MONOMIAL RESOLUTIONS

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CHAPTER 1

INTRODUCTION

1.1 Overview

In this thesis we introduce and explore the minimal free resolutions of dominant, 1-semidominant and 2-semidominant ideals, three families of monomial ideals that are easy to describe and have strong combinatorial properties.

For over half a century mathematicians have tried to obtain the minimal resolutions of families of ideals in closed form with little success. A common mark in the construction of these classes of ideals and their corresponding resolutions has been the use of a monomial ordering or, at least, an ordering of the variables. Groebner bases, mapping cones, Borel ideals and the (usually nonminimal) Lyubeznik resolution [No,Pe,Me] are some examples of this phenomenon.

Dominant, 1-semidominant and 2-semidominant ideals, as well as the technique that resolves them minimally, are distinguished from the objects mentioned above in that they do not require an ordering of the variables; instead, they are characterized by the exponents with which the variables appear in the factorization of the monomial generators. The concept of dominance resembles the definition of generic ideal [BPS,BS] as we will explain in Section 2.2.

We will show that the minimal free resolutions of these classes of ideals have some important properties. In particular, the Taylor resolution of a monomial ideal is minimal if and only if the ideal is dominant. In other words, dominant ideals give a full and explicit characterization of when the Taylor resolution is minimal.

The minimal resolutions of 1-semidominant ideals are also remarkably simple;

they are given by the Scarf complex. Thus it would be fair to say that we know everything about them. Although not as easy to decode as in the first two cases, the minimal resolutions of 2-semidominant ideals can also be expressed in simple terms: informally speaking, they can be obtained from their Taylor resolutions eliminating pairs of face and facet of equal multidegree in arbitrary order, until exhausting all possibilities.

The concepts of dominant and 1-semidominant ideal extend those of complete and almost complete intersection in a natural way, and the transition from dominant to 1-semidominant ideal is smooth. The latter definition is obtained from the former via a minor modification. However, the combinatorial properties of dominant and 1-semidominant ideals can be radically different. For instance, in Section 2.3 we give a condition under which a dominant ideal and a 1-semidominant ideal (that look almost identical) have the largest and smallest possible projective dimensions, respectively.

That is why in Chapter 3 se introduce a class of monomial ideals, called 1-cancellations, whose combinatorial properties resemble those of dominant ideals. In the second part of Chapter 3 we focus our attention toward a particular subfamily of 1-cancellations and use it to give a partial answer to three open problems that appear in a paper of Peeva-Stillman.

1.2 Background and Notation

Throughout, the letter S denotes a polynomial ring in the variables $x_1, \ldots, x_n$, over a field k; that is, $S = k[x_1, \ldots, x_n]$. An expression of the form $x_1^{c_1}, \ldots, x_n^{c_n}$, $c_i \geq 0$, is referred to as a **monomial** in S (note that the multiplicative identity 1 is viewed as a monomial). A **monomial ideal** in S is an ideal generated by monomials. It is a corollary to Hilbert's Basis Theorem that monomial ideals are finitely generated. Moreover, monomial ideals are finitely generated by monomials. Thus, if M is a monomial ideal in S, it can be represented in the form $M = (m_1, \ldots, m_q)$, where each

m_i is a monomial.

1.3 Graded Modules

Definition 1.1 *Given a semigroup $(H, *)$, we say that S is **graded** (with respect to $(H, *)$) if there are k-vector spaces S_h, $h \in H$, such that*

(i) $S = \bigoplus_{h \in H} S_h$ *as a k-vector space.*

(ii) $S_h S_{h'} \subseteq S_{h*h'}$, *for all $h, h' \in H$.*

*An element $l \in S$ is called **homogeneous** if $l \in S_h$ for some $h \in H$.*

Definition 1.2 *Given a semigroup $(H, *)$, we say that an S-module M is graded if there exist k-vector spaces M_h, $h \in H$, such that*

(i) $M = \bigoplus_{h \in H} M_h$ *as a k-vector space,*

(ii) $S_h M_{h'} \subseteq M_{h*h'}$ *for all $h, h' \in H$.*

*An element $m \in M$ is called **homogeneous** if $m \in M_h$ for some $h \in H$.*

1.4 Standard Graded Modules

Below, we introduce a grading that will be used often. Consider the semigroup $(\mathbb{N}_0, +)$ of the nonnegative integers under ordinary addition. For every monomial in S define its degree by $\deg(x_1^{c_1} \ldots x_n^{c_n}) = c_1 + \ldots + c_n$. For every $i \in \mathbb{N}_0$, let S_i be the k-vector space spanned by all monomials of degree i. An element $l \in S$ is said to have degree i (that is, $\deg l = i$) if $l \in S_i$. Under these conditions, S is graded. This grading will be called the **standard grading** of the polynomial ring S.

Example 1.1 *Let M be a monomial ideal and $(\mathbb{N}_0, +)$ the semigroup of nonnegative integers with addition. We will define a grading on the S-modules M and S/M and call it the **standard grading** of M and S/M, respectively.*

For every $i \in \mathbb{N}_0$, let M_i be the k-vector space spanned by all monomials in M of degree i. An element $l \in M$ is said to have degree i (that is, $\deg l = i$) if $l \in M_i$, Under these conditions, M is graded. Now, the quotient S-module S/M inherits a grading via $S/M = \bigoplus_{i \in \mathbb{N}_0} (S/M)_i$, where $(S/M)_i = S_i/M_i$. The elements of $(S/M)_i$ are called homogeneous, of degree i.

Example 1.2 *Once again, we consider the semigroup* $(\mathbb{N}_0, +)$. *Let* $\sigma = \{m_1, \ldots, m_s\}$, *where* $m_1, \ldots, m_s$ *are monomials. Let* $[\sigma]$ *be a formal symbol. We define the degree of* $[\sigma]$ *as* $\deg[\sigma] = \deg(\text{lcm}(m_1, \ldots, m_s))$. *Let* $S[\sigma]$ *be the free* S*-module spanned by* $[\sigma]$. *We will make* $S[\sigma]$ *into a graded free* S*-module as follows. To simplify our notation, let us say that* $\deg[\sigma] = t$. *For every monomial* $m \in S$, *set* $\deg(m[\sigma]) = \deg(m) + t$. *Now, define* $(S[\sigma])_i$ *to be the* k*-vector space spanned by all elements* $m[\sigma] \in S[\sigma]$ *such that* m *is a monomial and* $\deg(m[\sigma]) = i$. *That is,*

$$(S[\sigma])_i = \begin{cases} S_{i-t}[\sigma], \text{ if } i \geq t \\ 0, \text{ if } i < t \end{cases}$$

The elements of $(S[\sigma])_i$ *will be said to have degree* i. *Then*

$$S[\sigma] = \left(\bigoplus_{i \geq 0} S_i \right) [\sigma] = \bigoplus_{i \geq 0} S_i[\sigma] = \bigoplus_{i \geq 0} (S[\sigma])_{i+t} = \bigoplus_{i \geq 0} (S[\sigma])_i .$$

Likewise,

$$S_j (S[\sigma])_i = S_j S_{i-t}[\sigma] \subseteq S_{j+i-t}[\sigma] = (S[\sigma])_{i+j} .$$

Therefore, we have endowed $S[\sigma]$ *with a grading, which will be called the* ***standard grading*** *of* $S[\sigma]$. *More generally, if* $\sigma_i = \{m_{i_1}, \ldots, m_{i_s}\}$, *with* $i = 1, \ldots, r$, *are sets of* s *monomials, we define* r *formal objects* $[\sigma_1], \ldots, [\sigma_r]$, *and set* $\deg[\sigma_i] = \deg(\text{lcm}(m_{i_1}, \ldots, m_{i_s}))$. *Let* $\bigoplus_{i=1}^{r} S[\sigma_i]$ *be the free* S*-module generated by* $[\sigma_1], \ldots, [\sigma_r]$. *For every* $j \in \mathbb{N}_0$, *let* $\left(\bigoplus_{i=1}^{r} S[\sigma_i] \right)_j = \bigoplus_{i=1}^{r} (S[\sigma_i])_j$. *The elements of* $\left(\bigoplus_{i=1}^{r} S[\sigma_i] \right)_j$ *will be called homogeneous of degree* j. *It can be verified that* *(i)* $\bigoplus_{i=1}^{r} S[\sigma_i] = \bigoplus_{j\geq 0} \left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_j$;

(ii) $S_k \left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_j \subseteq \left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_{j+k}$.

Thus, we have endowed $\bigoplus_{i=1}^{r} S[\sigma_i]$ *with a grading, which will be called the standard grading of* $\bigoplus_{i=1}^{r} S[\sigma_i]$.

1.5 Multigraded Modules

Let $\mathscr{S}$ be the set of all monomials in S (recall that $1 \in S$ is viewed as a monomial). For each $m \in \mathscr{S}$, let S_m be the k-vector space spanned by m. Then

(i) $S = \bigoplus_{m\in\mathscr{S}} S_m$

(ii) $S_m S_{m'} \subseteq S_{mm'}$.

Thus, S is a graded polynomial ring with respect to the semigroup $(\mathscr{S}, .)$. This grading of S will be called **multigrading**.

Example 1.3 *Let* M *be a monomial ideal in* S*. Let* $\mathscr{M}$ *be the set of all monomials in* M*. For each* $m \in \mathscr{M}$*, let* M_m *be the* k*- vector space spanned by* m*. Then*

(i) $M = \bigoplus_{m\in\mathscr{M}} M_m$

(ii) $S_m M_{m'} \subseteq M_{mm'}$.

Thus, M is a graded module with respect to the semigroup $(\mathscr{M}, .)$, and we will say that M is a **multigraded** module. The elements of each space M_m will be said to have multidegree m. The quotient S-module S/M inherits a grading via $S/M = \bigoplus_{m\in\mathscr{M}} (S/M)_m$, where $(S/M)_m = S_m/M_m$.

Example 1.4 *Let* $\sigma = \{m_1, \ldots, m_s\}$*, where* $m_1, \ldots, m_s$ *are monomials. Let* $[\sigma]$ *be a formal symbol. We define the multidegree of* $[\sigma]$*, denoted* $\mathrm{mdeg}[\sigma]$*, as* $\mathrm{mdeg}[\sigma] = \mathrm{lcm}(m_1, \ldots, m_s)$.

Let $S[\sigma]$ be the free S-module generated by $[\sigma]$. Let $\mathscr{S}$ be the set of all monomials in S. For every $m \in \mathscr{S}$, set $\operatorname{mdeg}(m[\sigma]) = m. \operatorname{mdeg}[\sigma]$. To keep our notation simple, let $\operatorname{mdeg}[\sigma] = l$. Now define $(S[\sigma])_m$ to be the k-vector space

$$(S[\sigma])_M = \begin{cases} S_{ml^{-1}}[\sigma], \text{ if } l \mid m \\ 0, \text{ if } l \nmid m \end{cases}$$

(We will say that every element of a component $(S[\sigma])_m$ has multidegree m.) Then

$$S[\sigma] = \left(\bigoplus_{m \in \mathscr{S}} S_m\right)[\sigma] = \bigoplus_{m \in \mathscr{S}} S_m[\sigma] = \bigoplus_{m \in \mathscr{S}} (S[\sigma])_{ml} = \bigoplus_{m \in \mathscr{S}} (S[\sigma])_m .$$

Likewise, $S_m\,(S[\sigma])_{m'} = S_m S_{m'l^{-1}}[\sigma] = S_{mm'l^{-1}}[\sigma] = (S[\sigma])_{mm'}$.

Therefore, we have defined a grading on $S[\sigma]$, which will be called multigrading. More generally, if $\sigma_i = \{m_{i_1}, \ldots, m_{i_s}\}$, with $i = 1, \ldots, r$, are sets of s monomials, we define r formal objects $[\sigma_1], \ldots [\sigma_r]$ and set their multidegree to be $\operatorname{mdeg}[\sigma_i] = \operatorname{lcm}(m_{i_1}, \ldots, m_{i_s})$. To keep our notation simple, let $l_i = \operatorname{mdeg}[\sigma_i]$ $\forall i + 1, \ldots, r$. We define a multigrading on the free S-module $\bigoplus_{i=1}^{r} S[\sigma_i]$ in the same fashion we did before; that is

$$\left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_m = \bigoplus_{i=1}^{r} (S[\sigma_i])_m .$$

With this definition it can be verified that

(i) $\bigoplus_{i=1}^{r} S[\sigma_i] = \bigoplus_{m \in \mathscr{S}} \left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_m$

(ii) $S_m \left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_{m'} \subseteq \left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_{mm'}$.

This grading will be called the multigrading of the free S-module $\bigoplus_{i=1}^{r} S[\sigma_i]$.

1.6 Graded Free Resolutions

Definition 1.3 *Let $f : M \to N$ be a homomorphism between two S-modules that are graded with respect to the same semigroup $(H, *)$. We say that f is homogeneous if $f(M_h) \subseteq N_h$, for all $h \in H$.*

Definition 1.4 *Let M be a monomial ideal in S. A **graded free resolution** of the S-module S/M is an exact sequence of the form*

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0,$$

where each F_i is a free S-module, and the following properties hold:

(i) S/M and the F_i are graded with respect to a fixed semigroup $(H, *)$,

(ii) each f_i is homogeneous with respect to $(H, *)$.

*The maps f_i are called differential maps, and the matrices (f_i) associated to these maps are called differential matrices. If $[\sigma]$ is a basis element of the free S-module F_i, we say that $[\sigma]$ has **homological degree** i, which we denote $\operatorname{hdeg}[\sigma] = i$.*

Throughout this work, we are only interested in free resolutions that are graded with respect to the semigroup that defines either the standard grading or the multigrading. Thus, when we speak of the standard graded free resolution, we make reference to the first kind of resolution, while the expression multigraded free resolution is reserved for the second case.

1.7 Minimal Resolutions

Let M be a monomial ideal. Let

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} \frac{S}{M} \to 0$$

be a free resolution of S/M. $\mathbb{F}$ is said to be minimal if for every i, the differential matrix (f_i) of $\mathbb{F}$ has no invertible entries.

The idea behind the definition of minimal resolution is this: when one matrix of a resolution $\mathbb{F}$ has an invertible entry, $\mathbb{F}$ can be expressed as a direct sum of the form $\mathbb{F} = \mathbb{G} \oplus \left(0 \to S \xrightarrow{(1)} S \to 0\right)$, where $\mathbb{G}$ is also a resolution of S/M. Since $\mathbb{G}$ is "smaller" than $\mathbb{F}$, $\mathbb{F}$ is not minimal. We will say that $\mathbb{G}$ is obtained from $\mathbb{F}$ by means of a **consecutive cancellation**.

There are two main reasons minimal resolutions are important:

(i) Although there are many graded free resolutions for a given monomial ideal, the minimal resolution is unique up to isomorphism.

(ii) Minimal resolutions encode important information about a monomial ideal. For example, the Betti numbers, which we introduce in the next section, can be read off as the ranks of the free modules in a minimal resolution.

1.8 The Taylor Resolution

Let $M = (m_1, \ldots, m_q)$ be a monomial ideal. For every subset $\{m_{i_1}, \ldots, m_{i_s}\}$ of $\{m_1, \ldots, m_q\}$, with $1 \leq i_1 < \ldots < i_s \leq q$, we create a formal symbol $[m_{i_1}, \ldots, m_{i_s}]$, called a **Taylor symbol**. The Taylor symbol associated to $\{\}$ will be denoted by $[\varnothing]$. For each $s = 0, \ldots, q$, set F_s equal to the free S-module with basis $\{[m_{i_1}, \ldots, m_{i_s}] : 1 \leq i_1 < \ldots < i_s \leq q\}$ given by the $\binom{q}{s}$ Taylor symbols corresponding to subsets of size s. That is, $F_s = \bigoplus\limits_{i_1 < \ldots < i_s} S[m_{i_1}, \ldots, m_{i_s}]$ (note that $F_0 = S[\varnothing]$). Define

$$f_0 : F_0 \to S/M$$

$$s[\varnothing] \mapsto f_0(s[\varnothing]) = s$$

For $s = 1, \ldots, q$, let $f_s : F_s \to F_{s-1}$ be given by

$$f_s\left([m_{i_1}, \ldots, m_{i_s}]\right) = \sum_{j=1}^{s} \frac{(-1)^{j+1} \operatorname{lcm}(m_{i_1}, \ldots, m_{i_s})}{\operatorname{lcm}(m_{i_1}, \ldots, \widehat{m_{i_j}}, \ldots, m_{i_k})} [m_{i_1}, \ldots, \widehat{m_{i_j}}, \ldots, m_{i_k}]$$

and extended by linearity. The **Taylor resolution** $\mathbb{T}_M$ of S/M is the exact sequence

$$\mathbb{T}_M : 0 \to F_q \xrightarrow{f_q} F_{q-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} S/M \to 0.$$

It can be proven that $\mathbb{T}_M$ is a multigraded free resolution of S/M.

Remark 1.1 *Suppose that* $\mathbb{F}$ *is a multigraded free resolution of* S/M*. Let* $f_j : F_j \to F_{j-1}$ *be an arbitrary differential map of* $\mathbb{F}$*. Let* $F_j = \bigoplus_{i=1}^{r} S[\sigma_i]$*. Fix an arbitrary number* $k \geq 0$*. Let* $\sum_{t=1}^{r} m_t[\sigma_t]$ *be an arbitrary element of* $\left(\bigoplus_{i=1}^{r} S[\sigma_i]\right)_k$*. By definition,* $m_t[\sigma_t] \in (S[\sigma_t])_k$*, for all* $t = 1, \ldots, r$*. Let* $l_t = \text{mdeg}\,(m_t[\sigma_t])$*. This implies that* $\deg l_t = \deg(\text{mdeg}\,(m_t[\sigma_t])) = \deg(m_t[\sigma_t]) = k$*. Since* f_j *is homogeneous,* $\text{mdeg}\,(f_j\,(m_t[\sigma_t])) = \text{mdeg}\,(m_t[\sigma_t]) = l_t$*. Thus*

$$\deg(f_j\,(m_t[\sigma_t])) = \deg\{\text{mdeg}\,(f_j\,(m_t[\sigma_t]))\} = \deg\{\text{mdeg}\,(m_t[\sigma_t])\} = \deg l_t = k.$$

In other words, $f_j\,(m_t[\sigma_t]) \in (F_{j-1})_k$*. Hence,* $f_j\left(\sum_{t=1}^{r} m_t[\sigma_t]\right) \in (F_{j-1})_k$*, which implies that* $f_j\left((F_j)_k\right) \subseteq (F_{j-1})_k$*. It follows that* f_j *is homogeneous with respect to the standard grading of* $S/M, F_0, F_1, \ldots,$ *and hence,* $\mathbb{F}$ *is also a standard graded free resolution of* S/M*. This means that multigraded free resolutions are particular cases of standard graded free resolutions.*

Example 1.5 *Let* $M = (x^2y^2, xz, yz)$*. The Taylor resolution* $\mathbb{T}_M$ *of* S/M *is*

$$0 \to S[x^2y^2, xz, yz] \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ xy \end{pmatrix}} \begin{array}{c} S[x^2y^2, xz] \\ \oplus \\ S[x^2y^2, yz] \\ \oplus \\ S[xz, yz] \end{array} \xrightarrow{\begin{pmatrix} -z & -z & 0 \\ xy^2 & 0 & -y \\ 0 & xy^2 & x \end{pmatrix}} \begin{array}{c} S[x^2y^2] \\ \oplus \\ S[xz] \\ \oplus \\ S[yz] \end{array}$$

$$\xrightarrow{\begin{pmatrix} x^2y^2 & xz & yz \end{pmatrix}} S[\varnothing] \to S/M \to 0.$$

Notice that $\mathbb{T}_M$ *is not minimal because one of the differentials has invertible entries. In chapter 2 we will explain how to obtain a minimal resolution of* S/M *by making a consecutive cancellation to* $\mathbb{T}_M$ *but, for now, let us just accept that the following is a minimal resolution* $\mathbb{F}$ *of* S/M*:*

$$0 \to \begin{array}{c} S[x^2y^2, yz] \\ \oplus \\ S[xz, yz] \end{array} \xrightarrow{\begin{pmatrix} -z & 0 \\ 0 & -y \\ xy^2 & x \end{pmatrix}} \begin{array}{c} S[x^2y^2] \\ \oplus \\ S[xz] \\ \oplus \\ S[yz] \end{array} \xrightarrow{\begin{pmatrix} x^2y^2 & xz & yz \end{pmatrix}} S[\varnothing] \to S/M \to 0.$$

1.9 The Scarf Complex

Let $M = (m_1, \ldots, m_q)$ be a monomial ideal. Let $\mathbb{T}_M$ be the Taylor resolution of S/M, and let A be the set of Taylor symbols whose multidegrees are not common to other Taylor symbols; that is, a Taylor symbol $[\sigma]$ is in A if and only if $\text{mdeg}[\sigma] \neq \text{mdeg}[\sigma']$, for every Taylor symbol $[\sigma'] \neq [\sigma]$. For each $s = 0, \ldots, q$, set G_s equal to the free S-module with basis $\{[m_{i_1}, \ldots, m_{i_s}] \in A : 1 \leq i_1 < \ldots < i_s \leq q\}$. For each $s = 0, \ldots, q$, let $g_s = f_s \restriction_{G_s}$. It can be proven that the g_s are well defined (more precisely, that $g_s(G_s) \subseteq G_{s-1}$) and that

$$0 \to G_q \xrightarrow{g_q} G_{q-1} \to \cdots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} \frac{S}{M} \to 0$$

is a subcomplex of $\mathbb{T}_M$, which will be called the Scarf complex of S/M. Although the Scarf complex itself is a chain complex, it is not exact in general, and thus it is not generally a resolution of S/M. Those ideals M for which the Scarf complex of S/M is exact (and thus, a resolution of S/M) are called Scarf ideals. It can be proven that whenever the Scarf complex is a resolution, it is minimal.

Example 1.6 *Let $M = (x^2y^2, xz, yz)$. The following is the Scarf complex $\mathbb{S}$ of S/M*

which can be easily obtained from $\mathbb{T}_M$, given in Example 1.5.

$$0 \to S[xz,yz] \xrightarrow{\begin{pmatrix} 0 \\ -y \\ x \end{pmatrix}} \begin{matrix} S[x^2y^2] \\ \oplus \\ S[xz] \\ \oplus \\ S[yz] \end{matrix} \xrightarrow{\begin{pmatrix} x^2y^2 & xz & yz \end{pmatrix}} S[\varnothing] \to S/M \to 0.$$

Remark 1.2 *Notice that $\mathbb{S}$ in Example 1.6 is a proper subcomplex of the minimal resolution $\mathbb{F}$ of S/M, given in Example 1.5. Thus $\mathbb{S}$ is not even a free resolution of S/M. However, it can be proven that whenever the Scarf complex is a resolution of a monomial ideal, it is a minimal free resolution of it.*

1.10 Betti Numbers

Definition 1.5 *Let M be a monomial ideal, and let*

$$\mathbb{F} : \cdots \to F_i \xrightarrow{f_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} \frac{S}{M} \to 0$$

be a minimal multigraded free resolution of S/M. For every i, we define the i^{th} ***total Betti number*** *(or just the i^{th} Betti number) of S/M, denoted $\mathrm{b_i}(S/M)$ (or just $\mathrm{b_i}$), to be $\mathrm{b_i}(S/M) = \mathrm{rank}(F_i)$.*

For every $i, j \geq 0$, we define the **graded Betti number** $\mathrm{b_{i,j}}(S/M)$ of S/M, in homological degree i and internal degree j , as

$$\mathrm{b_{i,j}}(S/M) = \#\{\text{basis elements } [\sigma] \text{ of } F_i : \deg[\sigma] = j\}.$$

Let m be a monomial in S. For every i, we define the **multigraded Betti number** $\mathrm{b_{i,m}}(S/M)$ of S/M, as

$$\mathrm{b_{i,m}}(S/M) = \#\{\text{basis elements } [\sigma] \text{ of } F_i : \mathrm{mdeg}[\sigma] = m\}.$$

We define the **regularity** and **projective dimension** of S/M, denoted $\operatorname{reg}(S/M)$ and $\operatorname{pd}(S/M)$, respectively, to be

$$\operatorname{reg}(S/M) = \max\{r : \mathrm{b_{i,i+r}}(S/M) \neq 0, \text{ for some } i \geq 0\}.$$

$$\operatorname{pd}(S/M) = \max\{i : \mathrm{b_i}(S/M) \neq 0\}.$$

CHAPTER 2

MINIMAL RESOLUTIONS OF DOMINANT AND SEMIDOMINANT IDEALS

2.1 Foundational Results

The results in this section are foundational in character because they deal with the basic concepts of change of basis and consecutive cancellation, which are natural avenues leading to the minimal free resolution of a monomial ideal. Most of these results are known in some form to experts, yet we have decided to include statements with full proofs because the material is essential to the development of this thesis and, as far as we know, nobody has published these particular facts with careful explanations.

The reader will find that the underlying ideas have the strong familiar flavor of linear algebra.

Definition 2.1 *Let M be a monomial ideal and let*

$$0 \to F_q \xrightarrow{f_q} \cdots \to F_{j+2} \xrightarrow{f_{j+2}} F_{j+1} \xrightarrow{f_{j+1}} F_j \xrightarrow{f_j} F_{j-1} \to \cdots \to F_0 \to S/M \to 0$$

be a free resolution of S/M.

Let $U = \{[u_1], \cdots, [u_h]\}$ be a basis of F_{j+1} and let $V = \{[v_1], \cdots, [v_g]\}$ be a basis of

F_j. *Suppose a_{rs} is an invertible entry of the differential matrix*

$$(f_{j+1})_{U,V} = \begin{pmatrix} a_{11} & \cdots & a_{1s} & \cdots & a_{1h} \\ \vdots & & \vdots & & \vdots \\ a_{r1} & \cdots & a_{rs} & \cdots & a_{rh} \\ \vdots & & \vdots & & \vdots \\ a_{g1} & \cdots & a_{gs} & \cdots & a_{gh} \end{pmatrix}.$$

The change of basis $U' = \{[u_1]', \cdots, [u_h]'\}$, where $[u_s]' = [u_s]$ and $[u_i]' = [u_i] - \frac{a_{ri}}{a_{rs}}[u_s]$ for all $i \neq s$; and $V' = \{[v_1]', \ldots, [v_g]'\}$, where $[v_r]' = \sum_{i=1}^{g} a_{is}[v_i]$ and $[v_i]' = [v_i]$, for all $i \neq r$ will be called ***the standard change of basis*** *(around a_{rs}).*

Lemma 2.1 *With the notation used in Definition 2.1, if we make a standard change of basis around a_{rs}, the following properties hold:*

(i) $\mathrm{mdeg}[u_i]' = \mathrm{mdeg}[u_i]$, *for all* $i = 1, ..., h$; $\mathrm{mdeg}[v_i]' = \mathrm{mdeg}[v_i]$, *for all* $i = 1, ..., g$.

(ii) The differential matrix $(f_{j+1})_{U',V'}$ is of the form

$$(f_{j+1})_{U',V'} = \begin{pmatrix} b_{1,1} & \ldots & b_{1,s-1} & 0 & b_{1,s+1} & \ldots & b_{1,h} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{r-1,1} & \ldots & b_{r-1,s-1} & 0 & b_{r-1,s+1} & \ldots & b_{r-1,h} \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ b_{r+1,1} & \ldots & b_{r+1,s-1} & 0 & b_{r+1,s+1} & \ldots & b_{r+1,h} \\ \vdots & & \vdots & \vdots & & & \vdots \\ b_{g,1} & \ldots & b_{g,s-1} & 0 & b_{g,s+1} & \ldots & b_{g,h} \end{pmatrix}.$$

(iii) Let $1 \leq c \leq g$ and $1 \leq d \leq h$. If $c \neq r$ and $d \neq s$, then $b_{cd} = a_{cd} - \frac{a_{rd}a_{cs}}{a_{rs}}$.

(iv) The differential matrix $(f_{j+2})_{T,U'}$ is obtained from $(f_{j+2})_{T,U}$ by turning the s^{th} row into a row of zeros, and the differential matrix $(f_j)_{V',W}$ is obtained from

$(f_j)_{V,W}$ by turning the r^{th} column into a column of zeros. (Here we assume that T and W are bases of F_{j+2} and F_{j-1}, respectively.)

Proof.

(i) This part is essentially a consequence of the fact that f_{j+1} is a graded map of degree 0.

First, notice that since $f_{j+1}([u_s]) = \sum_{i=1}^{g} a_{is}[v_i]$, we must have

$$\text{mdeg}[u_s] = \text{mdeg}(a_{is}[v_i]) = \text{mdeg}\, a_{is}\, \text{mdeg}[v_i] \text{ for all } i.$$

In particular, since $\text{mdeg}\, a_{rs} = 1$, we have that $\text{mdeg}[u_s] = \text{mdeg}[v_r]$. On the other hand,

$$\text{mdeg}[u_s] = \text{mdeg}(f_{j+1}([u_s]) = \text{mdeg}\left(\sum_{i=1}^{g} a_{is}[v_i]\right) = \text{mdeg}[v_r]'.$$

Combining these facts, we get that $\text{mdeg}[v_r]' = \text{mdeg}[v_r]$. In addition to this, it is clear that for all $i \neq r$, $\text{mdeg}[v_i]' = \text{mdeg}[v_i]$, which proves the first part of (i).

Now given that $f_{j+1}([u_i]) = \sum_{p=1}^{g} a_{pi}[v_p]$, we must have that $\text{mdeg}[u_i] = \text{mdeg}\,(a_{pi}[v_p])$, for all $i = 1, ..., h$ and $p = 1, ..., g$. In particular, $\text{mdeg}[u_i] = \text{mdeg}\,(a_{ri}[v_r])$. Therefore, $\text{mdeg}\left(\frac{a_{ri}}{a_{rs}}[u_s]\right) = \text{mdeg}\left(\frac{a_{ri}}{a_{rs}}\right)\text{mdeg}[u_s] = \text{mdeg}\, a_{ri}\, \text{mdeg}[v_r] = \text{mdeg}(a_{ri}[v_r]) = \text{mdeg}[u_i]$, which shows that $[u_i]' = [u_i] - \frac{a_{ri}}{a_{rs}}[u_s]$ is homogeneous and $\text{mdeg}[u_i]' = \text{mdeg}[u_i]$. Finally, it is clear that $\text{mdeg}[u_s]' = \text{mdeg}[u_s]$.

(ii) $f_{j+1}([u_s]') = f_{j+1}([u_s]) = \sum_{i=1}^{g} a_{is}[v_i] = [v_r]'$. Therefore, the s^{th} column of $(f_{j+1})_{U',V'}$ is as stated in the lemma.

On the other hand, for all $i \neq s$,

$$
\begin{aligned}
f_{j+1}([u_i]') &= f_{j+1}\left([u_i] - \frac{a_{ri}}{a_{rs}}[u_s]\right) \\
&= f_{j+1}([u_i]) - \frac{a_{ri}}{a_{rs}} f_{j+1}([u_s]) \\
&= \sum_{p=1}^{g} a_{pi}[v_p] - \frac{a_{ri}}{a_{rs}} \sum_{p=1}^{g} a_{ps}[v_p] \\
&= \sum_{p \neq r} \left(a_{pi} - \frac{a_{ri}}{a_{rs}} a_{ps}\right)[v_p] + 0[v_r] \\
&= \sum_{p \neq r} \left(a_{pi} - \frac{a_{ri}}{a_{rs}} a_{ps}\right)[v_p]' + 0[v_r]'.
\end{aligned}
$$

Hence, the r^{th} row of $(f_{j+1})_{U',V'}$ is as stated.

(iii) If $c \neq r$ and $d \neq s$, we have

$$
\begin{aligned}
f_{j+1}([u_d]') &= f_{j+1}\left([u_d] - \frac{a_{rd}}{a_{rs}}[u_s]\right) \\
&= \sum_{i=1}^{g} a_{id}[v_i] - \frac{a_{rd}}{a_{rs}} \sum_{i=1}^{g} a_{is}[v_i] \\
&= \sum_{i \neq c \ i \neq r} \left(a_{id} - \frac{a_{rd}}{a_{rs}} a_{is}\right)[v_i] + \left(a_{cd} - \frac{a_{rd}}{a_{rs}} a_{cs}\right)[v_c] + 0[v_r] \\
&= \sum_{i \neq c \ i \neq r} \left(a_{id} - \frac{a_{rd}}{a_{rs}} a_{is}\right)[v_i]' + \left(a_{cd} - \frac{a_{rd}}{a_{rs}} a_{cs}\right)[v_c]'.
\end{aligned}
$$

This implies that $b_{cd} = a_{cd} - \dfrac{a_{rd}a_{cs}}{a_{rs}}$.

(iv) We will denote by A_{ip} the entries of $(f_{j+2})_{T,U}$ and by B_{ip} the entries of $(f_{j+2})_{T,U'}$.
If $[t_p]$ is a basis element in T, $f_{j+2}([t_p]) = \sum_{i=1}^{h} A_{ip}.[u_i]$.
Given that for all $i \neq s$, $[u_i] = [u_i]' + \dfrac{a_{ri}}{a_{rs}}[u_s]'$, it follows that

$$
\begin{aligned}
f_{j+2}([t_p]) &= \sum_{i \neq s} A_{ip}\left([u_i]' + \frac{a_{ri}}{a_{rs}}[u_s]'\right) + A_{sp}[u_s]' \\
&= \sum_{i=1}^{h} A_{ip}[u_i]' + \left[\left(\sum_{i \neq s} A_{ip}\frac{a_{ri}}{a_{rs}}\right) + A_{sp}\right][u_s]'.
\end{aligned}
$$

This implies that, for all $i \neq s$, $B_{ip} = A_{ip}$.

On the other hand, the entry $B_{sp} = \left(\sum_{i \neq s} A_{ip} \frac{a_{ri}}{a_{rs}} \right) + A_{sp}$ must be zero, as we show below.

Since $\operatorname{Im} f_{j+2} = \operatorname{Ker} f_{j+1}$, we must have $(f_{j+1} \circ f_{j+2})([t_p]) = 0$; that is,

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$= (f_{j+1})_{U',V'} (f_{j+2})_{T,U'} ([t_p])$$
$$= \begin{pmatrix} b_{1,1} & \dots & b_{1,s-1} & 0 & b_{1,s+1} & \dots & b_{1,h} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_{r-1,1} & \dots & b_{r-1,s-1} & 0 & b_{r-1,s+1} & \dots & b_{r-1,h} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ b_{r+1,1} & \dots & b_{r+1,s-1} & 0 & b_{r+1,s+1} & \dots & b_{r+1,h} \\ \vdots & & \vdots & \vdots & & & \vdots \\ b_{g,1} & \dots & b_{g,s-1} & 0 & b_{g,s+1} & \dots & b_{g,h} \end{pmatrix} \begin{pmatrix} A_{1p} \\ \vdots \\ A_{s-1p} \\ \left(\sum_{i \neq s} A_{ip} \frac{a_{ri}}{a_{rs}} \right) + A_{sp} \\ A_{s+1p} \\ \vdots \\ A_{hp} \end{pmatrix}.$$

Notice that the s^{th} entry of the resulting column vector is $0 = \left(\sum_{i \neq s} A_{ip} \frac{a_{ri}}{a_{rs}} \right) + A_{sp}$. This proves our statement regarding $(f_{j+2})_{T,U'}$.

The proof of the second statement is as follows: for all $i \neq r$, $[v_i]' = [v_i]$, which means that $f_j([v_i]') = f_j([v_i])$. In turn, this implies that the i^{th} columns of $(f_j)_{V',W}$ and $(f_j)_{V,W}$ are equal. Finally, since $[v_r]' = f_{j+1}([u_s]') \subseteq \operatorname{Im} f_{j+1} = \operatorname{Ker} f_j$, we must have $f_j([v_r]') = 0$, which means that the r^{th} column of $(f_j)_{V',W}$ is a column of zeros, as stated. ■

Lemma 2.1 has several important implications that we discuss next. We continue to use the notation introduced in that lemma.

Remark 2.1 *It is obvious that when we make a standard change of basis, some of the basis elements $[u_i]$ and $[v_i]$ change. However, since the free modules $S[u_i]$ and $S[u_i]'$ (respectively $S[v_i]$ and $S[v_i]'$) are isomorphic, and given that by Lemma 2.1 (i), $[u_i]$ and $[u_i]'$ (respectively $[v_i]$ and $[v_i]'$) are abstract objects with the same multidegree, we can assume that the basis elements $[u_i]$ and $[v_i]$ do not change. Therefore, after making a standard change of basis, we can interpret that we have two different representations*

$$\cdots \to \bigoplus S[u_i] \xrightarrow{(f)} \bigoplus S[v_i] \to \cdots$$

and

$$\cdots \to \bigoplus S[u_i]' \xrightarrow{(f)'} \bigoplus S[v_i]' \to \cdots$$

of the same free resolution of S/M, or we can interpret that we have two representations

$$\cdots \to \bigoplus S[u_i] \xrightarrow{(f)} \bigoplus S[v_i] \to \cdots$$

and

$$\cdots \to \bigoplus S[u_i] \xrightarrow{(f)'} \bigoplus S[v_i] \to \cdots$$

of two different free resolutions of S/M. We will choose the second interpretation. This way, if we identify the basis of $\mathbb{T}_M$ with a simplicial complex, when we make a standard change of basis or a consecutive cancellation, the basis of the new resolution can be identified with a subset of the simplicial complex and we can still speak in terms of faces and facets.

Remark 2.2 *In the same fashion that we identified the differential map f_{j+1} with the differential matrix $(f_{j+1})_{U,V} = (a_{rs})$, we can identify the s^{th} basis element $[u_s]$ of F_{j+1} with the column vector (δ_{is}), where $\delta_{is} = 0$ if $i \neq s$, and $\delta_{ss} = 1$. Similarly, the image $f_{j+1}([u_s]) = \sum_{i=1}^{g} a_{is}[v_i]$ of $[u_s]$ can be identified with the s^{th} column vector $(f_{j+1})_{U,V} \cdot (\delta_{is}) = (a_{is})$ of (a_{rs}). Thus each entry a_{rs} is the coefficient of $[v_r]$ when $f_{j+1}([u_s])$ is expressed in terms of the basis $V = \{[v_1], \ldots, [v_g]\}$. Notice that there is* *a bijective correspondence between the entries* a_{rs} *of* $(f_{j+1})_{U,V}$ *and the ordered pairs* $([u_s],[v_r])$ *of basis elements* $[u_s]$ *and* $[v_r]$ *in homological degrees* $j+1$ *and* j*, respectively. This means that the entry* a_{rs} *of* $(f_{j+1})_{U,V}$ *can be written* $a_{\tau\sigma}$*, where* $[\sigma]$ *is the* s^{th} *basis element of* U *and* $[\tau]$ *is the* r^{th} *basis element of* V*. That is, instead of using subscripts that denote the number of row and column where the entry is placed, we can use subscripts that identify the basis elements that generate this entry. Most of the time we will choose the notation* $a_{\tau\sigma}$ *over* a_{rs} *and will say that* $a_{\tau\sigma}$ *is determined by* $[\sigma]$ *and* $[\tau]$*.*

Remark 2.3 *Since* f_{j+1} *is graded of degree* 0*, if* $a_{rs} \neq 0$ *we must have*

$$\begin{aligned} \text{mdeg}[u_s] &= \text{mdeg}\, f_{j+1}([u_s]) \\ &= \text{mdeg}\left(\sum_{i=1}^{g} a_{is}[v_i]\right) \\ &= \text{mdeg}\,(a_{rs}[v_r]) \\ &= \text{mdeg}\, a_{rs}\, \text{mdeg}[v_r]. \end{aligned}$$

Hence, $a_{rs} = 0$ *or* $\text{mdeg}\, a_{rs} = \dfrac{\text{mdeg}[u_s]}{\text{mdeg}[v_r]}$.

With the notation introduced in Remark 2.2: $a_{\tau\sigma} = 0$ *or* $\text{mdeg}\, a_{\tau\sigma} = \dfrac{\text{mdeg}[\sigma]}{\text{mdeg}[\tau]}$*. In particular, if* $a_{\tau\sigma}$ *is invertible then* $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$*.*

Now let $b_{\tau\sigma}$ *be the entry determined by* $[\sigma]$ *and* $[\tau]$ *in* $(f_{j+1})_{U',V'}$*. Reasoning as before, we get* $b_{\tau\sigma} = 0$ *or* $\text{mdeg}\, b_{\tau\sigma} = \dfrac{\text{mdeg}[\sigma]}{\text{mdeg}[\tau]}$.

(Informally speaking, the multidegrees of the entries do not change under standard changes of bases.) In particular, if $a_{\tau\sigma}$ *is invertible, then* $b_{\tau\sigma} = 0$ *or* $b_{\tau\sigma}$ *is also invertible.*

Remark 2.4 *It follows from Lemma 2.1 (ii) and (iv) that after making a standard change of basis around* a_{rs}*, it is possible to make the consecutive cancellation* $0 \to S[u_s]' \to S[v_r]' \to 0$*. With the interpretation we adopted in Remark 2.1 and the notation we introduced in Remark 2.2, the preceding observation can be restated as follows: after making a standard change of basis around* $a_{\tau\sigma}$*, the resulting resolution admits the consecutive cancellation* $0 \to S[\sigma] \to S[\tau] \to 0$*.*

We close this section introducing the following terminology. After making a standard change of basis around an invertible entry $a_{\tau\sigma}$ of a resolution $\mathbb{F}$, we obtain a new resolution $\mathbb{F}'$ such that $\mathbb{F} = \mathbb{F}' \oplus (0 \to S[\sigma] \to S[\tau] \to 0)$. From now on, the consecutive cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ will be called **standard cancellation**, and we will say that $\mathbb{F}'$ is obained from $\mathbb{F}$ by means of a standard cancellation.

2.2 Dominant Ideals

We are ready to address the study of our first family of monomial ideals, the dominant ideals. This study includes the construction of their minimal free resolutions as well as an analysis of their combinatorial properties.

Definition 2.2 *Given a set G of monomials in S, we say that*

- *An element $m \in G$ has a* ***dominant variable*** *x (with respect to G) if for all $m' \in G \setminus \{m\}$, the exponent with which x appears in the factorization of m is larger than the exponent with which x appears in the factorization of m'; that is, there exists a positive k such that $x^k \mid m$ and $x^k \nmid m'$, for all $m' \neq m$.*
- *An element $m \in G$ is a* ***dominant monomial*** *(with respect to G) if it has a dominant variable.*
- *The set G is a* ***dominant set*** *if every $m \in G$ is dominant.*
- *A monomial ideal M is a* ***dominant ideal*** *if its minimal generating set is dominant.*

Example 2.1 *The ideals $M_1 = (x^3y, xy^2z, xz^2)$ and $M_2 = (wx, y^3, z^2)$ are dominant, while $M_3 = (x^2, y^2, xy)$ is not.*

Some comments are in order. First, notice that the concept of dominant monomial always depends on a reference set. For example, the ideal M_3 introduced above is not dominant because xy is not dominant in the minimal generating set $\{x^2, y^2, xy\}$; however, xy is dominant in the proper subset $\{x^2, xy\}$.

Second, the definitions of dominant ideal and generic ideal are based on properties of the exponents of the monomial generators. (Recall that an ideal is generic if no variable appears with the same nonzero exponent in more than one monomial generator.) Despite this similarity, dominant and generic ideals are generally different. In Example 2.1, for instance, M_1 is dominant but not generic, while M_3 is generic but not dominant.

Finally, observe that if a monomial ideal is a complete intersection, its monomial generators are dominant because they do not have variables in common (such is the case with M_2). It follows that the ideal itself is dominant. Thus, monomial complete intersections are a subset of the family of dominant ideals.

Let us now study some properties derived from the concept of dominance. The following lemma will be quoted often throughout this work.

Lemma 2.2 *Let M be a monomial ideal with minimal generating set G. If $[\sigma_1]$ and $[\sigma_2]$ are two basis elements of $\mathbb{T}_M$ with $\text{mdeg}[\sigma_1] = \text{mdeg}[\sigma_2]$, then $[\sigma_1]$ and $[\sigma_2]$ contain the same dominant monomials of G.*

Proof. Let L_1 and L_2 be the sets of monomials contained in $[\sigma_1]$ and $[\sigma_2]$, respectively. Then $\text{lcm}(L_1) = \text{lcm}(L_2)$. If neither L_1 nor L_2 contains dominant elements of G, there is nothing to prove.

Suppose now that one of these sets, call it L_i, contains a dominant monomial m of G. We will show that the other set, call it L_j, contains m as well. Since m has a dominant variable x, there is a positive k such that $x^k \mid m$ and $x^k \nmid m'$, for all m' in $G \setminus \{m\}$. In particular, $x^k \nmid m'$ for all m' in $L_j \setminus \{m\}$. That is, $x^k \nmid \text{lcm}(L_j \setminus \{m\})$. On the other hand, $x^k \mid \text{lcm}(L_i) = \text{lcm}(L_j)$.

Hence, $L_j \neq L_j \setminus \{m\}$, which means that m is in L_j. We have proven that each dominant element m of G which is in one of $[\sigma_1]$ and $[\sigma_2]$ is also contained in the other. ■

In the following theorem we construct the minimal resolutions of dominant ideals. This theorem yields, in addition, an explicit characterization of when the Taylor resolution is minimal.

Theorem 2.1 *Let M be a monomial ideal. Then $\mathbb{T}_M$ is minimal if and only if M is dominant.*

Proof. ($\Rightarrow$) Suppose that M is not dominant. Then its minimal generating set G contains a nondominant monomial n. Let $\sigma = G$ and $\tau_m = G \setminus \{m\}$. This means that $n \mid lcm\,(\tau_n)$ and thus, $\text{mdeg}[\sigma] = \text{mdeg}\,[\tau_n]$. So, the top differential map sends $[\sigma] \mapsto \sum\limits_{m \neq n} a_m\,[\tau_m] \pm 1\,[\tau_n]$. Since the coefficient ± 1 of $[\tau_n]$ is invertible, $\mathbb{T}_M$ is not minimal, a contradiction.

($\Leftarrow$) If $[\sigma] = [m_1, \ldots, m_j]$ and $[\tau_i] = [m_1, \ldots, \widehat{m_i}, \ldots, m_j]$ for all i, then

$$f_j\,([\sigma]) = \sum_{i=1}^{j} a_{\tau_i \sigma}[\tau_i],$$

where $a_{\tau_i\sigma} = (-1)^{i+1} \dfrac{\text{mdeg}[\sigma]}{\text{mdeg}[\tau_i]}$. Since m_i is dominant, it follows from Lemma 2.2 that $a_{\tau_i\sigma}$ is not invertible. This means that the differential matrices of $\mathbb{T}_M$ do not have invertible entries, and hence $\mathbb{T}_M$ is minimal. ■

Corollary 2.1 *Dominant ideals are Scarf.*

Proof. If two basis elements $[\sigma_1], [\sigma_2]$ of $\mathbb{T}_M$ have the same multidegree, according to Lemma 2.2, they contain the same dominant monomials. Since all monomials of the minimal generating set are dominant, $[\sigma_1] = [\sigma_2]$. ■

It follows from Lemma 2.2 that if M is dominant, no facet $[\tau_i]$ of $[\sigma]$ has the same multidegree as $[\sigma]$. However, Corollary 2.1 shows that an even stronger statement is true: if M is dominant, all basis elements of $\mathbb{T}_M$ have different multidegrees.

Example 2.2 *Let $M = (x^2, xy, y^3)$. The Taylor resolution $\mathbb{T}_M$ of S/M is*

$$0 \to S[x^2, xy, y^3] \xrightarrow{\begin{pmatrix} x \\ -1 \\ y^2 \end{pmatrix}} \begin{matrix} S[xy, y^3] \\ \oplus \\ S[x^2, y^3] \\ \oplus \\ S[x^2, xy] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -y^3 & -y \\ -y^2 & 0 & x \\ x & x^2 & 0 \end{pmatrix}} \begin{matrix} S[x^2] \\ \oplus \\ S[xy] \\ \oplus \\ S[y^3] \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}}$$

$$S[\varnothing] \to S/M \to 0.$$

Notice that M is not a dominant ideal since xy is nondominant. It follows from Theorem 2.1 that $\mathbb{T}_M$ is not minimal, which is consistent with the fact that one of the differential matrices contains an invertible entry -1.

In contrast to the previous example, the next one contains a Taylor Resolution which is minimal.

Example 2.3 *Let $M = (x^2, xz, y^3)$. the Taylor resolution $\mathbb{T}_M$ of S/M is*

$$0 \to S[x^2, xz, y^3] \xrightarrow{\begin{pmatrix} x \\ -z \\ y^3 \end{pmatrix}} \begin{matrix} S[xz, y^3] \\ \oplus \\ S[x^2, y^3] \\ \oplus \\ S[x^2, xz] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -y^3 & -z \\ -y^3 & 0 & x \\ xz & x^2 & 0 \end{pmatrix}} \begin{matrix} S[x^2] \\ \oplus \\ S[xz] \\ \oplus \\ S[y^3] \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xz & y^3 \end{pmatrix}}$$

$$S[\varnothing] \to S/M \to 0.$$

In this example, M is dominant. According to Theorem 2.1, the Taylor Resolution $\mathbb{T}_M$ is minimal, which is consistent with the fact that none of the differential matrices contains invertible entries.

Having obtained the minimal free resolutions of the dominant ideals, we can now study some combinatorial and homological properties of the family.

Theorem 2.2 *(Regularity of Dominant Ideals)*
Let M be a dominant ideal with minimal generating set $G = \{m_1, \dots, m_q\}$. Let $h = \deg(\text{mdeg}[m_1, \dots, m_q])$. Then $\text{reg}(S/M) = h - q$.

Proof. Since $[m_1, \dots, m_q]$ is a basis element in homological degree q, it follows that $b_{qh} \neq 0$. Thus, $\text{reg}(S/M) \geq h - q$. We will prove that if $b_{ij} \neq 0$, then $h - q \geq j - i$, which will complete the proof.

Let $[\sigma] = [m_{r_1}, \dots, m_{r_i}]$ be a basis element of $\mathbb{T}_M$ with $\deg(\text{mdeg}[\sigma]) = j$. Let $m \in G \setminus \{m_{r_1}, \dots, m_{r_i}\}$. Since different monomial generators have different dominant variables, it follows that

$$\deg(\text{mdeg}[m_{r_1}, \dots, m_{r_i}, m]) \geq \deg(\text{mdeg}[m_{r_1}, \dots, m_{r_i}]) + 1.$$

Then, after applying the preceding reasoning $q - i$ times, we get

$$\begin{aligned} h &= \deg(\text{mdeg}[m_1, \dots, m_q]) \\ &= \deg\left(\text{mdeg}[m_{r_1}, \dots, m_{r_i}, m_{s_1}, \dots, m_{s_{q-i}}]\right) \\ &\geq \deg(\text{mdeg}[m_{r_1}, \dots, m_{r_i}]) + (q - i) \\ &= j + q - i. \end{aligned}$$

This implies that $h - q \geq j - i$. ■

Corollary 2.2 *(Characterization of the minimal Taylor Resolution)*
Let M be a monomial ideal minimally generated by q monomials. The following statements are equivalent:

(i) $\mathbb{T}_M$ *is minimal.*

(ii) M *is dominant.*

(iii) $\text{b}_\text{i}(S/M) = \binom{q}{i}$ *for all* i.

(iv) $\mathrm{pd}(S/M) = q$.

(v) The LCM lattice of M is Boolean.

Proof. The equivalence of (i), (ii), (iii) and (v) is immediate, as is (iii) $\Rightarrow$ (iv). We complete the proof by showing that (iv) $\Rightarrow$ (i).

Assume that the Taylor Resolution is not minimal. Then, by Theorem 2.1, M is not dominant. Thus there exists a nondominant monomial m in the minimal generating set G of M. Let $\sigma = G$ and $\tau = G \setminus \{m\}$. Then $m \mid \mathrm{lcm}(\tau)$ and hence, $\mathrm{mdeg}[\sigma] = \mathrm{mdeg}[\tau]$. Since $[\sigma]$ and $[\tau]$ are face and facet in homological degrees q and $q-1$ respectively, it follows that the q^{th} differential matrix (d_q) of $\mathbb{T}_M$ contains an invertible entry. After making a consecutive cancellation in homological degrees q and $q-1$, we obtain a new resolution $\mathbb{F}$ of S/M. But the rank of the free module in homological degree q of $\mathbb{T}_M$ is 1, which implies that the rank of the free module in homological degree q of $\mathbb{F}$ is 0. Hence, the length of $\mathbb{F}$ is less than q, a contradiction. ■

The following two remarks are now trivial but show that dominant ideals are as good as we could expect. First, note that the Taylor resolution of S/M agrees with the Scarf complex of S/M if and only if M is dominant. This is interesting because the Taylor resolution is usually highly nonminimal, while the Scarf complex is often strictly contained in the minimal free resolution of S/M. Second, two dominant ideals whose minimal generating sets have the same cardinality must have the same projective dimension and the same total Betti numbers. This is immediate from Corollary 2.2 (iii) and (iv).

2.3 Semidominant Ideals

In this section we introduce the semidominant ideals by slightly modifying the definition of dominance in such a way that the resulting family does not overlap with the family of dominant ideals and yet retains some of its rich properties.

Definition 2.3 *Let G be a set of monomials in S. We say that G is* ***semidominant*** *if exactly one monomial of G is not dominant. A monomial ideal M is called a* ***semidominant ideal*** *if its minimal generating set is semidominant. When a semidominant set G is expressed in the form $G = \{m_1, \ldots, m_q, n\}$ we will assume that $m_1, \ldots, m_q$ are dominant and n is nondominant.*

Example 2.4 *The ideals $M_1 = (x^2, y^3, xy)$ and $M_2 = (xy, z^2, yz)$ are semidominant, $M_3 = (x^2z, y^3, yz^3)$ is dominant, and $M_4 = (xy, yz, xz)$ is neither dominant nor semidominant.*

Note that the concept of semidominance is obtained from that of dominance in the same way as the definition of almost complete intersection is derived from that of complete intersection; namely, by relaxing the defining conditions. In the next proposition we explain how the former concepts extend the latter.

Proposition 2.3 *Monomial almost complete intersections are either dominant or semidominant ideals.*

Proof. Let $M = (l_1, ..., l_q, l)$ be a monomial almost complete intersection, where $l_1, ..., l_q$ form a regular sequence and hence have no variable in common. Note that for all i, $l_i \nmid l$. Then there is a variable x_i that appears with a larger exponent in the factorization of l_i than in that of l. Therefore, x_i is a dominant variable for l_i, which means that l_i is a dominant monomial. ■

Observe that the two cases stated in the proposition are feasible (consider M_2 and M_3 in Example 2.4). Later, we will prove that semidominant ideals are Scarf which, combined with Corollary 2.1 and Proposition 2.3, implies that monomial almost complete intersections are Scarf too.

Now we are ready to construct the minimal free resolutions of semidominant ideals. The idea is simple: if M is semidominant and we identify the basis of $\mathbb{T}_M$ with the full simplex on M, we will prove that the basis of the minimal free resolution of S/M can be obtained by eliminating pairs $([\sigma],[\tau])$ of face and facet of equal multidegree from the simplicial complex in arbitrary order until we exhaust all such pairs. We begin with a lemma.

Lemma 2.3 *Let M be a semidominant ideal. Let $\mathbb{F}$ be a free resolution of S/M obtained from $\mathbb{T}_M$ by means of standard cancellations. If two basis elements of $\mathbb{F}$ have the same multidegree, then they are face and facet.*

Proof. Let $[\sigma]$ and $[\tau]$ be two basis elements of $\mathbb{F}$. If $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$ then, according to Lemma 2.2, $[\sigma]$ and $[\tau]$ contain the same dominant monomials, and thus they must differ in the nondominant monomials that define them. Since the minimal generating set of M contains exactly one nondominant monomial n, we conclude that one of these basis elements contains n while the other does not. That is, $[\sigma]$ and $[\tau]$ are face and facet. ■

The next two results show that, in the context of semidominant ideals, the process of eliminating pairs of face and facet of equal multidegree is equivalent to that of making standard cancellations.

Note: We will say that two pairs of basis elements $([\sigma],[\tau])$ and $([\theta],[\pi])$ of $\mathbb{T}_M$ are "disjoint" if $[\sigma] \neq [\theta],[\pi]$ and $[\tau] \neq [\theta],[\pi]$.

Lemma 2.4 *Let M be a semidominant ideal. Let $\mathbb{F}$ be a free resolution of S/M obtained from $\mathbb{T}_M$ by means of standard cancellations. Let $a_{\tau\sigma}$ and $a_{\pi\theta}$ be two invertible entries of $\mathbb{F}$, determined by two disjoint pairs of basis elements $([\sigma],[\tau])$ and $([\theta],[\pi])$ of $\mathbb{F}$, respectively.*

Then after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, it is possible to make the standard cancellation $0 \to S[\theta] \to S[\pi] \to 0$.

Proof. $[\sigma]$ and $[\tau]$ are basis elements in homological degrees j and $j-1$, respectively, for some j. Thus $a_{\tau\sigma}$ is an entry of the differential matrix (f_j) of $\mathbb{F}$. Similarly, $[\theta]$ and $[\pi]$ are basis elements in some homological degrees k and $k-1$, and $a_{\pi\theta}$ is an entry of the differential matrix (f_k) of $\mathbb{F}$.

In order to prove the lemma, it is enough to show that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, the entry $a'_{\pi\theta}$ of the differential matrix (f'_k) of the new resolution $\mathbb{F}'$ is invertible.

Given that only (f_{j+1}), (f_j) and (f_{j-1}) are affected by the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, if $k \neq j-1, j, j+1$, then $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible. Therefore, we only need to prove that $a'_{\pi\theta}$ is invertible in the following cases:
$k = j$; $k = j-1$, and $k = j+1$.

First, suppose $k = j$. Since $a_{\pi\theta}$ is invertible, $\text{mdeg}[\pi] = \text{mdeg}[\theta]$. Then $a'_{\pi\theta} = 0$ or $a'_{\pi\theta}$ is invertible. Let us assume that $a'_{\pi\theta} = 0$. By Lemma 2.1 (iii), we have that $0 = a'_{\pi\theta} = a_{\pi\theta} - \dfrac{a_{\pi\sigma}a_{\tau\theta}}{a_{\tau\sigma}}$. It follows that $a_{\pi\theta}a_{\tau\sigma} = a_{\pi\sigma}a_{\tau\theta}$ and, since $a_{\pi\theta}$ and $a_{\tau\sigma}$ are invertible, $a_{\pi\sigma}$ and $a_{\tau\theta}$ must be invertible too. In particular, the fact that $a_{\pi\sigma}$ is invertible implies that $\text{mdeg}[\sigma] = \text{mdeg}[\pi]$ which, combined with the hypothesis $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, implies that $\text{mdeg}[\tau] = \text{mdeg}[\pi]$. It follows from Lemma 2.3 that one of $[\tau]$ and $[\pi]$ is a face and the other is its facet. Then they must appear in consecutive homological degrees, which is absurd because $k = j$. We conclude that $a'_{\pi\theta}$ is invertible.

Now suppose $k = j-1$. In this case $[\tau]$ and $[\theta]$ appear in homological degree $j-1$. Let $[\tau]$ and $[\theta]$ be the r^{th} and s^{th} basis elements, respectively. It follows from Lemma 2.1 iv) that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, the matrix $\left(f'_{j-1}\right)$ of the new resolution $\mathbb{F}'$ is obtained from (f_{j-1}) by eliminating its r^{th} column. Since the entry $a'_{\pi\theta}$ is placed in the s^{th} column of $\left(f'_{j-1}\right)$, we have that $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible.

Finally, suppose $k = j+1$. In this case $[\sigma]$ and $[\pi]$ appear in homological degree

j. Let $[\sigma]$ and $[\pi]$ be the u^{th} and v^{th} basis elements, respectively. It follows from Lemma 2.1 (iv) that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, the matrix $\left(f'_{j+1}\right)$ of the new resolution $\mathbb{F}'$ is obtained from (f_{j+1}) by eliminating its u^{th} row. Since the entry $a'_{\pi\theta}$ is placed in the v^{th} row of $\left(f'_{j+1}\right)$, we have that $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible. ■

Theorem 2.4 *Let M be a semidominant ideal. Let $([\sigma_1],[\tau_1]),\ldots,([\sigma_k],[\tau_k])$ be k pairs of basis elements of $\mathbb{T}_M$, satisfying the following properties:*

(i) $([\sigma_i],[\tau_i])$ *and* $([\sigma_j],[\tau_j])$ *are disjoint, if* $i \neq j$.

(ii) $[\tau_i]$ *is a facet of* $[\sigma_i]$*, for all* $i = 1,\ldots,k$.

(iii) $\text{mdeg}[\sigma_i] = \text{mdeg}[\tau_i]$*, for all* $i = 1,\ldots,k$.

Then, starting with $\mathbb{T}_M$ it is possible to make the following sequence of standard cancellations:

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad , 0 \to S[\sigma_k] \to S[\tau_k] \to 0.$$

Proof. The proof is by induction on k.
If $k = 2$, the statement holds by Lemma 2.4, with $\mathbb{F} = \mathbb{T}_M$. (The fact that $a_{\tau_1\sigma_1}$ and $a_{\tau_2\sigma_2}$ are invertible follows from the fact that in $\mathbb{T}_M$ faces and facets of equal multidegree always determine an invertible entry.)

Assume that the theorem holds for $k = j-1$. Let $k = j$. Then it is possible to make either of the following two sequences of standard cancellations:

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad , 0 \to S[\sigma_{j-1}] \to S[\tau_{j-1}] \to 0$$

and

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad , 0 \to S[\sigma_{j-2}] \to S[\tau_{j-2}] \to 0, 0 \to S[\sigma_j] \to S[\tau_j] \to 0.$$

This means that after making the first $j-2$ cancellations

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad , 0 \to S[\sigma_{j-2}] \to S[\tau_{j-2}] \to 0$$

either of the following two cancellations can be made:

$$0 \to S[\sigma_{j-1}] \to S[\tau_{j-1}] \to 0$$

and

$$0 \to S[\sigma_j] \to S[\tau_j] \to 0.$$

In other words, after making the first $j-2$ standard cancellations, we obtain a free resolution $\mathbb{F}$, where the entries $a_{\tau_{j-1}\sigma_{j-1}}$ and $a_{\sigma_j\tau_j}$ determined by $([\sigma_{j-1}],[\tau_{j-1}])$ and $([\sigma_j],[\tau_j])$, respectively, are invertible. Therefore, it follows from Lemma 2.4, that after making the cancellation $0 \to S[\sigma_{j-1}] \to S[\tau_{j-1}] \to 0$, the cancellation $0 \to S[\sigma_j] \to S[\tau_j] \to 0$ is still possible. ■

Note. In Theorem 2.4, the pairs $([\sigma_1],[\tau_1]), \ldots, ([\sigma_k],[\tau_k])$ are indistinguishable, which implies that the standard cancellations can be made in arbitrary order.

Lemma 2.5 *Let $M = (m_1, \ldots, m_q, n)$ be a semidominant ideal. Let $A = \left\{\left([m_{i_1}, \ldots, m_{i_j}, n], [m_{i_1}, \ldots, m_{i_j}]\right) : n \mid \operatorname{lcm}(m_{i_1}, \ldots, m_{i_j})\right\}$. Then the following properties are satisfied:*

(i) If $([\sigma_1],[\tau_1])$ and $([\sigma_2],[\tau_2])$ are distinct ordered pairs of A, then they are disjoint.

(ii) $[\tau]$ is a facet of $[\sigma]$, for all $([\sigma],[\tau]) \in A$.

(iii) $\operatorname{mdeg}[\sigma] = \operatorname{mdeg}[\tau]$, for all $([\sigma],[\tau]) \in A$.

(iv) If $([\sigma],[\tau])$ is an ordered pair of basis elements of $\mathbb{T}_M$ such that $[\tau]$ is a facet of $[\sigma]$ and $\operatorname{mdeg}[\sigma] = \operatorname{mdeg}[\tau]$, then $([\sigma],[\tau]) \in A$.

Proof. (i) Since $[\sigma_1]$ and $[\sigma_2]$ contain n and $[\tau_1]$ and $[\tau_2]$ do not contain n, it follows that $[\sigma_1] \neq [\tau_2]$ and $[\tau_1] \neq [\sigma_2]$. Let us assume that $[\sigma_1] = [\sigma_2]$. Then, by construction, $[\tau_1] = [\tau_2]$ and thus $([\sigma_1], [\tau_1]) = ([\sigma_2], [\tau_2])$, a contradiction. Let us now assume that $[\tau_1] = [\tau_2]$. Then, by construction, $[\sigma_1] = [\sigma_2]$ and thus $([\sigma_1], [\tau_1]) = ([\sigma_2], [\tau_2])$, a contradiction. (ii) Trivial. (iii) Since $n \mid \text{lcm}\left(m_{i_1}, \ldots, m_{i_j}\right)$, it follows that $\text{lcm}\left(m_{i_1}, \ldots, m_{i_j}\right) = \text{lcm}\left(m_{i_1}, \ldots, m_{i_j}, n\right)$. (iv) If $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$ then, by Lemma 2.2, $[\sigma]$ and $[\tau]$ contain the same dominant monomials, and therefore they differ in the nondominant monomials that define them. But the minimal generating set of M contains exactly one nondominant monomial and $[\tau]$ is a facet of $[\sigma]$, which implies that $[\sigma]$ and $[\tau]$ must be of the form $[\sigma] = [m_{i_1}, \ldots, m_{i_j}, n]$; $[\tau] = [m_{i_1}, \ldots, m_{i_j}]$. ■

Theorem 2.5 *Let $M = (m_1, \ldots, m_q, n)$ be a semidominant ideal. Let $A = \left\{\left([m_{i_1}, \ldots, m_{i_j}, n], [m_{i_1}, \ldots, m_{i_j}]\right) : n \mid \text{lcm}(m_{i_1}, \ldots, m_{i_j})\right\}$. Then the minimal free resolution of S/M can be obtained from $\mathbb{T}_M$ by doing all standard cancellations $0 \to S[\sigma] \to S[\tau] \to 0$, with $([\sigma], [\tau]) \in A$. In other words, if $\mathbb{F}$ is the minimal free resolution of S/M, then*

$$\mathbb{T}_M = \mathbb{F} \oplus \left(\bigoplus_{([\sigma],[\tau]) \in A} 0 \to S[\sigma] \to S[\tau] \to 0 \right).$$

Proof. Notice that the ordered pairs of A satisfy the hypotheses of Theorem 2.4, by Lemma 2.5. Therefore, starting with $\mathbb{T}_M$, it is possible to make all standard cancellations $0 \to S[\sigma] \to S[\tau] \to 0$, with $([\sigma], [\tau]) \in A$. We claim that the free resolution $\mathbb{F}$, obtained after making all these cancellations, is minimal.

Let us assume that $\mathbb{F}$ is not minimal. Then there exists an invertible entry $a_{\tau\sigma}$ of $\mathbb{F}$, determined by two basis elements $[\sigma]$ and $[\tau]$ of $\mathbb{F}$. Hence, $[\sigma]$ and $[\tau]$ have the same multidegree. Thus by Lemma 2.3, $[\sigma]$ and $[\tau]$ are face and facet. It follows from Lemma 2.5 (iv) that $([\sigma], [\tau]) \in A$, a contradiction. ■

Corollary 2.3 *Semidominant ideals are Scarf.*

Proof. Let M be a semidominant ideal. If $[\sigma]$ and $[\tau]$are basis elements of $\mathbb{T}_M$ and $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, then by Lemma 2.3 we have that $[\sigma]$ and $[\tau]$ are face and facet. It follows from Lemma 2.5 (iv) and Theorem 2.5 that $[\sigma]$ and $[\tau]$ are excluded from the minimal free resolution of S/M. ■

Since the Scarf complex of an ideal is the intersection of all its minimal resolutions (as proved in [Me]), it follows that all minimal resolutions of semidominant ideals have the same basis.

Example 2.5 *Let* $M = (x^3y, y^2z, xz^2, xyz)$. *Note that* M *is semidominant,* xyz *being the nondominant generator. By Corollary 2.3,* M *is Scarf. Now, the multidegrees that are common to more than one basis element of* $\mathbb{T}_M$ *are* x^3y^2z, x^3yz^2, xy^2z^2, *and* $x^3y^2z^2$ *as one can determine by simple inspection. Hence, the basis of the minimal resolution* $\mathbb{F}$ *of* S/M *is obtained from the basis of* $\mathbb{T}_M$ *by eliminating the elements that have one of the multidegrees mentioned above. This leads to the following resolution:*

$$\mathbb{F}: \quad 0 \to \begin{array}{c} S[x^3y, xyz] \\ \oplus \\ S[y^2z, xyz] \\ \oplus \\ S[xz^2, xyz] \end{array} \xrightarrow{(f_2)} \begin{array}{c} S[x^3y] \\ \oplus \\ S[y^2z] \\ \oplus \\ S[xz^2] \\ \oplus \\ S[xyz] \end{array} \xrightarrow{(f_1)} S[\varnothing] \xrightarrow{(f_0)} S/M \to 0.$$

Corollary 2.4 *Let* M *be a semidominant ideal with minimal generating set* $G = \{m_1, \ldots, m_q, n\}$.

(i) The projective dimension of S/M *is the cardinality of the largest dominant subset of* G *that contains* n.

(ii) Let $B_j = \{[m_{t_1}, \ldots, m_{t_j}] : n \nmid \text{mdeg}[m_{t_1}, \ldots, m_{t_j}]\}$. *Then the total Betti numbers are given by the formula*

$$\mathrm{b_i}\left(S/M\right) = \#B_i + \#B_{i-1}.$$

Proof. Let $\mathbb{F}$ and A be as in Theorem 2.5.

(i) Let $r = \max\{\#(D) : D \text{ is a dominant subset of } G \text{ that contains } n\}$ and let $\{m_{t_1}, \ldots, m_{t_{r-1}}, n\}$ be a dominant subset of G. Then $n \nmid \text{lcm}\left(m_{t_1}, \ldots, m_{t_{r-1}}\right)$. Thus $\left([m_{t_1}, \ldots, m_{t_{r-1}}, n], [m_{t_1}, \ldots, m_{t_{r-1}}]\right)$ is not in A and, therefore, $[m_{t_1}, \ldots, m_{t_{r-1}}, n]$ is a basis element of the minimal resolution $\mathbb{F}$. Thus, $\text{pd}\left(S/M\right) \geq r$. Now, if $[\sigma]$ is a basis element of $\mathbb{T}$, in homological degree $k > r$, then $[\sigma]$ must be of the form: $[\sigma] = [m_{s_1}, \ldots, m_{s_k}]$ or $[\sigma] = [m_{s_1}, \ldots, m_{s_{k-1}}, n]$. If $[\sigma] = [m_{s_1}, \ldots, m_{s_k}]$, then $\{m_{s_1}, \ldots, m_{s_k}, n\}$ cannot be dominant because its cardinality is larger than r. Hence, $n \mid \text{lcm}(m_{s_1}, \ldots, m_{s_k})$, which means that $([m_{s_1}, \ldots, m_{s_k}, n], [\sigma]) \in A$, and thus $[\sigma]$ is not a basis element of $\mathbb{F}$. Similar reasoning shows that if $[\sigma] = [m_{s_1}, \ldots, m_{s_{k-1}}, n]$ then $\left([\sigma], [m_{s_1}, \ldots, m_{s_{k-1}}]\right) \in A$, and thus $[\sigma]$ is not a basis element of $\mathbb{F}$. Given that every basis element of $\mathbb{T}_M$ in homological degree $k > r$ is excluded from the basis of $\mathbb{F}$, we conclude that $\text{pd}\left(S/M\right) = r$. (ii) The basis elements of $\mathbb{T}_M$ in homological degree i are of the form $[m_{s_1}, \ldots, m_{s_{i-1}}, n]$ or $[m_{t_1}, \ldots, m_{t_i}]$. Since the basis elements of $\mathbb{F}$ are obtained from the basis of $\mathbb{T}_M$ by eliminating those elements which are the first or the second component of a pair $([\sigma], [\tau]) \in A$, it follows that the family of basis elements of $\mathbb{F}$ in homological degree i is: $\{[m_{t_1}, \ldots, m_{t_i}] : n \nmid \text{lcm}\left(m_{t_1}, \ldots, m_{t_i}\right)\} \cup \left\{[m_{s_1}, \ldots, m_{s_{i-1}}, n] : n \nmid \text{lcm}\left(m_{s_1}, \ldots, m_{s_{i-1}}\right)\right\}$. The statement of part (ii) is now clear. ■

Corollary 2.5 *Let* $M = (m_1, \ldots, m_q, n)$ *be a semidominant ideal. Then* $\text{pd}\left(S/M\right) = 2$ *if and only if for all* $i \neq j$, $n \mid \text{lcm}(m_i, m_j)$.

Proof. ($\Rightarrow$) If $\text{pd}\left(S/M\right) = 2$, then the largest dominant subset of $\{m_1, \ldots, m_q, n\}$

that contains n has cardinality 2 (Corollary 2.4). Thus every set $\{m_i, m_j, n\}$ is non-dominant, which implies that $n \mid \operatorname{lcm}(m_i, m_j)$.
($\Leftarrow$) If $k \geq 2$, then $n \mid \operatorname{lcm}(m_{i_1}, \ldots, m_{i_k})$. Therefore, the set $D = \{m_{i_1}, \ldots, m_{i_k}, n\}$ is not dominant and, according to Corollary 2.4, $\operatorname{pd}(S/M) \leq 2$. Now, $\{m_1, n\}$ is dominant, so $\operatorname{pd}(S/M) = 2$. ■

Corollary 2.5 is interesting because it tells us that an ideal M may have maximum projective dimension (i.e., $\operatorname{pd}(S/M) =$ number of generators of M) and another ideal M', obtained by adding one generator to the minimal generating set of M, may have minimum projective dimension (i.e., $\operatorname{pd}(S/M') = 2$). The next example illustrates this phenomenon.

Example 2.6 *Let* $M = (v^2xyz, vw^2yz, vwx^2z, vwxy^2, wxyz^2)$, *and let* $M' = (v^2xyz, vw^2yz, vwx^2z, vwxy^2, wxyz^2, vwxyz)$. *Since M is dominant,* $\operatorname{pd}(S/M) = 5$. *The semidominant ideal M' obtained from M by adding the generator $vwxyz$ satisfies the condition of Corollary 2.5 and thus* $\operatorname{pd}(S/M') = 2$.

Corollary 2.6 *Let M be a semidominant ideal with minimal generating set $G = \{m_1, \ldots, m_q, n\}$. Then*

$$\operatorname{reg}(S/M) = \max\{\deg(\operatorname{mdeg}[\sigma]) - \operatorname{hdeg}[\sigma] : \sigma \subset G,\ n \in \sigma,\ \text{and } \sigma \text{ is dominant}\}.$$

Proof. Let $\{m_{r_1}, \ldots, m_{r_t}, n\}$ be a dominant set such that

$$\deg(\operatorname{mdeg}[m_{r_1}, \ldots, m_{r_t}, n]) - (t+1) = c.$$

Then $\operatorname{reg}(S/M) \geq c$. We will prove that if $b_{ij} \neq 0$, then $c \geq j - i$, which will complete the proof. There are two ways in which we might have $b_{ij} \neq 0$:

(i) the minimal free resolution contains a basis element of the form $[m_{r_1}, \ldots, m_{r_i}]$ such that $\{m_{r_1}, \ldots, m_{r_i}, n\}$ is dominant and $\deg(\operatorname{mdeg}[m_{r_1}, \ldots, m_{r_i}]) = j$;

(ii) the minimal free resolution contains a basis element of the form $[m_{s_1}, \ldots, m_{s_{i-1}}, n]$ such that $\{m_{s_1}, \ldots, m_{s_{i-1}}, n\}$ is dominant and $\deg\big(\text{mdeg}[m_{s_1}, \ldots, m_{s_{i-1}}, n]\big) = j$.

If (i) happens, then $[m_{r_1}, \ldots, m_{r_i}, n]$ is also in the minimal free resolution and

$$\deg(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}, n]) \geq \deg(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}]) + 1.$$

It follows from the construction of c that

$$c \geq \deg(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}, n]) - (i+1) \geq \deg(\text{mdeg}[m_{r_1}, \ldots, m_{r_i}]) + 1 - (i+1) = j - i.$$

If (ii) happens, then it follows from the construction of c that

$$c \geq \deg\big(\text{mdeg}[m_{s_1}, \ldots, m_{s_{i-1}}, n]\big) - i = j - i.$$

■

Example 2.7 *Let* $M = (x^3y, y^2z, xz^2, xyz)$ *as in Example 2.5. Since we already know the minimal free resolution* $\mathbb{F}$ *of* S/M*, we can read off the numbers* $\text{pd}(S/M)$*,* $\text{b}_\text{i}(S/M)$*, and* $\text{reg}(S/M)$ *from* $\mathbb{F}$*. However, we will calculate these numbers using Corollary 2.4 and Corollary 2.6 which, in some cases, turns out to be a faster alternative.*

Observe that the largest dominant sets containing the nondominant generator xyz *are* $\{x^3y, xyz\}$*,* $\{y^2z, xyz\}$*, and* $\{xz^2, xyz\}$*. It follows from Corollary 2.4 (i) that* $\text{pd}(S/M) = 2$*.*

Besides that, according to Corollary 2.4 (ii), $\text{b}_2(S/M)$ *is given by the formula:*

$$\text{b}_2(S/M) = \#\{[m_i, m_j]/n \nmid \text{mdeg}[m_i, m_j]\} + \#\{[m_i, n]/n \nmid \text{mdeg}[m_i]\} = \#\{\} + \#\{[x^3y, xyz]; [y^2z, xyz]; [xz^2, xyz]\} = 3.$$

($\text{b}_1(S/M)$ *and* $\text{b}_0(S/M)$ *are always easily obtained from* $\mathbb{T}_M$*.) Finally, by Corollary 2.6 we have* $\text{reg}(S/M) = \max\{\deg(\text{mdeg}[x^3y, xyz]) - 2; \deg(\text{mdeg}[y^2z, xyz]) - 2; \deg(\text{mdeg}[xz^2, xyz]) - 2\} = \max\{5-2;\ 4-2;\ 4-2\} = 3.$

All our calculations are consistent with the information encoded in $\mathbb{F}$, as we can easily verify.

2.4 2-semidominant Ideals

The concepts of dominance and semidominance lead in a natural way to the more general definition of p-semidominance, which we give next.

Definition 2.4 *A set of monomials is called p-**semidominant** if it contains exactly p nondominant monomials. A monomial ideal is called p-**semidominant** if its minimal generating set is p-semidominant.*

With this definition, dominant and semidominant ideals can be thought of as being 0-semidominant and 1-semidominant, respectively. Sometimes, the word semidominant is used to denote 1-semidominant ideals while other times it makes reference to p-semidominant ideals in general (as in the title of this thesis). The meaning will be clear from the context.

In this section we will construct the minimal free resolution of 2-semidominant ideals; that is, monomial ideals M with minimal generating set $G = \{m_1, \ldots, m_q, n_1, n_2\}$ where $m_1, \ldots, m_q$ are dominant and n_1 and n_2 are nondominant. First, we want to know the character of the entries of the differential matrices of $\mathbb{T}_M$.

Lemma 2.6 *Let M be a 2-semidominant ideal. If two basis elements of a resolution of S/M, in consecutive homological degrees, have the same multidegree, then they are face and facet.*

Proof. Let $[\sigma]$ and $[\tau]$ be basis elements in homological degrees $j + 1$ and j, respectively. If $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, then $[\sigma]$ and $[\tau]$ must be generated by the same dominant monomials. Given that $[\sigma]$ has one more generator than $[\tau]$, if $[\tau]$ contains no nondominant generator, $[\sigma]$ must contain exactly one. On the other hand, if $[\tau]$ contains one nondominant generator, then $[\sigma]$ must contain both nondominant generators. The possibilities are four:

(i) $[\tau] = [m_{i_1}, \ldots, m_{i_j}]$; $[\sigma] = [m_{i_1}, \ldots, m_{i_j}, n_1]$;

(ii) $[\tau] = [m_{i_1}, \ldots, m_{i_j}]$; $[\sigma] = [m_{i_1}, \ldots, m_{i_j}, n_2]$;

(iii) $[\tau] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1]$; $[\sigma] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1, n_2]$;

(iv) $[\tau] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_2]$; $[\sigma] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1, n_2]$.

In every case we see that $[\tau]$ is a facet of $[\sigma]$. ■

Our next goal is to prove that the basis of the minimal free resolution of S/M can be obtained from the basis of its Taylor resolution by eliminating pairs of basis elements $[\sigma]$, $[\tau]$ in an arbitrary order, where $[\tau]$ is a facet of $[\sigma]$ and $\text{mdeg}[\sigma] = \text{mdeg}[\tau]$, until exhausting all possibilities.

If this idea is going to succeed, we need first to confirm that the following dangerous scenario never occurs. Suppose that $([\sigma_1], [\tau_1])$ and $([\sigma_2], [\tau_2])$ are disjoint pairs of face and facet with $\text{mdeg}[\sigma_i] = \text{mdeg}[\tau_i]$. Let $([\sigma_1], [\tau_1])$ determine the invertible entry a_{rs} of the differential matrix (f_{j+1}) of $\mathbb{T}_M$. Then eliminating $[\sigma_1]$ and $[\tau_1]$ from the basis of $\mathbb{T}_M$ is equivalent to making the standard change of basis around a_{rs}, followed by the standard cancellation $0 \to S[\sigma_1] \to S[\tau_1] \to 0$.

Similarly, $([\sigma_2], [\tau_2])$ defines an invertible entry a_{cd} and eliminating $[\sigma_2], [\tau_2]$ from the basis of the Taylor resolution is equivalent to making a standard change of basis around a_{cd}, followed by the standard cancellation $0 \to S[\sigma_2] \to S[\tau_2] \to 0$.

However, when we make the standard change of basis around a_{rs}, the entries of the matrices change. In particular, the entry a_{cd} might become noninvertible, which would prevent us from doing the standard cancellation $0 \to S[\sigma_2] \to S[\tau_2] \to 0$.

In the next lemma, which is analogous to Lemma 2.4, we show that this scenario is not possible for 2-semidominant ideals.

Lemma 2.7 *Let M be a 2-semidominant ideal. Let $\mathbb{F}$ be a free resolution of S/M obtained from $\mathbb{T}_M$ by means of standard cancellations. Let $a_{\tau\sigma}$ and $a_{\pi\theta}$ be two invertible entries of $\mathbb{F}$, corresponding to two disjoint pairs of basis elements $([\sigma],[\tau])$ and $([\theta],[\pi])$ of $\mathbb{F}$, respectively. Then after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, it is possible to make the standard cancellation $0 \to S[\theta] \to S[\pi] \to 0$.*

Proof. $[\sigma]$ and $[\tau]$ are basis elements in homological degrees j and $j-1$, respectively, for some j. Thus $a_{\tau\sigma}$ is an entry of the differential matrix (f_j) of $\mathbb{F}$. Similarly, $[\theta]$ and $[\pi]$ are basis elements in some homological degrees k and $k-1$, and $a_{\pi\theta}$ is an entry of the differential matrix (f_k) of $\mathbb{F}$.

In order to prove the lemma, it is enough to show that after making the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, the entry $a'_{\pi\theta}$ of the differential matrix (f'_k) of the new resolution $\mathbb{F}'$ is invertible.

Given that only (f_{j+1}), (f_j) and (f_{j-1}) are affected by the standard cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, if $k \neq j-1, j, j+1$ then $a'_{\pi\theta} = a_{\pi\theta}$; that is, $a'_{\pi\theta}$ is invertible. Therefore, we only need to prove that $a'_{\pi\theta}$ is invertible in the following cases:
$k = j$; $k = j-1$, $k = j+1$.

Suposse $k = j$. Since $a_{\pi\theta}$ is invertible, $\text{mdeg}[\pi] = \text{mdeg}[\theta]$. Then $a'_{\pi\theta} = 0$ or $a'_{\pi\theta}$ is invertible. Let us assume that $a'_{\pi\theta} = 0$. By Lemma 2.1 (iii), we have that $0 = a'_{\pi\theta} = a_{\pi\theta} - \dfrac{a_{\pi\sigma}a_{\tau\theta}}{a_{\tau\sigma}}$. It follows that $a_{\pi\theta}a_{\tau\sigma} = a_{\pi\sigma}a_{\tau\theta}$ and, since $a_{\pi\theta}$ and $a_{\tau\sigma}$ are invertible, $a_{\pi\sigma}$ and $a_{\tau\theta}$ must be invertible too. In particular, the fact that $a_{\pi\sigma}$ is invertible implies that $\text{mdeg}[\sigma] = \text{mdeg}[\pi]$ which, combined with the hypothesis

$\mathrm{mdeg}[\sigma] = \mathrm{mdeg}[\tau]$, implies that $\mathrm{mdeg}[\tau] = \mathrm{mdeg}[\pi]$.

In particular, $[\tau]$ and $[\pi]$ contain the same dominant monomials and thus they differ in the nondominant monomials that define them. Since $[\tau]$ and $[\pi]$ appear in the same homological degree, they must contain exactly one nondominant generator each. Then $[\tau]$ and $[\pi]$ are of the form $[\tau] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1]$; $[\pi] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_2]$. Given that $\mathrm{mdeg}[\tau] = \mathrm{mdeg}[\theta]$, and the fact that $[\tau]$ and $[\theta]$ appear in homological degrees $j-1$ and j, respectively, it follows from Lemma 2.6 that $[\tau]$ is a facet of $[\theta]$. Thus θ must be of the form $[\theta] = [m_{i_1}, \ldots, m_{i_{j-1}}, n_1, n_2]$. Since $[\tau]$ is also a facet of $[\sigma]$, the same reasoning applies to $[\sigma]$, which means that $[\sigma] = [\theta]$, a contradiction. We conclude that $a'_{\pi\theta}$ is invertible.

The cases $k = j-1$ and $k = j+1$ are as in the proof of Lemma 2.4. ■

Theorem 2.6 *Let M be a 2-semidominant ideal. Let $([\sigma_1], [\tau_1]), \ldots, ([\sigma_k], [\tau_k])$ be k pairs of basis elements of $\mathbb{T}_M$, satisfying the following properties:*

(i) $([\sigma_i], [\tau_i])$ *and* $([\sigma_j], [\tau_j])$ *are disjoint, if $i \neq j$.*

(ii) $[\tau_i]$ *is a facet of* $[\sigma_i]$ *for all $i = 1, \ldots k$.*

(iii) $\mathrm{mdeg}[\sigma_i] = \mathrm{mdeg}[\tau_i]$ *for all $i = 1, \ldots k$.*

Then, starting with $\mathbb{T}_M$, it is possible to make the following sequence of standard cancellations:

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \quad \cdots \quad , 0 \to S[\sigma_k] \to S[\tau_k] \to 0.$$

Proof. Identical to the proof of Theorem 2.4. ■

Theorem 2.7 *Let M be a 2-semidominant ideal. Let $A = \{([\sigma_1], [\tau_1]), \ldots, ([\sigma_k], [\tau_k])\}$ be a family of pairs of basis elements in $\mathbb{T}_M$, having the following properties:*

(i) $([\sigma_i], [\tau_i])$ *and* $([\sigma_j], [\tau_j])$ *are disjoint, if $i \neq j$.*

(ii) $[\tau_i]$ *is a facet of* $[\sigma_i]$ *for all* $i = 1, \dots k$.

(iii) $\text{mdeg}[\sigma_i] = \text{mdeg}[\tau_i]$ *for all* $i = 1, \dots k$.

(iv) A is maximal with respect to inclusion among the sets satisfying i), ii) and iii).

Then a minimal free resolution $\mathbb{F}$ *of* S/M *can be obtained from* $\mathbb{T}_M$ *by doing all standard cancellations* $0 \to S[\sigma] \to S[\tau] \to 0$, *with* $([\sigma], [\tau]) \in A$. *In symbols,*

$$\mathbb{T}_M = \mathbb{F} \oplus \left(\bigoplus_{([\sigma],[\tau]) \in A} 0 \to S[\sigma] \to S[\tau] \to 0 \right).$$

Proof. By Theorem 2.6, $\mathbb{F}$ is a resolution of S/M. We claim that $\mathbb{F}$ is minimal. If $\mathbb{F}$ were not minimal, one of its differential matrices would contain an invertible entry. That, in turn, would mean that there exists a pair $([\sigma], [\tau])$ of basis elements of $\mathbb{T}_M$, such that $A \bigcup \{([\sigma], [\tau])\}$ satisfies conditions (i), (ii), and (iii), which contradicts (iv). ■

We have explained that all minimal resolutions of 1-semidominant ideals, obtained from $\mathbb{T}_M$ by eliminating faces and facets of equal multidegree, have a common basis. However, the bases of the minimal resolutions of 2-semidominant ideals, obtained in the same way, are not unique, as the next example shows.

Example 2.8 *Let* $M = (x^2y^2, xz, yz)$. *The only repeated multidegree is* $m = x^2y^2z$, *which is common to the three basis elements* $[\sigma] = [x^2y^2, xz, yz]$, $[\tau_1] = [x^2y^2, xz]$, *and* $[\tau_2] = [x^2y^2, yz]$. *By eliminating the pair* $[\sigma]$, $[\tau_1]$ *from the basis of* $\mathbb{T}_M$, *we obtain the basis of a minimal resolution of* S/M. *By eliminating the pair* $[\sigma]$, $[\tau_2]$ *from the basis of* $\mathbb{T}_M$, *we obtain a different basis of another minimal resolution of* S/M.

Theorem 2.8 *(Characterization of the Scarf 2-semidominant Ideals)*

Let M *be a 2-semidominant ideal.*

Let $B = \{m : m$ *is the multidegree of more than one basis element of* $\mathbb{T}_M\}$. *For each*

$m \in B$, let $B_m = \{[\sigma] \in \mathbb{T}_M : \text{mdeg}[\sigma] = m\}$. Then M is Scarf if and only if $\#(B_m)$ is even for all $m \in B$.

Proof. Let $G = \{m_1, \ldots, m_q, n_1, n_2\}$ be the minimal generating set of M. Let us denote with $\mathbb{F}$ the minimal resolution of S/M.

($\Rightarrow$) Let $m \in B$. Because M is Scarf, all elements of B_m are excluded from the basis of $\mathbb{F}$, but the elements of B_m are eliminated in pairs, making standard cancellations. It follows that $\#(B_m)$ is even.

($\Leftarrow$) Let $m \in B$. We need to prove that no element of the basis of $\mathbb{F}$ has multidegree m. Given that basis elements of $\mathbb{T}_M$ with the same multidegree contain the same dominant monomials, what distinguishes these elements is the nondominant monomials that define them. Thus there are at most four basis elements of multidegree m; namely,

$$[\sigma_1] = [m_{i_1}, \ldots, m_{i_r}]; \quad [\sigma_2] = [m_{i_1}, \ldots, m_{i_r}, n_1];$$

$$[\sigma_3] = [m_{i_1}, \ldots, m_{i_r}, n_2]; \quad [\sigma_4] = [m_{i_1}, \ldots, m_{i_r}, n_1, n_2].$$

The fact that $\#(B_m)$ is even implies that either

(i) $\#(B_m) = 4$ or (ii) $\#(B_m) = 2$.

(i) In this case $([\sigma_2], [\sigma_1])$, $([\sigma_4], [\sigma_3])$ and $\mathbb{T}_M$ satisfy the hypotheses of Lemma 2.7, which means that after making the standard cancellation $0 \to S[\sigma_2] \to S[\sigma_1] \to 0$ in $\mathbb{T}_M$, it is still possible to make the cancellation $0 \to S[\sigma_4] \to S[\sigma_3] \to 0$. Hence, the basis of $\mathbb{F}$ does not contain elements of multidegree m.

(ii) We will show that the two basis elements with multidegree m are face and facet. There are exactly two pairs of basis elements that are not face and facet; these pairs are $[\sigma_2], [\sigma_3]$ and $[\sigma_1], [\sigma_4]$. If we assume that $\text{mdeg}[\sigma_2] = \text{mdeg}[\sigma_3] = m$, then $n_2 \mid \text{mdeg}[\sigma_3] = \text{mdeg}[\sigma_2]$. It follows that $\text{mdeg}[\sigma_4] = \text{mdeg}[\sigma_2]$ and thus $[\sigma_4], [\sigma_2]$ and $[\sigma_3]$ have multidegree m, which is not possible because $\#(B_m) = 2$.

Similarly, if $\text{mdeg}[\sigma_1] = \text{mdeg}[\sigma_4]$, then $n_2 \mid \text{mdeg}[\sigma_4] = \text{mdeg}[\sigma_1]$, which implies that $[\sigma_3]$, $[\sigma_1]$ and $[\sigma_4]$ have multidegree m, which is not possible. Therefore, if $\text{mdeg}[\sigma_i] = \text{mdeg}[\sigma_j] = m$, then $[\sigma_i]$ and $[\sigma_j]$ must be face and facet. Thus they determine an invertible entry of $\mathbb{T}_M$, and it is possible to eliminate $[\sigma_i]$ and $[\sigma_j]$ from the basis of $\mathbb{T}_M$ by means of a standard cancellation. This means that no element of the basis of $\mathbb{F}$ has multidegree m. ■

Theorem 2.8 gives a complete characterization of the Scarf 2-semidominant ideals. This characterization, however, is difficult to verify in practice because it requires several calculations. In order to have a good mix between theoretical and practical results, we include two criteria to help determine whether a 2-semidominant ideal is Scarf. These two tests, although weaker than the preceding theorem, are easy to implement in concrete cases.

Corollary 2.7 *Let $M = (m_1, \ldots, m_q, n_1, n_2)$ be 2-semidominant. If M is Scarf, then $n_1, n_2 \mid \text{lcm}(m_1, \ldots, m_q)$.*

Proof. Let $m = \text{mdeg}[m_1, \ldots, m_q, n_1, n_2]$. Since n_1 is nondominant, $n_1 \mid \text{lcm}(m_1, \ldots, m_q, n_2)$, which means that $m = \text{mdeg}[m_1, \ldots, m_q, n_2]$. Similarly, since n_2 is nondominant, we must have that $n_2 \mid \text{lcm}(m_1, \ldots, m_q, n_1)$ and this implies that $m = \text{mdeg}[m_1, \ldots, m_q, n_1]$. This means that at least three basis elements of $\mathbb{T}_M$ have multidegree m. Now, in the proof of Theorem 2.8 we showed that for 2-semidominant ideals, there are at most four basis elements of $\mathbb{T}_M$ with a given multidegree. In our case, the fourth candidate is $[m_1, \ldots, m_q]$. If M is Scarf, it follows from Theorem 2.8 that the number of basis elements of $\mathbb{T}_M$ with multidegree m is even. Thus, we must have that $m = \text{mdeg}[m_1, \ldots, m_q]$. The last two equations imply that $n_1 \mid \text{lcm}(m_1, \ldots, m_q)$. Similarly, $n_2 \mid \text{lcm}(m_1, \ldots, m_q)$. ■

Corollary 2.8 *Let $M = (m_1, \ldots, m_q, n_1, n_2)$ be 2-semidominant. If no variable appears with the same nonzero exponent in n_1 and n_2, then M is Scarf.*

Proof. If we assume that M is not Scarf, then by Theorem 2.8, there is a multidegree m which is common to an odd number $k > 1$ of basis elements of $\mathbb{T}_M$. By the proof of Theorem 2.8, there are at most four basis elements with multidegree m. They are of the form $[\sigma_1] = [m_{i_1}, \ldots, m_{i_r}]$; $[\sigma_2] = [m_{i_1}, \ldots, m_{i_r}, n_1]$; $[\sigma_3] = [m_{i_1}, \ldots, m_{i_r}, n_2]$; $[\sigma_4] = [m_{i_1}, \ldots, m_{i_r}, n_1, n_2]$. Now given that $k > 1$ and odd, we must have $k = 3$. It is easy to verify that if exactly three of the four elements $[\sigma_1]$, $[\sigma_2]$, $[\sigma_3]$, $[\sigma_4]$ have multidegree m, these elements must be $[\sigma_2]$, $[\sigma_3]$, $[\sigma_4]$ (in any other case, that three of these elements have multidegree m would imply that the fourth one has multidegree m as well).

The fact that $\text{mdeg}[\sigma_1] \neq \text{mdeg}[\sigma_2]$ implies that $n_1 \nmid \text{lcm}(m_{i_1}, \ldots, m_{i_r})$. In particular, there is a variable x such that x appears with exponent $\alpha > 0$ in the factorization of n_1, and $x^\alpha \nmid \text{lcm}(m_{i_1}, \ldots, m_{i_r})$. On the other hand, the fact that $\text{mdeg}[\sigma_2] = \text{mdeg}[\sigma_3]$ implies that $x^\alpha \mid \text{lcm}(m_{i_1}, \ldots, m_{i_r}, n_2)$. Therefore, $x^\alpha \mid n_2$.

Let β be the exponent with which x appears in the factorization of n_2. Notice that if we had that $\alpha < \beta$ or $\alpha > \beta$, then we would also have that $\text{mdeg}[\sigma_2] \neq \text{mdeg}[\sigma_3]$. Thus x appears with the same nonzero exponent in the factorization of n_1 and n_2, a contradiction. ∎

In the context of 2-semidominant ideals, Corollary 2.8 extends a beautiful theorem by Bayer, Peeva and Sturmfels [BPS], that states the following: If M is a generic ideal, then M is Scarf.

Let us see how Corollaries 2.7 and 2.8 work in practice.

Example 2.9 *Let $M_1 = (x^3y, y^2z, yz^4, xz^2w, x^2zw)$ and $M_2 = (x^3y, y^2z, yz^4, xz^2, x^2z)$. Notice that M_1 is 2-semidominant, $n_1 = xz^2w$ and $n_2 = x^2zw$ being the nondominant generators. Since w appears in the factorization of n_1 but not in the factorization of any of the dominant monomials $m_1 = x^3y$, $m_2 = y^2z$, $m_3 = yz^4$, we have that $n_1 \nmid lcm(m_1, m_2, m_3)$. Thus, by Corollary 2.7, we have that M_1 is not Scarf. Now* *observe that M_2 is also 2-semidominant, $n_1 = xz^2$ and $n_2 = x^2z$ being the nondominant generators. Since neither x nor z appears with the same nonzero exponent in the factorization of n_1 and n_2, it follows from Corollary 2.8 that M_2 is Scarf. Incidentally, note that M_2 is not generic. We chose two very similar ideals M_1 and M_2 to show how sensitive monomial resolutions are.*

2.5 Standard Cancellations in Arbitrary Order

In this last section of Chapter 2 we depart from the concept of p-semidominance, and study certain conditions under which the minimal resolution of S/M can be obtained from $\mathbb{T}_M$ by making consecutive cancellations in arbitrary order.

Theorem 2.9 *Let M be a monomial ideal. Let us assume that for every basis element $[\tau]$ of $\mathbb{T}_M$, which is a common facet of two faces $[\sigma_1]$ and $[\sigma_2]$, such that $\text{mdeg}[\sigma_1] = \text{mdeg}[\sigma_2] = \text{mdeg}[\tau] = m$, the following property holds:*

$$\textit{whenever } [\tau'] \neq [\tau] \textit{ is a facet of } [\sigma_1] \textit{ or } [\sigma_2], \ \text{mdeg}[\tau'] \neq m.$$

Then the basis of the minimal resolution of S/M can be obtained from the basis of $\mathbb{T}_M$, eliminating pairs of face and facet of equal multidegree in arbitrary order, until exhausting all possibilities. ■

The proof of this theorem follows from the next three lemmas.

Lemma 2.8 *Under the hypotheses of Theorem 2.9, if $\mathbb{F}$ is a resolution of S/M, obtained from $\mathbb{T}_M$ by means of consecutive cancellations, then an entry $b_{\tau\sigma}$ of a differential of $\mathbb{F}$ is invertible if and only if $[\tau]$ is a facet of $[\sigma]$ and $\text{mdeg}[\tau] = \text{mdeg}[\sigma]$.*

Proof. The proof is by induction on the number k of consecutive cancellations made to obtain $\mathbb{F}$. If $k = 0$, the statement is true because $\mathbb{F} = \mathbb{T}_M$. Assume that the lemma is true for $k = l - 1$. Let us prove that the lemma holds for $k = l$. Let

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \cdots, 0 \to S[\sigma_l] \to S[\tau_l] \to 0$$

be the sequence of consecutive cancellations made to obtain $\mathbb{F}$. By induction hypothesis, when we make the first $l-1$ cancellations

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \cdots, 0 \to S[\sigma_{l-1}] \to S[\tau_{l-1}] \to 0,$$

we obtain a free resolution $\mathbb{F}'$ whose differential matrices have the following property: $a_{\tau\sigma}$ is an invertible entry if and only if τ is a facet of σ, and $\text{mdeg}[\tau] = \text{mdeg}[\sigma]$.

($\Rightarrow$) Let us assume that when we make the consecutive cancellation $0 \to S[\sigma_l] \to S[\tau_l] \to 0$ in $\mathbb{F}'$, one of the entries $b_{\tau\sigma}$ of a differential matrix of $\mathbb{F}$ is invertible but $[\tau]$ is not a facet of $[\sigma]$ (the fact that $b_{\tau\sigma}$ is invertible implies that $\text{mdeg}[\tau] = \text{mdeg}[\sigma]$). We derive a contradiction. Let $a_{\tau\sigma}$ be the entry determined by $[\sigma]$ and $[\tau]$ in $\mathbb{F}'$. Then

$$b_{\tau\sigma} = a_{\tau\sigma} - \frac{a_{\tau_l\sigma}a_{\tau\sigma_l}}{a_{\tau_l\sigma_l}}.$$

Since $[\tau]$ is not a facet of $[\sigma]$, $a_{\tau\sigma}$ is not invertible by induction hypothesis. On the other hand, since $\text{mdeg}[\tau] = \text{mdeg}[\sigma]$, we must have that $a_{\tau\sigma} = 0$. Thus, $b_{\tau\sigma}a_{\tau_l\sigma_l} = -a_{\tau_l\sigma}a_{\tau\sigma_l} \Rightarrow a_{\tau_l\sigma}$ and $a_{\tau\sigma_l}$ are invertible, implying that $[\tau_l]$ is a facet of both $[\sigma]$ and $[\sigma_l]$, while $[\tau]$ is a facet of $[\sigma_l]$ with $\text{mdeg}[\sigma] = \text{mdeg}[\tau_l] = \text{mdeg}[\sigma_l] = \text{mdeg}[\tau]$. This contradicts the hypotheses of Theorem 2.9.

($\Leftarrow$) Let us assume that when we make the consecutive cancellation $0 \to S[\sigma_l] \to S[\tau_l] \to 0$ in $\mathbb{F}'$, one of the entries of a differential of $\mathbb{F}$ is $b_{\tau\sigma} = 0$, where $[\tau]$ is a facet of $[\sigma]$ and $\text{mdeg}[\tau] = \text{mdeg}[\sigma]$. We derive a contradiction. Let $a_{\tau\sigma}$ be the entry determined by $[\tau]$ and $[\sigma]$ in $\mathbb{F}'$. Then

$$0 = b_{\tau\sigma} = a_{\tau\sigma} - \frac{a_{\tau_l\sigma}a_{\tau\sigma_l}}{a_{\tau_l\sigma_l}}.$$

It follows that $a_{\tau\sigma}a_{\tau_l\sigma_l} = a_{\tau_l\sigma}a_{\tau\sigma_l}$. By induction hypothesis, $a_{\tau\sigma}$ is invertible and hence, the left hand side is invertible. This implies that $a_{\tau_l\sigma}a_{\tau\sigma_l}$ must be invertible. This means that $[\tau]$ is a facet of both $[\sigma]$ and $[\sigma_l]$, while $[\tau_l]$ is a facet of $[\sigma]$, and $\text{mdeg}[\sigma] = \text{mdeg}[\tau] = \text{mdeg}[\sigma_l] = \text{mdeg}[\tau_l]$. This contradicts the hypotheses of Theorem 2.9. ■

Lemma 2.9 *Under the hypotheses of Theorem 2.9, let* $\mathbb{F}$ *be a resolution of* S/M*, obtained from* $\mathbb{T}_M$ *by means of consecutive cancellations. If* $a_{\tau\sigma}$ *and* $a_{\pi\theta}$ *are two invertible entries of* $\mathbb{F}$*, determined by two disjoint pairs* $([\sigma], [\tau])$ *and* $([\theta], [\pi])$*, then after making the consecutive cancellation* $0 \to S[\sigma] \to S[\tau] \to 0$*, it is possible to make the consecutive cancellation* $0 \to S[\theta] \to S[\pi] \to 0$.

Proof. Since $a_{\pi\theta}$ is invertible, it follows from Lemma 2.8 that $[\pi]$ is a facet of $[\theta]$ and $\text{mdeg}[\pi] = \text{mdeg}[\theta]$. Then, by Lemma 2.8 again, after making the cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{F}$, the entry $b_{\pi\theta}$ of the resulting resolution is invertible and, therefore, it is possible to make the cancellation $0 \to S[\theta] \to S[\pi] \to 0$. ■

Lemma 2.10 *Under the hypotheses of Theorem 2.9, assume that* $([\sigma_1], [\tau_1]), \cdots, ([\sigma_k], [\tau_k])$ *are* k *pairs of basis elements of* $\mathbb{T}_M$*, satisfying the following properties:*

(i) $([\sigma_i], [\tau_i])$ *and* $([\sigma_j], [\tau_j])$ *are disjoint if* $i \neq j$.

(ii) $[\tau_i]$ *is a facet of* $[\sigma_i]$*, for all* $i = 1, \cdots, k$.

(iii) $\text{mdeg}[\sigma_i] = \text{mdeg}[\tau_i]$*, for all* $i = 1, \cdots, k$.

Then starting with $\mathbb{T}_M$*, it is possible to make the following sequence of consecutive cancellations:*

$$0 \to S[\sigma_1] \to S[\tau_1] \to 0, \cdots, 0 \to S[\sigma_k] \to S[\tau_k] \to 0.$$

Proof. Identical to the proof of Theorem 2.4 (semidominant case) and the proof of Theorem 2.6 (2-semidominant case). ■

The proof of Theorem 2.9 is now a simple consequence of the preceding corollaries.

Proof. [of Theorem 2.9]

By Lemma 2.10, after eliminating pairs of face and facet of equal multidegree in arbitrary order, until exhausting all possibilities, we obtain the basis of a free resolution $\mathbb{F}$ of S/M. If we assume that $\mathbb{F}$ is not minimal, then there is a differential matrix of $\mathbb{F}$ that contains an invertible entry $a_{\tau\sigma}$. By Lemma 2.8, $[\sigma]$ and $[\tau]$ are face and facet of equal multidegree, which means that not all possibilities have been exhausted, a contradiction. ∎

Example 2.10 *Let* $m_1 = x_1x_4x_7x_9$*;* $m_2 = x_2x_5x_7x_8$*;* $m_3 = x_3x_6x_8x_9$*;* $m_4 = x_1x_2x_3$*;* $m_5 = x_4x_5x_6$*. Let* $M = (m_1, m_2, \ldots, m_5)$*. (Notice that* M *is 5-semidominant.) It is easy to verify that the only multidegree that is common to more than one basis element of* $\mathbb{T}_M$ *is* $m = x_1x_2 \ldots x_9$*. The following table shows all basis elements of multidegree* m*, and their corresponding homological degrees.*

Table 2.1: Elements of Multidegree m

homological degree	basis elements
3	$[\tau] = [m_1, m_2, m_3]$
4	$[\sigma_i] = [m_1, \ldots, \widehat{m_i}, \ldots, m_5]$; $i = 1, \ldots, 5$
5	$[\theta] = [m_1, \ldots, m_5]$

Note that the only instance in which we have two faces of multidegree m *with a common facet of multidegree* m *is when the faces are* $[\sigma_4]$ *and* $[\sigma_5]$*, and the common facet is* $[\tau]$*. Since neither* $[\sigma_4]$ *nor* $[\sigma_5]$ *have other facets of multidegree* m*, the hypotheses of Theorem 2.9 are satisfied, and we can obtain a basis of a minimal resolution of* S/M *by eliminating pairs of faces and facets of multidegree* m*. By simple inspection, we conclude that in every case, this process consists of two eliminations of the form* $([\theta], [\sigma_i])$ *and* $([\sigma_j], [\tau])$*, where* $i \in \{1, 2, \ldots, 5\}$*;* $j \in \{4, 5\}$*, and* $i \neq j$*. For example, the basis of a minimal resolution of* S/M *can be obtained from the basis of* $\mathbb{T}_M$ *by* *eliminating* $([\theta],[\sigma_2])$ *and* $([\sigma_5],[\tau])$*.*

Example 2.11 *Let* $m_1 = x_1x_2x_3$*;* $m_2 = x_1x_4x_6$*;* $m_3 = x_3x_5x_6$*;* $m_4 = x_2x_4x_5$*;* $m_5 = x_3x_7$*. Let* $M = (m_1, m_2, \ldots, m_5)$*. (Notice that* M *is 4-semidominant.) In order for* M *to violate the hypothesis of Theorem 2.9, there must exist four basis elements of* $\mathbb{T}_M$ *with a common multidegree, two of them in some homological degree* k *and the other two in homological degree* $k+1$*. We will show that this does not happen.*

Notice that $m = x_1x_2\ldots x_6$ *and* $m' = x_1x_2\ldots x_7$ *are the only two multidegrees that are common to more than one Taylor symbol. Now, the basis elements of* $\mathbb{T}_M$ *with multidegree* m *are* $[m_1,\ldots,m_4]$*, in homological degree* 4*, and its four facets, in homological degree* 3*. On the other hand, the basis elements of* $\mathbb{T}_M$ *having multidegree* m' *are* $[m_1,\ldots,m_5]$*, in homological degree* 5*; four of its facets, in homological degree* 4*, and* $[m_1,m_4,m_5]$ *in homological degree* 3*. Therefore, it is impossible to find four basis elements with a common multidegree; two in homological degree* k *and two in homological degree* $k{+}1$*. By Theorem 2.9, the basis of a minimal resolution of* S/M *can be obtained from* $\mathbb{T}_M$ *by removing pairs of face and facet of equal multidegree in arbitrary order until exhausting all possibilities. For instance, remove* $([m_1,\ldots,m_4],[m_1,\ldots,m_3])$*;* $([m_1,\ldots,m_5],[m_2,\ldots,m_5])$ *and* $([m_1,m_2,m_4,m_5],[m_2,m_4,m_5])$*.*

CHAPTER 3

APPLICATIONS

3.1 1-cancellations

Since the concept of semidominance is obtained from that of dominance via a minor modification, it is reasonable to think of 1-semidominant ideals as objects that are close to being dominant. However, when we studied the combinatorial properties of dominant and 1-semidominant ideals, we observed a radically different behavior (see Corollary 2.5 and Example 2.6). The similarity between these two classes of monomial ideals lies on the way we construct them, not on their combinatorial properties.

In this section we define and study new monomial ideals which are very close to being dominant from a combinatorial point of view.

Definition 3.1 *a monomial ideal M is called a **1-cancellation** ideal (or simply, a **1-cancellation**), if a minimal resolution of S/M can be obtained from $\mathbb{T}_M$ by means of exactly one consecutive cancellation.*

Note that if M is a 1-cancellation ideal, $\mathbb{T}_M$ is not minimal. Then, by the equivalence between the statements (i) and (iv) of Corollary 2.2, it follows that the only consecutive cancellation in $\mathbb{T}_M$ occurs in the last two homological degrees.

Theorem 3.1 *Let M be a monomial ideal minimally generated by $G = \{m_1, \ldots, m_q\}$. Then M is a 1-cancellation ideal if and only if G is not dominant but every subset $G \setminus \{m_i\}$ is.*

Proof. ($\Rightarrow$) Since M is 1-cancellation, M is not dominant and, hence, G is not dominant. On the other hand, if we assume that a subset $G \setminus \{m_i\}$ is not dominant, there is a monomial $m_j \in G \setminus \{m_i\}$, such that $m_j \mid \mathrm{lcm}(G \setminus \{m_i, m_j\})$. It follows that $\mathrm{lcm}(G \setminus \{m_i, m_j\}) = \mathrm{lcm}(G \setminus \{m_i\})$. Let $[\sigma]$ and $[\tau]$ be the Taylor symbols defined by $G \setminus \{m_i\}$ and $G \setminus \{m_i, m_j\}$, respectively. Then $[\sigma]$ and $[\tau]$ are face and facet of equal multidegree, and they appear in homological degrees $q-1$ and $q-2$. Thus, it is possible to make the cancellation $0 \to S[\sigma] \to S[\tau] \to 0$ in $\mathbb{T}_M$, which contradicts the fact that the the only cancellation occurs in the last two homological degrees of $\mathbb{T}_M$.

($\Leftarrow$) Suppose that $[\sigma]$ and $[\tau]$ are basis elements of $\mathbb{T}_M$, such that $[\tau]$ is a facet of $[\sigma]$ and $\mathrm{mdeg}[\sigma] = \mathrm{mdeg}[\tau]$. Then σ is not a dominant set. If $\#(\sigma) < q$, then there is a set of the form $G \setminus \{m_i\}$, such that $\sigma \subseteq G \setminus \{m_i\}$. This implies that $G \setminus \{m_i\}$ is not dominant, a contradiction. Thus, $\#(\sigma) = q$.

We have proved that if $\mathbb{T}_M$ admits a consecutive cancellation, it must take place in homological degrees q and $q-1$. The fact that G is not dominant, implies that $\mathbb{T}_M$ admits such a cancellation. Since $\mathbb{T}_M$ contains only one basis element in homological degree q, after making that consecutive cancellation, we obtain a minimal resolution of S/M. ■

Example 3.1 *Let $M = (x^2, y^2, xy)$. Note that $G = \{x^2, y^2, xy\}$ is nondominant, but $G \setminus \{x^2\}$; $G \setminus \{y^2\}$; $G \setminus \{xy\}$ are. Thus, by Theorem 3.1 M is a 1-cancellation.*

Note: in general, if M is minimally generated by three monomials, either $\mathbb{T}_M$ is minimal or the minimal resolution of S/M is obtained from $\mathbb{T}_M$ by making exactly one cancellation. That is, M is either dominant or a 1-cancellation.

The preceding example is a particular case of a more general construction which, in turn, is a corollary to Theorem 3.1.

Corollary 3.1 *Let M be a 1-semidominant ideal, minimally generated by $G = \{m_1, \ldots, m_q, m'\}$. If $(m_1, \ldots, m_q)$ is a complete intersection and for all $i = 1, \ldots, q$, $\gcd(m_i, m') \neq 1$, then M is a 1-cancellation.*

Example 3.2 *Let* $M = (x_1^2, \ldots, x_q^2, x_1x_2 \ldots x_q)$.

By Corollary 3.1, M *is a* 1*-cancellation.*

Example 3.3 *Let* M *be minimally generated by* $G = \{m_1 = x_1x_2x_3; m_2 = x_1x_4x_5; m_3 = x_2x_4x_6; m_4 = x_3x_5x_6\}$. *Notice that* G *is 4-semidominant (which means that* G *is not dominant). However, each set* $G \setminus \{m_i\}$ *is dominant. By Theorem 3.1,* M *is a 1-cancellation.*

In the next theorem we study some combinatorial properties of 1-cancellations.

Theorem 3.2 *Let* M *be a 1-cancellation minimally generated by* $G = \{m_1, \ldots, m_q\}$. *Let* $\mathbb{F}$ *be the minimal resolution of* S/M *obtained from* $\mathbb{T}_M$*, after making one consecutive cancellation. Then*

(i) $\text{pd}\,(S/M) = q - 1$.

(ii) $\text{reg}\,(S/M) = \max\{\deg[\sigma] - (q-1)$, *with* $[\sigma]$ *a basis element of* $\mathbb{F}$, *and* $\text{hdeg}[\sigma] = q - 1\}$.

Proof. (i) Trivial. (ii) Let $[\tau]$ be a basis element of $\mathbb{F}$. Then there is a basis element $[G \setminus \{m_i\}]$ of $\mathbb{F}$, such that $\tau \subseteq G \setminus \{m_i\}$. This means that τ is of the form $\tau = G \setminus \{m_{i_1}, \ldots, m_{i_s}\}$, and $\text{hdeg}[\tau] = q - s$. Now, by Theorem 3.1, $G \setminus \{m_i\}$ is dominant, which implies that $\deg[G \setminus \{m_{i_1}\}] \geq \deg[G \setminus \{m_{i_1}, m_{i_2}] + 1 \geq \cdots \geq \deg[G \setminus \{m_{i_1}, \ldots, m_{i_s}\}] + (s - 1) = \deg[\tau] + (s - 1)$ Hence

$$\begin{aligned}
&\max\{\deg[\sigma] - (q-1), \text{ with } [\sigma] \text{ a basis element of } \mathbb{F}, \text{ and } \text{hdeg}[\sigma] = q-1\} \\
&\quad \geq \deg[G \setminus \{m_{i_1}\}] - (q-1) \\
&\quad \geq \deg[\tau] + (s-1) - (q-1) \\
&\quad = \deg[\tau] - (q-s) \\
&\quad = \deg[\tau] - \text{hdeg}[\tau].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\max\{\deg[\sigma] - (q-1), \text{ with } [\sigma] \text{ a basis element of } \mathbb{F}, \text{ and } \text{hdeg}[\sigma] = q-1\} \\
&\quad \geq \max\{\deg[\tau] - \text{hdeg}[\tau], \text{ with } [\tau] \text{ a basis element of } \mathbb{F}\} \\
&\quad = \text{reg}\,(S/M).
\end{aligned}$$

■

3.2 A Special Family of 1-cancellations

We now construct an explicit family of 1-cancellations, which is realated to three interesting open problems.

Theorem 3.3 *Let $p \geq 1$. Let $S = k[x_1, \ldots, x_k]$, where $k = \binom{p+1}{2}$. Then there exist $p+1$ square-free monomials $m_1, \ldots, m_{p+1}$, of degree p, such that*

(i) Each variable x_i divides exactly two of the monomials $m_1, \ldots, m_{p+1}$.

(ii) For every pair of monomials m_s, m_t, there is exactly one variable x_i that divides both m_s and m_t.

Proof. Let $A = \{y_{i,j}$, with $1 \leq i < j \leq p+1\}$ be a set of formal objects. Let $B = \{x_1, \ldots, x_k\}$. Since $\#A = \binom{p+1}{2} = \dfrac{p(p+1)}{2} = \#B$, there is a bijection $f : A \to B$. For all $i = 1, \ldots, p+1$, let $m_i = \prod_{h=i+1}^{p+1} f(y_{i,h}) \prod_{h=1}^{i-1} f(y_{h,i})$. Then $m_1, \ldots, m_{p+1}$ are $p+1$ square-free monomials of degree p. We claim that this monomials satisfy properties (i) and (ii) of this theorem.

(i) Let $x_i \in B$. Let $y_{s,t}$ be the (only) element in A such that $f(y_{s,t}) = x_i$. Then x_i appears in the factorization of $m_s = \prod_{h=s+1}^{p+1} f(y_{s,h}) \prod_{h=1}^{s-1} f(y_{h,s})$ (when $h = t$, we obtain $f(y_{s,t}) = x_i$). Similarly, x_i appears in the factorization of $m_t = \prod_{h=t+1}^{p+1} f(y_{t,h}) \prod_{h=1}^{t-1} f(y_{h,t})$ (when $h = s$, we obtain $f(y_{s,t}) = x_i$). Moreover, by construction of m_j, x_i does not appear in the factorization of m_j, if $j \neq s$ and $j \neq t$. (ii) Let us say that $1 \leq s < t \leq p+1$. Then $f(y_{s,t})$ is a factor of $m_s = \prod_{h=s+1}^{p+1} f(y_{s,h}) \prod_{h=1}^{s-1} f(y_{h,s})$ (corresponding to $h = t$). Similarly, $f(y_{s,t})$ is a factor of $m_t = \prod_{h=t+1}^{p+1} f(y_{t,h}) \prod_{h=1}^{t-1} f(y_{h,t})$ (corresponding to $h = s$). Thus, $f(y_{h,t})$ divides both m_s and m_t. Now, suppose that $f(y_{u,v})$ is a variable that divides m_s and m_t. By construction of m_s, either $u = s$ or $v = s$. Likewise, by construction of m_t, either $u = t$ or $v = t$. Then $\{u, v\} = \{s, t\}$. It follows that $f(y_{u,v}) = f(y_{s,t})$ or $f(y_{u,v}) = f(y_{t,s})$. Since $s < t$, $y_{t,s} \notin A$. Hence, $f(y_{u,v}) = f(y_{s,t})$, which means that there exists exactly one variable that divides both m_s and m_t. ∎

Since the proof of the theorem is not constructive, we explain how to construct $m_1, \ldots, m_{p+1}$, explicitly, using a simple diagram.

Consider the following right triangle containing the variables $x_1, \ldots, x_k$

$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & \cdots & x_p \\
 & x_{p+1} & x_{p+2} & x_{p+3} & \cdots & x_{2p-1} \\
 & & x_{2p} & x_{2p+1} & \cdots & x_{3p-(1+2)} \\
 & & & x_{3p-2} & \cdots & x_{4p-(1+2+3)} \\
 & & & & \ddots & \vdots \\
 & & & & & x_{(p+1)p-(1+2+\ldots+p)}
\end{array}$$

Now the monomials $m_1, \ldots, m_q$ are obtained from the union of this right triangle and its reflection across the hypothenuse:

$$\begin{array}{cccccccc}
m_1 & = & x_1 & x_2 & x_3 & x_4 & \cdots & x_p \\
m_2 & = & x_1 & x_{p+1} & x_{p+2} & x_{p+3} & \cdots & x_{2p-1} \\
m_3 & = & x_2 & x_{p+1} & x_{2p} & x_{2p+1} & \cdots & x_{3p-(1+2)} \\
m_4 & = & x_3 & x_{p+2} & x_{2p} & x_{3p-2} & \cdots & x_{4p-(1+2+3)} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
m_p & = & x_{p-1} & x_{2p-2} & \cdots & \cdots & \cdots & x_{(p+1)p-(1+\ldots+p)} \\
m_{p+1} & = & x_p & x_{2p-1} & x_{3p-(1+2)} & x_{4p-(1+2+3)} & \cdots & x_{(p+1)p-(1+\ldots+p)}
\end{array}$$

Example 3.4 $p = 4 \Rightarrow k = \binom{5}{2} = 10$

$$\begin{array}{cccccc}
m_1 & = & x_1 & x_2 & x_3 & x_4 \\
m_2 & = & x_1 & x_5 & x_6 & x_7 \\
m_3 & = & x_2 & x_5 & x_8 & x_9 \\
m_4 & = & x_3 & x_6 & x_8 & x_{10} \\
m_5 & = & x_4 & x_7 & x_9 & x_{10}
\end{array}$$

Theorem 3.4 *Let $p \geq 1$. Let $S = k[x_1, \ldots, x_k]$, where $k = \binom{p+1}{2}$. Let $M = (m_1, \ldots, m_{p+1})$ be the ideal generated by the $p+1$ monomials of Theorem 3.3. Then*

M is a 1-cancellation, and a minimal resolution of S/M can be obtained from $\mathbb{T}_M$ by making the cancellation $0 \to S[\sigma] \to S[\tau] \to 0$, where $[\sigma]$ is the only Taylor symbol in homological degree $p+1$, and $[\tau]$ is an arbitrary Taylor symbol in homological degree p.

Proof. Let $G = \{m_1, \ldots, m_{p+1}\}$ and let $m_i \in G$. By Theorem 3.3 (i), each variable x that divides m_i, is also a divisor of some other monomial $m_x \in G \setminus \{m_i\}$. Hence, m_i is not dominant in G and then, G itself is not dominant. On the other hand, the set $G \setminus \{m_i\}$ is dominant because, for each $m \in G \setminus \{m_i\}$ there is a variable x_m that divides both m_i and m (Theorem 3.3 (ii)). Now, by Theorem 3.3 (i), x_m does not divide any of the monomials of $G \setminus \{m_i, m\}$. Hence, $\text{mdeg}[G \setminus \{m_i, m\}] \neq \text{mdeg}[G \setminus \{m_i\}]$. This implies that each m is dominant in $G \setminus \{m_i\}$ and thus, $G \setminus \{m_i\}$ is a dominant set. By Theorem 3.1, M is then a 1-cancellation. It only remains to prove that in the consecutive cancellation that leads to the minimal resolution of S/M, we can choose an arbitrary $[\tau]$ in homological degree p. Notice that every $[\tau]$ is of the form $[G \setminus \{m_i\}]$, while $[\sigma] = [G]$. Since m_i is nondominant in G, it follows that $\text{mdeg}[G] = \text{mdeg}[G \setminus \{m_i\}]$. ■

3.3 Three Open Problems

We suggest that 1-cancellations are easy to manipulate and represent a useful tool for making computations. Indeed, we will now show through some easy computations how the class of 1-cancellation ideals defined in the last section gives a partial solution to three open problems, simultaneously.

The following open problems were posed by Peeva-Stillman in their article "Open problems on Syzygies and Hilbert functions". (Here we respect the numbers with which they appear in that paper.)

Problem 3.5 Let $a_1 \geq a_2 \geq \ldots \geq a_q \geq 2$ be the degrees of the elements in a minimal system of homogeneous generators of M. Set $r = \operatorname{codim}(S/M)$. Find nice sufficient conditions on M so that $\operatorname{reg}(S/M) \leq a_1 + \ldots + a_r - r$.

Problem 3.6 Assuming the ideal M satisfies some specials conditions, find a sharp upper bound for $\operatorname{reg}(M)$, in term of the maximum degree of an element in a minimal system of homogeneous generators of M.

Problem 6.3 Let M be a monomial ideal generated by q monomials of degree p. Let W be the monomial ideal generated by the first q square-free monomials of degree p in reverse lex order. Find conditions of M that imply $b_i^S(S/W) \leq b_i^S(S/M)$, for every $i \geq 0$.

In order to solve these problems, we need to study the combinatorial properties of the ideals defined in the last section.

Theorem 3.5 *Let $k = \dfrac{p(p+1)}{2}$ and $S = k[x_1, \ldots, x_k]$. Let $M = (m_1, \ldots, m_{p+1})$ be the ideal generated by the monomials of Theorem 3.3. Then $\operatorname{reg}(S/M) = k - p$.*

Proof. Let $G = \{m_1, \ldots, m_{p+1}\}$. By Theorem 3.4, M is a 1-cancellation, and $x_1 \ldots x_k$ $= \operatorname{mdeg}[G] = \operatorname{mdeg}[G \setminus \{m_i\}]$, for all i. Then by Theorem 3.2 (ii), $\operatorname{reg}(S/M) = \deg(x_1, \ldots, x_k) - p = k - p$. ■

Now we can give an answer to Problem 3.6 (within the context that we are considering). Notice that Problem 3.6 asks for an upper bound of the regularity in terms of the maximum degree a_1 of an element in a minimal generating set of M. We will do more than that. We will express the regularity as a function of a_1.

Corollary 3.2 *For every $p \geq 1$, let $M = (m_1, \ldots, m_{p+1})$ be the ideal generated by the $p+1$ monomials of Theorem 3.3. Then every monomial generator m_i has degree $a_1 = p$, and $\operatorname{reg}(S/M) = \frac{1}{2}a_1^2 - \frac{1}{2}a_1$.*

Proof. By Theorem 3.5, $\operatorname{reg}(S/M) = k - p = \dfrac{p(p+1)}{2} - p = \dfrac{1}{2}p^2 - \dfrac{1}{2}p$. ■

Theorem 3.6 *Let $M = (m_1, \ldots, m_{p+1})$ be the ideal generated by the $p+1$ monomials of Theorem 3.3. Then*

$$\operatorname{codim}(S/M) = \begin{cases} \dfrac{p+1}{2} & \text{if } p \text{ is odd} \\ \dfrac{p+2}{2} & \text{if } p \text{ is even}. \end{cases}$$

Proof. Let $G = \{m_1, \ldots, m_{p+1}\}$. Suppose p is odd. For all $i = 1, \ldots, \dfrac{p+1}{2}$, let $y_i \in \{x_1, \ldots, x_k\}$ be the (only) variable that divides both m_{2i-1} and m_{2i}. Then every monomial $m \in G$ is divisible by at least one of the variables of $L = \{y_1, \ldots, y_{\frac{p+1}{2}}\}$. Therefore, $\operatorname{codim}(S/M) \leq \dfrac{p+1}{2}$. Now, suppose that there is a set $L' = \{x_{i_1}, \ldots, x_{i_l}\} \subseteq \{x_1, \ldots, x_k\}$, such that every monomial $m \in G$ is divisible by some variable in L'. Then $p + 1 = \#\{m \in G : m \text{ is divisible by some variable in } L'\} \leq \sum\limits_{j=1}^{l} \#\{m \in G : m \text{ is divisible by } x_{i_j}\}$
$= 2l \Rightarrow \dfrac{p+1}{2} \leq l$. Thus, $\operatorname{codim}(S/M) = \dfrac{p+1}{2}$.
Suppose now that p is even. For all $i = 1, \ldots, \dfrac{p}{2}$, let $y_i \in \{x_1, \ldots, x_k\}$ be the (only) variable that divides both m_{2i-1} and m_{2i}. In addition, let $y_{\frac{p}{2}+1}$ be a variable that divides m_{p+1}. Then every monomial $m \in G$ is divisible by at least one of the variables of $L = \{y_1, \ldots, y_{\frac{p}{2}+1}\}$. Therefore, $\operatorname{codim}(S/M) \leq \dfrac{p}{2} + 1 = \dfrac{p+2}{2}$. If there is a set $L' = \{x_{i_1}, \ldots, x_{i_l}\} \subseteq \{x_1, \ldots, x_k\}$, such that every monomial $m \in G$ is divisible by some variable in L', then $p+1 = \#\{m \in G : m \text{ is divisible by some variable in } L'\} \leq \sum\limits_{j=1}^{l} \#\{m \in G : m \text{ is divisible by } x_{i_j}\}$
It follows that $p + 1 \leq 2l$ but since p is even, we must have $p + 2 \leq 2l$. This implies that $\dfrac{p+2}{2} \leq l$. ■

Having studied the combinatorial properties of $M = (m_1, \ldots, m_{p+1})$, we only need to put the pieces together to prove Problem 3.5 for our particular family. We do so in the next corollary.

Corollary 3.3 *For every $p \geq 1$, let $M = (m_1, \ldots, m_{p+1})$ be the ideal generated by the $p+1$ monomials of Theorem 3.3 . Let $r = \operatorname{codim}(S/M)$, and let $a_1 \geq a_2 \geq \ldots \geq$*

$a_{p+1} \geq 2$ *be the degrees of the* $m_1, \ldots, m_{p+1}$, *respectively. Then*

$$\operatorname{reg}(S/M) \leq a_1 + \ldots + a_r = r.$$

Proof. By construction, $a_1 = a_2 = \ldots = a_r = p$. By Theorem 3.5, $\operatorname{reg}(S/M) = k - p = \dfrac{p(p+1)}{2} - p$. Suppose first that p is odd. By Theorem 3.6, $r = \dfrac{p+1}{2}$. Then $a_1 + \ldots + a_r = pr = \dfrac{p(p+1)}{2}$. Hence,

$$\operatorname{reg}(S/M) = \frac{p(p+1)}{2} - p = a_1 + \ldots + a_r - p \leq a_1 + \ldots + a_r - \frac{p+1}{2} = a_1 + \ldots + a_r - r.$$

Suppose now that p is even. By Theorem 3.6, $r = \dfrac{p+2}{2}$. Then $a_1 + a_2 + \ldots + a_r = pr = \dfrac{p(p+2)}{2}$. Hence,

$$\operatorname{reg}(S/M) = \frac{p(p+1)}{2} - p \leq a_1 + a_2 + \ldots + a_r - p \leq a_1 + \ldots + a_r - \frac{p+2}{2} = a_1 + \ldots + a_r - r.$$

■

Finally, we will prove that the family $\{(m_1, \ldots, m_{p+1}), p \geq 1\}$, where $m_1, \ldots, m_{p+1}$ are the monomials defined in Theorem, satisfies the inequality of Problem 6.3.

Theorem 3.7 *For* $p \geq 1$, *let* $M = (m_1, \ldots, m_{p+1})$ *be the ideal generated by the* $p+1$ *monomials of Theorem 3.3. Let* W *be the ideal generated by the first* $p+1$ *square-free monomials of degree* p *in reverse lex order. Then*

$$b_i(S/W) \leq b_i(S/M), \text{ for all } i \geq 0.$$

Proof. Let $G' = \{m'_1, \ldots, m'_{p+1}\}$ be the minimal generating set of W. Notice that

$m'_1 = x_1 x_2 \ldots x_{p-1} x_p$.

$m'_2 = x_1 x_2 \ldots x_{p-1} x_{p+1}$

$m'_3 = x_1 x_2 \ldots x_{p-2} x_p x_{p+1}$.

Therefore, m'_1 is not dominant in G'. This means that $\operatorname{mdeg}[G'] = \operatorname{mdeg}[G' \setminus \{m'_1\}]$ and $\mathbb{T}_W$ admits the consecutive cancellation $0 \to S[G'] \to S[G' \setminus \{m'_1\}] \to 0$. On the other hand, M is a 1-cancellation. Hence, for all $i = 0, 1, \ldots, p-1$, $b_i\,(S/W) \leq \binom{p+1}{i} = b_i\,(S/M)$. Besides that, $b_p = (S/W) \leq \binom{p+1}{p} - 1 = b_p\,(S/M)$ and $b_{p+1}\,(S/W) = b_{p+1}\,(S/M) = 0$. ■

Remark 3.1 *Notice that in Problem 6.3, the ideal M is generated by q monomials of degree p, while in Theorem 3.7, M is generated by $p+1$ monomials of degree p. This means that our solution, though infinite, is far from being the most general solution.*

CHAPTER 4

CONCLUSIONS

The thread that runs through the entire study of dominant, 1-semidominant and 2-semidominant ideals is the fact that their minimal resolutions can be obtained eliminating pairs consisting of face and facet of equal multidegree, in arbitrary order. Of course, this principle is trivial in the case of dominant ideals because their Taylor resolution is already minimal and in the case of semidominant ideals, this rule is eclipsed by an even stronger fact; namely, semidominant ideals are Scarf.

In both cases, however, the principle is implicit. In order to prove that $\mathbb{T}_M$ is minimal whenever M is dominant, all we have to do is show that it is impossible to find a face and a facet of equal multidegree (see Theorem 2.1 ($\Leftarrow$)). Thus we do not apply the rule to $\mathbb{T}_M$ but we certainly study $\mathbb{T}_M$ in light of it. Similarly, the proof that semidominant ideals are Scarf is based on the fact that when we make random standard cancellations involving faces and facets of equal multidegree, all basis elements with a repeated multidegree are eliminated.

Having understood the common theme in the study of these three classes of ideals, it is natural to wonder whether 3-semidominant ideals can be resolved in the same way. Unfortunately, the answer is no, as the next example shows.

With the assistance of a software system (for instance, Macaulay 2 [GS]) it is easy to verify that the 3-semidominant ideal $M = (x^2y^2z^2, xw^2, yw^2, zw)$ is Scarf. Now, there are six basis elements of $\mathbb{T}_M$ with multidegree $m = x^2y^2z^2w^2$ which, therefore, are excluded from the basis of the minimal resolution of S/M. However, if we eliminate pairs of face and facet of equal multidegree as follows:

$([x^2y^2z^2, xw^2, yw^2, zw], [x^2y^2z^2, xw^2, zw])$ first, and $([x^2y^2z^2, xw^2, yw^2], [x^2y^2z^2, yw^2])$ next, then the remaining basis elements of multidegree m, $[x^2y^2z^2, yw^2, zw]$ and $[x^2y^2z^2, xw^2]$, cannot be eliminated in this way because they are not face and facet. This proves that the basis of the minimal resolution of S/M cannot be obtained eliminating pairs of face and facet of equal multidegree, at random.

It remains an open problem to determine the family of all monomial ideals the basis of whose minimal resolutions can be obtained following the rule that we are discussing. What we know though is that the family contains more ideals than the dominant, 1-semidominant, and 2-semidominant ideals (for instance, the 3-semidominant and 4-semidominant ideals $M_3 = (xy, xz, yz)$ and $M_4 = (xz, yz, xw, yw)$ are in the family). In order to expand our knowledge of this class, in the last section of Chapter 2 we set aside the concept of p-semidominance and studied monomial ideals under different hypotheses (see Theorem 2.9). This means that the minimal resolutions of all monomial ideals in Chapter 2 can be obtained making standard cancellations in arbitrary order. It would be nice to obtain other results in the same line of reasoning.

In Chapter 3 we studied the 1-cancellation ideals in general, and then we worked with a particular class of them to solve three open problems. The natural continuation in this study is to define the p-semidominant ideals as those whose minimal resolutions can be obtained from their Taylor resolutions by means of exactly p standard cancellations. The next step would be the characterization of the p-semidominant ideals. The characterization of the 2-cancellations seems to be rather simple. However, for larger values of p, characterizing the p-cancellations appears to be challenging.

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We construct the minimal resolutions of three classes of monomial ideals: dominant, 1-semidominant, and 2-semidominant ideals. The families of dominant and 1-semidominant ideals extend those of complete and almost complete intersections. We show that dominant ideals give a precise characterization of when the Taylor resolution is minimal, 1-semidominant ideals are Scarf, and the minimal resolutions of 2-semidominant ideals can be obtained from their Taylor resolutions by eliminating faces and facets of equal multidegree, in arbitrary order. We study the combinatorial properties of these classes of ideals and explain how they relate to generic ideals. We also give a partial solution to three open problems on syzygies.

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