m+1}\mathrm{D}F_\mu(z)^{-1}[F_\mu(z)\nabla det(\mathrm{D}F_\mu(z))^t + det(\mathrm{D}F_\mu(z))\mathrm{D}F_\mu(z)\\ &\qquad - \mathrm{D}_z(\mathrm{D}F_\mu(z))\Phi_{A,\mu}(z)]\\ &= (-1)^{m+1}\mathrm{D}F_\mu(z)^{-1}F_\mu(z)\nabla det(\mathrm{D}F_\mu(z))^t - |det(\mathrm{D}F_\mu(z))|I\\ &\qquad + (-1)^m\mathrm{D}F_\mu(z)^{-1}\mathrm{D}_z(\mathrm{D}F_\mu(z))adj(\mathrm{D}F_\mu(z))F_\mu(z).\end{aligned}$$

Corollary 2.1.4. *If $p \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is a fixed point of $\frac{dz}{dt} = \Phi_{A,\mu}(z)$ and $\mathrm{D}F_\mu(z)$ is of full rank in a neighborhood about $p, \mathrm{D}\Phi_{A,\mu}(p) = -1|\det(\mathrm{D}F_\mu(p))|I$ and hence p is a sink.*

Proof: From Theorem 2.1.1, if $\mathrm{D}F_\mu(p)$ is of full rank, $\Phi_{A,\mu}(p) = 0$ if and only if $p \in E_{F_\mu}$. Hence, $F_\mu(p) = 0$ and therefore the corollary follows from Theorem 2.1.3.

Chapter 7 contains a formulation for $\mathrm{D}\Phi_{A,\mu}(z)$ at an arbitrary point $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Also included is the analysis of the eigenvalues of $\mathrm{D}\Phi_{A,\mu}(z)$ at a fixed point z, corresponding to $\mu = 0$, for which $\mathrm{D}F_\mu(z)$ is rank deficient.

Theorem A.4 provides a basis for the study of solutions of differential equations. We need to verify the existence and uniqueness of solution curves for the above trajectories in our particular setting.

Proposition 2.1.5. *$\Phi_{A,\mu}(z)$ is C^1 for all $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: From the definition of $F_\mu(z)$, the components of $F_\mu(z)$ are polynomials in the components of z. Also, the entries of $adj(\mathrm{D}F_\mu(z))$ are determinants of

$(2n+2m-1) \times (2n+2m-1)$ submatrices of $\mathrm{D}F_\mu(z)$. Let Z_p^q denote the diagonal matrix with diagonal entries $z_p, z_{p+1}, \ldots z_q$. Then

$$\mathrm{D}F_\mu(z) = \begin{pmatrix} I_{mm} & 0 & A & 0 \\ 0 & -I_{nn} & 0 & A^t \\ 0 & Z_{n+m+1}^{2n+m} & Z_{m+1}^{m+n} & 0 \\ Z_{2n+m+1}^{2n+2m} & 0 & 0 & Z_1^m \end{pmatrix}.$$

It follows that the entries of $adj(\mathrm{D}F_\mu(z))$ are also polynomials in the components of z. Hence $\Phi_{A,\mu}(z)_i$ is a polynomial in z for every i. It follows that $\Phi_{A,\mu}(z)$ is C^1 on $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.

Corollary 2.1.6. *At every point $z_0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ there exists a unique solution to*

$$\textbf{(IVP)} \qquad \frac{dz}{dt} = \Phi_{A,\mu}(z(t)), \qquad z(t_0) = z_0.$$

Proof: Proof follows directly from Theorem A.4 and Proposition 2.1.5.

Note the relationship of the trajectories of the Adjoint, Newton and Smale vector fields. Recall $\Phi_{S,\mu}(z) = \frac{\Phi_{A,\mu}(z)}{\|\Phi_{A,\mu}(z)\|}$.

Lemma 2.1.7. *Given $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that $\Phi_{N,\mu}(z)$ or $\Phi_{S,\mu}(z)$ exists then $\Phi_{N,\mu}(z) = \lambda_1 \Phi_{A,\mu}(z)$ or $\Phi_{S,\mu}(z) = \lambda_2 \Phi_{A,\mu}(z)$ for some $\lambda_i > 0$. Note that if both exist then both λ_i exist.*

Proof: The case for $\Phi_{S,\mu}(z)$ is clear. As before, $sgn(det(\mathrm{D}F_\mu(z))) = (-1)^m$. If $\Phi_{N,\mu}(z)$ exists, it follows that $\mathrm{D}F_\mu(z)^{-1} = \frac{1}{det(\mathrm{D}F_\mu(z))} adj(\mathrm{D}F_\mu(z))$. Hence

$$\begin{aligned} \Phi_{N,\mu}(z) = (-1)\mathrm{D}F_\mu(z)^{-1}F_\mu(z) &= (-1)\frac{1}{det(\mathrm{D}F_\mu(z))} adj(\mathrm{D}F_\mu(z))F_\mu(z) \\ &= (-1)^{m+1}\left|\frac{1}{det(\mathrm{D}F_\mu(z))}\right| adj(\mathrm{D}F_\mu(z))F_\mu(z) \\ &= \lambda_1 \Phi_{A,\mu}(z) \end{aligned}$$

where $\lambda_1 = \left|\frac{1}{det(\mathrm{D}F_\mu(z))}\right|$. It follows that the directions of the $\Phi_{A,\mu}(z), \Phi_{N,\mu}(z)$, and $\Phi_{S,\mu}(z)$ in this setting are identical.

Finally, we note a property that is important for using the continuous Newton method.

Lemma 2.1.8. *Given* $\mu \geq 0, \Phi_{A,\mu}(z) \in ker(\mathrm{D}g_{F_\mu}(z))$.

Proof:

$$\mathrm{D}F_\mu(z)\Phi_{A,\mu}(z) = \mathrm{D}F_\mu(z)(-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z) = (-1)|det(\mathrm{D}F_\mu(z))|F_\mu(z).$$

Hence the lemma holds from Proposition 2.0.5

§2.2 Smale Vector Field

Next we outline the work of [Sm] and [HSm] involving the construction of Smale's vector field. The setting for [Sm] was the following.

Consider a C^2 function $F : \bar{M} \subset \mathbf{R^N} \to \mathbf{R^N}$ where M is a bounded nonempty subset such that $\partial\bar{M}$ is smooth (hence $\partial\bar{M}$ is a submanifold of dimension N-1). The idea behind the work of Smale was to define an IVP for which the solution curve traces a path to a root of F. In particular he considered the IVP

$$\frac{d\xi}{dt} = \Phi_S(\xi(t)), \qquad \xi(t_0) = z^0, \qquad g_F(\xi) = g_F(z^0),$$

where $z^0 \in Reg(g_F)$ and $\Phi_S(z)$ is constructed so that $\Phi_S(z) \in ker(\mathrm{D}g_F(z))$, $||\Phi_S(z)|| = 1$, and $\Phi_S(z)$ is tangent to the connected component through z in $A(z) = g_F^{-1}(g_F(z))$.

For this vector field Smale [Sm] stated an Existence and Convergence Theorem for the following setting.

Consider the ODE

$$\mathrm{D}F(z)\frac{dz}{dt} = -\lambda F(z)$$

Let M be given as above. Suppose that we have the *boundary condition* on F given by,

Definition 2.2.1 Boundary Condition (BC). *For each* $z \in \partial\bar{M}$, $\det(\mathrm{D}F(z)) \neq 0$, *and there exists a choice, (a)* $sgn(\lambda(z)) = sgn(\det(\mathrm{D}F(z)))$, $\forall z \in \partial\bar{M}$, *or (b)* $sgn(\lambda(z)) = -sgn(\det(\mathrm{D}F(z)))$, $\forall z \in \partial\bar{M}$, *which makes* $-\lambda(z)\mathrm{D}F(z)^{-1}F(z)$ *point into* M *at each* $z \in \partial\bar{M}$.

Note, Smale used the fact that F was actually C^2 on some open neighborhood, Ω, of $\bar{M}$. In this setting, with $\det(\mathrm{D}F(z)) \neq 0$, and given that F is C^2, it follows that $sgn(\det(\mathrm{D}F(z)))$ is constant on $\partial\bar{M}$. Hence, Smale's boundary condition, in fact, implies that either (a) is chosen **for all** $z \in \partial\bar{M}$ or (b) is chosen **for all** $z \in \partial\bar{M}$.

Note, the work done in section 6.2 below provides expanded proofs of the following theorems of Smale.

Theorem 2.2.2. *Let* $F : \bar{M} \to \mathbf{R}^\mathbf{N}$ *be* C^2 *and satisfy* **BC**. *There exists a canonically defined subset* Σ *of measure 0 in* $\partial\bar{M}$ *such that if* $z^0 \in \partial\bar{M}, z^0 \notin \Sigma$, *then there exists a unique* C^1 *solution* $\xi : [t_0, t_1) \to \bar{M}$ *of*

$$\frac{d\xi}{dt} = \Phi_S(\xi(t)), \qquad \xi(t_0) = z^0, \qquad g_F(\xi) = g_F(z^0),$$

starting at ξ_0 *with* $\|\frac{d\xi}{dt}\| = 1$, *and* t_1 *maximal,* $t_1 \leq \infty$. *This solution converges to* E_F *as* $t \to t_1$.

Some explanation of Smale's use of the term *canonically* is in order. Σ is defined as

$$\Sigma = \{z \in \partial\bar{M} : \exists \tilde{z} \in Crit(g_F), g_F(z) = g_F(\tilde{z})\}.$$

It follows that $z \in \Sigma \Leftrightarrow \exists \tilde{z} \in Crit(g_F)$ such that $F(z) = \lambda F(\tilde{z})$ for some $\lambda > 0$. It is based on this geometry that Smale uses the term canonically.

Definition 2.2.3. *A function $F : \bar{M} \to \mathbf{R}^{\mathbf{N}}$ is said to satisfy a* **Non-Singularity (NS)** *condition if* $\forall z \in E_F,\ \mathrm{D}f(z)$ *is non-singular.*

Under the non-singularity condition **NS**, Smale [Sm] stated the following theorem.

Theorem 2.2.4. *Let $F : \bar{M} \to \mathbf{R}^{\mathbf{N}}$ be C^2 and satisfy* **BC** *and* **NS**. *There exists a canonically defined closed subset Σ of measure 0 in $\partial\bar{M}$ such that if $z^0 \in \partial\bar{M}$, $z^0 \notin \Sigma$, then there exists a unique C^1 solution $\xi : [t_0, t_1) \to \bar{M}$ of*

$$\frac{d\xi}{dt} = \Phi_S(\xi(t)), \qquad \xi(t_0) = z^0, \qquad g_F(\xi) = g_F(z^0),$$

starting at ξ_0 with $\|\frac{d\xi}{dt}\| = 1$, and t_1 maximal, $t_1 \leq \infty$. This solution converges to a single point $\xi^* \in E_F$ as $t \to t_1$.

CHAPTER 3

REGULAR POINTS AND REGULAR VALUES OF g_{F_μ} ON $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$

We have seen in Chapter 2 the important relationship between $Crit(g_{F_\mu})$ and the fixed points of $\frac{dz}{dt} = \Phi_{A,\mu}(z)$. Clearly one of the questions that arises is that of identifying $Reg(g_{F_\mu})$. This chapter provides analysis of the regular points of g_{F_μ}. It is shown that $Crit(g_{F_\mu})$ is of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Also, the structure of the regular values of g_{F_μ} is discussed and it is established that for $\mu > 0$, the set of points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $g_{F_\mu}(z)$ is a critical value is of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.

Proposition 3.1. *If* $(s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$, $DF_\mu(s, r, x, y)$ *is of full rank and therefore* $(s, r, x, y) \in Reg(g_{F_\mu})$.

Proof: The proof follows directly from Corollary 2.0.5 and Proposition 2.1.2.

We now turn our attention to determining the structure of regular points in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. The orthant $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is formed by intersecting the half-spaces given by $z_i \geq 0$. We will classify a point $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ by the number of components z_i for which $z_i = 0$.

Definition 3.2. *Given I_0, a nonempty subset of $\{1, \ldots, 2n + 2m\}$ and $\#(I_0) = k$, we say the set*

$$\mathcal{F}(I_0) = \{z \in \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}} : i \in I_0 \Rightarrow z_i = 0\}$$

is a **face of codimension** k **in** $\partial\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$.

Definition 3.3. *Given I_0, a nonempty subset of $\{1, \ldots, 2n+2m\}$ such that $\#(I_0) = k$, a point $z = (z_i) \in \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$ is in the* **relative interior of** $\mathcal{F}(I_0)$ *if*

$$z_i = 0 \Leftrightarrow i \in I_0.$$

Hence each of the polyhedral boundary faces of $\partial\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$ can be given in the form

$$\{z \in \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}} : i \in I_0 \Rightarrow z_i = 0\} = \mathcal{F}(I_0)$$

where I_0 is a nonempty subset of $\{1, \ldots, 2n + 2m\}$.

Again, recall that the Remark following (1.21) holds.

Proposition 3.4. *Given $I_0 \subset \{1, \ldots, 2n + 2m\}$ such that $\#(I_0) = 1$, if $z = (s, r, x, y)$ is in the relative interior of $\mathcal{F}(I_0)$, then $\mathrm{D}F_\mu(z)$ is of full rank and hence $z \in Reg(g_{F_\mu})$.*

Proof: Since we are in the interior of a face of codimension 1 it follows that $z_i = 0$ for exactly one i. We have seen already that the rank of $\mathrm{D}F_\mu$ is completely determined by

$$\begin{pmatrix} R & -XA^t \\ YA & S \end{pmatrix}.$$

The following can all be formed by elementary matrix operations.

If $r_j = 0$ for some j then we can form

$$\begin{pmatrix} R + XA^tS^{-1}YA & 0 \\ S^{-1}YA & I_{mm} \end{pmatrix}.$$

If $x_j = 0$ for some j then we can form

$$\begin{pmatrix} I_{nn} & -R^{-1}XA^t \\ 0 & S + YAR^{-1}XA^t \end{pmatrix}.$$

If $y_j = 0$ for some j then we can form

$$\begin{pmatrix} R + XA^tS^{-1}YA & 0 \\ S^{-1}YA & I_{mm} \end{pmatrix}.$$

If $s_j = 0$ for some j then we can form

$$\begin{pmatrix} I_{nn} & -R^{-1}XA^t \\ 0 & S + YAR^{-1}XA^t \end{pmatrix}.$$

All the above matrices are of full rank. Hence, $\mathrm{D}F_\mu$ is of full rank on the interior of faces of codimension 1. From Corollary 2.0.5, the proposition holds.

We now turn our attention to a specific type of face of codimension 2 in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. In particular, we consider special faces which contain points z^0 for which $x_i^0 = r_i^0 = 0$ or $y_j^0 = s_j^0 = 0$, that is points where the complementary pairs are equal to zero.

Proposition 3.5. *Given $I_0 \subset \{1, \ldots, 2n+2m\}$ such that $\#(I_0) = 2$, if $z = (s, r, x, y)$ is in the relative interior of $\mathcal{F}(I_0)$ and there exists an i such that $r_i = x_i = 0$ or $s_i = y_i = 0$, then $rank(\mathrm{D}F_\mu(z)) = 2n + 2m - 1$.*

Proof: First,consider the case where $x_i = r_i = 0$ for some i and all other components are positive. Set $l = m + n + i$. It follows that the lth row of $\mathrm{D}F_\mu(z)$ is a zero row. Hence $\mathrm{adj}(\mathrm{D}F_\mu(z))$ is a matrix of all zeros except for the lth column.

Now consider $\det(\mathrm{D}F_\mu(l|l)(z)$ where $\mathrm{D}F_\mu(i|j)(z)$ is the submatrix formed by removing the ith row and the jth column from $\mathrm{D}F_\mu(z)$. Given a matrix B let B'

denote the matrix formed by removing the jth row and column of B for a given j. Let $\tilde{B}$ denote the matrix formed by removing the jth row of B for a given j. Let $\hat{B}$ denote the matrix formed by removing the jth column of B for a given j. It follows that

$$\mathrm{D}F_\mu(l|l)(z) = \begin{pmatrix} -I_{mm} & 0 & \hat{A} & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & \tilde{X} & R' & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

By doing elementary row and column operations we form the matrix

$$\begin{pmatrix} -I_{mm} & 0 & \hat{A} & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & 0 & R' & -X'\widetilde{A^t} \\ 0 & 0 & Y\hat{A} & S \end{pmatrix}.$$

Once again using elementary row and column operations we form that matrix

$$\begin{pmatrix} -I_{mm} & 0 & \hat{A} & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & 0 & I_{n-1n-1} & -(R')^{-1}X'\widetilde{A^t} \\ 0 & 0 & 0 & S + Y\hat{A}(R')^{-1}X'\widetilde{A^t} \end{pmatrix}.$$

Note that $\widetilde{A^t} = (\hat{A})^t$. Now $S + Y\hat{A}(R')^{-1}X'\widetilde{A^t} = S + Y\hat{A}(R')^{-1}X'(\hat{A})^t$. $SY^{-1} + \hat{A}(R')^{-1}X'(\hat{A})^t$is symmetric positive definite, hence $\det(S + Y\hat{A}(R')^{-1}X'\tilde{A}^t) > 0$. Therefore $(\det(\mathrm{D}F_\mu(l|l)(z))) \neq 0$ and therefore $rank(\mathrm{D}F_\mu(z)) = 2n + 2m - 1$.

Now we consider the case where $y_i = s_i = 0$ for some i and all other components are positive. Let $l = m + 2n + i$. It follows that the lth row of $\mathrm{D}F_\mu(z)$ is a zero row. Hence $\mathrm{adj}(\mathrm{D}F_\mu(z))$ is a matrix of all zeros except for the lth column. Using notation as above, it follows that

$$\mathrm{D}F_\mu(l|l)(z) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & \widehat{A^t} \\ 0 & X & R & 0 \\ \tilde{Y} & 0 & 0 & S' \end{pmatrix}.$$

By doing elementary row and column operations we form the matrix

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & \widehat{A^t} \\ 0 & 0 & R & -X\widehat{A^t} \\ \tilde{0} & 0 & \tilde{Y}A & S' \end{pmatrix}.$$

Notice that $\tilde{Y}A = Y'\tilde{A}$ and $(\tilde{A})^t = \widehat{A^t}$. Now using elementary operations we form the matrix

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & \widehat{A^t} \\ 0 & 0 & I_{nn} & -R^{-1}X\widehat{A^t} \\ \tilde{0} & 0 & 0 & S' + Y'\tilde{A}R^{-1}X(\tilde{A})^t. \end{pmatrix}$$

Since $S'(Y')^{-1} + \tilde{A}R^{-1}X(\tilde{A})^t$ is symmetric positive definite, it follows that $\det(S' + Y'\tilde{A}R^{-1}X\hat{A}^t) > 0$. Therefore $\det(\mathrm{D}F_\mu(l|l)(z)) \neq 0$ and hence $rank(\mathrm{D}F_\mu(z)) = 2n + 2m - 1$.

Proposition 3.6. *If* $\mathrm{D}F_\mu(s, r, x, y)$ *has a row of zeros and* $\mu > 0$, *then*

$$range(\mathrm{D}F_\mu(z)) \cap L = \{0\}$$

where $L = \{\alpha \cdot F_\mu(z) : \alpha \in \mathbf{R}\}$.

Proof: From the structure of $\mathrm{D}F_\mu(z)$, we may only have a zero row in the last $n + m$ rows. Suppose that we have (WLOG) $x_1 = r_1 = 0$. Then

$$\mathrm{D}F_\mu = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & \begin{pmatrix} 0 \\ \tilde{X} \end{pmatrix} & \begin{pmatrix} 0 \\ \tilde{R} \end{pmatrix} & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

Suppose there exists $\alpha \neq 0$ such that $\mathrm{D}F_\mu(z)u = \alpha \cdot F_\mu(z)$. That is,

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & \begin{pmatrix} 0\ldots0 \\ \tilde{X} \end{pmatrix} & \begin{pmatrix} 0\ldots0 \\ \tilde{R} \end{pmatrix} & 0 \\ Y & 0 & 0 & S \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_{2n+2m} \end{pmatrix} = \alpha \begin{pmatrix} A\begin{pmatrix} 0 \\ x \end{pmatrix} - s - b \\ A^t y + \begin{pmatrix} 0 \\ r \end{pmatrix} - c \\ \begin{pmatrix} 0 \\ \tilde{X} \end{pmatrix}\begin{pmatrix} 0 \\ \tilde{r} \end{pmatrix} - \mu e^n \\ Ys - \mu e^m \end{pmatrix}.$$

Expanding we have,

$$-\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} + A \begin{pmatrix} u_{m+n+1} \\ \vdots \\ u_{m+2n} \end{pmatrix} = \alpha(A \begin{pmatrix} 0 \\ x \end{pmatrix} - s - b)$$

$$\begin{pmatrix} u_{m+1} \\ \vdots \\ u_{m+n} \end{pmatrix} + A^t \begin{pmatrix} u_{2n+m+1} \\ \vdots \\ u_{2n+2m} \end{pmatrix} = \alpha(A^t y + \begin{pmatrix} 0 \\ r \end{pmatrix} - c)$$

$$0 = \alpha(-\mu)$$

$$\tilde{X} \begin{pmatrix} u_{m+2} \\ \vdots \\ u_{m+n} \end{pmatrix} + \tilde{R} \begin{pmatrix} u_{m+n+2} \\ \vdots \\ u_{m+2n} \end{pmatrix} = \alpha(\tilde{X}\tilde{r} - \mu e^n)$$

$$Y \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} + S \begin{pmatrix} u_{2n+m+1} \\ \vdots \\ u_{2n+2m} \end{pmatrix} = \alpha(Ys - \mu e^m).$$

It follows that we must have $\mu = 0$. This contradicts our assumption on μ and therefore $range(\mathrm{D}F_\mu(z)) \cap L = \{0\}$.

Corollary 3.7. *Suppose that $I_0 \subset \{1, \ldots, 2n+2m\}$, $\#(I_0) = 2$, and $\mu > 0$. If $z = (s, r, x, y)$ is in the relative interior of $\mathcal{F}(I_0)$ and $r_i = x_i = 0$ or $s_i = y_i = 0$, then $z \in Reg(g_{F_\mu})$.*

Proof: Follows from Propositions 3.5, 3.6 and Corollary 2.0.5.

We will now identify one of the effects of μ on the vector field.

Proposition 3.8. *Suppose that $I_0 \subset \{1, \ldots, 2n+2m\}$, $\#(I_0) = 2$, and $\mu = 0$. If $z = (s, r, x, y)$ is in the relative interior of $\mathcal{F}(I_0)$ and $r_i = x_i = 0$ or $s_i = y_i = 0$, then $z \in Crit(g_{F_\mu})$.*

Proof: As in the proof of Proposition 3.5, if $x_i = r_i = 0$ ($s_j = y_j = 0$) then for $l = m + n + i$ ($l = m + 2n + j$), $\mathrm{adj}(\mathrm{D}F_\mu(z))$ is a matrix of all zeros except for the lth column. It follows that

$$\Phi_{A,\mu}(z) = (-1)^{m+1}(F_\mu)_l(z)(lth\ col\ of\ \mathrm{adj}(\mathrm{D}F_\mu(z)).$$

But, $(F_\mu)_l(z) = 0$, and therefore $\Phi_{A,\mu}(z) = 0$. Hence, from Theorem 2.1.1 it follows that $z \in Crit(g_{F_\mu}) \cup E_{F_0}$. Now, if $z \in E_{F_0}$, $x_j r_j = 0$ $\forall j$. It follows that there exists some $\hat{I}_0 \subset \{1, \ldots, 2n+2m\}$ such that $\#(\hat{I}_0) = n$ and $z \in \mathcal{F}(\hat{I}_0)$. Since z is in the relative interior of $\mathcal{F}(I_0)$, no such $\hat{I}_0$ exists. Hence $z \in Crit(g_{F_\mu})$.

Proposition 3.9. *$Crit(g_{F_\mu}) \cap \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: By Proposition 3.4, points in the relative interior of a face of codimension 1 are regular points of g_{F_μ}. It follows that critical points of g_{F_μ} may only occur in faces of codimension ≥ 2 in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Such faces are of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.

Proposition 3.10. *The set $Crit(g_{F_\mu})$ is closed in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$.*

Proof: By Proposition 2.1.5, $\Phi_{A,\mu}|_{\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}}(s,r,x,y)$ is continuous. Also, from Theorem 2.1.1, a point $(s,r,x,y) \in (\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}) \cap Crit(g_{F_\mu})$ if and only if $\Phi_{A,\mu}|_{\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}}(s,r,x,y) = 0$. Therefore $Crit(g_{F_\mu}) = \Phi_{A,\mu}|^{-1}_{\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}}(0)$ is closed and the proposition follows.

Since $Crit(g_{F_\mu})$ is closed, $(\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}) \setminus Crit(g_{F_\mu})$ is a $2n+2m-1$-dim manifold. In this setting we have Morse-Sard's Theorem to classify critical values. The following version is from [H].

Theorem 3.11 (Morse-Sard Theorem). *Let M, N be differentiable manifolds of dimension m, n respectively and $f : M \to N$ a C^r map. If $r > \max\{0, m-n\}$ then the set of critical values of f is of measure zero in N.*

It follows that for our setting we have the following corollary.

Corollary 3.12. *The set of critical values of g_{F_μ} is of measure zero in $S^{2n+2m-1}$.*

Definition 3.13. *Let $\mu \geq 0$, denote*

$$\Lambda_\mu = \{w \in S^{2n+2m-1} : \exists z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \cap Crit(g_{F_\mu}), g_{F_\mu}(z) = w\}.$$

Hence, Λ_μ is the set of measure zero of critical values of g_{F_μ} given in Corollary 3.12 corresponding to μ.

Definition 3.14. *Let $\mu \geq 0$, denote*

$$\Sigma_\mu = \{z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} : \exists z \in \bar{C}_\mu(z^0), g_{F_\mu}(z) \in \Lambda_\mu\}.$$

Define

$$\Sigma_\mu^\partial = \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \cap \Sigma_\mu$$

and

$$\Sigma_\mu^+ = \mathbf{R}_+^{\mathbf{2n+2m}} \cap \Sigma_\mu.$$

Note, $\bar{C}_\mu(z^0)$ denotes the closure of $C_\mu(z^0)$. For a fixed μ, Σ_μ is the set of all points $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which the exists a z in the closure of the connected component of $A_\mu(z^0)$ such that $g_{F_\mu}(z)$ is a critical value of g_{F_μ}. It is important to have an understanding of the structure of the critical values. In what follows, conditions on $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ are given which ensure that $g_{F_\mu}(z)$ is a regular value of g_{F_μ}. This will also provide a constructive proof that for $\mu > 0$, Σ_μ^∂ is of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. The following proposition is important in the work that is to follow.

Proposition 3.15. *Let $z_1, z_2 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ be such that $F_\mu(z_1) \neq 0 \neq F_\mu(z_2)$. Then $g_{F_\mu}(z_1) = g_{F_\mu}(z_2) \Leftrightarrow F_\mu(z_1) = kF_\mu(z_2)$ for some $k > 0$.*

Proof: Suppose that $F_\mu(z_1) = kF_\mu(z_2)$ for some $k > 0$. It follows that

$$g_{F_\mu}(z_1) = \frac{F_\mu(z_1)}{||F_\mu(z_1)||} = \frac{kF_\mu(z_2)}{||kF_\mu(z_2)||} = \frac{kF_\mu(z_2)}{k||F_\mu(z_2)||} = g_{F_\mu}(z_2).$$

Now suppose that $g_{F_\mu}(z_1) = g_{F_\mu}(z_2)$. Then

$$\frac{F_\mu(z_1)}{||F_\mu(z_1)||} = \frac{F_\mu(z_2)}{||F_\mu(z_2)||}.$$

It follows that

$$F_\mu(z_1) = \frac{||F_\mu(z_1)||}{||F_\mu(z_2)||} F_\mu(z_2)$$

and the proposition holds.

Proposition 3.16. *Let $\mu > 0$. Suppose that there exist sets $\bar{I}_0 \subset \{1, \ldots, n\}, \bar{I}_1 \subset \{1, \ldots, m\}$ for which $\#(\bar{I}_0 \cup \bar{I}_1) > n + m - 2$. If (s, r, x, y) is such that*

$$i \in \bar{I}_0 \;\Rightarrow x_i r_i - \mu > 0$$

$$j \in \bar{I}_1 \;\Rightarrow y_j s_j - \mu > 0,$$

then $(s, r, x, y) \in Reg(g_{F_\mu})$ and $g_{F_\mu}(s, r, x, y)$ is a regular value of g_{F_μ}.

Proof: Let $\mu > 0, \bar{I}_0 \subset \{1, \ldots, n\}, \bar{I}_1 \subset \{1, \ldots, m\}, \#(\bar{I}_0 \cup \bar{I}_1) > n + m - 2$, and $z = (s, r, x, y)$ be such that

$$i \in \bar{I}_0 \;\Rightarrow x_i r_i - \mu > 0$$

$$j \in \bar{I}_1 \;\Rightarrow y_j s_j - \mu > 0.$$

From Proposition 3.4, for $\bar{z} \in Crit(g_{F_\mu})$, there exists a subset I_0 of $\{1, \ldots, 2n + 2m\}$ with $\#(I_0) \geq 2$ such that $\bar{z}_i = 0 \Leftrightarrow i \in I_0$. Since $\mu > 0$, from Corollary 3.7, if $\bar{z} \in Crit(g_{F_\mu})$ and $\bar{z}$ is in the relative interior of a face of codimension 2 in $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ then

$$\#\{\{(\bar{x}_i, \bar{r}_i) : \bar{x}_i \bar{r}_i = 0\} \cup \{(\bar{y}_j, \bar{s}_j) : \bar{y}_j \bar{s}_j = 0\}\} = 2.$$

It follows that for $\bar{z} \in Crit(g_{F_\mu}) \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$,

$$\#\{\{(\bar{x}_i, \bar{r}_i) : \bar{x}_i \bar{r}_i - \mu < 0\} \cup \{(\bar{y}_j, \bar{s}_j) : \bar{y}_j \bar{s}_j - \mu < 0\}\} \geq 2.$$

Since

$$i \in \bar{I}_0 \ \Rightarrow x_i r_i - \mu > 0$$

$$j \in \bar{I}_1 \ \Rightarrow y_j s_j - \mu > 0,$$

and $\#(\bar{I}_0 \cup \bar{I}_1) > n + m - 2$, it follows that

$$\#\{\{(x_i, r_i) : x_i r_i - \mu < 0\} \cup \{(y_j, s_j) : y_j s_j - \mu < 0\}\} < 2.$$

and therefore $(s, r, x, y) \in Reg(g_{F_\mu})$. Now suppose there exists $\bar{z} \in Crit(g_{F_\mu})$ such that

$$g_{F_\mu}(\bar{z}) = g_{F_\mu}(z).$$

From above,

$$\#\{\{(\bar{x}_i, \bar{r}_i) : \bar{x}_i \bar{r}_i - \mu < 0\} \cup \{(\bar{y}_j, \bar{s}_j) : \bar{y}_j \bar{s}_j - \mu < 0\}\} \geq 2.$$

and

$$\#\{\{(x_i, r_i) : x_i r_i - \mu < 0\} \cup \{(y_j, s_j) : y_j s_j - \mu < 0\}\} \leq 1.$$

It follows that there exists (WLOG) some $j \in \{1, \ldots, m\}$ such that $y_j s_j - \mu > 0, \bar{y}_j \bar{s}_j - \mu < 0$. It follows from Proposition 3.15 that there exists some $k > 0$ such that

$$F_\mu(z) = k F_\mu(\bar{z}).$$

Therefore,

$$0 < y_j s_j - \mu = k(\bar{s}_j \bar{y}_j - \mu) < 0.$$

Hence no such k exists and the proposition holds.

Proposition 3.17. *Suppose $\mu = 0, (s^0, r^0, x^0, y^0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$, and define $w^0 = g_{F_\mu}(s^0, r^0, x^0, y^0)$. Then $g_{F_\mu}^{-1}(w^0) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$ and hence w^0 is a regular value of g_{F_μ}.*

Proof: Let $\mu = 0$. For all $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, by Definition 1.8,

$$F_\mu(s,r,x,y) = F_0(s,r,x,y) = \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr \\ Ys \end{pmatrix}.$$

Let $z^0 = (s^0, r^0, x^0, y^0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and $w^0 = g_{F_\mu}(s^0, r^0, x^0, y^0)$. Suppose there exists $\bar{z} = (\bar{s}, \bar{r}, \bar{x}, \bar{y}) \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that

$$g_{F_\mu}(\bar{s}, \bar{r}, \bar{x}, \bar{y}) = g_{F_\mu}(s^0, r^0, x^0, y^0).$$

So there exists (WLOG) some i such that $\bar{x}_i = 0$. Now since $g_{F_\mu}(z^0) = g_{F_\mu}(\bar{z})$, from Proposition 3.15, there exists some $k > 0$, such that

$$F_\mu(s^0, r^0, x^0, y^0) = kF_\mu(\bar{s}, \bar{r}, \bar{x}, \bar{y}).$$

That is,

$$\begin{pmatrix} Ax^0 - s^0 - b \\ A^t y^0 + r^0 - c \\ X^0 r^0 \\ Y^0 s^0 \end{pmatrix} = k \begin{pmatrix} A\bar{x} - \bar{s} - b \\ A^t \bar{y} + \bar{r} - c \\ \bar{X}\bar{r} \\ \bar{Y}\bar{s} \end{pmatrix}.$$

In particular,

$$0 < x_i^0 r_i^0 = k(\bar{x}_i \bar{r}_i) = 0.$$

Hence, no such $(\bar{s}, \bar{r}, \bar{x}, \bar{y})$ exists. It follows that $g_{F_\mu}^{-1}(w^0) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. Now, if $\bar{z} \in Crit(g_{F_\mu})$, it follows from Proposition 3.1, $\bar{z} \notin \mathbf{R}_+^{\mathbf{2n+2m}} \subset Reg(g_{F_\mu})$, and hence $\bar{z} \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Therefore, $Crit(g_{F_\mu}) \cap g_{F_\mu}^{-1}(z^0) = \emptyset$. From Definition 2.0.3, $g_{F_\mu}(s,r,x,y)$ is a regular value of g_{F_μ}.

Proposition 3.18. *Suppose $I_0 \subset \{1, \ldots, 2n+2m\}, \#(I_0) = 1$, and $\mu > 0$. If z is in the relative interior of $\mathcal{F}(I_0)$, $g_{F_\mu}(z)$ is a regular value of g_{F_μ}.*

Proof: From Proposition 3.4, there are no critical points in the interior of faces of codimension 1 in $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Also, from Proposition 3.7, any critical point in the interior of faces of codimension 2 in $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ must have

$$\#\{\{(x_i, r_i) : x_i r_i = 0\} \cup \{(y_j, s_j) : y_j s_j = 0\}\} = 2.$$

Hence, for any $z^* \in Crit(g_{F_\mu}) \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$,

$$\#\{\{(x_i^*, r_i^*) : x_i^* r_i^* = 0\} \cup \{(y_j^*, s_j^*) : y_j^* s_j^* = 0\}\} \geq 2.$$

Therefore, for any $z^* \in Crit(g_{F_\mu}) \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$,

$$\#\{\{(x_i^*, r_i^*) : x_i^* r_i^* - \mu = -\mu\} \cup \{(y_j^*, s_j^*) : y_j^* s_j^* - \mu = -\mu\}\} \geq 2.$$

Now suppose that $I_0 \subset \{1, \ldots, 2n+2m\}, \#(I_0) = 1$, z^0 in the relative interior of $\mathcal{F}(I_0)$ in $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and $g_{F_\mu}(z^0) = g_{F_\mu}(z^*)$ for some $z^* \in Crit(g_{F_\mu}) \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. From Proposition 3.15, $F_\mu(z^0) = kF_\mu(z^*)$ for some $k > 0$. Note,

$$\#\{\{(x_i^0, r_i^0) : x_i^0 r_i^0 = 0\} \cup \{(y_j^0, s_j^0) : y_j^0 s_j^0 = 0\}\} = 1.$$

Hence, there exists some (WLOG) i such that

$$x_i^0 r_i^0 > 0, \qquad x_i^0 r_i^0 - \mu = k(x_i^* r_i^* - \mu) = k(0 - \mu) = k(-\mu).$$

Solving for $x_i^0 r_i^0$ we have $x_i^0 r_i^0 = \mu(1-k)$. Since $x_i^0 r_i^0 > 0$, it follows that $0 < k < 1$. Now there also exists (WLOG) some j such that $x_j^0 r_j^0 = 0$. If $x_j^* r_j^* = 0$, then $g_{F_\mu}(z^0) = g_{F_\mu}(z^*) \Rightarrow -\mu = k(-\mu)$ and therefore $k = 1$. If $x_j^* r_j^* > 0$, then $g_{F_\mu}(z^0) = g_{F_\mu}(z^*) \Rightarrow -\mu = k(x_j^* r_j^* - \mu)$. Solving for $x_j^* r_j^*$ we have $x_j^* r_j^* = \mu(1 - \frac{1}{k})$ and since $x_j^* r_j^* > 0$, $k > 1$. In either case we have a contradiction on the value of k and so the proposition holds.

Corollary 3.19. *Let $\mu > 0$. Σ_μ^∂ is of measure zero in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: From Proposition 3.18, points z for which $g_{F_\mu}(z)$ is a critical value are contained in the relative interior of faces of codimension $k \geq 2$. Hence the corollary holds.

Proposition 3.20. *Let $z = (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$. Then there exists $\bar{\mu} > 0$ such that $g_{F_\mu}(s, r, x, y)$ is a regular value of g_{F_μ} for every $0 \leq \mu < \bar{\mu}$.*

Proof: Set

$$\bar{\mu} = \min\{\{x_i r_i\} \cup \{y_j s_j\}\}.$$

Then for $0 \leq \mu < \bar{\mu}$,

$$\#\{\{(x_i, r_i) : x_i r_i - \mu < 0\} \cup \{(y_j s_j) : y_j s_j - \mu < 0\}\} < 2.$$

Hence, from Proposition 3.16 the proposition holds.

We can prove the existence of points $z \in Reg(g_{F_\mu})$ for which $g_{F_\mu}(z)$ is a critical value of g_{F_μ}. The following theorem is from Wright [WS].

Theorem 3.21. *Given (1.18)-(1.20), the Central Path exists and is in fact a C^1 trajectory. Hence, for every $\mu \geq 0$, there exists $z^* \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that $F_\mu(z^*) = 0$. For $\mu > 0, z^*$ is unique.*

Proposition 3.22. *Given (1.18)-(1.20), for every $\mu > 0$, there exists some $z_\mu \in Reg(g_{F_\mu})$ such that $g_{F_\mu}(z_\mu)$ is a critical value for g_{F_μ}.*

Proof: Let $\mu > 0$. $(0, 0, 0, 0) \in Crit(g_{F_\mu})$ and $F_\mu(0, 0, 0, 0) = (-b, -c, -\mu e, -\mu e)$. Given (1.18)-(1.20), from Theorem 3.21, the Central Path exists and hence there exists some $z \in \mathbf{R}_+^{2n+2m}$ such that

$$\begin{pmatrix} 0 \\ 0 \\ \mu e \\ \mu e \end{pmatrix} = F_\mu(z) = \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr - \mu e \\ Ys - \mu e \end{pmatrix}.$$

It follows that $\frac{1}{2}z \in \mathbf{R}_+^{2n+2m} \subset Reg(g_{F_\mu})$. Now,

$$F_\mu(\frac{1}{2}z) = \begin{pmatrix} \frac{1}{2}(Ax - s) - b \\ \frac{1}{2}(A^t y + r) - c \\ \frac{1}{4}Xr - \mu e \\ \frac{1}{4}Ys - \mu e \end{pmatrix} = \begin{pmatrix} \frac{1}{2}b - b \\ \frac{1}{2}c - c \\ \frac{1}{4}(2\mu)e - \mu e \\ \frac{1}{4}(2\mu)e - \mu e \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -b \\ -c \\ -\mu e \\ -\mu e \end{pmatrix} = \frac{1}{2}F_\mu(0,0,0,0).$$

From Definition 2.0.3, $g_{F_\mu}(\frac{1}{2}z)$ is a critical value for g_{F_μ}.

As before, Σ_μ^+ is the set of points in $\mathbf{R}_+^{2n+2m}$ which have critical values. The following proposition shows that Σ_μ^+ is of measure zero for $\mu > 0$.

Proposition 3.23. *Let $\mu > 0$. Σ_μ^+ is of measure zero in $\mathbf{R}_+^{2n+2m}$.*

Proof:. Suppose that $z^0 \in \Sigma^+$ is such that

$$g_{F_\mu}(z^0) = g_{F_\mu}(z^*)$$

for $z^* \in Crit(g_{F_\mu})$. From Proposition 3.4 and Corollary 3.7,

$$\#\{\{(x_i^*, r_i^*) : x_i^* r_i^* - \mu = -\mu\} \cup \{(y_j^*, s_j^*) : y_j^* s_j^* - \mu = -\mu\}\} \geq 2.$$

Then, from Proposition 3.15, there exists some $k > 0$ such that

$$F_\mu(z^0) = kF_\mu(z^*).$$

Hence, (WLOG), there exists some i, j such that

$$x_i^0 r_i^0 - \mu = k(-\mu)$$

$$y_j^0 s_j^0 - \mu = k(-\mu).$$

It follows that $x_i^0 r_i^0 = y_j^0 s_j^0$ for any $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$ for which $g_{F_\mu}(z^0)$ is a critical value. Hence, Σ_μ^+ is a closed set and of measure zero in $\mathbf{R}_+^{\mathbf{2n+2m}}$.

Note, Σ_μ^+ is a set that would be excluded by Smale and Hirsch [Sm], [HSm] for the function $F_\mu(z)$. In Chapter 6 it is established that the trajectories which are solutions to (1.17) for which $z(0) \in \Sigma_\mu^+$ need not be excluded from consideration.

CHAPTER 4

TRANSVERSALITY OF TRAJECTORIES TO $\partial \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$

In the approach that we will be using it is important to determine whether the solution curve is transverse to the the boundary of the domain. We are considering $F_\mu(s,r,x,y)$ defined on the $\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$. Therefore we are interested in the structure of the trajectories on $\partial \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$. This chapter establishes whether the solution trajectories of (1.17) are transverse to $\partial \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$. In it, the terminology of transversality is defined based on considering how vectors are transverse to closed half-spaces. It is shown that for $\mu \geq 0$, $\Phi_{A,\mu}(z)$ is not outward transversal to $\partial \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$ at any point $z \in \partial \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$. The work done here is based various matrix manipulations. The terminology used is based on the following definitions.

Definition 4.0.1. *Given a nonzero vector d and a constant K, denote by*

$$H^+ = H^+_{d,K} = \{z \in \mathbf{R}^{\mathbf{N}} : d^t z \leq K\}$$

the **closed half-space bounded by** *the hyperplane*

$$H = H_{d,K} = \{z \in \mathbf{R}^{\mathbf{N}} : d^t z = K\}$$

with $-d$ being the inward pointing normal of the half-space.

Definition 4.0.2. *Given a finite collection of closed half-spaces,* $\{H_j^+ = H_{d_j,K_j}^+\}$, *the set*

$$\Pi = \cap_j H_j^+$$

is called a **polyhedral convex set.**

Definition 4.0.3. *Let* $\{H_j^+ = H_{d_j,K_j}^+\}_{j=1}^M$ *be a collection of closed-half spaces bounded by* $\{H_j\}_{j=1}^M \subset \mathbf{R}^\mathbf{N}$ *respectively. Let* $\Pi = \cap_j H_j^+$. *Suppose that* Π *has a nonempty interior and* $z^0 \in \partial\Pi$. *A nonzero vector* d *is* **inward pointing to** $\partial\Pi$ **at** z^0 *if and only if there exists some* $\alpha_0 > 0$ *such that* $\forall \alpha \in [0, \alpha_0], z = z^0 + \alpha d \in int(\Pi)$. d *is* **parallel to** $\partial\Pi$ **at** z^0 *if and only if there exists some* $\alpha_0 > 0$ *such that* $\forall \alpha \in [0, \alpha_0], z = z^0 + \alpha d \in \partial\Pi$. d *is* **outward pointing to** $\partial\Pi$ **at** z^0 *if and only if there exists some* $\alpha_0 > 0$ *such that* $\forall \alpha \in [0, \alpha_0], z = z^0 + \alpha d \in \mathbf{R}^\mathbf{N} - \Pi$.

Note that $int(\Pi) = \cap_j int(H_j^+)$. We will use the following theorem as a basis to discuss transversality.

Theorem 4.0.4. *Let* $\{H_j^+ = H_{d_j,K_j}^+\}_{j=1}^M$ *be a collection of closed-half spaces bounded by* $\{H_j\}_{j=1}^M$ *respectively. Let* $\Pi = \cap_j H_j^+$. *Suppose that* Π *has a nonempty interior. Suppose that* $z^0 \in \partial\Pi$ *and is such that there exists a set* $J \subset \{1, \ldots, M\}$ *such that* $z^0 \in H_j \Leftrightarrow j \in J$. *Then the following hold.*

1. *A nonzero vector* d *is inward pointing to* $\partial\Pi$ *at* $z^0 \Leftrightarrow -d_j^t d > 0\ \forall j \in J$.
2. *A nonzero vector* d *is outward pointing to* $\partial\Pi$ *at* $z^0 \Leftrightarrow \exists j \in J$ *such that* $-d_j^t d < 0$.
3. *A nonzero vector* d *is parallel to* $\partial\Pi$ *at* $z^0 \Leftrightarrow -d_j^t d \geq 0\ \forall j \in J$ *and* $\exists j \in J$ *such that* $-d_j^t d = 0$.

Proof: Let $z^0 \in \partial\Pi$ such that $z^0 \in \cap_{j\in J} H_j$. Let d be a nonzero vector.

1. Suppose that $-d_j^t d > 0 \; \forall j \in J$. Let $\alpha > 0$. Given $j \in J$,

$$d_j^t(z^0 + \alpha d) = d_j^t z^0 + \alpha d_j^t d = K_j + \alpha d_j^t d < K_j.$$

Hence, $z^0 + \alpha d \in int(H_j^+)$ for some appropriately small α. Since this is true for all j, it follows that there exists some $\alpha_0 > 0$ such that $z^0 + \alpha d \in int(\Pi)$ for $\alpha \in [0, \alpha_0]$. Now suppose that there exists $\alpha_0 > 0$ such that for $\alpha \in [0, \alpha_0]$, $z^0 + \alpha d \in int(\Pi)$. It follows that $z^0 + \alpha d \in \cap_{j=1}^M int(H_j^+)$. Therefore $d_j^t(z^0 + \alpha d) < K_j \; \forall j$. Now for $j \in J, d_j^t z^0 = K_j$. Hence, $\forall j \in J, -d_j^t d > 0$.

2. Suppose $\exists j \in J$ such that $-d_j^t d < 0$. Let $\alpha > 0$. Then

$$d_j^t(z^0 + \alpha d) = d_j^t z^0 + \alpha d_j^t d = K_j + \alpha d_j^t d > K_j.$$

It follows that $z^0 + \alpha d \in \mathbf{R^N} - \Pi$. Now suppose that there exists $\alpha_0 > 0$ such that for $\alpha \in [0, \alpha_0], z^0 + \alpha d \in \mathbf{R^N} - \Pi$. Now $\mathbf{R^N} - \Pi = \mathbf{R^N} - \cap_j H_j^+ = \cup_j(\mathbf{R^N} - H_j^+)$. Hence, there exists some j such that $z^0 + \alpha d \in \mathbf{R^N} - H_j^+$. That is, $d_j^t(z^0 + \alpha d) > K_j$. Since $d - j^t z^0 = K_j$, it follows that $d_j^t d > 0$.

3. Suppose that $-d_j^t d \geq 0 \; \forall j \in J$ and $\exists j \in J$ such that $-d_j^t d = 0$. Let $\alpha > 0$. It follows that $d_j^t(z^0 + \alpha d) = d_j^t z^0 + \alpha d_j^t d = d_j^t z^0 = K_j$. So, $z^0 + \alpha d \in H_j$. Also, since $-d_j^t d \geq 0 \; \forall j \in J, d_j^t(z^0 + \alpha d) \leq K_j, \; \forall j \in J$. It follows that there exists $\alpha_0 > 0$ such that for $\alpha \in [0, \alpha_0], \; z^0 + \alpha d \in \partial\Pi$. Now suppose there exists $\alpha_0 > 0$ such that for $\alpha \in [0, \alpha_0], \; z^0 + \alpha d \in \partial\Pi$. Since Π is closed, it follows that $d_j^t(z^0 + \alpha d) \leq K_j \forall j$. Also, since $z^0 + \alpha d \in \partial\Pi$, it follows that there exists some j such that $d_j^t(z^0 + \alpha d) = K_j$. Hence, $-d_j^t d \geq 0 \; \forall j$ and $\exists j \in J$ such that $-d_j^t d = 0$.

Given Theorem 4.0.4, we use the following terminology for integral curves of vector fields.

Definition 4.0.5. *Let $\{H_j^+ = H_{d_j,K_j}^+\}_{j=1}^M$ be a collection of closed-half spaces bounded by H_j respectively. Let $\Pi = \cap_j H_j^+$. Suppose that Π has nonempty interior. Suppose that $z^0 \in \partial\Pi$ and is such that there exists a set $J \subset \{1,\dots,M\}$ such that $z^0 \in H_j \Leftrightarrow j \in J$. A solution curve, $z(t)$, of*

$$\text{(IVP)} \qquad \frac{dz}{dt} = \Phi(z(t)), \qquad z(t_0) = z^0$$

is **transverse to $\partial\Pi$ at z^0** *if $-d_j^t\Phi(z(t_0)) \neq 0 \ \forall j \in J$. If in fact $-d_j^t\Phi(z(t_0)) > 0 \ \forall j \in J$ then the trajectory actually enters the interior of the domain and we say the solution curve is* **inward transverse to $\partial\Pi$ at z^0**. *A solution curve is* **outward transverse to $\partial\Pi$ at z^0** *if there exists some $j \in J$ such that $-d_j^t\Phi(z(t_0)) < 0$. A solution curve is* **parallel to $\partial\Pi$ at z^0** *if $-d_j^t\Phi(z(t_0)) \geq 0 \ \forall j \in J$ and $\exists j \in J$ such that $-d_j^t\Phi(z(t_0)) = 0$.*

§4.1 Faces of Codimension 1

We first consider the relative interior of a face of codimension 1 in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. From Proposition 3.4, $\mathrm{D}F_\mu(z)$ is of full rank. In the full rank setting, from Lemma 2.1.7, $\Phi_{A,\mu}(z), \Phi_{S,\mu}(z)$ are inward transversal to a given boundary if and only if $\Phi_{N,\mu}(z)$ is inward transversal to the given boundary. We will use the following method. If v is the inward normal to the face then

$$\Phi_{A,\mu}(z) \cdot v = v^t\Phi_{A,\mu}(z) = v^t(-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z).$$

Set $w^t = (-1)^{m+1}v^t\mathrm{adj}(\mathrm{D}F_\mu(z))$. Then $\Phi_{A,\mu}(z) \cdot v = w^tF_\mu(z)$. Also

$$v^t = w^t(\frac{-1}{|\det(\mathrm{D}F_\mu(z))|}\mathrm{D}F_\mu(z)).$$

So it becomes a question of showing that for a good choice of w, we get v back (or a scaled version of it) and that $w^t F_\mu(z) > 0$.

Through out this work, e_i denotes the standard position vector of dimension appropriate to the given setting.

Proposition 4.1.1. *Let $\mu = 0, I_0 \subset \{1, \ldots, 2n+2m\}$, and $\#(I_0) = 1$. If z is in the relative interior of $\mathcal{F}(I_0)$, $\Phi_{A,\mu}(z)$ is nonzero and parallel to $\mathcal{F}(I_0)$.*

Proof: Let z be in the relative interior of a face of codimension 1. Then by Proposition 3.4 and Theorem 2.1.1, since $F_\mu(z) \neq 0, \Phi_{A,\mu}(z)$ is nonzero. From above, if $w^t F_\mu(z) = 0$, $\Phi_{A,\mu}(z)$ is normal to v. Since v is the normal of a face of codimension 1, it follows from above that $\Phi_{A,\mu}(z)$ parallel to the face and therefore the proposition holds. We have four cases to address.

For $s_i = 0$ let $l = m + 2n + i$. The normal vector for this codimension 1 face is e_i. Now $(F_\mu)_l = 0$ where $(F_\mu)_l$ is the lth component function of F_μ. Set $w = -e_l$. Then $w^t F_\mu(z) = 0$ and

$$\begin{aligned} v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} e_l^t \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix} \\ &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} (y_i {e_i}^t + s_i e^t_{2n+m+i}) \\ &= \frac{y_i}{|\det(\mathrm{D}F_\mu(z))|} {e_i}^t. \end{aligned}$$

Hence we have that $\Phi_{A,\mu}(z)$ is parallel to the face determined by $s_i = 0$.

Similar arguments hold for the other faces of codimension 1 and are given below.

For $r_i = 0$ let $l = m + n + i$. Then the corresponding normal vector is e_{m+i}

and $(F_\mu)_l = 0$. Set $w = -e_l$. Then $w^t F_\mu(z) = 0$ and

$$v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) = \frac{x_i}{|\det(\mathrm{D}F_\mu(z))|} {e_{m+i}}^t.$$

Hence, $\Phi_{A,\mu}(z)$ is parallel to the face determined by $r_i = 0$.

For $x_i = 0$ let $l = m + n + i$. The corresponding normal vector is e_l and $(F_\mu)_l = 0$. Set $w = -e_l$. Then $w^t F_\mu(z) = 0$ and

$$v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) = \frac{r_i}{|\det(\mathrm{D}F_\mu(z))|} {e_l}^t.$$

Hence, $\Phi_{A,\mu}(z)$ is parallel to the face determined by $x_i = 0$.

For $y_i = 0$ let $l = m + 2n + i$. The corresponding normal vector is e_l and $(F_\mu)_l = 0$. Set $w = -e_l$. Then $w^t F_\mu(z) = 0$ and

$$v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) = \frac{s_i}{|\det(\mathrm{D}F_\mu(z))|} {e_l}^t.$$

Hence, $\Phi_{A,\mu}(z)$ is parallel to the face determined by $y_i = 0$.

Proposition 4.1.2. *Let $\mu > 0, I_0 \subset \{1, \ldots, 2n + 2m\}$, and $\#(I_0) = 1$. If z is in the relative interior of $\mathcal{F}(I_0)$, $\Phi_{A,\mu}(z)$ is inward transversal to $\mathcal{F}(I_0)$.*

Proof: As before, we have four cases.

For $s_i = 0$ let $l = m+2n+i$. Then $(F_\mu)_l = -\mu$. Set $w = -e_l$. Then $w^t F_\mu(z) = \mu > 0$ and

$$\begin{aligned} v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} e_l^t \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix} \\ &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} (y_i {e_i}^t + s_i e_{2n+m+i}^t) \\ &= \frac{y_i}{|\det(\mathrm{D}F_\mu(z))|} {e_i}^t. \end{aligned}$$

Hence we have inward transversality for the face determined by $s_i = 0$ when $\mu > 0$.

For $r_i = 0$ let $l = m+n+i$. Then $(F_\mu)_l = -\mu$. Set $w = -e_l$. Then $w^t F_\mu(z) = \mu$ > 0 and

$$v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) = \frac{x_i}{|\det(\mathrm{D}F_\mu(z))|} {e_{m+i}}^t.$$

Hence inward transversality for the face holds when $\mu > 0$.

For $x_i = 0$ let $l = m+n+i$. Then $(F_\mu)_l = -\mu$. Set $w = -e_l$. Then $w^t F_\mu(z) = \mu$ > 0 and

$$v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) = \frac{r_i}{|\det(\mathrm{D}F_\mu(z))|} {e_l}^t.$$

Hence inward transversality for the face holds when $\mu > 0$.

For $y_i = 0$ let $l = m+2n+i$. Then $(F_\mu)_l = -\mu$. Set $w = -e_l$. Then $w^t F_\mu(z) = \mu$ > 0 and

$$v^t = \frac{-1}{|\det(\mathrm{D}F_\mu(z))|} w^t \mathrm{D}F_\mu(z) = \frac{s_i}{|\det(\mathrm{D}F_\mu(z))|} {e_l}^t.$$

Hence inward transversality for the face holds when $\mu > 0$.

We can now prove stronger results about the transversality of the trajectories on $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.

Proposition 4.1.3. *Let $\mu = 0$. There are no points $z \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $\Phi_{A,\mu}(z)$ is outward (inward) transversal to $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: Suppose $I_0 \subset \{1, \ldots, 2n+2m\}, \#(I_0) = k$, and $\tilde{z}$ in the relative interior of $\mathcal{F}(I_0) \subset \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is a point for which $\Phi_{A,\mu}(\tilde{z})$ is outward transversal to $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Note, from Proposition 4.1.1, $k \geq 2$. Since $\Phi_{A,\mu}(\tilde{z})$ is outward transversal to $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ there exists some nonempty set $\eta \subset I_0$ such that if $i \in \eta$ then $\Phi_{A,\mu}(\tilde{z})_i = \Phi_{A,\mu}(\tilde{z}) \cdot e_i < 0$. Now $\Phi_{A,\mu}(z)_j$ is a polynomial in z for every j. So there exists $\epsilon > 0$ such that if $||z - \tilde{z}|| < \epsilon$ then $\Phi_{A,\mu}(z)_i < 0$ for $i \in \eta$. Pick some $l \in \eta$. Define a point z^* by

$$z_i^* = \begin{cases} \tilde{z}_i & \text{if } \tilde{z}_i > 0; \\ \frac{\epsilon}{2k} & \text{if } i \neq l; \\ 0 & \text{if } i = l. \end{cases}$$

Then

$$||z^* - \tilde{z}|| = \sqrt{\sum_{i \in \eta} (z_i^* - \tilde{z}_i)^2} = \sqrt{\sum_{i=1}^{k-1} (\frac{\epsilon}{2k})^2} = \frac{\epsilon}{2} \frac{\sqrt{k-1}}{k} < \frac{\epsilon}{2}.$$

It follows from continuity (Proposition 2.1.5) that $\Phi_{A,\mu}(z^*)_i < 0$ for $i \in \eta$. In particular, $\Phi_{A,\mu}(z^*)_l < 0$. But, z^* is in the relative interior of a face of codimension 1 in $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ given by $z_l = 0$, which has been shown to have $\Phi_{A,\mu}(z)$ parallel to it. That is $\Phi_{A,\mu}(z^*)_l = \Phi_{A,\mu}(z^*) \cdot e_l = 0$. Therefore no such $\tilde{z}$ exists. A similar argument holds for an inward transversal point.

Corollary 4.1.4. *Let $\mu = 0$. Given $z \in Reg(g_{F_\mu}) \cap \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, $\Phi_{A,\mu}(z)$ is nonzero and parallel to $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: Since $z \in Reg(g_{F_\mu}) \cap \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, from Theorem 2.1.1, $\Phi_{A,\mu}(z) \neq 0$. Hence, from Proposition 4.1.3, the corollary follows.

Proposition 4.1.5. *Let $\mu > 0$. There are no points $z \in \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $\Phi_{A,\mu}(z)$ is outward transversal to $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: Suppose $I_0 \subset \{1, \ldots, 2n + 2m\}, \#(I_0) = k$, $\tilde{z}$ in the relative interior of $\mathcal{F}(I_0) \subset \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is a point for which $\Phi_{A,\mu}(\tilde{z})$ is outward transversal to $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Note, from Proposition 4.1.2, $k \geq 2$. Since $\Phi_{A,\mu}(\tilde{z})$ is outward transversal to $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ there exists some nonempty set $\eta \subset I_0$ such that if $i \in \eta$ then $\Phi_{A,\mu}(\tilde{z})_i = \Phi_{A,\mu}(\tilde{z}) \cdot e_l < 0$. Now $\Phi_{A,\mu}(z)_j$ is a polynomial in z for every j. So there exists $\epsilon > 0$ such that if $|z - \tilde{z}| < \epsilon$ then $\Phi_{A,\mu}(z)_i < 0$ for $i \in \eta$. Pick some $l \in \eta$. Define a point z^* by

$$z_i^* = \begin{cases} \tilde{z}_i & \text{if } \tilde{z}_i > 0; \\ \frac{\epsilon}{2k} & \text{if } i \neq l; \\ 0 & \text{if } i = l. \end{cases}$$

Then

$$|z^* - \tilde{z}| = \sqrt{\sum_{i \in \eta} (z_i^* - \tilde{z}_i)^2} = \sqrt{\sum_{i=1}^{k-1} (\frac{\epsilon}{2k})^2} = \frac{\epsilon}{2} \frac{\sqrt{k-1}}{k} < \frac{\epsilon}{2}.$$

It follows from continuity (Proposition 2.1.5) that $\Phi_{A,\mu}(z^*)_i < 0$ for $i \in \eta$. In particular, $\Phi_{A,\mu}(z^*)_l < 0$. But, z^* is contained in the interior of a face of codimension 1 given by $z_l = 0$, which has been shown to be inward transversal. That is $\Phi_{A,\mu}(z^*)_l = \Phi_{A,\mu}(z^*) \cdot e_l > 0$. Therefore no such $\tilde{z}$ exists and the proposition holds.

Corollary 4.1.6. *Given $\mu > 0$ and $z \in Reg(g_{F_\mu}) \cap \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, $\Phi_{A,\mu}(z)$ is nonzero and either parallel or inward transversal to $\partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$.*

Proof: Since $z \in Reg(g_{F_\mu}) \cap \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, from Theorem 2.1.1, $\Phi_{A,\mu}(z) \neq 0$. Hence, from Proposition 4.1.5, the corollary follows.

§4.2 Faces of Codimension k

If $\mathrm{D}F_\mu(z)$ is of full rank at a point which is in the relative interior of a face which is the intersection of k $(2n + 2m - 1)$-dimensional faces, then by showing inward transversality to each $(2n + 2m - 1)$-dimensional face we obtain inward transversality for the face of codimension k. The argument is as above and inward transversality holds in this setting. If $\mathrm{D}F_\mu(z)$ is not of full rank then we must switch our approach. If $z \in Crit(g_{F_\mu}) \cup E_{F_\mu}$, then $\Phi_{A,\mu}(z) = 0$ and hence $\Phi_{A,\mu}(z)$ is not outward transversal to the face. Finally, if $z \in Reg(g_{F_\mu})$ and $\mathrm{D}F_\mu(z)$ is of rank $2n + 2m - 1$ then $Rank(\mathrm{adj}(\mathrm{D}F_\mu(z))) = 1$. We need only consider $\mu > 0$, as the case for $\mu = 0$ is completely determined above.

§4.2.1 Codimension 2

We will consider a point z in the relative interior of a face of codimension 2 in $\partial\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$ for which $x_i = r_i = 0$ or $y_j = s_j = 0$. This is a particularly interesting case as, from Proposition 3.5, $\Phi_{N,\mu}(z)$ is undefined at such a point, yet, from Corollary 3.7 and Theorem 2.1.1, $\Phi_{A,\mu}(z) \neq 0$.

Proposition 4.2.1.1. *Let* $\mu > 0, I_0 = \{j,k\}$ *such that* $(z_j, z_k) = (r_i, x_i)$ *or* $(z_j, z_k) = (s_i, y_i)$ *for some* i. *If* z *is in the relative interior of* $\mathcal{F}(I_0)$, $\Phi_{A,\mu}(z)$ *is inward transversal to* $\bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}$.

Proof: First we turn our attention to the case where $x_i = r_i = 0$ for some i and all other components are positive. Set $l = m + n + i$. It follows that the lth row of $\mathrm{D}F_\mu(z)$ is a zero row. Hence $\mathrm{adj}(\mathrm{D}F_\mu(z))$ is a matrix of all zeros except for the lth column. Recall that to show we have transversality at the intersection of faces it is enough to show that we in fact have transversality for each face individually.

First consider the face determined by $x_i = 0$. In this case the inward normal vector is $v^t = [0, 0, e_i, 0]$. It follows that

$$\begin{aligned} v^t\Phi_{A,\mu}(z) =& [0,0,e_i,0](-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z) \\ =& (-1)^{m+1}[0,0,e_i,0]\mathrm{adj}(\mathrm{D}F_\mu(z)) \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr - \mu e \\ Ys - \mu e \end{pmatrix} \end{aligned}$$

Therefore,

$$v^t\Phi_{A,\mu}(z) = (-1)^{m+1}(lth \ row \ of \ \mathrm{adj}(\mathrm{D}F_\mu(z))) \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr - \mu e \\ Ys - \mu e \end{pmatrix}$$

However, $\operatorname{adj}(\mathrm{D}F_\mu(z))$ is a matrix of all zeros except for the lth column. Hence,

$$\begin{aligned}
v^t\Phi_{A,\mu}(z) = (-1)^{m+1}\operatorname{adj}(\mathrm{D}F_\mu(z))_{ll}F_{\mu_l} &= (-1)^{m+1}\operatorname{adj}(\mathrm{D}F_\mu(z))_{ll}(-\mu) \\
&= (-1)^{m+1}(-1)^{l+l}(-\mu)\det(\mathrm{D}F_\mu(l|l)(z)) \\
&= (-1)^m\mu\det(\mathrm{D}F_\mu(l|l)(z))
\end{aligned}$$

where $\mathrm{D}F_\mu(i|j)(z)$ is the submatrix of $\mathrm{D}F_\mu(z)$ formed by removing the ith row and the jth column of $\mathrm{D}F_\mu(z)$. Given a matrix B let B' denote the matrix formed by removing the jth row and column of B for a given j. Let $\tilde{B}$ denote the matrix formed by removing the jth row of B for a given j. Let $\hat{B}$ denote the matrix formed by removing the jth column of B for a given j. It follows that

$$\mathrm{D}F_\mu(l|l)(z) = \begin{pmatrix} -I_{mm} & 0 & \hat{A} & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & \tilde{X} & R' & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

By doing elementary row and column operations we form the matrix

$$\begin{pmatrix} -I_{mm} & 0 & \hat{A} & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & 0 & R' & -X'\widetilde{A^t} \\ 0 & 0 & Y\hat{A} & S \end{pmatrix}.$$

Note that $\widetilde{A^t} = (\hat{A})^t$. Once again using elementary row and column operations we form that matrix

$$\begin{pmatrix} -I_{mm} & 0 & \hat{A} & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & 0 & I_{n-1n-1} & -(R')^{-1}X'(\hat{A})^t \\ 0 & 0 & 0 & S + Y\hat{A}(R')^{-1}X'(\hat{A})^t \end{pmatrix}.$$

Note that $SY^{-1} + \hat{A}(R')^{-1}X'(\hat{A})^t$is symmetric positive definite. It follows that $\det(S + Y\hat{A}(R')^{-1}X'\tilde{A}^t) > 0$. Therefore $\det(\mathrm{D}F_\mu(l|l)(z)) \neq 0$ and

$$sgn(\det(\mathrm{D}F_\mu(l|l)(z))) = (-1)^m = sgn(\det(\mathrm{D}F_\mu(\zeta)))$$

where ζ is a regular point of $\mathrm{D}F_\mu(z)$. So

$$sgn(v^t\Phi_{A,\mu}(z)) = sgn((-1)^m\mu\det(\mathrm{D}F_\mu(l|l)(z))) = (-1)^{2m} = 1.$$

Hence inward transversality holds.

Now we consider the face defined by $r_i = 0$. In this case the inward normal vector is $v^t = [0, e_i, 0, 0]$. It follows as before that

$$\begin{aligned} v^t\Phi_{A,\mu}(z) &= [0, e_i, 0, 0](-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z) \\ &= (-1)^{m+1}(-1)^{l+m+i}(-\mu)\det(\mathrm{D}F_\mu(l|m+i)(z)) \\ &= (-1)^{m+1}(-1)^{n}(-\mu)\det(\mathrm{D}F_\mu(l|m+i)(z)) \\ &= (-1)^{m+n}\mu\det(\mathrm{D}F_\mu(l|m+i)(z)). \end{aligned}$$

Now

$$\mathrm{D}F_\mu(l|m+i)(z) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & \hat{I}_{nn} & 0 & A^t \\ 0 & X' & \tilde{R} & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

By doing elementary row and column operations we form the following sequence of matrices.

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & \hat{I}_{nn} & 0 & A^t \\ 0 & X' & \tilde{R} & 0 \\ Y & 0 & 0 & S \end{pmatrix}$$

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & \hat{I}_{nn} & 0 & A^t \\ 0 & X' & \tilde{R} & 0 \\ 0 & 0 & YA & S \end{pmatrix}$$

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & \hat{I}_{nn} & 0 & A^t \\ 0 & X' & \tilde{R} & 0 \\ 0 & 0 & S^{-1}YA & I_{mm} \end{pmatrix}$$

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & \hat{I}_{nn} & -A^tS^{-1}YA & 0 \\ 0 & X' & \tilde{R} & 0 \\ 0 & 0 & 0 & I_{mm} \end{pmatrix}$$

It follows that the determinant of the submatrix is completely determined by the determinant of

$$\begin{pmatrix} \hat{I}_{nn} & -A^t S^{-1} Y A \\ X' & \tilde{R} \end{pmatrix}.$$

Using elementary operations we form the matrices

$$\begin{pmatrix} \hat{I}_{nn} & -A^t S^{-1} Y A \\ I_{n-1n-1} & (X')^{-1}\tilde{R} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -(A^t S^{-1} Y A + I^*_{nn} X^{-1} R) \\ I_{n-1n-1} & (X')^{-1}\tilde{R} \end{pmatrix}$$

where I^*_{nn} is I_{nn} with the ith row zeroed out. Now $A^t S^{-1} Y A > 0$ and $I^*_{nn} X^{-1} R \geq 0$. Therefore $A^t S^{-1} Y A + I^*_{nn} X^{-1} R$ is positive definite, $\det(\mathrm{D}F_\mu(l|m+i)(z)) \neq 0$, and $sgn(\det(\mathrm{D}F_\mu(l|m+i)(z))) = (-1)^{m+n}$. Therefore,

$$sgn(v^t \Phi_{A,\mu}(z)) = (-1)^{m+n} sgn(\mu \det(\mathrm{D}F_\mu(l|m+i)(z))) = 1.$$

Hence inward transversality holds.

Now we consider the case where $y_i = s_i = 0$ for some i and all other components are positive. Let $l = m + 2n + i$. It follows that the lth row of $\mathrm{D}F_\mu(z)$ is a zero row. Hence $\mathrm{adj}(\mathrm{D}F_\mu(z))$ is a matrix of all zeros except for the lth column.

First consider the face determined by $y_i = 0$. In this case the inward normal vector is $v^t = [0, 0, 0, e_i]$. It follows that

$$\begin{aligned} v^t \Phi_{A,\mu}(z) &= [0, 0, 0, e_i](-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z) \\ &= (-1)^{m+1}(lth\ row\ of\ \mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z). \end{aligned}$$

However, the only nonzero entry of the lth row is in the lth column. Hence,

$$\begin{aligned} v^t \Phi_{A,\mu}(z) = (-1)^{m+1}\mathrm{adj}(F_\mu(z))_{ll} F_{\mu l} &= (-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))_{ll}(-\mu) \\ &= (-1)^{m+1}(-\mu)(-l)^{l+l}\det(\mathrm{D}F_\mu(l|l)(z)) \\ &= (-1)^m(\mu)\det(\mathrm{D}F_\mu(l|l)(z)). \end{aligned}$$

Using notation as above, it follows that

$$\mathrm{D}F_\mu(l|l)(z) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & \widehat{A^t} \\ 0 & X & R & 0 \\ \tilde{Y} & 0 & 0 & S' \end{pmatrix}.$$

By doing elementary row and column operations we form the matrix

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & \hat{A}^t \\ 0 & 0 & R & -X\hat{A}^t \\ \tilde{0} & 0 & \tilde{Y}A & S' \end{pmatrix}.$$

Notice that $\tilde{Y}A = Y'\tilde{A}$ and $(\tilde{A})^t = \widehat{A^t}$. Now using elementary operations we form the matrix

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & \hat{A}^t \\ 0 & 0 & I_{nn} & -R^{-1}X\hat{A}^t \\ \tilde{0} & 0 & 0 & S' + Y'\tilde{A}R^{-1}X(\tilde{A})^t \end{pmatrix}.$$

Since $S'(Y')^{-1} + \tilde{A}R^{-1}X(\tilde{A})^t$ is symmetric positive definite, it follows that $\det(S' + Y'\tilde{A}R^{-1}X\hat{A}^t) > 0$ and $\det(\mathrm{D}F_\mu(l|l)(z)) \neq 0$. Therefore,

$$sgn(\det(\mathrm{D}F_\mu(l|l)(z))) = (-1)^m = sgn(\det(\mathrm{D}F_\mu(\zeta)))$$

where ζ is a regular point of $\mathrm{D}F_\mu(z)$. It follows that

$$sgn(v^t\Phi_{A,\mu}(z)) = sgn((-1)^m(\mu)\det(\mathrm{D}F_\mu(l|l)(z))) = (-1)^{2m} = 1.$$

Hence inward transversality holds.

Now we consider the face determined by $s_i = 0$. In this case the inward normal vector is $v^t = [e_i, 0, 0, 0]$. Hence,

$$\begin{aligned} v^t\Phi_{A,\mu}(z) &= [e_i, 0, 0, 0](-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z) \\ &= (-1)^{m+1}(-1)^{i+l}(-\mu)\det(\mathrm{D}F_\mu(l|i)(z)) \\ &= (-1)^{m+1}(-1)^m(-\mu)\det(\mathrm{D}F_\mu(l|i)(z)) \\ &= \mu\det(\mathrm{D}F_\mu(l|i)(z)) \end{aligned}$$

Now,

$$\mathrm{D}F_\mu(l|i)(z) = \begin{pmatrix} -\hat{I}_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y' & 0 & 0 & \tilde{S} \end{pmatrix}.$$

By doing elementary row and column operations we form the following sequence of matrices.

$$\begin{pmatrix} -\hat{I}_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y' & 0 & 0 & \tilde{S} \end{pmatrix}$$

$$\begin{pmatrix} -\hat{I}_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ I_{m-1m-1} & 0 & 0 & (Y')^{-1}S'\tilde{I}_{mm} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & A & \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm} \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ I_{m-1m-1} & 0 & 0 & (Y')^{-1}S'\tilde{I}_{mm} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & A & \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm} \\ 0 & I_{nn} & 0 & A^t \\ 0 & I_{nn} & X^{-1}R & 0 \\ I_{m-1m-1} & 0 & 0 & (Y')^{-1}S'\tilde{I}_{mm} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & A & \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm} \\ 0 & 0 & -X^{-1}R & A^t \\ 0 & I_{nn} & X^{-1}R & 0 \\ I_{m-1m-1} & 0 & 0 & (Y')^{-1}S'\tilde{I}_{mm} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & A & \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm} \\ 0 & 0 & -I_{nn} & R^{-1}XA^t \\ 0 & I_{nn} & X^{-1}R & 0 \\ I_{m-1m-1} & 0 & 0 & (Y')^{-1}S'\tilde{I}_{mm} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & AR^{-1}XA^t + \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm} \\ 0 & 0 & -I_{nn} & R^{-1}XA^t \\ 0 & I_{nn} & X^{-1}R & 0 \\ I_{m-1m-1} & 0 & 0 & (Y')^{-1}S'\tilde{I}_{mm} \end{pmatrix}.$$

It follows that

$$\det(\mathrm{D}F_\mu(l|i)(z)) = (-1)^{m(m-1)+n(n-1)+n+n}\det(AR^{-1}XA^t + \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm}).$$

Now $\hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm} = (\tilde{I}_{mm})^t(Y')^{-1}S'\tilde{I}_{mm}$ is symmetric positive semi-definite and $AR^{-1}XA^t$ is symmetric positive definite. Hence,

$$\det(AR^{-1}XA^t + \hat{I}_{mm}(Y')^{-1}S'\tilde{I}_{mm}) > 0$$

and $det(\mathrm{D}F_\mu(l|i)(z))) > 0$. Therefore, $sgn(det(\mathrm{D}F_\mu(l|i)(z))) = 1$,

$$sgn(v^t\Phi_{A,\mu}(z)) = sgn((\mu)\det(\mathrm{D}F_\mu(l|i)(z)))) = 1,$$

and inward transversality holds.

§4.2.3 Codimension k=m+1

For $\mu > 0$, it is possible to have $\Phi_{A,\mu}(z)$ parallel to $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. In this case we may have what is called an **r-deficiency**. Recall that

$$\mathrm{D}F_\mu(z) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}$$

Suppose that $k = m + 1$ and

$$r_i = 0 \Leftrightarrow i \in I \subset \{1, \ldots n\}, \#(I) = k.$$

Suppose that A is such that any set of m columns of A are linearly independent and $m < n$. Then from the structure of $\mathrm{D}F_\mu(z)$ it follows that $\mathrm{D}F_\mu(z)$ has rank $2m + 2n - 1$. In particular, by using elementary column operations on

$$\begin{pmatrix} A \\ 0 \\ R \\ 0 \end{pmatrix}$$

it is possible to form a zero column. By the assumption that any set of m columns of A are linearly independent, it follows that there is one and only one zero column possible at any given time. Hence, the rank is $2n + 2m - 1$. For $\Phi_{A,\mu}(z)$

to be transversal on this face we must have positive entries for $\Phi_{A,\mu}(z)$ in those places that correspond to $r_i = 0$. The entries of $\Phi_{A,\mu}(z)$ that correspond to r are $(\Phi_{A,\mu}(z))_{m+1}, \ldots, (\Phi_{A,\mu}(z))_{m+n}$. The rows of $\mathrm{adj}(\mathrm{D}F_\mu(z))$ are formed from the corresponding columns of $\mathrm{D}F_\mu(z)$. Consider any column numbered $m+1$ to $m+n$ of

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

In particular we are looking at

$$\begin{pmatrix} 0 \\ I_{nn} \\ X \\ 0 \end{pmatrix}.$$

If we consider any maximal submatrix formed from these columns then we see that we are simply removing a row from

$$\begin{pmatrix} A \\ 0 \\ R \\ 0 \end{pmatrix}.$$

It follows that by elementary column operations we can still form a zero column in

$$\begin{pmatrix} A \\ 0 \\ R \\ 0 \end{pmatrix}$$

after a row has been removed. Hence the determinant of the maximal submatrix is 0. It follows that $\mathrm{adj}(\mathrm{D}F_\mu(z))$ has all zero entries in the rows numbered $m+1$ to $m+n$. In fact, this same argument applies to the rows numbered 1 to m and $m+2n+1$ to $2m+2n$. Therefore $(\Phi_{A,\mu}(z))_i = 0$ for $i \in \{1, \ldots, m+n\}$ or $i \in \{m+2n+1, \ldots, 2m+2n\}$. Hence $\Phi_{A,\mu}(z)$ is NOT transversal to the given face. Note that $\mathrm{D}F_\mu(z)$ has rank $2n+2m-1$ and therefore there is at least one nonzero row in $\mathrm{adj}(\mathrm{D}F_\mu(z)$. It follows that, in general, this is not a fixed point of $\Phi_{A,\mu}(z)$

and that nonzero rows occur only in rows numbered $m+n+1$ to $m+2n$ which correspond to the $x's$. Therefore $\Phi_{A,\mu}(z)$ is **parallel** to the given face.

Definition 4.2.3.1. *Let $\mu > 0, \Sigma^p_\mu$ denotes the set of all $z \in \partial\bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$ for which there exists a set $I_0 \subset \{1,\ldots,2n+2m\}$, such that z is in the relative interior of $\mathcal{F}(I_0)$ and for which there exists $i \in I_0$ such that $(\Phi_{A,\mu})_i(z) = 0$.*

That is, Σ^p_μ is the set of all points $z \in \partial\bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$ for which $\Phi_{A,\mu}(z)$ is NOT inward transversal to $\partial\bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$. Note, from Theorem 2.1.1, $(E_{F_\mu} \cup Crit(g_{F_\mu})) \subset \Sigma^p_\mu$.

Proposition 4.2.3.2. *Let $\mu > 0, \Sigma^p_\mu$ is a nowhere dense set of measure zero in $\partial\bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$.*

Proof: Suppose $z \in \Sigma^p_\mu$. From Propositions 4.1.2 and 4.2.1.1, there exists $I_0 \subset \{1,\ldots,2n+2m\}, \#(I_0) \geq 2$ such that z is in the relative interior of $\mathcal{F}(I_0)$. It follows that Σ^p_μ is contained in the relative interior of a collection of subsets $\{\mathcal{F}(I^k_0)\}$ for which $\#(I^k_0) \geq 2$ for every k. Hence the proposition holds.

§4.2.4 Codimension n+m+1

In this setting we must have at least one strict complementary failure. Also, this highest rank of $\mathrm{D}F_\mu(z)$ possible is $2n+2m-1$. If rank is $2n+2m-2$ then we have a fixed point for $\Phi_{A,\mu}$. Suppose the rank is $2n+2m-1$. Then since we have a complementary pair failure, $\mathrm{adj}(\mathrm{D}F_\mu(z))$ has only one nonzero column. Suppose (WLOG) that $x_i = r_i = 0$. Then the nonzero column of $\mathrm{adj}(\mathrm{D}F_\mu(z))$ is the $n+m+ith$ column. Also $F_{\mu_{n+m+i}}(z) = -\mu$. Therefore, $\Phi_{A,\mu}(z) = (-1)^m\mu(n+m+ith\ column\ of \mathrm{adj}(\mathrm{D}F_\mu(z))$. It follows that this is not a fixed point of the vector field as the column is nonzero.

CHAPTER 5

TRANSVERSALITY OF TRAJECTORIES ON BOUNDARIES OF OTHER SETS

We need to describe and detail the behavior of the vector field on the structures that will form the boundary of the subset we will be working on. The boundary will consist of three different types of surfaces. The first surface is $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and has been covered in Chapter 4. The other surfaces will be constructed to bound the trajectory in $\mathbf{R}_+^{\mathbf{2n+2m}}$. This chapter provides constructions of other sets, M, for which $\Phi_{A,\mu}(z)$ is inward transversal at $z \in \partial M$. Some of the sets are "hyperbolic" sets and transversality of $\Phi_{A,\mu}(z)$ to ∂M at $z \in \partial M$ is verified by showing that $\Phi_{A,\mu}(z)$ is transversal to the supporting closed half-spaces to the M at $z \in \partial M$. Conditions are established for which $\Phi_{A,\mu}(z)$ is not outward transversal to a collection of the various sets and $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Note, throughout this section we will only consider points $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. The work done here provides a basis for bounding the trajectory $z(t)$ away from $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, as needed in Chapter 6, or to bound an individual component of $z(t)$, as needed in Chapter 7.

Definition 5.0.1. *Given a matrix B, let B_i denote the ith row of B.*

§5.1 Transverse Trajectories on Closed Half-Spaces

We will be considering four different types of closed half-spaces.

Definition 5.1.1. *Given $b \in \mathbf{R^m}$ in (LP), let $i \in \{1, \ldots, n\}$ and $\hat{M} > b_i$. A* **Type 1** *closed half-space is defined as*

$$H^+ = \{(s, r, x, y) : A_i x - s_i \leq \hat{M}\}.$$

Now we must check the transversality of $\Phi_{A,\mu}(z)$ on ∂H^+ for a Type 1 closed half-space. Given a point $z \in \partial H^+$, the inward normal vector to $\partial H^+$is $v_i = -[-e_i, 0, A_i, 0]$ where A_i is defined as above. Recall that we are only concerned with points $(s, r, x, y) \in \bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$. For the moment we will only consider a point (s, r, x, y) such that $(s, r, x, y) \in \bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$ and $\mathrm{D}F_\mu(s, r, x, y)$ is of full rank.

Theorem 5.1.2. *Let $i \in \{1, \ldots, n\}$ and H^+ be a Type 1 closed half-space. Suppose that $z = (s, r, x, y) \in \partial H^+ \cap \bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$ is such that $\mathrm{D}F_\mu(z)$ is of full rank. Then $\Phi_{A,\mu}(z)$ is inward transversal to ∂H^+ at z.*

Proof: Let $z \in \partial H^+ \cap \bar{\mathbf{R}}^{\mathbf{2n+2m}}_+$ and suppose that $\mathrm{D}F_\mu(z)$ is of full rank. Then $z = (s, r, x, y)$ is such that $A_i x - s_i = \hat{M}$ for $\hat{M} > b_i$. From section 4.1, we need to find w^t so that

$$\begin{aligned} -v_i^t = [-e_i, 0, A_i, 0] &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} w^t(\mathrm{D}F_\mu(z)) \\ &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} w^t \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}. \end{aligned}$$

We see that by choosing the *ith* row of $\mathrm{D}F_\mu(z)$ we get a scaled version of the desired vector v. Hence we set $w^t = |\det(\mathrm{D}F_\mu(z)|[e_i, 0^n, 0^n, 0^m]$ where 0^m is the 0 m-vector. Now we need to check $w^t F_\mu(z)$. We have

$$w^t F_\mu(z) = |\det(\mathrm{D}F_\mu(z))|[e_i, 0^n, 0^n, 0^m] \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr \\ Ys \end{pmatrix}$$

Therefore,

$$v^t \Phi_{A,\mu}(z) = |\det(\mathrm{D}F_\mu(z))|(A_i x - s_i - b_i) = |\det(\mathrm{D}F_\mu(z))|(\hat{M} - b_i) > 0.$$

It follows that the $\Phi_{A,\mu}(z)$ is inward transversal to ∂H at z.

Definition 5.1.3. *Given $c \in \mathbf{R^n}$ in (LP), let $j \in \{1, \ldots, m\}$ and $\hat{M} > c_i$. A **Type 2** closed half-space is defined as*

$$H^+ = \{(s, r, x, y) : A^t y_j + r_j \leq \hat{M}\}.$$

If H^+ is a Type 2 closed half-space, $z \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, the inward normal vector to ∂H^+ at z is $v_j = -[0, e_j, 0, A_j^t]$. As above, we will only consider $(s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that $\mathrm{D}F_\mu(s, r, x, y)$ is of full rank.

Theorem 5.1.4. *Let $j \in \{1, \ldots, m\}$ and H^+ be a Type 2 closed half-space. Suppose that $z = (s, r, x, y) \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is such that $\mathrm{D}F_\mu(z)$ is of full rank. Then $\Phi_{A,\mu}(z)$ is inward transversal to ∂H^+ at z.*

Proof: Let $z \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and suppose that $\mathrm{D}F_\mu(z)$ is of full rank. Then $z = (s, r, x, y)$ is such that $A_j^t y + r_j = \hat{M}$ for $\hat{M} > c_j$. It follows that the inward normal vector is $v^t = -[0, -e_j, 0, A_j^t]$. Now,

$$\begin{aligned} -v^t = [0, e_j, 0, A_j^t] &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} w^t(\mathrm{D}F_\mu(z)) \\ &= \frac{1}{|\det(\mathrm{D}F_\mu(z))|} w^t \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}. \end{aligned}$$

We see that by choosing the $m + jth$ row of $\mathrm{D}F_\mu$ we get a scaled version of v back. So set

$$w^t = |\det(\mathrm{D}F_\mu(z))|[0^m, e_j, 0^n, 0^m].$$

It follows that

$$
\begin{aligned}
w^t F_\mu(z) &= |\det(\mathrm{D}F_\mu(z))|[0^m, e_j, 0^n, 0^m] \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr \\ Ys \end{pmatrix} \\
&= |\det(\mathrm{D}F_\mu(z))|(A_j^t y + r_j - c_j) \\
&= |\det(\mathrm{D}F_\mu(z))|(\hat{M} - c_j) > 0.
\end{aligned}
$$

It follows that $\Phi_{A,\mu}(z)$ is inward transversal to ∂H^+ at z.

Definition 5.1.5. *Given $b \in \mathbf{R^m}$ in (LP), let $i \in \{1, \ldots, n\}$ and $\hat{M} < b_i$. A* **Type 3** *closed half-space is defined as*

$$H^+ = \{(s, r, x, y) : -A_i x + s_i \leq -\hat{M}\}.$$

If H^+ is a Type 3 closed half-space and $z \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, the inward normal vector to ∂H^+ at z is $v_i = [-e_i, 0, A_i, 0]$.

Theorem 5.1.6. *Let $i \in \{1, \ldots, n\}$ and H^+ be a Type 3 closed half-space. Suppose that $z = (s, r, x, y) \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is such that $\mathrm{D}F_\mu(z)$ is of full rank. Then $\Phi_{A,\mu}(z)$ is inward transversal to ∂H^+ at z.*

Proof: The proof is identical to the proof of theorem 5.1.2 with

$$w^t = |\det(\mathrm{D}F_\mu(z))|[-e_i, 0^n, 0^n, 0^m].$$

Definition 5.1.7. *Given $c \in \mathbf{R^n}$, let $j \in \{1, \ldots, m\}$ and $\hat{M} < c_j$. A* **Type 4** *closed half-space is defined as*

$$H^+ = \{(s, r, x, y) : -A^t y_j - r_j \leq -\hat{M}\}.$$

If H^+ is a Type 4 closed half-space and $z \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$, the inward normal vector to ∂H^+ at z is is $v_j = [0, e_j, 0, A_j^t]$.

Theorem 5.1.8. *Let $j \in \{1, \ldots, m\}$ and H^+ a Type 4 closed half-space. Suppose that $z = (s, r, x, y) \in \partial H^+ \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is such that $\mathrm{D}F_\mu(z)$ is of full rank. Then $\Phi_{A,\mu}(z)$ is inward transversal to ∂H^+ at z.*

Proof: The proof is identical to the proof of theorem 5.1.4 with

$$w^t = |\det(\mathrm{D}F_\mu(z))|[0^m, -e_j, 0^n, 0^m].$$

Corollary 5.1.9. *Let $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ be a feasible point such that $\mathrm{D}F_\mu(z)$ is of full rank, then $\Phi_{A,\mu}(z)$ is parallel to Type 1 (Type 3) closed half-spaces with $\hat{M} = b_i$ and parallel to Type 2 (Type 4) closed half-spaces with $\hat{M} = c_i$.*

Proof: The proof follows from the proofs of theorems 5.1.2, 5.1.4, 5.1.6, 5.1.8 with the appropriate value of $\hat{M}$.

§5.2 Transverse Trajectories on Hyperbolic Boundaries

The next types of surfaces we will consider will be those of a hyperbolic structure that is formed by the complementary pairs. These sets will be important in providing bounds on the trajectories.

Given $\hat{M} > \mu > 0$ and $i \in \{1, \ldots, n\}$, consider the set

$$\{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \leq \hat{M}\}.$$

To this point, the term *transverse* has been related to polyhedral convex sets. Therefore we need to relate the above set in some way to polyhedral sets in order to discuss transversality. Convexity will provide the basis to relate the hyperbolic structure to closed half-spaces. The set

$$\mathcal{K} = \{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \geq \hat{M}\}$$

is a convex set. Now,

$$\partial\mathcal{K} = \{(s,r,x,y) : (s,r,x,y) \in \bar{\mathbf{R}}_{+}^{\mathbf{2n+2m}}, x_i r_i = \hat{M}\}$$

Suppose that $z^0 = (s^0, r^0, x^0, y^0) \in \partial\mathcal{K}$. The supporting closed half-space to $\mathcal{K}$ at z^0 is

$$H_{\mathcal{K}}^{+} = \{z : -[0, x_i^0 e_i, r_i^0 e_i, 0]^t z \leq -2x_i^0 r_i^0\}$$

with corresponding supporting hyperplane to $\mathcal{K}$ at z^0 given by

$$H_{\mathcal{K}} = \{z : -[0, x_i^0 e_i, r_i^0 e_i, 0]^t z = -2x_i^0 r_i^0\}.$$

The following definition is a generalization of Definition 5.0.3.

Definition 5.2.1. *Given a convex set $K \subset \mathbf{R}^{\mathbf{N}}$ and $z^0 \in \partial K$. A nonzero vector d is outward transversal to K at z^0 if $\exists\, \alpha_0 > 0$ such that $\forall\, \alpha \in [0, \alpha_0]$, $z^0 + \alpha d \in \mathbf{R}^{\mathbf{N}} \setminus K$.*

Given the above definition we now turn to the specific set $\mathcal{K}$ as defined above.

Theorem 5.2.2. *Let $z^0 \in \partial\mathcal{K}$. A nonzero vector d is outward transversal to $\partial\mathcal{K}$ at z^0 if and only if d is outward transversal to $H_{\mathcal{K}}^{+}$ at z^0*

Proof: Let $z^0 \in \partial\mathcal{K}$. Suppose that d is a nonzero vector that is outward transversal to $\partial\mathcal{K}$ at z^0. Then there exists $\alpha_0 > 0$ such that $z^0 + \alpha d \in \mathbf{R}^{\mathbf{N}} \setminus \mathcal{K}$ for all $\alpha \in [0, \alpha_0]$. Let d_{r_i}, d_{x_i} denote the components of d corresponding to r_i, x_i respectively. Now

$$\begin{aligned} z^0 + \alpha d \in \mathbf{R}^{\mathbf{N}} \setminus \mathcal{K} &\Leftrightarrow (r_i^0 + \alpha d_{r_i^0})(x_i^0 + \alpha d_{x_i^0}) < \hat{M} \\ &\Leftrightarrow r_i^0 x_i^0 + \alpha(x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} + \alpha d_{r_i^0} d_{x_i^0}) < \hat{M} \\ &\Leftrightarrow x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} + \alpha d_{r_i^0} d_{x_i^0} < 0 \\ &\Leftrightarrow x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} < -\alpha d_{r_i^0} d_{x_i^0}. \end{aligned}$$

Note that $x_i^0, r_i^0 > 0$. Since the above is true for all $\alpha \in [0, \alpha_0], x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} < 0$. It follows that

$$-[0, x_i^0 e_i, r_i^0 e_i, 0]^t d = -(x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0}) > 0.$$

Hence, d is outward transversal to $H_{\mathcal{K}}^+$ at z^0.

Now suppose that a nonzero vector d is outward transversal to $H_{\mathcal{K}}^+$ at z^0. It follows that

$$x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} = [0, x_i^0 e_i, r_i^0 e_i, 0]^t d < 0.$$

As before $x_i^0, r_i^0 > 0$. If $d_{x_i^0} d_{r_i^0} > 0$ then

$$\frac{x_i^0}{d_{r_i^0}} + \frac{r_i^0}{d_{x_i^0}} = \frac{x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0}}{d_{x_i^0} d_{r_i^0}} < 0.$$

Pick $\alpha_0 > 0$ such that

$$\frac{x_i^0}{d_{r_i^0}} + \frac{r_i^0}{d_{x_i^0}} < -\alpha_0 < 0.$$

Then for all $\alpha \in [0, \alpha_0]$,

$$\frac{x_i^0}{d_{r_i^0}} + \frac{r_i^0}{d_{x_i^0}} < -\alpha_0 < -\alpha$$

and hence

$$x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} < -\alpha d_{r_i^0} d_{x_i^0}.$$

It follows that d is outward transversal to $\partial\mathcal{K}$ at z^0. If $d_{x_i^0} d_{r_i^0} \leq 0$ then for any $\alpha > 0$,

$$x_i^0 d_{r_i^0} + r_i^0 d_{x_i^0} < -\alpha d_{r_i^0} d_{x_i^0}$$

and as before the theorem holds.

Now at $z^0, \Phi_{A,\mu}(z^0)$ is inward transversal to the set

$$\{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \leq \hat{M}\}.$$

if and only if $\Phi_{A,\mu}(z^0)$ is outward transversal to

$$\mathcal{K} = \{(s,r,x,y) : (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \geq \hat{M}\}.$$

It follows that we need only consider the supporting closed half-spaces $H_{\mathcal{K}}^+$ when we are considering inward transversality.

The inward normal vector to the set

$$\{(s,r,x,y) : (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \leq \hat{M}\}$$

is $v^t = -[0, x_i e_i, r_i e_i, 0]$.

Theorem 5.2.3. *Let $\hat{M} > \mu \geq 0$ and $i \in \{1, \ldots, n\}$. Suppose that $z = (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ is such that $x_i r_i = \hat{M}$. Then $\Phi_{A,\mu}(z)$ is inward transversal to the set*

$$\{(s,r,x,y) : (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \leq \hat{M}\}.$$

Proof: Since $z \in \mathbf{R}_+^{\mathbf{2n+2m}}, \mathrm{D}F_\mu(z)$ is of full rank. We need to solve

$$-v^t = [0, x_i e_i, r_i e_i, 0] = \frac{1}{|\det(\mathrm{D}F_\mu(z))|} w^t \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

The $m+n+ith$ row of $\mathrm{D}F_\mu(z)$ is $-v^t$. Therefore we set $w^t = |\det(\mathrm{D}F_\mu(z))| e_{m+n+i}$.

It follows that

$$\begin{aligned} v^t F_\mu(z) &= |\det(\mathrm{D}F_\mu(z))| e_{m+n+i}^t \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr - \mu e \\ Ys - \mu e \end{pmatrix} \\ &= |\det(\mathrm{D}F_\mu(z))| (x_i r_i - \mu) \\ &= |\det(\mathrm{D}F_\mu(z))| (\hat{M} - \mu) > 0. \end{aligned}$$

It follows that $\Phi_{A,\mu}(z)$ is inward transversal to

$$\{(s,r,x,y) : (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \le \hat{M}\}$$

at z.

Now, given $\hat{M} > \mu > 0$ and $j \in \{1,\ldots,n\}$, consider the set defined as

$$\{(s,r,x,y) : (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, y_j s_j \le \hat{M}\}.$$

Let $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}, y_j s_j = \hat{M}$. As above, given $j \in \{1,\ldots,m\}$, we form the set

$$\mathcal{K} = \{(s,r,x,y) : (s,r,x,y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}, y_j s_j \ge \hat{M}\}.$$

The corresponding closed half-space is

$$H_{\mathcal{K}} = \{z : -[y_j e_j, 0, 0, s_j e_j]^t z \le -2y_i^0 s_i^0\}.$$

Theorem 5.2.4 corresponds to Theorem 5.2.2.

Theorem 5.2.4. *Let $z^0 \in \partial\mathcal{K}$. A nonzero vector d is outward transversal to $\partial\mathcal{K}$ at z^0 if and only if d is outward transversal to $H_{\mathcal{K}}^+$ at z^0*

The inward normal vector for this surface is $v^t = -[y_j e_j, 0, 0, s_j e_j]$.

Theorem 5.2.5. *Let $\hat{M} > \mu \ge 0$ and $j \in \{1,\ldots,n\}$. Suppose that $z = (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ is such that $y_j s_j = \hat{M}$. Then $\Phi_{A,\mu}(z)$ is inward transversal to the set*

$$\{(s,r,x,y) : (s,r,x,y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, y_j s_j \le \hat{M}\}.$$

Proof: The proof is as in Theorem 5.2.3 using $w^t = |\det(\mathrm{D}F_\mu(z))| e_{m+2n+i}$.

There are analogous theorems for when $\hat{M} < \mu$. These will be of importance when analyzing our trajectories for the case of $\mu > 0$.

Theorem 5.2.6. *Let $\mu > \hat{M} \geq 0$ and $i \in \{1, \ldots, n\}$. Suppose that $z = (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ is such that $x_i r_i = \hat{M}$. Then $\Phi_{A,\mu}(z)$ is inward transversal to the set*

$$\{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \geq \hat{M}\}.$$

Proof: Since $z \in \mathbf{R}_+^{\mathbf{2n+2m}}$, $\mathrm{D}F_\mu(z)$ is of full rank. We need to solve

$$-v^t = -[0, x_i e_i, r_i e_i, 0] = \frac{1}{|\det(\mathrm{D}F_\mu(z))|} w^t \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & X & R & 0 \\ Y & 0 & 0 & S \end{pmatrix}.$$

The $m+n+ith$ row of $\mathrm{D}F_\mu(z)$ is v^t. Therefore we set $w^t = -|\det(\mathrm{D}F_\mu(z))| e_{m+n+i}$.

It follows that

$$\begin{aligned} v^t F_\mu(z) &= -|\det(\mathrm{D}F_\mu(z))| e^t_{m+n+i} \begin{pmatrix} Ax - s - b \\ A^t y + r - c \\ Xr - \mu e \\ Ys - \mu e \end{pmatrix} \\ &= -|\det(\mathrm{D}F_\mu(z))|(x_i r_i - \mu) \\ &= -|\det(\mathrm{D}F_\mu(z))|(\hat{M} - \mu) > 0. \end{aligned}$$

It follows that $\Phi_{A,\mu}(z)$ is inward transversal to

$$\{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i \leq \hat{M}\}$$

at z.

Theorem 5.2.7. *Let $\mu > \hat{M} \geq 0$ and $j \in \{1, \ldots, n\}$. Suppose that $z = (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ is such that $y_j s_j = \hat{M}$. Then $\Phi_{A,\mu}(z)$ is inward transversal to the set*

$$\{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, y_j s_j \geq \hat{M}\}.$$

Proof: Proof follows method used in Theorem 5.2.6.

Definition 5.2.8. *Given $\hat{M} > \mu$ and*

$$z^0 \in \{(s, r, x, y) : (s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}, x_i r_i = \hat{M}\},$$

the closed half-space

$$\{z : [0, x_i^0 e_i, r_i^0 e_i, 0]^t z \leq 2x_i^0 r_i^0\}$$

is called a **type 5** *closed half-space. If*

$$z^0 \in \{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, y_j s_j = \hat{M}\},$$

the closed half-space

$$\{z : [y_j e_j, 0, 0, s_j e_j]^t z \leq 2y_i^0 s_i^0\}$$

is called a **type 6** *closed half-space. Similarly, given $\hat{M} < \mu$ and*

$$z^0 \in \{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, x_i r_i = \hat{M}\},$$

the closed half-space

$$\{z : [0, x_i^0 e_i, r_i^0 e_i, 0]^t z \geq 2x_i^0 r_i^0\}$$

is called a **type 7** *closed half-space. If*

$$z^0 \in \{(s, r, x, y) : (s, r, x, y) \in \mathbf{R}_+^{\mathbf{2n+2m}}, y_j s_j = \hat{M}\},$$

the closed half-space

$$\{z : [y_j e_j, 0, 0, s_j e_j]^t z \geq 2y_i^0 s_i^0\}$$

is called a **type 8** *closed half-space.*

Suppose that $\hat{M} > (n + m)\mu$. Consider the set

$$\{(s, r, x, y) : (s, r, x, y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}, \sum x_i r_i + \sum y_j s_j \leq \hat{M}\}.$$

Let $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that $\sum x_i^0 r_i^0 + \sum y_j^0 s_j^0 = \hat{M}$. As above, form the set

$$\mathcal{K} = \{(s,r,x,y) : (s,r,x,y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}, \sum x_i r_i + \sum y_j s_j \geq \hat{M}\}.$$

The corresponding closed half-space is

$$H_{\mathcal{K}} = \{z : -[y^0, x^0, r^0, s^0]^t z \leq -2(\sum x_i^0 r_i^0 + \sum y_i^0 s_i^0)\}.$$

Given this closed half-space and the above arguments, we have the following theorem.

Theorem 5.2.9. *Given $\mu \geq 0$, let $\hat{M} > (n+m)\mu$. Suppose that $z = (s,r,x,y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is such that $\sum x_i r_i + \sum y_j s_j = \hat{M}$ and $\mathrm{D}F_\mu(z)$ is of full rank. Then $\Phi_{A,\mu}(z)$ is inward transversal to the set*

$$\{(s,r,x,y) : (s,r,x,y) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}, \sum x_i r_i + \sum y_j s_j \leq \hat{M}\}.$$

Proof: The proof is as in theorem 5.2.3 using $w^t = |\det(\mathrm{D}F_\mu(z)))| \sum_{i=1}^{n+m} e_{n+m+i}$.

Definition 5.2.10. *Given $\hat{M} > (n+m)\mu, z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that $\sum x_i^0 r_i^0 + \sum y_j^0 s_j^0 = \hat{M}$, the closed half-space*

$$\{z : [y^0, x^0, r^0, s^0]^t z \leq 2(\sum x_i^0 r_i^0 + \sum y_i^0 s_i^0) = \hat{M}\}$$

is a **type 9** *closed half-space.*

§5.3 Transversality on Intersections of Boundary Structures

We have already shown that $\Phi_{A,\mu}(z)$ is inward transversal to the individual hyperplanes and hyperbolic structures at points $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $\mathrm{D}F(z^0)$ is of full rank, in particular for all $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$. Also the structure is known for $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Therefore, we must turn our attention to those points lying on the intersection of these different structures.

Proposition 5.3.1. *Let $\{H_j^+\}_{j=1}^M$ be a collection of type 1-9 closed half-spaces. Let $\Pi = \cap_j H_j^+$. Suppose that $z^0 \in \mathbf{R}_+^{2n+2m} \cap \Pi$. Given (1.20), $\Phi_{A,\mu}(z^0)$ is inward transversal to $\partial\Pi$ at z^0.*

Proof: Since $z^0 \in \mathbf{R}_+^{2n+2m}$, from Proposition 2.1.2, $\mathrm{D}F_\mu(z^0)$ is of full rank. Now at z^0, from Theorems 5.1.2, 5.1.4, 5.1.6, 5.1.8, 5.2.3, 5.2.5, 5.2.6, 5.2.7, and 5.2.9, $\Phi_{A,\mu}(z^0)$ is inward transversal to every individual closed half- space H_j^+ for which $z^0 \in H_j$. It follows that $\Phi_{A,\mu}(z^0)$ is inward transversal to $\partial\Pi$ at z^0.

Theorem 5.3.2. *Let $\{H_j^+\}_{j=1}^M$ be a collection of type 1-4,9 closed half-spaces. Let $\Pi = \cap_j H_j^+$. Suppose that $z^0 \in \partial\bar{\mathbf{R}}_+^{2n+2m} \cap \Pi$. Given (1.18)-(1.20), $\Phi_{A,\mu}(z^0)$ is not outward transversal to $\partial\Pi$ at z^0.*

Proof: If $z^0 \in Crit(g_{F_\mu})$ then $\Phi_{A,\mu}(z^0) = 0$ and hence is not outward transversal. Suppose $z^0 \in Reg(g_{F_\mu})$. Then if $\mathrm{D}F_\mu(z^0)$ is of full rank it follows from Corollaries 4.1.4 and 4.1.6 that $\Phi_{A,\mu}(z^0)$ is either parallel or inward transversal to $\partial\bar{\mathbf{R}}_+^{2n+2m}$. From Theorems 5.1.2, 5.1.4, 5.1.6, 5.1.8, and 5.2.9, $\Phi_{A,\mu}(z^0)$ is also inward transversal to each of the closed half-spaces H_j^+ for which $z^0 \in H_j$. It follows that $\Phi_{A,\mu}(z^0)$ is not outward transversal to $\partial\Pi$ at z^0. Finally, if $\mathrm{D}F_\mu(z^0)$ is not of full rank and $\Phi_{A,\mu}(z^0) \neq 0$, (that is $rank(\mathrm{D}F_\mu(z)) = 2n + 2m - 1$) then from Corollaries 4.1.4 and 4.1.6, $\Phi_{A,\mu}(z^0)$ is either parallel or inward transversal to $\partial\bar{\mathbf{R}}_+^{2n+2m}$.

Suppose that $\Phi_{A,\mu}(z^0)$ is outward transversal to $\partial\Pi$ at z^0. Then it follows that there exists some j such that $z^0 \in H_j$ and $\Phi_{A,\mu}(z^0)$ is outward transversal to H_j^+. Being outward transversal means that $-d_j^t\Phi_{A,\mu}(z^0) < 0$ where $-d_j^t$ is the inward normal vector. Now $-d_j^t\Phi_{A,\mu}(z)$ is continuous in z. It follows that there exists $\epsilon > 0$ such that if $\|z^0 - z\| < \epsilon$ then $-d_j^t\Phi_{A,\mu}(z) < 0$. z^0 is such that $d_j^t z^0 = K_j$.

From (1.18)-(1.20), there exists a $\hat{z} = (\hat{s}, \hat{r}, \hat{x}, \hat{y}) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ such that $A\hat{x} - \hat{s} = b, A^t\hat{y} + \hat{r} = c$.

Suppose that H_j^+ is a Type 3 closed half-space with $K_j = -\hat{M}$ and $\hat{M} < b_i$ for some i. It follows that $d_j^t = [e_i, 0, -A_i, 0]$. Set $\gamma = b_i - \hat{M}$. Then $\bar{z} = \hat{z} + \gamma(e_i, 0, 0, 0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and is such that $d_j^t\bar{z} = d_j^t\hat{z} + \gamma = -b_i + b_i - \hat{M} = K_j$. Define $l(t) = (1-t)z^0 + t\bar{z}$. It follows that for all $t \in [0,1], d_j^t l(t) = K_j$. Pick $\kappa \geq 3$ such that $\frac{\epsilon}{\kappa\|z^0 - \bar{z}\|} < 1$. Let $z = l(\frac{\epsilon}{\kappa\|z^0-\bar{z}\|})$. Then, $\|z - z^0\| < \epsilon$. Since $z \in \mathbf{R}_+^{\mathbf{2n+2m}}, \Phi_{A,\mu}(z)$ is inward transversal to H_j^+ at z. That is, $-d_j^t\Phi_{A,\mu}(z) > 0$. But $\|z^0 - z\| < \epsilon$ means that $-d_j^t\Phi_{A,\mu}(z) < 0$. Hence H_j^+ is not a Type 3 closed half-space.

Suppose that H_j^+ is a Type 2 closed half-space with $K_j = \hat{M} > c_i$ for some i. It follows that $d_j^t = [0, e_i, 0, A_i^t]$. Set $\gamma = \hat{M} - c_i$. Then $\bar{z} = \hat{z} + \gamma(0, e_i, 0, 0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and is such that $d_j^t\bar{z} = d_j^t\hat{z} + \gamma = c_i + \hat{M} - c_i = K_j$. Define $l(t) = (1-t)z^0 + t\bar{z}$. It follows that for all $t \in [0,1], d_j^t l(t) = K_j$. Pick $\kappa \geq 3$ such that $\frac{\epsilon}{\kappa\|z^0-\bar{z}\|} < 1$. Let $z = l(\frac{\epsilon}{\kappa\|z^0-\bar{z}\|})$. Then, $\|z - z^0\| < \epsilon$. Since $z \in \mathbf{R}_+^{\mathbf{2n+2m}}, \Phi_{A,\mu}(z)$ is inward transversal to H_j^+ at z. That is, $-d_j^t\Phi_{A,\mu}(z) > 0$. But $\|z^0 - z\| < \epsilon$ means that $-d_j^t\Phi_{A,\mu}(z) < 0$. Hence H_j^+ is not a Type 2 closed half-space.

Suppose that H_j^+ is a Type 1 closed half-space with $K_j = \hat{M} > b_i$. It follows that $d_j^t = [-e_i, 0, A_i, 0]$. Define $l(t) = (1-t)z^0 + t\hat{z}$. Then for any $z_\tau = l(\tau)$,

$$\bar{M} = d_j^t z_\tau = (1-t)d_j^t z^0 + t d_j^t \hat{z} = (1-t)\hat{M} + tb_i.$$

It follows that for any $\tau \in (0,1), z_\tau \in \mathbf{R}_+^{\mathbf{2n+2m}}$, and $d_j^t z_\tau = \bar{M} > b_i$. Hence, from the proof of Theorem 5.1.2, $-d_j^t\Phi_{A,\mu}(z_\tau) > 0$. Pick $\kappa \geq 3$ such that $\frac{\epsilon}{\kappa\|z^0-\hat{z}\|} < 1$. Set $z = l(\frac{\epsilon}{\kappa\|z^0-\hat{z}\|})$. Then $z \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and $\|z^0 - z\| < \epsilon$. It follows that $-d_j^t\Phi_{A,\mu}(z) < 0$. This can't happen from above, hence H_j^+ is not a Type 1 closed half-space.

Suppose that H_j^+ is a Type 4 closed half-space with $K_j = -\hat{M}$ and $\hat{M} < c_i$. It follows that $d_j^t = [0, -e_i, 0, -A_i^t]$. Define $l(t) = (1-t)z^0 + t\hat{z}$. Then for any $z_\tau = l(\tau), \tau \in (0,1)$,

$$-\bar{M}_\tau = d_j^t z_\tau = (1-t)d_j^t z^0 + td_j^t\hat{z} = (1-t)(-\hat{M}) - tc_i.$$

It follows that $\bar{M}_\tau < c_i$. Hence from the proof of Theorem 5.1.8, for $\tau \in (0,1)$, $-d_j^t\Phi_{A,\mu}(z_\tau) > 0$. Pick $\kappa \geq 3$ such that $\frac{\epsilon}{\kappa\|z^0-\hat{z}\|} < 1$. Set $z = l(\frac{\epsilon}{\kappa\|z^0-\hat{z}\|})$. Then $z \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and $\|z^0 - z\| < \epsilon$. It follows that $-d_j^t\Phi_{A,\mu}(z) < 0$. This can't happen from above, hence H_j^+ is not a Type 4 closed half-space.

Finally, suppose that H_j^+ is a Type 9 closed half-space. It follows that

$$H_j = d_j^t z = [y^0, x^0, r^0, s^0]z = 2(\sum x_i^0 r_i^0 + \sum y_i^0 s_i^0) = K_j > (n+m)\mu.$$

Now, by continuity, since $-d_j^t\Phi_{A,\mu}(z^0) < 0$, there exists $\epsilon > 0$ such that if $\|z - z^0\| < \epsilon$ then

$$-d_j^t\Phi_{A,\mu}(z) \leq \frac{-1}{2}d_j^t\Phi_{A,\mu}(z^0) < 0.$$

Let $\gamma = \max\{z_i, 1\}, \tau \in [0, \epsilon)$. Define z^1 by

$$z_i^1 = \begin{cases} z_i^0 & \text{if } z_i^0 > 0 \\ \frac{\tau}{4\gamma(n+m)} & \text{else .} \end{cases}$$

Hence, $z^1 \in \mathbf{R}_+^{\mathbf{2n+2m}}$, $\|z^1 - z^0\| \leq \frac{\epsilon}{2}$, and $K_j < 2(\sum x_i^1 r_i^1 + \sum y_i^1 s_i^1) < K_j + \epsilon$. It follows that for all $\tau \in [0, \epsilon)$,

$$-d_j^t\Phi_{A,\mu}(z^1) \leq \frac{-1}{2}d_j^t\Phi_{A,\mu}(z^0) < 0.$$

Let v be the vector having components of 0's and 1's (by construction of z^1) such that $-(d_j^t + \frac{\tau}{4\gamma(n+m)}v^t)$ is the inward normal vector to the closed half-space given by

$$\{z : [y^1, x^1, r^1, s^1]z \leq 2(\sum x_i^1 r_i^1 + \sum s_i^1 y_i^1) > K_j > (n+m)\mu\}.$$

From Theorem 5.2.9, $-(d_j^t + \frac{\tau}{4\gamma(n+m)}v^t)\Phi_{A,\mu}(z^1) > 0$ for any $\tau \in (0,\epsilon)$. Now, $-(d_j^t + \frac{\tau}{4\gamma(n+m)}v^t)\Phi_{A,\mu}(z^1)$ is continuous with respect to the parameter τ. Hence, by continuity, for τ small enough,

$$-d_j^t\Phi_{A,\mu}(z^1) \geq 0.$$

This can't happen and therefore H_j^+ is not a type 9 closed half-space. It follows that the theorem holds.

CHAPTER 6

PROPERTIES OF TRAJECTORIES AND CONVERGENCE THEOREMS

§6.1 Exact Solutions

There are instances when closed solutions to (1.17) are readily available. Given Theorem A.5, these solutions determine the behavior of trajectories containing points in a neighborhood of points lying on the trajectory of the closed solution.

Theorem 6.1.1. *Let $\mu = 0$. Suppose that $z^* \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ is such that $F_\mu(z^*) = 0$ and $DF_\mu(z^*)$ is of full rank. For $t_0 \in (0,1]$,let $z^0 = t_0 z^*$. Define $z(t) = tz^*$ for $t \in [t_0, 1]$. Then $z(t)$ is a solution to (1.17).*

Proof:

$$DF_\mu(z(t)) = \begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & tX^* & tR^* & 0 \\ tY^* & 0 & 0 & tS^* \end{pmatrix}.$$

Since $DF_\mu(z^*)$ is of full rank, it follows that $DF_\mu(z(t))$ is of full rank. Hence, we need only consider $\Phi_{N,\mu}(z)$. Now, $DF_\mu(z(t))\frac{dz}{dt} =$

$$\begin{pmatrix} -I_{mm} & 0 & A & 0 \\ 0 & I_{nn} & 0 & A^t \\ 0 & tX^* & tR^* & 0 \\ tY^* & 0 & 0 & tS^* \end{pmatrix} \begin{pmatrix} s^* \\ r^* \\ x^* \\ y^* \end{pmatrix} = \begin{pmatrix} b \\ c \\ 0 \\ 0 \end{pmatrix}.$$

Also,

$$DF_\mu(z(t))\Phi_{N,\mu}(z(t)) = -F_\mu(z(t)) = -\begin{pmatrix} t(Ax^* - s^*) - b \\ t(A^t y^* + r^*) - c \\ t^2(X^* r^*) \\ t^2(Y^* s^*) \end{pmatrix} = -\begin{pmatrix} (t-1)b \\ (t-1)c \\ 0 \\ 0. \end{pmatrix}$$

It follows that for $t \in (0,1]$, that $DF_\mu(z(t))\frac{dz}{dt}$ and $DF_\mu(z(t))\Phi_{N,\mu}(z(t))$ differ only by a positive multiple and hence the theorem holds.

§6.2 Properties of Trajectories

Since we will be using the preimage of a regular value of a given mapping, the following theorem from [H] allows us to classify pull-backs of mappings in general

Theorem 6.2.1 (Preimage Theorem). *Given $F : \Omega \subset \mathbf{R}^{\mathbf{M}} \to \mathbf{R}^{\mathbf{M}}$, suppose y is a regular value of the C^2 map $g_F : \Omega - E_F \to S^{M-1}$. Then $g_F^{-1}(y)$ is a 1-dimensional submanifold of $M - E_f$.*

Proposition 6.2.2. *Let $\mu \geq 0$, $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and $w^0 = g_{F_\mu}(z^0)$. If $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (\Sigma_\mu^\partial \cup \Sigma_\mu^+)$ then $g_{F_\mu}^{-1}(w^0)$ is a 1-dimensional submanifold in $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$. If $\mu > 0$, $z^0 \in \Sigma_\mu^+ \subset \mathbf{R}_+^{\mathbf{2n+2m}}$ then $g_{F_\mu}^{-1}|_{\mathbf{R}_+^{\mathbf{2n+2m}}}(w^0)$ is a 1-dimensional submanifold in $\mathbf{R}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$.*

Proof: Suppose that $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (\Sigma_\mu^\partial \cup \Sigma_\mu^+)$. Then w^0 is a regular value of g_{F_μ}. Since $E_{F_\mu} = F_\mu^{-1}(0)$, E_{F_μ} is a closed set in $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. For $\mu > 0$, by Theorem 3.21, E_{F_μ} is of measure zero in $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. For $\mu = 0$, $E_{F_0} \subset \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and hence is of measure zero in $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. It follows that $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$ is a $2n+2m$-dimensional manifold. From Theorem 6.2.1, $g_{F_\mu}^{-1}(w^0)$ is a 1-dimensional submanifold in $(\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}) \setminus E_{F_\mu}$.

For $\mu > 0$, suppose that $z^0 \in \Sigma_\mu^+ \subset \mathbf{R}_+^{2n+2m}$. Since $\mathrm{D}F_\mu(z)$ is of full rank on $\mathbf{R}_+^{2n+2m}$, w^0 is a regular value of $g_{F_\mu}|_{\mathbf{R}_+^{2n+2m}}$. As above, $\mathbf{R}_+^{2n+2m} \setminus E_{F_\mu}$ is a $2n+2m$-dimensional manifold. Hence, $g_{F_\mu}^{-1}|_{\mathbf{R}_+^{2n+2m}}(w^0)$ is a 1-dimensional submanifold in $(\mathbf{R}_+^{2n+2m}) \setminus E_{F_\mu}$. Note, for $\mu = 0$, from Proposition 3.17 and Definition 3.14, $\Sigma_0^+ = \emptyset$.

We will now give an orientation for $C_\mu(z^0)$. Recall from Lemma 2.1.8 that $\Phi_{A,\mu}(z) \in ker(\mathrm{D}g_{F_\mu}(z))$. If $z^0 \in Reg(g_{F_\mu})$, from Theorem 2.1.1, $\Phi_{A,\mu}(z^0) \neq 0$. Also, for $z^0 \in Reg(g_{F_\mu})$, since $\mathrm{D}g_{F_\mu}(z)$ is of rank $2n + 2m - 1$, it follows that $dim(ker(\mathrm{D}g_{F_\mu}(z^0)) = 1$. It follows that $\Phi_{A,\mu}(z)$ spans $ker(\mathrm{D}g_{F_\mu}(z^0))$. Let $z(t)$ be a parameterization of $C_\mu(z^0)$ which moves in the direction of $\Phi_{A,\mu}(z)$ as t increases. That is we orientate curve $C_\mu(z^0)$ so that the angle between $\frac{dz}{dt}|_{t=0}$ and $\Phi_{A,\mu}(z^0)$ is 0. Note, under this orientation, $\frac{dz}{dt} = \Phi_{A,\mu}(z)$. We call $\{z(t) : t \geq 0\}$ the **forward orbit** of $C_\mu(z^0)$. Let $C_\mu^+(z^0) = C_\mu(z^0)|_{t \geq 0}$. That is, $C_\mu^+(z^0)$ is the forward orbit of z^0 in $C_\mu(z^0)$. Note also that if $C_\mu(z^0) \subset Reg(g_{F_\mu})$ then $C_\mu(z^0)$ is a 1-dimensional C^1 manifold.

Proposition 6.2.3. *For $\mu \geq 0$, if $z^0 \in Reg(F_\mu(z))$ then $||F_\mu(z)||$ is strictly decreasing at z^0 along the forward orbit of $C_\mu(z^0)$. For all $z \in \bar{\mathbf{R}}_+^{2n+2m}$, $||F_\mu(z)||$ is non-increasing along the forward orbit of $C_\mu(z^0)$.*

Proof: Suppose $z(t)$ this the parameterization of $C_\mu(z^0)$ as given above with $z(0) = z^0 \in Reg(F_\mu(z))$. Then $g_{F_\mu}(z(t)) = g_{F_\mu}(z^0)$. Differentiating with respect to t we get $\mathrm{D}g_{F_\mu}(z)\frac{dz}{dt} = 0$ where $\frac{dz}{dt}$ is the tangent vector to $C_\mu(z^0)$ at $z(t)$. Hence, $\frac{dz}{dt} \in ker(\mathrm{D}g_{F_\mu}(z))$. From Proposition 2.0.4, $\mathrm{D}F_\mu(z)\frac{dz}{dt} = \lambda(z)F_\mu(z)$ for some $\lambda(z) \in \mathbf{R}$. Now $\Phi_{A,\mu}(z^0) \neq 0$. Also

$$sgn(det(\mathrm{D}F_\mu(z(t))) = (-1)^m$$

and

$$\mathrm{D}F_\mu(z(t))\Phi_{A,\mu}(z(t)) = (-1)|\det(\mathrm{D}F_\mu(z(t)))|F_\mu(z(t))$$

for all $z(t)$. Now it follows that along the forward orbit of $C_\mu(z^0)$,

$$\begin{aligned}
\frac{d}{dt}(||F_\mu(z(t))||) = \frac{F_\mu(z)^t DF_\mu(z)}{||F_\mu(z)||}\frac{dz}{dt} &= \frac{F_\mu(z)^t DF_\mu(z)}{||F_\mu(z)||}\Phi_{A,\mu}(z) \\
&= \frac{F_\mu(z)^t DF_\mu(z)}{||F_\mu(z)||}(-1)^{m+1}\mathrm{adj}(\mathrm{D}F_\mu(z))F_\mu(z) \\
&= \frac{1}{||F_\mu(z)||}\left[F_\mu(z)^t\lambda(z)F_\mu(z)\right] \\
&= \lambda(z)||F_\mu(z)|| < 0
\end{aligned}$$

where $\lambda(z) = (-1)|det(\mathrm{D}F_\mu(z))|$. It follows that $||F_\mu(z)||$ is strictly decreasing along $C_\mu(z^0)$ at z^0 for this orientation. From Proposition 2.1.2, $det(\mathrm{D}F_\mu(z)) = (-1)^m|det(\mathrm{D}F_\mu(z))|$ for all $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Hence $\lambda(z) \leq 0$ for all $z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and so the proposition holds.

Proposition 6.2.4. $\Sigma_\mu^\partial \subset \Sigma_\mu^p$.

Proof: For $\mu = 0$, from Proposition 4.1.3, $\Sigma_\mu^p = \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Hence for $\mu = 0, \Sigma_\mu^\partial \subset \Sigma_\mu^p$. Suppose $\mu > 0$, and that there exists $z^0 \in \Sigma_\mu^\partial$ such that $z^0 \notin \Sigma_\mu^p$. It follows from the definition of Σ_μ^∂ that $z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Hence, there exists $I_0 \subset \{1,\ldots,2n+2m\}, \#(I_0) = k > 0$, such that z^0 is in the relative interior of $\mathcal{F}(I_0)$. Since $z^0 \notin \Sigma_\mu^p$, from Proposition 4.1.5, for every $i \in I_0$, $\Phi_{A,\mu}(z^0)\cdot e_i > 0$. It follows that $-\Phi_{A,\mu}(z^0)\cdot e_i < 0$ for all $i \in I_0$. Note, since $\Phi_{A,\mu}(z^0) \neq 0$, from Theorem 2.1.1, $z^0 \in Reg(g_{F_\mu})$. Also, since there exists a natural extension to g_{F_μ} in a neighborhood of z^0, z^0 is a boundary point for $C_\mu(z^0)$. Let $z(t)$ be the parameterization of $C_\mu^+(z^0)$ as given above. Since $z \in \Sigma_\mu^\partial$ and z^0 is a boundary point of $C_\mu(z^0)$, it follows that there exists some $z^1 \in C_\mu^+(z^0) \cap Crit(g_{F_\mu})$. Since $z^0 \notin \Sigma_\mu^p$, there exists some $\beta > 0$, such that $z(t) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$ for all $t \in (0, \beta]$. Hence, $z^1 \notin \{z(t) : t \in [0, \beta]$. Therefore, $z^1 \in C_\mu^+(z(\beta))$. Set $\gamma = \frac{1}{2}\min\{x_j(\beta)r_j(\beta), y_j(\beta)s_j(\beta), \mu\}$. If follows that $\gamma > 0$. Hence, from Theorems 5.2.6, 5.2.7, $x_j(t)r_j(t) > \gamma, s_j(t)y_j(t) > \gamma$ for all $t \geq \beta$. It follows that $C_\mu^+(z(\beta)) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. Hence, $z^1 \in Crit(g_{F_\mu}) \cap \mathbf{R}_+^{\mathbf{2n+2m}} = \emptyset$. So the proposition holds.

Let $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$. Since $C_\mu(z^0)$ is a connected 1-dimensional manifold, it follows that $C_\mu(z^0)$ is diffeomorphic to S^1 or some interval. First we show that $C_\mu(z^0)$ is not a closed (periodic) orbit. That is, $C_\mu(z^0)$ is not diffeomorphic to S^1.

Proposition 6.2.5. *If $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$ then $C_\mu(z^0)$ is not diffeomorphic to S^1.*

Proof: Suppose $C_\mu(z^0)$ is diffeomorphic to S^1 under the diffeomorphism $\Delta : \mathbf{S^1} \to C_\mu(z^0)$. Now $\psi : [0, 2\pi] \to (\cos(t), \sin(t))$ is a parameterization of $\mathbf{S^1}$. Hence, $\Delta(\psi(t))$, $\Delta(\psi(-t))$ are parameterizations of $C_\mu(z^0)$ with different orientations. It follows that one of these corresponds to the forward orbit of $C_\mu(z^0)$. Suppose (WLOG) that $\Delta(\psi(t))$ corresponds to the forward orbit. Note, for $\mu = 0$, from Proposition 4.1.3, $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_0^p = \mathbf{R}_+^{\mathbf{2n+2m}}$. Hence, for $\mu = 0, z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$. Since $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$, either $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$ or, for $\mu > 0$, $z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$such that $z^0 \notin \Sigma_\mu^p$. In either case, there exists $\beta > 0$ such that $\Delta(\psi(t)) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ for all $t \in (0, \beta)$. It follows from Proposition 3.1, that $\Delta(\psi(t)) \in Reg(F_\mu)$ for all $t \in (0, \beta)$. Since $C_\mu(z^0)$ is diffeomorphic to S^1, it follows that there exists $\gamma \in (0, 2\pi]$, such that $\Delta(\psi(\gamma)) = \Delta(\psi(0)) = z^0$. Since $\|F_\mu(z)\|$ is decreasing along the forward orbit of $C_\mu(z^0)$, for $t \in (0, \beta)$, it follows that $\|F_\mu(z^0)\| > \|F_\mu(z(t))\|$ for all $t \in (0, 2\pi]$. It follows that no such γ exists and hence $C_\mu(z^0)$ is not diffeomorphic to S^1.

As before, suppose $z(0) = z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$ where $z(t)$ is a parameterization of $C_\mu(z^0)$. Therefore z^0 must be a boundary point for $C_\mu^+(z^0)$. Hence, $C_\mu^+(z^0)$ is diffeomorphic to some closed or half-closed interval. Now we establish that $C_\mu^+(z^0)$ is not a 1-dimensional manifold with boundary. That is, $C_\mu^+(z^0)$ is not diffeomorphic to a closed interval.

Proposition 6.2.6. *If* $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$, *then* $C_\mu^+(z^0)$ *is diffeomorphic to* $[0, \infty)$.

Proof: Suppose $C_\mu^+(z^0)$ is diffeomorphic to a closed interval and its parametrization $z(t)$ is defined for $t \in [0, t_1]$ where t_1 is maximal, that is $z(t_1)$ is the boundary (endpoint) of the 1-dimensional manifold $C_\mu^+(z^0)$. Then $z^1 = z(t_1)$ is such that $g_{F_\mu}(z^1) = g_{F_\mu}(z^0) = w^0$. Suppose $\mu > 0$. Since $z^0 \notin \Sigma_\mu^p$, there exists, as before, some $\beta > 0$, such that $z(t) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ for all $t \in (0, \beta]$. Note that $\beta \leq t_1$. Define $\gamma = \frac{1}{2}\min\{x_j(\beta)r_j(\beta), y_j(\beta)s_j(\beta), \mu\}$. It follows from Theorems 5.2.6, 5.2.7, that $x_j(t)r_j(t) > \gamma, y_j(t)s_j(t) > \gamma$ for all $t \geq \gamma$. Hence $z^1 \in \mathbf{R}_+^{\mathbf{2n+2m}}$. It follows that $\Phi_{A,\mu}(z^1)$ is C^1 and nonzero. So by Theorem A.4, there exists an $\alpha > 0$ such that a solution $\tilde{z}(t)$ of $\frac{dz}{dt} = \Phi_{A,\mu}(z)$ is defined for $t_1 - \alpha < t_1 < t_1 + \alpha$ such that $\tilde{z}(t_1) = z^1$. Note that $\tilde{z}(t) \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$ from Proposition 4.1.5 and Corollary 4.1.6. From uniqueness and using the correct orientation, it follows that $z(t)$ may be extended to $[0, \tilde{t}_1)$ where $\tilde{t}_1 > t_1$. Hence $z(t_1)$ is not the boundary for $C_\mu^+(z^0)$ and therefore $C_\mu^+(z^0)$ is not diffeomorphic to a closed interval. It follows that $C_\mu^+(z^0)$ is diffeomorphic to $[0, \infty)$. Now suppose that $\mu = 0$. From Proposition 4.1.3, $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p = \mathbf{R}_+^{\mathbf{2n+2m}}$. Since $g_{F_\mu}(z^1) = g_{F_\mu}(z^0)$, it follows from Proposition 3.15 that $F_\mu(z^1) = kF_\mu(z^0)$ for some $k > 0$. In particular, since $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$,

$$x_i^1 r_i^1 = k(x_i^0 r_i^0) > 0$$

$$y_j^1 s_j^1 = k(y_j^0 s_j^0) > 0.$$

Hence, $z^1 \in \mathbf{R}_+^{\mathbf{2n+2m}}$. The above arguments holds and hence $C_\mu^+(z^0)$ is diffeomorphic to $[0, \infty)$.

Let z^* be an ω-limit point of $C_\mu^+(z^0)$, that is suppose that there exists a sequence $\{\tau_n\}$ of real numbers such that $\tau_n \to \infty$ and $z(\tau_n) \to z^*$ where $z(\tau_n) \in C_\mu^+(z^0)$.

Proposition 6.2.7. *If* $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$ *then* $z^* \notin Crit(g_{F_\mu}) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu})$.

Proof: Suppose $z^* \in Crit(g_{F_\mu}) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu})$. Then, from Proposition 3.1, $z^* \in \partial \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Suppose that z^* is an ω-limit point of $C_\mu^+(z^0)$ for $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$. Now $g_{F_\mu}(z^0) = g_{F_\mu}(z(t))$ for every t. Since $z^* \in Crit(g_{F_\mu})$, $g_{F_\mu}(z^*) \in \Lambda_\mu$. Also, $g_{F_\mu}(z^*) = g_{F_\mu}(z(t)) = g_{F_\mu}(z^0)$ as $g_{F_\mu}(z(t))$ is continuous since $F_\mu(z^*) \neq 0$ on $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$. Hence, $z^0 \in \Sigma_\mu = (\Sigma_\mu^\partial \cup \Sigma_\mu^+)$. But, z^0 was chosen so that $z^0 \notin \Sigma_\mu^\partial$. It follows that $z^0 \in \Sigma_\mu^+ \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. If $\mu = 0$, from Proposition 3.17 and the definition of $\Sigma_\mu^+, \Sigma_\mu^+ = \emptyset$. Hence no such z^* exists. For $\mu > 0$, we make an argument based on the hyperbolic transversality Theorems 5.2.6, 5.2.7. Define $\gamma = \frac{1}{2}\min\{x_i^0 r_i^0, y_j^0 s_j^0, \mu\}$. If $z^0 \in \Sigma_\mu^+$, $\gamma > 0$. From Theorems 5.2.6, 5.2.7, $x_i(t)r_i(t) > \gamma, y_j(t)s_j(t) > \gamma$ for all $t \in [0, \infty)$. It follows that $x_i^* r_i^* > 0, y_j^* s_j^* > 0$. Hence, $z^* \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and therefore, from Proposition 3.1, $z^* \in Reg(g_{F_\mu}) \cap Crit(g_{F_\mu})$. Therefore no such z^* exists.

Proposition 6.2.8. *If* $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$ *then* $z^* \notin (Reg(g_{F_\mu}) \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}})$.

Proof: By way of a contradiction, let $z^* \in Reg(g_{F_\mu}) \cap \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. It follows from Theorem 2.1.1 that $\Phi_{A,\mu}(z^*) \neq 0$. Note that $z^0 \in Reg(g_{F_\mu})$. Suppose $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$. For $\mu = 0$, it follows from Theorem 3.17 that $C_0^+(z^0) \subset g_{F_0}^{-1}(g_{F_0}(z^0)) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. Hence $\|F_0(z(t))\|$ is strictly decreasing along $C_0^+(z^0)$. Since $z^* \in Reg(g_{F_0})$, $z^* \notin E_{F_0}$ and therefore, by continuity, $g_{F_0}(z^*) = g_{F_0}(z^0)$. Hence, from Proposition 3.15, there exists some $k > 0$ such that $F_0(z^*) = kF_0(z^0)$. In particular,

$$x_i^* r_i^* = k(x_i^0 r_i^0) > 0$$

$$y_j^* s_j^* = k(y_j^0 s_j^0) > 0.$$

It follows that $z^* \in \mathbf{R}_+^{\mathbf{2n+2m}}$. If $\mu > 0$, we use the same transversality argument as in the proof of Proposition 6.2.7. Define $\gamma = \frac{1}{2}\min\{x_i^0 r_i^0, y_j^0 s_j^0, \mu\}$. Since $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}}$, $\gamma > 0$. From Theorems 5.2.6, 5.2.7, $x_i(t)r_i(t) > \gamma, y_j(t)s_j(t) > \gamma$ for all $t \in [0, \infty)$. It follows that $x_i^* r_i^* > 0, y_j^* s_j^* > 0$. Hence, $z^* \in \mathbf{R}_+^{\mathbf{2n+2m}}$ Since $\Phi_{A,\mu}(z)$ is C^1 and $\Phi_{A,\mu}(z^*) \neq 0$, it follows that there exists a solution $\hat{z}(t)$ of

$$\frac{d\hat{z}}{dt} = \Phi_{A,\mu}(\hat{z}) \qquad \hat{z}(0) = z^*$$

which is defined for $-\hat{\alpha} < t < \hat{\alpha}$ for some $\hat{\alpha} > 0$. Since $z^* \in \mathbf{R}_+^{\mathbf{2n+2m}}$, $\exists \epsilon > 0$ such that, from Proposition 6.2.3,

$$\|F_\mu(\hat{z}(\hat{\alpha}))\| \leq \|F_\mu(\hat{z}(0))\| - \epsilon = \|F_\mu(z^*)\| - \epsilon.$$

Since $z(\tau_n) \to z^*$ (by the ω-limit property) and $F_\mu(z)$ is continuous at z^*, it follows that there exists $K_1 < \infty$ such that $\forall\ n \geq K_1$, $\|F_\mu(z^*)\| \leq \|F_\mu(z(\tau_n))\| + \frac{\epsilon}{4}$. Now, $\tilde{z}_m(t) = z(\tau_m + t)$ is a solution of

$$\frac{d\tilde{z}}{dt} = \Phi_{A,\mu}(\tilde{z}) \qquad \tilde{z}_m(0) = z(\tau_m)$$

Since $\tilde{z}_m(0) = z(\tau_m) \to z^* = \hat{z}(0)$ as $m \to \infty$, from Theorem A.5,

$$z(\tau_m + \hat{\alpha}) = \tilde{z}_m(\hat{\alpha}) \to \hat{z}(\hat{\alpha})$$

as $m \to \infty$. Hence $\hat{z}(\hat{\alpha})$ is an ω-limit point of $z(t)$ with $z(\tau_m + \hat{\alpha}) \to \hat{z}(\hat{\alpha})$ as $\tau_n + \hat{\alpha} \to \infty$. From the continuity of $F_\mu(z)$ at $\hat{z}(\hat{\alpha})$, Proposition 6.2.3, and the

ω-limit property, that is, $z(\tau_m + \hat{\alpha}) \to \hat{z}(\hat{\alpha})$, there exists $K_2 \geq K_1$ such that for $m \geq K_2$,

$$\|F_\mu(z(\tau_m + \hat{\alpha}))\| = \|F_\mu(\tilde{z}_m(\hat{\alpha}))\| < \|F_\mu(\hat{z}(\hat{\alpha}))\| + \frac{\epsilon}{4}.$$

Hence, for all $m, n \geq K_2$,

$$\begin{aligned}\|F_\mu(z(\tau_m + \hat{\alpha}))\| < \|F_\mu(\hat{z}(\hat{\alpha}))\| + \frac{\epsilon}{4} &< \|F_\mu(z^*)\| - \epsilon + \frac{\epsilon}{4} \\ &< \|F_\mu(z(\tau_n)\| + \frac{\epsilon}{4} - \frac{3\epsilon}{4} \\ &= \|F_\mu(z(\tau_n)\| - \frac{\epsilon}{2}.\end{aligned}$$

However, since $\|F_\mu(z(t))\|$ is strictly decreasing along $z(t)$, for any m, n such that $\tau_n < \tau_m + \hat{\alpha}$, $\|F_\mu(z(\tau_n))\| < \|F_\mu(z(\tau_m + \hat{\alpha}))\|$ and hence this cannot happen. It follows that no such z^* exists.

Now suppose that, for the case $\mu > 0$, $z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Since $z^0 \notin \Sigma_\mu^p$, it follows, from Definition 4.2.3.1 and Proposition 4.1.5, that there exists $\beta > 0$ for which $z(t) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ for all $t \in (0, \beta]$ and such that $\{z(t) : t \in [0, \beta]\} \subset C_\mu^+(z^0)$. Since $z^0 \notin \Sigma_\mu^p$, it follows that $C_\mu(z(\beta)) = C_\mu^+(z^0)$. Therefore, we need only consider $C^+(z(\beta))$. Hence the above argument holds using as a initial point $z(\beta) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ and therefore the proposition holds.

Corollary 6.2.9. *Let $\mu \geq 0$. Suppose that z^* is an ω-limit point of $C_\mu^+(z^0)$ for $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$. Then $F_\mu(z^*) = 0$.*

Proof: Let $\mu \geq 0, z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$. If z^* is an ω-limit point of $C_\mu^+(z^0)$ for $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$, from Propositions 6.2.7, 6.2.8, $z^* \notin Reg(g_{F_\mu}) \cup Crit(g_{F_\mu})$. From Definitions 2.0.2 and 1.15, $\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} = Reg(g_{F_\mu}) \cup Crit(g_{F_\mu}) \cup E_{F_\mu}$ where, by definition, this is a disjoint union. Hence the corollary holds.

Proposition 6.2.10. *Let $\mu \geq 0$, and suppose that $z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$, then*

$$C_\mu^+(z^0) = \{z(t) : t \geq 0\}$$

where $z(t)$ is the solution to (1.17).

Proof: Now $C_\mu^+(z^0) \subseteq \{z(t) : t \geq 0\}$. If $C_\mu^+(z^0) \neq \{z(t) : t \geq 0\}$ then since $C_\mu^+(z^0)$ is diffeomorphic to $[0, \infty)$, it follows that there exists $\beta > 0$ such that $C_\mu^+(z^0) = \{z(t) : t \in [0, \beta)\}$. Hence, $z(\beta)$ is an ω-limit point of $C_\mu^+(z^0)$ under the appropriate parameterization. From Corollary 6.2.9, $F_\mu(z(\beta)) = 0$. However,

$$\mathrm{D}F_\mu(z(t))\frac{dz}{dt} = \mathrm{D}F_\mu(z(t))\Phi_{A,\mu}(z(t)) = -|det(\mathrm{D}F_\mu(z(t)))|F_\mu(z(t)).$$

From Proposition 2.1.2, $|det(\mathrm{D}F_\mu(z(t)))| = (-1)^m det(\mathrm{D}F_\mu(z(t)))$. Given the structure of $\mathrm{D}F_\mu(z)$ from (1.10), $det(\mathrm{D}F_\mu(z(t)))$ is polynomial in the components of $z(t)$ and is therefore C^1 for all t. It follows that

$$F_\mu(z(t)) = e^{-\int_0^t |det(\mathrm{D}F_\mu(z(\tau)))|d\tau} F_\mu(z^0).$$

Hence,

$$0 = F_\mu(z(\beta)) = e^{-\int_0^\beta |det(\mathrm{D}F_\mu(z(\tau)))|d\tau} F_\mu(z^0).$$

Since $z^0 \notin \Sigma_\mu^p$, it follows from Definition 4.2.3.1, $\exists\ \alpha > 0$ such that $z(t) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ for all $t \in (0, \alpha)$. It follows from Proposition 2.1.2 that $|det(\mathrm{D}F_\mu(z(t)))| > 0$ for all $t \in (0, \alpha)$. Therefore, either $F_\mu(z^0) = 0$ and z^0 is a fixed point of $\Phi_{A,\mu}(z)$ or no such β exists. In either case, it follows that $C_\mu^+(z^0) = \{z(t) : t \geq 0\}$.

The following lemma establishes the boundedness of the trajectories. A similar result for numerical (discrete) methods is found in Mizuno and Jarre [MJ].

Lemma 6.2.11. *Given (1.18)-(1.20), $\mu \geq 0$ and $z(0) = (s^0, r^0, x^0, y^0) = z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$, $C^+(z^0)$ is bounded.*

Proof: First note that if $F_\mu(z^0) = 0$ then $C_\mu^+(z^0) = \{z^0\}$ and clearly the lemma holds. Suppose the $\mu \geq 0$ and $\bar{z} = (\bar{s}, \bar{r}, \bar{x}, \bar{y}) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ is such that

$$A\bar{x} - \bar{s} = b$$

$$A^t\bar{y} + \bar{r} = c.$$

Suppose that $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$. If $\mu = 0$, from Proposition 3.17, $C_0^+(z^0) \subset g_{F_0}^{-1}(g_{F_0}(z^0)) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. If $\mu > 0$, using a transversality argument based on Theorems 5.2.6, 5.2.7, as in Propositions 6.2.7, 6.2.8, $C_\mu^+(z^0) \subset \mathbf{R}_+^{\mathbf{2n+2m}}$. Therefore, regardless of μ, $\mathrm{D}F_\mu(z)$ is of full rank for all $z \in C_\mu^+(z^0)$. Let $z(t)$ be the (re)parametrization of $C^+(z^0)$ induced by the solution of

$$\frac{dz}{dt} = \Phi_{N,\mu}(z) \qquad z(0) = z^0.$$

Now,

$$\mathrm{D}F_\mu(z)\Phi_{N,\mu}(z) = -F_\mu(z).$$

It follows that

$$F_\mu(z(t)) = e^{-t}F_\mu(z^0).$$

Therefore,

$$\begin{aligned}
Ax(t) - s(t) &= b(t) = e^{-t}(Ax(0) - s(0) - b) + b, \\
A^t y(t) + r(t) &= c(t) = e^{-t}(A^t y(0) + r(0) - c) + c, \\
x(t)^t r(t) + y(t)^t s(t) &= e^{-t}(x(0)^t r(0) + y(0)^t s(0) - (n+m)\mu) + (n+m)\mu \\
&= e^{-t}(x(0)^t r(0) + y(0)^t s(0)) + (1 - e^{-t})(n+m)\mu.
\end{aligned}$$

It follows that

$$A[(1-e^{-t})\bar{x}+e^{-t}x(0)-x(t)]-[(1-e^{-t})\bar{s}+e^{-t}s(0)-s(t)]$$

$$=(1-e^{-t})b+e^{-t}b(0)-b(t)=0$$

and

$$A^t[(1-e^{-t})\bar{y}+e^{-t}y(0)-y(t)]+[(1-e^{-t})\bar{r}+e^{-t}r(0)-r(t)]$$

$$=(1-e^{-t})c+e^{-t}c(0)-c(t)=0.$$

Hence,

$$\begin{aligned}
&-((1-e^{-t})\bar{x}+e^{-t}x(0)-x(t))^t((1-e^{-t})\bar{r}+e^{-t}r(0)-r(t))\\
&-((1-e^{-t})\bar{y}+e^{-t}y(0)-y(t))^t((1-e^{-t})\bar{s}+e^{-t}s(0)-s(t))\\
&=((1-e^{-t})\bar{x}+e^{-t}x(0)-x(t))^tA^t((1-e^{-t})\bar{y}+e^{-t}y(0)-y(t))\\
&\quad-((1-e^{-t})\bar{y}+e^{-t}y(0)-y(t))^t((1-e^{-t})\bar{s}+e^{-t}s(0)-s(t))\\
&=((1-e^{-t})\bar{y}+e^{-t}y(0)-y(t))^t(A[(1-e^{-t})\bar{x}+e^{-t}x(0)-x(t)]\\
&\qquad\qquad-[(1-e^{-t})\bar{s}+e^{-t}s(0)-s(t)])\\
&=0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&((1-e^{-t})\bar{x}+e^{-t}x(0))^tr(t)+((1-e^{-t})\bar{r}+e^{-t}r(0))^tx(t)\\
&\quad+((1-e^{-t})\bar{y}+e^{-t}y(0))^ts(t)+((1-e^{-t})\bar{s}+e^{-t}s(0))^ty(t)\\
&=[(1-e^{-t})\bar{x}+e^{-t}x(0)]^t[(1-e^{-t})\bar{r}+e^{-t}r(0)]+r(t)^tx(t)\\
&\quad+[(1-e^{-t})\bar{y}+e^{-t}y(0)]^t[(1-e^{-t})\bar{s}+e^{-t}s(0)]+s(t)^ty(t)\\
&=(1-e^{-t})^2[(\bar{x})^t\bar{r}+(\bar{y})^t\bar{s}]\\
&\quad+e^{-t}(1-e^{-t})[x(0)^t\bar{r}+r(0)^t\bar{x}+y(0)^t\bar{s}+s(0)^t\bar{y}]\\
&\quad+e^{-t}(1+e^{-t})[x(0)^tr(0)+y(0)^ts(0)]+(1-e^{-t})(n+m)\mu\\
&\le e^{-t}K_1+(1-e^{-t})K_2
\end{aligned}$$

for some $K_i > 0$. Let ζ be such that $0 < \zeta < \min\{\bar{x}_i, \bar{r}_i, \bar{s}_i, \bar{y}_i\}$. Now,

$$\begin{aligned}
&((1-e^{-t})\bar{x} + e^{-t}x(0))^t r(t) + ((1-e^{-t})\bar{r} + e^{-t}r(0))^t x(t) \\
&\quad + ((1-e^{-t})\bar{y} + e^{-t}y(0))^t s(t) + ((1-e^{-t})\bar{s} + e^{-t}s(0))^t y(t) \\
&\geq (1-e^{-t})[(\bar{x})^t r(t) + (\bar{r})^t x(t) + (\bar{y})^t s(t) + (\bar{s})^t y(t)] \\
&\geq (1-e^{-t})\zeta[\sum_{i=1}^{n} x_i(t) + \sum_{i=1}^{n} r_i(t) + \sum_{i=1}^{m} y_i(t) + \sum_{i=1}^{m} s_i(t)] \geq 0.
\end{aligned}$$

Since $z^0 \in \mathbf{R}_+^{\mathbf{2n+2m}} \setminus E_{F_\mu}$, $z^0 \in Reg(g_{F_\mu})$. It follows from Proposition 2.1.4 and Theorem A.4 that there exists $\tau > 0$ such that $z(t)$ is defined on $[0, \tau]$ and $\Phi_{A,\mu}(z^0) \neq 0$. Now $\{z(t) : t \in [0, \tau]\}$ is compact as the continuous image of a compact set. Also, for $t \geq \tau$,

$$0 \leq [\sum_{i=1}^{n} x_i(t) + \sum_{i=1}^{n} r_i(t) + \sum_{i=1}^{m} y_i(t) + \sum_{i=1}^{m} s_i(t)] \leq \frac{1}{\zeta}[\frac{e^{-t}}{(1-e^{-t})}K_1 + K_2].$$

It follows that for $t \geq \tau, z_i(t)$ is bounded for every i. Hence, $C^+(z^0)$ is bounded. Now, for the case $\mu > 0$, if $z^0 \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus \Sigma_\mu^p$, let $z(t)$ be the parametrization of $C^+(z^0)$ given by (1.17). Since $z^0 \notin \Sigma_\mu^p$, there exists $\beta > 0$, such that $z(t)$ is defined for all $t \in [0, \beta]$ and for which $z(t) \in \mathbf{R}_+^{\mathbf{2n+2m}}$ for all $t \in (0, \beta]$. Now, $\{z(t) : t \in [0, \beta]\}$ is compact and $C^+(z(\beta))$ is bounded as above (since $z(\beta) \in \mathbf{R}_+^{\mathbf{2n+2m}}$). Hence the proposition holds.

Lemma 6.2.12. *Let $\mu = 0, z(0) \in \mathbf{R}_+^{\mathbf{2n+2m}}$, and suppose that $z^* = (s^*, r^*, x^*, y^*)$ exists such that $F_0(z^*) = 0$. Then $C_0^+(z^0)$ is bounded.*

Proof: Note, from Proposition 3.17, $C_\mu^+(z^0) \subset \mathbf{R}_+^{\mathbf{2n+2m}} \subset Reg(F_0)$. Hence $\mathrm{D}F_\mu(z)$ is of full rank for all $z \in C_\mu^+(z^0)$. Let $z(t)$ be the (re)parametrization of $C_\mu^+(z^0)$ induced by the solution of

$$\frac{dz}{dt} = \Phi_{N,\mu}(z) \qquad z(0) = z^0.$$

Note that $z(t) \in \mathbf{R}_+^{2n+2m}$ for all $t > 0$. Now,

$$\mathrm{D}F_\mu(z)\frac{dz}{dt} = -F_\mu(z).$$

It follows that

$$F_\mu(z(t)) = e^{-t}F_\mu(z^0).$$

Therefore,

$$Ax(t) - s(t) = b(t) = e^{-t}(Ax(0) - s(0) - b) - b,$$

$$A^t y(t) + r(t) = c(t) = e^{-t}(A^t y(0) + r(0) - c) + c,$$

$$x(t)^t r(t) + y(t)^t s(t) = e^{-t}(x(0)^t r(0) + y(0)^t s(0)).$$

It follows that

$$A[(1-e^{-t})x^* + e^{-t}x(0) - x(t)] - [(1-e^{-t})s^* + e^{-t}s(0) - s(t)]$$
$$= (1-e^{-t})b + e^{-t}b(0) - b(t) = 0$$

and

$$A^t[(1-e^{-t})y^* + e^{-t}y(0) - y(t)] + [(1-e^{-t})r^* + e^{-t}r(0) - r(t)]$$
$$= (1-e^{-t})c + e^{-t}c(0) - c(t) = 0.$$

Hence,

$$\begin{aligned}
&-((1-e^{-t})x^* + e^{-t}x(0) - x(t))^t((1-e^{-t})r^* + e^{-t}r(0) - r(t))\\
&-((1-e^{-t})y^* + e^{-t}y(0) - y(t))^t((1-e^{-t})s^* + e^{-t}s(0) - s(t))\\
&= ((1-e^{-t})x^* + e^{-t}x(0) - x(t))^t A^t((1-e^{-t})y^* + e^{-t}y(0) - y(t))\\
&\quad -((1-e^{-t})y^* + e^{-t}y(0) - y(t))^t((1-e^{-t})s^* + e^{-t}s(0) - s(t))\\
&= ((1-e^{-t})y^* + e^{-t}y(0) - y(t))^t(A[(1-e^{-t})x^* + e^{-t}x(0) - x(t)]\\
&\qquad - [(1-e^{-t})s^* + e^{-t}s(0) - s(t)])\\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&((1-e^{-t})x^* + e^{-t}x(0))^t r(t) + ((1-e^{-t})r^* + e^{-t}r(0))^t x(t) \\
&\quad + ((1-e^{-t})y^* + e^{-t}y(0))^t s(t) + ((1-e^{-t})s^* + e^{-t}s(0))^t y(t) \\
&= [(1-e^{-t})x^* + e^{-t}x(0)]^t[(1-e^{-t})r^* + e^{-t}r(0)] + r(t)^t x(t) \\
&\quad + [(1-e^{-t})y^* + e^{-t}y(0)]^t[(1-e^{-t})s^* + e^{-t}s(0)] + s(t)^t y(t) \\
&= e^{-t}(1-e^{-t})[x(0)^t r^* + r(0)^t x^* + y(0)^t s^* + s(0)^t y^*] \\
&\quad + e^{-t}(1+e^{-t})[x(0)^t r(0) + y(0)^t s(0)] \\
&\leq e^{-t}K
\end{aligned}$$

for some $K > 0$. Let ζ be such that $0 < \zeta < \min\{x_i(0), r_i(0), s_i(0), y_i(0)\}$. Now,

$$\begin{aligned}
&((1-e^{-t})x^* + e^{-t}x(0))^t r(t) + ((1-e^{-t})r^* + e^{-t}r(0))^t x(t) \\
&\quad + ((1-e^{-t})y^* + e^{-t}y(0))^t s(t) + ((1-e^{-t})s^* + e^{-t}s(0))^t y(t) \\
&\geq e^{-t}[x(0)^t r(t) + r(0)^t x(t) + y(0)^t s(t) + s(0)^t y(t)] \\
&\geq e^{-t}\zeta[\sum_{i=1}^{n} x_i(t) + \sum_{i=1}^{n} r_i(t) + \sum_{i=1}^{m} y_i(t) + \sum_{i=1}^{m} s_i(t)] \geq 0.
\end{aligned}$$

It follows that

$$0 \leq e^{-t}\zeta[\sum_{i=1}^{n} x_i(t) + \sum_{i=1}^{n} r_i(t) + \sum_{i=1}^{m} y_i(t) + \sum_{i=1}^{m} s_i(t)] \leq e^{-t}K$$

and therefore

$$0 \leq [\sum_{i=1}^{n} x_i(t) + \sum_{i=1}^{n} r_i(t) + \sum_{i=1}^{m} y_i(t) + \sum_{i=1}^{m} s_i(t)] \leq \frac{1}{\zeta}K$$

It follows that $z_i(t)$ is bounded for every i and hence $C_0^+(z^0)$ is bounded.

§6.3 Proofs of Convergence Theorems

Proof of Theorem 1.23: Let $z^0 = (s^0, r^0, x^0, y^0) \in \mathbf{R}_+^{2n+2m}$ and $\mu = 0$. By Corollary 2.1.5, there exists a unique solution, $z(t)$,to

$$\text{(IVP)} \qquad \frac{dz}{dt} = \Phi_{A,\mu}(z(t)), \qquad z(0) = z^0.$$

From Lemma 6.2.12, $\{z(t) : t \in [0,\infty)\}$ is bounded. Let M be such that $\|z(t)\| \leq M$ for all $t \in [0,\infty)$. Let $\Gamma_M = \{z : z \in \bar{\mathbf{R}}_+^{2n+2m}, \|z\| \leq M\}$. It follows that $z(t)$ must have an ω-limit point z^* in Γ_M. From proposition 3.17, $g_{F_\mu}(z^0)$ is a regular value of g_{F_μ}. It follows from Corollary 6.2.9 and Proposition 6.2.10 that $z^* \in E_{F_\mu}$. Now, if $z(t) \not\to z^*$, there exists a sequence of points, $z^k = z(t_k)$, such that $t_k \to \infty$, and $z^k \not\to z^*$. But, $z^k \in \Gamma_M$ which is compact. If follows that $\{z^k\}$ has an ω-limit point $\hat{z}^* \in \Gamma_M$. By Corollary 6.2.9 and Proposition 6.2.10, $\hat{z}^* \in E_{F_0}$. Now from (1.21), $E_{F_0} = \{z^*\}$. Hence $z(t) \to z^*$.

Proof of Theorem 1.24: Let $\mu > 0$ and Σ_μ^p be given as defined. Suppose that $z^0 = (s^0, r^0, x^0, y^0) \in \bar{\mathbf{R}}_+^{2n+2m} \setminus \Sigma_\mu^p$. By Corollary 2.1.5, there exists a unique solution, $z(t)$,to

$$\text{(IVP)} \qquad \frac{dz}{dt} = \Phi_{A,\mu}(z(t)), \qquad z(0) = z^0.$$

From Lemma 6.2.11, there exists $M > 0$ such that $\|z(t)\| \leq M$ for all $t \in [0,\infty)$. Let $\Gamma_M = \{z : z \in \bar{\mathbf{R}}_+^{2n+2m}, \|z\| \leq M\}$. It follows that $z(t)$ must have a limit point, z^*,in Γ_M. It follows from Corollary 6.2.9 and Proposition 6.2.10 that $z^* \in E_{F_\mu}$. From Theorem 3.21, $\Gamma_M \cap E_{F_\mu} = z^*$. Since Γ_M is compact and $\{z(t) : t \in [0,\infty)\}$ is connected, as in the proof of Theorem 1.23, from Corollary 6.2.9 and Proposition 6.2.10, $z(t) \to z^*$. Also, from Proposition 4.2.3.3, Σ_μ^p is as described.

CHAPTER 7

PROPERTIES OF CRITICAL TRAJECTORIES

Chapter 7 has preliminary results on the properties of trajectories $z(t)$ which are solutions of (1.17) for which $\mu = 0$ and for which there exists some $\beta > 0$ such that $z(\beta) \in Crit(g_{F_\mu})$. These trajectories are a special type included in a larger class of trajectories known as **critical trajectories.**

Definition 7.1. *Given* $\mu \geq 0, z^0 \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}, w^0 = F_\mu(z^0)$, *a trajectory,* $z(t)$, *through* z^0, *which parameterizes* $g_{F_\mu}^{-1}(w^0)$ *is called a* **critical trajectory** *if there exists* $z^1 \in Crit(g_{F_\mu}) \cap \bar{C}_\mu^+(z^0)$.

Note, in Definition 7.1 with z^0, z^1 as given, it follows that $g_{F_\mu}(z^1) = g_{F_\mu}(z^0)$. Points $z^0 \in Reg(g_{F_\mu})$ through which a critical trajectory went were excluded from consideration by Smale [Sm], and Hirsch and Smale [HSm]. By Definition 7.1, trajectories satisfying (1.17) through points $z \in \Sigma_\mu$ are critical trajectories. It has also been shown that for $\mu > 0, \Sigma_\mu^\partial, \Sigma_\mu^+, \Sigma_\mu^p$ are all non-empty while for $\mu = 0, \Sigma_\mu^\partial$, and Σ_μ^p are non-empty. Hence, given that in Chapter 6 results for $z^0 \in \Sigma_\mu^+$ were established for $\mu > 0$, we have already provided results for some critical trajectories. The work done here begins to provide a basis for establishing the existence of C^0 trajectories $z(t)$ for which $z(t) \in \partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ and $z(t) \rightarrow z^* \in F_\mu^{-1}(0)$. In particular, we will analyze trajectories corresponding to $\mu = 0$ and for which $z^0 = z(0)$ is in the relative interior of a face of codimension 1 in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. Recall from Proposition 3.4, in this setting $\mathrm{D}F_\mu(z^0)$ is of full rank. Future work will involve the study of stable and unstable manifolds at points $z \in Crit(g_{F_\mu})$ which are limit points of the above defined critical trajectories. The goal will be to establish that critical trajectories "pass through" points $z \in Crit(g_{F_\mu}) \setminus E_{F_\mu}$ and "continue" to points $z \in E_{F_\mu}$.

Let $\mu = 0, I_0 \subset \{1, \dots, 2n+2m\}, \#(I_0) = 1$, $z^0 = z(0)$ in the relative interior of $\mathcal{F}(I_0)$, and $C_\mu(z^0)$ be the connected component of $g_{F_\mu}^{-1}(g_{F_\mu}(z^0))$.

Proposition 7.2. *$C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (Crit(g_{F_\mu})) \cup E_{F_\mu})$ is diffeomorphic to $[0, \infty)$.*

Proof: Let $w^0 = g_{F_\mu}(z^0)$. Note, since z^0 is in the relative interior of $\mathcal{F}(I_0), z^0 \notin E_{F_\mu}$. Also, $\Phi_{A,\mu}(z)$ is C^1 on $\mathbf{R}^{\mathbf{2n+2m}}$ and from Theorem 2.1.1, $Crit(g_{F_\mu}) \cup E_{F_\mu} = \Phi_{A,\mu}^{-1}(0)$. Hence, w^0 is a regular value of $g_{F_\mu}|_{\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (Crit(g_{F_\mu}) \cup E_{F_\mu})}$ and therefore Theorem 6.2.1 and Propositions 6.2.2-6.2.4 hold. It follows that $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (Crit(g_{F_\mu}) \cup E_{F_\mu}))$ is diffeomorphic to $[0, \infty)$.

Proposition 7.3. *Given (1.18)-(1.20), $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (Crit(g_{F_\mu}) \cup E_{F_\mu}))$ has an ω-limit point z^* for which either $F_\mu(z^*) = 0$ or $z^* \in Crit(g_{F_\mu})$ and there exists $I_0 \subset \{1, \dots, 2n+2m\}, \#(I_0) = 2$ such that z^* is in the relative interior of $\mathcal{F}(I_0)$ and there exists i such that $x_i^* = r_i^* = 0$ or $y_i^* = s_i^* = 0$.*

Proof: Now, $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (Crit(g_{F_\mu}) \cup E_{F_\mu}))$ is in the relative interior of $\mathcal{F}(I_0)$. It follows, from Proposition 3.4 and Lemma 6.2.12, that $C^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus (Crit(g_{F_\mu}) \cup E_{F_\mu}))$ is bounded and therefore has a limit point. Suppose that $z^* \in Reg(g_{F_\mu})$. It follows that $g_{F_\mu}(z^*) = g_{F_\mu}(z^0)$. Suppose (WLOG) that $x_i^0 = 0$. If follows from Proposition 3.15 that there exists some $k > 0$, such that $k(x_i^* r_i^*) = x_i^0 r_i^0 = 0$. It follows that either $x_i^* = r_i^* = 0$ or exactly one of x_i^*, r_i^* is 0. From

Proposition 3.8, if $x_i^* = r_i^* = 0$ then $z^* \in Crit(g_{F_\mu})$. It follows that exactly one of x_i^*, r_i^* is 0. Since $g_{F_\mu}(z^*) = g_{F_\mu}(z^0)$ and z^0 is in the relative interior of $\mathcal{F}(I_0)$, it follows that z^* is in the relative interior of a face of codimension 1 in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$. From Proposition 3.4, $z^* \in Reg(F_\mu)$. It follows from the proof used for Proposition 6.2.8, either $F_\mu(z^*) = 0$ or $z^* \in Crit(g_{F_\mu})$. If $F_\mu(z^*) \neq 0, z^* \in Crit(g_{F_\mu})$ and $C^+(z^0) \subset \{z(t) : t \in [0, \infty)\}$. It follows that $C^+(z^0) = \{z(t) : t \in [0, \beta)\}$ for some $\beta > 0$. Since $F_\mu(z^*) \neq 0, g_{F_\mu}(z^*) = g_{F_\mu}(z^0)$. Suppose (WLOG) that there exists i for which z^0 is such that $x_i^0 = 0$. By construction and continuity,

$$F_\mu(z^*) = F_\mu(z(\beta)) = e^{-\int_0^t |det(\mathrm{D}F_\mu(z(\tau)))| d\tau} F_\mu(z^0).$$

Hence, since there exists $0 < \alpha \leq \beta$ such that $|det(\mathrm{D}F_\mu(z(\tau))) \neq 0$ for all $\tau \in (0, \alpha)$, $x_i^* r_i^* = e^{-\int_0^\beta |det(\mathrm{D}F_\mu(z(\tau)))| d\tau} x_i^0 r_i^0 = 0$ and $\forall j \neq i$,

$$x_j^* r_j^* = e^{-\int_0^\beta |det(\mathrm{D}F_\mu(z(\tau)))| d\tau} x_j^0 r_j^0 \neq 0$$

and $\forall k$,

$$y_k^* s_k^* = e^{-\int_0^\beta |det(\mathrm{D}F_\mu(z(\tau)))| d\tau} y_k^0 s_k^0 \neq 0.$$

Since $z^* \in Crit(g_{F_\mu})$, it follows that Proposition 3.4 that $x_i^* = r_i^* = 0$.

Given the above proposition, we are interested in the properties of fixed points that are in faces of codimension 2 in $\partial\bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ for which $x_i = r_i = 0$ or $y_i = s_i = 0$. Given that $\mathrm{D}F_\mu(z)$ is rank deficient in this setting, the formulation of $\mathrm{D}\Phi_{A,\mu}(z)$ given in Theorem 2.1.2 does not hold. The following theorem provides a general formulation of $\mathrm{D}\Phi_{A,\mu}(z)$.

Theorem 7.4. $\mathrm{D}\Phi_{A,\mu}(z) = (-1)|det\mathrm{D}F_\mu(z)|I + (-1)^{m+1}\Delta$ *where* $\Delta = (\delta_{jk})$ *is such that* $\delta_{jk} =$

$$F_{\mu_1}(z)(-1)^{1+j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{l=n+m+1}^{2n+2m}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_{\alpha_1(l)})}\partial z_k}\prod_{\substack{p=2\\p\neq l}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_1(p)})}}\right)\right]$$

$$+$$

$$F_{\mu_2}(z)(-1)^{2+j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{l=n+m+1}^{2n+2m}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_{\alpha_2(l)})}\partial z_k}\prod_{\substack{p=1\\p\neq l,2}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_2(p)})}}\right)\right]$$

$$\vdots$$

$$+$$

$$F_{\mu_q}(z)(-1)^{q+j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{\substack{l=n+m+1\\l\neq q}}^{2n+2m}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_{\alpha_q(l)})}\partial z_k}\prod_{\substack{p=1\\p\neq l,q}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_q(p)})}}\right)\right]$$

$$\vdots$$

$$+$$

$$F_{\mu_{2n+2m}}(z)(-1)^{j}\left[\sum_{\sigma\in S_\rho} sg(\tau(\sigma)) \sum_{l=n+m+1}^{2n+2m-1}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_l)}\partial z_k}\prod_{\substack{p=1\\p\neq l}}^{2m+2n-1}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_p)}}\right)\right]$$

where $S_\rho = S_{2n+2m-1}, \tau_j(\sigma)$ *is defined by*

$$\tau_j(\sigma_k) = \begin{cases} \sigma_k & \sigma_k < j \\ \sigma_k + 1 & \sigma_k \geq j \end{cases}$$

where σ_k *is the image of* k *in* σ *and*

$$\alpha_i(l) = \begin{cases} l & l < i \\ l-1 & l \geq i. \end{cases}$$

Proof: Define $B(i|j)$ to the be submatrix of $B \in \mathbf{R}^{\mathbf{N}\times\mathbf{N}}$ formed by removing the ith row and the j column of B. It follows that the jth column of $adj(B)$ is

$$\begin{pmatrix} (-1)^{j+1}det(B(j|1)) \\ \vdots \\ (-1)^{j+n}det(B(j|N)) \end{pmatrix}.$$

Therefore

$$\begin{aligned}\Phi_{A,\mu}(z) =&(-1)^{m+1}adj(\mathrm{D}F_\mu(z))F_\mu(z)\\ =&(-1)^{m+1}F_{\mu_1}(z)\begin{pmatrix} (-1)^{1+1}det(\mathrm{D}F_\mu(1|1)) \\ \vdots \\ (-1)^{1+2n+2m}det(\mathrm{D}F_\mu(1|2n+2m)) \end{pmatrix} + \dots \\ &+ (-1)^{m+1}F_{\mu_{2n+2m}}(z)\begin{pmatrix} (-1)^{2n+2m+1}det(\mathrm{D}F_\mu(2n+2m|1)) \\ \vdots \\ (-1)^{2(2n+2m)}det(\mathrm{D}F_\mu(2n+2m|2n+2m)) \end{pmatrix}.\end{aligned}$$

It follows from Lemma 2.0.1 that $(-1)^{m+1}\mathrm{D}\Phi_{A,\mu}(z) =$

$$\begin{aligned}&\begin{pmatrix} (-1)^{1+1}det(\mathrm{D}F_\mu(1|1)) \\ \vdots \\ (-1)^{1+2n+2m}det(\mathrm{D}F_\mu(1|2n+2m)) \end{pmatrix} \nabla F_{\mu_1}(z)^t \\ &+ F_{\mu_1}(z)\mathrm{D}_z(\begin{pmatrix} (-1)^{1+1}det(\mathrm{D}F_\mu(1|1)) \\ \vdots \\ (-1)^{1+2n+2m}det(\mathrm{D}F_\mu(1|2n+2m)) \end{pmatrix}) \\ &+ \dots \\ &+ \begin{pmatrix} (-1)^{2n+2m+1}det(\mathrm{D}F_\mu(2n+2m|1)) \\ \vdots \\ (-1)^{2(2n+2m)}det(\mathrm{D}F_\mu(2n+2m|2n+2m)) \end{pmatrix} \nabla F_{\mu_{2n+2m}}(z)^t \\ &+ F_{\mu_{2n+2m}}(z)\mathrm{D}_z(\begin{pmatrix} (-1)^{2n+2m+1}det(\mathrm{D}F_\mu(2n+2m|1)) \\ \vdots \\ (-1)^{2(2n+2m)}det(\mathrm{D}F_\mu(2n+2m|2n+2m)) \end{pmatrix}).\end{aligned}$$

Hence,

$$
\begin{aligned}
(-1)^{m+1}\mathrm{D}\Phi_{A,\mu}(z) =& adj(\mathrm{D}F_\mu(z))\mathrm{D}F_\mu(z) \\
& + F_{\mu_1}(z)\mathrm{D}_z(\begin{pmatrix} det(\mathrm{D}F_\mu(1|1)) \\ \vdots \\ (-1)det(\mathrm{D}F_\mu(1|2n+2m)) \end{pmatrix}) \\
& + \dots \\
& + F_{\mu_{2n+2m}}(z)\mathrm{D}_z(\begin{pmatrix} (-1)det(\mathrm{D}F_\mu(2n+2m|1)) \\ \vdots \\ det(\mathrm{D}F_\mu(2n+2m|2n+2m)) \end{pmatrix}) \\
=& det(\mathrm{D}F_\mu(z)I \\
& + F_{\mu_1}(z)\mathrm{D}_z(\begin{pmatrix} det(\mathrm{D}F_\mu(1|1)) \\ \vdots \\ (-1)det(\mathrm{D}F_\mu(1|2n+2m)) \end{pmatrix}) \\
& + \dots \\
& + F_{\mu_{2n+2m}}(z)\mathrm{D}_z(\begin{pmatrix} (-1)det(\mathrm{D}F_\mu(2n+2m|1)) \\ \vdots \\ det(\mathrm{D}F_\mu(2n+2m|2n+2m)) \end{pmatrix}).
\end{aligned}
$$

For $\sigma \in S_{2m+2n-1}$ define σ_i to be the image of i in σ and define $\tau_j(\sigma_k)$ as

$$
\tau_j(\sigma_k) = \begin{cases} \sigma_k & \sigma_k < j \\ \sigma_k + 1 & \sigma_k \geq j. \end{cases}
$$

Also define $\alpha_i(\cdot)$ as

$$
\alpha_i(l) = \begin{cases} l & l < i \\ l-1 & l \geq i \end{cases}
$$

Then $det(\mathrm{D}F_\mu(i|j)) =$

$$
\sum_{\sigma \in S_{2m+2n-1}} sg(\tau_j(\sigma)) \frac{\partial F_{\mu_1}}{\partial z_{\tau_j(\sigma_1)}} \cdots \frac{\partial F_{\mu_{i-1}}}{\partial z_{\tau_j(\sigma_{i-1})}} \frac{\partial F_{\mu_{i+1}}}{\partial z_{\tau_j(\sigma_i)}} \cdots \frac{\partial F_{\mu_{2m+2n}}}{\partial z_{\tau_j(\sigma_{2m+2n-1})}}.
$$

It follows that $\dfrac{\partial}{\partial z_v}(det(\mathrm{D}F_\mu(i|j)))$

$$=\frac{\partial}{\partial z_v}\Big(\sum_{\sigma\in S_{2m+2n-1}} sg(\sigma)\frac{\partial F_{\mu 1}}{\partial z_{\tau_j(\sigma_1)}}\cdots\frac{\partial F_{\mu i-1}}{\partial z_{\tau_j(\sigma_{i-1})}}\frac{\partial F_{\mu i+1}}{\partial z_{\tau_j(\sigma_i)}}\cdots\frac{\partial F_{\mu 2m+2n}}{\partial z_{\tau_j(\sigma_{2m+2n-1})}}\Big)$$
$$=\sum_{\sigma\in S_{2m+2n-1}} sg(\sigma)\frac{\partial}{\partial z_v}\Big(\frac{\partial F_{\mu 1}}{\partial z_{\tau_j(\sigma_1)}}\cdots\frac{\partial F_{\mu i-1}}{\partial z_{\tau_j(\sigma_{i-1})}}\frac{\partial F_{\mu i+1}}{\partial z_{\tau_j(\sigma_i)}}\cdots\frac{\partial F_{\mu 2m+2n}}{\partial z_{\tau_j(\sigma_{2m+2n-1})}}\Big)$$
$$=\sum_{\sigma\in S_{2m+2n-1}} sg(\sigma)\left(\sum_{\substack{l\le 2m+2n\\ l\neq i}}\left(\frac{\partial^2 F_{\mu l}}{\partial z_{\tau_j(\sigma_{\alpha_i(l)})}\partial z_v}\prod_{\substack{p=1\\ p\neq i,l}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_i(p)})}}\right)\right).$$

Therefore

$$\mathrm{D}_z\left(\begin{pmatrix}(-1)^{i+1}det(\mathrm{D}F_\mu(i|1))\\ \vdots\\ (-1)^{i}det(\mathrm{D}F_\mu(i|2n+2m))\end{pmatrix}\right)=\bar{\Delta}$$

where $\bar{\Delta}=(\bar{\delta}_{jk})$ is such that

$$\bar{\delta}_{jk}=(-1)^{i+j}\sum_{\sigma\in S_{2m+2n-1}} sg(\tau_j(\sigma))\left(\sum_{\substack{l\le 2m+2n\\ l\neq i}}\left(\frac{\partial^2 F_{\mu l}}{\partial z_{\tau_j(\sigma_{\alpha_i(l)})}\partial z_k}\prod_{\substack{p=1\\ p\neq i,l}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_i(p)})}}\right)\right)$$

Now, $\frac{\partial^2 F_{\mu_i}}{\partial z_j \partial z_k}=0$ for $1\le i\le n+m$. Therefore,

$$\bar{\delta}_{jk}=(-1)^{i+j}\sum_{\sigma\in S_{2m+2n-1}} sg(\tau_j(\sigma))\left(\sum_{\substack{l=n+m+1\\ l\neq i}}^{2n+2m}\left(\frac{\partial^2 F_{\mu l}}{\partial z_{\tau_j(\sigma_{\alpha_i(l)})}\partial z_k}\prod_{\substack{p=1\\ p\neq i,l}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_i(p)})}}\right)\right)$$

Also, from Proposition 2.1.2, $\det(\mathrm{D}F_\mu(z) = (-1)^m|\det(\mathrm{D}F_\mu(z))|$.

It follows that $\mathrm{D}\Phi_{A,\mu}(z)=(-1)|\det(\mathrm{D}F_\mu(z))|I+(-1)^{m+1}\Delta$ where $\Delta=(\delta_{jk})$

is such that $\delta_{jk} =$

$$F_{\mu_1}(z)(-1)^{1+j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{l=n+m+1}^{2n+2m}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_{\alpha_1(l)})}\partial z_k}\prod_{\substack{p=2\\ p\neq l}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_1(p)})}}\right)\right]$$

$$+$$

$$F_{\mu_2}(z)(-1)^{2+j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{l=n+m+1}^{2n+2m}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_{\alpha_2(l)})}\partial z_k}\prod_{\substack{p=1\\ p\neq l,2}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_2(p)})}}\right)\right]$$

$$\vdots$$

$$+$$

$$F_{\mu_q}(z)(-1)^{q+j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{\substack{l=n+m+1\\ l\neq q}}^{2n+2m}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_{\alpha_q(l)})}\partial z_k}\prod_{\substack{p=1\\ p\neq l,q}}^{2m+2n}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_q(p)})}}\right)\right]$$

$$\vdots$$

$$+$$

$$F_{\mu_{2n+2m}}(z)(-1)^{j}\left[\sum_{\sigma\in S_\rho} sg(\tau_j(\sigma)) \sum_{l=n+m+1}^{2n+2m-1}\left(\frac{\partial^2 F_{\mu_l}}{\partial z_{\tau_j(\sigma_l)}\partial z_k}\prod_{\substack{p=1\\ p\neq l}}^{2m+2n-1}\frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_p)}}\right)\right]$$

for $S_\rho = S_{2n+2m-1}$.

In the following proposition we have $x_i = r_i = 0$ for some i. An analogous result holds for $y_i = s_i = 0$.

Proposition 7.5. *Suppose that* $\mu = 0, I_0 \subset \{1,\ldots,2n+2m\}, \#(I_0) \geq 2, z \in \mathcal{F}(I_0)$, *and there exists* i, j, k *such that* $j, k \in I_0, (z_j, z_k) = (r_i, x_i)$. *Then either* $\mathrm{D}\Phi_{A,\mu}(z)$ *has* $2n+2m$ *zero eigenvalues and is of rank 0 or it has exactly 2 nonzero eigenvalues,* λ_1, λ_2, *such that* $\lambda_1\lambda_2 < 0$ *and* $\mathrm{D}\Phi_{A,\mu}(z)e_j = \lambda_i e_j$ *for some* $j \in \{m+i, m+n+i\}$.

Proof: Suppose that $x_i = r_i = 0$ for some i. It follows that $(F_\mu)_{n+m+i}(z) = 0$ and $\frac{\partial}{\partial z_k}(F_\mu)_{n+m+i}(z) = 0$ for all k. Also,

$$\frac{\partial^2 (F_\mu)_{n+m+i}}{\partial z_k \partial z_j} = \begin{cases} 1 & z_k = x_i,\ z_j = r_i \text{ or } z_k = r_i, z_j = x_i, \\ 0 & \textit{else}. \end{cases}$$

Let Δ be as given in Theorem 7.3. It follows that $\delta_{jk} = 0$ for $k \neq m+i, n+m+i$. Note also that $det(\mathrm{D}F_\mu(z)) = 0$. Hence, $\mathrm{D}\Phi_{A,\mu}(z)$ has at most 2 non-zero columns. Now, in the expansion of Δ, if $l \neq n+m+i$, the corresponding term of the summation is zero as $\frac{\partial}{\partial z_k} F_{\mu_{n+m+i}}(z) = 0$ for all k and $F_{\mu_{n+m+i}}(z) = 0$. Therefore, Δ simplifies so that $(\delta_{jk}) =$

$$F_{\mu_1}(z)(-1)^{1+j}\left[\sum_{\sigma \in S_{2m+2n-1}} sg(\tau_j(\sigma)) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_j(\sigma_{m+n+i-1})} \partial z_k} \prod_{\substack{p=2 \\ p \neq m+n+i}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_1(p)})}}\right]$$

$$+F_{\mu_2}(z)(-1)^{2+j}\left[\sum_{\sigma \in S_{2m+2n-1}} sg(\tau_j(\sigma)) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_j(\sigma_{m+n+i-1})} \partial z_k} \prod_{\substack{p=1 \\ p \neq m+n+i,2}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_2(p)})}}\right]$$

$$\vdots$$

$$+F_{\mu_q}(z)(-1)^{q+j}\left[\sum_{\sigma \in S_{2m+2n-1}} sg(\tau_j(\sigma)) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_j(\sigma_{\alpha_q(m+n+i)})} \partial z_k} \prod_{\substack{p=1 \\ p \neq m+n+i,q}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_{\alpha_q(p)})}}\right]$$

$$\vdots$$

$$+F_{\mu_{2n+2m}}(z)(-1)^{j}\left[\sum_{\sigma \in S_{2m+2n-1}} sg(\tau_j(\sigma)) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_j(\sigma_{m+n+i})} \partial z_k} \prod_{\substack{p=1 \\ p \neq m+n+i}}^{2m+2n-1} \frac{\partial F_{\mu_p}}{\partial z_{\tau_j(\sigma_p)}}\right].$$

Note,

$$\alpha_q(m+n+i) = \begin{cases} m+n+i-1 & \text{if } q < m+n+i, \\ m+n+i & \text{if } q > m+n+i. \end{cases}$$

We have four cases to consider. Suppose that $j = m+i, k = m+n+i$. It follows that

$$\frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_{m+i}(\sigma_{\alpha_q(m+n+i)})} \partial z_{m+n+i}} \neq 0 \Leftrightarrow \tau_{m+i}(\sigma_{\alpha_q(m+n+i)}) = m+i.$$

However, by the definition of $\tau_j, \tau_j(p) \neq j \forall p$. It follows that $\delta_{m+i,m+n+i} = 0$. Similarly, if $j = m+n+i, k = m+i$, then

$$\frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_{m+n+i}(\sigma_{\alpha_q(m+n+i)})} \partial z_{m+i}} \neq 0 \Leftrightarrow \tau_{m+n+i}(\sigma_{\alpha_q(m+n+i)}) = m+n+i.$$

Hence, $\delta_{m+n+i,m+i} = 0$.

Now suppose that $j = m+i, k = m+i$. As above,

$$\frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_{m+i}(\sigma_{\alpha_q(m+n+i)})} \partial z_{m+i}} \neq 0 \Leftrightarrow \tau_{m+i}(\sigma_{\alpha_q(m+n+i)}) = m+n+i.$$

For $q < m+n+i$,

$$\tau_{m+i}(\sigma_{\alpha_q(m+n+i)}) = m+n+i \Leftrightarrow \tau_{m+i}(\sigma_{m+n+i-1}) = m+n+i$$
$$\Leftrightarrow \sigma_{m+n+i-1} = m+n+i-1.$$

For $q > m+n+i$,

$$\tau_{m+i}(\sigma_{\alpha_q(m+n+i)}) = m+n+i \Leftrightarrow \tau_{m+i}(\sigma_{m+n+i}) = m+n+i$$
$$\Leftrightarrow \sigma_{m+n+i} = m+n+i-1.$$

Recall that $\tau_j(\sigma_{2n+2m})$ was previously undefined. Now, since $j = m+i$, it follows that $\tau_j(\sigma)$ is a mapping from $\{1, \ldots, 2n+2m-1\}$ to the set $\{1, \ldots, m+i-1, m+i+1, \ldots 2n+2m\}$. Define $\tau_{m+i}(\sigma_{2n+2m}) = m+i$. It follows that $\tau_{m+i}(\sigma) \in S_{2m+2n}$. Now, from the definition of $\tau_j, sg(\tau_{m+i}(\sigma) = (-1)^{2n+m-i} sg(\sigma)$. It follows that $\delta_{m+i\ m+i} =$

$$F_{\mu_1}(z)(-1)^1 \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+n+i} \partial z_{m+i}} \prod_{\substack{p=2 \\ p \neq m+n+i}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{m+i}(\sigma_{\alpha_1(p)})}} \right]$$
$$+F_{\mu_2}(z)(-1)^2 \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+n+i} \partial z_{m+i}} \prod_{\substack{p=1 \\ p \neq m+n+i,2}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{m+i}(\sigma_{\alpha_2(p)})}} \right]$$
$$\vdots$$
$$+F_{\mu_q}(z)(-1)^q \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+n+i} \partial z_{m+i}} \prod_{\substack{p=1 \\ p \neq m+n+i,q}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{m+i}(\sigma_{\alpha_q(p)})}} \right]$$
$$\vdots$$
$$+F_{\mu_{2n+2m}}(z) \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+n+i} \partial z_{m+i}} \prod_{\substack{p=1 \\ p \neq m+n+i}}^{2m+2n-1} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{m+i}(\sigma_p)}} \right].$$

Now consider the case $j = m+n+i, k = m+n+i$.

$$\frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{\tau_{m+n+i}(\sigma_{\alpha_q(m+n+i)})}\partial z_{m+n+i}} \neq 0 \Leftrightarrow \tau_{m+n+i}(\sigma_{\alpha_q(m+n+i)}) = m+i.$$

For $q < m+n+i$,

$$\tau_{m+n+i}(\sigma_{\alpha_q(m+n+i)}) = m+i \Leftrightarrow \tau_{m+n+i}(\sigma_{m+n+i-1}) = m+i$$

$$\Leftrightarrow \sigma_{m+n+i-1} = m+i.$$

For $q > m+n+i$,

$$\tau_{m+n+i}(\sigma_{\alpha_q(m+n+i)}) = m+i \Leftrightarrow \tau_{m+n+i}(\sigma_{m+n+i}) = m+i$$

$$\Leftrightarrow \sigma_{m+n+i} = m+i.$$

Again, since $j = m+n+i$, it follows that $\tau_{m+n+i}(\sigma)$ is a mapping from $\{1, \ldots, 2n+2m-1\}$ to the set $\{1, \ldots, m+n+i-1, m+n+i+1, \ldots 2n+2m\}$. As above, define $\tau_{m+n+i}(\sigma_{2n+2m}) = m+n+i$. Hence, $\tau_{m+n+i}(\sigma) \in S_{2m+2n}$. Now, $sg(\tau_{n+m+i}(\sigma) = (-1)^{n+m-i} sg(\sigma)$. Therefore, $\delta_{n+m+i\ n+m+i} =$

$$F_{\mu_1}(z)(-1)^1 \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+i} \partial z_{m+n+i}} \prod_{\substack{p=2 \\ p \neq m+n+i}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{m+n+i}(\sigma_{\alpha_1(p)})}} \right]$$

$$+F_{\mu_2}(z)(-1)^2 \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+i} \partial z_{m+n+i}} \prod_{\substack{p=1 \\ p \neq m+n+i,2}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{n+m+i}(\sigma_{\alpha_2(p)})}} \right]$$

$$\vdots$$

$$+F_{\mu_q}(z)(-1)^q \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+i} \partial z_{m+n+i}} \prod_{\substack{p=1 \\ p \neq m+n+i,q}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{n+m+i}(\sigma_{\alpha_q(p)})}} \right]$$

$$\vdots$$

$$+F_{\mu_{2n+2m}}(z) \left[\sum_{\sigma \in S_{2m+2n-1}} sg(\sigma) \frac{\partial^2 F_{\mu_{m+n+i}}}{\partial z_{m+i} \partial z_{m+n+i}} \prod_{\substack{p=1 \\ p \neq m+n+i}}^{2m+2n-1} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{n+m+i}(\sigma_p)}} \right].$$

Now, for any $\sigma \in S_{2n+2m-1}$,

$$\prod_{\substack{p=1 \\ p \neq m+n+i,q}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{m+i}(\sigma_{\alpha_q(p)})}} = \prod_{\substack{p=1 \\ p \neq m+n+i,q}}^{2m+2n} \frac{\partial F_{\mu_p}}{\partial z_{\tau_{n+m+i}((m+i\ m+n+i)\sigma_{\alpha_q(p)})}}$$

where

$$sg(\tau_{m+i}(\sigma_{\alpha_q(p)}))sg(\tau_{n+m+i}((m+i\ m+n+i)\sigma_{\alpha_q(p)})) = -1.$$

Since this is true for every term it follows that

$$\delta_{m+i\ m+i}\delta_{m+n+i\ m+n+i} \leq 0.$$

If $\delta_{m+i\ m+i} \neq 0, \delta_{m+i\ m+i} = (-1)\delta_{m+n+i\ m+n+i}$ and $\mathrm{D}\Phi_{A,\mu}(z)$ has exactly two nonzero eigenvalues $\delta_{m+i\ m+i}, \delta_{m+n+i\ m+n+i}$ corresponding to eigenvectors e_{m+i}, e_{m+n+i}. If, in fact, $\delta_{m+i\ m+i} = 0$, then $\delta_{kk} = 0 \forall k$, and $rank(\mathrm{D}\Phi_{A,\mu}(z)) = 0$ and $\mathrm{D}\Phi_{A,\mu}(z)$ has $2n+2m$ zero eigenvalues.

Given the above theorem, it follows that points z such that $x_i = r_i = 0$ or $y_j = s_j = 0$ are non-hyperbolic fixed points.

Suppose that $I_0 = \{1, \ldots, 2n+2m\}, \#(I_0) = 1$, and z^0 is in the relative interior of $\mathcal{F}(I_0)$. We will now show that using the results of Chapter 5, we can establish the boundedness of $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus Crit(g_{F_\mu}))$ under conditions less restrictive than those in Proposition 7.3. The method used here is important as this type of argument will be used in future work regarding the critical trajectories.

Proposition 7.6. *Let $\mu = 0, I_0 = \{1, \ldots, 2n+2m\}, \#(I_0) = 1$, z^0 is in the relative interior of $\mathcal{F}(I_0)$ and there exists $z^* \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}}$ such that $F_0(z^*) = 0$. Then $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus Crit(g_{F_\mu}))$ is bounded.*

Proof: Suppose that z^0 is as given. Suppose (WLOG) that there exists $i \in \{1, \ldots, n\}$ for which $x_i^0 = 0$. Hence, $y_j^0 s_j^0 > 0\ \forall j, x_l^0 r_l^0 > 0\ \forall l \neq i$ and clearly $x_i^0 r_i^0 = 0$. Now from Proposition 3.15, for any $\bar{z} \in C_\mu^+(z^0)$, there exists some $\bar{k} > 0$ such that

$$x_l^0 r_l^0 = \bar{k}(\bar{x}_l \bar{r}_l)$$

$$y_j^0 s_j^0 = \bar{k}(\bar{y}_j \bar{s}_j)$$

for all l, j. It follows that $\bar{y}_j > 0, \bar{s}_j > 0\ \forall j, \bar{x}_l > 0, \bar{r}_l > 0\ \forall l \neq i$ and at least one of $\bar{x}_i, \bar{r}_i$equals 0. Now consider the proof of Lemma 6.2.13. The proof holds, for $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus Crit(g_{F_\mu}))$, up to the point where the value of ζ is chosen. Now choose ζ such that $0 < \zeta < \min\{z_k^0 : k \notin I_0\}$. Now, using the notation of the proof of Lemma 6.2.13,

$$
\begin{aligned}
&((1-e^{-t})x^* + e^{-t}x(0))^t r(t) + ((1-e^{-t})r^* + e^{-t}r(0))^t x(t) \\
&\quad + ((1-e^{-t})y^* + e^{-t}y(0))^t s(t) + ((1-e^{-t})s^* + e^{-t}s(0))^t y(t) \\
&\geq e^{-t}[x(0)^t r(t) + r(0)^t x(t) + y(0)^t s(t) + s(0)^t y(t)] \\
&\geq e^{-t}\zeta[\sum_{i=1}^{n} x_j(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} r_j(t) + \sum_{j=1}^{m} y_j(t) + \sum_{j=1}^{m} s_j(t)] \geq 0.
\end{aligned}
$$

It follows that

$$0 \leq e^{-t}\zeta[\sum_{j=1}^{n} x_j(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} r_j(t) + \sum_{j=1}^{m} y_j(t) + \sum_{j=1}^{m} s_j(t)] \leq e^{-t}K$$

and therefore

$$0 \leq [\sum_{j=1}^{n} x_j(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} r_j(t) + \sum_{j=1}^{m} y_j(t) + \sum_{j=1}^{m} s_j(t)] \leq \frac{1}{\zeta}K$$

It follows that $z_j(t)$ is bounded for every j except possibly the component $r_i(t)$. Suppose that there exists some sequence $\{t_k\}$ for which $t_k \to \infty, r_i(t_k) > 0$ and $r_i(t_k) \to \infty$. Note, by Proposition 3.4, for $r_i(t_k) > 0, \mathrm{D}F_\mu(z(t_k))$ is of full rank. Define $b^0 = Ax^0 - s^0, c^0 = A^t y^0 + r^0$. Choose $\bar{M} > 0$ such that $\bar{M} > \{b_j^0, c_k^0, b_j, c_k\}$. Define a set $\Gamma_{\bar{M}}$ by

$$\Gamma_{\bar{M}} = \{z \in \bar{\mathbf{R}}_+^{\mathbf{2n+2m}} : -\bar{M} \leq A_j x - s_j \leq \bar{M}\ \forall j, -\bar{M} \leq A_k^t y + r_k \leq \bar{M}\ \forall k\}.$$

Note that $z^0 \in \Gamma_\mu$. Suppose that there exists some t_l for which $z(t_l) \notin \Gamma_\mu$. It follows that the trajectory $z(t)$ is outward transversal to $\partial\Gamma_\mu$ at some point $z(\hat{t})$.

If $r_i(\hat{t}) > 0, \mathrm{D}F_\mu(z(\hat{t}))$ is of full rank. It follows from Theorem 5.1.2, 5.1.4, 5.1.6, 5.1.8, and Corollaries 4.1.4, and 4.1.6 that $\Phi_{A,0}(z(\hat{t}))$ is inward transversal to $\partial\Gamma_\mu$ at $z(\hat{t})$. Hence it must be true that $r_i(\hat{t}) = 0$. But since $r(t_l) > 0$, there must exist some $\tilde{M} > \hat{M}$ for which $z(t)$ is outward transversal to $\partial\Gamma_{\tilde{M}}$ at a point $z(\tilde{t})$ for which $r_i(\tilde{t}) > 0$. Again we would have a contradiction. Hence, no such t_l exists. It follows that $z(t_k) \in \Gamma_{\tilde{M}}$ for some $\tilde{M} > \bar{M}$. Now since $A_i^t y(t) - r_i(t) \leq \tilde{M}$ and for any $j, y_j(t)$ is bounded for all $t \geq 0$, it follows that $r_i(t)$ is bounded for all $t \geq 0$ as well. Hence, for all $j, z_j(t)$ is bounded for all $t \geq 0$ and therefore $C_\mu^+(z^0) \cap (\bar{\mathbf{R}}_+^{\mathbf{2n+2m}} \setminus Crit(g_{F_\mu}))$ is bounded.

CHAPTER 8

RESULTS FOR STANDARD FORM

In this chapter we will show results for the standard form that are analogous to results for the symmetric form. Particular attention is paid to the differences in the inherent geometry of the different forms. One particular problem is the existence of the non-sign constrained variable y. This along with the fact that there is no complementarity condition for y will make it necessary to have some added conditions if we hope to obtain results similar to those for the symmetric form. Recall that the standard form of the linear programming problem is given by

$$\begin{aligned} &\max c^t x \qquad \text{subject to } Ax = b \\ (LP) \qquad &A \in \mathbf{R^{m\times n}}, m \le n, x \in \mathbf{R^n_{\geq 0}}. \end{aligned}$$

The function based on the optimality conditions is

$$F^s_\mu(r,x,y) = \begin{pmatrix} A^t y + r - c \\ Xr - \mu e \\ Ax - b \end{pmatrix}.$$

It follows that

$$\mathrm{D}F^s_\mu(r,x,y) = \begin{pmatrix} I & 0 & A^t \\ X & R & 0 \\ 0 & A & 0 \end{pmatrix}.$$

First we will address the work done in Chapter 3. Of particular interest is the measure of the $Crit(g_{F^s_\mu}(z)) \cap \partial\bar{\mathcal{R}}^{2n+m}_+$.

Proposition 8.1. *If* $(r,x,y) \in \mathcal{R}_+^{2n+m}$, *then* $det(\mathrm{D}F_\mu^s(r,x,y)) > 0$ *and it follows that* $(r,x,y) \in Reg(g_{F_\mu^s})$.

Proof: Proof is as in proposition 3.1.

Proposition 8.2. *Let* $\mu = 0$. *If* $(r,x,y) \in \mathcal{R}_+^{2n+m}$, $g_{F_\mu^s}(r,x,y)$ *is a regular value of* $g_{F_\mu^s}$.

Proof: Proof is as in proposition 3.17.

We now turn our attention to the case for which $\mu > 0$. For the symmetric form, the properties that were fundamental to the results were that for any point z in the relative interior of a face of codimension 1 in $\partial\bar{\mathcal{R}}_+^{2n+m}$, $z \in Reg(g_{F_\mu^s})$ and $g_{F_\mu^s}(z)$ is a regular value of $g_{F_\mu^s}$. Much of the work was based on the fact that for such faces $\mathrm{D}F_\mu^s(s,r,x,y)$ is of full rank. In fact, given the difficulty of verifying the rank-deficient regularity condition, having $Rank(\mathrm{D}F_\mu^s(z)) = 2n+m$ is key to having a convergence theorem for boundary points.

For $\partial\bar{\mathcal{R}}_+^{2n+m}$, faces of codimension 1 are given by $x_i = 0$ or $r_i = 0$. Suppose that z is in the relative interior of a face of codimension 1 given by $x_i = 0$. By doing elementary matrix operations on $\mathrm{D}F_\mu^s(z)$, we produce the matrix

$$\begin{pmatrix} I & 0 & A^t \\ 0 & I & -R^{-1}XA^t \\ 0 & 0 & AR^{-1}XA^t \end{pmatrix}.$$

Let v be an m-vector. Then $AR^{-1}XA^tv = 0 \Leftrightarrow A^tv = \alpha e_i$. Let a_j denote the jth column of A. It follows that $A^tv = \alpha e_i \Leftrightarrow a_j \perp v \forall j \neq i$. To guarantee the regularity of $\mathrm{D}F_\mu^s(z)$, we need to insure that for every $i, A^tv = \alpha e_i \Leftrightarrow v = 0 (\alpha = 0)$. Note, if $\{b_j\}_{j=1}^m$ is a collection of linearly independent m-vectors and $b_j \perp v = 0 \forall j$, then $v = 0$. With this in mind, we gave the assumption (1.35).

Proposition 8.3. *Suppose that (1.31), (1.35) hold. If $z = (r, x, y)$ is in the relative interior of a face of codimension 1 in $\partial\bar{\mathcal{R}}_+^{\mathbf{2n+m}}$, then $\mathrm{D}F_\mu^s(z)$ is of full rank and hence $z \in Reg(g_{F_\mu^s})$.*

Proof: First suppose that the face of codimension one is determined by $x_i = 0$. As above, we need only consider the rank of $AR^{-1}XA^t$. Now

$$AR^{-1}XA^t v = 0 \Leftrightarrow A^t v = \alpha e_i \Leftrightarrow a_j \perp v \forall j \neq i.$$

By (1.35), $m \leq n-1$, and there exists some collection of linear independent columns $\{a_j\}_{j \in J}$, such that $i \notin J, \#(J) = m$. It follows that $v = 0$. Hence $AR^{-1}XA^t, \mathrm{D}F_\mu^s(z)$ are of full rank.

Now suppose that the face of codimension 1 is determined by $r_i = 0$. By doing elementary matrix operations on $\mathrm{D}F_\mu^s(z)$, we form the matrix

$$\begin{pmatrix} I & 0 & A^t \\ 0 & RX^{-1} & A^t \\ 0 & A & 0 \end{pmatrix}.$$

Suppose that there exists vectors v_1, v_2 such that

$$\begin{pmatrix} RX^{-1} & A^t \\ A & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that $0 = v_1^t(RX^{-1}v_1 + A^t v_2) = v_1^t RX^{-1}$. Hence $v^t RX^{-1} v_1 = 0$ and $v_1 = \alpha e_i$. Now,

$$0 = Av_1 = A(\alpha e_i) = \alpha A e_i = \alpha a_i \Leftrightarrow \alpha = 0.$$

Hence $v_1 = 0$. Also, as above, $A^t v_2 = 0 \Leftrightarrow v_2 = 0$. It follows that $\mathrm{D}F_\mu^s(z)$ is of full rank.

Proposition 8.4. *Suppose that (1.31), (1.35) hold and $\mu > 0$. If $z = (r, x, y)$ is in the relative interior of a face of codimension 2 in $\partial\bar{\mathcal{R}}_+^{2n+m}$ such that $x_i = r_i = 0$, then $z \in Reg(g_{F_\mu^s})$.*

Proof: The proof uses the framework used in Propositions 3.5, 3.6 and the properties of $AR^{-1}XA^t$ identified in Proposition 8.3.

We now turn our attention to the transversality of our solutions curves.

Proposition 8.5. *Let $\mu = 0$ and $z^0 \in \partial\bar{\mathcal{R}}_+^{2n+m}$. Then $\Phi_{A,\mu}(z^0)$ is not outward (inward) transversal to $\partial\bar{\mathcal{R}}_+^{2n+m}$.*

Proof: Given Proposition 4.1.3, we need only show that if z is in the relative interior of a face of codimension 1, then $\Phi_{A,\mu}(z)$ is not outward (inward) transversal to the face. We have only two types of faces of codimension 1. Those for which some $x_i = 0$ and those for which some $r_i = 0$.

Suppose that the face of codimension 1 is determined by $x_i = 0$ for some i. Set $l = n + i$. In this case the inward normal vector is $v^t = [0, e_i, 0]$. Therefore,

$$v^t\Phi_{A,\mu}(z) = [0, e_i, 0](-1)adj(\mathrm{D}F_\mu^s(z))F_\mu^s(z) = (-1)(lth\ row\ of\ adj(\mathrm{D}F_\mu^s(z)))F_\mu^s(z).$$

Now, the lth row of $adj(\mathrm{D}F_\mu^s(z))$ is all zeros except for possibly the lth component. It follows that

$$v^t\Phi_{A,\mu}(z) = (-1)det(\mathrm{D}F_\mu^s(l|l)(z))(F_\mu^s)_l(z))$$

where $(F_\mu^s)_l(z)$ is the lth component of $F_\mu^s(z)$. But $x_i = 0 = \mu$, and therefore $(F_\mu^s)_l(z) = 0$. Hence $v^t\Phi_{A,\mu}(z) = 0$. It follows that either $\Phi_{A,\mu}(z) = 0$ or $\Phi_{A,\mu}(z)$ is parallel to the face of codimension 1 given by $x_i = 0$.

A similar argument holds for faces of codimension 1 given by $r_i = 0$.

Proposition 8.6. *Suppose that (1.31), (1.35) hold and $\mu > 0$. There are no points $z \in \partial\bar{\mathcal{R}}_+^{2n+m}$ for which $\Phi_{A,\mu}(z)$ is outward transversal to $\partial\bar{\mathcal{R}}_+^{2n+m}$.*

Proof: The proof follows that proof of Proposition 4.1.5.

We now give the construction of the closed half-spaces that are used for bounding the trajectories $z(t)$.

Definition 8.7. *Let $i \in \{1, \ldots, n\}$. A* **Type 1** *closed half-space is defined as*

$$H^+ = \{(r, x, y) | A_i x \leq \hat{M}\}$$

for some $\hat{M} > b_i$. A **Type 3** *closed half-space is defined as*

$$H^+ = \{(r, x, y) | -A_i x \leq -\hat{M}\}$$

for some $\hat{M} < b_i$. Let $j \in \{1, \ldots, m\}$. A **Type 2** *closed half-space is defined as*

$$H^+ = \{(r, x, y) | (A^t y)_j + r_j \leq \hat{M}$$

for some $\hat{M} > c_j$. A **Type 4** *closed half-space is defined as*

$$H^+ = \{(r, x, y) | -(A^t y)_j - r_j \leq -\hat{M}$$

for some $\hat{M} < c_j$. Given $i \in \{1, \ldots, n\}$ and a point

$$z_0 \in \{(r, x, y) | (r, x, y) \in \mathcal{R}_+^{2n+m}, x_i r_i = \hat{M} > \mu\},$$

the closed half-space

$$\{z | [x_i^0 e_i, r_i^0 e_i, 0]^t z \leq 2 x_i^0 r_i^0\}$$

is called a **type 5** *closed half-space. Given $i \in \{1, \ldots, n\}$ and a point*

$$z_0 \in \{(r, x, y) | (r, x, y) \in \mathcal{R}_+^{2n+m}, x_i r_i = \hat{M} < \mu\},$$

the closed half-space

$$\{z|[x_i^0 e_i, r_i^0 e_i, 0]^t z \geq 2x_i^0 r_i^0\}$$

is called a **type 6** *closed half-space.*

Based on the above definitions we have the following propositions that are crucial to the behavior of the trajectories.

Proposition 8.8. *Let $\{H_j^+\}_{j=1}^M$ a collection of type 1-6 closed half-spaces. Let $\Pi = \cap_j H_j^+$. Suppose that $z^0 \in \mathcal{R}_+^{2n+m} \cap \partial\Pi$. Given (1.31), $\Phi_{A,\mu}(z^0)$ is inward transversal to $\partial\Pi$ at z^0.*

Proof: The proof follows the framework of the proof for Proposition 5.3.1.

Proposition 8.9. *Let $\{H_j^+\}_{j=1}^M$ be a collection of type 1-5 closed half-spaces. Let $\Pi = \cap_j H_j^+$. Suppose $z^0 \in \partial\bar{\mathcal{R}}_+^{2n+m} \cap \partial\Pi$, given (1.29)-(1.31), $\Phi_{A,\mu}(z^0)$ is not outward transversal to $\partial\Pi$ at z^0.*

Proof: The proof follows the framework of the Theorem 5.3.2 proof.

At this point, we have identified the distinguishing characteristics of the standard form of (LP) verses the symmetric form. In fact, the results given in Chapter 6 are not based on the particular form used. Therefore, given the above theorems, and the work done in Chapter 6, the proofs of theorems 1.34 and 1.36 follow the proofs of theorems 1.23 and 1.24.

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APPENDIX A

DIFFERENTIAL EQUATIONS

Recall the following definitions from differential equations.

Definition A.1. *For the differential equation $\frac{dz}{dt} = \bar{F}(z(t))$, a point p is called a* **fixed point** *if $\bar{F}(p) = 0$.*

Definition A.2. *A fixed point p is called* **hyperbolic** *if all the eigenvalues of $D\bar{F}(p)$ have non-zero real parts.*

Definition A.3. *Suppose p is a hyperbolic fixed point of $\bar{F}(x)$. We say p is a* **sink** *if the real parts of all eigenvalues are negative. p is called a* **source** *if the real parts of all eigenvalues are positive. p is called a* **saddle** *if it is neither a sink nor a source.*

The following two known theorems from Differential Equations will be of importance in this work. From Robinson [R] we have,

Theorem A.4. *(Existence and Uniqueness of Differential Equations) Let $U \subset \mathbf{R^n}$ be an open set and $f : U \to \mathbf{R^n}$ be a Lipschitz or C^1 function. Let $z_0 \in U$ and $t_0 \in \mathbf{R}$. Then there exists $\alpha > 0$ and a solution, $z(t)$ of $\frac{dz}{dt} = f(z(t))$ defined for $t_0 - \alpha < t < t_0 + \alpha$ such that $z(t_0) = z_0$. Also if $y(t)$ is another solution with $y(t_0) = z_0$ then $z(t) = y(t)$ on a common interval of definition about t_0.*

From [HSm1] we have,

Theorem A.5. *(Continuity of Solutions with Respect to Initial Conditions) Let $U \subset \mathbf{R^n}$ be an open set and $f : U \to \mathbf{R^n}$ be a C^1 function. Let $z_0 \in U$ and $t_0 \in \mathbf{R}$. Let $z(t)$ be a solution of $\frac{dz}{dt} = f(z(t))$ defined on the closed interval $[t_0, t_1]$ with $z(t_0) = z_0$. Then there exists a neighborhood $V \subset U$ of z_0 and a constant κ such that if $\hat{z}_0 \in V$, then there is a unique solution $\hat{z}(t)$ also defined on $[t_0, t_1]$ with $\hat{z}(t_0) = z_0$; and $\hat{z}$ satisfies*

$$\|\hat{z}(t) - z(t)\| \leq \|\hat{z}_0 - z_0\| e^{\kappa(t-t_0)}$$

for all $t \in [t_0, t_1]$.

Given that we are going to use an IVP with vector field $\Phi(z)$ to find a solution to $f(z) = 0$ for $f : \mathbf{R^n} \to \mathbf{R^n}$, we need to be able to analyze $\Phi(z)$. One important question is that of the structure of the Jacobian, $D\Phi(z)$, of $\Phi(z)$. Since in some cases $\Phi(z)$ involves $Df(z)$ we must address the issue of finding the second derivative of $f : \mathbf{R^n} \to \mathbf{R^n}$.

Given $f : \mathbf{R^n} \to \mathbf{R^n}$, $D_z(f(z)) = Df(z) \in \mathbf{R^{n\times n}}$ where D_z is the differential operator. Hence $Df(z) \in L(\mathbf{R^n}, \mathbf{R^n})$. It follows that the second derivative of $f(z)$, $D_z(Df(z))$, is an element of $L(\mathbf{R^n}, L(\mathbf{R^n}, \mathbf{R^n}))$. Hence,

$$D_z(Df(z)) \in L(\mathbf{R^n}, L(\mathbf{R^n}, \mathbf{R^n})) \cong L(\mathbf{R^n} \otimes \mathbf{R^n}, \mathbf{R^n}).$$

So for $u \in \mathbf{R^n}, D_z(Df(z))u \in L(\mathbf{R^n}, \mathbf{R^n}) = \mathbf{R^{n\times n}}$. It follows that $[D_z(Df(z))u]v \in \mathbf{R^n}$ for $u, v \in \mathbf{R^n}$.

The question that remains is that of the structure of $D_z(Df(z))$. Suppose that

$$f = \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(x) \end{pmatrix}$$

where $f_i(z) \in \mathbf{C^2}$ for every i. Then

$$Df(z) = \begin{pmatrix} \frac{\partial f_1(z)}{\partial z_1} & \cdots & \cdots & \frac{\partial f_1(z)}{\partial z_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n(z)}{\partial z_1} & \cdots & \cdots & \frac{\partial f_n(z)}{\partial z_n} \end{pmatrix}.$$

Now consider $f_i(z) : \mathbf{R^n} \to \mathbf{R}$. Then

$$D_z(f_i(z)) = \nabla f_i(z) = \begin{pmatrix} \frac{\partial f_i(z)}{\partial z_1} \\ \vdots \\ \frac{\partial f_i(z)}{\partial z_n} \end{pmatrix}.$$

Therefore,

$$D_z(Df_i(z)) = \begin{pmatrix} \frac{\partial^2 f_i(z)}{\partial z_1 \partial z_1} & \cdots & \cdots & \frac{\partial^2 f_i(z)}{\partial z_n \partial z_1} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f_i(z)}{\partial z_1 \partial z_n} & \cdots & \cdots & \frac{\partial^2 f_i(z)}{\partial z_n \partial z_n} \end{pmatrix} = H_{f_i(z)}$$

where $H_{f_i(z)}$ is the Hessian of $f_i(z)$. Since f_i was a component of f, it follows that

$$[D_z(Df(z))u]v = \begin{pmatrix} v^t H_{f_1(z)} u \\ \vdots \\ v^t H_{f_n(z)} u \end{pmatrix} \in \mathbf{R^n}$$

where $H_{f_i(z)}$ is the Hessian matrix of $f_i(z)$. It follows that to understand the structure of $D_z(Df(z))$ we need only understand the structure of $H_{f_i(z)}$ for every i.

APPENDIX B

DUAL-SYMMETRIC FORM OF LP

It is shown here that every linear programming problem given in the symmetric-dual form (LP), (DP) can be given in the form of (LP) with the added condition that $m \leq n$. This is done by simply multiplying by (-1) the appropriate equations.

Suppose that we have,

$$\min c^t x \qquad \text{subject to } Ax - s = b$$

$$(LP) \qquad A \in \mathbf{R^{m\times n}},\ m > n,\ x \in \bar{\mathbf{R}}^{\mathbf{n}}_{+}, s \in \bar{\mathbf{R}}^{\mathbf{m}}_{+},$$

and

$$\max b^t y \qquad \text{subject to } A^t y + r = c$$

$$(DP) \qquad A^t \in \mathbf{R^{n\times m}},\ y \in \bar{\mathbf{R}}^{\mathbf{m}}_{+}, r \in \bar{\mathbf{R}}^{\mathbf{n}}_{+},$$

Now $Ax - s = b \Leftrightarrow (-A)x + s = (-b)$. Set $B^t = -A$. Also,

$$\min c^t x \qquad \text{subject to } \ B^t x + s = -b,$$

$$B^t \in \mathbf{R^{m\times n}}, m > n,\ x \in \bar{\mathbf{R}}^{\mathbf{n}}_{+}, s \in \bar{\mathbf{R}}^{\mathbf{m}}_{+},$$

is equivalent to

$$\max\ (-c^t)x \qquad \text{subject to } \ B^t x + s = -b,$$

$$B^t \in \mathbf{R^{m\times n}},\ m > n,\ x \in \bar{\mathbf{R}}^{\mathbf{n}}_{+}, s \in \bar{\mathbf{R}}^{\mathbf{m}}_{+},$$

If we consider this the dual format, then the corresponding primal form is

$$\min\ (-b^t)y \qquad \text{subject to } \ By - r = (-c),$$

$$B \in \mathbf{R^{n\times m}}, n < m,\ y \in \bar{\mathbf{R}}^{\mathbf{m}}_{+}, r \in \bar{\mathbf{R}}^{\mathbf{n}}_{+},$$

Hence we need only consider (LP) with the additional condition that $m \leq n$.

VITA

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