AN INVESTIGATION OF STUDENTS' REASONING WITH EXAMPLES AND NON-EXAMPLES OF FUNCTION IN ABSTRACT ALGEBRA

By

ROSAURA USCANGA LOMELI

Bachelor of Science in Mathematics
The University of Texas at Arlington
Arlington, Texas
2012

Master of Science in Mathematics
Oklahoma State University
Stillwater, Oklahoma
2015

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Dissertation Approved:

Dr. John Paul Cook

Dissertation Adviser

Dr. Michael Oehrtman

Dr. Michael Tallman

Dr. Edward Richmond

Dr. Jennifer Cribbs

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Abstract: The focus of this dissertation is on students' thinking regarding the function concept in the context of abstract algebra, focusing on the properties of well- and everywhere-definedness. This dissertation follows a three-paper format, where each paper has a different yet related focus.

In the first paper, I analyze (non-)examples found in textbooks and those used during instruction related to function in abstract algebra to gain insight into how students are expected to reason about functions in this context. I investigate experts' thinking about functions by conducting a textbook analysis and semi-structured clinical interviews with mathematicians. I then elaborate the essential properties of well- and everywhere-definedness into four categories of non-examples that students are expected to successfully reason about in an introductory abstract algebra course.

In the second paper, I explore students' reasoning with examples and non-examples of function related to the four categories by conducting task-based clinical interviews. I provide characteristics of a *coordinated way of understanding functions* in abstract algebra and illustrate how such a coordinated way of understanding functions enables students to reason productively with function tasks and the properties of well- and everywhere-definedness. This paper addresses what is entailed in reasoning productively about well-definedness and everywhere-definedness.

In the last paper, I present a functions activity focused on recovering an example from a non-example of a function by prompting students to modify the domain, the codomain, and/or the rule. I argue that such an activity can help instructors have a clearer image of how they might support their students in developing *a coordinated view of function* and is a tool that abstract algebra instructors can use to help students attend to well-definedness and everywhere-definedness.

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CHAPTER I

INTRODUCTION

In this dissertation, I investigate students' reasoning regarding the function concept in the context of abstract algebra—in particular, the characteristic properties of well-definedness and everywhere-definedness. This dissertation follows a three-paper format, where each paper has a different yet related focus. Before introducing the three papers, I present an overview of (1) the importance of function in abstract algebra, (2) the challenges and difficulties documented in the literature that students experience with the function concept, and (3) discuss the scarcity of the literature on well-definedness and everywhere-definedness—key properties of the function concept.

1.1 Why are Functions Important in Abstract Algebra?

The function concept is considered of key importance in mathematics and a core topic in the secondary and undergraduate mathematics curriculum (Bagley, Rasmussen, & Zandieh, 2015; Dubinsky & Wilson, 2013; Hitt; 1998). In abstract algebra, students encounter various important classes of functions in settings that are different from what they have previously experienced. Melhuish (2019) found that topics like binary operations, homomorphisms, and isomorphisms—all of which are examples of function—are some of the most important concepts in abstract algebra courses. Binary operations (e.g., addition on $\mathbb{R}$, multiplication on $\mathbb{R}$), which appear throughout the secondary mathematics curriculum, tend not to be formally defined until advanced undergraduate mathematics courses like abstract algebra. A binary operation $\star$ on a set G is a function mapping $G \times G$ into G (Dummit & Foote, 2004). Binary operations play a central role in abstract algebra because they are used to define algebraic structures like groups and rings. Another important class of functions are homomorphisms which show up in abstract algebra for the first time. A homomorphism is a function between two algebraic objects of the same type (e.g., two groups, two rings) which preserves the algebraic structure.[1] Isomorphisms are a third important class of functions that students encounter in abstract algebra. An isomorphism is a homomorphism that is both injective and surjective. Numerous undergraduate abstract algebra courses also cover permutations which are another example of functions that arise in abstract algebra. A permutation of a set S is a function $\phi: S \to S$ that is injective and surjective (Dummit & Foote, 2004, p. 3). Students may also encounter a combination of classes of functions in a single instance in abstract algebra. For example, the set of all permutations of a nonempty set S forms a group (i.e., the symmetric group on S) under function composition. Thus the elements of the group—the permutations—are one class of functions and the operation—function composition—is a binary operation which is a different class of function (discussed above). Thus, functions formalize several important concepts in this course. Yet, the majority of research on functions in the mathematics education literature does not account for the ways in which students must reason about the function concept in this context. The research that exists tends to focus on general student understandings about particular kinds of functions like binary operations (e.g., Melhuish, Ellis, & Hicks, 2020) or isomorphisms (e.g., Larsen, 2009; Leron, Hazzan, & Zazkis, 1995) but does not delve into specific aspects of the function concept that are important in abstract algebra. This highlights the need for research that specifically addresses the function concept in abstract algebra.

[1] For example, a ring homomorphism from a ring $(R, +_R, \cdot_R)$ to a ring $(S, +_S, \cdot_S)$ is a function ϕ from R into S, satisfying $\phi(a +_R b) = \phi(a) +_S \phi(b)$ and $\phi(a \cdot_R b) = \phi(a) \cdot_S \phi(b)$ for all elements a and b in R (Dummit & Foote, 2004).

1.2 Functions are Challenging and Nuanced

Due to their importance, functions are one of the most studied topics in the literature pertaining to courses through calculus. Indeed, the literature on student thinking and learning of the function concept is extensive, with most of it either identifying common student difficulties with the concept (both at the secondary and undergraduate level) or characterizing productive ways of reasoning about functions (like quantitative and covariational reasoning (e.g., Carlson et al., 2002; Saldanha & Thompson, 1998; Smith & Thompson, 2007; Thompson, 1994a, 2011)).

1.2.1 View of functions in the math education literature. The majority of function research in the mathematics education literature focuses on a covariational approach to functions (e.g., Carlson, 1998; Carlson, Persson, & Smith, 2003; Oehrtman, Carlson, & Thompson, 2008). From this perspective, students are expected to reason about how a change in one variable relates to the change in another (Oehrtman, Carlson, & Thompson, 2008). Such an approach provides a productive lens through which to reason about functions in algebra, trigonometry, and calculus. For example, this view is productive for understanding modeling and application problems (e.g., Carlson et al., 2002), constant and average rate of change (e.g., Musgrave & Carlson, 2016), derivatives (e.g., Thompson, 1994b, Zandieh, 2000), limits (e.g., Oehrtman, Swinyard, & Martin, 2014), as well as particular properties of certain kinds of functions like exponential functions (e.g., Ellis et al., 2016) or trigonometric functions (e.g., Moore, 2014). However, most of this research has focused on functions from the real numbers to the real numbers. While in this setting, functions admit meaningful quantitative and covariational interpretations, this is not the case in abstract algebra. For example, students in abstract algebra encounter functions between sets of matrices, polynomials, or equivalence classes. In this context, the objects in question cannot be thought of as quantities and therefore a covariational approach is not productive for reasoning about functions defined on such discrete structures. Abstract algebra requires a different treatment of functions which could be one of the reasons why undergraduate students find function-related concepts in this course to be so challenging (e.g., Brown et al., 1997; Leron, Hazzan, & Zazkis, 1995; Melhuish & Fagan, 2018; Melhuish et al., 2020; Rupnow, 2017, 2019).

1.2.2 Student difficulties with the function concept. The literature on student understanding of functions identifies multiple student difficulties with the concept. For example, students struggle to recognize whether particular correspondences satisfy the definition of function (e.g., Markovitz, Eylon, & Bruckheimer, 1986). Students also hold misconceptions about the definition of function, like assuming that functions can only be defined by a single formula (e.g., Carlson, 1998; Hitt, 1998; Thompson, 1994c; Vinner & Dreyfus, 1989) or thinking of functions as always being continuous (e.g., Carlson, 1998; Hitt, 1998; Vinner & Dreyfus, 1989). Additionally, students encounter difficulty with domains and codomains as well as images and preimages (e.g., Even, 1990; Hitt; 1998; Markovitz, Eylon, & Bruckheimer, 1986), recognizing different representations of a function (e.g., Carlson, 1998; Dubinsky & Wilson, 2013) and transitioning between these different representations (e.g., Hitt, 1998; Markovitz, Eylon, & Bruckheimer, 1986). They also tend to overrely on the vertical line test as an identifying criterion for function (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Wilson, 1994).

With relation to the abstract algebra literature, several student struggles are documented with particular kinds of function, but this literature does not delve into the specific aspects of function that are important to understand these concepts. For example, Leron, Hazzan, and Zazkis (1995) documented students' struggles with constructing specific isomorphisms and formulating definitions of isomorphism. Similarly, Larsen (2009) showed difficulties that arise with the concept of isomorphism in the context of a teaching experiment. Rupnow (2017) on the other hand, documented students' struggles with homomorphisms when students did not have flexibility with metaphors and Melhuish, Ellis, and Hicks (2020) documented that students struggle to understand binary operations because they bring some of the same difficulties with the function concept mentioned above into abstract algebra. However, none of these difficulties were explicitly tied to the general difficulties students experience with functions.

General difficulties with the function concept can hinder students from reasoning productively about functions in abstract algebra. For example, students need to be able to distinguish whether a particular correspondence is or is not a function which is relevant when deciding whether a proposed rule is a binary operation or when defining a homomorphism or an isomorphism—all of which are common tasks in abstract algebra courses. Such tasks involve attending to whether the proposed correspondence is well-defined and everywhere-defined. Another difficulty identified above is that students often think functions are defined by a single formula. However, functions in abstract algebra are not always given in terms of a formula and students who think of functions in this way may overlook certain binary operations, homomorphisms, or isomorphisms that are not given in terms of a specific formula. Additionally, students who have a hard time attending to the domain or codomain of a function (or identifying particular preimages and images) may overlook the requirement for a binary operation to have two inputs. Finally, an overreliance on the vertical line test presents problems in abstract algebra because functions in this context do not always lend themselves to useful graphically representations since they are often defined on discrete sets. Even when students do not experience difficulties with functions in other contexts, they do not always carry the productive conceptions they possess into abstract algebra (Melhuish et al., 2020).

1.3 Scarcity of Literature on Well-definedness and Everywhere-definedness

The notions of everywhere-definedness (i.e., the condition that each element of the domain is mapped to at least one element of the codomain) and well-definedness (i.e., the condition that each element of the domain is mapped to at most one element of the codomain) are of particular importance in abstract algebra. For example, in working with groups and rings, students often have to check whether a proposed binary operation is well-defined. If asked to define their own homomorphism, they first have to check whether their proposed correspondence is well- and everywhere-defined before checking the homomorphism property. In this study, I explore students' reasoning with these two key properties of functions. In what follows, I discuss the relevant literature on well-definedness and everywhere-definedness. An important theme I identified in this literature is that students struggle with well-definedness. Indeed, students "may not identify all required properties in their concept images and, for example, miss the requirement of well-definedness" (Melhuish & Fagan, 2018, p. 23), thus further emphasizing the need for a detailed exploration of students' reasoning about well-definedness.

I found no studies directly examining notions of everywhere-definedness which could perhaps be attributed to everywhere-definedness often not being emphasized prior to advanced undergraduate mathematics courses. However, there have been some studies that examine well-definedness (commonly referred to in the literature as *univalence*). As previously mentioned, research generally reports that students struggle with well-definedness (e.g., Bailey et al., 2019; Dorko, 2017; Even & Bruckheimer, 1998; Even & Tirosh, 1995). They have difficulties articulating what well-definedness means and why it is important (e.g., Even, 1993; Even & Tirosh, 1995) and often associate it with procedural conceptions of the vertical line test (e.g., Clement, 2001; Kabael, 2011; Thomas, 2003). Additionally, they struggle to adapt well-definedness (and the vertical line test) to functions whose domains are not the real numbers (e.g., Dorko, 2017; Even & Tirosh, 1995). For example, Dorko (2017) explains that many students have difficulties generalizing notions of well-definedness to functions whose graphs are difficult to visualize. It is important to reemphasize that the vertical line test is of limited use in abstract algebra (previously discussed in section 1.2.2) because many functions in this context have domains that are not easily ordered like the real numbers. In abstract algebra courses, it is common to define functions between sets of matrices, sets of polynomials, sets of equivalence classes (like $\mathbb{Z}/n\mathbb{Z}$) or to have multidimensional domains (e.g., the domain of a binary operation is a Cartesian product). Thus functions in this context do not tend to have useful graphical illustrations and the vertical line test does not easily generalize here. As a result, much of the literature on well-definedness which focuses on students' use and understanding of the vertical line test is of little use in this context.

I also found no studies that directly examine the notion of well-definedness (and as before, of everywhere-definedness) and its use in abstract algebra settings; research emphasizes the importance of issues related to well- and everywhere-definedness but does not directly examine these properties. The current function literature, and in particular the abstract algebra literature, does not delve into the specifics of what these concepts entail, how they interact, and how they might be reasoned about productively. For example, Melhuish and colleagues (2020) explored abstract algebra students' personal concept definitions and examples of functions and noted whether well- and everywhere-definedness were included as part of the students' concept definitions but did not investigate students' reasoning with these properties. Additionally, as I suggested above, courses up thorough the calculus sequence do not often emphasize everywhere-definedness which leaves well-definedness as the sole defining feature of a function. This general lack of emphasis on everywhere-definedness together with the focus on the vertical line test of the research on well-definedness calls attention to the need for studies that examine students' reasoning with both well-definedness and everywhere-definedness and underscores the importance of developing a clearer image of how well-definedness and everywhere-definedness are used in abstract algebra in particular.

1.4 Overview of the Dissertation

As mentioned at the beginning of this chapter, the overall focus of this dissertation is on students' reasoning regarding the function concept in the context of abstract algebra with particular focus on the properties of well-definedness and everywhere-definedness. In the subsections that follow, I give a brief overview of each of the three papers that form this dissertation.

1.4.1 Summary of the first paper. In Paper 1, I examine the contents and the structure of the *instructional example space* (Watson & Mason, 2005; Zazkis & Leikin, 2008) for function in abstract algebra which includes the examples found in textbooks as well as those used during instruction as a means to gain insight into how students are expected to reason about functions in this context. I interpret 'example' holistically to mean examples, exercises, representations, diagrams, and non-examples. In order to explore the instructional example space, I investigated experts' thinking about functions (with particular focus on well-definedness and everywhere-definedness) by conducting a textbook analysis and conducting semi-structured clinical interviews (Fylan, 2005) with five algebraists. An essential theme that emerged in (and shaped) my analysis was the importance of non-examples in the instructional example space.

The main contributions of this paper are (1) the identifications of the non-examples that are contained in the instructional example space (*the contents*) and (2) the identification of four key categories of non-examples that students are expected to successfully reason about in an introductory abstract algebra course (*the structure*). In particular, these categories are elaborations of the essential properties of well- and everywhere-definedness. I argue that these elaborations provide additional insight into the nature of the function concept and contribute viable explanations into the successes and difficulties that students experience with functions in abstract algebra. In particular, students are likely to be familiar with two of these categories from secondary mathematics and courses up through the calculus sequence. However, the other two categories of non-examples are relatively unfamiliar for students prior to advanced undergraduate mathematics courses but important for reasoning productively about functions in abstract algebra.

1.4.2 Summary of the second paper. In this paper, I explore students' reasoning with examples and non-examples of function related to the four key categories of non-examples of function I identified in Paper 1. I frame students' reasoning in terms of Harel's (2008a) *ways of understanding* and *ways of thinking*. In particular, I provide characteristics of a *coordinated way of understanding functions* in abstract algebra and illustrate how such a coordinated way of understanding functions enables students to reason productively with function tasks and, in particular, with the important properties of well- and everywhere-definedness. To infer students' ways of understanding particular function tasks, I conducted task-based clinical interviews (Clement, 2000; Goldin, 2000) with five students. I present the reasoning of four of these students as they work on determining whether particular correspondences are examples or non-examples of function. The main contribution of this paper is that it addresses what is entailed in reasoning productively about well-definedness and everywhere-definedness in abstract algebra. This is a key contribution because, as discussed in sections 1.2 and 1.3 above, there has been no direct detailed investigations of what is entailed in reasoning productively about well- and everywhere-definedness in advanced mathematical settings.

1.4.3 Summary of the third paper. The last paper is a practitioner-oriented piece in which I share the results of this study and present a functions activity focused on recovering an example from a non-example of a function. In particular, I present a summary of the main contributions for instruction from the first two papers of this dissertation: (1) a productive and coherent way to choose/structure the examples and non-examples that instructors in abstract algebra use in their classrooms, and (2) a way of understanding functions that supports students in reasoning productively about these examples and non-example. Additionally, I present a functions activity and example tasks that address both (1) and (2) above. This activity focuses on recovering an example from a non-example of a function by prompting students to modify the domain, the codomain, and/or the rule. The main contribution of this paper is the activity of recovering an example of a function from a non-example which I argue can help instructors have a clearer image of how they might support their students in developing *a coordinated view of function* (from Paper 2). This kind of task is a tool that abstract algebra instructors can use to help students attend to the two defining properties of the function concept—well-definedness and everywhere-definedness.

CHAPTER II

PAPER 1 [THE INSTRUCTIONAL EXAMPLE SPACE]

Analyzing the Structure of the Non-examples Contained in the Instructional Example Space for Function in Abstract Algebra

2.1 Introduction

The function concept is considered of key importance in mathematics and a core topic in the secondary and undergraduate mathematics curriculum (Bagley, Rasmussen, & Zandieh, 2015; Dubinsky & Wilson, 2013; Even & Tirosh, 1995; Hitt, 1998; Oehrtman, Carlson, & Thompson, 2008). In abstract algebra, functions play a fundamental role in many topics in the course. A nationally representative sample of experts recently concluded that topics like homomorphism, isomorphism, and binary operations—all of which are examples of function—are some of the most important concepts in the course (Melhuish, 2019). Additionally, functions are an important connection pre-service teachers should make between abstract algebra and secondary mathematics (Melhuish & Fagan, 2018).

Due to its importance, the function concept is one of the most studied topics in the mathematics education literature pertaining to courses through calculus. This research has centered on characterizing productive ways of reasoning about functions (such as quantitative and covariational reasoning (e.g., Carlson, 1998; Carlson, et al., 2002; Saldanha & Thompson, 1998, Thompson, 2011; Thompson & Carlson, 2017)) but there are still reasons to investigate the function concept in abstract algebra. For example, research on functions through the calculus sequence has generally focused on functions from the real numbers to the real numbers (the domain of a function is generally assumed to be the largest possible subset of $\mathbb{R}$ for which the given rule is defined and the codomain is assumed to be $\mathbb{R}$). In this context, functions admit meaningful quantitative and covariational interpretations. However, in abstract algebra, students encounter functions between sets that are not continuous (the elements of these sets can be sets, polynomials, functions, etc.) and thus have no quantitative interpretation. Functions also formalize lots of important concepts like binary operations, homomorphisms, and isomorphisms in abstract algebra and are used in those cases for the purpose of imposing structure or identifying sameness, all of which are new uses for students. In other words, abstract algebra requires students to grapple with unfamiliar aspects of this familiar topic.

While there has been a fair amount of research examining students' reasoning about functions in abstract algebra, nearly all of it has examined particular types of functions, including homomorphisms (e.g., Melhuish et al. 2020; Rupnow, 2017, 2019), isomorphisms (e.g., Larsen, 2009, 2013; Leron, Hazzan, & Zazkis, 1995), and binary operations (e.g., Brown et al., 1997; Melhuish & Fagan, 2018; Melhuish, Ellis, & Hicks, 2020). Aspects of the function concept itself, such as what students might need to attend to when determining what is and is not a function, has been relatively unexplored.

The function properties of well-definedness and everywhere-definedness are of particular importance in abstract algebra. For example, quotient rings are an important yet difficult concept for students in this course and a question that remains open is how to develop the concept from students' own activity and informal knowledge. If we consider forming, say, the quotient ring $\mathbb{Z}/2\mathbb{Z}$ from $\mathbb{Z}$ using the instructional theory articulated in Larsen and Lockwood (2013) for quotient groups, students would first have to partition the integers into the subsets $2\mathbb{Z} =$

$\{\ldots, -2, 0, 2, \ldots\}$ and $2\mathbb{Z} + 1 = \{\ldots, -3, -1, 1, 3, \ldots\}$, and then create addition and multiplication tables using these two subsets as the elements of the quotient structure. However, an everywhere-definedness issue occurs when defining the same operation used in Larsen and Lockwood (2013)—multiplying $2\mathbb{Z}$ by itself yields a third subset ($2\mathbb{Z} \odot 2\mathbb{Z} = 4\mathbb{Z}$) which is not one of the original two elements of the new structure (i.e., the element $(2\mathbb{Z}, 2\mathbb{Z})$ does not have a corresponding image in the codomain $\{2\mathbb{Z}, 2\mathbb{Z} + 1\}$). Well-definedness and everywhere-definedness are similarly impactful in more conventional treatments of quotient rings—particularly, defining a binary operation on a set of cosets R/I involves verifying that (1) every ordered pair in $R/I \times R/I$ maps to at least one coset in R/I (everywhere-definedness), and (2) every ordered pair in $R/I \times R/I$ maps to at most one coset in R/I (well-definedness). So, if students instead attempted to define multiplication in terms of representatives (rather than as set multiplication as seen above), they would then run into issues regarding whether or not their operation is well-defined.

The properties of well-definedness and everywhere-definedness also arise in the context of discussing homomorphisms and isomorphisms. For example, Rupnow (2021) details a metaphor for homomorphism and isomorphism related to these two properties which involves specifically using "a function property to draw conclusions about isomorphisms and homomorphisms, such as everywhere-defined and well-defined" (p. 5). Additionally, Melhuish and colleagues (2020) highlight the importance of well- and everywhere-definedness for homomorphisms. In their study, zero of their 18 participants' definitions of function addressed everywhere-definedness and only two out of the 18 addressed well-definedness. This signals a lack of attention from students to these two properties (perhaps due to a lack of emphasis of these properties prior to advanced undergraduate mathematics courses). They discuss the illustrative case of one student who identified a particular correspondence as an example of a homomorphism when it was not even a function and raise the question of how students might or should think about well-definedness and everywhere-definedness. Hence, relying on the current abstract algebra literature available is not enough to solve particular issues like the ones just discussed. This calls for the need of characterizations of productive conceptions of well- and everywhere-definedness, particularly in the context of advanced undergraduate mathematics courses like abstract algebra.

In short, despite the breadth and depth of the functions literature in general, there is a need for more specific research into the function concept in abstract algebra because (1) abstract algebra requires students to reason about functions in new ways that cannot be accounted for by much of the functions literature, and (2) research on the function concept itself (as opposed to specific types of functions) is scarce in abstract algebra. My ultimate goal in this line of research is to characterize a productive way of thinking about functions in abstract algebra. But in order to do so, I propose that it is first helpful to develop a clear image of the various examples of functions and non-functions that students encounter and are expected to reason about in an introductory abstract algebra course. In this paper, I investigate the contents and structure of the *instructional example space* (Watson & Mason, 2005; Zazkis & Leikin, 2008) which involves the examples found in textbooks and used during instruction. I do so by (1) conducting a textbook analysis and (2) conducting semi-structured clinical interviews with mathematicians. A key theme that emerged in and shaped my analysis was the importance of the non-examples that are included in the instructional example space. The primary contribution of this paper, in addition to identifying the non-examples contained in the instructional example space (the contents), is the identification of four key categories of non-examples that students are expected to be able to reason about productively in an introductory abstract algebra course (the structure). Importantly, two of these categories are likely to be somewhat familiar to students from their previous experiences with function, but two are relatively new and unfamiliar. I conclude with a discussion of the implications of these findings and the implications for future research.

2.2 Literature Review

In this section, I discuss the relevant literature on functions in abstract algebra in order to (1) outline constructs that clarify the perspective I take on functions in this paper, and (2) highlight the need for a more thorough examination of the essential properties of function: well-definedness and everywhere-definedness.

2.2.1 Characterizations of function. According to Weber and colleagues (2020), there are two ways to define a function in both the mathematics and mathematics education literature. The first characterization involves conceptualizing a function as containing three parts: a domain, a codomain, and the correspondence between these two sets. As Weber and colleagues (2020) describe, "a function is defined as consisting of a domain, a codomain, and a correspondence between the domain and the codomain such that each member of the domain is assigned exactly one element of the codomain" (p. 2).[2] The second characterization involves viewing a function as a set of ordered pairs which satisfy the condition: given (x_1, y_1) and (x_2, y_2) in f, if $x_1 = x_2$ then $y_1 = y_2$. Under this conception, the domain of f is the set of all of the first coordinates of these ordered pairs and there is no single unique codomain—a codomain of f can be any set that contains its image (the set of all of the second coordinates).

In this paper, I adopt the first characterization of function because it is the most commonly used in abstract algebra. In practice, it is common to define a mapping between a familiar, well-understood algebraic structure and one that is unfamiliar in order to familiarize oneself with the latter (this is one of the many uses of the First Isomorphism Theorem, for example). In this view of function, changing the domain or codomain changes the function.

[2] This definition follows Bourbaki, where a function f is considered to be a triple "(F, A, B), where F is a relation from a set A to a set B (i.e. $F \subseteq A \times B$) satisfying the following condition: For all x in A, there exists a unique y in B such that the ordered pair (x, y) is in F" (Weber et al., 2020, p. 2). In Bourbaki's definition, A is the domain of f and B is the codomain.

Weber and colleagues (2020) give the example of the squaring function from $\mathbb{R}$ to $\mathbb{R}$ and note that it is different than the squaring function from $\mathbb{R}$ to $[0, \infty)$ since the latter is surjective but the former is not. Furthermore, using this first characterization, in addition to checking that a proposed mapping is well-defined (i.e., that each element of the domain maps to *at most* one element of the codomain), one must also check that the proposed mapping is everywhere defined (i.e., that each element of the domain maps to *at least* one element of the codomain). As noted above, this is not the case with the second characterization in which functions defined or viewed in that way are automatically everywhere-defined. I thus view both well-definedness and everywhere-definedness as critical aspects of the concept definition of function.

2.2.2 Ways of reasoning about functions. There are two primary ways of reasoning about functions documented in the mathematics education literature: covariational and correspondence/relational. The overwhelming majority of function research focuses on a covariational approach to functions (e.g., Carlson, 1998; Carlson et al., 2002; Carlson, Persson, & Smith, 2003; Oehrtman, Carlson, & Thompson, 2008). This covariational perspective (Carlson, 1998) involves understanding how a change in one variable relates to the change in another (Oehrtman, Carlson, & Thompson, 2008). A covariational approach to functions provides a productive lens through which to reason about modeling and application problems (e.g., Carlson et al., 2002) as well as foundational concepts like constant rate of change and derivatives (e.g., Carlson, Persson, & Smith, 2003). However, I note that this quantitative, covariational approach is only productive for functions whose domain and codomain are $\mathbb{R}$ (or similarly behaving subsets therein) and is thus of limited use in abstract algebra, where the domain and codomain of a function can be sets which have limited (if any) quantitative meaning, including: of matrices, polynomials, or equivalence classes. The covariational approach is not productive in this context since it “superimposes an ordinal system on function, which does not underlie many of the discrete structures in abstract algebra” (Melhuish & Fagan, 2018, p. 22). Thus, the majority of research on functions in the math education literature does not account for the ways in which students must reason about functions in abstract algebra, highlighting the need for research that specifically addresses the function concept in abstract algebra and, particularly, well- and everywhere-definedness.

Instead, I adopt a correspondence/relational view of function (Slavit, 1997). According to Slavit (1997), "relationships between input-output pairs comprise the essence of a relational view" of function and it encompasses both relationships "between individual inputs and outputs" as well as an "entire set of input-output pairs" (p. 262). I focus on this correspondence/relational approach because it is the view needed at the advanced undergraduate level, especially in abstract algebra, and this way of reasoning is compatible with the first characterization of function above since its focus is on sets of inputs and sets of outputs (thus providing a way to think about how changing one of these sets changes the function).

2.2.3 Research on well-definedness and everywhere-definedness. I identified three themes regarding the abstract algebra functions literature. First, consistent with my argument above, all of the studies of function-related ideas in abstract algebra have employed a correspondence/relational view of functions. Second, these studies emphasize the importance of issues related to well- and everywhere-definedness but do not directly examine these properties in detail—what they entail, how they interact, and how they might be reasoned about productively has not been clearly explicated in the literature in general, and in abstract algebra in particular. For example, Melhuish and colleagues (2020) prompted abstract algebra students to state their personal concept definition for function and list several examples of functions. While these researchers noted whether or not well- and everywhere-definedness were included in some form in the students' definitions, they did not investigate students' reasoning with or conceptions of these two properties. They concluded that, with respect to reasoning coherently about homomorphisms, "a fractured or rich understanding of function may serve as a hindrance or support, respectively" (p. 14). Many other studies have also generally called attention to the importance of the underlying function concept for understanding such topics as binary operation (e.g., Brown et al., 1997; Melhuish, Ellis, & Hicks, 2020; Melhuish & Fagan, 2018), homomorphism (e.g., Hausberger, 2017; Rupnow, 2021), and isomorphism (e.g., Leron, Hazzan, & Zazkis, 1995; Larsen, 2009; Nardi, 2000) but have similarly stopped short of explicitly addressing the definitive function properties of well- and everywhere-definedness.

I found no studies directly examining notions of everywhere-definedness. There have, however, been some studies that examine well-definedness (also referred to in the literature as univalence). I note two themes from these studies. First, students generally struggle with the nuanced concept of well-definedness. As Melhuish and Fagan (2018) explain "students may not identify all required properties in their concept images and, for example, miss the requirement of well-definedness" (p. 23). Additionally, students struggle to articulate what it means and why it is important (e.g., Even, 1993; Even & Tirosh, 1995) and typically associate it with procedural conceptions of the vertical line test (e.g., Clement, 2001; Kabael, 2011; Thomas, 2003). Second, students have difficulties adapting well-definedness (and the vertical line test) to functions whose domains are not the real numbers (e.g., Dorko, 2017; Even & Tirosh, 1995). I note that the vertical line test is of limited use in abstract algebra as many functions have domains that are not easily ordered like the real numbers (e.g., matrices, polynomials, $\mathbb{Z}/n\mathbb{Z}$) or multidimensional domains (e.g., the domain of a binary operation is a Cartesian product)—in either case, these kinds of functions do not usually lend themselves to a useful graphical illustration (which is required for the vertical line test). Thus, much of the literature on well-definedness focuses on students' use and understanding of a procedure that is of very limited use in abstract algebra. I also found no studies directly examining the notion of well-definedness and its use in abstract algebra settings. Yet, the complexities of well- and everywhere-definedness and the reports of students' difficulties with them call attention to the importance of and need for a thorough examination of these notions and how they manifest in abstract algebra.

Finally, the last theme I identified in the literature is that recent studies have focused on the importance of a *unified* notion of function—initially identified by Zandieh, Ellis, and Rasmussen (2017) to describe the thinking and activity of linear algebra students. A unified function essentially involves understanding functions in such a way that one can readily see prototypical examples of functions from secondary mathematics (e.g., polynomial functions on the real numbers) and undergraduate mathematics (e.g., linear transformations, homomorphisms) as instances of the same overarching function concept. Melhuish and colleagues (2020) adapted this notion to abstract algebra, noting that in this context "students may or may not see important concepts such as homomorphism as belonging to the larger category of function" (p. 4). This highlights that abstract algebra does indeed build upon and reinforce notions of function from previous courses. This is one of the reasons why function has been identified as a key connection that pre-service teachers should make between their abstract algebra coursework and secondary mathematics (e.g., Melhuish & Fagan, 2018; Wasserman, 2017). But I also note that abstract algebra emphasizes parts of the function concept that are different from those in previous courses. For example, as noted by Weber and colleagues (2020), in courses leading up to calculus, the domain and codomain for many functions (provided that they are discussed at all) is typically assumed to be the real numbers. Relatedly, as I suggested above, much of students' previous experience with function does not emphasize a correspondence/relational view of function. Additionally, under the second characterization of function mentioned in section 2.2.1, everywhere-definedness is automatic, leaving well-definedness as the sole defining feature. As this characterization is not uncommon in courses up to the calculus sequence, I infer that everywhere-definedness is an idea that has relatively little basis in students' experience. (Most versions of the vertical line test, for example, focus only on well-definedness and not on everywhere-definedness.) These differences underscore the importance of developing a clearer image of how well-definedness and everywhere-definedness are used in abstract algebra and, further, how these uses resonate with or are different from students' experience with functions in previous courses.

2.2.4 Summary and implications. Despite the many impactful and enlightening studies that have investigated students' reasoning with binary operations, homomorphisms, and isomorphisms, research that has directly investigated issues related to well- and everywhere-definedness in abstract algebra is scarce, and what research does exist points to the fact that these concepts are difficult for students. This paper contributes to the literature by directly focusing on and elaborating the essential function properties of well-definedness and everywhere-definedness. Furthermore, the current literature emphasizes that the examples and uses of function in abstract algebra reinforce and build upon students' previous experiences with functions, but also emphasize new, possibly unfamiliar aspects of the function concept. While this review points out the existence of these similarities and differences, it also calls attention to the need for research to characterize these differences in more detail, particularly with respect to well- and everywhere-definedness. Thus, in this paper I focus explicitly on the properties of well-definedness and everywhere-definedness and propose two elaborations that I argue (1) provide additional insight into the nature of the function concept, and (2) provide viable explanations regarding the successes and difficulties that students experience with functions in abstract algebra.

2.3 Theoretical Perspective

2.3.1 What is an example? In this paper, I use examples of the function concept to gain insight into well-definedness and everywhere-definedness. Following Watson and Mason (2005), I interpret 'example' inclusively to mean any specific illustration of an abstract mathematical principle, concept, or idea. This might include exercises, representations, diagrams, and, importantly for this study, non-examples. My use of the word ‘might’ reflects the theoretical principle that “exemplariness resides not in the example, but in how the example is perceived” (Mason, 2006, p. 62), a stance that is consistent with the constructivist epistemology (von Glasersfeld, 1995) that underpins this study. From this perspective, mathematical conceptions are (1) cognitive structures that emerge and develop as an individual continually reorganizes their experiences in internally coherent ways, and (2) exist only in relation to the individual who has constructed them. In other words, the phrase “f is a non-example of a function” makes sense only in relation to the person for whom f is a non-example of function. Other considerations include how the person for whom f is a non-example conceives of functions and what aspects of their conceptions of function are highlighted by the non-example. That is, in addition to providing important context, by examining a person’s reasoning about examples and non-examples, “researchers may draw inferences about their knowledge” (Zazkis & Leikin, 2008, p. 132). In this paper, I examine the examples—specifically, the non-examples—of function generated by experts in order to gain insight into productive ways of knowing and reasoning about well-definedness and everywhere-definedness.

2.3.2 The importance and utility of examples. Examples are important in mathematical reasoning because they help illustrate concepts and principles as well as contextualize mathematical ideas (Watson & Mason, 2005). Examples in general afford key insights into the concepts they exemplify. Non-examples are particularly insightful because they “demonstrate the boundaries or necessary conditions of a concept” (Watson & Mason, 2005, p. 65). That is, non-examples illustrate how far a particular concept or idea can be generalized (Watson & Mason, 2005). They showcase the essential aspects and features of definitions (such as well-definedness and everywhere-definedness) by showing what fails when these conditions are not met and thus help illustrate the role and importance of such features. In this paper, I investigate the ideas of well-definedness and everywhere-definedness using non-examples.

In addition to examples being a useful window of insight into mathematical concepts in general, the literature suggests that they could be particularly so for functions at the advanced undergraduate level. Melhuish and colleagues (2020) noted that, for the students in their study, "a lack of unification between the general function [concept] and *specific AA functions* was pervasive" (p. 15, emphasis added). Similarly, Even (1993) identified some challenges that undergraduate students face when distinguishing between functions and non-functions and proposed that, in particular, having students consider well-chosen examples of functions and non-functions could be beneficial in helping them develop a clearer image of this distinction. Even and Tirosh (1995) later elaborated this recommendation by noting that the undergraduate students in their study adapted their thinking about functions when presented with specific "problematic *cases*" (p. 17, emphasis added). The problematic cases proposed[3] were primarily *non*-examples—for instance, the non-examples $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(a, b) = \frac{a}{b}$ (which is well-defined but not everywhere-defined) and $g: \mathbb{Q} \to \mathbb{Q}$ given by $g\left(\frac{m}{n}\right) = (-8)^{m/n}$ (which is neither well-defined nor everywhere-defined). Taken together, I interpreted the insights of these researchers as an indication of the power of both examples and non-examples of function for illustrating key issues related to well- and everywhere-definedness.

2.3.3 Example spaces. I operationalize examples in this study using Watson and Mason's (2005) notion of *example space*—the content and structure of the examples that one associates to a particular concept. The construct of example space was initially proposed as a means of explaining the nature of mathematical thinking and learning. More recently, researchers have found that constructing models of both students' and experts' example spaces affords valuable insights into their thinking because one's example spaces "mirror their understanding of particular mathematical concepts" (Zazkis & Leikin, 2008, p. 131). Watson and Mason (2005)

[3] I inferred reasonable domains and codomains and added the names f and g.

distinguished between different kinds of example spaces, two of which are important for my objectives here. A *personal example space* is the collection and organization of examples (and non-examples, etc.) that a person associates with a particular mathematical topic. The *conventional example space* is the collection of examples "as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students" (Watson & Mason, 2005, p. 76). Zazkis and Leikin (2008) proposed a useful refinement of the conventional example space, distinguishing between *expert example spaces* and *instructional example spaces*. Expert example spaces display the "rich variety of expert knowledge" whereas instructional example spaces involve what is "displayed in textbooks" and used in instruction (Zazkis & Leikin, 2008, p. 132). Given the cognitive approach to examples I outlined in section 2.3.1, I consider the contents of the instructional example space to be the union of the instructional examples used by specific, individual experts. Thus, to say that an example is in the instructional example space for function in abstract algebra is to say that there is a specific individual (in this case, an abstract algebra instructor or abstract algebra textbook author) who (1) views the proposed object as an example of function, and (2) considers it to be useful in their instruction. In this study, I examine the non-examples in the instructional example space to gain insight into how students could, should, or might be thinking about functions.

Example spaces, including the instructional example space, are not just lists of examples (the *contents*), but also include the means of organizing these examples (the *structure*). Indeed, I note that many papers that operationalize example spaces are fine-grained analyses of collections of examples; the analyses are then based upon researchers' perceptions and inferences of how these collections are organized. A key point here is that the structure of an example space is based upon a researcher's inferences—that is, hypotheses about how the contents of the example space in question might be productively and coherently organized. The researcher bases these inferences on the explanations and rationale that an individual uses to describe particular (non-

)examples. Inferences might include, for instance, a researcher's view of (1) the purpose served by an example (such as the attributes that make it exemplary), or (2) important distinctions between (non-)examples in a given collection (and what aspects of the associated topic these distinctions might correspond to). In particular, in this paper I infer a *structure* for the instructional example space for function by considering the explanations and rationale offered by abstract algebra instructors (textbook authors and other algebraists) to describe its *contents*.

The instructional example space provides useful insight into how the function concept might be productively understood in abstract algebra. Watson and Mason (2005) noted that one of the ways in which students can extend their personal example space—and thus also extend their understanding of the associated concept—is by reasoning about the contents of the conventional example space. This calls attention to the potential for examining the instructional example space for function in abstract algebra to both (1) identify key aspects of the definitive characteristics of well-definedness and everywhere-definedness, and, eventually, (2) characterize the elements of a productive way of reasoning about functions that takes these definitive characteristics into account. This paper focuses on (1) while laying the groundwork for (2). So, in this paper I examine the contents (by conducting a textbook analysis) *and* structure (by conducting interviews with mathematicians) of the non-examples in the instructional example space for function in abstract algebra in order to gain insight into the nature of well-definedness and everywhere-definedness.

2.4 Methods

I employed two different methodologies to examine the instructional example space for function in abstract algebra. I first conducted a textbook analysis, the primary purpose of which was to identify the non-examples for function in the instructional example space (the *contents*). One limitation of textbook analyses, however, is that one cannot ask clarifying or follow-up questions about how the examples might be organized, which are useful ways to gain insight into the *structure*. Thus, I complemented the textbook analysis with a series of semi-structured interviews (Fylan, 2005) with mathematicians. The primary purpose of the interviews was to develop a clearer image of the structure of the non-examples in the instructional example space in order to identify important aspects of the key function characteristics of well-definedness and everywhere-definedness. I elaborate on the important methodological principles and procedures for the textbook analysis in section 2.4.1 and the interviews in section 2.4.2.

2.4.1 Textbook analysis. I first conducted a textbook analysis because (1) the instructional example space, by definition, contains the examples in textbooks (see section 2.3.3), and (2) textbook analyses can provide insight into "how experts in a field … define and frame foundational concepts" (Lockwood, Reed, & Caughman, 2017, p. 389). While the primary purpose was to identify the non-examples in the instructional example space (the contents), I was also attentive to insights in the textbooks regarding how experts might organize these non-examples (the structure). Consistent with my theoretical perspective, instead of considering the results of my analysis as a one-to-one representation of the contents and structure of the instructional example space, it is important to note that these are contents and structure I *inferred* from my analysis of the content presentation in these textbooks.

2.4.1.a Data collection. In total, I collected data from 14 abstract algebra textbooks (see Table 1). Four of the textbooks came from Melhuish (2019) which identified the four most popular abstract algebra textbooks used in the United States. Additionally, I compiled a list of the top 25 ranked universities in the United States (National University Rankings, n.d.) and identified the first abstract algebra course offered at each of these institutions by looking at their course catalogs. I proceeded to find titles of the required texts by checking the university bookstore's website and instructors' course pages or syllabi (all but two were within one year and the earliest date listed was 2016). Unfortunately, I was unable to obtain access to the textbooks used at three of these universities (one of which used their own packet). This process added a total of five new books to my list—some of the textbooks used by the top universities were among the four books previously mentioned and thus already on the list. Lastly, I included five textbooks from my personal library. For each of the textbooks, I analyzed the most recent edition I had access to and omitted all other editions.

Table 1: Textbooks used in the textbook analysis

Author(s) (Year)	*Title (Edition)*
Artin, M. (2011)	Algebra (2nd ed.)
Beachy, J. A. & William, D. B. (2019)	Abstract Algebra (4th ed.)
Birkhoff, G. & Mac Lane, S. (1977)	A Survey of Modern Algebra (4th ed.)
Davidson, N. & Gulick, F. (1976)	Abstract Algebra: An Active Learning Approach
Dummit, D. S. & Foote, R. M. (2004)	Abstract Algebra (3rd ed.)
Fraleigh, J. B. (2002)	A First Course in Abstract Algebra (7th ed.)
Gallian, J. A. (2017)	Contemporary Abstract Algebra (9th ed.)
Gilbert, L. & Gilbert, J. (2015)	Elements of Modern Algebra (8th ed.)
Herstein, I. N. (1975)	Topics in Algebra (2nd ed.)
Herstein, I. N. (1996)	Abstract Algebra (3rd ed.)
Hodge, J. K., Schlicker, S., & Sundstrom, T. (2014)	Abstract Algebra: An Inquiry-Based Approach
Hungerford, T. W. (2014)	Abstract Algebra: An Introduction (3rd ed.)
Pinter, C. C. (1990)	A Book of Abstract Algebra (2nd ed.)
Rotman, J. J. (2006)	A First Course in Abstract Algebra (3rd ed.)

Once the list was compiled, I identified the places in each textbook that I suspected would discuss functions and different aspects of function explicitly. To begin, I created a list of terms (informed by the literature and my personal knowledge of abstract algebra) related to function like different names used in abstract algebra for functions (e.g., map/mapping, correspondence) as well as the key aspects of function identified by Melhuish and Fagan (2018) (e.g., well-definedness, everywhere-definedness, domain, range), variable aspects of function (e.g., injective, surjective), and examples of function that arise in abstract algebra (e.g., binary operation, homomorphism, isomorphism). I first looked at the table of contents of each book and identified the sections I believed would discuss these terms. Then, I turned to the index and searched there, noting each section where the terms showed up. Once I had a complete list of sections to focus on for each book, I collected those sections (including the exercises that accompanied each section) from either the PDF files (when those were available) or by digitally scanning those sections.

2.4.1.b Data analysis. I followed Creswell's (2012) method for identifying and interpreting themes in qualitative data. I began by looking at all of the data collected, checked that it did indeed discuss functions (data that was not relevant was removed from the analysis), and made notes of my first impressions of the data. There were two kinds of excerpts I sought to identify: (1) those containing non-examples of function and (2) the authors' associated descriptions and explanations related to a given non-example. The list of non-examples provides the specific contents of the instructional example space and the explanations—and any inferences drawn from the non-examples and explanations—contributed to the inferences I drew concerning the *structure* of the example space.

For the first round of coding, the notions of well-definedness and everywhere-definedness served as a useful means of categorizing the non-examples I identified. Specifically, I coded each excerpt containing a non-example by classifying each non-example as a "well-definedness" non-example (a proposed correspondence that fails to be a function because at least one element of the domain maps to more than one element of the codomain) and/or as an "everywhere-definedness" non-example (a proposed correspondence that fails to be a function because at least one element of the domain maps to no element of the codomain). Similarly, I coded each excerpt containing a description or explanation in the same way based upon my image of what issue or aspect of function the author was pointing out. I then administered additional codes that described particular characteristics of these non-examples. Following Creswell (2012), I looked for new codes that arose as well as elaborations to existing codes; these additional codes were continually refined and revised in this manner as coding progressed.

2.4.2 Semi-structured interviews with mathematicians. I conducted a series of semi-structured interviews (Fylan, 2005) with mathematicians as a way to follow up on conjectures I developed—as well as points that needed clarification—in the textbook analysis. Semi-structured interviews were important for my objectives because they allow the interviewer to "address aspects that are important to individual participants" (Fylan, 2005, p. 66). In this type of interview, the researcher has a list of topics to cover and a set of questions to ask (my preset list was informed by the textbook analysis—see below) but the format of the interview allows for the conversations to vary and provides flexibility for the discussion to change considerably between participants (Fylan, 2005). It is in this way that I was able to flexibly pursue emerging themes I inferred related to the structure of the instructional example space.

2.4.2.a Data collection. I began by conducting an hour-and-a-half-long open semi-structured group interview with five mathematicians as a way to pursue conjectures I developed about potential key structural features in the textbook analysis as well as to identify rich areas to follow up on. The five mathematicians (whom I refer to as Professor A, B, C, D, and E)[4] were all tenured or tenure-track faculty members at a midwestern Research 1 university. I was interested in faculty members who had taught at least one abstract algebra course in the last five years. I invited all professors who met this criterion; the five participants in this study were those who accepted this invitation.

[4] I use the gender-neutral pronoun "they" when referring to individual professors throughout the paper.

The prompts used in this interview were informed by the textbook analysis. Mathematicians were asked in advance to think about the question: “When you are teaching the introductory unit on functions at the beginning of an introductory abstract algebra course, what are three examples/non-examples that you like to use to illustrate the concept?” During the interview, they shared and discussed their examples and explained why they liked the examples they chose. The main prompts and questions I asked the mathematicians were:

1) In your experience, how do you think students typically view or think about functions when they begin abstract algebra?
2) What examples do you like to use at the beginning of the course to illustrate the importance of well-definedness of a function?
3) What examples do you like to use at the beginning of the course to illustrate the importance of everywhere-definedness of a function?

Consistent with the semi-structured interview methodology, I used these three prompts as a starting point to ask more targeted follow-up questions. The follow-up questions often centered on specific non-examples; these questions were often based on hypotheses regarding the emerging primary codes (see section 2.4.1.b) I was using to organize the collection of non-examples.

The individual interviews that followed were about an hour to an hour-and-a-half in length and allowed me to ask clarifying questions about themes and comments that emerged in the textbook analysis and group interview. The following list contains prototypical questions related to the structure of non-examples that I asked the mathematicians:

1) What do you see as key aspects of the following non-example of function $f: \mathbb{Q} \to \mathbb{Z}$ given by $f\left(\frac{a}{b}\right) = a + b$?

2) What do you see as key aspects the following non-example of function $g: \mathbb{Z} \to \mathbb{N}$ given by $g(x) = x^3$?
3) What do you see as key aspects of the following non-example of function, $\phi: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ given by $\phi(a, b) = \frac{a}{b}$?
4) What do you see as key aspects of the following non-example of function, $p: (0, \infty) \to \mathbb{R}$ given by $p(x) = \pm\sqrt{x}$?

All mathematicians were invited to participate in additional follow-up interviews to explore in greater detail emerging themes related to my hypotheses about the structuring of the instructional example space that emerged in previous interviews (I provide more information about my analysis of these interviews in the next section). All mathematicians[5] participated in at least one individual interview; Professor B participated in two and Professors A and E participated in three.

2.4.2.b Data analysis. Analysis for the semi-structured interviews proceeded in two phases. Phase 1 was conducted during and between interview sessions and primarily sought to identify potential points of clarification and additional detail pertaining to the mathematicians' thoughts on how to productively structure the collection of non-examples that I identified in the textbook analysis. For Phase 2, I again used Creswell's (2012) method (outlined in 2.4.1.b above). One distinction, though, was that this analysis was more targeted and made use of the codes from the textbook analysis. Iterating Creswell's (2012) procedures enabled me to clarify, refine, and elaborate these existing codes. I had emerging hypotheses about the key structural elements of the non-examples in the instructional example space that were then continually refined into the following final codes:

[5] Two mathematicians were unable to participate in more than one individual interview due to scheduling constraints.

Table 2: Final revised list of codes

Well-definedness: Equivalence/Representation A non-example was assigned this code if there was at least one element in the domain that had different equivalent representations and the rule assigned different images in the codomain to each of these representations.	*Well-definedness: Multiple Rules* A non-example was assigned this code if there was at least one element in the domain that was assigned different images in the codomain because the rule was ambiguous (not due to different representations of the domain element).
Everywhere-definedness: Some Set A non-example was assigned this code if there was at least one element in the domain that was assigned an output that was not contained in the proposed codomain (but this output was contained in a larger accessible set).	*Everywhere-definedness: No Set* A non-example was assigned this code if there was at least one element in the domain that did not get assigned any value (and any possible value was not contained in an accessible set).

2.5 Results

This section is structured based upon the categories of non-examples that I perceived in the instructional example space. The two initial categories—well-definedness and everywhere-definedness—emerged as useful categories in the textbook analysis; the subcategories (well-definedness equivalence, well-definedness multiple rules, everywhere-definedness some set, and everywhere-definedness no set) emerged as hypotheses as the result of the textbook analysis and were explored further and continually refined in the interviews with the mathematicians. For each category (and subcategory), I describe my classification criteria and provide a list of the non-examples that populate it—using textbook excerpts and comments from mathematicians.

2.5.1 Well-definedness. I classified a non-example in the "well-definedness" category if there exists an element of the (proposed) domain for which there are at least two corresponding images contained in the (proposed) codomain. For simplicity, I focus on non-examples that meet this criterion (and not on non-examples that might have both well-definedness and everywhere-definedness issues). For example, let $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$, and consider the correspondence given by the subset $\{(1,6),(1,2),(1,4)\}$ of $A \times B$ (this is an exercise in Fraleigh, 2002, p. 8). This correspondence is not a function from A to B because the element 1 is assigned three different images in the specified codomain, namely 2, 4, and 6. So the rule or assignment is ambiguous. The element 1 maps to more than one element of the codomain and therefore this proposed correspondence has a well-definedness issue. As another example, consider the rule $f: \mathbb{Z}_6 \to \mathbb{Z}_7$ given by $f([a]_6) = [a+1]_7$ (this is a special case of an exercise in Hodge, Schliker, & Sundstrom, 2014, p. 134). Then $f([0]_6) = [0+1]_7 = [1]_7$ and $f([6]_6) = [6+1]_7 = [7]_7 = [0]_7$. Since $[1]_7 \neq [0]_7$ in $\mathbb{Z}_7$, we have that $f([0]_6) \neq f([6]_6)$ but $[0]_6 = [6]_6$ in $\mathbb{Z}_6$. Thus, f is not well-defined. Other non-examples that I classified in this category are displayed in Table 3.

Table 3: Non-examples of function with well-definedness issues[6]

1 $g: \mathbb{Q} \to \mathbb{Q}$ given by $$g\left(\frac{a}{b}\right) = ab$$ (Rotman, 2006, p. 91)	**4** A is the union of two subsets A_1 and A_2 f from A to the set $\{0, 1\}$ where f maps everything in A_1 to 0 and everything in A_2 to 1 (Dummit & Foote, 2004, p. 1-2)
2 S is the set of all people in the world T is the set of all countries in the world f from S to T is the rule that assigns to every person his or her country of citizenship (Herstein, 1996, p. 8)	**5** $f: \mathbb{Z}_4 \to \mathbb{Z}_6$ given by $$f([x]_4) = [x]_6$$ (Beachy & Blair, 2019, p. 57)
3 $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $$\phi\left(\frac{a}{b}\right) = a + b$$ (Gallian, 2017, p. 21)	**6** □ defined on the set ℝ by: a□b is the number whose square is ab (Pinter, 2010, p. 20)

[6] These non-examples were slightly modified for clarity and simplicity.

Generally, I noticed a key distinction regarding the way these well-definedness non-examples might be productively structured. For example, when comparing non-example 3 (on Table 3) and a similar version[7] of non-example 6 (on Table 3), Professor A said,

> the first one, the addition, there is a clean procedure … the formula as written, it looks like it's cut and dried and well-defined. Um, it's not, but it looks like you're getting, an unambiguous output. Uh, whereas the second one, the formula doesn't even look like that. There's no way you could write this formula down and as you were writing it, think your output is unambiguous.

This was similar to a distinction made by Professor E, who mentioned "So, the first problem … it's the way the sets are defined. Uh, the second one is the way the function is defined."

In the context of discussing different non-examples of function, all with well-definedness issues, Professor B said,

> They are two different types of problems … your function could be, um, not well-defined because, the value in the domain is not well-defined, or that you have to make a choice in the value of the domain. Or they could be, not well-defined because the value of the output is not well-defined and you have to make a choice of that value of the output.

Taken together, I interpreted these comments to mean that some non-examples in this category (like non-example 3 on Table 3) cause well-definedness issues because of matters due to different yet equivalent representations of the same element in the domain (consider, for example, Professor E's comment about "it's the way the sets are defined" and Professor B's comment that "you have to make a choice in the value of the domain"). On the other hand, some non-examples in this category (like non-example 6) seem to cause well-definedness issues because of the rule

[7] The square root correspondence: $p: (0, \infty) \to \mathbb{R}$ given by $p(x) = \pm\sqrt{x}$.

(consider, for example, Professor A's comment that there is "no way you could write this formula down and … think your output is unambiguous" and Professor E's comment that the issue is related to "the way the function is defined"). While I shall explain the importance and implications of this distinction in greater detail in section 2.6 (Discussion); here I shall simply note that prior to advanced mathematics, students have generally had experience with "multiple rules" (such as non-examples 2, 4, and 6 in Table 3) but not "equivalence/representation" (such as non-examples 1, 3, and 5 on Table 3). I elaborate on these notions in the subsections that follow.

2.5.1.a Well-definedness equivalence/representation. The *well-definedness equivalence/representation* non-examples are non-examples of function where (1) elements in the domain can be represented in multiple ways and (2) the rule maps these different representations of the same element to different outputs. Textbook authors attended to this distinction as well, as Beachy and Blair (2019) explain, "Problems arise when the element x can be described in more than one way, and the rule or formula for $f(x)$ depends on how x is written" (p. 56). Indeed, if "there are multiple ways to represent elements in the domain (like in $\mathbb{Z}_n$ or $\mathbb{Q}$), then we need to know whether our mapping is well-defined before we worry about any other properties the mapping might possess" (Hodge, Schlicker, & Sundstrom, 2014, p. 129). This holds true for any function. With specific focus on binary operations (which are a class of examples of function in abstract algebra), Gilbert and Gilbert (2015) caution,

> If the defining rule for a possible binary operation is stated in terms of a certain type of representation of the elements, then the rule does not define a binary operation unless the result is independent of the representation for the elements—that is, unless the rule is well-defined. (p. 305)

Attending to this kind of well-definedness issues—related to *equivalence/representation* of elements in the domain—is of particular importance in abstract algebra because, as Professor B explained, “a large, uh, an important, uh, aspect of abstract algebra is to construct things, by means of, equivalence relations. And, uh, so, the validity of your constructions, depends on checking, that equivalent things are used in the same way.” This is a key issue in proofs, especially those involving functions on quotient structures (e.g., $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$).

Non-examples 1, 3, and 5 on Table 3 are all examples of rules or correspondences that have a *well-definedness equivalence/representation* issue. On non-example 1 (on Table 3), we see that there are infinitely many ways one can represent an element of the domain, $\mathbb{Q}$. For example, $\frac{1}{2}$ is equivalent to $\frac{2}{4}$, but also to $\frac{3}{6}, \frac{4}{8}, \frac{5}{10}$, and many others. As Rotman (2006) points out, “since $\frac{1}{2} = \frac{3}{6}$, we see that $g\left(\frac{1}{2}\right) = 1 \cdot 2 \neq 3 \cdot 6 = g\left(\frac{3}{6}\right)$, and so g is not a function” (p. 91). The issue here is that g maps the element $\frac{1}{2} \in \mathbb{Q}$ and all of its equivalent representations to different values in the codomain and thus it is not well-defined.

A similar issue occurs in non-example 3 (on Table 3). Gallian (2017) explains that ϕ “does not define a function since $1/2 = 2/4$ but $\phi(1/2) \neq \phi(2/4)$” (p. 21). As Professor D explained, “on one half and two fourths you get different answers. So if you get different answers for the same input, it’s not a function.” They described further, “Any rational number that you pick, has non-unique representations as a fraction.” The issue is therefore on the representation of the elements in the domain, in coordination with the rule. Professor E pointed to this as the issue in this non-example,

> the function is deliberately taking, a particular presentation … takes a, a particular presentation of the rationals … That’s the issue … that’s a problem. Like if you’re going to, if you’re gonna use a representative … then you have to be extra careful.

Professor C suggested that this was an issue for students because "students tend to look at fractions as a fixed thing and not a class of a quo-, an equivalence class, a representative of an equivalence class."

Non-example 5 (on Table 3) deals with a different domain where issues of equivalence/representation also occur. In this case, "the formula $f([x]_4) = [x]_6$ does not define a function from $\mathbb{Z}_4$ into $\mathbb{Z}_6$. To see this, we only need to note that although $[0]_4 = [4]_4$, the formula specifies that $f([0]_4) = [0]_6$, whereas $f([4]_4) = [4]_6$, giving two different values in $\mathbb{Z}_6$, since $[0]_6 \neq [4]_6$" (Beachy & Blair, 2019, p. 57). When discussing this non-example, Professor B mentioned,

> definitely, you know, um, in the section where you're describing modular arithmetic $\mathbb{Z}_n$, we certainly discuss that kind of function, that it's not a function. That it's not well-defined. And, yeah, those are definitely, um, uh, worth having.

Thus signaling that this was a common task in abstract algebra. They continued, focusing on why it is a useful non-example for students,

> it's so easy to think that [f] of x equals x is always a function, you know? … and it's hard to get used to, um, it not being well-defined based on the definition of the domain … the idea is that, umm, if I take, um, if I take x equal to, one, or the-the equivalence class of one mod four, well that's equivalent to five. And so, if I, if I were to choose five, it would map to five, and if I were to choose one, it would map to one. And one and five are not equivalent in the … codomain.

Professor C brought up, "So, understanding that $\mathbb{Z}_4$, is not simply something with four elements in it, but that four elements that are just numbers zero, one, two, three, but that … these are, divisor classes, as it were."

I interpreted this to mean that it is critical to understand that the elements of $\mathbb{Z}_4$ (the domain) are equivalence classes—and not simply numbers—and therefore can be represented in different ways. (See Table 4 for other non-examples in this subcategory.)

Table 4: Non-examples with *well-definedness equivalence/representation* issues[8]

$\mathbb{Z}/\langle 3\rangle \to \mathbb{Z}_6$ $x + \langle 3\rangle \mapsto 3x$ (Gallian, 2017, p. 195)	$f: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by $f\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a+c}{b+d}$ (Rotman, 2006, p. 105)
$f: \mathbb{Q}^{\geq 0} \to \mathbb{Z}$ given by $f\left(\frac{m}{n}\right) = 2^m 3^n$ (Herstein, 1996, p. 14)	$\phi: \mathbb{Z}_6 \to \mathbb{Z}_5$ given by $\phi(a_6) = a_5$ (Davidson & Gulick, 1976, p. 94)
$\diamond$ defined on $\mathbb{Z}_n$ as follows: $[a] \diamond [b] = \begin{cases} [1] \text{ if } a \text{ and } b \text{ have the same parity} \\ [0] \text{ if } a \text{ and } b \text{ have opposite parity} \end{cases}$ (parity refers to whether an integer is even or odd) (Hodge, Schlicker, & Sundstrom, 2014, p. 53)	$\star$ on $\mathbb{Z}_n$ given by: $[a] \star [b] = \begin{cases} [1] \text{ if } a = b \text{ (mod 5)} \\ [0] \text{ if } a \neq b \text{ (mod 5)} \end{cases}$ (Hodge, Schlicker, & Sundstrom, 2014, p. 61)
$f: \mathbb{Q} \to \mathbb{Z}$ given by $f\left(\frac{a}{b}\right) = a$ (Dummit & Foote, 2004, p. 4)	$f: \mathbb{Q} \to \mathbb{Q}$ given by $f\left(\frac{m}{n}\right) = \frac{m+1}{n+1}$ (Beachy & Blair, 2019, p. 64)
$f: \mathbb{Z}_4 \to \mathbb{Z}_8$ given by $f([a]_4) = [3a]_8$ (Hodge, Schlicker, & Sundstrom, 2014, p. 134)	$p: \mathbb{Z}_{12} \to \mathbb{Z}_5$ given by $p([x]_{12}) = [2x]_5$ (Beachy & Blair, 2019, p. 65)

2.5.1.b Well-definedness multiple rules. The *well-definedness multiple rules* non-examples are non-examples where the definition of the rule is ambiguous because it has two or

[8] These non-examples were slightly modified for clarity and simplicity.

more choices of images in the specified codomain for an element of the domain. It differs from well-definedness equivalence/representation in that the rule is not dealing with different representations of elements of the domain in question, but rather the rule is simply outputting two or more values and a choice has to be made as to which element in the codomain to map to. The main distinction that I inferred between this category and well-definedness equivalence/representation is on where the choice is being made. Professor A, for instance, said, "Where is the choice taking place? Is it in your, uh, input? Or is it, uh, in the execution of the rule?" Thus, in this category, the choice being made is in the execution of the rule. The distinction between these two types of well-definedness is important because as Professor A explained,

> in both cases a choice needs to be made. In the [well-definedness equivalence/representation] case, it's actually worse because there are infinitely many choices. Um, but somehow, just the way they're presented and the way that I imagine, uh, the student getting there, really does demand a different treatment.

The name 'multiple rules' was inspired by textbooks and mathematicians referring to well-defined functions as single-valued or using the term multiply-valued functions in discussing non-functions with well-definedness issues. For example, in discussing well-definedness, Rotman (2006) mentions that "one sometimes says that [a function] is *single-valued*" (p. 91). In one of the interviews, Professor B said, "if the representative in the domain is well-defined the problem is the [value] in the codomain, then I would call that the problem of a multiply-valued function." Thus I adopted the term *well-definedness multiple rules* as a way to reference this notion of a non-example that takes on multiple values for a single input.

Non-examples 2, 4, and 6 on Table 3 illustrate this subcategory. The first of these, non-example 2 (on Table 3), helps clarify the difference between this subcategory of well-definedness and the previous one. As Herstein (1996) explains,

> there are people in the world that enjoy a dual citizenship; for such people there would not be a unique country of citizenship. Thus, if Mary Jones is both an English and French citizen, f would not make sense, as a mapping, when applied to Mary Jones. (p. 8)

So here f would assign to Mary Jones both England and France as countries of citizenship. Notice that this is not a result of different representations (the input, Mary Jones, is not being represented in a different way) rather it is the definition of the rule itself that assigns more than one image to this element. Therefore, f has a *well-definedness multiple rules* issue.

On non-example 4 (on Table 3), we would only have an issue if the union of the sets that make up the domain are not disjoint. Indeed, Dummit & Foote (2004) explain,

> This unambiguously defines f unless A_1 and A_2 have elements in common (in which case it is not clear whether these elements should map to 0 or to 1). Checking that this f is well defined therefore amounts to checking that A_1 and A_2 have no intersection. (p. 1-2)

Professor E pointed out that this example does not depend on representation of an element of the domain, but rather the rule itself just assigns different values to certain elements of the domain,

> it's a matter of like, sending objects to too many things sometimes, potentially … but there's no issue of representative, you know, like, you're not, you're not invoking, a uh, a presentation of elements of A or B. You're just saying, you got something in A or B.[9]

Professor B further clarified, "I wouldn't say that equivalence is the heart of, uh, [it.]" They continued,

[9] By A and B here, Professor E means A_1 and A_2; similarly for Professor B.

> the problem with a union is that they could have overlap … the definition has, two possible values on the intersection of A and B … you have to clarify which value you're gonna choose … that's a problem with, the multiple values of the rule again.

Lastly, non-example 6 (on Table 3) displays a non-example of a binary operation—a specific kind of function in abstract algebra—with a well-definedness multiple rules issue. This particular non-example actually showed up in three different textbooks: as a non-example of a binary operation in Pinter (2010), as a non-example of a function in Rotman (2006), and as an exercise in Hungerford (2014). I will first discuss it as a non-example of a binary operation and then switch to discussing it as a non-example of a function like the one found in Rotman (2006). The issue here, as Pinter (2010) points out, it "is ambiguous because 2□8, let us say, may be either 4 or -4. Thus, □ does not qualify as an operation on $\mathbb{R}$" (p. 20). It is important to note here again that the issue is not related to different representations of the elements 2 or 8, or any elements of the domain for that matter. Professor E explained, "you're not taking advantage of any strange representation … the problem is just like with the function itself … so, that to me is a key difference." Discussing a similar non-example (written as a function rather than a binary operation[10]), Rotman (2006) points out "that there are two candidates for $\sqrt{9}$, namely, 3 and -3. In order that there be exactly one number assigned to 9, one must select one of the two possible values ± 3" (p. 87). It later continues, "$f(a) = \pm\sqrt{a}$ is not single-valued, and hence it is not a function" (p. 91). Professor A further explained, "we need a unique output, which means we've got to make a choice." The choice in this case seems to happen in the codomain; a choice needs to be made whether the square root maps to the positive square root or to the negative square root, but this rule does not deal with different representations of the real numbers in the domain. (See Table 5 for other non-examples in this subcategory.)

[10] The square root correspondence $f\colon \mathbb{R}^+ \to \mathbb{R}$ given by $f(a) = \pm\sqrt{a}$.

Table 5: Non-examples with *well-definedness multiple rules* issues[11]

$*$ on $\mathbb{Z}^+$ defined by: $a * b = c$ where c is at least 5 more than $a + b$ (Fraleigh, 2002, p. 27)	$h: \{1, 2\} \to \{1, 2, 3\}$ given by $h(1) = 1, h(2) = 2, h(2) = 3$ (Davidson & Gulick, 1976, p. 3)
$f: \mathbb{R}^+ \to \mathbb{Z}$ given by f maps a real number r to the first digit to the right of the decimal point in a decimal expansion of r (Dummit & Foote, 2004, p. 4)	$S = \{1, 2, 3\}$ and $T = \{4, 5, 6\}$ The correspondence F_4 given by the subset of $S \times T$: $F_4 = \{(1, 4), (2, 4), (2, 5), (3, 6)\}$ (Beachy & Blair, 2019, p. 54-55)
"If you actually look at the football roster, there are a bunch of guys, I guess on the practice team or something who never play, but share numbers with, regular players." (Professor A)	"I think graphically speaking, [students] probably know that you break the vertical line test, if you want it to not be a function. So they'll do a squiggle or they'll give you a circle, or something like that." (Professor D)
"A straight line that's a vertical line." (Professor D)	"They'd probably draw a sideways parabola." (Professor D)

2.5.2 Everywhere-definedness. I classified a non-example in the everywhere-definedness category if there exists at least one element of the domain that does not have a corresponding image contained in the suggested codomain. Just like I did with the previous category, as a way to simplify matters, I focus in this section only on non-examples with only an everywhere-definedness issue (and not on non-examples that might have both everywhere-definedness and well-definedness issues). As an example, consider the formula $f: \mathbb{C} \to \mathbb{C}$ given by $f(x) = 1/(x^2 + 1)$ (this is an exercise in Beachy & Blair, 2019, p. 64). Notice that for $i \in \mathbb{C}$, we get $f(i) = \frac{1}{i^2+1} = \frac{1}{-1+1} = \frac{1}{0}$, which is not an element of the codomain, $\mathbb{C}$. So f does not assign the element i of the domain any element in the proposed codomain (the same is also true for $-i \in \mathbb{C}$)

[11] These non-examples were slightly modified for clarity and simplicity.

and thus f is not everywhere-defined. As another example, consider the proposed correspondence from the set of nonnegative integers to itself given by $f(s) = s - 1$ from is another illustration of this (this is an exercise in Herstein, 1996, p. 13). The element 0 of the domain (which is the set of nonnegative integers) does not get mapped to an element in the proposed codomain (which is also the set of nonnegative integers) since $f(0) = 0 - 1 = -1$ which is not a nonnegative integer. Therefore, this rule has an everywhere-definedness issue. The everywhere-definedness category also includes the following non-examples on Table 6:

Table 6: Non-examples of function with everywhere-definedness issues[12]

1 $+$ on the set of all odd integers defined as: the usual addition on $\mathbb{R}$ (Davidson & Gulick, 1976, p. 10)	**4** $+$ on the set $M(\mathbb{R})$ of all matrices with real entries defined as: the usual matrix addition $+$ (Fraleigh, 2002, p. 21)
2 Division on the set $\mathbb{R}$ of the real numbers (Pinter, 2010, p. 19)	**5** $g: \mathbb{Z} \to \mathbb{N}$ given by $$g(x) = x^3$$ (modification of an example in Gallian (2017, p. 23))
3 $f: \mathbb{R} \to \mathbb{R}$ given by $$f(x) = \sqrt{x}$$ (Beachy & Blair, 2019, p. 56)	**6** $g: \mathbb{R} \to \mathbb{R}$ given by $$g(x) = \frac{1}{x}$$ (Professor E)

Similar to the well-definedness category, I offer a refinement of the everywhere-definedness category based on a distinction I inferred regarding the way that these everywhere-definedness non-examples can be productively structured. For instance, consider what Professor B commented about non-example 3 (on Table 6): “the outputs are not contained in the

[12] These non-examples were slightly modified for clarity and simplicity.

codomain." I interpreted this to mean that the outputs correspond to some elements that exist in some set that contains the proposed codomain. Contrast this with what they said about division (similar to non-example 2 on Table 6): "some changes are kind of forced on you … there's no division by zero, ever." I interpreted this to mean that there is no element that corresponds to division by zero. In other words, unlike in the previous example, there is not a set (or, at least, a set accessible to students in introductory abstract algebra) containing the proposed codomain in which the desired output resides. Professor A implied something similar, "there are situations where you just can't execute the instructions and there are situations where you could execute the instructions but only by a margin target." I call the first of these subcategories "everywhere-definedness some set" to refer to cases in which the output is contained in some accessible set that contains the proposed codomain; this includes non-examples 1, 3, and 5 on Table 6 above. I call the second "everywhere-definedness no set" to refer to cases in which the output is generally undefined and therefore not contained in any accessible set. This includes non-examples 2, 4, and 6 on Table 6 above. I explore these subcategories in the next two subsections.

2.5.2.a Everywhere-definedness some set. The *everywhere-definedness some set* non-examples are non-examples in which an element in the domain has no output contained in the proposed codomain but the codomain can be slightly modified or enlarged to an accessible set so that the output specified by the rule is contained in this new set. Professor A hinted to this when they mentioned that sometimes you have to "make the codomain big enough" to make certain rules into functions. Similarly, Professor E mentioned that some issues are about the "codomain not being large enough." I inferred from these comments that the input in question does correspond to some element, but that element is currently outside of the specified codomain.

Non-examples in this *everywhere-definedness some set* category include non-examples 1, 3, and 5 on Table 6. Consider first non-example 1 on Table 6. Davidson & Gulick (1976) explain "the operation of addition on $\mathbb{R}$ does not induce a binary operation on the set of all odd integers since this set is not closed under addition" (p. 64). As Hungerford (2014) explains, "The sum of two odd integers is not odd" (p. 45). Therefore, viewing this as a rule from the cross product of the set of odd integers with itself into the set of odd integers, it has an everywhere-definedness issue. The outputs are all even integers—that is, they exist in some set (the integers) but not in the currently specified codomain (the odd integers). since the rule specifies an output for every pair of odd integers that is not contained in the set of odd integers.

Non-example 3 on Table 6 shows another instance of this particular everywhere-definedness issue. As Beachy and Blair (2019) describe, "if we attempt to use the square root to define a function $f: \mathbb{R} \to \mathbb{R}$, we immediately run into a problem: the square root of a negative number cannot exist in the set of real numbers" (p. 56). However, there exists some (accessible) set that would contain the outputs as they further explain, "we can enlarge the codomain to the set $\mathbb{C}$ of all complex numbers" (p. 56).[13] Professor D hinted to this as well by specifying, "it's not a function because, that formula … is not defined on every element of domain … So [we're] having to adjust the codomain." Similarly, Professor E commented "there is a way to extend the codomain to make them functions." Indeed, the set of complex numbers, $\mathbb{C}$, contains the outputs that are specified by the rule. Lastly, in non-example 5 on Table 6, we see a similar issue where the codomain specified does not contain the outputs given by the rule. As Professor C explained, "the problem is that, where does minus one go? 'Oh, it's not a natural number, okay.' Yeah, actually, that's a nice one." Here, if we look at the output of the element -1 of the domain, we see that $g(-1) = (-1)^3 = -1$ but -1 is not in the proposed codomain, $\mathbb{N}$. As Professor D explained, "it's only problematic because you have labeled the codomain incorrectly. Those outputs can be negative as well as positive." They continued,

[13] Beachy and Blair (2019) note that they are using the convention that if "x is a nonnegative real number, then $\sqrt{x}$ is the nonnegative real number whose square is x, and if x is a negative real number, then $\sqrt{x}$ is $i\sqrt{|x|}$" (p. 56).

> [students] should know that $\mathbb{Z}$ is the set of integers, which include both positive and negatives, and the natural numbers are positives only. So they should certainly know that before they start to reason about something like this … so I think the thing that they have to remember to think about there is a negative integer … because it is only in the presence of the negative integers that $\mathbb{Z}$ differs from $\mathbb{N}$.

I inferred from Professors C and D's comments that the proposed codomain, namely $\mathbb{N}$, does not contain the outputs specified by the rule. However, this can be remedied by enlarging $\mathbb{N}$ to $\mathbb{Z}$ to have a perfectly fine function (and therefore I categorized this as an *everywhere-definedness some set* non-example. (See Table 7 below for more non-examples included in this subcategory.)

Table 7: Non-examples with *everywhere-definedness some set* issues[14]

Subtraction $-$ on $\mathbb{Z}^+$ (Dummit & Foote, 2004, p. 16)	Addition defined on $\{\pm n^2 : n \in \mathbb{Z}\}$ (Davidson & Gulick, 1976, p. 10)
S and T are both the set of positive integers $f: S \to T$ given by $f(s) = s - 1$ (Herstein, 1996, p. 13)	Addition defined on $\{0\} \cup \{\pm 2^n : n \in \mathbb{Z}^+\}$ (Davidson & Gulick, 1976, p. 10)
$*$ defined on $\mathbb{Z}^+$ by: $a * b = c$ where c is the largest integer less than the product of a and b (Fraleigh, 2002, p. 27)	"They're so used to … being allowed to divide by any nonzero number that they want that […] now you're dealing with integers, and […] you have to be extra careful." (Professor E)

2.5.2.b Everywhere-definedness no set. The *everywhere-definedness no set* non-examples are non-examples where an element in the domain has no output defined or has an output that is not contained in any set that might be accessible to a beginning abstract algebra student. This is different from the previous everywhere-definedness distinction in that the

[14] These non-examples were slightly modified for clarity and simplicity.

language used before focused on the output values specified by the rule simply not being contained in the codomain that is being proposed, whereas in this case, the focus is on the rule not being defined on certain domain elements (i.e., there being no output for such elements) or the rule being completely undefined.

Non-examples 2, 4, and 6 on Table 6 are non-examples in the *everywhere-definedness no set* category. On non-example 2 on Table 6, division of real numbers has this problem. Pinter (2010) explains,

> For example, division does not qualify as an operation on the set $\mathbb{R}$ of the real numbers, for there are ordered pairs such as $(3, 0)$ whose quotient $3/0$ is undefined. In order to be an operation on $\mathbb{R}$, division would have to associate a real number a/b with *every* ordered pair (a, b) of elements of $\mathbb{R}$. No exceptions allowed! (p. 19)

If we define this operation on $\mathbb{Q}$ instead (a subset of $\mathbb{R}$), it is also not defined everywhere as Fraleigh (2002) states, "On $\mathbb{Q}$, let $a * b = a/b$. Here $*$ is *not everywhere-defined* on $\mathbb{Q}$, for no rational number is assigned by this rule to the pair $(2, 0)$" (p. 25) Professor D alluded to the same issue by saying "you have to know that you can't divide by zero" and Professor A explained more generally, "when there's a formula, there are, at least potentially, uh, things that you can't put into the formula without breaking it." I infer the following theme from these excerpts: they all allude to the fact that division by 0 is not simply not defined in the proposed codomain but it is not defined in any proposed codomain. Of course, as Professor A pointed out in one of the interviews, "you absolutely can pass to the Riemann sphere or something and, um, have it make sense." However, they even specified that this was not something they would necessarily do with less mathematically advanced students (which would be the case with students in an introductory abstract algebra course). Thus, even in cases where we might be able to define a value for these elements in the domain, such a value would not necessarily exist or be able to be defined in sets that are accessible to students at this level.

Non-example 6 on Table 6 is slightly similar in that the issue is caused by dividing by zero. In general, Professor E commented that this tends to be a typical non-example, "denominator equaling zero, style examples, is a very common first thing. You know like, one over x. Uhh, things like this … that's a very common sort of like [everywhere-definedness] error." Professor A explained the issue there, "[it] fails because zero goes in there and it doesn't work." Again, the language of the value not being defined showed up in Professor B's explanation, "is not a function because it's not defined, yeah, because of the domain issue, because it's not defined at, uh, x equals zero." Thus, this non-example also has an *everywhere-definedness output in no set* issue.

Lastly, non-example 4 on Table 6 has the same issue of the operation not being defined everywhere. In fact, this issue is more prevalent here since the operation is not defined for most pairs as Fraleigh (2002) explains, "The usual matrix addition $+$ is *not* a binary operation on [the set of all matrices with real entries] since $A + B$ is not defined for an ordered pair (A, B) of matrices having different numbers of rows or of columns" (p. 21). Professor B explained that, for the related case of matrix multiplication (with the same proposed domain and codomain), the "issue [is] that matrix multiplication is only defined, if the, number of columns … in A equals the number of rows in B … then you'd have to say, normally we just say it's undefined." They continued, "it's undefined, if, the number of rows, of B is not equal to the number of columns of A." It is clear from this excerpt that this multiplication (without specifying the sizes of the rows and columns of the matrices) is being treated as undefined—that is, if the number of columns of A is not equal to the number of rows of B, there is simply no corresponding output that exists in any set. Thus I classified this example as a non-example of a function in the *everywhere-*

definedness no set category. (See Table 8 below for more non-examples included in this subcategory.)

Table 8: Non-examples with *everywhere-definedness no set* issues[15]

S is the set of all women T is the set of all men $f: S \rightarrow T$ given by $f(s) =$ husband of s (Herstein, 1996, p. 13)	$S = \{1, 2, 3\}$ and $T = \{4, 5, 6\}$ $f_3: S \rightarrow T$ given by $f_3(1) = 4, f_3(3) = 6$ (Beachy & Blair, 2019, p. 55)
S is the set consisting of 20 people, no two of whom are of the same height $*$ defined on S by: $a * b = c$, where c is the shortest person in S who is taller than both a and b (Fraleigh, 2002, p. 25)	$A = \{1, 2, 3\}$ $B = \{2, 4, 6\}$ The relation: {(1, 2), (2, 6), (2, 4)} (Fraleigh, 2002, p. 8)
"They might do factorial … where the domain is the real, you can't do factorial everywhere." (Professor E)	"[The] absolute value … doesn't exist … in spaces of polynomials and in $\mathbb{Z}_n$. (Professor E)

2.6 Discussion

Using a textbook analysis and semi-structured interviews with mathematicians, I have identified key categories that describe the types of non-examples that exist in the instructional example space. In this section, I revisit the research question and discuss the implications of these findings.

2.6.1 Revisiting the research question. In this paper I focused on answering the research question: what are key elements of the structure of the instructional example space? As we saw in the results, I elaborated two key aspects of the function concept: well-definedness and everywhere-definedness. In particular, I introduced an elaboration to the well-definedness

[15] These non-examples were slightly modified for clarity and simplicity.

category—*well-definedness equivalence/representation* and *well-definedness multiple rules*. *Well-definedness equivalence/representation* refers to non-examples of function where the elements in the domain have equivalent representations and the rule assigns different images in the codomain to these representations, thus resulting in a single element of the domain being mapped to multiple values. On the other hand, *well-definedness multiple rules* is used to refer to non-examples where the rule is ambiguous as a result of assigning two or more different choices of elements in the codomain to a single element of the domain. The main distinction between these two elaborations is that *well-definedness multiple rules* does not deal with different representations of elements of the domain in question, the rule simply outputs two or more values and a choice has to be made as to which element in the codomain to map to. I also elaborated the everywhere-definedness category into *everywhere-definedness no set* and *everywhere-definedness some set*. The *everywhere-definedness no set* non-examples are non-examples where an element in the domain is not assigned an output or the rule assigns to it an output that is not contained in any set (accessible to a student at this mathematical level). On the contrary, the *everywhere-definedness some set* non-examples are those in which an element in the domain is assigned an output by the rule, but such an output is not contained in the proposed codomain. However, the codomain can be enlarged to some (student accessible) set so that the output specified by the rule is indeed contained in this set. (I summarize these elaborations in Table 9 below.)

These elaborations are important for multiple reasons. First, the "outer" columns of Table 9 are the ones that secondary mathematics students generally have limited experience with. However, these outer parts of the table are critical for reasoning about functions in undergraduate mathematics, especially more advanced courses like abstract algebra. The non-examples represented by the "inner" columns of the table are of course still important but, I propose, are more likely to resonate with students' experiences with functions in previous courses. Thus, this elaboration is important because (1) it points out aspects of the function concept that students must attend to in order to successfully reason about functions in abstract algebra, while also (2) providing a viable explanation for why students struggle with functions (they have experience with only two out of the four columns, leaving lots of key examples and non-examples of function out). Second, a practical suggestion that emerges from this analysis is that productive learning experiences for students should involve non-examples of each of these four kinds. Hence, these elaborations can inform instructional design and selection of non-examples for abstract algebra lessons.

Table 9: Summary of well-definedness and everywhere-definedness elaborations

Well-definedness		Everywhere-definedness	
Equivalence: There exists at least one element in the domain that has different equivalent representations and the rule assigns different images in the codomain to these representations.	Multiple Rules: There exists at least one element in the domain that gets assigned different images in the codomain due to the rule being ambiguous.	No set: There exists at least one element in the domain that does not get assigned any value or this value is contained in an inaccessible set.	Some set: There exists at least one element in the domain that gets assigned an image that is not contained in the proposed codomain (but this image is contained in an enlarged accessible set).
$\phi: \mathbb{Q} \to \mathbb{Q}$ given by $\phi\left(\frac{a}{b}\right) = a + b$	$f: \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = \pm\sqrt{x}$	$g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$	$h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$
Relatively unfamiliar to students	Familiar from previous courses	Familiar from previous courses	Relatively unfamiliar to students

A third reason for the importance of these elaborations is that there are mathematical benefits with gaining experience with each of these four categories, which lays the foundation for reasoning with more advanced ideas that follow in abstract algebra. In the introduction, I pointed out the general importance of well- and everywhere-definedness for reasoning about functions in this context. For example, *well-definedness equivalence/representation* is critical for proofs and key concepts like functions defined on sets of equivalence classes or quotient structures, the First Isomorphism Theorem, or the formal construction of the rational numbers. In particular, students need to check well-definedness issues related to equivalence (represented by *well-definedness equivalence/representation* in my framework) every time they define a function whose domain involves a quotient structure (e.g., when defining a binary operation on $\mathbb{Z}/2\mathbb{Z}$ as discussed in section 2.1)—a common task in abstract algebra. Furthermore, as seen in Rupnow (2021), *well-definedness equivalence/representation* also plays a role in discussions of homomorphisms, as one of the metaphors that arose in her analysis involves using "knowledge of the structure of groups to find similar elements in the domain that can be mapped to the same place in the range … which reveals sameness of elements via mapping" (p. 5). *Everywhere-definedness some set* is also important in abstract algebra when determining, for example, closure of a binary operation (as discussed in 2.1) or when defining a homomorphism or isomorphism. In these cases, students need to verify that the image of each input is in the codomain. In abstract algebra, defining functions between two structures is common practice and oftentimes one has to "fix" a proposed correspondence by either restricting the domain (typical of *everywhere-definedness no set* non-examples) or broadening the codomain (typical of *everywhere-definedness some set* non-examples). Therefore, being able to determine the root cause of the everywhere-definedness issue and classifying it as *everywhere-definedness some set* or *everywhere-definedness no set* is helpful for defining functions because it can support students in developing an awareness of how such issues can be fixed.

Non-examples in the *well-definedness multiple rules* category are those that are possibly most familiar to students from previous courses yet occur far less frequently in abstract algebra. I suspect that a student who only has experience with *well-definedness multiple rules* non-examples would struggle significantly with some of the mathematical activity discussed in this section (defining binary operations on quotient structures, for example). The main takeaway with respect to the *well-definedness multiple rules* category therefore is to reinforce the notion that abstract algebra requires students to reason about function in fundamentally different ways than in previous courses and to underscore the need for the broadening of students' example space via the other categories. Fourth, this is a study of the rationale behind the (non-)examples that are or can be used in instruction, but it can inform studies of students' thinking and learning as well. For example, though I have argued above that students should gain experience with all four categories of non-examples in abstract algebra, this does not account for or explain the context or nature of these experiences that potentially make them impactful. For example, what kinds of experiences with each of these four categories support students' development of a productive conception of function? In what contexts and for what purposes should these non-examples appear in an instructional sequence, and why? These nuanced questions could be addressed in future research via task-based clinical interviews, conceptual analyses, and hypothetical learning trajectories.

2.6.2 Contributions. In addition to the contributions already mentioned, these elaborations provide a rational frame of reference or context for why functions in abstract algebra are so difficult for students and why students see functions in advanced mathematics as different from what they have experienced before. Prior discussions of univalence and vertical line test touch only briefly on issues of well-definedness; well-definedness is not a prominent focus of instruction prior to number theory, discrete mathematics, or modular arithmetic (typically sophomore or junior level mathematics major courses). This paper examines well-definedness from a perspective much more broad than what is afforded by the vertical line test. One of the main takeaways is that we should not just assume that introductory abstract algebra students have a productively structured example space for function. This paper calls attention to the fact that there is much nuance in the function concept in abstract algebra (especially for aspects of the function concept that students do not have much prior experience with). I therefore see it as important for abstract algebra instructors and researchers to think carefully about—and not assume that students will be immediately able to adapt their knowledge of—functions in an abstract algebra context.

Additionally, this paper contributes to the literature on example spaces since it is one of only a few investigations of the instructional example space for functions. Textbook analyses and semi-structured interviews with mathematicians were extremely useful in investigating the instructional example space. They allowed me to explore the non-examples and explanations that students might see when reading their class textbook while at the same time explore the non-examples that are used in practice in a classroom and aspects of these examples that are important for students' reasoning. In particular, pairing a textbook analysis with semi-structured interviews with mathematicians was helpful because it allowed me to ask questions of these mathematicians, which was particularly useful for clarifying aspects of the structure of the instructional example space. Lastly, this paper is also one of only a few analyses of non-examples, which are a key element of examples spaces that have not received much attention in the literature.

2.6.3 Limitations and future research. One limitation of this study is that is focuses on non-examples only and thus leaves open the question of whether there are similar categories for examples. Additionally, I do not focus on how students reason about these kinds of non-examples or how they should reason about these non-examples. In particular, I do not investigate what a productive way of reasoning about functions in abstract algebra might be. In future research, I hope to empirically refine these categories via task-based clinical interviews with students. I will also conduct teaching experiments to test and refine possible learning trajectories. The data from this study indicated that tasks based upon non-examples could be a particularly useful tool in examining students' thinking about these ideas, as they were certainly useful points of discussion in both the textbook analysis and the interviews with the mathematicians. Particularly, the data from this study suggests that tasks focusing on "recovering an example from a non-example" could be particularly illuminating.

Another limitation is that I have not directly addressed specific kinds of functions, but rather what is involved in determining whether or not a proposed correspondence is indeed a function. This is crucial and necessary, of course, but ultimately abstract algebra students must also apply the function concept to other ideas like inverse function, binary operation, homomorphism, and isomorphism. Future research could explore the implications of these categories for the instructional example space on binary operation, homomorphism and isomorphism, among other classes of function typically encountered in abstract algebra. Research could also explore how students whose personal example spaces are structured in this way reason with these subsequent topics. Additionally, in advanced mathematics, proving that a homomorphism or an isomorphism candidate is well-defined is extremely important, and multiple representations are common. Moreover, Melhuish and colleagues (2020) have documented some students thinking that a given relation was a homomorphism but not a function. Thus, future research could more clearly spell out the implications of the ways of reasoning documented here for binary operations, homomorphisms, and isomorphisms, or connect this framework to existing frameworks for these other concepts.

CHAPTER III

PAPER 2 [A COORDINATED WAY OF UNDERSTANDING]

Defining and Illustrating a Coordinated Way of Understanding Function in Abstract Algebra

3.1 Introduction

Functions are important in mathematics and particularly in abstract algebra, where functions are the core concept underlying key ideas like binary operation, homomorphism, and isomorphism. However, research has underscored how a students' understanding of the function concept—particularly well-definedness and everywhere-definedness—can support or constrain their reasoning with these subsequent ideas.

The function concept in abstract algebra—like many concepts in advanced mathematics—presents an interesting dichotomy. On one hand, researchers have emphasized the importance of a unified view of function, enabling students to see examples of functions in abstract algebra and examples of functions from calculus (and previous courses) as instances of the same overarching concept. This recommendation emphasizes that the uses of function in abstract algebra reinforce and build upon the familiar notions of function from previous courses. On the other, abstract algebra requires reasoning with functions in ways that are new to students: the domains and codomains of specific functions in abstract algebra, as opposed to being the real numbers or similarly behaving subsets (as is common in courses up to calculus), can be discrete (e.g., polynomials with integer coefficients or matrices with integer entries), unorderable (e.g., groups of symmetries or sets of equivalence classes modulo n), or Cartesian products of such sets (e.g., a binary operation on a quotient group). Students are not likely to have much experience with these kinds of constructions, which helps to provide context for the widespread reports of difficulties that students experience with function and function-related concepts in abstract algebra (e.g., Brown et al., 1997; Larsen, 2009; Leron, Hazzan, & Zazkis, 1995; Melhuish & Fagan, 2018; Rupnow, 2017, 2019).

Viewing the function concept through the lens of the structure of the instructional example space that I proposed in Paper 1 helps to clarify why it can be so challenging for abstract algebra students. In particular, the elaborations of well- and everywhere-definedness I presented in Paper 1 are of key importance in this context. For instance, *well-definedness equivalence/representation* is important for proofs and reasoning with functions defined on domains that involve equivalence classes or a quotient structure and *everywhere-definedness some set* is relevant when dealing with, for example, closure of a binary operation or when defining homomorphisms. Additionally, defining functions between two structures is common practice in abstract algebra and often involves modifying a proposed correspondence by either restricting the domain (typical of *everywhere-definedness no set* non-examples) or expanding the codomain (typical of *everywhere-definedness some set* non-examples). The *well-definedness multiple rules* category is the category that students are likely most familiar with from previous courses and occurs less frequently in abstract algebra, however, a student who only has experience with non-examples in this category would likely struggle with some of the mathematical ideas discussed in abstract algebra (defining binary operations on quotient structures, for example). However, given these new uses of function in abstract algebra, little is known about how students reason productively with functions in this context. My central objective in this paper is to use the task-based clinical interview methodology to examine (1) what might be involved in a *coordinated way of understanding functions* in abstract algebra, and (2) how a coordinated way of understanding might support students in reasoning productively with examples and non-examples of function in abstract algebra.

3.2 Literature Review

In this literature review, I aim to (1) define and point out key characteristics of the function concept in abstract algebra (section 3.2.1), (2) underscore the need for careful, detailed examinations of students' reasoning about everywhere-definedness and well-definedness (section 3.2.2), and (3) use the literature to develop initial insight into what a productive way of understanding function might entail (section 3.2.3).

3.2.1 Definition and key aspects of the function concept. I begin by outlining the definition of function that informs this work and then explicate key aspects of this definition. A function f from a set A to a set B (usually denoted $f: A \to B$) is a rule that assigns to each element a of A exactly one element b in B. Melhuish and Fagan (2018) identified several aspects of this definition that are critical for reasoning productively with function in abstract algebra: well-definedness, everywhere-definedness, domain, and codomain. Briefly, the set A is called the domain and the set B is called the codomain. *Well-definedness* is the condition that every a in A gets assigned *at most one* element of B, and *everywhere-definedness* is the condition that the rule has to assign *at least one* element of B to each element of A. There are two points to make here related to the components of these frameworks that shape the current study in meaningful ways; these points also provide the structure for the remainder of my literature review. The objective of this paper is to characterize a way of understanding that supports students in reasoning productively about examples and non-examples of function in abstract algebra. Framing the function concept in terms of both[16] well-definedness and everywhere-definedness implies there are two possible ways in which a proposed correspondence can fail to be a function: it can fail the well-definedness condition (i.e., when one element of the domain A is assigned to more than one element of the codomain), or it can fail the everywhere-definedness condition (i.e., when one element of the domain A is not assigned to any element of the codomain B).

In Paper 1, I further organized the set of non-examples of function into four categories as shown in Table 10. These categories are important for our purposes because they outline the structure of the instructional example space and thus reflect the kinds of non-examples that students should be able to reason with successfully. An immediate implication for my purposes in this paper is that any productive way of understanding function in abstract algebra should support students in reasoning with multiple categories in this framework.

Table 10: Categorizations of non-examples of function

Well-definedness		Everywhere-definedness	
Equivalence: There exists at least one element in the domain that has different equivalent representations and the rule assigns different images in the codomain to these representations.	Multiple Rules: There exists at least one element in the domain that gets assigned different images in the codomain due to the rule being ambiguous.	No set: There exists at least one element in the domain that does not get assigned any value or this value is contained in an inaccessible set.	Some set: There exists at least one element in the domain that gets assigned an image that is not contained in the proposed codomain (but this image is contained in an enlarged accessible set).
$\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$	$p: (0, \infty) \to \mathbb{R}$ given by $p(x) = \pm\sqrt{x}$	$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$	$h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$

[16] I note that, in some characterizations of the function concept, everywhere-definedness is automatically guaranteed. This is not the case in this paper. I adopt this perspective because it is common and conventional in abstract algebra. See Paper 1 as well as Weber et al. (2020) for more information about the two characterizations of function and their associated implications.

Additionally, the components of these two frameworks correspond to important gaps in the research on students' thinking about functions in abstract algebra. First, although the framework above outlines the kinds of non-examples that students should be able to reason with and the key aspects of the function concept they should account for, it does not provide insight into how they might do so. Generally, although certain function concepts (such as binary operation and homomorphism) have seen a flurry of recent activity in the abstract algebra literature, the underlying concept of function itself and how students reason about well-definedness and everywhere-definedness to determine whether an example is or is not a function has received considerably less attention. For example, Melhuish, Ellis, and Hicks (2020) focus on what students perceive to be the key aspects of the binary operation concept. Notably, the notions of well-definedness and everywhere-definedness are not amongst the aspects that emerge in their analysis. Similarly, Melhuish and colleagues (2020) focus on students' concept images of function and how they relate to special kinds of functions like homomorphisms. These researchers do attend to whether or not well-definedness and everywhere-definedness are elements of students' concept images for function, illustrating that attention to well-definedness and everywhere-definedness in some capacity is necessary for productively reasoning about homomorphism. However, they do not attend to how and in what ways students might productively reason about these ideas. The same is true for other investigations of specific types of function in abstract algebra (e.g. Brown et al., 1997; Hausberger, 2017; Larsen, 2009; Leron, Hazzan, & Zazkis, 1995; Nardi 2000): the core notions of well-definedness and everywhere-definedness are either identified as important but are not examined much further or are not discussed. To be clear, this is an important and insightful body of research that does indeed emphasize the importance of reasoning productively with well-definedness and everywhere-definedness, but it stops short of explicating *how* students might do so. Outside of the context of abstract algebra, though, function researchers have set forth (but not directly examined) some useful hypotheses; I examine these hypotheses in section 3.2.3.

3.2.2 General functions research on everywhere- and well-definedness. In this section, I discuss the relevant literature on well-definedness and everywhere-definedness in contexts other than abstract algebra. A key theme is that students struggle with well-definedness, and their primary tool for determining it (the vertical line test) is inherently limited in scope. This emphasizes even more the need for detailed examinations of students' reasoning about well-definedness in more general mathematical settings.

I found no studies that directly examine students' reasoning about everywhere-definedness, perhaps because everywhere-definedness is often not emphasized prior to advanced mathematics (consider, for example, that the vertical line test[17]—the most pervasive procedure for determining what is and is not a function—focuses exclusively on well-definedness). Regarding well-definedness, researchers have generally reported that students struggle to attend to it[18] (e.g., Bailey et al., 2019; Dorko, 2017; Even & Bruckheimer, 1998; Even & Tirosh, 1995). Even and Tirosh (1995) proposed that, for many students, the concept of well-definedness is often viewed in a rote and procedural way. As a result, "further probing often reveal[s] that their knowledge of this requirement was rather superficial" (Even & Tirosh, 1995, p. 7). Indeed, the most common manifestation—procedural or otherwise—of well-definedness at the K-12 level is the vertical line test. On one hand, with proper context and accompanied by other meanings for well-definedness, the vertical line test can be a useful tool. For example, students typically evaluate the univalence condition for a graphical representation by using the vertical line test (e.g., Clement, 2001; Thomas, 2003). They also sometimes convert non-graphical representations to graphical representations to be able to use the vertical line test when evaluating whether well-definedness is violated (e.g., Kabael, 2011; Thomas, 2003). On the other hand, and particularly important for

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[17] The vertical line test states that a proposed correspondence (in which the domain and codomain are usually the real numbers) is not a function if there exists a vertical line that intersects the function's graph more than once.

[18] Well-definedness is sometimes referred to in this body of research as *univalence* (e.g., Dorko, 2017; Even, 1993). For consistency, I will continue to use the term well-definedness throughout this paper.

reasoning about functions in more advanced contexts, researchers have also noted that students overrely on the vertical line test. This can be problematic for students. For example, Dorko (2017) explained that many students found it challenging to generalize notions of well-definedness to functions of two variables (e.g. functions that have a domain of $\mathbb{R}^2$). One of the reasons was that, due to the inherent challenges in visualizing the graph of such a function (a 3-dimensional surface), the vertical line test does not easily generalize beyond the original context for which it was designed (graphical representations of correspondences between the real numbers). Additionally, researchers have noted that students conflate well-definedness and injectivity (e.g., Clement, 2001; Even, 1993; Markovits, Eylon, & Bruckheimer, 1986; Vinner & Dreyfus, 1989).

A key theme from this literature is that many (if not most) of these studies investigate students' reasoning about well-definedness via the vertical line test, which is simply not a tool with much practical use in abstract algebra (as many common functions in abstract algebra simply do not admit tractable graphical representations). As a result, this body of research establishes that students find well-definedness challenging while also calling attention to the fact that their primary tool for reasoning about well-definedness from previous courses is insufficient for use in more general contexts like abstract algebra. Combined with the general lack of research on everywhere-definedness, this calls attention to the need for research that examines students' reasoning with both well-definedness and everywhere-definedness in broader function contexts.

3.2.3 A hypothesis about a productive way of understanding functions. In addition to highlighting the challenges students experience with well-definedness, researchers have ventured (but not directly examined) hypotheses about how students might overcome these challenges. These hypotheses all derive from a common observation in the functions literature: that students often think about functions in terms of a formula (Bailey et al., 2019; Carlson, 1998; Clement, 2001; Breidenbach et al., 1992; Thompson & Carlson, 2017). For example, Thompson (1994c) explained that the "predominant image evoked in students by the word 'function' is of two written expressions separated by an equal sign" (p. 24). This observation provides the context needed to understand the nature of the hypotheses researchers have offered about what is involved in a productive way of understanding functions. Somewhat informally, a common theme running through these hypotheses is that it is advantageous for students to know *a function is more than a formula*. For example, Weber and colleagues (2020) clearly illustrated this notion by pointing out that, if one is reasoning from the characterization of function outlined above in 3.2.1, changing either the domain or codomain of a function changes the function itself—*even if the rule/formula remains the same*. They explain that, from this perspective, the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is different than the function $g\colon [0, \infty) \to \mathbb{R}$ given by $g(x) = x^2$. Another implication is that a formula might be considered an example or non-example of function based upon its domain and codomain. For example, $h\colon \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ is a non-example because it is not everywhere-defined ($x = -1$ has no image in the codomain). However, $j\colon \mathbb{N} \to \mathbb{N}$ given by the same formula $j(x) = x^3$ *is* a function (see Table 11 for similar instances related to the non-examples framework in section 3.2.1). I might hypothesize that a student for whom a function is only a formula would consider h to be an example of function simply because $h(x) = x^3$ is a familiar formula. Indeed, recognizing that h is a non-example requires attending to the fact that the proposed domain is $\mathbb{Z}$ (to select an appropriate element like -1) and the proposed codomain is $\mathbb{N}$ (to conclude that $(-1)^3$ is not a natural number).

Table 11: Examples of function – Modifications to non-examples in framework

Well-definedness		Everywhere-definedness	
Well-definedness: Equivalence	Well-definedness: Multiple Rules	Everywhere-definedness: No set	Everywhere-definedness: Some set
$\phi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a, b) = a + b$	$p\colon (0, \infty) \to \mathbb{R} \times \mathbb{R}$ given by $p(x) = (\sqrt{x}, -\sqrt{x})$	$f \to \mathbb{R}$ given by $f(x) = \dfrac{1}{x}$	$h\colon \mathbb{N} \to \mathbb{N}$ given by $h(x) = x^3$

Indeed, researchers have hinted on how critical it is for students to attend to the domain and range/codomain when determining when a proposed correspondence is or is not a function (e.g., Kabael, 2011; Oehrtman, Carlson, & Thompson, 2008; Zandieh & Knapp, 2006). For example, Dorko (2017) suggested that domain and range may "play a role in students' ability to generalise univalence, as testing for univalence requires identifying an element of the domain and checking that it is paired with exactly one element of the range" (p. 2). Thus, *it is advantageous for students to view function not simply as a formula/rule but as a coordination of the formula/rule with the domain and codomain*. This literature-based hypothesis—which has not yet been directly and explicitly examined—provided the foundation for my characterization of a coordinated way of understanding function in section 3.3.2 and served as my major research hypothesis.

3.3 Theoretical Perspective

3.3.1 Ways of understanding and ways of thinking. I frame students' reasoning in terms of Harel's (2008a) *ways of understanding* and *ways of thinking*. These constructs allowed me to distinguish and characterize students' mathematical thinking at various levels. Informally, ways of understanding are the interpretations, explanations, and descriptions that are brought forth in one's mathematical reasoning, while ways of thinking are the broader, more general perspectives, reasoning patterns, and mathematical habits of mind (Lockwood & Reed, 2020). More formally, a way of understanding is "a *cognitive* product of a mental act" (Harel, 2008a, p. 272)—the interpretations, explanations, and descriptions referred to above, for example, are the products of the mental acts of interpreting, explaining, and describing.

Once a researcher has identified a mental act of interest, they infer "common cognitive characteristics among [a] multitude of products of each act; these are the ways of thinking associated with the respective act" (Harel, 2008b, p. 497). In this way, ways of thinking are constructs by which researchers identify common elements of students' reasoning across a multitude of tasks. For example, a student who is reasoning about inverses in a variety of algebraic contexts might describe the inverse of a function as "switching x and y and solving for y," the additive inverse of a real number as "making the number negative," and the multiplicative inverse of a real number as "taking the reciprocal of that number." These descriptions represent the student's ways of understanding inverses within each of these particular contexts. More generally, the researcher might also infer that, across contexts, the student thinks about inverses primarily in terms of the (sequence of) manipulations by which the original element is transformed into its inverse element. This is what might be called an *inverse as a manipulation of the original element* way of thinking (Cook, Melhuish, & Uscanga, under review) because it captures key aspects of the students' inverse reasoning that are common across these algebraic contexts. I note that it is important for researchers to develop detailed descriptions of both ways of understanding and ways of thinking. However, operationally, in order to develop such descriptions of ways of thinking, researchers must first do so for ways of understanding because they are the basis on which ways of thinking are inferred. As I pointed out, in the literature review, neither ways of understanding nor ways of thinking about function in abstract algebra have been clearly detailed or explicitly addressed in the literature. Therefore, as a necessary precursor towards characterizing the elements of a productive way of thinking, in this paper I further examine what might be involved in a coordinated way of understanding functions in abstract algebra.

3.3.2 A coordinated way of understanding functions. In this paper I will infer students' *ways of understanding* particular tasks. As previously described, a way of understanding is an aspect of students' thinking. My initial characterization of a way of understanding function as being "coordinated" was built upon the inferences I drew from the literature (see section 3.2.3) and centered on (1) viewing a function as more than a formula, and, more specifically, (2) viewing a function in terms of a relationship between the domain, codomain, and the rule. Here, I explicate some initial indicators (in the form of students' observable behaviors) that I associate with this provisional characterization of a "coordinated" way of understanding. I used these behaviors to identify instances of students' activity that would enable me to make inferences about the ways of understanding function that underlie them. The mental act on which I focus is *explaining*: I will conclude that *a student has demonstrated a coordinated way of understanding* a particular function task if, when explaining why a proposed correspondence is or is not a function, they make explicit reference (in their language, gestures, or inscriptions) to (1) an element (or collection of elements) in the domain, (2) an element (or collection of elements) in the codomain, and (3) the rule. I return to this initial characterization in the discussion section, in which I explicate in greater detail the characteristics of a coordinated way of understanding and the various ways in which it might manifest in students' reasoning.

3.4 Methods

In this study, I conducted task-based clinical interviews (Clement, 2000; Goldin, 2000), a methodology that provides researchers with opportunities to interact with students and examine their mathematical activity related to a particular topic as they engage with carefully designed mathematical tasks. The objective of the task-based clinical interview is to use these interactions as a means to explain and characterize particular aspects of students' thinking that contributes to their successes or difficulties with particular tasks. A key principle of this methodology—consistent with the individual, cognitive perspective underpinning this project (von Glasersfeld, 1995)—is that researchers can only make inferences about what students' thinking might entail. As a result, researchers construct and continually refine *models* of students' thinking. A model is a researcher's attempt to outline a cognitive frame of reference that renders students' behaviors sensible; as noted by Clement (2000), this involves proposing and continually refining "hypotheses concerning the … reasoning patterns responsible for correct answers" (p. 552). This methodology therefore provided a means of achieving my primary objective: to further examine what a *coordinated way of understanding functions* might entail and how it might manifest in students' reasoning.

There are two other relevant points about the task-based clinical interview methodology that provide additional context for this study. First, it affords the flexibility necessary to answer nuanced questions about students' reasoning about a particular idea. Instead of simply determining whether or not a response to a task is correct or incorrect, task-based clinical interviews enable researchers to develop more detailed descriptions of students reasoning because it promotes nuanced and responsive mathematical interactions between students and researchers. This flexibility is often necessary to describe the ways of thinking and understanding that underlie and inform students' mathematical activity, especially for challenging, complex concepts like function in abstract algebra. Second, it is useful to note that it is similar to, but different from, the teaching experiment methodology (Steffe & Thompson, 2000). Whereas a teaching experiment aims to examine both students' mathematical reasoning and the nature of the instructional experiences that might occasion changes in this reasoning, a task-based clinical interview aims only to examine students' reasoning. This is an important point to make because it underscores that my objective in this paper is *not* to examine how students' might develop a productive way of understanding functions in abstract algebra, but to examine further what a coordinated way of understanding might entail (such as examples of a coordinated way of understanding and how it might be recognized in students' reasoning). Such instructional design efforts are certainly worthwhile and necessary but to do so, researchers must first develop a clear image of what such a way of understanding might entail. Thus, another way to frame my objective in this paper is that I aim to develop a clearer image of productive reasoning about functions in abstract algebra by using the task-based clinical interview methodology to document and illustrate a coordinated way of understanding by examining the reasoning of students who have already developed it.

3.4.1 Participants. Since my objective in this paper was to document and illustrate a coordinated way of understanding from students who had already developed it, I sought participants who (1) had already completed a course in abstract algebra or another algebra-related advanced mathematics course (e.g., cryptography, number theory), (2) successfully reasoned about basic "is this a function?" tasks, and (3) students who seemed to be enthusiastic about the study and seemed to be willing and able to articulate their mathematical thinking. To identify such students, I reached out to several algebra-related advanced mathematics courses to ask for participants. Students who were interested were asked to fill out a selection survey. An example of the type of tasks included in the selection survey is displayed in Figure 1 (please see Appendix A for the remaining tasks from the selection survey). In this paper, I focus on the mathematical activity of four students. I refer to these four students as Student 1, Student 2, Student 3, and Student 4. I also use the gender-neutral pronoun "they" when referring to individual students throughout the paper.

Figure 1: Example of task in the student selection survey

Consider ϕ defined on the integers as follows:

$$\phi(a) = \text{remainder when } a \text{ is divisible by } 3.$$

a) Does this define a function on $\mathbb{Z}$?

b) Explain why or why not.

3.4.2 Data collection and task-design. I conducted two to three interviews with each student individually for about an hour and a half each. I conducted one session per week to allow time for ongoing analysis (explained in the next section). On account of the COVID-19 pandemic, each interview was conducted and video-recorded over Zoom (both the students and I had our webcams on for the duration of each interview). This enabled me to capture students' language and gestures. Students' written work was obtained by screen capture: students with access to a tablet were asked to use Zoom's "screen share" feature while working (so that the written work would be captured automatically by the video recorder); students who did not have access to a tablet were asked to write on their own paper and bring this up to the screen regularly to capture their work.

The instructional tasks were informed by the structure of the non-examples in the instructional example space (outlined in section 3.2.1 as well as in Paper 1); in particular, the tasks were created with each category of non-examples from Paper 1 being represented (sometimes in the modified form of an example[19]). I created four tasks (corresponding to the four categories in the framework) as the basis for each interview that focused on having students identify whether or not the proposed correspondence was an example or a non-example of a function. Each task created was included because (1) they elicited student thinking about function in an abstract algebra context and (2) they served the overarching purpose of providing students with an opportunity that would allow me to observe their *way of understanding* a particular task as a first step in characterizing a *way of thinking* about functions in abstract algebra. The following is a list of the core tasks:

1) Consider $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$. Is g a function? Explain why or why not.
2) Consider $f: \mathbb{R} \to \mathbb{C}$ given by $f(x) = \sqrt{x}$. Is f a function? Explain why or why not.
3) Consider $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$. Is h a function? Explain why or why not.

[19] Recall from Section 3.2.3 that a rule can be considered an example or a non-example depending upon the proposed domain and codomain. For instance, consider the non-example $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$, which falls into the *everywhere-definedness no set* category. Changing the domain to $(0, \infty)$ "fixes" this non-example and enables us to associate the non-example f with the example $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$. For the purposes of task design, I therefore associated an example with a particular category if the example could be recovered from a non-example in that category.

4) Consider $\phi: \mathbb{Q} \rightarrow \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$. Is ϕ a function? Explain why or why not.

Throughout the interviews, I designed additional tasks that enabled me to test and refine the hypothesis I established for each student as a way to help me stabilize these so they were adequately supported or refuted based on my observations.

3.4.3 Data analysis. I engaged in both ongoing and retrospective analysis. Ongoing analysis can be thought of as a "local" form of analysis that informed the decisions I made during the interview sessions. Retrospective analysis can be thought of as a "global" form of analysis that aimed to identify themes and reasoning patterns across multiple interview sessions. Ongoing analysis occurred during and between sessions. During sessions, this involved intuitively responding to students in the moment to devise and ask follow-up questions (such as requests for clarification and elaboration) and between sessions, transcribing the previous session and summarizing what happened, calling attention to and making comments on potential ways of understanding that I inferred might explain the observable components of students' activity. This served two purposes, it informed my identification of the potential characteristics of a productive way of understanding functions, and, since the only tasks set in stone were the four to five indicated above for use in the first session, it also informed designing tasks for the upcoming session. These tasks were typically designed to test specific hypotheses about the students' reasoning.

Retrospective analysis centered on Clement's (2000) cycle for interpretive analysis; this involved an iterative five-step process for each interview. First, organizing the data set by grouping together video data and written work associated with each task the student was given. Second, along with notes made during ongoing analysis, generating observations and descriptions of the students' observable mathematical behaviors. Third, hypothesizing and drawing inferences about the ways of understanding that might explain these observations. Goldin (2000) pointed to the importance of being "as explicit as possible about the criteria for drawing particular sorts of inferences from particular sorts of observations" (p. 528). Thus, this step involved formulating (and refining) explicit criteria for the kinds of observations I associated with particular ways of understanding. (It also involved clarifying in detail the criterion for identifying a way of understanding that I outlined in section 3.3.2). Fourth, re-analyzing the data to look for confirming and disconfirming observations related to these hypotheses. And lastly, refining the hypotheses as needed (and restarting this five-step analysis cycle; this process continued until the hypotheses became stable and were adequately supported or refuted by additional observations). It is important to note that I did not assume that the students who successfully completed these function tasks would reason in a particular way or exhibit a coordinated way of understanding.

3.5 Results

3.5.1 Student 1. In what follows I detail Student 1's reasoning with different examples and non-examples of function related to the four different categories of non-examples outlined in section 3.2.1 (see Table 12 for these examples and non-examples). Specifically, I present Student 1's reasoning with three examples (one of which they came up with) and one non-example that I presented them.

Table 12: Tasks for Student 1

Well-definedness: Multiple Rules	Everywhere-definedness: No set	Everywhere-definedness: Some Set	Well-definedness: Equivalence
A vending machine where the inputs are the buttons and the outputs are the type of snacks it gives.	$g\colon (0,\infty) \to \mathbb{R}$ given by $$g(x) = \frac{1}{x}$$	$f\colon \mathbb{R} \to \mathbb{C}$ given by $$f(x) = \sqrt{x}$$	$\phi\colon \mathbb{Q} \to \mathbb{Z}$ given by $$\phi\left(\frac{a}{b}\right) = a + b$$
Example	Example	Example	Non-example

3.5.1.a Well-definedness multiple rules (example). When asked to come up with three to four examples of function, Student 1 came up with an example of a vending machine as a function. In this example, the letter and number combination of the buttons are the inputs of the function and the type of snacks it gives out are the outputs. They explained,

> like a vending machine, that's not broken. A-a not broken vending machine would be a good example of a function. Uh, you hit the letter like B5, and it gives you out a chocolate bar. You hit the letter B5 again, and it gives you out a chocolate bar. Right? Um, but maybe you hit A3 and there's like a bag of chips. Right? Uh, so, in this case we have two inputs, the number and the letter. But … for each combination in the vending machine that you have, unique combination of letter and number, you get out a unique, type of object from the vending machine.

In this excerpt, Student 1 demonstrated coordination in their explanation of this non-example which is made clear by their focus on the role that the domain, the codomain, and the rule play here. This student argued that for every "unique combination of letter and number" (which refers to the domain) you get a "unique, type of object from the vending machine" (which are the elements of the codomain). This shows attention to the types of elements contained in the domain and the codomain. Note that they also attended to the rule by discussing the process of clicking a button and getting a snack from the vending machine—which is the rule in their example. Thus, their main argument for why this example was well-defined (that is, for every input they have a unique output) involves coordinating of all three pieces of a function—the domain, the codomain, and the rule.

Student 1 also discussed the case in which their example would no longer be an example and would instead become a non-example of a function,

> [It's a function] provided, you know, somebody stocked it correctly, like, I guess technically it wouldn't be a function if someone did that weird thing that they do in vending machines, where they put a chocolate bar and then like, behind it, they try to hide like a-a nutrition bar, and then another chocolate bar. Then you'd hit B5, you'd get a chocolate bar, you hit B5, you get like a nutrition bar and you're very angry because it's not a function anymore. Right? But then you hit B5 again and there's a chocolate bar, so.

In their description of how their *well-definedness multiple rules* example could be a non-example, they again attended to all three parts of a function by first picking an element of the domain and then focusing on how the rule would map such an element into the codomain. More specifically, they mentioned clicking 'B5' and the vending machine giving you a chocolate bar but then clicking 'B5' again and getting out a nutrition bar, thus changing the snack associated to that letter/number combination. Student 1's argument was that this rule fails to be a function because it maps an element of the domain to more than one element in the codomain and thus indicated attention to and coordination between the domain, the codomain, and the rule.

3.5.1.b Everywhere-definedness no set (example). In a later part of the interview, I asked students to identify whether the following was a function and explain their reasoning: $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$. In working on this problem, Student 1 said,

> So is g a function? If I send x over to one over x, no, the answer, the answer is no. Uhh, yes, the answer is no. Alright. Oh, no, wait, wait, whoa whoa whoa, whoa whoa, whoa, whoa whoa whoa.

Since g is indeed a function, I asked Student 1 why they said "no" and then seemed to change their mind. They explained,

> Okay, so what went in my head, was I heard, 'is, uh, f from zero to infinity, goes to $\mathbb{R}$ a function?' Right? And … this is what I heard in my head. But it is not what is written on the paper, right? This is a, this is a-an open parenthesis, I heard, I saw the closed parenthesis, and so I said no, because, uh zero would map to some, to one over zero, what is that? That is not a real number. Right? And so, if it's not a real number, well then, this can't be an int-, so this input would have zero things.

In their explanation of why they originally thought g was a function, they focused on picking an element of the domain and plugging it into the rule to figure out what element you get back and whether it was in the specified codomain. That is, they thought about the element zero in the domain (which is in zero to infinity if we had closed parentheses like they mentioned) and what the rule does to it, namely maps it to $\frac{1}{0}$. They then mentioned that $\frac{1}{0}$ is not a number in $\mathbb{R}$ (the suggested codomain and thus concluded that g was not a function. Their ways of understanding this non-function displayed coordination since they attended to the domain, codomain, and rule and used all three in coordination with each other when deciding whether g was a function.

Something similar happened after they realized that they had misread the task and explained why g was indeed a function:

> I have some inputs, and each input is assigned some output … so as long as a is between zero and infinity, it maps, to one over a. Well, one over a, is going to be, in $\mathbb{R}$. And … this is the only output, for the input a. One divided by a has a specific definition. Um, so like, I don't know, if a was seven, then it would map to one over seven, and one divided by seven has a specific definition, there's only one number in the real numbers that is one over seven, which is, you know, one over seven … And, I know that I get to use this entire, I know I get to use this entire line for $\mathbb{R}$. Uh, but, each input is only going to have one output.

Student 1 again demonstrated coordination in their ways of understanding this *everywhere-definedness no set* example. They attended to an element in the domain, this time in more generality, by picking 'a' to be any number in the interval $(0, \infty)$. Then they coordinated with the rule to obtain the output of $\frac{1}{a}$ and further coordinated with the codomain when deciding that indeed if a was in $(0, \infty)$, then $\frac{1}{a}$ would be a unique element of $\mathbb{R}$.

3.5.1.c Everywhere-definedness some set (example). In another task, I asked the students to consider $f: \mathbb{R} \to \mathbb{C}$ given $f(x) = \sqrt{x}$ and decide whether it was a function and explain their thinking. Student 1 began,

> So is f a function? From $\mathbb{R}$ to $\mathbb{C}$. Ohh, from $\mathbb{R}$ to $\mathbb{C}$. That's nice. So I get to do, I get to have imaginary numbers, so I can take like the square root of negative one. That'll be good. And get i. That will give something … Okay, so if I restrict, if I restrict to just, um, positive things. I know, if x, is in, the positive real numbers, then, x goes to root of seven only means one thing.

Student 1 attended to the domain here by picking an element in the specified domain, namely x (which they wrote down as 7 on their paper as a way to exemplify to themselves what an element of this domain might look like). They coordinated with the rule when they figured out that the rule would map that to $\sqrt{7}$. Lastly, coordinated with the codomain when they concluded that this was indeed an element in $\mathbb{C}$ without multiple representations.

Focused later on the negative numbers, they continued,

> My problem, is if x is a negative number. Oh, I know if x is zero, then, uh, the square root of zero is just zero. There's no other zero. Uh, so what if x is like negative seven? So if x-x is a negative number … Uh, so, negative seven maps to the square root negative

> seven. This is like, the square root of negative one times seven. So this is seven i … There's only one seven i. Ah, so it's good. So it is a function.

This is another illustration of Student 1 exhibiting coordination in their explanation of why this *everywhere-definedness some set* example was a function. They did a similar thing when discussing the negative numbers, namely picking an element x in the domain and seeing what it was mapped to in the codomain. That is, they chose the element -7 in $\mathbb{R}$ and concluded it mapped to $\sqrt{-7} = 7i$ which was indeed an element of the codomain, $\mathbb{C}$.[20]

3.5.1.d Well-definedness equivalence (non-example). I also asked Student 1 if $\phi\colon \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ was a function. In this excerpt, Student 1 uses colors when referring to the domain (green), the codomain (red), and the rule (black) as can be seen by Figure 2. In thinking through the task, Student 1 said,

> Alright, nope … Okay, the reason I'm saying no, okay … the relation somehow says stuff on the green has to live in the green side and stuff on the red side has to live on the red side. Okay, so then stuff on the green side lives in this thing called $\mathbb{Q}$, the-the rational numbers, fractions … And then $\mathbb{Z}$ is on this side. Right? … Alright, so things that live in $\mathbb{Q}$, like three over four is a green thing. And also like, um, six over eight is a green thing. And, this-this green space has rules, right? Its rule says that these two things are the same. They're the same thing. Right?

They continued,

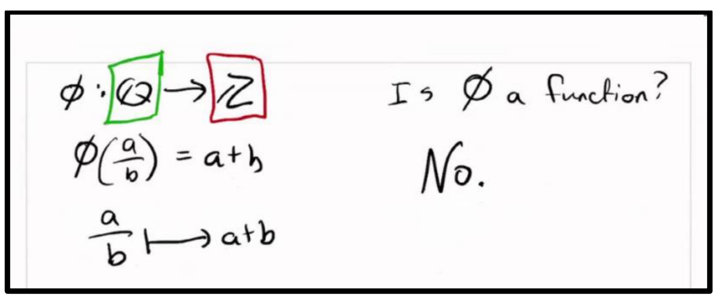
[20] Note that Student 1 is clearly using a branch of the square root function. In fact, at some point in the interview they stated that they were taking the square root function as being the one that maps a number to its "positive" square root. For example, -4 could possibly be mapped to $2i$ or $-2i$ since $(2i)^2 = 4i^2 = -4$ and $(-2i)^2 = 4i^2 = -4$ but Student 1 is making the choice to map it to -4 maps to $2i$.

And so, well, um, but my black, my black phi here, phi says that if I have three over four, well the red thing that it has to go to is three plus four, right? That's the name for seven in the red place, right? But the-the green thing, said, or, but six over eight is in the green thing … The black rule says that this has to go to six plus eight, which in the red place is fourteen. And these, these are not the same. Right? Not, not the same. Right?

Student 1 is referencing the equivalence in the domain and the fact that two equivalent elements are going to different things in the codomain. They explained further,

this element $\mathbb{Q}$ here, this input, has, at least, two outputs. Right? It has seven and it has fourteen. It, probably, has infinitely many different outputs, lots of other ones. But I just picked two, I-I was able to find two. And so there was one input, that had not exactly one output … Alright. It broke this second part, right? It had more than, it had more than one. So it-it's not a function, anymore.

Figure 2: Student 1's work on the *well-definedness equivalence* non-example



Student 1's explanations for why $\phi\colon \mathbb{Q} \to \mathbb{Z}$ is a non-example illustrate coordination in their ways of understanding this task. Student 1 again coordinated the domain, the codomain, and the rule. In general, they kept referencing "the green side" (referring to the domain), "the red side" (referencing the codomain), and what the "black phi" (which was the rule) does to elements in the domain to transform them into elements of the proposed codomain. Thus, showing some level of coordination between these three parts of the function. Furthermore, their main argument for why ϕ was not a function was that two equivalent elements of the domain map to two different elements in the codomain, using their awareness that "the green side" has the "rule [that] says that these two things are the same," meaning the elements $\frac{3}{4}$ and $\frac{6}{8}$. They explained that these two equivalent representations map to two different things in the codomain, namely 7 and 14, and made it clear that these two elements were not equivalent in $\mathbb{Z}$. Their awareness of the elements in the domain and codomain and the role equivalence played in those sets, together with their coordination with the rule and what it did to elements of the domain illustrates a coordinated way of understanding.

3.5.1.e Summary. Student 1 reasoned successfully about all these examples and non-examples. I claim that, in reasoning successfully about these tasks, they demonstrated *coordinated* ways of understanding these (non-)examples from all four categories (outlined in section 3.2.1): *well-definedness equivalence*, *well-definedness multiple rules*, *everywhere-definedness some set*, and *everywhere definedness no set*.

3.5.2 Student 2. In this section I describe Student 2's reasoning with examples and non-examples of function, one from each of the four different categories of non-examples outlined in section 3.2.1 (see Table 13 for these examples and non-examples). In what follows I discuss Student 2's reasoning with one example and three non-examples that I presented them.

Table 13: Tasks for Student 2

Everywhere-definedness: No Set	Everywhere-definedness: Some set	Well-definedness: Multiple Rules	Well-definedness: Equivalence
$g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$	$h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$	$q: 2\mathbb{Z} \cup 3\mathbb{Z} \to \{0,1\}$ given by $q(x) = \begin{cases} 1, x \in 2\mathbb{Z} \\ 0, x \in 3\mathbb{Z} \end{cases}$	$\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$
Example	Non-example	Non-example	Non-example

3.5.2.a Everywhere-definedness no set (example). I presented Student 2 with the rule $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$. In discussing their thinking about why this is an example of a function, they said,

> so our domain is, um, zero to infinity but, not including zero. Um, because every time you see one over x you think of zero is going to be an issue. Umm, but we're not using zero here, everything, you know, greater than zero. And so, you know, you can, you know, kind of a simple way to kind of get started is just put, plugging in numbers, you know, one, two, three, and four. Um, you're getting real numbers. You're always going to get real numbers. And, um, you know, you're not going to get the same output value either.

This student began by first attending to the rule and possible values that might not be able to be put into the rule and then coordinated with the domain to check whether such values were elements of the stated domain. For example, they identified zero as an element that could cause issues when plugging it into the rule but then mentioned that zero is not actually included in the specified domain. They furthered specified that no matter what number you plug in to the rule "you're getting real numbers" back, thus hinting to coordination with the codomain as well since they identified that the outputs obtained were indeed elements of the specified codomain.

In a later interview, I asked why the element zero would be an issue. Student 2 responded,

> If we were allowed to put zero in our function here, we would get one over zero as a potential input or a potential value, which is undefined. That's not a real number, so we kind of get an issue there with it being a function.

In this second excerpt coordination with the codomain becomes clearer, as Student 2 mentioned that if they plugged in zero into the rule they would get one over zero and this value is "undefined" and "not a real number." Hence, Student 2 exhibited coordination in their understanding of function when dealing with this particular *everywhere-definedness no set* example.

3.5.2.b Everywhere-definedness some set (non-example). I asked Student 2 if the rule $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ was a function. They stated,

> so negative three cubed is what? Nine times negative three, which is negative twenty-seven. And so our, uh, range was just the-the natural numbers, you know. Um, positive whole numbers. And, so with this one we're getting our output value of I, I just put in the, I just put in negative three in there … And you're getting negative twenty-seven and that's not a natural number.

Just like with the previous case, they began by looking at what might cause issues when plugged into the rule and identified negative numbers as causing possible issues. This is made clear by their choice of picking -3 as an element to examine from the specified domain and cubing it to obtain -27, hence showing a coordination of the domain and rule. They further coordinated with the codomain and correctly identified that -27 is not an element of the natural numbers since the natural numbers are positive whole numbers. Thus concluding that this was not a function.

They further clarified what the issue was,

> so our two sets, here, is the integers and the natural numbers. And so this, function that we wrote here, the x cubed, um, you know, there are some things that 'okay, yeah we can put some number in for x and we can get some natural numbers, that's fine.' Um, but

> there's also some numbers, you know, our-our negative numbers, if we put negative numbers in here, we're not getting output values in the natural numbers.

In further clarifying their thinking about why negative numbers cause issues, they resorted to coordination as well. Notice that in their explanation they attended to both the domain and codomain by mentioning their two sets, "the integers and the natural numbers." Furthermore, they discussed the rule, "the x cubed," and the fact that there are some numbers you can plug into the rule and get values in the natural numbers but there are also ones you cannot plug into the rule because "we're not getting output values in the natural numbers," thus making it not a function. This is a clear example of the student attending to and coordinating all three parts of the function—the domain, the codomain, and the rule. Thus, I classified Student 2's ways of understanding as *coordinated* in this *everywhere-definedness some set* non-example.

3.5.2.c Well-definedness multiple rules (non-example). In a second interview, I asked Student 2 if the rule $q(x) = \begin{cases} 1, \text{if } x \in 2\mathbb{Z} \\ 0, \text{if } x \in 3\mathbb{Z} \end{cases}$ from $2\mathbb{Z} \cup 3\mathbb{Z}$ into the set $\{0, 1\}$ was a function. Discussing their thinking aloud, they explained,

> $2\mathbb{Z}$ and $3\mathbb{Z}$, so that's, um, like multiples of two and multiples of three, basically. And, union, so, in both of those sets. And so, I kind of started writing what that set looks like, so I started writing zero, two, three, four, six. Umm, and then, when I saw, I mean, zero too, but, it mostly stuck out when I saw six. I was like, um, 'Well, that's in $2\mathbb{Z}$ and $3\mathbb{Z}$, um, so we would get one and zero there, um.' And so that's kind of where we have an issue … well, I guess zero too could be a problem, but I first noticed it at six.

Here they first attended to the domain, by focusing on what the elements of the union of $2\mathbb{Z}$ and $3\mathbb{Z}$ might be. As they mentioned, they began writing what the sets look like, "I started writing zero, two, three, four, six." It was in this way that they noticed the element 6 (and as they mentioned, the element 0 as well) is an element of both $2\mathbb{Z}$ and $3\mathbb{Z}$, and thus it is contained in the union (in other words, it is an element of the domain). To examine what was occurring with this element of the domain, Student 2 coordinated with the rule. It was in this way that they noticed that this element gets mapped to both 0 and 1 in the codomain. In fact, this also displays a slight coordination with the codomain.

They explained the issue in a bit more detail,

> so I mean zero, I guess, um, but I again, I first noticed it at six, that's a multiple of two and three, um, so it would map to one and zero. Umm, so we kind of have an issue there, uh, the mapping to two different, values.

So they stated that the element 6 (as well as 0) is a multiple of both 2 and 3 (hence contained in the domain) and the rule therefore maps it to both 0 and 1. This is an issue because one element of the domain is "mapping to two different, values" in the specified codomain. While the coordination with the codomain was not made explicit here, there was hints of coordination with the codomain in the sense that the student was aware about the output values that were allowed. Hence the student exhibited coordination when reasoning about this *well-definedness multiple rules* non-example.

3.5.2.d Well-definedness equivalence (non-example). Student 2 also discussed the non-example $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$. However, they incorrectly identified it as an example of a function as the following excerpt shows:

> Um, so, our rational numbers, um, can be described as like an, basically an integer over an integer, in its simplest form. Umm, yeah, so if we can write it as a fraction it's a rational number. And so, it's an integer over an integer. And so, if we just add integers, together. So a would be an integer, b would be an integer, and the integers are closed

under addition, so if you add integers, you're going to get another integer. Umm, and then, that output value, well, hold on.

Student 2 clearly coordinated the rule and the codomain since they discussed closure of the integers; they attended to the idea that the rule sends $\frac{a}{b}$ to $a + b$ and that if a and b are integers, $a + b$ will always be an integer as a result of closure. There was also coordination with the domain to some extent (although not explicit) since they were able to focus on what happened to the domain elements when pugging them into the rule. Notice that they displayed an understanding of elements of $\mathbb{Q}$ as being necessarily reduced or in "its simplest form." Thus not noticing equivalence in the domain as is the case of the elements $\frac{1}{2}$ and $\frac{2}{4}$, for example.

To make a final conclusion on whether this rule was a function or not, Student 2 resorted to working with specific elements,

> I was just kind of thinking if, um, like three fifths, and then five, thirds. Umm, oh that's still fine, because different input values can have the same output value. Um, so yeah, I would still say it was a function. I was just thinking because a, like, again, the example five thirds over three, and then three fifths, you know, if you're adding three and five you're going to get the same answer. Um, but that's still, that's still okay, it's still function.

In this excerpt I thought they had possibly realized their error but again, while coordinating with the domain, their focus was mainly on what was occurring in the codomain. They chose two elements of the domain, namely $\frac{3}{5}$ and $\frac{5}{3}$, and coordinated with the rule to obtain different outputs. They argued that this was still a function since the elements being observed in the domain were different and it is okay that they map to the same thing. I note here that while Student 2 still exhibited coordination in their ways of understanding this particular task, they were unable to solve it correctly. By imposing that elements of $\mathbb{Q}$ be reduced, they missed that equivalence was an issue and classified this *well-definedness equivalence* non-example as an example of a function.

3.5.2.e Summary. Student 2 reasoned successfully about three out of the four examples and non-examples. In reasoning successfully about these tasks, I claim that they demonstrated *coordination* in their ways of understanding function. However, they were unable to reason successfully about the *well-definedness equivalence* non-example. I argue that coordination is required but not sufficient; knowledge of the sets that compose the domain and codomain is necessary.

3.5.3. Student 3. In this section I detail Student 3's mathematical activity related to different examples and non-examples of function from the four different categories of non-examples outlined in section 3.2.1 (see Table 14 for these examples and non-examples). In particular, I present Student 3's reasoning with one example and one non-example I presented them and two non-examples that they came up with.

Table 14: Tasks for Student 3

Well-definedness: Equivalence	Everywhere-definedness: No set	Everywhere-definedness: Some Set	Well-definedness: Multiple Rules
The rule from $\mathbb{Q}$ to $\mathbb{Q}$ given by $\frac{a}{b} \mapsto \frac{a+1}{b+1}$	$g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$	$h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$	A correspondence written as a set of ordered pairs where there are two pairs that have the same first coordinate
Non-example	Example	Non-example	Non-example

3.5.3.a Well-definedness equivalence (non-example). When asked to give three to four non-examples of function, Student 3 suggested a rule from the rational numbers to the rational numbers where $\frac{a}{b}$ gets mapped to $\frac{a+1}{b+1}$. They explained their non-example,

> So, yeah, like some of the examples with rational numbers is-is like a great way to create, uh, ambiguities, because we can write rational numbers in many different ways. So if you want to have, functions from rational numbers to rational numbers. Um, then you take a by b, and then you take this to, um, let's say, let me try this, not sure if it's gonna work. Okay, so I want to find out whether this is a function, meaning that I want to find out if one number can map to two different numbers. So if I map zero by one, which is zero, then this will be one by two. But zero can also be written as zero by two. And that will map to one by three.

Notice they first note knowledge of the domain as a set. That is, they discussed why rational numbers can cause ambiguities; any rational number can be written "in many different ways." They then consider an element in the domain, and one of its equivalent representations. So they explore the element 0 written as both $\frac{0}{1}$ and $\frac{0}{2}$. They then coordinated with the rule to figure out possible output values for both of these elements of the domain. Student 3 decided that $\frac{0}{1}$ maps to $\frac{1}{2}$ but $\frac{0}{2}$ would map to $\frac{1}{3}$. They followed up, "really, zero maps to one by two and one by three both, according to this assignment. Um, and therefore, this is not a function." Student 3 thus demonstrated coordination when describing why their non-example was indeed not an example of a function. While the coordination with the codomain is not explicit, notice that in the second excerpt, they mentioned that the rule maps the element 0 to both $\frac{1}{2}$ and $\frac{1}{3}$. This, coupled with the fact that they seemed to understand equivalences in $\mathbb{Q}$ led me to conclude that they coordinated with the codomain and were aware that $\frac{1}{2}$ and $\frac{1}{3}$ are not equivalent elements in $\mathbb{Q}$. Hence, this explanation was taken as showing coordination in their understandings about this *well-definedness equivalence* non-example.

3.5.3.b Everywhere-definedness no set (example). I presented Student 3 with the rule $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$ and asked them to determine whether it was a function. Explaining their thinking, they began,

> Zero is not included and infinity is not included. So the way I'm going to first try to think about it is try to see algebraically what's happening. So, what I see, or want to check is that if, two things are equal to each other then is their output equal? … so if x_1 is equal to x_2, then, if we invert them, those are equal to, and we can invert them because x, these are both non-zero. So, I would, I would say that, algebraically this looks like a function.

Notice that they started by trying to grasp what the domain was, "Zero is not included and infinity is not included." They followed up by explaining that they were going to approach it algebraically and chose two general elements of the domain which they called x_1 and x_2. They coordinated with the rule since they discussed inverting the elements. That is, they saw the rule as a rule that tells them to invert elements. While performing these steps, it became clear they also coordinated with both the domain and codomain. They mentioned that x_1 and x_2 were non-zero (thus showing coordination with the domain) and discussed being allowed to invert them due to that. Additionally, they coordinated with the codomain because they were able to see that if x had been zero then you would get an element that is not part of the proposed codomain.

I followed-up by asking them why it was important that x_1 and x_2 are non-zero. They responded, zero "was not included, so we can actually invert these numbers … You're picking from, zero to infinity … Otherwise, we might get something undefined here and then, all bets are off.

In this *everywhere-definedness no set* non-example, Student 3 again exhibited coordination in their reasoning. This *coordinated* way of understanding is made clearer in this second excerpt when they expressed that zero "was not included, so we can actually invert these numbers" and followed up by saying that if zero had been part of the specified domain then "we might get something undefined." Thus, they showed coordination with the codomain and an understanding that $\frac{1}{0}$ is not considered an element of $\mathbb{R}$.

3.5.3.c Everywhere-definedness some set (non-example). I also asked Student 3 to determine whether the assignment $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ was a function. They explained,

> Okay, so my first instinct here is to just, see what it does to some, randomly chosen numbers, just to get used to the idea of this map and what it's doing. So the easiest number, I mean it's zero, but, you know, I'm going to say one. So one goes to one cubed, which is one, but now I'm going to try minus one because all integers are permissible.

Their strategy was to pick some numbers in the domain $\mathbb{Z}$ and see where those got mapped to by the rule. They picked the number one (presumably trying to choose a positive number) which demonstrated attention to the domain and then coordinated with the rule, "one goes to one cubed," to figure out an output value for 1. While they did not explicitly coordinate with the codomain in this excerpt, their coordination with the codomain was made clear when they suggested trying a negative number.

As shown in their previous comments, Student 3 was interested in trying different numbers in $\mathbb{Z}$ to see what the rule did to them:

> Umm, so minus one goes to minus one cubed, uhh, which is minus one. So we are in a bit of trouble here, because, um, minus one is in the domain, and the formula suggests that it should map to minus one, but that's not in the range. So are we, restricting our function

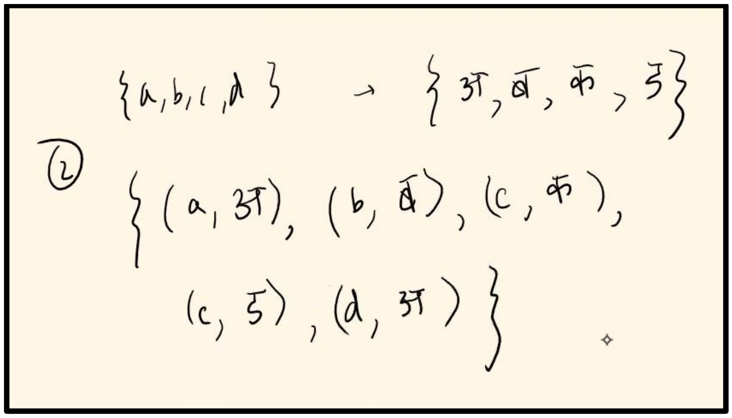
> here, like, because if we are not, then this is not a function. Because minus one does not have anywhere to go to in the range.

I characterized Student 3's ways of understanding as *coordinated* in trying to determine whether this *everywhere-definedness some set* non-example was a function. As shown above, they proceeded to try to find a negative number in $\mathbb{Z}$ to see what the rule did to it. They chose -1 which shows attention to the domain again and then coordinated with the rule, "so minus one goes to minus one cubed." Furthermore, they proceeded to explicitly coordinate with the rule as seen by their comment that "minus one is in the domain, and the formula suggests that it should map to minus one, but that's not in the range." Thus they took their output suggested by the rule and coordinated with the codomain to check whether this output would be in the codomain, $\mathbb{N}$. Hence, concluding that this rule was not a function as the output suggested by the rule does not live in the proposed codomain.

3.5.3.d Well-definedness multiple rules (non-example). I asked Student 3 to provide three to four non-examples of function and one of their non-functions was an ordered pair example as seen in Figure 3 below. They explained why this was a non-example as follows:

> So you want to define a function between a, b, c, d, and, um, let me use Hindi letters, *aa* <Hindi letter>, *ba* <Hindi letter>, *ga* <Hindi letter>, and *da* <Hindi letter>. Okay. So the Greeks have monopolized this enough, so it's time for the, Hindi letters to take over. So let me map a … to *aa*. And *b*, to *ba*. So far so good, c to *ga*, and c to *da*. And d to *aa*. So, the pathology here is introduced by the fact that c does not have a unique output, it has two different, outputs, namely *ga* and *da*, and, that is, not good. That's something you don't want.

Figure 3: Student 3's *well-definedness multiple rules* non-example



As we can see in their description, Student 3 coordinated with the domain, the rule, and the codomain. However, the coordination occurring here is a bit different than before due to the nature of the function's presentation. Rather than having an actual "rule," the rule in this case is the assignment of the letters. Notice that they coordinated the rule or assignment with the domain and codomain by attending to the fact that the letter c of the domain gets mapped to two different outputs in the codomain. They pointed out that this makes this a non-example of a function since one element is getting mapped to more than one output and "That's something you don't want." Thus Student 3 displayed *coordination* in their way of understanding this *well-definedness multiple rules* non-example.

3.5.3.e Summary. Student 3 was able to reason successfully about all of these examples and non-examples. I claim that, in reasoning successfully about these tasks, they showed a *coordination*, particularly when it comes to reasoning about (non-)examples from all four categories in the framework (outlined in section 3.2.1).

3.5.4 Student 4. In this last section of the results I present Student 4's reasoning about examples and non-examples of function related to the four categories of non-examples outlined in section 3.2.1 (see Table 15 for these examples and non-examples). I discuss Student 4's reasoning with one example I presented them and three non-examples, two of which they came up with.

Table 15: Tasks for Student 4[21]

Well-definedness: Equivalence	Everywhere-definedness: No set	Everywhere-definedness: Some Set	Well-definedness: Multiple Rules
$f\colon \mathbb{Q} \to \mathbb{Z}$ given by $f\left(\frac{a}{b}\right) = a^2 + b^2$	$g\colon (0,\infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$	$h\colon \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$	An elliptic curve given by the equation $y^2 = x^3 + ax + b$
Non-example	Example	Non-example	Non-example

3.5.4.a Well-definedness equivalence (non-example). I asked Student 4 to provide three to four non-examples of function. As one of their non-examples, they gave the rule $f\colon \mathbb{Q} \to \mathbb{Z}$ specified by $f\left(\frac{a}{b}\right) = a^2 + b^2$. When asked to explain why this was a non-example, they said, "This is not an expression in which I took my input and did something mathematically to it." They continued, "I split up my input first, I'm not doing something to the input. I broke the input and I'm doing something to different parts of the input." They began by hinting at coordination of the domain and the rule in very general terms. They discussed their rule as not performing mathematical operations on the input, thus signaling that coordination between the domain and the rule was occurring to some extent.

They then further clarified,

> The a over b is equal to a squared plus b squared. So here we broke up our input and we're no longer actually operating on the input. So, there's an issue there. Also, I believe

[21] For the two non-examples that the student provided, I made inferences from their mathematical activity about suitable domains and codomains. These inferences were consistent with all of their language and actions when discussing these two non-examples.

> that this, uh, particular input could map to various y's, because if I say a over b is going to be one over two, I can also express this as four over eight. In which case, if I'm calling this a and this b, for the first one here, I would get my one squared plus two squared, so five. For the second I would get four squared plus the eight squared, which would be something like ninety … So this maps to very, oh, infinitely many y's. So this is not a function. Any possible value for our input would map to infinitely many y's or many outputs.

In explaining this *well-definedness equivalence* non-example, Student 4 coordinated the domain and codomain, as well as the rule. They explained that if they pick $\frac{1}{2}$ then it is the same thing as $\frac{4}{8}$ (because they are equivalent in $\mathbb{Q}$). Afterward, they coordinated the domain with the rule to obtain output values for these two input values, getting an output of 5 for the element $\frac{1}{2}$ and an output of 80 for the input $\frac{4}{8}$. Their awareness that you get multiple input values for this function—that is, 5 and 80 are not equivalent elements in the codomain—suggests that they were indeed also coordinating with the codomain. This is further exemplified by their comment that you in fact get "infinitely many y's or many outputs" for any one input. Thus their explanation illustrates the role that coordination played in their understandings this non-example.

3.5.4.b Everywhere-definedness no set (example). In a later part of the interview, I asked students to identify whether $g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \frac{1}{x}$ was a function. Student 4 correctly identified it as a function, "So you're taking as input anything between zero and infinity. This is going to map to the real numbers, given by our function g of x is equal to 1 over x. Real numbers, yes, this is a function." When probed about why it was a function, they replied,

> You are transforming your input. You're basically just taking its inverse. And you are mapping to the real numbers, so all, your entirety of your domain is already in this field

> and just taking the inverse of it will leave it in the field. And since you're not considering negative numbers, although the negative numbers should work too. Although, you're not including zero. So you will not end up with infinity, so you're good there. And, you are assured only one y for every x … so, you are assured one output for every input and your expression here is taking your input and just taking the inverse of it. So it's just, applying some mathematical transformation to it.

Here they began by displaying coordination in a very general sense; they discussed the rule as taking inverses and coordinated the domain, codomain, and rule generally by saying "your entirety of your domain is already in this field and just taking the inverse of it will leave it in the field."

In the excerpt above, Student 4 seemed to suggest that zero would cause issues so I followed-up with them about that and they replied,

> So you have a vertical asymptote there which, means that you just wouldn't have any, um, any x value that you could use there. So the inp-, the, math isn't defined for division by zero … you would just have to specify that your domain could not include zero … In other words, you would not be able to get a valid y value for that particular value.

I attempted to clarify, "so if I plugged in zero into g, I wouldn't be able to get anything?" And they responded, "Correct, especially not something in the real numbers. And you're mapping to the real numbers." Thus Student 4 showed a coordinated way of understanding this *everywhere-definedness no set* example. As they explained further, they made the coordination more explicit by identifying the element 0 in the domain as an element that would case issues with the rule since it gives you a vertical asymptote (when looking at the graph). As they mentioned, "the math isn't defined for division by zero." Additionally, they coordinated with the codomain by mentioning that you are "especially not [getting] something in the real numbers. And you're mapping to the real numbers." Thus they showed awareness that the output given by their rule does not land in the real numbers and therefore this would not be a function if zero was included as part of the domain.

3.5.4.c Everywhere-definedness some set (non-example). In another task, I asked Student 4 to consider $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ and decide whether it was a function and explain their thinking. Student 4 explained,

> So this, is a function because for every x value, you have only one y value it can be. It's kind of a bijection, so, kind of, alright. And then, you're mapping from all integers, and then, all is are, in our domain here, let's see, and then you're mapping to natural numbers. Hold up, natural numbers, count, okay, so we don't consider the negative ones in the counting numbers.

They paused for a bit, then continued,

> you're restricting your range. Is this a function? This is not a function … Because your x values, some of them do not have corresponding y values. Uh, so … If you took the integer negative three … This relation would map it to, negative twenty-seven, which is not in our range. It does not exist in the set of elements that we're mapping to.

In explaining their thinking about this *everywhere-definedness some set* non-example, Student 4 attended to the domain, codomain, and the rule (and coordinated all three) since they noticed that negative numbers are part of the integers but the rule suggests mapping them to negative numbers which are not elements of the codomain, $\mathbb{N}$. Notice that they specifically picked an element in the suggested domain, namely -3 and knew that the rule says to map that to -27, however that "is not in our range." Thus showing coordination of the domain, codomain, and the rule.

After going back to a different task, they continued discussing this non-example:

> Negative anything would not map to anything … Negative anything would not map to a valid output … I think that I'm kind of stuck in thinking that this, since this is a function from real to real that's why I'm thinking, I'm getting mixed up. But it's not a function for integers to natural numbers. But it is a function for reals to reals.

This is another illustration where I characterized Student 4's ways of understanding as being *coordinated*. I want to point out that in their explanation, they seemed to be a little confused on whether this was actually a function or not, but I argue that thinking in a coordinated way allowed them to sort out their confusion. As they mentioned, "since this is a function from real to real that's why … I'm getting mixed up." But notice that once they coordinated the domain, codomain, and the rule they were able to clear up their confusion and realized that indeed "it's not a function for integers to natural numbers."

3.5.4.d Well-definedness multiple rules (non-example). In the task where I asked students to come up with three to four examples and three to four non-examples of function, Student 4 came up with a *well-definedness multiples rules* non-example. They explained,

> how about an elliptic curve? Uh, it's an expression but it's not really a function. So by inspection an elliptic curve would look something like this <drawing a graph of it> … And, here you can see again that since I have y squared, we'll just give this formula <$y^2 = x^3 + ax + b$>. I would be able to get two different y values from any one x … you-you can see this just by looking at, since we have y squared.

I asked about x equal to 0 to be able to understand their general statements and they replied, "Yep, and then what if b was negative. And then what if b was positive." While Student 4 did not pick a specific element of the domain to explore, it is clear that they attended to the domain and coordinated with the graph as well as the codomain as they were explaining that for a specific x value they always get two different y values. I provided them with a specific element in the proposed domain as a way to double check that I was understanding their explanation and they coordinated that value with the rule to obtain y squared equal to b and then explained that you have to pay attention to the fact that you get both positive and negative values for that x, which displays coordination with the codomain.

3.5.4.e Summary. Student 4 reasoned successfully about all four of these examples and non-examples from all four categories (outlined in section 3.2.1). I claim that, in reasoning successfully about these tasks, they showed displayed *coordination* in the ways they were understanding these tasks and want to note that while they were slightly confused about the *everywhere-definedness some set* non-example, coordination allowed them to successfully reason through the issue.

3.6 Discussion

In this paper I have provided initial indicators of and then further examined what is involved in a *coordinated* way of understanding functions in abstract algebra and illustrated how it can support students in reasoning productively. In particular, students demonstrated different ways of understanding function that are well documented in the literature and exhibited coordination when explaining why the correspondences in question were examples or non-examples of function. However, I note that, even though many of the ways of understanding enacted by the students in this study are documented in some form in the literature, I claim that many of these characterizations do not explicitly coordinate notions of the domain, the codomain, and the rule. To make this point clear, I provide examples that illustrate how a student is understanding function in particular instances and the reason(s) why I characterized those ways of understanding as *coordinated* in these instances. The following table includes some of the ways of understanding displayed by students in the results and makes explicit why I characterized those ways of understanding as “coordinated.”

Table 16: Coordinated ways of understanding

Way of Understanding	Description	Examples from Data	Coordination
Function as inputs and outputs	Involves viewing a function as taking in an input and returning an output.	"so negative three cubed is what? Nine times negative three, which is negative twenty-seven. And so our … range was just the-the natural numbers, you know. Um, positive whole numbers. And, so with this one we're getting our output value of I, I just put in the, I just put in negative three in there … And you're getting negative twenty-seven and that's not a natural number."	This student exhibited coordination in thinking about functions as inputs and outputs. In particular, they picked a specific element of the domain, the element -3, and coordinated with the rule to see what the rule assigns this value to and concluded that the output is -27. Then further coordinated with the codomain to conclude that -27 is not contained in the codomain specified.
Function as a machine	Involves viewing a function as a machine that acts on an object (either a single element or a set of elements) to produce another object.	"a vending machine … you hit the letter B5 … and it gives you out a chocolate bar … but maybe you hit A3 and there's like a bag of chips … for each combination in the vending machine that you have, unique combination of letter and number, you get out a unique … type of object from the vending machine."	This student focused on the role of the domain, codomain, and the rule. They argued that "for each … unique combination of letter and number" (referencing elements of the domain) you get a "unique … type of object from the vending machine" (referencing elements of the codomain). Additionally they coordinated with the rule which in this example is the process of a person clicking a button and getting a snack back.
Function as traveling or moving	Involves viewing a function as moving elements in the domain to elements in the codomain.	"So one goes to one cubed, which is one, but now I'm going to try minus one because all integers are permissible … so minus one goes to minus one cubed, uhh which is minus one. So we are in a bit of trouble here … So are we, restricting our function here, like, because if we are not, then this is not a function. Because minus one does	This student looked at specific elements in the domain $\mathbb{Z}$ (specifically, 1 and -1) to see where those got sent to in the codomain. In particular, when looking at -1 in the domain, they concluded that it "goes to minus one cubed." This shows coordination with the rule. They further coordinated with the codomain when they concluded that this equals -1 and that this is not an element of the specified codomain

		not have anywhere to go to in the range."	since "minus one does not have anywhere to go to in the range."
Function as an operation or procedure	Involves viewing a function as an operation or mathematical procedure that is performed on an input.	"from $\mathbb{R}$ to $\mathbb{C}$ … I get to have imaginary numbers, so I can take like the square root of negative one … My problem … is if x is a negative number … so what if x is like negative seven? … This like … the square root of negative one times seven. So this is seven i … There's only one seven i, Ah, so it's good."	This student exhibited coordination in their way of understanding function because when discussing the negative numbers, they picked an element in the domain to see what it was assigned to in the codomain. Namely, they choose -7 in $\mathbb{R}$ and concluded that when you take the square root of it you get $\sqrt{-7}$ as the output, and that this was an element of the codomain. Thus displaying coordination between the domain, the codomain, and the rule.
Function as a dynamic transformation, change, replacement, or morphing	Involves viewing a function as a way of transforming, converting, or changing inputs into outputs.	"You are transforming your input. You're basically just taking its inverse. And you are mapping to the real numbers, so all … your entirety of your domain is already in this field and just taking the inverse of it will leave it in the field … Although you're not including zero. So you will not end up with infinity, so you're good there. And … you are assured only one y for every x."	This student displayed coordination generally. They discussed the rule by saying they are just "transforming their input" by "taking its inverse." They then coordinated with the domain and codomain, mentioning that "your entirety of your domain is already in this field" (attending to the fact that the domain $(0, \infty)$ is a subset of $\mathbb{R}$) and then commenting that "taking the inverse of it will leave it in the field" (thus attending to the codomain). Additionally, they identified a possible problematic element but coordination with the domain allowed them to notice that it was not in their domain.
Function as a mapping	Involves viewing function as involving a domain, codomain, and the correspondence between the two.	"we can write rational numbers in many different ways … I want to find out whether this is a function, meaning that I want to find out if one number can map to two different numbers. So if I map zero by one, which is zero … then this will be one by two. But	In their understanding of function as a mapping, this student coordinated the domain, codomain, and the rule. They picked an element of the domain, $\frac{0}{1}$, and knew that this could also be written as $\frac{0}{2}$ in the domain $\mathbb{Q}$. Then coordinated with the rule to

		zero can also be written as zero by two. And that will map to one by three … really, zero maps to one by two and one by three both … therefore, it is not a function."	get that this maps to either $\frac{1}{2}$ or $\frac{1}{3}$. This signals coordination between the domain and codomain. Their understanding of $\mathbb{Q}$ as a set of equivalence classes together with their statement that this element (and its representations) maps to two elements led me to conclude that they were also coordinating with the codomain.

As illustrated in the table above, students demonstrated different ways of understanding function while working through different tasks that were representative of the four different categories of examples and non-examples from Paper 1. In the coordination column, I give details of why these particular ways of understanding were considered "coordinated" for these different examples and non-examples. I want to re-emphasize the point made above that the ways of understanding demonstrated by the students do not on their own point to coordination. That is, a student might view function as "inputs and outputs" or as a "dynamic transformation" but not necessarily attend to the relationship between the domain, the codomain, and the rule when describing why a proposed correspondence is or is not a function.

This paper addresses what is entailed in reasoning productively about well-definedness and everywhere definedness—the two characteristic properties of function—in abstract algebra. This is a key contribution because while researchers had previously examined various facets of reasoning productively with functions in mathematics in general (e.g., Carlson, 1998; Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008) and in advanced mathematics in particular (e.g., Melhuish et al., 2020; Zandieh, Ellis, & Rasmussen, 2017), there had previously been no direct, detailed investigations of what is entailed in reasoning productively about well-definedness and everywhere-defined in advanced mathematical settings. In other words, it was previously understood (and perhaps taken for granted) that understanding the function concept was critical for understanding, for example, binary operations and isomorphism, but it remained somewhat unclear exactly *how* one should do so. This paper makes this clear and provides many examples. In abstract algebra, the presumably familiar concept of function presents students with several unfamiliar aspects that are foundational for understanding key ideas like binary operation and isomorphism. I illustrated how a coordinated way of understanding can support students in reasoning about all four categories in the instructional example space (that I discussed in Paper 1): *well-definedness equivalence* (such as $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$), *well-definedness multiple rules* (such as $p: (0, \infty) \to \mathbb{R}$ given by $p(x) = \pm\sqrt{x}$), *everywhere-definedness some set* (such as $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$), *and everywhere definedness no set* (such as $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$).

Research in abstract algebra has established that the function concept is challenging and problematic for students (e.g., Brown, et al., 1997; Leron, Hazzan, & Zazkis, 1995; Melhuish & Fagan, 2018; Melhuish et al., 2020; Rupnow, 2017, 2019), and thus there was a need for this kind of research (which outlines the nature of the ways of understanding that might help students overcome these difficulties). I also note that significant portions of both the functions literature and abstract algebra literature focus on the challenges and difficulties students experience. Such research that identifies challenging topics and the nature of students' difficulties is certainly important but needs to be balanced by a commensurate amount of research that identifies ways to help students overcome them. I note that, generally, much of the functions literature as well as the abstract algebra literature has focused on the former. Thus, a contribution of this paper is that it offers "a positive counterpoint" (Bagley & Rabin, 2016, p. 84) to much of this literature.

3.6.1 Implications for practice. I have outlined a way of understanding in this paper. A key implication for practice is that instructors should carefully design instructional experiences for students in order to elicit productive ways of understanding. This could include, for example, ensuring that students' experience examples and non-examples in each of the four categories outlined above: *well-definedness equivalence/representation*, *well-definedness multiple rules*, *everywhere-definedness some set*, and *everywhere-definedness no set*. An important consideration when selecting examples and non-examples from these categories is to modify them (if needed) so that the lack of use of a coordinated way of understanding is likely to result in an incorrect answer (for example, it seems probable that a student who is not attending to the domain and codomain could conclude that $f\colon (0,\infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not a function). Such tasks could promote the cognitive conflict and intellectual necessity necessary for developing a coordinated way of understanding (I include more thoughts on this below).

3.6.2 Limitations and future research. A key element of the design of this study was that I examined the reasoning of students who had already developed the coordinated way of understanding I describe and illustrate in this paper. Future research could therefore employ the teaching experiment methodology to devise, test, and refine a hypothetical learning trajectory that outlines how students might be able to develop it. Additionally, while this paper takes a necessary first step towards outlining a productive way of thinking about functions in abstract algebra (by outlining what is involved in a productive way of understanding), it stops short of characterizing this productive way of thinking. One of the reasons is that, in an effort to make the examples and non-examples of functions in the task sequence more accessible (so that there would be minimal interference caused by unfamiliar concepts), the tasks were not sufficiently varied enough to conclude that these students were demonstrating a way of thinking. Now that a coordinated way of understanding has been outlined and illustrated, however, future research will be better situated to investigate students' ways of thinking about function by examining their reasoning in more varied function contexts.

The implications in the subsection above (the importance of and recommendations for careful instructional design) provide some guidance for how the instructional tasks in such a sequence might be designed. My reflection on the students' activity in this study provide another hypothesis about the kinds of tasks that could play a key role in such a learning process: the recovering of an example from a given non-example. For example, consider the non-example $h\colon \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$. In discussing this non-example, one student suggested modifying the domain, "So minus one gives us minus one, which does not lie in the range. Uh, so if we restrict the map to, um, things which are positive … integers then we are going to get, um, you know, we're-we're going to get a relation and it is going to be a function." Thus, they are changing this non-example into the example: $h\colon \mathbb{Z}^+ \to \mathbb{N}$ given by $h(x) = x^3$. Another student focused on making modifications to the codomain instead,

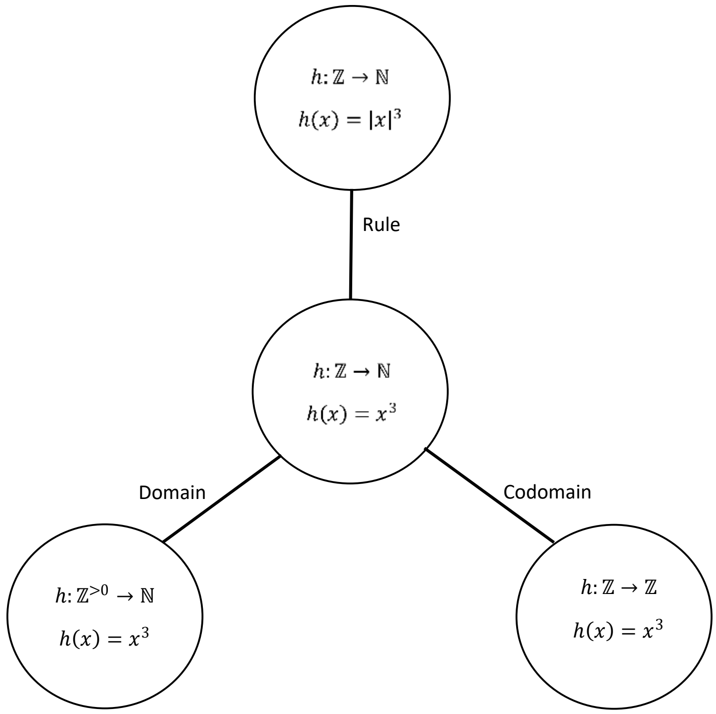
> when we put in negative values here, so -1 for x, we're getting $(-1)^3$, which is -1, and that's not a natural number. We can make it from $\mathbb{Z}$ to $\mathbb{Z}$ … If we put in an integer, we're going to get integers back.

Therefore creating the example $h\colon \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = x^3$. Lastly, a third student focused on modifying the rule,

> So the issue here is that negatives don't exist in … the natural numbers, right? … so your issue is, you have nowhere to go for all the negative integers … You could also say the absolute value of x^3 … This is just going to make it whatever it was positive. So if your x was larger than zero, then it's just mapping to x^3 and if your x was smaller than zero, it's just mapping to $-x^3$ and if your x is zero then, actually I'll just put that under there … Taking the absolute value of it.

Notice that in their mathematical activity, all three students coordinated the domain, the codomain, and the rule. They attended to the fact that the negative numbers cause issues and attempted to fix it by either modifying the domain, the codomain, or the rule. A visual of these three recovering examples can be seen in the following figure:

Figure 4: Non-example pod



While testing this hypothesis was beyond the scope of the current study, I offer it here as an interesting and potentially productive idea to pursue in future research. I also note that a coordinated way of understanding can support students in productively reasoning about functions but on its own is insufficient. Consider, for example, the fact that Student 3 was enacting a coordinated way of understanding (see the Results section 3.5) but was not able to reason successfully about examples and non-examples in the *well-definedness equivalence* category. Put another way, the coordinated way of understanding outlined here is useful but does not account for all of the key aspects of the function in concept in abstract algebra. This calls attention to the need for more research into the nature of students' thinking and reasoning about the definitive function characteristics of well-definedness and everywhere-definedness.

While one of the strengths of this paper, I propose, is that instead of jumping ahead to more advanced function-related concepts (like homomorphism) it focuses on the foundational properties of well-definedness and everywhere-definedness. Melhuish and colleagues (2020), for example, illustrated *that* the concepts of well-definedness and everywhere-definedness are important for understanding homomorphism, but left room to investigate in greater detail *how* and *in what ways* reasoning about these concepts might influence students' subsequent reasoning. Thus, it is currently unclear how the coordinated way of understanding functions might influence students' reasoning about these more advanced concepts—this could be an interesting line of future research.

CHAPTER IV

PAPER 3 [RECOVERING TASK]

Reasoning Productively about Functions in Abstract Algebra: Recovering an Example from a Non-example

4.1 Introduction

Functions are important in mathematics in general, and abstract algebra in particular because they underlie several important concepts in this course. Indeed, it is considered to be a core topic in secondary and undergraduate mathematics (Bagley, Rasmussen, & Zandieh, 2015; Dubinsky & Wilson, 2013; Hitt, 1998). In particular, students in abstract algebra encounter different important classes of functions like binary operations, homomorphisms, and isomorphisms in more abstract settings than they have previously experienced. Despite the prevalence of functions throughout the abstract algebra curriculum, little is known about how students reason with the concept and how they should understand it in this context—particularly how they understand well-definedness and everywhere-definedness.

To help abstract algebra instructors teach this important but challenging concept, in this research-to-practice paper, I share the results of a recent research project into students' reasoning about functions in abstract algebra. In particular, the purpose of this research project was to investigate the nature of the function concept in abstract algebra and how students reason about it.

This includes (1) starting with a productive and coherent way to choose/structure the examples and non-examples that instructors use in abstract algebra classrooms, (2) a way of understanding functions that supports students in reasoning productively about these kinds of examples and non-examples, and (3) a kind of functions task/activity that addresses both (1) and (2). Specifically, this kind of task involves recovering an example from a non-example of a function by prompting students to modify the domain, the codomain, and/or the rule.

4.2 Functions in Abstract Algebra

4.2.1 The importance of functions in abstract algebra. As previously mentioned, functions are a foundational concept in mathematics in general and in abstract algebra in particular. They are the foundation for such key concepts as binary operation (and therefore notions of algebraic structure) as well as homomorphism and isomorphism. Furthermore, in abstract algebra, students encounter functions between sets of polynomials, sets of functions, or even sets of sets. As a result, abstract algebra requires students to deal with unfamiliar aspects of this familiar topic from secondary mathematics. Many studies have generally called attention to the importance of the underlying function concept for understanding such topics as binary operation, homomorphism, and isomorphism (e.g., Brown et al. 1997; Hausberger, 2017; Melhuish & Fagan, 2018; Melhuish, Ellis, & Hicks, 2020; Nardi, 2000; Rupnow, 2021). Additionally, function has been identified as an important connection that pre-service teachers should make between their abstract algebra coursework and secondary mathematics (e.g., Melhuish & Fagan, 2018; Wasserman, 2017). Indeed, research has shown that the ways in which students understand the function concept exerts a considerable influence over their reasoning in subsequent ideas. In this research-to-practice paper, I synthesize what I have learned from my recent line of research and propose tasks that can support students in reasoning productively about the core function concept.

4.2.2 Functions in calculus versus functions in abstract algebra. Of course, although the overarching function concept remains invariant across contexts, the way that it manifests in particular circumstances can be quite different. On one hand, in algebra, trigonometry, and calculus, it can be productive to view function as a relationship between two covarying quantities (e.g., Carlson, 1998; Carlson et al. 2002; Oehrtman, Carlson, & Thompson, 2008). Focusing on how changes in one quantity correspond to changes in the other provides a solid foundation for understanding key concepts like constant and average rate of change (e.g., Musgrave & Carlson, 2016), derivatives (e.g., Thompson, 1994a; Zandieh, 2000), limits (e.g., Oehrtman, Swinyard, Martin, 2014), and characteristic properties of certain kinds of functions (for example, exponential (e.g., Ellis et al. 2016) or trigonometric (e.g., Moore, 2014) functions). It is also worth noting that, in many (if not most) of these situations, the domain and codomain of a function are conventionally assumed to be the real numbers (or a subset of the real numbers that behaves very similarly).

On the other hand, in more advanced courses like abstract algebra, a covariational view of function is often not relevant. Not only does abstract algebra not typically focus on how quantities change together, the objects in question often cannot even be thought of as quantities. (Consider, for example, functions whose domains and/or codomains are square matrices over the integers, or the set of equivalence classes modulo n.) Rather than a quantitative perspective, therefore, abstract algebra requires another way to view and think about functions. This is perhaps one reason why undergraduate students find function and function-related concepts to be so challenging (e.g. Brown, DeVries, Dubinsky, & Thomas, 1997; Leron, Hazzan, & Zazkis, 1995; Melhuish & Fagan, 2018; Rupnow, 2017, 2019): while the function concept is indeed familiar, abstract algebra requires that students reason about it in new ways. Such ways of reasoning (e.g., covariational and quantitative reasoning) have been identified for uses of function up to calculus, but this has not been done for abstract algebra.

4.2.3 Overview of this paper. The discussion above highlights the importance of (1) outlining what is involved in a productive view of function in abstract algebra, and (2) identifying instructional activities that might support students in developing it. My objectives in this research-to-practice paper are to share the results of some of my recent line of research, the purpose of which was to develop a clearer image of the function concept—how it manifests and how students might reason with it—in abstract algebra. In section 4.4, I address (1) by outlining what is involved in a *coordinated view of function*. In Section 4.5, I address (2) by proposing that the task of *recovering an example from a non-example* is a potentially effective way to engage and support students in developing this view. To provide some initial support for this idea, I showcase the reasoning of students engaged in this task in section 4.5.2.

4.3 Characterizing and Exemplifying the Function Concept

4.3.1 Key components of the function concept. For this paper, I adopt the following characterization of function as described by Weber and colleagues (2020), "a function is defined as consisting of a domain, a codomain, and a correspondence between the domain and the codomain such that each member of the domain is assigned exactly one element of the codomain" (p. 2). I follow other researchers (e.g., Melhuish & Fagan, 2018) as well as prominent textbook authors (e.g. Fraleigh, 2002; Gallian, 2017) who split this criterion into two conditions:

1) Well-definedness: each element of the domain is mapped to at most one element of the codomain.
2) Everywhere-definedness: each element of the domain is mapped to at least one element of the codomain.

There are two implications that result from framing function in this way. The first concerns the idea of a non-example of a function: a proposed correspondence can fail to be a function because it is (1) well-defined but not everywhere-defined (such as $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$), (2) everywhere-defined but not well-defined (such as $\phi: \mathbb{Q} \to \mathbb{Q}$ given by $\phi\left(\frac{a}{b}\right) = a + b$), or (3) neither well-defined nor everywhere-defined (such as the function that assigns to every person their favorite food). While I elaborate more on a productive way to structure non-examples and examples of function (as well as why it is useful to focus on non-examples) in what follows, for now I will only note that for simplicity, I will restrict my focus to non-examples of types (1) and (2). The second implication is that, as noted by Weber and colleagues (2020), changing the domain/codomain changes the function. The same formula can be associated with an example or a non-example depending on the stipulated domain and codomain. For example, $f(x) = \frac{1}{x}$ can either be a non-example (as shown above) or an example if we consider the domain to be $(0, \infty)$.

4.3.2 Non-examples and their relationship to examples. The central goal of this line of research was to characterize a productive view of function in abstract algebra. In addition to explicating exactly what the function concept entails and what its key components are (section 4.3.1), a necessary precursor to this endeavor was to identify what kinds of function tasks introductory abstract algebra students are expected to reason about. In other words, we can begin to develop a clearer image of what "productive reasoning" entails simply by identifying the kinds of tasks that instructors expect their students to be able to successfully complete. In order to do this, I (1) conducted a textbook analysis to identify the non-examples of function that are used to teach an introductory unit on functions in a first semester course in abstract algebra, and then (2) conducted interviews with mathematicians to gain insight into how these non-examples might be coherently organized and structured. I focused on non-examples because they have the potential to "demonstrate the boundaries or necessary conditions of a concept" (Watson & Mason, 2005, p. 65) in a way that examples by themselves might not. Along these lines, I found that non-examples can isolate the essential features of the function concept by showing what happens when these features are not met, illustrating the role and importance of such features.

Additionally, I noticed that the mathematicians intuitively wanted to try to "fix" the non-examples by modifying some combination of the domain, codomain, and/or the rule. For example, consider the non-example $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{1}{x}$ given above in section 4.3.1. Changing the domain to $(0, \infty)$ "fixes" this non-example and enables us to associate it with the example $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$; similarly, changing the rule to a piecewise function (in which $f(0)$ is defined to be 0) such as $f(x) = \begin{cases} \frac{1}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$ "fixes" the non-example. Thus, focusing on non-examples did not, in fact, exclude examples but instead drew a direct link to them: to discuss a non-example of a function was also to discuss the multiple examples that might be recovered from it. This activity—in which one makes reasonable modifications to the proposed domain, codomain, and/or rule of a non-example of a function in order to obtain an example—is what I refer to as *recovering an example from a non-example*. In this paper, I provide examples of tasks based upon this activity and illustrate how they might support students in developing a coordinated view of function.

4.3.3 Structuring the non-examples of function in abstract algebra. The textbook analysis and interviews with mathematicians highlighted a structure on the space of non-examples that are commonly used in instruction. The primary categories in this structuring involved the key aspects of well-definedness and everywhere-definedness. The well-definedness category includes all non-examples for which there is a well-definedness issue: that is, an element of the (proposed) domain that maps to more than one element of the (proposed) codomain. For example, $\phi: \mathbb{Q} \to \mathbb{Q}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ and $p: (0, \infty) \to \mathbb{R}$ given by $p(x) = \pm\sqrt{x}$. Here, ϕ maps equivalent elements in the proposed domain to different elements of the suggested codomain. As we can see, $\frac{1}{2} = \frac{2}{4}$ in the domain, but $\phi\left(\frac{1}{2}\right) = 3 \neq 6 = \phi\left(\frac{2}{4}\right)$. Similarly, p maps an element of the proposed domain to multiple elements of the stated codomain since say, 1, maps to both 1 and -1. This category can be further refined into two subcategories: *well-definedness equivalence* and *well-definedness multiple rules*. *Well-definedness equivalence* includes all non-examples for which the well-definedness issue is caused by equivalent, non-unique representations of an element. For instance, this category includes ϕ because as we saw above, $\frac{1}{2}$ (an element of $\mathbb{Q}$, the domain) maps to both $1 + 2 = 3$ and $2 + 4 = 6$ in the codomain since $\frac{1}{2}$ can be represented as both $\frac{1}{2}$ and $\frac{2}{4}$. *Well-definedness multiple rules* includes all non-examples for which the well-definedness issue is related to the rule being ambiguous, rather than issues with equivalence. For example, p above maps 1 (an element of $(0, \infty)$, the domain) to 1 and -1 in the codomain. But this is a result of the rule being ambiguous rather than equivalent representations of the element 1 of the domain.

The everywhere-definedness category includes all non-examples for which there is an everywhere-definedness issue: that is, an element of the (proposed) domain that does not map to at least one element of the (proposed) codomain. For example, $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ and $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. In the first non-example, the negative numbers have nowhere to map to in the codomain $\mathbb{N}$ (for example, -1 gets sent to -1 by the rule, but -1 is not an element of $\mathbb{N}$). In the second case, the element 0 of $\mathbb{R}$ does not map to anything in the proposed codomain since the rule sends 0 to $\frac{1}{0}$ which is undefined. This category can also be further refined into two subcategories: *everywhere-definedness some set* and *everywhere definedness no set*. *Everywhere-definedness some set* includes all non-examples for which the everywhere-definedness issue is related to the output assigned by the rule (to a particular element of the domain) not being contained in the proposed codomain. However, these outputs are contained in some (student accessible) enlarged set that contains the proposed codomain. For example, $h: \mathbb{Z} \to \mathbb{N}$ above is in this category because the rule sends -1 in $\mathbb{Z}$ to -1, but -1 is not an element of the codomain $\mathbb{N}$. However, the codomain can be slightly enlarged to $\mathbb{Z}$ to make this a function. *Everywhere-definedness no set* includes all non-examples for which the everywhere-definedness issue is caused by an element of the domain not being mapped to anything by the rule, or not mapping to an element in a student accessible set. This category includes $f: \mathbb{R} \to \mathbb{R}$ because as we saw above, 0 (an element of $\mathbb{R}$, the domain) gets sent to an undefined value.[22] Table 17 summarizes this discussion and provides additional non-examples that populate these categories:

Table 17: Well-definedness and everywhere-definedness elaborations

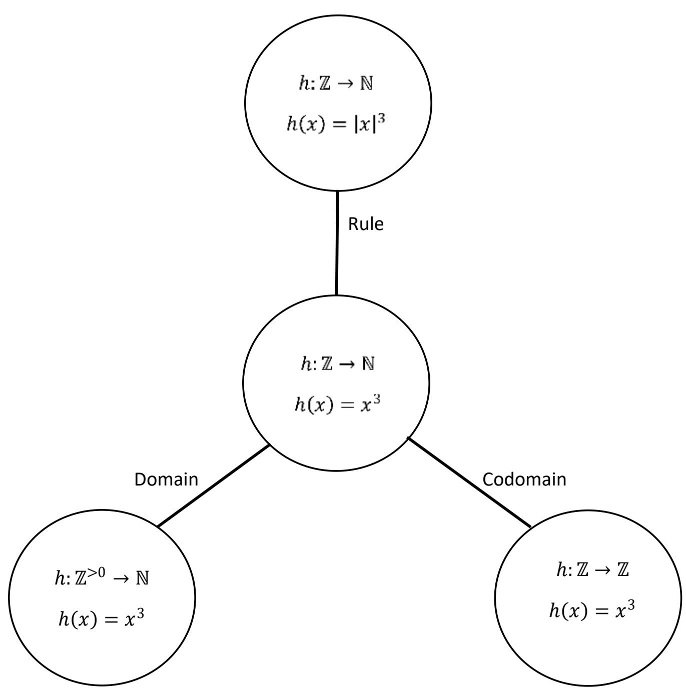
Well-definedness		Everywhere-definedness	
Equivalence: There exists at least one element in the domain that has different equivalent representations and the rule assigns different images in the codomain to these representations.	Multiple Rules: There exists at least one element in the domain that gets assigned different images in the codomain due to the rule being ambiguous.	No set: There exists at least one element in the domain that does not get assigned any value or this value is contained in an inaccessible set.	Some set: There exists at least one element in the domain that gets assigned an image that is not contained in the proposed codomain (but this image is contained in an enlarged accessible set).
$\phi: \mathbb{Q} \to \mathbb{Q}$ given by $\phi\left(\frac{a}{b}\right) = ab$ $\psi: \mathbb{Z}_4 \to \mathbb{Z}_6$ given by $\psi([x]_4) = [x]_6$	$q: 2\mathbb{Z} \cup 3\mathbb{Z} \to \{0, 1\}$ given by $q(x) = \begin{cases} 1, \text{if } x \in 2\mathbb{Z} \\ 0, \text{if } x \in 3\mathbb{Z} \end{cases}$ $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$ The correspondence: $\{(1, 6), (1, 2), (1, 4)\}$	$g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ given by $g(a, b) = \frac{a}{b}$ $*: M(\mathbb{R}) \times M(\mathbb{R}) \to M(\mathbb{R})$ given by $*(A, B) = A * B$	$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ $h: \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $h(x) = x - 1$
Relatively unfamiliar to students	Familiar from previous courses	Familiar from previous courses	Relatively unfamiliar to students

These categories inform my objectives in this paper in several ways. First, this can be useful because it can inform the design of instruction: instructors can use this framework to ensure that the collection of non-examples they are presenting or assigning to students is sufficiently varied. Second, and, importantly for my objectives in this paper, it provides a clear image of the kinds of tasks that students are expected to be able to complete and thus provides a

[22] We could define division by zero by passing to the Riemann sphere, but this set might not be accessible to a student in an introductory abstract algebra course.

clear benchmark for what qualifies as a "productive" view of function in abstract algebra. That is, I consider a view of function to be "productive" if students can use it to reason about non-examples of function in each of these four categories. Third, in section 4.3.2, I proposed that the task of recovering an example from a non-example provides a means to relate key non-examples of function with examples. That is, the framework can be used to classify, identify, and select non-examples, and the task of recovering can generate multiple related examples. For example, the non-example $h: \mathbb{Z} \rightarrow \mathbb{N}$ where $h(x) = x^3$ can be associated with the following examples: $h: \mathbb{Z} \rightarrow \mathbb{Z}$ where $h(x) = x^3$ (by changing the codomain), $h: \mathbb{N} \rightarrow \mathbb{N}$ where $h(x) = x^3$ (by changing the domain), and $h: \mathbb{Z} \rightarrow \mathbb{N}$ where $h(x) = |x|^3$ (by changing the rule). Throughout this paper, I use the term *non-example pod* refer to a core non-example and the various examples that might be recovered by "fixing" that non-example (see Figure 5 for a sample diagram that illustrates this structure). "Recovering" therefore provides a means to extend this framework beyond non-examples to include (at least some) examples as well.

Figure 5: Non-example pod for $h: \mathbb{Z} \rightarrow \mathbb{N}$ given by $h(x) = x^3$



4.4 A Coordinated View of Functions in Abstract Algebra

The view of function that I propose is productive for abstract algebra students to have is called a *coordinated view of function*. While I provide a more precise definition in the next subsection, the essential idea is that students view functions as a coordination of the domain, the codomain, and the rule. My focus on this idea was motivated both by the research literature and by comments made by the mathematicians in the interviews I conducted with them (for Paper 1). For example, Thompson (1994c) noted that the primary meaning that most students have for function is “two written expressions separated by an equal sign” (p. 24). Many other researchers have made similar observations (e.g. Carlson, 1998; Hit, 1998; Vinner & Dreyfus, 1989). While a preoccupation with the formula is a conception that is not desirable at any level, it is particularly limiting in abstract algebra because, as noted by Weber and colleagues (2020), “changing the codomain changes the function” (p. 2). It is not surprising, therefore, that many researchers (e.g., Kabael, 2011; Oehrtman, Carlson, & Thompson, 2008; Zandieh & Knapp, 2006) have made suggestions centering on how students should attend to the domain and codomain in addition to the rule/formula. As one example, Dorko (2017) suggested that reasoning productively about well-definedness “requires identifying an element of the *domain* and checking that it is paired with exactly one element of the *range*” (p. 2, emphasis added).

This theme also emerged in my interviews with the mathematicians. For example, one stated,

> if you say what the rule is, that’s not a function. A function has a domain, it has a codomain, and a rule that foes from one to the other. And all three need to be literally visible … So you really need to put all those ingredients right out there. And, the arrow from the domain and codomain, makes it really clear what the function is.

I interpreted the comment that "the rule is … not a function" to mean that the rule *by itself* is not a function. Similarly, I interpreted the comment that "all three need to be literally visible" as an indication that reasoning productively about functions requires attention to all three of these components, harmonizing with the themes I called attention to in the literature above.

Similarly, another professor stated that the notation $f: A \to B$ is critical and explained their reasoning:

> It's a shorthand for a whole bunch of information that you're giving the students about the function. You're telling them that the first set, after the colon, is the domain. So everything in there has to have an output value. And the arrow, and then the second set following the arrow, is the codomain. All of your outputs had better land in there. If they're landing outside of there, you have not done it properly.

What I found notable about this comment is that it does not even call explicit attention to the rule. Rather, the focus is only on the domain and codomain, emphasizing the critical nature of these two features. The recurring theme here is: it is beneficial for students to know that *a function is more than a formula*. Or, more specifically, function is a coordinated relationship between the domain, codomain, and rule.

A *coordinated view*[23] *of function* provides a specific way to operationalize this idea. As suggested above, it involves viewing the function concept in a way that involves explicit attention to the relationship between the domain, codomain, and rule. I define this idea more precisely in terms of a criterion that researchers and instructors can use to determine if a student is

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[23] I have previously framed the coordinated "view" of function in terms of Harel's (2008a) *way of understanding* and *way of thinking* constructs. Broadly, a *way of understanding* is a meaning that a student uses to make sense of and reason about a particular episode or situation, whereas a *way of thinking* can be thought of as a mathematical habit of mind or a general orientation in one's mathematical thinking that manifests across multiple episodes or events (Lockwood & Reed, 2020). The interested reader may consult Paper 2 for specific details about how I used this perspective as well as Harel (1998), Harel (2008), and Lockwood and Reed (2020) for general information on it.

demonstrating a coordinated view of function. The criteria focus on the observable components of a students' activity as they explain why a proposed correspondence is or is not a function (i.e., is an example or a non-example of function). Specifically, a student is demonstrating signs of having a *coordinated view of function* if, when explaining why a proposed correspondence is or is not a function, they make explicit reference (in their comments, gestures, or writing) to (criterion 1) an element (or collection of elements) in the domain, (criterion 2) an element (or collection of elements) in the codomain, *and* (criterion 3) the rule.

As an example of how this criterion might be used, consider the following excerpt from one of the interviews I conducted with mathematicians (for Paper 1). When reasoning about why the proposed correspondence $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$ was not a function, they said:

> you operate on, an element of your domain, [and it] doesn't go to anything in the codomain, that is being specified. So, negative one goes to, I mean negative one cubed is not a natural number, so it doesn't go anywhere, in its codomain.

Notice that, in addition to describing their general approach in more abstract terms at the beginning of the excerpt, they go on to specify how it can be used to conclude that g is not a function: they explicitly mentioned an element of the domain ("negative one," criterion 1), a collection of elements in the codomain ("is not a *natural number*," criterion 2), and the rule ("negative one *cubed*," criterion 3). On the other hand, we might expect someone who does not have such a view of function to perhaps conclude that g is a function because, after all, $g(x) = x^3$ is a formula that (1) is likely to be familiar from previous courses, and (2) is typically unproblematic in these courses (because it does not cause more obvious definedness issues like the formulas $f(x) = \frac{1}{x}$ or $p(x) = \sqrt{x}$ does). In this paper, we take for granted that a coordinated view of function is productive (see Paper 2) and use the criteria above to illustrate how tasks centering on *recovering an example from a non-example* can elicit this productive view of function.

4.5 Recovering Examples from Non-examples

4.5.1 Mathematician's recovering activity. As previously mentioned, the idea of recovering an example of a function from a non-example organically arose in interviews with mathematicians. When presented with tasks about determining whether a particular assignment was a function, the mathematicians approached the non-examples by attempting to "fix" the assignment to make it a function; such a process seemed helpful in reasoning through the task. This "fixing" of the assignment resulted in changing the proposed domain or codomain, or sometimes making slight modifications to the rule. The mathematicians appeared to coordinate the domain, codomain, and the rule to figure out which of their proposed changes actually resulted in a function. As previously noted, coordination is important in thinking about functions and in the process of analyzing the mathematicians' activity, I conjectured that coordination of the domain, the codomain, and the rule was a necessary part of the process of recovering an example of a function from a non-example. Thus the activity of recovering an example from a non-example can support productive reasoning about functions. To illustrate this, I focus on the recovering work done by mathematicians on a *well-definedness multiples rules* non-example and an *everywhere-definedness no set* non-example since these are the two categories that students are generally familiar with from secondary mathematics.

4.5.1.a Well-definedness multiple rules non-example. One of the *well-definedness multiple rules* non-examples I introduced to the mathematicians was $p: (0, \infty) \to \mathbb{R}$ given by $p(x) = \pm\sqrt{x}$. Working on recovering a function from this non-example, Professor B said,

> it's not a function because you have to decide which-which square root do you mean. The positive one or the negative one? … If you just leave it as $\mathbb{R}$ you're gonna have to

choose, you're gonna have to make a choice of the positive square root or the negative square root, at each number x … And there-there's infinitely many ways, infinitely many ways to make that choice … You could choose, you could make a different choice at different values of x … making the choice, would be related to changing the rule, you know, or clarifying the rule.

This suggestion focused on changing the rule to "fix" the non-example. They followed-up by providing a different suggestion for recovering an example of a function from this non-function by focusing on changing the codomain:

the other idea I had which I would never do would, well, I would do it in an algebra class. I mean, if you're, you're dealing with higher roots like cubed roots, etcetera, fourth roots, whatever … it's logical to talk about the set of all possible, fourth roots, or fifth roots, or something like that … you could change the codomain by saying it's sets of numbers, and the-the rule would be, p takes x to the set of all possible square roots, of x, something like that. So that-that would be a change in, changing the codomain, and, also the rule.

As seen above, Professor B provided two different suggestions to recover a function. The first was to modify the rule so to clearly map every element x of the domain to either its positive square root or its negative square root (by making a choice at every element x of the domain). They exhibited coordination by attending to the domain and codomain and coordinating with the rule so that the new rule they chose resulted in an element of the codomain for every element in the domain. Their second suggestion was to change the codomain to be sets of numbers rather than just $\mathbb{R}$. Thus, p would map into the power set of $\mathbb{R}$ and would take x to the set $\{\sqrt{x}, -\sqrt{x}\}$. In recovering an example in this way, Professor B again coordinated the domain, the codomain, and the rule by working with a general element x of the domain, attending to what the rule does to that element (that is, maps it to $\pm\sqrt{x}$, or more precisely $\{\sqrt{x}, -\sqrt{x}\}$), and then checking that this new output was indeed an element of the new proposed codomain (hence their change that the codomain be sets of numbers in $\mathbb{R}$).

Professors A and E suggested similar fixes to Professor B—focused on changing the rule. For example, Professor A suggested,

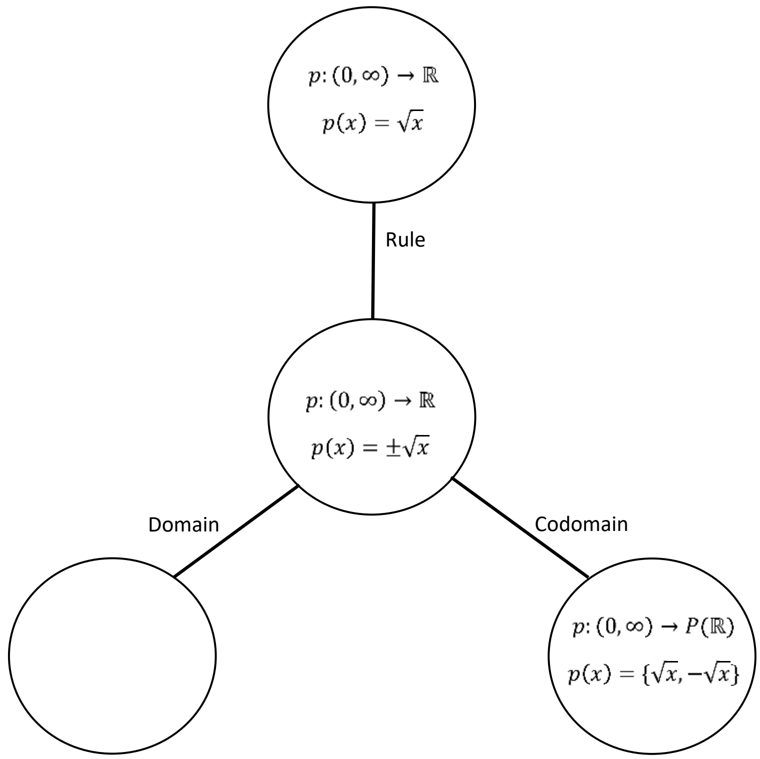
> [for p] to be a function, we need a unique output, which means we've got to make a choice. Um, and then once we make that choice, we've changed the rule of course, uh, but we're done. Um, we could have some complicated mechanism for choosing … the positive square root if you had an integer and the negative on if you didn't, or, something … like that.

However, they did not suggest changing the codomain. Professor E also made a change to the rule, as they explained, "just get rid of the minus." Additionally, they suggested changing the codomain,

> if I really wanted to keep plus or minus there then, the only way I can think of, changing it, is to, uhh, change the codomain to like, the power set of $\mathbb{R}$. And then you would send p of x to, to two elements.

In the process of recovering an example from this non-example, Professor A and E coordinated the domain, codomain, and rule, just like Professor B. For example, in order to know that changing the rule would "fix" the non-example, they necessarily had to be aware that an element of the domain would result in an element of the proposed codomain when plugged into their new rule; thus exhibiting coordination among these three aspects of function. In fixing the codomain, Professor E (just like Professor B) had to be aware about what the rule does to an element of this specific codomain and realize that this element would be landing in their proposed codomain. The following is a possible non-example pod from the mathematicians' recovery suggestions:

Figure 6: Non-example pod for *well-definedness multiple rules* (mathematicians)



4.5.1.b Everywhere-definedness no set non-example. Another non-example I presented to the mathematicians was the following *everywhere-definedness no set* non-example: $q: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ given by $q(a, b) = \frac{a}{b}$. In discussing how to recover a function from this non-example, Professor E said, "so, I would fix this by, changing the domain to, the second, the second $\mathbb{Z}$ to be, uhh, $\mathbb{Z}$ subtract zero … Would be how I would fix this." Later in the interview they came back to this non-example and explained,

> the first thing I did was I definitely said like, uh, 'What's the issue with this function?' You know, like, if you plug in two integers, you know, you are, like, given most integers, I am gonna get a unique, a unique rational function despite division … so, that was

definitely my first thought, and my second thought, umm, was uh, yeah, like, 'oh yeah wait, this is not actually defined for all integers,' you know? And actually, that definitely came second for me."

Notice that in thinking about how to "fix" this non-example, Professor E coordinated the domain, the codomain, and the rule by taking elements of the domain, plugging them into the formula, and deciding whether the outputs were contained in the specified codomain. So in order to "fix" the non-example or recover a function from it, Professor E had to coordinate these three aspects of function.

Professor A suggested a similar fix,

> So you just have to, umm, throw out the possibility that b is a denominator … so, yeah you modify the domain, to, get rid of zero on the bottom … you just have to fix the domain.

Notice that they coordinated the domain, the codomain, and the rule by identifying b equal to zero—or more specifically pairs of the form $(a, 0)$—as domain elements that would cause issues when plugged into the rule because they would result in an output that is not defined in the codomain, $\mathbb{Q}$. Additionally, they encouraged coordination later in the interview with relation to this problem, "the map from … $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Q}$, where you have to throw out the zero. And yeah, you want to talk about that, and you want to talk about why you're throwing out the zero." Thus further suggesting that they were indeed coordinating and also that they believe coordination is important.

Lastly, Professor B also suggested a similar approach to fixing this non-example, namely changing the domain. As they explained,

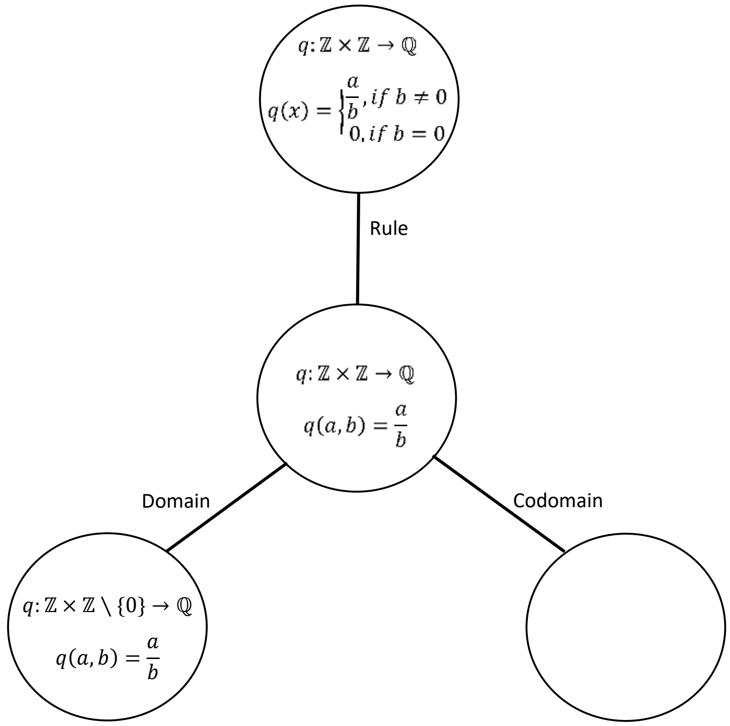
> yeah, so that gets back to the pairs of integers saying. So, um, okay, so a cross b, alright, so the only problem with that is the b not equal to zero. That's it, so, you just have to fix the domain by saying that b has to not be zero. And then it makes sense … the major problem is you, you have to, you have to not divide by zero.

They also suggested a second "fix" for this non-example. Their second fix focused on modifying the rule rather than changing the domain associated to this specific non-example. They mentioned,

> you could leave the domain the same, and then define, and then change the rule when b equals zero. So you could just say, you make it a split thing. You say, uh, it's a over b if b is not equal to zero, and then it's equal to some other value when b, um, is equal to zero. So … if you wanted to extend it to all of $\mathbb{Z}$ cross $\mathbb{Z}$, you'd have to define the rule, for the, differently for the case when b is equal to zero.

Professor B displayed coordination of the domain, the codomain, and the rule in both of their suggested "fixes" to this non-example of a function. In their first suggestion, Professor B first noticed that elements of the form $(a, 0)$ in $\mathbb{Z} \times \mathbb{Z}$ (that is, when $b = 0$) causes issues when plugged in to the rule since the output is not an element of $\mathbb{Q}$. Thus showing coordination between these three important aspects of function. Similarly, in their second suggestion they identified elements where b equal to zero caused issues when coordinating with the rule and proposed codomain. However, they decided to modify the rule this time, showing further coordination since they had to mentally check that their modification did indeed result in an example of a function. As with the previous non-example, a non-example pod from the mathematicians' recovery suggestions is displayed here:

Figure 7: Non-example pod for *everywhere-definedness no set* (mathematicians)



4.5.2 Students' recovering activity. In what follows, I present student work related to explicitly recovering examples from non-examples of function. As previously noted, students tend to have experience from secondary mathematics with two out of the four categories of non-examples of function identified in Paper 1, namely *well-definedness multiple rules* and *everywhere-definedness no set* non-examples. However, *well-definedness equivalence/representation* and *everywhere-definedness some set* non-examples are relatively new for students. As argued in Paper 1, these two categories of non-examples are crucial in productively reasoning about functions in an abstract algebra context. Thus in this paper, I focus on students recovering examples of function from non-examples in these two categories, specifically one non-example from each category.

4.5.2.a Well-definedness equivalence/representation non-example. I presented students with the non-example $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$ and asked them to attempt to recover a function from it to allow me to explore their thinking related to coordination. I focus here on two different students' recovery suggestions and what these indicate about their understanding of function. In response to this task, Student A's suggestion was to expand the domain. They explained,

> I do not accept this as a good fundamental rule. So I-I-I cannot think of a way to salvage it that would stay with a plus b. And for that weird reason, right, like if I have, I cannot think of a way to make one half and three sixths have the same answer if I'm going to do one plus two and three plus six … So, I think, I could think of expanding my domain … I want to expand it. I want to do like, $\mathbb{Z}$ cross $\mathbb{Z}$.

They continued by explaining why their proposed "fix" would work,

> So I could take an a and a b pair and then map it to a plus b … Would that be a function? I don't know … What, what do we have? So now I'm, now I'm thinking about, uh, a double pair, or not a double pair, a single pair, look a and b. There, they go, I got a pair, and so any pair that I choose, one and two, would map to one plus two. But <audio cuts off but they are writing $(3, 6) \mapsto 3 + 6$ down> that's a different pair. So this is fine, yes. Yes, this is a function … And why, if I choose some a' and a b', that's the same as a and b. So, if I choose a pair that looks the same … then I'm gonna get a' plus b', that's gonna be a plus b … $\mathbb{Z}$ doesn't have any weird things … so if a' comma b' i- so if this equality is true, well that's gonna mean that a equals a' and that b equals b'. Because $\mathbb{Z} \times \mathbb{Z}$ doesn't say anything about a and b having some sort of relationship with each other.

Notice Student A's mathematical activity as they recovered this example by expanding the domain. They identified the elements of the domain that were problematic due to equivalent representations, in particular $\frac{1}{2}$ and $\frac{3}{6}$. To “fix” the fact that the rule did not yield the same answer for both of these elements, they modified the domain and changed it from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$. They exhibited coordination when recovering this example by focusing on the elements of the domain that they had considered problematic before, now written as $(1, 2)$ and $(3, 6)$, and figuring out the outputs that the rule assigns to these two elements and their role in the codomain. In particular, they knew that in their new domain the elements $(1, 2)$ and $(3, 6)$ are no longer equivalent and thus the fact that they result in different elements of the proposed codomain is fine. Therefore, convincing themselves that this was indeed an example of a function.

Student D suggested a different approach where they changed the rule instead of changing the domain, as they explained,

> if I had for example, uh, the equivalence class of one half, uh, other elements, two fourths, three sixths, and so on. Now all of these have, uh, wait if you add them, if you, uh apply ϕ of them … this one is three, this one is uh, six, this one is nine … If I redefine, I think I, I’m not sure if this will work. I’m, but I’m almost positive, uh, that this will work. Redefine ϕ to send a over b, uh, a plus b over the greatest common factor of a and b. So now, I’ve taken, uh, an element in the, any element in the equivalence class, um, so every element in the equivalence class is a, is a number, you know, one over, one a over two a or whatever … um, where a is the greatest common factor.

Notice that Student D is trying to fix the issue of equivalent elements mapping to different things by working with the greatest common factor of the two integers that form the fraction. They continued,

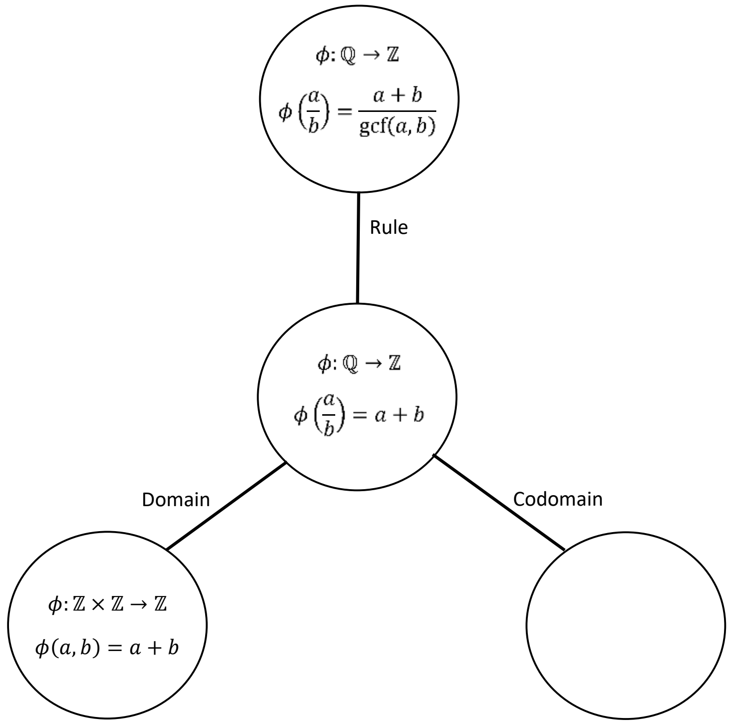
> Oh, ok, this is very confusing that I used the letter a, this is terrible … Um, greatest common factor is what that is … Let’s call that greatest common factor n. Then a over b gets sent to, uh, a plus b over n, right? […] But a plus b is equal to uh, c n plus d n over

> n … where the greatest common factor of c and d is equal to one … If that was not true, then, uh, that would be something that I could also factor out … And then that would be a part of the n … Uh, this is n times c plus d over n, um, which is c plus d and the greatest common factor of c and d is one [This is] a unique output. So we're just dividing out by the ambiguity here … I'm allowing the ambiguity but then I'm dividing out by it afterward.

While this is a different "fix" for this non-example, notice that their coordinated way of understanding this problem as a result of the recovery process is similar to that of Student A. Student D here recovered an example by modifying the rule rather than the domain.[24] However, just like Student A, they identified the elements of the domain that were problematic due to equivalent representations, in particular $1/2$, $2/4$, $3/6$, etcetera. Then coordinated with the rule to point out that these equivalent elements get assigned to different elements in the suggested codomain. To "fix" this issue, they modified the rule so that it would instead take equivalent elements of the domain and yield the same answer for all of these elements. In recovering this example, Student D attended to the domain (in particular the role of equivalence in $\mathbb{Q}$) and clearly coordinated with the rule and codomain in checking that their new rule indeed fixed the issue of assigning distinct outputs in the codomain to equivalent domain elements. By dividing out by the greatest common factor of a and b, they ensured that equivalent elements in $\mathbb{Q}$ were mapped by their new rule to the same codomain element. The following is a non-example pod representing the two proposed "fixes" explored in this subsection for this *well-definedness equivalence/representation* non-example:

[24] In a different "fix" they suggested, they focused on modifying the domain just like Student A but in a different way.

Figure 8: Non-example pod for *well-definedness equivalence/representation* (students)



4.5.2.b Everywhere-definedness some set non-example. I asked students if they could try to recover a function from the non-example $h: \mathbb{Z} \to \mathbb{N}$ given by $h(x) = x^3$. I discuss here three students' recovery suggestions to explore the role of coordination in their mathematical activity. Student B's suggestion is to modify the domain,

> So minus one gives us minus one, which does not lie in the range. Uh, so if we restrict the map to, um, things which are positive or n- non negative, or, well, do you include zero in your natural numbers? Anyway, I don't. So for me, natural numbers is ... are one, two, three … Okay. Natural numbers, for us, is just gonna be one, two and blah, blah, blah … So if we restrict to our positive, uh, integers then we are going to get, um, you know,

> we're- we're going to get a relation and it is going to be a function … And here, I'm making a restriction on the domain.

Student B recovered a function from this non-example by changing the domain. They demonstrated coordination by identifying specific domain elements, namely the negative integers, which might cause issues when plugged into the rule (as they mentioned, zero is also an issue if the convention being used for the natural numbers is that they do not include zero). They coordinated with the rule to figure out what was happening with the negative numbers. In particular, the negative numbers get sent to negative numbers and thus the outputs are not contained in the codomain since $\mathbb{N}$ does not contain negative numbers. This shows coordination with the codomain. To fix this issue, Student B thought of getting rid of the negative numbers in the domain, displaying coordination with the domain again. They chose to restrict the domain to be just positive numbers. In this way, the negative numbers are no longer an issue since they no longer get mapped since they are not part of the domain anymore. This way of "fixing" the non-example to recover a function from it shows how students necessarily have to coordinated to be able to successfully recover an example of function.

Student C also correctly identified the reason this was a non-example of a function but focused on making modifications to the codomain instead,

> If we made this go from $\mathbb{Z}$ to $\mathbb{R}$, that would be a function, $\mathbb{Z}$ to $\mathbb{R}$, x goes from x cubed. That would be a function … The issue before was, when we put in negative values here, so negative one for x, we're getting negative one cubed, which is negative one, and that's not a natural number. Here, we fix that with making it real numbers … We can put negative and positives in there … We can make it from $\mathbb{Z}$ to $\mathbb{Z}$ as well. If we put in an integer, we're going to get integers back, so it doesn't even have to be real numbers for the range.

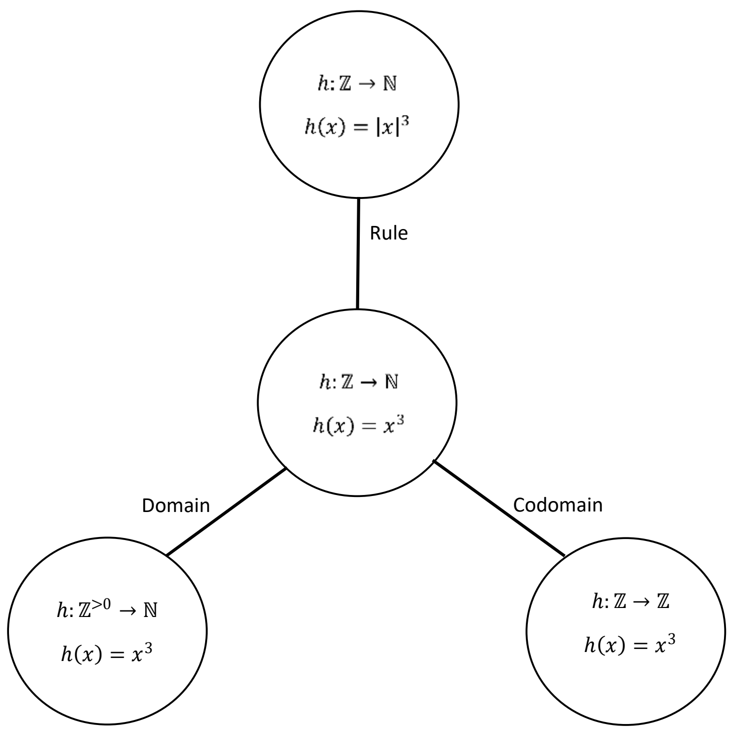
Notice that in their mathematical activity they coordinated the domain, the rule, and the codomain. They began by noticing that negative numbers caused issues. As they explained, negative one gets mapped to negative one by the rule but “that’s not a natural number.” To fix this issue, they expanded their codomain to $\mathbb{R}$ as a way to make sure the outputs (that is, negative numbers) were included in their suggested codomain. Their fixed focused on making sure that they could plug in negative and positive numbers. After, they were able to narrow down their codomain a little since coordination allowed them to see that if they plug in any integer into the rule, they get an integer back. Thus they did not need the real numbers, and $\mathbb{Z}$ would suffice. Therefore, they demonstrated coordination in their mathematical activity.

Lastly, Student E focused on modifying the rule which is different from the two modifications discussed above in which students changed the domain or codomain. Student E explained,

> So the issue here is that negatives don’t exist in the counting numbers, or the natural numbers, right? So if $\mathbb{Z}$ does contain all of those negatives, so your issue is, you have nowhere to go for all the negative integers … You could also say the absolute value of x cubed … This is just going to make it whatever it was positive. So if your x was larger than zero, then it’s just mapping to x cubed and if your x was smaller than zero, it’s just mapping to negative x cubed and if your x is zero then, actually I’ll just put that under there … Taking the absolute value of it. Uhh, another way to explain it is you’re just taking the magnitude of whatever that answer was and you’re just giving the magnitude.

Student E’s strategy for recovering an example here was to slightly modify the rule. In modifying the rule to define a function, they demonstrated a coordinated way of understanding the problem. First, they identified what the problematic elements were, namely the negative integers (note that they include 0 in their convention of the natural numbers). Then they coordinated with the rule to figure out exactly what the issue was; that is, the rule assigns negative numbers to negative numbers and thus the outputs are not contained in the codomain since $\mathbb{N}$ does not contain negative numbers, which shows coordination with the codomain as well. They decided that the absolute value fixes that issue because it converts negative numbers into positive numbers, thus every negative element would be cubed and the answer converted to a positive number so that it lies in $\mathbb{N}$. Hence making this a function. Figure 9 show a non-example pod created from the three proposed “fixes” discussed in this subsection for this *everywhere-definedness some set* non-example.

Figure 9: Non-example pod for *everywhere-definedness some set* (students)



4.6 Discussion

In this research-to-practice paper I provided a structure for how examples, and in particular non-examples, of function in abstract algebra can be organized. This framework provides insight into the kinds of tasks that students are expected to reason about in abstract algebra courses. Instructors can use the framework when designing their lessons to ensure that the collection of non-examples they present to students is varied. Students tend to generally have experience with *well-definedness multiple rules* and *everywhere-definedness no set* non-examples from secondary mathematics. However, they generally lack experience with non-examples in the *well-definedness equivalence/representation* and *everywhere-definedness some set* categories. These last two subcategories are critical for reasoning about functions in more advanced undergraduate mathematics courses like abstract algebra. Therefore, instructors should present students non-examples from each of the four categories discussed in section 4.3.3. I also outlined what is involved in a coordinated view of function—in particular, it requires viewing functions in a way that involves explicit attention to the relationship between the domain, the codomain, and the rule—and provided a description of criteria that instructors can use to determine whether a student is exhibiting a coordinated view of function. Lastly, I proposed the activity of recovering an example from a non-example and laid out examples of tasks that illustrate this activity. These tasks can help instructors have a clearer image of how they might support students in developing *a coordinated view of function*. Furthermore, the results of this recent research project detailed here address what is entailed in reasoning productively about well-definedness and everywhere-definedness which are the two characteristic properties of function. I want to issue a call for instructors to devote more attention to everywhere-definedness and well-definedness in abstract algebra courses. The recovering examples from non-examples tasks provided in this paper are some of the tools that abstract algebra instructors can use to help students attend to these two defining characteristics of the function.

CHAPTER V

CONCLUSION

4.1 Contributions

The goal of this dissertation was to explore students' reasoning about the function concept in abstract algebra and, in particular, the characteristic properties of well-definedness and everywhere-definedness. Paper 1 focused on the identifying the types of (non-)examples that exist in the instructional example space and the key elements of the structure of the instructional example space. I elaborated the two key aspects of the function concept—well-definedness and everywhere-definedness—into four categories: *well-definedness equivalence/representation*, *well-definedness multiple rules*, *everywhere-definedness no set*, and *everywhere-definedness some set*. These elaborations are important because they point out aspects of the function concept that students must attend to in order to successfully reason about functions in abstract algebra. Therefore, they can inform instructional design and selection of non-examples for abstract algebra lessons and, particularly important for my objectives, instructional design for task-based clinical interviews, conceptual analyses, and hypothetical learning trajectories.

In Paper 2, I defined *a coordinated way of understanding function* in abstract algebra and illustrated how it can support students in reasoning productively about functions. The results from Paper 1 were used for the purpose of developing tasks that would enable me to infer students' ways of understanding particular function tasks. This coordinated way of understanding functions is an important contribution because, as noted in different sections of this dissertation, the function literature previously lacked detailed investigations into what is entailed in reasoning productively about well-definedness and everywhere-definedness in general, and in advanced undergraduate mathematics courses in particular. A remaining question is *how* students might come to develop a coordinated way of understanding functions. To this end, in Paper 3 I introduced an activity that focuses on recovering examples from non-examples of function and laid out examples of tasks that illustrate the activity. I argue that this activity has the potential to enable students to develop such a way of understanding. Additionally, this activity can be used by abstract algebra instructors and researchers to help students attend to well-definedness and everywhere-definedness.

4.2 Future Research

One interesting question not addressed in this dissertation is “what are possible learning trajectories that might enable students to develop a coordinated way of understanding function?” While I conjectured that the task of recovering an example from a non-example could be potentially useful in helping students develop such a coordinated way of understanding functions in abstract algebra, these tasks have not been used with students with a focus on exploring the development of ways of understanding. To address this, I plan to conduct teaching experiments to develop, test, and refine a hypothetical learning trajectory that outlines how students might be able to develop *a coordinated understanding of function*.

While this dissertation takes a necessary first step in outlining what a productive way of thinking about functions in abstract algebra might look by establishing what is involved in a productive way of understanding functions, it does not characterize such a productive way of thinking. Thus, in future research I plan to examine students’ ways of thinking about function by investigating their reasoning in varied contexts.

Lastly, in this study I focused on exploring what is involved in determining whether or not a proposed correspondence is a function in abstract algebra. However, I did not address specific kinds of function typically encountered in abstract algebra courses like binary operations, homomorphisms, and isomorphisms. In future research, I plan to explore the implications of the four categories I identified here for the instructional example space of these kinds of functions.

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APPENDICES

Appendix A: Tasks from student selection survey

1) Consider $f: \mathbb{Q} \to \mathbb{Z}$ defined by $f\left(\frac{a}{b}\right) = a + b$.
   a) Does this define a function?
   b) How do you know? Explain.

2) Consider the ϕ which maps $\left(\frac{a}{b}, \frac{c}{d}\right)$ to $\frac{a+c}{b+d}$, i.e., $\phi\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{a+c}{b+d}$.
   a) Does this define a function?
   b) How do you know? Explain.

3) Consider $f: \mathbb{Q} \to \mathbb{Q}$ defined by $f\left(\frac{a}{b}\right) = \frac{2a+b}{2b}$.
   a) Does this define a function on $\mathbb{Q}$?
   b) How do you know? Explain.

4) Consider ϕ defined on the integers as follows:

   $$\phi(a) = \text{remainder when } a \text{ is divided by } 3.$$

   a) Does this define a function on $\mathbb{Z}$?
   b) Explain why or why not.

5) Consider g defined on $\{0, 1, 2\}$ as follows:

   $$g(0) = \text{an integer that has remainder 0 when divided by 3,}$$
   $$g(1) = \text{an integer that has remainder 1 when divided by 3,}$$
   $$g(2) = \text{an integer that has remainder 2 when divided by 3.}$$

   a) Does this define a function?
   b) How do you know? Explain.

VITA

Rosaura Uscanga Lomeli

Candidate for the Degree of

Doctor of Philosophy

Dissertation: AN INVESTIGATION OF STUDENTS' REASONING WITH EXAMPLES AND NON-EXAMPLES OF FUNCTION IN ABSTRACT ALGEBRA

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in July, 2021.

Completed the requirements for the Master of Science in Mathematics at Oklahoma State University, Stillwater, Oklahoma in 2015.

Completed the requirements for the Bachelor of Science in Mathematics at The University of Texas at Arlington, Arlington, Texas in 2012.

Professional Memberships:

Mathematical Association of America
The American Mathematical Society
Association for Women in Mathematics