PROPER MAPS AND INVOLUTIONS OF UNIT BALLS IN EUCLIDEAN LEVI-FLAT SPACES

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Abstract: As models of strictly pseudoconvex domains, we consider holomorphic functions on the unit ball $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$. In particular, we focus on proper holomorphic maps $\mathbb{B}_n \to \mathbb{B}_N$. In the equidimensional case $N = n$, proper holomorphic maps are automorphisms. We discuss the parameters associated to automorphisms, and more generally involutions and their higher-order analogues.

We then define the mixed spaces $\mathbb{B}_{n,k} = \{(z, s) \in \mathbb{C}^n \times \mathbb{R}^k : |z|^2 + |s|^2 < 1\}$, and address similar questions regarding proper maps, automorphisms, and involutions in the new setting. In particular, we show how to recover the parameters that determine an automorphism of $\mathbb{B}_{n,k}$ using the germ at $z = 0$. We also specify necessary conditions on involutions in both the $\mathbb{B}_n$ and $\mathbb{B}_{n,k}$ settings.

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CHAPTER I

INTRODUCTION

We start with a brief introduction to the difficulties distinguishing the study of one complex variable from the study of several complex variables.

Recall that a function $f : \mathbb{C} \to \mathbb{C}$ of a complex variable z is called **holomorphic** (or complex-analytic) if it is complex-differentiable. One way to rigorously define this notion is to start by identifying $\mathbb{C}$ with $\mathbb{R}^2$ (topologically), and write $z = x + iy$ as (x, y) (with $x, y \in \mathbb{R}$). Similarly writing $f : \mathbb{C} \to \mathbb{C}$ as $u + iv$, or instead $f : \mathbb{R}^2 \to \mathbb{R}^2$ as (u, v), so that the original function $f(z)$ is now interpreted as $f(x, y) = (u(x, y), v(x, y))$. The Cauchy-Riemann equations say that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

An attractive alternative notation is to define two operators, called **Wirtinger operators**, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Using this notation, a C^1 function $f : \mathbb{C} \to \mathbb{C}$ is complex-differentiable if and only if $\frac{\partial f}{\partial \bar{z}} = 0$. *All functions of complex variables are assumed to be holomorphic.*

For functions in higher complex dimensions, $f : \mathbb{C}^n \to \mathbb{C}^N$, which are functions of several complex variables $z_1, \ldots, z_n$, we will call f holomorphic if f is holomorphic in each complex variable separately, and is jointly continuous. This is sufficient (see e.g. [Leb19] or [GR65]) to conclude that f enjoys a power series expansion in a ball of positive radius at every point where it is complex-differentiable.

We will frequently make use of biholomorphic functions. A function $f : U \to V$ is **biholomorphic** if both f and f^{-1} are holomorphic. This implies, in particular, that f is one-to-one, and that the matrix of first derivatives of f has full rank at every point.

1.1 Domains of Holomorphy

An essential result in complex analysis (of one variable) is the Riemann Mapping Theorem, which shows, among other things, that every connected, simply-connected, open subset of the complex plane is biholomorphically equivalent to the unit disk $\mathbb{D}$, with the exceptions of the empty set and the whole complex plane itself. That is, if $U \subsetneq \mathbb{C}$ is a non-empty open set that is both connected and simply-connected, then there is a pair of holomorphic functions Φ, Ψ such that $\Phi(U) = \mathbb{D}$, $\Psi(\mathbb{D}) = U$, and both compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity. As a consequence, if one wants to understand the functions say $U \to X$, it suffices to understand the functions $f : \mathbb{D} \to X$, since $f \circ \Phi : U \to X$.

In a substantial sense, all open, connected, simply-connected non-empty proper subsets of $\mathbb{C}$ are thus equivalent. By simply comparing the function algebras on these domains, they cannot be distinguished. One important, but easily overlooked, quality of these domains is that they are examples of domains of holomorphy. A **domain of holomorphy** is a connected open set $D \subset \mathbb{C}^n$ that is the natural domain of definition for some holomorphic functions, in that for each point p in the boundary of D, there exists a holomorphic function on D which does not extend past p. When $n > 1$, not all open, connected, simply-connected, non-empty proper subsets of $\mathbb{C}^n$ are the natural domain of definition for a holomorphic function, in contrast to the situation in one complex variable, where the functions $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ and $g(z) = \sum_{n=0}^{\infty} z^{n!}$ are both holomorphic, but do not extend to any open set containing the unit disk, illustrating that the disk is a domain of holomorphy. By contrast, consider the following example, which is intimately related to the fact that the zero set of a holomorphic function cannot have real codimension one.

Example 1.1.1 *The unit ball $\mathbb{B}_2 = \{(z, w) \in \mathbb{C}^n : |z|^2 + |w|^2 < 1\}$ is a domain of holomor-*

phy. For any point $p \in \mathbb{B}_2$, the open set $\mathbb{B}_2 \setminus \{p\}$ is not *a domain of holomorphy.*

We refer the interested reader to Krantz's excellent introductory article [Kra87] for a proof. The idea is to expand any holomorphic function $f : \mathbb{B}_2 \to \mathbb{B}_2$ as a Laurent series in w, and argue that the coefficients (which are functions of the other variable) must vanish by examining them on one-variable complex disks lying in the complex planes of the form $z = a$ where a is nonzero. In particular, any holomorphic function defined on $\mathbb{B}_2 \setminus \{p\}$ must extend holomorphically to a function on $\mathbb{B}_2$.

Lest the reader feel unsatisfied that the domain, though simply-connected, is not "connected enough" in the sense that its nth fundamental group[1] is nontrivial, we refer the reader to Kaup and Kaup [KK83] for a wonderfully illustrated version of the example below, which is known as a Hartog's figure (as well as an illustration of a camel which, surprisingly, does provide intuition about why this example works).

Example 1.1.2 *The topologically trivial set H which is the union of $\{(z,w) \in \mathbb{D} \times \mathbb{D} : \frac{1}{2} < |z| < 1\}$ with $\{(z,w) \in \mathbb{D} \times \mathbb{D} : |w| < \frac{1}{2}\}$ is* not *a domain of holomorphy. Given* any *function f that is holomorphic on H, there is a holomorphic function F on the set $\mathbb{D} \times \mathbb{D}$ such that $F|_H = f$.*

As a consequence of the existence of domains that are not domains of holomorphy, the astute reader will have surmised that the Riemann Mapping Theorem fails for functions of several (i.e. two or more) complex variables. Even more surprisingly, Poincaré showed [Car31] that two of the natural generalizations of the disk to $\mathbb{C}^2$, namely $\mathbb{B}_2 = \{(z,w) : |z|^2 + |w|^2 < 1\}$ and $\Delta_2 = \{(z,w) : |z| < 1 \text{ and } |w| < 1\}$, are not biholomorphically equivalent domains, despite the fact that both *are* domains of holomorphy, and are even homeomorphic. Consequently, it is not sufficient to study the function theory on the reader's favorite domain of holomorphy in $\mathbb{C}^2$ (or beyond).

[1] We will not need the definition of this in this dissertation, so we omit it.

1.2 Pseudoconvexity

The unit ball $\mathbb{B}_n$ defined as $\{z \in \mathbb{C}^n : |z|^2 = \sum |z_k|^2 < 1\}$ is a reasonable first domain of holomorphy to study. The primary alternative, the so-called polydisc $\Delta_n = \{z \in \mathbb{C}^n : \max_k |z_k| < 1\}$, has a product structure, and so enjoys considerably fewer symmetries. Further, $\mathbb{B}_n$ lacks any notion of 'preferred' or 'distinguished' direction. As an added bonus that we will not make use of, the unit ball (or its boundary) is the appropriate set to consider when computing norms of linear operators on, say, finite-dimensional vector spaces.

To be more formal, we must first introduce one of multiple equivalent definitions of convexity as relevant to the area. Namely, we will describe the notion of Levi-pseudoconvexity.

First recall a standard definition of convexity. Suppose $U \subseteq \mathbb{R}^n$ is an open set with boundary prescribed by a smooth function $\rho : \mathbb{R}^n \to \mathbb{R}$ with derivative which doesn't vanish near the boundary, in the sense that $\rho(x) < 0$ if and only if $x \in U$, and $\rho(x) = 0$ if and only if x is in the boundary of U, denoted ∂U. (We say that ρ is a **defining function** for U.) The **Hessian** matrix of the defining function at a point $x \in \partial U$ is the matrix of second-partial derivatives

$$H = \begin{bmatrix} \frac{\partial^2 \rho}{\partial x_1^2} & \frac{\partial^2 \rho}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 \rho}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \rho}{\partial x_2 \partial x_1} & \frac{\partial^2 \rho}{\partial x_2^2} & \cdots & \frac{\partial^2 \rho}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \rho}{\partial x_n \partial x_1} & \frac{\partial^2 \rho}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 \rho}{\partial x_n^2} \end{bmatrix}.$$

An open set with smooth boundary is **convex** if and only if the Hessian matrix is positive semi-definite (as a form when restricted to the tangent space) at every point of the boundary. That is, if for every vector v based at point $p \in \partial U$ that is tangent to ∂U, $v^T H v \geq 0$, then U is convex.

Now to define pseudoconvexity, we can alter the above definition in a natural way. Suppose that $U \subseteq \mathbb{C}^n$ is an open set with boundary prescribed by a smooth defining function $\rho : \mathbb{C}^n \to \mathbb{R}$ with derivative which doesn't vanish near the boundary, so $\rho(z) < 0$ if and only if $z \in U$, with $\rho(z) = 0$ if and only if z is in the boundary ∂U. Define the **complex Hessian** matrix of the defining function at a point $z \in \partial U$ to be the matrix

$$L = \begin{bmatrix} \frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_1} & \frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_2} & \cdots & \frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_n} \\ \frac{\partial^2 \rho}{\partial \bar{z}_2 \partial z_1} & \frac{\partial^2 \rho}{\partial \bar{z}_2 \partial z_2} & \cdots & \frac{\partial^2 \rho}{\partial \bar{z}_2 \partial z_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \rho}{\partial \bar{z}_n \partial z_1} & \frac{\partial^2 \rho}{\partial \bar{z}_n \partial z_2} & \cdots & \frac{\partial^2 \rho}{\partial \bar{z}_n \partial z_n} \end{bmatrix}.$$

An open set with smooth boundary is **pseudoconvex** if and only if the complex Hessian matrix is positive semi-definite (as a form when restricted to the complex tangent space) at every point of the boundary. That is, if for every vector v based at point $p \in \partial U$ that is tangent to ∂U in the sense that $\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}\Big|_p v_j = 0$, we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial \bar{z}_j \partial z_k}\Big|_p \bar{v}_j v_k \geq 0$$

(which is more easily expressed as $v^\dagger L v \geq 0$), then U is pseudoconvex.

The unit ball stands as a prototypical example among the most fruitful class of domains known, namely the (strictly) pseudoconvex domains. In fact, every pseudoconvex domain with smooth boundary is locally biholomorphic to the unit ball up to second order (see [Leb19, Lemma 2.3.8] for discussion of the technical details), and so the unit ball locally models the boundaries of such domains.

1.3 Proper Maps

Definition 1.3.1 *A map between subsets of Euclidean spaces*[2] $f : X \to Y$ *is called proper if the preimage of any compact set* $K \subseteq Y$ *is also compact.*

This generalizes invertibility by declaring that "f^{-1} is continuous", were it a function. This definition is equivalent to the following standard characterization.

[2]We will use $\mathbb{R}^n$ with the usual dot product, or $\mathbb{C}^n$ with the sesquilinear inner product $\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle$ given by $\bar{w}_1 z_1 + \ldots + \bar{w}_n z_n$.

Proposition 1.3.1 *For a map between bounded open subsets X and Y of $\mathbb{R}^n$, the map $f : X \to Y$ is called proper if the image of any sequence which approaches the boundary of X approaches the boundary of Y.*

Thorough discussions of the importance of proper maps in the setting of complex manifolds and varieties can be found in [GR65] and especially the comprehensive survey of Forstneric [For93].

1.4 Background on the Unit Ball

Proper maps between unit balls $\mathbb{B}_n$ and $\mathbb{B}_N$ have been extensively studied over the past 50 years; see Section 2.8 for a small sample of results. In the case $\mathbb{B}_n \to \mathbb{B}_N$ with $N < n$, i.e., when the target space has lower dimension than the domain, there are no proper maps. If proper $f : \mathbb{B}_n \to \mathbb{B}_N$ were to exist, then the preimage of a point p would be a compact complex variety contained strictly inside the domain $\mathbb{B}_n$ (recall that since f is proper, there can be no sequence $\{x_n\}$ approaching the boundary of the domain for which $f(x_n) - p$ approaches zero), and hence a finite set (see [Rud08, Theorem 14.3.1] for a proof). On the other hand, if N is less than n, then the preimage of $p \in f(\mathbb{B}_n)$ would have to have positive dimension. Therefore only the cases when $N \geq n$ are interesting.

Alexander [Ale77a] showed that proper maps between unit balls in equidimensional complex spaces are automorphisms, that is, if $n > 1$ and $f : \mathbb{B}_n \to \mathbb{B}_n$ is proper, then f is an automorphism. By contrast, a similar statement for $\Delta^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| < 1, \ldots, |z_n| < 1\}$ fails to be true: since Δ^n is a product domain, with each factor being $\mathbb{B}_1$, and since [CV13] proper maps on product domains decompose as (permutations of) products of proper maps, we can easily see that maps such as

$$(z_1, z_2, z_3) \mapsto \left(z_2^4, \frac{\frac{1}{2} - z_1^5}{1 - \frac{1}{2} z_1^5}, z_3 \right)$$

are clearly not invertible, but are indeed proper[3]. See Section 2.1 below for discussion of

[3] To verify properness, we can compute the three quantities $1 - |z_2^4|^2$, $1 - \left| \frac{\frac{1}{2} - z_1^5}{1 - \frac{1}{2} z_1^5} \right|^2$, and $1 - |z_3|^2$. The first

what the factors may look like.

We will see more proper maps Section 2.8.

1.5 Outline of Thesis

In chapter two, we give an overview of the relevant literature on proper maps, and give results needed for chapter three. In Section 2.1, we restate the classical single complex variable material on proper maps. In Section 2.2, we discuss specifics about the proper self-maps of the unit disk which are linear fractional transformations. In Section 2.3, we fully classify those self-maps of the unit disk that are their own inverses. In Section 2.4, we generalize this to give conditions on the form for self-maps of the unit disk that generate finite groups under composition.

Sections 2.5-7 replicate as much as possible the material of Sections 2.2-4, but now in the context of the unit ball in $\mathbb{C}^n$. We note that the material of Section 2.1 does not need to be replicated, due to the result cited from [Ale77a] in the last section.

The third chapter consists of entirely new material defining the domains $\mathbb{B}_{n,k}$ as the unit balls in $\mathbb{C}^n \times \mathbb{R}^k$. We consider the results of chapter two as adapted for these new domains. In Section 3.2, we discuss homotopy equivalence and spherical equivalence. In Section 3.3, we discuss some results about extending proper maps to the closed ball $\overline{\mathbb{B}}_{n,k}$, as well as illustrate the fundamental imbalance between the treatment of the real and complex coordinates. In Section 3.4, we adapt work from Sections 2.2 and 2.5 to the new domains. In Section 3.5, we discuss some restrictions on these maps when we impose some rationality conditions on the real coordinates. In Sections 3.6 and 3.7, we again adapt the material from Sections 2.3 and 2.6 to this new setting.

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factors as the product of $1 - |z_2|^2$ with $1 + |z_2|^2 + |z_2|^4 + |z_2|^6$; the second factors as $1 - |z_1|^2$ multiplied by

$$3\frac{|z_1|^8 + |z_1|^6 + |z_1|^4 + |z_1|^2 + 1}{|z_1|^{10} - 4|z_1|^5 + 4}.$$

CHAPTER II

PROPER MAPS $\mathbb{B}_n \to \mathbb{B}_N$

In this chapter, we will provide an introductory discussion of proper holomorphic maps whose domain is $\mathbb{B}_n$ and codomain is $\mathbb{B}_N$ with particular emphasis on the $N = n$ case. The details differ for the two domain cases $\mathbb{B}_1$ and $\mathbb{B}_n$ $(n > 1)$.

2.1 Proper Maps $\mathbb{B}_1 \to \mathbb{B}_1$

Proper maps from the unit disk to itself include examples such as $z \mapsto e^{i\theta} \frac{a-z}{1-\bar{a}z}$ where $a \in \mathbb{B}_1$ and $\theta \in \mathbb{R}$, and the maps $z \mapsto z^\ell$ where $\ell \in \{0, 1, \ldots\}$. In fact, by multiplying and composing such maps, this is a complete characterization.

Proposition 2.1.1 *Suppose f is a proper holomorphic map $\mathbb{D} \to \mathbb{D}$. There exists a finite sequence of (not necessarily distinct) points $a_1, \ldots, a_L$ in $\mathbb{D}$, and a real number θ, such that*

$$f(z) = e^{i\theta} \prod_{\ell=1}^{L} \frac{a_\ell - z}{1 - \bar{a}_\ell z}.$$

This can be proven by recognizing that $f^{-1}(0)$ must be a finite set, dividing f by the finite product, and then arguing that the quotient must be a constant.

Note that when $L = 1$, the form is a linear fractional transformation of $\mathbb{D}$, which is in fact a one-to-one function. When the points $a_1, \ldots, a_L$ take the common value of zero, $f(z)$ reduces to $e^{i\theta} z^L$.

2.2 Linear Fractional Transformations on $\mathbb{B}_1$

It is well known (see e.g. [Ull08]) that $f : \mathbb{D} \to \mathbb{D}$ is an automorphism if and only if f can be written in the form

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

where θ is a real number and a is a point in $\mathbb{D}$. If we have a function $f : \mathbb{D} \to \mathbb{D}$ that is known to be an automorphism, then we can extract the parameters θ and a in a variety of ways. One such choice is by noting that $f(a) = 0$ implicitly defines a. Subsequently, we can find $e^{i\theta}$ using the fact $f(0) = e^{i\theta}a$ (at least if a is nonzero). This does not determine θ uniquely, since for any solution θ_0 for θ, the choice $\theta_1 = 2\pi + \theta_0$ is also a solution.

An alternative means of finding the parameters is as follows. Notice that $f(0) = e^{i\theta}a$ and that because

$$\frac{d}{dz} \frac{a - z}{1 - \bar{a}z} = \frac{-1 + |a|^2}{(1 - \bar{a}z)^2}$$

we have the simple expression

$$f'(0) = -e^{i\theta}(1 - |a|^2).$$

Since $|f(0)| = |a|$, we can write $f'(0)$ as $-e^{i\theta}(1 - |f(0)|^2)$, and so we conclude that

$$e^{i\theta} = -\frac{f'(0)}{1 - |f(0)|^2}$$

which also implies that

$$a = \frac{f(0)}{e^{i\theta}} = -\frac{f(0)}{f'(0)}(1 - |f(0)|^2).$$

Note that these expressions for a and $e^{i\theta}$ are valid no matter the values of $f(0) \in \mathbb{D}$ and $f'(0)$ (if nonzero). However, observe that $f'(0)$ cannot be zero, as evidenced by the form $f'(0) = -e^{i\theta}(1 - |a|^2)$ above.

The non-uniqueness of such an expression can be illustrated by giving an alternative form for a. Notice first that $f'(0) = -e^{i\theta}(1 - |a|^2)$ and $|a| < 1$ together imply that $|f'(0)| = 1 - |a|^2$

which can in turn be written as $1 - |f(0)|^2$. Now

$$\begin{aligned} a &= -\frac{f(0)}{f'(0)}(1 - |f(0)|^2) \\ &= -\frac{f(0)\overline{f'(0)}}{|f'(0)|^2}(1 - |f(0)|^2) \\ &= -\frac{f(0)\overline{f'(0)}}{(1 - |f(0)|^2)^2}(1 - |f(0)|^2) \\ &= -\frac{f(0)\overline{f'(0)}}{1 - |f(0)|^2} \end{aligned}$$

These formulas and computations will be generalized later in Section 2.5.

2.3 Involutions of $\mathbb{B}_1$

An interesting subclass of these functions that will be discussed further later is the class of *involutions*. Recall the definition.

Definition 2.3.1 *A function $f : X \to X$ is called an involution of X if $f \circ f$ is the identity function id.*

More generally, one can study functions $f : X \to X$ that satisfy *Babbage's equation* $f \circ \cdots \circ f = \text{id}$. Define $f^{\circ(1)} = f$ and for any positive integer n, set $f^{\circ(n)} = f \circ f^{\circ(n-1)}$. We use the notation $f^{\circ n}$, or $f^{\circ(n)}$. (The reader may be familiar with a common notation $f^{(n)}$ in the literature, but we avoid this to prevent confusion with a common notation for derivatives.) With this notation, Babbage's equation says $f^{\circ n} = \text{id}$, or if we want to emphasize the input, $f^{\circ n}(z) \equiv z$.

Solutions to Babbage's equation generate finite subgroups of maps (with composition as the operation) within the larger automorphism or endomorphism groups. In particular, involutions are natural candidates to use to conjugate other maps, since $f^{-1} = f$, and thus $f^{-1} \circ g \circ f = f \circ g \circ f$ is easy to calculate.

Lemma 2.3.1 *The involutions of the unit disk $\mathbb{D} = \mathbb{B}_1$ are precisely those linear fractional* *transformations which can be written in the form*

$$f(z) = \frac{a-z}{1-\bar{a}z} \quad \textit{or} \quad f(z) = z.$$

Proof. As stated at the start of 2.2, automorphisms of $\mathbb{D}$ are rotations of linear fractional transformations

$$f(z) = f(z; a, \theta) = e^{i\theta}\frac{a-z}{1-\bar{a}z}$$

where $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

Direct computation allows us to compute conditions on a and θ such that f is an involution. If $z \equiv f(f(z))$, then we must have

$$z \equiv e^{i\theta}\frac{a - \left(e^{i\theta}\dfrac{a-z}{1-\bar{a}z}\right)}{1-\bar{a}\left(e^{i\theta}\dfrac{a-z}{1-\bar{a}z}\right)}.$$

Rearranging,

$$ze^{-i\theta} \equiv \frac{a(1-\bar{a}z) - e^{i\theta}(a-z)}{(1-\bar{a}z) - \bar{a}e^{i\theta}(a-z)} = \frac{(-|a|^2 + e^{i\theta})z + (1-e^{i\theta})a}{(1-|a|^2e^{i\theta}) - \bar{a}(1-e^{i\theta})z}.$$

By cross-multiplying, we see the condition that

$$(1-|a|^2e^{i\theta})ze^{-i\theta} - \bar{a}(e^{-i\theta}-1)z^2 \equiv (-|a|^2 + e^{i\theta})z + (1-e^{i\theta})a,$$

or equivalently,

$$-\bar{a}(e^{-i\theta}-1)z^2 + (e^{-i\theta} - |a|^2 + |a|^2 - e^{i\theta})z - (1-e^{i\theta})a \equiv 0.$$

Since this must hold for all $z \in \mathbb{D}$, then in particular we know that it holds at $z = 0$, and hence either $a = 0$ or $e^{i\theta} = 1$.

In the case that $a = 0$, the quadratic condition reduces to $(e^{-i\theta} - e^{i\theta})z \equiv 0$. We can evaluate this condition at nonzero values of z, and so conclude that $e^{-i\theta} - e^{i\theta}$ must vanish. As a consequence, $e^{i\theta}$ must be either 1 or -1. Hence, we have $f(z) = \pm\dfrac{0-z}{1-0} = \mp z$.

In the case that $e^{i\theta} = 1$, the quadratic condition reduces to $0 \equiv 0$, so a is not further restricted. Hence $f(z) = \dfrac{a-z}{1-\bar{a}z}$ for any $a \in \mathbb{D}$ works in the case that $e^{i\theta} = 1$. Note that if $a = 0$ here, we recover the function $f(z) = -z$. ■

2.4 Higher-order self-maps of the Disk

While the content of this section certainly exists somewhere in the work of Klein, among others, and the study of $\mathrm{SL}(2, \mathbb{C})$, we include these results not to claim that they are new, but for the parallels in Section 2.7.

A strategy similar to that of the last section works to classify the linear fractional transformations $f : \mathbb{D} \to \mathbb{D}$ satisfying $f(f(f(z))) = z$. Indeed, this method can be generalized for the general Babbage's equation, and partial results are given.

Assume that $f : \mathbb{D} \to \mathbb{D}$ is a linear fractional transformation, so $f(z)$ can be written in the form $f(z) = e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$ for some real parameter θ and some $a \in \mathbb{D}$. (Note the slight change in form, which will simplify the signs in the computation below.) We consider the situation that a composition of f with itself N times is the identity function:

$$z = f^{\circ N}(z) := \underbrace{f \circ f \circ \cdots \circ f}_{N \text{ times}}(z).$$

If $a = 0$, then $f(z)$ has the simple form $e^{i\theta}z$, and the compositions $f^{\circ N}(z)$ have the form

$$e^{i\theta}(e^{i\theta}(\cdots(e^{i\theta}z)\cdots))$$

which simplifies to $e^{iN\theta}z$. Consequently, in the case that f fixes zero, the condition $f^{\circ N}(z) \equiv z$ is equivalent to saying that $e^{iN\theta} = 1$. Hence $e^{i\theta}$ must be an Nth root of unity. Further, we can state that f does not satisfy a lower-order identity of the form $f^{\circ n}(z) \equiv z$ if and only if $e^{i\theta}$ is a primitive Nth root of unity, i.e. a root of the Nth cyclotomic polynomial (commonly denoted Φ_N).

If the parameter a is nonzero, the form for $f^{\circ N}$ becomes significantly more complicated.

Lemma 2.4.1 *If $f : \mathbb{D} \to \mathbb{D}$ is given by $f(z) = e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$, then we can write $f^{\circ n}(z)$ in the form*

$$e^{i\theta}\frac{\alpha_n z - \beta_n a}{\gamma_n - \bar{a}\delta_n z}$$

where the coefficients $\alpha_n, \beta_n, \gamma_n, \delta_n$ *are determined according to the recursive equations*

$$\begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \\ \gamma_{n+1} \\ \delta_{n+1} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 & 0 & |a|^2 \\ 0 & 1 & e^{i\theta} & 0 \\ 0 & 1 & |a|^2 e^{i\theta} & 0 \\ e^{i\theta} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \\ \delta_n \end{bmatrix}$$

along with the initial values $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 1$.

Proof. The proof is a straight-forward induction argument. When $n = 1$, we have $f^{\circ 1}(z)$ just representing $f(z)$ itself, and so the initial values of the coefficients are verified.

Assume that $f^{\circ n}(z)$ has the form

$$f^{\circ n}(z) = e^{i\theta} \frac{\alpha_n z - \beta_n a}{\gamma_n - \bar{a}\delta_n z}$$

and consider now $f(f^{\circ n}(z))$.

$$\begin{aligned} f(f^{\circ n}(z)) &= f\left(e^{i\theta} \frac{\alpha_n z - \beta_n a}{\gamma_n - \bar{a}\delta_n z} \right) \\ &= e^{i\theta} \frac{\left(e^{i\theta} \frac{\alpha_n z - \beta_n a}{\gamma_n - \bar{a}\delta_n z} \right) - a}{1 - \bar{a} \left(e^{i\theta} \frac{\alpha_n z - \beta_n a}{\gamma_n - \bar{a}\delta_n z} \right)}. \end{aligned}$$

Multiplying through the top and bottom on the right-hand side, and regrouping terms in order to factor out z appropriately,

$$\begin{aligned} f(f^{\circ n}(z)) &= e^{i\theta} \frac{e^{i\theta}\alpha_n z - e^{i\theta}\beta_n a - a\gamma_n + |a|^2 \delta_n z}{\gamma_n - \bar{a}\delta_n z - \bar{a} e^{i\theta}\alpha_n z + \beta_n |a|^2 e^{i\theta}} \\ &= e^{i\theta} \frac{(e^{i\theta}\alpha_n + |a|^2\delta_n) z - (e^{i\theta}\beta_n + \gamma_n) a}{(\gamma_n + |a|^2 \beta_n e^{i\theta}) - \bar{a}(\delta_n + \alpha_n e^{i\theta}) z}. \end{aligned}$$

Comparing this with $f^{\circ(n+1)}(z) = e^{i\theta} \frac{a_{n+1} z - \beta_{n+1} a}{\gamma_{n+1} - \bar{a}\delta_{n+1} z}$, we see that we need

$$\alpha_{n+1} = e^{i\theta}\alpha_n + |a|^2 \delta_n$$

$$\beta_{n+1} = e^{i\theta}\beta_n + \gamma_n$$

$$\gamma_{n+1} = \gamma_n + |a|^2 \beta_n e^{i\theta}$$

$$\delta_{n+1} = \delta_n + \alpha_n e^{i\theta}$$

This agrees with the matrix formulation stated above. ∎

It is worth observing that the system of four equations above can be decoupled into two pairs of equations. One such formulation is

$$\begin{bmatrix} \alpha_{n+1} \\ \delta_{n+1} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & |a|^2 \\ e^{i\theta} & 1 \end{bmatrix} \begin{bmatrix} \alpha_n \\ \delta_n \end{bmatrix}$$

and

$$\begin{bmatrix} \gamma_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & |a|^2 e^{i\theta} \\ 1 & e^{i\theta} \end{bmatrix} \begin{bmatrix} \gamma_n \\ \beta_n \end{bmatrix}$$

In this presentation, it is much clearer that the two inductive processes are intimately related. Indeed, one can write the transition matrices as follows:

$$\begin{bmatrix} e^{i\theta} & |a|^2 \\ e^{i\theta} & 1 \end{bmatrix} = \begin{bmatrix} 1 & |a|^2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & |a|^2 e^{i\theta} \\ 1 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 1 & |a|^2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

In order to focus on the solutions of Babbage's equation, $f^{\circ n}(z) \equiv z$, we prove the following.

Lemma 2.4.2 *If $f : \mathbb{D} \to \mathbb{D}$ is a linear fractional transformation that does not preserve the origin, and if $f(z) \equiv z$, then when writing $f(z)$ in the form $e^{i\theta}\dfrac{\alpha z - \beta a}{\gamma - \bar{a}\delta z}$, we have $\delta = 0$, $\beta = 0$, and $e^{i\theta}\alpha/\gamma = 1$.*

Proof. Suppose that $f(z) \equiv z$. We can write

$$f(z) = e^{i\theta}\frac{\alpha z - \beta a}{\gamma - \bar{a}\delta z}.$$

Since f does not preserve the origin, then a is nonzero (otherwise $f(0) = a$ implies that $f(0) = 0$, i.e. f does preserve the origin). If $\delta \neq 0$, then clearly $f(z)$ has a pole at $\gamma/\bar{a}\delta$, i.e., f does not simplify to the entire function z.

Thus $\delta = 0$ is necessary. Consequently, we consider $f(z) = e^{i\theta}\dfrac{\alpha z - \beta a}{\gamma}$. In order for this to be defined, certainly $\gamma \neq 0$. We observe now that we have

$$z \equiv f(z) = \left(e^{i\theta}\frac{\alpha}{\gamma}\right) z + \left(-\frac{\beta a}{\gamma}\right)$$

and so (say, by the Identity Theorem[1], or simply by using the linear independence of monomials) it follows that $e^{i\theta}\alpha/\gamma$ is 1, and $-\beta a/\gamma = 0$. Since we have supposed that both $a \neq 0$ and $\gamma \neq 0$, we conclude further that $\beta = 0$. ■

We can actually say more about the terms appearing in the sequences of Lemma 2.4.1.

Lemma 2.4.3 *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a linear fractional transformation. Write $f^{\circ n}(z)$ as $e^{i\theta}\dfrac{\alpha_n z - \beta_n a}{\gamma_n - \bar{a}\delta_n z}$. The four sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ satisfy the same recurrence relation (with different initial conditions), and $\beta_n = \delta_n$ for all n.*

Proof. Rewrite the pair of equations

$$\alpha_{n+1} = e^{i\theta}\alpha_n + |a|^2\delta_n$$
$$\delta_{n+1} = \delta_n + \alpha_n e^{i\theta}$$

from the proof of Lemma 2.4.1 in the form

$$\alpha_{n+1} - e^{i\theta}\alpha_n = |a|^2\delta_n$$
$$\delta_{n+1} - \delta_n = \alpha_n e^{i\theta}$$

and take turns eliminating the variables to find the pair of second-order equations

$$e^{-i\theta}(\delta_{n+2} - \delta_{n+1}) - (\delta_{n+1} - \delta_n) = |a|^2\delta_n$$

and

$$(\alpha_{n+2} - e^{i\theta}\alpha_{n+1}) - (\alpha_{n+1} - e^{i\theta}\alpha_n) = |a|^2\alpha_n e^{i\theta}.$$

It is not hard to see that $\{\alpha_n\}$ and $\{\delta_n\}$ both satisfy the equation

$$x_{n+2} - (1 + e^{i\theta})x_{n+1} + e^{i\theta}x_n = |a|^2 e^{i\theta}x_n.$$

[1] In one complex variable, the following standard result holds: if $f \equiv 0$ on an open set, then all coefficients of f in a power series expansion (around a point in the open set) must vanish.

In the case appearing in the theorem, we compare two linear polynomials z and $mz + b$, and use the Identity Theorem applied to the difference $f(z) := z - (mz + b)$.

Similarly, the pair of equations

$$\gamma_{n+1} = \gamma_n + |a|^2 e^{i\theta} \beta_n$$
$$\beta_{n+1} = \gamma_n + e^{i\theta} \beta_n$$

can be rewritten as

$$\gamma_{n+1} - \gamma_n = |a|^2 e^{i\theta} \beta_n$$
$$\beta_{n+1} - e^{i\theta} \beta_n = \gamma_n$$

whence

$$(\beta_{n+2} - e^{i\theta}\beta_{n+1}) - (\beta_{n+1} - e^{i\theta}\beta_n) = |a|^2 e^{i\theta}\beta_n$$

and

$$e^{-i\theta}(\gamma_{n+2} - \gamma_{n+1}) - (\gamma_{n+1} - \gamma_n) = |a|^2 \gamma_n.$$

These also both satisfy

$$x_{n+2} - (1 + e^{i\theta})x_{n+1} + e^{i\theta} x_n = |a|^2 e^{i\theta} x_n.$$

The initial conditions $\alpha_1 = 1$, $\beta_1 = 1$, $\gamma_1 = 1$, and $\delta_1 = 1$ found in Lemma 2.4.1 imply that $\alpha_2 = e^{i\theta} + |a|^2$, $\beta_2 = 1 + e^{i\theta}$, $\gamma_2 = 1 + |a|^2 e^{i\theta}$, and $\delta_2 = 1 + e^{i\theta}$. The first two terms of a linear second-order recurrence relation characterize the solutions. Notice that $\beta_n = \delta_n$ for all n. ■

The recurrence relation can be explicitly solved. For fixed θ and a, the equation $x_{n+2} - (1 + e^{i\theta})x_{n+1} + e^{i\theta}(1 - |a|^2)x_n = 0$ is a constant coefficient equation, so we have auxiliary equation

$$r^2 - (1 + e^{i\theta})r + e^{i\theta}(1 - |a|^2) = 0.$$

With a bit of work, we could show that the discriminant of this quadratic equation,

$$(1 + e^{i\theta})^2 - 4e^{i\theta}(1 - |a|^2) = (1 - e^{i\theta})^2 + 4e^{i\theta}|a|^2,$$

is zero if and only if $|a|^2 = \sin^2 \frac{1}{2}\theta$.

In the case that the discriminant is not zero, there are two distinct roots

$$r_\pm = \frac{1}{2}(1+e^{i\theta}) \pm \frac{1}{2}\Big((1-e^{i\theta})^2 + 4e^{i\theta}|a|^2\Big)^{1/2}$$

and the quantity $\dfrac{\alpha_n}{\gamma_n}$ can be shown to be given by

$$\frac{(r_-r_+^{n-1} - r_+r_-^{n-1}) + \alpha_2(r_+^{n-1} - r_-^{n-1})}{(r_-r_+^{n-1} - r_+r_-^{n-1}) + \gamma_2(r_+^{n-1} - r_-^{n-1})} = \frac{(r_-r_+^{n-1} - r_+r_-^{n-1}) + (e^{i\theta}+|a|^2)(r_+^{n-1} - r_-^{n-1})}{(r_-r_+^{n-1} - r_+r_-^{n-1}) + (1+|a|^2e^{i\theta})(r_+^{n-1} - r_-^{n-1})}.$$

In the case that the discriminant is zero, we can similarly show that $\dfrac{\alpha_n}{\gamma_n}$ is given by

$$\frac{(2-n)r^{n+1} - (e^{i\theta}+|a|^2)(1-n)r^n}{(2-n)r^{n+1} - (1+|a|^2e^{i\theta})(1-n)r^n}.$$

Example 2.4.1 *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a linear fractional transformation $z \mapsto e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$ which is not the identity function. Then $f(f(f(z))) \equiv z$ if and only if $-|a|^2 = 1 + 2\cos\theta$.*

Proof. If $a = 0$, we know from the discussion at the beginning of the section that $(e^{i\theta})^3 = 1$ is required. In this case, $e^{i\theta}$ is a root of the equation $c^3 - 1 = 0$. If f is not the identity function, then $e^{i\theta} \neq 1$, so is a solution to $c^2+c+1 = 0$. This is easily verified to be equivalent to solving $1 + 2\cos\theta = 0$, by writing $2\cos\theta$ as $c + c^{-1}$.

Assume that a is nonzero. From Lemma 2.4.1 above, we know that

$$\begin{bmatrix} \alpha_3 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} e^{i\theta} & |a|^2 \\ e^{i\theta} & 1 \end{bmatrix} \begin{bmatrix} e^{i\theta} & |a|^2 \\ e^{i\theta} & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \delta_1 \end{bmatrix}.$$

Using the initial values of $\alpha_1 = 1$ and $\delta_1 = 1$, we compute

$$\begin{bmatrix} \alpha_3 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} e^{i\theta}(e^{i\theta}+|a|^2) + |a|^2(e^{i\theta}+1) \\ e^{i\theta}(e^{i\theta}+|a|^2) + (e^{i\theta}+1) \end{bmatrix}.$$

Hence δ_3 is given by

$$(e^{i\theta})^2 + (1+|a|^2)e^{i\theta} + 1.$$

From

$$|a|^2e^{i\theta} + e^{2i\theta} + e^{i\theta} + 1 = 0$$

we can see that

$$|a|^2 = -\frac{e^{2i\theta} + e^{i\theta} + 1}{e^{i\theta}}.$$

Since the left-hand side (recall that $a \neq 0$) is a real number in the interval $(0,1)$, the right side must also lie in $(0,1)$. Write $-(e^{i\theta} + 1 + e^{-i\theta})$ as $-(1 + 2\cos\theta)$. The appropriate values of θ are restricted so that $-1 - 2\cos\theta$ is positive. If we use the convention that $\theta \in [0, 2\pi)$, then $\theta \in (\frac{2}{3}\pi, \frac{4}{3}\pi)$ and such choice of θ must be compatible with $|a|^2 = -1 - 2\cos\theta$.

We compute similarly that

$$\begin{bmatrix} \gamma_3 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 + 2|a|^2 e^{i\theta} + |a|^2 e^{2i\theta} \\ 1 + e^{i\theta} + |a|^2 e^{i\theta} + e^{2i\theta} \end{bmatrix}$$

and observe that β_3 does equal zero.

It is a direct computation to show that

$$\frac{e^{i\theta}\alpha_3}{\gamma_3} = \frac{e^{3i\theta} + 2|a|^2 e^{2i\theta} + |a|^2 e^{i\theta}}{1 + 2|a|^2 e^{i\theta} + |a|^2 e^{2i\theta}}.$$

Substituting in that $|a|^2 = -e^{-i\theta} - 1 - e^{i\theta}$, we find that the fraction above simplifies naturally to

$$\frac{(e^{i\theta} + 1)^3}{(e^{i\theta} + 1)^3} = 1.$$

■

Example 2.4.2 *For the choice $\theta = \frac{5}{6}\pi$, i.e., $e^{i\theta} = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$, we may choose any $a \in \mathbb{D}$ satisfying*

$$|a|^2 = -\frac{(-\frac{1}{2}\sqrt{3} + \frac{1}{2}i)^2 + (-\frac{1}{2}\sqrt{3} + \frac{1}{2}i) + 1}{-\frac{1}{2}\sqrt{3} + \frac{1}{2}i} = -\frac{-\sqrt{3} + 3 - \sqrt{3}i + i}{\sqrt{3} - i} = -1 + \sqrt{3}.$$

For any real number ϕ, set

$$a = e^{i\phi}\sqrt{\sqrt{3} - 1}$$

The skeptical but interested reader can verify that for these choices of a and θ, the linear fractional transformation $f(z) = e^{i\theta}\dfrac{z - a}{1 - \bar{a}z}$ indeed satisfies $f(f(f(z))) \equiv z$.

By a similar computation, we can compute the $n = 4$ case.

Example 2.4.3 *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is a linear fractional transformation, written as in Example 3, which is not the identity function. Then $f^{\circ 4}(z) \equiv z$ if and only if $|a|^2 = -\cos\theta$. Consequently, for every choice of θ in the interval $(\frac{1}{4}\pi, \frac{3}{4}\pi)$, there is a corresponding linear fractional transformation satisfying $f(f(f(f(z)))) = z$.*

The $n = 5$ case is already much harder. We can find

$$\delta_5 = (1 + e^{i\theta})^4 - 3e^{i\theta}(1 + e^{i\theta})^2(1 - |a|^2) + e^{2i\theta}(1 - |a|^2)^2$$

which is biquadratic in $|a|^2$. As n increases, the degree of the polynomial $\delta_n = 0$ relating $e^{i\theta}$ and $|a|^2$ increases.

2.5 Linear Fractional Transformations of $\mathbb{B}_n$ $(n > 1)$

The linear fractional transformations of $\mathbb{B}_n$ are well-understood; see [D'A93] or [Rud08] for example. We will use the formulation of D'Angelo,

$$f(z) = U\frac{a - L(z)}{1 - \langle z, a\rangle}$$

where $L(z)$ denotes the map

$$L(z) = \sigma z + \frac{\langle z, a\rangle}{1 + \sigma}a$$

with $\sigma = \sqrt{1 - |a|^2}$. To emphasize the choice of a appearing in L, we may write $L_a(z)$ or $L(z; a)$ for $L(z)$.

The expression $\langle z, w\rangle$ denotes the conjugate-linear inner product on $\mathbb{C}^n$; in the usual coordinates, $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ have inner product

$$\langle z, w\rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n.$$

If points in $\mathbb{C}^n$ are denoted by column vectors,

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix},$$

then $\langle z, w \rangle$ takes the convenient form $w^\dagger z$ where the dagger $\dagger$ indicates the conjugate of the transpose.

Without much difficulty, we can write $f(z)$ in a matrix formulation. We utilize the fact that the product of scalars c with vectors v is commutative to rewrite $\langle z, a \rangle a$ as $a \langle z, a \rangle = a(a^\dagger z)$. Hence

$$\begin{aligned} f(z) &= U \frac{a - \sigma z - \dfrac{\langle z, a \rangle}{1+\sigma} a}{1 - \langle z, a \rangle} \\ &= U \frac{a - \left(\sigma + \dfrac{1}{1+\sigma} a a^\dagger \right) z}{1 - a^\dagger z}. \end{aligned}$$

Similar to the work near the end of Section 2.2, we can find expressions for U and a (and A) in terms of the values of f and its derivatives at 0.

Theorem 2.5.1 *Let $n > 1$ be an integer, and f be a linear fractional transformation $\mathbb{B}_n \to \mathbb{B}_n$. Then $f(z)$ has the form*

$$U \frac{a - L(z; a)}{1 - \langle z, a \rangle}$$

where a is given by

$$a = -\frac{1}{1 - |f(0)|^2} J^\dagger f(0),$$

the unitary matrix U is given by

$$U = -\frac{1}{1+\sigma} \frac{f(0) f(0)^\dagger}{\sigma^2} J - \frac{1}{\sigma} J,$$

and where J is the (complex) Jacobian matrix of f at the origin, and σ denotes $\sqrt{1 - |a|^2}$, as usual.

The form for a here matches the alternative form for a given at the end of Section 2.2 in the $n = 1$ case. Additionally, the form for U agrees with the formula $e^{i\theta} = f'(0)/(-1 + |f(0)|^2)$ by replacing U by $e^{i\theta}$, $f(0)^\dagger$ by $\overline{f(0)}$, and J by $f'(0)$.

Proof. Verifying the explicit formulas for a and U in terms of the germ of f (i.e., the values of f and its derivatives) at the origin is easy. (Discovering the formulas was the real work.) We must first give explicit formulas for $f(0)$ and J, and compute with them.

From $f(z) = U\dfrac{a-(\sigma+\frac{1}{1+\sigma}aa^\dagger)z}{1-a^\dagger z}$, we see that $f(0) = Ua$. An immediate consequence is that $|f(0)| = |a|$.

In order to compute the Jacobian matrix for $f = (f_1, \ldots, f_n)^T$,

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \frac{\partial f_2}{\partial z_1} & & & \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix},$$

we need to compute the columns $\frac{\partial}{\partial z_\ell}f$ of J. Let e_ℓ denote the unit vector pointing in the z_ℓ direction, so that $z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^T$ decomposes as $\sum_{\ell=1}^{n} z_\ell e_\ell$. We proceed to compute the columns of $U^{-1}J$ (as U is a constant with respect to each of the z_ℓ variables, we are allowed this slight ease of notation). Using the usual quotient rule,

$$U^{-1}\frac{\partial f}{\partial z_\ell} = \frac{\partial}{\partial z_\ell}\frac{a-(\sigma+\frac{1}{1+\sigma}aa^\dagger)z}{1-a^\dagger z}$$

$$U^{-1}\frac{\partial f}{\partial z_\ell} = \frac{(1-a^\dagger z)\frac{\partial}{\partial z_\ell}(a-(\sigma+\frac{1}{1+\sigma}aa^\dagger)z) - (a-(\sigma+\frac{1}{1+\sigma}aa^\dagger)z)\frac{\partial}{\partial z_\ell}(1-a^\dagger z)}{(1-a^\dagger z)^2}.$$

By choice of the e_ℓ vectors, $\frac{\partial}{\partial z_j}z$ is $\delta_{j\ell}e_\ell$ where δ_{km} denotes the usual Kronecker delta function, which takes the value 1 if $k = m$ and 0 otherwise. This allows an easy simplification of the derivative terms:

$$U^{-1}\frac{\partial f}{\partial z_\ell} = \frac{(1-a^\dagger z)(0-(\sigma+\frac{1}{1+\sigma}aa^\dagger)e_\ell) - (a-(\sigma+\frac{1}{1+\sigma}aa^\dagger)z)(0-a^\dagger e_\ell)}{(1-a^\dagger z)^2}.$$

We evaluate both sides at $z = 0$, as that is the only point where we claim to need the

Jacobian, and then simplify:

$$\begin{aligned}
U^{-1}\frac{\partial f}{\partial z_\ell}\Big|_{z=0} &= \frac{(1-0)(0-(\sigma+\frac{1}{1+\sigma}aa^\dagger)e_\ell)-(a-0)(0-a^\dagger e_\ell)}{(1-0)^2} \\
&= -(\sigma+\frac{1}{1+\sigma}aa^\dagger)e_\ell + aa^\dagger e_\ell \\
&= -\sigma e_\ell + \left(-\frac{1}{1+\sigma}+1\right)aa^\dagger e_\ell \\
&= -\sigma e_\ell + \frac{\sigma}{1+\sigma}aa^\dagger e_\ell .
\end{aligned}$$

Multiplying both sides by U (on the left) and assembling the columns into a matrix, we get

$$J|_{z=0} = U\left(\frac{\sigma}{1+\sigma}aa^\dagger - \sigma I\right)\begin{bmatrix} e_1 & e_2 & \cdots & e_n\end{bmatrix} = U\left(\frac{\sigma}{1+\sigma}aa^\dagger - \sigma I\right)$$

since $\begin{bmatrix} e_1 & e_2 & \cdots & e_n\end{bmatrix}$ is the identity matrix.

Because we are only concerned with the Jacobian matrix at $z=0$, we will simply write J for $J|_{z=0}$. Hence, we can write

$$J = \frac{\sigma}{1+\sigma}Uaa^\dagger - \sigma U$$

Now we verify the two expressions stated in the theorem.

$$\begin{aligned}
&-\frac{1}{1-|f(0)|^2}J^\dagger f(0) \\
&= -\frac{1}{1-|Ua|^2}\left(\frac{\sigma}{1+\sigma}Uaa^\dagger - \sigma U\right)^\dagger Ua \\
&= -\frac{1}{1-|a|^2}\left(\frac{\sigma}{1+\sigma}aa^\dagger U^\dagger - \sigma U^\dagger\right)Ua \\
&= -\frac{1}{\sigma^2}\left(\frac{\sigma}{1+\sigma}aa^\dagger - \sigma\right)U^\dagger Ua \\
&= -\frac{1}{\sigma}\left(\frac{1}{1+\sigma}aa^\dagger - 1\right)a \\
&= -\frac{1}{\sigma}\left(\frac{1}{1+\sigma}aa^\dagger a - a\right) \\
&= -\frac{1}{\sigma}\left(\frac{1}{1+\sigma}a|a|^2 - a\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\sigma}\left(\frac{1}{1+\sigma}a(1-\sigma^2) - a\right) \\
&= -\frac{1}{\sigma}\left(a(1-\sigma) - a\right) \\
&= a
\end{aligned}
$$

and

$$
\begin{aligned}
&- \frac{1}{1+\sigma}\frac{f(0)f(0)^\dagger}{\sigma^2}J - \frac{1}{\sigma}J \\
&= -\frac{1}{1+\sigma}\frac{(Ua)(Ua)^\dagger}{\sigma^2}\left(\frac{\sigma}{1+\sigma}Uaa^\dagger - \sigma U\right) - \frac{1}{\sigma}\left(\frac{\sigma}{1+\sigma}Uaa^\dagger - \sigma U\right) \\
&= -\frac{1}{1+\sigma}\frac{Uaa^\dagger U^\dagger}{\sigma}\left(\frac{1}{1+\sigma}Uaa^\dagger - U\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= -\frac{1}{1+\sigma}\frac{1}{\sigma}\left(\frac{1}{1+\sigma}Uaa^\dagger U^\dagger Uaa^\dagger - Uaa^\dagger U^\dagger U\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= -\frac{1}{1+\sigma}\frac{1}{\sigma}\left(\frac{1}{1+\sigma}Uaa^\dagger aa^\dagger - Uaa^\dagger\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= -\frac{1}{1+\sigma}\frac{1}{\sigma}\left(\frac{1}{1+\sigma}Ua|a|^2a^\dagger - Uaa^\dagger\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= -\frac{1}{1+\sigma}\frac{1}{\sigma}\left(\frac{1}{1+\sigma}Ua(1-\sigma^2)a^\dagger - Uaa^\dagger\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= -\frac{1}{1+\sigma}\frac{1}{\sigma}\left(Ua(1-\sigma)a^\dagger - Uaa^\dagger\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= -\frac{1}{1+\sigma}\frac{1}{\sigma}\left(Ua(-\sigma)a^\dagger\right) - \frac{1}{1+\sigma}Uaa^\dagger + U \\
&= \frac{1}{1+\sigma}\left(Uaa^\dagger\right) - \frac{1}{1+\sigma}Uaa^\dagger + U
\end{aligned}
$$

which simplifies down directly to U, thus concluding our proof. ■

An example illustrating a more complicated version of this result is provided in Section 3.4.

2.6 Involutions of $\mathbb{B}_n$ ($n > 1$)

As analogy with Section 2.3, we consider the involutions of the unit ball $\mathbb{B}_n$ in higher dimensions. As in the case $n = 1$, involutory proper maps in the $n > 1$ case are biholomorphisms, and therefore automatically linear fractional transformations. While several references, e.g.

[Rud08], discuss the following lemma as a source of involutions, they do not give a complete classification of all involutions (Proposition 2.6.1 below).

A common technique for finding normal, or normalized, forms for a function is to conjugate by some map (e.g. in order to pick a preferred fixed point). Involutions are particularly nice maps with which to conjugate, because the inverse map is known. In complex dimensions $n > 1$, computing the inverse map, even for a linear fractional transformation, is considerably more tedious, so involutions are even more attractive.

Lemma 2.6.1 *For each choice of $a \in \mathbb{B}_n$, the linear fractional transformation*

$$\varphi_a(z) = \varphi(z; a) = \frac{a - L_a(z)}{1 - \langle z, a \rangle}$$

is an involution of $\mathbb{B}_n$. Here, $L_a(z)$ denotes the map

$$z \mapsto \sqrt{1 - |a|^2} z + \frac{\langle z, a \rangle}{1 + \sqrt{1 - |a|^2}} a$$

For shorthand, we can denote $\sqrt{1 - |a|^2}$ by σ_a, which we write as σ when there is no potential for confusion.

Proof. We want to show that $\varphi_a(\varphi_a(z)) = z$ for any vector $z \in \mathbb{B}_n$. Fix z and decompose it into a component parallel to a and a component in the plane orthogonal to a, i.e., as $ca + B$ with c explicitly given by $\langle z, a \rangle / |a|^2$, and so B is a vector in $\mathbb{B}_n$ such that $\langle B, a \rangle = 0$. Since L_a is a linear map, we have $L_a(z) = cL_a(a) + L_a(B)$. It is easy to see that $L_a(B) = \sigma B$, and that

$$\begin{aligned} L_a(a) &= \sigma a + \frac{|a|^2}{1 + \sigma} a \\ &= \left(\sigma + \frac{1 - \sigma^2}{1 + \sigma} \right) a \\ &= (\sigma + 1 - \sigma) a = a \end{aligned}$$

so

$$L_a(z) = L_a(ca + B) = ca + \sigma B.$$

We can now compute $\varphi_a(z)$ as follows.

$$\begin{aligned}
\varphi_a(z) &= \varphi_a(ca+B) \\
&= \frac{a - L_a(ca+B)}{1 - \langle ca+B, a\rangle} \\
&= \frac{a - ca - \sigma B}{1 - c\langle a, a\rangle - \langle B, a\rangle} \\
&= \frac{(1-c)a - \sigma B}{1 - c|a|^2} \\
&= \frac{1-c}{1-c|a|^2}a - \frac{\sigma}{1-c|a|^2}B.
\end{aligned}$$

As shorthand, we denote this by $c_1 a + c_2 B$.

Since $\varphi_a(z)$ is in a very similar form as z, namely a linear combination of a and B, similar calculations show the following.

$$\begin{aligned}
\varphi_a(\varphi_a(z)) &= \varphi_a(c_1 a + c_2 B) \\
&= \frac{a - L_a(c_1 a + c_2 B)}{1 - \langle c_1 a + c_2 B, a\rangle} \\
&= \frac{a - c_1 L_a(a) - c_2 L_a(B)}{1 - c_1\langle a, a\rangle - c_2\langle B, a\rangle} \\
&= \frac{a - c_1 a - c_2 \sigma B}{1 - c_1|a|^2} \\
&= \frac{1-c_1}{1-c_1|a|^2}a - \frac{c_2\sigma}{1-c_1|a|^2}B.
\end{aligned}$$

It only remains to calculate the two new coefficients. The common denominator $1-c_1|a|^2$ is

$$1 - \frac{1-c}{1-c|a|^2}|a|^2 = \frac{1-c|a|^2-|a|^2+c|a|^2}{1-c|a|^2} = \frac{1-|a|^2}{1-c|a|^2}$$

so the coefficient of a in $\varphi_a(\varphi_a(z))$ is

$$\frac{1 - \dfrac{1-c}{1-c|a|^2}}{\dfrac{1-|a|^2}{1-c|a|^2}} = \frac{1-c|a|^2-(1-c)}{1-|a|^2} = c$$

and the coefficient of B is

$$-\frac{-\frac{\sigma}{1-c|a|^2}}{\frac{1-|a|^2}{1-c|a|^2}}\sigma = \frac{\sigma^2}{1-|a|^2} = 1$$

thus proving that indeed $\varphi_a(\varphi_a(ca+B)) = ca+B$. ■

Remark 2.6.1 *An alternative proof of this result can be centered on the multivariable analogue of the Schwarz Lemma that is known as Cartan's (Uniqueness) Theorem. Cartan's Theorem states (see e.g. [Rud08] or [Leb19]) that if f is a self-map of a bounded, connected, open set $U \subset \mathbb{C}^n$ with a fixed point $f(p) = p$ such that $f'(p)$ is the identity matrix, then f is the identity function on the entire domain U. The proof above is then reduced to observing that the composition $\varphi_a \circ \varphi_a$ satisfies the hypotheses of the theorem, in particular with $p = 0$ (or with $p = a$).*

Though the usual proof of Cartan's Theorem (see also the books [BM48], [Sch05], as well as the original paper [Car30]) conceals the geometric understanding of the above computations, it is worth mentioning. The proof considers Cauchy estimates of iterated compositions $f^{\circ j}$ of our self-mapping, and concludes that the higher-degree terms in a power series expansion around the fixed point must vanish. The appearance of iterated functions almost a century ago, especially in the proof of a result whose statement does not indicate iteration, is strong evidence that iterated functions deserve focused study.

As in the $\mathbb{B}_1$ case, there are involutions where the unitary matrix (here generalizing the lead $e^{i\theta}$ coefficient) is not the identity. The following lemma introduces a notational convenience for dealing with the more general linear fractional transformations $\mathbb{B}_n \to \mathbb{B}_n$.

Lemma 2.6.2 *The linear fractional transformations φ_a satisfy the following functional relationship with unitary matrices U:*

$$\varphi(Ux; a) = U\varphi(x; U^\dagger a).$$

Proof. We can start with the right side, $U\varphi(x;U^\dagger a)$. We know that

$$\varphi(x;U^\dagger a)=\frac{1}{1-\langle x,U^\dagger a\rangle}\left(U^\dagger a-\sqrt{1-|U^\dagger a|^2}x-\frac{\langle x,U^\dagger a\rangle}{1+\sqrt{1-|U^\dagger a|^2}}U^\dagger a\right)$$

and multiplying by U on the left, coupled with the fact that $UU^\dagger a=Ia=a$ because U is unitary, we see that

$$U\varphi(x;U^\dagger a)=\frac{1}{1-\langle x,U^\dagger a\rangle}\left(a-\sqrt{1-|U^\dagger a|^2}Ux-\frac{\langle x,U^\dagger a\rangle}{1+\sqrt{1-|U^\dagger a|^2}}a\right).$$

The length of any vector a is unchanged under the action of a unitary matrix, so $|U^\dagger a|=|a|$, and

$$U\varphi(x;U^\dagger a)=\frac{1}{1-\langle x,U^\dagger a\rangle}\left(a-\sqrt{1-|a|^2}Ux-\frac{\langle x,U^\dagger a\rangle}{1+\sqrt{1-|a|^2}}a\right).$$

Since $U^\dagger$ is the adjoint of U, we can rewrite the inner products $\langle x,U^\dagger a\rangle$ as $\langle Ux,a\rangle$. Finally, we have

$$U\varphi(x;U^\dagger a)=\frac{1}{1-\langle Ux,a\rangle}\left(a-\sqrt{1-|a|^2}Ux-\frac{\langle Ux,a\rangle}{1+\sqrt{1-|a|^2}}a\right).$$

Of course, the right-hand side here is precisely $\varphi(Ux;a)$. ∎

Using the above lemma to help streamline our work, we can now prove a result of particular interest, namely the classification of all involutions of $\mathbb{B}_n$. We will see a generalization of this result to a new setting at the end of the next chapter. This result is new, as far as the author is aware.

Proposition 2.6.1 *Suppose $f:\mathbb{B}_n\to\mathbb{B}_n$ is a holomorphic map. The composition $f\circ f$ is the identity function if and only if f can be written in the form $U\dfrac{a-L_a(z)}{1-\langle z,a\rangle}$ with $Ua=a$ and $U^2=I$.*

Proof. We must prove both directions.

($\Rightarrow$) If f is an involution of $\mathbb{B}_n$, then it is an automorphism. As in Section 2.5, it must have the form $f(z)=U\varphi(z;a)$ for some U and a. Now if $f(f(z))=z$, then in particular $f(f(a))=a$. We can explicitly describe $f(f(a))$ in terms of U and a, and it has a simple form. First, we note that $f(a)$ is zero. Thus $f(f(a))=f(0)$. Plugging 0 into f immediately gives $f(0) = Ua$. Consequently, $f(f(a)) = a$ implies that $Ua = a$. Hence, a is either the zero vector, or a is an eigenvector of U (with eigenvalue one).

If a is zero, then $f(z) = U\varphi(z; 0)$ simplifies to $-Uz$. Here, $f(f(z)) = z$ is easy to study: $f(f(z)) = -U(-Uz) = U^2 z$. Since $U^2 z = z$ must hold for all choices of $z \in \mathbb{B}_n$, the unitary U^2 is the identity matrix.

If a is nonzero, we know that $Ua = a$. Here our lemma $\varphi(Ux; a) = U\varphi(x; U^\dagger a)$ helps us; $f(f(z)) = U\varphi(U\varphi(z; a); a)$ becomes $U^2\varphi(\varphi(z; U^\dagger a); a)$ after one application. We notice that $U^\dagger a = a$ because U is unitary (to be clear: $Ua = a$ is equivalent to $U^{-1}Ua = U^{-1}a$, which in turn says $a = U^{-1}a$; i.e. $a = U^\dagger a$). As a consequence of $U^\dagger a = a$, we see that $U^2\varphi(\varphi(z; U^\dagger a); a)$ simplifies to $U^2\varphi(\varphi(z; a); a)$. We can directly use that $\varphi(-; a)$ is an involution to further simplify to $U^2 z$. That is, if $a \neq 0$ and $Ua = a$, then $f(f(z))$ simplifies to $U^2 z$. Since $f(f(z)) = z$ identically, we again find that $U^2 = I$.

($\Leftarrow$) Suppose that $Ua = a$ and $U^2 = I$. We want to show that $f(f(z))$ is the identity function. Since f is a map of the form $f(z) = U\varphi(z; a)$, we can simplify $f(f(z)) = U\varphi(U\varphi(z; a); a)$. Applying the lemma, we can write $U\varphi(U\varphi(z; a); a)$ as $U^2\varphi(\varphi(z; U^\dagger a); a)$. Applying the fact that $U^2 = I$ and that $U^\dagger a = a$, $f(f(z))$ simplifies to $\varphi(\varphi(z; a); a)$. But $\varphi(-; a)$ is an involution (Lemma 2.6.1), so we are done. ■

2.7 Higher-order self-maps of $\mathbb{B}_n$

Calculations similar to those in Section 2.4 can be performed to make progress towards finding solutions to $f^{\circ n}$ in the situation that f is a proper map $\mathbb{B}_n \to \mathbb{B}_n$. We will see that difficulties arise because of the non-commutative nature of the components U and a of the linear fractional transformations when $n > 1$. Of course, we assume $n > 1$ for the remainder of the section.

Any proper map $f : \mathbb{B}_n \to \mathbb{B}_n$ is a linear fractional transformation, so has the form

$$U \frac{a - \sqrt{1 - |a|^2}\, z - \dfrac{\langle z, a\rangle}{1 + \sqrt{1 - |a|^2}}\, a}{1 - \langle z, a\rangle}.$$

In order to simplify the calculations going forward, we will leverage the well-known correspondence (which works for $n \geq 1$) between linear fractional transformations and matrices in the special linear group $\mathrm{SL}(n+1, \mathbb{C})$, which consists of the multiplicative collection of $(n+1) \times (n+1)$ matrices with complex entries, modulo nonzero scalar multiples.

To be more precise, we note that the general linear fractional transform has form

$$z \mapsto \frac{Mz + v}{w^T z + c}$$

where M is an an $n \times n$ matrix, v and w are $n \times 1$ (column) vectors, c is a scalar, and the product $w^T z$ denotes the usual bilinear inner product $w \cdot z$ (as opposed to the sesquilinear inner product used in other sections). This linear fractional transformation is unchanged when the parameters M, v, w, and c are all simultaneously rescaled by nonzero scalar $\lambda \in \mathbb{C}$, since

$$z \mapsto \frac{\lambda M z + \lambda v}{(\lambda w) z + \lambda c}$$

is the same function.

This identification is especially useful because composition of linear fractional transformations is modeled by multiplication of (block) matrices. For example, if $F(z)$ is given by $\dfrac{Mz + v}{w^T z + c}$ and $G(z)$ is given by $\dfrac{\tilde{M} z + \tilde{v}}{\tilde{w}^T z + \tilde{c}}$, then we can clearly identify the composition

$$F(G(z)) = \frac{M \dfrac{\tilde{M} z + \tilde{v}}{\tilde{w}^T z + \tilde{c}} + v}{w^T \dfrac{\tilde{M} z + \tilde{v}}{\tilde{w}^T z + \tilde{c}} + c} = \frac{M \tilde{M} z + M \tilde{v} + \tilde{w}^T z v + \tilde{c} v}{w^T \tilde{M} z + w^T \tilde{v} + c \tilde{w}^T z + c \tilde{c}}$$

with the matrix product

$$\begin{bmatrix} M & v \\ w^T & c \end{bmatrix} \begin{bmatrix} \tilde{M} & \tilde{v} \\ \tilde{w}^T & \tilde{c} \end{bmatrix} = \begin{bmatrix} M \tilde{M} + v \tilde{w}^T & M \tilde{v} + \tilde{c} v \\ w^T \tilde{M} + c \tilde{w}^T & w^T \tilde{v} + c \tilde{c} \end{bmatrix}.$$

Under this identification, the form for a linear fractional transformation $f : \mathbb{B}_n \to \mathbb{B}_n$ found in section 2.5 can be written as a matrix

$$\begin{bmatrix} -\sigma U - \dfrac{1}{1+\sigma} U a a^\dagger & Ua \\ -a^\dagger & 1 \end{bmatrix}$$

where we have again used the notation $\sigma = \sqrt{1-|a|^2}$. If desired, this matrix representation of f can also be factored as

$$\begin{bmatrix} -U & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma I + \frac{1}{1+\sigma} aa^\dagger & -a \\ -a^\dagger & 1 \end{bmatrix}.$$

In the case that a is the zero vector, the matrix representation for f simplifies drastically to $\left[\begin{smallmatrix} -U & 0 \\ 0 & 1 \end{smallmatrix}\right]$, so the matrix representation for $f^{\circ n}(z)$ is given by

$$\begin{bmatrix} -U & 0 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} (-1)^n U^n & 0 \\ 0 & 1 \end{bmatrix}.$$

In the case $a = 0$, $f^{\circ n}(z) \equiv z$ if and only if $(-1)^n U^n$ is the identity matrix. (The negative sign distinguishing this result from the analogous discussion when in the case of maps $\mathbb{D} \to \mathbb{D}$ arises because of the alternative format used here. Compare $e^{i\theta}\dfrac{z-a}{1-\bar{a}z}$ and $e^{i\theta}\dfrac{a-z}{1-\bar{a}z}$.)

In the case that a is not the zero vector, difficulty in the computations arises quite quickly. Consider, for instance, the matrix representation of $f \circ f(z)$. Here, we have

$$\begin{bmatrix} \sigma^2 U^2 + \frac{\sigma}{1+\sigma} U^2 aa^\dagger + \frac{\sigma}{1+\sigma} Uaa^\dagger U + \frac{1}{(1+\sigma)^2} Uaa^\dagger Uaa^\dagger - Uaa^\dagger & -\sigma U^2 a - \frac{1}{1+\sigma} Uaa^\dagger Ua + Ua \\ \sigma a^\dagger U + \frac{1}{1+\sigma} a^\dagger Uaa^\dagger - a^\dagger & -a^\dagger Ua + 1 \end{bmatrix}.$$

We notice that expressions of the form $a^\dagger U^\ell a$ appear, and are scalars, so we denote these by μ_ℓ. This simplifies our matrix representation of $f^{\circ 2}$ to

$$\begin{bmatrix} \sigma^2 U^2 + \frac{\sigma}{1+\sigma} U^2 aa^\dagger + \frac{\sigma}{1+\sigma} Uaa^\dagger U + \frac{\mu_1}{(1+\sigma)^2} Uaa^\dagger - Uaa^\dagger & -\sigma U^2 a - \frac{\mu_1}{1+\sigma} Ua + Ua \\ \sigma a^\dagger U + \frac{\mu_1}{1+\sigma} a^\dagger - a^\dagger & -\mu_1 + 1 \end{bmatrix}$$

which allows us to group terms more effectively

$$\begin{bmatrix} \sigma^2 U^2 + \left(\frac{\sigma}{1+\sigma} + \frac{\mu_1}{(1+\sigma)^2} - 1\right) Uaa^\dagger + \frac{\sigma}{1+\sigma} Uaa^\dagger U & -\sigma U^2 a + \left(1 - \frac{\mu_1}{1+\sigma}\right) Ua \\ \sigma a^\dagger U + \left(\frac{\mu_1}{1+\sigma} - 1\right) a^\dagger & 1 - \mu_1 \end{bmatrix}.$$

Generally, we can write the matrix representing $f^{\circ n}$ in the form

$$\begin{bmatrix} -U & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_n & -\beta_n a \\ -a^\dagger \delta_n & \gamma_n \end{bmatrix}$$

where $\alpha_1 = \sigma I + \frac{1}{1+\sigma} aa^\dagger$ (denote this matrix R), $\beta_1 = \delta_1 = I$, and $\gamma_1 = 1$, and we have the recursion relations

$$\alpha_{n+1} = -RU\alpha_n + aa^\dagger \delta_n$$

$$\beta_{n+1} = -RU\beta_n + \gamma_n$$

$$\gamma_{n+1} = \gamma_n - a^\dagger U \beta_n a$$

$$\delta_{n+1} = \delta_n - U\alpha_n.$$

When $a = 0$, R is the identity matrix.

2.8 Some Rational Proper Maps $\mathbb{B}_n \to \mathbb{B}_N$ where $N > n > 1$

Generically, proper maps can be quite wild. A result of Dor [Dor90] shows that there exists proper maps (which are not rational) $\mathbb{B}_n \to \mathbb{B}_{n+1}$ for each $n \geq 2$ which cannot be extended to be C^2 in a neighborhood of the boundary. By contrast, Forstnerič [For89] showed that any map $\mathbb{B}_n \to \mathbb{B}_N$ which extends to the boundary with sufficient regularity (specifically, if the function is class C^{N-n+1}) is necessarily rational.

In this section, we state some known results regarding rational maps $\mathbb{B}_n \to \mathbb{B}_N$ that are proper. While these maps are certainly more 'interesting' than automorphisms, their classification is considerably more subtle, and requires more assumptions with regards to regularity of the maps as they extend to the boundary of the domain (if they do indeed extend to the boundary). Any maps appearing in this section are potential fodder for constructing examples of the type which appear in Chapter 3.

A fruitful notion of equivalence is the following.

Definition 2.8.1 *Two proper maps $f, g : \mathbb{B}_n \to \mathbb{B}_N$ are called spherically equivalent if there exists automorphisms (see Section 2.5) $\varphi : \mathbb{B}_n \to \mathbb{B}_n$ and $\psi : \mathbb{B}_N \to \mathbb{B}_N$ such that $\psi \circ g = f \circ \varphi$.*

This notion is a common choice for distinguishing proper maps. The maps $\mathbb{B}_2 \to \mathbb{B}_2$ are already known, so we state the next case.

Example 2.8.1 (Faran's maps) *According to Faran [Far82], the only proper maps $\mathbb{B}_2 \to \mathbb{B}_3$ which extend to the boundary as C^3 functions are*

- $(z, w) \mapsto (z, w, 0)$,
- $(z, w) \mapsto (z, zw, w^2)$,
- $(z, w) \mapsto (z^2, w^2, \sqrt{2}zw)$, *and*
- $(z, w) \mapsto (z^3, w^3, \sqrt{3}zw)$,

up to spherical equivalence.

For $n > 2$, the behavior (at least when $N = n + 1$) is simpler.

Example 2.8.2 *For $n \geq 3$, Webster [Web79] showed that there is precisely one rational proper map $\mathbb{B}_n \to \mathbb{B}_{n+1}$, up to spherical equivalence. That is, every rational map $\mathbb{B}_n \to \mathbb{B}_{n+1}$ is spherically equivalent to the linear embedding $z \mapsto (z, 0)$.*

For $n \geq 3$, there is a nice trichotomy. Faran [Far86] showed that there is only one spherical equivalence class for proper maps $\mathbb{B}_n \to \mathbb{B}_N$ when $N \leq 2n - 2$. Huang and Ji [HJ01] showed that there are two spherical equivalence classes $\mathbb{B}_n \to \mathbb{B}_{2n-1}$, represented by the linear embedding and the Whitney map

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, z_n z_1, \ldots, z_n^2).$$

Finally, D'Angelo [D'A88] showed that there are infinitely many spherically inequivalent maps, as illustrated by the family of maps

$$(z', z_n) \mapsto (z', z_n \cos\theta, z_n z \sin\theta)$$

where θ takes some fixed value in $(0, \pi/2)$.

CHAPTER III

PROPER MAPS $\mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$

We are ready to define our principal object of study. In recent years, Jiri Lebl, Alan Noell, and Sivaguru Ravisankar have studied extensions of functions that are defined on a relatively open subset of $\mathbb{C}^n \times \mathbb{R}$ to neighborhoods in $\mathbb{C}^n \times \mathbb{C}$ [LNR17b, LNR17a, LNR19]. In particular, in [LNR17b], the authors specify the space $\{(z, s) \in \mathbb{C}^n \times \mathbb{R} : \|z\|^2 + |s|^2 < 1\}$ [their notation] as the model case.

In earlier work, John D'Angelo and Jiri Lebl [DL16] studied homotopies between rational proper maps, which introduces the additional real parameter t. Expressions like their example $(z^2, \sqrt{2-t^2}zw, tw, \sqrt{1-t^2}w^2)$ will look similar to expressions below. In that paper, the examples given are proper for every value of t in the interval $[0, 1]$.

In this chapter, we abstract the two avenues above by considering proper maps defined between spaces of the following form.

Definition 3.0.1 *We denote the unit ball in $\mathbb{C}^n \times \mathbb{R}^k$ by $\mathbb{B}_{n,k}$. That is, $\mathbb{B}_{n,k}$ denotes the set* $\{(z, s) = (z_1, \dots, z_n, s_1, \dots, s_k) \in \mathbb{C}^n \times \mathbb{R}^k : |z_1|^2 + \dots + |z_n|^2 + s_1^2 + \dots + s_k^2 < 1\}$.

When $k = 0$, we may continue writing $\mathbb{B}_n$ for $\mathbb{B}_{n,0}$ especially when we have reason to think of $\mathbb{B}_{n,0}$ in the context of Chapter 2 (for example, in Theorem 3.1.1 below).

In general, maps f from $\mathbb{C}^n \times \mathbb{R}^k$ to $\mathbb{C}^N \times \mathbb{R}^K$ can be written as having $n + k$ input variables, which we will call $z_1, \dots, z_n, s_1, \dots, s_k$. The function f has $N + K$ components; we aggregate the first N components into a vector-valued function, say g, and the last K components into a vector-valued function, say h. We will occasionally write $\pi_{\mathbb{C}^N} f$ for g and $\pi_{\mathbb{R}^K} f$ for h.

Observe that *a priori* both g and h have domain $\mathbb{C}^n \times \mathbb{R}^k$. We will insist that all functions of the complex coordinates $z_1, \ldots, z_n$ are holomorphic. Since h is thus a holomorphic function of complex variables with its image lying in $\mathbb{R}^K$, the function h must be constant with respect to the complex variables (an elementary consequence of the Cauchy-Riemann equations, for example). That is, any such map f can be written as a pair of maps $g : \mathbb{C}^n \times \mathbb{R}^k \to \mathbb{C}^N \times \mathbb{R}^K$ and $h : \mathbb{R}^k \to \mathbb{R}^K$.

Throughout this chapter, we will see various ways to construct proper maps using proper maps $\mathbb{B}_n \to \mathbb{B}_N$ along with additional data, such as auxiliary maps $\mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$. Let us start with the following simple example.

Example 3.0.1 *The maps* $\mathbb{B}_{n,1} \to \mathbb{B}_{n,1} : (z,s) \mapsto \left(z \exp\left(\frac{i}{1-s}\right), s\right)$ *and* $\mathbb{B}_{n,k} \to \mathbb{B}_{n,k} : (z,s) \mapsto \left(z \exp \frac{i}{1-|s|}, s\right)$ *are proper.*

With this example, we can illustrate the usual way to check if a map $\mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$ is proper. We compute $1 - |f(z,s)|^2$ and check if this can be factored as a product of $1 - |z|^2 - |s|^2$ (a defining function for the domain) multiplied by an expression in terms of z and s which is bounded on the closed ball $\overline{\mathbb{B}}_{n,k}$. Indeed, we can compute here that the squared norm of these maps is $|z|^2 + |s|^2$, so $1 - |f(z,s)|^2$ is $1 - |z|^2 - |s|^2$ on the nose.

The following is a nontrivial example built using Example 2.8.1 from the last chapter.

Example 3.0.2 *The map* $f : \mathbb{B}_{2,1} \to \mathbb{B}_{3,1}$ *defined by mapping* (z,w,s) *to*

$$\left(e^{is}\sqrt{1-s^4}\frac{z^3}{\sqrt{1-s^2}^3},\ \sqrt{1-s^4}\frac{w^3}{\sqrt{1-s^2}^3},\ \sqrt{3}\sqrt{1-s^4}\frac{zw}{\sqrt{1-s^2}^2},\ s^2 \right)$$

is proper.

It is an exercise in algebra (or the use of a computer algebra system) to show that $1 - |f(z,w,s)|^2$ factors as a product of

$$(1-s^4)\left(1 - \frac{|z|^2}{1-s^2} - \frac{|w|^2}{1-s^2}\right)$$

with the rational function

$$1 + \frac{|z|^2}{1-s^2} + \frac{|z|^4}{(1-s^2)^2} + \frac{|w|^2}{1-s^2} + \frac{|w|^4}{(1-s^2)^2} - \frac{|w|^2|z|^2}{(1-s^2)^2}$$

which we will write as $X(z,w,s)$. The product of the first two factors can be written as $(1+s^2)(1-s^2-|z|^2-|w|^2)$, so it suffices to show that the rational function $X(z,w,s)$ is bounded. Observe that because the points $(z,w,s) \in \mathbb{B}_{2,1}$ satisfy $|z|^2+|w|^2+s^2<1$ and thus in particular $\dfrac{|z|^2+|w|^2}{1-s^2}<1$, we can bound X above:

$$\begin{aligned}
&1+\frac{|z|^2}{1-s^2}+\frac{|z|^4}{(1-s^2)^2}+\frac{|w|^2}{1-s^2}+\frac{|w|^4}{(1-s^2)^2}-\frac{|w|^2|z|^2}{(1-s^2)^2}\\
&\le 1+\frac{|z|^2+|w|^2}{1-s^2}+\left(\frac{|z|^2+|w|^2}{1-s^2}\right)^2+\frac{|z|^2+|w|^2}{1-s^2}+\left(\frac{|z|^2+|w|^2}{1-s^2}\right)^2+\left(\frac{|z|^2+|w|^2}{1-s^2}\right)^2<6.
\end{aligned}$$

While the construction of this last example is opaque (for the moment), the reader can recognize the numerators of each component function. One expects that we should be able to recover the example from last chapter, somehow. The first definition in the next section clarifies the relationship.

3.1 Relating Mixed-Type Proper Maps to Complex Proper Maps

To distinguish proper maps between spaces of the form $\mathbb{B}_n$ from proper maps between spaces of the form $\mathbb{B}_{n,k}$, we will refer to the former as *proper holomorphic maps* or *complex proper maps*, and the latter as *mixed-type proper maps*. Usually, the distinction will be perfectly clear from context, as in the following theorem, and we could refer to both as proper maps without confusion.

At times, we will be concerned with the regularity of our proper maps with respect to their real variables. We say that a mixed-type proper map $f:\mathbb{B}_{n,k}\to\mathbb{B}_{N,K}$ is of class Prop_{C^0} if it is holomorphic and (jointly) continuous; more generally we say f is of class Prop_{C^ℓ} if it is proper, holomorphic, and of class C^ℓ. In the cases where a mixed-type proper map f in class Prop_{C^ℓ} extends continuously to the boundary with ℓ derivatives, we say that f is of class $\overline{\mathrm{Prop}}_{C^\ell}$. Finally, we say that the mixed-type proper map f is $\mathrm{Rat}_{\mathrm{C}}$ or $\mathrm{Rat}_{\mathrm{CR}}$ if it is a rational in z or rational in z and s, respectively, and is (jointly) continuous.

Recall that the map f can always be written as a pair of maps (g,h) where h does not depend on the complex variable(s) z. Because of this decomposition, holomorphicity of f is completely encoded by holomorphicity of g. In particular, the map g, when viewed correctly (Theorem 3.1.1 below), is itself a proper map. In order to have explicit notation, we make the following definition.

Definition 3.1.1 *For a mixed-type proper map $f = (g, h) : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$ of any class, we can associate at each $s \in \mathbb{B}_{0,k}$ a map (which we call a* leaf map*)*

$$\Lambda_s\big[f\big] : z \mapsto \frac{g(z\sqrt{1-|s|^2}, s)}{\sqrt{1-|h(s)|^2}}$$

It is not difficult to show, but it is of extreme importance for this chapter, that each such associated leaf map is in fact a complex proper map, i.e. is proper in the sense of the previous chapter.

Theorem 3.1.1 *The leaf maps $\Lambda_s\big[f\big]$ associated to a mixed-type proper map $f : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$ of any class are proper holomorphic maps $\mathbb{B}_n \to \mathbb{B}_N$.*

Proof. Checking the domain and target is trivial.

Assume that f is proper, so whenever $1 - |z|^2 - |s|^2$ tends to zero, so does $1 - |g(z,s)|^2 - |h(s)|^2$. Fix $s \in \mathbb{B}_{0,k}$. To show that $\Lambda_s\big[f\big]$ is proper, we want to show that whenever $1 - |w|^2$ tends to zero, so too does $1 - |\Lambda_s\big[f\big](w)|^2$. For each point $w \in \mathbb{B}_n$ the point $(w\sqrt{1-|s|^2}, s)$ is in $\mathbb{B}_{n,k}$, and vice versa, for every point $(z, s) \in \mathbb{B}_{n,k}$ the point $w = z/\sqrt{1-|s|^2}$ is in $\mathbb{B}_n$ since

$$\left|\frac{z}{\sqrt{1-|s|^2}}\right|^2 = \frac{|z|^2}{1-|s|^2} < \frac{1-|s|^2}{1-|s|^2} = 1.$$

Now let $\{w_n\}_n$ be a sequence in $\mathbb{B}_n$ which tends to the boundary. Define the sequence $\{(w_n\sqrt{1-|s|^2}, s)\}_n$ in $\mathbb{B}_{n,k}$. This sequence tends to the boundary, too. We now look at the behavior of the sequence $\{\Lambda_s\big[f\big](w_n)\}_n$. The quantity $1 - |\Lambda_s\big[f\big](w_n)|^2$ can be written as

$$\frac{1 - |h(s)|^2 - |g(z_n, s)|^2}{1 - |h(s)|^2}.$$

The numerator tends to zero because the map f is proper. ■

The next lemma will help with algebraic book-keeping in the rest of the chapter.

Lemma 3.1.1 *Let $f = (g,h) : \mathbb{B}_{n,k} \to \mathbb{B}_{m,\ell}$ and $F = (G,H) : \mathbb{B}_{m,\ell} \to \mathbb{B}_{N,K}$ be mixed-type proper maps of any class. For any $s \in \mathbb{B}_{0,k}$, the leaf of the composition taken at s is given by the composition of leaf maps in the following way:*

$$\Lambda_s\big[F \circ f\big] = \Lambda_{h(s)}\big[F\big] \circ \Lambda_s\big[f\big].$$

Proof. This is a straightforward computation. Observe that plugging $f(z,s) = (g(z,s), h(s))$ into $F(z,s) = (G(z,s), H(s))$ yields $F \circ f(z,s) = \Big(G\big(g(z,s), h(s)\big), H\big(h(s)\big)\Big)$ so the leaf map of the composition is

$$\Lambda_s\big[F \circ f\big] : z \mapsto \frac{G\Big(g\big(z, \sqrt{1-|s|^2}, s\big), h(s)\Big)}{\sqrt{1 - |H\big(h(s)\big)|^2}}.$$

By comparison, the separate leaf maps are

$$\Lambda_{h(s)}\big[F\big] = \frac{G\Big(z\sqrt{1-|h(s)|^2}, h(s)\Big)}{\sqrt{1-|H\big(h(s)\big)|^2}} \quad \text{and} \quad \Lambda_s\big[f\big] = \frac{g\big(z\sqrt{1-|s|^2}, s\big)}{\sqrt{1-|h(s)|^2}}$$

so in the composition, the factors $\sqrt{1-|h(s)|^2}$ cancel appropriately. ■

One can morally reverse the process in the definition above in order to define a 'promotion' of a proper holomorphic map $g : \mathbb{B}_n \to \mathbb{B}_N$ along a map $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$ to produce the mixed-type proper map

$$\Lambda^{-1}(g,h) : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K} : (z,s) \mapsto \left(g\Big(\frac{z}{\sqrt{1-|s|^2}}\Big)\sqrt{1-|h(s)|^2}, h(s)\right)$$

This promotion inverts the leaf map in the sense that

$$\Lambda_s\big[\Lambda^{-1}(g,h)\big] = g(z) \quad \text{and} \quad \Lambda^{-1}\big(\Lambda_s\big[f\big], h\big) = f(z,s).$$

Remark 3.1.1 *For any orthogonal matrix Q, the function $\Gamma_Q(g,h)$ defined by taking (z,s) to*

$$\left(g\Big(\frac{z}{\sqrt{1-|s|^2}}\Big)\sqrt{1-|h(s)|^2}, Qh(s)\right)$$

satisfies $\Lambda_s\big[\Gamma_Q(g,h)\big](z) = g(z)$. However, we note that $\Gamma_Q(\Lambda_s\big[f\big], h(s))$ is the map taking z to $(g(z,s)Qh(s))$.

Other variations are possible.

Despite not being a completely canonical construction, many examples in this chapter are constructed by using modifications of this Λ^{-1}, including Example 3.0.2 above.

The correspondence between mixed-type proper maps $\mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$ and leaf maps $\mathbb{B}_n \to \mathbb{B}_N$ allures the reader to consider the role of functors in our discussion. There is an awkward way to define such a functor, but Lemma 3.1.1 illustrates why the most direct attempt at definition of a functor fails.

Remark 3.1.2 *If we insist that $h(0) = 0$, then Λ_0 is a functor.*

More generally, if we insist that $h(s_*) = s_*$ for some fixed $s_* \in \mathbb{B}_{0,k}$, then Λ_{s_*} is a functor.

We now turn to discussing the real components of our functions. More lurks here than one might first imagine.

Example 3.1.1 *It is not necessary for the projection onto the real coordinates to take values near one, as the mixed-type proper map*[1] *example $f : \mathbb{B}_{n,1} \to \mathbb{B}_{n+1,1}$ defined by*

$$f(z, s) = \left(\frac{z}{\sqrt{5}}, \frac{2+s}{\sqrt{10}}, \frac{2-s}{\sqrt{10}} \right)$$

illustrates. The map $s \mapsto \dfrac{2-s}{\sqrt{10}}$ is not a proper map $\mathbb{B}_{0,1} \to \mathbb{B}_{0,1}$.

There are two natural potential ameliorations to this problem. One could refactor this example, and instead view the map as an example $\mathbb{B}_{n,1} \to \mathbb{B}_{n,2}$. That is, the 'real' portion of the map can be viewed as $s \mapsto \left(\dfrac{2+s}{\sqrt{10}}, \dfrac{2-s}{\sqrt{10}} \right)$. This is indeed a proper map $\mathbb{B}_{0,1} \to \mathbb{B}_{0,2}$, as can be verified by checking that $1 - |h(s)|^2$ is given by $\frac{1}{5}(1 - s^2)$. This will not generally work.

The second reasonable consideration, especially in light of the definition of the leaf maps above, is to consider rescaling the image of the h function. Or, instead, to ask if h need be proper *onto its image*. This also fails. See Theorem 3.3.1 to see that h can be chosen quite freely.

[1] We can verify that f is proper by computing $1 - |f(z,s)|^2 = \frac{1}{5}(1 - |z|^2 - s^2)$.

Despite this promised failure, we have this comforting result, which says that the associated map h is indeed a proper map in its own right when the complex dimensions are equal.

Theorem 3.1.2 *If $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n,K} : (z,s) \mapsto (g(z,s), h(s))$ is a mixed-type proper map of any class, then the map $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$ is proper.*

Proof. For each s, we have a proper holomorphic map $\tilde{g}_s := \Lambda_s\big[f\big] : \mathbb{B}_n \to \mathbb{B}_n$. Since such $\tilde{g}_s$ is necessarily an automorphism of the ball by the result of [Ale77a], there is a preimage $a = a(s)$ of 0 under $\tilde{g}_s$. That is, for each s, $g(a(s)\sqrt{1-|s|^2}, s)$ is zero. We note that the map $s \mapsto a(s)\sqrt{1-|s|^2} =: A(s)$ must be continuous if f is, by Theorem 2.5.1. Now consider the limit as (z,s) approaches the boundary of $\mathbb{B}_{n,k}$ along the specific path $s \mapsto (A(s), s)$ to see that

$$1 = \lim |f(A(s), s)| = \lim |(0, h(s))| = \lim |h(s)|$$

as needed. ■

We can also intuit from this why h will not necessarily need not be proper when the complex codimension increases. Crucial for the proof was the existence of the path along which $\lim |h(s)| = 1$ as s approaches the boundary. In particular, when the complex dimension of the target is larger than the complex dimension of the source, it is possible that the image under $\Lambda_s\big[f\big]$ avoids the origin completely. See, for instance, Theorem 3.3.1.

3.2 Equivalences of Proper Maps

In this section, we discuss two usual notions equivalence as relevant in this setting.

Example 3.2.1 *The map $f : \mathbb{B}_{1,k} \to \mathbb{B}_{1,k}$ given by*

$$f(z,s) = \begin{cases} (z, 0) & s = 0 \\ \left(\frac{z^2}{\sqrt{1-|s|^2}}, s\right) & s \neq 0 \end{cases}$$

is a mixed-type proper map. For any choice of nonzero s, the leaf maps $\Lambda_0\big[f\big]$ and $\Lambda_s\big[f\big]$ are inequivalent (in the sense of definition below) as maps $\mathbb{B}_1 \to \mathbb{B}_1$. Indeed, we observe that f is not even continuous inside the domain.

Definition 3.2.1 *We say that two proper maps (of the same class) $f_0, f_1 : X \to Y$ are homotopy equivalent if there exists a function F continuous on $I \times X$ such that $F(t, -)$ is a proper map $X \to Y$ for every value of $t \in [0, 1]$, and such that $z \mapsto F(0, z)$ is the same function as f_0 and similarly $F(1, -) = f_1$.*

As far as we are concerned, only proper maps f which are (at least) continuous will be of interest. (See Example 3.2.1 above.) In this case, we can consider F as a continuous map from the product space $[0, 1] \times \mathbb{B}_{n,k}$ to the codomain $\mathbb{B}_{N,K}$, and further we assume that F is holomorphic with respect to the complex variables.

Proposition 3.2.1 *Promotion of g_1 along h_1 is homotopy equivalent to promotion of g_2 along h_2 if and only if g_1 is homotopy equivalent to g_2 and h_1 is homotopy equivalent to h_2.*

Proof. ($\Rightarrow$) Assume that G_t is a homotopy connecting g_1 and g_2, and that H_t is a homotopy connecting h_1 and h_2. Then $\Lambda^{-1}(G_t, H_t)$ is clearly a homotopy connecting $\Lambda^{-1}(g_1, h_1)$ and $\Lambda^{-1}(g_2, h_2)$.

($\Leftarrow$) If F_t is a homotopy connecting $\Lambda^{-1}(g_1, h_1)$ and $\Lambda^{-1}(g_2, h_2)$, then $\pi_{_{\mathbb{R}K}} F_t$ is a homotopy connecting h_1 and h_2, and $\Lambda_s\big[F_t\big]$ is a homotopy connecting g_1 and g_2. ■

The following is an example illustrating how one might utilize information from proper holomorphic maps (chapter two) to understand the forms of mixed-type proper maps.

Proposition 3.2.2 *Let f be a mixed-type proper map $\mathbb{B}_{2,k} \to \mathbb{B}_{3,K}$ of class* $\mathrm{Rat}_{\mathbb{C}}$*, i.e., every leaf map $z \mapsto f(z, s)$ is a rational function and f is continuous. All leaf maps $\{\Lambda_s\big[f\big] : s \in \mathbb{B}_{0,k}\}$ are the same, up to spherical equivalence.*

Proof. Let $f : \mathbb{B}_{2,k} \to \mathbb{B}_{3,K}$ be such a map. For any choices s and s' in $\mathbb{B}_{0,k}$, convexity of $\mathbb{B}_{0,k}$ tells us that $(1 - t)s + ts'$ is a path within $\mathbb{B}_{0,k}$ connecting s (when $t = 0$) to s' (when

$t=1$). The map G_t defined by $\Lambda_{(1-t)s+ts'}\big[f\big]$ is clearly continuous in t, and so is a homotopy equivalence of the rational proper holomorphic maps $\mathbb{B}_2 \to \mathbb{B}_3$ given by $\Lambda_s\big[f\big]$ and $\Lambda_{s'}\big[f\big]$. Further, for each value of t in the interval $[0,1]$, the proper holomorphic map $G_t : \mathbb{B}_2 \to \mathbb{B}_3$ is rational. By [DL16, Corollary 3.2], the spherically inequivalent maps enumerated by Faran are homotopically inequivalent. ∎

This leads to a natural question. If every leaf of a map $\mathbb{B}_{2,k} \to \mathbb{B}_{3,k}$ is spherically equivalent, is it possible to aggregate these leaf-level spherical equivalence maps in a nice way? The answer, sadly, is 'no'.

In order to construct an example that does not work well, it is helpful to use a proper holomorphic map $\mathbb{B}_n \to \mathbb{B}_N$ where the degrees of the component mappings are not all equal.

Example 3.2.2 *Recall from Example 3.0.2 the mixed-type proper map $\mathbb{B}_{2,1} \to \mathbb{B}_{3,1}$ defined by sending (z,w,s) to*

$$\left(e^{is}\sqrt{1-s^4}\frac{z^3}{\sqrt{1-s^2}^3},\ \sqrt{1-s^4}\frac{w^3}{\sqrt{1-s^2}^3},\ \sqrt{3}\sqrt{1-s^4}\frac{zw}{\sqrt{1-s^2}^2},\ s^2\right)$$

This is constructed by considering $\Lambda^{-1}\big((z^3,w^3,\sqrt{3}zw),s^2\big)$ and appropriately (or, inappropriately*) 'twisting' the first coordinate. There are no mixed-type proper bijections (of class* Prop_{C^0}*) $\Psi : \mathbb{B}_{3,1} \to \mathbb{B}_{3,1}$ and $\Phi : \mathbb{B}_{2,1} \to \mathbb{B}_{2,1}$ such that $\Lambda_s\big[\Psi \circ f \circ \Phi\big]$ is independent of s.*

Proof. The map Φ is completely determined by $\phi_s := \Lambda_s\big[\Phi\big]$ and the real map $\eta(s) := \pi_{\mathbb{R}^2}\Phi$. (Put another way, $\Phi = \Lambda^{-1}(\phi_s, \eta)$. See the discussion immediately preceding Remark 3.1.1.) We first show that η is a bijection. Suppose that s_1 and s_2 in $\mathbb{B}_{0,k}$ are mapped to the same image under η, i.e. $\eta(s_1) = \eta(s_2)$. We consider the leaf maps at these points. Observe that $\phi_{s_1} = \Lambda_{s_1}\big[\Phi\big]$ and $\phi_{s_2} = \Lambda_{s_2}\big[\Phi\big]$ are both holomorphic proper maps $\mathbb{B}_2 \to \mathbb{B}_2$. By Alexander's result, both ϕ_{s_1} and ϕ_{s_2} are thus automorphisms of $\mathbb{B}_2$; in particular, they are surjective. Choose z_1 and z_2 such that $\phi_{s_1}(z_1)$ and $\phi_{s_2}(z_2)$ are zero. For these choices we have

$$\Phi(z_j\sqrt{1-|s_j|^2}, s_j) = \left(\phi_{s_j}\left(\frac{z_j\sqrt{1-|s_j|^2}}{\sqrt{1-|s_j|^2}}\right)\sqrt{1-|\eta(s_j)|^2}, \eta(s_j)\right) = (0, \eta(s_j)).$$

Since Φ was injective by hypothesis, the points $(z_j\sqrt{1-|s_j|^2}, s_j)$ are equal for $j = 1, 2$. In particular, $s_1 = s_2$. If η were not surjective, then Φ would not be surjective.

Pick any $t \in (0, 1)$ and set $s_1 = \eta^{-1}(\sqrt{t})$, $s_2 = \eta^{-1}(-\sqrt{t})$. Note that s_1 and s_2 are distinct, as verified by $\eta(s_1) = \sqrt{t} \neq -\sqrt{t} = \eta(s_2)$. We now compare the leaf maps at these values of the real parameter. The composition rule 3.1.1 says

$$\Lambda_s\big[\Psi \circ f \circ \Phi\big] = \Lambda_{h\circ\eta(s)}\big[\Psi\big] \circ \Lambda_{\eta(s)}\big[f\big] \circ \Lambda_s\big[\Phi\big].$$

By choice of s_1 and s_2, we see that $h \circ \eta(s_1)$ and $h \circ \eta(s_2)$ share the common value t, and so $\Lambda_{h\circ\eta(s_1)}\big[\Psi\big] = \Lambda_{h\circ\eta(s_2)}\big[\Psi\big]$. Thus by composing on the left by the inverse of this common map, we see that $\Lambda_{s_1}\big[\Psi \circ f \circ \Phi\big] = \Lambda_{s_2}\big[\Psi \circ f \circ \Phi\big]$ reduces to

$$\Lambda_{\eta(s_1)}\big[f\big] \circ \Lambda_{s_1}\big[\Phi\big] = \Lambda_{\eta(s_2)}\big[f\big] \circ \Lambda_{s_2}\big[\Phi\big].$$

If we denote $\Lambda_{s_2}\big[\Phi\big] \circ (\Lambda_{s_1}\big[\Phi\big])^{-1}$ by (α, η), we can rewrite the desired equality as

$$\Lambda_{\eta(s_1)}\big[f\big] = \Lambda_{\eta(s_2)}\big[f\big] \circ (\alpha, \beta).$$

Explicitly, $\Lambda_{\eta(s_1)}\big[f\big]$ for our choice of f is

$$\begin{aligned}
\Lambda_{\eta(s_1)}\big[f\big] &= \frac{1}{\sqrt{1-|h(\eta(s_1))|^2}} g((z,w)\sqrt{1-|\eta(s_1)|^2}, \eta(s_1)) \\
&= \frac{1}{\sqrt{1-|h(\eta(s_1))|^2}} \Bigg(e^{i\eta(s_1)}\sqrt{1-\eta(s_1)^4}\frac{z^3\sqrt{1-|\eta(s_1)|^2}^3}{\sqrt{1-\eta(s_1)^2}^3}, \\
&\qquad\qquad \sqrt{1-\eta(s_1)^4}\frac{w^3\sqrt{1-|\eta(s_1)|^2}^3}{\sqrt{1-\eta(s_1)^2}^3}, \\
&\qquad\qquad \sqrt{3}\sqrt{1-\eta(s_1)^4}\frac{zw\sqrt{1-|\eta(s_1)|^2}^2}{\sqrt{1-\eta(s_1)^2}^2}\Bigg) \\
&= \frac{1}{\sqrt{1-|h(\sqrt{t})|^2}}\left(e^{i\sqrt{t}}\sqrt{1-\sqrt{t}^4}z^3, \sqrt{1-\sqrt{t}^4}w^3, \sqrt{3}\sqrt{1-\sqrt{t}^4}zw\right) \\
&= \frac{1}{\sqrt{1-|\sqrt{t}^2|^2}}\left(e^{i\sqrt{t}}\sqrt{1-t^2}z^3, \sqrt{1-t^2}w^3, \sqrt{3}\sqrt{1-t^2}zw\right) \\
&= (e^{i\sqrt{t}}z^3, w^3, \sqrt{3}zw)
\end{aligned}$$

and similarly

$$\begin{aligned}\Lambda_{\eta(s_2)}\big[f\big]\circ(\alpha,\beta) &= (e^{-i\sqrt{t}}z^3, w^3, \sqrt{3}zw)\circ(\alpha,\beta)\\ &= (e^{-i\sqrt{t}}\alpha^3, \beta^3, \sqrt{3}\alpha\beta).\end{aligned}$$

To have equality of these two maps, there must be valid choices of t, α, β such that we simultaneously have $e^{i\sqrt{t}}z^3 = e^{-i\sqrt{t}}\alpha^3$, $w^3 = \beta^3$, and $\sqrt{3}zw = \sqrt{3}\alpha\beta$. First rewrite these equations: $e^{2i\sqrt{t}}z^3 = \alpha^3$, $w^3 = \beta^3$, and $zw = \alpha\beta$. Cubing the last condition, we see that

$$\begin{aligned}(zw)^3 &= (\alpha\beta)^3\\ &= \alpha^3\beta^3\\ &= e^{2i\sqrt{t}}z^3w^3\end{aligned}$$

and hence we require that $e^{2i\sqrt{t}} = 1$. This can only occur if $2\sqrt{t} \in 2\pi\mathbb{Z}$, i.e. if t is of the form π^2n^2 for some integer n. This contradicts the choice of $t \in (0, 1)$. ∎

By tweaking the above construction, we can see that there are infinitely many maps f such that $\Lambda_s\big[\Psi \circ f \circ \Phi\big]$ is not independent of s for any choice of Ψ, Φ.

The maps Φ and Ψ in the previous proof play the role of the automorphisms appearing in the definition of *spherical equivalence* in Section 2.8. It is natural to introduce the corresponding definition here.

Definition 3.2.2 *Two mixed-type proper maps $f_1, f_2 : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$ of class $\mathcal{C}$ are called spherically equivalent (in class $\mathcal{C}$) if there exists mixed-type proper maps Φ and Ψ (in class $\mathcal{C}$) such that $\Psi \circ f_2 = f_1 \circ \Phi$.*

We note how the spherical equivalence classes interact with the action of 'taking a leaf' in the remainder of this section.

Let $[g]_{\mathrm{C}}$ denote the spherical equivalence class of g, where g is a complex proper map, and similarly let $[f]_{\mathrm{CR}}$ denote the spherical equivalence class of f, where f is a mixed-type proper map. Additionally, we define $[f]_{\mathrm{CR}'}$ to be the subset of $[f]_{\mathrm{CR}}$ whose elements are of the form $\Psi \circ f \circ \Phi$ with the additional restriction that $\pi_{\mathbb{R}^k}\Phi$ is the identity map.

Theorem 3.2.1 *For any mixed-type proper map $f : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$, and for all $s \in \mathbb{B}_{0,k}$, we have $\Lambda_s\big[[f]_{CR'}\big] = [\Lambda_s\big[f\big]]_C$.*

Proof. Fix a proper map $f = (g,h) : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$.

Let $\Psi \circ f \circ \Phi$ be a representative in $[f]_{\mathrm{CR}'}$. Then, by Lemma 3.1.1,

$$\Lambda_s\big[\Psi \circ f \circ \Phi\big] = \Lambda_{h(s)}\big[\Psi\big] \circ \Lambda_s\big[f\big] \circ \Lambda_s\big[\Phi\big].$$

Clearly, for each $s \in \mathbb{B}_{0,k}$, $\Lambda_s\big[\Psi\big]$ and $\Lambda_s\big[\Phi\big]$ are automorphisms of the appropriate balls in the pure complex spaces.

Conversely, let $\tilde{\psi} \circ f \circ \tilde{\phi}$ be a representative in $[\Lambda_s\big[f\big]]_{\mathrm{C}}$, so that $\tilde{\psi}$ is an automorphism of $\mathbb{B}_N$ and $\tilde{\phi}$ is an automorphism of $\mathbb{B}_n$. We construct Φ as $\Lambda^{-1}\Big(\tilde{\phi}, \mathrm{id}\Big)$ and similarly Ψ is the promotion of $\tilde{\psi}$ using the identity function. These maps are both injective and surjective, so indeed $\Psi \circ f \circ \Phi$ is in $[f]_{\mathrm{CR}'}$. Further, $\Lambda_s\big[\Psi \circ f \circ \Phi\big]$ is $\tilde{\psi} \circ f \circ \tilde{\phi}$. ∎

The more natural question is how the spherical equivalence classes relate. Of course, the above proposition shows that we cannot expect the leaf map to commute with taking the equivalence class, in general. We state the result.

Theorem 3.2.2 *For any mixed-type proper map $f : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$, and for all $s \in \mathbb{B}_{0,k}$, we have the containment $[\Lambda_s\big[f\big]]_C \subseteq \Lambda_s\big[[f]_{CR}\big]$.*

Proof. We need only note that $[f]_{\mathrm{CR}'}$ is a proper subset of $[f]_{\mathrm{CR}}$. ∎

3.3 Some Constructions

In this section, we make some observations regarding regularity of mixed-type proper maps and their extensions to the boundary of $\mathbb{B}_{n,k}$. We will see that, in contrast to some nice results such as Faran's $\mathbb{B}_2 \to \mathbb{B}_3$ case, regularity in the real variables is expectedly difficult to control.

To begin, we give a variant of an earlier example (Example 3.0.1).

Example 3.3.1 *Define the function $f : \mathbb{B}_{n,2} \to \mathbb{B}_{n,2}$ by writing w for $s + it$ and sending (z, s, t) to*

$$\left(z, \operatorname{Re}\{w \exp \frac{i}{1 - |w|}\}, \operatorname{Im}\{w \exp \frac{i}{1 - |w|}\}\right)$$

This function is proper, since $|f(z, s, t)|^2 = |z|^2 + |w \exp \frac{i}{1-|w|}|^2 = |z|^2 + |w|^2 = |z|^2 + |s|^2 + |t|^2$ and so $1 - |f(z, s, t)|^2$ tends to zero as $1 - |(z, s, t)|^2$ does.

We can also note that the map above is a bijection of $\mathbb{B}_{n,2}$ with itself.

Remark 3.3.1 *Example 3.0.1 is interesting in that it extends continuously to the boundary (by the squeeze theorem), but its derivative does not. That is, the map is of class $\overline{\mathrm{Prop}}_{C^0} \setminus \overline{\mathrm{Prop}}_{C^1}$.*

Similar to a result of Catlin and D'Angelo [CD96] which says that for each homogeneous vector-valued polynomial p defined on $\mathbb{C}^n$, there is a constant $M > 0$ and a vector-valued polynomial q such that $(\frac{1}{M}p, q)$ is a proper holomorphic mapping of balls $\mathbb{B}_n \to \mathbb{B}_N$, we can look for results allowing us to suitably "add dimensions" in order to make a map proper.

Proposition 3.3.1 *Let $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$ be a continuous map. If there exists a function $\beta : \mathbb{B}_{0,k} \to \mathbb{B}_{0,1}$ that satisfies*

$$\limsup_{|s| \to 1^-} \frac{1 - \beta(s)^2 - |h(s)|^2}{1 - |s|^2} < \infty,$$

then there exists a mixed-type proper map $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n+1,K}$ (which can also be viewed as a map $\mathbb{B}_{n,k} \to \mathbb{B}_{n,K+1}$) such that $\pi_{\mathbb{R}^k} f$ is the prescribed function h.

Proof. The construction is quite elementary, but does not necessarily extend nicely to the boundary, i.e. the resulting map is of class $\overline{\mathrm{Prop}}_{C^0}$ but not necessarily in $\overline{\mathrm{Prop}}_{C^\ell}$ for $\ell > 0$. Define

$$f(z, s) = \left(\sqrt{\frac{1 - \beta(s)^2 - |h(s)|^2}{1 - |s|^2}} z, \beta(s), h(s)\right).$$

This map is proper, since

$$1 - |f(z, s)|^2 = (1 - \beta(s)^2 - |h(s)|^2)\frac{1 - |s|^2 - |z|^2}{1 - |s|^2}.$$

For this example, note that the map $s \mapsto (\beta(s), h(s))$ must be proper. In order for the limit from the hypothesis to exist, the numerator must approach zero, since the denominator does.

Continuity of the extension to the boundary will follow from the squeeze theorem. Both β and h are bounded *a priori*, so we need only check that the leading expression vanishes as (z, s) approaches the boundary. But z approaches zero at the only potential points of discontinuity (i.e., when $|s| \to 1$). At those points, we can bound the modulus of the first term using the product of the given bounded limit superior together with the norm of z. ■

Similarly, we have the following.

Proposition 3.3.2 *Let* $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$ *be any* C^ℓ *real map. Then* $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n+1,K}$ *defined by*

$$(z, s) \mapsto \left(\sqrt{1+|s|^2}\sqrt{1-|h(s)|^2}z, |s|^2\sqrt{1-|h(s)|^2}, h(s)\right)$$

is a mixed-type proper map of class $\overline{\mathrm{Prop}}_{C^0}$. *This can also be viewed as a mixed-type proper map* $\mathbb{B}_{n,k} \to \mathbb{B}_{n,K+1}$ *of class* $\overline{\mathrm{Prop}}_{C^0}$.

Proof. This map is proper, as $1 - |f(z, s)|^2$ is calculated to be

$$(1 - |z|^2 - |s|^2)(1 + |s|^2)(1 - |h(s)|^2).$$

We again observe that the map $\tilde{h} : s \mapsto |s|^2\sqrt{1-|h(s)|^2}, h(s))$ is a proper map, as $1 - |\tilde{h}(s)|^2 = (1 - |h(s)|^2)(1 - |s|^4)$. ■

Proposition 3.3.3 *If* $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$ *is a map for which the quotient* $\dfrac{(1-|h(s)|^2)(1-|h(s)|^{2\ell})}{1-|s|^2}$ *is bounded, then*

$$(z, s) \mapsto \left(\sqrt{\frac{(1-|h(s)|^2)(1-|h(s)|^{2\ell})}{1-|s|^2}}z, |h(s)|^\ell\sqrt{1-|h(s)|^2}, h(s)\right)$$

is a mixed-type proper map.

This is a particular case of the more general following construction.

Proposition 3.3.4 *If* $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,K}$ *and* $B : \mathbb{B}_{0,k'} \to \mathbb{B}_{0,K'}$ *are maps for which* $\dfrac{(1-|h(s)|^2)(1-|B(s)|^2)}{1-|s|^2}$ *is bounded, then*

$$f : (z,s) \mapsto \left(\sqrt{\frac{(1-|h(s)|^2)(1-|B(s)|^2)}{1-|s|^2}}z, B(s)\sqrt{1-|h(s)|^2}, h(s) \right)$$

is a mixed-type proper map.

Proof. It is straightforward algebra to verify that $1-|f(z,s)|^2$ factors as

$$\frac{(1-|h(s)|^2)(1-|B(s)|^2)(1-|s|^2-|z|^2)}{1-|s|^2}.$$

■

The above constructions do not guarantee nice regularity up to the boundary. For example, the derivative of the term $|h(s)|^\ell\sqrt{1-|h(s)|^2}$ with respect to (any component of) s is guaranteed to diverge as $|s| \to 1$ if h is proper.

For some specific functions h, we can construct mixed-type proper maps $f : \mathbb{B}_{n,k} \to \mathbb{B}_{N,K}$ such that the real projection $\pi_{_{\mathbb{R}^K}} f$ is h. Sometimes, this new function extends to the boundary with nice regularity.

Proposition 3.3.5 *If* h *is an affine linear function* $\mathbb{B}_{0,1} \to \mathbb{B}_{0,1}$, *say* $h(s) = h_0 + h_1 s$, *and if there exists* $\beta_0, \beta_1 \in \mathbb{R}$ *satisfying* $\beta_0\beta_1 = -h_0h_1$ *and* $\beta_0^2 + \beta_1^2 = 1 - h_0^2 - h_1^2$, *then the map*

$$\mathbb{B}_{n,1} \to \mathbb{B}_{n,2} : (z,s) \mapsto (\sqrt{\beta_1^2 + h_1^2}z, \beta_0 + \beta_1 s, h_0 + h_1 s)$$

is a mixed-type proper map. Further, the map $\mathbb{B}_{n,1} \to \mathbb{B}_{n,2}$ *extends smoothly to the boundary, i.e., is of class* $\overline{\mathrm{Prop}}_{C^\omega}$.

Proof. It is a direct computation to check that for the function $f : \mathbb{B}_{n,1} \to \mathbb{B}_{n,2}$ as defined, $1-|f(z,s)|^2$ is given by

$$1 - (\beta_1^2 + h_1^2)|z|^2 - \beta_0^2 - 2\beta_0\beta_1 s - \beta_1^2 s^2 - h_0^2 - 2h_0h_1 s - h_1^2 s^2.$$

The condition $\beta_0\beta_1 = -h_0h_1$ causes the linear terms to cancel. The summand $1 - \beta_0^2 - h_0^2$ can be replaced by $\beta_1^2 + h_1^2$, and thus

$$1 - |f(z,s)|^2 = (\beta_1^2 + h_1^2)(1 - |z|^2 - s^2)$$

which verifies that f is proper. Smoothness of the extension is clear because each component of f is a polynomial. ■

We note that the real numbers β_0 and β_1 in the proposition always exist. The condition that $h(\mathbb{B}_{0,1}) \subset \mathbb{B}_{0,1}$ simply says that $|h_0 \pm h_1| \leq 1$. If h_0 happens to be zero, then we can pick $\beta_0 = 0$ and $\beta_1 = \sqrt{1 - h_1^2}$, for example, and hence the map $(z, \sqrt{1-h_1^2}s, h_1 s)$ is a valid choice of mixed-type proper map. Similarly if $h_1 = 0$, the map $(\sqrt{1-h_0^2}z, \sqrt{1-h_0^2}s, h_0)$ is proper. For the more general case, we only note that by viewing the map's codomain as $\mathbb{B}_{n+1,1}$, we can even allow the choice of β to have complex coefficients, as can the coefficient $\alpha(s) = \left((1-h(s)^2)(1-\beta(s)^2 - h(s)^2)/(1-s)^2\right)^{1/2}$ of z.

Theorem 3.3.1 *Let $N > n$ and $g : \mathbb{B}_n \to \mathbb{B}_N$ be the linear embedding $z \mapsto (z, 0)$. Let $h : \mathbb{B}_{0,1} \to \mathbb{B}_{0,K}$ be an arbitrary real map, and $U(s)$ be an arbitrary map into the set of $N \times N$ unitary matrices. Let $a : \overline{\mathbb{B}_{0,1}} \to \overline{\mathbb{B}_N}$ be a function satisfying*

$$\lim_{s \to \pm 1} a(s) \in \partial\mathbb{B}_N \setminus \overline{g(\mathbb{B}_N)}.$$

The following map $\mathbb{B}_{n,1} \to \mathbb{B}_{N,K}$ is a mixed-type proper map:

$$f(z,s) = \left(U(s)\sqrt{1-|h(s)|^2}\frac{a(s) - \sqrt{1-|a(s)|^2}w - \frac{\langle w, a(s)\rangle}{1+\sqrt{1-|a(s)|^2}}a(s)}{1 - \langle w, a(s)\rangle}, h(s) \right)$$

where w denotes $g(z/\sqrt{1-s^2})$.

We should remark before proving this existence result that if g is the linear embedding, such an a function always exists: let $a(s) = (0, \ldots, 0, s)$, for example.

Proof. To check that f is proper, it suffices to verify that $1 - |f(z,s)|^2$ tends to zero as $|(z,s)|$ approaches 1. A lengthy but direct computation shows that $1 - |f(z,s)|^2$ can be factored as

$$(1-|h(s)|^2)(1-|a(s)|^2)(1-|w|^2)\frac{2 - 2\sqrt{1-|a(s)|^2} - |a(s)|^2}{|1 - \langle w, a(s)\rangle|^2(1+\sqrt{1-|a(s)|^2})}.$$

If we can show that the denominator is well-behaved, we will be done, since the factor $1-|w|^2$ tends to zero (because g is proper) except possibly as $|s| \to 1$. However, as $|s|$ tends to 1, we instead rely on the fact that $|a(s)|$ tends to 1, by choice of a.

The factor $1 + \sqrt{1-|a(s)|^2}$ is bounded below, so we focus on the other factor. Because of the Cauchy-Schwarz inequality, we know that $|\langle w, a(s)\rangle|$ is bounded above by $|w||a(s)|$. For all points (z, s) in the ball, $|w| = |g(z/\sqrt{1-s^2})| < 1$ and $|a(s)| < 1$. Hence the limit of $|\langle w, a(s)\rangle|$ is bounded above by 1. The Cauchy-Schwarz inequality further tells us that equality can only be achieved if some vector in the set of accumulation points of $g(z/\sqrt{1-s^2})$ is parallel to the limit of $a(s)$. Note that if $g(z/\sqrt{1-s^2}) \to \lambda a(1)$ (for example) along some path, then $\langle w, a(s)\rangle \to \langle \lambda a(1), a(1)\rangle = \lambda$, so unless $\lambda = 1$, the denominator cannot be zero. But we know that $\lim_n g(z_n/\sqrt{1-s_n^2})$ cannot equal $a(1)$, otherwise $a(1)$ would be in the closure of the image of g. Consequently, for any path $(z, s) \to (0, 1)$, we do not have $g(z/\sqrt{1-s^2}) \to a(1)$. ■

3.4 Statement of form for proper maps $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n,K}$

Let $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n,K}$ be a map which is complex-differentiable in its complex coordinates. That is, we write $f(z, s) = (g(z, s), h(s))$ and insist that g be complex-differentiable in the $z = (z_1, \ldots, z_n)$ coordinates. By applying $\Lambda_s\big[f\big]$ given in Definition 3.1.1, we know that $\Lambda_s\big[f\big] = \frac{g(z\sqrt{1-|s|^2}, s)}{\sqrt{1-|h(s)|^2}}$ is a proper holomorphic map $\mathbb{B}_n \to \mathbb{B}_n$. Recall that [Ale77a] showed that proper holomorphic maps in the equidimensional case are automorphisms, so $\Lambda_s\big[f\big]$ is a linear fractional transformation. If $n = 1$, this means that there exist parameters $a = a(s)$ and $\theta = \theta(s)$ so that

$$\frac{g(z\sqrt{1-|s|^2}, s)}{\sqrt{1-|h(s)|^2}} = e^{i\theta(s)} \frac{a(s) - z}{1 - \overline{a(s)}z}$$

and so

$$g(z, s) = e^{i\theta(s)} \frac{a(s) - \frac{z}{\sqrt{1-|s|^2}}}{1 - \overline{a(s)}\frac{z}{\sqrt{1-|s|^2}}} \sqrt{1-|h(s)|^2}.$$

For $n > 1$, the linear fractional transform takes a more complicated form, but from

$$\frac{g(z\sqrt{1-|s|^2}, s)}{\sqrt{1-|h(s)|^2}} = U(s)\frac{a(s) - L(z; a(s))}{1 - \langle z, a(s)\rangle}$$

we can just as easily conclude that

$$g(z, s) = \sqrt{1-|h(s)|^2}U(s)\frac{a(s) - L(z/\sqrt{1-|s|^2}; a(s))}{1 - \langle z, a(s)\rangle / \sqrt{1-|s|^2}}$$

or more explicitly

$$g(z, s) = \sqrt{1-|h(s)|^2}U(s)\frac{a(s) - \sqrt{1-|a(s)|^2}\dfrac{z}{\sqrt{1-|s|^2}} - \dfrac{\left\langle z/\sqrt{1-|s|^2}, a(s)\right\rangle}{1+\sqrt{1-|a(s)|^2}}a(s)}{1 - \langle z, a(s)\rangle / \sqrt{1-|s|^2}}.$$

We introduce a small change in notation. The quantities $a(s)$ appearing in the $n = 1$ and $n > 1$ answer the question: what value of X makes $\Lambda_s\big[f\big](X) = 0$ true? However, when $k \neq 0$ they do not answer the question: what value of X makes $f(X, s) = 0$? To correct for this, we introduce $A(s) = \sqrt{1-|s|^2}a(s)$. In some sense, these new quantities now play the role that a did in the $\mathbb{B}_n \to \mathbb{B}_n$ cases, since $f(A(s), s) = (0, h(s))$ and $f(0, s) = (U(s)A(s), h(s))$.

Similar to Theorem 2.5.1, we can find explicit forms for the parameters a (or A) and U in terms of the germ of f at $z = 0$.

Theorem 3.4.1 *Let $n > 1$ be an integer, and f be a mixed-type proper map $\mathbb{B}_{n,k} \to \mathbb{B}_{n,k}$ of class* Prop_{C^0}*. Then $f(z, s)$ has the form*

$$\left(\sqrt{1-|h(s)|^2}U(s)\frac{\dfrac{A(s)}{\sqrt{1-|s|^2}} - L\left(\dfrac{z}{\sqrt{1-|s|^2}}; \dfrac{A(s)}{\sqrt{1-|s|^2}}\right)}{1 - \left\langle \dfrac{z}{\sqrt{1-|s|^2}}, \dfrac{A(s)}{\sqrt{1-|s|^2}}\right\rangle}, h(s)\right)$$

where $A(s)$ is given by

$$-\frac{1-|s|^2}{1-|f(0,s)|^2}J(s)^\dagger \pi_{\mathbb{C}^n} f(0, s)$$

and $U(s)$ is given below; $J(s)$ denotes the Jacobian matrix (with respect to the complex coordinates) of $\pi_{\mathbb{C}^n} f(z, s)$ evaluated at $z = 0$.

Proof. Since $\Lambda_s\big[f\big]$ is a proper holomorphic map $\mathbb{B}_n \to \mathbb{B}_n$, we can use the results of Section 2.5 to conclude that $a(s)$ can be written in the form

$$-\frac{1}{1-|\Lambda_s\big[f\big](0)|^2}\left(D_0\Lambda_s\big[f\big]\right)^\dagger \Lambda_s\big[f\big](0)$$

where $D_0\psi$ is used to denote the Jacobian of ψ evaluated at 0.

Recall that the functions $\Lambda_s\big[f\big]$ and $g := \pi_{\mathbb{C}^n} f$ are related by

$$\Lambda_s\big[f\big](z) = \frac{g(z\sqrt{1-|s|^2}, s)}{\sqrt{1-|h(s)|^2}}.$$

By a simple application of the chain rule,

$$D_z(\Lambda_s\big[f\big]) = \frac{(D_z g)(z\sqrt{1-|s|^2}, s)\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}}$$

and so in particular,

$$J(s) = D\Big|_{z=0}(\Lambda_s\big[f\big]) = \frac{\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}} D\Big|_{(z,s)=(0,s)} g.$$

We can also observe that $\pi_{\mathbb{C}^n} f(0,s)$ is simply written as $\sqrt{1-|h(s)|^2}U(s)a(s)$, and is related to $\Lambda_s\big[f\big](0)$ by

$$\Lambda_s\big[f\big](0) = \frac{\pi_{\mathbb{C}^n} f(0,s)}{\sqrt{1-|h(s)|^2}}.$$

We have

$$\begin{aligned}
a(s) &= -\frac{1}{1-|\Lambda_s\big[f\big](0)|^2}\left(D_0\Lambda_s\big[f\big]\right)^\dagger \Lambda_s\big[f\big](0) \\
&= -\frac{1}{1-\dfrac{|g(0,s)|^2}{1-|h(s)|^2}}\left(\frac{\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}}J(s)\right)^\dagger \left(\frac{\pi_{\mathbb{C}^n} f(0,s)}{\sqrt{1-|h(s)|^2}}\right) \\
&= -\frac{\sqrt{1-|s|^2}}{1-\dfrac{|g(0,s)|^2}{1-|h(s)|^2}}\frac{1}{1-|h(s)|^2}\left(J(s)\right)^\dagger \left(\pi_{\mathbb{C}^n} f(0,s)\right) \\
&= -\frac{\sqrt{1-|s|^2}}{1-|h(s)|^2-|g(0,s)|^2} J(s)^\dagger \pi_{\mathbb{C}^n} f(0,s) \\
&= -\frac{\sqrt{1-|s|^2}}{1-|f(0,s)|^2} J(s)^\dagger \pi_{\mathbb{C}^n} f(0,s) \\
A(s) &= -\frac{1-|s|^2}{1-|f(0,s)|^2} J(s)^\dagger \pi_{\mathbb{C}^n} f(0,s)
\end{aligned}$$

Further using the results of Section 2.5, we can also compute the form for $U(s)$ based on the form of U, namely

$$U = -\frac{1}{(1+\sigma)\sigma^2} f(0)f(0)^\dagger J - \frac{1}{\sigma} J$$

in Section 2.5. Note that previously σ denoted $\sqrt{1-|a|^2}$, so now denotes

$$\sqrt{1-|a(s)|^2} = \sqrt{1 - \frac{|g(0,s)|^2}{1-|h(s)|^2}} = \sqrt{\frac{1-|f(0,s)|^2}{1-|h(s)|^2}}.$$

Consequently, we can calculate $U(s)$ as

$$\left(-\frac{1}{(1+\sigma)\sigma^2}\Lambda_s[f](0)\Lambda_s[f](0)^\dagger - \frac{1}{\sigma}I\right)\left(\frac{\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}}J(s)\right)$$

$$= \left(\frac{g(0,s)g(0,s)^\dagger/\big(1-|h(s)|^2\big)}{(1+\sqrt{1-\frac{|g(0,s)|^2}{|h(s)|^2}})\left(1-\frac{|g(0,s)|^2}{|h(s)|^2}\right)} - \frac{1}{\sqrt{1-\frac{|g(0,s)|^2}{|h(s)|^2}}}I\right)\frac{\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}}J(s)$$

$$= \left(-\frac{1}{1+\sqrt{1-\frac{|g(0,s)|^2}{|h(s)|^2}}}\frac{g(0,s)g(0,s)^\dagger}{1-|f(0,s)|^2} - \frac{1}{\sqrt{1-\frac{|g(0,s)|^2}{|h(s)|^2}}}I\right)\frac{\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}}J(s)$$

$$= -\left(\frac{1}{\sqrt{1-|h(s)|^2}+\sqrt{1-|f(0,s)|^2}}\frac{g(0,s)g(0,s)^\dagger}{1-|f(0,s)|^2} + \frac{I}{\sqrt{1-|f(0,s)|^2}}\right)\sqrt{1-|s|^2}J(s)$$

■

Example 3.4.1 *The function $f : \mathbb{B}_{2,1} \to \mathbb{B}_{2,1}$ given by*

$$\begin{bmatrix} f_1(z,w,s)M(z,w,s) \\ f_2(z,w,s)M(z,w,s)/t \\ h(s) \end{bmatrix}$$

with explicit components

$$f_1(z,w,s) = -2s\left(15sw - 5t(\sqrt{3}u+10) + \sqrt{3}uz(\sqrt{3}u+10) + 25z\right)$$
$$+ t^2\left(-3st(\sqrt{3}u+10) + 3s\,(3sw+5z) + \sqrt{3}uw(\sqrt{3}u+10)\right)$$

$$
\begin{aligned}
f_2(z,w,s) &= -2s\left(-3st^2(\sqrt{3}u+10)+3st\,(3sw+5z)+\sqrt{3}tuw(\sqrt{3}u+10)\right)\\
&\qquad - t^2\left(-5t^2(\sqrt{3}u+10)+\sqrt{3}tuz(\sqrt{3}u+10)+t\,(15sw+25z)\right)\\
M(z,w,s) &= \frac{\sqrt{(1+s^2)^4-16s^2t^4}}{(1+s^2)^3\left(\sqrt{3}u+10\right)(-3sw+10t-5z)}\\
h(s) &= \frac{4s\,(s^2-1)}{(s^2+1)^2}
\end{aligned}
$$

where t denotes $\sqrt{1-s^2}$ and u denotes $\sqrt{25-3s^2}$ is a mixed-type proper[2] *map of class* Prop_{C^0}*, and application of the theorem correctly recovers*

$$
a(s) = \begin{bmatrix} 1/2 \\ 3s/10 \end{bmatrix}
$$

and

$$
U(s) = \frac{1}{s^2+1}\begin{bmatrix} 2s & s^2-1 \\ 1-s^2 & 2s \end{bmatrix}.
$$

Corollary 3.4.1 *If $f:\mathbb{B}_{n,k}\to\mathbb{B}_{n,k}$ is a mixed-type proper map of class* $\mathrm{Rat}_{\mathrm{C}}$ *(and $n>1$), then the associated map A is also a rational function.*

We can similarly state the following for the case $n=1$, which was excluded above.

Theorem 3.4.2 *Let f be a mixed-type proper map $\mathbb{B}_{1,k}\to\mathbb{B}_{1,k}$. Then $f(z,s)$ has the form*

$$
\left(\sqrt{1-|h(s)|^2}e^{i\theta(s)}\frac{\dfrac{A(s)}{\sqrt{1-|s|^2}}-\dfrac{z}{\sqrt{1-|s|^2}}}{1-\dfrac{\overline{A(s)}z}{1-|s|^2}},h(s)\right)
$$

where $A(s)$ is given by

$$
-\frac{1-|s|^2}{1-|f(0,s)|^2}\pi_{\mathbb{C}^n}f(0,s)\pi_{\mathbb{C}^n}\overline{f_z(0,s)}
$$

[2]Indeed, $1-|f(z,w,s)|^2$ is a product of $1-s^2-|z|^2-|w|^2$ and

$$
\frac{3\left(3s^2-25\right)\left(s^2-2s-1\right)^2\left(s^2+2s-1\right)^2\left(9s^2-20\sqrt{3}\sqrt{25-3s^2}-175\right)}{\left(s^2+1\right)^4\left(\sqrt{3}\sqrt{25-3s^2}+10\right)^2\left(3sw+5z-10\sqrt{1-s^2}\right)\left(3s\overline{w}-10\sqrt{1-s^2}+5\overline{z}\right)}
$$

and $e^{i\theta(s)}$ is given by

$$-\frac{\sqrt{1-|s|^2}}{\sqrt{1-|h(s)|^2}}\frac{\pi_{\mathbb{C}^n} f(0,s)}{1-\dfrac{|\pi_{\mathbb{C}^n} f(0,s)|^2}{1-|h(s)|^2}}.$$

Proof. The proof works the same as the proof above. By work of Section 2.2, we have

$$A(s) = -\frac{\Lambda_s[f](0)\overline{\Lambda_s[f]'(0)}}{1-|\Lambda_s[f](0)|^2}$$

and

$$e^{i\theta(s)} = -\frac{\Lambda_s[f]'(0)}{1-|\Lambda_s[f](0)|^2}$$

and the result follows. ■

3.5 Rational Self-maps of $\mathbb{B}_{n,k}$

We observe that mixed-type proper maps $\mathbb{B}_{n,k} \to \mathbb{B}_{n,k}$ are more restricted than those mixed-type proper maps that arise as restrictions of holomorphic proper maps $F : \mathbb{B}_{n+k} \to \mathbb{B}_{n+k}$ whose restrictions $f = F|_{\mathbb{B}_{n,k}}$ are mixed-type proper maps $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n,k}$. Note that F is a linear fractional transformation by Alexander's result (see Section 1.4). We can write F as a pair of functions (G, H) with G a map $\mathbb{B}_{n+k} \to \mathbb{B}_n$ and H a map $\mathbb{B}_{n+k} \to \mathbb{B}_k$. The map H must restrict to a map $h = H|_{\mathbb{B}_{n,k}}$ which, as in the discussion at the beginning of the chapter, must be a map $h : \mathbb{B}_{0,k} \to \mathbb{B}_{0,k}$, since it is a real-valued holomorphic function of the complex variables. The fact that F is a linear fractional transformation implies that F can be written as a fractional with a single denominator, i.e. G and H can be written as P_1/Q and P_2/Q, respectively. In particular, the map h, which is the restriction of P_2/Q to a $\mathbb{B}_{0,k} \to \mathbb{B}_{0,k}$ map, is a linear fractional transformation. Because the denominator Q does not depend on $z \in \mathbb{B}_n$, we know that the function G is not only a linear fractional transformation, it is in fact an affine linear function of z. Concrete statements about the form of F can be made, but these results are not used in the remainder of this chapter, so we omit them. In the remainder of this section, we see rational mixed-type proper maps whose leaf projections $\Lambda_s\big[f\big]$ are linear fractional transformations in the complex variables, as opposed to affine linear transformations.

The following cute result about real functions can be used to help interpret the results of the two theorems which follow.

Lemma 3.5.1 *Suppose that h is a real-valued bijection from the interval $(-1,1)$ to itself. If both h and h^{-1} are rational functions, then h is a linear fractional transformation. Further, $h(z)$ is a function of the form $\pm\dfrac{z-a}{1-az}$ where $a \in (-1,1)$.*

Proof. For the duration of the proof, we write g for the function given by h^{-1} on the interval $(-1,1)$. Note that both h and g are rational functions, and thus extend to the entire Riemann sphere meromorphically. Additionally, the identities $h \circ g(z) \equiv z$ and $g \circ h(z) \equiv z$ hold on the set $(-1,1)$, and this set has limit points (namely all points in the closed interval $[-1,1]$). By the Identity Theorem, the two identities hold on the entire Riemann sphere; that is, h and g are rational inverses on the Riemann sphere. Of course, both h and g must have at most one zero and at most one pole, so it follows that h and g are linear fractional transformations.

Since h is a linear fractional transformation on the Riemann sphere, it is determined uniquely by the image of three points. We require that $h(-1)$ is in the set $\{1,-1\}$ and that $h(1)$ takes the other value, i.e., $h(1)$ is in the set $\{1,-1\} \setminus \{h(-1)\}$. Our third point will be $a \in (-1,1)$ such that $h(a) = 0$. This is possible, since h is surjective. The case that $h(1) = 1$, $h(-1) = -1$, and $h(a) = 0$ is satisfied by the function $h(z) = \dfrac{z-a}{1-az}$. The case that $h(1) = -1$, $h(-1) = 1$, and $h(a) = 0$ is similarly satisfied by $h(z) = -\dfrac{z-a}{1-az}$. ■

For the following result, we may assume $k \geq 1$.

Theorem 3.5.1 *Let $f : \mathbb{B}_{1,k} \to \mathbb{B}_{1,k}$ be a mixed-type proper map, so f has the form*

$$\left(\sqrt{1-|h(s)|^2}\, e^{i\theta(s)} \frac{\dfrac{A(s)}{\sqrt{1-|s|^2}} - \dfrac{z}{\sqrt{1-|s|^2}}}{1 - \dfrac{\overline{A(s)}z}{1-|s|^2}}, h(s) \right).$$

Then f is rational with rational inverse if and only if the following four functions are rational: h, h^{-1}, A, and $s \mapsto \dfrac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}}e^{i\theta(s)}$.

Proof. Let $f : \mathbb{B}_{1,k} \to \mathbb{B}_{1,k}$ be an invertible map. If f is rational in both z and s, then in particular

$$f(0,s) = (-\sqrt{1-|h(s)|^2}e^{i\theta(s)}\frac{A(s)}{\sqrt{1-|s|^2}}, h(s))$$

is rational from which both $\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}}e^{i\theta(s)}A(s)$ and $h(s)$ must be rational. Additionally, $\pi_{\mathbb{C}^1}\frac{\partial f}{\partial z}(0,s)$ must be rational. We compute that $\pi_{\mathbb{C}^1}\dfrac{\partial f}{\partial z}$ is given by

$$\sqrt{1-|h(s)|^2}e^{i\theta(s)}\frac{\left(1-\frac{\overline{A(s)}z}{1-|s|^2}\right)\frac{1}{\sqrt{1-|s|^2}}+\left(\frac{z}{\sqrt{1-|s|^2}}-\frac{A(s)}{\sqrt{1-|s|^2}}\right)\frac{\overline{A(s)}}{1-|s|^2}}{\left(1-\frac{\overline{A(s)}z}{1-|s|^2}\right)^2}.$$

Evaluating this at $z = 0$ simplifies the result considerably

$$\begin{aligned}\pi_{\mathbb{C}^1}\frac{\partial f}{\partial z}(0,s) &= \sqrt{1-|h(s)|^2}e^{i\theta(s)}\left(\frac{1}{\sqrt{1-|s|^2}}-A(s)\frac{\overline{A(s)}}{\sqrt{1-|s|^2}^3}\right)\\ &= \frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}}e^{i\theta(s)}\left(1-\frac{|A(s)|^2}{1-|s|^2}\right).\end{aligned}$$

We note that

$$A(s) = \frac{\pi_{\mathbb{C}}f(0,s)}{\pi_{\mathbb{C}}f_z(0,s)}\frac{1-|f(0,s)|^2}{1-|\pi_{\mathbb{R}}f(0,s)|^2} = \frac{\pi_{\mathbb{C}}f(0,s)}{\pi_{\mathbb{C}}f_z(0,s)}\frac{1-f(0,s)^\dagger f(0,s)}{1-\left(\pi_{\mathbb{R}^k}f(0,s)\right)^2}$$

is rational because of Theorem 3.4.2. It follows, then, that

$$\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}}e^{i\theta(s)}$$

is also rational, as it can be written as the quotient of two rational functions.

The other direction of the proof is almost obvious. If the four specified functions are rational, the construction for f, namely

$$f(z,s) = \left(\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}}e^{i\theta(s)}\frac{z-A(s)}{1-\frac{\overline{A(s)}z}{1-|s|^2}}, h(s)\right),$$

is clearly rational. To conclude that the inverse function is rational, we can observe that the explicit form of the inverse function

$$f^{-1}(z,s) = \left(\frac{A\big(h^{-1}(s)\big) - z\dfrac{\sqrt{1-|h^{-1}(s)|^2}}{\sqrt{1-|s|^2}} e^{-i\theta\left(h^{-1}(s)\right)}}{1 - \dfrac{\overline{A\big(h^{-1}(s)\big)}}{1-|h^{-1}(s)|^2} z \dfrac{\sqrt{1-|h^{-1}(s)|^2}}{\sqrt{1-|s|^2}} e^{-i\theta\left(h^{-1}(s)\right)}},\ h^{-1}(s) \right)$$

is formed as a composition of rational functions. ■

We note that in the case that $n = 1$ and $k = 1$, we can say the following. By Lemma 3.5.1, $h(s)$ must of the form $\pm\dfrac{s-a}{1-as}$, and so $\dfrac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}}$ simplifies to $\dfrac{\sqrt{1-a^2}}{1-as}$ which is rational. In this case, we conclude that $e^{i\theta(s)}$ must be a rational function of s. There are many such examples: for instance, $P(s) + iQ(s)$ where the pair (P, Q) is a rational parameterization of an arc unit circle, i.e., (P, Q) is of the form $\left(\frac{2R}{1+R^2}, \frac{1-R^2}{1+R^2}\right)$ where R is any rational function.

If f is a mixed-type proper map $\mathbb{B}_{n,k} \to \mathbb{B}_{n,k}$ then it has the form

$$f(z,s) = \left(\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-s^2}} U(s) \frac{A(s) - \sqrt{1-\dfrac{|A(s)|^2}{1-|s|^2}}\, z - \dfrac{\langle z, A(s)\rangle}{1+\sqrt{1-\frac{|A(s)|^2}{1-|s|^2}}} \dfrac{A(s)}{1-|s|^2}}{1 - \dfrac{\langle z, A(s)\rangle}{1-s^2}}, h(s) \right).$$

If f is of class $\mathrm{Rat}_{\mathrm{CR}}$, then in particular $\pi_{\mathbb{C}^n} f(0,s) = \dfrac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}} U(s)A(s)$ and $\pi_{\mathbb{R}} f(0,s) = h(s)$ are rational.

Theorem 3.5.2 *Assume $n > 1$. Let $f : \mathbb{B}_{n,1} \to \mathbb{B}_{n,1}$ be a mixed-type proper map, so $f(z,s)$ has the form*

$$\left(\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-|s|^2}} U(s) \frac{A(s) - \sqrt{1-\dfrac{|A(s)|^2}{1-|s|^2}}\, z - \dfrac{\langle z, A(s)\rangle}{1+\sqrt{1-\frac{|A(s)|^2}{1-|s|^2}}} \dfrac{A(s)}{1-|s|^2}}{1 - \dfrac{\langle z, A(s)\rangle}{1-|s|^2}}, h(s) \right).$$

Suppose that f is rational with rational inverse. Then $A(s)$ is rational, $h(s)$ is a linear fractional transformation of the form $\pm\frac{s-a}{1-as}$ for some $a \in (-1, 1)$, and $U(s)$ is rational.

Proof. The form for h is given by Lemma 3.5.1. As observed in the discussion prior to this proof, we know that $\dfrac{\sqrt{1-|h(s)|^2}}{\sqrt{1-s^2}}$ simplifies to $\dfrac{\sqrt{1-a^2}}{1-as}$, which is itself rational.

The vector-valued $A(s)$ is a rational function if f is, because by Theorem 3.4.1, it can be expressed as a rational function of f and its derivatives evaluated at $z=0$. ■

3.6 Automorphisms and Involutions of $\mathbb{B}_{1,k}$

Recall from Theorem 3.4.2 that a mixed-type proper map $f : \mathbb{B}_{1,k} \to \mathbb{B}_{1,k}$ can be written as

$$\left(\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-s^2}} e^{i\theta(s)} \frac{A(s)-z}{1-\frac{\overline{A(s)}z}{1-s^2}},\ h(s) \right)$$

Proposition 3.6.1 *Suppose that $f : \mathbb{B}_{1,1} \to \mathbb{B}_{1,1}$ is a mixed-type proper map, so $f(z,s)$ can be written in the form*

$$f(z,s) = \left(\frac{\sqrt{1-|h(s)|^2}}{\sqrt{1-s^2}} e^{i\theta(s)} \frac{A(s)-z}{1-\frac{\overline{A(s)}z}{1-s^2}},\ h(s) \right)$$

The map f is an involution $f(f(z,s)) \equiv (z,s)$ if and only if f is the identity function, or $f(z,s)$ has the form

$$\left(\frac{A(s)-z}{1-\frac{\overline{A(s)}z}{1-s^2}},\ s \right)$$

where $A(s)$ is a real-valued function satisfying the bound $|A(s)| \le \sqrt{1-s^2}$.

Proof. From Lemmas 2.3.1 and 3.5.1, we know that $h(s)$ must either be in the form $h(s)=s$ or $h(s) = \dfrac{p-s}{1-ps}$ for some $p \in (-1,1)$. In the case $h(s)=s$, the composition $f(f(z,s))$ is identically (z,s) if and only if the three quantities $(1-e^{i\theta(s)})A(s)$, $(1-s^2)(1-e^{2i\theta(s)})$, and $A(s)e^{i\theta(s)}(1-s^2)(1-e^{i\theta(s)})$ all vanish. From the second condition, we see that $e^{2i\theta(s)}$ must be identically one. If θ is continuous, then we must have $e^{i\theta(s)} \equiv 1$ or $e^{i\theta(s)} \equiv -1$. If $e^{i\theta(s)} \equiv -1$, then $A(s) \equiv 0$, so we have

$$f(z,s) = \left(-\frac{0-z}{1-0},\ s \right) = (z,s).$$

If $e^{i\theta(s)} \equiv 1$, then $A(s)$ is undetermined, and we get

$$f(z,s) = \left(\frac{A(s) - z}{1 - \frac{\overline{A(s)}z}{1-s^2}},\ s \right).$$

In the case $h(s) = \dfrac{p-s}{1-ps}$, the composition $f(f(z,s))$ is identically (z,s) if and only if the three conditions $(1-ps)(1-ps-e^{i\theta(s)}\sqrt{1-p^2})\overline{A(s)}$,

$$(1-s^2)(1-e^{2i\theta(s)} + p^2s^2 + p^2e^{2i\theta(s)} - 2ps),$$

and

$$(1-s^2)e^{i\theta(s)}A(s)\Big((1-ps)\sqrt{1-p^2} - (1-p^2)e^{i\theta(s)}\Big)$$

all vanish. The first condition can only vanish if $A(s) \equiv 0$, since $1-ps$ can vanish only when $s = 1/p$, and the factor $1 - ps - e^{i\theta(s)}\sqrt{1-p^2})$ likewise can only vanish when

$$\left| \frac{1-ps}{\sqrt{1-p^2}} \right| = 1,$$

which occurs at at most two values of s; by continuity, $A(s)$ vanishes everywhere. The third condition is automatically satisfied. The second condition vanishing implies that

$$e^{2i\theta(s)} = \frac{(1-ps)^2}{1-p^2}$$

which cannot hold for all s, because the right-hand side is real and non-constant.

The other direction is a direct computation. ∎

3.7 Involutions of $\mathbb{B}_{n,k}$

In the following statement, we use $a(s)$ instead of $A(s)$ in order to simplify the form of the second condition.

Theorem 3.7.1 *Assume $n > 1$.*

Each mixed-type proper map $f : \mathbb{B}_{n,k} \to \mathbb{B}_{n,k}$ depends on three auxiliary functions U, A, and h as in the statement of Theorem 3.4.1. Recall $a(s) = A(s)/\sqrt{1-|s|^2}$. Write $f_{U,a,h}$ for f.

The map $f_{U,a,h}$ is an involution if and only if the three following conditions hold.

(a) $h \circ h = id$

(b) $U(s)a(s) = a(h(s))$

(c) $U(s)U(h(s)) = I$

Remark 3.7.1 *It is important to note that there is at least one mixed-type proper map which satisfies this set of conditions before taking the time to prove this. Observe that* $U(s) \equiv I$*,* $a(s) \equiv 0$*, and* $h(s) \equiv s$ *does satisfy (a)-(c).*

Proof. ($\Leftarrow$) Assume that the three conditions hold. We will show that

$$f_{U,a,h}(f_{U,a,h}(z,s)) = (z,s).$$

For convenience, we can write $f(z,s)$ as $(g(z,s),h(s))$. In this notation, the composition $f \circ f$ looks like

$$(g(z,s),h(s)) \circ (g(z,s),h(s)) = (g(g(z,s),h(s)),h(h(s))).$$

Condition (a) tells us that the $h(h(s))$ portion of the composition works as intended, so we need only show that $g(g(z,s),h(s))$ simplifies to simply z.

From definition 3.1.1, we know that $\Lambda_s[f] = \frac{g(z\sqrt{1-|s|^2},s)}{\sqrt{1-|h(s)|^2}}$ is a proper holomorphic map $\mathbb{B}_n \to \mathbb{B}_n$, and hence has form $U\varphi(z;a)$.

$$\begin{aligned}
&g(g(z,s),h(s)) \\
&= \sqrt{1-|h(h(s))|^2}U(h(s))\varphi\left(\frac{\sqrt{1-|h(s)|^2}U(s)\varphi\left(\frac{z}{\sqrt{1-|s|^2}};a(s)\right)}{\sqrt{1-|h(s)|^2}};a(h(s))\right) \\
&\overset{\text{(a)}}{=} \sqrt{1-|s|^2}U(h(s))\varphi\left(U(s)\varphi(\frac{z}{\sqrt{1-|s|^2}};a(s));a(h(s))\right) \\
&= \sqrt{1-|s|^2}U(h(s))U(s)\varphi\left(\varphi(\frac{z}{\sqrt{1-|s|^2}};a(s));U(s)^{\dagger}a(h(s))\right)
\end{aligned}$$

$$\overset{(c)}{=} \sqrt{1-|s|^2}\varphi\left(\varphi\big(\frac{z}{\sqrt{1-|s|^2}}; a(s)\big); U(s)^\dagger a(h(s))\right)$$

$$\overset{(b)}{=} \sqrt{1-|s|^2}\varphi\left(\varphi\big(\frac{z}{\sqrt{1-|s|^2}}; a(s)\big); a(s)\right).$$

Finally, we can use the fact that $\varphi(-, a(s))$ is an involution for each s, so the right-hand side simplifies to

$$\sqrt{1-|s|^2}\frac{z}{\sqrt{1-|s|^2}} = z$$

as needed.

($\Rightarrow$) Assume that the map $f_{U,a,h}$ is an involution. We want to show that the three conditions (a) $h(h(s)) = s$, (b) $U(s)a(s) = a(h(s))$, and (c) $U(s)U(h(s)) = I$ are all satisfied. The first condition, $h(h(s)) = s$, is immediately satisfied by composing $f \circ f$ and looking at the last k components.

Consider $f(f(0, s))$. The inside $f(0, s)$ is given by

$$\left(\sqrt{1-|h(s)|^2}U(s)\varphi\left(\frac{0}{\sqrt{1-|s|^2}}; a(s)\right), h(s)\right) = \left(\sqrt{1-|h(s)|^2}U(s)a(s), h(s)\right)$$

and plugging this into f again yields

$$\sqrt{1-|h(h(s))|^2}U(h(s))\varphi\left(\frac{\sqrt{1-|h(s)|^2}U(s)a(s)}{\sqrt{1-|h(s)|^2}}; a(h(s))\right)$$

in the first components. As we already know that $h(h(s)) = s$, we can simplify

$$f(f(0, s)) = \left(U(h(s))\varphi\Big(U(s)a(s); a(h(s))\Big), s\right).$$

This is supposed to equal $(0, s)$ because f is an involution, so we see that we have the condition

$$U(h(s))\varphi\Big(U(s)a(s); a(h(s))\Big) = 0.$$

The only preimage of 0 under $\varphi(-, B)$ is B itself, so we must have that $U(s)a(s) = a(h(s))$, hence showing (b).

To see (c), for all $w \in \mathbb{B}_n$, we have

$$
\begin{aligned}
w &= U(h(s))\varphi\Bigg(U(s)\varphi\Big(w; a(s)\Big); a(h(s))\Bigg) \\
&= U(h(s))U(s)\varphi\Bigg(\varphi\Big(w; a(s)\Big); U(s)^{-1}a(h(s))\Bigg)
\end{aligned}
$$

where the last equality follows from Lemma 2.6.2. By multiplying by the inverses of the unitary matrices, this tells us that

$$
U(s)^{-1}U(h(s))^{-1}w = \varphi\Bigg(\varphi\Big(w; a(s)\Big); U(s)^{-1}a(h(s))\Bigg).
$$

Observe that the left side is a unitary transformation acting on w, so the right side is as well. Temporarily denote this common unitary transformation by $V(s)$, and abbreviate $U(s)^{-1}a(h(s))$ by $B(s)$. In this notation, we have $\varphi(\varphi(w; a(s)); B(s)) = V(s)w$. Apply the inverse of $\varphi(-; B(s))$, i.e. $\varphi(-; B(s))$ itself, to both sides to get

$$
\varphi(w; a(s)) = \varphi(V(s)w; B(s)).
$$

In particular, evaluating at $w = 0$ tell us that $a(s) = B(s)$. Evaluating at $w = a(s)$ tells us that $V(s) = I$. That is, $U(s)^{-1}U(h(s))^{-1} = I$, or equivalently $U(h(s))U(s) = I$. ■

We also observe that condition (b) of the theorem can be stated in terms of the rescaled functions $A(s)$ as

$$
U(s)A(s) = \frac{\sqrt{1 - |s|^2}}{\sqrt{1 - |h(s)|^2}}A(h(s)).
$$

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