INVESTIGATION INTO THE DEVELOPMENT OF A QUANTITATIVELY BASED SUMMATION CONCEPTION OF THE DEFINITE INTEGRAL

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Abstract: Research has shown the majority of students who have completed a university calculus course reason about the definite integral primarily in terms of prototypical imagery or in purely algorithmic and non-quantitative ways. This dissertation draws on the framework of Emergent Quantitative Models to identify how calculus students might develop a quantitative understanding of definite integrals which supports them in modeling activity when the differential from is not a multiplicative product between a rate of change and a differential quantity (e.g. gravitational force). The Emergent Quantitative Models framework describes three quantitative schemes basic, local, and global models which students draw on when reasoning about definite integral tasks. Basic models are quantitative relationships that apply to a contextual situation if the quantities involved are constant values, a local model is a localized version of the basic model applied to a subregion of the overall context, and a global model is derived from an accumulation process applied to the local model, whose underlying quantitative reasoning is encoded in the differential form.

To characterize the mental activity that supports the productive development of a quantitative understanding of definite integrals I outlined a conceptual analysis, designed a hypothetical learning trajectory, and conducted an eight-week teaching experiment with five freshman calculus students to test and refine my hypotheses. The results of this dissertation research provide insight into foundational constructs which position students' development of primitive basic, local, and global models and outlines the mental activity which engenders the progressive coevolution of those models towards a quantitative understanding of the definite integral. I offer two constructs to the Emergent Quantitative Models framework which were identified as productive in such a development: a gross basic model and a generalized local model. Additionally, I provide a refined hypothetical learning trajectory and pedagogical implications towards the development of curriculum which supports a quantitative understanding of definite integrals.

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CHAPTER I

INTRODUCTION

Research has shown the majority of students primarily reason about the definite integral in terms of either prototypical imagery (e.g. area beneath a function, above an x-axis, and between two boundary lines at $x = a$ and $x = b$), or in purely algorithmic and non-quantitative ways (e.g. antiderivative). While these ways of reasoning may provide the means to complete the definite integral tasks typically found on US Calculus I final exams (Tallman, Carlson, Bressoud, & Pearson, 2016), they do not prepare students for applying definite integrals in context. Over the past two decades, a growing body of research has identified that connecting definite integrals to Riemann sums and quantitative reasoning provides students with robust ways to reason about contextual tasks. These studies have primarily focused on constructs in which the differential form represents a multiplicative product between a rate of change and a differential quantity, what I refer to as a Riemann product. However, limiting students' definite integral reasoning to Riemann product structures has been identified as potentially inadequate for a successful transition to other STEM coursework in which the integrand does not naturally decompose into a rate of change or density, such as electrostatic(Meredith & Marrongelle, 2008; Oehrtman, 2015). This study aims to contribute to the mathematics education field by offering insight into how students can develop an understanding of force definite integration which supports the quantitative reasoning necessary to productively engage in such tasks.

Using constructivism as the epistemological foundation, I drew on Simmons and Oehrtman's (2017, 2019) characterization of the Emergent Quantitative Model's framework to engage participants in an eight-week teaching experiment (L. P. Steffe & Thompson, 2000) focused on influencing and characterizing their emerging schemes for integration consistent with what I refer to as a Quantitatively Based Summation conception of integration. Specifically, I was interested in answering the following questions:

RQ1.) How might students develop a quantitative understanding of definite integration in a Calculus I course.

RQ2.) What are the limitations and affordances of a quantitative understanding of definite integration? In particular, how does a quantitative understanding of definite integration impact Calculus I students' ability to reason about physics-based integration tasks in which the varying quantity is not a rate of change or density function?

CHAPTER II

REVIEW OF LITERATURE

In the following sections, we will explore some of the most relevant contributions to the literature on definite integrals in mathematics and physics education research.

Quantitative Reasoning

Thompson's (1990, 2011) theory of quantitative reasoning is the study of mental actions involved in the conceptualization of quantities and relationships between quantities. Quantities are measurable qualities of objects which are formed by individuals engaging in a dialectic between an object, an attribute of that object which is of interest, and a way in which to measure that attribute to solve a problem (e.g. height as tallness of a person) (Thompson, 2012, p. 143). Quantities are quantified through a process of "settling what it means to measure a quantity, what one measures to do so, and what a measure means after getting one" (Thompson, 2011, p. 38). Note that, although the concept of number does emerge from reasoning about the measure of a quantity, quantity is not equivalent to number (Thompson & Carlson, 2017, p. 425). Thompson refers to a quantity that has been measured as a value.

The act of creating a new quantity from two or more already-conceived quantities is referred to as a quantitative operation. Quantitative operations can be described as structures among quantities: the original quantities, the new quantity, and the relationship between them. Trying to reason with these structures and communicate their meanings is highly demanding for students cognitively (Smith & Thompson, 2007; Thompson, 1995), and is the primary focus of most quantitative reasoning research. Research shows that reasoning quantitatively is nontrivial, does not necessarily develop quickly, is non-canonical, and that there are varying levels of complexity in the types of quantitative structures students must construct (Smith & Thompson, 2007; White Brahmia, 2019).

It must be emphasized that just as quantities are not equivalent to numbers, quantitative operations are not equivalent to numerical operations (such as addition, subtraction, multiplication, and division). Quantities do not need to be calculated to be used productively:

> You employ quantitative operations at the first moment of thinking of a situation quantitatively. Quantitative operations are the conceptual operations one uses to imagine a situation and to reason about a situation— often independently of any numerical calculations. (Thompson, 1995)

Students (and teachers) often conflate quantitative and numerical operations due, in part, to their shared symbolic notation (Thompson, 1994). For example, a typical way in which the concept of slope is communicated in algebra courses in the US is as a change in height (Δy) divided by a change in width (Δx). Often the take-away from this treatment is that there is no relationship between slope, Δx, and Δy outside of "slope is defined to be," and results in a conception of slope as a numerical operation between two quantities. On the other hand, a quantitative operation between these three quantities could be to conceive of slope as a multiplicative comparison between Δy and Δx. That is, a slope's measure, m, represents how many times greater Δy is than Δx ($\Delta y = m \cdot \Delta x$). When reasoning quantitatively, it is the characterization and relationships between quantities that are of primary focus, not computations. Numerical values and operations come as a natural byproduct (Thompson, 1993).

An inability to separate a quantity from its measure hinders students' ability to reason about quantitative relationships, especially when dealing with quantities of unknown value (Thompson, 1988). A focus on numerical (or symbolic) operations is not intrinsically harmful but should not be the focus of instruction until the quantitative operations that they represent are firmly established. Thompson (1995) laments, "too often [we] let students use numbers and operations meaninglessly, to the point where meaningless use of numbers and operations becomes their habitual activity." Smith and Thompson (2007) reinforce this sentiment, "when students do not attend to quantities and relationships, their problem-solving quickly becomes a matter of ungrounded debate about choosing numbers and operations." By introducing symbolic operations before students are ready, we are turning their attention away from conceptual meaning to instead learning to play the "school-math game" where math is just about producing the answer teachers want to see (Thompson, 1995). Ellis (2007) showed that students' generalizing activity was enhanced by students focusing on quantitative relationships rather than numeric patterns. For example, there is a distinction between a recognition that rate of change is the proportional relationship between two quantities and a view that every time you increase x by 1 unit, y increases by y' units. If the goal of mathematics is to understand structural relationships between mathematical objects which numerical operations can represent, then quantitative reasoning serves as an important base from which to extract that structure (Ellis, 2007; Mayes, 2011; Smith & Thompson, 2007).

According to Thompson, co-variation is characterized "in terms of conceptualizing individual quantities' values as varying and then conceptualizing two or more quantities as varying simultaneously" (Thompson & Carlson, 2017). That is, covariational reasoning is reasoning quantitatively in dynamic situations. Just as quantitative reasoning is classified as difficult for students, covariational reasoning is non-trivial and a great deal of research has been conducted to identify the productive ways in which students can reason about dynamic quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Oehrtman, & Engelke, 2010; Moore, 2013, 2014; Moore & Carlson, 2012; Moore, Silverman, Paoletti, & Laforest, 2014; Oehrtman, Carlson, & Thompson, 2008).

The theoretical underpinning of quantitative reasoning is that of Piaget's (1972) genetic epistemology. From this perspective, the interpretations of our experiences, including quantities, are not true reflections of some ontological reality but are schemes that are constructed and reside within the minds of individuals (Thompson, 1994, 2011, 2013). Ellis (2007) explains,

> Quantitative operations originate in actions, or activities of the mind (Piaget, 1967). As a learner interiorizes actions, creating mental operations, these operations allow one to comprehend situations representationally. They enable the learner to draw inferences, for example, about relationships that may not be present in the situation itself. If all mental actions are tied to experience, then any meaningful learning in mathematics must be grounded in quantitative referents. (p. 441)

Not attending to the individualized nature of interpretation can become problematic when trying to engage students in activities aimed at quantitative understandings. For instance, Thompson (1996) gave an example of a seventh-grade teacher providing the following prompt for their students to create a discussion as to when rounding is appropriate:

> A grocer buys Sara Lee cakes from his distributor in packages of 8 cakes per package. Each package costs $4.25. The grocer figures he needs 275 cakes for the next week. How much money should he plan on paying for cakes?

One student in this class fundamentally could not understand the question being asked. This was because, from his perspective, going to the grocery store simply entailed picking out the items you need and then taking them to the cashier. This student had no conception for restocking shelves, and because, from his experience, the cashier tells you how much you owe there was also no need to calculate totals yourself. From this student's point of view, the situation was non-problematic and there was no object in need of quantification. This case exemplifies that how an individual conceives of an object to be quantified is just as important as how they conceive of measuring it (Thompson, 2012). Thompson takes the stance that any "[framework] that puts meaning outside of individuals is less helpful for purposes of instructional and curricular design, teacher preparation, and teacher development than [one] that puts meaning within individuals" (Thompson, 2013). Although approaching quantitative reasoning through Piaget's genetic epistemology takes seriously the contention that everyone's interpretation of situations and quantities are unique, this does not imply that students cannot move towards understandings consistent with learning goals. The point is that instructors should be sensitive to these differing perspectives in their classrooms and anticipate experiences that facilitate students' mental actions towards those objectives (Moore, Carlson, & Oehrtman, 2009).

An ever-increasing demand for the ability to understand highly sophisticated quantitative problems has highlighted the demand for reasoning quantitatively to be a primary focus of mathematics education (Hatfield, 2011). However, while some may interpret this as a call to include extremely complex application tasks in curriculum, quantitatively rich problems should not be conflated with real-world or applied problems. Tasks are quantitatively rich based on the type of engagement in which the students participate with the reasoning, not necessarily on the difficulty or number of quantifications required. What is important is that students actively attend to identifying and characterizing the relationships between quantities while engaging in tasks designed to promote the extraction of mathematical structure (Ellis, 2007; Eric, Amy, Torrey, & Zekiye, 2014; Moore et al., 2014)

A great deal of quantitative reasoning research is based within primary education, particularly arithmetic and elementary algebra. However, the theory has also been utilized in secondary and tertiary education including geometry (Thompson, 1999), precalculus (Carlson et al., 2002; Carlson et al., 2010; Moore & Carlson, 2012; Moore et al., 2009; Oehrtman et al., 2008; Thompson, 1994), secondary teacher preparation (Moore et al., 2014; L. Steffe & Izsak, 2002; Tallman & Frank, 2020; Thompson, Carlson, & Silverman, 2007), trigonometry (Moore, 2013, 2014), and calculus (Bajracharya & Thompson, 2014; Mkhatshwa, 2019a, 2019b; Mkhatshwa & Doerr, 2018; Thompson & Silverman, 2008). According to Moore, a primary benefit of quantitative reasoning at these higher levels of education is that "it enables exploring mathematical ideas in non-canonical representations (e.g., input on the vertical axis) and in a variety of settings (e.g. polar coordinate systems)" (Moore et al., 2014).

Symbolic Forms

> The use of formal expressions in physics is not just a matter of the rigorous and routinized application of principles, followed by the formal manipulation of expressions to obtain an answer. Rather, successful students learn to understand what equations say in a fundamental sense; they have a feel for expressions, and this understanding guides their work. (Sherin, 2001, p. 482)

Working with equations is an integral part of students' interactions with mathematics and physics. Considering this, researchers have spent decades seeking to characterize how experts interpret various equations and how students learn to work with those equations throughout their school curriculum. One of these studies, Bruce Sherin's *How Students Understand Physics Equations* (2001), introduced the symbolic forms knowledge system which linked the naive (or intuitive) physics knowledge literature with college students' creation and interpretations of physics equations.

The naive physics knowledge literature is centered on the idea that students of physics, even at a young age, are not blank slates with which we can directly imprint correct 'knowledge' through instruction. Instead, "students enter physics instruction with quite a lot of knowledge about the physical world, and that this knowledge has a strong impact on their learning of formal physics" (Sherin, 2001, p. 484). While much of the early research characterized this prior informal learning as simply "preconceptions" or "misconceptions" that could be easily corrected with valid principles (e.g. Clement, 1993), diSessa argued that there is a *sense of mechanism,* "a sense of how things work, what sorts of events are necessary, likely possible, or impossible" (1993, p. 106), which remains resilient to change and which should be accounted for in a theory of learning. Other researchers acknowledged the importance of naive knowledge (e.g. McCloskey, 1983) and hypothesize that these "preconceptions" constitute a fully formed (though possibly incorrect) theory of the physical world. However, diSessa's view was that "intuitive physics is a fragmented collection of ideas, loosely connected and reinforcing, having none of the commitment or systematicity that one attributes to theories" (diSessa, 1993, p. 50). He called his framework "knowledge in pieces," which focused on his classification of dozens of *phenomenological primitives* (p-prims) that are "simple abstractions from common experiences that are taken as relatively primitive in the sense that they generally need no explanation; they simply happen" (p. 52). These p-prims vary in levels of connectivity and are thus activated in varying circumstances; that is to say, just because someone holds a certain conception (or p-prim) does not mean that it is evident in every situation. Using this knowledge system as a base, Sherin sought to identify how naive knowledge impacted students' formulation of physics equations.

Like the "knowledge in pieces" framework, Sherin was not necessarily trying to identify how students develop the ability to construct equations. Instead, recognizing that intuitive knowledge plays a role in development, Sherin wanted to uncover the underlying meanings elicited by students when constructing unfamiliar equations, a sort of naive equation knowledge, and how those meanings impacted their constructions. Through his analysis of third-semester physics students, Sherin identified *symbolic forms* as pieces of knowledge that can be activated by students in equation creation activity. A symbolic form is comprised of two parts: a symbol template and a conceptual schema. The symbol template is a sort of placeholder for symbols within an equation, while the conceptual schema are the underlying meanings associated with that particular arrangement of symbols. For example, Sherin identified a parts-of-a-whole symbolic form; the symbol template for which is represented by the pattern of symbols [$\square + \square + \square +$ ...]. For Sherin, "[t]he $\square$ refers to a term or group of symbols, typically a single symbol or a product of two or more factors. The brackets around the whole pattern indicate that the entity corresponding to the entire pattern is an element of the schema" (Sherin, 2001, p. 491). The conceptual schema for parts-of-a-whole is that each of the generic parts, $\square$, is a contributing factor to a whole entity. That is, if you were to remove one of the parts the entity would no longer be whole. It is important to note that while the symbol template does represent much of the observable output in students' finished work, the underlying meanings for the arrangement of symbols can be very distinct. To illustrate this point, I will expand on an additional symbolic form that looks similar to, but is distinct from, parts-of-a-whole. The symbolic form 'competing terms' has the template $\square \pm \square \pm \square$ ... [1] for which the underlying conceptual schema is that of influences in competition. Sherin notes that competing influences are often in association

1 Note that Sherin did not include brackets in this symbol template, denoting that you do not necessarily have to consider the whole entity at the time of activation. Instead he defined another symbolic form for identity ($x = [...]$; anything true on the right is true for the x) which could account for this. Sherin mentions that the identity form is rarely reflected in student language, but more implicit in their actions.

with directional quantities such as forces or energy in diagrams and that the term competition is used because the influences are fighting to "have their way." While the observable product of student conceptions (e.g. A+B) could have been developed under either conception, the meanings with which the students work have significant influence over how they develop and interpret the result. For instance, one could easily imagine a student with a reasonably productive image of vector addition speaking in terms of competing influences, one vector 'pulls' you in one direction, while the other 'pushes' you in a different direction. It is much less likely, however, that they would speak of vectors in terms of the parts-of-a-whole. How can each component vector be 'part' of the resultant vector when the magnitude of the resultant vector is likely smaller than that of the combined parts? In addition, the resultant vector often isn't even aligned with one, let alone both, of the directions of its 'components.' What is particularly important about this example is that it is not infeasible for students to naively attempt to add vectors in terms of parts-of-a-whole. Reasoning about addition in this way is not inherently wrong and is productive in many situations, such as computing Riemann sums. However, what some might consider a 'misunderstanding' about vector addition, is an activation of a specific symbolic form that happens to be unproductive in this particular circumstance.

In his 2001 work, Sherin introduced 21 symbolic forms that he arranged by cluster (Figure 1). Similar to diSessa's knowledge in pieces, Sherin built this framework to categorize small-grained, not necessarily connected nor hierarchical, schemata which stem from experience of the physical (and school-based) world noting that this was not anywhere near an exhaustive list. Sherin was also clear to indicate that this work did not claim to describe the genesis of symbolic forms. However, Sherin did claim that some symbolic forms are established through experiences in early math and physics classrooms, while others, such as balancing, may have developed from associated p-prims (Sherin, 2001, pp. 504-505). In addition, while Sherin's work was based in physics content and limited to relatively simple equation constructions, he did note that this framework could be extended to other areas; a call which was taken up by researchers across mathematical fields.

Symbolic Forms by Cluster

Cluster	*Symbolic form*	*Symbol pattern*
Competing terms cluster	Competing terms	$\square \pm \square \pm \square \ldots$
	Opposition	$\square - \square$
	Balancing	$\square = \square$
	Canceling	$0 = \square - \square$
Terms are amounts cluster	Parts-of-a-whole	$[\square + \square + \square \ldots]$
	Base ± change	$[\square \pm \Delta]$
	Whole – part	$[\square - \square]$
	Same amount	$\square = \square$
Dependence cluster	Dependence	$[\ldots x \ldots]$
	No dependence	$[\ldots]$
	Sole dependence	$[\ldots x \ldots]$
Coefficient cluster	Coefficient	$[x \square]$
	Scaling	$[n \square]$
Multiplication cluster	Intensive–extensive	$x \times y$
	Extensive–extensive	$x \times y$
Proportionality cluster	Prop+	$\left[\frac{\ldots x \ldots}{\ldots}\right]$
	Prop–	$\left[\frac{\ldots}{\ldots x \ldots}\right]$
	Ratio	$\left[\frac{x}{y}\right]$
	Canceling (b)	$\left[\frac{\ldots x \ldots}{\ldots x \ldots}\right]$
Other	Identity	$x = \ldots$
	Dying away	$\left[e^{-x \ldots}\right]$

Figure 1: Sherin's Symbolic Forms (Sherin, 2001, p. 506)

Now, some twenty years later, there are numerous examples of Sherin's symbolic forms as additions to the framework and tools for analysis from multiple disciplines and differing education levels. For example, Dorko and Speer (2015) identified two symbolic forms relating to units of measurement while examining how introductory calculus students think about area and volume problems, Rodriguez, Santos-Diaz, Bain, and Towns (2018) used symbolic forms as a basis for developing a construct called *graphical forms* to analyze students' reasoning about graphs in chemical kinetics, and two research groups have made headway in identifying symbolic forms in physics-based vector contexts (Dreyfus, Elby, Gupta, & Sohr, 2017; Schermerhorn & Thompson, 2017). While there is certainly diversity in the application and adaptation of symbolic forms, one area of mathematics and physics that stands out as being particularly dense in symbolic form research is integral and differential calculus (typically within physics contexts).

In 2008, Meredith and Marrongelle identified how two of Sherin's symbolic forms, dependence[2] and parts-of-a-whole, impacted students' perceived need for integration while working on electrostatic force problems. In particular, they observed students cued to integrate through a dependency relationship characterized the need to integrate due to the variation of the quantity x within the integrand. This resulted in what Meredith and Marrongelle deemed a "misapplied" symbolic form when students were engaged in modeling contexts of a more complicated nature. Specifically, Meredith and Marrongelle characterized the dependence symbolic form a "dead end" for contexts in which the

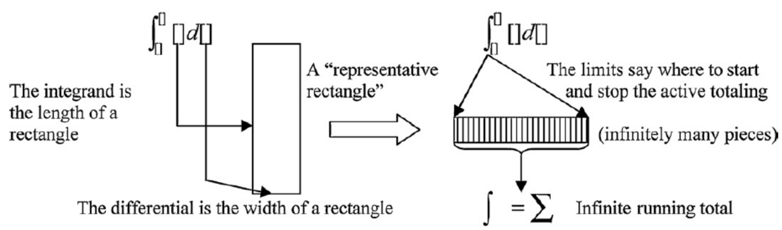
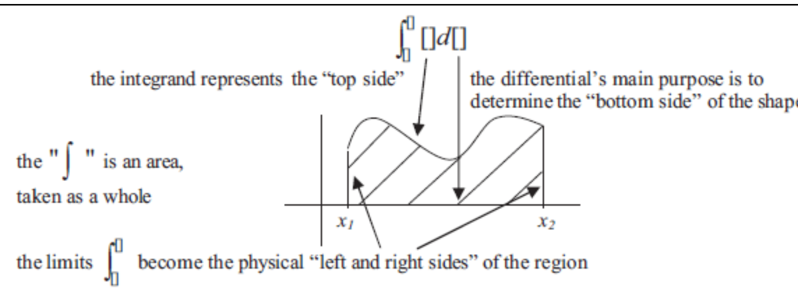
2 Dependence symbolic form: $[\ldots x \ldots]$; the whole depends on a quantity dependent upon an individual symbol (Sherin, 2001)

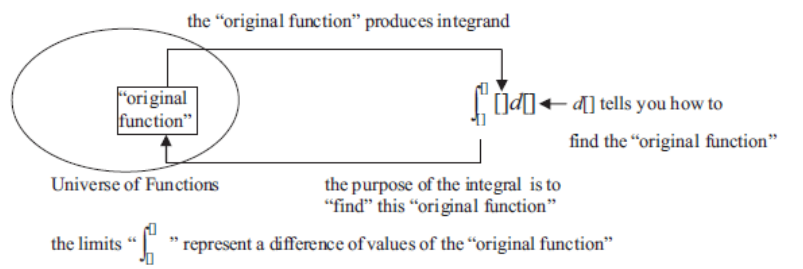
integrand does not represent a density or rate of change. This was a strong stance, considering Meredith and Marrongelle had earlier noted that mathematics education researchers previously identified coordinating a dependency relationship as "essential in understanding functions and integrals" (p. 574). However, Meredith and Marrongelle stressed the productivity of participants who viewed the differential form as a part-of-a-whole quantity. That is, students who viewed a definite integral as an accumulation of parts were more successful in writing integral expressions which accurately characterized more complicated differential form structures. They noted that even though only 7 out of 144 students were cued to integrate by the parts-of-a-whole symbolic form (compared to 53 of 144 for dependence), those students were all successful when presented with an integral task of computing the total electrostatic field due to a bar of charge. This finding supported Meredith and Marrongelle's assertion that definite integrals should be framed in terms of a parts-of-a-whole symbolic form.

Extending Sherin's symbolic forms to integral calculus, a mathematics education researcher, Steven Jones, set out to characterize numerous symbolic forms surrounding integration (2013, 2014, 2015a, 2015b). Notably, Jones identified three symbolic forms that were of particular consequence while analyzing pairs of students working on integral problems in mathematics and physics contexts: adding up pieces (later reframed as a Multiplicatively Based Summation conception), area and perimeter, and function matching which all share the same symbol template: $\int_{\square}^{\square} \square \, d\square$.

I begin by discussing the function matching symbolic form as it is the most straightforward and most often seen result of calculus courses (Bressoud, 2011). Function matching is closely linked with the anti-derivative process and within this symbolic form. The integrand represents the derivative of an "original function," while the differential serves as a dependency cue aiding students in identify how to compute the "original function." The limits of integration serve as a representation of a computation between two original values to calculate a difference (see Table 1). Even though the activity involved in performing these calculations may appear to be mindlessly procedural, Jones argues that there is persistent meaning behind the symbols for students which are consistent across groups and situations solidifying its place as a symbolic form.

Table 1: Jones' depiction of the conceptual schema for selected symbolic forms (from Jones, 2013, pp. 127, 129-130)

Symbolic Form	Conceptual Schema
Adding up pieces; Multiplicatively Based Summation conception	
Area and perimeter	

Function matching	

The area and perimeter symbolic form is also straight forward but instead of relying upon a link between the integral and an original function, it is highly geometric in nature. Here the conceptual schema for the entire integrand represents an area, namely the area bounded above by the curve of the integrand function, below by the *x*-axis, and on the left and right by vertical lines indicated by the limits of integration[3]. The most complex of the three symbolic forms for definite integrals discussed here is adding-up-pieces. This conceptual schema for adding-up-pieces draws upon the parts-of-a-whole symbolic form in which the entire integral represents a total amount obtained by the accumulation of the area of rectangles: the integrand represents the height of a rectangle and the differential its width. The limits of integration indicate the starting and stopping place of the totaling process. The complexity of this symbolic form comes in the form of the infinite addition. Jones noted,

3 For integrands that take on a negative value Jones notes that some students utilized a cognitive resource he called “facing the other way” to explain why it represents a negative area.

> in most instances where the students had activated the adding up pieces symbolic form, there was strong evidence that they viewed the rectangles as "infinitely thin" and the addition as happening over "infinitely many" rectangles…it seems clear that for these students, the adding up pieces symbolic form has embedded in it an inherent notion that the rectangles have already achieved the status of being infinitely thin and that the addition process requires an infinite summation over the infinitely many pieces[4] (Jones, 2013, pp. 126-127).

Jones observed that while every group in his study made productive use of the function matching and/or perimeter and area symbolic forms within pure mathematics contexts, students unable to adapt to an adding-up-pieces symbolic form when confronted with physics-based tasks were unproductive in finding a solution or providing meaning for their actions. Jones concluded that "it appears that the choice[5] of symbolic form activation may have either enhanced or inhibited the students' ability to work with integrals in applied physics problems" (p. 136). After subsequent analysis to identify what made this symbolic form more productive for physics tasks, Jones reframed adding-up-pieces to a Multiplicatively Based Summation conception of definite integrals to shift

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4 Jones also described a "miscompilation" of the adding up pieces symbolic form in which students conceived as the integral totaling pieces of the integrand and the result is multiplied by the variable of the differential (Jones, 2013, p. 135).

5 Note: I find the phrase "choice" to be highly suspect in this instance. I do not believe Sherin (or even Jones) would contend there is a conscious choice for which symbolic forms activate when presented a specific task. Certainly, if multiple symbolic forms were activated within a given context then an individual could make such a choice. However, it is also possible that a student possesses a symbolic form, has demonstrated proficiency with that symbolic form, but the symbolic form was not reactivated when presented a task for which that symbolic form might be advantageously applied.

focus to the multiplicative relationship between the integrand and differential, rather than necessarily one of the generalized rectangles.

While this was a significant step forward in the accounting for differing variations of parts-of-a-whole symbolic forms, the Multiplicatively Based Summation did not fully account for the situation that Meredith and Marrongelle described in which the integrand is not in the form of a density or rate. In light of this, Oehrtman gave evidence for a new symbolic form comprised of the symbol template $\int_{\square}^{\square}\boxed{}$. This symbolic form had a similar conceptual schema to that of Jones' multiplicatively based conception, but with the generality that the integrand differential relationship simply indicates a generalized piece of the whole quantity being totaled which could later take on the form of $[\;]d[\;]$. Further elaborated on by Simmons and Oehrtman (2017, 2019), this new symbolic form, a Quantitatively Based Summation conception of the definite integral, involved subsequent layers of quantitative reasoning which established the differential form as a representation of a part of a whole quantity which mirrors the quantitative relationship which would hold if the quantities involved were constant. The justification students provided for the invocation of this quantitative structure was, at a small enough scale, the variation of the quantities within the parts was 'nearly constant'. In addition, Simmons and Oehrtman (2018), described a (potentially problematic) symbolic form associated with the symbol template $\int_{\square}^{\square}\boxed{}\,d[\;]$ called an Integral as a Transformer. The underlying conceptual schema for this symbolic form involved an interpretation that the definite integral transforms a quantitative relationship that holds for constant quantities into an expression which allows the varying quantity to take on every value within the range described by the limits of integration. This was achieved by placing the quantitative relationship which holds for constant quantities directly into the integrand. Similar to the function matching symbolic form, the differential variable simply denotes the varying quantity. This new symbolic form is potentially problematic as it only produces correct solutions when the variation in question is a rate of change or density.

While mathematics education researchers took on the task of identifying symbolic forms for integrals, physics education researchers focused their efforts on the conceptual meanings for differentials. Investigating how introductory calculus-based physics engineering students connected their mathematical knowledge with physics concepts, Hu and Rebello (2013) observed three conceptual schemas for the symbol template of a differential ($d\square$) contained within integrals. The first, which they called differential as the variable of integration, describes the differential term as a variable of integration similar to Meredith and Marrongelle's dependency queue. The second symbolic form, differential as an operation, centers on a conceptual schema that the differential term indicates the operation of taking a derivative. This is somewhat similar to Jones' function mapping, however, Hu & Rebello were observing this in the context of the more general structure $\int_0^L dR$ where total resistance is the integral of the differential resistance rather than the differential term being a queue for the variable with which the derivative was taken. The last symbolic form involves the differential representing a small amount of some quantity. This could take the form of a small bit of the independent variable in an integrand differential multiplicative relationship, or a more general construct in which the total amount of something is comprised of small (or infinitesimal) pieces of that quantity. In addition to these three constructs, von Korff and Rebello (2014) elaborated on three additional symbolic forms for the differential, but in this case, they used symbol templates involving Δ to distinguish between three interpretations of infinitesimal quantities: amount, change, and product. The first symbolic form, amount, is very similar to Hu and Rebello's small change symbolic form but differs in the subtlety that the stand-alone symbol template Δ represents an infinitesimal amount of a quantity[6]. The change symbolic form, with the template $\square - \square = \Delta$, is special in the sense that the terms on the left side of the equality are finite, while the Δ continues to represent an infinitesimally small quantity. Von Korff and Rebello used position and mass as examples to illustrate that the distinction between these two symbolic forms is of cognitive significance; they note that while it makes sense that the difference between two positions makes cognitive sense from a change symbolic form (position 2 minus position 1), it is not as productive to think about small amounts of mass in the same manner. The final symbolic form von Korff & Rebello describe involves the product of a finite term with an infinitesimally small quantity which is useful in thermodynamics contexts.

While the majority of this section has been dedicated to listing various researchers' descriptions of symbolic forms, I hope that the significance of the compilation of results, especially within the context of calculus, has become apparent. Assessing a particular arrangement of symbols is worthless without some indication of the underlying meaning

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6 While von Korff and Rebello acknowledge the different representations of d or Δ in the mathematics literature, they chose not to give a mathematical definition of what it means to be infinitesimal or vanishingly small; instead including the differing meanings in an appendix of their 2014 paper.

which constitutes its parts. Researchers have shown that there are numerous ways to interpret and/or produce the same symbol template, even when the resultant product looks the same. This means that researchers, and educators, must be sensitive to the underlying meanings that are being associated with specific arrangements of mathematical symbols, including their strengths and weaknesses for given situations. While this means that individuals should be attentive to the meanings they wish to convey when discussing certain aspects of a quantity or equation, for instance the different components of a definite integral, it means educators should also be intentional about eliciting and analyzing students' meanings through conversation or written explanations. In addition, fully accepting the construction of symbolic forms constitutes an acknowledgment that students bring in previous conceptions which inform their interpretation and construction of symbolic forms; simply relaying a clearly articulated meaning for a construct and why it is applicable is not enough. Learners build their symbolic forms through experiences. Students must be engaged in experiences which reinforce the productivity of certain symbolic forms, while also being challenged with problems that illuminate their limitations. For instance, the research indicates that students are entering physics classrooms with an overreliance on dependency queues, perimeter and area, and function mapping symbolic forms for definite integrals which are hindering their ability to adapt to problems that need to be partitioned and approximated using generalized pieces. However, students are passing their calculus courses, so what is going wrong? Simply put: Students are not being challenged by the limitations of these symbolic forms before physics instruction. They have never (or rarely) been exposed to tasks in which the integrand is not a preformed rate of change. Furthermore, many of the studies listed have noted that, during interviews, a posed question about a particular component of a symbol template caused some students to reframe their reasoning about a particular situation and activate new symbolic forms. This acknowledgment of the activation of dormant symbolic forms sheds light on the fact that students are often not placed in situations in which they must reflect on the effectiveness of their underlying meanings. When students are forced to articulate their reasoning it can cause a shift that allows for sense-making to take place.

I would like to make one point explicit: I am not purporting that one symbolic form, or a specific way of thinking, is superior and should be prioritized at the cost of all others. Every single symbolic form listed above was productive for students in some way. Students' symbolic forms are not an arbitrary hodge-podge of meaningless symbols that serve no purpose, nor are students procedural robots who work with no meaning behind their actions. In almost all cases, students have a reason for saying/writing/thinking the way they did, which is directly tied to productive prior experience. This means that those function mapping and perimeter and area symbolic forms were useful enough to be solidified in students' cognitive arsenals. However, I am suggesting that based on the large body of evidence indicating a lack of flexibility in the activation of symbolic forms for definite integrals, we acknowledge that most calculus students are from other STEM fields which require a broader spectrum of meanings for the definite integrals and differentials than the calculus curriculum (as a whole) is currently offering. If we want students to have more robust and alternative symbolic forms for definite integrals we must provide opportunities for those forms to develop.

Definite Integrals in Mathematics and Physics

One of the earliest investigations into definite integrals was Orton's (1983) study involving 110 students ranging from 16-22 years of age. Orton asked these students a series of 38 elementary calculus questions (18 of which related to integration) and classified the errors students made into three categories: executive errors (failed computational manipulations), structural errors (not attending to relationships of the mathematicians within the task—most frequent), and arbitrary errors (not attending to the constraints of the task— least frequent). A critical result of Orton's study was the identification that most students were not connecting integration with the limit of Riemann sums. Even after thirty years, this lack of connection between definite integrals and Riemann sums is observed in mathematics students' reasoning (Jones, 2013; Rasslan & Tall, 2002). In particular, students are found to have a strong tendency to interpret definite integrals in terms of anti-differentiation or area under a curve (Jones, 2015a).

Numerous examples of this same reasoning can be found in the physics education literature. In particular, Meredith and Marrongelle (2008), building off the work of Sherin's symbolic forms, observed an over-reliance on what they referred to as a dependence cue for the definite integral (anti-differentiation), as opposed to a more robust parts-of-a-whole cue (Riemann sum), when physics students were working through contextual integration tasks. Nguyen and Rebello (2011) identified that physics students often did not reinterpret physics tasks in terms of area under a curve, instead relying on the algebraic manipulations of anti-differentiation. When students were confronted with an interpretation for definite integrals as an area under a curve, they were often unable to identify the quantity which the area represented within the given context.

In the following sections, we will outline a few key findings of research into students' understandings regarding definite integrals.

Integration in Terms of Covariational Reasoning

Because this text focuses on quantitative reasoning and its implications for frameworks involving the definite integral, I would be remiss if I did mention Thompson and Silverman's (Thompson & Silverman, 2008) covariational conception of indefinite integrals. Thompson and Silverman do not provide a framework for integration, but rather a model for a specific understanding of an indefinite integral (or accumulation function) set within the theory of quantitative reasoning. In particular, Thompson and Silverman classify an indefinite integral in terms of accumulation involving a quantitative understanding of rate of change (Carlson, Smith, & Persson, 2003; Thompson, 1994b).

While the particulars of how a student might construct Thompson and Silverman's image for a productive understanding of indefinite integrals is not necessarily relevant to our discussion, the authors' reflections on Riemann sums are. When discussing the difficulties student's face when only regarding integration as an area under a curve, in regards to Riemann sums Thompson and Silverman noted,

> If f is a function whose values provide measures of a quantity, and x also is a measure of a quantity, then $f(c)\Delta x$, where $c \in [x, x + \Delta x]$, is a measure of a derived quantity. The simplest case is when $f(x)$ is a measure of length and x is a measure of length. Then $f(c)\Delta x$ is a measure of area. If $f(x)$ is a measure of speed and x is a measure of time, then $f(c)\Delta x$ is a measure of distance. If $f(x)$ is a measure of force and x is a measure of distance, then $f(c)\Delta x$ is a measure of work. If $f(x)$ is a measure of cross-sectional area and x is a measure of height, then $f(c)\Delta x$ is a measure of volume. If $f(x)$ is a measure of electric current and

> x is a measure of time, then $f(c)\Delta x$ is a measure of electric charge. A Riemann sum, then, made by a sum of incremental bits each of which is made multiplicatively of two quantities, represents a total amount of the derived quantity whose bits are defined by $f(c)\Delta x$. Therefore, for students to see "area under a curve" as representing a quantity other than area, it is imperative that they conceive of the quantities being accumulated as being created by accruing incremental bits that are formed multiplicatively.

That is, for Thompson and Silverman, the integrand and differential are quantities. The differential form is a quantification of the multiplicative relationship between the integrand and differential which measures "incremental bits" of the desired quantity.

The Riemann Integral Framework

Sealey (2014) provided a framework for characterizing students' understanding of Riemann sums and definite integrals in contextual situations. This framework consists of five layers—orienting, product, summation, limit, and function—which align with the mathematical components which comprise the Riemann integral (see Table 2).

Table 2: Symbolic representation of R-I framework. (from Sealey, 2014, p. 242)

Riemann Integral Framework.

Layer	Symbolic representation
Pre-layer: Orienting	$\left[\frac{1}{c} \cdot f(x_i)\right]$ and/or $[c \cdot \Delta x]$
Layer 1: Product	$\left[\frac{1}{c} \cdot f(x_i)\right] \cdot [c \cdot \Delta x]$
Layer 2: Summation	$\sum_{i=1}^{n} f(x_i)\Delta x$
Layer 3: Limit	$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
Layer 4: Function	$f(b) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

The orienting layer, which Sealey calls a pre-layer, accounts for instances of students attempting to make sense of individual quantities in a scenario. That is, when students attend to either the $f(x)$ or Δx individually without coordinating the two through a product. Sealey noted that the orienting layer was an important addition to the framework, as students spent a great deal of time orienting themselves in the initial stages of problems but they also frequently returned to this layer throughout their problem-solving activity. In the product layer, students focus on the multiplicative product of the differential form. This layer consists of quantifying the product of a function times a small change in the independent value of that function, $f(x_i) \cdot \Delta x$. Sealey notes that this was often difficult for students in the context of a Riemann sum due to the need to reason about each product over intervals:

> For example, understanding distance as a product of velocity and time requires one to coordinate the quantities of velocity and time in a specific way. Moreover, it requires one to understand the precise meaning of "time" and "velocity" within that context. Specifically, "time" does not refer to the time at which the velocity calculation was taken, but instead refers to the elapsed time over which a calculation is being made. Similarly, "velocity" refers to a constant velocity on a given interval (Sealey, 2014, p. 238).

In addition, Sealey observed that certain contextual problems, such as the total amount of force on a dam, provided base quantities which did not promote students' quantification into an $f(x_i) \cdot \Delta x$ structure. For this case, students were more productive when quantifying the total force into pressure × area, rather than pressure-lengths × change in width. To account for this Sealey included constants C and 1/C into the product and orienting layers.

The summation layer represents an accumulation of the product layer into an approximation of the desired quantity: a way to account for the entire interval, rather than just subsections of it. Sealey observed that within the summation layer was when students recognized the Riemann sum as a partition of a whole into parts. Few students in Sealey's study struggled with this layer, but it was most often observed as their entry point into reasoning about the tasks. In the function layer, students attend to improving their approximation using limits. In other words, this layer represents the link between Riemann sums and Riemann integrals. Sealey notes that "distinctions are made within this layer between obtaining better approximations to the limit value and obtaining the exact value of the limit" but acknowledges that students in her study were only asked to approximate the limits and did not make a direct connection to definite integrals. Taking into account Thompson and Silverman's (2008) work on integrals as accumulation functions, Sealey included the function layer as a "logical step in the mathematics curriculum."

Sealey's framework draws heavily on Oehrtman's (2004) approximation framework and was born out of the epistemological perspective of Piaget's structuralism (1970, 1975). Sealey writes, "structure consists not only of elements or aggregates but also of the operations on these elements and the relationships between them. Together, these elements and operations form an entire structure" which, within the definite integral, "would be represented by an understanding of the structure of Riemann sums including the structure of the terms in the summation and an understanding of the relationship between Riemann sums and definite integrals." These structures are "self-regulating and are subject to the laws of reversibility, transformation, and wholeness" (Sealey, 2008, pp. 39-46).

As Sealey was not attending directly to the limits involved in Riemann integrals for this study, she did not explicitly attend to the concept of infinitesimals or differentials. However, it is clear by her attention to quantities throughout the framework, that the differential within the limit and function layers have units of measure. That is, the differential represents a quantity and the differential form represents measures of small partitions of the overall quantity desired.

Layers and Representations

Physics education researchers Von Korff and Rebello (2012) provided a framework for definite integrals set within the context of physics courses. Much of this framework mirrors that of Sealey's (2014) Riemann integral framework, however, Von Korff and Rebello characterize Riemann sum and definite integrals with a sequence of levels which are differentiated by a distinction between macroscopic quantities (Δt) and infinitesimal quantities (dt). The levels include four macroscopic levels for Riemann sums, macroscopic quantity (Δt), macroscopic product ($v \cdot \Delta t$), sum ($\sum v\Delta t$), and sum function $\left(\Delta x(t) = \sum_{i=0}^{\frac{t}{\Delta t}} v(i\Delta t)\Delta t\right)$, along with four infinitesimal levels related to integrals, infinitesimal quantity (dt), infinitesimal product (vdt), integral ($\int vdt$), and integral function $\left(\Delta x(t) = \int_{t_0}^{t} v(t')dt'\right)$ (Figure 2). Just like Sealey's framework, the product of the differential form represents pieces of the desired quantity, although specific attention to approximations is not made.

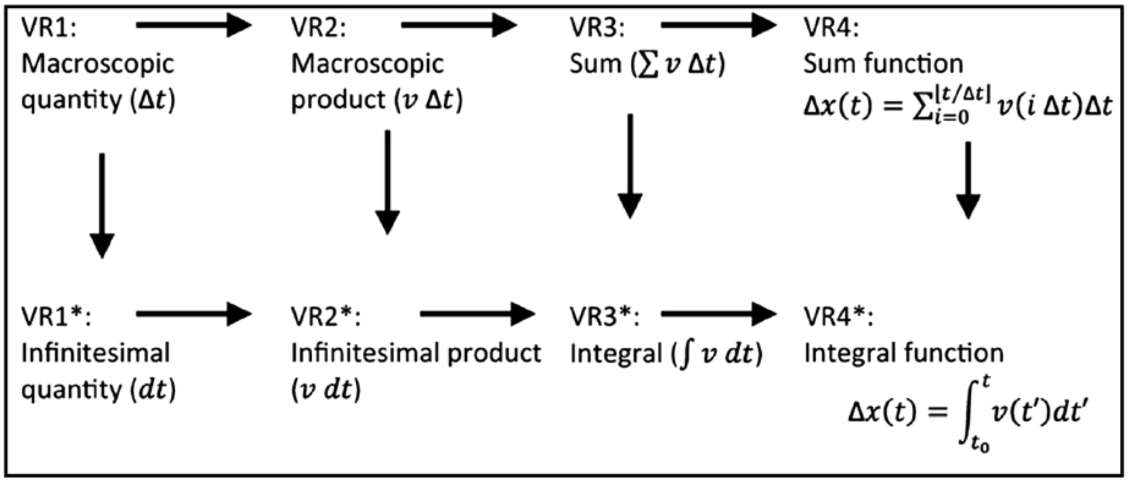


Figure 2: Layers and Representations framework (from von Korff & Rebello, 2012, p. 3)

Von Korff and Rebello noted that distinct paths taken through this framework can be indicative of ways students understand integration. For example, during one task a student transitioned through the framework in a way described as "limit of the sums" (VR1 → VR2 → VR3 → VR3*), and then later took a different route in another task (VR1* → VR2* → VR3*) which the authors described as "sum of small quantities." Von Korff and Rebello also provided a classification between differential (and consequently integral) types based on the type of quantity the differential represents. In particular, they make a distinction between change differentials (e.g. *dt* represent a change in time), and amount differentials (e.g. *dm* represents a small amount of mass), asserting that clearly identifying the type of quantity which is desired allows students to reason more easily with the quantities which comprise the differential form.

What is particularly important about Von Korff and Rebello's framework is their emphasis that it is not harmful for physics students to reason about differentials in terms of infinitesimals (or even just really small changes). This allows physics students to productively reason about integration without explicitly discussing limits and Riemann sums.

Resources and Conceptual Metaphors for Differentials

Hu and Rebello (2013) investigated students' conceptions of the differential in physics-based integration tasks. Drawing on two conceptual frameworks, resources and conceptual metaphors, their study found four different mathematical resources and four conceptual metaphors which students draw on to reason about the differential within a definite integral as they worked through various integration tasks.

The four mathematical resources were the differential as a small amount, as a point, as associated with differentiation, or as the variable of integration. When reasoning about the differential in terms of a small amount, the differential represents a measurable quantity, specifically an infinitesimally small quantity. Drawing on the notion of the differential as a point conjures imagery of differentials as points on a line or something with zero dimension similar to the collapse metaphor described by Oehrtman (2009). Reasoning about the differential in terms of differentiation is to associate it with the act of the process of differentiation. That is, the symbol *d* serves as a cue to take a derivative. Viewing the differential as a variable of integration was observed when students did not articulate a quantitative meaning to the differential. This was often observed when they forgot to include the differential in their definite integral at all.

The four metaphors students drew on for differentials while working through physics-based integration tasks were differentials as objects, locations, motion along a path, and machine. The objects metaphor is similar to viewing differentials as a small amount. Here differentials are objects which assist students in the ability to reason quantitatively. Invoking a location metaphor involves students' reasoning about the differential term as representing a location in space, while the motion along a path metaphor involves the variable of the differential term as representing a traveler along a line. The final metaphor, differential as machines, involves thinking about the integral as a function machine that performs an algorithm. The symbols $\int$ and d represent operators which execute those algorithms.

Summary

Through a review of the literature, I have outlined how mathematics and physics education researchers have identified significant advantages afforded to students through reasoning about the differential form in terms of quantitative relationships. Specifically, students were most productive in contextual physics integration tasks when they ascribed quantitative meaning to the differential. Therefore, to engender a quantitative understanding for the definite integral as students' schemes were in development, it would be important to draw on quantitative interpretations of the differential as part of that construction. Specifically, the Quantitatively Based Summation conception of the definite integral, which served as a basis for the underlying conceptual understanding I hoped to engender in students through this study, was born from an extension of these frameworks. Sealey's observation that students reasoned about fluid force tasks in terms of pressure × area, rather than pressure-lengths × change in width, indicated that students were more productive when reasoning about the quantitative relationships within the differential form by drawing on the same quantitative structure as the desired quantity as opposed to re-quantifying these relationships in terms of a rate of change or density function. Hu and Rebello's correlation between viewing the differential as an object and quantitative interpretations for that notation reinforced the quantitative relationships within the differential form. That is, a differential should represent a measurable quality of an object within a differential form: an amount of mass, a small height, a length of a piece of a rod. Finally, Von Korff and Rebello's observation that physics students could productively reason about differentials in terms of small changes, with no necessity to invoke limits, suggested constructing a scheme for definite integrals in which the differential form represents a macroscopic relationship or "bits" as Thompson and Silverman described. That is, by framing the differential form in terms of a measurable part of a whole students would be positioned to coordinate the symbolic form of a definite integral as the result of a limiting process, rather than the limiting process itself.

CHAPTER III

THEORETICAL PERSPECTIVE

The underlying theoretical perspective I have adopted for this dissertation research is that of radical constructivism (von Glasersfeld, 1995) which is rooted in Piaget's genetic epistemology (Piaget, 1972; Piaget & Duckworth, 1970). From this perspective, learning is not a matter of developing cognitive structures in increasing consistency with an external objective truth. From a constructivist approach, knowledge is an adaptive construct of human minds which is actively created consistent within an individual's conceptual structure through interactions with the outside world (von Glasersfeld, 1995, p. 51).

Assimilation and Accommodation

The theory of radical constructivism is centered on a cognitive organization of key aspects generalized from individuals' past experiences, called schemes, and how individuals interpret and adapt to new situations in terms of those schemes. In Piaget's learning theory, there are two ways an individual can react to a new stimulus, assimilation or perturbation (entering a state of disequilibrium). To assimilate a situation to a scheme is simply "treating new material as an instance of something known" (von Glasersfeld, 1995, p. 62). That is, assimilation represents a lack of reaction. Assimilating a new experience does not mean an individual understands a situation in the same way as someone else, or even that they are interpreting the situation in ways compatible with their own schemes if pushed. Assimilating an experience simply implies an individual interpreted the experience to fit into an already existing scheme along with the expectations implied by that scheme. A perturbation on the other hand will induce a reaction, whether visible to an outside party or not. A stimulus creates a perturbation when something about the situation does not align with (and is noticed as being distinct from) a previous scheme which often takes the form of an unexpected result. Perturbations are the first step in the progression of scheme development, and for a perturbation to result in an adaptation of a scheme it must be accommodated. Accommodation is accomplished through abstractions, a semi-hierarchical set of cognitive generalizations of an individual's actions (including mental actions). Piaget defined two general types of abstractions, empirical and reflective, and further subdivided reflective abstractions into three subcategories, pseudo-empirical, reflecting, reflected. According to von Glasersfeld,

> one is called 'empirical' because it abstracts sensorimotor properties from experiential situations. The first of the three reflective abstractions projects and reorganizes, on another conceptual level, a coordination or pattern of the subject's own activities or operations. The next is similar in that it also involves patterns of activities or operations, but it includes the subject's awareness of what has been abstracted and is therefore called 'reflected abstraction.' The last is called 'pseudo-empirical' because, like empirical abstractions, it can take place only if suitable sensorimotor material is available (1995, p. 105).

Models and Modeling

In 1991, Blum and Niss surveyed the mathematics education literature identifying five different categories of rationale for the inclusion of modeling in mathematics curriculum: formative, critical competence, utility, promoting mathematics learning, and picture of mathematics arguments (Blum & Niss, 1991, pp. 42-44). The first category, formative, focuses on the affective nature in developing and strengthening students' self-confidence regarding creativity and problem-solving abilities. The next categories, critical competence and utility, highlight the need for students to be able to function in a world that requires analysis and solutions. Critical competence arguments focus on the social aspect of a learner, while utility is concerned with directly linking mathematics to 'real-world' situations. Similar to the utility argument, the promoting mathematics learning arguments category acknowledges that linking mathematics to 'real-world' situations is important, however, the distinction lies in the focus on how access to modeling tasks provides motivation to learn and retain the mathematical ideas themselves. While the first four categories characterize modeling as an independent skill that utilizes mathematics in some way, the final argument, picture of mathematics, adopts the stance that modeling is an integral part of mathematical practice and should therefore naturally be included in the mathematics curriculum.

Emergent Models of Realistic Mathematics Education

For this study, I adopted the stance that modeling is an inherent part of the mathematics learning process. I drew on realistic mathematics education (RME) which views modeling as an integral aspect of what it means to do mathematics and sees "mathematics as a human activity" in which contextual problems play a crucial role

(Freudenthal, 1973, 1986). RME is a domain-specific instructional theory originating out of the Netherlands with three central design heuristics: guided reinvention (Freudenthal, 1973), didactical phenomenology (Freudenthal, 1986), and emergent models (Gravemeijer, 1999). While other instructional design theories, such as modeling cycles (e.g. Blum & Leiß, 2007; Blum & Niss, 1991; Niss, Blum, & Galbraith, 2007) or the models and modeling perspective (Lesh & Doerr, 2003; Lesh & Sriraman, 2005), frame modeling as a process which provides context allowing students to develop deeper understandings of already established mathematical ideas, RME emphasizes the epistemological role of modeling in the development of mathematical ideas themselves.

When Gravemeijer references emergent models he is referring to the process by which an individual constructs mathematical ideas through the broad act of progressive mathematization, abstraction, and generalization. Gravemeijer (2007) explains "the process of constructing models is one of progressively reorganizing situations" where "the model and the situation being modeled co-evolve and are mutually constituted in the course of modeling activity." RME views modeling as "a form of organizing, instead of an act of translation" and Gravemeijer notes the aim of RME is not just to connect informal and formal knowledge but to identify and evoke the interplay between informal and formal knowledge which allows for new constructions to develop (1997, 1998). Additionally, the goal of obtaining 'a specific model' or 'a specific way of modeling' is not the focus of RME and often the notion of a 'completed' model does not make sense. In RME modeling is the process by which mathematical learning happens and can manifest itself in the form of a model of a situation, a scheme, a description, or even a way of notation (Gravemeijer, 1998). By viewing modeling in this light, there is no 'real' or ‘correct’ model of a given situation; instead, there are numerous iterations, or sub-models, which evolve as an individual’s understanding of a situation evolves towards more formal mathematical reasoning. In this way, models are emergent from one’s activity in experientially real situations and are generalized of an evolving level of ‘common sense.’ From the RME perspective, experientially real does not necessarily mean that there must be a real-world context associated with a given task. What is important about an experientially real situation is that the meanings associated with the models (which are used and developed in the process of working on a task) emerge from the interplay between the situation, the student’s activity, and the student’s reasoning in relation to the situation; i.e. that models are rooted in student experiences which they both reflect on and abstract from. This means abstract mathematical tasks can also be experientially real for students and by anchoring the development within student experience the resultant knowledge (i.e. model) is regarded as their own. Gravemeijer notes that the emergent view of modeling, in alignment with Freudenthal's view that mathematics was not a ‘ready-made-system,’ was

> initially developed as an alternative for the common use of what we may call 'didactical models’, manipulative materials and visual models that are meant to make abstract mathematics more accessible for the students… in order to interpret these models correctly, students should already have at their disposal, the knowledge and understanding that is to be conveyed by the concrete models (Gravemeijer, 2007, p. 139).

Due to the evolutionary nature of emergent models, Gravemeijer identified four levels of activity—situational, referential, general, and formal—students encounter as models shift from being ‘models of’ specific situations to ‘models for’ more abstract mathematical thinking. The situational level references knowledge and strategies that are highly connected to specific contexts, be it a task set within a context, a more abstract mathematical query, or a real-world problem. In this layer, students are making sense of the task using whatever domain-specific knowledge they can to mathematize, or organize, the situation. Here they may act out the activity by physically counting or altering the environment. As students begin to organize information in ways that allow them to make inferences about the situation, or can use generalizations for the physical acts, they have moved into the referential layer. Here models are still situated within the context but are abstracted in the sense that they are referential rather than a direct generalization of specific situations. As models shift from being tied to specific situations, to being able to represent a class of situations it is said to be moving to the general level. At this point there the model makes a shift from being limited to a solution for a problem, to a potential solution strategy (i.e. a model for a class of situations). Transitioning to the general level occurs when students can reason about aspects of their models in ways that are not tied to the originating context. In the final layer, formal activity, new mathematical realities emerge in the sense that students no longer require the need to refer to the activity involved in the model for classes of situations. That is, the models themselves become objects and tools in new modeling activity and need not be deconstructed to be used.

In Gravemeijer's 1999 paper he describes a sequence of tasks designed to promote the reconstruction of a ruler as a measurement apparatus. Students are first given a context in which they are encouraged to physically act out a measuring process of counting how many 'heel-to-toe' iterations it would take to traverse a specific length.

This task is tied heavily to the situational context. Additional tasks are introduced which use slightly different methods of measuring (such as measuring distances using small creatures called Smurfs feet length) which are closely related enough that students can refer back to their own physical ‘heel-to-toe’ counting. Beginning to recognize the need for more standard ways to measure, students generalize their model of measurement. In this case, students might begin to use a standardized object to generalize measurement (e.g. cans or blocks) but can also transition to an instrument such as a ruler which can be generalized to any number of units. As students utilize their generalized model of a ruler to measure various situations, a transition can occur in which they begin to reason with the idea of measurement itself, rather than the process of measuring. That is, a ruler no need not be tied to the actual act of measuring, but can stand in for the idea of what it means to measure. This allows students to reason more formally about processes involving measurement such as the difference between two measures.

While many of Gravemeijer’s examples of RME’s emergent models focus on elementary mathematical concepts, such as that of counting or measuring, researchers have utilized this design heuristic when researching the learning of higher division mathematics such as calculus constructs (e.g. Gravemeijer & Doorman, 1999), abstract algebra (e.g. Larsen, 2004), differential equations (Rasmussen & King, 2000), linear algebra (e.g. Wawro, Rasmussen, Zandieh, & Larson, 2013), and defining as a mathematical practice (Zandieh & Rasmussen, 2010). This evolution of what RME encompasses is an intrinsic aspect of its theoretical consistency. RME is not considered as a fully formed, fixed, instructional theory, but is a way to understand and describe students’ reasoning about mathematics. Just as mathematical ideas are shaped through the interactions of human activity, the constructs of RME shall too be shaped by this interplay and continue to be refined.

Emergent Quantitative Models Framework

In line with Gravemeijer's approach to modeling as an emergent process, Simmons and Oehrtman (2017, 2019) described an emergent quantitative models framework for students' reasoning about definite integrals. In particular, this framework extends previous characterizations of students' reasoning about definite integrals when the differential form does not naturally decompose into a multiplicative relationship between a function and small change in the variable of that function (a quantitative relationship Simmons and Oehrtman call a Riemann product). This framework can be utilized to understand, for example, how students reason about physics-based tasks such as Sealey's total force on the dam problem in which students utilized the quantities of pressure × area, instead of pressure-lengths × length (or linear force densities × length). The framework relies on three conceptual models, basic, local, and global, which students draw on when reasoning about definite integrals;

> The basic model represents the quantitative relationship which would apply to the situation if the quantities involved were constant values, the local model is a localized version of the basic model applied to a sub region of the original situation (typically within a partition), and the global model is derived from an accumulation process applied to the local model, whose underlying quantitative reasoning is encoded in the differential form. (Simmons & Oehrtman, 2019).

Each of these models interacts with one another in significant ways leading to powerful, yet nonlinear development of, understandings of integration in a quantitative way.

For example, Simmons & Oehrtman describe two students, Matt and Julia, who differed in their conceptions of the same energy task:

> Suppose a 10-meter chain with a total uniform mass of 15kg is freely hanging from the roof of a building. Write an integral that represents the total energy required to lift the chain to the top of the building.

When constructing the integral for this task, both participants drew on the same basic model ([energy]=[acc_of_gravity]×[mass]×[height]) and had similar overarching conceptions (global models) that a definite integral is the accumulation of small partitioned bits of the desired quantity. However, these students differed in the way they partitioned the situation to construct their global models, influenced by their interpretations of the quantities within the basic models and resulted in two quantitatively distinct local models for the task.

As Matt was constructing his local model, he anticipated the integral summing the energy required to pull up the remaining chain over small increments. This interpretation resulted in Matt developing a local model that was quantitatively a local Riemann product. Julia, on the other hand, conceived of the integral summing the energies required to lift each small section of chain the entire distance to the roof. This interpretation of the context required Julia to partition the mass along small portions of the chain and resulted in the differential being an intrinsic component of a local model that was not a local Riemann product, as the differential was quantitatively conceptualized as part of the mass.

CHAPTER IV

METHODOLOGY

To track the development of a Quantitatively Based Summation conception of integration I engaged university calculus students in an eight-week-long teaching experiment with accompanying task-based clinical interviews. In the following sections, I describe what a teaching experiment is, its distinction from other types of learning environments and research studies, and how data was collected and analyzed as a part of this dissertation research.

Task-based Clinical Interview Methodology

For this study, I employed a combination of teaching experiments (L. P. Steffe & Thompson, 2000) and clinical interviews (Clement, 2000). A clinical interview can be described as a documented (usually video recorded) conversation between researcher and participant with the purpose of a researcher constructing second-order models of the participants' schemes for a specific topic. A second-order model is a collection of underlying reasons which provides rationale, from the researcher's perspective, for the subjects' observational behavior in a research setting along with the hypothesized implications of that way of reasoning. These models do not reflect the actual mathematical knowledge of the participant (which would constitute a first-order model), but serve as a reflection of what the researcher believes the student is capable of understanding from the researcher's perspective. An important aspect of building second-order models of students' reasoning involves testing the viability of those models. For this reason, clinical interviews are meant to be open-ended in the sense that researchers can adapt to interesting developments within the interview setting. This allows researchers to seek additional information adding clarification and nuance to their models.

A task-based clinical interview introduces mathematical tasks into the research setting as a catalyst for discussion related to specific mathematical topics. Participants engaged in a task-based clinical interview are asked to reason out loud, and the interviewer asks clarifying and investigative questions throughout the problem-solving process.

Teaching Experiment Methodology

Steffe and Thompson note that "a primary purpose for using teaching experiment methodology is for researchers to experience, firsthand, students' mathematical learning and reasoning" and construct second-order models of students' mathematical reasoning (pp. 267, 269). Unlike a clinical interview, where the goal is to capture students' understanding at a particular instant in time, the goal of a teaching experiment is to characterize the development of students' schemes as those understandings evolve over time in an effort to test hypothesized learning trajectories. It should be noted that teaching experiments are not analogous to classroom teaching in the traditional sense. Teaching experiments are a research tool designed to create models of students' thinking and map the changes in students' mathematical schemes, not to statistically evaluate the effect of a treatment. In the following sections, I will outline the aspects of the teaching experiment methodology which are most pertinent to this study.

Conceptual Analysis

According to Steffe, a conceptual analysis is an "answer to the question: 'What mental operations must be carried out to see the presented situation in the particular way one is seeing it?' (von Glasersfeld, 1995, p. 78)" (from Thompson, 2008). A conceptual analysis serves as a tool to orient a researcher's reasoning about what a particular understanding of a concept might entail and a foundation for the creation of curriculum with which to facilitate that development. An important aspect of a conceptual analysis is not a list of facts or procedures, it is a way to discuss the cognitive processes underlying particular ways of reasoning (O'Bryan, 2018, p. 125). Thompson details four ways in which conceptual analysis can be used by researchers:

> in building models of what students actually know at some specific time and what they comprehend in specific situations,
>
> in describing ways of knowing that might be propitious for students' mathematical learning, and
>
> in describing ways of knowing that might be deleterious to students' understanding of important ideas and in describing ways of knowing that might be problematic in specific situations.
>
> in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support. (Thompson, 2008, p. 46)

In constructivist research, a conceptual analysis serves as a foundation for, and product of, teaching experiments. A conceptual analysis is performed before a study to inform the desired outcomes and mental actions an individual could engage in to achieve those outcomes. As the study progresses, a conceptual analysis is constantly tested and modified against new information, and post study analysis produces a formalized refined conceptual analysis.

Hypothetical Learning Trajectory

There are nuanced ways in which hypothetical learning trajectories are discussed throughout the mathematics education literature (O'Bryan, 2018). For this study, I adopted Simon's (1995) hypothetical learning trajectory which is comprised of three components: a learning goal, learning activities, and a hypothetical learning process. The process of creating a hypothetical learning trajectory begins with a learning goal along with a model of a student's (possibly epistemic) current mathematical conceptions. The researcher hypothesizes a specific set of conceptions a student should develop to transition from their current understanding towards the learning goal. The researcher then develops a sequence of experiences that students could engage in to facilitate this transition based on their evolving mathematical schemes. These experiences primarily take the form of mathematical tasks developed to engender specific types of perturbations, problem-solving, and progressive mathematization. It is important to note that a hypothetical learning trajectory is not a generalized organization of tasks such as classroom curriculum, it is a research hypothesis which must be tested.

Teaching Experiment

The purpose of a teaching experiment is to test the viability of a hypothetical learning trajectory. A teaching experiment begins with an episode(s) of exploratory teaching. Exploratory teaching is how researchers construct second-order models of students' thinking and is akin to a clinical interview. This involves discussions and/or tasks between a researcher and research participants aimed at characterizing their current mathematical schemes. Exploratory teaching serves as an important aspect of the methodology to trace the changes in students' schemes throughout the study and can be a catalyst for new hypotheses for student reasoning. Steffe and Thompson emphasize that from a constructivist perspective in exploratory teaching "

> the teacher-researcher must attempt to put aside his or her own concepts and operations and not insist that the student learn what he or she knows. Otherwise, the researcher might become caught in what Stolzenberg (1984) called a "trap"—focusing on the mathematics the researcher takes as given instead of focusing on exploring students' ways and means of operating (L. P. Steffe & Thompson, 2000, p. 274).

While this task is impossible to achieve in its entirety, researchers can draw on their advanced understanding of student psychology and mathematics to create and test viable second-order models. However, they must also remain sensitive to not imposing their own mathematical interpretations or reasoning onto students throughout the interview and analysis process.

As the teaching experiment progresses through the tasks outlined in the hypothetical learning trajectory, hypotheses begin to arise regarding the models of student reasoning. These new hypotheses must be constantly tested by the researcher through pointed questions and/or tasks. Testing these hypotheses marks a transition between what Steffe and Thompson call responsive/intuitive interactions and analytical interactions. Steffe and Thompson note that in responsive and intuitive interactions,

> the teacher-researcher is usually not explicitly aware of how or why he or she acts as he or she does, and the action appears without forethought… we “lose” ourselves in our interactions. We make no intentional distinctions between our knowledge and the students’ knowledge…. In essence, we become the students and attempt to think as they do, (2000, pp. 279-280).

In contrast, analytical reactions are “an interaction with students initiated for the purpose of comparing their actions in specific contexts with actions consonant with the hypotheses” (2000, p. 283). During a teaching experiment episode, teacher-researchers will repeatedly transition back and forth between responsive/intuitive and analytical interactions to refine their second-order model of student reasoning.

Throughout the teaching experiment process, the overall hypothetical learning trajectory must also be refined in light of new information.

Teaching Experiment Analysis

During a teaching experiment there are two phases of analysis, on-going and retrospective, which are both analyzed using the constant comparative method. The constant comparative method is,

> [a] cyclical, interpretive analysis cycle of segmenting the protocol, making observations from each segment, formulating a hypothesized model of mental processes that can explain the observations (and suggest others to look for), returning to the data to refine and look for more confirming or disconfirming

observations, criticizing and modifying or extending the model (Clement, 2000, pp. 566-567).

While the teaching experiment is being performed, a teacher-researcher must constantly refine their model of students' reasoning, as well as the hypothetical learning trajectory and conceptual analysis. This takes the form of both adapting in the moment, if possible, and a routine of intimate review of the recorded video data to prepare for subsequent interviews. Throughout the teaching experiment, protocols and tasks should be adapted to the emerging model of students' reasoning.

At the conclusion of the teaching experiment, the complete data set must be reexamined through retrospective analysis. Throughout the teaching experiment a researcher's model of students' conceptions should consistently evolve. This means that at the conclusion of the experiment a teacher-researcher has a clearer picture of a participant's ways of reasoning which might not have been apparent in the early stages of the study. By reexamining all data, researchers are positioned to identify critical interactions with students missed in the moment. During this analysis researchers go through many iterative cycles of data analysis to both support and refute aspects of their models of student thinking as well as the impact of the hypothetical learning trajectory on students' evolving schemes.

Preliminary Conceptual Analysis and Hypothetical Learning Trajectory

In Spring 2020 I developed a preliminary conceptual analysis and hypothetical learning trajectory for a Quantitatively Based Summation conception of the definite integral in service of preparing a Summer 2020 calculus I course sequence. This Summer 2020 course would serve as the basis for an exploratory study to refine the conceptual analysis and hypothetical learning trajectory for the Fall 2020 teaching experiment. The preliminary conceptual analysis is provided in this section and the accompanying hypothetical learning trajectory task sequence can be viewed in Appendix A. All of the following must eventually fall into place and the ways in which that may happen are largely the subject of this study.

- Definite integrals describe the accumulation of infinitesimal (or very small) quantities. As such, students must be positioned to assimilate a total quantity as being the aggregation of small quantities which share the same quantitative properties and relationships as the whole (a global model).
- To recognize the necessity of a definite integral, rather than the application of a direct quantitative relationship in the form of an equation (a basic model), a student must have access to the basic model (including the quantitative relationships it represents between constant quantities), an awareness that the situation involves a varying quantity making a basic model inappropriate, and anticipation that if the situation is partitioned into small enough segments, the basic model can be used to approximate the small quantities with negligible error[7] (a local model). These approximations can then be accumulated to

7 Negligible error refers to the concept that as one partitions the situation into smaller and smaller pieces, the error tends towards zero.

provide an approximation (with negligible error) or exact value for the entire desired quantity (a global model).

- The symbolic form of the definite integral represents these ways of reasoning in the following ways. The $\int$ symbol conveys an infinite summing process; whether this is through a limit of approximations or the accumulation of infinitesimal quantities. The differential form represents a local model in which the quantitative relationship is modeled through an approximation of a partition using a basic model. The differential represents one quantity (or a component of one quantity) which constitutes the basic model's quantitative relationships. The limits of integration are inherently tied to the differential as representing the total measure and location of the quantity which was utilized in the partitioning of the situation. This typically represents the varying quantity which necessitates a definite integral in the problem-solving activity.

This way of reasoning was identified as advantageous for students working in physics tasks in which the differential form is not a Riemann product (Simmons & Oehrtman, 2017, 2019), but is also compatible with the typical area under a curve application of definite integrals in an elementary university calculus course.

For the design of the hypothetical learning trajectory I primarily drew upon the Emergent Models Framework as my explanatory tool for describing the evolution students' schemes. However, I was intentional in incorporating elements of the RME design heuristics, such as experientially real tasks and progressive mathematization into the task sequence. That is, I designed the task sequence to engage students in tasks that were not simply representations of tasks already performed. To this end, the majority of course lessons required students to work through tasks that went beyond those ideas directly addressed in video lectures as part of their course notes that were to be turned in weekly. For example, in the section which covered the application of definite integrals in geometric tasks, the video lectures covered the use of definite integrals to identify the total volume of a pyramid, while students were expected to use similar techniques to use definite integrals to identify the volume of a cone and a sphere. Similarly, students were shown how to identify the total work against gravity for building a cement column but were tasked with identifying the work against gravity to build a pyramid, or to lift a chain to the top of the building on their own (see Figure 3; note: no similar example was provided for the chain task although the concept of linear density was described at the beginning of the section). By providing students with tasks that required more than simply replacing numbers, the tasks become experientially real. That is the tasks introduced perturbations that required students to reason about previous exercises in relation to new experiences. In addition, many activities in the task sequence required students directly reference their previous problem-solving activity from an earlier assigned task or section. By providing an opportunity to reflect on their previous problem-solving activity, I positioned students to draw on previous situational models as referential experiences towards the construction of a generalized understanding of definite integrals.

Drawing on the emergent models of RME, it was important to situate the construct of definite integrals into a space more meaningful to students than abstractions of area under a curve. As many students in the Summer 2020 course were engineering majors, this took the form of basic energy tasks, density problems, and identifying the total fluid force of water on a dam. Tasks of this concrete contextual nature would serve to provide students with a sense that the mathematical models they were developing had meaning beyond this particular course. Situating definite integrals within context also provided situational models with which students could reflect on as their emergent models for definite integrals transitions through iterations of referential, general, and formal levels.

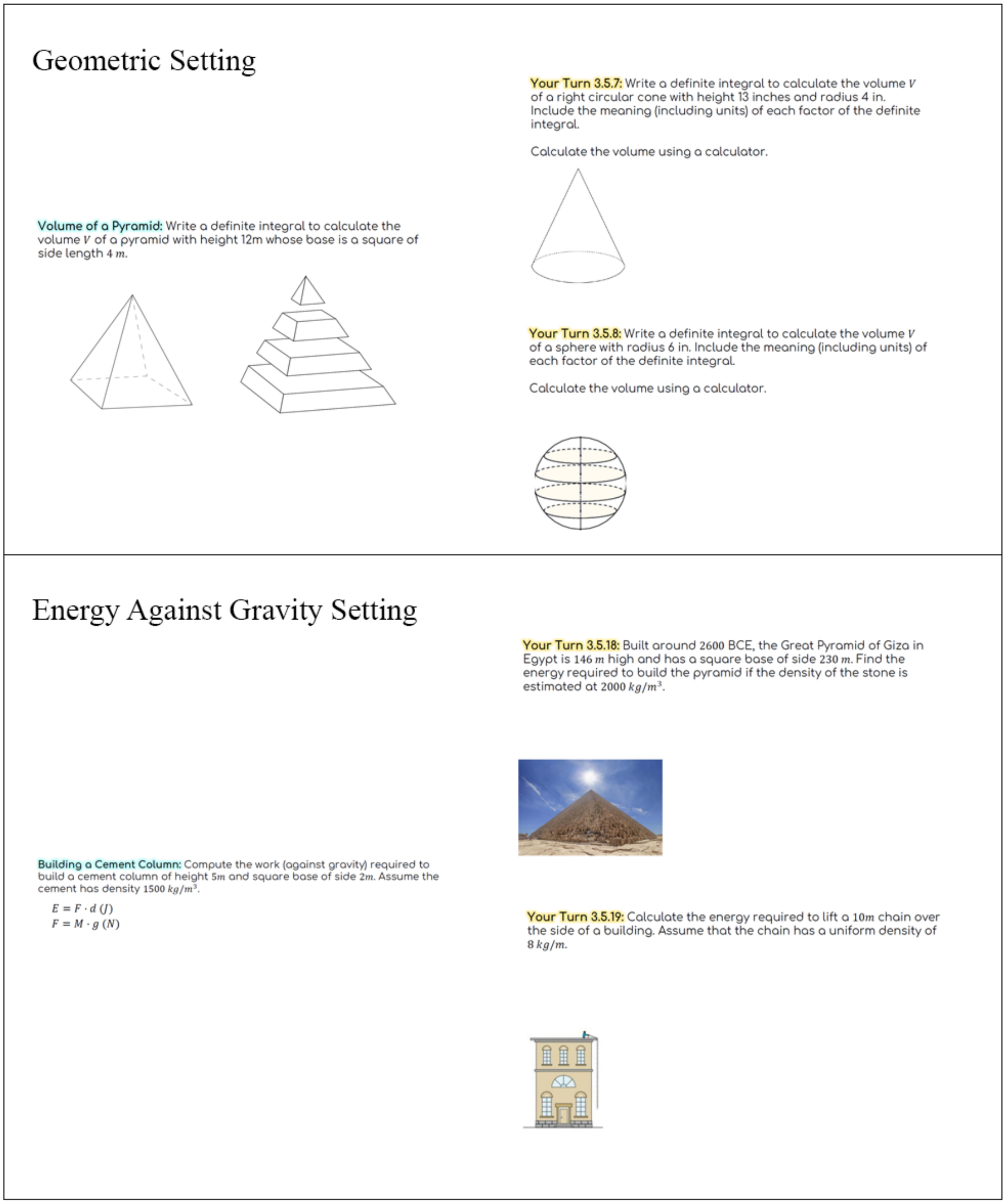


Figure 3: Example tasks for Hypothetical Learning Trajectory

The general trajectory for the structure of the 8-week course is given in Table 3. A significant feature of this course design was the introduction of definite integrals as models of accumulation for varying quantities at the beginning of the course. The

Table 3: Summer 2020 Course Outline

Week 1	• Some Basics
	  ○ Introduction
	  ○ Quantities
	  ○ The Spread of Disease
	• Accumulation
	  ○ Introduction to Accumulation
	  ○ Left and Right Riemann Sums

Week 2	• Accumulation
	  ○ Continuously Varying Rates
	  ○ Limits
	  ○ Modeling with Definite Integrals – Geometry

Week 3	• Accumulation
	  ○ Modeling with Definite Integrals – Density, Work/Energy, Force, Reinterpreting Accumulation as Area Under a Curve
	• Rates of Change
	  ○ Approximating Instantaneous Rates of Change; Limits

Week 4	• Rates of Change ○ The Derivative – Rules

Week 5	• Rates of Change
	  ○ The Derivative – Techniques
	  ○ The Derivative – Modeling: Basic Applications, Related Rates, Limiting Values, Differential Equations

Week 6	• Rates of Change ○ The Derivative – Modeling: Extreme Values, IVT and Monotonicity, Concavity, Drawing Graphs, Applied Optimization.

Week 7	• Bringing It All Together
	  ○ Fundamental Theorem of Calculus Part 1
	  ○ Antiderivatives
	  ○ Fundamental Theorem of Calculus Part 2

Week 8	• Bringing It All Together ○ Accumulation Functions • Review

conceptual analysis for a definite integral described in the previous section is separate of conceptions for antidifferentiation. That is, it was the intention of the hypothetical learning trajectory that definite integrals and antiderivatives be conceived of as two separate mathematical concepts which are connected through the Fundamental Theorem of Calculus. To that end, there was an intentional decision to reorder the course structure to separate these two constructs by as much as possible. To compute definite integrals in the first weeks of the course, students were shown how to calculate their modeled definite integrals using TI-83/TI-84 calculators.

In addition to the order of the course, the ordering of topic introduction was intentional to best align with the conceptual analysis for definite integrals. A chapter on quantities, including the distinction between varying quantities and constant quantities, the relationship between quantities, and function notation was covered at the beginning of the course. An introduction to accumulation focused on approximation methods using Riemann sums and the associated error was included early in the hypothetical learning trajectory. This early introduction of error provided an opportunity for reflection on how the error could be reduced by refining a partition of a global model. The introduction of limits into the course was in service of identifying the value which the Riemann sum approximations approach. Definite integrals were introduced as the notation used for this limiting value.

Exploratory Study and Implications

In Summer 2020, I conducted a study with 6 students enrolled in the accelerated 8-week asynchronous Calculus I course at a large southern university taught by me with curriculum based on the hypothetical learning trajectory described in the previous section. The primary aim of this study was to investigate students' reasoning associated with differentials as infinitesimal quantities, however, it also provided exploratory results towards the refinement of the conceptual analysis and hypothetical learning trajectory for the development of a Quantitatively Based Summation conception of integration prior to the Fall 2020 teaching experiment. Students were offered the chance to participate in the study with the incentive of a raffle for a 10$ gift card. Data collected included an introductory questionnaire which served as a baseline for students' understandings for calculus concepts, all written notes for the course, quizzes, exams, written homework, and hour-long task-based clinical interviews mid-semester (immediately following the accumulation section) and post final exam. The interviews were conducted through zoom using an online collaborative whiteboard. Two students also participated in a series of follow-up clinical interviews in Fall 2020 to identify the retention of their conceptions from the previous semester. Additionally, there were a series of 25 required short surveys throughout the semester, one at the end of each subsection module, in which students provided immediate feedback regarding their evolving understanding of calculus constructs.

The course sequence on accumulation was designed to emphasize the quantities comprising definite integrals with an emphasis on accumulation as the addition of parts-of-a-whole. The overall course curriculum followed a non-traditional sequence beginning with an emphasis on quantitative reasoning before moving to accumulation and definite integrals. The accumulation section was followed by lessons on rates of change and the course concluded with the fundamental theorem of calculus and antidifferentiation. Because the course was online and asynchronous the primary method of material delivery was through recorded videos which were posted in modules in an online Canvas course. The material was organized with the intent that the primary learning occurred through students' own problem-solving activity during what I called 'Your Turn' activities.

Throughout the semester Your Turn activities were routinely referenced back when new topics were introduced to allow students to reflect on previous problem-solving activity as they learned new skills (e.g. Recall Your Turn X.X). For example, once students learned the fundamental theorem of calculus they revisited all tasks from the accumulation section to rework those solutions by hand. Students were required to turn in scans of their handwritten work on the Your Turn activities every Friday for completion which accounted for 10% of their overall course grade. Solution keys were posted the following day allowing them to check their work. The entire task sequence for this course can be viewed in Appendix A.

For brevity, in this section I will include how the Summer 2020 study directly influenced the conceptual analysis and hypothetical learning trajectory for the Fall 2020 teaching experiment. It should be noted that because I served as an instructor for this course, it is likely that anecdotal data from students who did not participate in the study also influenced my decision-making processes.

Time Limitations

Changes were made to the teaching experiment task sequence due to time and coverage needs. In particular, the overall task sequence in the Summer 2020 study served as students 'in-class' lessons as well as their primary homework source. That is, while limited in number, students were presented with tasks covering similar skills to allow for additional practice. A primary example of this can be seen in the use of both the total energy to compress a spring and total force on a dam adapted from the CLEAR calculus curriculum which were included as Your Turns 3.3.6 and 3.3.7 respectively. While these two tasks engage students with different basic models, the general task sequence and primary learning objectives were identical to allow for generalization across activities. When transitioning to the teaching experiment duplicate tasks of this nature would be adapted to include additional cognitive constructs due to time limitations.

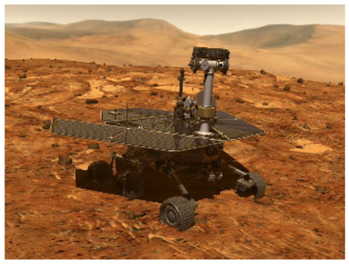
Additionally, because the Fall 2020 teaching experiment would run concurrently with students' Calculus I courses, there was no need to include sections covering rates of change or limits in the hypothetical learning trajectory. It would be assumed that students covered these concepts through their normal coursework. Introductory questions would be included in the initial clinical interview to assess students' incoming reasoning pertaining to quantities and rates of change, and a review of limits would only be included if deemed necessary for individual participants.

Expansion of the Rover Task

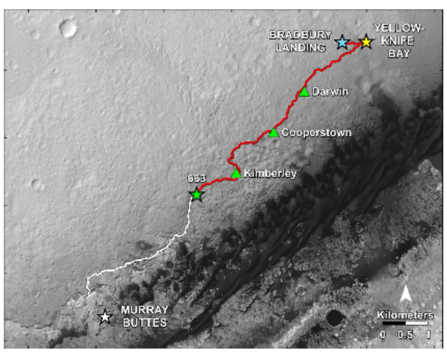
One of the key activities early in the hypothetical learning trajectory for the Summer 2020 course had students approximate the amount of dust that would accumulate on a rover's solar panels as it traveled on Mars. Students were provided a table of data which provided how far along the path the rover was at various sites, along with rates of dust accumulation at those sites (Figure 4). The primary goal of this task was to have students develop an image of the total approximate amount of dust as being comprised of 6 layers of dust that would accumulate on the rover's solar panels as it traveled between neighboring sites. These six layers could then be added to approximate the whole value.

While students in the course did not find the rover task overly challenging, it did not necessarily serve as an initiation to an adding-up-pieces conception in the way it was envisioned. In particular, data from handwritten work and quizzes showed that students

Your Turn 3.1.2: The Opportunity rover landed on Mars in 2004 and has been actively exploring the planet ever since. It is powered by solar cells. As the rover travels across the Martian surface, it kicks up dust, which accumulate on its solar cells. The amount of dust that it kicks up depended on the composition of the surface it was traveling over - a rockier surface kicks up less dust than a softer surface. When planning a path for the rover to follow, scientists need to know how far it might travel before too much dust accumulates on its solar panels.



Your Turn 3.1.2: The scientists have mapped out a 100-km path for the rover to follow (shown below) and have collected satellite data about the composition of the Martian surface at various points along the route using a LiDAR Spectrometer. Based on the following table, approximate the amount of dust accumulated on the rover's solar panels.



Composition	Position along path (km)	Amount of dust per distance traveled (mg/km)
Very sandy	0	6
Moderatley sandy	20	3.5
Slightly sandy	40	2.5
Slightly rocky	60	2
Moderately rocky	80	1.5
Very rocky	100	1

Figure 4: Summer 2020 rover task

primarily reasoned about this task in a purely computational fashion. Although results were entangled with difficulties of sigma notation (discussed in the next section), some students displayed evidence of not attending to all of the relevant quantities. For example, for at least one participant in the study, distance was not a meaningful component of total accumulation for the rover. In the survey immediately following this subsection he described accumulation as, “taking the rate of change and adding the changes together throughout a period of time.”

As part of the task design for the Summer 2020 course I also left the decision for how to identify the approximation of dust between two sites up to students. In this choice I anticipated some students would make overestimates while others would compute underestimates. Because the rover task followed an instructor-led example, I anticipated students would model their computations to mimic that work. Therefore, a solution strategy I did not account for was the possibility of students using average values to construct estimates. Because I aimed to coordinate definite integral notation with the limiting value of overestimates and underestimates through a refinement process, in the Fall 2020 hypothetical learning trajectory I would specifically ask students to identify both an overestimate and an underestimate value to detour this averaging solution strategy.

Many design decisions for the Summer 2020 course were made due to the constraints of an asynchronous and accelerated course. When preparing for the Fall 2020 teaching experiment the goal of the hypothetical learning trajectory was for the concept of accumulation to develop from students’ problem-solving activity—not an extension or reproduction of an instructor-led task. Therefore, I redesigned the rover task with targeted objectives of having students develop an adding up pieces conception of accumulation, attend meaningfully to all quantities, and clearly identify under what assumptions their approximations would hold true. In particular, to provide students with the impetus to attend to quantities I removed the table of data from the task. In its place students were provided with an applet that included a map of the rover’s path and a slider. As students moved the slider a miniature rover icon would traverse along the path and a readout listing travel distance and rate of dust accumulation for that specific site would be displayed. To ensure distance was a meaningful quantity the space between locations along the path would be non-uniform. Because students would be familiar with the concept of rate in the Fall 2020 teaching experiment, the 'amount of dust per distance traveled' would be identified as 'the rate of dust accumulation'. I requested students create both over and underestimates for the total dust on the rover's solar panels to avoid the impulse to identify approximations using average values and to provide better motivation precise mission parameters would be provided. In addition to these adjustments, the rover task was expanded to include additional data points so students would be provided the opportunity to refine their results. The fully redesigned rover task is described in the section on task design.

De-emphasis of Sigma Notation

One particularly relevant result of the Summer 2020 study was an observation of students' conflation between Riemann sum and definite integral symbolic forms. Two students in the study demonstrated difficulties coordinating the relationship between the limits of integration for a definite integral and index notation within a Riemann sum. That is, these students imposed the conceptual schema for a Riemann sum onto both templates. For one student, this was acutely influential as a strong correlation with index notation increasing by increments of one unit influenced his reasoning that partitions of a definite integral must also be subdivided into increments of one unit. In the same interview session, this student also described the bases of the graphical generalized rectangles of a Riemann sum tending towards zero for definite integrals. Due to the similar formatting of symbolic templates for these two structures and the resulting mapping of a conceptual schema to both templates, this student was unable to reconcile these conflicting facts despite being obviously perturbed.

In light of students' demonstrated difficulty transitioning between summation and definite integral notation, I made an intentional decision to deemphasize the need for student-generated summation notation in the teaching experiment. I would not dissuade students from introducing summation notation into their problem-solving process, but it would not be required. Although I first considered introducing students to specialized summation notation (e.g. the values above and below a summa symbol would represent values consistent with the upper and lower limits of a definite integral) I decided against it due to participants' current enrollment in Calculus I courses. All students in the study would be expected to learn summation notation as a part of their required coursework at some point during (or after) the teaching experiment and I did not want to introduce potential issues with their symbolic templates for Riemann sums which might adversely affect their course grade.

Instead, I decided to provide all participants with tools that would support their problem-solving process without the need to compute summations using formal notation. These tools would include a google sheets spreadsheet for the rover task, and a GeoGebra applet for other tasks summation tasks in the teaching experiment. The GeoGebra applet would act as a sum calculator (Figure 5) which would allow students to input starting and ending values, the number of partitions they wished to compute, and an expression that would measure the magnitude of the quantity of a generic piece within the partition. I anticipated students would be familiar with delta notation from their coursework and therefore incorporated its use into the GeoGebra sum calculator.

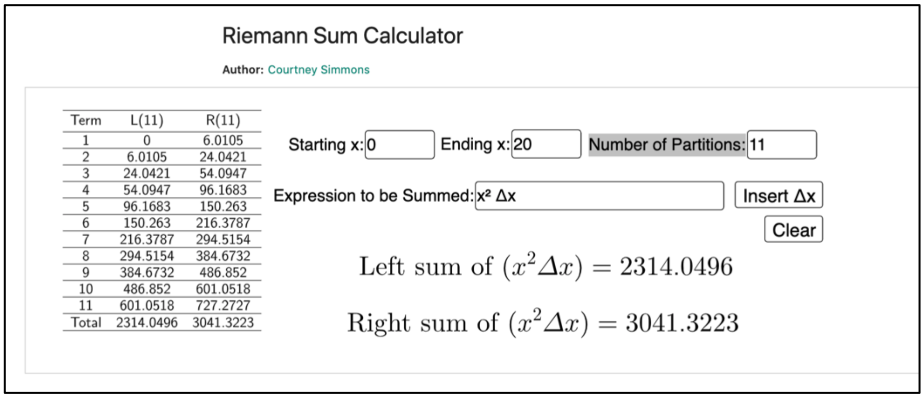


Figure 5: GeoGebra summation applet

To introduce the summation applet to students I would use their unique solution to the second task of the hypothetical learning trajectory to explain how the calculator works. Therefore the calculator would be a natural extension of their previous problem-solving activity. Once students input the required information, the calculator would compute a left sum and a right sum and would list the values for each term in the summation. By formatting the inputs and outputs of the Riemann sum calculator in this way I aimed to highlight the prevalent quantitative structures for students which would directly map to the symbolic template for a definite integral. This included the limits of integration representing the beginning and ending of a quantity that was partitioned, the differential representing a small change in that partition quantity, and an explicit expression for a local model which shares a quantitative structure with a basic model and maps directly to the notation for the differential form. The list of individual terms on the left side of the GeoGebra applet would act as reinforcement for a parts-of-a-whole global model.

Students in the Summer 2020 course were provided instruction on using TI83/84 calculators for the computation of a definite integral prior to the introduction of antiderivatives and the Fundamental Theorem of Calculus. To reduce the time needed to instruct students in such a skill I would provide the teaching experiment participants a GeoGebra-based integral calculator applet to compute solution totals (Figure 6).

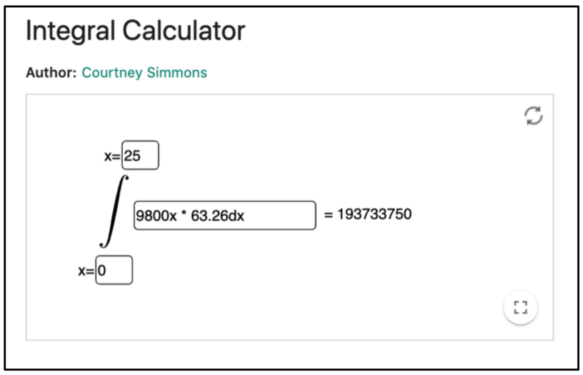


Figure 6: Definite integral GeoGebra applet

The Resilience of an Antiderivative Symbolic Form

The majority of Summer 2020 participants had previous experience with calculus topics, most through secondary education. This is a common occurrence in U.S. universities, however, it means, prior to my course, students in the Summer 2020 study already had schemes for accumulation, Riemann sums, and integrals as they engaged with lesson sequence. I did not continuously interview participants as they engaged in the task sequences for accumulation, so it is not possible to meaningfully speculate on the interactions between their incoming schemes with my hypothetical learning trajectory. However, by comparing results of the clinical interviews midway through the course it was clear that those who displayed a strong association with integrals as antiderivatives faced more difficulty with productively constructing local models for definite integrals when the differential form was not a Riemann product. Specifically, students who displayed an initial antiderivative conception for integrals tended to revert to non-quantitative meaning for the differential. For instance, one student, who was enrolled in a college calculus course the previous semester, gave specific quantitative meaning for the differential in his mid-semester interview and asserted that it must contain units. In fact, based on his quantitative reasoning in the mid-semester interview I was hopeful this student would be a primary example of how effective the protocol was. However, in his post-course interview, after being reintroduced to antidifferentiation, this student claimed that differential did not have units and did not contribute any substantive meaning to the expression. This association was persistent enough that I spent more than 30 minutes of the post-semester interview exploring and probing his schemes before he would even consider a differential having a unit of measurement. Because of my intentional design in this course to promote a distinction between symbolic forms for antiderivatives and definite integrals, continued reference in video recordings to the differential containing units, and assigned tasks which had students explicitly write units beneath all terms of definite integrals (including differentials), I strongly hypothesize that this students' previous schemes involving integration overrode any development that may have been achieved in the accumulation task sequence[8]. When unchallenged, antidifferentiation was his default way of reasoning about the symbolic forms for definite integrals.

In addition to the case described, participants who previously completed a calculus course displayed a tendency to attempt to solve definite integral tasks in the mid-semester interview using antidifferentiation. This was unexpected as the mid-semester interview prompts did not request solutions and neither derivatives nor antiderivatives had yet been introduced in the course, raising my awareness that continued reliance on antidifferentiation schemes could be an issue. Specifically, I anticipated there may be a need to address the correspondence between antiderivatives and definite integrals more directly than planned in the teaching experiment. It also influenced a decision to move away from the tasks in which the differential form was a Riemann product as quickly as possible. To provide students with an opportunity to coordinate their developing schemes of definite integrals with more generic, non-self-generated, definite integral expressions I also developed a task for the hypothetical learning trajectory midway through the teaching experiment to have students identify the appropriateness of definite integral expressions for a quantitative situation (see Task 5 in the sixth section of this chapter for more detail). The aim of including a task requiring students to interpret preconstructed

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8 I must note that this student solicited a significant amount of private mathematics tutoring during the summer semester. It is likely that experiences with the tutor also contributed to his understanding of definite integrals, although I was not privy to the content of their sessions. He did not mention any specific instances in his clinical interviews.

definite integral expressions was to reinforce quantitative meaning for the elements within a symbolic template for a definite integral.

Early Inclusion of Dam Tasks

The Summer 2020 tasks most often referenced by students as being helpful for their reasoning involved fluid force on dams. More than one participant emphasized how dam tasks were directly applicable to their engineering major, providing a sense of real purpose for mastering the skills presented in the accumulation chapter. Additionally, from the quiz and written homework data, the majority of students in the study were able to productively reason about the relevant quantities in rectangular dam prompts. In light of this, I adapted the CLEAR Calculus dam task (see the end of Appendix A) to be the second major task in the Fall 2020 hypothetical learning trajectory.

One major adaptation I made from the task, as written in the CLEAR Calculus curriculum, was to provide students additional scaffolding in refining their results to be within a specified error bound. While identifying the error of an estimation and the ability to refine that estimate to be as close to the real value as necessary is a critical aspect of a productive conception of a definite integral, the CLEAR Calculus curriculum is intentional to predispose students to identify error bounds as meaningful quantities throughout its entire design. The primary scaffolding for these constructs is presented early in the curriculum and was no longer necessary in the later sections which covered integration. Students coming into the teaching experiment were not afforded the opportunity to engage in this type of activity in their calculus courses, and therefore it would be unreasonable to expect them to engage in such reasoning spontaneously. Similarly, I decided to completely remove prompts which asked students to identify "a formula indicating how to find an approximation accurate to within any pre-determined error bound, ϵ," as my goal was not necessarily to have students build a construct for definite integrals that was computationally consistent with the formal definition of a limit.

Continued Difficulty Developing Local Models

Finally, although all but one participant in the Summer 2020 study showed significant attention to quantities and quantitative relationships which comprise definite integrals, every student interviewed exhibited continued difficulty developing a local model when presented with integration tasks where the differential form was not a Riemann product. This was true even when participants demonstrated evidence of working similar problems in notes, homework, and quizzes. For instance, only one student interviewed was able to correctly identify a definite integral expression modeling the total population within 15 miles of a city center given a radial density function. Although participants were not necessarily successful in constructing a productive local model, all other participants who attempted this task were aware that there was something wrong, often citing units being incorrect. However, they were unable to rectify the issue within their expression. While some difficulty stemmed from nonquantitative meaning for the differential, students also did indicate that they had developed a general solution strategy for constructing their local modes.

To address the lack of a generalized solution strategy for identifying an appropriate local model expression, I supplemented the teaching experiment with an early emphasis on creating these expressions which persisted throughout the entire task sequence. Additionally, I included a task at the conclusion of the teaching experiment which would require students to explicitly describe how to solve definite integral tasks. The point of this capstone prompt was not to evaluate participants' solutions but to engage students in active reflection of their problem-solving activity aimed at extracting commonalities across different contexts within the teaching experiment task sequence.

Summary

While not isomorphic to a teaching experiment, the preliminary study of the Summer 2020 calculus course served to greatly improve the overall hypothetical learning trajectory. The most significant contribution was identifying notational issues that could serve to impede students' reasoning as their conceptions developed. This was something I was forced to spend many weeks thinking about and decide what was important for the integrity of the study. Deciding to forgo summation notation was in the best interest of the development of a Quantitatively Based Summation conception of integration for this particular group of students due to conflicts that may have arisen for them in their coursework, however, an alternative solution might be to create notational conventions in the classroom which complete the same overall objective.

The preliminary study also provided clarity on tasks that seemed to work particularly well for students and problems which still posed challenges. This allowed me to develop a more robust protocol for the limited number of tasks I would be able to engage students in during the Fall 2020 teaching experiment.

Conceptual Analysis for a Quantitatively Based Summation Conception of the Definite Integral

When developing the conceptual analysis which characterizes a Quantitatively Based Summation conception of integration, I must first emphasize that any specified interrelationship between basic, local, and global models represents a series of highly non-linear developments which build upon one another, in incremental stages, as the full conception evolves and coalesces. These relationships are greatly influenced not only by incoming conceptions surrounding quantitative relationships which constitute basic models, but also through individuals' schemes associated with notation and ideas involving other calculus constructs such as Riemann sums, rates of change, and limits. That is to say, due to the complex nature of definite integrals, no single conceptual analysis can be viewed as being correct, or even appropriate, for all individuals in an instructional situation. I built this conceptual analysis based on my image of the incoming knowledge of participants for my study as introductory Calculus I students, along with the perceived effectiveness of the initial protocol from the Summer 2020 study. This image included anticipation that participants would enter the study with conceptions of a rate of change which include a proportional relationship between two quantities, an ability to conceive of a limit as a sequence of values that are approaching arbitrarily close to some fixed value, and working knowledge of fundamental quantitative relationships such as area and volume.

For a participant to be considered as having constructed a Quantitatively Based Summation Conception of integration they must be able to:

(1) Justify the inappropriateness of a basic model through the variation of one or more quantities in the situation they are engaged in modeling.

(2) Construct a quantitative image of a situation as small fragments of a quantity which, through a process of accumulation, represents an entire desired quantity. That is, construct a global model consistent with a parts-of-a-whole symbolic form.

(3) Justify the estimation of elements within their global model through a quantification process that shares the same fundamental structure as the basic model for the desired quantity. That is, students must be able to develop a local model and coordinate the elements within the global model with this construct. As part of a local model construction they must:

   (a) Identify an appropriate quantity within the quantitative structure of the basic model which can serve as a differential quantity, dx or Δx. This differential should represent the measure of a quantity over which the varying quantity which made a basic model inadequate, x, is of negligible variation.

   (b) Conceive of the accuracy of the local model estimation as dependent upon the magnitude of the differential quantity. An advanced conception of this would include an image of the error in the estimate tending towards zero as the differential quantity tends towards zero.

(c) Conceive of the global model as being comprised of the number of local model elements necessary to cover the entire measure of the unpartitioned differential quantity without overlap.

(d) Coordinate that each element within their global model represents a different generalized fragment of the overall quantity, but that each element shares the quantitative structure of the local model.

(4) Coordinate the refinement of a global model with a reduction in the magnitude of the local model. This refinement process must increase the accuracy of the global model so that a limiting value of an increasingly refined global model results in the exact desired quantity.

(5) Establish a symbolic form for a definite integral which assigns the following conceptual schemas to the symbolic template

$$\int_{[A]}^{[B]} [C]:$$

A and B are the values representing the beginning and end of the measure for the quantity defined to be the differential respectively. C represents an algebraic representation of their local model which shares quantitative structure with the desired quantity and must include a measurable differential quantity, dx. The differential quantity must be a multiplicative element within the local model, cannot be duplicated, and is inherently tied to the limits of integration A and B. The $\int$ symbol represents a limiting process of accumulations of increasingly refined global models through the reduction of the magnitude of the local model which results in an exact value.

What I have presented here represents an image of a fully-developed Quantitatively Based Summation conception of integration, however, such a construct cannot be built linearly. As part of the development of a Quantitatively Based Summation conception of integration an individual must progress through various stages of complexity within their basic-local-global models, sometimes focusing on individual aspects of a specific model and other times coordinating relationships between models[9]. As the interrelationships between models increase, an individual is then positioned to reason bilaterally about various components of their model relationships. That is, while the design of the teaching experiment might influence the development of a construct in one direction (e.g. progressive adding of small estimates (local model) to lead to an estimate for an entire quantity (global model)), in later activities students can draw on this relationship in reverse (e.g. to anticipate partitioning a global whole into local model estimates). Positioning students to reason bilaterally between models provides an opportunity to add further nuance to a particular model's utility as well as its interdependencies within the rest of the framework.

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9 Note that a nonlinear invocation of the interrelationships between basic, local, and global models is also a hallmark of students' problem-solving when drawing on a mature Quantitatively Based Summation conception of integration.

The hypothetical learning trajectory for the development of a Quantitatively Based Summation involves engaging participants in a continued cycle of progressive accommodations to relationships between their basic, local, and global models. These cycles would be designed to promote the evolution from a discrete set of model constructs used to produce rough estimates for unknown quantities up through a complex system of interrelationships described within the conceptual analysis. In the following paragraphs, I provide a quick overview of the hypothetical learning trajectory, but a detailed description of task design and the hypothesized influence on precise evolution to emergent models can be found in the next section of the text.

As an entry point, I engaged participants in what I call an approximation phase for their emergent models. This phase began with tasks that asked participants to identify both underestimates and overestimates for unknown quantities with increasing accuracy as the task sequence progresses. In the initial task, participants were provided a discrete set of data that can only be approximated when using a basic model and requires an association that the unknown data must be bounded for estimates to represent useful approximations. Participants were encouraged to construct a global model for estimation through the progressive addition of smaller estimates. This would create overall overestimates and underestimates for the entire dataset (the global model). By introducing additional data points I would position participants to establish a correlation between more components of the global model and more accurate parameters.

The next task in the sequence required participants to construct a local model. I provided a quantitative scenario in which an original global model estimate did not provide accurate enough data to make an informed decision. I requested participants partition the global situation into pieces in service of creating more accurate over and underestimates due to the decreased variation between endpoints of the partitions. This partitioning process would promote coordination between participants' basic models, their anticipated quantification of those partitions, an image that the collection of partitions represents the entire quantity, and anticipation that partitioning leads to a smaller error bound between over and underestimates. I encouraged participants to continue this refining process to establish a link between the accuracy of their global models and the size/number of local model elements. Within this same context, participants were asked to identify how many partitions would be necessary to reduce their parameters to within a specified tolerance, reinforcing an image that the actual value for the desired quantity is somewhere within these two estimates. Finally, to re-establish the connection between the basic model quantitative structure with that of both the local and global models, I made a slight adaption to the quantitative context, changing the geometric shape of the object, which requires participants to adapt their local model in light of a new global image.

A transition from the approximation phase to an exact phase was encouraged in the next task. I provided a geometric context in which participants were privy to a way to measure a quantity without the use of approximation, however, I would ask participants to apply their approximation models to the situation. Making observations about estimate values when a real value was known positioned students to draw inferences between ever-increasing accuracy and that real value, providing an opportunity to expand a global model from an image of an estimation process to one that can provide an exact result. Once the relationship between refined approximations and the real value was established, I introduced the notation and symbolic form for a definite integral. To reinforce this connection, I asked participants to revisit previous tasks to associate an exact image of a global model with their previous approximation activity. Moving forward in the task sequence I provided students more complex contexts which required multiple layers of quantitative operations to develop a local model expression.

During the final phase of the hypothetical learning trajectory, I asked participants to create a 'How-To' guide which required participants to reflect on their previous problem-solving activity as a way to abstract a definite integral, along with its corresponding models, as a generalized problem-solving tool for measuring quantities in which one, or more, of the components within the quantitative relationship is varying. The 'How-To' guide prompt involved participants defining precisely what their interpretation of a definite integral is, how it works, and what it is used for along with a detailed description of an appropriate problem-solving strategy for using a definite integral to solve novel tasks. The point of this exercise was not to have participants produce an exemplary description of a 'complete' view of definite integrals but to engage them in reflective activity and position them to generalize across contexts. The reflection activity would provide an opportunity to further enrich the underlying meanings participants have for their basic, local, and global models including relationships between them.

To investigate the impact of the hypothetical learning trajectory, I engaged participants in task-based clinical interviews involving two physics-based integration tasks in which the differential form was not a Riemann product: kinetic energy and gravitational force. To productively reason about these tasks, participants must have a conception of integration consistent with the conceptual analysis described above.

Additionally, both tasks required an additional level of quantitative reasoning, as participants had to coordinate the quantitative relationship for density as part of the quantification of a local model.

Hypothetical Learning Trajectory and Task Design

As part of the Fall 2020 teaching experiment, I developed a hypothetical learning trajectory and corresponding task sequence aimed at engendering a Quantitatively Based Summation conception of integration. The overall teaching experiment included six major tasks as elements of the overarching hypothetical learning trajectory—Curiosity Rover, Fluid Force on a Dam, Geometric Volume, Energy to Build a Pyramid, Grading Definite Integrals, and Design a 'How-To' Guide—along with 2 additional prompts as a part of the follow-up task-based clinical interviews. Most major tasks included multiple subsections, and sub-prompts, aimed at developing specific aspects of participants' emergent quantitative models. In the following sections, I provide detail for each major task, including all sub-prompts and hypothesized learning trajectory, which characterized my hypothesized image of how participants would engage with the teaching experiment material.

Task 1: Curiosity Rover

In the Curiosity Rover task, students were presented with the goal-orientated activity, in four parts, of identifying whether the rover would complete its mission on Mars. The task provided specific readings for rates of dust accumulation at different geographic locations and had a limitation that the rover could not continue operating when it had over 400 mg of dust on its solar cells. Students were provided an applet with a slider that presented data for specific sites which included location, total distance along the path from the landing site, composition of the surface, and the corresponding rate of dust accumulation (Figure 7).

The Curiosity Rover task was chosen and adapted because of its ability to provide a meaningful context for students to motivate the need for identifying an overestimate using the accumulation of local estimates (i.e. a global model consistent with an adding up pieces conception). I also felt it was important to provide students with a context other

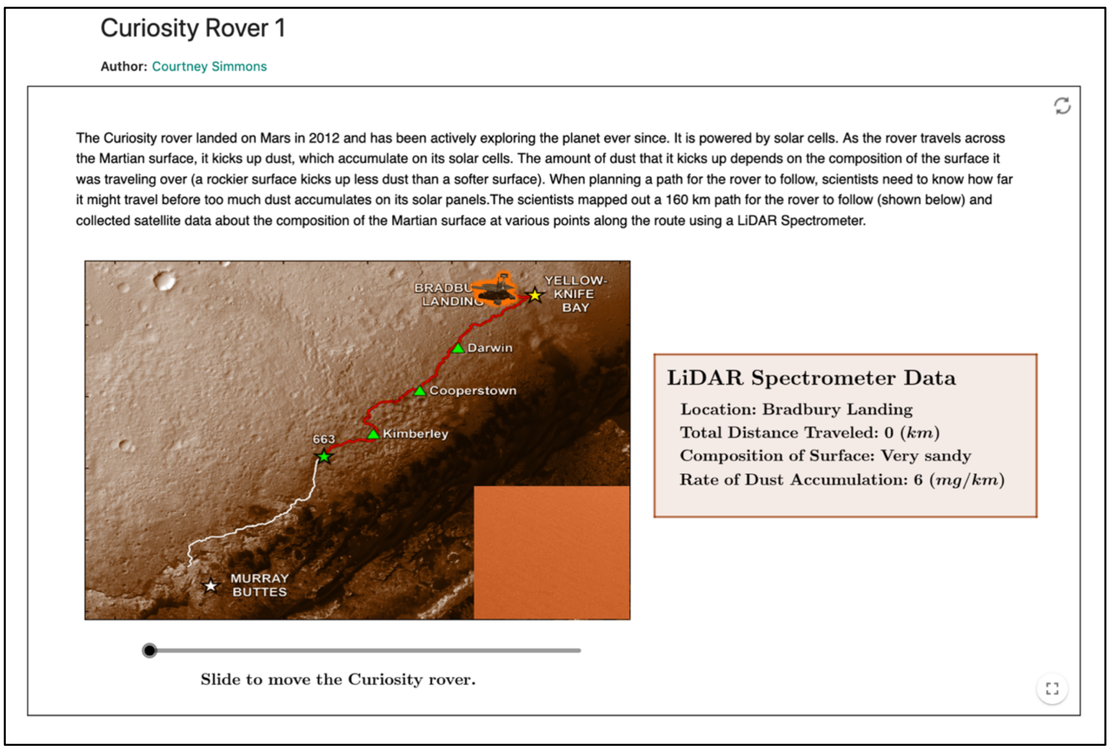

Figure 7: Curiosity rover applet

then the typical position-velocity-acceleration relationship often used to introduced Riemann sums. While velocity problems can provide powerful meaning which provides opportunities for sense-making, there is a danger of students over-relying on this relationship as a heuristic which can impede their ability to adapt to accumulation problems involving more advanced basic models. I anticipated participants would be exposed to a Riemann sum example involving velocity in the course of their regular calculus course, so I did not include such an example in the teaching experiment. In the following paragraphs, I will describe the hypothetical learning trajectory for the Curiosity Rover task. A detailed summary providing the major goals, specific question prompts, and hypothesized model development for each prompt is available in Table 4.

The anticipated conception the Curiosity Rover task aimed to engender most closely represents a finite Riemann Sum, however, no formal summation notation or language was introduced. Because this task represented the foundation for the rest of the teaching experiment, my hypothetical learning trajectory dictated prompts were that were open-ended yet also steered students' reasoning towards a specific way of reasoning. By this I mean that while many questions within the Curiosity Rover protocol were aimed at exploring students' incoming basic, local, and global models and allowed them to demonstrate any associated schemes, the questions were also relatively short and direct providing students with clear goals at each step. I planned to provide sub-prompts one at a time, and, depending on each groups' progress through the sequence, decide in the moment which to include for each group.

As an intentional design choice, the Martian sites provided on the applet were not uniformly spaced along the rover's path to require students to attend to the change in distance (Δx) as a meaningful quantity during their problems solving activity. While the function which modeled the rate of dust accumulation was non-linear and monotonic, I did not explicitly inform students of these facts, only providing data for specific locations along the path. The initial applet displayed information for 7 major sites (Parts 1 & 2), a subsequent applet adjustment allowed students to see data for the midpoints between sites (Part 3), and a final applet adjustment provided additional readings every 2.5 kilometers for a total of 65 data-points (Part 4). In addition to the final applet, I provided participants with a google sheets spreadsheet listing the total distance traveled along with corresponding rates of dust accumulation. Because the goal of providing the spreadsheet was to mitigate transcription errors for the larger data set, during the actual teaching experiment I assisted groups by entering spreadsheet formulas to compute values they requested.

In Part 1, I presented groups with a set of seven orienting questions through an editable google document. These prompts asked participants to identify and discuss rates of dust accumulation at different sites, measure distances between sites, and identify how one could approximate the total dust on the rover's solar cells as it traveled between sites if the rate of dust accumulation was constant. Specifically, the inclusion of these introductory questions served to (1) familiarize students with using the Curiosity Rover GeoGebra applet (2) bring attention to quantities within the task that would be necessary to complete their goals in later steps (3) provide an opportunity to explore students' basic models involving rates of change which would be necessary for Parts 3-4.

Part 2 of the curiosity rover task involved students identifying an overestimate and an underestimate for the total dust accumulated on the rover's solar panels. Through the applet, I provided participants information about the rate of dust accumulation at 7 different sites along the rover's path and asked questions that built towards having students explicitly identify both an overestimate and an underestimate for the total amount of dust at the end of the rover's journey. Despite only needing an overestimate to satisfy the task's main objective (whether or not Curiosity would be able to complete its journey on Mars) I included the need to identify an underestimate due to my anticipation of providing participants an opportunity, in Parts 3 and 4, to coordinate that more data provides a more accurate global model. Having students identify both these estimations also allowed me to naturally deter students from attempting to estimate the 'actual amount of dust' using average values. I want to clarify that the rest of the Curiosity Rover task required students to identify both overestimates as well as underestimates, for the reader's ease I will only reference prompts in terms of overestimates (unless significant) in this section.

In the first two prompts in Part 2, I asked participants to identify an overestimate for neighboring locations along the rover's path which, unlike the sites in Part 1, did not share identical rates of dust accumulation. Through these prompts I hoped to extend students' basic models to what I will describe as a gross basic model. A gross basic model represents applying a quantitative relationship that holds for constant quantities (a basic model) to a quantitative relationship in which one, or more, of the quantities is varying. An important aspect of a gross basic model is the recognition that the quantity obtained is only an approximation, and that the varying quantity within the gross basic model must be bounded (either above or below depending on the desired approximation). To support students in the development of a gross basic model I included additional prompts asking them to justify any assumptions which must be made to assert their estimations are accurate. Following this I hoped to engender an initial conception of a global model as being the aggregation of two, or more, values produced by a gross basic model applied to each subsection of the journey. Specifically, students would be asked to identify an overestimate for the amount of dust on the rover's solar panels if its journey was extending to the next location along its path. To extend this image of a global model I asked groups to progressively identify overestimate values for the amount of dust amassed over the next leg of the journey until the entire global model from Bradbury Landing to Murray Bates was constructed.

The final prompt in Part 2 asked students to make a recommendation to NASA based on their findings. This question was left open-ended to allow students the opportunity to demonstrate any prior basic-local-global model relationship schemes which may exist from previous experience. I did not anticipate participants in the study suggesting to send the rover as-is because the expected overestimate for the total amount of dust, 471.25 mg, was far over the 400 mg limit. However, as I aimed to direct students towards the creation of a local model by obtaining additional data, I wanted to steer students away from an attempted Curiosity redesign by including a disclaimer in the final prompt, "Note: rebuilding Curiosity would cost a considerable amount of time and money. Redesign should be suggested only as an absolute last resort." Although I did not necessarily anticipate spontaneous requests for more data in Part 2, I wanted to provide participants the opportunity to demonstrate any schemes associated with a refining process before I introduced prompts that would directly invoke the concept into the teaching experiment. I also hoped that the expected underestimate, 295.75 mg, being nearly 200 mg less than the overestimate value would provide students with the impetus to identify estimate values that were closer together. To directly introduce this notion, if not prompted by students, Part 3 provided context that an intern suggested using additional readings from the LiDAR spectrometer to obtain more information. A new

Geogebra applet was provided containing readings for the midpoints between each original site along the rover's path with corresponding rates of dust accumulation. Within the applet these midpoint locations were referenced as 'En route to ___.'

The first question in Part 3 asked students to evaluate the intern's suggestion, which was included to evaluate any schemes students may have for the refinement of an approximation now that the idea was introduced directly. Following this participants would be asked to again identify overestimates and underestimates for the amount of dust the rover would accumulate by the end of its journey on Mars (this time without scaffolding questions). The goal of having students rework the task using the new dataset was twofold. First, it provided an opportunity to initiate the development of a local model as a refinement of a gross basic model through a partitioning process rather than an appending process. In this case the partition was created for them, providing data for the rate of dust accumulation between each of the 7 major sites, and therefore would not yet represent a true local model.. Additionally, the partitioning process would provide an opportunity for participants to observe, through their goal-oriented activity, that additional data, obtained through a refinement of the original dataset, resulted in a smaller difference between their new underestimate (332.325 mg) and their new overestimate (420.075 mg). This would begin to inform the development of a critical local-global model relationship; that the refinement of a partition will result in a global model with less error.

For the final task in Part 3 I asked for students for new recommendations for NASA. By design the overestimate would still lie outside mission parameters, however, the approximations were orchestrated so that the overestimate would be closer to the 400 mg limit than the underestimate. I made this choice to, once again, provide students with an expectation that they could find a way to guarantee mission success. I anticipated that because participants were just provided additional data which resulted in a lower overestimate, they would suggest getting additional data about the surface on Mars which would act as an additional step towards building the local-global model refinement relationship.

In Part 4 groups were provided an updated applet and Google Sheets spreadsheet which contained data for rates of dust accumulation every 2.5 kilometers along the rover's entire path and a single prompt to provide a new recommendation to NASA. The overestimate of total dust accumulation with the provided data would fall below the critical threshold of 400 milligrams, allowing students to assert that the mission would succeed. There were three primary reasons I decided to provide students a spreadsheet rather than an applet alone: (1) to mitigate transcription errors, (2) to expedite computations, and (3) an expectation that students would be afforded the opportunity to observe patterns in their computations. In particular, I anticipated that providing a spreadsheet in which underestimates and overestimates for each segment of the journey would be computed individually, students would be positioned to identify that when distances between data points are a fixed value, the segment terms for their underestimate and overestimate share a majority of common values. While not crucial for the completion of the Curiosity rover prompts, this observation would serve to help students' problem-solving in the next major task of the teaching experiment in which they would need to identify how small they needed to make distances in their local model to obtain an approximation within a specified error bound. The primary reason for Part 4's inclusion into the sequence was to reinforce participants' schemes for their gross basic, local, and global models developed in Parts 2 & 3. Specifically, I wanted participants to coordinate that a reduction in the distance between data points (a refined global model through a decrease in the distance for the gross basic models) continued to reduce the error bound of their global model. Allowing the underestimate (336.7 mg) and overestimate (381.25 mg)[10] to be well within the 400 mg limit provided a 'feel-good' stopping point for the task.

With the inclusion of Part 4 I acknowledged students may not be proficient at making computations within a Google Sheets spreadsheet, and therefore anticipated helping with that aspect of the task. However, I planned to be careful in only making the specific computations requested by participants. The reason for my desire to carefully attend to instructions was because, occasionally, when Riemann sums are presented to students through regular lecture/coursework, computational shortcuts may be demonstrated. A common strategy, anecdotally observed, is when students factor out the quantity which is traditionally notated as a Δx, a measure of distance in this case, from a string of summations, $[\,]\Delta x + [\,]\Delta x + \cdots + [\,]\Delta x = ([\,] + [\,] + \cdots + [\,])\Delta x$, where which each [ ] represents a rate of change. While such manipulations are algebraically equivalent, they

10 Note that both the underestimate and overestimate in Part 4 are lower than the average value of the underestimate and overestimate from Part 2 (383.5 mg). This was part of the data design in case students' displayed a propensity for assuming linearity throughout the rover task.

are not quantitatively equivalent. That is, for this context, the desired quantitative relationship on the left represents an accumulation of small amounts of dust that were collected as the rover traveled along different sections of the overall path, all 2.5 km in length. The expression on the right loses this quantitative meaning, changing the construct from a parts-of-a-whole symbolic form to some multiplicatively based symbolic form. In addition to not representing the quantitative construction, I was aiming to engender in students, I also recognized that there is potential for students to conflate an algebraically derived symbolic template with the conceptual scheme underlying the basic model for a Riemann product (i.e. $([\,] + [\,] + \cdots + [\,])$ would collectively represent a single rate of change due to its placement within the symbolic form for a Riemann product $[\,]\Delta x$).

Table 4: Curiosity Rover prompts and hypothetical learning trajectory

	Primary Context Prompt	Sub-prompts	Goal/Hypothesized model development
Part 1	Before deploying the rover for launch, NASA scientists were required to identify the parameters that would ensure that Curiosity remained operational for the entire mission schedule. This included best-case and worst-case scenarios for the accumulation of dust on the solar panels as the rover traveled across the Martian surface. According to mission parameters, Curiosity's solar panels cannot handle more than 400 mg of accumulated dust during its trip from Bradbury Landing to Murray Buttes. **Major Task:** Identify relevant quantities by using the Curiosity rover Geogebra applet.	What was the rate of dust accumulation at Bradbury Landing? What does this mean?	Establish a baseline for participant's basic model of total dust on the rover.
		What was the composition of the surface of Mars at Yellow-Knife Bay?	Build an association between the magnitude of rate of dust accumulation and a quality of the Martian surface.
		How far did the rover travel to get from Bradbury to Yellow-Knife Bay?	Bring attention to, and evaluate schemes for, the quantification of distance.
		What was the rate of dust accumulation at Yellow-Knife Bay? Is that rate higher, lower, or the same as the rate of dust accumulation at Bradbury Landing?	Establish a baseline for participant's basic model of total dust on the rover.
		If we assume that the rate of dust accumulation between Bradbury Landing and Yellow-Knife Bay was constant, how much dust would accumulate on the rover as it traveled between those two sites?	Establish a baseline for participant's basic model of total dust on the rover.
		What was the composition of the surface of Mars at Kimberly?	Build an association between the magnitude of rate of dust accumulation and a quality of the Martian surface.
		How far did the rover travel to get from Yellow-Knife Bay to Kimberly?	Bring attention to, and evaluate schemes for, the quantification of distance.
Part 2	Same as Part 1 **Major Task:** Identify an overestimate and an underestimate for the total dust on the Curiosity rover's solar panels after traveling from Bradbury Landing to Murray Bates using datapoints for 7 major sites along its path.	Based on the limited data available from the LiDAR Spectrometer, identify an overestimate for the amount of dust accumulated on Curiosity's solar panels as it traveled from Yellow-Knife Bay to Darwin. Clearly articulate any assumptions that must be made to justify your overestimate as an accurate worst-case scenario.	Extend a basic model [total dust]=[constant rate of dust accumulation]·[distance traveled] to a gross basic model for an overestimate when a rate is non constant, [approximate total dust]=[highest rate of dust accumulation]·[distance traveled]. Coordinate requirement of boundedness for the use of a gross basic
		Identify an underestimate for the amount of dust accumulated on Curiosity's solar panels as it traveled from Yellow-Knife Bay to Darwin. Clearly articulate any assumptions that must be made to justify your underestimate as an accurate best-case scenario.	Extend a basic model [total dust]=[constant rate of dust accumulation]·[distance traveled] to a gross basic model for an underestimate when rate is non-constant, [approximate total dust]=[lowest rate of dust accumulation]·[distance traveled]. Coordinate requirement of boundedness for the use of a gross basic model.
		… Darwin to Cooperstown?	Same as above.

	Primary Context Prompt	Sub-prompts	Goal/Hyptohesized model development
Part 2 Cont.		Provide the worst-case and best-case scenario for the amount of dust accumulated on Curiosity's solar panels as it traveled from Yellow-Knife Bay to Cooperstown.	Establish a global model for accumulation as the aggregation of two gross basic models through an addive process.
		from Yellow-Knife Bay to Kimberly? to 663? …	Same as above.
		Provide the worst-case and best-case scenario for the amount of dust accumulated on Curiosity's solar panels for its entire mission from Bradbury Landing to Murray Buttes.	Establish a global model for accumulation as the aggregation of two, or more, gross basic models through an additive process.
		Provide a recommendation for NASA based on your results. Include the methods by which you arrived at the parameters that informed your recommendation. Note: rebuilding Curiosity would cost a considerable amount of time and money. Redesign should be suggested only as an absolute last resort.	Allow for reflection on magnitudes of underestimate and overestimates in relation to outlined limitations to support global model development (i.e. establish an understanding of participants' gobal model schemes in relation to the 'actual' amount dust being between these two estimate values). Provide opportunity for participants to demonstrate any prior local/global model relationship through the request of additional data/mention of partitioning.
Part 3	An intern suggests using additional surface data from the LiDAR Spectrometer. She ran some initial numbers and thinks that getting information for the surface composition at midpoints between each site will allow for better mission projections. **Major Task:** Identify an overestimate and an underestimate for the total dust on the Curiosity rover's solar panels after traveling from Bradbury Landing to Murray Bates utilizing additional information provided through the applet for the rate of dust accumulation at midpoints between the 7 major sites.	What do you think of the intern's suggestion? What affordances would this give the team? What are the limitations of her suggestion? Under what conditions will her suggestion result in a more accurate assessment of the situation?	Initiate the development of a local model as a partitioning process. Provide opportunity for participants to demonstrate any prior local/global model relationship through the request of additional data/mention of partitioning.
		All of the data suggests that the rate of dust accumulation never increases along Curiosity's path. Under this assumption, what is the worst-case scenario for the amount of dust accumulated on Curiosity's solar panels for its entire mission from Bradbury Landing to Murray Buttes. What is the best-case scenario?	Establish a local model as an extension of a gross basic model. Directly introduce the neesessary assumption of monotonicity.
		Provide a recommendation for NASA based on your results. Include the methods by which you arrived at the parameters that informed your recommendation including a comparison to your previous recommendation.	Same as Part 2. Provide opportunity for participants to demonstrate any developing local/global model relationship through the request of additional data/mention of additional partitioning.

	Primary Context Prompt	Sub-prompts	Goal/Hyptohesized model development
Part 4	You've sent the intern back for more data! She comes back a little haggard, but now there is now data every 2.5 kilometers along the rover's path. Based on the new data, provide a recommendation for NASA based on your results. Include the methods by which you arrived at the parameters that informed your recommendation including a comparison to your previous recommendations. **Major Task:** Identify an overestimate and underestimate for the total dust on the Curiosity rover's solar panels after traveling from Bradbury Landing to Murray Bates utilizing additional information about the rate of dust accumulation every 2.5km along Curiosity's path.		Reinforce development of a local model as an extension of a basic model through a partitioning process. Emphasize that a refinement of a local model, by lessoning the magnitude along the partitioned interval, is in service of lowering the error of a global model. Provide opportunity to observe equivalent values within neighboring terms of a Left/Right Riemann sum when distance is partitioned uniformly. Have participants reflect on their problem-solving process.

Task 2: Fluid Force on a Dam

The Fluid Force on a Dam task's primary objective was to encourage the development of a local-global model relationship for accumulation with particular attention to the requirements to develop, and impact of, a refined local model. In the Fluid Force on a Dam problem, I tasked students with identifying over and underestimates for the total fluid force exerted on both rectangular and trapezoidal-shaped dams. The Fluid Force on a Dam task was situated so that identifying over and underestimates was in service of providing parameters that allowed a superior to minimize the total cost of the dam. One reason for the choice of context was its ability to provide its own intrinsic motivation for identifying estimate values. Based on feedback from the summer course, engineering students emphasized that contexts in which they could easily see connections with their course majors provided more motivation than those which did not. This was particularly true for the dam tasks, which many students referenced specifically.

In addition to the purely motivational, there was a far more important reason for including the Total Force on a Dam task early in the teaching experiment sequence. As students moved forward in their construction of a scheme for integration, I felt it was important to quickly introduce them to tasks involving basic models which did not represent a prototypical Riemann product quantitative structure. This would (1) provide students an opportunity to reason about such structures as their basic-local-global models were in development, and (2) provide a challenge to those students who may have already constructed a scheme for integration that was based in antidifferentiation or consistent with schemes for the differential as a Riemann product. The Total Fluid Force on a Dam task was a good candidate for this transition because, while possible, previous studies indicated that it was unlikely for Calculus I students to reason about this quantitative relationship as a Riemann product. Additionally, while the quantitative relationship within the Fluid Force on a Dam task is more complex, the basic model still represents a relatively simple multiplicative product, [force]=[pressure]·[area]. In the following paragraphs, I describe the hypothetical learning trajectory for the Fluid Force on a Dam task. A detailed summary providing the major goals, specific question prompts, and hypothesized model development for each prompt is available in Table 5.

Because no prerequisite physics knowledge was required for participating in this teaching experiment, I included a small preamble outlining fluid force and fluid pressure at the beginning of this task providing access to the quantitative structures for these basic models. Students were encouraged to voice any concerns/questions they had about these quantities which I would clarify. In addition to the introductory paragraph, I prepared a GeoGebra applet (Figure 8) and series of prompts, which I called the Box Underwater activity, to assist students in familiarizing themselves with the basic models. I planned to only introduce The Box Underwater activity if the unfamiliarity of force and/or pressure caused too much difficulty for the groups as they worked through the primary task sequence. I hypothesized that because students did not quantitatively construct the basic models for fluid force and fluid pressure themselves, unfamiliarity with the quantities involved may result in improperly assigned values, or an inability to recognize that a partitioned area of the dam at a shallow depth would be subject to less force than the same sized partition at a lower depth. If such a situation arose, I anticipated students would be unable to productively develop an appropriate local model. The GeoGebra applet was a relatively simple construction that allowed students to adjust a slider to change the size of a cubed box and to drag a corner of the box below the surface of a blue translucent box, representing water, to the desired depth. The applet rendered 2D and 3D models of the box and water and displayed the depth and pressure at the bottom edge of the cubed box. The Box Underwater task, and corresponding applet, was purely supplemental and not planned to be presented to every group.

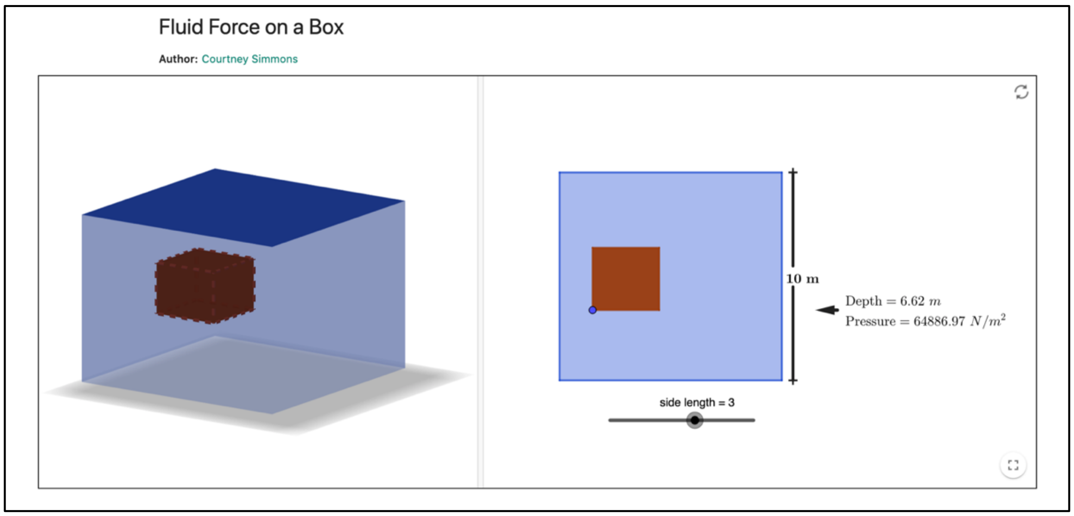

Figure 8: Fluid Force on a Box GeoGebra applet

I introduced participants to the Fluid Force on a Dam task through a request to articulate why a basic model was not appropriate in this situation. This prompt served two purposes (1) to draw attention to, and give students an opportunity to engage with, the basic model for fluid force, and (2) to engage students in reasoning about the inappropriateness of a basic model when one comprising quantity was varying. Following this, I asked participants to identify an overestimate and an underestimate for the total fluid force that would be exerted on a rectangular dam. I left this prompt vague in terms of precision to allow participants the opportunity to introduce any evidence of local-global model development spontaneously. As a part of this prompt, groups were also presented with a corresponding GeoGebra applet (Figure 9). Participants could use a slider to view pressure at specific depths along with the dam's vertical height below the waterline. The applet was relatively simple, as its only real purpose was to allow students to coordinate the information provided in the introductory paragraph a visualization of how pressure increases with depth. By visually highlighting the dam's height along with the interval of depths, it also served to provide students with the relevant quantity which would need to be partitioned in later prompts.

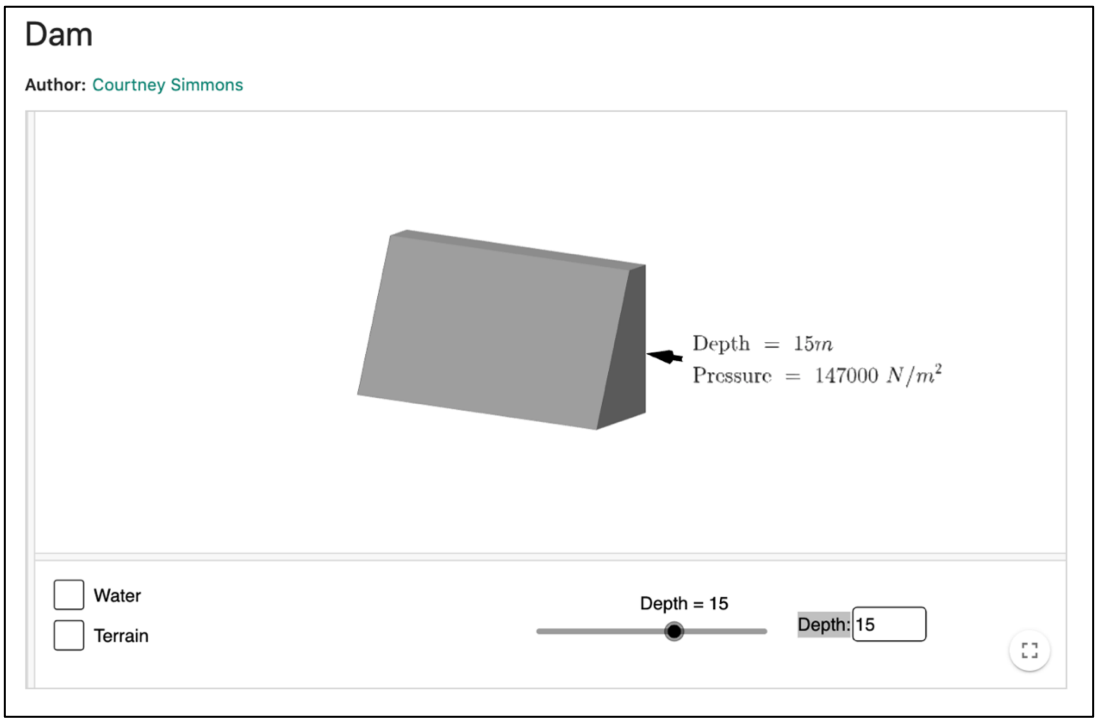

Figure 9: Fluid Force on a Dam GeoGebra applet

After identifying initial over and underestimates, I encouraged students to improve their estimations based on a supervisor's feedback to partition the dam into at least two pieces. In particular, because in the previous task I provided all data refinement, I anticipated students would provide initial estimates based on the gross basic models developed in the Curiosity Rover task. Because students were provided a formula for pressure at a given depth, rather than discrete data, the direction to create a refinement was to have students extend their gross basic schemes from the previous task from discrete to continuous data in which they performed the partitioning and measurement process. The required division into two sections also served to engage participants in coordinating the total height of the dam with a depth *d*. In essence, this prompt encouraged students to explicitly construct a local model for each half of the dam which required them to reason about the quantitative relationships involved in relation to their basic and global models. The choice to limit the number of partitions to 2 was to allow for direct connections between this task and their previous problem-solving activity in the Curiosity Rover task. What precisely was meant by 'partitioning the dam' was left to students to interpret for themselves as I did not want to explicitly articulate the mental activity of assuming constant pressures over two sections of the dam each ½ the original area. I wanted this to be an accommodation to the developing scheme from the first task as an extension of a gross basic model to a local model. In addition, while it was not anticipated to cause difficulty for a rectangular dam, the groups would have to reason about a supplied basic model for force by drawing on their area component of the basic model as a part of the problem-solving process.

The next prompt asked students to explicitly describe what informed their partitioning process. I anticipated students would bisect the area of the dam at a depth of 12.5m without much active consideration for why they made that choice, and by having students articulate and justify their precise method of division and computations, I aimed to provide the opportunity to make connections between their intuitive response and the varying quantities and partitioned depth. I also wanted the participants to coordinate that pressure itself is dependent upon the depth below the waterline which justifies why a horizontal, rather than vertical, bisection of the dam area is warranted. Students may have reasoned through a similar process internally, but it was important to have a record of that reasoning as part of the study. I also left the number of partitions to the participants themselves, so as not to limit their solution strategies. I asked participants to compute their new totals and draw a diagram of the quantities involved in the task, reinforcing the quantitative nature of the values involved in their calculations.

Part 3 was designed to direct students' goal-oriented activity towards making a connection between the refinement of a local model and the decreased magnitude of the global model error bound. I began by asking students to provide a "better set of parameters" by partitioning the dam into 5 pieces. This prompt directly informed the development of a refinement relationship between their local and global models, while allowing participants to extend their previous computations for a previous local model to one in which the height of the partition was smaller. I anticipated this would allow participants to observe that the overall quantitative structure, and underlying scheme, for the local model remains the same under a refinement process, but that the overall magnitude of each component of the subsequent global model would decrease.

Following these computations I asked students, within the given context, to identify over and underestimates that are within 50,000 Newtons of the actual value for the amount of fluid force that would be exerted on the dam. This prompt aimed to more directly coordinate the implication that decreasing the magnitude of the interval length in the local model (through additional partitions) permits one to create a global model that is accurate to within a predefined error bound. I believed that such connections had the potential to be rapidly produced through productive conversations between participants following this prompt, but also anticipated they may find difficulty in noticing the values within their previous computations which would aid in the formalization of those connections. Specifically, because the dam's upper edge was at a depth of 0, the underestimate value for the fluid force acting on the 'first' partition within an underestimate global model should be 0. This fact results in the overestimate value for the 'last' partition within an overestimate global model being equivalent to the error bound providing a means to computationally identify how small they must make the height of the partitions to have an error bound of less than 50,000 Newtons. To assist students in making this connection, I created a supplementary prompt that explicitly draws attention to these quantities. I then asked students to identify the number of partitions necessary to compute such a value, providing an opportunity to coordinate the magnitude of a local model with the number of elements that comprise a global model.

As a result of the preliminary study, I made the conscious decision to avoid introducing summation notation into the teaching experiment and therefore provided participants a means of computing the requested over and underestimates through the use of the GeoGebra summation applet. The primary purpose of having students compute their overestimate and underestimates using the sum calculator was (1) to provide computational assurance that their methodology for identifying the number of partitions required, reinforcing the link between the refinement process and improved global model, and (2) to begin to introduce components of a symbolic template for a definite integral that was inherently tied to their developing basic-local-global models. Although I put care into the GeoGebra applet, by design it was not generalizable enough to accept any computation/variable. To be able to effectively use the calculator for the dam task, students would have to coordinate the quantities and variables they created during their problem-solving activity with more traditionally used variables and notation. Specifically, the applet introduced the construct of $\Delta x.$ I anticipated students would be familiar with Δx from their calculus course but would bring different underlying conceptions such as "change in x" vs "amount of x." I designed the applet and task sequence to encourage the latter, and, as I will elaborate on in the discussion section, the critical conceptual schema students must adopt into their models is that Δx represents the magnitude of a quantity within the local model that is constant across all components of a global model. This would correspond to the fact that x was the quantity partitioned in the creation of the local and global models and therefore provides the meaning for "starting x" and "ending x" to represent the two values whose difference measures the magnitude of the entire quantity x. By requiring the inclusion of the Δx notation, the "expression to be summed" entry represents an algebraic representation of their local model (which matches the quantitative structure of the basic model). As a final prompt in Part 3, I asked students if it was possible to identify estimates accurate to within 1 N of the real amount of force on the dam. This question was aimed at eliciting the flexibility of their new local-global model refinement relationship for improving the accuracy of a global model.

In Part 4, I required students to adapt to a similar, but new, global structure by changing the shape of the dam to a trapezoid. I included this prompt to require engagement with the quantitative reasoning necessary to adapt to a local model in which more than one quantity of the corresponding basic model varies. In this case, through the rectangular dam activity, participants had worked with local models in which pressure changed between components of the global model but area remained a fixed value. By altering the shape of the dam, for two different elements within the global model, the values for both pressure and area would be distinct. The prompt itself was left relatively open-ended to allow participants to draw on their previous problem-solving activity. For the final prompt of the Fluid Force on a Dam task, I asked students to hypothesize a method by which they could identify the exact Fluid Force on a Dam, hoping students may recognize a connection between their refinement process and the concept of limits which they would have covered in their calculus class. This would serve to further develop their global models, and potentially their local-global model refinement relationship.

Table 5: Fluid Force on a Dam prompts and hypothetical learning trajectory

	Primary Context Prompt	Sub-prompts	Goal/Hypothesized model development
Part 1	Fluid force is the force on an object submerged in a fluid. This is the force divers feel as they descend below the water surface. For a uniform fluid pressure, P, across a surface area A, the total fluid force is F=PA Newtons*, Fluid pressure, P, is proportional to the depth of an object and does not act in a specific direction. Rather, fluid exerts pressure on each side of an object in the perpendicular direction. The pressure Pat a depth d, can be measured by P=gd where is the density of the liquid and g is acceleration due to gravity (9.8 m/s^2). The density of water is 1000 kg per cubic meter, so for an object submerged in water, fluid pressure can be modeled by P=9800d N/m^2. *A Newton is a standard unit for force, and 1 N = 1 $kg \cdot m/s^2$ which is the force required to accelerate a 1 kg mass at 1 m/s^2. **Major Task:** Preamble provides access to the quantiative structures for pressure and fluid force. Subprompts, and corresponding applet, only introduced if deemed necessary.	Create a box with side lengths 4m×4m×4m and lower it to a depth of 25 meters. What is the fluid pressure at this depth?	Coordinate fluid pressure as a quantity varying with depth.
		What is the total force acting upwards on the bottom of the box?	Establsih use of a basic model for fluid force at fixed depth.
		What is the total force acting downwards on the top of the box?	Establsih use of a basic model for fluid force at fixed depth. Quantify differing forces on top/bottom of box.
		Change the box to be 3m×3m×3m . What is the total force acting upwards on the box? What is the total force acting downwards on the box? What changed between this computation and the last?	Coordinate the basic model for fluid force as a quantity varying with area for a fixed depth.
		Move the box so that the base is at a depth of 75 meters. What is the total force acting upwards on the box? What is the total force acting downwards on the box? What changed between this computation and the last?	Coordinate the basic model for fluid force as a quantity varying with depth for a fixed area.
		Compute the total force acting downwards on a box that is 1.7 meters tall, 4.26 meters wide, and 2.45 meters long, that has been lowered to a depth of 63 meters.	Reflect on the goel-oriented activity to support basic models quantiative structure without support of applet.
		Can you use the same computations to find total force acting on the sides of the box? Explain why or why not.	Reflect on varying quantities for force. Identify that direct application of the baisc model is not warranted.
Part 2	Initial plans for a rectangular dam in Argentina are being drawn up as we speak. The dam is planned to be 63.26 meters wide and reach a total depth of 25 meters. Decisions must be made in order to minimize costs of such a massive project. In particular, the type of concrete used to build the dam from will reflect a significant portion in the overall budget. In order to know which material is appropriate for the job, it will be important to identify the parameters for the total fluid force that the dam must withstand. You've been tasked with identifying this figure. Your goal is to provide under and overestimates for the total fluid force that is likely to be exerted on the dam. **Major Task:** Identify an over and underestimate for the total fluid force on a rectangular dam.	Explain why we cannot just multiply a pressure times an area to compute the fluid force acting on the dam.	Coordinate the inappropriateness of a basic model for varying quantities. Motivate creation of a local model.
		Provide an overestimate for the total fluid force on the dam. Provide an underestimate of the total fluid force	Motivate gross basic model as appropriate tool for rough estimate when one a quantitty in basic model varies.
		You've taken these numbers to your boss and he balked! According to him this range of values is entirely too large, and you'll need to provide more accurate parameters if he's going to be able to make an informed decision about materials. He suggests that at the very least you could partition the dam into 2 pieces before computing any approximations for over and underestimates. He then waves you away to complete the task and returns to an important phone call. Does your boss's suggestion have merit? Why or why not?	Promote the development of a local model as a partitioning of a whole quantity which is comprised of two, or more, pieces which can be approximated through a gross basic model. Coordinate a global model comprised of local models as being more accurate than a gross basic model applied to the entire situation.

	Primary Context Prompt	Sub-prompts	Goal/Hypothesized model development
Part 2 Cont.		How will you partition the dam into two pieces? Explain what informed your decision to partition the dam in this way.	Motivate a gross basic model as an appropriate tool for providing a rough estimate when one a quantitty within a basic model is varying.
		Identify new over and underestimates for the total force acting on the dam. Draw a picture of your dam and label it with all the quantities involved.	Reinforce connection between the local and global models.
Part 3	While you have these new computations ready to go, you've decided to make up for your initial report by providing even better parameters than your boss suggested. **Major Task:** Create an increasingly refined partition of the dam in an effort to provide a solution accurate to within a predfined error bound.	In an effort to impress, identify over and underestimates for the total force on the dam if you partition it into 5 pieces.	Strengthen connection that lowering the magnitude of the local models improves the global model. Motivate quantiative relationship for Δd as the measure of the entire depth divided by the number of partitions.
		You've presented your new report to your boss and he's happy with your work so far. However, he notes that this range of possible forces doesn't completely narrow down the choice of materials. He'd like you to run some additional numbers and get back to him with a range of possible forces that is accurate to within 50,000 N.	Motivate an explicit coordination that by reducing the magnitude of a local model by a significant enough amount, the global model can be made as accruate as a predefined bound.
		* This seems like an impossible task, but as you're fiddling with computations from your most recent report you notice that the parameters have an error bound of at most 77493500 N. This also happens to be the exact overestimate for the deepest partition of the dam from that same report. Why are these values the same?	*Supplmental task preprepared. Anticipated students will have productive discussions, but will find it difficult to move forward computationally. Motivates the comparison of individual values produced by local model within the over/under global models to provide a means to identify the maximum magnitude of the local model.
		How many partitions do you need to make to identify the total fluid force on the dam accurate to within 50,000 N?	Create explicit coordination that increasing the number of partitions directly impacts the accuracy of a global model.
		Use the GeoGebra Sum Calculator to identify the over and underestiamtes.	Build connections for quantities which will be important to the template of a definite integral symbolic form within the context of their basic, local, and global models for accumulation.

	Primary Context Prompt	Sub-prompts	Goal/Hypothesized model development
Part 3 Cont.		Could you identify the total fluid force acting on the dam accurate to within 1 N? How?	Reinforce explicit coordination that by reducing the magnitude of a local model by a significant enough amount, the global model can be made as accruate as a predefined bound.
Part 4	More information has come in, and it turns out the dam won't be in the shape of a rectangle after all. The canyon the dam will be built in is narrower at the base than at the top. This means that while the top of the dam will be 63.26 meters wide, the base will only span 37.92, creating a trapezoid. **Major Task:** Engage in the quantiative reasoning necessary to adapt their solution from the previous task to a similar situation in which the quantitative structure of the basic model has changed.	Your boss would like you to draw up a new report with the force information for this new design.	Reinforce the required quantitative structure of a local model matching that of a basic model by changing the way in which local model width is quantified along depth. Coordinate need for local model to provide accurate approximation for any depth and magnitude of partition necessary.
		Is there a way to identify the exact force on the dam?	Draw out any connections between a refinement process and existing conceptions of a limiting process.

Task 3: Geometric Volume

The primary objective of the Geometric Volume task was to motivate the introduction of the conception of a limiting value to the global model refinement process which included the notational components which constitute the template for the symbolic form of a definite integral (Table 6). The shift in the quantitative structure of the corresponding basic model for this task would also allow participants an opportunity to continue constructing interrelationships between their basic, local, and global models.

In this task, participants began with the goal of identifying the volume of a right rectangular pyramid measuring 10in tall with a 4in×4in base. Before having participants engage in creating over/underestimates for the volume, I first provided an acknowledgment that they already had working access to a formula for a right rectangular pyramid, $V = \frac{(l \cdot w \cdot h)}{3}$, and indicated that one way to arrive at the quantification of that formula is through a method of estimates similar to one they had been engaging with. By introducing these constructs together, I wanted to bring explicit attention to the connection between a real amount of a quantity and their previous goal-oriented activity of identifying estimates extending beyond simple estimations. This would prime participants for the incorporation of a limiting process into their local-global model relationship as they worked through the prompts.

I began the task by asking participants to identify an overestimate and an underestimate for the provided pyramid dimensions without restriction. This would provide insight into their anticipation of the required error bound of an acceptable approximation. Based on their response I would ask participants to refine their solution further, likely numerous times. By continuing to prompt additional refinements to their global model to decrease the magnitude of the error bound, participants would be given the opportunity to reinforce connections between their image of the magnitude of their local model, the number of partitions/elements within their global model, and the accuracy of their global model. Finally, I would prompt participants to compute the 'real' volume of the pyramid using the provided formula and ask them to make a connection between the 'real' value for the volume and their global model estimates.

I did not design any tasks in the teaching experiment to engender a specific scheme associated with limiting values prior to this task, however, because I anticipated that all participants would enter the study with at least some concept of limits from their calculus coursework this part of the task was left-relatively open-ended. Through experience with the participants' classroom coursework including the textbook and associated homework activities, such as creating limit tables, I anticipated that one image of limits participants may have constructed would involve a pseudo-convergence of a sequence of values. These values would be obtained by identifying the outputs of a function at inputs with an ever-decreasing distance from a point of interest. While I acknowledge that this is not a mature view of a limiting value, because I believed it to be a likely construct associated limits, I drew upon this idea to promote an association between local/global models and limits. In particular, by having participants create a series of overestimates and underestimates as they were decreasing the magnitude of their local model (through the reduction of a quantity Δx), they were developing a sequence with which they could connect the limiting value of the 'real volume' to within their limit scheme. Such a listing process would also allow participants with a scheme for limits that was more inherently tied with an error-bound conception to make similar connections. While I constructed the prompts to elicit the activation of a limiting scheme, I wanted this problem-solving tool to be generated from the participants' problem-solving activity rather than my explicit introduction of the construct. This would position participants to view limits as a productive tool for identifying the anticipated outcome of a continued refinement process, rather than a complicated structure superimposed on their emergent models. If necessary, a direct introduction of limits would be made.

Following a connection to limits, I introduced the symbolic notation associated with definite integrals such as the integration sign, limits of integration, and differential form (Figure 10). This introduction included rough explanations and I made explicit connections between the symbolic template and the meanings participants had constructed for aspects of their local/global models when working with the sum calculator. I provided an example of definite integral notation for the total fluid force on the rectangular dam from Task 2, before prompting participants to create a definite integral for the total volume of the pyramid and the total fluid force on a trapezoidal dam. For these interactions I took on a primarily instructive role in which I aided students in incorporating their basic-local-global models to the symbolic form for a definite integral. By revisiting previous prompts I hoped to encourage reflection on previous problem-solving activity to incorporate an exactness construct upon their global model which had previously only exclusively dealt with estimates. By providing a definite integral calculator I also provided a means by which (1) participants could check whether their constructions were consistent with previous findings, and (2) would promote an association with the symbolic form of a definite integral with a single-valued answer promoting a single global model for definite integrals, in contrast to the two global models (over/under) for estimates.

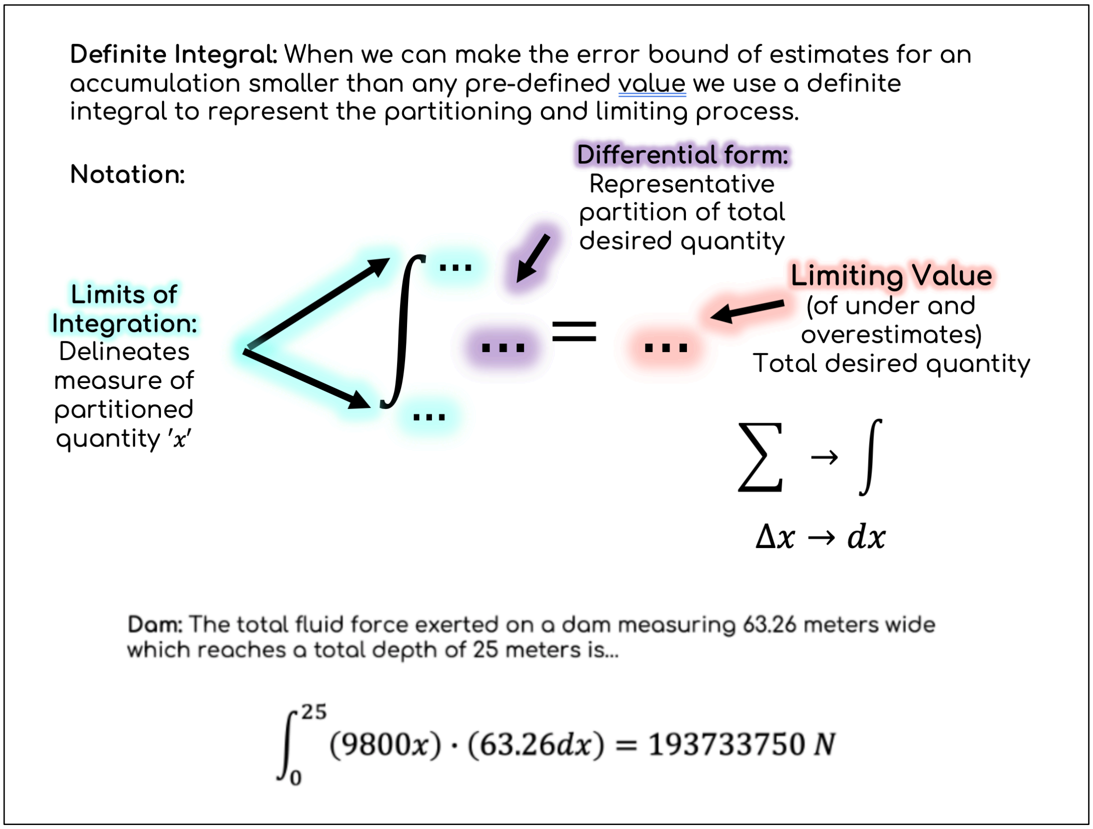

Figure 10: Introduction of definite integral notation

As a final prompt in this task sequence participants would be asked to write a definite integral that represented the volume of a sphere with 6in radius. By providing this task I hoped to allow participants additional opportunity to coordinate their models with the symbolic form for definite integrals by directing their goal-oriented activity to the construction of an explicit global model for definite integrals. I chose to ask another volume question because this would allow participants to direct the bulk of their problem-solving energy towards the creation of the definite integral because, although distinct, the quantification of a local model should be similar enough to their previous task that they would not be too terribly taxed.

Table 6: Geometric Volume prompts and hypothetical learning trajectory

	Primary Context Prompt	Sub-prompts	Goal/Hypothesized model development
Part 1	We know the volume of a right rectangular pyramid can be found using the equation V=(l*w*h)/3, but where does that formula come from? One way to find this volume is through your method of approximations.	Use your method of approximations to find overestimates and underestimates for the total volume for a right pyramid that is 10 inches tall with a square base of 4 inches.	Coordinate the development of a basic-local-global model system formatted upon a new basic model quantitative structure.
		How accurate can you make your estimates to the real value of the volume of the pyramid? Make an approximation accurate to within ___ (repeated).	Motivate reflection on estimates in relation to a real value for the volume. Create a sequence of underestimate and overestimate values through increased refinement of local-global models.
		Compute the volume of this pyramid using the provided formula. What do you notice about this value in relation to your estimates?	Establish an explicit connection between under/overestimates, a limiting process, and the real value of a quantity into basic-local-global models. Identify exact volume as the limiting value of the sequence of both the underestimate and overestimate values. Motivate notational need to represent exact value of this process.
Part 2	Introduction of definite integral notation.	Write the exact volume of a right pyramid that is 10 inches tall with. A square base of 4 inches using definite integral notation.	Establish a symbolic form for definite integrals that is rooted in the basic-local-global models for estimates. Expand global model to incorporate the construct of exactness.
		Write the exact total fluid force acting on a trapezoidal dam measuring 63.26 m at the top, 37.92 m at the bottom, with a depth of 25 m.	Coordinate the link between the fledgling symbolic form with previous goal-oriented activity.
Part 3		Write a definite integral that represents the total volume of a sphere with radius 6 in.	Coordinate the development of a basic-local-global model system which has incorporated limiting process to a slightly different quantitative situation.
		Include the meaning of each element in the definite integral.	Explicitly establish conceptual schema for the symbolic template which are connected to the quantitatively constructed local-global model relationships.
		Use the definite integral calculator to identify the total volume of the sphere.	Reinforce image that definite integral represents a single value.

Task 4: Energy to Build a Pyramid

I introduced the Energy to Build a Pyramid task with the intention of engaging participants in the goal-oriented activity of developing an exact global model, in the form of a definite integral, for a novel quantitative context. In this task, participants were asked to identify the total energy (against gravity) to build the Great Pyramid of Giza which measures 146m high with a square base of side-length 230m under the assumption that the density of the stone within the construction was 2000 kg/m^3 (Table 7). The basic models for energy, [energy]=[force]·[vertical distance traveled], and force, [force]=[mass]· [acceleraition due to gravity], were provided based on participants' demonstrated comfort with those quantities.

Table 7: Energy to Build a Pyramid prompt and hypothetical learning trajectory

Context Prompt	Goal/Hypothesized model development
Built around 2600 BCE, the Great Pyarmid of Giza in Egypt is 146m high and has a square base of side length 230m. Find the energy (against gravity) required to build the pyramid if the density of the stone is estimated at 2000 kg/m^3	Coordinate the refinment of an exact basic-local-global model system by providing context in which an adaptation of the local model is required. Coordinate the development of a local model when multiple layers of quantification are necessary due to increased complexity of basic models involved. Encourage reflection problem-solving activity for the volume of a pyramid task. Provide opportunity to coordinate the incorperation of a previously established local model into a new local model requiring a similar quantification process.
*Supplemental information Work/Energy against gravity: On the earth's surface, work against gravity is equal to the force (mass·acceleration due to gravity) times the veritcal distance through which the object is lifted. No work against gravity is done when an object is moved sideways. E=F·d F=M·g	Provided if participants unframiliar with quantification of basic model for energy Requires quantification of mass - also provided to participants if deemed necessary.

Due to the multiple layers of quantification involved in the basic model for energy, this task would also pose additional challenges to participants' quantification of their local model. In particular, as part of the expected solution strategy, I hoped to navigate participants towards assigning the differential quantity to the height component of volume buried deep within the volume component of the quantitative structure of their local model (see Figure 11). It was for this reason that the total volume of a pyramid was selected as one of the contexts in Task 3. Specifically, I wished to reduce part of the burden of identifying the best quantitative candidate with which to partition their global models global model as they were anticipating the development of a local model. By having participants draw their previous problem-solving activity of quantifying approximations for volume along the vertical height of the pyramid, they would already have an image of how such a partitioning could occur.

$$
\begin{aligned}
&[\text{Energy}]\\
&[\text{Force}] \cdot [\text{vert. Dist. traveled}]\\
&[\text{Mass}] \cdot [\text{acc. due to Grav}] \cdot [\text{vert. Dist. traveled}]\\
&[\text{Volume}] \cdot [\text{Density}] \cdot [\text{acc. due to Grav}] \cdot [\text{vert. Dist. traveled}]\\
&[\text{Length}] \cdot [\text{Width}] \cdot [\text{Height}] \cdot [\text{Density}] \cdot [\text{acc. due to Grav}] \cdot [\text{vert. Dist. traveled}]
\end{aligned}
$$

Figure 11: An expanded basic model for energy

By lessening the burden of this choice I hoped to accomplish two goals: (1) position students to spend more time coordinating the new notation, along with the accompanying limit conceptions within a global model, with the development of their local model, and (2) because volume was not an immediate component of the provided basic model, I hoped to engender sensitivity to this quantity in their problem-solving process. One concern that motivated my sensitivity to making the volume a salient feature of this task was anticipation that, because the volume component of the differential was buried deep within the quantitative structure of energy, participants overwhelmed by the new notation may attempt to assign the variable quantity of distance the differential quantity[11].

Despite preparing to limit the total burden for students' cognitive processes, I still anticipated the quantification of a local model would remain highly non-trivial and would require a considerable amount of effort on the part of the students.

Task 5: Grading Definite Integrals – Mass of Oil Slick

Task 5 was devoted to strengthening participants underlying basic-local-global model relationship with the symbolic form for a definite integral. In this task, my goal was to (1) reinforce the image of a definite integral as representing a quantitative object, and (2) provide opportunities for participants to solidify their image of specific elements within the template through an assessment of work they did not produce themselves in coordination with their image of the expected quantities within those template spots.

In this task, I provided a collection of potential solutions to an oil slick prompt which involved writing a definite integral which represented the total amount of oil in a circular oil slick with a given radial density. The provided solutions were posed as recreations of

11 I would like to quickly acknowledge that this task can be productively quantified using vertical distance traveled as the differential quantity within a local model. In fact, one group in the study did complete the task in this way. However, as part of the task design this outcome was not an expected result.

student work, and while some were, I added additional solutions which broke traditional symbolic template rules such as two differential quantities. Participants were asked to evaluate the solutions and to identify those which did not represent the correct quantitative representation and justify their response. In particular, this task was designed to provide participants access to definite integral expressions and values which did not align with their image of the proper template schemas. Such a perturbation would allow participants to evaluate their own conception of the proper template schemas, re-evaluate the solutions in light of those fresh conceptions, and then either make an accommodation into their models/symbolic form or reject the expression as incorrect.

Table 8: Grading Definite Integral prompt and hypothetical learning trajectory

Context Prompt	Goal/Hypothesized model development
Below are recreations of student provided solutions to the following prompt: **Oil Spill:** For a particular circular oil spill, the density of oil on the surface of the ocean at a distance r meters from the center of the slick is given by $\rho(r) =$ [illegible] kg/m^2. Find the total mass of the oil in the slick if it extends from $r = 0$ to $r = 18{,}000$ m  Identify all incorrect responses. Justify your choices.	Firmly relate the symoblic form of a definite integral with participants basic-local-global models. Evaluation of definite integral structure reinforces definite integral representation of a qantiative object, rather than a purely algebraic process.

Task 6: Design a 'How-To' Guide for Definite Integrals

In the final task of the teaching experiment sequence (Table 9) I asked participants to provide writeups describing the exact purpose of a definite integral and how it works to someone unfamiliar with calculus concepts. By asking participants to perform this task I was engaging them in reflecting on the relationship between their basic, local, and global models as a construct of a Quantitatively Based Summation conception of integration. In particular, this prompt required participants to reflect on how they used definite integrals throughout the teaching experiment and to explicitly coordinate the development of the definite integral construct from a rough approximation of a quantity through their complex interrelationships between basic, local, and global models to identify exact values.

Following this initial prompt, I asked groups to provide a write-up that would enable a reader to construct a definite integral expression for novel tasks. By engaging participants in this task, I was again positioning them to reflect on their problem-solving activity across the entire teaching experiment, but in this case, I was having them focus on the specific constructs which go into the process of creating a generalized local model partition which constitutes the differential form within the symbolic form of a definite integral. By asking participants to be general in their suggestions I was encouraging engagement in abstracting the commonalities in practice across different tasks which enabled their development of explicit local model construction.

One aspect I would like to make clear was that I did not anticipate evaluating the participants' write-ups for correctness. While the final artifact of the activity would provide insight into an image of participants' conceptions of their basic, local, and global models, the enhancements to participants’ emergent quantitative models were in the process of the activity, not the product.

Table 9: Design 'How-To' Guide prompts and hypothetical learning trajectory

Context Prompt	Goal/Hypothesized model development
Provide a write-up that describes exactly what a definite integral is, and how it works, to someone who has never taken calculus before. It is not necessary that the reader be able to compute definite integrals by hand, but your write-up should enable them to be able to understand the quantities involved for definite integrals such as the ones you've worked on over the past few weeks. Be sure to include specific descriptions for the notations you use.	Promote explicit reflection and articulation on symoblic form for definite integrals including coordination between early interpretation of basic-local-global models as estimates and the precise nature of basic-local-global models encaptulated by the symbolic form.
Provide a write-up that would enable a reader to construct a definite integral for tasks such as the ones you've worked on over the past few weeks. This write-up should be specific in its directions, but general enough that it can apply to novel tasks.	Promotes explicit reflection and abstraction for the development of a basic-local-global model system across contexts. Provides insight into participants image of their exact problem solving process and what aspects of the basic-local-global model system they place priority upon.

Task-Based Clinical Interviews

The final two tasks presented to students in this study were positioned to serve as tasks within a clinical interview setting after the conclusion of the teaching experiment sessions. From the participants' point of view nothing would change, however, I planned to take on the role of a researcher, rather than instructor-researcher, to investigate how participants were able to productively work through the provided tasks. Because these tasks were chosen due to their non-trivial nature, participants' engagement in these sessions would undoubtedly contribute to their overall basic-local-global model relationships, however, the tasks were not chosen to contribute any specific accommodation to the participants' schemes.

For the first task in this session, participants would be asked to identify the total kinetic energy of a rotating rod (Figure 12). This task was chosen due to its non-standard nature within a typical calculus I course, particularly as it does not naturally decompose into a Riemann product structure. No specific pre-amble for Kinetic Energy was prepared for students to evaluate their adaptability to quantitative structures they might not be familiar with. If requested, or it became obvious that unfamiliarity proved to be a barrier to progress, additional explanation for the quantitative structure of kinetic energy would be provided.

Kinetic Energy: The kinetic energy of an object with mass M and constant speed v is $K = \frac{1}{2}Mv^2$, at least in the case where the entire object is moving at the same speed. Suppose a $0.1m$ rod has a mass of $0.03\ kg$ with uniform density. It is rotating around one of its ends at a rate of one revolution per minute (like the second hand of a clock). What is the total kinetic energy of the rod?

Figure 12: Kinetic Energy of a Rotating Rod task

While the Kinetic Energy task included specific values for the relevant quantities, the second task, Gravitational Force Between a Rod and a Particle (Figure 13), did not. This introduced an additional layer of difficulty in which participants were required to reason about the quantities in the contextual situation devoid of precise measurement values as they developed their global and local models. Specifically, without values, participants would need to be intentional with reasoning about the method by which they measured different quantities. As with the kinetic energy task, the gravitational force task was chosen due to its non-standard nature within a typical Calculus I course.

Gravitational Force

The gravitational force between two particles of mass m_1 and m_2 at a distance r apart is $F(r) = \frac{Gm_1m_2}{r^2}$, where G is the gravitational constant $6.67\times10^{-11} Nm^2/kg^2$. Write a definite integral that gives the gravitational attraction between a thin uniform rod of mass M and length l and a particle of mass m lying in the same line as the rod at a distance a from one end.

Figure 13: Gravitational Force Between a Rod and a Particle task

Both the kinetic energy and gravitational force tasks required that participants develop a quantitative relationship for the density of a piece of a rod as an element of the mass component within the basic model. Because the expected solution for both tasks would situate the differential quantity within the mass component of the local model these tasks would demonstrate the success of their development of a Quantitatively Based Summation conception of integration and its influence on their ability to model definite integral tasks in which the differential form is not a Riemann product.

Teaching Experiment Data Collection and Analysis

Note that due to the outbreak of COVID-19 in Spring 2020, data for this study was conducted entirely remotely.

For this dissertation study, I recruited six students from a large southwestern university to take part in an eight-week-long teaching experiment near the middle of their Calculus I course. The interviews began roughly two weeks before the introduction of summation notation and Riemann sums as students were learning about the graphical implications of derivative functions. Due to the longitudinal nature of data collection which anticipated follow-up studies with these students in future calculus courses, I requested all Fall 2020 Calculus I instructors submit recommendations for students who they feel would be appropriate for a year-long study. Instructors were asked to make these assessments based on the students' engagement in the course, including, but not necessarily limited to, interactions in class/office hours, completion of homework, attendance, and exam scores. All recommended students were sent emails asking if they would like to participate in the study, laying out the time dedication, expectations, how the results would be used, and compensation information. Due to the large time commitment necessary, students were paid $20 per hour for their participation. I asked interested parties to fill out a small questionnaire listing their course plans for the rest of the calculus sequence (spring, summer, fall) and weekly availability. Students were then matched into groups of two based on availability. Nine parties requested interest, but only six provided availability in time to be a part of the study.

Due to social distancing restrictions, I interviewed and recorded study participants through the Zoom platform. Zoom allows for the recording of the web camera conversations, as well as shared screens, and provides a rough transcript of the interaction which was used for coding purposes. There were two types of interviews throughout the eight-week teaching experiment: paired (approximately one hour) and individual (approximately a half-hour). Both types of interviews were planned to take place twice a week for a total of 3 hours per week in interviews per participant. Most weeks all interviews took place as planned, although due to scheduling conflicts and the decision of one participant to drop from the study, adjustments to the schedules were made for all groups at some point during the eight-week study.

During the paired interviews participants worked on teaching experiment task sequence, talking with one another through zoom and writing on a collaborative online whiteboard, AWWApp.com. During some tasks participants were asked to type on a shared google document or google spreadsheet. To have access to what aspects of the task participants were referencing, one group member was asked to share their screen during the interview. The individual sharing their screen often changed from group one interview to the next. I occasionally asked clarifying or directive questions while transitioning between responsive/intuitive and analytical interactions as the students progressed through the tasks. After most interviews, I conducted follow-up individual interviews to elaborate on students' reasoning demonstrated in the group interview. These individual interviews served as intersubjectivity[12] checks and aided in the refinement of hypotheses with regard to students' reasoning. The group interviews focused on students engaging in the task sequence described in the previous section.

The decision to conduct paired interviews, rather than individuals, was made for the following reasons:

(1) The goal of the teaching experiment is to hypothesize the nature of students' thinking which requires that they produce utterances and artifacts of that reasoning. Having students work in pairs provides an outlet to communicate naturally, minimizing the role of an authority figure.

(2) Having the students engage primarily with the tasks and each other parallels classroom interactions more closely than a researcher consistently prompting their

12 On intersubjectivity: conversations enter a state of intersubjectivity when participants have no reason to question whether they understand one another; it does not matter if participants 'actually' understood each other's meaning. Conversations enter a state of intersubjectivity, not people" (Thompson, 2013, p. 65). By individually interviewing students I can further analyze conversations that appeared to be in a state of intersubjectivity, but that were not consistent with my current models of one (or both) participants.

thinking. This could provide a better implication as to the scalability of a task sequence designed to elicit a quantitative understanding of integration.

(3) Because the retention rate of STEM majors through the calculus sequence is poor, and a follow-up study is planned to revisit students during their future calculus coursework, the number of research participants must be larger than a standard teaching experiment. This will increase the probability of participation in that follow-up study. However, performing a standard teaching experiment with six students individually is not as feasible, timewise. Having the students work in pairs will allow for fewer hours of overall data collection per week while still having the same amount of screen time with each student.

(4) It is recommended that teaching experiments have a witness present in all interviews because a teacher-researcher can face difficulties in switching between roles in the moment and might miss an important interaction or utterance which implies a particular way of reasoning. However, due to the large amount of time commitment I believed it to be infeasible to request an additional party to commit 15+ hours a week to my dissertation project. Structuring the interviews in two parts allowed me to more easily attend to these roles individually. While I still transitioned between responsive and analytical interactions for both types of interviews, in the paired interviews I took on more of a teaching role and in the individual interviews I took on more of a research role. This structure of dual interviews also allowed me to review the paired interviews and prepare before individual sessions.

I analyzed the data for the teaching experiment in two phases: on-going and retrospective. As I was engaged in the process of the teaching experiment, I was in a mode of continually developing and refining my models of the participants' understandings. This involved taking notes during and after each group interview, noting significant interactions, and reviewing clips before conducting the follow-up individual interview when possible. Follow-up interviews provided an opportunity to test my hypothesized models of participants' reasoning, build a more coherent image of their evolving schemes, and let them elaborate on constructs they might not have provided enough detail on in the group interview. Such follow-up interviewers were particularly helpful in characterizing the evolving schemes of a participant when their partner took more of a leading role in the group session. When I developed specific hypotheses regarding students' reasoning that would not be investigated through the normal course of the hypothetical learning trajectory, I designed and introduced supplementary prompts and tasks which I provided to groups/individuals on an as-needed basis. Specific supplementary prompts and activities are described in further detail in the results section.

At the conclusion of the teaching experiment, I analyzed the data using constant comparative analysis using the MaxQDA analytic software. This analysis included refining the hypothetical learning trajectory and conceptual analysis based on additional passes through the teaching experiment data which characterized participants' emergent quantitative models. I began the retrospective analysis with an initial passthrough of the entire dataset writing open-ended memos which described specific interactions, implications for participants' emergent models, and any questions raised about those models through those interactions. I incorporated notes made through the on-going analysis, supplementing additional information that might not have been clear to me during the time of the teaching experiment. I used this information to enhance my overall image of the participants' emergent models as they progressed through the task sequence. Follow-up reviews of the dataset were performed to identify episodes that supported, or refuted, my evolving image of the participants' schemes until I felt that the data was no longer able to provide additional meaning to questions raised. I concluded with a cross-comparison between the emergent quantitative models of the different groups to identify commonalities and distinctions as they engaged in the teaching experiment task sequence.

As part of the data collection for this project, there were limitations to conducting the experiment online. First, the need for participants to have access to a computer, high-speed internet, and a web camera placed a potential handicap to the generalizability of my study. In particular, participants needed to have access to a private computer with internet access for at least three hours a week which means that it is likely my participants were of an above-average socioeconomic status. While I devoted a great deal of time identifying ways to alleviate this potential economic disparity, such as providing equipment at no cost to participants, it was an unavoidable consequence of the semester I collected data in.

In addition to limitations remote interviews placed on participation, I was also limited in the ways in which I could capture students' reasoning which would normally be evident in gestures, demeanor, and written work. By only viewing students' upper bodies the recordings I often missed out on slight hand movements, fidgeting, and quick scribbles made as they were problem-solving. Additionally, slow internet connections sometimes resulted in choppiness in videos and an inability to clearly identify what the participant was relaying.

Finally, it must be acknowledged that while qualitative data of this nature can assist in the development of materials for a generalized population, the participants in my study may not be an accurate representation of the general calculus population. In particular, these students were handpicked by their professors as being highly engaged and likely to succeed through to the next calculus course. There also may have been implicit biases in the instructors' recommendations which affected the outcome of the final participant population. This means that while the results of this dissertation can be used in the development of larger-scale studies and can serve to inform a generalized curriculum. Additional research should be conducted to confirm these results.

In the following section, I provide an overview of the results for my participants' engagement in the teaching experiment, including significant episodes which characterized the development of their emergent models.

CHAPTER V

RESULTS

In the following chapter, I provide an overview of the Fall 2020 teaching experiment and the evolution of the participants' emergent models as they engaged in various aspects of the overall task design. To provide an image of how the teaching experiment proceeded, I begin by providing a detailed account of Group A's progression through the entire task sequence. This includes accounts of individual interviews, as well as the geneses and implementation of supplemental tasks developed as part of the ongoing analysis. In subsequent sections, I provide analytical results for other participants. This chapter concludes with an analysis of the supplemental task effectiveness, the task-based clinical interview results, and a summary of important relationships between models during development.

I would like to make the reader aware that an original member of Group B withdrew from the study in week 3 for personal reasons—I will not include an analysis of this participant in this document. The same week a new, irreconcilable, scheduling conflict arose for a member from Group C, C2. Coincidently, C2 and the remaining Group B member were the only two participants who had not taken a calculus course prior to this semester. I originally wanted to pair these two students but was prevented by the availability of other participants. As I will describe later in the chapter, C2's original partner, C1, had a strong understanding of Riemann sums and Reiman integrals when joining the study. C1 also displayed an ability to effectively think aloud during problem-solving without a partner in the first two weeks of the teaching experiment. For these reasons, I decided to move C2 from Group C to Group B.

Group A's Progression Through Task Sequence

Group A consisted of two freshman students. A1 was a Caucasian female statistics major pursuing a minor in music. She described herself as someone for whom numbers "make sense" and felt choosing statistics as a major would prepare her to work in the healthcare industry. A1 took Advanced Placement Calculus her senior year of high school but described feeling insecure about the second half of the class due to the transition to online learning due to the COVID-19 pandemic. In her initial clinical interview, A1 displayed an inclination to rely on procedural knowledge without giving much thought to the underlying mechanics of a problem unless faced with direct questions. For example, she described a "House of Calculus" mnemonic she relied on to identify the graphical relationships between a function f, its derivative f', and its second derivative f''. A1 described her high school and college calculus experience as different because in high school it was about "how" you work the problem, while she felt her current instructor placed more emphasis on "why" you work the problem.

A2 was a Caucasian male architectural engineering major who also took a calculus course in high school. A2 agreed that it felt like he "missed like the last half of calculus" due to COVID. In the initial interview, A2 was much quieter than his partner, so it was difficult to ascertain his specific schemes until later in the teaching experiment. From those observations, I believe that, like A1, during his high school experience A2 developed primarily procedural schemes associated with calculus concepts. For example, while A2 demonstrated schemes associated with Riemann sums and limits, at the beginning of the teaching experiment the primary scheme activated whenever A2 discussed integration was that of an antiderivative (described in more detail in upcoming sections).

Group A worked well together and were open about what they were thinking throughout the full teaching experiment. As a result, I developed a detailed image of the precise development of their basic, local, and global models and corresponding relationships between these schemes as they constructed a quantitative understanding of definite integrals. In the following sections, I describe Group A's learning trajectory in detail to provide the reader an image of each development within their overall scheme for definite integrals.

Group A: Curiosity Rover

When answering the first prompt of the task sequence, identify the rate of dust accumulation at Bradbury Landing, A1 noted, "So, the rate of dust accumulation was six milligrams per kilometer. So every kilometer, they're getting six, average, six milligrams of dust on the solar panels." A1's inclusion of the word "average" demonstrated that she did not necessarily interpret the rate of dust accumulation as a precise measurement at Bradbury Landing, but rather as an approximation itself. More importantly, by correcting herself, A1 indicated a recognition that this situation is somehow different than a basic multiplicative model which would hold for constant rate. That is, the rate would be quantified using an averaging process. When attending to distances, A1 and A2 discussed the numeric operations necessary to measure that quantity with ease. While A1 appeared to work slightly quicker than A2 early in the interview, the task was unproblematic for either subject.

When transitioning to Part 2, A1 and A2 clarified for themselves, and each other, the goal of the task. A1 was particularly diligent to carefully read prompts and identify the precise wording for the information requested of her. Within moments of reading the first sub-prompt, the link between the requested total amount of dust and the inclusion of a rate of change activated an integration scheme for A2 who interjected "so, if we're going from a rate to an amount, are we taking the integral of it?" Although A1 did not reject this suggestion for a conceptual reason, she was hesitant to continue because their current calculus course had not yet covered integrals, making their use off-limits. Interested in exploring A2's integration schemes, I prompted him to continue by acknowledging both participants' prior calculus experience—providing a sort of authoritative approval to draw on that knowledge. I asked A2 to describe what he would do if he wanted to "take an integral of it," to which he replied, very quietly, "A Riemann sum? I don't know. I was always so bad at those." When A1 expressed confusion about what they would take an integral of, A2 continued,

> Well, so it's one, like it's one interval [referring to the segment of path from Yellow-Knife Bay to Darwin in terms of the overall path]. I don't, I don't necessarily want to take the derivative. I mean, the integral, but going backwards from a rate to an amount would be like the opposite of a derivative. So, we'd have to take the integral of something to get the amount from the two rates we have. And that would be like, the whole overestimate thing just reminds me of how you

> do the bar graph thing. Where you set it up, and then go over. And you have a little bit of excess triangle, that is your overestimate.

From his response, it was clear that A2 already had a number of different schemes associated with integration developed during his high school calculus course. A2's primary triggering scheme coordinated integration with, what I interpreted to be, an antiderivative conception—integrals are the way you go from a rate to an amount. This scheme, along with the goal of identifying an overestimate, activated related imagistic schemes of a visual representation of quantifying area beneath a curve using rectangles, or the "bar graph thing"—the excess triangle equating to a larger area and therefore an overestimate. A2's explanation prompted a similar estimation scheme for A1 which aided her developing understanding;

> Like right-hand and left-hand? Yeah. Oh, wow, that takes me back. I don't know. Because, so, if we're traveling. The rover is traveling. And, at the beginning, it's kind of getting dust at six milligrams per kilometer, but slowly, by the end of it, it's only getting dust at 3.5 milligrams per kilometer. So, this rate is going to be decreasing. Like, if we were to draw a graph. You know what I'm saying? So, I think it's wanting us to overestimate. Like, for example, a gigantic overestimate for this problem would be 6 times 30. If we just say, well, it just keeps the 6 milligrams rate the entire time it travels, times 30, which is the kilometers, you know, we would get like, what 180? That would be like a massive overestimate, because we know it changes because the soil changes. And then like, obviously, an underestimate would be the opposite to do it by 30 times 3.5.

While it is unclear precisely what underlying schemes A1 had for "right-hand" and "left-hand" sums, they must have contained elements of the imagery A2 described to be evoked. This graphical depiction prompted A1 to evaluate the overall behavior of the changes in the Martian surface which allowed her to develop an assumption that the rate of dust accumulation was decreasing along the rover's path. Under this presupposition A1 directly applied her basic model for total dust accumulation, [total dust accumulation]=[constant rate of dust accumulation]·[total distance traveled] to that distance to create a "massive" overestimate and "massive" underestimate. Thus creating a gross basic model.

Although from my perspective, A1 had developed the underlying scheme I aimed to evoke and I assumed the group would move to the next prompt, both A1 and A2 both voiced dissatisfaction with A1's solution. A2 noted that they seemed to be stuck because, "without any other points in between, we can't really find anything more specific. I'm sure we can or else she wouldn't be asking us this problem." This utterance was interesting to me for two reasons: (1) the first sentence indicated that within A1's schemes for integration/Riemann sums there was potential anticipation of a partitioning process, and (2) the second sentence implied some threshold of required difficulty for calculus tasks which the sub-prompt did not meet. Interested, I chose not to interject here and let A1 and A2's thoughts fully play out. After about 30 seconds they began a two and a half minute conversation as to whether they could assume that the rate of dust accumulation was decreasing linearly. A2 identified that if the rate of dust accumulation was decreasing at a constant rate, then they could assume the rate of dust accumulation at the midway point was the average of the two rates. He added that this assumption would allow them to make two rectangles for overestimates, pairing his explanation with hand gestures indicating that the generalized rectangles would be at different heights which reinforced my image of A2's desire to utilize some form of a local model for this context. A2 was hesitant to adopt his strategy noting that the task prompts did not assume linearity. Deciding I obtained enough insight, I reminded A1 and A2 they were only asked to find an overestimate, and they reverted back to their original response.

When transitioning to the prompt asking to identify an under/overestimate for the total dust accumulated on the rover between Yellow-Knife Bay and Cooperstown, creating a travel pattern from Yellow-Knife Bay (10 km, 6 mg/km) → Darwin (40 km, 3.5 mg/km) → Cooperstown (60 km, 3 mg/km) , A1 and A2 began by directly applying their gross basic model to the entire path, rather than my expected strategy of utilizing the computations they had already made for the shorter distance between Yellow-Knife Bay and Darwin. That is, they multiplied the values 6 mg/km and 3 mg/km each by 50 km (the entire distance from Yellow-Knife Bay to Cooperstown) to obtain 300 mg and 150 mg respectively. A1 continued by making a critical association that “if we’re saying those are the over and under’s, then we’re saying the actual value is going to be somewhere in there.” At first, I took this to indicate A1 had coordinated a boundedness component to the gross basic model. However, when A1 and A2 began discussing their assumptions explicitly, A1 brought that assumption into question:

> For a second I was like, that means [the rate of dust accumulation] can never be bigger than six and or ever smaller than 3.5. But there's a lot of space there… there's a lot of space in there for averages to fluctuate… but that would still equal out to somewhere between 150 and 300… so that's why I was like never mind.

Specifically, although there are indications that A1 wanted to assert boundedness on the rate, she had not yet fully coordinated its absolute requirement to use a gross basic model to make any assertions about the actual quantity of dust accumulated.

At this point, I began to push on A1's reasoning by asking her to consider scenarios in which the rate fluctuates to 7 mg/km everywhere except right at Yellow-Knife Bay, Darwin, and Cooperstown which caused her to conjure images of what the Martian surface must be like to satisfy these rate requirements. A1 asserted that the ground might be relatively rocky at these sites with looser sand between, which provided more evidence that the rate of dust accumulation was a meaningful quantity that was intrinsically related to the type of soil (possibly due to the applet design). Before A1 considered this scenario further, A2 stepped in and indicated that allowing the rate of dust accumulation taking on any value would not allow for them to provide any sort of meaningful approximation for the total accumulation of dust; "we can say that it's like 70 and not 7. Because if we assume it could be more [than 6 mg/km] it could be anything over. Making assumptions doesn't really make sense if we don't use the numbers." From his phrasing, I did not necessarily view A2 as viewing boundedness as a required component of a gross basic model, but rather as a trait imposed by the constraints of a mathematical task.

Because I wanted students to develop a global model through the summation of two values by applying a gross basic to subsections of the rover's path, I introduced a new prompt for A1 and A2 to consider.

> While NASA wants the best and worst-case scenarios it is also important to provide the most accurate parameters possible. Is there any way you can improve on your best and worst-case scenarios by taking into account you know the precise rate of dust accumulation at Darwin?

When presented with the new prompt, A1 stated “So basically, we can take from Darwin to Cooperstown, find our overestimate of that and then add it,” and A2 agreed. This ease indicated that the concept of engendering a global model through the accumulation of two gross basic models[13] was a natural progression when prompted more directly, at least for students who had previously taken a calculus course. After making those computations, A1 acknowledged “So that’s good. It means we’re getting closer… we’re narrowing it down,” which I took to mean that A1 was reflecting on their previous computation using the gross basic model applied to Yellow-Knife → Cooperstown and recognized that the error bound (the difference between the underestimates and overestimates) was decreasing in magnitude. A2’s short explication indicating a sense of narrowing in on the real value implied that the real amount of dust accumulated on the rover along that path must be trapped between the underestimate and overestimate. Finishing the prompt, A1 and A2 added their assumption that within each interval the rate of dust accumulation can not exceed the highest and lowest values of the data points provided.

When moving to identify over and underestimates for the amount of dust on the rover after its entire journey, Bradbury Landing→ Murray Bates, A1 stated, “I think we need to

13 Note that in this case A1 has not performed any partitioning process. Therefore this construct does not satisfy the precise meaning of a local model.

do it in chunks though" extending their summation to the 6 values needed to represent the amount of dust for the whole journey. In doing so A1 recognized that, although the wording asked for the best/worst cases, she was expected to find a solution with the smallest error bound. A2 agreed, "yeah, it's just a bunch of intervals." I took A2's reply to be a reference to his underlying scheme for Riemann sums. Rather than both work on the same computation, A1 and A2 decided to divide and conquer. A1 took on the task of computing the 'worst-case' scenario, while A2 took on the 'best-case' scenario. The coordination of 'best-case' scenario corresponding to an underestimate and 'worst-case' scenario corresponded to an overestimate caused A1 confusion, as both group members computed the underestimate for the rover's journey. When I asked A1 about this in her individual interview, she asserted that for her there was a correlation between the term 'best' and the most or highest value, while 'worst' means least or fewest. This meant that even though A1 understood that within this context of the Curiosity rover a worst-case scenario involved accumulating too much dust which would result in the rover breaking down, she was not really thinking about this aspect as she identified the quantities needed to compute her solution. She was computing a worst-case scenario, and therefore set out to find the lowest values.

When asked for recommendations to NASA, A1 and A2 demonstrated an understanding that if the rate of dust accumulation was non-zero any unnecessary distance would add to the overall total dust on the rover's solar by suggesting more direct routes between sites. A1 even suggested not visiting Yellow-Knife Bay "because Bradbury and Yellow-Knife have the same soil type. That's why I'm like… well, if we're studying soil, I don't know, they seem similar." Although not unexpected, neither A1 nor

A2 suggested obtaining additional data for this task. However when presented the opportunity both agreed that more information would be beneficial. A1 started by analyzing the data points midway between sites to verify that the new data was consistent with their assumption that the rate of dust accumulation between sites was bounded above and below by the primary data points. Although A1 and A2 agreed that more data would be helpful, neither explicitly remarked whether the new data points would result in new estimate values nor how those estimations might compare with their previous computations. Interested in how A1 and A2 might proceed I decided to skip the prompt asking them to identify another set of approximations for individual segments of the journey and instead asked what their new recommendation to NASA would be. A2 suggested calculating "another set of intervals… how we did intervals last time, right. Yeah, it's the exact same thing, just with more of them, because we can use the midpoints." I took this to mean that A2's global model included an expectation that a refined partition would adjust the overall estimate values. A1 added, "Yeah, we could make an even more precise over and underestimate, and see how far away from 400 that is… to see if how much like our recommendations need to change," indicating that A1 was also developing, or already possessed, an image of a partitioning process, which when refined produces a more accurate global model. In other words, I anticipated A1 and A2 might already possess a local-global model relationship that coordinated the number of partitions, and the corresponding size, with the accuracy of the global model as it closed in on a real value. However, this assumed association would be challenged as A1 and A2 moved onto computations.

Once again, A1 and A2 decided to split the workload; A1 undertook to compute an underestimate while A2 was to identify an overestimate. While both students were working offscreen with physical calculators and pen/paper, A2 voiced unhappiness with his result; "So, I think we did it wrong…. Yeah, I got a way higher number," and identified that his overestimate increased from 471.25 mg to 540 mg. Despite A2's unhappiness, A1 was not bothered by an increase in the overestimate, rationalizing that because both the over and underestimate increased by roughly the same amount it might not necessarily be an issue; "So the good thing is that they both went up by about the same amount. Right? You know what I mean? Like, you didn't get 1000, and I got 338. I mean, they went up like proportionally. So that's good." This was in direct conflict with my image of her previous statement and signaled that A1 did not possess robust coordination between a refinement of the global model through a partitioning process and a decrease in an error bound. That is, this relationship between an exact value for the total dust accumulation and her global overestimates and underestimates was still in a developmental state for A1.

A2, on the other hand, displayed obvious coordination of these relationships. He was displeased that an increase in the number of partitions "didn't really narrow the interval" and that "the smaller value should have gotten a bit bigger and the bigger value should have gotten smaller." He operated with the expectation that more intervals equate to a more accurate result and that this phenomenon should act "kind of like limits that should approach the actual [value]." I was aware that the overestimate within this section of the task should have totaled to 420.075 mg. Based on A1 and A2's clear articulation of their computational methods, and was pleased with what I assumed to be calculator/transcription error was providing such insight into A1 and A2's developing schemes for accumulation. Unfortunately, this unanticipated result led A2 to begin suggesting algebraic manipulations to force his computations to match expectations (e.g. dividing by the number of intervals). Wanting to redirect A2 from this trajectory, I interjected that I was having difficulty following and requested A2 type his computation in a google document so we could discuss the values more easily. While there was clearly some sort of calculator error involved in his original solution, the expression A2 typed outmatched the expected summative computations for the underestimate (Figure 5). A1 immediately voiced concern;

> Okay, I'm confused again, because I feel like those are the numbers for the underestimate. Because like, if you're overestimating it, when it's you never need this [gestures to +30(0.2) at the end of the expression], you would need this as your last rate, right [moves the slider on GeoGebra applet to the En route to Murray Bates position displaying the 0.41 mg/km reading]? Like, you wouldn't need that as your last rate [moves the slider on GeoGebra applet to Murray Bates position in which 0.2 mg/km is visible].

5(6)+5(6)+15(4.59)+15(3.5)+10(3.19)+10(3)+12.5(2.58)+12.5(1.75)+7.5(1.22)+7.5(1)+30(0.41)+30(0.2)
=540.6

Figure 5: Curiosity rover part 3 - A2 computations

The coordination of A1's global and gross basic[14] models, along with the recognition that the last value in A2's computation was identical to hers, supported A1's recognition that A2's expression would compute the underestimate. Specifically, A1 anticipated that for an overestimate the final term in the summed expression should represent an overestimate of the total dust the rover gathered along the last leg of the journey. This value would be identified through a gross basic model which means the rate of dust accumulated over those last 30 km should be the larger of the two possibilities, 0.41 mg/km and 0.2mg/km. A2 quickly agreed that he made the wrong computation, and, after writing out the full expression for the overestimate, both participants expressed satisfaction with their new result because the overestimate was smaller than the value they obtained on the previous section of the task.

When A1 and A2 directed their attention towards recommendations they noted that there was more of a chance of the mission succeeding than not, referencing the overestimate laying only 20 mg outside their prescribed limitation while there was approximately 70 mg of wiggle room in the underestimate. However, A1 and A2 did not use this observation to assert mission success. Indicating she was continuing to assume a non-linear decrease in the rate of dust accumulation, A1 noted "that's not the way chance works." Instead, supported by the construct of a global model as an aggregation of

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14 Possibly local models.

smaller approximations, A1 and A2 returned their attention to creating a more direct path for the rover between sites. By limiting the number of kilometers the rover traveled along each subsection, or interval, of the overall path they could limit the total amount of dust represented in each of the terms of their summation. This would ultimately reduce the total amount of dust.

As A1 and A2 discussed why a certain subsection of the path was more meandering than the others, A2 mentioned that NASA would probably have more data at their disposal. When asked to elaborate further, A1 interjected that she wasn't sure this would be true, "I don't know if they necessarily would, because that's kind of the point is like they're exploring it… but also, I'm not a scientist, I don't know all the tools they have for like imaging, like what they think different images mean." A2 agreed adding "it could go either way, just because we don't know the tools that are available." From this interaction, it became clear that A1 and A2's unfamiliarity with tools available imposed limitations on their ability to productively suggest additional strategies for mission success. I suggested it would be worth looking into, by drawing their attention to the fact that the LiDAR spectrometer was at least accurate to within 5 km as this was the smallest distance between data points in their last computation.

When I provided A1 and A2 the Google Sheets spreadsheet for the final subsection of the Curiosity Rover task, A1 first checked to identify whether the data followed the decreasing behavior. I found this pattern of immediately verifying whether her expectations were correct to be an important aspect in the development of A1's models. It indicated a strong association between the boundedness of the rate and the ability to utilize her gross basic model on subsections of the journey. Although not part of the task design, I am confident that if provided non-monotonic data, A1 would have noticed any anomalies and would have been driven by a need to address them directly. In practice, I anticipate she would have still drawn on a gross basic model for her computations by adapting which ‘side’ of the interval provided the appropriate rate to identify an overestimate or underestimate.

When identifying a computational strategy, A1 began, “So, basically we need to set it to make this column [motions to the column B which listed the rates every 2.5 km along the path] multiply everything by 2.5, and then get that total. Right? Because if each of those is 2.5 increments… Oh, no, not everything, every other one.” Here A1 hesitated, showing confusion. A2 quickly stepped in and assured her that she was right the first time, and they should multiply 2.5 by every value in the rates column. A1’s momentary assertion to only multiply every other cell by 2.5 was caused because she was attempting to utilize a gross basic model for a generic stretch of a path that was 2.5km long. However, this use of a gross basic model was not in service of explicitly identifying one of either an underestimate or an overestimate. Therefore, it became unclear what the resultant list of calculations would represent. Would it be a list of overestimates for the amount of dust that would accrue along 2.5 km segments, or a list of underestimates? Based on previous computations a 2.5 km stretch of the path should have both an overestimate and an underestimate. This caused A1 to question whether she should somehow skip every other computation for the summation.

As mentioned in the section on task design, I did not have an expectation that A1 and/or A2 would be proficient with spreadsheets and assisted with their computations but was careful to follow A1 and A2’s explicit instructions. Keeping with their original strategy, A1 directed me to create Column C by multiplying every cell in Column B (the list of rates of dust accumulation) by 2.5. Following this, she wanted to sum all the values in Column C. Upon arriving at a single solution, both A1 and A2 showed reservations. A2 conjectured, “All right, so that is, umm. Is that just an estimate of the actual?...Yeah, cause that’s not an over-under.” However, because the question prompts continued to situate students’ goal-oriented activity toward identifying both an under and overestimate, A1 remained perturbed. Drawing on her earlier correction of A2’s overestimate from Part 3 of the task, A1 recognized that the last term in Column C, 0.45, would represent an underestimate for the total amount of dust on the rover if it only traveled the last 2.5 km of the journey. Excited, A1 explained,

> I was thinking like, if we subtract 0.45 away from [the summed value], we get an overestimate. If we subtract 15 from it, we get an underestimate. If we subtract 0.45 we can overestimate it I'm pretty sure… Because when we found our underestimate, we um, we just didn't do the very first-rate. We went all the way to the bottom. And when we found our overestimate we didn't do the very last rate. Remember, like you had 0.2 and I was like the last one should be 0.41. So I think for our underestimate, we need to subtract 15 from this answer at the bottom. Okay. And then for overestimate, we need to subtract 0.45. Does that make sense?

Continuing to work under the assumption that their original spreadsheet computation, 381.7 mg, was as an approximation of the actual value, A2 voiced concern that their newly computed overestimate, 381.25 mg, would be smaller in value. This objection reinforced my observations that A2 had a strong association between a partitioning processes and reducing the error of an approximation which informed his expectation that the real value for the amount of dust must be trapped between their underestimate and overestimates. Drawing on her experience working with spreadsheets at a previous job, A1 was able to provide an alternative way to justify why 381.7 mg could not represent the total amount of accumulated dust;

> Because if we add up, okay, let's see, two minus, okay 65. Because 65 times 2.5 is 162.5, and see, we have 65 cells here and our total distance is only 160. So, whenever we just look at this, this is too much distance. Does that make sense? That's why you have to subtract one or the other to get your over and underestimate….So the underestimate is gonna be 364.7, and the over will be 381.25. Since we've narrowed it to that interval, we know that the rover should be able to handle the load of the milligrams of dust.… That's exciting! This is exciting, our rover works! At least according to our spreadsheet math.

This attention to the total distance represented in the 381.7 mg computation was not something specifically set out by the task design to highlight, but such an observation helped to support A2 in the recognition that 381.7 mg could not represent an approximation of the quantity desired—evident by A2 reiterating the same language as A1 in his follow-up interview. Because A1 provided the majority of the solution strategies within the main session, a large portion of A2's follow-up interview centered on making sure that, based on their problem-solving activity, A2 adopted A1's explanations into his own models and schemes. In particular, he and I spent time working on additional spreadsheets to draw correlations between the computations A1 and A2 made in part 2 of the task. A2's ability to reproduce similar conclusions led me to believe that he had accommodated A1's explanations into his emergent models.

Summary

When A1 and A2 began the Curiosity Rover task they seemed confident in their ability to reason quantitatively with the basic model, [rate of dust accumulation]·[distance]=[total dust accumulation], and the process by which one quantifies distance traveled. For A2 there was more evidence that he was able to relate the current tasks to his previous experiences in high school calculus. His interjections of terms and constructs involving integration, Riemann sums, and limits hinted at complex interconnections between schemes. A1 also demonstrated an antiderivative conception of integration, however, any additional schemes were not as obvious due to her hesitation to utilize such a tool for this context. Specifically, A1 associated an integral with having a function that would allow one to compute something. However, because A1 questioned A2 as to "what he'd take the integral of" and demonstrated recognition of A2's imagistic description of a graphical Right-hand sum, I did anticipate A1 entered the teaching experiment with additional schemes for sums and integrals.

As a result of the task sequence, both A1 and A2 were able to construct gross basic models. A1 demonstrated a stronger association that the varying quantity within a gross basic model must be bounded, although A2 never displayed behavior that suggested he reasoned differently. When transitioning to the construction of a global model, A1 and A2 first applied their gross basic models to the rover's entire journey, likely due to wording in the task design. This required an additional prompt to identify a more accurate under and overestimate. A1 and A2 easily transitioned to a parts-of-a-whole conception of a global model in which each part of the totality represented a value for the total amount of dust obtained by applying their gross basic model to a single portion of the path. A1 consistently demonstrated reasoning coherent with a parts-of-a-whole global model, while A2's assertion that their initial solution in Part 4 represented an approximation of the actual value for the amount of dust raised questions as to how dominant this scheme was for him. It is likely A2's initial observation that this task would involve integration engendered an expectation that eventually you would be able to obtain a single approximation for the total amount of dust, priming A2 to correlate that single computational solution with such a value.

Finally, while neither A1 nor A2 ever explicitly suggested partitioning the data set, they did begin the development of a local model as a partitioning process, along with a relationship between additional refinement with a lowering of the error bound. A2's incoming schemes supporting this relationship were much stronger than A1's initially, as he was perturbed whenever this expectation was challenged. Additionally, early in the task sequence A2 showed signs of attempting to utilize a local model to identify a better approximation, but was limited by not having access to a function that modeled the data. While I do not necessarily believe A1 developed a true local model throughout the course of this task, I am confident in asserting that A1 began to develop an association that more data provides you with a lower error bound.

Group A: Fluid Force on a Dam

When A1 and A2 began the Fluid Force on a Dam problem, it became clear that, although they read the preamble, they did not interpret the quantitative structure describing the basic models for fluid force and fluid pressure in the way expected. A1 and A2's first task was to provide an overestimate and underestimate for the total fluid force acting on the dam and, working with the values for height and width of the dam provided in the prompt, they quickly made a correct computation of 9800*25*63.25*25 to represent an overestimate and 0 for an underestimate. However, when I asked why the first value represented the maximum force possible A2 replied that anything over "wouldn't be a horizontal force on the dam" but rather a "vertical force, because it'd be more than the height of the dam." Similar descriptions from A1 indicated that, despite the accuracy of their computations, A1 and A2 were envisioning this scenario as different levels of water behind the dam at different periods of time. A maximum force would represent a water level at the full height of the dam. When there was no water behind the dam there would be no pressure so the computational value of force would be 0. In other words, the basic model they were drawing on was [force]=[9800*[height of dam's surface area in contact with water]]·[dam's surface area in contact with water].

While I acknowledge this could be an interesting scenario to investigate once participants had a more solid foundation of a global model comprised of local models, a focus on adjustments to what would eventually become limits of integration was not the aim of this task. To effectively draw on (or construct) a parts-of-a-whole global model, I needed A1 and A2 to identify that the pressure acting horizontally on the dam deep below the waterline was greater than that same phenomenon near the surface level. Verifying their reasoning pattern, I introduced the Fluid Force on a Dam GeoGebra applet and provided a brief explanation that for this task the water behind the dam would be at a fixed height of 25 meters at all times. The imagery of the dam within the GeoGebra applet, along with their initial basic model, caused A1 and A2 to become concerned about the depiction of shallow water at the front of the dam acting backward. This was not anticipated, and in an effort to direct them towards an image of the basic model consistent with the hypothetical learning trajectory, I acknowledged that the issues they raised regarding multiple forces acting on the dam would be important to consider in a real-world scenario, but that we would start off simply so the number of different possibilities would not overwhelm us. I indicated that the only quantity we would concern ourselves with would be the horizontal force of the water acting on the back of the dam. In response, A1 demonstrated a key component of her developing gross-basic/global models for estimates,

> Because, if we look at our, I mean, our area won't change… So yeah, I mean, gravity is not going to change. The density of the water won't change. So what? Like, we need to find a variable that will change to make an over and underestimate. Does that make sense? And I guess maybe I'm not understanding the problem, but if we have a constant height, water doesn't change ever.

Specifically, A1's image of the reason to invoke a gross basic model is tied to variability. To produce an overestimate and an underestimate there must be some quantity that is non-constant. In an effort to identify the varying quantity A1 had her partner adjust the slider on the Fluid Force on a Dam applet as she spoke aloud:

> And then, let's just slowly move the slider. Okay, so this applet gives us pressure as we move down. Like, the pressure is increasing as the depth is increasing. Which, because it's just a formula. So let's see what they get at 25… Okay, so I guess maybe I was thinking about it like the pressure on the entire dam. But didn't I just point out that it makes sense that there's different pressures on different parts of the dam? So at the bottom, there's more pressure than there is at the top. So maybe that's the over and underestimate we calculate. Like so, at the top of the dam there's not a whole lot of pressure, because there's not a lot of water weight at the very top…. That's what I'm thinking.

While A1 did identify a varying pressure, she justified the variance because the pressure was "just a formula." This indicated that while pressure was a quantitative component of the force equation, in the sense that it some quality she had a means to measure, the pressure itself did not represent a true quantitative relationship between the acceleration due to gravity, fluid density, and depth. This is not necessarily surprising, as pressure is generally quantified as a rate of change of force per unit of area, which is beyond the scope of my expectations for someone who has never had a physics course.

Attempting to support A1's observation, A2 revealed he had experiences scuba diving which made pressure a quality independent of the provided formula;

> I'm not for certain, but I know. I'm a scuba diver. So, in scuba school, they teach us about how there's a ton of forces acting on your ear. Whenever you're under the water it increases as you go down. There's certain, like, atmospheres I guess is how they measure it. So the further you go down, the larger the forces on your head. So you have to like equalize the pressure and stuff. And I guess that kind of makes sense to me that the further down you do have more pressure horizontally on the dam at those points.

Based on these interactions I decided to supplement A2's justification with the Box Underwater activity to provide A1 an opportunity to reinforce her brief image of pressure as a quantity that changes with depth. The supplemental applet and questions were not created to support an image of pressure as a rate of force per unit area. In fact, because the aim of this hypothetical learning trajectory was to allow participants to engage with reasoning about definite integrals that do not naturally decompose into a Riemann product, I specifically avoided any mention of pressure as a rate of change. The aim of having students work with the applet was to support a meaning for pressure which includes an image that (1) the numerical value for pressure increases with depth, and (2) support a view that a productive local model for fluid pressure in this task attends to horizontal (rather than vertical) surfaces. During the supplemental activity, the pair were asked to identify the pressures at various heights, the total force pushing up vertically on the base of a box and pushing down on the top of the box at various heights, and to compare those values. A1 and A2 were able to identify these quantities with little difficulty, which supported them in concluding that the pressure of water at the surface level would indeed be a smaller magnitude than at a depth of 25 meters. This allowed A1 and A2 to observe that their previous computations provided the correct values but that they "just didn't know why." Of course, their original computations did make sense as a gross basic model applied to the quantitative scenario they envisioned. A1's observation just referenced an acknowledgment that while the solution values remained identical, the quantitative meaning which constituted the construction of those values had changed.

I instructed A1 and A2 to provide new estimates by partition the dam into two pieces. A2, very briefly, considered partitioning the dam into two pieces of equivalent force, however, he reasoned that "this is what we're already trying to find" indicating that he recognized an inability to identify an exact value for force on a subsection of the dam. Instead, the pair decided to divide the dam horizontally at a depth of 12.5m as a simple starting point. A2 drew a diagram of the dam with a horizontal line through its middle. From here A1 and A2 worked incredibly efficiently. They drew on their diagram, identifying over and underestimates for the force on each part of their partitioned dam. To accomplish this they directly ascribed the basic model structure to the partitioned quantity. That is, they accommodated their gross basic model to be a true local model. Although the accompanying board work only displayed their final computation values, A1 and A2 clearly articulated the quantitative structure of their local models through their solution process. First identifying the pressure at a depth of 12.5 meters, A1 calculated the overestimate for the top piece of the dam, "9800 times 12.5 is 122,500. And so, if you multiply that by the area of that rectangle, which is 790.75, you get the force to be 996,875." By creating a connection from the Curiosity Rover task in which she observed that the overestimate for one 2.5km leg of the journey would be identical to the underestimate for the subsequent 2.5km section, A1 commented, "but for the bottom rectangle, our underestimate won't be zero. You know what I'm saying? It'll be 996,875." She continued to draw on their previous session's work, "remember with our rover… what we did was we did the underestimate for this distance [creates hand motion indicating a small gap] plus the underestimate of this distance [lowers the height of her hand motion]." It was clear that in this instance A1 did not believe she was adding distances together to get a force. Rather she was reflecting on her previous problem-solving activity to accommodate the global model developed in that task, which constituted a whole quantity as being an accumulation of gross basic models, into a global model comprised of local model elements.

As A1 and A2 attempted to identify an overestimate value for the bottom partition of the dam, ranging in depth from 12.5m to 25m, they voiced uncertainty as to which value of 25 should be included in their current local model, [force acting on partition]=[pressure acting on partition]·[area of partition]. What was interesting about this exchange was that A1 and A2 were associating that value of 25 to different components of their local model. A2 was considering whether to use the value of 12.5 or 25 for the pressure component of their local model. Specifically, he was attempting to rationalize whether or not their local model for the bottom partition should be considered in isolation of the overall context. If they were looking at two partitions of the dam in complete isolation then the appropriate depth of water would be 12.5m, as this would be the depth of water for a dam that was only 12.5m tall. However, within context, the actual depth of the lower edge of the bottom partition was at a depth of 25m. Deciding that the local model cannot be in isolation of the global context, A2 voiced support for the value of 25 because “it still has the weight of everything on top of it, even if we do break it into two parts,” indicating he had associated pressure with a weight which included some form of a compounding process. A1 disagreed with A2’s assertion that they should use the value of 25, but this was because she was considering whether the value of 25 would be an appropriate height of the dam within the area component of their local model. Specifically, based on a relationship between a refinement process and the reduction of a global model overestimate, A1 had anticipation that creating a local model through a partitioning process should similarly result in an overestimate of smaller magnitude. A1 had already assigned a value of 25 to depth within the pressure component so also assigning a value of 25 to height within the area component would have resulted in the exact same value as their previous computation which measured an overestimate of the force on the dam without partitions. Perturbed with this possibility A1 interjected, “If we add the same thing it gets bigger instead of smaller,” indicated that she had envisioned adding this overestimate for the lower part of the dam to the overestimate of the top half of the dam as part of a global model. Adding an additional, nonzero, value to this computation would result in an overall larger overestimate than their previous result. In the same instance, A1 also voiced, “the gap between the over and underestimates doesn’t change if we just add the same amount to that number.” This statement emphasized A1’s explicit expectation that you could lower the magnitude of an error bound through the invocation of a partitioning process on the global model. This served to aid A1 in rejecting assigning a quantitative value which would lead to a conflict with this expectation and aided her in the adaptation of her local model to [overestimate of force on bottom portion of dam]=[9800*[pressure at bottom of dam]]*[area of bottom partition]. Bolstered by this computation leading to an expected result, A1 noted, "I feel like I might kind of like that. Because then if we add the other, like 96 million… that means our total force could potentially be maxed out at 290 [trails off reading calculator]." Specifically, she was pleased with the new, refined, overestimate being smaller in magnitude than their original estimate. Not mentioning, or likely not realizing, that they were discussing different aspects of their local model, A2 agreed with A1's reasoning, allowing the pair to identify new parameters for the total force on the dam (Figure 14).

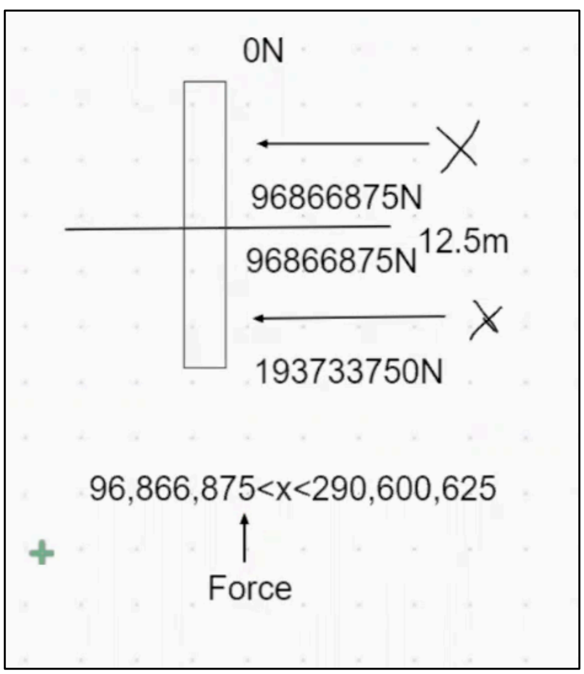

Figure 14: Group A's whiteboard work for the total force on a rectangular dam in two pieces

Before I had a chance to ask follow-up questions, A2 voiced anticipation of the next task in the sequence; "So, I guess just to kind of look ahead. Are we just going to keep breaking this up into smaller parts and adding these together to get narrower?... We did that before," clearly indicating that he also shared a [refinement]=[smaller error bound] local-global model relationship that was extended from the Curiosity Rover task. Feeling like this was a natural transition into the next prompt, I asked Group A to provide a better approximation by partitioning the dam into 5 pieces. A1 and A2 were quick to adapt their global model using an accumulation of local models with smaller magnitude, taking only 7 minutes to complete the entire prompt. Despite the brevity, there were many key interactions between A1 and A2 during their problem-solving activity.

First, A1 and A2 did not make adjustments to their drawing to represent five partitions instead of two, instead of jumping to writing a summation form of their global model which matched a parts-of-a-whole symbolic form. As A1 listed off calculations that would identify underestimates for pressure, A2 noted them down on the shared whiteboard. A2 began typing "0(9800)+5(" before, prompted by an expectation that the local model also represents a force, he spontaneously interjected the phrase "times the area" into the conversation. This caused A1 to pause and consider, "oh, I understand. So like 5 times 9800 multiplied by the area there. So that we just have the force equations." When A2 agreed A1 provided the area value of 316.3 by reading from a computation she already made on her own paper. As A2 began to type the next term in for their global model "+10(9800)(63," A1 saved him the effort of finishing the 63.25*5 computation by observing the area will always be "5 times the length [of the dam]."

This entire interaction, which only lasted two minutes, was an important glimpse into A1 and A2's developing models for a number of reasons: (1) the ease of transition into computations without the support of a diagram indicated that both A1 and A2 had strong parts-of-a-whole schemes for their global model as being an accumulation of values produced by a local model, (2) the ability to efficiently input the value for depth into each element of the global model indicated the generalized nature of their local model to estimate the total force on a single element of the global model. That is, they adopted using the measurement of the upper or bottom edge of the piece of the dam dependent upon whether they were looking for an underestimate or overestimate respectively, (3) A2's interjection of "times the area" indicated that each element within the symbolic form for the global model must represent a force, which (4) indicated A2's local model shared the quantitative structure of his basic and global models, (5) A1's quick adaptation to A2's short utterance hinted that, although not initially an inherent part of her global model, a quantitative structure of [force]+[force]+[force]+…+[force] was likely to be accommodation into her global model for fluid force on a dam, (6) A1's pre-prepared value for the appropriate area of 1/5th the entire dam signified that she had not disregarded the area component of her local model, so (7) her image of a local model included a way to quantify the height of that partitioned section.

A second key interaction occurred immediately following A1's observation that the area component of the local model remained constant. As a means of shortening the expression, A2 asked "does that mean we can just multiply it at the end?... Okay, so we'll do that, 316.3 times everything… I guess we could have multiplied 9800 outside also." I illustrate this episode as a separate point because I want to make clear to the reader that, despite the algebraic structure of their final computation, A1 and A2 amassed their global model through the progressive addition of local models. I also address this point to foreshadow that such algebraic manipulations will introduce difficulty for A1 and A2 as they progress further into the teaching experiment.

316.3(0(9800)+5(9800)+10(9800)+15(9800)+20(9800)+25(9800))

Figure 15: A2's underestimate global model for the rectangular dam in 5 pieces

After A2 wrote out all the terms in his global model (see Figure 15), A1 identified an issue of too many components within his expression;

> Okay, so 1, 2, 3, 4, 5, 6. Wait… So, if we want to do our underestimate, we've got to get rid of the 25 times 9800 because we only need five terms if we're breaking it up into five parts. Right? Yeah, and then when we want to do our overestimate we won't include that zero. Well, zero wouldn't be in there anyway, but like, that's because it's zero. But you know, we'd want the 25 back for the overestimate. Does that make sense?

Due to a similar observation in the previous session, I could conclude that A1 had a strong local-global model relationship which coordinated the size and number of partitions. In the Curiosity Rover task A1 had been perturbed when the cumulative distance of 65 local models with length 2.5km added to "too much distance," and she was now perturbed because the number of terms within the global model was too high. Similarly, from this second instance of remaining unperturbed, it was clear that such a link was not yet part of A2's local-global model relationship.

Finally, the last observation from this particular prompt was just a quick acknowledgment from A1 and A2 that their new computations narrowed the parameters more than their last estimates. While not a new development, a recurrence of the same observation did supply additional support of the existence of the desired refinement relationship.

The timing of their completion of this prompt fell near the end of the session. To aid in the construction of A1 and A2's models I asked them to reflect on and summarize their activity. This allowed A1 to reiterate her interpretation of their solution strategy, which reinforced my image of her current local and global models;

> So basically what we did was we did, um, we just kind of followed the formula for force, which is pressure times area. And I think the only kind of like, tricky part… like, when we just split [the dam] into two segments, we couldn't just assume that the pressure for both of them was the same. … In our computations, that's why we had to change the numbers… That's why we have like, the 5 and then the 10, and the 15. … we have to multiply [pressure] by the area and the area stays the same, you know, whenever you're dividing it into equal segments…. It was like the force for each segment added to the next force, all to make the total force.

For the next prompt, I asked A1 and A2 to identify a range of forces accurate to within 50,000 N. Before proceeding A1 checked that she understood the implications by directly referencing the need to reduce the difference between the over and underestimates to less than 50,000. A1 and A2 then checked to see what this difference would be for their previous computations noting that it definitely was not close to the 50,000 mark. A2 stated they could do the same thing they previously tried "but that'd be an insane amount of intervals to add," implying that he did connect a refinement of the local model would impact the number of terms within the global model and result in a smaller error bound. Both participants felt stuck, noting that there would be "too many intervals to feasibly do." At this point I interjected that if there was something that they would like to try, I could help identify a tool to help;

> So, if there's something that you want to try. If you think, 'okay, if I could have a tool that would do *this*, then that would solve the problem.' I might be able to help you identify a tool that would help with that… like the spreadsheet last time when we just made it do the computations for us.

By providing A1 and A2 this prompt open-ended I was attempting to gain access to how their current models influenced a solution strategy. My suggestion caused A2 to interject a thought, "I was thinking, I guess if there's some way to relate depth and force kind of like we already have but an equation, then we could use that equation to kind of get a more accurate feel for the estimations." His comment was a little vague, but I took it to mean that A2 wanted to develop an explicit formula for a generalized local model. That is, A2 wanted to create a generalized quantitative expression that could represent any element of a global model. This would allow A1 and A2 to identify an estimate for the force acting on a single partition at any specified depth with any specified size. A1 also took on this meaning, asking "Do you want to write a formula… using a tool to compute it?" As they began engaging in the process of creating such a formula, A1 reflected on the local model within their broader global model expression, noting the repeating pattern in their process; "we know we're just multiplying area times depth times 9800, right? And then we're adding that again. We're just going to keep going for each interval that we decide to do." This observation spurred A2 to recall his professor's introduction of a sigma sign as "the summation thing" with a recollection that you wrote a formula next to the sigma which would repeat if you were to write the whole summation expression. From A2's memory the sigma notation required that the accompanying formula should be written in terms of one variable, impacting his formula construction;

> Because if we write area in terms of depth. So break area down into what is it, 63.26? Times d. Times d again, so d squared… and then multiply by 9800. That gives us our same equation, but it's in terms of one variable now… and then we can plug any value in for d and get the force at that depth.

A1 followed A2's explanation but voiced a slight protest "yeah, that makes sense to me. But we'd still have to do it for every single interval. Does [the formula] do that?" A2 replied, "if we do the summation thing, so like, the full sigma." A2 then began to attempt to fill in the elements of the symbol template for Sigma notation while combining numerical values (Figure 16). In this instance, A1 was indicating a recognition that, although each element within her global model shared a fixed value for area, these same values within the global model were distinct across terms. However, while she was uneasy with A2's proffered formula, she was unable to immediately articulate why. Although it is possible that A2 also shared this scheme for a global model, either (1) such a connection was not strong enough to contend with the predominant scheme centered on creating a formula with only one variable, or (2) because he had not yet considered a specific number of partitions he was not envisioning a set of local elements. In either case, A2's conviction in the accuracy of his formula dissuaded his partner from pursuing her uncertainty.

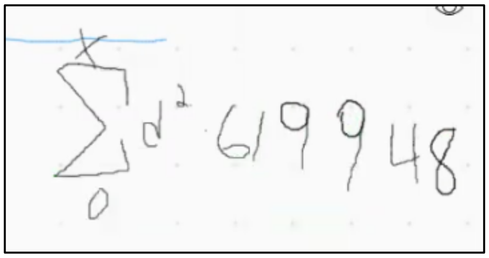

Figure 16: A2's sigma notation for fluid force on a rectangular dam

To aid A2 in recognizing the same issue A1 hinted at, I suggested using the sum calculator to check their formula by comparing it to the values obtained using five partitions that they had just completed. Discussing the limitations of the calculator, such as needing to use x for the variable took a few moments of explanation. A1 and A2 input their desired formula into the calculator (Figure 17) and observe a discrepancy between their previous values and the new calculator computations with A2 remarking “oh, that’s not quite right.”

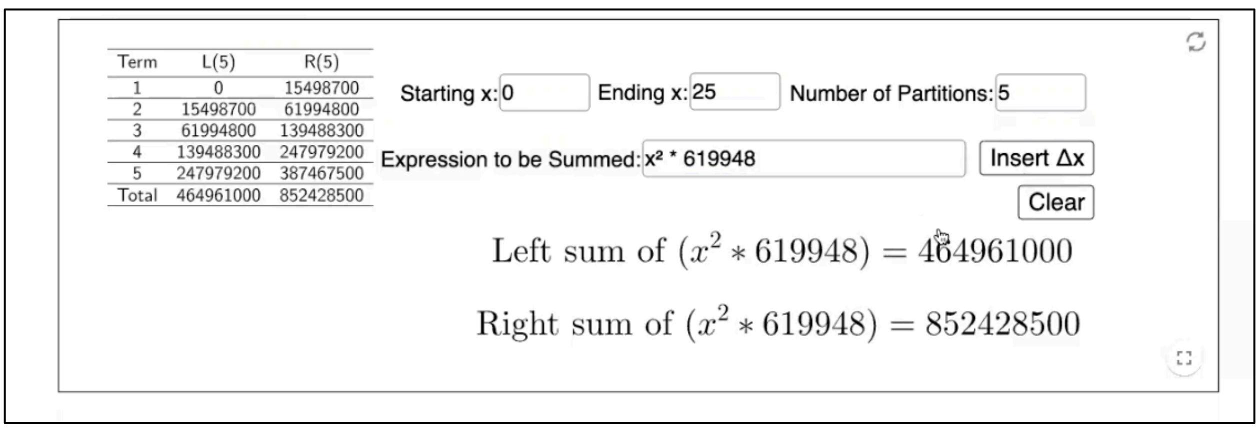

Figure 17: A2's initial local model expression input in GeoGebra summation applet

Because A1 had demonstrated concern with the formula, she quickly identified the problem;

> Oh, wait, I know what it is. It's because our depth is not the same as our area. Like we don't, because when we're doing the area, we would do like five times 63.26. But then when we are needing to calculate our depths, we would change that to like, 10, but the area would stay the same.

A1's predisposition to attend to which quantities remained fixed versus those that changed across elements of her global model allowed her to pinpoint a formulaic error in the area component of their local model expression. Specifically, this observation allowed her to identify that the quantity for height in the area component of their local model was not equivalent to the quantity for the depth component within their local model. Therefore they could not use the same variable to represent both quantities. Replacing one of the "x" terms and the "63.26" in their initial local model expression with 316.3, the value for area, resulted in the GeoGebra applet producing results in line with their expectations.

Bolstered by their success, A1 suggested "do you want to just play around with how many partitions it takes to get it down?" A2 agreed, but observed, "I feel like it's going to be a slightly unreasonable amount of partitions." These acknowledgments from A1 and A2 raised my confidence that I had correctly identified a link between a refinement process, the number of partitions, and the accuracy of the global model. Although I would have typically allowed A1 and A2 to follow their own trajectory, I came across a limitation of the GeoGebra applet when working with another group earlier in the week. I originally believed that the processing power for computations within GeoGebra applets was performed on GeoGebra's servers and that the ability to display the outputs of the applet was the only resource requirement from the participant's own computers. I was incorrect. During testing my personal computer had enough processing power to handle computations involving relatively large numbers, however, less powerful machines did not. This resulted in resource issues on the participants' end, often involving long computer freezes and crashed applets. I relayed this issue to Group A and suggested "so it might be better if we can come up with a way to figure out how many partitions we need, instead of just playing around with the numbers," directing them towards my expected solution path[15].

In response, A2 fell back on his local-global model refinement scheme as a way to move forward, reflecting "we can just see that more intervals equals a narrower approximation, like we knew from the last project. We just need a lot more." They started to compare the error bounds for their different computations, presumably to identify a pattern but started discussing their initial strategy of wanting to break the dam into 100 pieces. I relayed that the calculator would be able to handle 100 just fine. As they turned their attention to this line of inquiry A1 observed, "so, 63.26 times .25 to get a new area. We'll have to put a new one in. A new, little, formula in." A2's immediate exclamation of "oh, yeah you're right" indicated that, unlike his partner, A2 was still reasoning primarily with the area component of the basic model and had not yet fully taken on the same fixed vs variable aspect of the quantities within the local model.

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15 Fully acknowledging that this was not the only, nor necessarily most intuitive, strategy for this group.

While entering their new area value into the applet, A1 and A2 mused that there had to be some way to write the formula so that they didn't have to find the area every time. After verifying their precise computational method, the magnitude for the full height of the dam divided by the number of partitions, I let A1 and A2 know that I programmed the calculator applet to automatically find the height of each partition through the computations they had described using the symbol Δx. This would allow them to use the expression $9800 \cdot x \cdot 63.26 \cdot \Delta x$ in the applet. By introducing this symbol in the moment of their expressed need I was hoping they would take on the conceptual schema for $\Delta[\ ]$ to take on the quantitative meaning of the magnitude of the partitioned quantity within the local model.

Feeling as though the interactions between A1 and A2 had already been fruitful, and not wanting them to get hung up on trying to read my thoughts for a solution strategy, I directly pointed out that by scrolling down on the list of values produced in the sum calculator (1) they could observe that the cell immediately preceding the final total overestimates was exactly the same as the difference between the overestimate and underestimate, (2) that this cell's value represented an overestimate for the total force on the bottom partition of the dam, and (3) that the fact that these two quantities being equal might help them identify the exact number of partitions they needed. Drawing on previous observations she'd made, A1 was quick to realize why the values matched, explaining that the last cell was included in the overestimate computation but not in the underestimate, so the difference would be equal. A2, less convinced, indicated that the first cell in the left-hand column should have been included in the underestimate computation. This indicated A2's scheme for his under vs over global models was beginning to incorporate the same pattern A1 had pointed out: that you can list all of the local model values and then “toss out” either the first or last computation depending on whether you wanted an overestimate or an underestimate. Wanting to immediately validate and reinforce A2’s line of reasoning, I stepped in to note that the very first value absolutely plays a role in the error bound, but because it is 0 in this case it just doesn’t look like it.

From here, A1 and A2 again reverted to a guess and check method. While the interactions did not add any new developments to their models, their comfort coordinating the number of partitions with the height of an individual partition reinforced my analysis of their local-global models. When A1’s computer began to struggle with the number of partitions (roughly around 1000) I again suggested we try something different, this time specifically indicating that they needed to find when the force for the last partition was less than or equal to $50{,}000\ N$. Based on my suggestion, A2 realized that he could solve for Δx in the expression “$619949 \cdot 25 \cdot \Delta x \leq 50{,}000\ N$"[16];

> We have a long hairy decimal here, and if it needs to be less than or equal to 50,000 we would probably want to go a little bit smaller of interval. So we can round off at .003. And that should give us just under 50,000…. So now that we have that interval length, we need to know how many intervals are in 25 meters… That should be 8333.33

16 Note that 619948 is the product 9800*63.26.

A2's ability to coordinate the value for $\Delta x \leq 0.003$ and the number of partitions indicated Δx was solidly considered a length of an interval. That is, the vertical side of the diagram he'd drawn earlier (a rectangle representing the dam) acted as a number line ranging from 0 to 25 which could be partitioned into equal subintervals. How small those subintervals needed to be could be solved by isolating the Δx in the inequality (in this case 0.003). In contrast, A1 was reasoning about Δx only as the height of one generalized partition of the dam. She was far less systematic in her computations and, until A2 had identified a value, was keeping with the trial and error method by changing Δx to be specific heights. I entered 8334 partitions, as directed, into the GeoGebra sum applet on my computer and copy/pasted a screenshot into the shared whiteboard so that A1 and A2 had a satisfying conclusion to this prompt.

When moving onto the Trapezoidal Dam task, A1's first response was "That's okay. We can also do this, it's just going to change our area." However, A1 was not yet coordinating "change our area" with the value for area as varying across elements of her global model. She'd simply recognized that the area component of the basic model had changed to a different quantitative structure. This was evident because A1 and A2's first method of attack was to first recall/google the formula for the area of a trapezoid before computing what I would refer to as "the area of the entire trapezoidal dam," but which they called "our area." A1 then continued, "I mean, it's just the area times the depth times 9800" and that they could "set this to 50000." Noting that "height is the depth" A2 typed, "(63.26+37.92)/2*d*24500=50,000 N" on the shared whiteboard before he observed "we need Δd, right?...that's what we're solving for" making the adjustment from d to Δd in the expression. Feeling like something was missing A1 raised an issue stating "this area formula is incomplete," however A2 managed to justify his reasoning by attending to the impact of changing the d to a Δd, "no, because we're doing it based on the interval length.… we're just doing the area of the interval we choose." We were at the end of the session time, so I stepped in and pointed out that if they were only looking at a single interval for the dam, then the 63.26 wouldn't necessarily be the measurement for the top of the trapezoid. A1 and A2 acknowledged that this would cause a problem and planned to work on it in the next session. While I feel it is important to note exactly how A1 and A2 were thinking when first introduced to the total force on a trapezoidal dam prompt, due to scheduling issues the group didn't meet again for a full week.

When we gathered again, A1 and A2 tried to recall their problem-solving strategy for the trapezoid problem based on the expression they had written the previous week. After recalling the various components of the expression (and their previous issue involving area) A1 suggested they "go back to being like, super broad with it," recognizing there was something missing from their problem-solving strategy that had been present in the previous task. Attempting to identify what was different, A1 and A2 wrote an equation intending to identify an overestimate for the whole dam (1 partition). However, the fact that the height of the partition (or interval), Δd, and the depth, d, were the same value resulted in A1 and A2 leaving off one factor of 25 when computing their broad overestimate. Compounding upon this issue was a change in language between sessions. Previously A1 had been fairly consistent using terms such as "height" or "height of partition," but now occasionally interchanged "depth" for that same quantity. Based on the area context in which she was using the terms, it was clear she was not referring to the depth of the water but rather some quality of the dam. However, there was also a quantity for depth in the task, and A2 had not demonstrated quite as strong of an association with the delta notation representing a fixed quantity correlating with the size of a partition. Therefore, A1's misuse of these terms greatly influenced A2's understanding of the expression as they were composing it, particularly when they began to identify an overestimate for a trapezoidal dam with two partitions. Quickly referencing the quantitative structure for the global model of their overestimate for a rectangular dam in five partitions, A2 began to type out a new expression attempting to match the same structure,

[9800][area]([depth at bottom of partition[17]]+…+[depth at bottom of partition]).

This led to an interesting interaction in which A2 was attempting to write out their new expression, "9800*," and then mentioned, "first we need our base area, right? Because we're factoring it out." Due to her recollection of the previous session's problem, A1 responded that they couldn't factor area out "cus it's gonna be different" for each partition, but that "our depth can be factored out." In this context, A1 is only talking about the area component of the basic model for force, and the 'depth' she is referring to is the height of a partition of the dam. Assigning it a value of 12.5, and expanding the expression to "9800*12.5(." A2 did not adopt this same interpretation for the value 12.5 which became clear a few moments later. As A2 was finishing the expression he

17 A2 might have called this "depth at bottom of interval."

observed that now they would "do our area of each interval" and that if finding an underestimate they would multiply the first 'area' term "by zero because that's the top part of the interval," referencing the depth value for an underestimate of the pressure component of the local model. While it appeared as if A2's computation would lead to a correct expression for an underestimate when he reached the second factor for 'area' the previous inclusion of a 'depth' of 12.5 caused A2 to pause. He concluded, "I don't think that goes there," and moved the 12.5 from the beginning of the expression to the last term, "9800((50.59+63.26)/2)(0)+(50.59+37.92)/2)(12.5))." That is, A2 moved the 12.5 which represented a fixed height of every partitioned segment to instead represent a single depth measurement for the underestimate of the force on the second piece of the dam. As A2 was calculating the result, I asked the pair to tell me what each of the terms represented if you did not factor anything out (Figure 18).

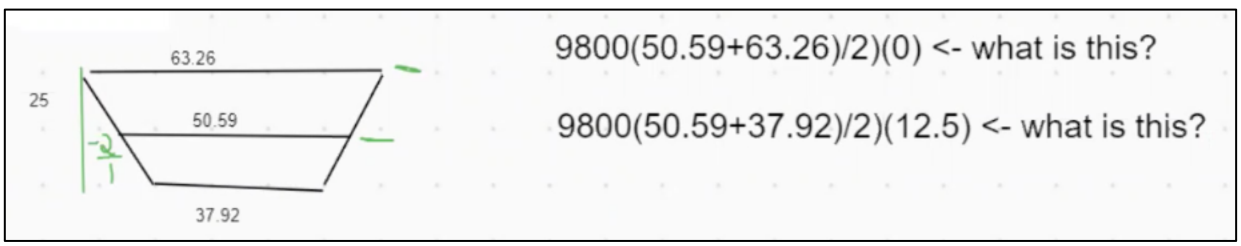

Figure 18: A1 and A2's global model elements with 'incomplete' area

I anticipate that if Group A had finished their computation they may have encountered a conflict between their new estimates, the estimates for a dam with 1 partition, and their local-global model refinement scheme. I decided not to follow this course of action in the interview because (1) their previous computations were based on a similar structural inconsistency, so I was unaware whether the resultant values would be positioned to produce a perturbation, and (2) A1 and A2 had, more than once, already demonstrated that such a conflict would cause them to reevaluate their expressions. Instead, I wanted to draw attention to a different aspect of their local-global model relationship. In particular, I wanted A1 and A2 to investigate the terms within their global models and while drawing on the underlying parts-of-a-whole conception. By redistributing the factored out 9800 pressure component and separating the terms, I aimed to draw A1 and A2's attention to the quantitative structure of their local model to provide an opportunity for observing their current model's conflict with the basic model for force.

In response to my questions, A2 described the first term as "the force for the first interval" and the reason it is zero is that it is "the least possible force to underestimate because there's the least pressure at zero." In this case, even when separated from the global model A2 was not fully evaluating or attending to the quantitative structure for the area component within his local models for force. This was unsurprising as A2 had not explicitly demonstrated a strong recognition that the local model must share the quantitative structure as the basic model, however, I do not want to frame A2's reasoning as completely negligent towards the incorporation of this construct into his schemes. In going through the effort of identifying a correct quantitative structure for the area of a trapezoid and algebraically identifying the width of the dam at a depth of 12.5, A2 was operating under the assumption that he'd already attended to the varying area component of the dam. He was no longer focused on this aspect of the basic model and therefore did not notice that as he made a 'correction' to his generalized local model he was deleting the quantity for height for the area component of that structure. It also had been a full week since he and A2 had referenced the quantitative structure for the area of a trapezoid online so it is possible he did not have a strong image of his basic model elements. A1, on the other hand, had shown more compelling evidence that the structure of her local models were tied directly to the quantitative relationships within the basic model. A1 observed, “We're not doing the area yet… we were trying to do the area, but it is incomplete. Does that make sense? Like, because we have the bases divided by two, but we haven’t multiplied by height.” This allowed A2 to recognize “that’s why we had the extra 12.5.”

Even with this recognition, when I referred A1 and A2 back to the missing height component of their expression for over and underestimates using only one partition, A2 reinterpreted the value of 25, which had represented a depth as part of the pressure component of the local model, to the height of the area component within the local model. Coordinating the local models for both pressure and area within a local model for force, A1 recognized that this would leave the pressure component of the local model incomplete, “we have to multiply area, times that gravity number, times the depth, so we have to multiply by 25 again” which A2 acknowledged was correct.

Moving on, A2 motioned to their overestimate expressions and stated, “That's for one interval. This is our 2-interval. So, now we need an equation so that we can find it any interval.” This indicated that the reason A1 and A2 started by “being super broad with it” was to position themselves to be able to focus on each element of their global model discretely before trying to coordinate a formulaic representation of their local model across elements. However, a lack of clarity in identifying the quantities to which they were referring, conflation due to identical values across elements of local models, and mismatched quantitative structure of their global models continued to plague A1 and A2’s ability to productively reason about the structure of their generalized local model. In fact, after identifying the expression for an overestimate of the final partition from the previous rectangular dam task, 9800*25*63.26*Δx, A1 demonstrated that the factored form of their current global model, 9800·[area of partitioned dam]·[depth of pressure measurement], influenced her to reinterpret their previous work, “I think that this in this equation that we copied and pasted 25 times 63.26 was our area.” Under this hypothesis, I stepped in to remind A1 and A2 of the quantitative structure they had previously described for this expression, [pressure]·[area]=[9800[depth x]]· [[width of dam]·[height of dam partition Δx]]. When this intervention failed to produce any productivity in the A1 and A2’s problem-solving, I also suggested that redistributing the quantities within their new overestimates for the trapezoidal dam may help them to notice a similar structure. In doing so I was able to contrast the distributed local model against the factored version and impart the need for A1 and A2 to be clear about which quantities they were referring to as they discussed the task;

> So, I think the fact that there's a 12.5 right here, which is representing a depth, and there's a 12.5 here, which is representing a Δx. Making sure that we keep straight, which one we're talking about, whether it's the depth or a length of an interval, is going to be important.

This intervention prompted A1 and A2 to discuss exactly how they wanted to define the quantities x and Δx, settling on x to represent the “relevant depth” and Δx as “length of interval” to be their current working definitions.

As evident in her follow-up interview, A1’s previous references to a potential relationship between her local and basic models in conjunction with my intervention allowed for a quick, and quite powerful, accommodation to her basic-local model relationship to incorporate a required [9800·[x]]·[area] shaped local model. When I asked what the quantities "9800*12.5*((50.59+63.26)/2)(0)" and "9800*12.5*((37.92+50.59)(12.5))" represented within this context (expressions in which I switched the values for x and Δx within a local model), rather than rearrange the quantities within the expressions, A1 completely reinterpreted what the meaning for the expressions should be[18].

> So this 9800 is just like the constant that we need to use in the pressure equation. Okay, I think everyone's clear on what that one is. So, and then this 12.5 is a depth, it's a depth because whenever we're looking for like our pressure equation, it's that constant times depth.… Then this 50.59 plus 63.26, those are both divided by two times zero. It's my understanding that those are our that's our area formula. … base one base plus base two divided by two times the height. Um, and since this one is zero, the reason like the height is zero right now is because we were getting an underestimate, so we were using, like, the smallest height. So if we wanted an overestimate for like, just this little piece, we would change the zero to 12.5 to get the largest height. Does that make sense?...

Bothered by inconsistency with their description from the previous session which did not involve an area of measure 0, A1 decided to reassign values within the expressions to

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18 Note that there was not a reference to [Force]=[Pressure]·[Area], visible on the shared whiteboard. A1 was drawing on this basic model relationship from her own schemes and was not directly attempting to match her explanations to a formula or expression from the group session.

properly fit her image of an underestimate for the amount of force on each partition (Figure 19). She explained,

> Yeah, but see, the thing is like, I don't know if, huh, okay. So I think that, let me change the color of my pen. I think that this one, depending on whether we're doing the underestimate or overestimate, might need to be 25 [writes 25 in purple beneath the first 12.5 in the second expression]. Because, like, if this was going to be the, this right here [places a star next to the second expression] is going to be the underestimate, it would be 12.5. But wouldn't this [circles the first 12.5 in the first expression], like, be zero in that case. Yeah, I don't know. Because I think maybe, I think maybe this should always be 12.5 [circles the zero at the end of the first expression]. Okay, because we want the area of the whole thing, like no matter what. We want this to be the depth [motions to the 12.5 in the first expression]. So, I think sometimes this, if we're doing underestimate that might need to be zero [writes 0 above the 12.5 in the first expression], and this [motions to the first 12.5 in the second expression] can be 12.5. And then if we're doing overestimate, this [motions to 12.5 in the first expression] would be 12.5, and that [motions to the first 12.5 in the second expression] would be 25…. Because like with our zero being right here [motions to the zero at the end of the first expression], like that makes this whole computation zero on our area's not going to be zero.

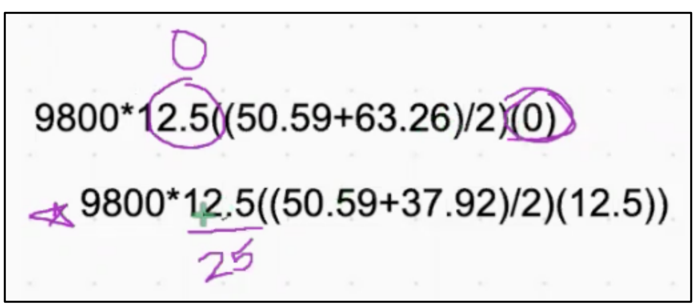

Figure 19: A1's reassignment of values to match her basic-local model quantitative structure

Something of note within this interaction was that A1 acknowledged that the values within the expression could be shifted around, however, their placement within the overall structure directly influenced her interpretation of their quantitative meaning. If a value was directly adjacent to 9800, then it must be in some way related to the quantitative relationship for pressure. Similarly, the value at the end of the expression must be a part of the quantitative relationship for area.

My intervention referencing the basic-local model relationship also influenced A2's ability to reason about how to move forward in their goal-oriented activity of identifying a generalized local model for the trapezoidal dam. Realizing they already completed the general structure of the pressure component of their local model, A2 observed that as part of their area component they would "need to write a function, like an equation, that gives you the width of the dam at any height." Because I was not as concerned with A1 and A2's ability to model this particular phenomenon, except in that it is a quantitative component critical in their construction of a generalized local model when A1 and A2 showed signs of struggle I offered suggestions on how they could identify such an expression. First I asked them to focus on identifying what the width of the dam would be at a specific depth, 7 m, and then drew on the shared whiteboard to bring attention to proportional triangles within the 'missing' parts of the dam. These small interventions allowed A1 and A2 to identify that for any depth, x, the expression $63.26 - 2 * \frac{12.67x}{25}$ would model the width of the dam.

Attempting to incorporate this expression into the same quantitative structure as the formula for the area of a trapezoid, $\frac{[\text{base 1}]+[\text{base 2}]}{2} \cdot [\text{height of trapazoid}]$, A1 realized that the value for the other base would have to be "based off our interval length." With the assistance of a quick diagram of a generalized partition, A1 and A2 were able to develop a generalized local model for the force acting on that partition at a depth x. Through this process, they first relied on the use of extra variables, such as x_1 to represent base 1 and x_2 to represent base 2, and identified that the variable used for depth would depend on whether you wanted to identify an overestimate or an underestimate (Figure 20). From here, additional suggestions centered on their definition for Δx allowed A1 to rewrite the generalized local model without the subscript notation—$x_1 = x, x_2 = x + \Delta x$, and $x_2 - x_1 = \Delta x$—although they left themselves a note that the quantity x within the symbolic template for pressure was still dependent on which type of estimate they were trying to find.

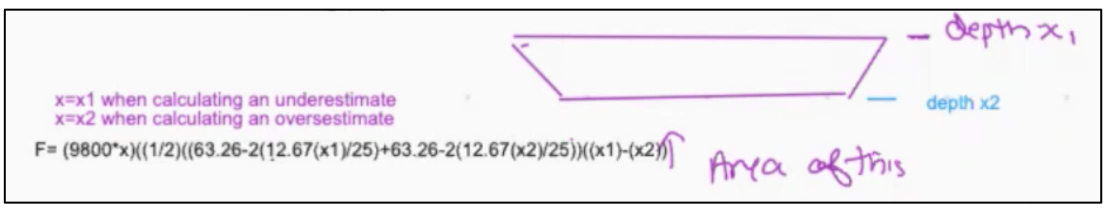

Figure 20: Group A's initial generalized local model for the trapezoidal dam

With their generalized local model in hand, A1 and A2 returned to their primary goal of identifying an over/underestimate within 50,000 N. In an earlier session I had given A1 that their previous strategy of using the magnitude of the last partition would not directly translate to this new dam shape. To investigate, A1 and A2 decided to use their generalized local model to find over and underestimates for a trapezoidal dam with five partitions and compare the values for each element within the global models for their over/underestimates to the value for the error bound. When A1 and A2 reported that they could not identify a pattern, I stepped in to explain why this was occurring—in short, while in the rectangle dam problem the values for x in both the pressure and area were the exact same, in the trapezoid problem they were, in essence, invoking a second value for x in the area component which was the midpoint between what they had originally called x_1 and x_2. I let them know there was a way to rewrite their formula so they were always using this midpoint, but that it would make their equation a little messier. At the same time I posed a question; "Is it okay, instead of using trapezoids, can you just make your approximations using rectangles like you did last time?" A2's reaction was immediate, "Ah, that's interesting," continuing while drawing a generalized partition in the shape of a trapezoid, he noted "and you just cut off those triangles. And that would be your under because the area is less than, and then your over would be if we did it from this corner [motioning to the corner of the longest side]."[19] What became clear relatively quickly was the different interpretations A1 and A2 had with regard to my suggestion. A1 envisioned making the over/underestimates using a single 63.26 m × 25 m rectangle exactly like their last prompt, while A2 interpreted my suggestion as individually applying rectangles as estimates to each partition. After realizing they were discussing estimating the dam in two different ways A1 decided to adopt A2's interpretation, explaining that by "making a specific size rectangle for each partition" they would be

19 Admittedly, I had not expected both an overestimate and an underestimate for the area component of the generalized local model, although in retrospect I should have.

“respecting the fact that it’s a trapezoid” which means the estimates would “get closer to being exactly right.” A2 added, “this way, as we increase the number of intervals, we'll be able to get a more specific value than the other one, that would just be that one block.” This indicated that both A1 and A2 had incorporated a new component to their global model. Specifically, a global model is more than a random approximation for an unknown quantity. If you could not continue to make your global model estimate closer to the actual value, then you were not respecting the global context and therefore it was not good enough. This adaptation to their global model was subtle and was not so fully formed that they could in any way characterize it as an error term tending towards zero. In fact, neither A1 nor A2 were able to articulate exactly why they could not just rely on the changes in pressure as they increased the number of subintervals. A2 just settled on stating that “it's more accurate because we're considering more information that we have available to us.”

Working with the new assumption that they would be using rectangles to approximate area, A1 and A2 rewrote a generalized local model for an ‘overestimate’ by drawing on their basic-local model relationship as sharing the same quantitative structure (see Figure 21). It was interesting that, because in that moment their goal-oriented activity had been centered on quantifying the area component of the generalized local model, they decided to label this expression an overestimate because the area component represented an overestimate of the trapezoidal area without any consideration for the pressure component of their generalized local model. Moving onto the underestimate, A2 observed, “[the area is] the exact same thing, but you add delta x… you add it because you're going down in interval, or going down the dam.” A1 assimilated his meaning to be that you'd add Δx to the quantity x in the area component of the generalized local model, however, again neither A1 nor A2 considered the pressure component as contributing to the type of estimation their generalized local model was measuring.

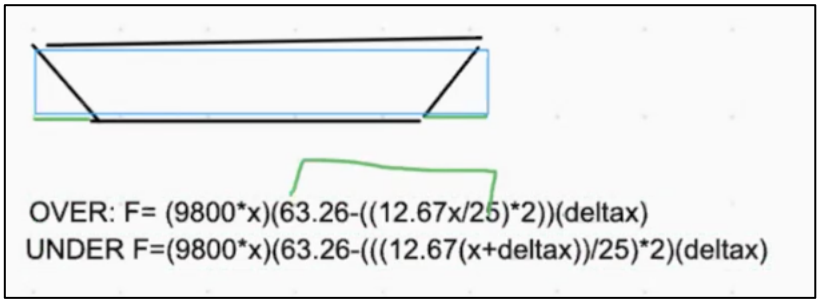

Figure 21: Group A's generalized local model using rectangles to approximate the area of a trapezoid

The session had already gone over by 5 minutes, so we had to stop the interview and return to this idea later in the week. Luckily this provided time to prepare additional support material for Group A. At this point, I decided the trajectory Group A was pursuing was not providing a means to develop their models further, however, I wanted to respect A1 and A2's time and commitment to this process. Therefore I decided to be more direct in helping them identify the quantitative relationships which would allow them to narrow down the height of a partition required for the estimates to be within 50,000 N while still allowing them to create the final expression themselves. Because A1 and A2 were using over/underestimates for the area component I anticipated that there would be a continued inability to effectively find this value. This was due to the value being invoked for the overestimate for pressure, once corrected, would be the value used for the underestimate in area and vice versa. To draw attention to this, in the next session I first asked A1 and A2 to fill in a table with general expressions that would model overestimates and underestimates for (1) pressure, (2) area, and (3) force for a single partition of the dam (Figure 22). In response, A2 wrote the quantitative structure for pressure and using this as a reference created a generalized local model expression corresponding to the overestimate and underestimate for that quantity. Moving onto area, A1 and A2 wrote the area component of a generalized local model for an overestimate as [width of the dam at a depth x]·[Δx], and their underestimate as [width of the dam at a depth $x + \Delta x$]·[Δx]. However, because the dam's width was decreasing as depth increased this would not provide the expected approximations. Based on the previous session's work I knew A1 and A2 had been able to quantify this expression, so I drew their attention to this inconsistency by having them draw a generic partition of the dam. Once they added values to their diagram, A1 and A2 noticed the discrepancy in the area component of their expression, but, in a fascinating turn of events, before interchanging the area expressions, A2 also interchanged the pressure expressions. A1 began to correct A2, "I think [pressure's] the same. It was how we had it because it would be like the depth," but then changed her mind, "wait, hold on. No, no, I think you're right." They continued back and forth for a full minute before A2 concluded that "something was fishy" and I stepped in to ask very targeted questions using their diagram of a general piece of the dam, such as "what x value would you want to use for an overestimate for pressure?" As I suspected, A1 voiced that she believed that the inclusion of either an "x" or an "$x + \Delta x$" into their expressions "had to match" in the respective columns. It is unclear how much of this assumption was born of my earlier suggestion to use rectangles instead of trapezoids versus an innate sense of symmetry. Regardless, the assumption was in direct conflict with the quantification of A1 and A2's generalized local models, resulting in their inability to productively move forward.

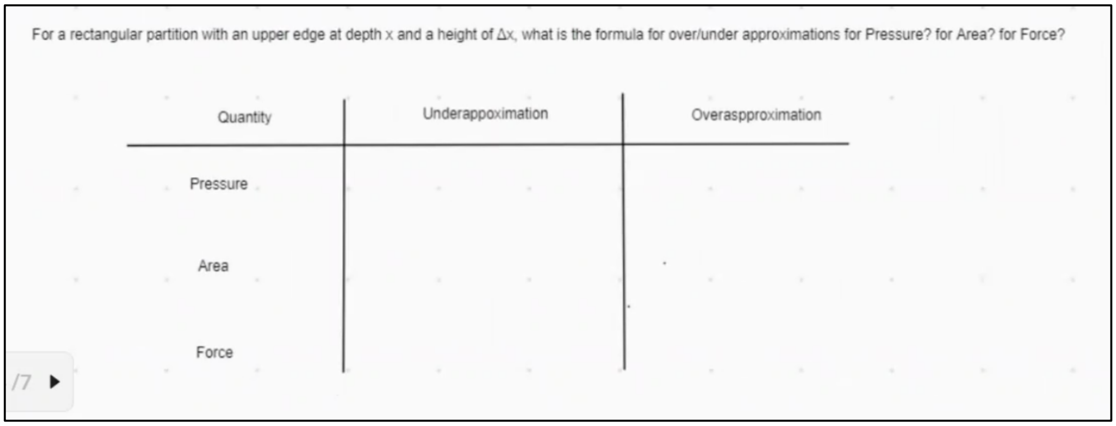

Figure 22: Supplemental activity for Group A during trapezoidal dam task

What is interesting about A1 and A2's notion that value used for both the pressure and area components within the resultant generalized local model for force must match is a key factor in being able to use the cancelation of terms which defines a 'proper' Riemann integral. Unfortunately, this did not match the quantitative situation they had constructed, so following this exercise I presented them with pre-computed values involving a dam with 3 partitions. This included a diagram with the dam partitions, along with a table of values for an estimate of the force acting on each partition using the under-under, under-over, over-under, and over-over expressions for pressure and area. This allowed A1 and A2 to observe that the values within the under-over and the over-under columns shared values, and positioned them to isolate an expression to which could identify how small to make their Δx element. While this second activity took nearly a full session, there were no noteworthy changes to their basic-local-global models beyond having them specifically attend to the quantitative structures of the elements within their generalized local models.

Summary

While working through the Fluid Force on a Dam task both A1 and A2 transitioned from a global model constructed through an accumulation of gross basic models with tentative coordination of refinement and accuracy constructed during the Curiosity Rover task to a dynamic system of basic-local-global model relationships for sums. This new scheme included an association that (1) the partitions of a global model are made in service of estimating a varying quantity (2) increasing the number of partitions leads to a global model with smaller error bound, (3) an image that the global model is composed of a finite number of elements identified using a local model, (4) the total number of elements is inherently linked with the magnitude of an element, Δx, within their local model quantification, (5) a generalized local model must share a similar quantitative structure to the basic model, (6) the local model must "respect" the global context as closely as possible in that the reduction in the magnitude of the local model must always lead to a more accurate global model, and (7) the whole system is tied to the variability that made the invocation of a local model necessary in the first place.

During this task, Group A also demonstrated the significance of the quantification process in their problem-solving on many occasions. For example, (1) A1 and A2's initial interpretation of the quantitative context produced an expected answer, but for unexpected reasons. Specifically, the values A1 and A2 produced were consistent with grossly inputting a pressure depth of 0 and a pressure depth of 25 into a basic model for force, however, I anticipated the application of the gross basic model would be in service of making estimations using the smallest and greatest amount of pressure which could be assumed across the entire surface of the dam. Instead, A1 and A2 interpreted the global context as varying levels of water making contact with the surface area of the dam. (2)

A1's unfamiliarity with pressure as a quantity in its own right resulted in difficulties in identifying the varying quantity, (3) that the specific symbolic template of basic models greatly influenced A1 and A2's understanding of which quantities specific values represented within expressions, and (4) A1's recognition that the constant quantity for the height of a partition and the varying quantity for the depth could not be represented by the same value within their generalized local model, despite the fact that they draw on the same range of numeric values for depth. While this last recognition may seem obvious to the reader, it was a significant development in Group A's ability to reason productively about their generalized local model and was only possible through quantitative reasoning instead of the formulaic approach A2 initially tried. In fact, because A2 had not fully developed the same distinction between fixed and varying elements across a global model as early as A1, Group A faced added difficulty of different meanings for notation and diagrams which slowed down their overall progress through the task. One particularly interesting thing I observed throughout this task was Group A's almost blinding focus on the part of an expression they were currently quantifying. Whenever A1 and A2 were focused on one component within the quantitative relationship for force, either pressure or area, they often didn't re-evaluate the other quantitative component at all. When they inevitably came up against an unexpected result A1 and A2 would quickly realize that the other component also needed to be adapted in some way, but this rarely happened as a holistic process for force, either pressure needed attention, or area needed attention.

In general, at the ending of the Fluid Force on a Dam task, I was confident that A1 had developed the expected reasoning laid out by the hypothetical learning trajectory, and had demonstrated nuances of the development of their basic-local-global models not fully anticipated prior to the teaching experiment. A2 had also developed a set of similar schemes, however, there was one clear distinction between the two—how they interpreted the notation Δx. Both were able to describe the computations to arrive at a value for Δx, however, A1 viewed this quantity as the height of a partition for the dam, while A2 viewed Δx as the "length of an interval" on a number line. While this distinction generally did not cause major problems for the group, having a more abstract notion of Δx made it more difficult for A2 to keep track of what precisely the Δx in their generalized local models quantified, sometimes assigning it to be a length of an interval and other times to be the total number of partitions. A2 even demonstrated difficulty consistently referencing which side of a generalized partition the Δx was referring to in an expression for area. Based on language A2 invoked during these instances, such as the full interval from 0 to 25 being "weird" because it was "vertical and not horizontal" I concluded that A2's primary source of conflict was due to the influence of an incoming scheme for Δx which associated the notation with an x-axis, traditionally ascribed to the horizontal axis in a graphical representation, along with an image that a Δx represents a small portion of that horizontal axis, i.e. the "interval."

Group A: Geometric Volume

Note: Due to scheduling conflicts and more time spent on the Fluid Force on a Dam task than expected, the teaching experiment with Group A was running roughly two and a half weeks behind schedule. Subsequently, A1 and A2's main calculus coursework had begun to introduce the notion of Riemann sums, antiderivatives, and definite integrals as area under a curve concurrent with Group A's interviews covering the Geometric Volume task. Both members of Group A had taken calculus coursework in high school and had mentioned both Riemann sums and integrals in earlier interviews, so I do not believe that the integrity of the teaching experiment was compromised as a result. However, because Group A drew on schemes involving integrals more readily than one may have expected otherwise, I wanted to make the reader aware of the overall timeline.

Moving into the Geometric Volume task, I asked A1 and A2 to identify overestimates and underestimates for the volume of a right pyramid measuring 10 in tall with a square base 4in×4in. When starting the task A2 mentioned that he "just kind of thought [volume] was the integral of area." When A1 questioned what "area" he was referring to A2 wasn't able to articulate his meaning, "I guess the area of a triangle, because length with height of a cube is x. Wait, that doesn't work," before deflecting "never mind, that's for something else." That is, an integral was A2's initial problem-solving tool for this task. An inability to immediately coordinate his image of "an integral of area" with the estimations required from the prompt caused him to disregard this construct, however, it was clear that A2 had at least some experience modeling volume in at least one of his calculus courses. Moving from his initial thought, A2 began drawing a diagram of the pyramid on the shared whiteboard and labeled "b=base=4in" and "h=height=10in" below. At first, A2 was stuck on a way to create an estimation for volume, however, his image that an area would somehow be involved positioned him to visualize the pyramid as being made up of lots of square areas. This influenced the development of a global model as being comprised of horizontal partitions of the pyramid ;

> I guess, I really don't know how to think about it in any way other than… Oh, oh, oh, oh, oh! Never mind, I do. Just kidding. So this is a four-sided pyramid, right? With a square base. Then we're looking at… that square base just gets smaller as

it goes up. We can still do intervals of area, right? Okay. So, we'll probably end up drawing little lines like this [draws four yellow horizontal lines distributed throughout the pyramid's height].

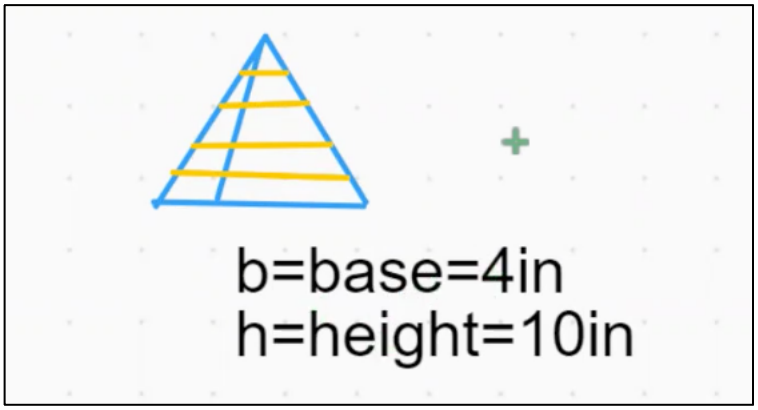

Figure 23: A2's horizontal partitioning of a right pyramid

I do want to be clear that A2's exact words in this instance were "intervals of area," and not "integrals of area," consistent with his reference to delta notation as "interval lengths" in the previous task. That is, while his image that volume, integrals, and area were connected influenced the way in which he partitioned his global model, his goal-oriented activity was directed at trying to quantify an under and overestimate for the volume of the pyramid—not to write an integral of any kind.

It was interesting that A2 did not attempt to immediately identify a gross estimate value for the volume of the pyramid using any formula available from recollection. That is, he didn't try to immediately apply a basic model. The anticipation of drawing on the parts-of-a-whole aspect of his global model motivated A2's need to coordinate the partitioning of the pyramid with a basic model which could accurately characterize the decreasing side-length along the pyramid's height.

While A1 made no objections to A2's partitions, she did make a reference to "us[ing] the volume equation" and that "only the top" partition would remain pyramid-shaped. This indicated that A1 wasn't yet anticipating using a formula other than the one for volume which was provided in the prompt, $V = (l \cdot w \cdot h)/3$. Feeling like A1's inability to recognize a need for estimates may have been caused by the wording of the prompt, I stepped in to make my intention for including the formula more clear;

> I just put [the formula for volume] in there because I probably shouldn't have because I think it just leads to more confusion… I wanted to acknowledge the fact that we know there's actually an equation out there for volume of a pyramid. But this equation came from somewhere, it didn't just come out of thin air. This process is one method for finding that equation.

Understanding that she would be expected to explicitly use estimates, A1 turned her attention to identifying a rectangular prism as an appropriate local model for volume,

> So what if we made it… if we just took it and made it into a rectangle like with a base of 4 and a height of 10? A rectangular prism… I'm totally blanking on what the formula would be… oh yeah, just length times width times height.

Agreeing A2 drew a green rectangle around his diagram which measured the full height of the pyramid, and, after making a quick computation using a local model[20], labeled the overestimate as 160 and the underestimate as 0.

In the next session, I asked A1 and A2 if there was a way they could make their original estimates better. In response, A2 began to describe a method of using his horizontal partitions to create a sequence of rectangular prisms with decreasing base lengths. Drawing generalized images of the boxes on his diagram (in green) it became clear that A2's image of the local model elements within his global model did not have a common fixed height, but stretched from the partitions original placement in the pyramid up through the full height of 10in (Figure 24). In fact, based on A2's language which included getting "smaller squares" and how "add[ing] more squares" would enable them to "get closer to our actual area," I did not believe that his diagram truly represented 3-dimensional shapes in the same sense as A1's rectangular prism basic model. A1 did not mention a need to adjust the diagram and instead focused on the fact that A2's description captured the decreasing side-length of the pyramid which would provide a way to model that behavior as closely as possible consistent with a global model which "respects" the global context; "Yeah, I see.… kind of like how we did with the dam, we want to make it more precise. So yeah, that makes sense to me." A1 added "So, I guess

20 Representing the single element within the current global model as they were already envisioning a partition process.

you'd probably want to write a formula?" anticipating her partners' desire to create an algebraic representation of their local model as they had in the previous task.

As A2 turned his attention to creating a generalized local model, the need to create an explicit formula led him to question the qualities of the objects he'd suggested for a local model; "Are these cubes or squares? I guess they're cubes, right?" A2's adjustment to his description of the local model from 2-dimensional to 3-dimensional objects represented a direct consequence of his desire to quantify a global model for volume which enforced a

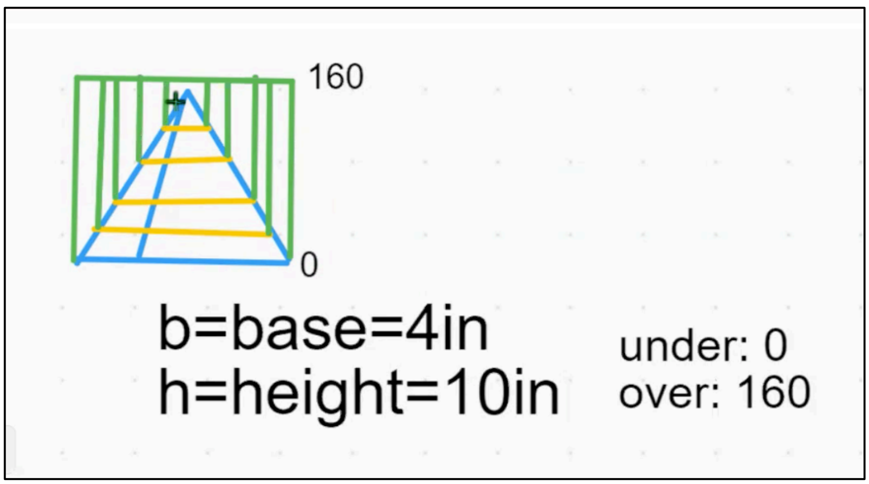

Figure 24: A2's diagram depicting his initial local model for the Volume of a Pyramid

need for the generalized local model to also represent a volume. In passing, A1 attempted to correct A2's use of the term "cubes" to "rectangular prisms" because cubes "have to be the same length on length on height." However, as A1 and A2 began to try and identify a way to algebraically capture the decreasing side-length, A2 was still operating with an image of local models spanning up to a height of 10in. This caused him to assert that "our height is going to change also." Being confronted directly with this conception, A1 squarely rejected A2's image of a local model, "our height will be Δ, or like, the Δx. It'll be the length of the interval… Δx will be the height divided by the number of intervals, so it'll be like the height of the individual partitions.” This description clearly indicated that A1’s global model included a common fixed height for all elements, represented by Δx, and therefore her local model must satisfy a compatible quantitative structure.

A2 followed A1’s explanation, and, recalling that not clearly defining their variables in the Fluid Force task caused difficulties in creating a generalized local model, A1 and A2 began making a list of quantities in the task, assigning variables, descriptions, and values along the way. A2’s first order of business was to change the “height of partitions” from Δx to ΔH, so that the H could “equal the height of everything.” A1 then assigned a value of 10 to H which indicated that, while within her local model there was a coordination between the height of the partitions, ΔH, with the full height of the pyramid, 10, H was just a label and not yet a true variable quantity. A2 defined the base to be “$B =$ 4in” before questioning whether they should also have ΔB. A1 responded, “yeah, it's like we need some sort of general formula that we can use to calculate each of the B’s, like for each rectangular prism,” indicating that at this time the symbolic form A1 and A2 were operating under for delta notation, Δ[ ], had the underlying schema of ‘magnitude of the quantity [ ] for a single partition.’ However, previous coursework involving rates of change brought in other schemes related to delta notation. Specifically, A2 introduced the idea of taking the derivative of the volume equation, $V = B^2H$, to obtain a formula involving ΔB and ΔH, likely drawing on images of implicit differentiation tasks. A1 did not like this approach, citing that they had not used strategies involving rates in previous tasks, and instead drew on their previous activity which involved quantifying the width of the trapezoidal dam; “We need to write a formula, kind of like how we wrote for our other rectangles, our rectangles on the trapezoid, to show how that changes. How the base changes." Meaning A1 was engaged in the activity of trying to quantify the base component of a generalized local model by adapting her local model. That is she was attempting to generalize the component [side-length of *a* partition] to an adaptable quantity [side-length of *any* partition] flexible enough to capture the varying base lengths across different elements of her global model.

Drawing on proportional reasoning between the full height and a side length of the pyramid, A1 and A2 were able to recognize a 10:4 ratio, but as they began to explicitly write a formula for ΔB their lack of a quantity that could capture a variable height caused difficulty. When A2 input an H into an expression, $\Delta B = 4 - \frac{10}{4}H$, A1 said "H is 10" causing A2 to question whether they should use ΔH instead. A1 did not approve of this change, saying that ΔH is the "size" of the partition and that "ΔH will be the same for every partition." They could not use ΔH in their expression, because that "would make our base be the same for every single partition." So at this point, while the ΔH represented a fixed value, it was not a connection with the notation itself, but a result of their choice in global model partitioning. This observation, paired with the inability to place one of their defined 'variables' into the expression engendered the need for A1 and A2 to redefine "$H = 10$in" to "Height of pyramid = 10in" and "B = 4in" to "Base of pyramid = 4in" and introduce a variable quantity "$h =$ height above ground." A1 noted that this would allow h to adapt to whether they were identifying an under or overestimate satisfying an aspect of the need for a generalized local model to adapt to these two scenarios.

As A1 and A2 began to discuss the exact structure of the ΔB formula, A2 noticed that they had not defined precisely what ΔB stood for. This caused A1 to voice an objection to using this notation,

> I don't know if we should call that ΔB or if we should just call that little b… It's just a measurement of the base. It's not necessarily, I just don't know if I feel comfortable calling it a rate. It probably is, but I don't know.

Just like A2, the use of delta notation was invoking schemes involving rates of change, and while her reference to ΔB as a "rate" was inconsistent with my image of what a calculus student should view as a rate of change, the disparity between the rate at which something changes and an expression which models a changing value was enough to make A1 discount the delta notation as inappropriate for their desired use. In A1's follow-up interview, after she described Δ to represent "change in," I asked her why she felt so strongly that they shouldn't use ΔB, but decided to keep ΔH. Through a few minutes of conversation, it became clear that for A1 there was a distinction between "a change in x" which represented a fixed value, and "a changing x" which represented a variable quantity; "our B would be different because it was getting smaller, as we went up our pyramid… but the ΔH was always the same." That is across elements of A1's global model the ΔH represented a fixed value, while the expression they wrote for ΔB represented a propositional relationship. Therefore giving them the same notation wasn't appropriate.

Constructing a new variable b to represent the "base based on height" or "frustum bases," A1 and A2 wrote the expression $b = 4 - \frac{4h}{10}$, which A1 noted should be squared because the base is "length times width." This illustrated a small discrepancy in A2's image that b represented an area versus A1 view that b was the length of one side of the base. This distinction did not cause too much difficulty as A1 and A2 coordinated the development of their generalized local model by assimilating each other's precise phrasing into their own schemes. Drawing on the quantitative structure for the volume of a rectangular prism, [volume]=[length]·[width]·[height], to construct their local model, A1 and A2 squared their new expression for b, and then multiplied it by ΔH because A1 said they needed "the height of our partition, which we decided was the big height divided by the number of intervals." This resulted in two compatible local models, A1:[volume of partition]=[length of partition base]^2 · [height of partition] and A2:[volume of partition]=[area of base of partition] · [height of partition], which they expressed through their algebraic representation $V = (4 - 4h/10)^\wedge 2\Delta h$ (Figure 25). A1's observation also illustrated the strong connection between the number of expressions in A1's global model with the number and size of partitions.

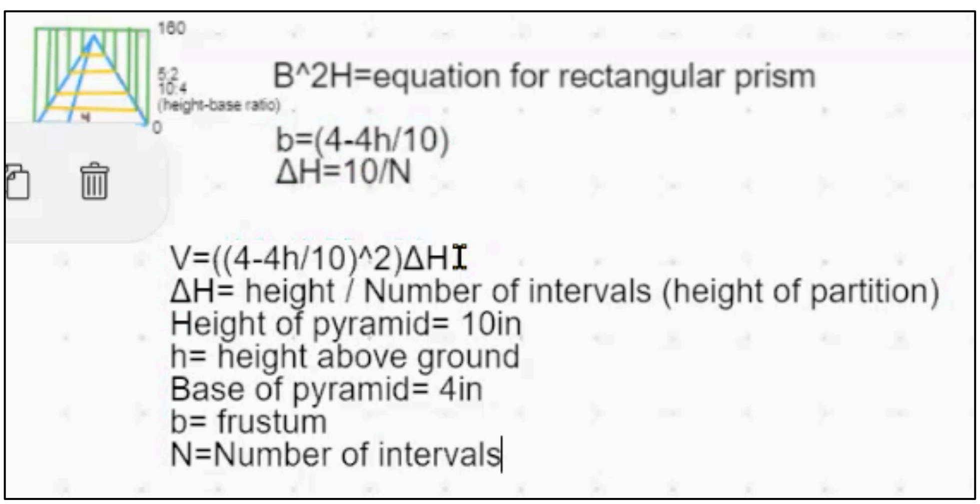

Figure 25: A1 and A2's defined variables along with their generalized local model for the Volume of a Pyramid

Before declaring victory, A1 and A2 decided to "test" their formula by using it to identify an overestimate using one partition to "see if we can get 160" (and 0). While checking an algebraic formula might just seem like good mathematical practice to engage in, doing so also reflects the direct coordination between A1 and A2's generalized local model and the need for it to be generalizable to whatever number of pieces they decide to partition the global model into. Having already done the work for 1 partition, they could quickly and easily verify this adaptability directly. Making the appropriate computations, and excited that their "equation works," A1 explicitly described what their expression measures; "it's the length times width times the height. So, that way we can find the volume of the different frustums." When I pointed out that length times width times height measures the volume of a box and not a frustum, A1 continued, that this was an "approximation formula" and that on their diagram they were "drawing rectangles" and they would "make more and more rectangles" to make their estimates "more specific." Explaining that the original green lines on their diagram were just their "preliminary thought process" A1 and A2 redrew their picture to display that their local model would identify overestimates (red boxes) or underestimates (purple boxes) by adjusting the variable h in their generalized local model (Figure 26).

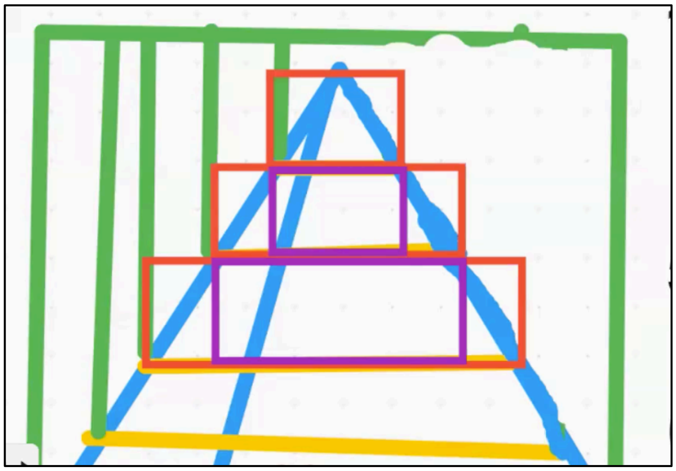

Figure 26: A1 and A2's updated diagram representing their final local models for the Volume of a Pyramid

Unprompted A1 and A2 reviewed the main task objective and reminded themselves that they were supposed to identify an overestimate and underestimate for the volume of the pyramid. Deciding to use the sum calculator to identify these values using 5 partitions, A1 and A2 identified new estimates of 70.4 and 38.4 noting that doing so "significantly narrows our approximation." Unsure of the required accuracy for the task, A2 asked "how specific does the pharaoh want his pyramid to be approximated?" Because they had already created a general formula for their local model, I prompted "what if he wanted to know the exact value? What would we do?" Referencing her individual interview from two days prior, A1 immediately responded "oh no, not this limit business again."

A1's Schemes Associated with Integrals, Sums, Antiderivatives, and Area

While the primary source of model development was meant to occur during the main group interviews, during an individual follow-up interview A1 and I explored her models for integrals and sums constructed through her normal calculus coursework which undoubtedly impacted her associated schemes in the main teaching experiment. As mentioned previously, a delay in the overall timeline meant her class was already moving into these constructs so, in addition to those developed in the teaching experiment, her schemes were a conglomeration of her images of calculus constructs developed in high school along with topics discussed more recently in class. The following few paragraphs detail those schemes she demonstrated through the course of the individual interview.

In her interview, A1 described a tentative link between sigma notation for a right-hand and left-hand Riemann sum with the overestimates and underestimates she and A2 had been identifying in the rover and fluid force tasks. Her schemes for sums included a symbol template that involved indices which started at 0 (left) or 1 (right) and a function f evaluated at $a + i\Delta x$ multiplied with a Δx. Flipping through her class notes she mused, "we either took the limit or the integral," indicating that there was a link between her schemes for sums, limits, and integrals but that this association was more a recollection of the order in which they covered the topic rather than a direct association between two distinct global models. Finding her place in the notes she identified that they "took the limit of the sums," but that she wasn't sure how that was related to integrals;

> I remember we did that. And then I'm like, 'Where did we ever do any of the actual limits, though?' You know what I mean? Like, what does? And then somehow we got into integrals. Which the integral is the accumulation. Like, if $f(x)$ is the rate, the integral is the accumulation. Yeah, because whenever you go backwards, the integral you go back. It's an antiderivative, so you would go back to like a rate. Okay. So I feel like maybe if we could take that if we could find that integral, we could find the accumulation because that's what we're doing now,

> we're doing accumulation.… Like, it's weird, because this is the part of understanding all the concepts or you're trying to, like weave them together, but I feel like I still have some roadblocks in understanding how they're all woven together. Does that make sense? I'm still trying to process through. Which is part of the reason why I'm like, 'Okay, so we've got that limit, and a sum.' And then I know the integral, I know that's the accumulation, and we take the derivative and we get back to, quote, unquote, the original equation, which would have been the rate of change. But I can't remember how the limit of the sum relates to that or if it even relates to that.

Ignoring[21] A1's non-standard interpretations of an antiderivative producing a rate of change or "tak[ing] a derivative" inside an integral, from this description it was clear that A1 has two separate schemes for a definite integral which were not working in harmony. On one hand, some schemes were associated with antiderivative processes, while on the other she was working with a global model which was tied to some form of an accumulation process. While she was able to associate sums (and potentially limits) to this second scheme, the first conception clouded the ability to do so fluidly. A1 continued thinking aloud about limits of a Riemann sum, which revealed an exactness condition of her global model for sums when a limit is involved, "since there is a limit it's not an approximation anymore, it's the actual," and that the limit of the left Riemann sum and

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21 A1's coursework had just started covering definite integrals and had not begun antiderivative techniques, so her descriptions of the antiderivative process were recalled from high school coursework over 6 months prior. Because the teaching experiment was not concerned with antiderivative techniques or encouraging a direct link between antiderivatives and definite integrals I did not interrupt her train of thought to further investigate these non-standard interpretations.

the right Riemann sum produce the “same answer.” This observation of exactness, along with her image that both Riemann sums and integrals involved accumulation allowed A1 to conclude that “[Riemann sums with limits] are related to the integral” because “the equation” that comes after the integral sign was an “accumulation equation.” Asking her to say more, A1 defined an “accumulation equation” to be the function that defines the upper boundary of a graphical area consistent with an area and perimeter conception of a definite integral. Working with the specific equation $y = x$, A1 was able to write a definite integral expression that would measure the area under a curve between 0 and 3, including a dx at the end[22], and even corrected her earlier derivative vs antiderivative procedural confusion. However, connecting her class coursework with the teaching experiment tasks she began to wonder,

> But my question now is that I'm like, looking at these things. We already talked about, like, we know, I'm fairly confident that [limit of sums and integrals] are related. But now I'm wondering, what does this equation look like? Like, how did we get from this equation [$f(a+i)\Delta x$ in the summation] to that equation [differential form in an integral]?… I’m trying to remember if we even talked about how we get from that equation. I'm trying to look back to figure it out. Because, the whole point is how can we get our rectangle slash trapezoids, how can we take the integral of that to get the exact [value]?”

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22 No specific quantification of dx was mentioned.

Having coordinated a relationship between Riemann sum and definite integrals, A1 now wanted to identify a direct correlation between the local models she and A2 had developed in the Fluid Force task and the "accumulation equations" that were a part of the symbol template for a definite integral. However, her image that "accumulation equations" (and therefore definite integrals) represent graphical areas prevented her from doing so.

> Because in my mind, I'm like, 'Okay, well, we know it's area under a curve,' but then you're like, 'Well, how do you put a trapezoid on a graph? You don't.' … the equation can't just be, I don't think at least, it can't just be the area. I guess it could be the area of a trapezoid? I don't know. I'm still trying to connect those dots.

A1's inability to coordinate these two ideas exemplifies the difficulty students face when attempting to attend to definite integral tasks in which the differential form is not a Riemann product. Throughout the teaching experiment, and this individual interview, A1 had continuously demonstrated a strong ability to reason quantitatively, however, an image of an integral as only measuring area blocked her from being able to use an integral as a tool to find the exact fluid force as she desired.

Returning to the main group interview A2 also quickly drew a connection between integrals and exact values. Because A1 and A2 were a few weeks behind in the teaching experiment a definite integral calculator was available at the very bottom of the interview page for other groups' use. When I asked about identifying an exact value A2 moved his screen to the integral calculator and asked "is that this thing?" while circling his mouse around the integral expression. Because A1 and I had already been discussing a link to integral notation, and A2 had clearly demonstrated pre-existing schemes for integration as exact values for volumes, I decided to let A1 and A2 pursue that line of inquiry. Building on our discussion for appropriate expressions for differential form, A1 asked A2, "what do you think we need to put in there as our formula?" He replied that he was not sure, but referenced back to the expression they had placed in the sum calculator saying he assumed they would put in that formula (their generalized local model). A1 agreed that they should try using that expression to "see what they get" and whether it provided a value "between our left and right sum." Because I wanted to engender a direct correlation between their generalized local model and the differential form, I was happy to let A1 and A2 approach this link through a method of trial and error with an expectation that the exact value must lay between their two estimates. Demonstrating some familiarity with the integral notation, along with an observation that there was a dx in the example expression, A2 decided to change the Δx in their generalized local model to a dx when he placed the expression in the integral calculator. This change, along with entering the value of 0 and 10 for the limits of integration, resulted in a value of 53.33 which A1 expressed excitement about (Figure 27);

> Oh, that's in between [160 and 0], and it's in between this one too [70.4 and 38.4]… I wonder how. Let's see… I think that might work. I think that might be what we're supposed to do… Should we ask the pharaoh?

This allowed me to inquire if they had already been introduced to integrals in their current calculus course and A1 verified they had begun that section earlier in the week. She now demonstrated that after this lesson her image of integration was more related to an "accumulation" which is just "adding" and therefore an integral was appropriate because they were trying to find “the accumulation of the volume.” When I asked why they were pleased with the value the calculator produced a solution between 70.4 and 38.4, A2 explained that he had taken the average of those two values as a sort of guess for the real value, noting that the average value “might be a little bit off.” A1 said she had similarly found the average value, but knows you “can’t just use the average” otherwise you would “never need to take an integral… the whole point of having an over and under[estimate] is that the real [value] is in the middle[23].” This indicated that A1 had a global model for integrals that included a link to nonlinear variation of quantities in a basic model. She added that comparing the integral value to the average was just “encouraging” because it was “in the right ballpark.” Following up on this I asked A1’s expectations if they were to increase the number of partitions in the sum calculator from 5 to 50 and she replied,

> I think it would narrow, yeah, narrow it down to closer to what we got with our integral. Narrow it down to the exact. Which is exactly what we said, the more partitions the closer we get to an accurate approximation. Or the accurate? Yes, the real accumulation. It’s not an approximation anymore.

Although A1 did not directly reiterate her earlier terminology involving limits, she demonstrated clear coordination between an increase in the number of elements in a global model with the error of her accumulation estimates converging upon an exact

23 Her usage of middle here is ‘trapped between’ and not ‘equidistant.’

value. Asking in a more direct fashion I revoiced A1's mention of "oh no not that limit business again," and asked, "where is the limit in this situation?" A2 drew on a scheme for derivatives that involved limits to explain a similar process for integrals;

> So let me go back to that split-screen thing over here, so this is a Riemann sum calculator. And as we get closer, as we add more partitions, we're moving into like an infinite amount of partitions is I guess how I thought about it when A1 said limit. And we touched on in class—the limits of Riemann sums, notation, and all that. So, whenever we move towards infinity, we get closer and closer to our 53 and a third, because that's just, like, we're getting more accurate with more intervals. So that's the limit. That's the same way, like, the derivative works. As you get closer and closer to infinity. You get closer and closer to the actual derivative.

While A2 was not precise in what it means to get "close to infinity," he did demonstrate the existence of a relationship between the refinement of a global model for estimates and a global model for an exact value.

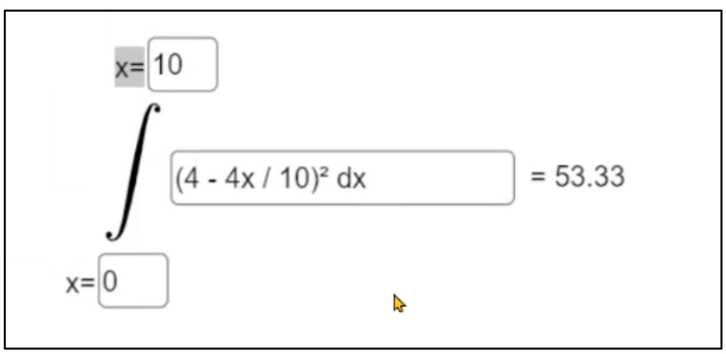

Figure 27: A1 and A2's definite integral expression for the Volume of a Pyramid

Following up on other aspects of the symbolic form for the definite integral, I asked A1 and A2 why they put specific elements in the places they chose. Describing their placement of the values 0 and 10, A1 explained that the "accumulation [was] only happening from 0 to 10." That is, A1 had coordinated the limits of integration with her global model for estimates in which 0 to 10 represented the full magnitude of the height of the pyramid. This full length was partitioned in service of creating estimates using a local model which was then accumulated through her global model. The 0 and 10, therefore, represented a starting and stopping point for the global model accumulation through progressive addition. Specifically, you would begin the global model accumulation by identifying an estimate for the first partition at a height of 0. To that, you would add the estimate for the second partition, and then the third, and so on. You would continue adding estimates for partitions until the top edge of a partition is 10 inches above the ground.

A1 continued, "and then the dx is there, because that's like, Δx is the change in x. And that's kind of that's why the *dx* is there." I found the phrase "change in x" to be an interesting divergence from A1's, previously consistent, descriptions of Δx as a "height" or "length of an interval." I was curious if this was just her way of measuring that "length" or if the transition to dx in the integral notation was retroactively imposing a schema for dx onto her image of Δx. Bringing attention to this directly I asked, "So, your Δx is a change in, it *was* a length of your interval, right?" A1 and A2 both agreed, so I continued "Is *dx* the length of an interval?" This caused an obvious perturbation in both participants indicating that their current scheme for dx did not include this association;

> A2: Yes, because it's still… [9-second pause] No, no it's not. It's the derivative. Or, it's…
>
> [2-second pause]
>
> A1: I think it's… [7-second pause]. Yeah, I don't know what it is actually.

A2 then expanded on his reference to the dx being a derivative, describing dx as a demarcation that a derivative process is complete;

> Whenever we do the derivative of like x^2, we always do it and then put dx at the end [types "$(x^2)dx$" on the shared whiteboard], I guess just to note that this is what we're taking the derivative of. I don't know if this is right, but I've always thought of it, like, whenever truckers use the radio, and then they say over at the end of it. That's just, how do you know that we're done taking the derivative…. So I guess this would just notate for me what we're done taking the integral of.

Needless to say, this scheme was incompatible with my view for a productive image of derivatives and one I believed would directly favor an antiderivative conception of definite integrals over a Quantitatively Based Summation conception. I did not want to derail the group interview, but A2 and I spent nearly his entire follow-up individual interview exploring his conception in more detail.

As a quick summary of this follow-up interview, I came to understand that throughout his calculus coursework A2 had developed two schemas for Δx: the first was consistent with calculus concepts often deemed productive in which Δ stood for "a change in," while his other scheme assigned the phrase "derivative of" along with all associated procedures to that same notation. Having never really been faced with a conflict between these schemes, A2 found it difficult to be able to describe whether the dx at the end of the differential form represented a height or just signified the end of a derivative;

> [the dx] signifies the end. Like we're done taking the derivative now, and this would signify to me that we're done taking the integral, I guess. But it doesn't… because it means something else [referencing the height from the local model].

> Maybe it meant something else up here too [motions to where he had written $(x^2)dx$]. I just never thought too hard about it.

Being familiar with A2's specific course curriculum, I knew that his instructor spent a great deal of class time devoted to illustrating rates of change as proportional relationships, $\Delta f = [\text{rate of change}] \cdot \Delta x$ and $df = [\text{rate of change}] \cdot dx$. Therefore, in the follow-up interview I was positioned to quickly verify my suspicion that somewhere along the way A2 had internalized the expression "$df = [\text{rate of change}] \cdot dx$ " to represent the phrase "the derivative of f is equal to [rate of change]" where the $dx's$ only role is to signify the termination of a derivative process. Because this way of reasoning about derivatives would directly conflict with A2's ability to productively view dx as a meaningful quantity I spent the rest of his individual interview session addressing the issue and being quite explicit that the "dx" within a derivative expression, $\frac{df}{dx} = [\text{rate of change}]$, still represented "a change in x." While he seemed to take on this information, I wouldn't really know if A2's derivative schemes had been accommodated until later sessions[24].

In the group session, I wanted to redirect the discussion away from A2's unconventional invocation of derivatives, so I asked A1 if she thought about the dx differently. A1 admitted, "I don't really know why it's there. I don't really think about it,

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24 Spoiler: Although this scheme did occasionally result in A2 not ascribing a quantitative meaning to dx, A2 never explicitly referred to a differential as representing "the end" of a derivative or integral again.

to be honest," so I followed up by asking what the "$(4 - 4x/10)^2$" represented. A2 chimed in noting that this was "the equation for the bases at any given height," which caused A1 to speculate, "I mean, maybe the dx is the height, or is the Δh…It has to be related because otherwise, we don't have the same equation… we're just taking like the accumulation of lines[25]" with A2 adding "Yeah, this is our third dimension." This interaction represented a critical first step in the accommodation to their conceptual schema for differential notation. Specifically, the image that a global model in this context is an accumulation of volumes required that the elements they were summing to also be volumes, even within the structure of a definite integral. This allowed for direct coordination between their generalized local model and the differential form within a definite integral and promoted a correlation between a Δx and dx representing the same quantity. A2 even tried to replace the dx within the definite integral calculator with a Δx just to see if it would work[26].

Feeling like A1 and A2 had modified their models in the desired ways I provided them with the short prepared writeup describing the symbolic form for definite integrals.

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25 Based on the coherence of A1's quantification earlier in the task I believe it's fair to assume she was referring to area in this case and not lines. The partitions within the diagram the pair drew had perfectly horizontal lines moving up the pyramid, so imagining the collapse of the height dimension to zero would result in only an image of a line rather than an area. I have complete confidence that had I pointed out the slip, A1 would have immediately corrected herself.

26 I had to inform A2 that I didn't build the calculator to be able to interchange dx and Δx. Although if I were to use the same calculator in future studies, it could be an interesting feature to incorporate.

I also provided an example expression that modeled the exact fluid force on a rectangular dam from Task 2 so they could associate the notation with their previous activity. Following this, I asked A1 and A2 to identify an appropriate definite integral expression for the exact fluid force acting on the trapezoidal dam from Task 2. Demonstrating the expansion of her local-global model relationship to include the exactness of a definite integral expression, A1 observed they just used the formula "they'd already written" referring to their generalized local model expression. While there was a little difficulty with mismatched parentheses when inputting the expression into the calculator, A1 and A2 were able to adapt their expression for the overestimate of the last partition into an appropriate expression for the differential form by changing the depth measurement from the pressure component of their local model to an arbitrary x, Δx to dx, and identifying their "bounds" as 0 and 25.

Moving onto the next task in the sequence, I asked A1 and A2 to find the volume of a sphere with a 6-inch radius. At the start of the task, A1 and A2 felt fairly relaxed, saying that "it was kind of the same concept." Both deciding to use "the volume of a rectangular prism, like how [they] did with the pyramid" A2 began drawing a diagram of a sphere with horizontal partitions and rectangular prisms for over and underestimates[27]. A1

27 I immediately knew that A1 and A2's choice to use rectangular prisms instead of cylinders would not result in a correct value for the volume of a sphere. However, wanting this conflict of expectations to represent a genuine perturbation rather than one superimposed by an authority figure, I let A1 and A2 proceed without indication that this local model would not lead to a correct solution.

observed that this problem "might be weird" because the symmetry of the sphere indicated that their estimates would be "getting bigger and then getting smaller," so A2 suggested just finding the volume for half the sphere. A1 agreed, observing they could just multiply their volume by 2.

Viewing this new task as similar to their previous work, A1 and A2 went back and copied their generalized local model for the volume of a pyramid and set forth on "adjusting" that formula to represent their sphere. A1 and A2's first attempt at this was to literally adjust the numbers in their previous local model, $V = \left(4 - \frac{4h}{10}\right)^2 \Delta\text{H}$, to the new values within the sphere equation, $V = \left(12 - \frac{2h}{6}\right)^2 \Delta H$. Following the prompt, A1 and A2 listed out their interpretations for each component of their local model (Figure 28).

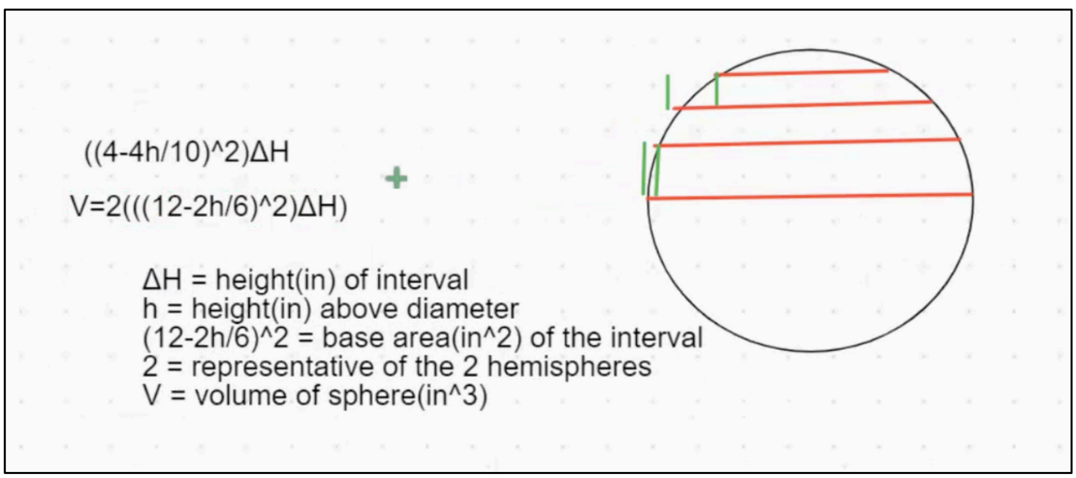

Figure 28: A1 and A2's adaptation of a local model for the Volume of a Pyramid to the local model for the Volume of a Sphere

After defining their 'factors,' A2 began writing out the values which would test their local model expression. That is she attempted to use their generalized local model to identify over and underestimates using a single partition. This led to an issue as their underestimate came out to be 1200 in^3 instead of the value of 0 in^3 they expected. A1 recognized that there must be an error in the "proportion," referencing the $2h/6$ element in their expression but wasn't sure what they should change it to. This resulted in A2 reanalyzing their expression and identifying an issue with their quantification of the width of their rectangular prisms;

> So I think if that proportion would work just fine if. Yeah, hold on. We used it in the triangles because the rule of similar triangles and stuff. So if the thing was like this [draws a chord from the left edge of the horizontal diameter to the top of the vertical diameter in the diagram of the sphere], that would work. But since this is a circle we gotta figure out something else.

Realizing A2 was correct, A1 identified that to move forward they need "to figure out how to find those lines," referencing the length of horizontal partition demarcations A2 had drawn on the diagram. A1 suggested they "start with 12" because that was the length of the diameter. To find the length of the next partition's base they should subtract "those little bitty pieces on the edges," but A1 also noticed that as you traveled up the sphere vertically the "pieces" were increasing in size; "look at how much those lines that your cursor is hovering over right now are barely any away from each other. And then like further up are quite a bit away from each other." This led A1 to reemphasize A2's point that they "probably should not be using a proportion" because the size of the "pieces" which were being deducted from the side length was "not constant."

I let A1 and A2 spend about 8 minutes trying to identify a way to quantify the phenomena they were noticing into a formula, before stepping in to offer a nudge in the right direction. They had already made a connection that they could somehow model the width of their partitions using trigonometry but were having difficulty recalling a helpful expression. I suggested they try using the Pythagorean Theorem. Both A1 and A2 were a little confused as to how they would proceed using my suggestion, so I followed up by drawing a right triangle on their diagram with the hypotenuse a radius length, drew two blue lines on their diagram indicating a correlation between the base of the triangle and half the length of one of their partitions, and reminded them that they knew the hypotenuse would always be 6 inches (Figure 29). With this hint, A1 was able to make a connection that they could identify the height of the triangle based on the number of partitions which would allow them to identify the length of the blue line. This motivated A1 and A2 to return to trying to write a general expression for the width of their

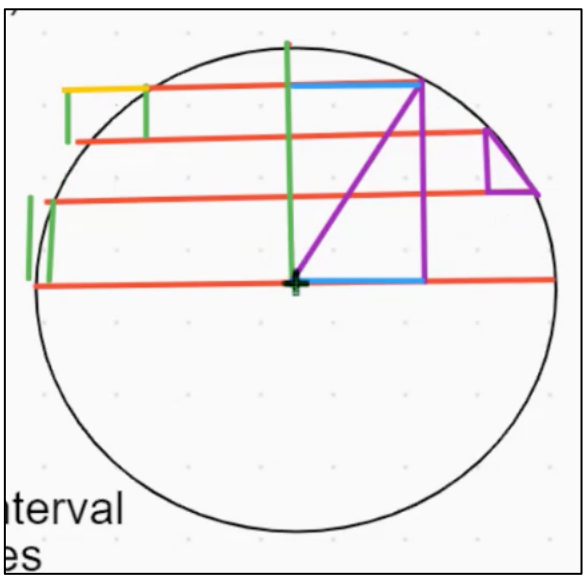

Figure 29: Interviewer's Pythagorean Theorem hint to A1 and A2

partitions, however, during this process A1 encountered a conflict with their quantification of the height of the triangle and their desire to create a generalized local model;

> If we knew how many partitions beforehand, we would know what our ΔH could be multiplied by… But that won't work either. Because that one is always going to be like five or four… I don't think that's right. Because like, for our that top partition part, or the very first Pythagorean Theorem triangle that we looked at with the blue lines. If we wanted that height, that height would be ΔH times 3. Because we were taking the height from three partitions and adding those together, but we'd still need that whole thing squared, the three heights squared.

Specifically, A1 was recognized that by relying on the height of each partition to identify the height of the triangle meant they would have to preemptively decide how many partitions they were breaking the sphere into. Trying to construct a local model in this way did not fit within her image of the generalized local model that would flexibly adapt to any partition height and could identify any partition's base length within the sphere.

Refocusing A1 and A2's attention on their list of variables I pointed out that they had already defined h to be the height above the diameter on the sphere. Identifying that this solved her dilemma, A1 assisted A2 in writing out the Pythagorean Theorem using h as the height of the triangle and acknowledging that they had to "unfortunately" define another variable, $x =$ half the width of a partition, but that "they weren't going to have x in [their] equation" because they were going to solve for it right away. Doing so they were able to quickly identify an expression for a generalized local model of a sphere, $V = \left(\left(2\sqrt{(36-h^2)}\right)^2\right) \cdot \Delta h$ which they multiplied by 2 to simultaneously capture the measurement for the reflected partition at the same time. A1 and A2 asserted that this expression must work because it fit into their image of a local model for volume, [volume of a rectangular prism-shaped partition]= [length of base partition]^2·[height of partition].

A1 and A2 placed their generalized local model into the integral calculator which resulted in an output of 576. Even though they "knew they were right," A1 wanted to check their solution using the known formula for the volume of a sphere[28]. When they identified that the volume of their sphere should have been 288π they realized something was not right. Reviewing their work, A1 wondered aloud "where are we wrong" searching through each expression in their local model. A1 then asked for a link to the sum calculator to check if that would provide a different solution, and, after receiving the same answer, concluded that they must have entered the right expression into the integral calculator because the exact value was between the two estimates. We were right up against the time for the session to end, so I mentioned that "the big, glaring obvious thing to me is that there's a π in this one and there's not a π in yours." This observation caused immediate exclamations from A2 of "oh my gosh" and A1 to note that "we weren't trying to do anything that required π though. We were trying to do our square over and underestimates." I assured A1 that the integral they wrote did compute the volume of a shape using boxes, just as they intended, but that the shape this would create would be a "bubble pyramid," not a sphere. This characterization was proffered to draw on A1 and

28 The 576 was off by a factor of 2 due to a misplaced parentheses (the 2 in their [length of partition] element didn't get squared). I didn't catch this during the interview, but because A1 and A2 were extremely clear about their quantification process I don't believe it had any impact on the outcome.

A2's desire for their local model to "respect" the global context and caused A2 to suggest that they should have used circles in their local model instead of squares.

When A1 and A2 returned to this task the following day, A2 set about making sure they were on the same page about what local model they were trying to quantify,

> So I'm going to get rid of this square, just to eliminate confusion [erases an earlier diagram showing a top view of the partitions with consecutively smaller squares]. All right. Then do a new circle, and this will be like our side view like this was [motions mouse to the main diagram they had been using]. Okay, so we have our diameter, and then we go up [draws a circle to represent the sphere, and then adds horizontal partition lines]. And so, these are circles now not squares. Yeah. But they're still they still have interval height.

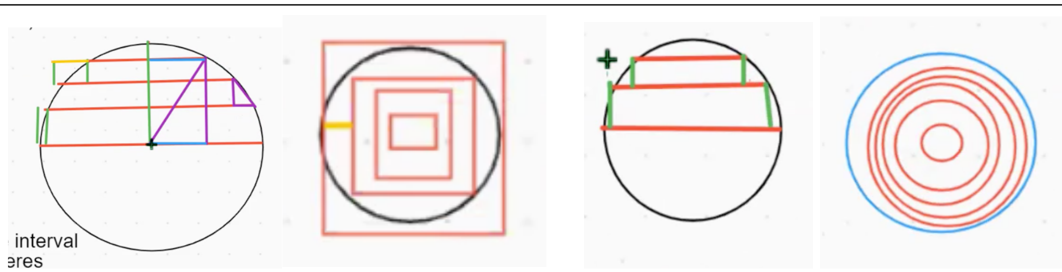

Diagram for rectangular prisms partitions

Diagram for cylindrical partitions

Figure 30: A2's diagrams for Group A's different local and global models for the Volume of a Sphere

What I found interesting about A2's sketch of a new diagram (Figure 30) was not the added view that they were using cylinders instead of prisms (although this was certainly important)—the day before A2 had already added an image depicting a top view of the partitions using cylinders which included consecutively smaller circles. What was important about this interaction was that A2 was going through full consideration of whether adjusting the base shape of the partitions would have an effect on the rest of the quantification of the local model, specifically the height. This represented an evolution of A2's adjustment process for the quantification of a local model. In the Fluid Force task, A1 and A2 would focus on specific elements of their local model without consideration for how changing those components may affect other components within their generalized local model. By redrawing the diagram from both vantage points A2 was making sure they were attending to that possibility.

As A1 and A2 were constructing their new local model, A2 emphasized an important shift in their structural thinking as part of this quantification process. Early in the Fluid Force on a Dam task, when working with strings of summed elements as a written artifact of their global model, A1 and A2 had factored elements such as the area from their expression. As discussed in the previous section, this led to difficulties attempting to write a formula as they shifted to a generalized local model. During the Geometric Volume tasks, with an expectation of using the online calculators, A1 and A2 were free to focus explicitly on their leveraged local model for structure. Not having to write out summation notation, simplify terms, or factor quantities to make computational inputs into a calculator easier allowed A1 and A2 to more effectively focus on their generalized local model in ways consistent with the quantitative structures they were trying to capture. For example, by drawing on the equation for the area of a circle, $A = \pi r^2$, A2 convinced A1 to square the $\sqrt{36 - h^2}$ term they had identified as half the width of their base using the Pythagorean Theorem; "We would have to square it since it's… that's π, and [sqrt(36-h^2)] will be our r right? So, we do have the square that because it's π r squared." Neither A1 nor A2 attempted to simplify $\left(\sqrt{36 - h^2}\right)^2$ to $36 - h^2$ at any point in the rest of the task, which positioned them to consistently identify what that quantity represented within their generalized local model structure.

After A1 and A2 corrected their integral, I asked why their first instinct was to use rectangular prisms. A2 explained that they had used boxes on all the previous tasks, and A1 added that even when A2 suggested using an arc length early in the task she figured "there could be other ways to do it, but that [rectangular prisms] should work too."

In my conceptual analysis, I indicated that an accurate local model would include coordination with an error term tending towards zero, however, I did not include this conception into the main hypothetical learning trajectory. This decision was made primarily due to time constraints and a focus on positioning participants to successfully construct a generalized local model. I anticipated that building a productive conception of an error term would require a considerable amount of tasks devoted to that one construct. While there is curriculum positioned to leverage such an understanding, such as Oehrtman's CLEAR Calculus which draws on an approximation framework as a unifying theme across calculus constructs, no participants would be enrolled in courses using that curriculum. I did not believe it would be feasible to condense these ideas into a few short tasks, however, A1 and A2's use of rectangular prisms instead of cylinders suggested there might be an opportunity to explore the idea of an error term for an approximation. Therefore I developed a supplemental activity for A1 and A2 to engage in with the goal of identifying the commonality between tasks where their use of an alternative expression as the quantitative structure for their local model had produced an accurate global model.

Summary

A1 and A2's basic-global-local models expanded a significant amount as they worked through the Geometric Volume tasks. First, both A1 and A2 demonstrated the applicability of the schemes they developed in the Fluid Force task. That is, when encountering a request to approximate the volume of the pyramid they were able to draw on their parts-of-a-whole global model and anticipation of creating partitions to identify an appropriate basic model, the volume of a rectangular prism, with which to construct their local model estimates. Having experienced difficulties associated with simplifying expressions, A1 and A2 began to format their local model expressions to be consistent with the same quantitative structure as their basic model. That is if the basic model is a length times a width times a height, then the local model is a length times a width times a height, and they would keep the ordering of these quantities. This allowed A1 and A2 to communicate effectively about their developing generalized local model expression as well as identify inconsistencies with their own expectations of the quantitative structure of individual local models.

In the Geometric Volume tasks, A1 demonstrated that she coordinated Δ notation to be a fixed value across elements of her global model, in this case, a fixed height. While increasing the number of elements of the global model impacts the size of this fixed value, it is always the same within a given partition. When A2 used ΔB to represent the base length of a partition, the image that base lengths were not constant across every element in her global model allowed A1 to reject this notational usage as part of their local model expression. A1 and A2 also engaged in more consistent practices of checking that adjustments to certain components of their generalized local model expressions would not have repercussions on the quantification of other elements within the local model.

Due to the introduction of an exactness condition the largest adaptation to their model development was to their global models and local-global model relationships. Specifically, through coordination of adapting their generalized local model to be the differential form, A1 and A2 coordinated their previously finite and approximate global model to one that contained an element of exactness. This was closely tied to the refinement relationship they had been developing in previous tasks and allowed them to anticipate that as you transitioned from approximate global models (sums) to an exact global model (integral) through progressively "zooming in" on the real value by using more and more partitions to find over and underestimates.

In terms of local model development, A1 and A2's decision to use rectangular prisms to find the Volume of a Sphere demonstrated an interesting limitation to their previous desire to "respect" the global situation in the trapezoidal task. In particular, A1's continued confusion about why using boxes did not work allowed her to be curious when you are allowed to invoke a local model that is not a perfect generalization of an original basic model quantitative relationship, such as using area of rectangles to approximate the area of trapezoids. While I did not expect such a course of action, reflecting on the ordering of the tasks within the teaching experiment I am not surprised by this result.

The most critical episode within Group A's Geometric Volume task was undoubtedly their accommodation to differential notation, dx. When A1 and A2 placed their generalized local model into the integral calculator and made the adjustments from Δx to

dx, their language surrounding that quantity changed. For A2, dx just marked the end of an integral and for A1 it changed from a "length" to "a change in x." These schemes were a part of A1 and A2's incoming understanding of differential notation which was incompatible with a Quantitatively Based Summation conception of integration. By noticing this distinction immediately and directly providing an opportunity for A1 and A2 to be perturbed by these conflicting schemes[29], I positioned them to draw on their problem-solving activity which had assigned Δx clear quantitative meaning to coordinate that dx must also share that same quantitative meaning. This allowed A1 and A2 to engage in further coordination between the generalized local model for approximations with the differential form within the symbol template of a definite integral.

Supplemental Task: Can we use a rectangle to approximate the area of an annulus?

In the supplemental task, I asked A1 and A2 to try and identify whether or not they could use a rectangle to approximate the area of an annulus (Figure 31). In an effort to draw their attention to the error term between an approximate local model and a real measurement I provided A1 and A2 with diagrams of generalized local model partitions, corresponding measurements, and error terms for each task they had already engaged in: the trapezoidal dam (Figure 32), the volume of a pyramid (Figure 33), and the volume of

29 By posing the question "So, your Δx is a change in, it was a length of your interval, right? Is dx the length of an interval?"

a sphere (Figure 34). Within the volume of a sphere diagram I compared the local models for both a rectangular prism and the cylinders they had just completed.

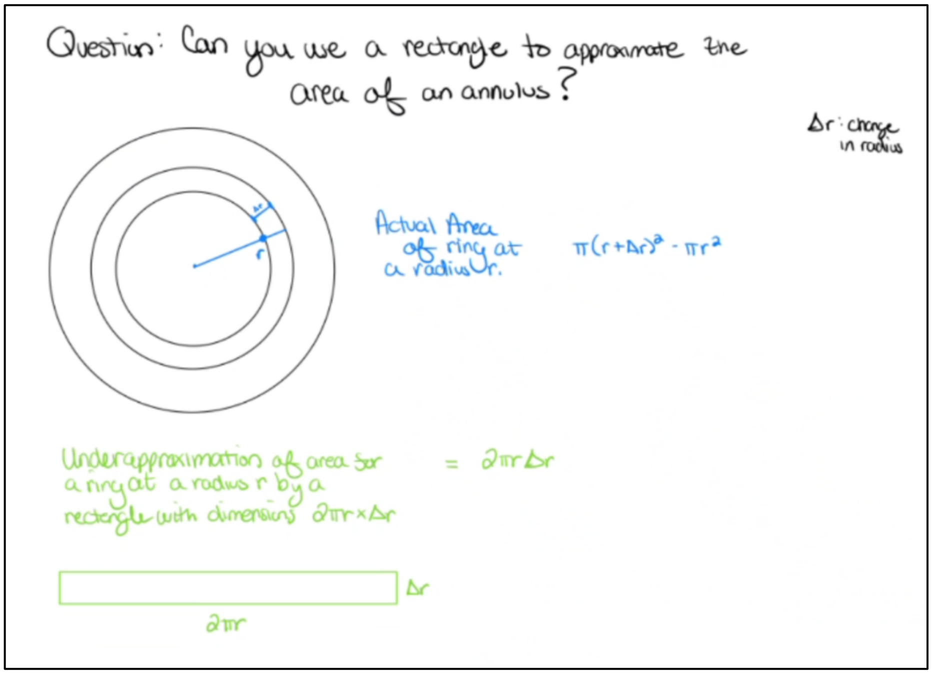

Figure 31: ‘Can you use a rectangle to approximate the area of an annulus?’ subtask

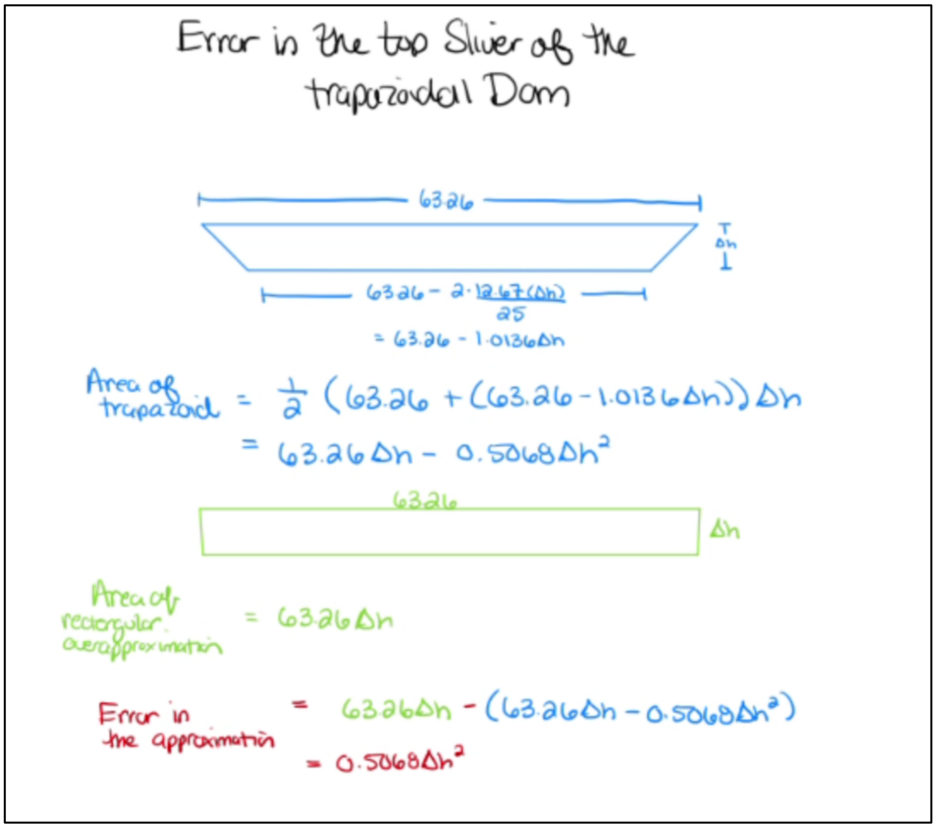

Figure 32: Error in trapezoidal dam local model

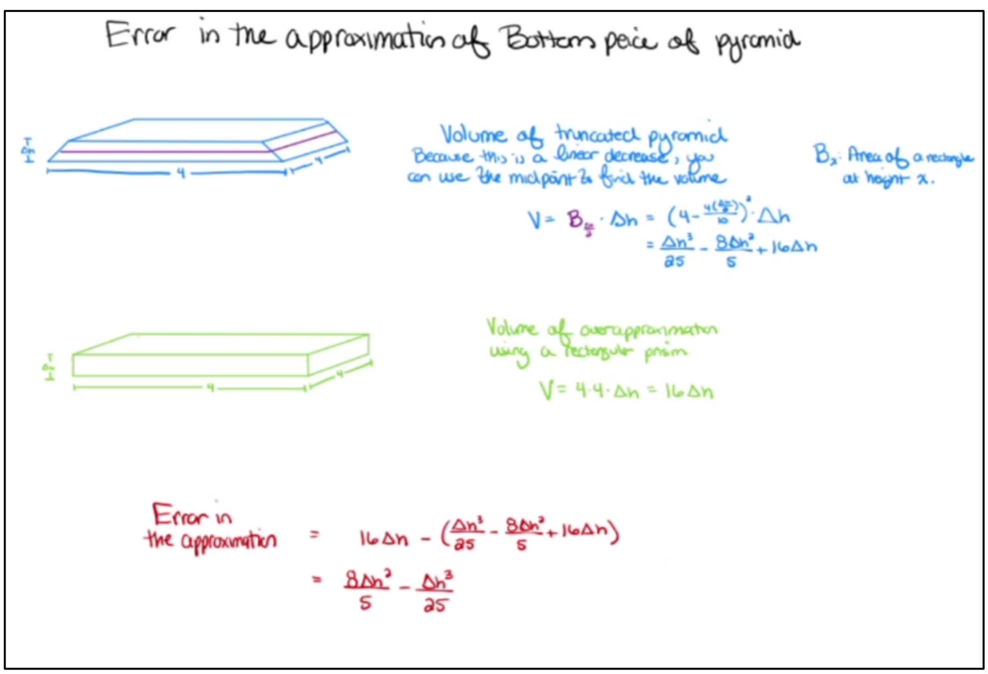

Figure 33: Error in Volume of a Pyramid local model

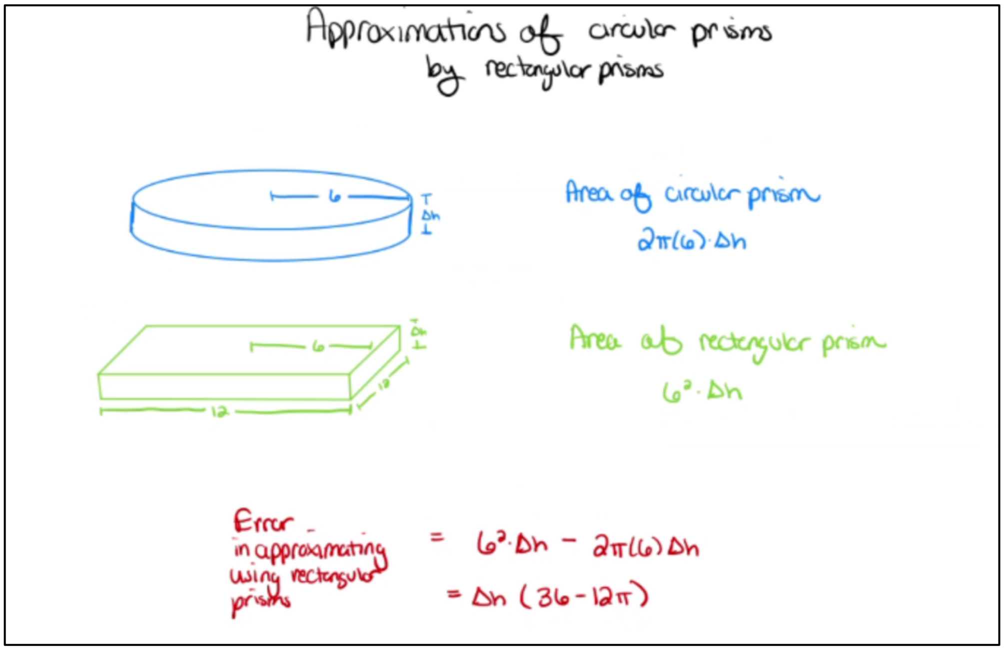

Figure 34: Error in Volume of a Sphere using rectangular prism local models

Because A1 and A2 showed evidence of proportional reasoning, I hoped to draw on the structural relationship between the error of their different local model approximations. Specifically, I wanted A1 and A2 to notice within every error expression that accurately modeled the quantitative situation, the error term did not contain any single factors of their Δ element. That is, I wanted them to identify that the error of a local model cannot have a linear relationship with their Δ quantity.

Despite drawing on A1 and A2's local models, there was a significant amount of information in this activity for A1 and A2 to unpack. Because they did not write the formulas personally, A1 and A2 spent a significant amount of time making sense of, and asking questions about, the expressions I wrote. However, A1 was able to identify the crux of the situation. With a gut feeling, A1 explained,

> I would say yeah, you could. I mean, if it's an under approximation. My thing is like, because last time when we did it with the rectangular prisms with a sphere, the problem was as we would have gotten more specific, we would have gotten away… our under and over, like, started out as like 1728 and 0… And so like, as we would have gotten more specific, we would have, passed 904. We would have zoomed in on something that was not 904. Does that make sense?… And so my question is like with this under approximation, is it specific? Would it be specific enough that as we started narrowing it, we would zoom in on the right one? Or would it be like whenever we use those rectangular prisms that wouldn't have zoomed in on the right one?

That is, A1 recognized that as they created a global model with more partitions, it was critical to know whether the value the overestimate and underestimate approached was truly the exact value they were seeking to find and that this was somehow tied to the local model estimates. This need to have both an underestimate and overestimate that can "zoom in" influenced A1 and A2 to spend a few minutes discussing how they could overestimate the area of an annulus using a rectangle, eventually deciding they would just add Δr to the quantity r in the length component of the area. Building back on A1's earlier statement I informed A1 and A2 that the question I was essentially asking was, without the use of the integral calculator or using antiderivatives, could they tell me whether the definite integral $\int_0^5 2\pi r dr$ measured the area of a circle with a radius of 5. When presented this way there was a direct conflict between A1's scheme for an expression that measures the area of a circle, $A_C = \pi r^2$, and the integrand expression which represented circumference of a circle. This caused her to change her mind, "our answer is no," but that they needed to figure out a way to justify it without using antiderivatives.

A1 and A2 then turned their attention to the other examples in the task, deciding to compare errors using a $\Delta[\ ]$ value of 1. This comparison did not really provide leverage to move forward in their problem-solving. As they were thinking A1 began reflecting on their previous day's activity and realized she still did not understand why their initial solution had not worked; "So, I'm gonna be honest. I'm still like, I'm so confused about why what happened yesterday happened. Why it didn't it at least, why didn't the rectangles work?" A2 responded by redrawing a picture of a circle. He then inscribed and circumscribing the circle with two squares and explained that "those can't get closer together. Just because, you can't add more and get a more specific than that right there [motioned between the difference between the corner of the smaller square and the corresponding corner of the larger square]." I took A2's "add more" to mean making more partitions, indicating that no matter how 'short' they made their partition as a result of having "more" there would still be an error in the base area between the circle and the square. This was the exact conception I wanted A1 and A2 to take on, so I revoiced A2's statement while making an explicit connection to Δh, "a good thing to point out is that it doesn't matter how small you, kind of what A2 said, no matter how small you make Δh, that difference doesn't get any smaller." With this in mind, A1 and A2 returned to the new task and, via a suggestion from me, engaged in comparing the errors of the different local models to try and identify similarities and differences. Noticing that the trapezoidal dam error had a ΔH^2 term, that the volume of a pyramid had ΔH^2 and ΔH^3, A1 observed that the error in between a cylindrical prism vs a rectangular prism had a term of only ΔH. This allowed me to introduce a link between A1's observation and A2's description of the overestimates of squares never getting close to the area of a circle. By leveraging specific values for Δh I engaged A1 and A2 in examining the non-simplified error expressions and discussions about what would happen to the terms which had a $(\Delta h)^2$ and $(\Delta h)^3$ in comparison to those terms that only had Δh as they used smaller $\Delta h's$ such as $\frac{1}{2}, \frac{1}{4}$, etc. This aided A1 in observing that those values would be much smaller comparatively, and allowed her to conjecture, "that probably has something to do with the fact that it doesn't get any closer." Not wanting to spend too much time on this side activity, I agreed with A1 and directly tied the error expression, $\Delta h(36 - 12\pi)$, to A2's diagram; "Yeah… because the area doesn't get any closer, you're always going to have an error that is proportional to 36-12π. That's just the error between the area of a circle and the square." This link enabled A1 to recognize that by calculating an error expression for an annulus approximated using a rectangle, they could see whether the delta terms "had an exponent." Working through the computations and canceling like terms, to obtain $\pi(\Delta r^2)$, A1 observed "maybe we can use a rectangle."

During some follow-up discussion, A2 made it clear that he had followed and agreed with A1's conclusion that they could use a rectangle to approximate the area of an annulus, however, he voiced confusion about "Why in the world you would ever need to do this?" Specifically, A2 felt it would be "so easy" to identify the actual area of the annulus by deducting the area of a circle with radius r from the area of a circle with radius $r + \Delta r$, so estimating this value with a rectangle was adding needless complication. This observation from A2 indicated that, despite the seeming opportunity to introduce a discussion about errors and their relationship to local models, this task design had not provided enough intellectual need for A2 to engage in the reasoning I envisioned as necessary for motivating these observations. It is possible that A2 simply did not need such an activity because he had realized the previous day that rectangular prisms would not model a sphere accurately enough, but I do not think this was the case. I did answer A2's question by quickly reviewing the upcoming oil slick task which used a radial density function (promoting the need to measure a circle using rings), and using the Fundamental Theorem of Calculus (which A1 and A2 were both familiar with) to show that using the ring method was one way to prove the area of a circle with radius r is πr^2. However, I believe that had I incorporated one of these activities as the motivating factor to investigate the ring method, the problem-solving activity involved would have been positioned to provide that intellectual need for A2.

Group A: Energy to Build a Pyramid

Based on Group A's basic-local-global model development in the previous tasks I predicted the Energy to Build a Pyramid task would be a relatively straightforward process. I had an expectation that they would draw on their image of a global model to anticipate breaking the pyramid into partitions, that this would inform their need to draw on basic models to develop a local model with which they could make a gross approximation, and that they would the quantification of their gross approximation to aid in the creation of a generalized local model. While my hypothesis was not far from how this task ultimately played out for Group A, the early aspects of the task did not go as smoothly as I anticipated.

When I introduced Group A to the Energy to Build a Pyramid task I was familiar with A1's slight anxiety with physics contexts. To ease that burden I provided some additional context such as covering the units for Joules and Newtons as well as providing the approximation for the acceleration due to gravity, $g \approx 9.8\ m/s^2$. Upon receiving this additional information, A1 began to dissect the expressions provided in the main prompt, $E = F \cdot d$ and $F = M \cdot d$, to decide what pieces of information the task provided versus what they needed to find; "So, we have our height, or we have our we have our vertical distance. We have g, so, we need our mass." Notice that at this early stage, due to her unfamiliarity with the concept of energy, A1 was not viewing vertical distance as a variable quantity. Instead, the d in the equation represented the 146m height of the pyramid. A2, on the other hand, was more comfortable with the context of energy and had constructed an image of the pyramid being built one stone at a time. Anticipating that he would need to identify "how high off the ground" the stones were a part of the quantification for energy, A2 was attempting to identify a way to figure out the exact height of each stone layer. That is, there was an interrelationship between A2's image of a pyramid built in layers, that these layers were at different heights, and that it takes more energy to lift a stone to a higher layer. By reasoning about this quantity in two different ways, A1 and A2's discussion about possible ways to solve the task was not progressing in a productive way. A1 wanted to plug values into the basic models and A2 wanted to figure out how many stones were in the pyramid. A discussion over whether d represented a variable distance came to the forefront of the conversation about ten minutes later when A2 observed that they "still need[ed] to find vertical distance." A1 objected that this would just be the height of the pyramid, but A2 defended his claim noting that "not every stone was carried 146m high… because if it was just at 146 that means the pyramid would sit up here on the two-dimensional plane and it would just be like all spread out at that height." The depiction of a whole set of blocks comprising the pyramid being 146m off the ground aided A1 in recognizing d as a variable quantity, and positioned A1 to "see why [they] need the number of stones." However, A1 thought they could "do it in a different way," because if they were not provided the number of stones in the prompt then they "can probably do it without." A1 went onto suggest that for this prompt maybe they were just supposed to just find a "massive overestimate and a massive underestimate" and that they would be provided the number of stones after they completed that step because "whenever you put in the layers; that's kind of like partitions." However, the anticipation of making partitions positioned A1 to draw on her local-global model relationship to suggest developing a generalized local model; "we don't need the number of blocks if we're chopping it up… we could just make a formula for the base." Beginning the quantification for their local model A1 drew on a quantitative structure for mass as part of the basic model for force, [mass]=[density]·[volume]. The need to construct a local model involving volume for a pyramid reminded A1 and A2 of the Geometric Volume task. A2 observed, "So, just like before, we're gonna make our underestimate a rectangular prism [draws a diagram of the pyramid with a generalized underestimate partition]. Okay, the overestimate is also rectangular prism [draws a generalized overestimate partition]."

After spending some time computing a "massive" overestimate for energy A1 reviewed their string of expressions and noted;

> I think I mean, all we need to mess with is this mass formula. Like, it's like, obviously, our force, we can do mass times 9.8. So all we have to figure out is the mass, and then plug that into our bigger formula.

In other words, as A1 was envisioning making a generalized local model she was attending to which of the quantities in the overestimate computations would need to be adaptable. Recognizing that the d component of the energy expression was the element that corresponded to the different local model overestimate or underestimate, A1 narrowed in on the mass as being the quantity that must contain the Δ element. A2 disagreed because “all the mass information was given,” since density was provided and they could identify the exact volume that they should “change distance.” A1 attempted to go with her partners’ suggestion, but by assuming a fixed mass she immediately ran into an issue when she envisioned finding an overestimate using two partitions;

> If we do 73 and say that's like an overestimate for our first partition, and then we do the same thing times 146. That's going to give us like, an even bigger number than what we have right now.

This conflict within A1’s local-global model refinement relationship caused her to reassert, “I think we need to split our mass up for like, this is the first partitioned mass and this is the second partition mass.”

An important note I would like to address is that when A1 and A2 made their “massive” overestimate for the amount of energy required to build the pyramid, they made their computation by drawing on a local model which used a rectangular prism as part of the volume component. They did not articulate their basic model when they made their volume computation, so I am unsure if this was a conscious decision or not. In either case, when A2 had mentioned making partitions using rectangular prisms, A1 wanted to adjust their overestimate computation to reflect this. Oddly, when they realized that their computation already represented the volume of a rectangular prism they decided to ‘fix’ this solution by dividing it by 3 so that the volume would be exact. I do not know what prompted this decision, but the desire to use exact volumes continued to permeate the discussion. Specifically, when they were trying to write a general expression for the mass component of their local model A2 suggested identifying the exact volume of the bottom partition by subtracting the volume of the top partition from the volume of the entire pyramid[30]. I am honestly unsure what prompted A2 to revert to this way of thinking. It is possible that this was a result of the issue that arose in the volume of a sphere task and supplemental task, but when I asked A2 about it he was unable to recall what he was thinking when he made this suggestion.

When A1 and A2 realized that there wouldn't be an easy way to generalize finding volume in this way, A1 suggested they look back at their old work to aid them in finding the different bases for their local model;

> Okay. So yeah, this is good we use this is where we use the proportion that I tried to use on everything since then… see we do need those boxes. So, we need to get our different bases and find a formula for any number of partitions.

Recalling their earlier use of proportionality, A1 identified an expression for the width of the pyramid at a height x above the ground to be $\frac{115x}{73}$. This quantity wasn't quite correct, actually representing the width of the pyramid at a vertical distance x from the pyramid's peak, however, I did not intervene hoping that something in their problem-solving would cause the issue to arise naturally. At this point A2 gave an excellent articulation of their

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30 Particularly interesting as A2 was the one to suggest the use of rectangular prisms to make approximations.

construction of a generalized local model by drawing on the quantitative structure for the volume of a rectangular prism;

> Good. Okay. So that's proportion, then we can insert it into, this will give you the base for anything. So, before we like, move on from that part. The base is so that we can find the area of the square of the prism. So area equals 115 times x divided by 73, and that's going to be squared.

In a bit of a light-hearted moment, when A1 and A2 recognized they would need to multiply this expression by Δx, A2 flipped back a few pages to find the delta symbol since they had been copy-pasted it in for each use and A1 joked “Let's see, we made it 43 minutes without finding our delta.” While made in jest, this comment does indicate that A1 and A2 had developed a sort of expectation that these tasks would always involve a quantity which they would label with a delta symbol which, so far, had always represented a height of a partition. Moving on, A1 and A2 worked together to find the rest of the generalized local model expression. The following interaction allows for a glimpse into how A1 and A2’s current local models allowed for them to quickly draw on their local model, [energy to lift a partition to N’th spot]=[density]·[volume] $\cdot$ [9.8] $\cdot$ [distance to get to N’th position], to construct their generalized expression. However, it also demonstrates a lingering limitation in their ability to coordinate their variable quantity x, which they had defined to be the height above the ground, with a way to measure the position of an element in the global model;

> **A1:** So cool. Now we can find our mass. This is good.
>
> **A2:** So mass, maybe 2000 thousand times this monstrosity.
>
> **A1:** Okay, and so now to find our force, we just multiply all of that by 9.8.
>
> …

A1: Okay, cool. So yes. Now our energy

…

A1: Also by Δx? Like another Δx don't you think?

A2: Energy is force times distance off the ground? So I would assume.

A1: Oh, off the ground?

A2: Off the ground, yeah.

A1 So, what should we do? We can't just do 146 minus Δx, that won't work. We have to do. It's like Δx times N, but we don't want to put N in there.

A2: i?

A1: So Δx is like, I have the height of our intervals and N is like the number of intervals. So like, if we said two times Δx, we'd have, like, the second partitions off the ground? Well, we probably could throw N in there, honestly, because that's easy enough to know how many intervals we're doing.

A2: What have we been doing?

Int: Usually, if you've had an i times Δx you've just been making that an x.[31]

A1: But that won't work. Because x in our problem right now is height. And so if we do, like say our height right now…

A2: It's always height.

Int: I'm having trouble keeping track. Are you talking about the height from the top of the pyramid to the bottom, or from the bottom of the pyramid to the top? Because those are two different measurements.

A1: The bottom to the top.

A2: So x works just fine.

A1: Oh. Yeah, it does. Okay.

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31 A2's reference to an index i caused me to confuse Group A with Group B who had been routinely using $i\Delta x$ notation in their problem-solving.

A2: Is that right? Just that?

A1: Yeah, I was definitely overthinking it on that one.

This indicated that while A1 and A2 were defining x to be a quantity in the sense that it could represent different heights between 0 and 146, they weren't really thinking of the x as a dynamic element within their generalized local model. It was simply a means for identifying the width of an arbitrary partition's base. To address this I include a supplemental activity at the end of their session, which I will cover in the next section.

With their final, slightly flawed, expression for their generalized local model, without performing their usual check against their initial under and overestimates, A1 and A2 immediately placed the expression into the derivative calculator. Recognizing that the issue would not arise naturally, I drew A1 and A2's attention to the error they made in quantifying their local model side-lengths by pointing out that if x represents the height above the ground, then at a height of 0 their equation says that the length of the green line in their diagram (representing the width of the pyramid) would be zero. Realizing that "x" in their formula should measure the distance from the top of the pyramid to that green line, A1 rewrote x as $146 - x$ and corrected their expression in the integral calculator.

Supplemental Task: Do These Integrals Measure the Energy to Build a Pyramid?

In their teaching experiment, Group B solved their Energy to Build a Pyramid task in an interesting, but highly unexpected, way. Believing that having A1 and A2 examine Group B's final definite integral expression would position them to develop better coordination between a variable quantity x and a differential quantity dx in their local model, I presented the expressions in Figure 35 in their individual sessions[32] and asked "whether these definite integrals made sense for this task" and "what exactly the x was measuring in each expression." In my presentation of the definite integral expressions, I made sure to keep the overall structure of the local model consistent with the expansion of the energy expression, $E = F \cdot d$.

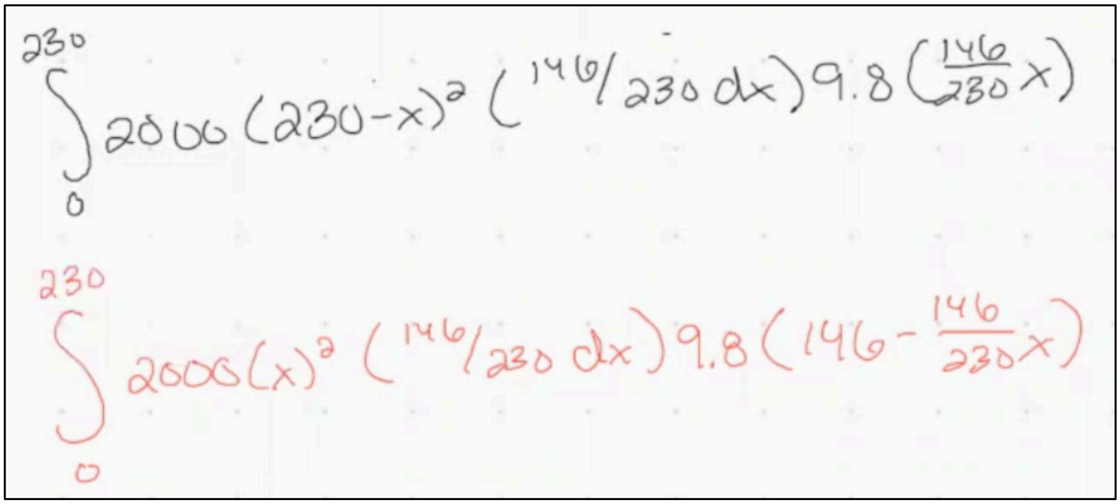

Figure 35: Group B's definite integral expression for the Energy to Build a Pyramid task (black) along with an adapted expression (red)

When A2 was considering the first integral, he found himself unsure of "what exactly is going on here," but by coordinating the basic model and his own local model, [energy to lift a partition to a height x]=[density]·[length of partition base]^2·[height of a

32 For both group members I initially only presented the first expression in Figure 35 which was Group B's actual solution. I created the second expression in A2's interview because I believed he'd be able to quickly identify what the x in the second expression measured despite difficulty with the first. I also presented the second expression to A1 after getting her initial thoughts on Group B's solution.

partition][acc. due to gravity]· [distance partition traveled above the ground], he was able to identify that "2000 is definitely density" but,

> they're doing 230 minus x, which assumes x is their height. But I guess that doesn't have to be the case. That's not the case. What are they doing is they're squaring this, like it's their base area. So our integral we did our area is the proportion of our base area. And so they did it by subtracting x. But I'm not sure what they're calling x.

I was not surprised that A2's initial inclination was to superimpose his own quantification of height above the ground onto the variable x, but his image that the second element should represent an area of a partition was not compatible with the expression $230 - x$. This allowed him to conclude that x must represent some other measurement although it was not immediately obvious what. By drawing on the limits of integration, 0 and 230, he concluded that x must have something to do with the side length, but still could not coordinate $230 - x$ with what he concluded must be the length of the base of a partition;

> Okay. So they're trying. I guess over here, ours is trying to find the length of a base at a specified height. And their equation is looking for the height at a specified base length [draws a diagram of a pyramid].… So they have, like, we're always trying to find our green line here [draws a line horizontally midway up the pyramid]. So, we do it based off of our height, which is our red line. So they are saying that if we have a base, that is, let's say we want to find what the height is at a base area, or base length, like the width of one side at x equals like 30. Right? So then. Why would they? Why have they subtracted the 30 though?

As I mentioned, Group B's local model construction was unexpected. As A2 pondered, I realized that although he had clearly been able to coordinate how one would develop a local model using a base length rather than height, identifying that x was the negative image of his green lines (i.e. x represented the amount of base length deducted from the full width) was likely too big of a stretch to recognize in someone else's work. To capitalize on his productive interpretation, I wrote the second integral in which x did represent a base length and asked him to interpret its meaning instead. His response was instant,

> So this would be the base length. Okay, so this whole x squared represents the base area, and to find volume, you have to multiply by height. And I guess that's this proportion. Okay, so this makes a lot more sense. But how is? I just don't understand what this one. Like this [circles the second integral expression] makes sense to me. This is force due to gravity or acceleration due to gravity is the 9.8. And the 2000 is your density. And this is your area, which means that this is your height. Which means that this together is your volume…. And they're trying to find mass in the beginning. Right? You do mass times your volume, or density. Density times volume to get mass is what we decided. So that's what this whole part is, is the mass and the mass times acceleration due to gravity to get force, and the force times distance to get energy and this distance is the vertical proportion.

The ability to coordinate an alternate quantification of a definite integral through its quantitative structure indicated that A2's symbolic form for integrals had incorporated the basic-local-global model relationships which characterize a Quantitatively Based Summation conception of integration.

Returning to the first integral A2 was still unable to quite identify what x measured, although he firmly stated that the $(230 - x)^2$ must be "a base area" and $\frac{146}{230}x$ "has to be a distance." Asking if I would tell him because he was "really curious," I explained that it might help to think about what the endpoint values for x would represent; when x is 0 the side-length is 230 and distance is 0, and when x is 230 the side-length is 0 and the distance is 146, so you can find which green lines that would represent on the diagram, the lowest and highest respectively. This discerption enabled A2 to recognize that if he were to draw a generalized partition on the diagram, the top edge of the rectangle not inside the pyramid was what x measured.

Unfortunately, A1's interpretation of the expressions did not demonstrate the same coordination with her symbolic form for definite integrals. Specifically, A1 recognized that the change in the limits of integration from 0 to146 to 0 to 230 represented a shift from x measuring height to x measuring side length. However, I believe a preexisting scheme involving integration along the y-axis prompted A2 to rotate her pyramid diagram so the base ran along a vertical axis which she then partitioned horizontally (Figure 36). That is, A1 has an association with changing the limits of integration to be the 'other' type of measurement in a situation that is linked with a graphical representation for finding an area where horizontal rectangles, rather than vertical rectangles, are employed.

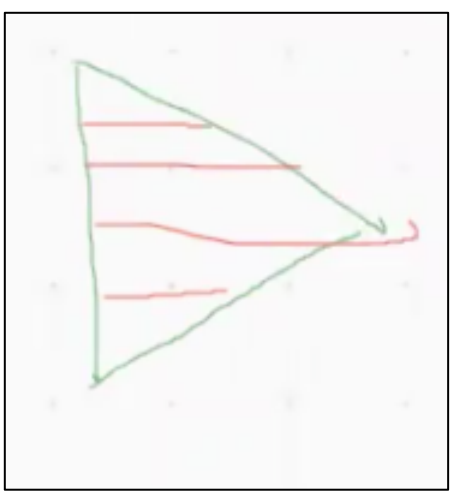

Figure 36: A1's rotated pyramid partitioned along with the base measurement

Attempting to draw A1's attention back to the context of this particular task and the implied quantification involved, I asked her whether or not this would work because when she and A2 had discussed the task they said that it takes more energy to take things to the top of the pyramid, so if you were to divide the pyramid in this way would it actually measure the energy required to build that piece? This question made A1 start to evaluate the quantitative structure of the differential form in comparison to her own local model. While A1 was able to make some connections, such as the links in the density and gravity constants, but in attempting to coordinate the quantities in the differential form, her own local model, and the rotated diagram A1 was not able to meaningfully discuss what the other elements represented.

Believing that maybe she would have better luck with the second expression I switched focus. She spent some time trying to figure out why the proportion was inverted from their expression. As she made her computations off-screen she said in a surprised tone, "x is a base. A base of what?" I had been keeping notes for her on the shared whiteboard, so I asked, "x is a base? Is that what you said?," which made A2 attempt to reason out what her computation implied,

> Well, it's like, it has to be. Not necessarily a base because. Yeah, so like because there's an x squared, and the only reason we need to square anything, at least me and A2's equation was because it's side-length or like a base. Like we squared it to make a base for our volume.

Working from here A2 was able to rationalize what each element within the differential form represented by coordinating it with her local model. However, when she attempted to demonstrate these values on the diagram it became clear that she was not fully drawing on the last element in the local model as being [height that a partition was lifted] but rather just [height to the top of the partition].

A2's desire to partition along the base remained consistent throughout her interview, even when rotating the pyramid to have a horizontal base she drew vertical partitions. Her image of the partitioning process also remained resilient to evaluating partitions at specific values for x such as 0 and 230. This told me that in addition to graphical imagery which prompted the initial 90-degree rotation, A2 had also developed a strong coordination between the limits of integration representing a length or distance and that partitions must be created by chopping that length into segments. To use the same partitioning method, but quantify it in a different way was not possible.

Group A: Grading Definite Integrals – Mass of Oil Slick

Due to a scheduling issue, A1 and A2 started task 5 individually and then gathered together to finish the task later in the week. Each had slightly different approaches to starting the task. A1 wanted to write her own definite integral first and then compare it to the other solutions, while A2 jumped into evaluating the definite integral expressions by trying to recognize whether the differential form represented the right type of expression.

When evaluating an expression of the form

$$\int_0^{100000} [\text{density at } r][\text{expression for the area of a circle with radius } 10000][dr]$$

A2 identified a basic model [mass]=[density]·[area]. As he began to identify whether it was an appropriate expression A2 observed,

> I'm wondering I guess, if this 10,000 is, or should be noted by r, because this [motions mouse at the variable r in the density equation] is variable. And since this [motions to the limits of integration] is the, these are r values, because it's the radius values… Yeah, because this r value [motions to the 10,000] that would make you only be able to put in 10,000 for this r in the denominator [of the density expression]…

While there were multiple issues with this particular expression, what allowed A2 to reject it as a viable candidate was that a local model which produced the expression would not be adaptable to a varying radius component in both the density and the area components. Recognizing that the next expression was the same exact values with the constants "pulled out" of the integral, A2 concluded that the 10,000 should still be an r. In addition to the need for adaptability, A2 demonstrated that the differential form had to have an r somewhere in it with an image that without the variable r there would be no need to go from 0 to 10000.

As A2 moved on, a pattern emerged in which he was primarily attending to the structure of a basic model and not a local model. This was particularly interesting when he inspected a correct, integral of the form $\int_0^{10000} \frac{50}{1+r} \cdot 2\pi r dr$;

> This is the density, and this $2\pi r$ is circumference. Right. So that's interesting. It's not area, so I'm not so sure. No, that shouldn't. That shouldn't work…. well. It gives you sort of circumferences from zero to 10,000. I guess that works. For circumferences, for radii from 0 to 10,000. I guess that would not work. I don't know. I'm gonna give it a yellow box.

With the image that the integrand should represent a mass, which is density times area in this case, A1 was conflicted because the second term he was considering was a circumference and not an area which clearly did not match the basic model. However, when he conjured an image of an infinite number of circumferences he could envision these concentric circles as an aggregation of elements within the interior of a circle—not as an accumulation of a quantity. This type of approach is not uncommon, however, it certainly did not reflect A2's inclusion of the differential quantity when he evaluated Group B's integral in the supplemental task.

As A2 continued, he came across an expression of the form $\int_0^{10000} \frac{50}{1+\pi r^2} dr$ which he rejected;

> I don't think that's right, and I'm not even sure why other than he doesn't have like. So I guess I'm probably comparing this against the standard of what I think should be right. I'm imagining integral from 0 to 10,000… 50 divided by one plus R, because that what's given to us is the density formula. And then, as I said up here, and M has to equal D times A, like we said originally, so mass equals density times area. So I think that you should then multiply by πr^2.

That is, when A2 was constructing his own definite integral expressions during the teaching experiment the differential represented a meaningful quantity, but not when evaluating preconstructed integral expressions. This marked the end of A2's session, so in the next few paragraphs, I will cover A1's progress through the task.

When A1 started the task she decided to write her own integral expression first which she could use to compare to the provided examples.

> Our integral needs to be density times area … and we have a density equation…Now we need a, we need to write an area… So the area of a circle is πr^2, which is good because we have r in our problem…. So we should have a mass because that would be density times an area. At least that's what I'm

> thinking. And then let's see if we're looking to find mass….then our accumulation equation is just those two multiplied… Okay, bounds are from zero to 10,000…

Similar to A2, A1's desire to match the quantitative structure for mass led her to draw on the corresponding algebraic expression for density (provided) and area of a circle, however, she was not really envisioning a local model and did not mention partitions. Indicating that the expression she just described, $\int_0^{10000} \frac{50}{1+r} \pi r^2$, was somewhere on the whiteboard prompted A1 to express uneasiness with there not being a differential,

> So we need to put a dr at the end, but then I was thinking. I was like, well, whenever we wrote our other equation like we had the Δx in there already, or Δh, and that's what we changed. Whereas for this, I feel like I'm kind of just like, throwing the dr in at the end.

The coordination between the differential and the Δ notation which represented a physical quantity was enough to cause A1 to pause and reevaluate her approach. Specifically, she asked herself "what's going to be changing in our formula?" Observing that the "obvious" answer is the density as the oil expands along the radius, so "the radius is what's changing in the formula." This changing value that A1 was trying to identify was the motivating factor for why there should be an integral in the first place. In explicitly identifying that quantity A1 was trying to anticipate how she could partition the situation so that the different densities could be captured.

Deciding to develop in an explicit local model, A1 began a 30-minute long quantification process which required a number of interventions. Specifically, despite knowing that she could not just "stick" the dx or Δx onto the end of an expression, she continued to draw on the quantitative structure for mass which involved an area component and the obvious area of a circle was πr^2. I was able to aid A1 in confronting this assumption by having her attend to the units that would be involved in each quantity. A1 was confident enough to assign a unit of meters to the differential, so when there was an extra component of mass in her definite integral expression she knew there must be something wrong with the area, but was unable to conceptualize what this might be. In post analysis, I identified this issue to be a non-recognition that the elements of her local model were overlapping. Recalling the work she and A2 had done with the supplemental error task, A1 returned to that page to observe that she could use circumference times a Δx to obtain an area. Finally arriving at an expression $\int_0^{10000} \frac{50}{1+r} \cdot 2\pi r dr$ A1 checked the units for each element of her local model to match her image that the differential form should represent a partition of mass; "then we have kilograms per meter squared times meters. And those will cancel. And we'll just get kilograms. Which is good. That's what we want."

Once A1 fully constructed a definite integral that was consistent with her image of a basic-local-global model relationship, she was able to draw on this structure along with an image of an appropriate local model to quickly classify the rest of the definite integral expressions. This included a need to have a differential quantity in the differential form, otherwise the integral was "incomplete," that r needed to be a variable quantity within the circumference element of the local model because there "needs to be a way for the r to change along with the dr, and that the integral sign marks the beginning of the elements you want to accumulate so any varying quantities cannot be factored out front of the integral sign. When A1 was confronted with local models which did not conform to her same quantification, such as $\int_0^{10000} \frac{50}{1+r}(\pi(r+dr)^2 - \pi(r)^2)$, she explicitly constructed a diagram of the corresponding local model to identify that the area of the ring in this definite integral was being measured by finding the area of an outer circle and then deducting the area of an inner circle. However, her image of the symbol template for an integral involving a single differential made her hesitant to conclude that this was an appropriate integral; “

> I guess it actually does work when they write it like this. But also this is dr squared…I mean, the dr can be squared, because we're. So, I think this one's okay. Because it gives us the area of each ring and multiplies it by the density. Which is what we want it to do. Like that's what we did with our $2\pi r$, we got an area of a ring, then we multiplied it by density. And so I'm thinking this one's okay.

That is, A1’s desire to connect the differential form with an appropriate local model superseded an image that a definite integral can only have one differential term.

When Group A gathered together for the next session A1 described her solution process to A2, which he said he understood. To draw a connection between A1’s explanation and A2’s initial reasoning, I pointed out they both began the problem in very similar ways—thinking about mass as density times area—but that A1’s inclusion of the *dr* allowed her to see that the expression could not be correct because the resulting units did not represent a mass. Then I emphasized that in her final expression the $2\pi rdr$ was still an area, so it matched the structure they wanted to find. Because A1 had gone through the entire task already I decided to move on to the next problem in the teaching experiment.

Group A: Design a 'How-To' Guide for Definite Integrals

A2 began trying to identify what a definite integral was and commented, "I don't even know. I can differentiate between indefinite integral, but we've done so many different things with it that I can't I don't know how to like conceptualize the entire subject, like, in a couple of sentences," which I felt was a pretty accurate reflection of the task I had taken on with this project. Recall that the point of this activity was not to evaluate the final product for correctness, but to engage A1 and A2 in the act of reflecting and generalizing the activity they had engaged in over the previous weeks. This means that the primary concern of this section was to map their evolving definitions and instructions through their constant revision of phrasing, word choice, and the order in which the constructs activated relevant schemes.

As A1 and A2 began to define what an integral was, they had difficulty finding the right word to use to note that an integral represents an accumulation process. They decided to begin by defining "the formula family for finding the exact accumulation of a quantity/value/thing/object." By "formula family" they were referring to the symbolic template which was adaptable to be applied in multiple situations. Not sure exactly how to improve their definition, A1 suggested "So, maybe if we talk a little bit about how it works that will help us like make our definition better." Switching to this mode of reasoning A1 coordinated the need for a definite integral being tied to variability; "we could say the exact accumulation of a changing. It's like, we always include something that's changing… like, equation for finding the exact accumulation of a changing quantity?" However, A2 was not completely satisfied with this classification because the integral does not “find the change” in a quantity, it finds the “exact value,” even though they “use changes” in the form of $dr, dh,$ etc.

Continuing to describe how an integral works A1 voiced her evolving conception by first drawing on her local-global model relationship, “we take a formula for a small piece of what we're trying to accumulate like we take a small part and we add those up. Like that's what the integral is. So we could say, ‘using tiny pieces of the whole thing’…” which conjured images of physical objects. Feeling like this was not generalized enough A1 had difficulty choosing the right phrasing because a definite integral is “not specifically for area or volume” which caused A2 to suggest “partitions of the whole.” A1 agreed was a good way to phrase it. Prompted to describe a local model by this image of elements within a global model, A2 typed “Using partitions of the whole added together to approximate” which caused A1 to object because a definite integral “is not approximating.” A2 defended the usage because “the basis of integral integration is the Riemann sum things” which allowed A1 to also identify that the basis of their global model were these local model elements. Wanting to capture the exact nature of a definite integral A2 continued typing “approximate the value of the whole with infinite partitions creating an exact value.” Reflecting on their sentence A2 observed that “it seems counterintuitive” because “we’re using approximations” to find an “approximation of the whole thing, infinitely.” The task setup asked A1 and A2 to try and describe this idea to someone who had never taken a calculus class before to dissuade them from just drawing on calculus jargon, but as a result, they found themselves tripping over how to incorporate a limit idea into their description. Specifically, they wanted to capture how creating more partitions led to the “exact” answer, so they changed the “infinite partitions creating an exact value" to "increasing partitions that move toward the exact value of the whole."

When trying to define the quantities which comprise a definite integral represented, A1 tried to distinguish between a variable, x, and differential quantity, dx;

> So when we make a partition, we're making some sort of shape in our thing. So we have a quantity that's like always going to be changing, which is the dx or whatever variable we're using for that problem. Right? And that is normally like it's a set number because like, it's the same for each partition whenever we make partitions like because it's the length… for each Riemann sum we calculate, it's the same for that for one Riemann sum…. So it's hard to explain that the dx is changing, but it's not changing at the same time. You know?

That is, prompted by her local-global refinement relationship, A1 was attempting to describe a dynamic relationship between the differential quantity dx and the accuracy of the approximation.

At this point, A1 and A2 decided to move on to the second task to see if would help them write a better description. This second prompt asked A1 and A2 to write general problem-solving guidelines someone has to go through to solve a novel definite integral task which engaged them in explicitly reflecting on previous tasks. Identifying that they usually identified the variables and formulas from the task prompt their next step was usually to draw a picture and envision "what the shape of the partitions will be." Through this particular phrasing, A1 and A2 were identifying that can use the context within the prompt to identify a varying quantity which will promote a partitioning in a specific order.

Flipping back to the Energy to Build a Pyramid Task to recall what steps they had taken, A2 noted that this task "was different because we had to go inside of four equations" in which he was referred to the multiple quantitative relationships they had to draw on to write their generalized local model expression (e.g. $E \rightarrow F \cdot d \rightarrow M \cdot g \cdot d \rightarrow V \cdot \rho \cdot g \cdot d$). A1 observed, "normally we have to write some sort of equation to help us find… the measurements of the partitions as they change." In this reflection, A1 and A2 were observing that as part of the quantification of their generalized local model they had to draw directly on the structure of the basic model along with other algebraic expressions as part of that process.

Building from this basic-local model relationship for structure, A1 appended coordinated a basic-global model relationship for structure,

> Maybe we should have a step that talks about, like how we have to think about the integral. Like, for example, if we're doing an area, we need to have the area formula. I don't know how to write 'decide what quantities need to be in your integral design.' Decide what the integral is supposed to be measuring.

Considering the differential form, A1 asked A2 to add something to the partitioning section of their write-up specifying a need to explicitly identify "what's changing" so you can find your dx. Because A2's differential scheme did not share the same dynamic property when A1 referred to a "changing" quantity he envisioned the x. Drawing on the same symbolic template he demonstrated in the previous task which did not give quantitative meaning to the differential, A2 wrote out a new step: "Add changing variable in the notation *d(x)*." A1's need for the generalized local model, which contained an inherent differential quantity, to have quantitative meaning caused her to object,

> My only issue with that is that normally… when we're writing the equation for the size of the partition, that already has the dx. Does that make sense? So that's why I'm like, I want to make it clear, because I know that's normally it's a really important step for us in writing our integral is deciding what is changing and where how that relates to the partitions measurements, or size, or whatever we want to call that.

Catching on to the different usage of "changes" in their language, along with his own image of a generalized local model, A2 revoiced A1's statement into his own words "So, dr changes when *N* changes, the number of intervals, but r changes as you switch partitions. So we have to recognize, this is so weird." Having difficulty identifying a succinct way to describe this they left it to be an aspect of the more general "write an equation" step which referenced the development of a generalized local model. Expanding on this step A2 also reflected on some tasks requiring the quantification of the local model in terms of the variable quantity, x, so they added, "solve for everything in terms of one variable."

Turning their attention to the limits of integration, A2 commented that defining these values was difficult because "it depends on the problem," specifically commenting on the different types of quantities the limits of integration could represent (e.g. heights, depths, radii). During this discussion, there was a reemergence of A1's chopping up and graphical conception between the limits of integration and the differential quantity, but the pair ultimately decided to just say "Set the limits of integration as relevant quantities for the whole being measured and the employed variables." By this phrasing, A1 and A2 expressed a desire to capture the adaptability of a definite integral to measure a quantity in more than one way like they had observed in the energy to build a pyramid task. That

someone who has never taken calculus before. It is not necessary that the reader be able to compute definite integrals by hand, but your write-up should enable them to be able to understand the quantities involved for definite integrals such as the ones you've worked on over the past few weeks. Be sure to include specific descriptions for the notations you use.

1) <u>Definite integral</u>: The formula family for finding the exact accumulation of a quantity/value/thing/object over a given or calculated interval.
2) <u>How it works:</u> Using partitions of the whole added together to approximate the value of the whole with increasing partitions that move toward an exact value for the whole(limit)(calculus).
3) <u>Quantities:</u> Interval is the region being measured in terms of one variable
    a) D(x) is the length of the subinterval
        i) Where x can be any variable

2. Provide a write-up that would enable a reader to construct a definite integral for tasks such as the ones you've worked on over the past few weeks. This write-up should be specific in its directions, but general enough that it can apply to novel tasks.

1) Identify variables and draw a picture
2) Label variable on the picture
3) Decide what measurements are needed and what is provided
4) Decide the shape of the partitions
5) Write an equation to find the size of the partition
    a) Solve for everything in terms of one variable
6) Write definite integral using the equation from the size of the partition
    a) Set limits of integration as relevant quantities for the whole being measured and the employed variables

Figure 37: Group A's "How-To" guide for definite integrals

is, at least A2 had incorporated an image of differing quantification processes into their local and global models. By employing the term "employed variables" A1 and A2 were attempting to capture a correspondence between the limits of integration and the corresponding generalized local model. For example, they mentioned that if your x and dx measure a height your limits of integration should be height values. Reflecting on their overall writeup (Figure 37) A2 commented that writing a definite integral "Sounds so simple but, it's definitely not."

Important Aspects of C1's Emergent Models

C1 was a Caucasian male freshman majoring in Mechanical and Aerospace Engineering currently enrolled in a Calculus I course along with a 1-hour honors course which focused on visual representations of calculus topics. As part of his secondary education, C1 attended a "pre-engineering academy" and passed the AP Calculus exam. He also took a calculus-based physics course and numerous engineering courses in high school which he often referenced when providing examples for how he was reasoning about particular tasks. C1 decided to enroll in Calculus I, rather than advancing immediately to Calculus II, because the transition to online coursework during the COVID-19 pandemic left him feeling less confident with his understanding of topics covered towards the end of his coursework.

As I will describe below, C1 entered the teaching experiment with a fairly robust understanding of integration which involved links between Riemann sums, area under a curve, rates of change, and antidifferentiation which, (1) dramatically impacted how he interpreted and reasoned through tasks within the teaching experiment, and (2) provided detailed insight into the affordances and limitations of this hypothetical learning trajectory to engender a broader Quantitatively Based Summation conception of integration for students who have already developed coherent schemes for integration.

C1's Incoming Schemes for Riemann sums and Integrals

When introduced to the Curiosity Rover task, despite only being asked to identify an overestimate for a single segment of the rover's journey, C1's coordination between schemes involving decreasing rates of change, an image of negative slope, and the need to identify an overestimates resulted in the recognition that this task was related to Riemann sums. This provoked C1 to immediately attempt to identify an overestimate for the rover's entire journey. C1's partner did not share these same schemes and so C1 spent time articulating exactly how his basic-local-global models were working in tandem to help him solve the task;

> Since we're finding an overestimate, we don't know exactly at what point the rate goes from 6 to 3.5. So, since we're getting an overestimate, we just want to grab the highest number there and say this is the worst-case scenario. This is how much sand it will have maximum… So it's going from 6 to 3.5. But for all we know, after that 30, it's 6 milligrams per kilometer all the way until the last bit, and that's when it becomes 3.5. And then it goes 40 to 60. So that's 20 kilometers from 3.5 to 3, so that's 20 kilometers at 5. And then that's 25 at 3, then that one is 15 at 1.75. And then 60 at 1. So let's see if we just take the sum here…

Here C1's global model was squarely centered on a parts-of-a-whole conception. That is, he obtained the whole by the summation of overestimates for the amount of dirt the rover obtained as it traveled between neighboring sites along its journey. From C1's immediate transition to identifying the global model, rather than just an estimate for the requested segment, it was clear that his image of those elements within the global model were local models. That is, his goal was to identify the whole and so he found the parts. Providing additional imagistic justification for his overestimate local models, C1 drew a graph along with a generalized rectangle (Figure 38) and explained that the rectangle represents an overestimate because the "little outcropping" provides "more than what it would be." He added that they "can't really safely get an average or middle of the way number" to get calculations because they don't have the function, so they "just use six and then take it as an overestimate."

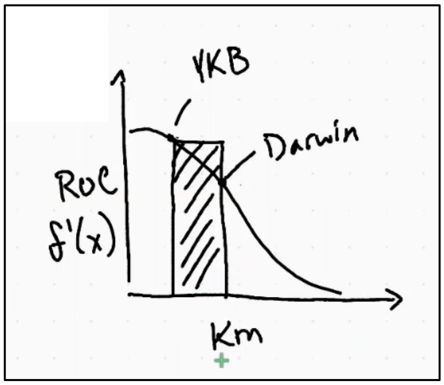

Figure 38: C1's generalized rectangles for his Curiosity Rover local models

By drawing this graphical representation C1 revealed that he envisioned some underlying function that could model the real rate of dust accumulation, supporting my image that his global model elements were local models. His diagram also invoked schemes associated with integration and allowed C1 to reveal important distinctions between integrals and sums.

> And so this is supposed to be, this is kind of representing the estimate that we've made. Since it's a rate of change graph, it's basically an f prime graph. We're mapping rates of change, instead of just values. And so, when you want to, when you want to know, like, what the, I guess displacement, or what the Δx is for the original graph, you would take the integral in calculus. But we don't have the actual equation. So, instead of being able to take the appropriate area under the curve, we have to take either an over or underestimate. And so it's uhh. Contextualizing it using physics again, if you have a velocity graph, so if like your velocity is like this, then the area under a velocity versus time graph is your displacement. And so this is kind of a rate of change. And so the rate at which you accumulate the dust if you take the area under that curve, that is how much dust there is.

Due to C1's mention of an integral shortly after reading the prompt for Part 2, along with his description's heavy reference to rates of change along with my image of his basic, local, and global models I suspected C1 may have entered the study with a Multiplicatively Based Summation conception of integration. However, his primary references to integrals in terms of an antiderivative heuristic and area cautioned me from being too hasty.

In C1's follow-up interview, I explicitly investigated this further by asking him to describe the difference between Riemann sums and integrals and how the constructs were related. His initial response was to again describe integration as area under a curve and referenced Riemann sums as "lower resolution integral[s] because you've got all these sharp corners that are going to poke out" or don't "quite reach." Continuing, C2 redemonstrated that when trying to precisely describe what an integral was would revert to antiderivative explanations,

> So, okay, I'm going to try to remember how you do integrals. And that's, you take the reverse derivative of it. And then you take. Then. You're not really getting a finer. Uh, it's not really finer. What am I trying to say? You're getting a block, you're getting the displacement here, you don't know how far it's gone when you just know what it's covered. And so what's happening is you're taking the end, and you're taking the beginning, and you're finding out how far you've gone… So you won't know exactly how much it moves it for each individual segment but you know how far you've gotten total.

C1's specific phrase "getting a block" referenced that when you were talking about an integral there were no longer generalized rectangles that measure displacement over subsets of the longer interval. You can only talk about displacement between endpoints. C1 went onto explain that Riemann sums were just the "stepping stone" to integrals indicating that it was a way to approximate the total area prior to obtaining a more powerful tool (antiderivatives) that would allow you to identify the exact value for the area.

However, in the same interview, C1 was able to describe how the progressively smaller error bounds, through the refinement of the global model (through creating more partitions) lead to an integral through a limiting process.

> We kind of briefly touched on limits, when we're talking about getting the over and the underestimate closer and closer to this correct integral value. Because the overestimate was getting closer, and it was decreasing, and the underestimate was getting closer and increasing. And so the limit there between the over and the underestimate would be that correct integral value… We just didn't really explore much further because there wasn't really anywhere to go from that. Because we weren't getting more and more points. But I did briefly mention it. If you the more points you get, the closer you get from your estimate being the real thing.… Well, you know, the more data points you have, the finer it's going to be. And there's only so many data points you can have before you're not really able to work something out. I mean, you know, there's a there's a limit to how a person how many times a person can plug some numbers into a calculator. But if you were to be able to plug those infinite number of points into this, this Riemann sum that we have set up, you would get that interval.

This description, along with C1's mention of integrals early in the task demonstrated that while the primary schemes activated for C1 when reasoning about integrals are those concerned with antidifferentiation and area, C1 did have a scheme akin to a Multiplicatively Based Summation conception. However, this scheme was only activated through graphical imagery of area under curves—not inherently tied to the symbol template for an integral[33].

The Development of Two Distinct Global Models for Integrals

Throughout the teaching experiment C1 increasingly drew on his image of a definite integral as area under a curve. Based on C1's first few sessions it was clear that these schemes were well established before the teaching experiment began, and their increase in use was tied to the progression of increasing difficulty in the tasks presented. During early group interviews, where C1 was not asked to identify exact values or find a definite integral expression, these schemes were not activated as often unless asked specific questions which prompted C1 to draw a graphical diagram. However, after integral notation was introduced, C1 consistently drew on these schemes as part of his problem-solving process. Due to the intentional design of the tasks in the teaching experiment, C1 was often unable to directly apply his antiderivative or area under a curve schemes to solve the tasks which led C1 to develop two separate global models for integration, one for the accumulation through a parts-of-a-whole conception of quantities alongside his global model for antiderivatives and area under a curve. These two global models had corresponding local models: with the global modal involving accumulation taking on a trivial local model generalized [small partition of desired quantity] and local model for

33 C1 also expressed that he "think[s] about derivatives graphically," and not really in terms of rates of change.

area under the curve taking on the form [function expression for dependent quantity – height]·[differential expression for independent quantity – width]. In the following section, I will refer to these sets of constructs as a sum-model and an integral-model respectively. As I will describe below, C1 became consciously aware of these two distinct systems of models and made comments when a task resulted in an integral expression that was more of a "sum" than a "real integral."

C1 became explicitly aware he was operating with two schemes for integrals during a variation of the Energy to Build Task in which he was asked to identify the energy required to construct a rectangular column measuring 5m high with a square base 2m×2m. This supplemental task was added to C1's task sequence because his honors course had covered the Volume of a Sphere the previous day. This resulted in C1 solving the Sphere task in a fraction of the time I had expected. Because C1 had completed the Volume of a Pyramid task earlier in the same session, I did not want him to progress immediately to the Energy to Build a Pyramid task. That is, by placing the Volume to Build a Sphere between these two tasks, I wanted to provide participants the opportunity to reflect on their problem-solving activity. For this to be a productive reflection, I desired at least some time to pass between these tasks to ensure any problem-solving activity wasn't directly copied from previous work. C1 was presented the Energy to Build a Pyramid Task in the following session. With no time to prepare such an activity, I selected an Energy Against Gravity task from C1's calculus textbook.

Prompted by the equation for work, C1's initial activity centered on drawing a force vs distance(height) graph. Shortly after constructing this image, C1 voiced confusion by the premise of the task. With a solid image of work measuring dynamic systems, C1 was having trouble coordinating energy or work with a stationary object,

> When I think of work, I think of moving something. Because of when I took the principles of engineering class, I think of, you know, pulley systems or levers or pushing or pulling it. It's weird to think of the work required to build a cement column. I mean, something like this would be, you know, a mold would be set up, and then some it would be poured in, and then you just have a column.

In other words, C1 couldn't identify a need to quantify this task in a way consistent with the need for a definite integral due to a lack of motion in the context. To realistically build a column you'd have to lift all the cement to at least 5m. After some discussion, I told him "if I remember right, the textbook authors were thinking of this being built layer by layer kind of like the pyramids." This phrasing was to introduce an image of motion into the context, through the partitions being raised to their appropriate heights.

After constructing an appropriate local model for force through horizontal partitions, each measuring Δh tall and raised to a height of h, C1 considered his expression in relation to his graph. Consistent with his previous graphical depictions C1 had constructed an image in which the independent quantity was along the horizontal axis. He had not coordinated this graphical representation with a partitioning process but voiced that there must be a linear relationship between the two quantities to justify his linearly increasing $F(h)$. Perturbed by the fact that the forces in his quantification were constant, but that his graphical representation showed an increasing relationship between force and distance, C1 tried to rectify these two images. He found himself unable to move forward, so I attempted to aid him in the recognition that Δh within his generalized local model was not the same quantity as the Δh in his graphical depiction by asking him to explicitly identify the various components of his generalized local model.

Figure 39: The quantities in C1's Energy to Build a Cement Column task

After completing this task (Figure 39), I asked C1 to identify those same quantities on his graph diagram, however, this did not play out as I expected. While C1 quickly noticed that the h_k and Δh quantities were "switched", he did not make an accommodation to his graphical depiction as I anticipated. Instead, C1 completely re-quantified how he envisioned the partitioning and accumulation process would occur within his sum-model. Specifically, he adapted from horizontal partitions of the column with height Δh being lifted a distance h, to summing an increasing sequence of larger and larger blocks of the column all being lifted a height of Δh.

This second conception was compatible with his integral-model and was therefore incorporated into that scheme as being the preferred method of partitioning if possible. In particular, C1 drew on this image directly when he wrote a definite integral for the energy to build a pyramid, imagining progressively larger pyramids all being lifted a height Δh.

Coordinating the distinction between these two concepts C1 drew diagrams of both scenarios. When I asked him why both of those quantifications could be written in terms of the same definite integral expression (Figure 40) C1 thought about it for a while, drew some graphs, and considered the labels he assigned to the axes. He then decided that the identical graphical representations he drew did not make sense for the sum-model and explained,

> Okay, so, in this case, we have an integration of the area under a curve of F with respect to h. And this scenario, this is more like a one of the Riemann sums that we were performing before, where we're taking volumes, and we're adding them up. And so we're not really taking, we're taking less of the area under a curve and we're doing more of an accumulation of values like we were doing with the dam, I got a dam. When we took an integration, we ended up with moment instead of force. When we were just adding up the values, we were able to get force like we needed. So in this case, this is less of a integration and more of a sum.

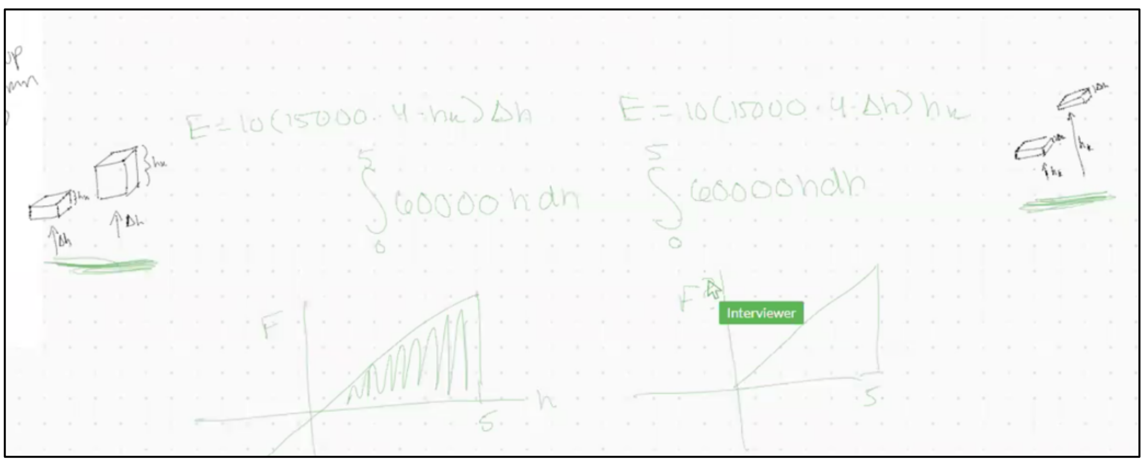

Figure 40: C1's diagrams for two emergent model systems (integral, left; sum, right)

When C1 entered the teaching experiment he was drawing on his integral-model to construct his summation expressions. It was not until he was asked to begin ascribing limits and definite integral notation to this model that it transitioned to the sum-models I described above. The most prominent feature of C1's two model systems was the activation order. C1 always attempted to apply his integral-model to a situation first. A primary example of this was in the "Grading Definite Integrals Task." Due to C1's ongoing preference for his integral-model, I presented the Total Mass of Oil Task before providing the pre-prepared definite integral expressions. When C1 began to work on the task his first inclination was to write a generalized local-integral-model, $\frac{50}{1+r}\pi r^2 dr$, matching a [Mass]· $[d[\ ]]$ structure. Recognizing that the units were incorrect, C1 drew a diagram of a circle representing the oil spill but did not draw any partitioning features on that diagram. He then drew a set of graphical axes using the quantity he identified as dependent ρ, on the vertical axis, leaving A, to be placed on the horizontal axis, indicating A must be the independent quantity, which prompted a local model of the form ρdA and motivated C1 to rewrite ρ to be dependent on A. C1 rewrote the density equation by using a symbolic manipulation of the equation for a circle to replace r in $\rho(r)$. Due to an alternate line of questioning, C1 did not attempt to solve this definite integral, but I anticipate he would have adjusted the limits of integration as he referenced their incorrectness. In summary, if it became obvious that the integral-model was inappropriate, C1 would attempt to modify the situation so that it fits within this image before resorting to his sum-model.

When creating his 'How-To' writeup C1 demonstrated that even at the end of the teaching experiment he still viewed a definite integral as an area under a curve. Specifically, the integral-model is what he referred to as "an integral." Although C1 recognized an ability to write the sum-model using definite integral notation, this global model was a limit of a general sum, not the limit of a Riemann sum. C1 also reflected on an instance when an invocation of his integral-model became problematic by referencing the Kinetic Energy of a Rotating Rod task,

> I guess just the way I was integrating, it was wrong…. I was integrating the area under the curve, which was the wrong way to do it…. I guess the problem was that I was taking the integral of a function. And I think this is kind of a different method. Because what I was doing was I was taking the integral of Kbar, which is to say, I did this [writes $\int K(r)dr$]. And that gives me the area that gives me the area of this curve. Which outputs jewel meters, which is not something I need.

That is, the distinction between global models had clearly different quantification processes and corresponding local models which C1 now recognized. C1 observed that by invoking an integral-model he was setting himself to obtain an inappropriate unit within his differential form which did was not consistent with either local model's requirements. C1 was cognizant of his continued reliance on his integral-model, noting frustration that the same problem occurred repeatedly:

> Let's take a closer look at the catastrophe. Let's see what happened here….And so that was where I went wrong… I just plugged numbers in and then put a *dr* at the end. Which was, I think what I did a lot… I mean, this was after this pyramid one, which I solved flawlessly. I don't understand what I forgot in the in-between times. Yeah, I leaked it into my pillow through my ear, I guess. For a lot of these questions, what I did was really wrong, I just wrote down the formula and then added *dx* at the end of it, and that's just not how you do it. I needed to find out what in the equation was the dx, if you just slap a dx on the end of it, then you're going to be ending up with, you know, the wrong integral. You're going to have the wrong units, joules becomes joule meters, you know? If you slap it onto the end of like a velocity equation, then you're going to end up with like, well, you're going to end up with displacement. But if you're trying to find velocity, it's going to be a problem.

While this self-reflection marks a significant step towards C1's ability to model these types of quantitative situations in the future without repeating the same misapplication of his integral-models, the unfortunate truth is that C1's upcoming calculus coursework will primarily focus on antiderivative techniques with only rare opportunity to reinforce the awareness he demonstrates in this statement. While C1 did not make the same error in his task-based clinical interview, his admission that he reverts to his image of an integral as an area under a curve means that it is unclear if any major adjustments resulted from this observation. Because I will be following up with this student in Fall 2021 I will have an opportunity to investigate a longer-term outcome of the final reflection task.

Collapse Metaphor Strengthened by Basic Model

When engaging in the Geometric Volume of a Pyramid Task, C1 created a finite-sum-model for over and underestimates, and when I asked if we could obtain an exact answer he was able to transition to definite integral notation without issue. Prior to this transition, C1 had described the elements of his finite-sum-model to be volumes, specifically, volumes of rectangular prisms which decreased in side-length as height increased above the ground. Each of these prisms shared a similar height, Δh. However, after C1 wrote his integral expression,

$$\int_0^{10} \left(\frac{2}{5}h\right)^2 dh\,,$$

I observed a change in his language. When describing the differential form of his definite integral, C1 explained,

> That's the length of one of our sides of our square base [motioning to $\frac{2}{5}h$]. So, you square that you get the area, and then you do change in height so that you can get

the area moving down the triangle. And then if you add all of those together, you'll get the volume.

Specifically, there was a clear distinction between 'multiply by a height to get a volume' which characterized his previous descriptions, to "do the change in height" to "get area moving down." So I asked how adding areas would produce a volume. He replied,

> You're multiplying them by Δh, and that's going to be your extra inches. The $\Delta h's$ are infinitesimally small. So, they don't change the number at all…. So the goal is to get as Δh approaches 0… If you're getting the limit, as it approaches zero, you're getting just as close to zero as possible. And so you'll end up getting kind of, basically zero, you won't change the number at all, but you won't make it zero like you won't be multiplying by zero.

When I asked what he meant by "not changing the number at all" and he continued,

> You're just getting the area, you're adding up the areas. But it's kind of like, the volume of a cylinder is the area of the base times the height. Right? And that just means that you have for that height, you have this area, and then you just have it all throughout.

C1 continued in this fashion, reaffirming that in the sum version "you're multiplying by a big chunk" so "you're adding up volume because you're taking these big building bricks, whereas when you're taking the integral you're adding up kind of slices."

This interaction was demonstrating that C1 was invoking a clear collapse metaphor for a volume integral (Oehrtman, 2009). In brief, a collapse metaphor characterizes a students' image of the objects which measure a quantity, such as rectangular areas to quantify area under a curve, collapsing in 1-dimension as a result of the limiting process. In other words, the rectangles "become" lines. C1's image of a basic model for volume, which characterizes geometric prisms as an aggregation of areas along a height h, was supporting, perhaps even engendering, this metaphor.

Although students can productively reason about many integration tasks by drawing on collapse metaphors, I was trying to engage students in the reasoning necessary to construct definite integrals where the differential quantity plays a key, quantitatively measurable, role. Therefore, I engaged C1 in a series of discussions aimed at accommodating the differential as being a measurable quantity, in this case, dx should continue to be a height. In particular, I entered the summation expression into an online calculator, WolframAlpha.com, in a way so it would provide a sequence of values that represented C1's summation expression evaluated at 1-partition, 2-partitions, 3-partitions, up through 15-partitions as this was the limitation of the calculator. Instead of viewing the limiting process as dynamically squeezing generalized rectangles until they are basically lines, I wanted C1 to envision discrete computations which continued indefinitely, but whose values approach a limiting number. This is the understanding I designed as a part of the hypothetical learning trajectory, but due to C1's past calculus experience, he did not fully engage in these aspects of the task sequence. For example, he did not create an estimate for the rectangular dam with 1-partition, 2-partitions, 5-partitions, 100-partitions, 1000-partitions, up through 8000-partitions, and then have to go through a similar process with the trapezoidal dam and volume of a pyramid. C1's use of sigma notation, and ability to create formulas, allowed him to make one quantitative operation that computed all these computations for him. Because the presentation of the sequence of values was, in essence, a similar use of notation, I accompanied it with an explanation explicitly describing this way I sometimes think about limits of sums, Int: So, it is two partitions, three partitions, four partitions, five partitions. And each one of these, you're still adding up volumes, and you're just getting new values every time.

C1: And the limit is actually what is the sequence going towards?

Int: Yeah, so it's not necessarily that you're suddenly adding up areas. It's that each one of those sums is adding up little tiny volumes.

C1: Yeah, yeah, I see. It's, it's approaching the value 53.33. So if you were to take the limit of the overestimates, so from k is one up to n, it would be getting smaller and smaller, closer and closer, that same limiting value.

Although this was just a verbal explanation with a diagram of values, C1 was able to accommodate this view of a limiting process to his sum-model, as was evident by his continued description of the differential quantity measuring an aspect of a partition.

Summation Notation – Affordances and Setbacks

When I provided C1 and C2 the general prompt for Fluid Force on a Rectangular Dam task, which only asked for an overestimate and underestimate for the fluid force on the dam, C1 introduced summation notation unprompted as a method of finding a formulaic approach to the task. In particular, C1 was unsatisfied with only applying a gross basic model to the dam, observing that doing so would give a "really over" and "really under" estimate, which he felt wasn't good enough. C1 voiced wanting to create a formula instead. That is, he had anticipation that identifying appropriate estimates for the total fluid force would require the use of a global model which was dynamic in nature and would require the coordination of writing an explicit expression for his representative local[34].

While summation notation did provide C1 an avenue to quickly compute values for global models, the necessity to adapt his reasoning to specific notation often complicated C1's problem-solving process. Compounded with a propensity for simplifying expressions by combining and rearranging terms within the expression to allow easy input into a TI 84 calculator, C1 spent a significant amount of time unpacking or rederiving entire sequences of expressions to locate a computational or quantitative error. For example, due to C1's strong coordination with physics quantities and units, a consistent method of introducing perturbations was to have him evaluate the units of his differential form. C1 always assigned units to his delta and differential quantities and had a clear expectation that his local model, regardless of global model association, should share units of the desired quantity. However, when such a perturbation was achieved, C1's effort was spent unpacking his simplifications to be able to map the corresponding quantities rather than making a direct connection to the initial quantities which comprised his local models. The added coordination required to reconcile the relationship between an index, k, and number of intervals, n, masking quantities that were needed as productive transitions to integral notation, as x_k and Δx, meant that I often had to step in

34 Interestingly, C1 also mentioned that he felt like he was missing something important that "had to do with integrals" as he was trying to create a dynamic formula, but he did not explore it further.

to aid C1 in locating the critical step of a computation despite his strong quantitative and algebraic skills.

One prominent instance of this occurred when C1 was working on the Kinetic Energy of a Rotating Rod Task[35]. Drawing on his sum-model, C1 created an appropriate generalized local model using summation notation.

$$K = \frac{1}{2}\left(\frac{0.03}{n}\right)\left(\frac{\pi}{60}k\left(\frac{r}{n}\right)\right)^2,$$

However, by accidently ascribing the $k\left(\frac{r}{n}\right)$ to dr, rather than r_k, C1 ended up with a generalized local model expression $K = \frac{\pi^2}{6000}dr^3$ (in integral notation) which was wholly incompatible with his image of the differential form of a definite integral. C1 then began reviewing each line of his work to identify why this made sense but was unsuccessful until I stepped in to ask directly if that $k\left(\frac{r}{n}\right)$ in the first equation was meant to be a dr. Furthermore, when, C1 attempted to verify the units of his corrected generalized local model, $K = \frac{\pi^2}{6000}r_k^2\Delta r$, he misassigned the units of mass, kg/s^2, to the constant quantity, $\frac{\pi^2}{6000}$, which led him to assume he had made an error in the quantification of his local

35 Due to C1's ability to find ways to interpret tasks in the teaching experiment in terms of area under a curve, I moved the Kinetic Energy of a Rotating Rod Task to come before Task 5: Grading Definite Integrals.

model. After a few minutes, I stepped in to help C1 identify that the $\frac{\pi^2}{6000}$ did not represent a scaled mass because the Δr was a part of his quantification of that element by pointing out the relevant step in which he moved the 1/m element out of that unit.

The transition to a definite integral was not the only problem that arose for C1 when using summation notation. When provided the initial fluid force on a dam prompt, C1 had voiced a desire to “get all the gold stars” by creating a dynamic formula that adapted to any number of partitions automatically. Although C1 did get a rough form of summation notation into that early assignment, his partners’ unfamiliarity with the notation and no anticipation of needing to create a global model with thousands of elements meant C1’s partner wanted to move on in the task sequence quicker than C1. When his partner switched groups I asked C1 if he wanted to return to that goal and he agreed. In a fortuitous[36] turn of events, C1’s calculus section had covered Riemann sum notation the previous day. C1 wanted to practice the $\sum_{k=0}^{N-1} f(x_i)\Delta x$ notation, so, despite having originally quantified the global model for the fluid force to be equivalent to,

$$\sum_{k=0}^{24} 619948 d_k \Delta d,$$

C1 wrote out a new expression for a left Riemann sum. By ascribing $f(x)$ to be the function of interest, force in this instance, along with an image that you append Δx to this

36 For my research. Not for C1’s understanding.

function when C1 rewrote the expression global model he included an extra factor of Δd. Because this adaptation was undoubtedly linked with C1's coordination between Riemann sums, definite integrals, and his integral-model, it provided evidence for a potential source of some students' misapplication of an extra differential quantity within a Riemann sum.

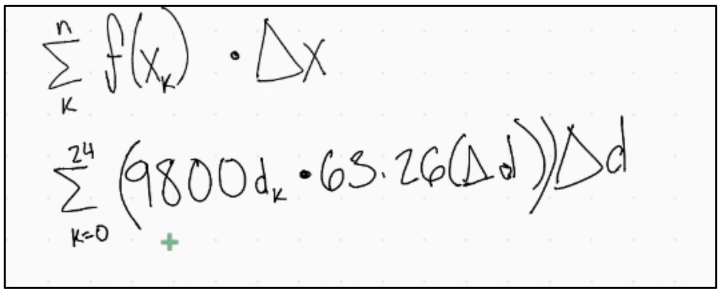

Figure 41: C1's rewritten global model expression for the rectangular dam task

C1's use of sigma notation and algebraic manipulation demonstrated considerable skill, which superseded that of most Calculus I students I have encountered both as an instructor and a researcher. The reason I reference the computational mistakes made by C1 when using summation notation is to emphasize that even for students who have a fluid understanding of notation and the ability to quantitatively construct formulaic representations, extraneous notation which students must keep track of adds to the cognitive burden of their problem-solving process and can have the potential to detract from the primary objective of a learning experience. That is not to say that integral tasks should be designed to avoid the use of sigma notation, as it is a highly useful mathematical construct, especially in calculus. However, when designing tasks to engender targeted reasoning the employ of this notation can interfere with that goal and must be attended to.

Important Aspects of Group B's Emergent Models

B1 and C2 were both Caucasian male freshmen majoring in Mechanical and Aerospace Engineering currently enrolled in a Calculus I course along with a 1-hour honors course that focused on visual representations of calculus topics. This was B1 and C2's first experience with calculus, neither having taken it during their secondary education. B1 and C2 were roommates and requested to be paired in the interview sessions, but due to scheduling conflicts with other study participants, they were originally paired in separate groups. Due to B1's partner dropping from the study in week 3, and new scheduling in C1 and C2's availability I decided to move C2 from Group C into Group B in week 3.

As I will discuss in more detail below, this decision made an impact on Group B's development of their emergent models which persisted throughout the teaching experiment. Specifically, C2's original partner introduced the notion of summation notation into the early task sequence. C2 took on this notation on as the 'proper' way to solve the tasks, and continued to rely on summation notation throughout the entire task sequence. Although C2 developed basic local and global model relationships consistent with a quantitative understanding of definite integrals, framing the final "How-To" guide in a notational framing impacted C2's ability to reason about definite integrals in which the differential form is not a Riemann Product. B1 on the other hand, built his generalized local model using notation from the sum calculator I provided, which kept his focus on the quantitative structure of his local models rather than trying to match a formulaic structure.

The Effect of the First Two Tasks on B1's Emergent Model Development

Note that B1's original partner only sporadically joined the sessions in the first two weeks before dropping from the study in week 3. Therefore, in the early stages of the study, I took on the role of acting as a sounding board and pseudo-partner for B1 more often than with other groups. This meant that occasionally I was more forthright with introducing alternate solution strategies, as B1 did not have a partner to naturally introduce discourse into his problem-solving.

During the Curiosity Rover task, B1 acknowledged that he had never identified overestimates and underestimates before. Therefore this task represented the development of a brand new construct. Throughout the task B1 consistently referenced average values, such as trying to find a better approximation by taking the averages of his overestimates and underestimates. I hypothesized that this could be the result of an attempt to coordinate the "rate of dust accumulation" with a scheme involving rates of change as a limit of average values (which would have been covered recently in his calculus course). Because I did not plan to use rates of change beyond the first task I decided did not to investigate this further unless it became problematic in later tasks. It is also possible that identifying an average value was simply a scheme evoked through coordinating an average with a type of approximation. Regardless of its origin, because identifying averages became the most prominent solution strategy, B1 did not fully adopt the refinement of the dataset as overestimates and underestimates converging on a 'real' amount of dust accumulated on the rover. That is, B1 did not fully coordinate that a gross basic model provided a boundary on his overestimate and underestimate values.

When dealing with the actual computations involved in these tasks, B1 had a strong command of the quantities which composed his gross basic model, and, from his responses in Part 1, he had constructed his gross basic model using the same quantitative structure of his basic model through proportional reasoning. That is, when asked how much dust would accumulate on the rover if we assume that the rate of dust accumulation between Bradbury Landing (0 km, 6 mg/km) and Yellow-Knife Bay (10 km, 6 mg/km) was constant, B1 stated, that "you would just multiply the milligrams times 10." When asked to elaborate further B1 continued,

> if the rate at which the dust is accumulating stays constant at six milligrams per kilometer, then if you increase by a product of 10, then you would also have to do the same to the milligrams. You'd do the same to the milligrams as you would to the kilometers. So the kilometers multiplied by 10, so you have to do milligrams multiplied by 10.

From this interaction, I understood the quantitative relationship for B1's basic model to be that of a relationship between milligrams of dust and kilometers traveled which scaled with magnitude. You begin with 6 milligrams of dust for every 1 kilometer traveled. Because you are scaling the distance (1 kilometer traveled) by a product of 10, you must also scale the dust (6 milligrams of dust) by a product of 10 to maintain the relationship between the two quantities. This provided the solution of 60 kilograms of dust per every 10 kilometers traveled.

B1 naturally created the development of a gross basic model when asked to identify overestimates for the neighboring sites (e.g. Yellow-Knife Bay (10 km, 6 mg/km) → Darwin (40 km, 3.5 mg/km)). Not having access to the precise behavior of the rate of dust accumulation along the route influenced B1's development of a gross basic model consistent with my hypothesized structure. Describing this process B1 explained,

> so to overestimate it, I would just. I'm not too familiar with that, but what I would do is take the distance between them, which is 30, and multiply that by, probably 6. Because I don't know when it changes from 3.5 to 6, or 6 to 3.5… the rate of accumulation cannot go past 6 milligrams per kilometer.

B1 was able to extend, or apply, his gross basic model to identify an underestimate along the same stretch of road stating he would "basically do the same thing but using Darwin's rate of accumulation," and that "for that to be true, the rate of accumulation cannot drop below 3.5" indicating B1 had developed the appropriate initial models intended by these two prompts.

As B1 moved to identifying an overestimate for longer stretches of the rover's journey, (e.g. (Yellow-Knife Bay (10km, 6 mg/km) → Darwin (40km, 3.5 mg/km) → Cooperstown (60 km, 3 mg/km)), he again drew on his gross basic model multiplying the rate of dust accumulation at Cooperstown and the 50 km distance traveled from Yellow-Knife Bay to Cooperstown to identify the underestimate. I had to intervene to have B1 view longer segments of the journey as the progressive addition of two values obtained by using a gross basic model on each segment. I introduced this alternate strategy by inquiring what the "likelihood" of the rover only gathering 90 mg of dust would be, and asked B1 to identify a "better" overestimate for the rover's journey from Yellow-Knife to Cooperstown. B1 responded coherently, but not in the way I hoped. Specifically, B1 did not feel that the underestimate of 90 mg was very probable, and responded with an anticipated solution strategy of taking average values to find a "better" approximation of the real value. However, at the same time, B1 hedged, “but I don't know how I would get an overestimate or an underestimate for it” which indicated the structure of the prompts requiring both an underestimate and overestimate dissuaded him from adopting an averaging strategy. From this interaction I realized that, despite the introductory context implying an overall task goal, B1’s goal-oriented activity was focused on each prompt of the task sequence discretely without anticipation of connection between them. NASA’s job was to identify whether the rover could complete the mission, B1’s job was to answer the individual prompts as they were provided.

Attempting a more direct route I explicitly asked B1 to identify the underestimates for the path from Darwin to Cooperstown. He computed this value easily but did not attempt to add this value to the underestimate from Yellow-Knife Bay to Cooperstown. Because a parts-of-a-whole symbolic form was a critical component of the global model, and B1 did not have a partner to work with, I decided explicitly introduce the solution strategy of summing the two underestimates for the shorter segments together, explaining that NASA would like to have a narrower range in the possible parameters. B1 acquiesced and computed the required value as well as a corresponding solution for the overestimate. As B1 moved on to the tasks of computing under/overestimates for the entire journey, Bradbury Landing (0 km, 6 mg/km) → Murray Bates (160 km, 0.2 mg/km), he continued to use a parts-of-a-whole method for computation, but it was clear from his language that he was simply following my strategy and that this was not yet a product of his global model. B1 described,

> Doing what you just told me, I could take the worst and best-case scenarios of each one of these, from one point to the other, all the way from Bradbury. So like

> going Bradbury to Yellow-Knife, from Yellow-Knife to Darwin, and Darwin to Cooperstown. Cooperstown to Kimberly, Kimberly to 603, and 603 to Murray. Do all those individually, and then add them together to get a more specific answer.

In particular, B1 was still attached to using an average value to identify a more accurate approximation as evident by his recommendations in the final prompt of Part 2;

> I think what I do… I would…take the… average amount that it drops per kilometer between these two or between any of these two, and like, use that to get the average. Which would be a number between these two, between the best and worst-case scenario. And depending on where that falls is where I would make my decision, because if it's pretty far below 400, then I would not rebuild this rover.

There was no time left in this session, and, because the next session began with B1 catching up his partner on what they had missed, B1 and I did not return to this particular idea. However, I anticipate that questions aimed at whether the data was linear could have challenged B1's desire to reason using average values. In particular, I would have asked B1 to compare the average rates for subsets of the path in comparison to longer stretches to have him identify the discrepancies between the two. The goal of such a prompt would serve to help B1 identify that the rate of dust accumulation was not linear over the rover's path. In addition, I would ask B1 to consider scenarios in which using an average value would produce a worse result than applying a gross basic model, which would reinforce his coordination between using a basic model as a shared template for an estimate with the required property of boundedness.

When I asked B1 to catch his partner up on what we had done in the previous interview, B1 did so in a very matter-of-fact manner, describing his aggregation of gross basic models to find total estimates. His phrasing during this explanation did include more self-identity talk, such as "what I did" versus questions "she asked." B1's use of "I" language when describing the subsequent addition of overestimates for subsection to identify an overestimate for the whole journey indicated he had taken a parts-of-a-whole (through progressive addition of gross-basic models) on as his global model.

Due to B2's addition into the task sequence at a natural breaking point, I had Group B move on to Part 3 of the task in which I introduced the additional context prompt and corresponding applet. When evaluating the intern's suggestion, B1 acknowledged noted that additional data will "allow our best and worst-case scenarios to be more precise. Because I also saw that the numbers for dust accumulation changed," while B2 added,

> since we have that dust accumulation of the midpoints we can adjust our best and worst-case scenarios because instead of taking the whole thing over-under, we can just take like halves of the sections we were doing before and over under them. So, that it'll be more accurate.

Because these two interactions happened so quickly together I am unsure whether B1's use of the term "more precise" was an indication that somehow this new data would in a lower overestimate and a larger overestimate resulting in a smaller error bound, or from his observation that the values for the rates of dust accumulation "changed" somehow implied the data was simply more accurate now. Based on the rest of the data I suspected the latter, but regardless, B2's assimilation of the addition of more data to a local-global model relationship allowed the two to move forward and quickly compute new under/overestimate pairs.

When asked if B1 or B2 had any new suggestions based on their new overestimate remaining 20 mg over the limitation of 400 mg, B1 and B2 conjectured ways to mitigate the total amount of dust through adding components to the rover (despite the disclaimer to avoid redesign within the prompt) such as adding a windshield wiper or compressed air cans. Neither participant considered the underestimate until I specifically reminded them that the intern's suggestion had allowed them to move forward. B2 observed, "Yeah, it's narrowing down the difference between the two," to which B1 interjected, "I'm sure if we took another, like, half…," and B2 finished "maybe closer together, yeah." By reminding B1 and B2 about the intern was asking them to reflect on what specifically about that suggestion allowed them to make progress. This helped them to identify that narrowing the difference between the over/under allowed them to shorten the distance between their over/under values and established an initial connection between a refining process and the narrowing of those estimates. I asked B1 to say more about how "the over and underestimates were getting closer together," to which he replied,

> because the first, whenever I look at it as just the first graph that you gave me, the numbers were so far spaced apart. That the over and under were, like, extremely far off from each other. But getting them closer together gets closer averages for the rate of dust.

I did not notice the significance of B1's reuse of the term "averages" in this case. Unlike B2, B1 did not appear to view the over and underestimates as converging on some 'real' value for the amount of dust that would accumulate on the rover but was instead trying to lower the error between the estimates and the average value between them.

Because B1's partner was not present in the first half of this task, B1 and B2's suggestion that partitioning the path in half again in Part 3, the pair's clear articulation of how they obtained their computations, and time constraints, I made an in-the-moment decision to provide B1 and B2 the total values for the over and underestimates for part 4 rather than have them compute those values themselves. B1 and B2 both agreed that based on the total over and underestimates provided by the new data points they would feel safe sending the Curiosity Rover to Mars as is.

When transitioning to the Fluid Force on a Dam task, I began by introducing B1 and his partner to the Fluid Force on a Box task because B1 had mentioned not knowing a lot about physics. This task took longer than expected because B1's partner was not attending to the prompt, "What is the fluid force acting on the bottom of a $4\text{m} \times 4\text{m} \times 4\text{m}$ whose base is at a depth of 25 meters?" instead of attempting to identify the force on the whole box rather than just the bottom face. However, by reading the prompt and referencing diagram I made of the box, B1 was able to gather the intended implications of pressure acting on horizontal surfaces would be dependent on the depth of the horizontal surface in question. He made suggestions for changes to his partner's computations, such as only including the computation for the area of the face (as opposed to the cubic meters of the entire box) and that the force acting on the top of the box and the force acting on the bottom of the box would be different due to these surfaces being located at different depths. I had B1 and his partner work a few additional computations, such as the force acting on the top of the box and the bottom of two additional boxes: one measuring 3m × 3m × 3m at the same depth of 25 meters, and another measuring 2.45m × 4.2m × 1.7m at a depth of 63 meters. I then asked if they could use this same strategy to identify the fluid pressure acting on the side of the box and B1 responded, "No. Because the depth is always changing for like the area."

Transitioning into the main task, I asked B1 and his partner if they could use the equation, force equals pressure times area, to find the fluid force acting on the dam. Both acknowledge that they could not, so I asked them to identify an overestimate and underestimate for the fluid force acting on the dam. B1's partner had two years of high school physics experience, so he introduced the notion of using the midpoint as a delineation between the overestimates and underestimate values[37]. That you could use any pressure value before 12.5 for an underestimate and any pressure value after 12.5 to be an overestimate. This reactivated B1's image of average values,

> Is the pressure that's being applied on here and linear. Cause if it is, then 12.5, would be the average pressure that's being applied throughout the whole thing. I think we can use that. Like, I know that the pressure is linear… but I also think the area times pressure formula is also linear.

Wanting to redirect B1 back to identifying overestimates and underestimates I interjected a comment referencing that if his boss asked for overestimates and underestimates and

37 As a reminder, I am not including an analysis of B2's reasoning, just the effect it had on B1.

you came back with the 12.5 number, you would probably need to have really good justification for why you can use this value since they would spend billions of dollars on the project.

B1's partner did not attend sessions again, so when returning to this prompt B1 did not return to the 12.5 linearity strategy. Instead, B1 explained that for the overestimate you would compute,

> 9800 times 25 for the pressure that's being applied. And then for the force on the dam, you'd have to take the surface area of the whole damn, … 245,000 times area of the dam 63.26 times 25 is the whole surface area.

And an underestimate is "just 0 because the smallest depth you can have is 0 and you'd multiply that by 9800 times surface area." That is B1 applied his gross basic model to the entire dam to find his estimates. When asked to identify an overestimate for the dam in two pieces B1 extended his gross basic model to a local model, calculating 9800*12.5*12.5*63.25 and 9800*25*12.5*63.25 and adding those values together to "bring it down to 290 million." That is B1 had an expectation that adding the values of these two values would result in a lower overestimate than his previous computation of 387 million. B1 noted that for an underestimate it would be "just zero for the top half" so the underestimate would just be 96.9 million. B1 did not perform any new computations on his calculator during this step, implying he recognized he had already computed the 9800*12*12.5*63.25 when calculating the overestimate. During this time B1 was working on his calculator off-screen so as he talked out loud I took notes for him on the shared whiteboard since he didn't have a partner.

I asked B1 why he so "naturally" cut the dam in half horizontally rather than vertically and he said, "because depth." When I asked if we could have done the same thing by cutting vertically and get the same answer he agreed we could. I asked him to go ahead and compute those numbers, and upon doing so he realized he got the same value as his original overestimate. This was unexpected and B1 reflected, "if you don't use depth, it definitely causes a big problem. Because then your overestimate and underestimate aren't gonna change." That is, B1 was recognizing that useful partitions for a global model should reduce the error between your underestimate and overestimate. When reflecting on the vertical partitions, B1 recognized that he was just dividing the original overestimate in two and added it back together, stating,

> It's not actually changing anything that you're multiplying. You're just cutting it in half. Like you're cutting the values that you're multiplying in half for the vertical cut into two pieces. But for the horizontal cut and changing the depth, you're actually changing the values that you're multiplying instead of just changing the surface area. The surface area doesn't really affect anything if you're still taking the whole surface area.

I then asked B1 to create cut the dam into five pieces instead of two. As he worked on his calculator finding the expected computations, I took notes for him on the shared whiteboard. After he finished, B1 explained how he made his calculations;

> I cut the damn and five individual parts. So the surface area is always going to stay the same for each individual part. So that's the 5 times 63.26. The 9800 is a constant that you gave to me. Not really sure what it means but…[I provided a quick reminder of the quantities making up that value]... Okay, yeah. All right. And then that's my constant for the fluid pressure formula. And then five is the

depth for the overestimate. So like, the deepest that this surface area would take on for the overestimate.

Notice that B1 is already noticing patterns in his computations, i.e. surface area is "always going to stay the same" and the constant 9800. When I introduced the next prompt,

5*9800*5*63.26=15,000,000 N	0*9800*5*63.26=0 N
10*9800*5*63.26 = 31,000,000 N	5*9800*5*63.26=15,000,000 N
15*9800*5*63.26 = 46,200,000 N	10*9800*5*63.26 = 31,000,000 N
20*9800*5*63.26= 62,000,000 N	15*9800*5*63.26 = 46,200,000 N
25*9800*5*63.26=77,000,000 N\	20*9800*5*63.26= 62,000,000 N
= 232,000,000 N	155,000,000 N

Figure 42: B1's original computations for the Fluid Force on a Rectangular dam in 5 pieces (typed by me – answers rounded by B1)

needing to identify an overestimate and underestimate accurate to within 50,000 N, B1, shocked and laughing he said "Wait, did you say 50,000? 50,000? Not 50,000,000?"

At first, B1 did not know exactly how to proceed and spent a few minutes looking at the expressions I wrote on the whiteboard with his computations (Figure 42). Then B1 drew on his proportional reasoning to recognize that "maybe there's like something plugging in a variable for 5 or something. To the 5 and maybe the first number as well. The depth." He continued that "all these numbers are proportional." That is, B1 recognized that the second expression was 2 times the first expression, the third was 3 times the first expression, and so on. With his recognition, I introduced the idea that the difference between the overestimate and underestimate was exactly equal to the

77,000,000 N value in the last of those underestimate values. This led B1 to conclude that by solving for x in the expression $5(9800x * 63.26x) = 50{,}000$ N he could identify the length of the interval, $x = 0.127$. B1 did his scratch work off-screen and did not talk aloud, so I am unsure exactly how he came to this solution. However, his x values were meant to represent a length of a partition, so I believe the 5 (rather than a 25/x which would represent the number of arbitrary partitions) was a consequence of working from the list of expressions for five subintervals. Although this answer was not quite correct (yet) I asked him how many times he'd have to partition the dam and he concluded 197 times. Recognizing that B1 had recognized the aspects of the equations necessary to construct a generalized local model, but that his goal was not to create such a formula, I used this as an opportunity to introduce the sum calculator so we could check the answer. Because he defined x to be the length of the subintervals I was more direct in explaining how the calculator worked than I might otherwise have been, explaining that the Δx was his x value, and the x was the changing depth that would be automatically taken care of with the calculator. This computation allowed B1 to see that his number was not quite right, which led him to reevaluate his expression and identify (again off-screen) that the 5 should be adjusted to $\left(\frac{25}{x}\right)$ leading to 7750 partitions. Entering this value into the sum calculator B1 was pleased his estimates were within 50,000 N.

In the trapezoidal dam task, B1 began by applying a gross basic model to find an initial overestimate and underestimate. He then continued to draw on his proportional reasoning as a way to quantify the lengths of a base for a partition in anticipation of generalizing as he had before. As he started this process B1 commented,

> I dunno, like on the last one that we did. I think I did like five for the intervals. I just had interval five on the base. But I'm gonna have to find each one of these [motions to the width of the dam], like, this is going to be different for each, like fifth interval. I'm gonna have to use the formula four times. So, I guess I can find a formula for it. See… how long the bottom side is.

That is, B1 recognized that for each of his dam pieces the top width and the bottom width would be changing. And he did not want to have to compute the answer by hand repetitively. He decided he would start with cutting the dam into 5 and identify the area for each piece of the dam. B1 spent a long time working these computations off-screen on his calculator, and only provided the final computations for the whiteboard. I attempted to suggest that we just write down what the values were, and not worry about the final totals quite yet, but he continued for roughly 7 minutes working through computations. After he provided the second overestimate I interjected more directly,

> I can almost tell just, kind of like by your defeatedness that this putting it into the calculator thing is getting old already. [B1: Yeah] So, maybe we can try and come up with a quicker way to do this, instead of doing them individually that will kind of give us maybe the exact force for any kind of piece that we want for any depth. Maybe we can try and work on like creating a formula like that… Let's start with the exact length of the purple line [the width of the dam] for any depth.

B1 agreed and worked on his own paper for about a minute and a half before announcing "All right, I have something," and dictated that the length of the purple line, at a depth d, would be equal to $37.92 + \left(2\left(12.67 * \frac{25-d}{25}\right)\right)$. To arrive at this formula B1 drew on proportional reasoning for similar triangles. This was a strategy I introduced when he was trying to identify the area of the second trapezoid section in the middle of the dam. Building off his formula, and acknowledging how difficult it would be to find the length of a second purple line, I suggested using a rectangle instead of a trapezoid, asking B1 if this was okay. B1 was hesitant because this was not the shape of the dam. However, he agreed that this would provide “a very rough overestimate for sure.” B1 was able to identify that to find the area of a rectangle at a depth d he would multiply the length of the purple line by the length of an interval (which I labeled L), and that to find an overestimate for force you would multiply that value by 9800*$(d + L)$ since you want to take your depth measurement at the bottom of a partition. As we transitioned to try to enter this into the sum calculator, so we didn’t have to compute the values by hand, B1 asked how certain values were input into the calculator. By drawing on computations he suggested for how 5 partitions would work,

$$9800*(0 + L) * \text{Area} + 9800 * (5 + L) * \text{Area} + 9800 * (10 + L) * \text{Area} + 9800 * (15 + L) * \text{Area} + 9800 * (20 + L) * \text{Area}.$$

I suggested using $1L$, $2L$, $3L$, $4L$, and $5L$ for the parentheses, which was a pattern B1 observed in the rectangular dam task. When he agreed, I informed B1 that the sum calculator recognized those values as just an x and would compute them automatically based on the number of partitions he entered.

B1’s computer had difficulty running the computation, so I shared my screen and entered it into the calculator. I then asked B1 how he was feeling about this and he replied “it seems very complicated.” When I asked what he meant, he just said “all the ideas coming together.” When I asked B1 if there would be a way to find the exact force on the dam, he said that he’s not so sure about the trapezoid, but,

> [for] the rectangle it'd be a lot easier to figure it out. You're just like taking the average of the pressures… Like the 12.5 would be like, the average over all of it because it's increasing like linearly. So the middle of it is going to be the average.

B1's revoicing of drawing on the linear relationship of the rectangular dam indicated that while he acknowledged that the overestimates and underestimates were improving, he had not yet coordinated a 'real' value being the number which these estimates were converging upon. This prevented him from feeling confident about making any assertions about the real value for force on the trapezoid dam.

Summary

As B1 worked through the first two tasks he obtained the primary basic, local, and global models intended by the task sequence. In the Curiosity Rover task, B1 constructed a global model through the successive addition of gross basic models, and in the Fluid Force task, he seamlessly extended his gross basic model to a local model which shared the same structural properties as the basic model. B1 demonstrated an expectation that partitioning the dam into more than one piece the expectation was that it should create more accurate overestimates and underestimates. That is, he developed a relationship between his local and global models that the shape of the local model should improve the accuracy of the global model. Because B1 reasoned about the task which required the number of partitions to be accurate to within 50,000 N proportionally he did not need to create an explicit generalized local model. However, B1 did engage in the same generalization across elements of his global model to identify what quantities were varying and which were constant to coordinate his recognition of the proportional relationships into an equation which would allow him to identify the number of partitions he needed to complete the task. In the trapezoidal dam task, B1 was able to engage in this ability to recognize values that were varying and those which were staying constant when I suggested he make an explicit formula for identifying the total force on an arbitrary subinterval of the dam. The "limitations" of the calculator also supported my effort to reframe his variable definitions for important quantities into those which would position him to be successful as he transitioned to definite integrals while drawing on his underlying quantitative reasoning which contracted those quantities (e.g. redefining the length of his intervals x into Δx and the proportional counters $1L$, $2L$, $3L,$ into x's).

Because B1 did not have a partner for the majority of these first sessions I often interjected solution strategies quicker than I would have for a grouped pair. For example, once B1 coordinated the need to quantify a length for the base for the trapezoid-shaped section of the dam, I explicitly suggested a method I believed to facilitate this quantification. I also wrote out the equations B1 voiced aloud down on the shared whiteboard, anticipating his future need to reference more than just his final computations. Despite my greater involvement, B1 was fully engaged in the activity of constructing his basic, local, and global model relationships.

The Effect of the First Two Tasks on C2's Emergent Model Development

As described in the section outlining C1's incoming schemes, during the first few tasks of the sequence C1 performed most of the primary problem solving for Group C. This meant that C2, for the most part, was just internalizing explanations C1 provided for why he solved a task a particular way and did not have a chance to fully engaged in the primary reasoning the task sequence was designed to evoke. This was through no fault of C2. He demonstrated a desire to work through the tasks and asked C1 many follow-up questions when he did not understand. He simply did not have the pre-existing schemes his partner demonstrated, and therefore could not quickly assimilate the goal of the tasks to pre-existing tools. In other words, C2 did was not provided the opportunity to authentically construct his basic, local, and global models through active engagement with the tasks themselves. Instead, C2 developed his early basic, local, and global models through the active engagement of observing and questioning C1's actions.

Early in the Curiosity Rover task C2 showed a propensity for thinking about the quantities involved. For example, when I asked whether C1's precise phrasing "for every kilometer [the rover] travels, it accumulates six milligrams of dust" implied the rover had to drive for a whole kilometer, C2 disagreed, explaining "because it wouldn't, like, as soon as you hit a kilometer six milligrams wouldn't just appear on the panel." In this case, C2 was reasoning about a proportionality between the amount of distance traveled and total dust on the panels. If the rover travels 1 kilometer at 6mg/km the total dust amount would be 6 mg, and if the rover only traveled ½ a kilometer he was able to identify the total amount of dust would only be 3 mg.

When asked to identify the overestimate and overestimate for one segment of the rover's journey C1 anticipated needing to identify an overestimate and underestimate for the entire journey and began making those computations immediately. When C2 showed confusion C1 described his justification for directly applying the basic model to each subsection of the path;

> Since we're finding an overestimate, we don't know exactly at what point or at what rate it goes from 6 to 3.5. So, since we're getting an overestimate, we just want to grab the highest number there and say this is the worst-case scenario.

During this period C2 began to adopt some of these constructs, building models from his understanding of C1's explanations. When asked about any assumptions that must be made to justify their overestimate C2 explained, "We know eventually, somewhere between Yellowknife Bay and Darwin [the rate] goes to 3.5, and we know it starts out at 6. Yes. So we just use the max rate that we have, cause we know it's an overestimate. It has to be the worst case." In this case, C2 was mostly rephrasing C1's earlier explanation into his own words, however, because this concept made sense to C2 in a way that he was confident enough to put forth his answer means that, at least on a single subsection of the overall journey, C2 had at minimum established a gross basic model to overestimate the total amount of dust by applying the larger of the two rates of change available to him.

As mentioned C1 had already completed listing the values necessary to compute estimates for the rover's entire journey, so when I decided to present that question next C2 immediately interjected "Okay, our assumption was right… Yes, this is what we were preparing for" to which C1 added "We were working towards this. I knew this was coming." While this response from Group C was delivered as light-hearted banter, it does clearly illustrate the anticipatory nature of their work. C1 likely did not need any scaffolding questions for this task and could have jumped straight to the prompt which asked for recommendations to NASA. However, because C2 had no experience with Riemann sums or integrals I believed it important to provide the opportunity to continue to identify both an over and underestimate. This would provide computations that could be directly compared to future calculations in Parts 3 and 4 and provide an opportunity to construct the local-global model relationship relating refinement of a partition with a smaller error bound. When discussing how to take an underestimate C2 described, "the inverse or the opposite of the other assumptions we made, instead of taking the highest rate, we take the lowest rate" illustrating that his gross basic (and/or local) model was adaptive to the goal of the task. C1 also introduced the idea that they "have to assume that [the rate of change] doesn't make any dramatic dips anywhere along the path," which was the first indication he gave that there was a strict boundedness aspect to these local models.

When I introduced the option for more data C2 was immediately able to identify the implications, "that way our overestimates and understate are closer to the actual values," implying that not only had he constructed a local model from C1's detailed explanations in Part 2 but that his local model was already tied closely with the image that lowering its magnitude directly improves its accuracy to the "actual" amount of dust on those subintervals.

When evaluating their overestimate, still larger than the limitation of 400 mg, C2 introduced the idea of averaging the underestimate and overestimate,

> What if we take the average between those two. That would be the closest that, it'll be close to the actual. It seems like approaching the actual value of the data. So wouldn't the actual value be like 376.2 milligrams?

Such an invocation of averages indicated that although C2 coordinated the underestimate and overestimates getting closer to one another, he did not necessarily appreciate the possibility that the average value could be significantly different than the real value. C2 also did not stand by this statement when I followed up with him by asking directly if he would approve the mission based on this observation he stated "I wouldn't say so. I think that's just the average. If you're like, say, run it like 10 times, it still could have a worst case of the project.”

During Part 4 both C2 and C1 expressed that they weren’t familiar with excel commands, so I had them direct me on what they would like to compute. C1 suggested calculations in the excel sheet which resulted in a new underestimate of 187.425, a value significantly lower than their previous underestimate. I had C2 and C1 explicitly remind me what that value had been. C1 acknowledged that the previous estimate was “a lot higher than that,” but C2 wasn’t bothered, noting, “I mean, I think we’re right. I think I think we are. I mean, if that's our underestimate our overestimate should be way under.” This illuminated a very subtle, but important distinction, between C2 and C1’s local-global[38] model relationships. C1 had an image that the refinement of any partition would result in an overestimate getting smaller and an underestimate getting larger and that this new interval would be a subset of the original range in values. While C2 equated the same refinement with a reduction of the magnitude between overestimate and underestimate, but no correspondence with the previous global model partition. C1’s understanding of this relationship allowed him to be perturbed by the new, lower, underestimate and prompted him to reevaluate the computations I had input into the

38 I am unsure if C2 currently constructed a local model or a gross basic model at this point. I hypothesize the later as he didn’t talk about partitions, subintervals, or similar constructs..

spreadsheet. C1 quickly corrected the error and C2 and C1 were satisfied that this new computation resulted in an overestimate lower than the limitation of 400 mg.

When beginning the Fluid Force on a Rectangular Dam task which asked for an overestimate and underestimate for the fluid force on the dam C2 commented,

> So is the over and under actually going to be the 25 meters is the highest amount of pressure. And then would you put in zero meters? Er, I guess we need to think about this, is that what they're asking?

That is, C2 recognized the variation of pressure along the dam's depth and his initial instinct was to apply a gross basic model to identify those estimates. C1 showed hesitancy to follow this strategy, anticipating that this would be a task he was used to which would require him to identify a better estimate, but C2 justified "Yeah. I mean, if you have that information, like, if you know the maximum amount of pressure it's going to take, so all you really need is an overestimate for this." C1 agreed to calculate those values, which C2 commented were "very large" numbers and asking if they seemed "reasonable." In reply, I commented that usually forces like these are written in scientific notation because they get so large, so it was not abnormal to get a value that large. C1 continued to think about this task aloud, injecting ideas of using specific areas for specific depths in anticipation of creating a formula. Building off C1's desire, I suggested C1 and C2 create overestimates and underestimates for two pieces of the dam, sort of like the rover task. Before computing these values C2 suggested writing the formula to be $F = 9800d * 63.26d$. This provided evidence that without a partitioned dam with at least two elements and corresponding expressions for force on each subsection (either explicitly written or anticipated) the creation of a generalized formula was non-trivial.

C1, anticipating writing expressions for two separate partitions recognized that the $d's$ in the formula did not measure the same value, which C1 explained to C2,

> If we were to cut the area there in half the kind of cross-sectional area that we're pushing up against… what we do is we take the force of the lowest times the area of the, like, the area that we would be multiplying… if they're equal segments, you know, the top one is going to be zero, and then the bottom ones just going to be the same area times the, you know, the most. So it would kind of be just half the force.

C1 then worked off-screen to compute the overestimate and observed that "it would change." This led C1 into providing C2 a very long explanation about breaking the dam into two pieces, the corresponding force equations on each piece, and linking it with a "better" estimate because it provided more data. Although these explanations were coherent, they removed a lot of the cognitive steps for C2 to have built himself. In other words, C2 was again just internalizing C1's explanations rather than fully engaging in the task sequence as intended. This is not to say that C2 did not construct schemes associated with C1's explanations. When C2 and I discussed this episode he was able to discuss the dam being split into different pieces and that their equation was to identify what the force acting on those pieces of the dam was. I had him calculate the overestimate and underestimate value for the partitioned dam he drew (25 pieces) and he was able to identify the expected underestimate, $9800 * 24 * 63.26 * 1$, and overestimate $9800 * 25 * 63.26 * 1$. I then asked what would change if he divided the dam into 50 pieces instead,

> So if you chopped it into 50 slices, instead of 25, the distance between your like sections. I guess the height of your sections would get smaller and smaller. And

> so the actual, the actual values of F would be a lot lower for the whole because the entire the forces for the entire section. So since the section is smaller, that would get smaller. And then but you'd also be more accurate. For each like, point, the more you divided the more accurate you get.

Based on C1's coordination of the height of the sections and the number of slices indicated that he had incorporated a refinement relationship between his local and global models. When C1 is referring to accuracy in this case he is not referencing the global model becoming more accurate with extra partitions, but the measurement of the force on the partition. When you make the sections smaller the total force on that section is smaller, but it is closer to the real force.

When C1 explained that he wanted to construct a formula that would calculate parts of the dam that were so thin they were basically zero, this evoked schemes for C2 related to rates of change and he suggested using Δd for the height of the subsections. C2's suggestion, along with his desire for a formula, motivated C1 to introduce summation notation into the task, writing general global model expressions,

$$\sum_{n=1}^{25} 9800n\Delta d(63.26\Delta d)$$

$$\sum_{n=1}^{25} 619948n\Delta d^2$$

Although in the group interview C2 did not object to C1's use of summation notation, in his individual interview C2 admitted to not fully understanding what all the parts referred to. When trying to make sense of it C2 described,

> So that's basically the force formula. In terms of like, it's for any change in d between 1 and 25. So if you're, like n, I don't necessarily get why we had the n and the Δd. Let me look and see if I can figure it out. So Δd would be the height that I have, n with the depth, pressure takes. Okay, so the n is for the pressure formula because you need to have the actual depth. Not just that, because Δd would technically be the height which you use for area. And so that's what you have in the area side of the formula, because like, so over here you have area, and over here you have pressure, pressure times area. So the $9800n$ times Δd is the pressure side of the formula. Yesterday, the Δd made a lot more sense to me.

C2 has coordinated that whatever is on the inside of this equation represents a force of one of the sections of the dam and that the structure must represent pressure times area. That is, C2's local model had taken on the property that it must have the same quantitative structure as the basic model. Additionally, although he was not sure precisely how summation notation works, C2 recognized that this set of symbols represents the total force, or the global model. C2's confusion over the n and its relationship to Δd as he's assimilating n to represent the depth, not a counter. When I asked C2 what the $\sum$ symbol means, he replied

> I have two different ideas. One is that maybe it means, like you take n 1 through n 25, add them all together and take the average of them. Or if it just is the notation of saying, this is like, the function next to it only goes from 1 to 25.

While I could have explained in the moment, I wanted to provide C1 the opportunity to realize some constructs he was drawing upon were tools C2 might not be familiar with. So, I suggested that we discuss this more during our next meeting to make sure we were all on the same page. Unfortunately, while I did bring this up in the group setting, C1 just offered a quick explanation that you would add things and n would be a "step" counter so you'd plug in $n = 1$ and add that to $n = 2$, etc. That is, they did not discuss the notation with relation to capturing the different depths for the pressure component. C1 also used his calculator off-screen to compute the overestimate and underestimate values for 25 partitions, later providing a verbal explanation to C1 how to do this computation on his own calculator.

When I asked C1 and C2 to identify an estimate accurate to within 50,000 N, C1 again took the lead off-screen quickly using proportional reasoning to identify that you'd need 8000 partitions to find an answer this accurate. When C1 explained that he recognized that when he changed his counter from 25 to 100 the difference in the overestimate and underestimate would reduce by a factor of 4 so he just used that same idea, C2 said he would have just "used trial and error."

When I introduced the trapezoid problem, C2 observed "we just have to divide it into sections again, it's just a bunch of trapezoids" indicated that his local models took on the shape of the global context and that he recognized that the basic structure of the local model would stay the same, $P \cdot A$. C1 was again able to quickly identify the problem posed by the new context, voicing that they needed to find the width of the dam for any step. C2 suggested identifying the force on two rectangles, one rectangle measuring $37.29\text{m} \times 25\text{m}$ which represented the middle section of the trapezoid added to a smaller rectangle measuring $12.985\text{m} \times 25\text{m}$ which represented two triangles placed together from the extra part of the trapezoid (Figure 43). C1 followed this strategy, but adapted C2's additive process into a subtraction between two summation expressions with five partitions,

$$\sum_{n=1}^{5} 619948n(\Delta d)^2 - \sum_{n=1}^{5} 127253n(\Delta d)^2 .$$

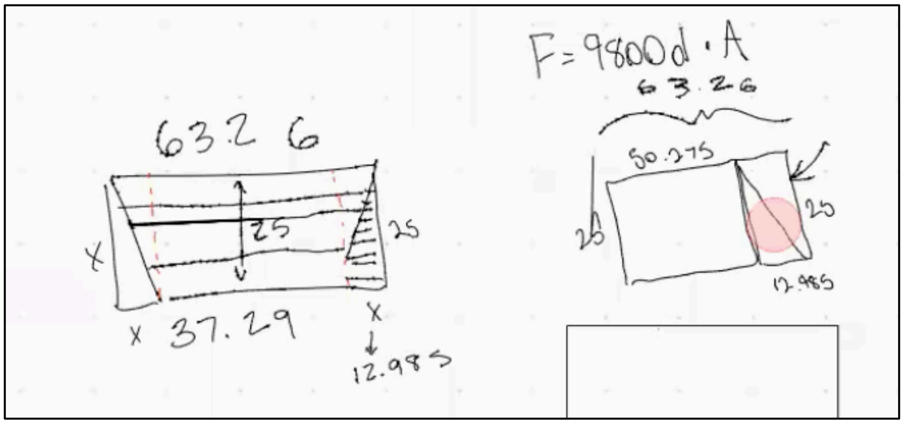

Figure 43: C1 and C2's two rectangle strategy for the trapezoidal dam task

More than once in this interview C2 voiced confusion about these two expressions, which led me to suggest that C1 write these expressions out the long way without combining terms. Above the summation, C1 wrote out,

$$(9800n\Delta d \cdot 63.26\Delta d) - (9800n\Delta d \cdot 12.985\Delta d).$$

After C2 wrote out this expression, I asked C1 and C2 to explain whether their expression was different than

$$\sum_{n=1}^{5} 9800n\Delta d \cdot 50.275\Delta d,$$

and if not, was that okay or did it pose a problem for computing the value of the forces for the trapezoid. When C1 attempted to run numbers and compare values I explained that I was just writing the expression for a dam that was 50.275 meters wide, which was the value of 63.26-12.985. C2 did not observe a problem with this computation, “I don't see why it wouldn’t work. I mean, we're just taking and moving certain things. I mean, obviously, you can't move the actual dam. But the math should add up." This indicated that the notation was clouding C2's ability to identify that subsections of the two triangles he moved to combine into one rectangle would not properly account for the increase in pressure along with the depth. C2 also recognized this difficulty as his next suggestion was to just try and write expressions for the dam in two pieces. This allowed C1 and C2 to recognize there was an issue with their overall method.

C1 and C2 returned to trying to quantify their trapezoid, and because they recognized that they needed to identify the width of a dam at any spot, but were having trouble identifying how I introduced the idea of using similar triangles. C1 was able to recognize the proportionally and translate it to summation notation (Figure 44), but C2 voiced confusion again and had to ask C1 what he was doing.

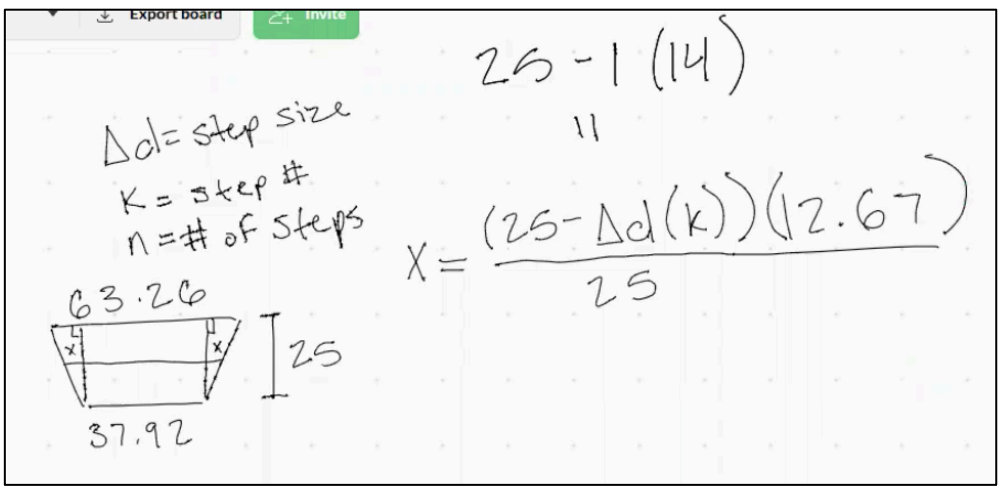

Figure 44: C1's summation notation for the side length of the triangles for the *kth* partition

My repeated reference to C2's confusion is not a judgment of something lacking in C2's computational or reasoning ability. During his individual interviews, C2 continued to demonstrate that he understood the quantities involved, was reasoning in terms of basic, local, and global models, and could take on C1's explanations with fairly good incorporation into his schemes. However, the use of notation was continuing to hinder C2's ability to recognize which quantities C1 was referring to. In C2's follow-up interview I noticed that when he wrote out his expressions it was always in the form of pressure times area, so I asked why this was a more comfortable way to write the expression, as opposed to C1's tendency to combine terms. C2 explained,

> Because it like derives from the original formula we got, like the force equals pressure times area. I was just dividing it into pressure and then the area… I like that in-between stuff a little better. I still wasn't completely comfortable, so I kinda needed a few extra steps.

Summary

During the first two tasks, C2 demonstrated that he had coordinated the structure of a local model with a basic model, that a global model was an accumulation of local models, and that an increase in the number of sections of the dam you increase the accuracy of the overestimate and underestimate values. C2 also displayed anticipation that you could adjust the summation formula to as many partitions as necessary, identifying that you would just need to adjust the value for Δd to be the total length divided by the number of sections you are computing.

C2's partner went through the task prompts much faster than C2 was necessarily comfortable with, often anticipating future questions before they were asked. This speed denied C2 an opportunity to fully engage in a lot of the prompts planned throughout the task sequence. Therefore, when the scheduling conflict arose and B1's partner dropped out I decided it was in both C1 and C2's best interest that I move C2 to Group B. This would allow C1 to move through the task sequence at his own pace, provide B1 a partner, and provide C2 the opportunity to actually engage in the tasks as planned rather than just trying to keep up with C1.

The "How-To" Guide Fiasco

Throughout the rest of the teaching experiment, B1 and C2 constructed their emergent models fairly robustly, however, they never became particularly proficient at creating generalized local models. Part of this difficulty was that often B1 and C2 reasoned about the quantification of their local models in different ways. For example, during the energy to build a pyramid task, C2 initially reasoned about quantifying the volume component of their local models using a variation of height, while B1 thought about measuring those same volumes as the portions of the side length were removed. That is, B1 envisioned different slabs of the pyramid being lifted to different heights and noticed that the quality that was changing between these partitions was a decrease in the width of the base length. Deciding to go with B1's quantification, C2 had to reason about the task in a way that wasn't inherently natural for him. On the other hand, C2 would regularly fall back on using summation notation for various tasks throughout the teaching experiment. B1 found the need to rewrite things in terms of summation notation tedious, wanting to write his generalized local models directly into definite integral notation. While these differences in approaches were not insurmountable obstacles, they did pose added challenges along the way and were, in part, a result of their different introduction into the task sequence. However, what was important about B1 and C2's emergent models is that they were built on an image that their basic model dictated the shape of their local models which informed the quantitative structure their differential forms needed to be in, they ascribed a meaningful quantity to the differential as a measure of the length of an interval for their local model and were persistent in their quantification process once a productive partition which would allow them to reduce variation was made. All of these elements helped them to produce definite integral expressions required throughout the teaching experiment, even if the path to each expression was not a straightforward quantification of a generalized local model.

It was for the reasons above that I was surprised by the "How-To" write-up activity. Like the other two groups, B1 and C2 were asked to answer the following two prompts:

1. Provide a write-up that describes exactly what a definite integral is, and how it works, to someone who has never taken calculus before. It is not necessary that the reader be able to compute definite integrals by hand, but your write-up should enable them to be able to understand the quantities involved for definite integrals such as the ones you've worked on over the past few weeks. Be sure to include specific descriptions for the notations you use.
2. Provide a write-up that would enable a reader to construct a definite integral for tasks such as the ones you've worked on over the past few weeks. This write-up should be specific in its directions, but general enough that it can apply to novel tasks.

This prompt was placed at the end of the learning trajectory to serve as a reflection activity. That is, its sole purpose was to encourage participants to review their work throughout the teaching experiment, identify common practices between different tasks, and position them to consciously recognize some of their emergent model structure (e.g. creating partitions, adding up quantities, over and underestimates, etc.) However, during this task B1 and C2 did not review any of their previous work instead of providing explanations for definite integrals in terms of “area under a curve,” velocity and position heuristics, and “$f(x)$ times dx.” Because we had not discussed area under a curve nor velocity at any point during the teaching experiment I understood that B1 and C2 were describing the schemes for definite integrals developed in their calculus coursework, not from the teaching experiment.

As B1 and C2 moved onto the second prompt, the activity seemed to be towards the intended trajectory because C2 said “I mean, no matter what we did, we always divided something up into a bunch of pieces. I think that’d be important for someone to know trying to set one up.” C2 followed this up by talking about needing to use summation notation before moving into integral notation, which I took to mean he was reflecting on needing to imagine actual physical partitions which he could measure, however, he followed this up with a generic Riemann Product description of summation notation, $f(x_i)\Delta x$;

> So where do we start with an integral? I still really can't just write an integral, I always go from summation notation to integral notation… if you go from some summation notation, you can just say, you know find Δx you got to find x_i and then you multiply Δx times $f(x_i)$."

I attempted to direct B1 and C2 to reflect on some of the specific activities they worked on, like the pyramid and the dam, and identify things in common such as C2’s example of “dividing things up.” However instead C2, without objection from B1, wrote steps that framed the aspects of the definite integral in terms of notation rather than quantities: e.g. find Δx which is $\frac{b-a}{N}$, write an equation $\Delta x\ f(x_i)$ (Figure 45).

Unfortunately, working with this list of steps in mind, C2's answer to the Kinetic Energy of a Rotating rod task resulted in a solution consistent with an Integral as a Transformer conception (i.e. the differential was appended to the end of the basic model). Although B1 was able to recognize an issue with appending a differential quantity to the basic model, C2's easy reversion to not attending to the quantitative structure that the local model must share with the basic model provides more evidence of the difficulties students face when they have an image of $f(x_i)\Delta x$ as their image for a local model. Following the Kinetic Energy task B1 and C2 returned to their "How-To" guide to rewrite their steps, as C2 realized they were not effective. This included B1 and C2

Common things in problems:
Breaking things apart
Need a Dx
(a,b)
f(x) in the inside

Step 1.
Find your interval (a,b)
Step 2.
Find a delta X which is (b-a)/N. N is the number of intervals
Step 3.
Find Xi which is the a+i(deltaX)
step 4.
Write an equation where f = (deltaX)*f(Xi)
Step 5.
Once you have the equation you replace (deltaX) with dx and Xi with X
Step 4.
Write the integral with a stop at a and b and put the found equation inside

Figure 45: B1 and C2's initial "How-To" write a definite integral task

discussing the *mental actions* they took to solve the tasks such as identifying variation, identifying a basic model, focusing on how to measure local model elements, etc. (Figure 46).

Common things in problems:
Breaking things apart
Need a Dx
(a,b)
f(x) in the inside

Step 1.
Find the interval (a,b) in the terms that you will be taking the integral with respect to. The interval will always be the value that is changing or it is connected to the value that is changing in some way.
Step 2.
Find a formula or use a given formula that directly relates to the asked question. Find a formula that models the quantity.
Step 3.
Make sure all values are in terms of the variable that is being used for the integral.
Step 4.
Find in terms of the interval where the value from the integral changes for each section ie. For a pyramid, using square prisms with varying lengths and widths and height dx. Or speed changes for a rod spinning around from one point as you increase the distance from the point. (The rod goes around faster as you reach the end of it rather
Whatever value you're looking for, you need to divide up into sections and find the value for individual sections and add it together.
Step 5.
Plug all terms into the formula with correct units.
Step 6.
Plug new formula with variable x and dx into integral from (a,b)

Figure 46: B1 and C2's "How-To" guide rewrite

As a direct result of reframing how they thought about working through definite integral tasks to focus on primary aspects of their model relationships, B1 and C2 were successful in solving the Gravitational Force task with little difficulty.

Supplementary Tasks as a Part of On-Going Analysis

As each group made their way through the planned task sequence I made changes to the hypothetical learning trajectory due to my ongoing analysis of each group's individual needs as planned. This often took the form of making slight changes to the ordering or inclusion of specific sub-prompts but also included occasional suppletory tasks for the participants to engage in. Many of these supplementary tasks were presented in the follow-up clinical interviews in service of developing a more nuanced image of their incoming and evolving schemes (e.g. "what is $\lim_{x\to\infty}\frac{1}{x}$?"), however, there were also 3 major changes that I made to the overall task sequence. This included the two suppletory tasks I designed for Group A, and the inclusion of one of the task-based clinical interview questions to be present for C1 prior to his final reflection task.

As described in the last chapter, the decision to include the Kinetic Energy of a Rotating Rod task in C1's hypothetical learning trajectory was due to his continued attempts to directly apply his integral-model to tasks or to conform a definite integral he created using his sum-model to a partitioning consistent with his integral-model. Because the hypothetical learning trajectory developed as a part of this study was in service of the *developing* of a quantitatively based summation conception of integration, and not *replacing* an already existing scheme for Riemann integrals, most tasks contexts within the task sequence did not require the use of such an understanding of integration. This was intentional because most contexts which require a Quantitatively Based Summation conception of integration are inherently more difficult to quantify.

By moving the Kinetic Energy of the Rotating Rod task into the task sequence I positioned C1 to be able to reflect on his problem-solving activity for this task during his "How-To" guide. This resulted in a direct identification that he had been unable to conform this sum-model into his existing integral-model and a conclusion that attempting to apply this second set of schemes to a context of this nature was "the wrong way to do it." As he reviewed more of his tasks he identified his common theme of always attempting to directly apply his integral-model to every task and that some contexts just aren't appropriate for his integral-model. While the design of this study was not positioned to aid C2 in identifying a way to immediately identify which of his schemes would be appropriate. It did serve to engender the development of a quantitatively based conception for definite integrals (his sum-model) and a recognition that this scheme is more appropriate for some types of tasks. This suggests that for students who enter calculus with a highly developed scheme for Reimann integration, this learning trajectory can serve to engender the development of a Quantitatively Based Summation conception of integration, possibly by the creation of a new scheme entirely. This result also provides an avenue to address a common misapplication of definite integral notation observed in students, appending a dx to a generic basic model as the differential form within definite integral notation. C1's integral-scheme is not the only reason students might append a dx to a basic model (e.g. students who do not ascribe units to a differential quantity or don't include one at all), but for those who display a similar understanding, this case-study provides a potential series of events which led to a critical perturbation for the correction of such an error. They must (1) develop a sum-model, (2) cognitively recognize that they have two separate schemes for accumulation (3) observe a repeated behavior of their direct application of an integral-model, and (4) observe that at least one of these examples could not be re-conformed to their integral-model.

Another adjustment I made was the inclusion of the "Can we use a rectangle to approximate the area of an annulus?" task into Group A's task sequence immediately following their Volume of a Sphere task. I included this task because, while A1 and A2 recognized that the answer their definite integral expression provided was not the same value as the "real" volume of a sphere, they could not identify why their method of using rectangular boxes didn't work. I had to inform them that the shape they actually found the volume for was a “bubble pyramid[39]” which allowed them to recognize they should use cylindrical prisms instead.

Believing that this might provide a good opportunity to have A1 and A2 investigate when it’s appropriate to use a basic model different than an exact shape, I provided A1 and A2 a synthesis of the error for their previous computations and asked them whether they could use a rectangle to approximate the area of an annulus if they were trying to create an integral modeling the area of a circle. Through quite a bit of guidance from me, A1 and A2 were eventually able to recognize that if the error had a linear relationship with the differential quantity then they could not use that shape to make the approximation, however, I do not believe this task was a productive addition to the teaching experiment. In particular, when A1 re-voiced confusion about why their rectangular prisms didn’t work during this activity, A2 succinctly explained that you could not estimate the area of a circle using a square. While an excellent question and amazing response, this exchange effectively removed the intellectual need to investigate this question any further. While A1 remained invested in the broader image of wondering when they could just ‘decide’ to replace a quantitative element within a local model, A2 was focused on the specific question I’d posed regarding the area of an annulus. Because A1 deemed this to be an unnecessary estimation to make, due to the ability to find the

39 I drew on the terminology from the Pyramid task immediately preceding this task

exact area of an annulus by finding the difference between the two corresponding circle areas, the task was not engaging. Additionally, and perhaps more importantly, the task sequence was not designed to meaningful leverage the connection to error in the later tasks. While A1 and A2 did reference this activity when they needed to model the Total Mass of a Circle Oil Spill, the geometric shapes A1 and A2 were invoking for the rest of the sequence had already been identified as acceptable. Therefore they did not have a need to investigate error throughout the rest of the sequence.

A much more successful addition to the hypothetical learning trajectory was the "Do These Integrals Model the Energy to Build a Pyramid?" task in which I provided, correct, integral expressions which were quantified through a partitioning process distinct from each group's approach. While this task was conceived of as a direct result of the discourse between B1 and B2 during their own pyramid task, I included it for Group A (individually) after they finished the corresponding task and for C1 within their "Grading Definite Integrals" task. The inclusion of this task sequence was particularly successful for Group A for 3 reasons.

(1) The task provided a concrete example that there is not a canonical way to set up definite integrals. Group A had been excellent about communicating throughout the semester and building the local models, including their quantification, was always a team effort. Although there was already planned inclusion of alternate expressions in the "Grading Definite Integrals" task, this example added a second context to observe that property.

(2) It positioned A1 and A2 to compare the symbol template of the definite integral for an expression by directly comparing to their work. By matching the differential form to their generalized local model structure, A1 and A2 were able to coordinate the precise quantification of different elements of the definite integral through their image of those components.

(3) Most important, this task revealed a critical element that was missing from my hypothetical learning trajectory. Specifically, this task was positioned to reveal the exact correspondence A1 and A2 had between a partitioning process and the limits of integration within the symbolic form of a definite integral. While their own choices for the limits of integration were obvious through their correspondence during the development of a generalized local model, A1's interview during this subsequent task revealed she had an incoming scheme for the limits of integration which were tied to graphical imagery of changing the axis which you were partitioning. Although A1 could meaningfully identify the purpose of each element within the differential form, changing the limits of integration from measuring heights to measuring widths imposed a scheme for the rotation of the partitions to run vertically rather than horizontally. More accurately, A1 envisioned the pyramid physically rotating so that she could envision the partitioning process as a form of subdividing the base of the pyramid rather than the height which they had done before. Although I tried to induce a perturbation of this image through coordination of vertical partitions not providing an appropriate way to estimate varying energy, A1's unfamiliarity with physics, made her resilient to such inquiries, as she assimilated specific values for

the partitioning process into her diagram. Unfortunately, due to limited time in the individual interviews and scheduling issues due to the holidays, A1 and I were not able to resolve this issue. Because I did not have a task of this nature, a chance to revisit this idea did not arise naturally. In a subsequent study I will have an opportunity to investigate A1's reasoning again, in which I plan to reintroduce this prompt to identify whether this conception is still present, and if so, have a preprepared task to attempt to induce a necessary perturbation.

Results of Task-Based Clinical Interviews

At the conclusion of the teaching experiment, I engaged participants in identifying definite integrals for two tasks in which the differential form was not a Riemann product: Total Kinetic Energy of a Rotating Rod, and Gravitational Force Between a Rod and a Particle. These tasks incorporated new challenges for the participants due to the presentation of a fixed quantity for the mass of a whole rod which required the need to conceive of the differential quantity, or fixed value across elements of their global model, as an element of their quantification of a constant mass for a local model. Although a direct comparison cannot be made between individual students, to serve as a baseline for what a student may have learned through the course of their normal college calculus course work, I interviewed one additional student who successfully completed a calculus course with the same instructor as four of the study participants. I discuss the results and implications for the students' emergent models below.

Total Kinetic Energy of a Rotating Rod

The Total Kinetic Energy task requested students model the kinetic energy for a rod rotating about one end. Students were provided a way to situate this knowledge within already existing schemes for motion by the additional context (like the second hand of a clock). By providing a radial unit to characterize the behavior of rotation (1 rev/min) an additional level of complexity existed in this task due to variability not being immediately obvious.

Group A's Kinetic Energy Task

The Kinetic Energy task posed a problem for A1 and A2 due to the perceived lack of variation in the context. A2 began by drawing a diagram of their interpretation of the situation, including quantities and values, as well as the basic model expression, $K = \frac{1}{2}Mv^2$. A2 observed, "I guess the first thing we have to do is find these speeds because we know our mass is constant." The need to identify a varying quantity but assimilating the 1 rev/min to be a constant speed made A1 question whether mass would indeed be constant in this context; "Is it? I thought our speed would be the 1 revolution per minute." Reorienting herself A1 noted,

> Okay, so we're trying to think about what to do, what we want our equation to be measuring. So, we know we needed to measure kinetic energy, and we have that formula.… So, we've decided that our mass is going to be constant no matter what. So our big M is a constant. So that means the only variable is v in that equation.

In this observation, A1 is demonstrating that she is attempting to identify exactly what could be varying in this context. A1's need to isolate the varying element is driven by the anticipation of creating a local model which will measure the kinetic energy of partitions of her global model. A2 added,

> Speed will be our changing variable. So I'm trying to figure out, it's one revolution per minute, right here at the end, right. So because the minute hand goes all the way around in one minute, but here, it goes around at a different speed, right? Because there's less distance travel that goes with this guy [drawing the path of rotation for a piece of the rod near the center of the circle].

A2's recognition that the end of the rod covers more distance than the center as it rotates is informing his image of where a partition should occur. That is, he anticipates that by breaking the rod into partitions along its length, they will be able to reduce the variation in speeds between the endpoints of each partition. However, by drawing out the paths of these different sections, A2's memory of two tasks involving concentric rings prompts him to suggest that their partitions should "be the ring." A1 contended that "they're cutting the length in pieces." After reviewing their work on the ring activity they realize they're trying to find kinetic energy, not area, so A2 returns to trying to capture the variable speed,

> Is it just the circumference, we need to find the distance that travels. And then we can read a proportion. Based on the length of one meter. It goes around one revolution per minute. So we need to find a base of 0.05 meters. It would go at, I guess two meters per second, right?

This observation caused A1 difficulty; "I don't know. To be honest, I'm a bit confused. Because it's like, the bottom part is still doing one revolution per minute." That is, A1 correlated revolutions per minute with the speed of the rod at all parts along its length. Even with a partitioned length, all pieces of the rod would travel the same ‘distance’ of 1 full revolution. This conflict poses a significant challenge for A1 who described her meaning using her pencil in the video screen;

> But let's think if we had partitions, like I guess, if we were to cut it into partitions, it would be like the bottom partition. Because what I'm thinking is if I'm taking my pencil and I'm moving it like this [rotates the pencil around its base], all my hands are moving at the same, like my hands are both moving the same…This is only one speed, my hand is only going one speed. And it's making the whole thing. Does that make sense?

It was clear that A1 was attempting to suggest that no matter how she chopped up the pencil her hand would always travel 1 full revolution in the same amount of time, it would go the same “speed.” After some back and forth A2 observed,

> Okay, so in our formula, here we have mass and speed. But mass is definitely not changing. Because it's constant. It's the rod. But speed is changing. Except our units are revolutions per minute. And every minute, every part of this thing makes one revolution. Except this part here at the middle isn't going as fast. So speed must not be one. I think we're defining our speed wrong… So right now, we said that speed equals one revolution per minute. And I don't think that's right.

This conflict between whether the speed was constant or changing persisted and required that I point out that velocity would be measured in a length of distance, like meters, over an interval of time. This prompted A2 to begin writing the algebraic expression for their generalized local model, $K = \frac{1}{2}(0.03)\left(\frac{d}{t}\right)^2$, noting,

> So in our like actual equation here, K equals one half 0.03 times distance divided by time squared. Because our mass is constant, and this defines our speed. Our time is constant too because it's one minute. Yeah. So it’s just divided by one.

Which means that speed in this case is just distance. But I'll leave [1] there for clarity.

Attempting to measure this speed, A1 and A2 commented that they needed to rewrite speed in terms of the length of the rod because that is how they made the partitions.

The diagram markings, which looked like rings, caused A1 and A2 to continue referencing back to the concentric ring method for finding area, but they could not coordinate this with needing to find a distance. A1 and A2 knew they were trying to quantify a variable speed, but due to not having familiarity with measuring velocities on circles they were unable to meaningfully measure that quantity. To aid them in moving forward, I drew a dot on their diagram at a distance r and asked how far that dot would travel if it went around the ring, to which they replied $2\pi r$ meters. I then asked long it took, which they replied was 1 minute. This allowed A1 and A2 to coordinate that the velocity of that dot would be $2\pi r/1$ m/min.

The introduction of the velocity element of their generalized local model raised a new issue for A1 and A2 because they realized that there was no differential quantity in their expression,

A2: So $2\pi l$ is the largest distance, which means it's traveling the fastest. At the very outside point. So if we do $2\pi l$ minus. I just can't figure how to do it without writing Δl on there. That doesn't make sense though.

A1: We need it in our problem.

A2: Yeah, I know, that has to be there somewhere. But I don't know why. Well, I do know why. So it is since it's

A1: It's the length of our interval. The length of the partition of the rod, like as the rod has been chopped.

I recognized that while A1 and A2 understood that Δl would represent the length of a partition, they were not thinking about a single partition. That is, they were focusing on attempting to write their generalized local model too abstractly and not considering single partitions as they had before. I verified this by asking what their K expression represented and A2 said "it's the kinetic energy of the whole rod." I countered by asking whether K or $\int_0^{0.1} K$ would be the kinetic energy of the whole rod. A2 acknowledged "this is just the equation for the kinetic energy of this rod at like this point." This refocused A1 and A2 on writing an expression that would represent a point on the rod, and not the entire rod itself. This led A1 to rewrite $K = \frac{1}{2}\frac{0.03}{dl}\left(\frac{2\pi l}{1}\right)^2$, however, A2 said that did not make sense because you'd divide the 0.03 by the number of partitions, n, not dl. This led to a series of attempts to "make the math work" where A1 threw out ideas of what they could divide by. Realizing she was suggesting random numbers and was not making progress, A1 reoriented herself,

> Because it would work if we said 0.03 divided by 2. Yeah, but [the number of partitions] is not what we decided dl is.… Okay, let's think about how do we even get dl to begin with? We do the total divided by n and that's how we get our dl… So we need to solve for dl in that equation.

A1's observation that $dl = \frac{\text{[length of whole rod]}}{\text{[number of partitions]}}$ positioned A1 and A2 to rewrite 0.03 to be $\left(\frac{0.03}{\left(\frac{0.01}{dl}\right)}\right)$ and conclude that their generalized local model expression was their "general version for the size of the partition." Which they entered into their K equation, rewriting it as $(0.03)\left(\frac{dl}{0.01}\right)$. There was not a method to input an integral sign into the whiteboard, so A1 and A2 directed me on how to write their final integral expression on the whiteboard, resulting in a global model expression, $\int_0^{0.1} \frac{1}{2}(0.03)\left(\frac{dl}{0.01}\right)\left(\frac{2\pi l}{1}\right)^2$.

When I asked A1 and A2 how they knew their expression was right, A2 suggested that they enter it in the integral calculator. When I asked how getting a specific value from the calculator would tell him he had the right answer, he realized he was not sure. A2 then worked on his own sheet of paper to "check" whether their expression was correct. A1 inquired "how" he was "checking it," asking "Are you writing like a Riemann sum or something?" A2 responded, "Well, I mean, I was just gonna do an overestimate for like, the big one. So a Riemann sum, yeah, but just like the biggest possible overestimate… To make sure we're ballparking it." I then asked A2 if he was using his K formula for the Riemann sum because if so then the definite integral value will be lower than an overestimate of that value. A2 clarified "I'm just making sure I have my variables correct," indicating he was verifying that the K equation would position him to identify the correct values for the quantities necessary to create an overestimate if they only used a single partition. A1 and A2 went on to discuss what values different quantities in their K equation would take on for an overestimate and underestimate with 1 partition, for example, the dl would be 0.1 "because *dl* is the length of the partition" but they would adjust the l within the speed component. This allowed A2 to conclude,

> So conceptually, I guess we know it's right because it tracks here [referencing the generalized local model expression]… once you have that equation, all you're doing is changing from a Δl to a dl. So because we know [the generalized local model] equation tracks, then we know [the global model expression] is accurate.

That is, the generalized local model A1 and A2 constructed effectively positioned them to be able to identify the kinetic energy of a piece of the rod for a specified partition that is compatible with the structure of the basic model. Therefore including the same expression within the integral, switching $\Delta l's$ to *dl*'s, would identify find the exact value that bounded by those over and underestimates.

Note that A1 and A2's final expression did not measure kinetic energy in Joules, as they measured their time in units of minutes. Due to the time limitation of the session, and A1 and A2's unfamiliarity with common physics units like Joules, I did not view it as productive to revisit this task the following day to have them rewrite their final expression in terms of Joules. I am confident that had I pointed out the error they would have quickly adjusted the 1 minute within their generalized local model, which was still clearly identifiable, to 60 seconds.

Group B's Kinetic Energy Task

When Group B began their kinetic energy task, C2 drew a diagram noting the end of the rod would have more kinetic energy than its base, however, there was tension over the existence of a varying speed. B1 was uncomfortable with partitioning along the rod because the units of 1 rev/min implied a constant speed, so he was "confused on how we find speed." Attempting to coordinate measurements of speed, C2 stumbled over his words, but eventually realized,

> Because like its circle has lesser circumference… Because this is the distance traveled over time. And so I've got to find the circumference of the circle with a radius *r*, and if we call r from there to the blue line. Speed equals distance over time.

Convinced speed was changing, and anticipating making the velocity the variable expression, B1 voiced that the limits of integration should be the minimum and maximum speed rather than lengths of the rod's measure as C2 depicted on the diagram. C2 did not understand B1's intentions. This caused him to discount B1's suggestion, who became frustrated and just supported C2's solution moving forward.

In anticipation of placing the expression he was constructing into summation notation, C2 prewrote wrote $\frac{.1i}{N}$ on the side of the shared whiteboard. This expression represented the Δx quantity of the symbolic template necessary for a Riemann sum $\sum f(a + i\Delta x)\Delta x$. C2 then focused on identifying the function component of this template, f, writing length of the rod as $(0 + x)$ and completed his expression for speed, $\frac{2\pi(0+x)}{1(\text{min})}$ which was simplified to $2\pi x$. Because C2 already accounted for the Δx as an element independent of his f element, he was unperturbed by the lack of Δx in the kinetic energy formula he created, $f = \frac{1}{2}(.03)(2\pi x)^2$. C2 finished his summation and definite integral expressions by adjoining his $\Delta x(dx)$ element and his constructed f(Figure 47).

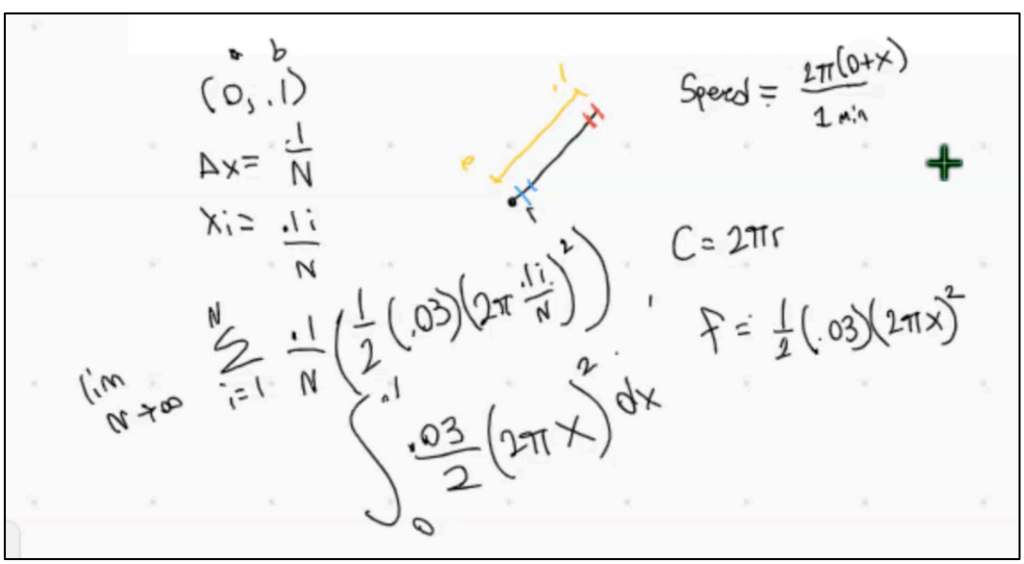

Figure 47: C2's solution to the Kinetic Energy task

I was disappointed that, by framing the steps in the "How-To" guide in symbolic terms such as "write an equation where $f = (\Delta x) * f(x_i)$" C2 lost track of the need for each element in his summation to represent an element of kinetic energy (i.e. that the 'parts' of the whole must be kinetic energy). That is, by focusing his attention on attempting to match the symbolic template of a Riemann sum he was not positioned to observe that appending a Δx to the basic equation could not result in kinetic energy. C2 had shown evidence throughout the teaching experiment of creating generalized local models for which the Δ element represented a quantity within the quantification of a generalized local model. Therefore, I attempted to engage B1 and C2's emergent models without summation notation interfering by appealing to B1's distaste for writing expressions in that manner;

> I know that [B1], I guarantee you probably wouldn't have written this in sum notation at all. So, maybe how would you have attacked this? To see if you would come up with the same answer.

Because B1 did not have a tablet, I wrote the expressions for him as he explained,

> I think I would have gotten the $(2\pi x)^2$ there. Then the mass I would have gotten 0.3 for. And then over 2 because of the one half. I guess I'm confused about the *dx*. I'm thinking about units. Like, we were messing up with the units on the other one. So like, this would be kinetic energy right here. And then adding on a *dx,* I don't know what that does to it.

This observation from B1 was enough to perturb C2 as well, so they both looked at the equation more. B1 then asked for clarification on what "a uniform mass of 0.03 kg" meant. From this suggestion, I inferred that B1 was attempting to identify how the differential, which is measured in meters, could be incorporated into their expression. I explained that a uniform mass of 0.03 kg meant that the rod had the same density throughout, so if you cut it in half each piece would be 0.015 kg. This intervention allowed B1 and C2 to identify that the definite integral expression C2 had written, $\int_0^{0.1} \frac{0.03}{2}(2\pi x)^2 dx$, should be adjusted to $\int_0^{0.1} \frac{0.3}{2}(2\pi x)^2 dx$ to account for the density $\frac{0.3}{0.1}$ kg/m. Recognizing that this second expression was the "correct" definite integral was no-trivial for B1 and C2, because they were not constructing this new global model expression in a vacuum. That is, B1 and C2 were attempting to adjust C2's original expression to represent the correct units, not construct a generalized local model from scratch. So, even though they identified replacing the mass with density $\frac{0.03}{0.1}$ would fix the units, B1 asserted they could not "just throw it in there." As they discussed the issue further, C2 verified the length of an "interval" was dx, but then said "but we can't have more than one dx" which indicated he was attempting to quantify the mass of one piece of one interval of the rod, but the preexistence of the dx at the end of the expression made the 0.3 density and length dx two distinct elements, not a mass. To check I asked B1 and B2 what the mass of a piece of the rod would be, which they verified was $0.3dx$ and then B1 said, "Oh, so then that [second expression] is right." That is, by identifying the structure of the mass component as a single expression he recognized that the two elements, density, and length, separated in the global model expression represented the mass. This was compatible with B1's expectation that the structure of the basic model dictates the structure of the generalized local model.

Before concluding, I informed B1 and C2 that their answer was not actually in terms of Joules. Their inattention to this detail was expected due to their unfamiliarity with physics units, so it was an easy fix for them to make once I made them aware.

As previously discussed, in light of their solution to the kinetic energy problem, and reflecting on how the two different approaches led to two different answers, C2 and B1 returned to their "How-To" guide to rework their reflection on how they solved problems.

C1's Kinetic Energy Task

Recall C1 was introduced to the Kinetic Energy task during his main learning trajectory. During that activity, C1 mentioned that the task was "easier" than it could have been because the rod had a uniform density. He was not sure if he would be able to solve a similar task with a non-uniform density. Therefore, instead of presenting the exact question again, I made a slight adjustment to have the density of the rod linearly increase, 0.05 kg/m at the center to 0.3 kg/m at the end. Because C1 already worked through creating a generalized local model, he did not have to go through those steps again. Instead, C1 began by writing down his basic model expression. His next expression expanded the quantitative relationships within the basic model. That is, he identified a way in which he could quantify mass, a density times a length $(\rho \cdot r)$ and recognized he already quantified the velocity element during his previous problem, $\left(\frac{\pi r}{30}\right)^2$. This new expression engendered a need to identify the quantitative relationship for the linear density. C2 described this process aloud,

> I think this, what this would change… [rereading prompt] okay linearly increasing density with .05 kilograms per meter at the center to .3 kilograms per meter at the

end. So it increases .25 over .1 meter. So, 2.5 is the rate of change there…. Okay, so last time I just had it be the density times the, what's it called? times the slice? Since it's you know, it was uniform. This time, I have to find the specific density. I'm at, you know, whatever point with that find me make sure.

C2's comment that the density expression should identify the density at a point caused him to realize he left off the initial value for his linear expression, which he corrected before moving on.

Finally, C1 created a generalized local model, simplified the expression, and created the algebraic representation of his global model—the symbolic form for the definite integral (Figure 48).

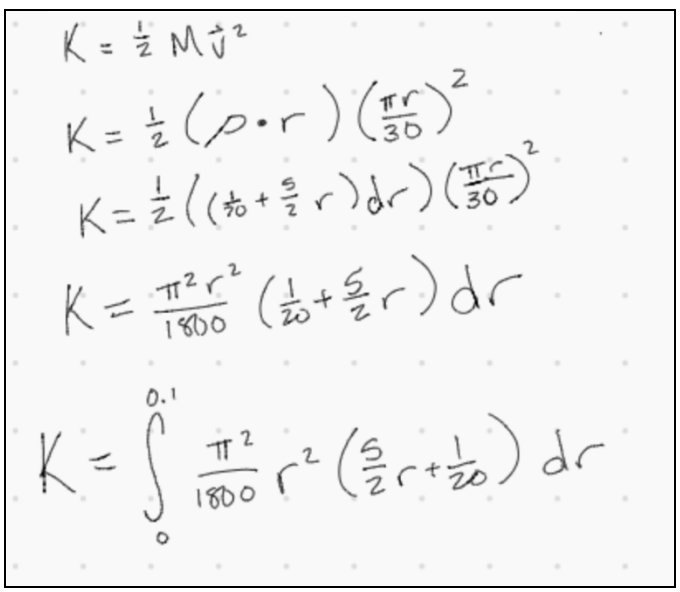

Figure 48: C1's Kinetic Energy of a Rotating Rod board work

C1 took less than 7 minutes to complete this entire prompt including the time to explain his reasoning and correct the density expression after realizing he left off the 1/20. When I asked him why he expressed concern about a non-uniform density C2 replied, I um, once I kind of sat down, looked at it, thought about it for a bit. I was able to figure out what I needed to do, which was, you know, just kind of find the density at each point and then multiply that by dr. Since I knew that the dr was going to go in there, I just kind of tackled it as its own kind of nested problem… I kind of dealt with it like, we were accumulating density first.

C1's mention of "accumulating density first" prompted me to ask if the expression

$$\int_0^{0.1}\left(\int_0^{0.1}\left(\frac{1}{20}+\frac{5}{2}r\right)dr\right)\left(\frac{1}{2}\right)\left(\frac{\pi r}{30}\right)^2$$

would work for this prompt. C1 quickly replied,

> I think the problem with this part is that you have the accumulation inside the accumulation. This right here is just the mass, right? And so, what you have is the accumulation of the mass times, you know, the one half and then the velocity. But it's not going to kind of it, it's going to do the accumulation on the inside before the outside. So every time you're just going to have the full mass.

What was important about this interaction is that it demonstrated C1's ability to (1) identify that the inner integral represented the mass of a rod of length 0.1, and (2) coordinate that if you were to compute that integral to get mass then your next integral would assume a constant mass for each r along its length. Such a quick observation means that he was able to assign quantitative meaning to definite integral expressions that were not constructed using his emergent models, but which he could anticipate would create a conflict with those models. Specifically, each piece of the rod would have the entire mass of the rod ascribed to it. This speaks to the integrity of C1's local model—the global model is not just an accumulation of every individual quantity within the generalized local model expression, the mass within the local model is a local measurement of mass. He also added that the second definite integral would be missing a dr, but if you were to add that dr you would get J·m which is not the units you want for energy. That is, the units of a generalized local model with this structure did not match his expectation that the units should be the same as the global and basic models.

C1 clarified that when he mentioned "accumulating density" he was just talking about realizing the mistake he made deriving $\rho(r)$ by not including his "starting point."

Total Gravitational Force Between a Rod and a Particle

The Gravitational Force task provided new challenges to students by providing variables for quantities that had previously been ascribed a specific value (e.g. instead of a length of 5 meters, participants were provided a generic length l). This provided students an opportunity to generalize their generalized local model construction and would require additional coordination between identifying the fixed and variable quantities across elements of their global model. An additional challenge was that this context was the first time the value of zero would not represent their lower limit of integration.

Group A Gravitational Force task

When A1 and A2 began the Gravitational Force task, their lack of familiarity with the Law of Universal Gravitation resulted in them imposing a sense of movement between the rod and the particle, as if they were being pulled into one another and would eventually collide. An additional challenge this task posed was the lack of values for specific quantities. A2 described,

> Because length is denoted as l, we don't have a measurement for it. So it's almost like, it may not matter. And our mass is also not given for either of them. But I feel like that does matter, and we don't have it. We only have the gravitational constant. Our distance is the measurement that the gravitation all energy will be dependent on…. I feel like I don't know because it's in the F equation. Length isn't, but mass one and mass two are. I would imagine they would matter.

After more discussion, A2 voiced that he "feel[s] like there's a lot of this question that we're just not understanding conceptually." I took this as a cue to introduce the general principles of The Law of Universal Gravitation, and that the objects in this context were fixed in place. Drawing on A2's example about the earth "holding the moon in place" I explained that objects pull on each other, but that the distance between them matters, so the gravitational force between the particle and the closest end of the rod will be a lot stronger than the pull between the particle and the farthest point on the rod. This allowed A2 to recognize,

> l does matter because the way that this whole distance thing works is the particle is a distance away from the rod right now. And we're looking at, like, how, how much gravitational forces is on the particle from the rod at this point? And then this point, and this point and this point.

This indicated to me that had A1 and A2 been more familiar with the physics principles targeted in the question prompt, they would have been better situated to identify the variation inherent in the context. Recognizing that the distance between the two points in question was the varying quantity, A1 and A2 partitioned their rod along its length (Figure 49). A1 described,

> So I think I think what we're gonna end up doing is like, we're going to partition the rod. Right? I think we're on the same page that we're going to partition the

> rod. Yeah. And so it'll be like a really little problem we did whenever we met last, we did, like the length, we had to find like the mass of each part of the partitions, and we wrote a kind of a general thing that could work that could find the mass depending on how many partitions like however long they were. I think we need to do that again, for this equation. And then like, I think you're on the right track. I think we're on the right track was saying like, okay, yes, we're gonna have to add whatever that partition length is, or like, however many however much of the rod we're using, add that to the distance *a*, so what we're just going to add *a* plus whatever the distance of the rod we're using. Okay, but then we're also going to have to look at the mass of the rod in our equation and change that for like, how much of the rod we're using. Does that make sense?

During the interview, I missed A1's language referencing "how much" of the rod they were "using," but it was something to which I should have been more attuned. That is, when A1 and A2 were trying to identify how much gravitational force there was between the point mass and the midpoint of the rod, A1 was envisioning also accounting for the entire first half of the rod in that single additive component of the global model. That the pull of this last little bit in the middle of the rod would push on the preceding half of the rod as well. It is not uncommon for students unfamiliar with the Law of Universal Gravitation to envision this type of compounding process for gravitational force. Had I caught it in the moment, I would have intervened. However, I interpreted A1's description as an indication that she wanted to define the variable and differential quantity to measure changes in the length of the rod, l and Δl, in contrast to A2's descriptions of measuring the situation in terms of the distance from the point mass and the subinterval of the partition, r and Δr40. Therefore I did not interject. What transpired, as a result, was an inability for A1 and A2 to create a generalized representative model because they were unable to generalize across elements of their global model. They could write an equation for the first subinterval of the partition of the rod in their diagram, but when they looked at the next partition element on their diagram they were unable to coordinate a way to account for represent the quantities that were changing and staying fixed as a generalized expression. For example, A1 and A2 wanted to write $(a + \Delta l)^2$ in for the dominator to account for the extra sections of the length being compounded at each step, but also had an expectation that a and Δl were fixed values. This conflicted with a need for $(a + \Delta l)$ to represent a varying quantity. They discussed adding an N into the equation somehow (as a counter to increase the number of $\Delta l's$ to add, but were unsure how that would work within the global model as they had never incorporated an N before. With no way to seemingly add on an extra element of Δl at each iteration, they were unable to make progress.

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40 To quantify this situation in terms of the length of the rod would result in limits of integration from 0 to l and a denominator of $(a + [\text{variable}])^2$ as opposed to in terms of the distance from the point mass which would result in limits of integration from a to $a + l$ and a denominator of $([\text{variable}])^2$.

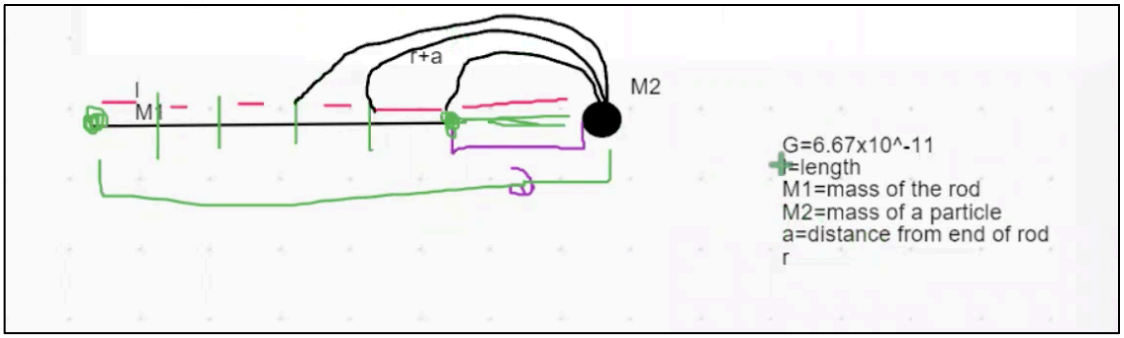

Figure 49: Group A's diagram for the Gravitational Force task

The next day A1 and A2 were talking about their expression and decided to use numbers instead of variables to see if that would help, however, they still ran into the same issue. Using a rod with a length of 5 meters (with 5 partitions) that was 1 meter from the point mass, A2 explained,

> but I think we still have the same issue at the bottom. It's still always going to be one plus one. Like the distance won't change. So I think maybe we do we need to have l? No. I think that's why we talked about switching it to r. Maybe we have to have r in the bottom.

Finally realizing what the issue might be, I asked A1 and A2 what the gravitational force between one subinterval of the partition and the point mass would be, shading in a single subinterval partition on their diagram in blue. They decided to create an underestimate and overestimate using the lines on each side of the subinterval. After creating their overestimate value, A1 voiced that their underestimate expression would be "the same thing. It's just the fact that r would be different." Writing these expressions, A1 realized that this might have been the intent of the task,

> I think yesterday, like, I was confused because I think I was thinking, in my mind that we needed to write an equation that would somehow include the mass of…

> the mass all of this stuff for r equals 3 [shades in the distance from the point mass to the third subinterval of the rod]. And then include the blue for r equals four… But this makes sense. Because it's like, well, we just want for the partitions to do like an accumulation situation. Yeah. I think we can probably adjust our integral now. To say what we just wrote down.

A1 and A2's quick adaptation verified that their ability to reason about this task in terms of their emergent models was not at fault for the difficulty in writing the generalized local model. Adjusting their expression, A2 declared that this solution "deserve[d] a green box."

A1 and A2 then reflected on their "How-To" guide and how they might make adjustments to make it better. A2 suggested, "Do you want to do a step in here somewhere that says, write a hypothetical over-under approximation to help, like, validate the question?" A1 replied, "Yes!" and A2 added "If confused, write an over/underestimate for one hypothetical partition" to their write-up. By this statement, A1 and A2 were recognizing that writing an explicit overestimate and underestimate expression for one subinterval of their partition had been a critical activity in the creation of their generalized local model. They also added "draw a picture," "redraw the picture," and "draw another picture," throughout their steps.

Group B Gravitational Force task

After having made a complete revision to their "How-To" guide for definite integrals, B1 and C2 approached the second clinical interview task without the burden of writing things in summation notation. C2 began by drawing a diagram (Figure 50) and observed,

> Okay, so I feel like what this is asking. So the gravitational force as you go, like from this point [motions mouse curser to m_2] to that point [draws red line closest to m_2] is more than this point [draws second red line] than this point [draws third red line]. And you're way down here [motions mouse cursor to the end of the rod]. So I feel like our integral for finding interval I would say, I don't know. I got it. I got it. I'll also start writing like this. Closest distance a farthest distance is $a + L$. I feel like that would be the interval.

By drawing his diagram, C2 positioned himself to identify where the variation is occurring within this context. Extracting the meaning for gravitational force from the task prompt, C2 recognized the distance between the particle and different parts of the rod was increasing along the length of the rod. This provoked him to envision a partitioning of the rod along its length.

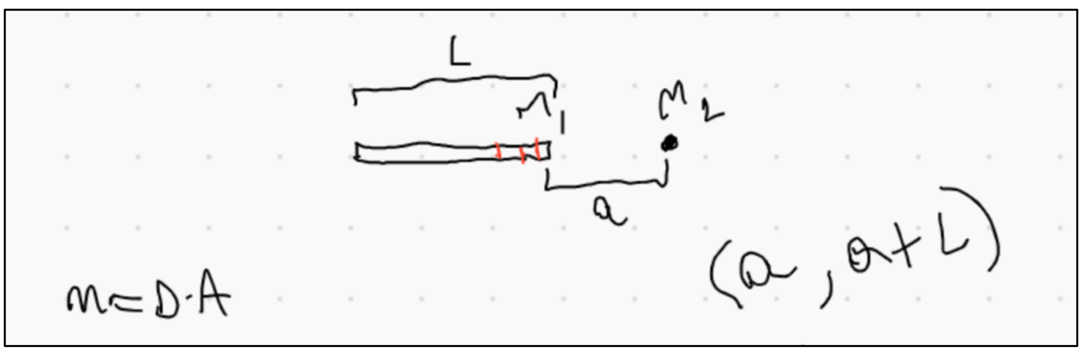

Figure 50: C2's diagram for the Kinetic Energy task

After some discussion, B1 and C2 decide to write keep their variable as an r, with B1 noting that this will capture the whole length of the rod between a and $a + L$. When trying to write their generalized local model, C2 noted he thought they could rewrite M's in terms of r's, cause "mass is just density times volume, or area or whatever" and B1 added that they know density is uniform. An unfamiliarity with linear density resulted in C2 trying to identify an area. Unsure how to proceed, B1 asked "if we throw a *dr* on end, it messes it up doesn't it?" Identifying that such an act would result in the wrong units, B1 said they

> need to relate m_1 to r somehow… wait, would it be m_1 over l times dr… the density times the interval... that'll get you like mass, at a certain instant[41]… that'll make our units correct. I'm pretty sure this is right.

B1 adjusted the m_1 in the integral expression he wrote earlier to

$$\int_{a}^{a+L} \frac{6.67 \times 10^{-11}\ Nm^2/kg^2 \left(\frac{M}{l}\right)\ m_2}{r^2}\,dr\,.$$

When I asked why they were so confident that this was the correct answer, C2 responded that the r was the distance and it would take on all the values of a to $a + L$. C2 continued,

> So it had to be our masses and the mass of the particle isn't going to change. That's going to stay there. So the only thing that's going to change is the mass of each individual section, like the section dr. So, we knew we had to divide the mass divided by the length to get the density, times the *dr* would get us the mass of that section.

That is, their global model was correct because their generalized local model reflected how you would quantify one subinterval the same structure in the same quantitative manner as the basic model.

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41 B1's use of certain instant in this case was referring to a specific subinterval.

Although there are slight distinctions between the way B1 and C2 referred to the partitioned rod, for B1 $\left(\frac{M}{l}dr\right)$ was the mass at an "instant" and for C2 it was a "mass of that section," they were both able to productively reason about the quantitative structure required of the mass element. Because they wrote the $\frac{M}{L}$ and dr separate within their global model I verified that these two elements were supposed to go together. I completed the interview asking B1 why the dr was at the end of the definite integral expression and he replied "I just did that for like uh, that's how it usually looks. Like that's standard, dr comes at the end." When I asked why he thought that was B1 said he did not know, but "it just looks nice" and C2 added,

> In different problems you don't have to necessarily, I don't know, like, volume problems are common sense. You just kind of find the formula put it in there and put dr next to it. Because dr is just like the height. So, that makes sense. So you take the area times the height, get the volume. Problems like this are a little bit more complex.

This statement from C2 is particularly powerful because it (1) indicates a recognition that problems such as gravitational force are quantitatively different than a Riemann Product structure which cannot be solved by appending a differential quantity, (2) that even within a Riemann Product structure, such as volume, the generalized local model still represents a piece of the desired whole, and (3) such an interpretation positioned him to generalize across all volume problems to identify that the differential form is just "area times the height."

C1 Gravitational Force task

Like the previous task, C1 completed this prompt in very little time (roughly 8 minutes total). He began by attempting to apply a general formula using the middle point along the rod before realizing the prompt asked for a definite integral. However, the diagram he drew as part of that process, along with his experience with the quantities involved, allowed C2 to fluidly reason through writing his generalized local model and the definite integral expression required (An important aspect of C1's "times *dx*" is that it was a part of the mass component of his generalized local model, as is evident in his final generalized local model expression. It was not conceived of as an integrand function times a *dx*. C1's symbolic template for definite integrals just carries the requirement to write the *dx* at the end of an expression, which he performs as a later algebraic manipulation, in anticipation of identifying an antiderivative.

> So in that case, I should… force is *GMm* over r^2. And in this case, r^2 would be… a plus, and then that would be L. Our current L, so we'll call that L_k since we're not in integral form yet. And why don't we write this in terms of x. I'll deal with that in a second. In this case, I just kind of want to do a plus, L would be $x\ \Delta x$. Um, no that would be, just equal to x… It would be a plus x, just x should be fine… Yeah, that should do. And then in that case, let's try to get I think capital M was the rod. Yes. In that case, that would be Gm over $(a+x)^2$. Um, and then I want to solve M as being the, um, how do I think about density. So it should just kind of be the same as last time, which would be *M* over *L*, that would be the density. Times *dx*.

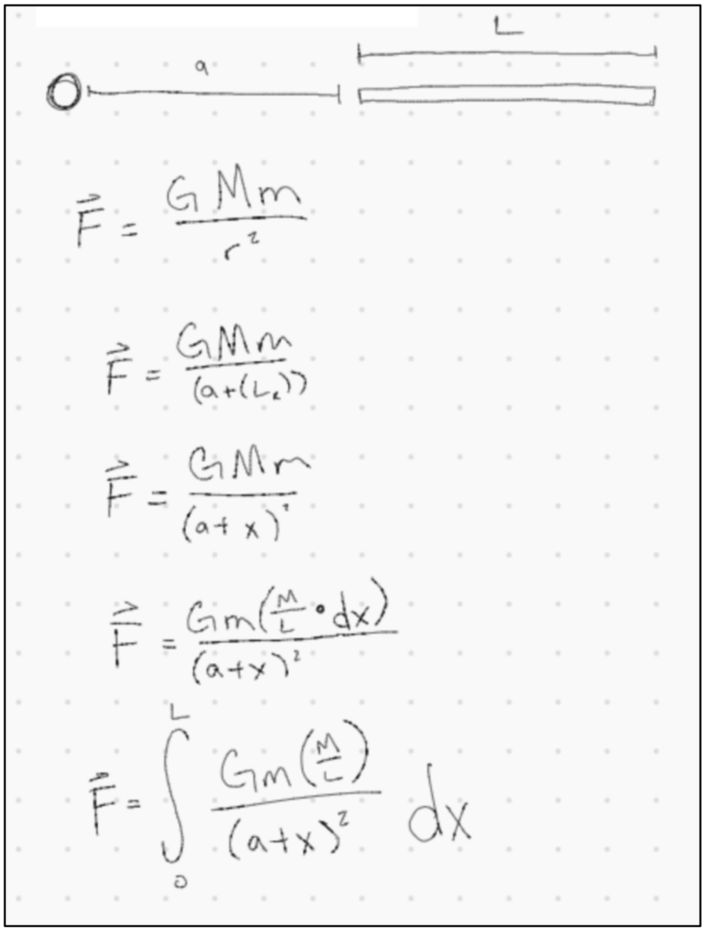

Figure 51: C1's board work for the Gravitational Force task

An important aspect of C1's "times *dx*" is that it was a part of the mass component of his generalized local model, as is evident in his final generalized local model expression. It was not conceived of as an integrand function times a *dx*. C1's symbolic template for definite integrals just carries the requirement to write the *dx* at the end of an expression, which he performs as a later algebraic manipulation, in anticipation of identifying an antiderivative.

Non-Teaching Experiment Task-Based Clinical Interview

Based on previous work, I had a reasonable expectation that a calculus student who engaged in a traditional calculus course that introduced integration in terms of graphical area and antiderivatives would be unsuccessful in solving the tasks laid out in this teaching experiment. Therefore, I made adjustments to the protocol to spend more time attending to participants' (1) schemes for definite integrals as abstract objects, (2) schemes for Riemann Integrals in context, and (3) the impact of 1 and 2 on their ability to productively reason about the Fluid Force on a Rectangular Dam task and the Kinetic Energy of a Rotating Rod task. In the protocol design, I did not include the supplementary activities for the Dam task however I would provide explanations of quantities involved if students (1) asked, or (2) it became clear that a non-standard interpretation or inability to reason about the quantities involved was presenting progress. I recruited 3 participants for this phase of data collection. Unfortunately, only 1 participant attended their interview. Subsequent requests for volunteers did not result in additional subjects. Therefore this section will only reflect the views of one student, D1 who participated in a single hour and a half long interview covering 5 prompts. D1 was a Latinx female who was a mathematics tutor at the university tutoring center. As mentioned, D1 was enrolled in a calculus course with the same instructor as four of the five teaching experiment participants during the same university semester. Therefore, her exposure to topics related to definite integrals at the university level is as close to the teaching experiment participants as feasibly possible.

When asked what the expression $\int_a^b f(x)dx$ mean to her, D1 first referenced that it was an antiderivative, but when explaining her meanings she primarily relied on an area and perimeter symbolic form. That is, she identified $f(x)$ as a prototypical function, sketching graphical axes, identifying the purpose of a definite integral to be finding the area bounded between a line at $x = a$, $x = b$, the graph of f, and the x-axis. The differential did not have any substantive meaning for D1, as she commented that you couldn't leave the dx off of the differential form expression but she was "not really sure why. I just know [the integral sign and the dx] have to go together."

Next, I asked D1 two prompts related to contextual Riemann Integrals:

(1) Watson is filling a huge beaker with water from a faucet. He is playfully turning the facet up and down so that the water's flow rate is continually changing. There is a flow meter on the faucet that tells him this flow rate, $r(t)$, in ounces/sec, where t is measured in seconds after Watson first turns on the faucet. What does $\int_3^8 r(t)dt$ measure?

(2) Acero owns a small maple grove that produces syrup. The rate at which his grove produced syrup steadily increased during the month of April, from 6 gal/day at the beginning of the month to 8 gal/day at the end. Write a definite integral that will identify how many gallons of syrup Acero produced in the month of April. (Note there are 30 days in the month of April).

For the flow rate prompt, D1 commented that normally integrals were about area under a curve, but that "this one was different." Not having a primary scheme to draw on for this task she drew on her two schemes for definite integrals to decide that this expression represented the "change in flow rate." That you are finding the "difference" between $r(8)$ and $r(3)$, and that this "difference" was kind of like an "average." Her use of the term "average" bothered her, and she added, "most of the time when you're taking an integral. It's not going to be something linear or constant." To clarify her meaning, I drew diagrams of two functions with different concavity but equivalent $r(8)$ and $r(3)$ values. I then asked D1 if the averages would be the same value. She replied,

> Um, no, I would think the averages are different because from what I remember when we learned about like the definition of integrals. It was more like the area under the curve. So between the curve and the x-axis and it looks like you have a much bigger area on the right graph than the left one.

That is, the faucet task activated D1's schemes associated with both antidifferentiation, and the Mean Value Theorem for Definite Integrals, $\overline{r}(t) = \frac{\int_3^8 r(t)dt}{8-3}$. Because D1 did not give quantitative meaning to the differential form, i.e. dt was not a base length of a rectangle[42], she was unperturbed by the missing denominator in the provided expression. Cued by the symbolic form of a definite integral to take a difference of values while coordinating a scheme for averages applied to a flow rate expression combined to become a difference of flow rates (i.e. "a change in flow rate"). Careful not to ask more questions that might perturb her schemes, I did not point out any further conflict in this reasoning.

An anecdotal scheme revealed by the syrup task was D1's decision to make the limits of integration go from 1 to 30 which she realized she had done because "there is no day zero" on a calendar. However, she mused that if it had been measured in hours she probably would have put 0. While she demonstrated difficulty writing the linear expression, she knew that the equation she was attempting to construct would represent

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42 Verified at the end of the task. As in the previous task, dt's only purpose was to inform the dependency of your function.

the inside of the integral and would create an upper boundary for the area under a curve the expression would find. This was consistent with her original description of a general definite integral and no new significant schemes were demonstrated.

During the Fluid Force task[43], D1 demonstrated an unfamiliarity with the equations and units for force and pressure. For example, after reading

$$F = P \cdot\ A\ N \text{ (Newtons; } 1\ N =\ 1\ kg/ms^2\text{)}$$

in the task prompt, D1 asked if force was "P times A times N." Therefore, I offered explanations for the quantities involved, including a diagram indicating she only needed to worry about the water on one face of the dam and an explanation that pressure would increase the deeper underwater you were. For this explanation, I drew on A2's diver analogy to provide a context she might be more familiar with.

As she thought about the prompt to find the fluid force, D1 observed, "I feel like we could solve this without an integral" and wrote the expression of the basic model with the largest depth in place of *x*. This implied that she had associated the task sequence with definite integrals but this problem did not conform to her schemes for the appropriate use of a definite integral. When I asked D1 why she did not need an integral, she expressed, "it already gives us the equation for force," so she just needed to "plug in all the knowns." When I asked if we cut the dam in half whether the top half of the dam and the

43 Note that D1 was measuring the force on a dam measuring 50 meters tall by 75.5 meters wide. This was an oversight from pulling the prompt wording from the summer 2020 interview protocol.

bottom half of the dam would have the same force acting on it, she replied that if “the water pressure is still the same, I would think yes… otherwise you could just change the formula.” Her mention of pressure being constant indicated that, despite my brief explanation that pressure increased with depth, her unfamiliarity with this quantity persisted and she was still unable to recognize the variation within the context.

To introduce the idea of variance, I asked D1 about what the expressions would look like for the top half of the dam and the bottom half of the dam. In her initial explanation, she said that the “equations would stay the same” but that you would need to change the area to reflect the new height of 25 meters instead of 50 meters. I then directly asked her about the pressure element of her expression, drawing a purple arrow below the 50 in her first expression, and whether that would need to change. D1 responded

> I guess that would change. So for the top half it would be, I guess if we’re splitting it down the middle it’s only 25 meters down from the surface, then in the second one, I guess it would have to be 50 because it’s the full 50 meters below the surface.

I then wrote two expressions below her original expression to represent the quantification she just described: $9800 \cdot 25 \times 25(75.5)$ and $9800 \cdot 50 \times 25(75.5)$. D1 acknowledged “I guess they would be different forces.” I then asked D1 if the fact that these were different forces was something she needed to attend to or if it was taken care of automatically in the first expression, $9800(50) \times 50(75.5)$. She replied, “It seems like they're kind of averaged out. I would say it's the pressure on the whole thing, not just the bottom specifically or the tops specifically.” This provided further evidence that the reason D1 did not see a need for an integral was because the force for formula would somehow account for these differences. While the prompt did state that the formula for force only worked for a uniform pressure P, D1 either did not notice or had assimilated that into an image of "continued" pressure in which the water was always pushing against the dam.

Interested in how D1 would attempt to solve the problem if she had she been cued to integrate via variation, I used an on-screen calculator to compute the values of the three expressions we discussed and pointed out that the sum of the two expressions for the dam thought about in two different pieces did not provide the same value as her original expression. Bothered, D1 replied that she thought that the equation for force should have somehow "averaged" things, but she was "not sure why" this did not work, but "maybe if we take the integral of that whole thing" it would work. However, looking at the expression she had written with a 50 in for pressure, she said "but we don't have a variable." She then started to examine the expressions I had introduced below her original equation and noticed which quantities were changing and which were staying the same, drawing black lines below them (Figure 52). By looking across the expressions for different partitions of a global model, D1 wrote the expression,

$$\int_0^{50} 9800d \cdot h \, 75.5 \, d.$$

As she wrote D1 identified one of the variables as acting more like depth, *d*, and the other as acting more like height, *h*. As she was writing the beginning of the differential *d* at the end of her definite integral expression D1 commented, "but now I don't know what to take it with respect to." D1 pondered this for a while but concluded that she was “stuck” because she did not know which variable to take the antiderivative with respect to.

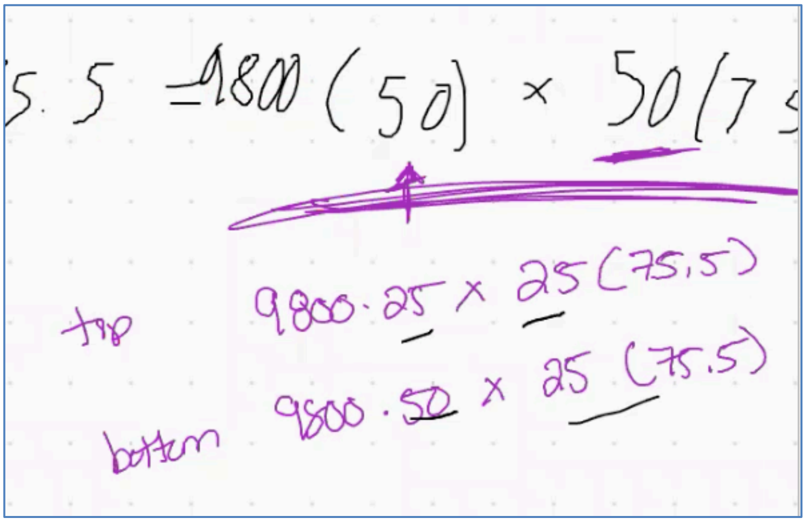


Figure 52: D1’s observation of changes across different partitions of a dam.

I did not pre-plan to introduce the notion of generalizing across different partitions into this interview. By writing expressions for the top part of the dam and the bottom part of the dam, I was simply attempting to position D1 to recognize that the equation for force would not automatically account for the variation in depth. However, by matching the structure of her original expression I also positioned D1 to recognize which elements in her expression would be “changing.” Based on D1’s solution, in the next task, I hypothesize her strategy would have been to write an expression equivalent to,

$$\int_0^{50} 9800x[\text{some expression for height: likely 50 or } x]\, 75.5dx.$$

However, attempting to incorporate her generalizations across different partitions of a global model into her already existing schemes led to a conflict of too many changing values which she could not reconcile. Because her schemes for definite integrals did not ascribe a meaningful quantitative purpose to the differential notation, D1 was not able to coordinate that one of those variables would be her differential element. This supports the need for the creation of a generalized local model to be a fundamental part of the genesis of students' conceptions of a definite integral and not an aspect that can be trivially incorporated at a later time.

When D1 began the Kinetic Energy of a Rotating Rod task, she quickly noticed that the units of 1 rev/min did not match the units for time provided in the kinetic energy formula, so she converted 1 rev/min to 1 rev/60 s. D1 then commented that her "instinct" was to "just solve this with what we have" but that she "has a feeling we're gonna have to take the integral somewhere." Listing her observations, she noted "mass is gonna be constant," "the length is gonna be constant," "that's moving at a constant speed too," and "we don't have anything that would tell me that we have bounds to take," concluding that "everything is going to be constant the whole time, so there's nothing to plug in. No variable that'll change." When I asked D1 why she mentioned using an integral she confirmed that it was just the previous tasks that prompted even looking for something that would change.

I asked D1 to explain a little more about what she meant about the rod moving at the same speed and she explained, "Well, I guess if it's like if we're thinking of it as like a hand on a clock rotating. And it does tell us it's a constant speed, so it'll be constantly moving." This once again indicated that if you are unable to recognize the variation in a task, integration schemes will not be activated. This was not unsurprising, so I asked D1 if the outermost half of the rod was moving at the same speed as the innermost half of the rod. Recognizing what I meant D1 observed, "No. So, I guess that's going to be our changing variable." D1 then drew a diagram of a circle with a line from the center representing the rod, noting it is length. Circling the 0.5 m on her diagram with her mouse curser, she said “that’s what’s changing,” and proceeded to write a definite integral expression consistent with an Integral as a Transformer conception. That is, she interpreted a definite integral as a way to transform the basic model for kinetic energy into an expression that would capture the variation along the rod. To accomplish this, she placed the formula for kinetic energy into the integrand of a definite integral expression (Figure 53).

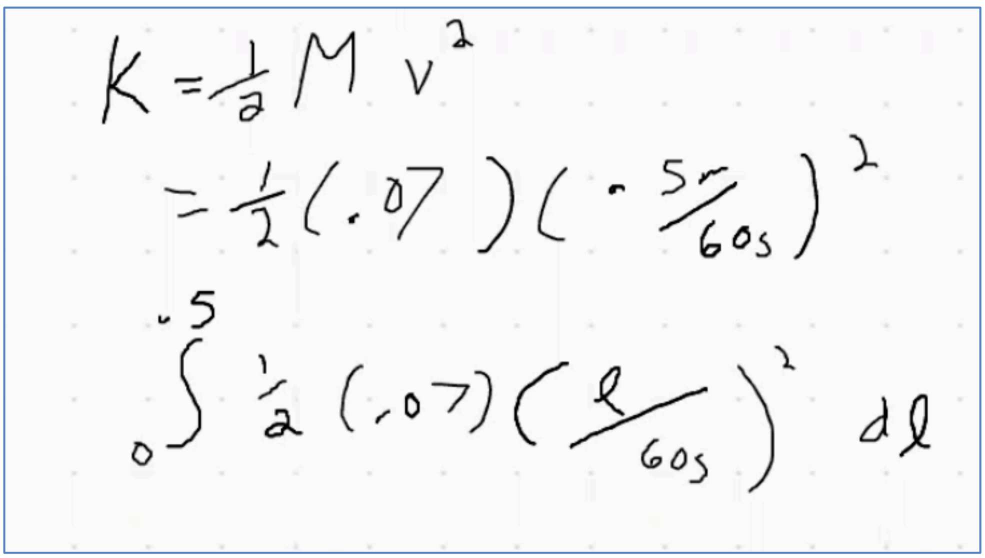


Figure 53: D1's board work for the Kinetic Energy of a Rotating Rod task

Implications

As I have demonstrated, all groups in the teaching experiment were successful in modeling definite integral tasks in which the differential form was not a Riemann Product. In claiming these students were successful, I am not suggesting these tasks, nor those in the teaching experiment itself, were non-problematic. To verify that students developed a quantitative understanding of definite integrals which would enable them to solve tasks in which the differential form is not naturally composed as a Riemann Product required the final tasks to introduce problematic elements. Similarly, due to the complex interrelationships which compose students' basic local and global models each task in the teaching experiment needed to problematize at least one aspect of their emergent model system. What I am claiming is, except for C2's initial response during the kinetic energy task, each participant was able to recognize when a global model expression was incorrect and could draw on the relationships between their basic, local, and global models to effectively quantify the scenarios presented to them. The largest difficulties presented by these tasks were the unfamiliarity of the physics quantities (and corresponding context) and recognizing variation when those quantities appear to be "constant."

The interview with the student who was not a part of the teaching experiment also offered insight into how students might approach these tasks without having engaged in the learning trajectory laid out by this study. I was not surprised D1 was unable to correctly solve the tasks which were presented to her. A long history of research shows that D1 is not alone in having a nonquantitative interpretation of differentials and primary relying on antiderivative and area under a curve schemes of definite integrals which have been shown to be inadequate for supporting students' ability to reason about tasks of this nature. However, D1's attempt to incorporate the mental activity of generalizing across elements within a partition and between two different partitions into her preexisting schemes for integrals indicated that this would be a non-trivial accommodation requiring an equivalent intervention to the hypothetical learning trajectory designed for this study. By not ascribing a meaningful quantity to the differential, D1 was not positioned to coordinate what those differences between over and underestimates might indicate about the structure of her differential form. This suggests that the development of a generalized local model should be a fundamental aspect of the development of students' schemes for definite integrals, rather than an addendum.

Summary of the Evolution of Students' Emergent Models

Throughout the design, implementation, and analysis of this study the significant foundational aspect of the students' emergent models was the relationships between models—not necessarily the models themselves. These relationships co-evolve, resulting in highly personal model systems.

One benefit of these results is the variety of calculus backgrounds of the participants. C1 came into the teaching experiment with a robust pre-existing system of models associated with Riemann Integrals. A1 and A2 had taken a calculus course before but reasoned about definite integrals in primarily procedural terms at outset of the study. B1 and C2, who were grouped in week 3, had never taken a calculus course. This level of variation allowed me to understand nuances of students' emergent models as they drew on different sets of incoming schemes. In the following sections, I will describe constructs that proved to be important aspects of the evolution of these students' emergent models.

Assimilation of a Global Context

An expected result of the study acknowledges that the fundamental ways students assimilate or accommodate the provided context greatly influenced their productive engagement in the tasks. Many of the tasks within the hypothetical learning trajectory involve physics contexts with quantities and expressions that are not familiar to all calculus students. For one student, prior physics coursework provided him alternate avenues for reasoning about the physical contexts themselves, such as drawing a force vs. distance graph to identify variational relationships. This background also presented challenges. For example, when tasked to identify the energy to build a column, this participant could not identify any meaningful movement within that context. While he could acknowledge a superimposed aspect of movement, he was derailed by the concern that you “wouldn’t build a column like that.” That is, his understanding of the underlying principles of energy subdued the activation of a global model.

Alternately, one group had little experience with physics contexts. When tasked with identifying the total fluid force on a rectangular dam, this group did not interpret the context in an anticipated way, as a fixed height of water behind a dam where the increasing pressure along the depth imposes different levels of force on the dam at varying depth. Instead, they assimilated the context to their own experience with dams, in that water levels do not remain fixed–they rise and fall. With this image of the context, an overestimate would measure a ‘full’ dam, while an underestimate represents an ‘empty’ one. With this interpretation of the global context, if you have not already constructed a system of models for a definite integral, what exactly is there to partition? How would you develop a local model?

I took care throughout the study to recognize issues with students’ familiarity with various quantities early and provide supplementary materials. However, the issue remains a critical aspect of students’ ability to develop desired aspects of an emergent model, such as a local model construct.

The Role of Variation

Within this study, variation played a number of different roles in students' reasoning in terms of their basic, local, and global models.

First, and fundamentally, how a quantity is changing within a basic model promotes the need for a method of estimation. This means that the variation of a quantity within the context should serve as a cue for the invocation of a global model. While later tasks in the sequence were positioned to engage students in this activity, due to their association with me, their partner, and the task setting, I cannot be sure if a quantity changing prompted the need to integrate. For similar reasons, I chose not to 'check' if variation cued the need to integrate during the final interview setting. As an intentional design choice, early tasks explicitly prompted participants' estimation activity. For tasks in which variation was not explicitly visible, this prompt served as a cue to identify a varying quantity. This suggests that variation and their model systems are connected, however, for my specific participants, I can only prove this relationship one way. Intertwined with this notion is the perceived lack of variation (e.g., the energy task I described in the previous section). When students were unable to identify a varying element within the context, their global and local models were only cued by the perceived expectation of the task sequence.

Variation also serves as a meaningful way with which to choose a partition (i.e. the shape of the local models). Global model partitions must be made in such a way that the increase in the number of partitions, or decrease in the magnitude of a local model, results in a more accurate estimate. There is more than one way to achieve this relationship. One method focuses this construct within a relationship between the basic model and local model, where a local model's size can be made so small the varying quantity is of negligible variation. In this study, I engendered this same meaning through a process of increased refinements to the global model which reduced the corresponding over and underestimates for that global model. This created a sequence of over and underestimate values that converged to the ‘real’ value of the desired quantity. This design was in service of supporting students’ image of a definite integral as the limiting value of this sequence of estimations. This means that the variation relationship my students coordinated was intertwined within the refinement relationship between local and global models.

Refinement Relationships - The Issue with Area and Volume

As described, a refinement relationship between local and global models was a specific target of the hypothetical learning trajectory. In early tasks, a refinement process was imposed on students to engender recognition that additional data reduces the discrepancy between their overestimate and underestimate values. This positioned students to later coordinate a partitioning act to produce a similar overestimate and underestimate relationship for their global model which was in service of coordinating an image of a definite integral as the limiting value of the refinement process.

One interesting consequence of my decision to place the focus on global accuracy as a relationship between global models and local models (as opposed to local models and basic models) was a nonstandard application of rectangular prisms as local models for identifying the volume of a sphere. Because these students justified their choice due to the previous tasks in the sequence (rectangles were used to estimate the area of a trapezoid and rectangular prisms for the volume of a pyramid) I did not immediately recognize the implications[44]. My analysis revealed an interesting issue. By having students construct their emergent model systems with a focus on the whole, they weren't positioned to immediately recognize that their square bases could never approximate a circle. This was compounded because students do not naturally draw on rectangles for their basic models of area. If students know a formula for computing a more complicated area or volume, they often draw on that equation as their basic model. For instance, all students engaged in the trapezoid dam task first used the volume of a trapezoid to measure the area of their local models. This means care must be made to (1) include coordination that a refinement process must also result in less error between the local model and basic model, (2) support an accommodation to the image of a basic model for volume to identify the 'varying' element of the shape in question, (3) an eventual adjustment to their local model to fit the Riemann sum structure, or (4) some combination thereof.

44 Although the other two groups completed this task, their honors calculus course was ahead of the primary calculus sequence and covered the same context the day before presented with the Volume of a Sphere task. This resulted in a primarily re-presented approach from both groups.

CHAPTER VI

DISCUSSION

This dissertation study was developed to characterize the development of a quantitative understanding of definite integrals which positions students to productively engage in definite integral tasks in which the differential form is not a Riemann product. To meaningfully capture such a development, I drew upon the design heuristics of Realistic Mathematics Education and Constructivist Teaching Experiment methodology to create a hypothesized learning trajectory aimed at engendering a Quantitatively Based Summation conception of definite integrals. Before conducting the primary 8-week study, I created a preliminary conceptual analyses and hypothetical learning trajectory and conducted an exploratory study to refine my hypotheses. The results of the exploratory study revealed unanticipated difficulties students faced during their construction of schemes related to definite integrals, such as continued difficulty constructing local models and conflation between sum and integral notation. Informed by these findings, I further clarified the conceptual analysis and redesigned the hypothetical learning trajectory to explicitly target the mental activity identified as productive in the exploratory study. This careful preparatory work resulted in the refined hypothetical learning trajectory which positioned 5 freshman calculus participants to successfully develop a quantitative understanding of definite integrals through their engagement in an 8-week teaching experiment. To accurately characterize students' model development, I supplemented the paired group task-based interviews with individual follow-up sessions, allowing further refinement and testing of my evolving second-order models of the participants' reasoning. Through careful ongoing and retrospective analysis, I was able to characterize students' emergent models for definite integrals throughout various stages of their development. Due to the diverse schemes demonstrated by participants at the outset of the teaching experiment, from no knowledge of integration through robust schemes for Riemann integrals, I am able to provide a refined hypothetical learning trajectory and implications for instruction which proved successful for a wide range in students' pre-existing calculus knowledge.

In this chapter, I begin by outlining two contributions to the Emergent Quantitative Models framework which (1) provided useful analytical insight into the evolution of my participants' emergent models, and (2) served as critical aspects of the mental activity which promoted their development of a quantitative understanding of definite integrals. Next, I provide an overview of adjustments to the hypothetical learning trajectory as a result of the teaching experiment with notes regarding implications for instruction. I conclude the discussion chapter with an outline of implications for future research.

Contributions to the Emergent Quantitative Models Framework

Recall that the Emergent Quantitative Models framework, which characterizes my design and analysis, relies on three quantitative schemes, basic, local, and global models, which students draw on when reasoning about definite integrals;

> The basic model represents the quantitative relationship which would apply to the situation if the quantities involved were constant values, the local model is a localized version of the basic model applied to a sub-region of the original situation (typically within a partition), and the global model is derived from an accumulation process applied to the local model, whose underlying quantitative reasoning is encoded in the differential form. (Simmons & Oehrtman, 2019).

The iteration of the framework I drew on for the design of this study was developed as a method of characterizing students' basic, local, and global models for definite integrals. While this framework described relationships between model systems as highly interconnected, the constructs of these models and relationships within the framework had already been developed through prior research. Therefore, the analysis of this dissertation characterizes additional nuance to the co-emergent influences as students' models are in their initial stages of development. In this section, I review two emergent model constructs established in service of describing the results of this study, gross basic models and generalized local models.

Gross Basic Model

I created the gross basic model construct to engender students' goal-oriented activity towards identifying estimations for a whole through the direct application of a basic model. In support of generating a global model as an accumulation of elements of the same quantitative type as the basic model, I engaged students in the act of progressive addition of gross basic models to create a global whole. This positioned students to create a local model, as a new construct, through an accommodation of applying a gross basic estimate to a partitioned element of a global model. The result of this design choice, and the development of a gross basic model, offered affordances and limitations to students' reasoning as they progressed throughout the rest of the task sequence.

(1) When asked to identify "an overestimate and underestimate" only one group spontaneously created a local model, and this was a result of a previously-constructed scheme for Riemann sums. This was not an unexpected result, as neither the hypothetical learning trajectory nor the gross basic model construct was designed to promote this development. However, it is a limitation of employing a gross basic model early in a task sequence.

(2) Due to a gross basic model's cultivation as a way to estimate when one element of a basic model was non-constant, its activation through a request for estimation prompted students to look for variation within the context. For some students, a prompt to seek 'something' to estimate aided in their ability to recognize the intended variation within the task context, while for others it imposed a need to ascribe an image of variation to that context.

(3) Participants showed evidence of extending the quantitative structure of their initial global model (created through progressive addition) to a global model comprised of local model elements. For example, students demonstrated a propensity for writing a global model for the fluid force on a dam as [force]+[force]+[force]+…+[force], either horizontally or vertically. This positioned students to make comparisons across elements of their global model to recognize which components of the quantitative relationship of force remain fixed and which varied and supported the productive development of a generalized

local model. Although I describe this construct in more detail in the next section, in short, a generalized local model is a result of the mental activity students engage in as they generalize the structure of a local model across elements of their global model. While the desire to create a generalized local model is often initiated through the goal-oriented activity of creating an explicit formula that can represent any element of a global model, including a change in partitioning, an algebraic representation is not required. Students can also engage in this same mental activity as they coordinate various actions, decisions, and observations across instantiations of their global model.

One unanticipated result of the introduction of a gross basic model was one group's extension of identifying a gross overestimate to create maximum and minimum estimates throughout the entire task sequence. In the Fluid Force on the Dam task, the gross basic model and global model with 1-partition took on the same algebraic structure, providing worst-case and best-case scenarios, which served as a method of checking whether their quantification of subsequent partitioning(s) of the global model did not extend beyond those initial bounds. When this group moved to contexts where the algebraic representation of a gross basic model and a global model with 1-partition were no longer identical, such as being asked to explicitly estimate the volume of a pyramid, they extended their gross basic model by identifying an appropriate local model, a rectangular prism, and identified their bounds using a single partition.

Generalized Local Model

I identified a generalized local model to be a critical element in the evolution of students' emergent models. This construct is a result of the mental activity students engage in as they make comparisons across elements of their global model to coordinate which quantitative components vary and which remain fixed. In this study, the genesis of a generalized local model emerged from students' need to create an explicit formula for their local (or global) model which would allow them to identify the value of any element of a partition, for any size partition, in service of identifying the number of partitions necessary to be within a given tolerance. The mental activity required for such an activity is cognitively distinct from that of computing explicit values for partitions through a measurement process and is at the crux of the difficulty in students' ability to productively model complex quantitative situations using definite integrals. If students cannot coordinate across partitioned elements to identify those components of the quantitative structure which vary or remain fixed, they will not be positioned to productively identify the variable and differential quantities for a differential form. Perhaps most critically, the results of the task-based clinical interviews suggest that a generalized local model is most productively quantified when only considering a local model. That is, whenever students attempted to write an expression for a differential form from a perspective of a global model, either implicitly or explicitly, they were less successful in identifying the differential quantity.

The most obvious outside behavior students engage in as part of their development of a generalized local model is the algebraic representation of this activity. However, as a generalized local model becomes a more engrained aspect of their emergent model system, students draw on this same mental activity to anticipate the applicability of a partition for a global model. That is, if participants cannot ascribe a way to meaningful notice variation across a partitioned quantity, then they will look for an alternative partitioning process. While this activity aided students in their ability to identify viable partitions, it also faced limitations. For example, in one group's final interview an unfamiliarity with the units of kinetic energy for a rotating rod resulted in an inability to reconcile their choice of partition with a lack of variation. That is, they wanted to partition the length of the rod, however, from their point of view, every element of that partition would be moving at the same "speed" of 1 revolution per minute. Therefore there was no variation across elements to capture with their estimation process. This conflict prompted a need to seek alternative partition options, such as creating concentric annuli. To allow this group to move on, I had to refocus their attention on the required units of the velocity element to allow them to reconcile their conflicting image of the partitions covering different distances but all traveling at the same "speed."

The hypothetical learning trajectory task sequence and supplemental materials were designed to engender the development of generalized local models as part of their emergent modeling, serving as both a "model of" their prior quantitative reasoning and as a "model for" subsequent generalized activity. As described, the employment of a gross basic model positioned students to write expressions in a format that supported their future generalization activity. That is, by explicitly writing the quantities which compose each element, students were positioned to coordinate which quantities varied or remained fixed across elements of the global model. Additionally, the employment of tools, such as spreadsheets and the GeoGebra sum calculator, situated students to keep their primary focus on the quantitative structure of their expressions. This allowed students' problem-solving effort to be engaged exclusively in the activity of creating a generalized local model.

During its conception, the task sequence was designed to lay a solid foundation for the creation of an explicit local model formula. The critical nature of the generalized local model reemphasizes this need. As a part of the study, an issue arose in which more than one student simplified their computational expressions which added difficulty to (1) coordinating the components of the local model remained fixed and variable across elements of a global model, and (2) superimposed different quantitative meaning for rearranged elements during this generalization process. This supports the need for algebraic expressions to be left in terms of the individual values which constitute the quantitative relationship of the local model. Such activity allows students to coordinate the components of their local model with the structure of the basic model and supports their eventual ability to interpret other definite integral structures in terms of those expected placeholders for quantitative relationships. Conversely, the rearrangement of elements within the differential form can result in students divorcing the differential from its quantitative interpretation.

Hypothetical Learning Trajectory Adjustments and Implication for Instruction

The mental activity required for the development of a quantitative understanding of integrals was based on a substantial body of preliminary groundwork, however, the teaching experiment identified significant improvements that could be made to the hypothetical learning trajectory. While many of these adjustments have been mentioned at various points throughout this dissertation, I will explicitly readdress my suggestions below:

Curiosity Rover Task

The Curiosity Rover task served as an important introduction to the overall task sequence to support students in developing a conception of a global model as an estimation produced through the progressive addition of gross basic model estimates. By providing limited data, the task successfully oriented participants' mental activity towards the actions necessary to construct gross basic and global models even when schemes for Riemann sums and antidifferentiation were activated. However, a great deal of scaffolding can be removed from this task. Specifically,

(1) Remove introductory tasks. Part 1 of the Curiosity Rover task was devoted to ensuring participants understood the quantities involved and the GeoGebra applet. These questions are best utilized as supplemental questions instructors can introduce if students demonstrate difficulty with the quantification of dust accumulation.

(2) Only ascribe the terms "best-case" and "worst-case" to global models for entire contexts, not subparts. Creating this association between the largest and smallest values an estimate can take on supports the creation of initial boundary conditions.

(3) Remove the prompts requesting recommendations for NASA. These prompts were included in the teaching experiment to provide participants an opportunity to display pre-existing schemes and did not result in significant advancements in their model development.

Fluid Force on a Dam Task

The primary function of the Fluid Force on a Rectangular Dam task served to (1) promote the creation of a local model, (2) cultivate a local-global model refinement relationship, and (3) promote the development of a generalized local model. The critical nature of a generalized local model suggests the inclusion of a prompt specifically devoted to the explicit creation for this expression. Therefore, after the prompt requesting students identify how small a partition needs to be to find estimates within 50,000 N of the real fluid force, the following prompt should be added to the learning trajectory:

> Create an expression that will allow you to identify the fluid force acting on a generic piece of the dam if it were cut up into N pieces.

Although students will likely develop their own notation for this prompt, the introduction of the GeoGebra sum calculator promotes a transition from their mental activity into a standardized notation which will support their successful progression to integral notation.

The Trapezoid Dam task was more problematic during the teaching experiment. This task served to engage students in the creation of a second generalized local model. However, by defining the top of the dam to be the longer base, participants were forced to coordinate overestimate and underestimate values using different sample points for each subinterval. Mathematical structures of this type are worthwhile to investigate as part of an overall learning trajectory for definite integrals, however, its inclusion so early in the task sequence obscured the primary purpose of this task. Therefore I suggest defining the top of the dam to be the shorter base and to request students identify overestimates and underestimates using a partition with 8000 pieces. Additional scaffolding should also be included to support students in adopting rectangles as an appropriate quantification for the area of each subsection.

Geometric Volume Tasks

As designed, the two Geometric Volume tasks served to support students' transition to definite integral notation. By activating their local-global refinement relationship and drawing on their expectation that a geometric shape has a precise, exact, volume, I positioned students to coordinate the definite integral with the limiting value of their over and underestimates. This series of tasks also introduced new basic models. I would make the following adjustments:

(1) Remove the expression for the volume of a pyramid from the main context prompt—it was unnecessary and caused confusion about the goal of the task. This expression should be provided at a later point as a means to 'verify' their limiting value.

(2) Remove the word "partitioned" from the introduction to the definite integral notation handout. "Partitioned" is not consistent with the conceptual analysis' characterization of the limits of integration.

The data supports the Volume of a Pyramid task as an effective transition to definite integral notation due to participants' familiarity with exact expressions for volume. However, instructors should be aware of the possible implications of using volume contexts for students' first experience with integral notation. Participants with previous calculus experience demonstrated nuanced changes in their language precisely at the introduction of definite integral notation. Specifically, Δx was a length, but dx was a "change in" a variable or simply a demarcation for the end of an integral. This is a non-trivial shift in reasoning and requires students to be confronted with their two separate meanings. This can usually be accomplished by introducing the question "Δx was a [quantitative description] is dx a [same quantitative description]?" Additionally, a new prompt should be appended to this task which asks students to explicitly classify each component of both their generalized local model and their definite integral expression. Prior to moving on, an accommodation to their scheme for dx to incorporate their current image of Δx must be made.

Instructors should also be prepared to intervene if a collapse metaphor is evoked (i.e. recognizing language about "adding areas"). Supported by his basic model (volume of a rectangular prism measured through an aggregation of area along the height) one student demonstrated such an understanding of volume integrals. Through the activity of creating a sequence of converging values for over and underestimates of the volume of a pyramid, students can be productively positioned to develop an image of a definite integral as the limiting value of a sequence.

Energy to Build a Pyramid

I would not make adjustments to this task. The Energy to Build a Pyramid task engaged participants in the activity of constructing a definite integral in which the differential form was not necessarily a Riemann Product. It also positioned students to draw on their previous problem-solving activity to support their recognition that the differential quantity was an element of the volume component of their local model.

Grading Definite Integrals

The Grading Definite Integrals task was designed to promote students' coordination between the quantities in a definite integral expression with the quantities of a basic model. Although this task did offer affordances I would replace it with the Supplemental Task: "Does this integral also measure the energy to build a pyramid?" I would only include the integral constructed using x as either a side-length or as a height dependent on students' solutions. This supplemental task served the same hypothetical purpose of the grading definite integrals task with two added advantages; it

(1) did not require students to quantify a new definite integral expression. At this stage of the learning trajectory, additional tasks were not necessary for generalization across context.

(2) is positioned to reveal a misapplication of the partitioning process and provide instructors an opportunity for intervention. This could be accomplished by including additional prompts to the task requiring students to draw a diagram and label the relevant quantities on both the diagram and the definite integral itself.

Additional tasks requiring students to evaluate preconstructed definite integral expressions would pair nicely with this task. In particular, including an incorrect quantification of a differential form. For example, the expression $\int_0^{10000} \rho(r) 2\pi r dr$ from the Mass of Oil Slick task can provide an opportunity to reinforce the need for the differential quantity to represent a quantitative component of the generalized local model.

Kinetic Energy of a Rotating Rod

As discussed in the results section, some students enter a calculus course with robust schemes for Riemann sums and Riemann integrals which can be resistant to accommodation. Therefore, before engaging students in generalizing across contexts, I recommend including a definite integral task that cannot be trivially conceived of in terms of a Riemann product such as the Kinetic Energy of a Rotating Rod task. My data suggested that this task was not required for the development of a quantitative understanding of definite integrals, however, it served as a task that provided a clear necessity for reasoning about definite integrals in a quantitative way rather than in terms of antidifferentiation and area under a curve.

Design a "How-To" Guide

I would only make surface-level adjustments to Design a "How-To" Guide task. This prompt served as an important capstone activity by engaging students in the active reflection and generalization across their problem-solving activity. To better position students to engage in this activity, I would remove the constraints of having them provide explanations to someone who has never had calculus before.

Future Work

As I began this study, I set out to answer two questions;

> RQ1.) How might students develop a quantitative understanding of definite integration in a Calculus I course.
>
> RQ2.) What are the limitations and affordances of a quantitative understanding of definite integration? In particular, how does a quantitative understanding of definite integration impact Calculus I students' ability to reason about physics-

based integration tasks in which the varying quantity is not a rate of change or density function?

I was able to answer these questions by designing, testing, and refining a conceptual analysis and hypothetical learning trajectory which was shown to promote the development of a quantitative understanding of definite integration. While some participants faced difficulty using physics quantities they were not familiar with, all participants were able to write correct definite integral expressions for novel physics-based integration tasks in which the varying quantity was not a rate of change or density. Although this learning trajectory was successful in engendering a quantitative understanding of definite integrals for my participants, I do not assert that it is the only path towards a quantitative understanding of integration. However, one benefit of this study was the wide variation in the participants' prior experience with definite integrals, suggesting curriculum founded on this conceptual analysis and hypothetical learning trajectory has the potential to productively engage a wide range of students in the activity necessary to construct a quantitative understanding of definite integrals.

Despite these results, there is more work that can be done to support students' development of quantitative reasoning for integration tasks. First, in service of promoting students' ability to productively construct a generalized local model, my hypothetical learning trajectory did not emphasize the basic-local model refinement relationship's importance in the overall accuracy of a global model. This refinement relationship between basic and local models is an important cornerstone of the emergent model system, and therefore its inclusion into the overall learning trajectory needs to be carefully orchestrated. The supplemental activity developed during the teaching experiment to address the basic-local model refinement relationship was not as successful as I would like. Therefore, moving forward, I plan to conduct additional teaching experiments with more meaningful incorporation of tasks and sub-prompts devoted to cultivating this relationship.

I also plan to revisit the dataset with an eye towards reflective and reflected abstraction. When designing this dissertation research I anticipated the need to actively engender the development of specific schemes, therefore I drew on constructivist epistemology for the teaching experiment methodology. However, due to the co-evolutionary nature of model development, I analyzed the data using the Emergent Quantitative Models framework. This means, that although a generalized local was a critical aspect of students' emergent models, I was only able to characterize it in terms of a relationship between models and the outward behavior of writing an algebraic expression. By framing this same phenomenon in terms of reflective and reflected abstraction, I anticipate being able to provide more explanatory power for how a generalized local model can later provide students' anticipatory expectations as they contemplate different global model partitioning and quantification possibilities.

Finally, this fall I will conduct a study that is a direct continuation of this dissertation work. Specifically, I will follow up with these research participants during the final course in their calculus sequence to investigate the resilience of their quantitative understandings of the definite integral as they progressed through their traditional calculus coursework, and how engagement in the experiment influences their development of multivariable integration and other calculus constructs.

REFERENCES

Bajracharya, R., & Thompson, J. (2014). Student understanding of the Fundamental Theorem of Calculus at the mathematics-physics interface. *Proceedings of the 17th special interest group of the Mathematical Association of America on research in undergraduate mathematics education*, 394-399.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical Modelling: Education, Engineering and Economics-ICTMA 12* (pp. 222-231): Elsevier.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects — State, trends and issues in mathematics instruction. *Educational Studies in Mathematics, 22*(1), 37-68. doi:10.1007/BF00302716

Bressoud, D. (2011). Historical reflections on teaching the fundamental theorem of integral calculus. *The American Mathematical Monthly, 118*(2), 99-115.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378. doi:10.2307/4149958

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: a tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113-145. doi:10.1080/07370001003676587

Clement, J. (1993). Using bridging analogies and anchoring intuitions to deal with students' preconceptions in physics. *Journal of Research in Science Teaching, 30*(10), 1241-1257. doi:10.1002/tea.3660301007

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Hillsdale, NJ: Lawrence Erlbaum.

diSessa, A. A. (1988). Knowledge in pieces. In *Constructivism in the computer age.* (pp. 49-70). Hillsdale, NJ, US: Lawrence Erlbaum Associates, Inc.

diSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and Instruction, 10*(2-3), 105-225. doi:10.1080/07370008.1985.9649008

Dorko, A., & Speer, N. (2015). Calculus students' understanding of area and volume units. *Investigations in Mathematics Learning, 8*(1), 23-46. doi:10.1080/24727466.2015.11790346

Dreyfus, B. W., Elby, A., Gupta, A., & Sohr, E. R. (2017). Mathematical sense-making in quantum mechanics: An initial peek. *Physical Review Physics Education Research, 13*(2), 020141. doi:10.1103/PhysRevPhysEducRes.13.020141

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439-478. doi:10.1080/07370000701632397

Eric, W., Amy, E., Torrey, K., & Zekiye, O. (2014). Six principles for quantitative reasoning and modeling. *The Mathematics Teacher, 108*(1), 24-30. doi:10.5951/mathteacher.108.1.0024

Gravemeijer, K. (1997). Solving word problems: A case of modelling? *Learning and Instruction, 7*(4), 389-397. doi:https://doi.org/10.1016/S0959-4752(97)00011-X
Gravemeijer, K. (1998). Developmental research as a research method. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity: An ICMI Study Book 1. An ICMI Study Book 2* (pp. 277-295). Dordrecht: Springer Netherlands.
Gravemeijer, K. (2007). Emergent modeling as a precuser to mathematical modeling. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 137-144). New York, NY: Springer+Business Media, LLC.
Hatfield, L. L. (2011). Perspectives and collaborations for quantitative reasoning in mathematics education and science education. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education*. Laramie, WY: University of Wyoming: WISDOMe Mongraphs.
Hu, D., & Rebello, N. S. (2013). Using conceptual blending to describe how students use mathematical integrals in physics. *Physical Review Special Topics - Physics Education Research, 9*(2), 020118. doi:10.1103/PhysRevSTPER.9.020118
Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141. Retrieved from http://www.sciencedirect.com/science/article/pii/S0732312312000612
Jones, S. R. (2014). Three conceptualizations of the definite integral in mathematics and physics contexts. *Proceedings of the 17th special interest group of the Mathematical Association of America on research in undergraduate mathematics education. Denver, CO*.
Jones, S. R. (2015a). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior, 38*, 9-28.
Jones, S. R. (2015b). *Promoting student's construction and activation of the multiplicatively-based summation conception of the definite integral.* Paper presented at the Proceedings of the 18th annual conference on research in undergraduate mathematics education, Pittsburgh, PA.
Lesh, R., & Doerr, H. M. (2003). In what ways does a models and modeling perspective move beyond constructivism? In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 519-556). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
Lesh, R., & Sriraman, B. (2005). Mathematics education as a design science. *ZDM, 37*(6), 490-505.
Mayes, R. (2011). Quantitative reasoning and mathematical modeling in the sciences: A virtual conversation. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *WISDOMe: New perspectives and directions for collaborative research in mathematics education* (pp. 149-168). Laramie, WY: University of Wyoming.
McCloskey, M. (1983). Intuitive physics. *Scientific american, 248*(4), 122-131.
Meredith, D. C., & Marrongelle, K. A. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics, 76*(6), 570-578.
Mkhatshwa, T. P. (2019a). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical Thinking and Learning*, 1-23. doi:10.1080/10986065.2019.1658055
Mkhatshwa, T. P. (2019b). Students' quantitative reasoning about an absolute extrema optimization problem in a profit maximization context. *International Journal of Mathematical Education in Science and Technology, 50*(8), 1105-1127. doi:10.1080/0020739X.2018.1562116

Mkhatshwa, T. P., & Doerr, H. M. (2018). Undergraduate students' quantitative reasoning in economic contexts. *Mathematical Thinking and Learning, 20*(2), 142-161. doi:10.1080/10986065.2018.1442642

Moore, K. C. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. *Educational Studies in Mathematics, 83*(2), 225-245. doi:10.1007/s10649-012-9450-6

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138. doi:10.5951/jresematheduc.45.1.0102

Moore, K. C., & Carlson, M. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior, 31*(1), 48-59. doi:10.1016/j.jmathb.2011.09.001

Moore, K. C., Carlson, M., & Oehrtman, M. (2009). *The role of quantitative reasoning in solving applied precalculus problems.* Paper presented at the Twelfth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Raleigh, NC: North Carolina State University.

Moore, K. C., Silverman, J., Paoletti, T., & Laforest, K. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator, 2*(2), 141. doi:10.5951/mathteaceduc.2.2.0141

Nguyen, D.-H., & Rebello, N. S. (2011). Students' difficulties with integration in electricity. *Physical Review Special Topics-Physics Education Research, 7*(1), 010113.

Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 3-32). New York, NY: Springer+Business Media, LLC.

O'Bryan, A. E. (2018). *Exponential growth and online learning environments: Designing for and studying the development of student meanings in online courses*. Unpublished dissertation. Arizona State University. Retrieved from https://repository.asu.edu/attachments/211325/content/OBryan_asu_0010E_18389.pdf

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 396-426.

Oehrtman, M. (2015). *The equation has particles! How calculus students construct definite integral models.* Paper presented at the Proceedings of the 18th annual conference on research in undergraduate mathematics education, Pittsburg, PA.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education, 27*, 42.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics, 14*(1), 1-18.

Piaget, J. (1972). Principles of genetic epistemology. In. New York, NY: Routledge.

Piaget, J., & Duckworth, E. (1970). Genetic epistemology. *American Behavioral Scientist, 13*(3), 459-480.

Rasslan, S., & Tall, D. (2002). *Definitions and images for the definite integral concept.* Paper presented at the PME CONFERENCE.

Rodriguez, J.-M. G., Santos-Diaz, S., Bain, K., & Towns, M. H. (2018). Using symbolic and graphical forms to analyze students' mathematical reasoning in chemical kinetics. *Journal of Chemical Education, 95*(12), 2114-2125. doi:10.1021/acs.jchemed.8b00584

Schermerhorn, B., & Thompson, J. (2017). *Physics students' use of symbolic forms when constructing differential elements in multivariable coordinate systems.* Paper presented at the Paper presented at the Proceedings of the 20th annual conference on research in undergraduate mathematics education, San Diego, CA.

Sherin, B. (2001). How students understand physics equations. *Cognition and Instruction, 19*(4), 479-541.

Simmons, C., & Oehrtman, M. (2017). *Beyond the product structure for definite integrals.* Paper presented at the Paper presented at the Paper presented at the Proceedings of the 20th annual conference on research in undergraduate mathematics education, San Diego, CA.

Simmons, C., & Oehrtman, M. (2019). *Quantitatively Based Summation: A framework for student conception of definite integrals.* Paper presented at the Paper presented at the Calculus in upper secondary and beginning university mathematics conference, Agder, Kristiansand, Norway.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132).

Steffe, L., & Izsak, A. (2002). *Pre-service middle-school teachers' construction of linear equation concepts through quantitative reasoning.* Paper presented at the Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Steffe, L. P., & Thompson, P. W. (2000). *Teaching experiment methodology: Underlying principles and essential elements*. Hillsdale, NJ: Erlbaum.

Tallman, M. A., Carlson, M., Bressoud, D., & Pearson, M. (2016). A Characterization of Calculus I Final Exams in U.S. Colleges and Universities. *International Journal of Research in Undergraduate Mathematics Education, 2*(1), 105-133. doi:10.1007/s40753-015-0023-9

Tallman, M. A., & Frank, K. M. (2020). Angle measure, quantitative reasoning, and instructional coherence: an examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education, 23*(1), 69-95. doi:10.1007/s10857-018-9409-3

Thompson, P. W. (1988). *Quantitative concepts as a foundation for algebraic reasoning: Sufficiency, Necessity, and Cognitive Obstacles.* Paper presented at the Annual Conference of the International Group for the Psychology of Mathematics Education, Dekalb, IL.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebra. *Manuscript. Center for Research in Mathematics & Science Education, San Diego State University.*

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208. doi:10.1007/bf01273861

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. In *Learning Mathematics* (pp. 125-170): Springer.

Thompson, P. W. (1995). Notation, convention, and quantity in elementary mathematics. In J. Sowder & B. Schapelle (Eds.), *Providing a foundation for teaching middle school mathematics* (pp. 199-221). Albany, NY: SUNY Press.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, B. Greer, P. Nesher, P. Cobb, & G. Goldin (Eds.), *Theories of learning mathematics*. Hillsdale, NJ: Erlbaum.

Thompson, P. W. (1999). What is required to understand fractal dimension? *The Mathematics Educator, 10*(2).

Thompson, P. W. (2008). *Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education.* Paper presented at the Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (pp. 33-57). Laramie, WY: University of Wyoming: WISDOMe Mongraphs.

Thompson, P. W. (2012). Advances in research on quantitative reasoning. In *Mayes, R., Bonillia* (pp. 143-148).

Thompson, P. W. (2013). In the absence of meaning…. In K. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57-93). New York, NY: Springer.

Thompson, P. W., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers Mathematics.

Thompson, P. W., Carlson, M., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*(4), 415-432. doi:10.1007/s10857-007-9054-8

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. *Making the connection: Research and teaching in undergraduate mathematics, 73*, 43-52.

von Glasersfeld, E. (1995). Radical constructivism: A way of learning and knowing. In. London: Falmer Press.

Von Korff, J., & Sanjay Rebello, N. (2014). Distinguishing between "change" and "amount" infinitesimals in first-semester calculus-based physics. *American Journal of Physics, 82*(7), 695-705.

White Brahmia, S. (2019). Quantification and its importance to modeling in introductory physics. *European Journal of Physics, 40*(4), 044001. doi:10.1088/1361-6404/ab1a5a

APPENDICES

APPENDIX A

SUMMER 2020 ACCUMULATION TASK SEQUENCE

§3.1: Introduction to Accumulation

Problem 1: Let V be the volume of water in a reservoir serving a small town, measured in millions of gallons. Because rainfall adds water to the reservoir, and evaporation & town consumption decreases water in the reservoir, we can think of V as a function of time t. Let V' be the rate at which water is flowing into the reservoir, in millions of gallons with respect to time measured in days. Suppose V' is measured every two days, and those measurements are recorded in the following table:

t (days)	$V'(t) = \frac{\Delta V}{\Delta t}$ ($10^6 \times$ gal. per day)
0	.34
2	.11
4	-.07
6	-.23
8	-.14
10	.03
12	.08

Problem 1: Approximate the total change in V over these 14 days.

t (days)	$V'(t) = \frac{\Delta V}{\Delta t}$ ($10^6 \times$ gal. per day)
0	.34
2	.11
4	-.07
6	-.23
8	-.14
10	.03
12	.08

Starting t	Current ΔV	Accumulated $\sum \Delta V$	Ending t
0			2
2			4
4			6
6			8
8			10
10			12
12			14

Your Turn 3.1.1: Madison decides to go for a run before school. She starts her run from home. The function $y = v(t)$ expresses the relationship between Madison's velocity (in meters per minute) as she runs and the number of minutes t elapsed since she started running.

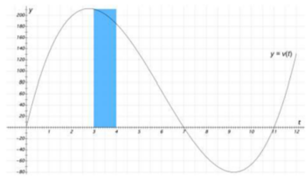
a) What does the product $v(t) \times \Delta t$ approximate?

b) What does the following sum approximate?

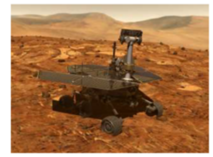
$$v(0) \times \frac{1}{2} + v\left(\frac{1}{2}\right) \times \frac{1}{2} + v(1) \times \frac{1}{2} + v\left(\frac{3}{2}\right) \times \frac{1}{2} + v(2) \times \frac{1}{2} + v\left(\frac{5}{2}\right) \times \frac{1}{2}$$

Your Turn 3.1.1: Madison decides to go for a run before school. She starts her run from home. The function $y = v(t)$ expresses the relationship between Madison's velocity (in meters per minute) as she runs and the number of minutes t elapsed since she started running.

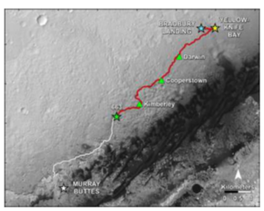
c) What does the area of the blue rectangle on the graph of the function $y = v(t)$ below represent?



Your Turn 3.1.2: The Opportunity rover landed on Mars in 2004 and has been actively exploring the planet ever since. It is powered by solar cells. As the rover travels across the Martian surface, it kicks up dust, which accumulate on its solar cells. The amount of dust that it kicks up depended on the composition of the surface it was traveling over - a rockier surface kicks up less dust than a softer surface. When planning a path for the rover to follow, scientists need to know how far it might travel before too much dust accumulates on its solar panels.



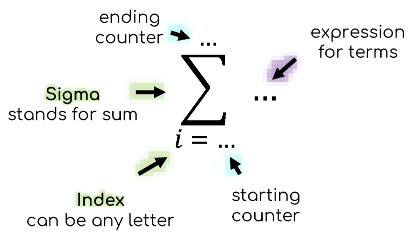
Your Turn 3.1.2: The scientists have mapped out a 100-km path for the rover to follow (shown below) and have collected satellite data about the composition of the Martian surface at various points along the route using a LiDAR Spectrometer. Based on the following table, approximate the amount of dust accumulated on the rover's solar panels.



Composition	Position along path (km)	Amount of dust per distance traveled (mg/km)
Very sandy	0	6
Moderatley sandy	20	3.5
Slightly sandy	40	2.5
Slightly rocky	60	2
Moderately rocky	80	1.5
Very rocky	100	1

§3.2: Riemann Sums
Sigma Notation

What is Sigma Notation?
Sigma notation is a compact way to describe sums.



Examples

$$\sum_{i=0}^{5} i =$$

$$\sum_{i=1}^{5} i =$$

$$\sum_{k=5}^{10} 3k =$$

Examples

$$\sum_{n=0}^{4} n(-1)^n =$$

$$\sum_{m=0}^{3} \frac{3m}{2} + 1 =$$

$$\sum_{k=2}^{6} 5 =$$

Some Properties of Sigma Notation

$$\sum_{i=a}^{b} A + B =$$

$$\sum_{i=a}^{b} A - B =$$

$$\sum_{i=a}^{b} cA =$$

Your Turn 3.2.1:

a) Determine the numerical value for

$$\sum_{j=1}^{5} \frac{1}{j} - \frac{1}{j+1}$$

b) Write "the sum of the first five positive even integers" in sigma notation.

c) Recall Your Turn 3.1.2 and write the following expression in sigma notation. What does this expression measure?

$$v(0)\times\frac{1}{2}+v\left(\frac{1}{2}\right)\times\frac{1}{2}+v(1)\times\frac{1}{2}+v\left(\frac{3}{2}\right)\times\frac{1}{2}+v(2)\times\frac{1}{2}+v\left(\frac{5}{2}\right)\times\frac{1}{2}$$

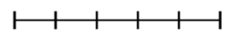
Riemann Sums

As described in previous videos a Reimann sum is a special sum in which you consider a product of a function value and a change in the varying quantity.

Left Riemann Sum (L_N):

Right Riemann Sum (R_N):

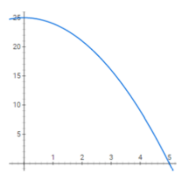
Composition	Position along path (km)	Amount of dust per distance traveled (mg/km)
Very sandy	0	6
Moderatley sandy	20	3.5
Slightly sandy	40	2.5
Slightly rocky	60	2
Moderately rocky	80	1.5
Very rocky	100	1



Your Turn 3.2.2: Estimating Area

a) Let $f(x) = 25 - x^2$ and suppose the interval $[0,2]$ has been divided into four equal subintervals Δx and let x_i be the right endpoint of the ith interval. Write and determine the value of the Left Riemann sum L_4 and the Right Riemann sum R_4 ($i.e. \sum(f(x)\Delta x)$).

b) What does each $f(x_i)\Delta x$ represent geometrically?



c) Draw 2 graphs of $f(x)$ over the interval $[0,2]$. Draw the quantities for L_4 and R_4 on these graphs with detailed labels for each quantity involved.

d) Notice that $f(x) = 25 - x^2$ is a decreasing function on $[0,2]$. Does this have any implications for whether the Left Riemann sum is an overestimate or an underestimate? What about the Right Riemann sum?

Your Turn 3.2.3: A gorilla (wearing a parachute) jumped off the top of a building. We were able to record the velocity of the gorilla with respect to time twice each second. The data is shown below. Note that the gorilla touched the ground just after 5 seconds.

a) Use what you've learned to approximate the total distance the gorilla fell from the time he jumped off the building until the time he landed on the ground. (note: distance = velocity · time)

b) Is your calculation an overestimate? an underestimate? an exact value? How can you tell?

Time (in seconds)	Velocity (in feet per second)
0	0
0.5	5
1.0	7
1.5	8
2.0	11
2.5	11.5
3.0	12
3.5	13
4.0	15.5
4.5	18
5.0	19

§3.2: Riemann Sums
Using a Calculator to Evaluate Sums

Examples

$$\sum_{n=0}^{4} n(-1)^n =$$

$$\sum_{k=0}^{500} 1 =$$

$$\sum_{m=1}^{30} \frac{3m}{2} + 1 =$$

$$\sum_{k=0}^{500} 1 =$$

Calculating Summations on a TI 83/84

LIST → MATH → 5: sum(→ LIST → OPS → 5: seq(

sum(seq(… , … , … , … , …))

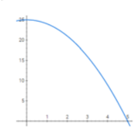
- expression (1st entry)
- index (2nd entry)
- starting value (3rd entry)
- ending value (4th entry)
- step size = 1 (5th entry)

Your Turn 3.2.4: (Recall Your Turn 3.2.2) Estimating Area

a) Let $f(x) = 25 - x^2$ and suppose the interval $[0,2]$ has been divided into 100 equal subintervals Δx and let x_i be the right endpoint of the ith interval. Write the sigma notation for the Left Riemann sum L_{100} and the Right Riemann sum R_{100}. (Warning: you cannot just write $25 - x^2$ as the expression)

b) Use your calculator to evaluate L_{100} and R_{100}. (You should get $L_{100} = 47.3732$ and $R_{100} = 47.2932$.)

c) Is this estimation better or worse than the one from YT 3.2.2? Explain why.



§3.3: Continuously Varying Rates

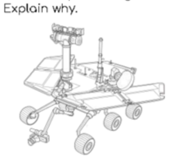
Linear Approximations

Your Turn 3.3.1: The NASA Q36 Robotic Lunar Rover can travel up to 3 hours on a single charge and has a range of 1.6 miles. After leaving its base and traveling for t hours, the speed of the Q36 is given by the function $v(t) = \sin\left(\sqrt{9-t^2}\right)$ in miles per hour. One hour into an excursion, the Q36 will have traveled 0.19655 miles. Two hours into a trip, the Q36 will have traveled 0.72421 miles.

a) Use your calculator to graph $v(t)$. Explain in words what the graph says about how the Q36 moves during a 3-hour trip starting with a full charge.

b) Approximate how far the Q36 traveled in the 10 minutes immediately following the t=1 hour mark. Is this an underestimate or overestimate? Explain why.

c) Controllers want to turn the Q36 around to head back to its base after traveling 0.75 miles. Use the linear approximation at $t = 2$ to determine approximately what time this will happen. Will the actual time be a little earlier or a little later than your estimate? Explain.



Linear Approximations
If $f'(x)$ is the rate at which the function $f(x)$ is changing with respect to the quantity x and Δx is small, then

$$\Delta f \approx f'(x)\Delta x$$

In other words, for small changes in x we can assume that f is changing at a constant rate. The larger the change in x the larger the error of our approximation.

Problem 1: A study suggests that the extinction rate $r(t)$ of marine animal families during the Phanerozoic Eon can be modeled by the function $r(t) = \frac{3130}{t+262}$ for $0 \le t \le 262$ where t is the time elapsed (in millions of years) since the beginning of the eon 544 million years ago. Compute R_{10} and L_{10} for the time period between 520 and 420 million years ago. What does this calculate? Why aren't R_{10} and L_{10} the same?

Problem 1 continued: A study suggests that the extinction rate $r(t)$ of marine animal families during the Phanerozoic Eon can be modeled by the function $r(t) = \frac{3130}{t+262}$ for $0 \le t \le 262$ where t is the time elapsed (in millions of years) since the beginning of the eon 544 million years ago.

Compute R_{33} and L_{33} for the time period between 520 and 420 million years ago.

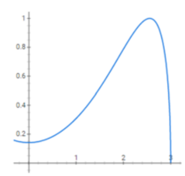
§3.3: Continuously Varying Rates

Refining Our Approximations

Your Turn 3.3.2: A factor produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week where t measures the amount of time since production began. Approximately how many bicycles were produced from halfway through week 2 until the end of week 3? How confident are you in your answer? Can you make your approximation better?

Recall YT 3.3.1 : The NASA Q36 Robotic Lunar Rover can travel up to 3 hours on a single charge and has a range of 1.6 miles. After leaving its base and traveling for t hours, the speed of the $Q36$ is given by the function $v(t) = \sin(\sqrt{9 - t^2})$ in miles per hour. Approximate the distance traveled by the Q36 in the first two hours.

A Rough Estimate:



A Rough Estimate:

2 hours at $v(0) = 0.14112$ mph is 0.28224 miles

2 hours at $v(2) = 0.78675$ mph is 1.57350 miles

A difference of 1.29126 miles

Error bound

Recall: We know from YT 3.3.1 that the actual distance traveled in the first two hours was 0.72421.

A Better Approximation: $v(t) = \sin(\sqrt{9 - t^2})$

Let's try ½ hour intervals.

Time t in hours	
Velocity $v(t)$ in mph	

A Better Approximation: $v(t) = \sin(\sqrt{9 - t^2})$

Because we have a function for velocity, we can compute the velocity at any time we want which means we can get a better approximation by assuming a constant speed, but using shorter time intervals.

An Even Better Approximation: $v(t) = \sin(\sqrt{9 - t^2})$

Let's try 10 minute intervals.

Time t in hours	0	1/6	2/6	3/6	4/6	5/6
Velocity $v(t)$ in mph	.14112	.14571	.15948	.18252	.21491	.25675

Time t in hours	6/6	7/6	8/6	9/6	10/6	11/6	12/6
Velocity $v(t)$ in mph	.30807	.36882	.438752	.51715	.60292	.69395	.78675

An Even Better Approximation: $v(t) = \sin(\sqrt{9 - t^2})$

Let's try 10 minute intervals.

Time t in hours	0	1/6	2/6	3/6	4/6	5/6
Velocity $v(t)$ in mph	.14112	.14571	.15948	.18252	.21491	.25675

Time t in hours	6/6	7/6	8/6	9/6	10/6	11/6	12/6
Velocity $v(t)$ in mph	.30807	.36882	.438752	.51715	.60292	.69395	.78675

Error Bound =

An Even Better Approximation: $v(t) = \sin(\sqrt{9 - t^2})$

Let's try 10 minute intervals.

Time t in hours	0	1/6	2/6	3/6	4/6	5/6
Velocity $v(t)$ in mph	.14112	.14571	.15948	.18252	.21491	.25675

Time t in hours	6/6	7/6	8/6	9/6	10/6	11/6	12/6
Velocity $v(t)$ in mph	.30807	.36882	.438752	.51715	.60292	.69395	.78675

Error Bound = $v_f \Delta t - v_0 \Delta t$

Actual Error < Error Bound

Your Turn 3.3.3: The NASA Q36 Robotic Lunar Rover can travel up to 3 hours on a single charge and has a range of 1.6 miles. After leaving its base and traveling for t hours, the speed of the $Q36$ is given by the function $v(t) = \sin(\sqrt{9 - t^2})$ in miles per hour.

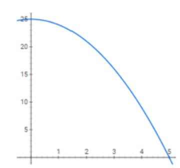
How small will you need to make the partitions for the two hour time period since leaving the base in order to insure that your approximation is within 1% of the actual distance traveled in two hours? What does this translate to in minutes?

Your Turn 3.3.4: (Recall YT 3.2.2 & 3.2.4)

Estimating Area: Let $f(x) = 25 - x^2$.

a) In YT 3.2.4 you found that for this function $L_{100} = 47.3732$ and $R_{100} = 47.2932$. How far off from the actual area was your approximation?

b) Describe how you can find an approximation within .01% of the actual area beneath this graph.



Your Turn 3.3.5: (Recall YT 3.3.2) A factory produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week where t measures the amount of time since production began. Approximately how many bicycles were produced from halfway through week 2 until the end of week 3?

How confident are you in your answer now? Can you make your approximation better? Can you provide an exact number of bicycles that were produced? Explain.

Total Mass of a Rod: A rod's linear mass density ρ is defined as the mass per unit length. If ρ is constant, then by definition,

$$\text{total mass} = \text{linear mass density} \times \text{length} = \rho \cdot l$$

Consider a 2-m rod with a linear density of $\rho(x) = 1 + x(2 - x)\ kg/m$ where x measures the distance from one of the rod in meters. Approximate the total mass of the rod. Note that the $\rho(x)$ is increasing on $[0,1]$ and decreasing on [1,2].

Your Turn 3.3.6: Energy required to compress a spring. See module 3.3.

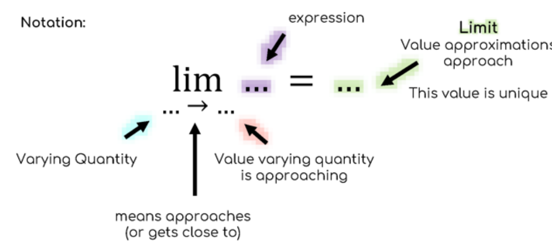
Your Turn 3.3.7: Force of water exerted on a dam. See module 3.3.

§3.4: Limits

What is a Limit?

What is a limit? When we can make the error of an approximation for an expression smaller than any pre-defined error bound, we call the value that our approximations approach a **limit (or limiting value)**.

Notation:



An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$|v(2) - v(0)|$

Error bound $= 0.64563 \cdot \Delta t$

$v(t) = \sin\sqrt{9 - t^2}$
[0,2]

underestimate (L_N), overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
12	$\frac{2-0}{12} = \frac{1}{6}$	0.671687	0.779292	0.107605
130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{N \to \infty} L_N \approx 0.72 \text{ mi.}$$

$$\lim_{N \to \infty} R_N \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

Error bound $= 0.64563 \cdot \Delta t$

$v(t) = \sin\sqrt{9 - t^2}$
[0,2]

underestimate (L_N), overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
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130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{N \to \infty} L_N = \lim_{N \to \infty} \sum_{i=0}^{N-1} v(0 + i \cdot \Delta t) \cdot \Delta t \approx 0.72 \text{ mi.}$$

$$\lim_{N \to \infty} R_N = \lim_{N \to \infty} \sum_{i=1}^{N} v(0 + i \cdot \Delta t) \cdot \Delta t \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$ [0,2]

Error bound $= 0.64563 \cdot \Delta t$

underestimate (L_N), overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
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130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{N \to \infty} L_N = \lim_{N \to \infty} \sum_{i=0}^{N-1} \left(v\left(0 + i \cdot \frac{2-0}{N}\right) \cdot \left(\frac{2-0}{N}\right) \right) \approx 0.72 \text{ mi.}$$

$$\lim_{N \to \infty} R_N = \lim_{N \to \infty} \sum_{i=1}^{N} \left(v\left(0 + i \cdot \frac{2-0}{N}\right) \cdot \left(\frac{2-0}{N}\right) \right) \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$ [0,2]

Error bound $= 0.64563 \cdot \Delta t$

underestimate (L_N), overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
12	$\frac{2-0}{12} = \frac{1}{6}$	0.671687	0.779292	0.107605
130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{N \to \infty} L_N = \lim_{N \to \infty} \sum_{i=0}^{N-1} \left(\sin\sqrt{9 - \left(0 + i \cdot \frac{2-0}{N}\right)^2} \cdot \left(\frac{2-0}{N}\right) \right) \approx 0.72 \text{ mi.}$$

$$\lim_{N \to \infty} R_N = \lim_{N \to \infty} \sum_{i=1}^{N} \left(\sin\sqrt{9 - \left(0 + i \cdot \frac{2-0}{N}\right)^2} \cdot \left(\frac{2-0}{N}\right) \right) \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$ [0,2]

Error bound $= 0.64563 \cdot \Delta t$

underestimate (L_N), overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
12	$\frac{2-0}{12} = \frac{1}{6}$	0.671687	0.779292	0.107605
130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$\Delta t = \frac{2-0}{N}$ $\quad N = \frac{2-0}{\Delta t}$

$$\lim_{\Delta t \to 0} L_{\left(\frac{2-0}{\Delta t}\right)} \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} R_{\left(\frac{2-0}{\Delta t}\right)} \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$ [0,2]

Error bound $= 0.64563 \cdot \Delta t$

underestimate (L_N), overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
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130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{\Delta t \to 0} L_{\left(\frac{2-0}{\Delta t}\right)} = \lim_{\Delta t \to 0} \sum_{i=0}^{\left(\frac{2-0}{\Delta t}\right)-1} \left(\sin\sqrt{9 - (0 + i \cdot \Delta t)^2} \cdot \Delta t \right) \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} R_{\left(\frac{2-0}{\Delta t}\right)} = \lim_{\Delta t \to 0} \sum_{i=1}^{\left(\frac{2-0}{\Delta t}\right)} \left(\sin\sqrt{9 - (0 + i \cdot \Delta t)^2} \cdot \Delta t \right) \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$
[0,2]

Error bound = $0.64563 \cdot \Delta t$

# of partitions N	Δt	$L_N \approx$ (underestimate)	$R_N \approx$ (overestimate)	Error Bound $\approx$
12	$\frac{2-0}{12} = \frac{1}{6}$	0.671687	0.779292	0.107605
130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{\Delta t \to 0} L_N = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$
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Error bound = $0.64563 \cdot \Delta t$

# of partitions N	Δt	$L_N \approx$ (underestimate)	$R_N \approx$ (overestimate)	Error Bound $\approx$
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130	$\frac{2-0}{130}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{N \to \infty} L_N = \lim_{N \to \infty} \sum_{i=0}^{N-1} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\lim_{N \to \infty} R_N = \lim_{N \to \infty} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

Your Turn 3.4.1: (Recall YT 3.3.2 & 3.3.5) A factory produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week where t measures the amount of time since production began.

What is the limiting value for the number of bicycles that were produced from halfway through week 2 until the end of week 3? Practice writing limit notation to express this solution.

§3.4: Limits

Limit Basics

Your Turn 3.4.2: What are the following limiting values?

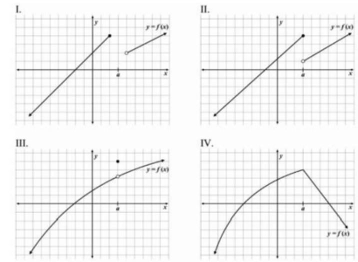
a) $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}$

b) $\lim_{x \to 1} \frac{x^2-4x+3}{x-1}$

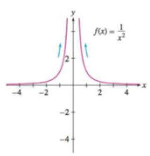
c) $\lim_{v \to 0} \frac{v}{|v|}$

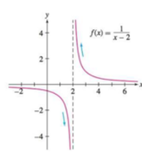
d) $\lim_{h \to 0} \frac{(5+h)^2-5^2}{h}$

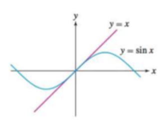
Your Turn 3.4.3: For which of the following graphs does $\lim_{x \to a} f(x)$ exist?



Infinite Limits: If the approximations for a limit increase without bound as $x \to c$ we say that the limit is ∞. If the approximations for a limit decrease without bound as $x \to c$ we say that the limit is $-\infty$. Note that ∞, and $-\infty$ are not real numbers, they are shorthand for "increases without bound" and "decreases without bound".







Implications for Limits Graphically - Asymptotes:

If $\lim_{x \to c} f(x)$, $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ increases (or decreases) without bound, then we say that there is a vertical asymptote for $f(x)$ at $x = c$.

If $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$, then we say that there is a horizontal asymptote for $f(x)$ at $y = c$.

Some Basic Limit Laws: Limits work the way you expect them to. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then,

$\lim_{x\to c}\big(f(x)+g(x)\big) =$

$\lim_{x\to c} kf(x) =$

$\lim_{x\to c}\big(f(x)\cdot g(x)\big) =$

$\lim_{x\to c}\frac{f(x)}{g(x)} =$

$\lim_{x\to c}\big(f(x)\big)^n =$

Your Turn 3.4.4: Try to identify the value of $\lim_{\theta\to 0}\sin\left(\frac{\pi}{\theta}\right)$. Does it exist? If so, identify how close you have to be to 0 in order for your error bound to be less than .01. Graph this function on your calculator and explain the implications for this type of function.

When do limits not exist?

We saw from the last video that if $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$ then we say that $\lim_{x\to c} f(x)$ does not exist.

Are there other instances where limits don't exist?

§3.4: LIMITS
CONTINUITY

Continuity: A function $f(x)$ is continuous at $x = c$ if,

$$\lim_{x \to c} f(x) = f(c).$$

A function $f(x)$ is continuous on $[a, b]$ if it is continuous at $x = c$ for every $c \in [a, b]$.

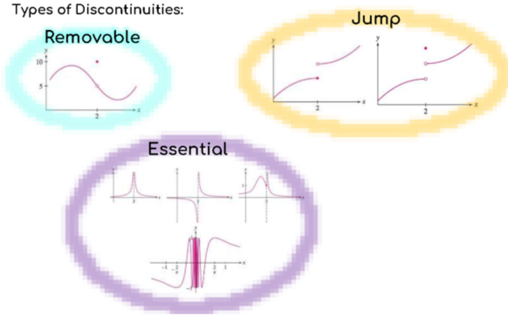
A function $f(x)$ is left-continuous at $x = c$ if,

$$\lim_{x \to c^-} f(x) = f(c).$$

A function $f(x)$ is right-continuous at $x = c$ if,

$$\lim_{x \to c^+} f(x) = f(c).$$

Types of Discontinuities:



Your Turn 3.4.6: Are the following functions continuous? If not, which condition(s) of continuity are not met?

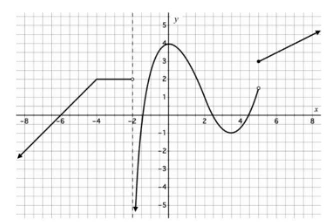
a) $f(x) = \frac{-x^2+5x-4}{x-4}; (-\infty, \infty)$

b) $g(l) = |3 - 2l|; (-\infty, \infty)$

c) $f(t) = \frac{1}{t}; (0, \infty)$

d) $f(t) = \frac{1}{t}; (-\infty, \infty)$

Your Turn 3.4.7: Consider the following graph of a function $y = f(x)$. Describe the intervals for which $f(x)$ is continuous.



§3.4: Limits
Definite Integrals

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

An example: Recall our NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$
[0,2]

Error bound $= 0.64563 \cdot \Delta t$

underestimate (L_N) overestimate (R_N)

# of partitions N	Δt	$L_N \approx$	$R_N \approx$	Error Bound $\approx$
12	$\frac{2-0}{12} = \frac{1}{6}$	0.671687	0.779292	0.107605
130	$\frac{2-0}{120}$	0.719255	0.729187	0.009933
1000	$\frac{2-0}{1000}$	0.723565	0.724856	0.001291

$$\lim_{\Delta t \to 0} L_N = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\int_0^2$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t\ ) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\int_0^2 v(t)$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t\ ) \cdot \Delta t) \approx \mathbf{0.72 \text{ mi.}}$$

$$\int_0^2 v(t) \cdot dt$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t\ ) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\int_0^2 v(t) \cdot dt \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t\ ) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\int_0^2 v(t) \cdot dt \approx 0.72 \text{ mi.}$$

Limits of integration

Integrand

Differential

Limit
Value approximations approach; i.e. the total quantity

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$Total\ Distance = \int_0^2 \underbrace{v(t) \cdot dt}_{\text{Differential form}} \approx 0.72 \text{ mi.}$$

For constant quantities, $Distance = velocity \cdot time$

Your Turn 3.4.8: In the context of the NASA Q36 robotic Lunar Rover problem, what does the definite integral

$$\int_1^2 \sqrt{9 - t^2} \cdot dt$$

measure?

$$\lim_{\Delta t \to 0} R_N = \lim_{\Delta t \to 0} \sum_{i=1}^{N} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

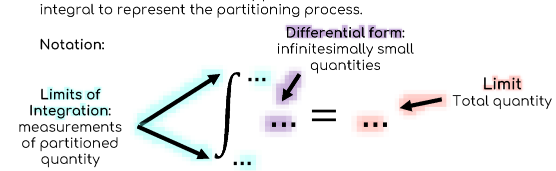
$$\int_0^2 v(t) \cdot dt \approx 0.72 \text{ mi.}$$

$$\lim_{\Delta t \to 0} L_N = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} (v(0 + i \cdot \Delta t) \cdot \Delta t) \approx 0.72 \text{ mi.}$$

$$\int_0^2 v(t) \cdot dt \approx 0.72 \text{ mi.}$$

Definite Integral: When we can make the error of an approximation for an accumulation smaller than any pre-defined error bound we use a definite integral to represent the partitioning process.

Notation:



Official Definition of a Definite Integral:

For a function $f(x)$ on the interval $[a,b]$ with arbitrary partitions $\Delta x_i = [x_{i-1}, x_i]$ and $c_i \in [x_{i-1}, x_i]$ then,

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

Some functions don't have definite integrals:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Official Definition of a Definite Integral:

For a function $f(x)$ on the interval $[a,b]$ with arbitrary partitions $\Delta x_i = [x_{i-1}, x_i]$ and $c_i \in [x_{i-1}, x_i]$ then,

$$\lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

If $f(x)$ has finitely many discontinuities and there are finitely many changes in monotonicity then we can always find the error bounds using left and right Riemann sums. If that error bound tends towards zero (i.e. you can make it smaller than any arbitrary number) then we can use a definite integral.

Total Mass of a Rod: A rod's linear mass density ρ is defined as the mass per unit length. If ρ is constant, then by definition,

total mass = linear mass density × length = $\rho \cdot l$

Consider a 2-m rod with a linear density of $\rho(x) = 1 + x(2-x)\ kg/m$ where x measures the distance from one of the rod in meters. Write a definite integral that represents the total mass of the rod. Note that the $\rho(x)$ is increasing on $[0,1]$ and decreasing on $[1,2]$.

Your Turn 3.4.9: (Recall YT 3.3.2, 3.3.5 & 3.4.1) A factor produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week where t measures the amount of time since production began.

Write the definite integral which represents the limiting value of how many bicycles were produced from halfway through week 2 until the end of week 3.

§3.4: Limits

Definite Integrals: Using a Calculator to Evaluate Definite Integrals

Properties of Definite Integrals:

Assume $f(x)$ and $g(x)$ are integrable, $a \le b \le c$, and k is a constant.

$\int_a^b (f(x) + g(x))dx =$

$\int_a^b kf(x)dx =$

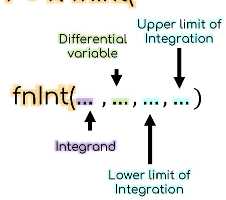
$\int_a^a f(x)dx =$

$\int_b^a f(x)dx =$

$\int_a^b f(x)dx + \int_b^c f(x)dx =$

Calculating Definite Integrals on a TI 83/84

MATH → 9: fnInt(



Recall the NASA Q36 Robotic Lunar Rover problem.

$v(t) = \sin\sqrt{9 - t^2}$

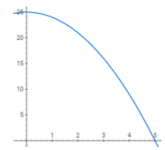
$[0,2]$

Your Turn 3.4.10: (Recall YT 3.3.2, 3.3.5, 3.4.1, & 3.4.8) A factor produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week where t measures the amount of time since production began.

Use your calculator to evaluate your definite integral from YT 3.4.8. Did you get the same value as your approximation from 3.4.1?

Your Turn 3.4.11: (Recall YT 3.2.2 & 3.2.4) Use a definite integral and a calculator to identify the area under the curve of $f(x) = 25 - x^2$ between the values of $x = 0$ and $x = 2$. How close were your approximations in 3.2.2 and 3.2.4?

What quantity does your integrand represent?
What quantity does your differential represent?
What quantity does your differential form represent?



Your Turn 3.4.12: (Recall YT 3.3.6) Use a definite integral and a calculator to compute the total energy required to compress the spring in YT 3.3.6.

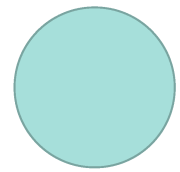
What quantity does your differential represent?
What quantity does your differential form represent?
What quantity does your definite integral represent?

Your Turn 3.4.13: (Recall YT 3.3.7) Use a definite integral and a calculator to compute the total force exerted on the dam in 3.3.7.

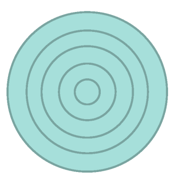
What quantity does your differential represent?
What quantity does your differential form represent?
What quantity does your definite integral represent?

§3.5: Modeling with definite integrals
Geometry – Part 1: Area

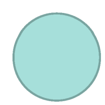
Area of a Circle:



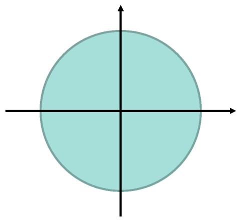
Area of a Circle: concentric rings



Your Turn 3.5.1: Show that the error for the concentric ring method can be made smaller than any predefined error bound.

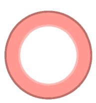


Area of a Circle: area beneath a curve

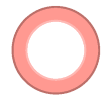


Your Turn 3.5.3: Area of an Annulus: area under a curve. Write an expression that measures the area of an annulus with inner radius $r = 3in$ and outer radius $R = 4\,in$ using the method of area under a curve. Include the meaning (including units) of each factor of the definite integral. (Hint: this expression will require two integrals)

Calculate the area using a calculator. Is it the same answer as YT 3.5.2?

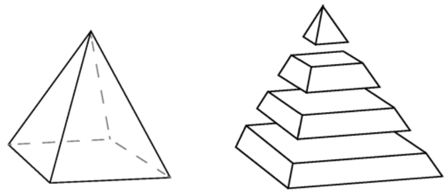


Your Turn 3.5.2: Area of an Annulus: concentric rings. Write an integral that measures the area of an annulus with inner radius $r = 3in$ and outer radius $R = 4\,in$ using the method of concentric rings. Include the meaning (including units) of each factor of the definite integral.
Calculate the area using a calculator.

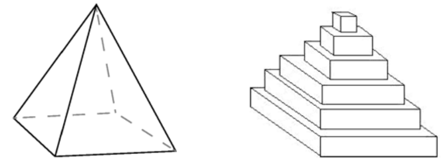


§3.5: Modeling with Definite Integrals
Geometry – Part 2: Introduction to Volume

Volume of a Pyramid: Write a definite integral to calculate the volume V of a pyramid with height 12m whose base is a square of side length $4\,m$.



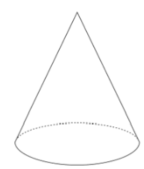
Volume of a Pyramid: Write a definite integral to calculate the volume V of a pyramid with height 12m whose base is a square of side length $4\,m$.



Volume of a Pyramid: Write a definite integral to calculate the volume V of a pyramid with height 12m whose base is a square of side length $4\,m$.

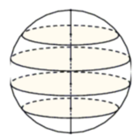
Your Turn 3.5.7: Write a definite integral to calculate the volume V of a right circular cone with height 13 inches and radius 4 in. Include the meaning (including units) of each factor of the definite integral.

Calculate the volume using a calculator.



Your Turn 3.5.8: Write a definite integral to calculate the volume V of a sphere with radius 6 in. Include the meaning (including units) of each factor of the definite integral.

Calculate the volume using a calculator.



Your Turn 3.5.9: Write a definite integral that will calculate the volume of a solid whose base is the unit circle $(x^2 + y^2 = 1)$ and whose cross sections perpendicular to the x-axis are triangles whose height and base are equal. Include the meaning of each factor of the definite integral.

Calculate the volume using a calculator.

Suppose we are given a information about a solid with a given base and cross sectional shapes.

Example: Find the volume of the solid whose base is the triangle enclosed by $x + y = 1$, the x-axis, and the y-axis and whose cross sections perpendicular to the y-axis are semicircles.

§3.5: Modeling with Definite Integrals Geometry – Part 3: Solids of Revolution

Solid of Revolution: A solid of revolution is a solid obtained by rotating a region on a graph about an axis.

Geogebra Applet

Your Turn 3.5.10: Write a definite integral to calculate the volume V of the solid of revolution created by rotating the function $f(x) = x^2 - 3x$ about the x-axis $[0,3]$. Include the meaning of each factor of the definite integral.

Graph the function $f(x)$ on your calculator and sketch the solid of revolution between $[0,3]$. For one partition, label the relevant quantities on your drawing.

Calculate the volume using a calculator.

Disk Method: Recall YT 3.5.7 Write a definite integral to calculate the volume V of a right circular cone with height 13 inches and radius 4 in.

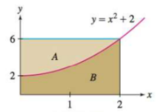
Disk Method:
Geogebra Applet

Your Turn 3.5.11: Write a definite integral to calculate the volume V of the solid of revolution created by rotating the function $y = f(x) = x^2$ about the y-axis for $1 \le y \le 4$. Include the meaning of each factor of the definite integral.

Graph the function $f(x)$ on your calculator and sketch the solid of revolution for $1 \le y \le 4$. For one partition, label the relevant quantities on your drawing.

Calculate the volume using a calculator.

Washer Method: Write a definite integral to calculate the volume V of the region A rotated about the x-axis.



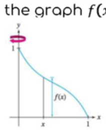
Washer Method:
Geogebra Applet

Your Turn 3.5.12: Write a definite integral to calculate the volume V of the solid of revolution created by rotating the region enclosed by the functions $y = x^2$, $y = 12 - x$, and $x = 0$ about the line $y = -2$. Include the meaning of each factor of the definite integral.

Sketch the solid of revolution. For one partition, label the relevant quantities on your drawing.

Calculate the volume using a calculator.

Shell Method: Write a definite integral to calculate the volume V of the graph $f(x) = 1 - 2x + 3x^2 - 2x^3$ over $[0,1]$, about the y-axis.



Shell Method:
Geogebra Applet

Your Turn 3.5.13: Write a definite integral using the shell method that will calculate the volume of a solid obtained by rotating the region enclosed by $y = 3x - 2$, $y = 6 - x$, $x = 0$ about the y-axis.

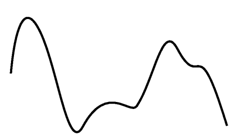
Sketch the solid of revolution. For one partition, label the relevant quantities on your drawing.

Calculate the volume using a calculator.

§3.5: Modeling with definite integrals

Geometry – Part 4: Arc length and surface area

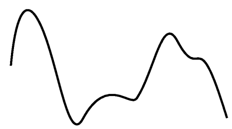
Arc Length (s) is the total length of a curved line.



Warning: We have been a little lax on checking to make sure our errors can be made smaller than any predetermined error bound. The techniques for area of two dimensional objects and volume of three dimensional objects that we've discussed do have nice errors which tend towards 0. However, we must be cautious whenever we're trying to model new phenomena.

Example: Why can't we just use horizontal lines for arc length?
Consider a straight line from $(0,0)$ to $(1,1)$.

Arc Length is measured by summing the lengths of infinitesimally small hypotenuses.



Your Turn 3.5.14: The surface area of a solid of revolution CANNOT be found using an analogue of the disk method (That is, $\int_a^b 2\pi f(x)dx$ does not calculate surface area.) Explain why not in terms of error.

You shouldn't have to do any additional computations. You can reason about this 3-dimensional case as an extension of the 2-dimensional case for arc length.

In case you're interested:
Surface area of a solid of revolution

$$S = \int_a^b 2\pi \cdot f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}\,dx$$

Example: But why did it work for area?
Consider a straight line from $(0,0)$ to $(1,1)$.

§3.5: Modeling with definite integrals
Density

Density describes the degree of compactness of a substance or object.

Linear Density: mass per unit length $(M = \rho \cdot L)$
Area Density: mass per unit area $(M = \rho \cdot A)$
Density: mass per unit volume $(M = \rho \cdot V)$
Population Density: population # per unit area $(P = \rho \cdot A)$

Your Turn 3.5.15: King Arthur's lance is 10 feet long and heavier on one end, decreasing from a linear density of $1.2 lb/ft$ at the base to $0.2\ lb/ft$ at the tip. Write an integral that gives the total mass of King Arthur's lance.

Recall: Total Mass of a Rod A rod's linear mass density ρ is defined as the mass per unit length. If ρ is constant, then by definition,

total mass = linear mass density × length = $\rho \cdot l$

Consider a 2-m rod with a linear density of $\rho(x) = 1 + x(2 - x)\ kg/m$ where x measures the distance from one of the rod in meters. Write a definite integral that represents the total mass of the rod. Note that the $\rho(x)$ is increasing on $[0,1]$ and decreasing on [1,2].

Deer Population: The density of deer in a particular forest can be calculated by the distance from a circular pond with diameter $1km$ at its center. The density function which models this phenomenon for this particular forest is $\rho(r) = \frac{150}{(r^2+2)^2}$ deer per square kilometer, where r is the distance (in kilometers) from the center of the pond. Calculate the number of deer within 5 km of the pond.

Your Turn 3.5.16: For a particular circular oil spill, the density of oil on the surface of the ocean at a distance r meters from the center of the slick is given by $\rho(r) = \frac{50}{1+r}\,kg/m^2$. If the oil slick extends from $r = 0$ to $r = 10{,}000$ m, write a definite integral that gives the total mass of the oil in the slick. Describe the meaning of each factor of your integral and give the units it's measured in. Evaluate the integral and explain the meaning of your result in context.

§3.5: Modeling with Definite Integrals

Work/Energy

"For those who want some proof that physicists are human, the proof is in the idiocy of all the different units which they use for measuring energy."

-Richard Feynman,
The Character of Physical Law

The amount of energy expended during a task is called work.

For a constant force F energy/work is defined as

Joule (J) calorie foot-pound

$$E = F \cdot d$$

Constant Force Example: Energy required to lift a 2-kg stone 3 m above the ground.

$E = F \cdot d$

Gravity pulls down on the stone of mass M with a force equal to $-M \cdot g$, where $g = 9.8\,m/s^2$.

So lifting the stone requires an upward vertical force of $F = M \cdot g$.

Recall YT 3.3.6 & 3.4.12: Energy required to compress a spring.

Spring arrives already compressed 5m.

Want to calculate the energy required to compress 5 additional meters.

Recall YT 3.3.6 & 3.4.12: Energy required to compress a spring.

$E = F \cdot d$

Why can't we just use $E = F \cdot d$ in this context?

Recall YT 3.3.6 & 3.4.12: Energy required to compress a spring.

$energy(J) = force(N) \cdot distance(m)$

Recall YT 3.3.6 & 3.4.12: Energy required to compress a spring.

$E = F \cdot d$

$force(N) = spring\ constant(N/m) \cdot distance\ from\ starting\ point(m)$

Recall YT 3.3.6 & 3.4.12: Energy required to compress a spring.

from 5m to an additional 5m.

$E = F \cdot d$

$F = k \cdot x$

$k = 155\ N/m$

Building a Cement Column: Compute the work (against gravity) required to build a cement column of height $5m$ and square base of side $2m$. Assume the cement has density $1500\ kg/m^3$.

$E = F \cdot d\ (J)$

$F = M \cdot g\ (N)$

Working against Gravity: On the earth's surface, work against gravity is equal to the force (mass · acceleration due to gravity) times the vertical distance through which the object is lifted. No work against gravity is done when an object is moved sideways.

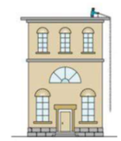
$$E = F \cdot d$$

$$F = M \cdot g$$

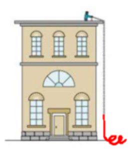
Your Turn 3.5.17: How much energy is required to build a cylindrical column structure of height $4\ m$ with radius $0.8\ m$ out of a lightweight material with density $600\ kg/m^3$?

Your Turn 3.5.18: Built around 2600 BCE, the Great Pyramid of Giza in Egypt is $146\ m$ high and has a square base of side $230\ m$. Find the energy required to build the pyramid if the density of the stone is estimated at $2000\ kg/m^3$.

Your Turn 3.5.19: Calculate the energy required to lift a $10m$ chain over the side of a building. Assume that the chain has a uniform density of $8\ kg/m$.



Your Turn 3.5.20: How much energy is required to lift a $12m$ hanging chain that has a uniform density of $3kg/m$ to a height of $10m$. (Note: 2 feet of the chain starts coiled on the ground)



Kinetic Energy: The kinetic energy of an object with mass M and constant speed v is $K = \frac{1}{2}Mv^2$, at least in the case where the entire object is moving at the same speed. Suppose a $0.1m$ rod has a mass of $0.03\ kg$ with uniform density. It is rotating around one of its ends at a rate of one revolution per minute (like the second hand of a clock). What is the total kinetic energy of the rod?

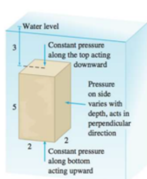
Your Turn 3.5.21: If the rod is only half as long, but moves twice as fast (still with a mass of .03kg), does the kinetic energy increase or decrease?

§3.5: Modeling with definite integrals

More on Force

Fluid Force is the force on an object submerged in a fluid.

- Fluid pressure p is proportional to depth.
- Fluid pressure does not act in a specific direction.



Recall YT 3.3.7 & 3.4.13: Using a definite integral to compute the total force exerted on the dam.

Force on an Inclined Surface

The side of a dam is inclined at an angle of $45°$. The dam has a height of $700\,ft$ and a width of 1500 ft. Calculate the force F on the dam if the reservoir is filled to the top of the dam. Water has density $\omega = 62.5\,lb/ft^3$.

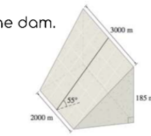
Gravitational Force

The gravitational force between two particles of mass m_1 and m_2 at a distance r apart is $F(r) = \frac{Gm_1m_2}{r^2}$, where G is the gravitational constant $6.67 \times 10^{-11} Nm^2/kg^2$. Write a definite integral that gives the gravitational attraction between a thin uniform rod of mass M and length l and a particle of mass m lying in the same line as the rod at a distance a from one end.

Your Turn 3.5.22: The massive Three Gorges Dam on China's Yangtze River has height $185\,m$. Write an integral that will calculate the force on the dam, assuming that the dam is a trapezoid of base $2000m$ and upper edge $3000m$, inclined at an angle of $55°$ to the horizontal. Include the meaning of each factor (including units) of the definite integral.

Draw a picture of the situation. For one partition, label the relevant quantities on your drawing.

Use your calculator to compute the total force on the dam.



Your Turn 3.5.23: By the end of this calculus course you'll be able to use the previous integral to show that if two objects of mass M and m are separated by a distance r_1, then the energy required to increase the separation to a distance r_2 is equal to $E = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$.

Use this result to calculate the energy required to place a 2000-kg satellite in an orbit 1200 km above the surface of the earth. Assume that the earth is a sphere of radius $R_e = 6.37 \times 10^6 m$ and mass $M_e = 5.98 \times 10^{24} kg$. Treat the satellite as a point mass.

§3.5: Modeling with definite integrals

Reinterpreting Accumulation as Area

Example: A plane travels from Los Angeles to San Francisco at a velocity of $v(t) = 880\sin(\pi \cdot t)$ km/hr. What does $\int_0^1 v(t)dt$ measure?

Rates: One of the really nice things about rates is that we can use graphs to visualize the accumulation process. I.e. we can use geometric properties to represent physical quantitative relationships.

Your Turn 3.5.24: A population of insects increases at a rate of $r(t) = 200 + 10t + 0.25t^2$ insects per day. Find the insect population after 3 days, assuming that there are 35 insects at day $t = 0$.

Draw a picture of the graph of this function. Partition the graph. For at least one partition, represent the geometric interpretation of your differential form. Label it with the relevant quantities. What does the product of $r(t)$ and dt represent geometrically? What does the product represent physically?

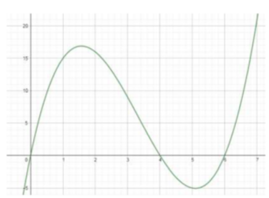
Your Turn 3.5.25: The heat capacity $C(T)$ of a substance is the amount of energy (in joules) required to raise the temperature of $1\,g$ by $1°C$ at temeperature T.

Explain why the energy required to raise the temperature from T_1 to T_2 is the area under the graph of $C(T)$ over $[T_1, T_2]$.

What would one infinitesimally small partition represent geometrically? Physically?

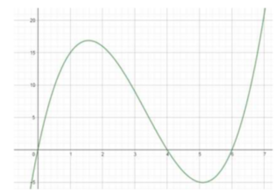
How much energy is required to raise the temperature from $50°C$ to $100°C$ if $C(T) = 6 + 0.2\sqrt{T}$.

Velocity specific:
Displacement vs. Distance Traveled



Speed in a given direction

Signed Area: A particle has velocity $v(t) = t^3 - 10t^2 + 24t$ m/s. How far away is the particle from its starting point at time t=6?



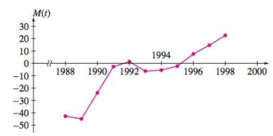
Your Turn 3.5.26: Write an integral to calculate the total distance a helicopter traveled in the first five minutes it was flying if the helicopter had a velocity of $v(t) = 4x - x^2$ mi/min. Draw a graph and label the relevant quantities for your definite integral.

Your Turn 3.5.27: Below is the migration rate $M(t)$ of Ireland in the period 1988 – 1998. This is the rate at which people (in thousands per year) move into or out of the country.

a) Is the following integral positive or negative? What does this quantity represent?

$$\int_{1988}^{1998} M(t)dt$$

b) Did migration in the period 1988 – 1998 result in a net influx or a net outflow of people from Ireland?



Your Turn 3.3.6: Total Energy Required to Compress a Spring

Sam is tired of walking up two flights of stairs to get to calculus class every day, so he enlists Kelli to help him build a giant spring to lift him perfectly up to the second floor window. They order a two-story tall spring from Katelyns Giant Spring Limited Liability Co. When it arrives, it is packaged already compressed down 5 m shorter than its resting length. They figure they need to compress it another 5 m in order to climb on from ground level before launch. Tony walks by and points out this will take a lot of energy, saying:

> For a constant force* F to move an object a distance d requires an amount of energy** equal to $E = Fd$. Hooke's Law says that the force exerted by a spring displaced by a distance x from its resting length (compressed or stretched) is equal to $F = kx$, where k is a constant that depends on the particular spring.
>
> > *The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s^2 or the force required to accelerate a 1 kg mass at 1 m/s^2. Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.
> >
> > **The standard unit of energy is Joules (J), where 1 J = 1 Nm or the energy required to move an object with a constant force of 1 N a distance of 1 m. Increasing either the force or the distance requires a proportional increase in energy.

Sam and Kelli's spring has a spring constant of $k = 155$ N/m.

A. Draw and label a picture of a spring initially compressed 5 m from its natural length then compressed to a displacement of 10 m.

B. Does it take less, the same, or more energy to compress the spring from 5 m to 7.5 m than it takes to compress the spring from 7.5 m to 10 m? Explain.

C. Explain why we cannot just multiply a force times a distance to compute the energy.

D. Use Riemann sums with 4 terms to find both an underestimate and overestimate for the energy required to compress the spring from 5 m to 10 m. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.

E. Write an algebraic expression for your error (use L to represent the quantity "Actual total energy required to compress the spring measured in Joules"). What is the bound on the error for your approximations? What is the range of possible values for the energy (in Nm) required to stretch the spring from 5 m to 10 m?

F. Find an approximation accurate to within 0.5 Joules.

G. Write a formula indicating how to find an approximation accurate to within any pre-determined error bound, ε.

Your Turn 3.3.7: Total Force Exerted on a Dam

Oh no! Chris accidentally broke Horsetooth Dam! Specifically, they cracked it loose from the canyon walls and floor, leaving nothing to hold back the tremendous force of water in the reservoir. Luckily, Erin is thinking quickly and braces herself against the dam to hold back the impending flood. Josh decides he should figure out exactly how much force Erin needs to exert to hold up the dam. Luckily Jarrod is able to provide the key information to figure this out:

> A uniform pressure P** applied across a surface area A creates a total force* of $F = PA$. The density of water is 1000 kg per cubic meter, so that under water the pressure varies according to depth, d, as $P = 9800d$. In this activity you will approximate the total force of the water exerted on a dam 63.26 meters wide and extending 25 meters under water.
>
> > *The standard unit of force is Newtons (N), where 1 N = 1 kg·m/s^2 or the force required to accelerate a 1 kg mass at 1 m/s^2. Increasing either the mass or the acceleration rate therefore requires a proportional increase in force.
> >
> > **Pressure is the force per unit area, $P = F/A$, so for example a force of 6 N applied over a 2 m^2 area would generate a pressure of 3 N/m^2. Increasing the force would increase the pressure proportionally. Increasing the area would decrease the pressure proportionally (an inverse proportion).

A. Draw and label a large picture of a dam 63.26 m wide and extending 25 m under water.

B. Is there less, the same, or more force on the top half of the dam or the bottom half? Explain.

C. Explain why we cannot just multiply a pressure times an area to compute the force.

D. Use a Riemann sum with 5 terms to find both an underestimate and overestimate for the total force of the water exerted on this dam. Write out your sums numerically and with summation notation. Illustrate the terms of your sum on your picture.

E. What is the error bound for each of these approximations?

F. Find an approximation accurate to within 50,000 N.

G. Write a formula indicating how to find an approximation accurate to within any pre-determined error bound, ε.

APPENDIX B

IRB APPROVAL LETTER

Oklahoma State University Institutional Review Board

Date:	03/18/2020
Application Number:	IRB-20-152
Proposal Title:	Investigations into the development and impact of a quantitatively based summation conception of integration
Principal Investigator:	Courtney Simmons
Co-Investigator(s):	
Faculty Adviser:	Michael Oehrtman
Project Coordinator:	
Research Assistant(s):	
Processed as:	Exempt
Exempt Category:	

Status Recommended by Reviewer(s): Approved

The IRB application referenced above has been approved. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in 45CFR46.

This study meets criteria in the Revised Common Rule, as well as, one or more of the circumstances for which continuing review is not required. As Principal Investigator of this research, you will be required to submit a status report to the IRB triennially.

The final versions of any recruitment, consent and assent documents bearing the IRB approval stamp are available for download from IRBManager. These are the versions that must be used during the study.

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be approved by the IRB. Protocol modifications requiring approval may include changes to the title, PI, adviser, other research personnel, funding status or sponsor, subject population composition or size, recruitment, inclusion/exclusion criteria, research site, research procedures and consent/assent process or forms.
2. Submit a request for continuation if the study extends beyond the approval period. This continuation must receive IRB review and approval before the research can continue.
3. Report any unanticipated and/or adverse events to the IRB Office promptly.
4. Notify the IRB office when your research project is complete or when you are no longer affiliated with Oklahoma State University.

Please note that approved protocols are subject to monitoring by the IRB and that the IRB office has the authority to inspect research records associated with this protocol at any time. If you have questions about the IRB procedures or need any assistance from the Board, please contact the IRB Office at 405-744-3377 or irb@okstate.edu.

Sincerely,

Oklahoma State University IRB

VITA

Courtney Simmons

Candidate for the Degree of

Doctor of Philosophy

Dissertation: INVESTIGATION INTO THE DEVELOPMENT OF A QUANTITATIVELY BASED SUMMATION CONCEPTION OF THE DEFINITE INTEGRAL

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in July, 2021.

Completed the requirements for the Master of Science in Mathematics at Oklahoma State University, Stillwater, Oklahoma in 2021.

Completed the requirements for the Bachelor of Science in Mathematics at University of Central Oklahoma, Edmond, Oklahoma in 2013.

Experience:

Graduate TA, Oklahoma State University, 2013-2017
Graduate RA, Oklahoma State University, 2017-2020
Graduate TA, Oklahoma State University, 2020-2021

Professional Memberships:

American Mathematical Society,
Research in Undergraduate Mathematics Education Special Interest Group of the Mathematical Association of America
Association for Women in Mathematics