EXCEPTIONAL REPRESENTATIONS OF THE METAPLECTIC DOUBLE COVER OF THE GENERAL LINEAR GROUP

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PREFACE

At least since the pioneering work of Shimura on modular forms of half-integral weight [Sh1] non-algebraic central extensions of algebraic groups have played a substantial rôle in number theory. The most well-known applications have involved the metaplectic double cover of the symplectic group, starting with Weil's preparation for his work on Siegel's formula in [Wei] and continuing with the theory of theta lifting and dual pairs. However, other examples of non-algebraic covers have also been useful. The metaplectic triple cover of SL(2) played a fundamental part in Patterson's work on cubic Gauss sums (see [Pa1] and [Pa2]) and the metaplectic double cover of GL(3) was employed in Patterson's and Piatetski-Shapiro's work on the symmetric square L-functions on GL(3) [PPS], later generalized to GL(r) by Bump and Ginzburg [BuG]. Numerous other examples could be cited involving both the general linear group and other algebraic groups.

At the same time the representation-theoretic point of view on modular forms was being developed and it became natural to study the local and the global representation theory of the metaplectic groups. Although the general outlines of the theory were the same as in the non-metaplectic case there were some surprises in store. For instance, it is a widely-known and much used fact that the vast majority of irreducible admissible representations of GL(r) have a unique Whittaker model. As soon as we consider a non-trivial metaplectic cover of $\mathrm{GL}(r)$, however, this becomes false. An irreducible admissible representation may now have many Whittaker models or none and it becomes an interesting problem to locate the rare "distinguished" representations which do have unique Whittaker models (see [GeP] and [KaP]). Since the theory over the real numbers can use the fact that, whilst not algebraic, the metaplectic groups are never the less Lie groups of a fairly reasonable type, it boasts a degree of completeness which is not matched by the non-Archimedean theory. For instance, the unitary dual of the metaplectic double cover of $\mathrm{GL}(n, \mathbb{R})$ has been classified by Huang [Hua].

At present, which representations of the metaplectic covers of $\mathrm{GL}(r)$ over non-Archimedean fields deserve detailed study has largely been decided on utilitarian grounds. In the works so far cited, and the others of which the author is aware, most attention has focussed on the so-called exceptional representations of these groups, first defined in generality in [KaP]. These will also be the subject of the current work. The interest which they evoke is largely justified by their importance in studying the symmetric square L-functions on $\mathrm{GL}(r)$ (for which see [BuG]) and also by the hope that they may provide an analogue for $\mathrm{GL}(r)$ of the justly famous Siegel-Shale-Weil representation of the metaplectic double covers of the symplectic groups.

We now turn to a brief description of the contents of this work, trying where possible to indicate its relationship to the already existing literature. In the first chapter we review the construction of the metaplectic cover of $\mathrm{GL}(n, F)$ associated with a Steinberg symbol $c : F^\times \times F^\times \to A$ where A is an abelian group. The corresponding construction with $\mathrm{GL}(n)$ replaced by a semisimple group is due to Matsumoto [Mat], and Milnor [Mil] gives a very clear account of Matsumoto's work in the case where the group is $\mathrm{SL}(n)$. In section 0 of [KaP] a 2-cocycle is exhibited which defines the metaplectic cover of $\mathrm{GL}(n)$ in the case where c is the m^{th} order Hilbert symbol on a local field F containing the m^{th} roots of unity. Further discussion of this construction may therefore seem superfluous. It is reviewed here for two reasons; first because this will serve to fix notation and secondly because there is an error in the formulæ of [KaP] in the case where $c(-1,-1) \neq 1$. We shall have to deal with this case and so it is necessary to correct the error.

In §1.1 we discuss the double covers of $\mathfrak{S}_n$, as these will play a role later on. This material is well-known and we merely put it in a form suitable to our purpose. In §1.2 we use Milnor's description of the central extension of $\mathrm{SL}(n)$ associated to c to find expressions for certain values of a 2-cocycle on $\mathrm{GL}(n)$. In §1.3 we relate this 2-cocycle to that of Kazhdan and Patterson. As far as the author is aware, the coboundary which connects the first of these cocycles with (the inverse of) the second has not appeared before. In §1.4 we study the lifts of the main involution on $\mathrm{GL}(n)$ to its metaplectic covers. The existence of a lift of this automorphism has not been dealt with sufficiently carefully before and it may come as a surprise to some readers familiar with the literature that the lift in question is far from unique. In §1.5, the last in Chapter 1, we briefly discuss some topological properties of the metaplectic covers in the case where F is a local field and c a Hilbert symbol.

Chapter 2 contains the principal results of this work. It is mainly devoted to the study of the exceptional representations of the metaplectic double cover of GL(r) over a non-Archimedean local field. In §2.1 we construct a metaplectic analogue of the tensor product functor. Here we work in a fairly general setting; the category on which the construction takes place is that of admissible representations of the metaplectic group of finite length possessing a central character. Having this functor in hand makes it possible to phrase many constructions in a much more natural way than has been possible previously. The author hopes to return to this topic and extend the construction to the n-fold covers of GL(r). In §2.2 we merely collect our conventions on modular characters, parabolic induction and the like and fix some notation.

The discussion of the exceptional representations in §2.3 relies upon Kazhdan's and Patterson's work in [KaP]. We extend the notion of an exceptional representation to cover representations of products of metaplectic groups, as was suggested but not systematically pursued in [BuG]. The metaplectic tensor product functor of section 1 turns out to be particularly convenient here. The section ends with a few technical results which will be necessary later. In §2.4 we undertake a systematic study of the *semi-Whittaker functions*, which provide models for the exceptional representations similar to the Whittaker models of non-metaplectic representations. As well as proving numerous results of mostly technical interest we discover that, of the two species of semi-Whittaker functions, one gives an analogue of Kirillov models and the other does not.

The next two sections address a problem suggested by the applications of exceptional representations in the construction of Rankin-Selberg integrals (see [PPS] and [BuG]). In those integrals the product of two functions derived from the exceptional representations occurs, multiplied by a non-metaplectic function coming from some representation of $\mathrm{GL}(r)$. In order to understand such an integral representation-theoretically it is natural to study the existence and uniqueness of invariant linear forms on the tensor product of two exceptional representations and a non-metaplectic representation. This problem was investigated by Savin on $\mathrm{GL}(3)$ and he obtained nearly definitive results when the third representation belongs to the principal series (see [Sav]). In §2.5 we study the uniqueness of such linear forms for general r and establish it in many cases. Our results include, for instance, uniqueness in the case of a cuspidal representation, which Savin did not address. In the course of proving one of the two main uniqueness results (Theorem 1 in §2.5) we take the opportunity to correct a serious error made by Bump and Ginzburg in their paper [BuG]. For representations of the principal series our results are not as precise as Savin's, but they do indicate a strong restriction on the induction datum if the tensor product of two exceptional representations with the given principal series representation is to support an invariant functional. This restriction is of exactly the kind to be expected if Savin's heuristics about lifting from orthogonal and symplectic groups are valid (see [Sav] for further discussion).

The existence result we are able to obtain in §2.6 suffices to establish the existence part of Savin's conjecture in [Sav]. We show (in rather different language from that used later in this work – we do not subsequently discuss lifting) that if an irreducible spherical principal series representation is lifted from the appropriate orthogonal or symplectic group then its tensor product with two suitable exceptional representations does carry a non-zero invariant functional. The methods in this section rely heavily on the use of semi-Whittaker functions and we hope that this will provide a partial justification for the lengthy technical preparation required in §2.4. Finally, §2.7 contains some suggestions for proving further results.

I would like to thank my advisor, Dr. James Cogdell, for suggesting the problem which led to the one I solved and for his help throughout the enterprise and Dr. David Wright for introducing me to *Basic Number Theory* without which none of this would have been possible.

Ms. Belinda Bruner provided all manner of practical, emotional and psychological support at more than one critical time. This work is dedicated to her with gratitude and affection.

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CHAPTER 1

THE METAPLECTIC COVERS OF GL(n)

1. The Double Covers of $\mathfrak{S}_n$

Let F be a field of characteristic 0 and put $K = F\left(\sqrt{2}\right)$. Take $n \geq 2$, $r \geq 0$ integers and put $V = K^{n+r}$. Let $\{e_1, \ldots, e_{n+r}\}$ be the standard basis for V and define a bilinear form Q on V by $Q(e_i, e_j) = -\delta_{ij}$.

If $W \leq \mathrm{GL}(n, F)$ denotes the group of permutation matrices then, provided that r is odd, $W \cong \mathfrak{S}_n$ may be embedded in $\mathrm{SO}(Q)$ via the map

$$\eta_r : w \mapsto \begin{pmatrix} w & 0 \\ 0 & \det(w)\mathrm{I}_r \end{pmatrix}$$

and we shall identify W with its image under this map. There is a central extension

$$1 \to \{\pm 1\} \to \mathrm{Spin}(Q) \xrightarrow{\phi} \mathrm{SO}(Q) \to 1$$

and we aim to identify the induced extension

$$1 \to \{\pm 1\} \to \widetilde{W} \to W \to 1 \quad .$$

where $\widetilde{W} = \phi^{-1}(W)$.

In order to be able to calculate conveniently we first recall the description of $\mathrm{Spin}(Q)$ in terms of the Clifford algebra $C(Q)$. The associative algebra $C(Q)$

is generated (as a K-algebra) by the elements of V and these are subject to the relation

$$u^2 = Q(u,u) \cdot 1 \ .$$

There is a subalgebra $C^+(Q)$ which is generated by all products of an even number of elements of V. For vectors $v_1, \ldots, v_p \in V$ we define

$$(v_1 \cdot \ldots \cdot v_p)^* = (-1)^p v_p \cdot \ldots \cdot v_1$$

and extend $*$ linearly to $C(Q)$. Then $*^2 = \mathrm{id}$ and $*$ is an anti-automorphism of $C(Q)$. We may now define

$$\mathrm{Spin}(Q) = \left\{ x \in C^+(Q) \,|\, xx^* = 1 \ \text{ and } \ xVx^* \subseteq V \right\}$$

and $\phi : \mathrm{Spin}(Q) \to \mathrm{SO}(Q)$ by

$$\phi(x)v = xvx^* \quad \text{for} \quad v \in V \ .$$

We shall identify $\mathrm{End}(V)$ with $\mathrm{M}((n+r) \times (n+r), K)$ using the standard basis. For distinct i, j satisfying $1 \le i, j \le n+r$ we define $m_{ij} \in \mathrm{End}(V)$ by

$$m_{ij} e_k = \begin{cases} -e_j & \text{if } k = i \\ e_i & \text{if } k = j \\ e_k & \text{if } k \notin \{i, j\} \end{cases}$$

and $w_{ij} \in C^+(Q)$ by

$$w_{ij} = \tfrac{1}{\sqrt{2}}(1 - e_i e_j) \ .$$

It is routine to check that $w_{ij} \in \mathrm{Spin}(Q)$ and $\phi(w_{ij}) = m_{ij}$.

Now let Φ denote the root system of $\mathrm{GL}(n)$ and Δ be the standard choice of positive simple system in Φ. We may identify Φ with the set of pairs

$\{(i,j)|1 \le i,j \le n,\ i \ne j\}$, whereupon Δ is identified with $\{(i,i+1)|1 \le i \le n-1\}$. If $\alpha \in \Delta$ then let s_α denote the corresponding simple reflection, thought of as an element of W. With this notation we have

$$\eta_r(s_\alpha)e_k = \begin{cases} e_{i+1} & \text{if } k = i \\ e_i & \text{if } k = i+1 \\ e_k & \text{if } k \in \{1,\dots,n\} \setminus \{i,i+1\} \\ -e_k & \text{if } k \in \{n+1,\dots,n+r\} \end{cases}$$

when $\alpha = (i,i+1)$ and hence

$$\eta_r(s_\alpha) = m^2_{n+r,n+r-1} \cdots m^2_{n+3,n+2} m^2_{i+1,n+1} m_{i,i+1} \,.$$

Now $w_{ij}^2 = -e_i e_j$ and so if we write

$$\begin{aligned} t_\alpha &= \tfrac{1}{\sqrt{2}} e_{n+r} e_{n+r-1} \cdots e_{n+3} e_{n+2} e_{i+1} e_{n+1} (1 - e_i e_{i+1}) \\ &= \tfrac{1}{\sqrt{2}} e_{n+r} \cdots e_{n+2} (e_{i+1} e_{n+1} + e_{n+1} e_i) \end{aligned}$$

then $t_\alpha \in \widetilde{W} \subseteq \mathrm{Spin}(Q)$ and $\phi(t_\alpha) = s_\alpha$. (Recall that $\phi(-1) = 1$.)

If $J \subseteq \{1,\dots,n+r\}$ and $e_J = \prod_{j \in J} e_j$ then (regardless of the order in which the product is arranged) we have

$$e_J^2 = (-1)^{\frac{1}{2}|J|(|J|+1)}$$

where $|J|$ denotes the cardinality of J. Let us put $\epsilon_r = (-1)^{\frac{1}{2}r(r-1)}$. Using the above observation we find that if $\alpha \in \Delta$ then

$$t_\alpha^2 = -\epsilon_r \,,$$

if $\alpha = (i-1,i)$ and $\beta = (i,i+1)$ then

$$t_\alpha t_\beta = \tfrac{1}{2}\epsilon_r (1 - e_{i-1}e_{i+1} + e_{i-1}e_i + e_i e_{i+1})$$

and if $\alpha = (i, i+1)$ and $\beta = (j, j+1)$ with $\langle \alpha, \beta \rangle = 0$ then

$$t_\alpha t_\beta = \tfrac{1}{2}\epsilon_r(e_i e_j - e_i e_{j+1} + e_j e_{i+1} + e_{i+1} e_{j+1})$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on Φ.

From these expressions we can further compute that if $\alpha = (i-1, i)$ and $\beta = (i, i+1)$ then

$$\begin{aligned}(t_\alpha t_\beta)^2 &= -\tfrac{1}{2}(1 + e_{i-1}e_{i+1} - e_{i-1}e_i - e_i e_{i+1}) \\ &= -\epsilon_r \, (t_\alpha t_\beta)^*\end{aligned}$$

and if $\alpha = (i, i+1)$ and $\beta = (j, j+1)$ with $\langle \alpha, \beta \rangle = 0$ then

$$(t_\alpha t_\beta)^2 = -1 \, .$$

Also if $\alpha = (i-1, i)$ and $\beta = (i, i+1)$ then

$$\begin{aligned}t_\beta t_\alpha &= \tfrac{1}{2}\epsilon_r(1 + e_{i-1}e_{i+1} - e_{i-1}e_i - e_i e_{i+1}) \\ &= (t_\alpha t_\beta)^*\end{aligned}$$

from which it follows that

$$(t_\beta t_\alpha)^2 = -\epsilon_r \, (t_\beta t_\alpha)^* \, .$$

If, as usual, we denote by $m(\alpha, \beta)$ the order of $s_\alpha s_\beta$ in W then the preceding formulæ may neatly be summarized as follows:

$$(t_\alpha t_\beta)^{m(\alpha,\beta)} = \begin{cases} -\epsilon_r & \text{if } \langle \alpha, \beta \rangle = 2 \\ -\epsilon_r & \text{if } \langle \alpha, \beta \rangle = -1 \\ -1 & \text{if } \langle \alpha, \beta \rangle = 0 \end{cases} \tag{1}$$

where $\alpha, \beta \in \Delta$. Using the fact that the corresponding relations among the s_α give rise to a presentation of W one may show that these relations together with $(-1)^2 = 1$ and $(-1)t_\alpha = t_\alpha(-1)$ for all $\alpha \in \Delta$ suffice to give a presentation of $\widetilde{W}$.

Note that ϵ_r depends only on the residue class of r modulo 4. Thus the equivalence class of the central extension of W which has just been constructed also depends only on this residue class. We have therefore obtained two central extensions of W, which we shall call the *1–spin* and *3–spin extensions* respectively. These extensions each correspond to a class in $\mathrm{H}^2(W, \mu_2)$ (where μ_2 denotes the group $\{\pm 1\}$), the *1–spin* and *3–spin classes.*

Observe that if $\alpha \in \Delta$ then the two elements of $\widetilde{W}$ mapping to the involution $s_\alpha \in W$ are $\pm t_\alpha$. When $r = 1$ these elements each have order four and when $r = 3$ they each have order two.. Thus the 1–spin class and the 3–spin class are always distinct. If $\alpha, \beta \in \Delta$ are orthogonal then the two elements $\pm t_\alpha t_\beta$ of $\widetilde{W}$ which map to the involution $s_\alpha s_\beta$ have order four and it follows that both spin classes are non-trivial. This conclusion holds for $n \geq 4$ since this is exactly the condition necessary for Δ to contain orthogonal roots. If $n = 2$ or 3 then (1) shows that in the 3–spin extension the elements $\{t_\alpha\}_{\alpha \in \Delta}$ are subject to the same relations as govern the elements $\{s_\alpha\}_{\alpha \in \Delta}$ of W. Thus in these cases the 3–spin class is trivial and the 1–spin class non-trivial.

It is well-known ([How], [Sch]) that

$$\mathrm{H}^2(\mathfrak{S}_n, \mu_2) \cong \begin{cases} C_2 & \text{if } n = 2, 3 \\ C_2 \times C_2 & \text{if } n \geq 4 \end{cases}$$

where C_2 denotes the cyclic group of order two. It follows from the remarks we have just made that the two spin classes always generate $\mathrm{H}^2(\mathfrak{S}_n, \mu_2)$. If $n \geq 4$ then there is a third non-trivial class in $\mathrm{H}^2(\mathfrak{S}_n, \mu_2)$, which is the product of the spin classes. Unlike them it is easy to describe; it is the class represented by the central extension of $\mathfrak{S}_n$ which carries a square-root of the sign character.

2. The Construction and a Partial Cocycle

Let $c : F^\times \times F^\times \to A$ be a Steinberg symbol with values in an abelian group A. Here F is any field, not necessarily of characteristic zero. We are going to construct a central extension of $\mathrm{GL}(n, F)$ by pulling back the extension of $\mathrm{SL}(n+1, F)$ constructed by Matsumoto via the embedding $\eta : \mathrm{GL}(n, F) \to \mathrm{SL}(n+1, F)$ given by

$$\eta(g) = \begin{pmatrix} g & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}.$$

Let H_n denote the group of diagonal matrices in $\mathrm{GL}(n)$, M_n^0 the group of monomial matrices in $\mathrm{GL}(n)$ all of whose entries are ± 1 and M_n the group of all monomial matrices. The subscript may sometimes be omitted. If G is any subgroup of $\mathrm{GL}(n)$ then we shall write $SG = G \cap \mathrm{SL}(n)$.

Following Milnor ([Mil], §12) we let $\widetilde{SH}_n$ be the set $SH_n \times A$ and $\phi : \widetilde{SH}_n \to SH_n$ be the projection onto the first factor. If $d = \mathrm{diag}(u_1, \ldots, u_n)$ and $d' = \mathrm{diag}(v_1, \ldots, v_n)$ lie in SH_n and $a, a' \in A$ then we define

$$(d, a)(d', a') = \big(dd', aa' \prod_{i \geq j} c(u_i, v_j)\big) \qquad \text{and}$$
$$(d, a)^{-1} = \big(d^{-1}, a^{-1} \prod_{i \geq j} c(u_i, u_j)\big).$$

With these definitions $\widetilde{SH}_n$ becomes a group and ϕ a homomorphism. Identifying A with the subgroup $\{(1, a) \mid a \in A\}$ of the center of $\widetilde{SH}_n$ we obtain a central extension

$$1 \to A \to \widetilde{SH}_n \xrightarrow{\phi} SH_n \to 1\,.$$

Given $u \in F^\times$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we define

$$d_{ij}(u) = \mathrm{diag}(1, 1, \ldots, 1, \underset{i}{u}, 1, \ldots, 1, \underset{j}{u^{-1}}, 1, \ldots, 1)$$

and

$$h_{ij}(u) = \begin{cases} (d_{ij}(u), 1) & \text{if } i < j \\ (d_{ij}(u), c(u,u)) & \text{if } i > j \end{cases}.$$

With this notation we are ready for the first of several results which we shall quote from [Mil] without proof.

Lemma 1: *We have*

(1) $h_{ji}(u) = h_{ij}(u)^{-1}$

(2) $h_{kj}(u)h_{ik}(u) = h_{ij}(u)$

(3) $h_{ij}(u)h_{ij}(v) = c(u,v)h_{ij}(uv)$

The next step is to define a group $\widetilde{SM}_n^0$, a homomorphism $\phi_0 : \widetilde{SM}_n^0 \to SM_n^0$ and certain elements $w_{ij}(1) \in \widetilde{SM}_n^0$. If $c(-1,-1) = 1$ then we let $\widetilde{SM}_n^0 = SM_n^0$, ϕ_0 be the identity map and $w_{ij}(1)$ be the matrix m_{ij} which was introduced in the last section. If $c(-1,-1) \neq 1$ then the field F necessarily has characteristic zero (this is a consequence of Steinberg's theorem that a Steinberg symbol on a finite field must be trivial). We regard SM_n^0 as a subgroup of $\mathrm{SO}(n)$, restrict the central extension

$$1 \to \mu_2 \to \mathrm{Spin}(n) \xrightarrow{\phi} \mathrm{SO}(n) \to 1$$

to SM_n^0 to obtain

$$1 \longrightarrow \mu_2 \longrightarrow \widetilde{SM}_n^0 \xrightarrow{\phi_0} SM_n^0 \longrightarrow 1$$

and let $w_{ij}(1)$ be the element w_{ij} of $\phi_0^{-1}(m_{ij})$ given in the previous section. In either case we also set $w_{ij}(-1) = w_{ij}(1)^{-1}$ and $h'_{ij}(-1) = w_{ij}(-1)^2$. Direct calculation shows that $w_{ij}(-1) = w_{ji}(1)$ regardless of the value of $c(-1,-1)$, so that $h'_{ij}(-1) = w_{ji}(1)^2$. These elements of $\widetilde{SM}_n^0$ satisfy $\phi_0(h'_{ij}(-1)) = d_{ij}(-1) \in SM_n^0$.

We are now ready to define a central extension

$$1 \to A \to \widetilde{SM}_n \xrightarrow{\phi} SM_n \to 1$$

of SM_n. The underlying set of $\widetilde{SM}_n$ is the quotient of $\widetilde{SH}_n \times \widetilde{SM}_n^0$ by the equivalence relation $\sim$ generated by the equivalences

$$\big(hh_{ij}(-1), w_0\big) \sim \big(h, h'_{ij}(-1)w_0\big)$$

and the map $\phi : \widetilde{SM}_n \to SM_n$ is given by

$$\phi\big([(h, w_0)]\big) = \phi(h)\phi_0(w_0)\,.$$

We may identify $\widetilde{SH}_n$ and $\widetilde{SM}_n^0$, respectively, with the subsets $\{[(h,1)] \mid h \in \widetilde{SH}_n\}$ and $\{[(1,w_0)] \mid w_0 \in \widetilde{SM}_n^0\}$ of $\widetilde{SM}_n$. Milnor shows that it is possible to define an operation on $\widetilde{SM}_n$ under which it becomes a group, which extends the multiplication on $\widetilde{SH}_n$ and $\widetilde{SM}_n^0$ and which satisfies $[(h,w_0)] = [(h,1)][(1,w_0)]$. This operation is completely determined by these conditions together with the following result.

Lemma 2: *Let $w \in \widetilde{SM}_n$ and suppose that $\phi(w) = p_\pi \mathrm{diag}(u_1, \ldots, u_n)$, where p_π is the permutation matrix corresponding to $\pi \in \mathfrak{S}_n$. Then*

(1) $w h_{ij}(v) w^{-1} = c(u_i u_j^{-1}, v) h_{\pi(i),\pi(j)}(v)$

(2) $w w_{ij}(1) w^{-1} = h_{\pi(i),\pi(j)}(u_i u_j^{-1}) w_{\pi(i),\pi(j)}(1)$.

Milnor now shows that there is a central extension

$$1 \longrightarrow A \longrightarrow \widetilde{\mathrm{SL}}(n, F) \xrightarrow{\phi} \mathrm{SL}(n, F) \longrightarrow 1 \tag{1}$$

which on restriction to SM_n gives the central extension which we have just constructed. We do not need to recall the proof of this result here as we are mainly concerned with the behavior of the extension over the monomial matrices.

We now arrive at our main purpose in this chapter, which is to discuss the central extension

$$1 \longrightarrow A \longrightarrow \widetilde{\mathrm{GL}}'(n, F) \xrightarrow{p'} \mathrm{GL}(n, F) \longrightarrow 1 \tag{2}$$

which is obtained by pulling back the extension (1) (with n replaced by $n+1$) under the map η specified earlier. (The reason for the $'$ will become clear in the next section.) Notice that if η is restricted to $W \subseteq \mathrm{GL}(n, F)$ then it agrees with the embedding used in section 1 with $r = 1$. Now $\eta(W) \subseteq SM^0_{n+1}$ and so we conclude that if $c(-1,-1) = 1$ then the sequence (2) is split over W, but if $c(-1,-1) \neq 1$ then the class of the restriction of (2) to W agrees with the spin class (with $r = 1$), or rather its image in $\mathrm{H}^2(W, A)$ where μ_2 is regarded as a subgroup of A by identifying -1 with $c(-1,-1)$, and hence may not be trivial.

Suppose that $\mathbf{s} : \mathrm{GL}(n, F) \to \widetilde{\mathrm{GL}}'(n, F)$ is a section of the map p' in (2). Corresponding to the choice of $\mathbf{s}$ we obtain a 2-cocycle τ representing the class of (2) in the group $\mathrm{H}^2(\mathrm{GL}(n, F), A)$. This cocycle is defined by the equation

$$\mathbf{s}(g_1)\mathbf{s}(g_2) = \tau(g_1, g_2)\mathbf{s}(g_1 g_2)\,.$$

In our situation it will be difficult to make an explicit choice of section $\mathbf{s}$. What we shall do is to specify a partial section and then extend it to the whole group in any way. This will lead to explicit formulæ for the value of $\tau(g_1, g_2)$ in those cases where $\mathbf{s}(g_1)$, $\mathbf{s}(g_2)$ and $\mathbf{s}(g_1 g_2)$ have been specified.

If $h \in H_n$ then $\eta(h) \in SH_{n+1}$ and since $\widetilde{\mathrm{SL}}(n+1, F)$ contains $\widetilde{SH}_{n+1}$ as a subgroup we may specify a section of p over H_n by

$$\mathbf{s}(h) = (\eta(h), 1) \in \widetilde{SH}_{n+1}\,.$$

If $c(-1,-1) = 1$ then for $w \in W$ we set $\mathbf{s}(w) = \eta(w) \in \widetilde{SM}^0_{n+1}$. If $c(-1,-1) \neq 1$ then we shall not specify $\mathbf{s}$ on W; we suppose it chosen in any way. Now every element of M_n may be written uniquely as the product of an element of H_n and an element of W. We may thus extend $\mathbf{s}$ to M_n by defining

$$\mathbf{s}(hw) = c(\det(h), \det(w))\mathbf{s}(h)\mathbf{s}(w)$$

for $h \in H_n$ and $w \in W$. Finally extend $\mathbf{s}$ to $\mathrm{GL}(n, F)$ arbitrarily.

Using the definition of the multiplication in $\widetilde{SH}_{n+1}$ it is easy to check that if $h = \mathrm{diag}(u_1, \ldots, u_n)$ and $h' = \mathrm{diag}(v_1, \ldots, v_n)$ lie in H_n then

$$\tau(h, h') = \prod_{i \geq j} c(u_i, v_j)\,. \tag{3}$$

Also, it follows from our choice of section over the monomial matrices and the fact that $\det(w)^2 = 1$ for all $w \in W$ that

$$\tau(h, w) = c(\det(h), \det(w)) . \tag{4}$$

Next we want to calculate $\tau(w, h)$ for $w \in W$ and $h \in H_n$. This is possible, even though $\mathbf{s}$ is not completely specified on W in all cases, because W normalizes H_n and the inner automorphism $\tilde{h} \to \mathbf{s}(w)\tilde{h}\mathbf{s}(w)^{-1}$ of $\widetilde{H}_n = (p')^{-1}(H_n)$ depends only on w and not on the choice of $\mathbf{s}$. With this in mind we define a map $\mu : W \times H_n \to A$ through the equation

$$\mathbf{s}(w)\mathbf{s}(h)\mathbf{s}(w)^{-1} = \mu(w, h)\mathbf{s}\big(h^{w^{-1}}\big) .$$

Lemma 3: *We have*

(1) If $h_1, h_2 \in H_n$ and $w \in W$ then

$$\mu(w, h_1 h_2) = \tau\big(h_1^{w^{-1}}, h_2^{w^{-1}}\big)\tau(h_1, h_2)^{-1}\mu(w, h_1)\mu(w, h_2) .$$

(2) If $w \in W$ and $h = \operatorname{diag}(1, \ldots, 1, \underset{p}{v}, 1, \ldots, 1) \in H_n$ then

$$\mu(w, h) = c(\det(w), \det(h)) .$$

(3) If $w \in W$ corresponds to the permutation $\pi \in \mathfrak{S}_n$ and h is the matrix $\operatorname{diag}(v_1, v_2, \ldots, v_n) \in H_n$ then

$$\mu(w, h) = \prod_{\substack{i<j \\ \pi(i)>\pi(j)}} c(v_i, v_j) \cdot c(\det(w), \det(h)) .$$

Proof:

(1) To derive this formula we shall calculate the quantity $\big(\mathbf{s}(w)\mathbf{s}(h_1)\mathbf{s}(w)^{-1}\big)\big(\mathbf{s}(w)\mathbf{s}(h_2)\mathbf{s}(w)^{-1}\big)$ in two different ways. First

$$\begin{aligned}
&\big(\mathbf{s}(w)\mathbf{s}(h_1)\mathbf{s}(w)^{-1}\big)\big(\mathbf{s}(w)\mathbf{s}(h_2)\mathbf{s}(w)^{-1}\big)\\
&= \big(\mu(w,h_1)\mathbf{s}\big(h_1^{w^{-1}}\big)\big)\big(\mu(w,h_2)\mathbf{s}\big(h_2^{w^{-1}}\big)\big)\\
&= \mu(w,h_1)\mu(w,h_2)\mathbf{s}\big(h_1^{w^{-1}}\big)\mathbf{s}\big(h_2^{w^{-1}}\big)\\
&= \mu(w,h_1)\mu(w,h_2)\tau\big(h_1^{w^{-1}},h_2^{w^{-1}}\big)\mathbf{s}\big((h_1h_2)^{w^{-1}}\big)
\end{aligned}$$

and secondly

$$\begin{aligned}
&\big(\mathbf{s}(w)\mathbf{s}(h_1)\mathbf{s}(w)^{-1}\big)\big(\mathbf{s}(w)\mathbf{s}(h_2)\mathbf{s}(w)^{-1}\big)\\
&= \mathbf{s}(w)\mathbf{s}(h_1)\mathbf{s}(h_2)\mathbf{s}(w)^{-1}\\
&= \mathbf{s}(w)\tau(h_1,h_2)\mathbf{s}(h_1h_2)\mathbf{s}(w)^{-1}\\
&= \tau(h_1,h_2)\mu(w,h_1h_2)\mathbf{s}\big((h_1h_2)^{w^{-1}}\big).
\end{aligned}$$

Comparing these expressions gives the formula.

(2) If we put $d = \mathrm{diag}(1,\ldots,1,\underset{p}{v},1,\ldots,1,v^{-1}) \in SH_{n+1}$ then by definition $\mathbf{s}(h) = (d,1) = h_{p,n+1}(v)$. Now we may write $w = p_\pi$ where $\pi \in \mathfrak{S}_n$ is a suitable permutation. Then $\phi(\mathbf{s}(w)) = p_\pi \mathrm{diag}(1,\ldots,1,\det(w))$ and so by the first formula

of Lemma 2 we have

$$\begin{aligned}\mathbf{s}(w)\mathbf{s}(h)\mathbf{s}(w)^{-1} &= \mathbf{s}(w)h_{p,n+1}(v)\mathbf{s}(w)^{-1} \\ &= c(\det(w), v)h_{\pi(p),n+1}(v) \\ &= c(\det(w), v)\mathbf{s}(h^{w^{-1}}) \\ &= c(\det(w), \det(h))\mathbf{s}(h^{w^{-1}}) .\end{aligned}$$

(3) We shall proceed by induction on the number of v_i which are not equal to 1. If all but one of the v_i equal 1 then the formula follows from (2). In general, take $p \in \{1, \ldots, n\}$ such that $v_p \neq 1$ but $v_i = 1$ for all $i > p$. Then $h = h'h''$ where $h' = \operatorname{diag}(v_1, \ldots, v_{p-1}, 1, \ldots, 1)$ and $h'' = \operatorname{diag}(1, \ldots, 1, v_p, 1, \ldots, 1)$. Using the inductive hypothesis we have

$$\mu(w, h') = \prod_{\substack{i<j<p \\ \pi(i)>\pi(j)}} c(v_i, v_j) \cdot c(\det(w), \det(h'))$$

and

$$\mu(w, h'') = c(\det(w), \det(h''))$$

and from (1) we get

$$\mu(w, h) = \tau\big((h')^{w^{-1}}, (h'')^{w^{-1}}\big)\tau(h', h'')^{-1}\mu(w, h')\mu(w, h'') .$$

The choice of p implies that $\tau(h', h'') = 1$ and therefore it only remains to calculate $\tau\big((h')^{w^{-1}}, (h'')^{w^{-1}}\big)$. Now

$$(h')_i^{w^{-1}} = \begin{cases} v_{\pi^{-1}(i)} & \text{if } \pi^{-1}(i) < p \\ 1 & \text{if } \pi^{-1}(i) \geq p \end{cases}$$

and

$$(h'')_j^{w^{-1}} = \begin{cases} v_p & \text{if } \pi^{-1}(j) = p \\ 1 & \text{if } \pi^{-1}(j) \neq p \end{cases}$$

and thus

$$\begin{aligned}
\tau\big((h')^{w^{-1}}, (h'')^{w^{-1}}\big) &= \prod_{i \geq j} c\big((h')_i^{w^{-1}}, (h'')_j^{w^{-1}}\big) \\
&= \prod_{i \geq \pi(p)} c\big((h')_i^{w^{-1}}, v_p\big) \\
&= \prod_{\substack{i > \pi(p) \\ \pi^{-1}(i) < p}} c(v_{\pi^{-1}(i)}, v_p) \\
&= \prod_{\substack{q < p \\ \pi(q) > \pi(p)}} c(v_q, v_p)
\end{aligned}$$

on setting $q = \pi^{-1}(i)$. Hence

$$\begin{aligned}
\mu(w, h) &= \prod_{\substack{i<j<p \\ \pi(i)>\pi(j)}} c(v_i, v_j) \cdot \prod_{\substack{q<p \\ \pi(q)>\pi(p)}} c(v_q, v_p) \cdot c(\det(w), \det(h)) \\
&= \prod_{\substack{i<j \\ \pi(i)>\pi(j)}} c(v_i, v_j) \cdot c(\det(w), \det(h))
\end{aligned}$$

and the induction step is complete. □

Lemma 4: *If $h, h' \in H_n$ and $w, w' \in W$ then*

$$\tau(hw, h'w') = \mu(w, h')\tau\big(h, (h')^{w^{-1}}\big)\tau(w, w')c(\det(h), \det(w'))c(\det(h'), \det(w)) .$$

Proof: By definition

$$
\begin{aligned}
&\mathbf{s}(hw)\mathbf{s}(h'w') \\
&= \mathbf{s}(h)\mathbf{s}(w)\mathbf{s}(h')\mathbf{s}(w')c(\det(h),\det(w))c(\det(h'),\det(w')) \\
&= \mathbf{s}(h)\mathbf{s}(w)\mathbf{s}(h')\mathbf{s}(w)^{-1}\mathbf{s}(w)\mathbf{s}(w')c(\det(h),\det(w))c(\det(h'),\det(w')) \\
&= \mathbf{s}(h)\mu(w,h')\mathbf{s}\big((h')^{w^{-1}}\big)\tau(w,w')\mathbf{s}(ww')\cdot \\
&\qquad\qquad c(\det(h),\det(w))c(\det(h'),\det(w')) \\
&= \mu(w,h')\tau(w,w')\tau\big(h,(h')^{w^{-1}}\big)\mathbf{s}\big(h(h')^{w^{-1}}\big)\mathbf{s}(ww')\cdot \\
&\qquad\qquad c(\det(h),\det(w))c(\det(h'),\det(w')) \\
&= \mu(w,h')\tau(w,w')\tau\big(h,(h')^{w^{-1}}\big)c\big(\det\big(h(h')^{w^{-1}}\big),\det(ww')\big)\mathbf{s}\big(h(h')^{w^{-1}}ww'\big)\cdot \\
&\qquad\qquad c(\det(h),\det(w))c(\det(h'),\det(w')) \\
&= \mu(w,h')\tau(w,w')\tau\big(h,(h')^{w^{-1}}\big)c(\det(h),\det(w'))\cdot \\
&\qquad\qquad c(\det(h'),\det(w))\mathbf{s}(hwh'w')
\end{aligned}
$$

and since

$$\mathbf{s}(hw)\mathbf{s}(h'w') = \tau(hw,h'w')\mathbf{s}(hwh'w')$$

the formula follows. □

Setting $h=1$, $h'=h$ and $w'=1$ in Lemma 4 we obtain

$$\tau(w,h) = \mu(w,h)c(\det(h),\det(w)) \tag{5}$$

and combining this with the third formula of Lemma 3 we find that if $h=\mathrm{diag}(v_1,\ldots,v_n)\in H_n$ and w corresponds to the permutation $\pi\in\mathfrak{S}_n$ then

$$\tau(w,h) = \prod_{\substack{i<j\\ \pi(i)>\pi(j)}} c(v_i,v_j)\,. \tag{6}$$

Also we may use (5) to eliminate the occurrence of μ in Lemma 4. This gives

$$\tau(hw, h'w') = \tau(w, h')\tau\big(h, (h')^{w^{-1}}\big)\tau(w, w')c(\det(h), \det(w')) . \tag{7}$$

3. The Cocycle of Kazhdan and Patterson

In this section we shall determine the relationship between the cocycle τ constructed above and the cocycle used in [KaP] and in the various papers, such as [BuG] and [BuH], which rely on it. This is somewhat awkward for two reasons. First, the choices made by Kazhdan and Patterson in their work and by Matsumoto in his original construction lead naturally to inverse classes in $\mathrm{H}^2(\mathrm{GL}(n)\,, A)$ being labelled as the metaplectic class. (Actually, since $\mathrm{GL}(n)$ is not perfect, there are several metaplectic classes in each case, those on one list being inverse to those on the other.) We followed Matsumoto above and so we must expect that Kazhdan's and Patterson's cocycle will be roughly τ^{-1}. Since a number of authors (notably Milnor) follow Matsumoto and a number (notably Bump) follow Kazhdan and Patterson this annoyance is now firmly embedded in the literature. A really neat solution is possible only when A has exponent two (as in the important special case where A is μ_2) since then $\mathrm{H}^2(\mathrm{GL}(n)\,, A)$ also has exponent two and the classes coincide. Secondly, Kazhdan and Patterson erroneously assume that the metaplectic extension is always split over the Weyl group. As we shall see, this is only true in general when $c(-1,-1) = 1$ and so only in this case can we expect to recover the cocycle of [KaP] exactly. When the metaplectic cover is not split over W no formula for the cocycle restricted to W is known at present. We shall therefore have to be content with identifying the induced cover with one of those constructed in section one; this at least makes it possible to perform calculations in the covering group if necessary.

In order to relate Kazhdan's and Patterson's cocycle to that already constructed we shall require a coboundary built from certain functions on $\mathrm{GL}(n, F)$. If $g \in \mathrm{GL}(n, F)$ and $1 \leq \ell \leq n$ then let $\mathrm{X}_\ell(g)$ denote the first non-zero $(\ell \times \ell)$–minor formed from the last ℓ rows of g, where the minors are ordered lexicographically according to the columns they involve. From Laplace's expansion of the determinant of g as a sum of products of these minors and their signed cominors it follows that not all the minors can be zero and hence $\mathrm{X}_\ell(g)$ is well-defined. Of course $\mathrm{X}_n(g)$ is simply the determinant of g.

If $w \in W$ then we let $\Phi^+(w) = \{\alpha \in \Phi^+ \,|\, w\alpha < 0\}$ and if $h \in H_n$ and $\alpha = (i, j) \in \Phi^+$ we write h^α for h_i/h_j.

Lemma 1: *Suppose that $w, w' \in W$ and $h, h' \in H_n$. Then*

(1) $\mathrm{X}_\ell(hh') = \mathrm{X}_\ell(h)\mathrm{X}_\ell(h')$ *for all* ℓ,

(2) $\mathrm{X}_\ell(hw) = \mathrm{X}_\ell(h)\mathrm{X}_\ell(w)$ *for all* ℓ,

(3)

$$\prod_{\ell=1}^{n-1} \frac{\mathrm{X}_\ell(wh)}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(h)} = \prod_{\alpha \in \Phi^+(w)} h^\alpha,$$

(4)

$$\frac{\mathrm{X}_\ell(hwh'w')}{\mathrm{X}_\ell(hw)\mathrm{X}_\ell(h'w')} = \frac{\mathrm{X}_\ell(wh')}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(h')} \cdot \frac{\mathrm{X}_\ell(ww')}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(w')}.$$

Proof:

(1) Since $X_\ell(h) = \prod_{i \geq n-\ell+1} h_i$ this is clear.

(2) The matrix hw is obtained from h by a permutation of the columns. If we fix our attention on the last ℓ rows of hw we shall see that the unique non-zero $(\ell \times \ell)$-minor in the last ℓ rows of h has undergone a corresponding permutation, π_0 say, of its columns. Thus $X_\ell(hw) = \mathrm{sgn}(\pi_0) X_\ell(h)$ and as $X_\ell(w) = \mathrm{sgn}(\pi_0)$, the identity follows.

(3) Making use of (2) we obtain

$$\frac{X_\ell(wh)}{X_\ell(w)X_\ell(h)} = \frac{X_\ell(h^{w^{-1}})}{X_\ell(h)}$$

and so we wish to show that

$$\prod_{\ell=1}^{n-1} \frac{X_\ell(h^{w^{-1}})}{X_\ell(h)} = \prod_{\alpha \in \Phi^+(w)} h^\alpha .$$

We shall do this by induction on the length of w. If $\ell(w) = 1$ then w is a simple reflection. Suppose that $w = s_\gamma$ where $\gamma = (p, p+1) \in \Delta$. We have $\Phi^+(w) = \{\gamma\}$ and

$$\left(h^{w^{-1}}\right)_i = \begin{cases} h_i & \text{if} \quad i \neq p, p+1 \\ h_{p+1} & \text{if} \quad i = p \\ h_p & \text{if} \quad i = p+1 \end{cases} .$$

Thus, using the formula from the proof of (1), we obtain

$$X_\ell(h^{w^{-1}}) = \begin{cases} X_\ell(h) & \text{if} \quad \ell \geq n-p+1 \\ X_{n-p}(h) h_p h_{p+1}^{-1} & \text{if} \quad \ell = n-p \\ X_\ell(h) & \text{if} \quad \ell \leq n-p-1 \end{cases}$$

and so

$$\prod_{\ell=1}^{n-1} \frac{X_\ell(h^{w^{-1}})}{X_\ell(h)} = h_p h_{p+1}^{-1} = h^\gamma ,$$

as required.

Now suppose that $\ell(w) > 1$ and choose $\gamma \in \Delta$ such that $w = s_\gamma w_1$ with $\ell(w) = \ell(w_1) + 1$. Then $\Phi^+(w) = \Phi^+(w_1) \cup \{w_1^{-1}\gamma\}$ and

$$\begin{aligned}
\prod_{\ell=1}^{n-1} \frac{\mathrm{X}_\ell(h^{w^{-1}})}{\mathrm{X}_\ell(h)} &= \prod_{\ell=1}^{n-1} \frac{\mathrm{X}_\ell((h^{w_1^{-1}})^{s_\gamma})}{\mathrm{X}_\ell(h^{w_1^{-1}})} \cdot \prod_{\ell=1}^{n-1} \frac{\mathrm{X}_\ell(h^{w_1^{-1}})}{\mathrm{X}_\ell(h)} \\
&= (h^{w_1^{-1}})^{\gamma} \cdot \prod_{\alpha \in \Phi^+(w_1)} h^\alpha \\
&= h^{w_1^{-1}\gamma} \cdot \prod_{\alpha \in \Phi^+(w_1)} h^\alpha \\
&= \prod_{\alpha \in \Phi^+(w)} h^\alpha
\end{aligned}$$

which completes the induction.

(4) Using (1) and (2) repeatedly we have

$$\begin{aligned}
\frac{\mathrm{X}_\ell(hwh'w')}{\mathrm{X}_\ell(hw)\mathrm{X}_\ell(h'w')} &= \frac{\mathrm{X}_\ell(hwh'w^{-1}ww')}{\mathrm{X}_\ell(hw)\mathrm{X}_\ell(h'w')} \\
&= \frac{\mathrm{X}_\ell(h)\mathrm{X}_\ell(wh'w^{-1})\mathrm{X}_\ell(ww')}{\mathrm{X}_\ell(h)\mathrm{X}_\ell(w)\mathrm{X}_\ell(h')\mathrm{X}_\ell(w')} \\
&= \frac{\mathrm{X}_\ell(wh'w^{-1})}{\mathrm{X}_\ell(h')} \cdot \frac{\mathrm{X}_\ell(ww')}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(w')} \\
&= \frac{\mathrm{X}_\ell(wh'w^{-1})\mathrm{X}_\ell(w)}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(h')} \cdot \frac{\mathrm{X}_\ell(ww')}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(w')} \\
&= \frac{\mathrm{X}_\ell(wh')}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(h')} \cdot \frac{\mathrm{X}_\ell(ww')}{\mathrm{X}_\ell(w)\mathrm{X}_\ell(w')},
\end{aligned}$$

as required. □

We now define

$$\sigma(g_1, g_2) = \tau(g_1, g_2)^{-1} c(\det(g_1), \det(g_2)) \cdot \prod_{\ell=1}^{n-1} c\left(-1, \frac{\mathrm{X}_\ell(g_1 g_2)}{\mathrm{X}_\ell(g_1)\mathrm{X}_\ell(g_2)} \right). \quad (1)$$

Notice that $\tau_0(g_1, g_2) = c(\det(g_1), \det(g_2))$ is a 2-cocycle on $\mathrm{GL}(n, F)$ and that the last factor in (1) is the coboundary derived from the 1-cochain

$$g \mapsto \prod_{\ell=1}^{n-1} c(-1, \mathrm{X}_\ell(g)).$$

Therefore σ is a 2-cocycle on $\mathrm{GL}(n, F)$ representing the cohomology class $[\tau]^{-1}[\tau_0]$ in $\mathrm{H}^2(\mathrm{GL}(n, F), A)$. We let

$$1 \to A \to \widetilde{\mathrm{GL}}(n, F) \xrightarrow{p} \mathrm{GL}(n, F) \to 1 \tag{2}$$

be the central extension corresponding to the class $[\sigma]$ and, by abuse of notation, denote by $\mathbf{s} : \mathrm{GL}(n, F) \to \widetilde{\mathrm{GL}}(n, F)$ a section with respect to which $[\sigma]$ is represented by σ.

Now we must identify the restriction of (2) to W. If $c(-1, -1) = 1$ then $\tau(w, w') = \tau_0(w, w') = 1$ and $c(-1, \mathrm{X}_\ell(w)) = 1$ for all $w, w' \in W$ and all ℓ. Thus $\sigma(w, w') = 1$ for all $w, w' \in W$ and the sequence (2) is split over W. Suppose now that $c(-1, -1) \neq 1$. We remarked in section 2 that $\mathrm{res}_W([\tau])$ is always equal to the image of the $r = 1$ spin class in $\mathrm{H}^2(W, A)$, where res_W denotes the restriction homomorphism from $\mathrm{H}^2(\mathrm{GL}(n, F), A)$ to $\mathrm{H}^2(W, A)$. Now one easily checks that $\mathrm{res}_W([\tau_0]) \in \mathrm{H}^2(W, A)$ is equal to the image of the non-trivial class in $\mathrm{H}^2(W, \mu_2)$ under extension of scalars when $n = 2$ or $n = 3$ and to the image of the product of the two spin classes when $n \geq 4$. From this it follows that $\mathrm{res}_W([\sigma]) \in \mathrm{H}^2(W, A)$ is trivial when $n = 2$ or $n = 3$ and equal to the image of the $r = 3$ spin class when $n \geq 4$. In particular, when $A = \mu_2$ and $n \geq 4$, (2) is never split over W.

From the identities for τ which we obtained in the previous section and Lemma 1 we arrive at certain identities for σ. If $h, h' \in H_n$ then (3) of section 2 and

Lemma 1 (1) give

$$\sigma(h,h') = \prod_{i<j} c(h_i, h'_j)\,. \tag{3}$$

If $h \in H_n$ and $w \in W$ then (4) of section 2 and Lemma 1 (2) give

$$\sigma(h,w) = 1 \tag{4}$$

and (6) of section 2 and Lemma 1 (3) give

$$\sigma(w,h) = \prod_{\alpha=(i,j)\in\Phi^+(w)} c(h_i,h_j)^{-1} c(-1,h^\alpha)\cdot c(\det(w),\det(h))\,. \tag{5}$$

Combining Lemma 1 (4) with (7) of section 2 we obtain

$$\sigma(hw,h'w') = \sigma(w,h')\sigma\big(h,(h')^{w^{-1}}\big)\sigma(w,w')\,. \tag{6}$$

When $n \leq 3$ or $c(-1,-1) = 1$ we also have $\sigma(w,w') = 1$ for all $w, w' \in W$ and so we have recovered the cocycle of [KaP]. If $n \geq 4$ and $c(-1,-1) \neq 1$ then we may assume that $\mathbf{s}$ has been chosen so that if $\mathbf{s}(s_\alpha) = t_\alpha$ for $\alpha \in \Delta$ then the t_α satisfy the relations

$$(t_\alpha t_\beta)^{m(\alpha,\beta)} = \begin{cases} 1 & \text{if} \quad \langle\alpha,\beta\rangle \neq 0 \\ -1 & \text{if} \quad \langle\alpha,\beta\rangle = 0\,. \end{cases} \tag{7}$$

The group $\widetilde{W} = p^{-1}(W)$ is generated by $\{t_\alpha \,|\, \alpha \in \Delta\}$ and A and the relations (7), the trivial relations recorded after (1) in section 1, all the relations in A and the relation $c(-1,-1) = -1$ suffice to give a presentation of $\widetilde{W}$.

4. Lifting the Main Involution

We retain the assumptions and notation of the previous section. Recall that the *main involution* on $\mathrm{GL}(n)$ is the automorphism $g \mapsto {}^\iota g$ given by ${}^\iota g = w_0\,{}^t g^{-1} w_0$, where $w_0 \in W$ is the longest Weyl element. It is our intention in this section to study the lifts of ι to $\widetilde{\mathrm{GL}}(n)$ (the formal definition is given below).

In order to recall some general facts about central extensions let us briefly adopt the following notation. We suppose that

$$1 \to A \to \widetilde{G} \xrightarrow{p} G \to 1 \tag{1}$$

is a central extension of groups, that $\mathbf{s} : G \to \widetilde{G}$ is any section of p and that σ is the 2–cocycle representing the class of (1) in $\mathrm{H}^2(G, A)$ with respect to $\mathbf{s}$. If $f : G \to G$ is an automorphism then a *lift* of f is an automorphism $\widetilde{f} : \widetilde{G} \to \widetilde{G}$ making the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \widetilde{G} & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} & & \\ 1 & \longrightarrow & A & \longrightarrow & \widetilde{G} & \xrightarrow{p} & G & \longrightarrow & 1 \end{array}$$

commute. By the 5–lemma any homomorphism $\widetilde{f}$ making this diagram commute is in fact a lift of f. We shall denote by $\mathcal{L}(f)$ the set of all lifts of f. Note that $\mathrm{Aut}(G)$ acts on $\mathrm{H}^2(G, A)$ by $f[\tau] = [f(\tau)]$ where τ is any 2–cocycle and $f(\tau) = \tau \circ (f \times f)$.

Lemma 1: *The set $\mathcal{L}(f)$ is non-empty if and only if $f[\sigma] = [\sigma]$.*

Proof: If $\widetilde{f} \in \mathcal{L}(f)$ then $\mathbf{s}' : G \to \widetilde{G}$ defined by

$$\mathbf{s}'(g) = (\widetilde{f}^{-1})[\mathbf{s}(f(g))]$$

is a section of p. A computation shows that the 2–cocycle representing $[\sigma]$ with respect to $\mathbf{s}'$ is $f(\sigma)$. Hence $f[\sigma] = [\sigma]$ if $\mathcal{L}(f) \neq \emptyset$.

Conversely suppose that $f[\sigma] = [\sigma]$ and let $\sigma = f(\sigma) \cdot \partial\kappa$ where $\kappa : G \to A$ and ∂ denotes the coboundary map. We define

$$\widetilde{f}(a\mathbf{s}(g)) = a\kappa(g)^{-1}\mathbf{s}(f(g))$$

for $a \in A$ and $g \in G$. Since $\mathbf{s}$ is a section of p this gives a well-defined map $\widetilde{f} : \widetilde{G} \to \widetilde{G}$ which is easily verified to be a lift of f. $\square$

In due course we shall use this Lemma to show that $\mathcal{L}(\iota) \neq \emptyset$. The next result calculates the size of $\mathcal{L}(f)$ when this set is non-empty.

Lemma 2: *Suppose that $\mathcal{L}(f) \neq \emptyset$. Then $\mathcal{L}(f)$ is a principal homogeneous space for the group* $\mathrm{Hom}(G, A)$.

Proof: We define an action of $\mathrm{Hom}(G, A)$ on $\mathcal{L}(f)$ by setting

$$(\varphi \cdot \widetilde{f})(\widetilde{g}) = \varphi(p(\widetilde{g}))\widetilde{f}(\widetilde{g})$$

for $\varphi \in \mathrm{Hom}(G, A)$, $\widetilde{f} \in \mathcal{L}(f)$ and $\widetilde{g} \in \widetilde{G}$. It is routine to check that this gives an action; we must show in addition that the action is transitive and that the point stabilizers are trivial.

If $\widetilde{f}_1, \widetilde{f}_2 \in \mathcal{L}(f)$ and $\mathbf{t} : G \to \widetilde{G}$ is any section of p then set

$$\varphi(g) = \widetilde{f}_1(\mathbf{t}(g))\widetilde{f}_2(\mathbf{t}(g)^{-1})$$

for $g \in G$. This function maps G to A since $\widetilde{f}_1$ and $\widetilde{f}_2$ are lifts of f and it is independent of the choice of $\mathbf{t}$. In addition if $g, h \in G$ then

$$\varphi(gh) = \widetilde{f}_1(\mathbf{t}(gh))\widetilde{f}_2(\mathbf{t}(gh)^{-1})$$

$$= \widetilde{f}_1(\sigma_{\mathbf{t}}(g,h)^{-1}\mathbf{t}(g)\mathbf{t}(h))\widetilde{f}_2(\sigma_{\mathbf{t}}(g,h)\mathbf{t}(h)^{-1}\mathbf{t}(g)^{-1})$$

$$= \widetilde{f}_1(\mathbf{t}(g))\varphi(h)\widetilde{f}_2(\mathbf{t}(g)^{-1})$$

$$= \widetilde{f}_1(\mathbf{t}(g))\widetilde{f}_2(\mathbf{t}(g)^{-1})\varphi(h)$$

$$= \varphi(g)\varphi(h)$$

and so $\varphi \in \mathrm{Hom}(G, A)$. If $\widetilde{g} \in \widetilde{G}$ then we may choose $\mathbf{t} : G \to \widetilde{G}$ a section such that $\mathbf{t}(p(\widetilde{g})) = \widetilde{g}$. With this choice we have

$$\begin{aligned}(\varphi \cdot \widetilde{f}_2)(\widetilde{g}) &= \varphi(p(\widetilde{g}))\widetilde{f}_2(\widetilde{g}) \\ &= \widetilde{f}_1(\widetilde{g})\widetilde{f}_2(\widetilde{g}^{-1})\widetilde{f}_2(\widetilde{g}) \\ &= \widetilde{f}_1(\widetilde{g})\end{aligned}$$

and hence $\varphi \cdot \widetilde{f}_2 = \widetilde{f}_1$. This shows that the action is transitive. Finally if $\varphi \cdot \widetilde{f} = \widetilde{f}$ for some $\varphi \in \mathrm{Hom}(G, A)$ then $\varphi(p(\widetilde{g})) = 1\ \forall \widetilde{g} \in \widetilde{G}$ and since p is onto, φ is the trivial homomorphism. $\square$

This Lemma implies that we cannot generally hope to obtain a unique lift of ι to $\widetilde{\mathrm{GL}}(n)$. However if we assume that F is infinite, as we shall henceforth, then $\mathrm{SL}(n+1, F)$ is perfect and hence any automorphism of this group which lifts to $\widetilde{\mathrm{SL}}(n+1, F)$ does so uniquely. This makes it more convenient to begin by studying an involution of $\mathrm{SL}(n+1, F)$ which induces ι on $\mathrm{GL}(n, F)$ embedded as in section two.

Let $w_1 = \mathrm{diag}(w_0, 1) \in \mathrm{GL}(n+1, F)$, where $w_0 \in \mathrm{GL}(n, F)$ is as above, and for $g \in \mathrm{GL}(n+1, F)$ put $\hat{g} = w_1{}^t g^{-1} w_1$. The map $g \mapsto \hat{g}$ is an involution of $\mathrm{GL}(n+1, F)$ which stabilizes $\mathrm{SL}(n+1, F)$. When restricted to the subgroup

$\mathrm{GL}(n) \leq \mathrm{SL}(n+1, F)$ it induces the involution ι. Let us denote by ν the 2−cocycle which represents the class of the extension (1) of section two (with n replaced by $n+1$) with respect to any section $\mathbf{s} : \mathrm{SL}(n+1, F) \to \widetilde{\mathrm{SL}}(n+1, F)$ whose restriction to SH_{n+1} is $\mathbf{s}(d) = (d, 1) \in \widetilde{SH}_{n+1}$. Thus if $d = \mathrm{diag}(u_1, \ldots, u_{n+1})$ and $d' = \mathrm{diag}(v_1, \ldots, v_{n+1})$ are in SH_{n+1} then

$$\nu(d, d') = \prod_{i \geq j} c(u_i, v_j)\,. \tag{2}$$

Lemma 3: *Suppose that* $n \geq 2$. *Then* $[\nu]\hat{} = [\nu]$ *in* $\mathrm{H}^2(\mathrm{SL}(n+1, F)\,, A)$.

Proof: It follows from the remark in §11 of [Mil] that the restriction map

$$\mathrm{res} : \mathrm{H}^2(\mathrm{SL}(n+1, F)\,, A) \to \mathrm{H}^2(SH_{n+1}, A)$$

is a monomorphism (it is here that we need $n \geq 2$; recall also that we are assuming F to be infinite). Since $g \mapsto \hat{g}$ stabilizes SH_{n+1} and we have $\mathrm{res}([\nu]\hat{}) = \mathrm{res}([\nu])\hat{}$ it is enough to show that $\mathrm{res}([\nu])\hat{} = \mathrm{res}([\nu])$. We shall henceforth abuse notation by omitting the restriction maps.

Let $d, d' \in SH_{n+1}$ be as above. Since $\prod_{j=1}^{n+1} v_j = 1$ we may rewrite (2) as

$$\nu(d, d') = \prod_{n+1 > i \geq j} c(u_i, v_j)\,. \tag{3}$$

Now $\hat{d} = \mathrm{diag}(u_n^{-1}, \ldots, u_1^{-1}, u_{n+1}^{-1})$ and hence, using (3),

$$\begin{aligned}\hat{\nu}(d, d') &= \nu(\hat{d}, \hat{d}') \\ &= \prod_{n+1 > i \geq j} c(u_{n-i+1}^{-1}, v_{n-j+1}^{-1})\end{aligned}$$

$$= \prod_{n+1>i\geq j} c(u_{n-i+1}, v_{n-j+1})$$
$$= \prod_{a\leq b<n+1} c(u_a, v_b)$$

on setting $a = n - i + 1$ and $b = n - j + 1$. Now using $\prod_{a=1}^{n+1} u_a = 1$ this may be rewritten as

$$\hat{\nu}(d, d') = \prod_{a\leq b} c(u_a, v_b)$$

or, using the skew-symmetry of Steinberg symbols, as

$$\hat{\nu}(d, d') = \nu(d', d)^{-1}.$$

Hence

$$\begin{aligned}(\nu \cdot \hat{\nu}^{-1})(d, d') &= \nu(d, d')\nu(d', d) \\ &= \prod_{i\geq j} c(u_i, v_j) \cdot \prod_{a\geq b} c(v_a, u_b) \\ &= \prod_{i>j} c(u_i, v_j) \cdot \prod_{a>b} c(v_a, u_b) \end{aligned} \tag{4}$$

on once again using the identity $c(x, y)c(y, x) = 1$.

Now let us define $\kappa : SH_{n+1} \to A$ by

$$\kappa(d) = \prod_{k>\ell} c(u_k, u_\ell). \tag{5}$$

Then

$$\begin{aligned}\kappa(dd') &= \prod_{k>\ell} c(u_k v_k, u_\ell v_\ell) \\ &= \kappa(d)\kappa(d') \cdot \prod_{i>j} c(u_i, v_j) \cdot \prod_{a>b} c(v_a, u_b) \\ &= \kappa(d)\kappa(d')(\nu \cdot \hat{\nu}^{-1})(d, d')\end{aligned}$$

by (4) and hence

$$(\nu \cdot \hat{\nu}^{-1})(d, d') = \kappa(dd')/\kappa(d)\kappa(d')$$
$$= (\partial\kappa)(d, d') .$$

The claim follows. $\square$

Proposition 1: *Suppose that $n \geq 2$. The involution $g \mapsto \hat{g}$ of $\mathrm{SL}(n+1, F)$ has a unique lift to $\widetilde{\mathrm{SL}}(n+1, F)$. This lift is itself an involution. Regarding $\widetilde{SH}_{n+1}$ as a subgroup of $\widetilde{\mathrm{SL}}(n+1, F)$ and denoting the lift of $\hat{}$ by the same symbol we have*

$$(d, a)^{\wedge} = (\hat{d}, a\kappa(d)^{-1})$$

where $d = \mathrm{diag}(u_1, \ldots, u_{n+1}) \in SH_{n+1}$, $a \in A$ and

$$\kappa(d) = \prod_{i>j} c(u_i, u_j) .$$

Proof: Combining Lemma 1 and Lemma 3 shows that $g \mapsto \hat{g}$ has a lift and Lemma 2 together with the discussion which follows it implies that the lift is unique. Also $\hat{} \circ \hat{} : \widetilde{\mathrm{SL}}(n+1, F) \to \widetilde{\mathrm{SL}}(n+1, F)$ is an automorphism of $\widetilde{\mathrm{SL}}(n+1, F)$ which lifts the identity automorphism of $\mathrm{SL}(n+1, F)$ and so the unicity of lifts implies that $\hat{} : \widetilde{\mathrm{SL}}(n+1, F) \to \widetilde{\mathrm{SL}}(n+1, F)$ is an involution. The formula for $\hat{}$ on $\widetilde{SH}_{n+1}$ follows by combining the proof of Lemma 1 with that of Lemma 3. $\square$

It may be of some service to the skeptical reader to prove directly that $\hat{}$ is an involution of $\widetilde{SH}_{n+1}$. This amounts to checking that $\kappa(d)\kappa(d') = 1$ for all $d \in SH_{n+1}$, which is accomplished by the following calculation:

$$\begin{aligned}
\kappa(d)\kappa(d') &= \prod_{i>j} c(u_i, u_j) \cdot \prod_{k<\ell<n+1} c(u_k^{-1}, u_\ell^{-1}) \cdot c(u_{n+1}^{-1}, u_1^{-1} \cdot \ldots \cdot u_n^{-1}) \\
&= \prod_{n+1>i>j} c(u_i, u_j) \cdot \prod_{n+1>\ell>k} c(u_\ell, u_k)^{-1} \cdot c(u_{n+1}, u_1 \cdot \ldots \cdot u_n)^2 \\
&= c(u_{n+1}, u_1 \cdot \ldots \cdot u_n)^2 \\
&= c(u_{n+1}, u_{n+1}^{-1})^2 \\
&= c(u_{n+1}, u_{n+1})^{-2} \\
&= c(-1, u_{n+1})^{-2} \\
&= c((-1)^{-2}, u_{n+1}) \\
&= c(1, u_{n+1}) \\
&= 1
\end{aligned}$$

where we have used the identity $c(x,x) = c(-1,x)$ valid for all Steinberg symbols. We also remark that it follows from Lemma 3 that there is a function $\kappa : \mathrm{SL}(n+1, F) \to A$ extending the one defined in Proposition 1 and satisfying $\nu = \hat{\nu} \cdot (\partial\kappa)$. We choose any such function and fix it for the rest of this section. We are now ready to return to the main involution itself.

Proposition 2: *For every $n \geq 1$ the set of lifts of the main involution of* $\mathrm{GL}(n)$ *to* $\widetilde{\mathrm{GL}}(n)$ *is non-empty. Each of these lifts stabilizes $\widetilde{H}_n$ and one of them satisfies*

$$^{\iota}\mathbf{s}(h) = \prod_{i>j} c(h_i, h_j) \cdot \mathbf{s}(^{\iota}h)$$

for all $h \in H_n$.

Proof: The case $n = 1$ is trivial and we shall henceforth assume that $n \geq 2$. The class of the metaplectic extension in $\mathrm{H}^2(\mathrm{GL}(n), A)$ is represented by the 2−cocycle σ given in equation (1) of the previous section. We seek a function $\lambda : \mathrm{GL}(n) \to A$ such that $\sigma = {}^{\iota}\sigma \cdot (\partial\lambda)$ and to find it we shall work separately with the three factors constituting σ.

Since τ is the pull-back of ν under the embedding $\eta : \mathrm{GL}(n) \to \mathrm{SL}(n+1)$ we know from the proof of Proposition 1 that

$$\begin{aligned}
\tau(g_1, g_2) &= \nu(\eta(g_1), \eta(g_2)) \\
&= (\partial\kappa)(\eta(g_1), \eta(g_2)) \cdot \hat{\nu}(\eta(g_1), \eta(g_2)) \\
&= (\partial\kappa)(\eta(g_1), \eta(g_2))\nu(\eta(g_1)\hat{}, \eta(g_2)\hat{}) \\
&= (\partial\kappa)(\eta(g_1), \eta(g_2))\nu(\eta({}^{\iota}g_1), \eta({}^{\iota}g_2)) \\
&= (\partial\lambda_1)(g_1, g_2)^{-1}\tau({}^{\iota}g_1, {}^{\iota}g_2) \\
&= (\partial\lambda_1)(g_1, g_2)^{-1}({}^{\iota}\tau)(g_1, g_2)
\end{aligned}$$

where we have set $\lambda_1(g) = \kappa(\eta(g))^{-1}$. Thus we have $\tau^{-1} = {}^{\iota}\tau^{-1} \cdot (\partial\lambda_1)$. The second factor in σ is invariant under ι. As for the third factor, if we set

$$x(g) = \prod_{\ell=1}^{n-1} c(-1, X_\ell(g))$$

then it is simply (∂x) and we have $(\partial x) = {}^{\iota}(\partial x) \cdot (\partial\lambda_3)$ where

$$\lambda_3(g) = \prod_{\ell=1}^{n-1} c\left(-1, \frac{X_\ell(g)}{X_\ell({}^{\iota}g)}\right) .$$

Combining these equations we find that if $\lambda : \mathrm{GL}(n) \to A$ is defined by $\lambda(g) =$

$\lambda_1(g)\lambda_3(g)$ then $\sigma = {}^\iota\sigma \cdot (\partial\lambda)$. It follows that ${}^\iota[\sigma] = [\sigma]$ and by Lemma 1 we have the first statement of the Propostion.

Examining the proof of Lemma 1 we see that there is a lift of ι satisfying ${}^\iota\mathbf{s}(g) = \lambda(g)^{-1}\mathbf{s}({}^\iota g)$. In order to verify the formula given in the statement it is thus sufficient to calculate $\lambda(h)$ for $h = \mathrm{diag}(h_1, \ldots, h_n) \in H_n$. We begin with $\lambda_3(h)$. Since $X_n(g) = \det(g)$ and

$$c\left(-1, \frac{X_n(g)}{X_n({}^\iota g)}\right) = c(-1, \det(g)^2) = 1$$

we may write

$$\lambda_3(g) = \prod_{\ell=1}^{n} c\left(-1, \frac{X_\ell(g)}{X_\ell({}^\iota g)}\right) .$$

We have remarked before that $X_\ell(h) = \prod_{i \geq n-\ell+1} h_i$ and since $({}^\iota h)_j = h_{n-j+1}^{-1}$ this gives

$$X_\ell({}^\iota h) = \prod_{j \geq n-\ell+1} h_{n-j+1}^{-1} = \prod_{k \leq \ell} h_k^{-1} .$$

From these identities and the formula for λ_3 just given it follows that

$$\begin{aligned}
\lambda_3(h) &= \prod_{\ell=1}^{n} c\Big(-1, \prod_{i \geq n-\ell+1} h_i \cdot \prod_{k \leq \ell} h_k \Big) \\
&= \prod_{\ell=1}^{n} \prod_{i \geq n-\ell+1} c(-1, h_i) \cdot \prod_{\ell=1}^{n} \prod_{k \leq \ell} c(-1, h_k) \\
&= \prod_{\ell=1}^{n} \prod_{i \geq \ell} c(-1, h_i) \cdot \prod_{\ell=1}^{n} \prod_{k \leq \ell} c(-1, h_k) \\
&= \prod_{\ell=1}^{n} c(-1, h_\ell) \cdot \prod_{\ell=1}^{n} \prod_{j=1}^{n} c(-1, h_j) \\
&= c(-1, \det(h)) \cdot c(-1, \det(h))^n \\
&= c(-1, \det(h))^{n+1} .
\end{aligned}$$

Turning now to $\lambda_1(h)$ we have, with $i,j \in [1,n+1]$ and $k,\ell \in [1,n]$,

$$\begin{aligned}
\lambda_1(h) &= \kappa(\eta(h))^{-1} \\
&= \prod_{i>j} c\big(\eta(h)_i, \eta(h)_j\big)^{-1} \\
&= \prod_{n+1>i>j} c(h_i, h_j)^{-1} \cdot \prod_{j=1}^{n} c(\det(h)^{-1}, h_j)^{-1} \\
&= \prod_{k>\ell} c(h_k, h_\ell)^{-1} \cdot c(\det(h), \det(h)) \\
&= c(-1, \det(h)) \cdot \prod_{k>\ell} c(h_k, h_\ell)^{-1} .
\end{aligned}$$

Thus

$$\lambda(h) = c(-1, \det(h))^n \cdot \prod_{k>\ell} c(h_k, h_\ell)^{-1}$$

and it follows that there is a lift of ι satisfying

$$^{\iota}\mathbf{s}(h) = c(-1, \det(h))^n \cdot \prod_{k>\ell} c(h_k, h_\ell)\, \mathbf{s}(^{\iota}h) . \tag{6}$$

To obtain the formula given in the statement it is only necessary to observe that according to Lemma 2 the group $\mathrm{Hom}(\mathrm{GL}(n), A)$ acts on the set of lifts of ι and applying the element $\varphi(g) = c(-1, \det(g))^n$ of this group to the lift of ι satisfying (6) gives another answering the requirements of the Proposition. □

Proposition 3: *Every lift of the main involution of* $\mathrm{GL}(n)$ *to* $\widetilde{\mathrm{GL}}(n)$ *is itself an involution.*

Proof: Let $\varphi \in \mathrm{Hom}(\mathrm{GL}(n), A)$. Since A is abelian and $[\mathrm{GL}(n), \mathrm{GL}(n)] = \mathrm{SL}(n)$ (recall that we are assuming the underlying field to be infinite) the map φ must have the form $\varphi(g) = \psi(\det(g))$ for some $\psi \in \mathrm{Hom}(F^\times, A)$. In particular it follows that $\varphi(g)\varphi({}^\iota g) = 1$ for all $g \in \mathrm{GL}(n)$. Using Lemma 2 it now follows that every lift of ι is an involution if and only if one of them is. We may thus restrict attention to the lift which was singled out in Proposition 2. We denote this particular lift by ι. The map $\widetilde{g} \mapsto {}^\iota({}^\iota\widetilde{g})$ is a lift of the identity map and so Lemma 2 implies that there is some $\varphi \in \mathrm{Hom}(\mathrm{GL}(n), A)$ such that ${}^\iota({}^\iota\widetilde{g}) = \varphi(p(\widetilde{g}))\widetilde{g}$. With $\psi \in \mathrm{Hom}(F^\times, A)$ as before this is equivalent to ${}^\iota({}^\iota\widetilde{g}) = \psi(\det(\widetilde{g}))\widetilde{g}$. But if $h = \mathrm{diag}(x, 1, \ldots, 1)$ for $x \in F^\times$ then the formula of Proposition 2 gives ${}^\iota\mathbf{s}(h) = \mathbf{s}({}^\iota h)$ and then ${}^\iota({}^\iota\mathbf{s}(h)) = {}^\iota\mathbf{s}({}^\iota h) = \mathbf{s}(h)$ from which it follows that $\psi = 1$. Hence ${}^\iota({}^\iota\widetilde{g}) = \widetilde{g}$, as required.

□

We shall refer to the lift of ι singled out in Proposition 2 as the *main involution* of $\widetilde{\mathrm{GL}}(n)$ and denote it again by ι.

Proposition 4: *If $z \in p^{-1}(\{\lambda I_n \mid \lambda \in F^\times\})$ then ${}^\iota z = z^{-1}$.*

Proof: It suffices to show that ${}^\iota\mathbf{s}(\lambda I_n) = \mathbf{s}(\lambda I_n)^{-1}$ for all $\lambda \in F^\times$. Using the formula in Proposition 2 and (3) of section 3 we have

$$\begin{aligned}
{}^\iota\mathbf{s}(\lambda I_n)\mathbf{s}(\lambda I_n) &= \prod_{i>j} c(\lambda, \lambda) \cdot \mathbf{s}(\lambda^{-1} I_n)\mathbf{s}(\lambda I_n) \\
&= c(\lambda, \lambda)^{n(n-1)/2}\sigma(\lambda^{-1} I_n, \lambda I_n) \\
&= c(\lambda, \lambda)^{n(n-1)/2} \prod_{i<j} c(\lambda^{-1}, \lambda) \\
&= \big(c(\lambda, \lambda)c(\lambda^{-1}, \lambda)\big)^{n(n-1)/2} \\
&= 1\,,
\end{aligned}$$

as required. □

Let us denote by N^+ the unipotent radical of the Borel subgroup of $\mathrm{GL}(n)$ which corresponds to the positive system Φ^+. If $\alpha \in \Phi^+$ then let $x_\alpha : (F, +) \to \mathrm{N}^+$ be the standard homomorphism whose image is the "root subgroup" N^α. Note that $\eta \circ x_\alpha : (F, +) \to \mathrm{SL}(n+1)$ is itself such a homomorphism with respect to some positive root of $\mathrm{SL}(n+1)$ and $\eta(\mathrm{N}^+)$ is a subgroup of the unipotent radical of the standard Borel in $\mathrm{SL}(n+1)$. Examining our construction of $\widetilde{\mathrm{GL}}(n)$ in the light of these remarks and of Lemme 5.1, Chapitre II of [Mat] and observing that the second and third factors of (1) of section 3 are identically 1 on $\mathrm{N}^+ \times \mathrm{N}^+$ we see that the section $\mathbf{s} : \mathrm{GL}(n) \to \widetilde{\mathrm{GL}}(n)$ may be chosen so that $\mathbf{s}|_{\mathrm{N}^+}$ is a homomorphism. We shall suppose below that this has been done.

Proposition 5: *If A has exponent m prime to the characteristic exponent of F then ${}^\iota\mathbf{s}(n) = \mathbf{s}({}^\iota n)$ for all $n \in \mathrm{N}^+$.*

Proof: With $\mathbf{s}$ chosen as above the map $n \mapsto {}^\iota\mathbf{s}(n)\mathbf{s}({}^\iota n)^{-1}$ is an element of the group $\mathrm{Hom}(\mathrm{N}^+, A)$ and so it suffices to show that this group is trivial. Let $\zeta \in \mathrm{Hom}(\mathrm{N}^+, A)$ and $\alpha \in \Phi^+$. Then for all $t \in F$ we have

$$\begin{aligned} \zeta(x_\alpha(t)) &= \zeta(x_\alpha(m^{-1}t)^m) \\ &= \zeta(x_\alpha(m^{-1}t))^m \\ &= 1 \end{aligned}$$

and so $\zeta(\mathrm{N}^\alpha) = \{1\}$. But the N^α with $\alpha \in \Phi^+$ generate N^+ and this completes the proof. □

5. Topological Considerations

Up to this point our discussion of the metaplectic groups has proceeded in almost complete generality; the only condition we have yet imposed is that the underlying field be infinite. In this section we shall come closer to the situation which will concern us in later chapters by imposing additional assumptions of a topological and arithmetic nature on F, A and c.

If G, A and $\widetilde{G}$ are Hausdorff topological groups then an extension

$$1 \to A \to \widetilde{G} \xrightarrow{p} G \to 1$$

is called *topological* if p is continuous and open and the inclusion map $A \hookrightarrow \widetilde{G}$ is continuous and closed. The following two results record some useful general facts about topological extensions of ℓ–groups.

Proposition 1: *Suppose that*

$$1 \to A \to \widetilde{G} \xrightarrow{p} G \to 1$$

is a topological extension of Hausdorff topological groups. If A and G are ℓ–groups then $\widetilde{G}$ is an ℓ–group. If in addition A is discrete then there is a compact open subgroup of G over which the sequence is split and a continuous section $\mathbf{s} : G \to \widetilde{G}$.

Proof: Suppose that A and G are ℓ–groups. We first observe that $\widetilde{G}$ is locally compact. Indeed it suffices to find a compact neighbourhood of the identity in $\widetilde{G}$. Let us choose compact open subgroups $K_1 \leq G$ and $G_2 \leq A$. Then $K = p^{-1}(K_1)$ is an open subgroup of $\widetilde{G}$ with K/K_2 homeomorphic to K_1 and hence compact. By I), section 19, Chapter 3 of [Pon] it follows that K is compact, as required. It is easy to check that since G and A are both totally disconnected, $\widetilde{G}$ is also totally disconnected. Applying Theorem 16, section 22, Chapter 3 of [Pon] we conclude that $\widetilde{G}$ is in fact an $\ell-$group.

If A is discrete then since the inclusion map $A \hookrightarrow \widetilde{G}$ is a homeomorphism of A onto its image there is an open set $W \subseteq \widetilde{G}$ with $A \cap W = \{e\}$. Since $\widetilde{G}$ is an $\ell-$group we may find a compact open subgroup $\widetilde{U}$ of $\widetilde{G}$ with $\widetilde{U} \subseteq W$. Then $\widetilde{U} \cap A = \{e\}$ and so if we set $U = p(\widetilde{U})$ then $p|_{\widetilde{U}} : \widetilde{U} \to U$ is an isomorphism of topological groups. The map $(p|_{\widetilde{U}})^{-1}$ then splits the sequence over the subgroup U.

Let S be a left transversal for U in G and for each $s \in S$ choose any $\widetilde{s} \in \widetilde{G}$ such that $p(\widetilde{s}) = s$. If we define $\mathbf{s} : G \to \widetilde{G}$ by $\mathbf{s}(su) = \widetilde{s}(p|_{\widetilde{U}})^{-1}(u)$ for $s \in S$ and $u \in U$ then $\mathbf{s}$ is a continuous section of p. $\square$

Propostion 2: *Let*

$$1 \to A \to \widetilde{G} \xrightarrow{p} G \to 1$$

be a topological central extension of $\ell-$groups with A discrete and of exponent m. Suppose that G has a neighbourhood base at the identity consisting of compact open subgroups U such that U^m is open, where U^m denotes the group generated by $\{u^m \mid u \in U\}$. If $f \in \mathrm{Aut}(G)$ is a homeomorphism then any lift of f is also a homeomorphism.

Proof: Using Proposition 1 we may find a compact open subgroup U_0 of G and a splitting $\psi : U_0 \to \psi(U_0)$. Using the hypotheses on f and G we may find a compact open subgroup $V_0 \subseteq U_0$ such that $f(V_0) \subseteq U_0$ and V_0^m is open. Let $\widetilde{f}$ be a lift of f and define $\zeta : p^{-1}(V_0) \to p^{-1}(V_0)$ by $\zeta = \widetilde{f}^{-1} \circ \psi \circ f \circ p$. Then $p \circ \zeta = p$ and so ζ is a lift of the identity map on V_0. From Lemma 2 of section 4 it follows that ζ differs from the identity map by the action of some element of $\mathrm{Hom}(V_0, A)$. But any such homomorphism is trivial on V_0^m and so $\zeta|_{p^{-1}(V_0^m)} = \mathrm{id}_{p^{-1}(V_0^m)}$. Composing this equation on the left with $\widetilde{f}$ we obtain $\widetilde{f} = \psi \circ f \circ p$ on $p^{-1}(V_0^m)$ and hence on $\psi(V_0^m)$. Now both $\psi(V_0^m)$ and $\psi(f(V_0^m))$ are neighbourhoods of the identity in $\widetilde{G}$ and it follows from what we have just done that $\widetilde{f} : \psi(V_0^m) \to \psi(f(V_0^m))$ is a homeomorphism. Since $\widetilde{f}$ is also an automorphism and the topology on $\widetilde{G}$ is homogeneous the claim follows. $\square$

Let us now assume that F is a local field, c is the m^{th} order Hilbert symbol on F and $A = \mu_m$, the group of m^{th} roots of unity in F, which because of the assumptions necessary to define the Hilbert symbol is a cyclic group of order m. When necessary we regard A as a topological group with the discrete topology. In [Mat] Matsumoto determines the exact condition which must be placed on a Steinberg symbol in order to make the corresponding central extension topological (see [Mat], Théorème 8.2 and also [Mil], Assertion 11.4). In the subsequent discussion Matsumoto observes that the Hilbert symbols satisfy the necessary condition and hence the extension (1) of section 2 is topological with the natural topology on $\mathrm{SL}(n, F)$. Since the map η introduced at the beginning of section 2 is a homeomorphism onto its image it follows that the extension (2) of that section is also topological. The 2−cocycle $\tau_0(g_1, g_2) = c(\det(g_1), \det(g_2))$ on $\mathrm{GL}(n, F)$ is easily seen to correspond to a topological extension; indeed the extension is split over the open subgroup $\{g \in \mathrm{GL}(n, F) \mid \det(g) \in (F^\times)^m\}$ which suffices for the claim. It follows from the results of Moore in [Mo1] that the Baer product and Baer inverse of topological extensions of locally compact groups are again topological. Hence the metaplectic extension (2) of section 3 is toplogical with the natural topology on $\mathrm{GL}(n, F)$.

It follows from Proposition 1 that when F is non-Archimedean the metaplectic group is an ℓ−group and the metaplectic extension is split over some compact open subgroup of $\mathrm{GL}(n, F)$. This justifies the first assertion of [KaP], Proposition 0.1.2 which, contrary to their claim, does not seem to be proved in [Mo2]. Furthermore $\mathrm{GL}(n, F)$ is easily seen to satisfy the hypotheses of Proposition 2 and it follows that the main involution of $\widetilde{\mathrm{GL}}(n)$ is a homeomorphism in this case. When F is Archimedean both $\mathrm{GL}(n)$ and $\widetilde{\mathrm{GL}}(n)$ are Lie groups. Only the case $F = \mathbb{R}$, $m = 2$ is interesting since otherwise $\widetilde{\mathrm{GL}}(n)$ is merely the direct product of $\mathrm{GL}(n)$ and A. In this case one may combine Propositions 2 and 5 of the previous section with the standard theory of such groups to see that the main involution of $\widetilde{\mathrm{GL}}(n)$ is an analytic automorphism.

CHAPTER 2

THE EXCEPTIONAL REPRESENTATIONS

1. Metaplectic Tensor Products

Let $\widetilde{\mathrm{G}}(r)$ be the metaplectic n-fold cover of $\mathrm{G}(r) = \mathrm{GL}(r)$ corresponding to the n^{th} order Hilbert symbol $(\cdot\,,\,\cdot)$ on a non-Archimedean local field F satisfying $|\mu_n(F)| = n$, where $\mu_n(F)$ is the group of n^{th} roots of unity in F. We denote the projection homomorphism by $p_r : \widetilde{\mathrm{G}}(r) \to \mathrm{G}(r)$. For $\gamma = (r_1, \ldots, r_k)$ with $r_1, \ldots, r_k \geq 1$ we put $\widetilde{\mathrm{G}}(\gamma) = p_r^{-1}(\mathrm{G}(\gamma))$ where $r = r_1 + \cdots + r_k = |\gamma|$ and $\mathrm{G}(\gamma) = \mathrm{G}(r_1) \times \cdots \times \mathrm{G}(r_k)$ is embedded in $\mathrm{G}(r)$ in the standard way. If $g \in \widetilde{\mathrm{G}}(\gamma)$ then we set $\det(g) = \det(p_r(g))$. For $H_1 \leq \widetilde{\mathrm{G}}(r)$ and $H_2 \leq \widetilde{\mathrm{G}}(s)$ satisfying $H_1 = p_r^{-1}(p_r(H_1))$ and $H_2 = p_s^{-1}(p_s(H_2))$ we define

$$H_1 \mathbin{\widetilde{\times}} H_2 = p_{r+s}^{-1}(p_r(H_1) \times p_s(H_2))$$

and similarly for more than two factors. With this definition we have

$$\widetilde{\mathrm{G}}(\gamma) = \widetilde{\mathrm{G}}(r_1) \mathbin{\widetilde{\times}} \ldots \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}(r_k)$$

and an easy calculation shows that if $g \in \widetilde{\mathrm{G}}(\gamma_1)$ and $g' \in \widetilde{\mathrm{G}}(\gamma_2)$ then

$$gg' = (\det(g), \det(g'))\, g'g$$

in $\widetilde{\mathrm{G}}(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}(\gamma_2)$.

If π_1 (resp. π_2) is a genuine admissible representation of $\widetilde{\mathrm{G}}(\gamma_1)$ (resp. $\widetilde{\mathrm{G}}(\gamma_2)$) of finite length then we aim to define a "tensor product" $\pi_1 \widetilde{\otimes} \pi_2$ which is to be a representation of $\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}(\gamma_2)$. The difficulty is that $\widetilde{\mathrm{G}}(\gamma_1)$ and $\widetilde{\mathrm{G}}(\gamma_2)$ do not commute and so some care is necessary. In [Hua] Huang addressed this problem for the real metaplectic group in the case where π_1 and π_2 are irreducible and in [FlK], §26.2 the problem is briefly discussed over a non-Archimedean local field, again for irreducible representations. Note that this latter reference contains some inaccuracies. We shall solve the problem in this section for the case where $n = 2$, which is our focus in later sections. The main ingredient will be a study of the decomposition of representations of $\widetilde{\mathrm{G}}(r)$ on restriction to certain subgroups of finite index, which will be carried out in somewhat greater generality.

We shall require several preliminaries of a technical nature; they must be well-known to algebraists but I have been unable to locate a suitable reference. Fix an algebraically closed field K (we shall eventually take $K = \mathbb{C}$). Actually algebraic closure is unnecessary for much of what we shall say but we have no use for the possible extra generality.

Definition 1: *Let A be a $K-$algebra. We call A local if $A \setminus A^{\times}$ is a two-sided ideal of A and strongly local if every element of A is either a unit or else is nilpotent.*

It is easy to check that every stongly local algebra is local. Also A is local if and only if $J(A) = A \setminus A^{\times}$ where $J(A)$ denotes the Jacobson radical. If A is local and Artinian then the radical of A is nilpotent and consequently A is strongly local. Thus local and strongly local coincide for Artinian algebras.

Lemma 1: *Let A be a $K-$algebra and E an $A-$module of finite length. Then E is indecomposable if and only if the algebra* $\mathrm{End}_A(E)$ *is strongly local.*

Proof: That the endomorphism algebra is strongly local when E is indecomposable is proved in [Lan] VI.9.4. Conversely it is immediate that a strongly local algebra cannot contain non-trivial idempotents. $\square$

Lemma 2: *Let A_1 and A_2 be finite-dimensional local $K-$algebras. Then $A_1 \otimes A_2$ is also local, where the tensor product is taken over K.*

Proof: Let $I_1 = A_1 \setminus A_1^\times$ and $I_2 = A_2 \setminus A_2^\times$. The $K-$algebra A_1/I_1 is a finite-dimensional division algebra over K and hence $A_1/I_1 \cong K$. Similarly $A_2/I_2 \cong K$. We define

$$I = A_1 \otimes I_2 + I_1 \otimes A_2$$

which is a two-sided ideal of $A = A_1 \otimes A_2$. Then

$$\begin{aligned} A/I &= \big(A_1 \otimes A_2\big)/\big(A_1 \otimes I_2 + I_1 \otimes A_2\big) \\ &\cong \big(A_1 \otimes (A_2/I_2)\big)/\big(I_1 \otimes (A_2/I_2)\big) \\ &\cong A_1/I_1 \otimes A_2/I_2 \\ &\cong K \end{aligned}$$

and it follows that $I = A \setminus A^\times$. Thus A is local. $\square$

Lemma 3: *Let A_1 and A_2 be $K-$algebras and D_1 and E_1 (resp. D_2 and E_2) be finite-dimensional A_1- (resp. A_2-) modules. Then*

$$\mathrm{Hom}_{A_1\otimes A_2}(D_1\otimes D_2, E_1\otimes E_2)\cong \mathrm{Hom}_{A_1}(D_1,E_1)\otimes \mathrm{Hom}_{A_2}(D_2,E_2)\,.$$

Proof: The natural map

$$\mathrm{Hom}_K(D_1,E_1)\otimes \mathrm{Hom}_K(D_2,E_2)\to \mathrm{Hom}_K(D_1\otimes D_2,E_1\otimes E_2)$$

is an isomorphism which carries

$$\mathrm{Hom}_{A_1}(D_1,E_1)\otimes \mathrm{Hom}_{A_2}(D_2,E_2)$$

into

$$\mathrm{Hom}_{A_1\otimes A_2}(D_1\otimes D_2,E_1\otimes E_2)\,.$$

Thus only its surjectivity is in question. Let $f\in \mathrm{Hom}_{A_1\otimes A_2}(D_1\otimes D_2,E_1\otimes E_2)$ and write

$$f=\sum_{i=1}^{m}\varphi_i\otimes\psi_i$$

with $\varphi_i\in \mathrm{Hom}_K(D_1,E_1)$ and $\psi_i\in \mathrm{Hom}_K(D_2,E_2)$. Rewriting this sum we may assume that the set $\{\psi_i\}$ is linearly independent. Fix $1\le q\le m$; since $\mathrm{Hom}_K(D_2,E_2)^*\cong D_2\otimes E_2^*$ it follows that there exist finite sets $\{\lambda_s\}\subseteq E_2^*$ and $\{\xi_t\}\subseteq D_2$ such that

$$\sum_{s,t}\lambda_s\big(\psi_i(\xi_t)\big)=\delta_{iq}$$

for all $1 \leq i \leq m$. Because f is an $(A_1 \otimes A_2)-$homomorphism the map

$$\zeta \mapsto \sum_{s,t} m_1(\mathrm{id}_{E_1} \otimes \lambda_s)\big[f(\zeta \otimes \xi_t)\big]$$

lies in $\mathrm{Hom}_{A_1}(D_1, E_1)$ where $m_1 : E_1 \otimes K \to E_1$ is the natural isomorphism. On the other hand

$$\begin{aligned}
&\sum_{s,t} m_1(\mathrm{id}_{E_1} \otimes \lambda_s)\big[f(\zeta \otimes \xi_t)\big] \\
=&\sum_{s,t,i} m_1(\mathrm{id}_{E_1} \otimes \lambda_s)(\varphi_i \otimes \psi_i)[\zeta \otimes \xi_t] \\
=&\sum_{s,t,i} m_1\big(\varphi_i(\zeta) \otimes \lambda_s(\psi_i(\xi_t))\big) \\
=&\ m_1\big(\varphi_q(\zeta) \otimes 1\big) \\
=&\ \varphi_q(\zeta)
\end{aligned}$$

and so $\varphi_q \in \mathrm{Hom}_{A_1}(D_1, E_1)$.

We may find a set $I \subseteq \{1, \ldots, m\}$ such that $\{\varphi_i \,|\, i \in I\}$ is linearly independent and $\{\varphi_i\} \subseteq \mathrm{span}_K\{\varphi_i \,|\, i \in I\}$. This done we may write

$$f = \sum_{i \in I} \varphi_i \otimes \psi_i'$$

for certain $\psi_i' \in \mathrm{Hom}_K(D_2, E_2)$. Arguing as before we see that

$$\psi_i' \in \mathrm{Hom}_{A_2}(D_2, E_2)$$

for all $i \in I$, as required. $\square$

Lemma 4: *If A_1 and A_2 are $K-$algebras and E_1 (respectively E_2) is a finite-dimensional indecomposable A_1- (respectively A_2-) module then $E_1 \otimes E_2$ is an* *indecomposable* $(A_1 \otimes A_2)-$*module.*

Proof: The finite-dimensional $K-$algebras $\text{End}_{A_1}(E_1)$ and $\text{End}_{A_2}(E_2)$ are both strongly local (Lemma 1) and hence local. Thus $\text{End}_{A_1 \otimes A_2}(E_1 \otimes E_2)$ is the tensor product of finite-dimensional local algebras (Lemma 3) and so is local (Lemma 2). Being finite-dimensional it is Artinian and hence strongly local. Thus (Lemma 1) $E_1 \otimes E_2$ is indecomposable. $\square$

We now recall some notation from [BZ1]. Let G be an $\ell-$group and N a compact open subgroup of G. We denote by $\mathcal{H}_N$ the Hecke algebra of bi$-N-$invariant compactly supported distributions on G. If π is an algebraic representation of G on the complex vector space E then we denote by π_N the representation of $\mathcal{H}_N$ on the space E^N of $N-$fixed vectors which corresponds to π.

Lemma 5: *Let* π *be an algebraic representation of* G *of finite length. Then* π *is indecomposable if and only if* π_N *is indecomposable for some sufficiently small open compact subgroup* N*.*

Proof: Suppose that N is so small that if ρ is any non-zero subquotient of π then $\rho_N \neq 0$. There are such N since π is assumed to be of finite length. If π is decomposable then it follows at once that π_N is decomposable. Now assume that π_N is decomposable and that $E^N = V_1 \oplus V_2$ where V_1 and V_2 are non-zero $\mathcal{H}_N-$submodules of E^N. Let E_i be the $G-$submodule of E generated by V_i. It follows from the arguments of [BZ1], §2.10 that $E_i^N = V_i$ and so $(E_1 \cap E_2)^N \subseteq V_1 \cap V_2 = \{0\}$ and $(E_1 + E_2)^N = V_1 + V_2 = E^N$. Recalling our assumption on N this means that $E_1 \cap E_2 = \{0\}$ and $E = E_1 + E_2$. Thus π is decomposable. $\square$

Proposition 1: *Let G_1 and G_2 be $\ell-$groups and π_1 (resp. π_2) be an indecomposable admissible representation of G_1 (resp. G_2) of finite length. Then $\pi = \pi_1 \otimes \pi_2$ is an indecomposable representation of $G_1 \times G_2$.*

Proof: For all sufficiently small compact open subgroups N_1 of G_1 and N_2 of G_2 we know from Lemma 5 that $\pi_1^{N_1}$ and $\pi_2^{N_2}$ are finite-dimensional indecomposable modules for $\mathcal{H}_{N_1}$ and $\mathcal{H}_{N_2}$ respectively. Thus with $N = N_1 \times N_2$ we see that $\pi^N \cong \pi_1^{N_1} \otimes \pi_2^{N_2}$ is an indecomposable $\mathcal{H}_N \cong \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$ module from Lemma 4. Since N may be made as small as required by making N_1 and N_2 small and π is admissible and of finite length the claim follows by a second application of Lemma 5. $\square$

Proposition 2: *Let G_1 and G_2 be $\ell-$groups and π_1 and ρ_1 (resp. π_2 and ρ_2) be admissible representations of G_1 (resp. G_2). Then*

$$\mathrm{Hom}_{G_1 \times G_2}(\pi_1 \otimes \pi_2, \rho_1 \otimes \rho_2) \cong \mathrm{Hom}_{G_1}(\pi_1, \rho_1) \otimes \mathrm{Hom}_{G_2}(\pi_2, \rho_2)\,.$$

Proof: It is routine to check that since all the representations are assumed algebraic we have

$$E_{\pi_1} \cong \varinjlim_{N_1} E_{\pi_1}^{N_1} \qquad\qquad E_{\rho_1} \cong \varinjlim_{N_1} E_{\rho_1}^{N_1}$$

and

$$\mathrm{Hom}_{G_1}(E_{\pi_1}, E_{\rho_1}) \cong \varinjlim_{N_1} \mathrm{Hom}_{\mathcal{H}_{N_1}}(E_{\pi_1}^{N_1}, E_{\rho_1}^{N_1})$$

where the limits are taken over the system of compact open subgroups of G_1 and similarly for G_2 and its representations. Combining the admissibility of the representations, Lemma 3 and the fact that direct limits commute with tensor products proves the Proposition. □

Definition 2: *If π is a representation of a group G with center Z and ω is a character of Z then we say that π admits ω if there is a non-zero subquotient of π on which Z acts via ω.*

Suppose that G is an $\ell-$group with center Z and π is an admissible representation of G of finite length. Let E_π denote the space of π and for each $\omega \in Z^\frown$ put

$$E_\pi(\omega) = \{\xi \in E_\pi \,|\, \exists m \geq 1 \text{ such that } (\pi(z) - \omega(z))^m \xi = 0 \;\forall z \in Z\}.$$

It follows easily from Schur's Lemma and Fitting's Lemma that $E_\pi = \oplus_\omega E_\pi(\omega)$ the sum being taken over the finite set of ω which are admitted by π. Moreover Z acts on every irreducible subquotient of $E_\pi(\omega)$ via ω and the set of characters admitted by π is the same as the set admitted by its socle. In particular, if π is indecomposable then π admits one and only one character of Z which we shall denote by ω_π.

We now need a little Clifford theory in a slightly more general setting than is usual. So let G be a group and H a normal subgroup of finite index. Recall that restriction to H and induction from H both preserve the condition of being of finite length (see [BZ1], §2.9).

Lemma 6: *Let π be an indecomposable representation of G. Let σ be any summand of $\pi|_H$ and suppose that if $g \notin H$ then*

$$\mathrm{Hom}_H({}^g\sigma, \sigma) = \{0\}$$

where ${}^g\sigma$ denotes the conjugate representation ${}^g\sigma(h) = \sigma(g^{-1}hg)$. Then $\pi \cong \mathrm{ind}_H^G(\sigma)$. Conversely if σ is an indecomposable representation of H of finite length which satisfies the above condition then $\mathrm{ind}_H^G(\sigma)$ is indecomposable.

Proof: The hypotheses imply that we may write $E_\pi = E_\sigma \oplus D$ where D is an $H-$submodule of E_π. Let $q : E_\pi \to E_\sigma$ be the corresponding projection. We claim that if $\xi \in E_\sigma$ and $g \notin H$ then $q(\pi(g)\xi) = 0$. Indeed the map $\xi \mapsto q(\pi(g)\xi)$ is an $H-$intertwining operator from ${}^g\sigma$ to σ and so is zero by hypothesis.

We define maps $T : \pi \to \mathrm{ind}_H^G(\sigma)$ and $S : \mathrm{ind}_H^G(\sigma) \to \pi$ by

$$(T\xi)(g) = q(\pi(g)\xi)$$

and

$$S(f) = \sum_{g \in H\backslash G} \pi(g^{-1})f(g)\,.$$

One checks that these are well-defined and G–intertwining. If $f \in \mathrm{ind}_H^G(E_\sigma)$ and $g_0 \in G$ then

$$\begin{aligned}
&(T \circ S)(f)(g_0) \\
&= q\big(\pi(g_0)S(f)\big) \\
&= q\big(\pi(g_0) \sum_{g \in H\backslash G} \pi(g^{-1})f(g)\big) \\
&= \sum_{g \in H\backslash G} q\big(\pi(g_0 g^{-1})f(g)\big) \\
&= q\big(\pi(g_0 g^{-1})f(g)\big) \qquad \text{where } g_0 g^{-1} \in H \\
&= \sigma(g_0 g^{-1})f(g) \\
&= f(g_0)
\end{aligned}$$

and so $T \circ S$ is the identity. From [Lan] Lemma VI.9.6 it follows that both T and S are isomorphisms. We now turn to the second statement. Combining the Mackey subgroup theorem, Frobenius Reciprocity and the hypothesis on σ we obtain

$$\mathrm{End}_G\big(\mathrm{ind}_H^G(\sigma)\big) \cong \mathrm{End}_H(\sigma)\,.$$

Now two applications of Lemma 1 complete the proof. $\square$

Returning now to the metaplectic groups we define

$$\widetilde{\mathrm{G}}^m(\gamma) = \{g \in \widetilde{\mathrm{G}}(\gamma) \mid \det(g) \in \big(F^\times\big)^m\};$$

this is a normal subgroup of $\widetilde{\mathrm{G}}(\gamma)$ and $\widetilde{\mathrm{G}}(\gamma)/\widetilde{\mathrm{G}}^m(\gamma) \cong F^\times/(F^\times)^m$. If π is a representation of a subgroup of $\widetilde{\mathrm{G}}(\gamma)$ containing $\widetilde{\mathrm{G}}^m(\gamma)$ then we shall denote by

π^m the restriction of π to $\widetilde{\mathrm{G}}^m(\gamma)$. We also set $\mathrm{Z}^m(r) = \{tI_r \mid t \in (F^\times)^m\}$, $\widetilde{\mathrm{Z}}^m(r) = p_r^{-1}(\mathrm{Z}^m(r))$, $\mathrm{Z}(r) = \mathrm{Z}^{n/(n,r-1)}(r)$ and $\widetilde{\mathrm{Z}}(r) = p_r^{-1}(\mathrm{Z}(r))$. It is shown in [KaP] §0.1.1 that

$$p_r^{-1}\left(\{\lambda I_r \mid \lambda^{r-1} \in (F^\times)^n\}\right)$$

is the center of $\widetilde{\mathrm{G}}(r)$ and one easily checks that this equals $\widetilde{\mathrm{Z}}(r)$.

It will be useful to identify the finite-dimensional admissible representations of $\widetilde{\mathrm{Z}}^m(r)$. For this purpose let us recall some well-known facts about the finite-dimensional admissible representations of $\mathrm{GL}(1, F)$. Let the symbol V_p denote the $\mathrm{GL}(1)$ −submodule of $C^\infty(F^\times)$ generated by the function $x \mapsto \mathrm{ord}_F(x)^p$ where $\mathrm{ord}_F : F^\times \to \mathbb{Z}$ is defined by $|x|_F = q^{-\mathrm{ord}_F(x)}$, q being the module of F. Denote by π_p the representation of $\mathrm{GL}(1)$ afforded by V_p. Then $\{\pi_p \mid p \geq 0\}$ is a complete set of representatives for the isomorphism classes of finite-dimensional admissible indecomposable representations of $\mathrm{GL}(1)$ which admit the trivial character. If π is such a representation of $\mathrm{GL}^m(1)$ then one easily sees that the units in $\mathrm{GL}^m(1)$ act trivially under π and hence that π extends to a representation of $\mathrm{GL}(1)$. It follows that π is isomorphic to the restriction of π_p for some p and all these restrictions are indecomposable. Thus the same classification holds for $\mathrm{GL}^m(1)$ as holds for $\mathrm{GL}(1)$. Evidently if π_p is restricted from $\mathrm{GL}^m(1)$ to $\mathrm{GL}^{ma}(1)$, $a \geq 1$, it remains indecomposable, and if $v \in V_p$ is a cyclic vector for π_p as a $\mathrm{GL}^m(1)$ −module then it is also a cyclic vector for π_p restricted to the smaller group.

Suppose now that ω is a character of $\widetilde{\mathrm{Z}}^m(r)$ and that ρ is a finite-dimensional admissible indecomposable representation of $\widetilde{\mathrm{Z}}^m(r)$ with $\omega_\rho = \omega$. Using the observations in the previous paragraph one sees that $\rho \cong \omega \otimes \pi_p$ for some $p \geq 0$ where π_p is regarded as a (non-genuine) representation of $\widetilde{\mathrm{Z}}^m(r)$. Also $\rho|_{\widetilde{\mathrm{Z}}^{ma}(r)}$ is indecomposable and a cyclic vector under $\widetilde{\mathrm{Z}}^m(r)$ remains cyclic under $\widetilde{\mathrm{Z}}^{ma}(r)$. Furthermore any finite-dimensional admissible representation of $\widetilde{\mathrm{Z}}^m(r)$ is a direct sum of certain $\omega \otimes \pi_p$ and it easily follows from this that if such a representation has a cyclic vector and admits only one character then it must be indecomposable.

At this point we make the assumption that $n \mid r(r-1)$. This assumption will be in force for the rest of this section and given it we factor n as $n = s \cdot (n/s)$ where $s \mid (r-1)$ and $(n/s) \mid r$. It is unlikely that this hypothesis has any real significance for the results to be proved; however it seems to represent the limit of the elementary methods used in the proofs. It already covers the two special cases of greatest interest, namely $n = 2$ and $n = r$.

Proposition 3: *Let $r = |\gamma|$, $a = 1$ or n/s and let π be an admissible indecomposable representation of $\widetilde{\mathrm{G}}^a(\gamma)$ of finite length. Then π^{sa} is indecomposable and*

$$\{\chi \in (\widetilde{\mathrm{G}}^a(\gamma)/\widetilde{\mathrm{G}}^{sa}(\gamma))^\wedge \mid \mathrm{Hom}_{\widetilde{\mathrm{G}}^a(\gamma)}(\chi \otimes \pi, \pi) \neq \{0\}\} = \{\chi_0\}$$

where χ_0 denotes the trivial character. Moreover

$$\mathrm{ind}_{\widetilde{\mathrm{G}}^{sa}(\gamma)}^{\widetilde{\mathrm{G}}^a(\gamma)}(\pi^{sa}) \cong \bigoplus_{\chi \in (\widetilde{\mathrm{G}}^a(\gamma)/\widetilde{\mathrm{G}}^{sa}(\gamma))^\wedge} \chi \otimes \pi\,.$$

Proof: Note that $\widetilde{\mathrm{Z}}(r) = \widetilde{\mathrm{Z}}^{n/s}(r) \subseteq \widetilde{\mathrm{G}}^a(\gamma)$ since $s = (n, r-1)$. For any $t \in F^\times$ we have $\mathbf{s}(t^{n/s}I_r) \in \widetilde{\mathrm{Z}}(r)$. Also $\det(\mathbf{s}(t^{n/s}I_r)) = t^{rn/s}$ and since $s \mid (r-1)$ it follows that s and rn/s are relatively prime. Hence we may choose coset representatives for $\widetilde{\mathrm{G}}^a(\gamma)/\widetilde{\mathrm{G}}^{sa}(\gamma)$ from $\widetilde{\mathrm{Z}}(r)$. The character admitted by $\chi \otimes \pi$ is $\chi|_{\widetilde{\mathrm{Z}}(r)} \cdot \omega_\pi$ and this equals ω_π if and only if $\chi|_{\widetilde{\mathrm{Z}}(r)}$ is trivial. These two observations combine to prove the second statement.

To show that π^{sa} is indecomposable, we shall in fact prove more. We claim that if D is a $\widetilde{\mathrm{G}}^{sa}(\gamma)$-submodule of E_π then D is stable under $\widetilde{\mathrm{G}}^a(\gamma)$. Suppose not; then D cannot be stable under $\pi(\widetilde{\mathrm{Z}}(r))$ and so we may choose $\xi \in D$ such that $V = \mathrm{span}_{\mathbb{C}}\{\pi(z)\xi \mid z \in \widetilde{\mathrm{Z}}(r)\}$ is not contained in D. There is a $p \geq 1$ such that $(\pi(z) - \omega_\pi(z))^p E_\pi = \{0\}$ and hence for every $z \in \widetilde{\mathrm{Z}}(r)$ the set $\{\pi(z^c) \mid c \in \mathbb{Z}\}$ spans a finite-dimensional subspace of $\mathrm{End}_{\widetilde{\mathrm{G}}^a(\gamma)}(E_\pi)$. Since π is admissible there is an $f \geq 1$ such that $\pi(z) = \zeta \cdot \mathrm{id}_{E_\pi}$ for some $\zeta \in \mu_n(\mathbb{C})$ if $p_r(z) \in (1 + \mathcal{P}_F^f)I_r$ where $\mathcal{P}_F$ denotes the prime ideal in $\mathcal{O}_F$. The group $F^\times/(1 + \mathcal{P}_F^f) \cdot I_r$ is finitely-generated and if we choose $z_1, \ldots, z_b \in \widetilde{\mathrm{Z}}(r)$ so that $\{p_r(z_1), \ldots, p_r(z_b)\}$ generates it then $\pi(\widetilde{\mathrm{Z}}(r))$ is generated by $\{\pi(z_1), \ldots, \pi(z_b)\}$ together with $\pi(\ker p_r)$, which is a finite set of scalar operators. Combining this with the previous observation we see that $\mathrm{span}_{\mathbb{C}}\{\pi(z) \mid z \in \widetilde{\mathrm{Z}}(r)\}$ is finite-dimensional and hence that V is finite-dimensional. Now V has a cyclic vector by definition and admits only the character ω_π. By the remarks made above on the representation theory of $\widetilde{\mathrm{Z}}^{n/s}(r)$ it follows first that V is indecomposable and secondly that $V = \mathrm{span}_{\mathbb{C}}\{\pi(z)\xi \mid z \in \widetilde{\mathrm{Z}}^n(r)\}$. But $\widetilde{\mathrm{Z}}^n(r) \subseteq \widetilde{\mathrm{G}}^{sa}(\gamma)$ and hence $V \subseteq D$, a contradiction. This proves the claim and so the first statement.

Finally the isomorphism is well-known, relying as it does simply on the fact that $\widetilde{G}^a(\gamma)/\widetilde{G}^{sa}(\gamma)$ is abelian. □

We remark that any divisor of s could replace s in the statement of the Proposition. This follows at once from Clifford theory and what has already been proved.

Proposition 4: *Let $r = |\gamma|$, $a = 1$ or s and π be a genuine admissible indecomposable representation of $\widetilde{G}^a(\gamma)$ of finite length. Suppose that σ is any of the indecomposable summands of $\pi^{na/s}$. Then*

$$\pi \cong \operatorname{ind}_{\widetilde{G}^{na/s}(\gamma)}^{\widetilde{G}^a(\gamma)}(\sigma),$$

$\chi \otimes \pi \cong \pi$ for all $\chi \in (\widetilde{G}^a(\gamma)/\widetilde{G}^{na/s}(\gamma))^\wedge$ and

$$\pi^{na/s} \cong \bigoplus_{g \in (\widetilde{G}^a(\gamma)/\widetilde{G}^{na/s}(\gamma))} {}^g\sigma .$$

Moreover if $g \notin \widetilde{G}^{na/s}(\gamma)$ then $\operatorname{Hom}_{\widetilde{G}^{na/s}(\gamma)}({}^g\sigma, \sigma) = \{0\}$.

Proof: Since $Z^1(r)$ is central in $G(r)$ the map from $F^\times \times G(r)$ to $\mu_n(F)$ given by

$$(t, \kappa) \mapsto [s(tI_r), s(\kappa)]$$

is bimultiplicative and hence there is a bicharacter ν of $F^\times$ such that

$$[s(tI_r), s(\kappa)] = \nu(t, \det(\kappa)).$$

A direct calculation using (3) of Chapter1, section 3 with $\kappa = \operatorname{diag}(x, 1, \ldots, 1)$ shows that $\nu(t, x) = (t, x)^{r-1}$ and hence

$$s(tI_r)g = (t, \det(g))^{r-1}\, g\, s(tI_r) \tag{1}$$

for every $t \in F^\times$ and $g \in \widetilde{\mathrm{G}}(r)$.

Now $\widetilde{\mathrm{Z}}^s(r) \subseteq \widetilde{\mathrm{G}}^{na/s}(\gamma)$ and in fact it follows from (1) that $\widetilde{\mathrm{Z}}^s(r)$ is central in $\widetilde{\mathrm{G}}^{na/s}(\gamma)$. Since σ is an indecomposable admissible representation of $\widetilde{\mathrm{G}}^{na/s}(\gamma)$ of finite length it admits a unique character ω_σ as above. Since π is genuine, so is σ and hence ω_σ. If $g \in \widetilde{\mathrm{G}}^a(\gamma)$ then for $t \in (F^\times)^s$ we have

$$\begin{aligned}\omega_{{}^g\sigma}(\mathbf{s}(tI_r)) &= \omega_\sigma(g^{-1}\mathbf{s}(tI_r)g)\\ &= \omega_\sigma((t,\det(g))^{r-1}\mathbf{s}(tI_r))\end{aligned}$$

on using (1). Thus if $\omega_{{}^g\sigma} = \omega_\sigma$ then $(t,\det(g))^{r-1} = 1$ for all $t \in (F^\times)^s$, which implies that $\det(g) \in (F^\times)^{n/s}$ by the non-degeneracy of the Hilbert symbol and the fact that $r-1$ and n/s are relatively prime. Thus

$$\det(g) \in (F^\times)^a \cap (F^\times)^{n/s} = (F^\times)^{na/s}$$

(recall that s and n/s are relatively prime) and so $g \in \widetilde{\mathrm{G}}^{na/s}(\gamma)$. From this we conclude that if $g \notin \widetilde{\mathrm{G}}^{na/s}(\gamma)$ then $\mathrm{Hom}_{\widetilde{\mathrm{G}}^{na/s}(\gamma)}({}^g\sigma,\sigma) = \{0\}$ and consequently an appeal to Lemma 6 shows that

$$\pi \cong \mathrm{ind}_{\widetilde{\mathrm{G}}^{na/s}(\gamma)}^{\widetilde{\mathrm{G}}^a(\gamma)}(\sigma)\,.$$

Now if $\chi \in (\widetilde{\mathrm{G}}^a(\gamma)/\widetilde{\mathrm{G}}^{na/s}(\gamma))^\wedge$ then

$$\begin{aligned}\chi\otimes\pi &\cong \chi\otimes\mathrm{ind}_{\widetilde{\mathrm{G}}^{na/s}(\gamma)}^{\widetilde{\mathrm{G}}^a(\gamma)}(\sigma)\\ &\cong \mathrm{ind}_{\widetilde{\mathrm{G}}^{na/s}(\gamma)}^{\widetilde{\mathrm{G}}^a(\gamma)}(\chi\otimes\sigma)\\ &= \mathrm{ind}_{\widetilde{\mathrm{G}}^{na/s}(\gamma)}^{\widetilde{\mathrm{G}}^a(\gamma)}(\sigma)\\ &\cong \pi\end{aligned}$$

giving the second claim. Finally the last isomorphism follows from the Mackey subgroup theorem. □

Note that when we speak of a representation being genuine there is an implicit choice of a faithful character of $\mu_n(F)$ (or equivalently of an isomorphism between $\mu_n(F)$ and $\mu_n(\mathbb{C})$) involved. For the preceding result it is irrelevant how that choice is made.

From now on we restrict to the case $n = 2$. If π is a genuine admissible indecomposable representation of $\widetilde{\mathrm{G}}\,(\gamma)$ of finite length then we wish to point out the consequences of Propositions 3 and 4 for π. If $r = |\gamma|$ is odd then $s = 2$ and so, according to Propostion 3, π remains indecomposable on restriction to $\widetilde{\mathrm{G}}^2\,(\gamma)$ and is not isomorphic to any of its twists by non-trivial characters of $\widetilde{\mathrm{G}}\,(\gamma)\,/\widetilde{\mathrm{G}}^2\,(\gamma)$. If $r = |\gamma|$ is even then $s = 1$ and so, according to Proposition 4, π decomposes as much as possible on restriction to $\widetilde{\mathrm{G}}^2\,(\gamma)$ and is isomorphic to all of its twists by characters of $\widetilde{\mathrm{G}}\,(\gamma)$ trivial on $\widetilde{\mathrm{G}}^2\,(\gamma)$. Much use will be made of these observations in what follows.

If $r, s \geq 1$ then $\widetilde{\mathrm{Z}}^2(r) \cong \widetilde{\mathrm{Z}}^2(s)$ via the map $\alpha : \widetilde{\mathrm{Z}}^2(r) \to \widetilde{\mathrm{Z}}^2(s)$ given by $\alpha(\mathbf{s}(tI_r)\epsilon) = \mathbf{s}(tI_s)\epsilon$ for $t \in F^\times$ and $\epsilon \in \mu_2$. (Note that $\widetilde{\mathrm{Z}}^1(r)$ and $\widetilde{\mathrm{Z}}^1(s)$ are not generally isomorphic.) Thus a character of $\widetilde{\mathrm{Z}}^2(r)$ may also be regarded as a character of $\widetilde{\mathrm{Z}}^2(s)$ via α and we shall do this without comment in what follows. The group $\widetilde{\mathrm{G}}^2\,(\gamma_1) \,\widetilde{\times}\, \widetilde{\mathrm{G}}^2\,(\gamma_2)$ is isomorphic to $(\widetilde{\mathrm{G}}^2\,(\gamma_1) \times \widetilde{\mathrm{G}}^2\,(\gamma_2))/B$ where $B = \{(\epsilon, \epsilon)\,|\,\epsilon \in \mu_2\}$. If π_1 is a genuine representation of $\widetilde{\mathrm{G}}\,(\gamma_1)$ and π_2 is a genuine representation of $\widetilde{\mathrm{G}}(\gamma_2)$ then B acts trivially on $\pi_1^2 \otimes \pi_2^2$ and so this may be regarded as a (genuine) representation of $\widetilde{\mathrm{G}}^2(\gamma_1) \tilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)$. If ω_1 is a genuine character of $\widetilde{\mathrm{Z}}^2(r)$ and ω_2 is a genuine character of $\widetilde{\mathrm{Z}}^2(s)$ then the same construction gives us a genuine character $\omega_1 \otimes \omega_2$ of $\widetilde{\mathrm{Z}}^2(r) \tilde{\times} \widetilde{\mathrm{Z}}^2(s)$. It must be distinguished from the (non-genuine) character $\omega_1 \cdot \omega_2$ obtained by regarding ω_1 and ω_2 as characters on the same $\widetilde{\mathrm{Z}}^2(s)$ and multiplying them. Note that $\pi_1^2 \otimes \pi_2^2$ admits the character $(\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{\mathrm{Z}}^2(|\gamma_1|+|\gamma_2|)}$.

Theorem 1: *Let π_1 (resp. π_2) be a genuine admissible indecomposable representation of $\widetilde{\mathrm{G}}(\gamma_1)$ (resp. $\widetilde{\mathrm{G}}(\gamma_2)$) of finite length and put $r = |\gamma_1| + |\gamma_2|$. Suppose that ω is a character of $\widetilde{\mathrm{Z}}(r)$ such that*

$$\omega|_{\widetilde{\mathrm{Z}}^2(r)} = (\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{\mathrm{Z}}^2(r)}. \tag{2}$$

Then the representation

$$\Pi = \mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma_1) \tilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1) \tilde{\times} \widetilde{\mathrm{G}}(\gamma_2)} (\pi_1^2 \otimes \pi_2^2)$$

has an indecomposable summand admitting ω. Any two indecomposable summands of Π admitting the same character are isomorphic. The restriction of an indecomposable summand of Π to $\widetilde{\mathrm{G}}^2(\gamma_1) \tilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)$ is isomorphic to the direct sum of $[F^\times : (F^\times)^2]$ copies of $\pi_1^2 \otimes \pi_2^2$ if both $|\gamma_1|$ and $|\gamma_2|$ are odd and to $\pi_1^2 \otimes \pi_2^2$ otherwise.

Proof: If G is a group, H a normal subgroup of finite index with and J is an intermediate group with J/H abelian then it is easy to check that

$$\mathrm{ind}_H^G(\rho|_H) \cong \bigoplus_{\chi \in (J/H)^\frown} \mathrm{ind}_J^G(\chi \otimes \rho)$$

where ρ is any representation of J. If we take H to be $\widetilde{\mathrm{G}}^2(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}^2(\gamma_2)$, J to be $\widetilde{\mathrm{G}}(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}^2(\gamma_2)$ and $\rho = \pi_1 \otimes \pi_2^2$ then the hypotheses are satisfied and we obtain

$$\Pi \cong \bigoplus_{\chi \in (\widetilde{\mathrm{G}}(\gamma_1)/\widetilde{\mathrm{G}}^2(\gamma_1))^\frown} \operatorname{ind}_{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}(\gamma_2)} \left((\chi \otimes \pi_1) \otimes \pi_2^2\right).$$

If $g_2 \in \widetilde{\mathrm{G}}(\gamma_2)$ then we denote by χ_{g_2} the character of $\widetilde{\mathrm{G}}(\gamma_1)/\widetilde{\mathrm{G}}^2(\gamma_1)$ given by $\chi_{g_2}(g_1) = (\det(g_1), \det(g_2))$. The non-degeneracy of the Hilbert symbol implies that every element of $(\widetilde{\mathrm{G}}(\gamma_1)/\widetilde{\mathrm{G}}^2(\gamma_1))^\frown$ arises in this way and we have $g_2^{-1} g_1 g_2 = \chi_{g_2}(g_1) g_1$ for all $g_1 \in \widetilde{\mathrm{G}}(\gamma_1)$ and $g_2 \in \widetilde{\mathrm{G}}(\gamma_2)$. From this it follows that if ρ_1 is a representation of $\widetilde{\mathrm{G}}(\gamma_1)$ and ρ_2 is a representation of $\widetilde{\mathrm{G}}^2(\gamma_2)$ then ${}^{g_1 g_2}(\rho_1 \otimes \rho_2) = (\chi_{g_2} \otimes {}^{g_1}\rho_1) \otimes {}^{g_2}\rho_2$ for $g_1 \in \widetilde{\mathrm{G}}(\gamma_1)$ and $g_2 \in \widetilde{\mathrm{G}}(\gamma_2)$.

If $g_2 \in \widetilde{\mathrm{G}}(\gamma_2)$ then

$$\begin{aligned} \operatorname{ind}_{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}(\gamma_2)} \left((\chi_{g_2} \otimes \pi_1) \otimes \pi_2^2\right) &\cong \operatorname{ind}_{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}(\gamma_2)} \left(\pi_1 \otimes {}^{g_2^{-1}}\pi_2^2\right) \\ &\cong \operatorname{ind}_{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}(\gamma_2)} \left(\pi_1 \otimes \pi_2^2\right) \end{aligned}$$

since ${}^{g_2^{-1}}\pi_2^2 \cong \pi_2^2$. Thus if we put

$$\Pi_1 \cong \operatorname{ind}_{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}(\gamma_2)} \left(\pi_1 \otimes \pi_2^2\right)$$

then $\Pi \cong \Pi_1^{\oplus [F^\times : (F^\times)^2]}$ and it follows that it is sufficient to prove our claims with Π_1 in place of Π.

If both $|\gamma_1|$ and $|\gamma_2|$ are odd then π_2^2 is indecomposable and so $\pi_1 \otimes \pi_2^2$ is indecomposable by Proposition 1. If $g_2 \in \widetilde{\mathrm{G}}(\gamma_2) \setminus \widetilde{\mathrm{G}}^2(\gamma_2)$ then

$$\operatorname{Hom}_{\widetilde{\mathrm{G}}(\gamma_1) \widetilde{\times} \widetilde{\mathrm{G}}^2(\gamma_2)} \left({}^{g_2}(\pi_1 \otimes \pi_2^2), \pi_1 \otimes \pi_2^2\right)$$

$$\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}\big((\chi_{g_2}\otimes\pi_1)\otimes\pi_2^2,\pi_1\otimes\pi_2^2\big)$$

$$\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma_1)}(\chi_{g_2}\otimes\pi_1,\pi_1)\otimes\mathrm{Hom}_{\widetilde{\mathrm{G}}^2(\gamma_2)}(\pi_2^2,\pi_2^2)\quad\text{by Proposition 2}$$

$$=\{0\}$$

from Proposition 3. Lemma 6 then implies that Π_1 is indecomposable. Since r is even, (2) determines ω uniquely and it is equal to the character admitted by Π_1. The last claim follows from the Mackey subgroup theorem.

If either $|\gamma_1|$ or $|\gamma_2|$ is even then we may assume without loss of generality that $|\gamma_2|$ is even. Let σ_2 be an indecomposable summand of π_2^2. Then from Proposition 4 we have

$$\begin{aligned}\Pi_1 &\cong \bigoplus_{g_2\in(\widetilde{\mathrm{G}}(\gamma_2)/\widetilde{\mathrm{G}}^2(\gamma_2))}\mathrm{ind}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\big(\pi_1\otimes{}^{g_2}\sigma_2\big)\\ &\cong \bigoplus_{\chi\in(\widetilde{\mathrm{G}}(\gamma_1)/\widetilde{\mathrm{G}}^2(\gamma_1))^\frown}\mathrm{ind}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\big((\chi\otimes\pi_1)\otimes\sigma_2\big).\end{aligned}$$

Let us put

$$\Sigma_\chi=\mathrm{ind}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\big((\chi\otimes\pi_1)\otimes\sigma_2\big).$$

Since all the conjugates of σ_2 are distinct it follows as above that Σ_χ is indecomposable. If $|\gamma_1|$ is also even then Proposition 4 implies that $\chi\otimes\pi_1\cong\pi_1$ for all $\chi\in(\widetilde{\mathrm{G}}(\gamma_1)/\widetilde{\mathrm{G}}^2(\gamma_1))^\frown$ and so $\Pi_1\cong\Sigma_{\chi_0}^{\oplus[F^\times:(F^\times)^2]}$. Again $r=|\gamma_1|+|\gamma_2|$ is even, (2) determines ω uniquely and it equals ω_Σ. Computing the restriction of Σ_{χ_0} to $\widetilde{\mathrm{G}}^2(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)$ in stages one finds that it equals $\pi_1^2\otimes\pi_2^2$ and the Theorem is proved in this case.

Finally suppose that $|\gamma_1|$ is odd. The character admitted by Σ_χ is equal to

$\chi|_{\widetilde{Z}(|\gamma_1|)} \cdot \omega_\Sigma$ and since $\widetilde{G}(\gamma_1) = \widetilde{Z}(|\gamma_1|) \cdot \widetilde{G}^2(\gamma_1)$ this character runs over

$$\{\omega \in \widetilde{Z}(r)^\frown \,|\, \omega|_{\widetilde{Z}^2(r)} = (\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{Z}^2(r)}\}$$

as χ runs over $(\widetilde{G}(\gamma_1)/\widetilde{G}^2(\gamma_1))^\frown$. This proves the first two statements and since we also have $(\Sigma_\chi)|_{\widetilde{G}^2(\gamma_1)\,\widetilde{\times}\,\widetilde{G}^2(\gamma_2)} \cong \pi_1^2 \otimes \pi_2^2$ we are done. $\square$

Definition 3: *Let π_1 (resp. π_2) be a genuine admissible indecomposable representation of $\widetilde{G}(\gamma_1)$ (resp. $\widetilde{G}(\gamma_2)$) of finite length and ω a character of $\widetilde{Z}(|\gamma_1| + |\gamma_2|)$ satisfying*

$$\omega|_{\widetilde{Z}^2(|\gamma_1|+|\gamma_2|)} = (\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{Z}^2(|\gamma_1|+|\gamma_2|)}.$$

We define $\pi_1 \widetilde{\otimes}_\omega \pi_2$ to be an indecomposable summand of the representation Π of the theorem admitting the character ω. (It follows from the theorem that $\pi_1 \widetilde{\otimes}_\omega \pi_2$ is well-defined.)

Proposition 5: *Let π be a genuine irreducible admissible representation of the group $\widetilde{G}(\gamma_1) \,\widetilde{\times}\, \widetilde{G}(\gamma_2)$ admitting the character ω of $\widetilde{Z}(|\gamma_1| + |\gamma_2|)$. Then there are genuine irreducible admissible representations π_1 and π_2 of $\widetilde{G}(\gamma_1)$ and $\widetilde{G}(\gamma_2)$ respectively such that $\pi \cong \pi_1 \widetilde{\otimes}_\omega \pi_2$. Conversely any such tensor product is irreducible.*

Proof: If π_1 and π_2 are irreducible then π_1^2 and π_2^2 are semisimple and so Π as in the Theorem is semisimple. Since $\pi_1 \widetilde{\otimes}_\omega \pi_2 \leq \Pi$ is indecomposable by construction it follows that it is irreducible.

Conversely suppose that π is as stated. Then the restriction $\pi^{2,2}$ of π to the group $\widetilde{G}^2(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{G}^2(\gamma_2)$ is semisimple and regarding it as a representation of $\widetilde{G}^2(\gamma_1) \times \widetilde{G}^2(\gamma_2)$ on which the group B acts trivially we see that we may find irreducible representations ρ_1 and ρ_2 of $\widetilde{G}^2(\gamma_1)$ and $\widetilde{G}^2(\gamma_2)$ respectively such that

$$\mathrm{Hom}_{\widetilde{G}^2(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{G}^2(\gamma_2)}(\pi^{2,2}, \rho_1 \otimes \rho_2) \neq \{0\}.$$

Let π_1 be an irreducible constituent of $\mathrm{ind}_{\widetilde{G}^2(\gamma_1)}^{\widetilde{G}(\gamma_1)}(\rho_1)$ such that π_1^2 contains ρ_1 and similarly with ρ_2 and π_2. If Π denotes the representation of the Theorem then

$$\begin{aligned}
&\mathrm{Hom}_{\widetilde{G}(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{G}(\gamma_2)}(\pi, \Pi) \\
&\cong \mathrm{Hom}_{\widetilde{G}^2(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{G}^2(\gamma_2)}(\pi^{2,2}, \pi_1^2 \otimes \pi_2^2) \\
&\neq \{0\}
\end{aligned}$$

and since π is irreducible it follows that it is a subrepresentation of Π. Since Π is semisimple in this case we obtain $\pi \cong \pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2$ from the definition. $\square$

Proposition 6: *Suppose that π_1 and ρ_1 (resp. π_2 and ρ_2) are genuine indecomposable admissible representations of $\widetilde{G}(\gamma_1)$ (resp. $\widetilde{G}(\gamma_2)$) of finite length such that*

$$(\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{Z}^2(|\gamma_1|+|\gamma_2|)} = (\omega_{\rho_1} \otimes \omega_{\rho_2})|_{\widetilde{Z}^2(|\gamma_1|+|\gamma_2|)}.$$

Let ω be a character of $\widetilde{Z}(|\gamma_1| + |\gamma_2|)$ whose restriction to $\widetilde{Z}^2(|\gamma_1| + |\gamma_2|)$ is equal to this common value. Then it is possible to find two characters

$$\chi_1 \in (\widetilde{G}(\gamma_1)/\widetilde{G}^2(\gamma_1))^\wedge \qquad \textit{and} \qquad \chi_2 \in (\widetilde{G}(\gamma_2)/\widetilde{G}^2(\gamma_2))^\wedge$$

such that

$$\begin{aligned}&\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}(\pi_1\ \widetilde{\otimes}_\omega\ \pi_2, \rho_1\ \widetilde{\otimes}_\omega\ \rho_2)\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma_1)}(\chi_1\otimes\pi_1,\rho_1)\otimes\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma_2)}(\chi_2\otimes\pi_2,\rho_2)\,.\end{aligned}$$

Proof: Suppose first that π and ρ are genuine indecomposable admissible representations of $\widetilde{\mathrm{G}}\,(\gamma)$ of finite length with $r = |\gamma|$ and $\omega_\pi|_{\widetilde{\mathrm{Z}}^2(r)} = \omega_\rho|_{\widetilde{\mathrm{Z}}^2(r)}$. If r is odd then let χ be the unique element of $(\widetilde{\mathrm{G}}\,(\gamma)\,/\widetilde{\mathrm{G}}^2\,(\gamma))^\frown$ such that $\chi|_{\widetilde{\mathrm{Z}}(r)}\cdot\omega_\pi = \omega_\rho$. In this case we have

$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}(\chi\otimes\pi,\rho) &\cong \bigoplus_{\chi'}\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}(\chi'\otimes\pi,\rho)\\ &\cong \bigoplus_{\chi'}\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}(\pi,\chi'\otimes\rho) \quad \text{on rearranging the sum}\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}\big(\pi,\mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma)}^{\widetilde{\mathrm{G}}(\gamma)}(\rho^2)\big) \quad \text{by Proposition 3}\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}^2(\gamma)}(\pi^2,\rho^2)\,.\end{aligned}$$

If r is even and ν is an indecomposable summand of ρ^2 then for any

$$\chi\in(\widetilde{\mathrm{G}}\,(\gamma)\,/\widetilde{\mathrm{G}}^2\,(\gamma))^\frown$$

we have

$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}(\chi\otimes\pi,\rho) &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}(\pi,\rho) \quad \text{by Propostion 4}\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}\big(\pi,\mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma)}^{\widetilde{\mathrm{G}}(\gamma)}(\nu)\big) \quad \text{by Proposition 4}\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}^2(\gamma)}(\pi^2,\nu)\,.\end{aligned}$$

It follows from the proof of the Theorem that $\pi_1 \widetilde{\otimes}_\omega \pi_2$ and $\rho_1 \widetilde{\otimes}_\omega \rho_2$ are isomorphic to representations induced from either $\widetilde{G}(\gamma_1) \widetilde{\times} \widetilde{G}^2(\gamma_2)$ or $\widetilde{G}^2(\gamma_1) \widetilde{\times} \widetilde{G}(\gamma_2)$ to $\widetilde{G}(\gamma_1) \widetilde{\times} \widetilde{G}(\gamma_2)$ and the inducing representation is explicit in each case. Thus the space of homomorphisms from $\pi_1 \widetilde{\otimes}_\omega \pi_2$ to $\rho_1 \widetilde{\otimes}_\omega \rho_2$ may be calculated using Frobenius reciprocity. On doing this and then using Proposition 2 and the isomorphisms which have just been established we obtain the Proposition. □

Corollary 1: *Suppose that π_1 and ρ_1 (respectively π_2 and ρ_2) are genuine irreducible admissible representations of $\widetilde{G}(\gamma_1)$ (resp. $\widetilde{G}(\gamma_2)$) and ω is a character of $\widetilde{Z}(|\gamma_1| + |\gamma_2|)$ such that both $\pi_1 \widetilde{\otimes}_\omega \pi_2$ and $\rho_1 \widetilde{\otimes}_\omega \rho_2$ are defined. Then $\pi_1 \widetilde{\otimes}_\omega \pi_2 \cong \rho_1 \widetilde{\otimes}_\omega \rho_2$ if and only if there are characters*

$$\chi_1 \in (\widetilde{G}(\gamma_1) / \widetilde{G}^2(\gamma_1))^\frown \quad \textit{and} \quad \chi_2 \in (\widetilde{G}(\gamma_2) / \widetilde{G}^2(\gamma_2))^\frown$$

such that $\chi_1 \otimes \pi_1 \cong \rho_1$ and $\chi_2 \otimes \pi_2 \cong \rho_2$.

Proof: The "if" direction is immediate from the definition and the "only if" direction follows from Proposition 6. □

Proposition 7: *Suppose that π_1 (resp. π_2) is a genuine indecomposable admissible representation of $\widetilde{G}(\gamma_1)$ (resp. $\widetilde{G}(\gamma_2)$) of finite length. Then*

$$\left(\pi_1 \widetilde{\otimes}_\omega \pi_2\right)^\frown \cong \widehat{\pi}_1 \widetilde{\otimes}_{\omega^{-1}} \widehat{\pi}_2$$

where $\frown$ is being used to denote the contragredient representations and ω is any suitable character.

Proof: We have

$$\mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\left(\pi_1^2 \otimes \pi_2^2\right)^\frown \cong \mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\left((\widehat{\pi}_1)^2 \otimes (\widehat{\pi}_2)^2\right)$$

and by using Lemma 5 to reduce to the case of a finite-dimensional module over an algebra we see that the contragredient of an indecomposable representation is indecomposable. Thus $(\pi_1 \,\widetilde{\otimes}_\omega\, \pi_2)^\frown$ is an indecomposable summand of

$$\mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\left((\widehat{\pi}_1)^2 \otimes (\widehat{\pi}_2)^2\right)$$

and since it admits the character ω^{-1} the claim follows. □

Proposition 8: *For $i = 1, 2, 3$ let π_i be a genuine admissible indecomposable representation of $\widetilde{\mathrm{G}}\,(\gamma_i)$ of finite length and put $r = |\gamma_1| + |\gamma_2| + |\gamma_3|$. Then for any character ω of $\widetilde{\mathrm{Z}}(r)$ such that*

$$\omega|_{\widetilde{\mathrm{Z}}^2(r)} = (\omega_{\pi_1} \otimes \omega_{\pi_2} \otimes \omega_{\pi_3})|_{\widetilde{\mathrm{Z}}^2(r)}$$

we have $\pi_1 \,\widetilde{\otimes}_\omega\, (\pi_2 \,\widetilde{\otimes}\, \pi_3) \cong (\pi_1 \,\widetilde{\otimes}\, \pi_2) \,\widetilde{\otimes}_\omega\, \pi_3$. Here the inner tensor products may be formed with respect to any suitable character.

Proof: Using the definition of $\widetilde{\otimes}_\omega$ and the transitivity of induction it is routine to check that both $\pi_1 \,\widetilde{\otimes}_\omega\, (\pi_2 \,\widetilde{\otimes}\, \pi_3)$ and $(\pi_1 \,\widetilde{\otimes}\, \pi_2) \,\widetilde{\otimes}_\omega\, \pi_3$ are isomorphic to any of the indecomposable summands of

$$\mathrm{ind}_{\widetilde{\mathrm{G}}^2(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_3)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_3)}\left(\pi_1^2 \otimes \pi_2^2 \otimes \pi_3^2\right)$$

admitting the character ω. $\square$

Finally we want to extend the definition of $\widetilde{\otimes}_\omega$ to include representations which are not necessarily indecomposable. Call a representation π of $\widetilde{\mathrm{G}}\,(\gamma)$ *homogeneous* if it admits only one character of $\widetilde{\mathrm{Z}}(|\gamma|)$; if π is homogeneous then it still makes sense to write ω_π. If π_1 (resp. π_2) is a genuine admissible homogeneous representation of $\widetilde{\mathrm{G}}\,(\gamma_1)$ (resp. $\widetilde{\mathrm{G}}\,(\gamma_2)$) of finite length and ω is a character of $\widetilde{\mathrm{Z}}(|\gamma_1| + |\gamma_2|)$ satisfying

$$\omega|_{\widetilde{\mathrm{Z}}^2(|\gamma_1|+|\gamma_2|)} = (\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{\mathrm{Z}}^2(|\gamma_1|+|\gamma_2|)}$$

then we define $\pi_1 \ \widetilde{\otimes}_\omega \ \pi_2$ by requiring $\widetilde{\otimes}_\omega$ to distribute over direct sums. It follows from the Krull-Schmidt Theorem that this extension is well-defined. The properties of $\widetilde{\otimes}_\omega$ given in Propositions 6, 7 and 8 remain valid in this setting.

2. Parabolic Induction and Jacquet Functors

Continuing with the notation of the previous section with $n = 2$ we let $\mathcal{R}(\widetilde{\mathrm{G}}\,(\gamma))$ denote the category of genuine admissible representations of $\widetilde{\mathrm{G}}\,(\gamma)$ of finite length for $\gamma = (r_1, \ldots, r_k)$ a k-tuple of positive integers. Let us introduce an equivalence relation $\sim$ on the objects of $\mathcal{R}(\widetilde{\mathrm{G}}\,(\gamma))$ by defining $\chi \otimes \pi \sim \pi$ if π is indecomposable and $\chi \in (\widetilde{\mathrm{G}}\,(\gamma)\,/\widetilde{\mathrm{G}}^2\,(\gamma))^\frown$ and then $\pi_1 \sim \pi_2$ if $\pi_i \cong \oplus_{j=1}^t \pi_{ij}$ with π_{ij} indecomposable and $\pi_{1j} \sim \pi_{2j}$ for all j. For $\pi \in \mathcal{R}(\widetilde{\mathrm{G}}\,(\gamma))$ we let $[\pi]$ denote the equivalence class of π under $\sim$. The set of equivalence classes in $\mathcal{R}(\widetilde{\mathrm{G}}\,(\gamma))$ may be regarded as the set of objects in a category $\mathcal{R}[\widetilde{\mathrm{G}}\,(\gamma)]$ where a morphism from $[\pi_1]$ to $[\pi_2]$ is simply a $\widetilde{\mathrm{G}}\,(\gamma)$-homomorphism from any element of $[\pi_1]$ to any element of $[\pi_2]$. We know from the previous section that if $\pi \in \mathcal{R}(\widetilde{\mathrm{G}}(\gamma))$ is indecomposable then $[\pi]$ consists of one element if $|\gamma|$ is even and of $[F^\times : (F^\times)^2]$ elements if $|\gamma|$ is odd. In the latter case the elements of $[\pi]$ are distinguished by the character of $\widetilde{\mathrm{Z}}(|\gamma|)$ which they admit. It follows from the definition of $\sim$ that we may set $[\pi_1] + [\pi_2] = [\pi_1 \oplus \pi_2]$ for $\pi_1, \pi_2 \in \mathcal{R}(\widetilde{\mathrm{G}}(\gamma))$.

For $\gamma = (r_1, \dots, r_k)$ and $r = |\gamma|$ we let $\mathrm{Q}(\gamma) \le \mathrm{G}(r)$ be the standard parabolic subgroup corresponding to the set of simple roots

$$\Delta(\gamma) = \Delta \backslash \{(r_1, r_1{+}1), (r_1{+}r_2, r_1{+}r_2{+}1), \dots, (r_1{+}\cdots{+}r_{k-1}, r_1{+}\cdots{+}r_{k-1}{+}1)\}.$$

Then $\mathrm{G}(\gamma)$ is a Levi subgroup of $\mathrm{Q}(\gamma)$ and we denote by $\mathrm{N}(\gamma)$ the unipotent radical of $\mathrm{Q}(\gamma)$. For any subgroup $H \le \mathrm{G}(r)$ let us put $\widetilde{H} = p_r^{-1}(H) \le \widetilde{\mathrm{G}}(r)$. Recall that it is possible to choose $\mathbf{s}$ so that $\mathbf{s}|_{\mathrm{N}(\gamma)}$ is a homomorphism and we shall always do this in what follows. Then $\mathrm{N}^*(\gamma) = \mathbf{s}(\mathrm{N}(\gamma))$ is a normal subgroup of $\widetilde{\mathrm{Q}}(\gamma)$ and $\widetilde{\mathrm{Q}}(\gamma) = \widetilde{\mathrm{G}}(\gamma) \cdot \mathrm{N}^*(\gamma)$ with $\widetilde{\mathrm{G}}(\gamma) \cap \mathrm{N}^*(\gamma) = \{1\}$.

If $\gamma = (r_1, \dots, r_k)$ and $\delta = (s_1, \dots, s_\ell)$ are both partitions of r then we say that γ *refines* δ and write $\gamma < \delta$ if there exist $k_1, \dots, k_\ell$ such that

$$s_i = \sum_{j=k_{i-1}+1}^{k_i} r_j$$

where k_0 is interpreted as 0. If $\gamma < \delta$ then we let $\mathrm{N}(\delta, \gamma) = \mathrm{G}(\delta) \cap \mathrm{N}(\gamma)$ and $\mathrm{N}^*(\delta, \gamma) = \mathbf{s}(\mathrm{N}(\delta, \gamma))$. Note that since $\mathrm{G}(\gamma) \le \mathrm{G}(\delta)$, $\mathrm{N}(\delta, \gamma)$ is normalized by $\mathrm{G}(\gamma)$ and $\mathrm{N}^*(\delta, \gamma)$ is normalized by $\widetilde{\mathrm{G}}(\gamma)$.

If $\gamma < \delta$ we now define $i_{\gamma,\delta}$ to be the normalized induction functor and $\varphi_{\delta,\gamma}$ to be the normalized Jacquet functor corresponding to the groups $\widetilde{\mathrm{G}}(\delta)$, $\widetilde{\mathrm{G}}(\gamma)$

and $\mathrm{N}^*(\delta,\gamma)$ and the trivial character on $\mathrm{N}^*(\delta,\gamma)$. The definitions of $i_{\gamma,\delta}$ and $\varphi_{\delta,\gamma}$ together with many properties enjoyed by these functors are given in [BZ2], beginning on page 444. In addition to the properties recorded there we want to mention that both $i_{\gamma,\delta}$ and $\varphi_{\delta,\gamma}$ take representations of finite length into representations of finite length. That is, $i_{\gamma,\delta}$ is a functor from $\mathcal{R}(\widetilde{\mathrm{G}}\,(\gamma))$ to $\mathcal{R}(\widetilde{\mathrm{G}}\,(\delta))$ and $\varphi_{\delta,\gamma}$ is a functor from $\mathcal{R}(\widetilde{\mathrm{G}}\,(\delta))$ to $\mathcal{R}(\widetilde{\mathrm{G}}\,(\gamma))$. In both cases this is simply a matter of adapting a proof available in the literature for $\mathrm{G}(\delta)$ and $\mathrm{G}(\gamma)$ to the situation of the covering groups. One first establishes along the lines of [BZ1], §4.1 that, for admissible representations of $\widetilde{\mathrm{G}}\,(\gamma)$, being of finite length is equivalent to being finitely-generated. Then imitation of [Cas], §3.3 shows that $i_{\gamma,\delta}$ preserves the condition of being finitely-generated and imitation of [BZ1], §3.13 establishes the corresponding property of $\varphi_{\delta,\gamma}$. (The development leading up to §3.3 in [Cas] is not quite accurate, but it can easily be repaired for the purpose at hand using results from [BZ1].) Proposition 1.9 (f) of [BZ2] applied to the functors $\varphi_{\delta,\gamma}$ and $i_{\gamma,\delta}$ shows that they respect the equivalence relation $\sim$ introduced above. Therefore they may also be considered as functors on the categories $\mathcal{R}[\widetilde{\mathrm{G}}\,(\delta)]$ and $\mathcal{R}[\widetilde{\mathrm{G}}\,(\gamma)]$.

It will be useful to make some remarks on the modular function which occurs in the definitions of $i_{\gamma,\delta}$ and $\varphi_{\delta,\gamma}$. Since $\widetilde{\mathrm{G}}\,(\gamma)$ is a locally compact Hausdorff topological group any closed subgroup of it carries a Haar measure and hence if $G_1, G_2 \leq \widetilde{\mathrm{G}}\,(\gamma)$ are closed subgroups with G_1 normalizing G_2 then there is a corresponding modular character $\mathrm{mod} : G_1 \to \mathbb{R}_+^\times$ defined by

$$\int_{G_2} f(g_1 g g_1^{-1})\, dg = \mathrm{mod}(g_1)^{-1} \int_{G_2} f(g)\, dg$$

for any $f \in \mathrm{L}^1(G_2)$ and $g_1 \in G_1$ where dg denotes any Haar measure on G_2. Suppose that $H_1, H_2 \leq \mathrm{G}(\gamma)$ and H_1 normalizes H_2. If mod_{H_2} denotes the modular character of H_1 acting on H_2 and $\mathrm{mod}_{\widetilde{H}_2}$ denotes the modular character of $\widetilde{H}_1$ acting on $\widetilde{H}_2$ then since μ_2 is central in $\widetilde{H}_1$ and $p : \widetilde{\mathrm{G}}(\gamma) \to \mathrm{G}(\gamma)$ is a local homeomorphism we have

$$\mathrm{mod}_{\widetilde{H}_2}(h) = \mathrm{mod}_{H_2}(p(h))$$

for every $h \in \widetilde{H}_1$.

Let $\gamma < \delta$ be partitions of r. With the considerations of the previous paragraph in mind we define $\mu_{\delta,\gamma} : \widetilde{\mathrm{G}}(\gamma) \to \mathbb{R}_+^\times$ to be the modular character of $\widetilde{\mathrm{G}}(\gamma)$ acting by conjugation on $\mathrm{N}^*(\delta, \gamma)$. It is the same as the modular character of $\widetilde{\mathrm{G}}(\gamma)$ acting on $\widetilde{\mathrm{N}}(\delta, \gamma)$ and, after composition with p, it also equals the modular character of $\mathrm{G}(\gamma)$ acting on $\mathrm{N}(\delta, \gamma)$. We shall identify all of these by the usual abuse of notation. A calculation in $\mathrm{G}(r)$ shows that if $\gamma = (r_1, \ldots, r_k)$ and $k_1, \ldots, k_\ell$ are as in the definition of $\gamma < \delta$ then

$$\mu_{\delta,\gamma}(g_1, \ldots, g_k) = \prod_{i=1}^{\ell} \prod_{j=k_{i-1}+1}^{k_i} |\det(g_j)|^{m_{ij}} \tag{1}$$

where $m_{ij} = \sum_{a=k_{i-1}+1}^{k_i} \varepsilon_{ja} r_a$,

$$\varepsilon_{ab} = \begin{cases} 1 & \text{if } a < b \\ 0 & \text{if } a = b \\ -1 & \text{if } a > b \end{cases}$$

and $|\cdot|$ denotes the standard absolute value on F. If β is a third partition of r and $\beta < \gamma < \delta$ then we have $\mathrm{N}(\delta, \beta) = \mathrm{N}(\delta, \gamma) \cdot \mathrm{N}(\gamma, \beta)$ and $\mathrm{N}(\delta, \gamma) \cap \mathrm{N}(\gamma, \beta) = \{1\}$. It follows from this that $\mu_{\delta,\beta} = \mu_{\delta,\gamma} \cdot \mu_{\gamma,\beta}$ as characters of $\mathrm{G}(\beta)$ or of $\widetilde{\mathrm{G}}(\beta)$.

Next we would like to define a "cross" product of representations as in [BZ2] and [Zel]. Unfortunately on the category $\oplus_{r\geq 0}\mathcal{R}(\widetilde{\mathrm{G}}\,(r))$ the definition is rather awkward and only gives rise to a partial operation. This is due to the role of the character of $\widetilde{\mathrm{Z}}(r)$ which must be chosen in order to define the tensor products. Although it will not be necessary in what follows we would also like to indicate how a more satisfactory definition can be made on the category $\oplus_{r\geq 0}\mathcal{R}[\widetilde{\mathrm{G}}\,(r)]$. If $r = r_1 + r_2$, $\pi_1 \in \mathcal{R}(\widetilde{\mathrm{G}}\,(r_1))$ and $\pi_2 \in \mathcal{R}(\widetilde{\mathrm{G}}\,(r_2))$ are homogeneous representations, ω is a character of $\widetilde{\mathrm{Z}}(r)$ satisfying

$$\omega|_{\widetilde{\mathrm{Z}}^2(r)} = (\omega_{\pi_1} \otimes \omega_{\pi_2})|_{\widetilde{\mathrm{Z}}^2(r)} \tag{2}$$

and $\chi \in (\widetilde{\mathrm{G}}\,(r)\,/\widetilde{\mathrm{G}}^2\,(r))^\frown$ then

$$\chi \otimes (\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2) \cong \pi_1 \mathbin{\widetilde{\otimes}_{\chi\cdot\omega}} \pi_2\,.$$

Every character of $\widetilde{\mathrm{Z}}(r)$ satisfying (2) has the form $\chi\cdot\omega$ for some χ and so the class of $\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2$ does not depend on ω. This allows us to define $[\pi_1] \mathbin{\widetilde{\otimes}} [\pi_2] = [\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2]$ for any homogeneous representations π_1 and π_2 and any ω satisfying (2). We may then extend this definition to general $[\pi_1] \in \mathcal{R}[\widetilde{\mathrm{G}}\,(r_1)]$ and $[\pi_2] \in \mathcal{R}[\widetilde{\mathrm{G}}\,(r_2)]$ by making $\widetilde{\otimes}$ distributive over $+$. Then given $[\pi_i] \in \mathcal{R}[\widetilde{\mathrm{G}}\,(r_i)]$ for $i = 1, \ldots, k$ we set

$$[\pi_1] \times \cdots \times [\pi_k] = i_{(r_1,\ldots,r_k),(r)}([\pi_1] \mathbin{\widetilde{\otimes}} \ldots \mathbin{\widetilde{\otimes}} [\pi_k])$$

with $r = r_1 + \cdots + r_k$. This operation has the same good properties as Bernstein and Zelevinsky's $\times$ operation; in particular, if $\mathcal{R}[\widetilde{\mathrm{G}}\,(r)]$ is replaced by its Grothendieck group with respect to short exact sequences then $\times$ becomes commutative. If

$\pi_i \in \mathcal{R}(\widetilde{\mathrm{G}}(r_i))$, with $i = 1, \ldots, k$, are homogeneous representations and ω is a character of $\widetilde{\mathrm{Z}}(r)$ with $r = r_1 + \cdots + r_k$ satisfying

$$\omega|_{\widetilde{\mathrm{Z}}^2(r)} = (\omega_{\pi_1} \otimes \cdots \otimes \omega_{\pi_k})|_{\widetilde{\mathrm{Z}}^2(r)}$$

then we also define

$$\pi_1 \times_\omega \pi_2 \times \cdots \times \pi_k = i_{(r_1,\ldots,r_k),(r)}(\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2 \mathbin{\widetilde{\otimes}} \ldots \mathbin{\widetilde{\otimes}} \pi_k)\,.$$

In view of Proposition 8 of Section 1 it is unnecessary to specify the order in which the tensor products are formed or the intermediate choices of central character. The operation $\times_\omega$ is associative and distributive over direct sums and $\pi_1 \times_\omega \pi_2 \times \cdots \times \pi_k$ is homogeneous, admitting the character ω. The first two of these claims are clear; for the last one observe that if $\pi = \pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2 \mathbin{\widetilde{\otimes}} \ldots \mathbin{\widetilde{\otimes}} \pi_k$ then we may find $m \geq 1$ such that $(\pi(z) - \omega(z))^m$ annihilates E_π for all $z \in \widetilde{\mathrm{Z}}(r)$. It then follows directly from the definition that $(\Pi(z) - \omega(z))^m$ with $\Pi = \pi_1 \times_\omega \pi_2 \times \cdots \times \pi_k$ annihilates the space of the induced representation for all $z \in \widetilde{\mathrm{Z}}(r)$, which implies the claim.

We close this section with a technical result which will be useful later on.

Proposition 1: *Let $\pi_1 \in \mathcal{R}(\widetilde{\mathrm{G}}(\delta_1))$ and $\pi_2 \in \mathcal{R}(\widetilde{\mathrm{G}}(\delta_2))$ be homogeneous representations and suppose that $\gamma_1 < \delta_1$, $\gamma_2 < \delta_2$. Then*

$$\varphi_{(\delta_1,\delta_2),(\gamma_1,\gamma_2)}(\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2) \cong \varphi_{\delta_1,\gamma_1}(\pi_1) \mathbin{\widetilde{\otimes}_\omega} \varphi_{\delta_2,\gamma_2}(\pi_2)$$

for every character ω of $\widetilde{\mathrm{Z}}(|\delta_1| + |\delta_2|)$ for which $\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2$ is defined.

Proof: Set $\delta = (\delta_1, \delta_2)$ and $\gamma = (\gamma_1, \gamma_2)$. We begin by applying Theorem 5.2 of [BZ2] to compute the composition of $\varphi_{\delta,\gamma}$ with induction from $\widetilde{\mathrm{G}}(\delta_1) \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}^2(\delta_2)$ to

$\widetilde{\mathrm{G}}(\delta)$. Since we may choose coset representatives for $\widetilde{\mathrm{G}}^2(\delta_2)\backslash\widetilde{\mathrm{G}}(\delta_2)$ from $\widetilde{\mathrm{G}}(\gamma_2)$ there is a single

$$(\widetilde{\mathrm{G}}(\delta_1) \,\widetilde{\times}\, \widetilde{\mathrm{G}}^2(\delta_2), \widetilde{\mathrm{G}}(\gamma)\cdot \mathrm{N}^*(\delta,\gamma))$$

double coset in $\widetilde{\mathrm{G}}(\delta)$. Using this observation we obtain an isomorphism of functors

$$\varphi_{\delta,\gamma}\circ \mathrm{ind}_{\widetilde{\mathrm{G}}(\delta_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\delta_2)}^{\widetilde{\mathrm{G}}(\delta_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\delta_2)} \cong \mathrm{ind}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\circ\left(\varphi_{\delta_1,\gamma_1}\otimes\varphi^2_{\delta_2,\gamma_2}\right) \tag{3}$$

where $\varphi^2_{\delta_2,\gamma_2}:\mathcal{R}(\widetilde{\mathrm{G}}^2(\delta_2))\to\mathcal{R}(\widetilde{\mathrm{G}}^2(\gamma_2))$ is the Jacquet functor with respect to the trivial character on $\mathrm{N}^*(\delta_2,\gamma_2)\leq\widetilde{\mathrm{G}}^2(\delta_2)$. Also

$$|_{\widetilde{\mathrm{G}}^2(\gamma_2)}\circ\varphi_{\delta_2,\gamma_2}\cong\varphi^2_{\delta_2,\gamma_2}\circ|_{\widetilde{\mathrm{G}}^2(\delta_2)}$$

where the vertical bars represent the restriction functors. If both $|\delta_1|=|\gamma_1|$ and $|\delta_2|=|\gamma_2|$ are odd then

$$\pi_1\,\widetilde{\otimes}_\omega\,\pi_2\cong\mathrm{ind}_{\widetilde{\mathrm{G}}(\delta_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\delta_2)}^{\widetilde{\mathrm{G}}(\delta_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\delta_2)}(\pi_1\otimes\pi_2^2)$$

and similarly for the representations of $\widetilde{\mathrm{G}}(\gamma_1)$ and $\widetilde{\mathrm{G}}(\gamma_2)$. In this case the Proposition follows directly from (3).

If either $|\delta_1|$ and $|\delta_2|$ is even then we shall assume that $|\delta_2|$ is even, the other case being similar. We may also assume that π_1 and π_2 are indecomposable since both sides of the proposed isomorphism respect direct sums. For each suitable ω there is an indecomposable summand σ_2 of π_2^2 such that

$$\pi_1\,\widetilde{\otimes}_\omega\,\pi_2\cong\mathrm{ind}_{\widetilde{\mathrm{G}}(\delta_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\delta_2)}^{\widetilde{\mathrm{G}}(\delta_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\delta_2)}(\pi_1\otimes\sigma_2)$$

and so

$$\varphi_{\delta,\gamma}(\pi_1\,\widetilde{\otimes}_\omega\,\pi_2)\cong\mathrm{ind}_{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(\gamma_2)}^{\widetilde{\mathrm{G}}(\gamma_1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(\gamma_2)}\left(\varphi_{\delta_1,\gamma_1}(\pi_1)\otimes\varphi^2_{\delta_2,\gamma_2}(\sigma_2)\right). \tag{4}$$

The representation $\varphi_{\delta_2,\gamma_2}(\pi_2)$ will generally have several indecomposable summands, say

$$\varphi_{\delta_2,\gamma_2}(\pi_2) = \rho_1 \oplus \cdots \oplus \rho_k .$$

The indecomposable summands of ρ_j^2 are distinguished by the character of $\widetilde{Z}^1(|\delta_2|)$ which they admit. We choose a particular character of this group from amongst those restricting to ω_{π_2} on $\widetilde{Z}^2(|\delta_2|)$ and let ν_j be the summand of ρ_j^2 admitting this character. It then follows from Proposition 4 of section 1 that

$$\rho_j^2 \cong \bigoplus_{g \in (\widetilde{G}(\gamma_2)/\widetilde{G}^2(\gamma_2))} {}^g\nu_j$$

and so if we set $\nu = \nu_1 \oplus \cdots \oplus \nu_k$ then

$$\varphi^2_{\delta_2,\gamma_2}(\pi_2^2) \cong \bigoplus_{g \in (\widetilde{G}(\gamma_2)/\widetilde{G}^2(\gamma_2))} {}^g\nu . \tag{5}$$

Moreover if the character admitted by ν was chosen correctly then we have

$$\varphi_{\delta_1,\gamma_1}(\pi_1) \mathbin{\widetilde{\otimes}_\omega} \varphi_{\delta_2,\gamma_2}(\pi_2) \cong \operatorname{ind}_{\widetilde{G}(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{G}^2(\gamma_2)}^{\widetilde{G}(\gamma_1) \mathbin{\widetilde{\times}} \widetilde{G}(\gamma_2)} \left(\varphi_{\delta_1,\gamma_1}(\pi_1) \otimes \nu\right) . \tag{6}$$

In addition

$$\pi_2^2 \cong \bigoplus_{g \in (\widetilde{G}(\gamma_2)/\widetilde{G}^2(\gamma_2))} {}^g\sigma_2$$

and so

$$\varphi^2_{\delta_2,\gamma_2}(\pi_2^2) \cong \bigoplus_{g \in (\widetilde{G}(\gamma_2)/\widetilde{G}^2(\gamma_2))} {}^g\varphi^2_{\delta_2,\gamma_2}(\sigma_2) . \tag{7}$$

Since σ_2 is indecomposable it is homogeneous for $\widetilde{Z}^1(|\delta_2|)$ and hence $\varphi^2_{\delta_2,\gamma_2}(\sigma_2)$ is also homogeneous for this group and each conjugate in (7) admits a different character of $\widetilde{Z}^1(|\delta_2|)$. Comparing homogeneous summands under $\widetilde{Z}^1(|\delta_2|)$ between

(5) and (7) we conclude that $\varphi_{\delta_2,\gamma_2}(\sigma_2) \cong {}^g\nu$ for some $g \in (\widetilde{\mathrm{G}}(\gamma_2)/\widetilde{\mathrm{G}}^2(\gamma_2))$. Now (4) and (6) combine to show that

$$\varphi_{\delta,\gamma}(\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2) \cong \chi \otimes \left(\varphi_{\delta_1,\gamma_1}(\pi_1) \mathbin{\widetilde{\otimes}_\omega} \varphi_{\delta_2,\gamma_2}(\pi_2)\right)$$

for some $\chi \in (\widetilde{\mathrm{G}}(\gamma)/\widetilde{\mathrm{G}}^2(\gamma))^\frown$ and since both sides admit ω we can conclude that

$$\varphi_{\delta,\gamma}(\pi_1 \mathbin{\widetilde{\otimes}_\omega} \pi_2) \cong \varphi_{\delta_1,\gamma_1}(\pi_1) \mathbin{\widetilde{\otimes}_\omega} \varphi_{\delta_2,\gamma_2}(\pi_2),$$

as required. $\square$

3. The Local Exceptional Representations

In this section we place the exceptional representations of [KaP] in the framework of the last two sections and prove various results about them which will subsequently be needed. From (3) of Chapter 1, section 3 we see that $\widetilde{\mathrm{G}}(1) \cong F^\times \times \mu_2$ and hence every character χ of $F^\times$ gives rise to a genuine character of $\widetilde{\mathrm{G}}(1)$ and conversely; we shall use the same symbol to denote both objects. If χ is such a character and we let $\widetilde{\mathrm{H}}^2_r = \widetilde{\mathrm{G}}^2(1) \mathbin{\widetilde{\times}} \ldots \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}^2(1)$ then

$$\chi^{[r]} = \left(\chi|_{\widetilde{\mathrm{G}}^2(1)}\right) \otimes \cdots \otimes \left(\chi|_{\widetilde{\mathrm{G}}^2(1)}\right) \qquad (r \text{ factors})$$

is a genuine character of $\widetilde{\mathrm{H}}^2_r$. Let us choose a character ω of $\widetilde{\mathrm{Z}}(r)$ which satisfies

$$\omega|_{\widetilde{\mathrm{Z}}(r)\cap\widetilde{\mathrm{H}}^2_r} = \chi^{[r]}|_{\widetilde{\mathrm{Z}}(r)\cap\widetilde{\mathrm{H}}^2_r}. \tag{1}$$

This is always possible because $\widetilde{\mathrm{Z}}(r)$ is an abelian group and $\widetilde{\mathrm{Z}}(r) \cap \widetilde{\mathrm{H}}^2_r = \widetilde{\mathrm{Z}}^2(r)$ is an open subgroup of finite index (this second remark means that any extension of a continuous character will again be continuous). Note also that any such ω is automatically genuine. When ω and χ as above satisfy condition (1) we shall call them *compatible*. We may define a character $(\omega \cdot \chi^{[r]})$ of $\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2$ by setting

$$(\omega \cdot \chi^{[r]})(zh) = \omega(z)\chi^{[r]}(h)$$

for every $z \in \widetilde{\mathrm{Z}}(r)$ and $h \in \widetilde{\mathrm{H}}_r^2$; it follows from the choice of ω that this is well-defined. If $w \in W$ and $g \in \widetilde{\mathrm{G}}(r)$ then we let $g^w = \mathbf{s}(w)^{-1} g \mathbf{s}(w)$. This defines an action of W on $\widetilde{\mathrm{G}}(r)$ which preserves the subgroup $\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2$ and hence induces an action of W on the characters of this subgroup given by $({}^w\theta)(g) = \theta(g^w)$.

For any χ and ω as above we set

$$\chi_{r,\omega} = \chi \mathbin{\widetilde{\otimes}_\omega} \chi \mathbin{\widetilde{\otimes}} \ldots \mathbin{\widetilde{\otimes}} \chi \qquad (r \text{ factors})$$

which is a genuine irreducible representation of $\widetilde{\mathrm{H}}_r = \widetilde{\mathrm{G}}(1, \ldots, 1) \leq \widetilde{\mathrm{G}}(r)$ on which $\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2$ acts by $\omega \cdot \chi^{[r]}$. It follows from Frobenius reciprocity that $\chi_{r,\omega}$ is an irreducible constituent of the representation

$$\mathrm{ind}_{\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2}^{\widetilde{\mathrm{H}}_r} \left(\omega \cdot \chi^{[r]}\right).$$

From the theory of representations of Heisenberg groups (see, for example, [KaP] §0.3) or by direct computation we see that any representation of $\widetilde{\mathrm{H}}_r$ obtained by inducing a genuine character of $\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2$ is isotypic. This remark may be used to good effect in proving facts about $\chi_{r,\omega}$. For instance suppose that $w \in W$; then it is clear from the definitions that ${}^w\omega = \omega$ and ${}^w(\chi^{[r]}) = \chi^{[r]}$ and hence

$$\begin{aligned} {}^w\mathrm{ind}_{\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2}^{\widetilde{\mathrm{H}}_r} \left(\omega \cdot \chi^{[r]}\right) &\cong \mathrm{ind}_{\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2}^{\widetilde{\mathrm{H}}_r} \left({}^w(\omega \cdot \chi^{[r]})\right) \\ &= \mathrm{ind}_{\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2}^{\widetilde{\mathrm{H}}_r} \left(\omega \cdot \chi^{[r]}\right). \end{aligned}$$

It follows at once from this that ${}^{w}\chi_{r,\omega} \cong \chi_{r,\omega}$ for all $w \in W$. We will use this kind of argument again in the proof of Proposition 2.

For any partition γ of r let us define

$$\pi_\gamma(\chi, \omega) = i_{(1,\dots,1),\gamma}\big(\mu_{\gamma,(1,\dots,1)}^{-1/4} \otimes \chi_{r,\omega}\big)$$

where $\mu_{\gamma,(1,\dots,1)}$ is the modular character introduced in section 2. From now on we shall let $\gamma_{0,r}$ stand for the partition $(1,\dots,1)$ of r and write γ_0 for $\gamma_{0,r}$ when no ambiguity will arise. For any partition γ of r we let $W(\gamma)$ be the subgroup of W generated by the set of simple reflections $\{s_\alpha \mid \alpha \in \Delta(\gamma)\}$. The group $W(\gamma)$ is naturally identified with the Weyl group of $\mathrm{G}(\gamma)$. We also let $\Phi_\gamma = \Phi \cap \mathrm{span}_{\mathbb{Z}}(\Delta(\gamma))$ and $\Phi_\gamma^+ = \Phi_\gamma \cap \Phi^+$; then Φ_γ is the root system of $\mathrm{G}(\gamma)$ and Φ_γ^+ is the positive system corresponding to the simple system $\Delta(\gamma)$.

If π is a representation of any ℓ−group and π has a chain

$$\pi = \pi_k \geq \pi_{k-1} \geq \cdots \geq \pi_1 \geq \pi_0 = 0$$

of subrepresentations such that $\pi_i/\pi_{i-1} = \tau_i$ for $i = 1, \dots, k$ then we follow [BZ2] in saying that π is *glued from* the representations $\tau_1, \dots, \tau_k$.

Lemma 1: *Let $\gamma < \delta$ be partitions of r and ρ be an algebraic representation of $\widetilde{\mathrm{H}}_r$. Then $\varphi_{\delta,\gamma}\big(i_{\gamma_0,\delta}(\rho)\big)$ is glued from the representations $i_{\gamma_0,\gamma}({}^{w}\rho)$ as w runs over $W(\gamma)\backslash W(\delta)$.*

Proof: We have a Bruhat decomposition

$$\mathrm{G}(\delta) = \bigcup_{w \in W(\gamma)\backslash W(\delta)} \big(\mathrm{H}_r \cdot \mathrm{N}(\delta, \gamma_0)\big) w^{-1} \big(\mathrm{G}(\gamma) \cdot \mathrm{N}(\delta, \gamma)\big)$$

and it follows that

$$\widetilde{\mathrm{G}}\,(\delta) = \bigcup_{w \in W(\gamma)\backslash W(\delta)} \big(\widetilde{\mathrm{H}}_r \cdot \mathrm{N}^*(\delta,\gamma_0)\big)\mathbf{s}(w)^{-1}\big(\widetilde{\mathrm{G}}\,(\gamma)\cdot \mathrm{N}^*(\delta,\gamma)\big)\,.$$

Given this observation the Lemma is simply a version of Bernstein and Zelevinsky's Geometric Lemma and it follows from Theorem 5.2 of [BZ2] just as in that paper. □

Lemma 2: *Let $\gamma < \delta$ be partitions of r and suppose that $w \in W(\delta)$ is such that ${}^w\mu_{\delta,\gamma_0} = \mu_{\delta,\gamma_0}$ on the group $\mathrm{Z}^2(\gamma) = \mathrm{Z}(\gamma)\cap \mathrm{H}^2_r$ where $\mathrm{H}^2_r = p_r\big(\widetilde{\mathrm{H}}^2_r\big)$ and $\mathrm{Z}(\gamma)$ is the center of $\mathrm{G}(\gamma)$. Then $w \in W(\gamma)$.*

Proof: Suppose that $\gamma = (r_1,\ldots,r_k)$ and $\delta = (s_1,\ldots,s_\ell)$ and let $k_0,\ldots,k_\ell$ be as in the definition of $\gamma < \delta$. A typical element of $\mathrm{Z}^2(\gamma)$ has the form

$$z = \mathrm{diag}(\underbrace{z_1,\ldots,z_1}_{r_1},\underbrace{z_2,\ldots,z_2}_{r_2},\ldots,\underbrace{z_k,\ldots,z_k}_{r_k})$$

where $z_i \in (F^\times)^2$ for all i. Using (1) of section 2 we obtain

$$\mu_{\delta,\gamma_0}\big(\mathrm{diag}(t_1,\ldots,t_r)\big) = \prod_{i=1}^{\ell}\ \prod_{j=S_{i-1}+1}^{S_i} |t_j|^{m_{ij}}$$

where $S_i = \sum_{a=1}^{i} s_a$ and

$$\begin{aligned} m_{ij} &= \sum_{a=S_{i-1}+1}^{S_i} \varepsilon_{ja} \\ &= \mathrm{card}\{a \mid j < a \le S_i\} - \mathrm{card}\{a \mid S_{i-1}+1 \le a < j\} \\ &= \big(S_i - j\big) - \big(j - (S_{i-1}+1)\big) \\ &= S_i + S_{i-1} - 2j + 1\,, \end{aligned}$$

which gives

$$\mu_{\delta,\gamma_0}(\mathrm{diag}(t_1,\ldots,t_r)) = \prod_{i=1}^{\ell} \prod_{j=S_{i-1}+1}^{S_i} |t_j|^{S_i+S_{i-1}-2j+1} . \tag{2}$$

Let us define $R_c = \sum_{b=1}^{c} r_b$; it follows from the definition of $\gamma < \delta$ that $S_i = R_{k_i}$ for all $i = 1,\ldots,\ell$ and hence

$$[S_{i-1}+1, S_i] = \bigcup_{m=k_{i-1}+1}^{k_i} [R_{m-1}+1, R_m] .$$

Now $w \in W(\delta)$ if and only if w, thought of as a permutation of $\{1,\ldots,r\}$, satisfies the condition

$$w[S_{i-1}+1, S_i] = [S_{i-1}+1, S_i]$$

for $i = 1,\ldots,\ell$ and $w \in W(\gamma)$ if and only if

$$w[R_{m-1}+1, R_m] = [R_{m-1}+1, R_m] \tag{3}$$

for $m = 1,\ldots,k$.

If we write $z = \mathrm{diag}(t_1,\ldots,t_r)$ with z as above then $t_j = z_m$ for $R_{m-1}+1 \leq j \leq R_m$ and since the $|z_m|$ may be varied in $q^{2\mathbb{Z}}$ independently of one another it follows from (2) that ${}^{w}\mu_{\delta,\gamma_0} = \mu_{\delta,\gamma_0}$ as characters of $\mathrm{Z}^2(\gamma)$ if and only if

$$\sum_{j=R_{m-1}+1}^{R_m} j = \sum_{j=R_{m-1}+1}^{R_m} w(j) \tag{4}$$

for all $1 \leq m \leq k$. Thus we are reduced to showing that condition (4) implies condition (3) for $w \in W(\delta)$. We do this by induction on m, the first case being included in the general case. So suppose that $w \in W(\delta)$ satisfies (4) for all m and (3) for $m = 1,\ldots,(c-1)$ with $c \leq k$. Then $w[1, R_{c-1}] = [1, R_{c-1}]$ and consequently $w[R_{c-1}+1, R_k] = [R_{c-1}+1, R_k]$. Now $\sum_{j=R_{c-1}+1}^{R_c} j$ is the sum of the smallest $(R_c - R_{c-1})$ distinct numbers in $[R_{c-1}+1, R_k]$ and hence it is strictly less than any sum of $(R_c - R_{c-1})$ distinct numbers in $[R_{c-1}+1, R_k]$ except itself. Using condition (4) now implies that $w[R_{c-1}+1, R_c] = [R_{c-1}+1, R_c]$ and this completes the induction step. $\square$

Proposition 1: *Let $\gamma < \delta$ be partitions of r. Then*

$$\varphi_{\delta,\gamma}\big(\pi_\delta(\chi,\omega)\big) \cong \bigoplus_{w\in W(\gamma)\backslash W(\delta)} i_{\gamma_0,\gamma}\big({}^w\mu_{\delta,\gamma_0}^{-1/4}\otimes\chi_{r,\omega}\big)\,.$$

Proof: Since $\pi_\delta(\chi,\omega) = i_{\gamma_0,\delta}(\mu_{\delta,\gamma_0}^{-1/4}\otimes\chi_{r,\omega})$ it follows from Lemma 1 that the representation $\varphi_{\delta,\gamma}\big(\pi_\delta(\chi,\omega)\big)$ is glued from the representations $i_{\gamma_0,\gamma}\big({}^w(\mu_{\delta,\gamma_0}^{-1/4}\otimes\chi_{r,\omega})\big)$ as w runs over $W(\gamma)\backslash W(\delta)$. We know that ${}^w\chi_{r,\omega}\cong\chi_{r,\omega}$ for all $w\in W$ and hence

$$i_{\gamma_0,\gamma}\big({}^w(\mu_{\delta,\gamma_0}^{-1/4}\otimes\chi_{r,\omega})\big)\cong i_{\gamma_0,\gamma}\big({}^w\mu_{\delta,\gamma_0}^{-1/4}\otimes\chi_{r,\omega}\big)\,.$$

It remains to see that there can be no non-trivial extensions between these constituents so that $\varphi_{\delta,\gamma}\big(\pi_\delta(\chi,\omega)\big)$ must in fact be their direct sum. To do this we observe that $\widetilde{\mathrm{Z}}^2(\gamma)$ is central in $\widetilde{\mathrm{G}}\,(\gamma)$ and the character of $\widetilde{\mathrm{Z}}^2(\gamma)$ admitted by $i_{\gamma_0,\gamma}\big({}^w\mu_{\delta,\gamma_0}^{-1/4}\otimes\chi_{r,\omega}\big)$ is ${}^w\mu_{\delta,\gamma_0}^{-1/4}\cdot\chi^{[r]}$. It follows from Lemma 2 that these characters of $\widetilde{\mathrm{Z}}^2(\gamma)$ are all distinct as w runs over $W(\gamma)\backslash W(\delta)$ and this settles the remaining point. $\square$

Suppose that η is a character of $\widetilde{\mathrm{H}}_r^2$. Then following [KaP] we let η^2 be the character of H_r defined by $\eta^2(h) = \eta(\mathbf{s}(h^2))$. From any character λ of H_r and root $\alpha = (i,j)$ of $\mathrm{G}(r)$ with respect to H_r we obtain a character λ_α of $F^\times$ by setting $\lambda_\alpha(x) = \lambda(h_\alpha(x))$ where

$$h_\alpha(x) = \mathrm{diag}(1, \ldots, \underset{i}{x}, \ldots, \underset{j}{x^{-1}}, \ldots, 1).$$

A character η of $\widetilde{\mathrm{H}}_r^2$ (or of $\widetilde{\mathrm{Z}}(r)\cdot\widetilde{\mathrm{H}}_r^2$) is called *exceptional* by Kazhdan and Patterson if it satisfies $\eta_\alpha^2 = |\cdot|$ for all $\alpha \in \Delta$.

Let us denote by w_0 the longest element of W with respect to the positive system Φ^+. We wish to show that ${}^{w_0}\big(\mu_{(r),\gamma_0}^{-1/4} \otimes \chi^{[r]}\big)$ is an exceptional character. Since for $\alpha \in \Delta$ and $x \in F^\times$ we have

$$\begin{aligned}({}^{w_0}\chi^{[r]})\big(h_\alpha(x^2)\big) &= \chi^{[r]}\big(\mathbf{s}(h_\alpha(x^2))\big) \\ &= \chi(x^2)\chi(x^{-2}) \\ &= 1\end{aligned}$$

this amounts to showing that ${}^{w_0}\mu_{(r),\gamma_0}^{-1/4}$ is exceptional. From the proof of Lemma 2 we have the formula

$$\mu_{(r),\gamma_0}\big(\mathrm{diag}(t_1, \ldots, t_r)\big) = \prod_{j=1}^{r} |t_j|^{r-2j+1}$$

and hence

$${}^{w_0}\mu_{(r),\gamma_0}\big(\mathrm{diag}(t_1, \ldots, t_r)\big) = \prod_{j=1}^{r} |t_j|^{-r+2j-1}.$$

If $\alpha = (i, i+1) \in \Delta$ then this gives

$$\begin{aligned}{}^{w_0}\mu_{(r),\gamma_0}\big(h_\alpha(x^2)\big) &= |x^2|^{-r+2i-1}|x^{-2}|^{-r+2(i+1)-1} \\ &= |x|^{-4}\end{aligned}$$

and so ${}^{w_0}\mu_{(r),\gamma_0}^{-1/4}\left(h_\alpha(x^2)\right) = |x|$, as claimed. Once this fact is available Theorem I.2.9 of [KaP] implies that for every character χ of $F^\times$ and every character ω of $\widetilde{\mathrm{Z}}(r)$ satisfying (1) the representation $\pi_{(r)}(\chi,\omega)$ has a unique irreducible subrepresentation. This subrepresentation is isomorphic to the unique irreducible quotient of

$$i_{\gamma_0,(r)}\left({}^{w_0}(\mu_{(r),\gamma_0}^{-1/4} \otimes \chi_{r,\omega})\right) \cong i_{\gamma_0,(r)}\left(\mu_{(r),\gamma_0}^{1/4} \otimes \chi_{r,\omega}\right)$$

and is called by Kazhdan and Patterson an *exceptional representation.* We shall denote the unique irreducible subrepresentation of $\pi_{(r)}(\chi,\omega)$ by $\vartheta_{(r)}(\chi,\omega)$.

We define a representation $\vartheta_\gamma(\chi,\omega)$ of $\widetilde{\mathrm{G}}\,(\gamma)$ for any partition γ of r by setting

$$\vartheta_\gamma(\chi,\omega) = \mu_{(r),\gamma}^{1/4} \otimes \varphi_{(r),\gamma}\left(\vartheta_{(r)}(\chi,\omega)\right). \tag{5}$$

Any representation of $\widetilde{\mathrm{G}}\,(\gamma)$ which arises in this way will be called a $\gamma-$*exceptional representation* or simply an *exceptional representation* if γ is either evident or unimportant.

Theorem 1: *Let χ be a character of $F^\times$, ω a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ and $\gamma < \delta$ be partitions of r.*

(a) We have

$$\varphi_{\delta,\gamma}\left(\vartheta_\delta(\chi,\omega)\right) \cong \mu_{\delta,\gamma}^{-1/4} \otimes \vartheta_\gamma(\chi,\omega).$$

(b) There is an isomorphism $\vartheta_{\gamma_0}(\chi,\omega) \cong \chi_{r,\omega}$.

(c) The representation $\vartheta_\gamma(\chi,\omega)$ is isomorphic to the unique irreducible subrepresentation of $\pi_\gamma(\chi,\omega)$.

(d) If γ is written as $\gamma = (\gamma_1, \gamma_2)$ where γ_1 is a partition of r_1 and γ_2 is a partition of r_2 then

$$\vartheta_\gamma(\chi, \omega) \cong \vartheta_{\gamma_1}(\chi, \omega_1)\ \widetilde{\otimes}_\omega\ \vartheta_{\gamma_2}(\chi, \omega_2)$$

where $\omega_j \in \widetilde{\mathrm{Z}}(|\gamma_j|)^\frown$ is compatible with χ for $j = 1, 2$.

(e) The contragredient of $\vartheta_\gamma(\chi, \omega)$ is isomorphic to $\vartheta_\gamma(\chi^{-1}, \omega^{-1})$.

(f) The representation $\vartheta_\delta(\chi, \omega)$ is isomorphic to the unique irreducible subrepresentation of $i_{\gamma,\delta}\big(\mu_{\delta,\gamma}^{-1/4} \otimes \vartheta_\gamma(\chi, \omega)\big)$.

Proof: (a) We have

$$\begin{aligned}
\varphi_{\delta,\gamma}\big(\vartheta_\delta(\chi, \omega)\big) &= \varphi_{\delta,\gamma}\big(\mu_{(r),\delta}^{1/4} \otimes \varphi_{(r),\delta}(\vartheta_{(r)}(\chi, \omega))\big) \\
&\cong \mu_{(r),\delta}^{1/4} \otimes \varphi_{(r),\gamma}\big(\vartheta_{(r)}(\chi, \omega)\big) \\
&= \mu_{\delta,\gamma}^{-1/4} \otimes \mu_{(r),\gamma}^{1/4} \otimes \varphi_{(r),\gamma}\big(\vartheta_{(r)}(\chi, \omega)\big) \\
&= \mu_{\delta,\gamma}^{-1/4} \otimes \vartheta_\gamma(\chi, \omega)\,.
\end{aligned}$$

(b) When rewritten in our terminology the Periodicity Theorem (Theorem I.2.9 (e) of [KaP]) asserts that

$$\varphi_{(r),\gamma_0}\big(\vartheta_{(r)}(\chi, \omega)\big) \cong \mu_{(r),\gamma_0}^{-1/4} \otimes \chi_{r,\omega}\,.$$

In order to verify this claim it is essential to recall that the Jacquet functors in [KaP] are unnormalized whereas ours are normalized and that the irreducible constituents of

$$\mathrm{ind}_{\widetilde{\mathrm{Z}}(r)\cdot\widetilde{\mathrm{H}}_r^2}^{\widetilde{\mathrm{H}}_r}\big(\omega \cdot \chi^{[r]}\big)$$

are constructed in [KaP] by first extending $\omega \cdot \chi^{[r]}$ to a maximal abelian subgroup of $\widetilde{\mathrm{H}}_r$ and then inducing, whereas here they appear as tensor products of characters of $\widetilde{\mathrm{G}}(1)$ in the sense of section 1. Combining the above isomorphism with the definition gives (b).

(c) Using parts (a) and (b) of this Theorem and Frobenius Reciprocity we obtain

$$\begin{aligned}&\mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}\left(\vartheta_\gamma(\chi,\omega), \pi_\gamma(\chi,\omega)\right)\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma_0)}\left(\mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}, \mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}\right).\end{aligned}$$

The identity map from $\mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}$ to itself corresponds under this isomorphism to an embedding of $\vartheta_\gamma(\chi,\omega)$ into $\pi_\gamma(\chi,\omega)$. Thus $\vartheta_\gamma(\chi,\omega)$ may be regarded as a subrepresentation of $\pi_\gamma(\chi,\omega)$. Now $\pi_\gamma(\chi,\omega)$ is an induced representation and it follows from (the metaplectic analogue of) [BZ2] Theorem 2.4 (a) and (d) that $\pi_\gamma(\chi,\omega)$ has no cuspidal constituents. In particular $\vartheta_\gamma(\chi,\omega)$ has no cuspidal constituents and since

$$\begin{aligned}\varphi_{\gamma,\gamma_0}\left(\vartheta_\gamma(\chi,\omega)\right) &\cong \mu_{\gamma,\gamma_0}^{-1/4} \otimes \vartheta_{\gamma_0}(\chi,\omega)\\ &\cong \mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}\end{aligned}$$

is irreducible it follows that $\vartheta_\gamma(\chi,\omega)$ is irreducible. Finally we must show that $\pi_\gamma(\chi,\omega)$ has no other irreducible subrepresentations. So let $\rho \leq \pi_\gamma(\chi,\omega)$ be an irreducible representation other than $\vartheta_\gamma(\chi,\omega)$. Using Proposition 1, the fact that Jacquet functors are exact and the observations just made we see that

$$\varphi_{\gamma,\gamma_0}(\rho) \cong {}^{w}\mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}$$

for some $w \in W(\gamma) \setminus \{1\}$. The transitivity property of induction implies that

$$\pi_{(r)}(\chi,\omega) \cong i_{\gamma,(r)}\big(\mu_{(r),\gamma}^{-1/4} \otimes \pi_\gamma(\chi,\omega)\big)$$

and so $i_{\gamma,(r)}(\mu_{(r),\gamma}^{-1/4} \otimes \rho)$ is a subrepresentation of $\pi_{(r)}(\chi,\omega)$. Since $\vartheta_{(r)}(\chi,\omega)$ is the unique irreducible subrepresentation of $\pi_{(r)}(\chi,\omega)$ it follows that

$$\mathrm{Hom}_{\widetilde{\mathrm{G}}(r)}\big(\vartheta_{(r)}(\chi,\omega), i_{\gamma,(r)}(\mu_{\gamma,(r)}^{-1/4} \otimes \rho)\big) \neq \{0\}\,.$$

However

$$\begin{aligned}
&\mathrm{Hom}_{\widetilde{\mathrm{G}}(r)}\big(\vartheta_{(r)}(\chi,\omega), i_{\gamma,(r)}(\mu_{(r),\gamma}^{-1/4} \otimes \rho)\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}\big(\mu_{(r),\gamma}^{-1/4} \otimes \vartheta_\gamma(\chi,\omega), \mu_{(r),\gamma}^{-1/4} \otimes \rho\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(\gamma)}\big(\vartheta_\gamma(\chi,\omega), \rho\big)
\end{aligned}$$

and consequently $\vartheta_\gamma(\chi,\omega) \cong \rho$. This isomorphism gives rise to a contradiction, since Lemma 2 shows that $\varphi_{\gamma,\gamma_0}\big(\vartheta_\gamma(\chi,\omega)\big)$ is not isomorphic to $\varphi_{\gamma,\gamma_0}(\rho)$.

(d) Using Proposition 1 of section 2 we obtain

$$\begin{aligned}
&\varphi_{\gamma,\gamma_0}\big(\vartheta_{\gamma_1}(\chi,\omega_1) \;\widetilde{\otimes}_\omega\; \vartheta_{\gamma_2}(\chi,\omega_2)\big)\\
&\cong \varphi_{\gamma_1,\gamma_{0,r_1}}\big(\vartheta_{\gamma_1}(\chi,\omega_1)\big) \;\widetilde{\otimes}_\omega\; \varphi_{\gamma_2,\gamma_{0,r_2}}\big(\vartheta_{\gamma_2}(\chi,\omega_2)\big)\\
&\cong \big(\mu_{\gamma_1,\gamma_{0,r_1}}^{-1/4} \otimes \vartheta_{\gamma_{0,r_1}}(\chi,\omega_1)\big) \;\widetilde{\otimes}_\omega\; \big(\mu_{\gamma_2,\gamma_{0,r_2}}^{-1/4} \otimes \vartheta_{\gamma_{0,r_2}}(\chi,\omega_2)\big)\\
&\cong \mu_{(\gamma_1,\gamma_2),\gamma_0}^{-1/4} \otimes \big(\chi_{r_1,\omega_1} \;\widetilde{\otimes}_\omega\; \chi_{r_2,\omega_2}\big)\\
&\cong \mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}
\end{aligned}$$

and so the representation $\vartheta_{\gamma_1}(\chi,\omega_1) \;\widetilde{\otimes}_\omega\; \vartheta_{\gamma_2}(\chi,\omega_2)$ may be regarded as a subrepresentation of $\pi_\gamma(\chi,\omega)$. It follows from the construction of $\widetilde{\otimes}_\omega$ in section 1 that

$\vartheta_{\gamma_1}(\chi, \omega_1) \ \widetilde{\otimes}_\omega \ \vartheta_{\gamma_2}(\chi, \omega_2)$ is also irreducible and hence isomorphic to $\vartheta_\gamma(\chi, \omega)$ by the previous part of the Theorem.

(e) From Proposition 1.9 (d) of [BZ2] we see that

$$\pi_{(r)}(\chi, \omega)\widehat{\ } \cong i_{\gamma_0,(r)}\big((\mu_{(r),\gamma_0}^{-1/4} \otimes \chi_{r,\omega})\widehat{\ }\big)$$
$$\cong i_{\gamma_0,(r)}\big(\mu_{(r),\gamma_0}^{1/4} \otimes \chi_{r,\omega}\widehat{\ }\big)$$

where $\widehat{\ }$ is being used to denote contragredients. By Proposition 7 of section 1 we have

$$\chi_{r,\omega}\widehat{\ } \cong \chi^{-1} \ \widetilde{\otimes}_{\omega^{-1}} \ \chi^{-1} \ \widetilde{\otimes} \ \ldots \ \widetilde{\otimes} \ \chi^{-1}$$
$$= (\chi^{-1})_{r,\omega^{-1}} .$$

These isomorphisms imply that

$$\pi_{(r)}(\chi, \omega)\widehat{\ } \cong i_{\gamma_0,(r)}\big(\mu_{(r),\gamma_0}^{1/4} \otimes (\chi^{-1})_{r,\omega^{-1}}\big) .$$

We know that $\vartheta_{(r)}(\chi^{-1}, \omega^{-1})$ is the unique irreducible quotient of

$$i_{\gamma_0,(r)}\big(\mu_{(r),\gamma_0}^{1/4} \otimes (\chi^{-1})_{r,\omega^{-1}}\big)$$

and under the pairing between $\pi_{(r)}(\chi, \omega)$ and $\pi_{(r)}(\chi, \omega)\widehat{\ }$ the irreducible subrepresentation $\vartheta_{(r)}(\chi, \omega)$ must be paired with an irreducible quotient. Therefore we have an isomorphism $\vartheta_{(r)}(\chi, \omega)\widehat{\ } \cong \vartheta_{(r)}(\chi^{-1}, \omega^{-1})$. Using part (d) repeatedly shows that if $\gamma = (r_1, \ldots, r_k)$ then

$$\vartheta_\gamma(\chi) \cong \vartheta_{(r_1)}(\chi) \ \widetilde{\otimes}_\omega \ \vartheta_{(r_2)}(\chi) \ \widetilde{\otimes} \ \ldots \ \widetilde{\otimes} \ \vartheta_{(r_k)}(\chi)$$

and using Proposition 7 of section 1 gives the claim in general.

(f) From (c) and the exactness of induction we know that the representation $i_{\gamma,\delta}(\mu_{\delta,\gamma}^{-1/4} \otimes \vartheta_\gamma(\chi, \omega))$ may be regarded as a subrepresentation of

$$\begin{aligned} i_{\gamma,\delta}(\mu_{\delta,\gamma}^{-1/4} \otimes \pi_\gamma(\chi, \omega)) &\cong i_{\gamma,\delta}(\mu_{\delta,\gamma}^{-1/4} \otimes i_{\gamma_0,\gamma}(\mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega})) \\ &\cong i_{\gamma,\delta}(i_{\gamma_0,\gamma}(\mu_{\delta,\gamma_0}^{-1/4} \otimes \chi_{r,\omega})) \\ &\cong i_{\gamma_0,\delta}(\mu_{\delta,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}) \\ &= \pi_\delta(\chi, \omega) . \end{aligned}$$

The claim now follows from another application of (c) together with the fact that all the representations concerned have finite length. □

Next we state a simple result which shows that all the exceptional representations may be derived from a small number of them simply by twisting with non-genuine characters. We let χ_0 denote the trivial character of $F^\times$ and γ be a partition of r. If ω is a character of $\widetilde{Z}(r)$ which is compatible with χ_0 then we denote the representation $\vartheta_\gamma(\chi_0, \omega)$ by $\vartheta_{\gamma,\omega}$. Note that any two characters of $\widetilde{Z}(r)$ compatible with χ_0 differ by a character trivial on $\widetilde{Z}^2(r)$. Since $\widetilde{Z}(r)/\widetilde{Z}^2(r)$ has order one if r is even and order $[F^\times : (F^\times)^2]$ if r is odd, there is a unique character of $\widetilde{Z}(r)$ compatible with χ_0 if r is even and $[F^\times : (F^\times)^2]$ such characters if r is odd.

Proposition 2: *Let χ be a character of $F^\times$, ω a character of $\widetilde{Z}(r)$ compatible with χ and γ a partition of r. Then*

$$\vartheta_\gamma(\chi, \omega) \cong \chi(\det) \otimes \vartheta_{\gamma,\omega_0}$$

where $\omega_0 = \chi^{-1}(\det) \cdot \omega$.

Proof: We have

$$\begin{aligned}
&\chi(\det) \otimes \operatorname{ind}_{\widetilde{Z}(r)\cdot \widetilde{H}_r^2}^{\widetilde{H}_r} \left(\omega_0 \cdot \chi_0^{[r]}\right) \\
&\cong \operatorname{ind}_{\widetilde{Z}(r)\cdot \widetilde{H}_r^2}^{\widetilde{H}_r} \left(\chi(\det) \otimes (\omega_0 \cdot \chi_0^{[r]})\right) \\
&\cong \operatorname{ind}_{\widetilde{Z}(r)\cdot \widetilde{H}_r^2}^{\widetilde{H}_r} \left((\chi(\det) \cdot \omega_0) \cdot (\chi(\det) \cdot \chi_0^{[r]})\right) \\
&\cong \operatorname{ind}_{\widetilde{Z}(r)\cdot \widetilde{H}_r^2}^{\widetilde{H}_r} \left(\omega \cdot \chi^{[r]}\right).
\end{aligned}$$

Since we know that these induced representations are isotypic this calculation implies that $\chi(\det) \otimes (\chi_0)_{r,\omega_0} \cong \chi_{r,\omega}$. From this we obtain

$$\begin{aligned}
\chi(\det) \otimes \pi_\gamma(\chi_0, \omega_0) &= \chi(\det) \otimes i_{\gamma_0,\gamma}\left(\mu_{\gamma,\gamma_0}^{-1/4} \otimes (\chi_0)_{r,\omega_0}\right) \\
&\cong i_{\gamma_0,\gamma}\left(\mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi(\det) \otimes (\chi_0)_{r,\omega_0}\right) \\
&\cong i_{\gamma_0,\gamma}\left(\mu_{\gamma,\gamma_0}^{-1/4} \otimes \chi_{r,\omega}\right) \\
&= \pi_\gamma(\chi, \omega)
\end{aligned}$$

and since tensoring with a character does not affect the submodule structure of a representation it follows from Theorem 1 (c) that $\chi(\det) \otimes \vartheta_{\gamma,\omega_0} \cong \vartheta_\gamma(\chi, \omega)$ as required. □

In our definition of a γ–exceptional representation it might be thought more natural to have allowed representations of the form

$$\vartheta_{r_1}(\chi_1, \omega_1) \mathbin{\widetilde{\otimes}_\omega} \ldots \mathbin{\widetilde{\otimes}} \vartheta_{r_k}(\chi_k, \omega_k) \tag{6}$$

where $\gamma = (r_1, \ldots, r_k)$, χ_j is a character of $F^\times$ for $j = 1, \ldots, k$ and ω_j is a character of $\widetilde{\mathrm{Z}}(r_j)$ compatible with χ_j. The Proposition which has just been proved shows that nothing essential was lost by making our more restrictive definition. Indeed the representation (6) is isomorphic to $\chi \otimes \vartheta_{\gamma,\nu}$ where χ is the character of $\widetilde{\mathrm{G}}\,(\gamma)$ given by

$$\chi(g_1, \ldots, g_k) = \prod_{j=1}^{k} \chi_j\big(\det(g_j)\big)$$

and $\nu = \chi^{-1} \cdot \omega$.

As a complement to Proposition 2 we would like to describe in detail the characters of $\widetilde{\mathrm{Z}}(r)$ which are compatible with χ_0. It is no harder to describe all genuine characters of $\widetilde{\mathrm{Z}}^1(r)$ and this description will be important later in matching the local and global situations. If $t_1, t_2 \in F^\times$ then a direct computation using (3) of Chapter 1, section 3 gives

$$\sigma(t_1 I_r, t_2 I_r) = (t_1, t_2)^{r(r-1)/2}$$

and it follows that when $r \equiv 0, 1 \pmod 4$ we have $\widetilde{\mathrm{Z}}^1(r) \cong \widetilde{\mathrm{G}}\,(1)$. When $r \equiv 2, 3 \pmod 4$ we have instead that $\widetilde{\mathrm{Z}}^1(r) \cong \widetilde{\mathrm{G}}'(1)$ where $\widetilde{\mathrm{G}}'(1) = \widetilde{\mathrm{GL}}'(1, F)$ is the central extension originally constructed in Chapter 1, section 2. Explicitly we may take

$$\widetilde{\mathrm{G}}'(1) = \big\{[t, \epsilon] \,\big|\, t \in F^\times, \epsilon \in \mu_2\big\}$$

with the multiplication law

$$[t_1, \epsilon_1]\,[t_2, \epsilon_2] = [t_1 t_2, (t_1, t_2)\epsilon_1 \epsilon_2]\,.$$

As we have already observed, the set of genuine characters of $\widetilde{\mathrm{G}}\,(1)$ is in one-to-one correspondence with the set of characters of $F^\times$ and this correspondence is independent of any additional choices. It is a remarkable fact that after a choice of additive character ψ on F a similar statement holds true for $\widetilde{\mathrm{G}}'(1)$. Recall that in [Wei] Weil associates to each additive character ψ of F a complex number $\gamma(\psi)$. If ψ is such a character and $a \in F^\times$ then we put $\psi_a(x) = \psi(ax)$ for all $x \in F$. The essential property of γ for our purposes is expressed by the equation

$$(a, b) = \frac{\gamma(\psi_{ab})\gamma(\psi)}{\gamma(\psi_a)\gamma(\psi_b)} \tag{7}$$

which holds true for all $a, b \in F^\times$. A very useful reference for this and other properties of γ, including the question of computing its value for a particular ψ, is the appendix to [Rao]. Following [GeP] we introduce a function on $F^\times$ by

$$\mu_\psi(a) = \frac{\gamma(\psi)}{\gamma(\psi_a)}\,.$$

With this notation equation (7) may be rewritten as

$$(a, b) = \mu_\psi(a)\mu_\psi(b)\mu_\psi(ab)^{-1}\,. \tag{8}$$

If χ is a character of $F^\times$ then let us define $\chi_\psi : \widetilde{\mathrm{G}}'(1) \to \mathbb{C}^\times$ by

$$\chi_\psi\big([t, \epsilon]\big) = \chi(t)\mu_\psi(t)\epsilon\,.$$

A routine calculation using (8) shows that χ_ψ is a genuine character of $\widetilde{\mathrm{G}}'(1)$ and every such character arises in this way from some χ. We have thus obtained a description of the genuine characters of $\widetilde{\mathrm{Z}}^1(r)$ for all values of r.

Proposition 3: *Let ψ be an additive character of F and $X_2 = \{\chi \in (F^\times)^\wedge \mid \chi^2 = \chi_0\}$. If r is even the unique character ω of $\widetilde{\mathrm{Z}}(r)$ compatible with χ_0 satisfies*

$\omega(\mathbf{s}(t^2)) = 1$ *for all* $t \in F^\times$. *If* r *is odd then the set of characters of* $\widetilde{\mathrm{Z}}(r)$ *compatible with* χ_0 *is* $\{\omega_\chi \mid \chi \in X_2\}$ *where* ω_χ *satisfies* $\omega_\chi(\mathbf{s}(t)) = \chi(t)$ *for all* $t \in F^\times$ *if* $r \equiv 1 \pmod 4$ *and* $\omega_\chi(\mathbf{s}(t)) = \chi(t)\mu_\psi(t)$ *for all* $t \in F^\times$ *if* $r \equiv 3 \pmod 4$.

Proof: This follows immediately from what we did above. □

We have seen in Chapter 1 that $\widetilde{\mathrm{G}}(r)$ has an involution $g \mapsto {}^\iota g$ which lifts the main involution of $\mathrm{G}(r)$. Our next task is to describe the interaction between this involution and the exceptional representations.

Lemma 3: *Let* χ *be a unitary character of* $F^\times$ *and* ω *a character of* $\widetilde{\mathrm{Z}}(r)$ *compatible with* χ. *Then there is a conjugate linear map* $\Lambda : E_{\chi_{r,\omega}} \to E_{\chi_{r,\omega}}$ *which satisfies* $\Lambda \circ \Lambda = \mathrm{id}_{E_{\chi_{r,\omega}}}$ *and*

$$\Lambda\big(\chi_{r,\omega}(h)\xi\big) = \chi_{r,\omega}({}^\iota h)\Lambda(\xi) \tag{9}$$

for all $h \in \widetilde{\mathrm{H}}_r$ *and* $\xi \in E_{\chi_{r,\omega}}$.

Proof: According to the theory of representations of Heisenberg groups, the irreducible genuine representations of $\widetilde{\mathrm{H}}_r$ are determined up to isomorphism by their central character. Let us consider the representation $\overline{({}^\iota\chi_{r,\omega})}$ of $\widetilde{\mathrm{H}}_r$. It is irreducible and its central character is $zh \mapsto \overline{\omega({}^\iota z)} \cdot \overline{\chi^{[r]}({}^\iota h)}$ for $z \in \widetilde{\mathrm{Z}}(r)$ and $h \in \widetilde{\mathrm{H}}_r^2$. Since ω satisfies

$$\omega|_{\widetilde{\mathrm{Z}}(r)\cap\widetilde{\mathrm{H}}_r^2} = \chi^{[r]}|_{\widetilde{\mathrm{Z}}(r)\cap\widetilde{\mathrm{H}}_r^2}$$

it is also unitary. According to Proposition 4 of Chapter 1, section 4 we have ${}^\iota z = z^{-1}$ and so $\overline{\omega({}^\iota z)} = \overline{\omega(z^{-1})} = \omega(z)$. Proposition 2 of the same section implies that ${}^{\iota}\mathbf{s}(p(h)) = \mathbf{s}({}^{\iota}p(h))$ and it follows from this that $\chi^{[r]}({}^{\iota}h) = \chi^{[r]}(h)^{-1}$ and hence $\overline{\chi^{[r]}({}^{\iota}h)} = \chi^{[r]}(h)$. Thus $\chi_{r,\omega} \cong \overline{({}^{\iota}\chi_{r,\omega})}$ and any choice of isomorphism gives a conjugate linear map $\Lambda : E_{\chi_{r,\omega}} \to E_{\chi_{r,\omega}}$ which satisfies (9).

The map Λ is a bijection and $\Lambda^{-1} : E_{\chi_{r,\omega}} \to E_{\chi_{r,\omega}}$ is conjugate linear. Since ι is an involution we may replace h by ${}^{\iota}h$ and ξ by $\Lambda^{-1}(\xi)$ in (9) to obtain

$$\Lambda^{-1}\big(\chi_{r,\omega}(h)\xi\big) = \chi_{r,\omega}({}^{\iota}h)\Lambda^{-1}(\xi)\,.$$

Thus Λ^{-1} is also an isomorphism from $\chi_{r,\omega}$ to $\overline{({}^{\iota}\chi_{r,\omega})}$ and it follows from Schur's Lemma that $\Lambda^{-1} = c\Lambda$ for some c. Thus $(c^{1/2}\Lambda)^{-1} = c^{1/2}\Lambda$ and replacing Λ by $c^{1/2}\Lambda$ we obtain a map with the required properties. $\square$

Proposition 4: *Let χ be a unitary character of $F^{\times}$ and ω a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. There is a conjugate linear map $\xi \mapsto \xi'$ from $E_{\vartheta_{(r)}(\chi,\omega)}$ to itself which satisfies $(\xi')' = \xi$ and*

$$\big(\vartheta_{(r)}(\chi,\omega)(g)\xi\big)' = \vartheta_{(r)}(\chi,\omega)({}^{\iota}g)\xi'$$

for all $g \in \widetilde{\mathrm{G}}\,(r)$ and $\xi \in E_{\vartheta_{(r)}(\chi,\omega)}$.

Proof: If $f \in E_{\pi_{(r)}(\chi,\omega)}$ then let us define $f' : \widetilde{\mathrm{G}}\,(r) \to E_{\chi_{r,\omega}}$ by $f'(g) = \Lambda(f({}^{\iota}g))$ where Λ is the map from Lemma 3. We claim that $f' \in E_{\pi_{(r)}(\chi,\omega)}$. In section 5 of Chapter 1 it was shown that the main involution of $\widetilde{\mathrm{G}}\,(r)$ is a homeomorphism. From this it follows that f' is smooth, since f is. For $h \in \widetilde{\mathrm{H}}_r$ we have

$$f'(hg) = \Lambda(f({}^{\iota}h\,{}^{\iota}g))$$

$$= \Lambda(\mu_{(r),\gamma_0}^{1/4}({}^\iota h)\chi_{r,\omega}({}^\iota h)f({}^\iota g))$$

$$= \mu_{(r),\gamma_0}^{1/4}({}^\iota h)\chi_{r,\omega}(h)\Lambda(f({}^\iota g))$$

$$= \mu_{(r),\gamma_0}^{1/4}(h)\chi_{r,\omega}(h)f'(g)$$

where we have used the fact that $\mu_{(r),\gamma_0}^{1/4}$ is a real-valued character of $\widetilde{\mathrm{H}}_r$ stabilized by ι. By Proposition 5 of section 4, Chapter 1 we know that ${}^\iota\mathrm{N}^*(\gamma_0) = \mathrm{N}^*(\gamma_0)$ and so $f'(ng) = f'(g)$ for $n \in \mathrm{N}^*(\gamma_0)$. These three observations establish the claim.

The map $f \mapsto f'$ is conjugate linear and it is routine to show that $(f')' = f$ and

$$\left(\pi_{(r)}(\chi,\omega)(g)f\right)' = \pi_{(r)}(\chi,\omega)({}^\iota g)f' \tag{10}$$

for all $g \in \widetilde{\mathrm{G}}\,(r)$ and $f \in E_{\pi_{(r)}(\chi,\omega)}$. Suppose that $E \leq E_{\pi_{(r)}(\chi,\omega)}$ is the space of a subrepresentation of $\pi_{(r)}(\chi,\omega)$ and let $E' = \{f' \mid f \in E\}$. It follows from (10) that E' is also the space of a subrepresentation of $\pi_{(r)}(\chi,\omega)$. If $E_1 \leq E_2$ then $E_1' \leq E_2'$ and $E'' = E$ for all such E. It follows that $E \mapsto E'$ is an order preserving bijection on the lattice of submodules of $E_{\pi_{(r)}(\chi,\omega)}$. Therefore $E'_{\vartheta_{(r)}(\chi,\omega)} = E_{\vartheta_{(r)}(\chi,\omega)}$ and the Proposition is proved. $\square$

Proposition 5: *Let χ be a character of $F^\times$ and ω a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. Then, provided that $r \geq 2$, the representation $\pi_{(r)}(\chi,\omega)$ has no non-zero vectors invariant under the group* $\mathrm{N}^*((r-1,1))$.

Proof: Suppose that $f : \widetilde{\mathrm{G}}\,(r) \to E_{\chi_{r,\omega}}$ is a non-zero $\mathrm{N}^*((r-1,1))$-invariant vector in the space of $\pi_{(r)}(\chi,\omega)$. Since f is smooth, it is necessarily non-zero at some point of the "big cell" and hence if we set

$$w_0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and use the transformation law of f on the left under $\widetilde{\mathrm{H}}_r \cdot \mathrm{N}^*(\gamma_0)$ we see that $f(\mathbf{s}(w_0)n_0) \neq 0$ for some $n_0 \in \mathrm{N}^*(\gamma_0)$. But $\mathrm{N}^*((r-1,1))$ is normal in $\mathrm{N}^*(\gamma_0)$ and hence if f is $\mathrm{N}^*((r-1,1))$-invariant then so is $\pi_{(r)}(\chi,\omega)(n_0)f$. Replacing f by this vector, we may assume that $f(\mathbf{s}(w_0)) \neq 0$. Let us put

$$n(x) = \mathbf{s}\begin{pmatrix} 1 & 0 & \dots & 0 & x \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{N}^*((r-1,1)) .$$

Calculation (in $\mathrm{G}(r)$) shows that when $x \neq 0$ we have

$$\mathbf{s}(w_0)n(x) = \epsilon(x)h(x)n(x)h_0\overline{n}(x^{-1})$$

where $\epsilon(x) \in \mu_2$, $h(x) = \mathbf{s}(\mathrm{diag}(x^{-1},1,\dots,1,x))$, $h_0 = \mathbf{s}(\mathrm{diag}(-1,1,\dots,1))$ and

$$\overline{n}(x) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ x & 0 & \dots & 0 & 1 \end{pmatrix} .$$

Therefore, when $x \neq 0$,

$$\begin{aligned} f\big(\mathbf{s}(w_0)\big) &= f\big(\mathbf{s}(w_0)n(x)\big) \\ &= f\big(\epsilon(x)h(x)n(x)h_0\overline{n}(x^{-1})\big) \\ &= \epsilon(x)\mu_{(r),\gamma_0}^{1/4}(h(x))\chi_{r,\omega}(h(x))\chi_{r,\omega}(h_0)f\big(\overline{n}(x^{-1})\big) . \end{aligned} \tag{11}$$

We shall now restrict x to be a square, so that $\chi_{r,\omega}(h(x))$ is the identity. As $x \to \infty$, $\overline{n}(x^{-1}) \to e$ and since f is smooth we may take x large enough that $f(\overline{n}(x^{-1})) = f(e)$. Substituting the value of the modular character in this situation we obtain

$$f\big(\mathbf{s}(w_0)\big) = \epsilon(x)|x|^{(1-r)/2}\chi_{r,\omega}(h_0)f(e)\,. \tag{12}$$

From equation (12) we conclude in turn that $f(e) \neq 0$, that $\epsilon(x)$ is constant for large x and that $|x|^{(1-r)/2}$ is constant for large x. However $r \neq 1$ and so we have reached a contradiction. $\square$

4. Derivatives and Semi–Whittaker Models

The theory of derivatives was introduced by Bernstein and Zelevinsky in its fully elaborated form in [BZ2] after having been prefigured in [BZ1] and [GeK]. It will be the major technical tool on which the results of this section rely and we shall recall the basic definitions of the theory. Most of the derivatives of the exceptional representations are computed in [BuG] and with the aid of the results in the previous section the list will be completed here. It has been established by Bump and Ginzburg (in [BuG]) that the exceptional representations, which do not generally possess Whittaker models, do have similar models with respect to certain degenerate characters of $\mathrm{N}(\gamma_0)$. This section will close with a number of results on these so-called semi-Whittaker models and their interaction with the derivatives. We seek to place the theory of the associated semi-Whittaker functions on a similar footing to the more familiar theory of Whittaker functions, which will be our guide. It will be possible to obtain somewhat more precise results than in that theory since we are dealing with specific representations. We shall also see that there are certain differences arising from the degeneracy of the character with respect to which the semi-Whittaker functions are defined.

We let $\mathrm{P}(r) \leq \mathrm{G}(r)$ be the subgroup of $\mathrm{G}(r)$ defined by

$$\mathrm{P}(r) = \left\{ \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} \middle| g \in \mathrm{G}(r-1), x \in F^{r-1} \right\}.$$

Equivalently $\mathrm{P}(r)$ is the stabilizer in $\mathrm{G}(r)$ of the vector $(0, \ldots, 0, 1) \in F^r$ when $\mathrm{G}(r)$ acts on F^r on the right in the standard way. The group $\mathrm{P}(r)$ is called the *mirabolic subgroup* of $\mathrm{G}(r)$. If $s > r$ then $\mathrm{P}(r)$ may also be regarded as a subgroup of $\mathrm{G}(s)$ via the usual embedding $\mathrm{G}(r) \to \mathrm{G}(s)$. Following the notation of the last section we let $\widetilde{\mathrm{P}}(r)$ be the pre-image of $\mathrm{P}(r)$ under the map $p_r : \widetilde{\mathrm{G}}(r) \to \mathrm{G}(r)$. If $r \geq \ell$ then we define $\mathrm{Y}_r(\ell) = \mathrm{N}((\ell, r-\ell), (\ell-1, 1, r-\ell))$ with $\mathrm{Y}_r(r)$ abbreviated to Y_r. Displayed schematically we have

$$\mathrm{Y}_r(\ell) = \left\{ \begin{pmatrix} I_{\ell-1} & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{r-\ell} \end{pmatrix} \middle| x \in F^{\ell-1} \right\}$$

and $\mathrm{Y}_r(\ell)$ is equal to the unipotent radical of $\mathrm{P}(\ell)$ regarded as a subgroup of $\mathrm{G}(r)$. The image of $\mathrm{Y}_r(\ell)$ under the homomorphism $\mathbf{s} : \mathrm{N}(\gamma_0) \to \mathrm{N}^*(\gamma_0)$ will be denoted by $\mathrm{Y}^*_r(\ell)$.

Let us fix a non-trivial continuous additive character ψ of the field F. This gives rise to a character ψ of $\mathrm{N}(\gamma_0)$ defined by

$$\psi((n_{ab})) = \psi\left(\sum_{i=1}^{r-1} n_{i,i+1}\right).$$

In addition we obtain characters θ^1 and θ^2 of $\mathrm{N}(\gamma_0)$ defined by

$$\theta^1((n_{ab})) = \begin{cases} \psi(n_{23} + n_{45} + \cdots + n_{r-2,r-1}) & \text{if } r \text{ is even} \\ \psi(n_{12} + n_{34} + \cdots + n_{r-2,r-1}) & \text{if } r \text{ is odd} \end{cases}$$

and

$$\theta^2((n_{ab})) = \begin{cases} \psi(n_{12} + n_{34} + \cdots + n_{r-1,r}) & \text{if } r \text{ is even} \\ \psi(n_{23} + n_{45} + \cdots + n_{r-1,r}) & \text{if } r \text{ is odd.} \end{cases}$$

Observe that we always have $\psi = \theta^1 \cdot \theta^2$ as characters of $\mathrm{N}(\gamma_0)$. Since $\mathbf{s} : \mathrm{N}(\gamma_0) \to \mathrm{N}^*(\gamma_0)$ is an isomorphism, any character of a subgroup of $\mathrm{N}(\gamma_0)$ may also be regarded as a character of the corresponding subgroup of $\mathrm{N}^*(\gamma_0)$. We shall do this for ψ, θ^1 and θ^2 without any change in notation.

For any ℓ-group G we let $\mathcal{A}(G)$ denote the category of all algebraic representations of G. The theory of derivatives for representations of $\widetilde{\mathrm{G}}\,(r)$ rests upon the properties of four functors

$$\mathcal{A}(\widetilde{\mathrm{P}}(r-1)) \xrightarrow{\Phi^+} \mathcal{A}(\widetilde{\mathrm{P}}(r)) \xrightarrow{\Phi^-} \mathcal{A}(\widetilde{\mathrm{P}}(r-1))$$
$$\mathcal{A}(\widetilde{\mathrm{G}}\,(r-1)) \xrightarrow{\Psi^+} \mathcal{A}(\widetilde{\mathrm{P}}(r)) \xrightarrow{\Psi^-} \mathcal{A}(\widetilde{\mathrm{G}}\,(r-1))$$

which are the exact analogues in the metaplectic setting of the eponomous functors introduced in [BZ2]. Thus Φ^- is the Jacquet functor with respect to (Y_r^*, ψ), Ψ^- is the Jacquet functor with respect to $(\mathrm{Y}_r^*, 1)$, Φ^+ is the composition of the ψ-twisted extension from $\widetilde{\mathrm{P}}(r-1)$ to $\widetilde{\mathrm{P}}(r-1) \cdot \mathrm{Y}_r^*$ with compactly supported induction and Ψ^+ is the trivial extension from $\widetilde{\mathrm{G}}\,(r-1)$ to $\widetilde{\mathrm{G}}\,(r-1) \cdot \mathrm{Y}_r^* = \widetilde{\mathrm{P}}(r)$. It is important to remember that both the Jacquet functors and the extension functors are normalized by a suitable modular character. Thus Ψ^+ includes a twist by the square-root of the modular character of $\widetilde{\mathrm{G}}\,(r-1)$ acting on Y_r^* and similar remarks apply to the other functors. The basic properties of $\Phi^{\pm}$ and $\Psi^{\pm}$ as recorded in §3 of [BZ2] remain unchanged in the metaplectic setting. Indeed they are formal consequences of the properties of induction and Jacquet functors which were proved in [BZ1] for general ℓ-groups. We also note that the four functors take genuine representations into genuine representations.

If $\tau \in \mathcal{A}(\widetilde{\mathrm{P}}(r))$ then following [BZ2] we define a sequence of representations $\tau^{(k)} \in \mathcal{A}(\widetilde{\mathrm{G}}(r-k))$ by

$$\tau^{(k)} = \Psi^{-} \circ \left(\Phi^{-}\right)^{k-1}(\tau)$$

for $k = 1, \dots, r$ and call $\tau^{(k)}$ the k^{th} *derivative* of τ. If $\pi \in \mathcal{A}(\widetilde{\mathrm{G}}(r))$ then the k^{th} derivative of the representation $\pi|_{\widetilde{\mathrm{P}}(r)}$ is also referred to as the k^{th} *derivative* of π and denoted by $\pi^{(k)}$. In this case the notation is naturally extended by setting $\pi^{(0)} = \pi$.

Proposition 1: *Let χ be a character of $F^{\times}$ and ω_r a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. If r is odd then*

$$\vartheta_r^{(1)}(\chi, \omega_r) \cong |\det|^{-1/4} \otimes \vartheta_{r-1}(\chi, \omega_{r-1})$$

where ω_{r-1} is the unique character of $\widetilde{\mathrm{Z}}(r-1)$ compatible with χ. If r is even then

$$\vartheta_r^{(1)}(\chi, \omega_r) \cong \bigoplus_{\omega_{r-1}} |\det|^{-1/4} \otimes \vartheta_{r-1}(\chi, \omega_{r-1})$$

where the sum is over all characters of $\widetilde{\mathrm{Z}}(r-1)$ compatible with χ.

Proof: By definition $\vartheta_r^{(1)}(\chi, \omega_r) = \Psi^{-}\left(\vartheta_r(\chi, \omega_r)|_{\widetilde{\mathrm{P}}(r)}\right)$ and since Jacquet functors and restriction functors commute this may be expressed alternatively as

$$\vartheta_r^{(1)}(\chi, \omega_r) = \varphi_{(r),(r-1,1)}\left(\vartheta_r(\chi, \omega_r)\right)|_{\widetilde{\mathrm{G}}(r-1)}.$$

By Theorem 1 (a) of the previous section

$$\varphi_{(r),(r-1,1)}\big(\vartheta_r(\chi,\omega_r)\big) \cong \mu_{(r),(r-1,1)}^{-1/4} \otimes \vartheta_{(r-1,1)}(\chi,\omega_r)$$

and using (d) and (b) of that Theorem

$$\begin{aligned}\vartheta_{(r-1,1)}(\chi,\omega_r) &\cong \vartheta_{r-1}(\chi,\omega_{r-1}) \mathbin{\widetilde{\otimes}_{\omega_r}} \vartheta_1(\chi,\omega_1)\\ &\cong \vartheta_{r-1}(\chi,\omega_{r-1}) \mathbin{\widetilde{\otimes}_{\omega_r}} \chi\end{aligned}$$

where ω_1 and ω_{r-1} are compatible with χ. Using (1) of section 2 we obtain, for $g \in \widetilde{\mathrm{G}}\,(r-1)$,

$$\mu_{(r),(r-1,1)}(g,1) = |\det(g)|$$

and so it remains to evaluate

$$\big(\vartheta_{r-1}(\chi,\omega_{r-1}) \mathbin{\widetilde{\otimes}_{\omega_r}} \chi\big)|_{\widetilde{\mathrm{G}}(r-1)}\,.$$

Now $\vartheta_{r-1}(\chi,\omega_{r-1}) \mathbin{\widetilde{\otimes}_{\omega_r}} \chi$ occurs as an indecomposable summand of

$$\mathrm{ind}_{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(1)}^{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(1)}\big(\vartheta_{r-1}(\chi,\omega_{r-1}) \otimes \chi^2\big) \tag{1}$$

where χ^2 denotes the restriction of χ to $\widetilde{\mathrm{G}}^2\,(1)$ and using the Mackey subgroup theorem we obtain

$$\begin{aligned}&\mathrm{ind}_{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(1)}^{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(1)}\big(\vartheta_{r-1}(\chi,\omega_{r-1}) \otimes \chi^2\big)|_{\widetilde{\mathrm{G}}(r-1)}\\ \cong &\bigoplus_{g\in\widetilde{\mathrm{G}}(1)/\widetilde{\mathrm{G}}^2(1)} {}^{g}\vartheta_{r-1}(\chi,\omega_{r-1})\\ \cong &\bigoplus_{\lambda\in(\widetilde{\mathrm{G}}(r-1)/\widetilde{\mathrm{G}}^2(r-1))^{\widehat{}}} \lambda\otimes\vartheta_{r-1}(\chi,\omega_{r-1})\end{aligned} \tag{2}$$

since conjugation by $g \in \widetilde{\mathrm{G}}\,(1)$ induces a character of $\widetilde{\mathrm{G}}\,(r-1)$ trivial on $\widetilde{\mathrm{G}}^2\,(r-1)$ and every such character arises in this way. If r is even then $(r-1)$ is odd and so the representation (1) is indecomposable and hence equal to $\vartheta_{r-1}(\chi, \omega_{r-1}) \mathbin{\widetilde{\otimes}_\omega} \chi$. Moreover, we have

$$\lambda \otimes \vartheta_{r-1}(\chi, \omega_{r-1}) \cong \vartheta_{r-1}(\chi, \lambda|_{\widetilde{\mathrm{Z}}(r-1)} \otimes \omega_{r-1})$$

by a double application of Proposition 2 of section 3. Since $\lambda \otimes \omega_{r-1}$ runs over all characters compatible with χ as λ runs over the range given in (2) we obtain the second isomorphism in the Proposition.

If r is odd then $(r-1)$ is even and so the representation (1) is the direct sum of the reprepresentations $\vartheta_{r-1}(\chi, \omega_{r-1}) \mathbin{\widetilde{\otimes}_\omega} \chi$ as ω runs over all the characters of $\widetilde{\mathrm{Z}}(r)$ for which this is defined. Alternatively these summands may be obtained by fixing $\omega = \omega_r$ and forming the representations $\nu \otimes \left(\vartheta_{r-1}(\chi, \omega_{r-1}) \mathbin{\widetilde{\otimes}_{\omega_r}} \chi\right)$ where ν runs over $\left(\widetilde{\mathrm{G}}\,(1)\,/\widetilde{\mathrm{G}}^2\,(1)\,\right)^\frown$. It follows that all these summands have isomorphic restrictions to $\widetilde{\mathrm{G}}\,(r-1)$, of which there are $[F^\times : (F^\times)^2]$ in all. On the other hand by Proposition 4 of section 1 all the summands in (2) are isomorphic to $\vartheta_{r-1}(\chi, \omega_{r-1})$ and there are $[F^\times : (F^\times)^2]$ of them. Combining these facts gives the first isomorphism in the Proposition. □

Next we would like to compute the second derivative of an exceptional representation. It is shown in [BuG] that this must again be an exceptional representation, but to identify precisely which one we must work a little harder. Note that since $\mathrm{N}^*(r) \leq \widetilde{\mathrm{G}}^2\,(r)$ for every r the conjugation action of $\widetilde{\mathrm{Z}}^1(r)$ on $\mathrm{N}^*(r)$ is trivial and hence $\widetilde{\mathrm{Z}}^1(r)$ stabilizes every character of $\mathrm{N}^*(r)$. It follows from this that if any Jacquet functor with respect to $\mathrm{N}^*(r)$ is applied to a representation of $\widetilde{\mathrm{G}}\,(r)$ the result may be regarded as a representation of $\widetilde{\mathrm{Z}}^1(r)$. This observation applies in particular to the derivatives. Recall from the discussion preceding Proposition 3 in the previous section that, since we have fixed an additive character of F, we have in every case a correspondence between characters of $F^\times$ and characters of $\widetilde{\mathrm{Z}}^1(r)$. After these remarks we can state the following result of Gelbart and Piatetski-Shapiro, which appears in [GeP].

Lemma 1: *Let χ be a character of $F^\times$ and ω_2 be the unique character of $\widetilde{\mathrm{Z}}(2)$ compatible with χ. Then the group $\widetilde{\mathrm{Z}}^1(2)$ acts on $\vartheta_2^{(2)}(\chi, \omega_2)$ via the character $(\chi^2)_\psi$ where ψ is the additive character of F with respect to which the derivative is formed.*

Proof: This is Theorem 2.2 of [GeP]. Note that it follows from the definition of $\vartheta_2(\chi, \omega_2)$ together with their Proposition 2.3.3 that $\vartheta_2(\chi, \omega_2)$ is the representation which they denote by r_{χ^2}. □

Proposition 2: *Let χ be a character of $F^\times$ and ω_r a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. Then*

$$\vartheta_r^{(2)}(\chi, \omega_r) \cong |\det|^{-1/2} \otimes \vartheta_{r-2}(\chi, \omega_{r-2})$$

where

$$\omega_r = \left(\omega_{r-2} \otimes (\chi^2)_\psi\right)|_{\widetilde{\mathrm{Z}}(r)}\,. \tag{3}$$

Proof: The second derivative functor factors through $\varphi_{(r),(r-2,2)}$ and we begin with the isomorphisms

$$\varphi_{(r),(r-2,2)}\big(\vartheta_r(\chi,\omega_r)\big) \cong \mu_{(r),(r-2,2)}^{-1/4} \otimes \vartheta_{(r-2,2)}(\chi,\omega_r) \tag{4}$$

and

$$\vartheta_{(r-2,2)}(\chi,\omega_r) \cong \vartheta_{r-2}(\chi,\nu)\ \widetilde{\otimes}_{\omega_r}\ \vartheta_2(\chi,\omega_2) \tag{5}$$

which are furnished by Theorem 1 of the previous section. Here ν is any character of $\widetilde{\mathrm{Z}}(r-2)$ compatible with χ and ω_2 is as in Lemma 1. For $g \in \widetilde{\mathrm{G}}\,(r-2)$ we have

$$\mu_{(r),(r-2,2)}(g,1) = |\det(g)|^2$$

and so it remains to apply the Jacquet functor

$$\mathcal{A}\big(\widetilde{\mathrm{G}}\,(r-2)\ \widetilde{\times}\ \widetilde{\mathrm{G}}\,(2)\big) \to \mathcal{A}\big(\widetilde{\mathrm{G}}\,(r-2)\big)$$

which corresponds to the character ψ of $\mathrm{N}^*(2)$ to the right-hand side of (5). This may be done in stages by first restricting the representation to $\widetilde{\mathrm{G}}\,(r-2)\ \widetilde{\times}\ \widetilde{\mathrm{G}}^2\,(2)$ and then applying the Jacquet functor with respect to ψ in the second factor.

We may choose an irreducible subrepresentation σ of $\vartheta_2(\chi,\omega_2)^2$ so that

$$\vartheta_{r-2}(\chi,\nu)\ \widetilde{\otimes}_{\omega_r}\ \vartheta_2(\chi,\omega_2) \cong \mathrm{ind}_{\widetilde{\mathrm{G}}(r-2)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(2)}^{\widetilde{\mathrm{G}}(r-2)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(2)}\big(\vartheta_{r-2}(\chi,\nu)\otimes\sigma\big)$$

and hence

$$\begin{aligned}&\big(\vartheta_{r-2}(\chi,\nu)\ \widetilde{\otimes}_{\omega_r}\ \vartheta_2(\chi,\omega_2)\big)\big|_{\widetilde{\mathrm{G}}(r-2)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(2)}\\ &\cong \bigoplus_{g\in\widetilde{\mathrm{G}}(2)/\widetilde{\mathrm{G}}^2(2)} \big(\chi_g\otimes\vartheta_{r-2}(\chi,\nu)\big)\otimes {}^g\sigma\end{aligned} \tag{6}$$

where, as usual, $\chi_g(h) = \big(\det(h), \det(g)\big)$. This implies that for all elements $g \in \widetilde{\mathrm{G}}\,(2)\,/\widetilde{\mathrm{G}}^2\,(2)$ we have

$$\omega_r = \big((\chi_g \otimes \nu) \otimes \omega_{g\sigma}\big)\big|_{\widetilde{\mathrm{Z}}(r)} . \tag{7}$$

We know from [GeP] that $\vartheta_2^{(2)}(\chi, \omega_2)$ is one-dimensional and the Lemma says that $\widetilde{\mathrm{Z}}^1(2)$ acts on it by $(\chi^2)_\psi$. Thus, after applying the $(\mathrm{N}^*(2)\,, \psi)$ Jacquet functor, all the summands in (6) give zero except for the one which satisfies $\omega_{g\sigma} = (\chi^2)_\psi$. This summand yields a representation isomorphic to $\chi_g \otimes \vartheta_{r-2}(\chi, \nu)$. Now this is an exceptional representation and it follows at once from (7) that its central character satisfies the stated condition. □

Note that since r and $r-2$ have the same parity

$$\widetilde{\mathrm{Z}}(r) \subseteq \widetilde{\mathrm{Z}}(r-2)\; \widetilde{\times}\; \widetilde{\mathrm{Z}}^1(2)$$

and hence equation (3) serves to determine ω_r from ω_{r-2} or vice versa.

Proposition 3: *Let χ be a character of $F^\times$ and ω a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. Then $\vartheta_r^{(3)}(\chi, \omega) = 0$.*

Proof: Proposition 2 of the previous section shows that it is sufficient to assume that $\chi = \chi_0$. Using the fact that the third derivative functor factors through $\varphi_{(r),(r-3,3)}$ and Theorem 1, parts (a) and (d) of the previous section we are reduced to showing that $\vartheta_3^{(3)}(\chi_0, \omega) = 0$. But this is true by Lemma 6, §4 of [FKS]. □

For non-dyadic local fields the fact that $\vartheta_r^{(r)}(\chi, \omega) = 0$ for $r \geq 3$ is proved in [KaP], Theorem I.3.5. Thus for these fields the Proposition may be strengthened to $\vartheta_r^{(k)}(\chi, \omega) = 0$ for $k \geq 3$. One expects this to remain true for dyadic fields also, but although it is stated in [FKS] that $\vartheta_r^{(r)}(\chi_0, \omega) = 0$ for $r \geq 3$ even in the dyadic case, the proof given there is only complete when $r = 3$. Thus we must at present restrict ourselves to Proposition 3.

We now turn to a discussion of the semi-Whittaker models of the exceptional representations.

Definition 1: *A semi-Whittaker functional of the first kind on a representation $\pi \in \mathcal{A}(\widetilde{\mathrm{G}}\,(r))$ is a complex linear functional λ on E_π which satisfies*

$$\lambda\big(\pi(n)\xi\big) = \theta^1(n)\lambda(\xi)$$

$$\lambda\big(\pi(\epsilon)\xi\big) = \epsilon\lambda(\xi)$$

for all $n \in \mathrm{N}^*(\gamma_0)$, $\epsilon \in \mu_2$ and $\xi \in E_\pi$. A semi-Whittaker functional of the second kind is defined similarly replacing θ^1 by θ^2.

As usual there is a one-to-one correspondence between semi-Whittaker functionals on π and embeddings of π into

$$\mathrm{Ind}_{\mu_2 \cdot \mathrm{N}^*(\gamma_0)}^{\widetilde{\mathrm{G}}(r)}\big(\eta \cdot \theta^j\big)$$

where η denotes the non-trivial character of μ_2 and $j = 1$ or 2 as appropriate. An embedding of this kind will be called a *semi-Whittaker model* (of the first or second kind). The space of the semi-Whittaker models of π may be identified with the dual of the image of π under the Jacquet functor corresponding to $\big(\mathrm{N}^*(\gamma_0)\,, \theta^j\big)$. Thus, as we remarked before Lemma 1, the space of semi-Whittaker models of π may be regarded as a representation of $\widetilde{\mathrm{Z}}^1(r)$.

Proposition 4: *Let χ be a character of $F^\times$ and ω a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. The space of semi-Whittaker models of $\vartheta_r(\chi,\omega)$ of the second kind is one-dimensional. The space of semi-Whittaker models of $\vartheta_r(\chi,\omega)$ of the first kind is one-dimensional if r is odd and of dimension $[F^\times : (F^\times)^2]$ if r is even. Furthermore if r is even and ω^1 is an extension of ω to $\widetilde{\mathrm{Z}}^1(r)$ then the space of semi-Whittaker functionals λ of the first kind which satisfy*

$$\lambda(\vartheta_r(z)\xi) = \omega^1(z)\lambda(\xi) \tag{8}$$

for all $z \in \widetilde{\mathrm{Z}}^1(r)$ and $\xi \in E_{\vartheta_r(\chi,\omega)}$ is one-dimensional.

Proof: From the remarks before the statement of the Proposition it follows that the dimension of the space of semi-Whittaker models of $\vartheta_r(\chi,\omega)$ is equal to the dimension of the image of $\vartheta_r(\chi,\omega)$ under the $(\mathrm{N}^*(\gamma_0), \theta^j)$–Jacquet functor, with $j = 1$ or 2 as appropriate. But from the definition of θ^j it is clear that this Jacquet functor may be regarded in all cases as the composition of a sequence of first and second derivative functors. Suppose first that r is odd. Then the Jacquet functor corresponding to θ^1 is equal to the composition $[2]^{(r-1)/2} \circ [1]$ and the Jacquet functor corresponding to θ^2 is equal to $[1] \circ [2]^{(r-1)/2}$, where $[1]$ and $[2]$ denote the first and second derivative functors respectively. Using Propositions 1 and 2 repeatedly we see that in either case the space of semi-Whittaker models is one-dimensional. Now suppose that r is even. The Jacquet functor corresponding to θ^2 is equal to $[2]^{r/2}$ (with notation as above) and from Proposition 2 we see that the space of semi-Whittaker models of the second kind is one-dimensional. The Jacquet functor corresponding to θ^1 is equal to $[1] \circ$

$[2]^{(r-2)/2} \circ [1]$ and again combining Propositions 1 and 2 shows that the dimension of the space of semi-Whittaker models of the first kind is equal to the cardinality of $\widetilde{\mathrm{G}}\,(r-1)\,/\widetilde{\mathrm{G}}^2\,(r-1)$, which has the stated value.

In order to obtain the last statement recall that

$$\vartheta_r(\chi,\omega)^2 \cong \bigoplus_{h\in\widetilde{\mathrm{G}}(r)/\widetilde{\mathrm{G}}^2(r)} {}^h\sigma \tag{9}$$

where σ is any one of the indecomposable summands of $\vartheta_r(\chi,\omega)^2$. Moreover as h runs over $\widetilde{\mathrm{G}}\,(r)\,/\widetilde{\mathrm{G}}^2\,(r)$, the central character of ${}^h\sigma$ runs over all the characters of $\widetilde{\mathrm{Z}}^1(r)$ which extend ω. Since there are $[F^\times : (F^\times)^2]$ of these we see from the results of the previous paragraph that all we need do is show that when the functor $[1] \circ [2]^{(r-2)/2} \circ [1]$ is applied to any of the summands on the the right of (9), the result is non-zero. From Proposition 1 we see that

$$\vartheta_r^{(1)}(\chi,\omega)^2 \cong \left(|\det|^{-1/4} \otimes \vartheta_{r-1}(\chi,\omega_{r-1})^2\right)^{\oplus[F^\times:(F^\times)^2]}, \tag{10}$$

where ω_{r-1} is compatible with χ, and we know already that if $[1] \circ [2]^{(r-2)/2}$ is applied to $\vartheta_{r-1}(\chi,\omega_{r-1})^2$, the result is non-zero. The proof will be completed by showing that the first derivative of each ${}^h\sigma$ is equal to $|\det|^{-1/4} \otimes \vartheta_{r-1}(\chi,\omega_{r-1})^2$. Comparing (9) and (10) we see that one of the ${}^h\sigma$ must have non-zero first derivative and we assume without loss of generality that it is σ. Since $(r-1)$ is odd, $\vartheta_{r-1}(\chi,\omega_{r-1})^2$ is irreducible and hence

$$\sigma^{(1)} \cong \left(|\det|^{-1/4} \otimes \vartheta_{r-1}(\chi,\omega_{r-1})^2\right)^{\oplus m} \tag{11}$$

for some m. But we may choose the coset representatives of $\widetilde{\mathrm{G}}\,(r)\,/\widetilde{\mathrm{G}}^2\,(r)$ to have the form $h = \mathbf{s}(\mathrm{diag}(1,\ldots,1,t))$ where $t \in F^\times$ and if this is done then each h commutes elementwise with $\widetilde{\mathrm{G}}^2(r-1)$. Also the first derivative is formed using the trivial character of Y_r^* and although h acts non-trivially on this group by conjugation it stabilizes this character. These facts imply that for all $h \in \widetilde{\mathrm{G}}(r)/\widetilde{\mathrm{G}}^2(r)$ we have $({}^h\sigma)^{(1)} \cong \sigma^{(1)}$ and from (10) and (11) it follows that

$$\left({}^h\sigma\right)^{(1)} \cong |\det|^{-1/4} \otimes \vartheta_{r-1}(\chi, \omega_{r-1})^2 ,$$

as required. □

Up to now we have allowed χ to be a general character of $F^\times$. However, as Proposition 2 of the previous section shows, the essential case is $\chi = \chi_0$ and it will be convenient to restrict attention to this case henceforth. It follows from Proposition 4 and the remarks before it that if r is even then $\widetilde{\mathrm{Z}}^1(r)$ acts on the space of semi-Whittaker functionals of the second kind by some character. This gives us a distinguished extension of the unique character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ_0 to $\widetilde{\mathrm{Z}}^1(r)$. In order to be able to make uniform statements encompassing the various cases, we shall define $\Omega^j(r)$ to be the set of characters, ω, of $\widetilde{\mathrm{Z}}^1(r)$ such that $\omega|_{\widetilde{\mathrm{Z}}(r)}$ is compatible with χ_0 and there is a non-zero semi-Whittaker functional of the j^{th} kind on $\vartheta_{r,\omega|_{\widetilde{\mathrm{Z}}(r)}}$ transforming under $\widetilde{\mathrm{Z}}^1(r)$ via ω. If r is odd then $\Omega^j(r)$ is simply the set of all characters of $\widetilde{\mathrm{Z}}(r)$ compatible with χ_0. If r is even then $\Omega^1(r)$ is the set of all extensions to $\widetilde{\mathrm{Z}}^1(r)$ of the unique character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ_0 and $\Omega^2(r)$ is the singleton set containing the distinguished extension of that character to $\widetilde{\mathrm{Z}}^1(r)$. Notice that, regardless of the parity of r and the value of j, the restriction of the characters in $\Omega^j(r)$ to $\widetilde{\mathrm{Z}}(r)$ always gives us exactly the set of characters of $\widetilde{Z}(r)$ compatible with χ_0. If $\omega \in \Omega^j(r)$ then we shall allow ourselves to write $\vartheta_{r,\omega}$ to mean $\vartheta_{r,\omega|_{\widetilde{Z}(r)}}$ in order to ease the notation. Notice also that, regardless of the parity of r, we have

$$\Omega^1(r) \cdot \Omega^2(r) = \{\rho \in Z(r)^\frown \mid \rho^2 = \chi_0\}.$$

Lemma 2: *If r is even then*

$$\Omega^2(r) = \{((\chi_0)_\psi \otimes \cdots \otimes (\chi_0)_\psi)|_{\widetilde{Z}^1(r)}\}$$

where $(\chi_0)_\psi$ is the character of $\widetilde{Z}^1(2)$ corresponding to χ_0 as in section 3 and there are $r/2$ factors in the tensor.

Proof: For r even let ω_r be the unique character of $\widetilde{Z}(r)$ compatible with χ_0. In the course of the proof of Proposition 2 we showed that there is a $\widetilde{G}\,(r-2) \,\widetilde{\times}\, \widetilde{Z}^1(2)$ isomorphism between $\vartheta^{(2)}_{r,\omega_r}$ and $\vartheta_{r-2,\omega_{r-2}} \otimes \vartheta^{(2)}_{2,\omega_2}$. Using this repeatedly we see that the space of semi-Whittaker functionals of the second kind on ϑ_{r,ω_r} is isomorphic to

$$\vartheta^{(2)}_{2,\omega_2} \otimes \cdots \otimes \vartheta^{(2)}_{2,\omega_2}$$

as a representation of $\widetilde{Z}^1(2) \,\widetilde{\times}\, \cdots \,\widetilde{\times}\, \widetilde{Z}^1(2)$. Since we know that $\widetilde{Z}^1(2)$ acts on $\vartheta^{(2)}_{2,\omega_2}$ via the character $(\chi_0)_\psi$, the Lemma follows. □

Definition 2: *Let $\omega \in \Omega^j(r)$ and λ be a non-zero semi-Whittaker functional of the j^{th} kind on $\vartheta_{r,\omega}$. Put*

$$\Xi^{j,\omega}_\xi(g) = \lambda(\vartheta_{r,\omega}(g)\xi)$$

for all $g \in \widetilde{\mathrm{G}}(r)$ *and all* $\xi \in E_{\vartheta_{r,\omega}}$. *We call* $\Xi^{j,\omega}_{\xi}$ *a semi-Whittaker function of the* j^{th} *kind.*

The map $\xi \mapsto \Xi^{j,\omega}_{\xi}$ intertwines the representation $\vartheta_{r,\omega}$ with the representation of $\widetilde{\mathrm{G}}(r)$ on the space

$$\{\Xi^{j,\omega}_{\xi} \mid \xi \in E_{\vartheta_{r,\omega}}\}$$

by right translation. Since $\vartheta_{r,\omega}$ is irreducible and $\Xi^{j,\omega}_{\xi}(e) \neq 0$ for some $\xi \in E_{\vartheta_{r,\omega}}$ by definition it follows that this map is an isomorphism. In particular if $\Xi^{j,\omega}_{\xi} \equiv 0$ then $\xi = 0$.

The semi-Whittaker function $\Xi^{j,\omega}_{\xi}$ transforms under $\widetilde{\mathrm{Z}}^1(r)$ via the character ω. This presents a momentary puzzle when r is even, since in that case it follows from the results of section 1 that the representation $\vartheta_{r,\omega}$ has a non-zero subspace transforming under $\widetilde{\mathrm{Z}}^1(r)$ by any given character whose restriction to $\widetilde{\mathrm{Z}}(r)$ is compatible with χ_0. To resolve this dilemma, let $\xi \in E_{\vartheta_{r,\omega}}$ satisfy $\vartheta_{r,\omega}(z)\xi = \eta(z)\xi$ for every $z \in \widetilde{\mathrm{Z}}^1(r)$, where η is some character of $\widetilde{\mathrm{Z}}^1(r)$ such that $\eta|_{\widetilde{\mathrm{Z}}(r)}$ is compatible with χ_0. With this notation we may calculate as follows:

$$\begin{aligned}
\omega(z)\Xi^{j,\omega}_{\xi}(g) &= \Xi^{j,\omega}_{\xi}(zg) \\
&= \Xi^{j,\omega}_{\xi}(gz) \cdot (t, \det(g)) \qquad \text{where } p_r(z) = tI_r \\
&= \Xi^{j,\omega}_{\vartheta_{r,\omega}(z)\xi}(g) \cdot (t, \det(g)) \\
&= \Xi^{j,\omega}_{\xi}(g) \cdot \eta(z)(t, \det(g))
\end{aligned}$$

where we have used equation (1) of section 1 to pass from the first to the second line. Thus $\Xi^{j,\omega}_{\xi}(g) = 0$ unless $\omega(z)\eta^{-1}(z) = (t, \det(g))$ for every $z \in \widetilde{\mathrm{Z}}^1(r)$. This equation determines $\det(g)$ as a class in $F^\times/(F^\times)^2$ and it follows that $\Xi_\xi^{j,\omega}$ is supported on a single coset of $\widetilde{\mathrm{G}}^2(r)$ in $\widetilde{\mathrm{G}}(r)$. Expressing a general $\xi \in E_{\vartheta_{r,\omega}}$ as a sum of vectors each of which transforms under $\widetilde{\mathrm{Z}}^1(r)$ by some character is therefore parallel to expressing $\Xi_\xi^{j,\omega}$ as the sum of its restrictions to the various cosets of $\widetilde{\mathrm{G}}^2(r)$ in $\widetilde{\mathrm{G}}(r)$.

We now wish to begin investigating the formal and analytic properties of the semi-Whittaker functions. Note, however, that although the set

$$\{\Xi_\xi^{j,\omega} \mid \xi \in E_{\vartheta_{r,\omega}}\}$$

has been defined unambiguously, the individual function $\Xi_\xi^{j,\omega}$ has not, since the semi-Whittaker functional used to produce it is at present defined only up to a scalar factor. Thus any equation between two semi-Whittaker functions must be understood as saying that the underlying semi-Whittaker functionals may be chosen so that the equality obtains. Alternatively, one may understand such an equality as expressing a proportionality between the two functions, with the constant depending only on the choice of semi-Whittaker functionals on either side.

In order to state our first result it will be convenient to introduce yet more notation. If $j = 1$ or 2 then we shall set

$$j' = \begin{cases} j & \text{if } r \text{ is even} \\ 3 - j & \text{if } r \text{ is odd.} \end{cases}$$

With this notation it is easy to check that we have

$$\theta^j({}^\iota n) = \overline{\theta^{j'}(n)}$$

for all $n \in \mathrm{N}^*(\gamma_0)$, where ι denotes the main involution of $\widetilde{\mathrm{G}}\,(r)$ defined in chapter 1, section 4.

Proposition 5: *Let $\omega \in \Omega^j(r)$. Then*

$$\Xi_{\xi}^{j,\omega}({}^{\iota}g) = \overline{\Xi_{\xi'}^{j',\omega}(g)}$$

for all $\xi \in E_{\vartheta_{r,\omega}}$ and $g \in \widetilde{\mathrm{G}}\,(r)$. Here $\xi \mapsto \xi'$ is the map whose existence was established in Proposition 4 of section 3.

Proof: Let λ be a non-zero semi-Whittaker functional of the j^{th} kind on $\vartheta_{r,\omega}$ transforming by ω under $\widetilde{\mathrm{Z}}^1(r)$. For $\xi \in E_{\vartheta_{r,\omega}}$ let us set $\lambda'(\xi) = \overline{\lambda(\xi')}$. Since $\xi \mapsto \xi'$ is conjugate linear, λ' is a non-zero linear functional on $E_{\vartheta_{r,\omega}}$. For $n \in \mathrm{N}^*(\gamma_0)$ we have

$$\begin{aligned}
\lambda'\big(\vartheta_{r,\omega}(n)\xi\big) &= \overline{\lambda\big((\vartheta_{r,\omega}(n)\xi)'\big)} \\
&= \overline{\lambda\big(\vartheta_{r,\omega}({}^{\iota}n)\xi'\big)} \\
&= \overline{\theta^j({}^{\iota}n)\lambda(\xi')} \\
&= \theta^{j'}(n)\overline{\lambda(\xi')} \\
&= \theta^{j'}(n)\lambda'(\xi)
\end{aligned}$$

and since ${}^{\iota}\epsilon = \epsilon$ for $\epsilon \in \mu_2$ we also have $\lambda'\big(\vartheta_{r,\omega}(\epsilon)\xi\big) = \epsilon\lambda'(\xi)$ by a similar calculation. Thus λ' is a semi-Whittaker functional of the $(j')^{\text{th}}$ kind. We know from chapter 1, section 4, Proposition 4 that ${}^{\iota}z = z^{-1}$ for $z \in \widetilde{\mathrm{Z}}^1(r)$ and since ω is a unitary character we get

$$\lambda'\big(\vartheta_{r,\omega}(z)\xi\big) = \overline{\lambda\big((\vartheta_{r,\omega}(z)\xi)'\big)}$$

$$= \overline{\lambda\big(\vartheta_{r,\omega}({}^{\iota}z)\xi'\big)}$$

$$= \overline{\lambda\big(\vartheta_{r,\omega}(z^{-1})\xi'\big)}$$

$$= \overline{\omega(z^{-1})\lambda(\xi')}$$

$$= \omega(z)\lambda'(\xi)$$

and so λ' transforms under $\widetilde{Z}^1(r)$ by ω. If we use λ' to form the semi-Whittaker function on the right of the proposed equation and λ to form that on the left we obtain

$$\Xi_{\xi'}^{j',\omega}(g) = \lambda'\big(\vartheta_{r,\omega}(g)\xi'\big)$$

$$= \overline{\lambda\big((\vartheta_{r,\omega}(g)\xi')'\big)}$$

$$= \overline{\lambda\big(\vartheta_{r,\omega}({}^{\iota}g)(\xi')'\big)}$$

$$= \overline{\lambda\big(\vartheta_{r,\omega}({}^{\iota}g)\xi\big)}$$

$$= \overline{\Xi_{\xi}^{j,\omega}({}^{\iota}g)}$$

from which the claim follows. □

Theorem 1 (Inductive Structure): *Let $r \geq 3$, $\ell \leq \lfloor r/2 \rfloor$, $\omega_r \in \Omega^2(r)$, $\Omega^2(2\ell) = \{\omega_{2\ell}\}$ and $\omega_{r-2\ell}$ be the character of $\widetilde{Z}^1(r-2\ell)$ determined by the equation*

$$\omega_r = (\omega_{r-2\ell} \otimes \omega_{2\ell})|_{\widetilde{Z}^1(r)} . \tag{12}$$

Let $\xi \in E_{\vartheta_{r,\omega_r}}$. Then there are two finite sequences of vectors $\{\xi_j^{r-2\ell}\}_{j=1}^{M}$ in

$E_{\vartheta_{r-2\ell,\omega_{r-2\ell}}}$ and $\{\xi_j^{2\ell}\}_{j=1}^{M}$ in $E_{\vartheta_{2\ell,\omega_{2\ell}}}$ such that

$$\Xi_{\xi}^{2,\omega_r}(g_1g_2) = |\det(g_1)|^{\ell/2}|\det(g_2)|^{-(r-2\ell)/4}\sum_{j=1}^{M}\Xi_{\xi_j^{r-2\ell}}^{2,\omega_{r-2\ell}}(g_1)\cdot\Xi_{\xi_j^{2\ell}}^{2,\omega_{2\ell}}(g_2) \qquad (13)$$

whenever $g_1, g_2 \in \widetilde{\mathrm{G}}\,(r)$ satisfy the following two conditions:

(1) We have

$$p(g_1) = \begin{pmatrix} * & 0 \\ 0 & I_{2\ell} \end{pmatrix} \qquad \text{and} \qquad p(g_2) = \begin{pmatrix} I_{r-2\ell} & 0 \\ 0 & * \end{pmatrix}$$

where $*$ denotes an arbitrary matrix of the appropriate size in either case.

(2) At least one of $\det(g_1)$ and $\det(g_2)$ is a square.

Here g_1 is being regarded indifferently as an element of $\widetilde{\mathrm{G}}\,(r)$ and of $\widetilde{\mathrm{G}}\,(r-2\ell)$ and similarly with g_2 and $\widetilde{\mathrm{G}}\,(2\ell)$.

Proof: From Theorem 1 of section 3 we know that

$$\varphi_{(r),(r-2\ell,2\ell)}(\vartheta_{r,\omega_r}) \cong \mu_{(r),(r-2\ell,2\ell)}^{-1/4} \otimes \vartheta_{r-2\ell,\omega_{r-2\ell}}\ \widetilde{\otimes}_{\omega_r}\ \vartheta_{2\ell,\omega_{2\ell}} \qquad (14)$$

and from Theorem 1 of section 1 it follows that

$$\varphi_{(r),(r-2\ell,2\ell)}(\vartheta_{r,\omega_r})|_{\widetilde{\mathrm{G}}^2(r-2\ell)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(2\ell)} \cong \mu_{(r),(r-2\ell,2\ell)}^{-1/4} \otimes \vartheta_{r-2\ell,\omega_{r-2\ell}}^2 \otimes \vartheta_{2\ell,\omega_{2\ell}} \qquad (15)$$

and

$$\varphi_{(r),(r-2\ell,2\ell)}(\vartheta_{r,\omega_r})|_{\widetilde{\mathrm{G}}(r-2\ell)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^2(2\ell)} \cong \mu_{(r),(r-2\ell,2\ell)}^{-1/4} \otimes \vartheta_{r-2\ell,\omega_{r-2\ell}} \otimes \vartheta_{2\ell,\omega_{2\ell}}^2 \qquad (16)$$

regardless of the parity of r. Note that the underlying spaces of the representations on the left in (14), (15) and (16) are equal and that the same is true of those on the right. Further, the linear map underlying all three isomorphisms is the same.

Let us denote by $\zeta \mapsto [\zeta]$ the map from $E_{\vartheta_{r,\omega_r}}$ to $E_{\vartheta_{r-2\ell,\omega_{r-2\ell}}} \otimes E_{\vartheta_{2\ell,\omega_{2\ell}}}$ given by composing the natural projection from $E_{\vartheta_{r,\omega_r}}$ to the space of $\varphi_{(r),(r-2\ell,2\ell)}(\vartheta_{r,\omega_r})$ with this linear map. Recalling that the Jacquet functor in (14) is normalized we have

$$\left[\vartheta_{r,\omega_r}(g_1 g_2)\zeta\right] = \mu^{1/4}_{(r),(r-2\ell,2\ell)}(g_1, g_2) \cdot \left(\vartheta_{r-2\ell,\omega_{r-2\ell}}(g_1) \otimes \vartheta_{2\ell,\omega_{2\ell}}(g_2)\right)[\zeta] \qquad (17)$$

provided that g_1 and g_2 satisfy conditions (1) and (2). Also $[\vartheta_{r,\omega_r}(n)\zeta] = [\zeta]$ for all $n \in \mathrm{N}^*((r-2\ell, 2\ell))$ by the definition of the Jacquet functor $\varphi_{(r),(r-2\ell,2\ell)}$.

It follows from Lemma 2 that $\omega_{r-2\ell} \in \Omega^2(r-2\ell)$. Hence we may choose a non-zero semi-Whittaker functional of the second kind, $\lambda_{r-2\ell}$, on $\vartheta_{r-2\ell,\omega_{r-2\ell}}$ transforming by $\omega_{r-2\ell}$ under $\widetilde{\mathrm{Z}}^1(r-2\ell)$. We may also choose a non-zero semi-Whittaker functional of the second kind, $\lambda_{2\ell}$, on $\vartheta_{2\ell,\omega_{2\ell}}$ transforming by $\omega_{2\ell}$ under $\widetilde{\mathrm{Z}}^1(2\ell)$. Let us define a functional on $E_{\vartheta_{r,\omega_r}}$ by

$$\lambda(\zeta) = (\lambda_{r-2\ell} \otimes \lambda_{2\ell})([\zeta])\,.$$

Since $\zeta \mapsto [\zeta]$ is onto, λ is non-zero. We have a factorization

$$\begin{aligned} \mathrm{N}^*(\gamma_0) &= \mathrm{N}^*((r-2\ell, 2\ell), \gamma_0) \cdot \mathrm{N}^*((r-2\ell, 2\ell)) \\ &\cong \left[\mathrm{N}^*(r-2\ell, \gamma_0) \times \mathrm{N}^*(2\ell, \gamma_0)\right] \cdot \mathrm{N}^*((r-2\ell, 2\ell)) \end{aligned} \qquad (18)$$

and θ^2 is trivial on the second factor and decomposes as $\theta^2_{r-2\ell} \cdot \theta^2_{2\ell}$ on the first. Also $\mathrm{N}^*(r-2\ell, \gamma_0) \leq \widetilde{\mathrm{G}}^2\,(r-2\ell)$ and $\mathrm{N}^*(2\ell, \gamma_0) \leq \widetilde{\mathrm{G}}^2\,(2\ell)$. If $n \in \mathrm{N}^*(\gamma_0)$ is written as $n = n_1 n_2 n_3$ as in (18) then

$$\lambda\left(\vartheta_{r,\omega_r}(n)\zeta\right) = \left(\lambda_{r-2\ell} \otimes \lambda_{2\ell}\right)\left[\vartheta_{r,\omega_r}(n_1 n_2 n_3)\zeta\right]$$

$$= \left(\lambda_{r-2\ell} \otimes \lambda_{2\ell}\right)\left(\vartheta_{r-2\ell,\omega_{r-2\ell}}(n_1) \otimes \vartheta_{2\ell,\omega_{2\ell}}(n_2)\right)\left[\vartheta_{r,\omega_r}(n_3)\zeta\right]$$

$$= \left(\theta^2_{r-2\ell}(n_1) \cdot \theta^2_{2\ell}(n_2)\right)\left(\lambda_{r-2\ell} \otimes \lambda_{2\ell}\right)[\zeta]$$

$$= \theta^2(n)\lambda(\zeta)\,.$$

Since both $\vartheta_{r-2\ell,\omega_{r-2\ell}}$ and $\vartheta_{2\ell,\omega_{2\ell}}$ are genuine it follows that λ is also. Hence λ is a semi-Whittaker functional of the second kind on ϑ_{r,ω_r}. Finally, the inclusions

$$\widetilde{\mathrm{Z}}^1(r) \leq \widetilde{\mathrm{Z}}^1(r-2\ell) \mathbin{\widetilde{\times}} \widetilde{\mathrm{Z}}^1(2\ell) \leq \widetilde{\mathrm{G}}\,(r-2\ell) \mathbin{\widetilde{\times}} \widetilde{\mathrm{G}}^2\,(2\ell)$$

and equation (12) serve to show, by a similar calculation, that λ transforms by ω_r under $\widetilde{\mathrm{Z}}^1(r)$. Using λ to define the semi-Whittaker function on the left hand side of (13), $\lambda_{r-2\ell}$ and $\lambda_{2\ell}$ to define the semi-Whittaker functions on the right hand side and observing that

$$\mu^{1/4}_{(r-2\ell,2\ell)}(g_1, g_2) = |\det g_1|^{\ell/2} |\det g_2|^{-(r-2\ell)/4}$$

now gives the equation with

$$[\xi] = \sum_{j=1}^{M} \xi_j^{r-2\ell} \otimes \xi_j^{2\ell}\,.$$

□

The next result is the analogue of Theorem 1 for semi-Whittaker functions of the 1st kind. Unfortunately it is very awkward to state, although the proof will be exactly analogous to the proof of Theorem 1.

Theorem 2 (Inductive Structure): *Let $r \geq 3$ and $\ell \leq r$ both be odd, $\omega_r \in \Omega^1(r)$, $\Omega^2(r-\ell) = \{\omega_{r-\ell}\}$ and $\omega_\ell \in \Omega^1(\ell)$ be the character of $\widetilde{\mathrm{Z}}^1(\ell)$ determined by the equation*

$$\omega_r = (\omega_{r-\ell} \otimes \omega_\ell)|_{\widetilde{\mathrm{Z}}^1(r)}\,. \tag{19}$$

Let $\xi \in E_{\vartheta_{r,\omega_r}}$. *Then there are finite sequences of vectors* $\{\xi_j^{r-\ell}\}_{j=1}^M$ *in* $E_{\vartheta_{r-\ell,\omega_{r-\ell}}}$ *and* $\{\xi_j^\ell\}_{j=1}^M$ *in* $E_{\vartheta_{\ell,\omega_\ell}}$ *such that*

$$\Xi_\xi^{1,\omega_r}(g_1g_2) = |\det(g_1)|^{\ell/4}|\det(g_2)|^{-(r-\ell)/4}\sum_{j=1}^M \Xi_{\xi_j^{r-\ell}}^{2,\omega_{r-\ell}}(g_1)\cdot\Xi_{\xi_j^\ell}^{1,\omega_\ell}(g_2) \qquad (20)$$

whenever $g_1, g_2 \in \widetilde{\mathrm{G}}\,(r)$ *satisfy condition (1) below and at least one of* $\det(g_1)$ *and* $\det(g_2)$ *is a square.*

(1) We have

$$p(g_1) = \begin{pmatrix} * & 0 \\ 0 & I_\ell \end{pmatrix} \qquad \text{and} \qquad p(g_2) = \begin{pmatrix} I_{r-\ell} & 0 \\ 0 & * \end{pmatrix}$$

where $*$ *denotes an arbitrary matrix of the appropriate size in either case.*

Let $r \geq 2$ *be even and* $\ell < r$ *be odd,* $\omega_r \in \Omega^1(r)$, $\omega_\ell \in \Omega^1(\ell)$ *and* $\omega_{r-\ell} \in \Omega^2(r-\ell)$. *Let* $\xi \in E_{\vartheta_{r,\omega_r}}$. *Then there are two finite sequences of vectors,* $\{\xi_{j,\chi}^{r-\ell}\}_{j=1,\chi\in\Omega^1(\ell)}^{M_\chi}$ *and* $\{\zeta_{j,\chi}^{r-\ell}\}_{j=1,\chi\in\Omega^2(r-\ell)}^{N_\chi}$, *in* $E_{\vartheta_{r-\ell,\omega_{r-\ell}}}$ *and two finite sequences of vectors,* $\{\xi_{j,\chi}^\ell\}_{j=1,\chi\in\Omega^1(\ell)}^{M_\chi}$ *and* $\{\zeta_{j,\chi}^\ell\}_{j=1,\chi\in\Omega^2(r-\ell)}^{N_\chi}$ *in* $E_{\vartheta_{\ell,\omega_\ell}}$ *such that whenever* $g_1, g_2 \in \widetilde{\mathrm{G}}\,(r)$ *satisfy condition (1) above and* $\det(g_1)$ *is a square we have*

$$\Xi_\xi^{1,\omega_r}(g_1g_2) = |\det(g_1)|^{\ell/4}|\det(g_2)|^{-(r-\ell)/4}\cdot \sum_{\chi\in\Omega^1(\ell)}\sum_{j=1}^{M_\chi} \Xi_{\xi_{j,\chi}^{r-\ell}}^{2,\omega_{r-\ell}}(g_1)\cdot\Xi_{\xi_{j,\chi}^\ell}^{1,\chi}(g_2) \qquad (21)$$

and whenever $g_1, g_2 \in \widetilde{\mathrm{G}}\,(r)$ *satisfy condition (1) above and* $\det(g_2)$ *is a square we have*

$$\Xi_\xi^{1,\omega_r}(g_1g_2) = |\det(g_1)|^{\ell/4}|\det(g_2)|^{-(r-\ell)/4}\cdot \sum_{\chi\in\Omega^2(r-\ell)}\sum_{j=1}^{N_\chi} \Xi_{\zeta_{j,\chi}^{r-\ell}}^{2,\chi}(g_1)\cdot\Xi_{\zeta_{j,\chi}^\ell}^{1,\omega_\ell}(g_2)\,. \qquad (22)$$

Proof: Since the structure of the proof is identical with that of the proof of Theorem 1 there seems to be little point in giving all the details. We begin with the isomorphism

$$\varphi_{(r),(r-\ell,\ell)}(\vartheta_{r,\omega_r}) \cong \mu_{(r),(r-\ell,\ell)}^{-1/4} \otimes \vartheta_{r-\ell,\omega_{r-\ell}} \mathbin{\widetilde{\otimes}_{\omega_r}} \vartheta_{\ell,\omega_\ell} \tag{23}$$

as before. If r is odd then $(r-\ell)$ is even and it follows from Theorem 1 of section 1 that

$$\varphi_{(r),(r-\ell,\ell)}(\vartheta_{r,\omega_r})|_{\widetilde{G}^2(r-\ell)\,\widetilde{\times}\,\widetilde{G}(\ell)} \cong \mu_{(r),(r-\ell,\ell)}^{-1/4} \otimes \vartheta_{r-\ell,\omega_{r-\ell}}^2 \otimes \vartheta_{\ell,\omega_\ell} \tag{24}$$

and

$$\varphi_{(r),(r-\ell,\ell)}(\vartheta_{r,\omega_r})|_{\widetilde{G}(r-\ell)\,\widetilde{\times}\,\widetilde{G}^2(\ell)} \cong \mu_{(r),(r-\ell,\ell)}^{-1/4} \otimes \vartheta_{r-\ell,\omega_{r-\ell}} \otimes \vartheta_{\ell,\omega_\ell}^2 . \tag{25}$$

On the other hand, if r is even then $(r-\ell)$ is odd and we have instead

$$\begin{aligned}
&\varphi_{(r),(r-\ell,\ell)}(\vartheta_{r,\omega_r})|_{\widetilde{G}^2(r-\ell)\,\widetilde{\times}\,\widetilde{G}(\ell)} \\
&\qquad \cong \mu_{(r),(r-\ell,\ell)}^{-1/4} \otimes \vartheta_{r-\ell,\omega_{r-\ell}}^2 \otimes \Bigg(\bigoplus_{\chi\in(\widetilde{G}(\ell)/\widetilde{G}^2(\ell))^\wedge} \chi \otimes \vartheta_{\ell,\omega_\ell} \Bigg) \\
&\qquad \cong \mu_{(r),(r-\ell,\ell)}^{-1/4} \otimes \vartheta_{r-\ell,\omega_{r-\ell}}^2 \otimes \Bigg(\bigoplus_{\chi\in\Omega^1(\ell)} \vartheta_{\ell,\chi} \Bigg)
\end{aligned} \tag{26}$$

on using Proposition 2 of section 3. Similarly we have

$$\begin{aligned}
&\varphi_{(r),(r-\ell,\ell)}(\vartheta_{r,\omega_r})|_{\widetilde{G}(r-\ell)\,\widetilde{\times}\,\widetilde{G}^2(\ell)} \\
&\qquad \cong \mu_{(r),(r-\ell,\ell)}^{-1/4} \otimes \Bigg(\bigoplus_{\chi\in\Omega^2(r-\ell)} \vartheta_{r-\ell,\chi} \Bigg) \otimes \vartheta_{\ell,\omega_\ell}^2
\end{aligned} \tag{27}$$

when r is even.

Regardless of the parity of r we have a decomposition

$$\begin{aligned}\mathrm{N}^*(\gamma_0) &= \mathrm{N}^*((r-\ell,\ell),\gamma_0)\cdot \mathrm{N}^*((r-\ell,\ell)) \\ &\cong \left[\mathrm{N}^*(r-\ell,\gamma_0)\times \mathrm{N}^*(\ell,\gamma_0)\right]\cdot \mathrm{N}^*((r-\ell,\ell)) \qquad (28)\end{aligned}$$

and since ℓ is odd it is easy to check that θ^1 is trivial on the second factor in (28) and factorizes as $\theta^2_{r-\ell}\cdot\theta^1_\ell$ on the first. With these observations in hand the proof is completed as in Theorem 1. □

Recall that Δ denotes the standard choice of positive simple system inside the root system of $\mathrm{G}(r)$ and that Δ may be identified with the set $\{(i,i+1)\mid i=1,\ldots,r-1\}$. We define two subsets, Δ_1 and Δ_2, of Δ by

$$\Delta_1=\begin{cases}\{(2,3),(4,5),\ldots,(r-2,r-1)\} & \text{if } r \text{ is even}\\ \{(1,2),(3,4),\ldots,(r-2,r-1)\} & \text{if } r \text{ is odd}\end{cases}$$

and

$$\Delta_2=\begin{cases}\{(1,2),(3,4),\ldots,(r-1,r)\} & \text{if } r \text{ is even}\\ \{(2,3),(4,5),\ldots,(r-1,r)\} & \text{if } r \text{ is odd.}\end{cases}$$

Notice that $\Delta=\Delta_1\cup\Delta_2$ and $\Delta_1\cap\Delta_2=\emptyset$. Corresponding to these sets of simple roots we define

$$\mathrm{T}_j(r)=\{h\in \mathrm{H}_r\mid h^\alpha=1 \quad\text{for}\quad \alpha\in\Delta_j\}.$$

We have $\mathrm{T}_1(r)\cap\mathrm{T}_2(r)=\mathrm{Z}^1(r)$ and $\mathrm{T}_1(r)\cdot\mathrm{T}_2(r)=\mathrm{H}_r$ for all r. The sets Δ_1 and Δ_2 have been chosen so that the torus $\mathrm{T}_j(r)$ stabilizes the (non-metaplectic) character θ^j. Moreover, the only class in $\mathrm{T}_j(r)/\mathrm{Z}^1(r)$ which stabilizes θ^{3-j} is the identity class. The factorizations

$$\mathrm{T}_2(r)=\mathrm{T}_2(r-2)\times\mathrm{Z}^1(2) \qquad (29)$$

and

$$\mathrm{T}_1(r) = \mathrm{T}_2(r-1) \times \mathrm{Z}^1(1) \tag{30}$$

will be useful below; they follow immediately from the definitions.

Let us set $\mathrm{T}_j^2(r) = \mathrm{T}_j(r) \cap \mathrm{H}_r^2$ where, as before,

$$\mathrm{H}_r^2 = \{h \in \mathrm{H}_r \mid h_j \in (F^\times)^2 \text{ for } j = 1, \ldots, r\}.$$

The metaplectic cocycle is identically 1 on the group $\widetilde{\mathrm{T}}_j^2(r) = p^{-1}(\mathrm{T}_j^2(r))$ and hence $\widetilde{\mathrm{T}}_j^2(r) \cong \mathrm{T}_j^2(r) \times \mu_2$. We let $\eta : \widetilde{\mathrm{T}}_j^2(r) \to \{\pm 1\}$ be the genuine character which is trivial on the first factor in this decomposition.

Proposition 6: *Let $\omega \in \Omega^j(r)$ and $\xi \in E_{\vartheta_{r,\omega}}$. Then*

$$\Xi_\xi^{j,\omega}(hg) = \eta(h)\mu_{(r),\gamma_0}^{1/4}(h)\, \Xi_\xi^{j,\omega}(g) \tag{31}$$

for all $h \in \widetilde{\mathrm{T}}_j^2(r)$ and $g \in \widetilde{\mathrm{G}}(r)$.

Proof: Note first that by replacing ξ with $\vartheta_{r,\omega}(g)\xi$ it suffices to prove the equation with $g = e$. We begin with $j = 2$; the proof will be by induction on r. If $r = 1$ then $\vartheta_{r,\omega}$ is simply a genuine character which agrees with η on $\widetilde{\mathrm{T}}_2^2(1)$, the modular character is trivial, $\Xi_\xi^{2,\omega}(g) = \vartheta_{r,\omega}(g)\xi \in \mathbb{C}$ and (31) follows. If $r = 2$ then $\widetilde{\mathrm{T}}_2^2(2) = \widetilde{\mathrm{Z}}^2(2)$ and so the modular character is trivial on $\widetilde{\mathrm{T}}_2^2(2)$ and since $\eta|_{\widetilde{\mathrm{Z}}^2(2)} = \omega|_{\widetilde{\mathrm{Z}}^2(2)}$ equation (31) follows from the transformation law for $\Xi_\xi^{j,\omega}$ under $\widetilde{\mathrm{Z}}^2(2)$. Now suppose that $r \geq 3$; we shall apply Theorem 1 with $\ell = 1$. Factor $h \in \widetilde{\mathrm{T}}_2^2(r)$ as $h = h'z$ where $h' \in \widetilde{\mathrm{T}}_2^2(r-2)$ and $z \in \widetilde{\mathrm{Z}}^2(2)$ as in the (square, metaplectic) version of (29). Then the conditions of Theorem 1 are satisfied and we have

$$\begin{aligned}
\Xi_{\xi}^{2,\omega}(h) &= \mu_{(r),(r-2,2)}^{1/4}(h)\sum_{j=1}^{M}\Xi_{\xi_j^{r-2}}^{2,\omega_{r-2}}(h')\cdot\Xi_{\xi_j^{2}}^{2,\omega_2}(z) \\
&= \mu_{(r),(r-2,2)}^{1/4}(h',z)\eta(h')\eta(z)\mu_{(r-2),\gamma_0}^{1/4}(h')\mu_{2,\gamma_0}^{1/4}(z)\cdot \\
&\qquad \sum_{j=1}^{M}\Xi_{\xi_j^{r-2}}^{2,\omega_{r-2}}(e)\cdot\Xi_{\xi_j^{2}}^{2,\omega_2}(e) \qquad \text{by the inductive hypothesis} \\
&= \eta(h)\mu_{(r),(r-2,2)}^{1/4}(h)\mu_{(r-2,2),\gamma_0}^{1/4}(h)\cdot\Xi_{\xi}^{2,\omega}(e) \\
&= \eta(h)\mu_{(r),\gamma_0}^{1/4}(h)\,\Xi_{\xi}^{2,\omega}(e) \qquad (32)
\end{aligned}$$

and the result follows in this case.

If $j=1$ and $r=1$ then (31) is trivially true. If $r\geq 2$ then applying Theorem 2, with $\ell=1$, reduces (31) to the same equation for $j=2$, which has already been proved, and the $r=1$, $j=1$ case. Thus we have (31) for $j=1$ also. □

Proposition 7: *Let $\omega\in\Omega^j(r)$ and $\xi\in E_{\vartheta_{r,\omega}}$. Then there is a constant $C_\xi>0$ such that $\Xi_{\xi}^{j,\omega}(h)=0$ whenever $h\in\widetilde{\mathrm{H}}_r$ and $|h^\alpha|\geq C_\xi$ for some $\alpha\in\Delta_j$.*

Proof: This result is proved just as for Whittaker functions in the non-metaplectic setting. If $\alpha=(i,i+1)$ then we set $n_\alpha(x)=I_r+xE_{i,i+1}$, where $E_{i,i+1}$ is the usual elementary matrix, and $n_\alpha^*(x)=\mathbf{s}(n_\alpha(x))$. Then if $h\in\widetilde{\mathrm{H}}_r$ we have $n_\alpha^*(x)h=hn_\alpha^*(h^{-\alpha}x)$. Since $\alpha\in\Delta_j$ we may fix some $x\in F$ with $\theta^j(n_\alpha^*(x))\neq 1$. The representation $\vartheta_{r,\omega}$ is smooth and so we may find $C_\xi>0$ such that $\vartheta_{r,\omega}\big(n_\alpha^*(bx)\big)\xi=\xi$ if $|b|\leq C_\xi^{-1}$. If $|h^\alpha|\geq C_\xi$ then we have

$$\theta^j(n_\alpha^*(x))\,\Xi_{\xi}^{j,\omega}(h)=\Xi_{\xi}^{j,\omega}\big(n_\alpha^*(x)h\big)$$

$$= \Xi_\xi^{j,\omega}\left(hn_\alpha^*(h^{-\alpha}x)\right)$$

$$= \Xi_\xi^{j,\omega}(h)$$

and so $\Xi_\xi^{j,\omega}(h) = 0$. □

Proposition 8: *Let $\omega \in \Omega^j(r)$ and $\xi \in E_{\vartheta_{r,\omega}}$. Then there is a constant $C_\xi > 0$ such that*

$$\left|\Xi_\xi^{j,\omega}(h)\right| \le C_\xi \mu_{(r),\gamma_0}^{1/4}(h) \tag{33}$$

for all $h \in \widetilde{\mathrm{H}}_r$.

Proof: Since we may choose a fixed set, S, of representatives for the finite group $\widetilde{\mathrm{H}}_r^2\backslash\widetilde{\mathrm{H}}_r$ and replace the constant C_ξ which we obtain below by

$$\max\{C_{\vartheta_{r,\omega}(s)\xi} \cdot \mu_{(r),\gamma_0}^{-1/4}(s) \mid s \in S\}$$

it suffices to assume that $h \in \widetilde{\mathrm{H}}_r^2$. We begin with $j = 2$ The inequality will be established by induction. Indeed, looking back at (32), it is clear that if we had (33) for $r = 1$ and $r = 2$ then by using Theorem 1 with $\ell = 1$ we would obtain (33) in general. The $r = 1$ case is the statement that the trivial character is bounded. Thus the heart of the proof is the inequality when $r = 2$.

In order to obtain this, observe that we may factor an element of $\widetilde{\mathrm{H}}_2^2$ as

$$\epsilon\,\mathbf{s}\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{s}\begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$$

where $\epsilon \in \mu_2$ and $x, y \in F^\times$. We know that

$$\left|\Xi_\xi^{2,\omega}\left(\epsilon\,\mathbf{s}\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{s}\begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}\right)\right| = \left|\Xi_\xi^{2,\omega}\left(\mathbf{s}\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}\right)\right|$$

and when $|x| \gg 1$ this vanishes by the previous Proposition. When $|x| \ll 1$ it follows from Proposition 3.3.4 of [GeP] that

$$\left|\Xi_{\xi}^{2,\omega}\left(\mathrm{s}\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}\right)\right| = k|x|^{1/2} = k\mu_{(2),(1,1)}^{1/4}\left(\mathrm{s}\begin{pmatrix} x^2 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

for some constant $k \geq 0$; this gives (33).

The case $j = 1$ then reduces to the case $j = 2$ on using Theorem 2 with $\ell = 1$.

□

These results do not exhaust what may be learnt about the form of the functions $\Xi_{\xi}^{j,\omega}$ on $\widetilde{\mathrm{H}}_r$ by using Theorems 1 and 2. For instance, a very similar inductive argument starting with Proposition 3.3.4 of [GeP] shows that if $h \in \widetilde{\mathrm{H}}_r^2$ and $|h^{\alpha}| \ll 1$ for all $\alpha \in \Delta_j$ then

$$\Xi_{\xi}^{j,\omega}(h) = k\eta(h)\mu_{(r),\gamma_0}^{1/4}(h)$$

for some constant k. From this we conclude that if $h \in \widetilde{\mathrm{H}}_r$ and $|h^{\alpha}| \ll 1$ for all $\alpha \in \Delta_j$ then $\Xi_{\xi}^{j,\omega}(h)$ is a linear combination of functions of the form $\mu_{(r),\gamma_0}^{1/4}(h)\chi(h)$, where χ is a genuine character of $\widetilde{\mathrm{H}}_r$ restricting to η on $\widetilde{\mathrm{H}}_r^2$.

We close this section on a different note by determining when the semi-Whittaker functions give rise to Kirillov models of the exceptional representations.

Proposition 9: *Suppose that F is not dyadic, let χ be a character of $F^{\times}$ and ω a character of $\widetilde{\mathrm{Z}}(r)$ compatible with χ. Then $\vartheta_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$ has a unique irreducible subrepresentation. The second derivative of this subrepresentation is equal to $\vartheta_r^{(2)}(\chi,\omega)$ and its other derivatives are zero.*

Proof: The claims are trivial if $r = 1$ and so we may assume in what follows that $r \geq 2$. We will allow ourselves to use without further comment various results on the functors $\Phi^{\pm}$ and $\Psi^{\pm}$ which have been proved by Bernstein and Zelevinsky for $\mathrm{G}(r)$ but which extend routinely to $\widetilde{\mathrm{G}}\,(r)$. By Corollary 5.22 in [BZ1], $\vartheta_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$ has finite length; let us consider an irreducible subrepresentation τ of $\vartheta_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$. We know from Proposition 5.12 in [BZ1] that there is a short exact sequence

$$0 \to \Phi^+\Phi^-\big(\vartheta_r(\chi,\omega)\big) \to \vartheta_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)} \to \Psi^+\Psi^-\big(\vartheta_r(\chi,\omega)\big) \to 0$$

and it follows from this and the irreducibility of τ that τ is either a subrepresentation of $\Phi^+\Phi^-\big(\vartheta_r(\chi,\omega)\big)$ or else that it is isomorphic to a subrepresentation of $\Psi^+\Psi^-\big(\vartheta_r(\chi,\omega)\big)$. However $\mathrm{N}^*((r-1,1))$ acts trivially on $\Psi^+\Psi^-\big(\vartheta_r(\chi,\omega)\big)$ and we know from Proposition 5 of section 3 that $\vartheta_r(\chi,\omega)$ does not contain any non-zero vectors fixed under this group. Thus τ is a subrepresentation of $\Phi^+\Phi^-\big(\vartheta_r(\chi,\omega)\big)$.

By Proposition 5.12 of [BZ1], $\Psi^-\Phi^+ = 0$ and the functor $\Phi^-\Phi^+$ is naturally equivalent to the identity functor. We have shown that $\tau \leq \Phi^+\Phi^-\big(\vartheta_r(\chi,\omega)\big)$ and it follows that $\tau^{(1)} = 0$ and that $\Phi^-(\tau) \leq \Phi^-\big(\vartheta_r(\chi,\omega)\big)$. As a consequence of this second observation we may regard $\tau^{(j)}$ as a subrepresentation of $\vartheta_r^{(j)}(\chi,\omega)$ for all j. From the hypothesis that F is not dyadic and the remarks following Proposition 3 we know that $\vartheta_r^{(j)}(\chi,\omega) = 0$ if $j \geq 3$ and hence $\tau^{(j)} = 0$ unless $j = 2$. By Corollary 5.14 of [BZ1], some non-trivial *iterated* derivative of τ must be non-zero and this is only possible if $\tau^{(2)} \neq 0$. But $\vartheta_r^{(2)}(\chi,\omega)$ is irreducible and hence $\tau^{(2)} = \vartheta_r^{(2)}(\chi,\omega)$.

If τ_1 and τ_2 were distinct irreducible subrepresentations of $\vartheta_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$ then we would have $\tau_1 \oplus \tau_2 \leq \dot{\vartheta}_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$ and $\tau_i^{(2)} = \vartheta_r^{(2)}(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$ for $i = 1, 2$. Since the second derivative is an additive functor, this leads to a contradiction. Thus $\vartheta_r(\chi,\omega)|_{\widetilde{\mathrm{P}}(r)}$ has a unique irreducible subrepresentation and the conclusions of the previous paragraph apply to it to complete the proof. $\square$

Theorem 3 (Existence of Kirillov Models): *Suppose that F is not dyadic. Let $\omega \in \Omega^2(r)$ and suppose that $\xi \in E_{\vartheta_{r,\omega}}$ satisfies $\Xi_\xi^{2,\omega}(p) = 0$ for all $p \in \widetilde{\mathrm{P}}(r)$. Then $\xi = 0$.*

Proof: Let us set

$$V = \left\{ \xi \in E_{\vartheta_{r,\omega}} \mid \Xi_\xi^{2,\omega}(p) = 0 \quad \forall \quad p \in \widetilde{\mathrm{P}}(r) \right\}.$$

Then V is a $\widetilde{\mathrm{P}}(r)$-submodule of $\vartheta_{r,\omega}|_{\widetilde{\mathrm{P}}(r)}$ and hence, if it is non-zero, it must contain the space of the unique irreducible subrepresentation of $\vartheta_{r,\omega}|_{\widetilde{\mathrm{P}}(r)}$ whose existence was proved in the previous Proposition. From that we would conclude that $V^{(2)} = E_{\vartheta_{r,\omega}}^{(2)}$. The functional $\zeta \mapsto \Xi_\zeta^{2,\omega}(e)$ on $E_{\vartheta_{r,\omega}}$ is non-zero and factors through the second derivative of $\vartheta_{r,\omega}$. Let us choose $\zeta \in E_{\vartheta_{r,\omega}}$ such that $\Xi_\zeta^{2,\omega}(e) \neq 0$ and then $\xi \in V$ such that ξ and ζ have the same image in $E_{\vartheta_{r,\omega}}^{(2)}$. Then $\Xi_\xi^{2,\omega}(e) = \Xi_\zeta^{2,\omega}(e) \neq 0$, a contradiction. Thus $V = \{0\}$, proving the Theorem. $\square$

Theorem 4 (Non-Existence of Kirillov Models): *Suppose that F is not a dyadic field. Let $\omega \in \Omega^1(r)$ and define V to be*

$$\{\xi \in E_{\vartheta_{r,\omega}} \mid \Xi_\xi^{1,\omega}(p) = 0 \quad \forall \quad p \in \widetilde{\mathrm{P}}(r)\,\}$$

if r is odd and

$$\{\xi \in E_{\vartheta_{r,\omega}} \mid \Xi_\xi^{1,\nu}(p) = 0 \quad \forall \quad p \in \widetilde{\mathrm{P}}(r) \ \ and\ \nu \in \Omega^1(r)\}$$

if r is even. Then V is the space of the unique irreducible subrepresentation of $\vartheta_{r,\omega}|_{\widetilde{\mathrm{P}}(r)}$.

Proof: Let $\tau \leq \vartheta_{r,\omega}|_{\widetilde{\mathrm{P}}(r)}$ be the unique irreducible subrepresentation. Suppose that $\xi \in E_\tau$. Then, since $\tau^{(1)} = 0$, the image of ξ in $E^{(1)}_{\vartheta_{r,\omega}}$ is zero. The functional $\zeta \mapsto \Xi_\zeta^{1,\nu}(e)$ (where $\nu \in \Omega^1(r)$ is ω if r is odd and arbitrary if r is even) factors through the first derivative and so $\Xi_\xi^{1,\nu}(e) = 0$. If $p \in \widetilde{\mathrm{P}}(r)$ then

$$\Xi_\xi^{1,\nu}(p) = \Xi^{1,\nu}_{\vartheta_{r,\omega}(p)\xi}(e) = 0$$

since $\vartheta_{r,\omega}(p)\xi \in E_\tau$ also. Thus $E_\tau \leq V$. The map $\xi \mapsto \Xi_\xi^{1,\nu}|_{\widetilde{\mathrm{P}}(r)}$ thus factors through the quotient V/E_τ. We know that $\vartheta_{r,\omega}|_{\widetilde{\mathrm{P}}(r)}/\tau \cong \Psi^+\big(\vartheta^{(1)}_{r,\omega}\big)$ and each $\Xi_\xi^{1,\nu}|_{\widetilde{\mathrm{G}}(r-1)}$ gives rise to a semi-Whittaker function of the 2^{nd} kind on one of the exceptional representations whose sum is $\vartheta^{(1)}_{r,\omega}$. If $\xi \in V$ then each of these semi-Whittaker functions is identically zero and it follows that the image of ξ in V/E_τ is zero; that is, $\xi \in E_\tau$. Thus $V = E_\tau$, as claimed. $\square$

We note that, by the remarks after Proposition 3, the hypothesis that F is not dyadic may be replaced in the last three results by the hypothesis that $r \leq 3$.

5. Tensor Products of Exceptional Representations I

In this section we shall study the (inner) tensor product of two exceptional representations. Since both representations are genuine, their tensor product is non-genuine and may be regarded as a representation of $\mathrm{G}(r)$. As such it is smooth but not generally admissible. Apart from the intrinsic interest of these representations, our main motivation comes from applications to the local symmetric square L-functions on $\mathrm{G}(r)$. From this point of view, the most interesting questions are the existence and the uniqueness of invariant pairings between the tensor product and a given irreducible, admissible representation of $\mathrm{G}(r)$ and our investigation will focus on these questions. When $r = 3$ they have been considered by Savin in [Sav] and we shall obtain extensions of some of his results to general r. His methods, relying as they do on explicit calculations with models of the exceptional representations on $\widetilde{\mathrm{G}}\,(3)$, do not (at least in the present state of knowledge) extend to general r and we shall have to employ other methods. The price to be payed for allowing general r is that our results will not be as complete as Savin's. Along the way we shall point out and correct two errors made by Bump and Ginzburg in one of the central arguments in [BuG]. This will necessitate a more thorough review of the properties of the functors $\Psi^{\pm}$ and $\Phi^{\pm}$ than was called for in the previous section.

As an aid to brevity, a character, ω, of $\widetilde{\mathrm{Z}}(r)$ will be called *suitable* if ω is compatible with χ_0. Thus whenever ω is a suitable character and γ is a partition of r we have a representation $\vartheta_{\gamma,\omega}$ defined as in section 3. We shall carry over all the associated notation from the earlier sections of this chapter. We observe that a suitable character takes its values in $\{\pm 1\}$ and hence is equal both to its complex conjugate and to its inverse. This implies that each of the representations $\vartheta_{\gamma,\omega}$ is self-conjugate and self-contragredient (see Theorem 1(e) of section 3).

We shall begin by determining the decomposition of a tensor product of exceptional representations when $\gamma = \gamma_0$. As well as being the easiest case of our problem, this information will be useful in what follows. If χ is a character of H_r then χ is the direct product of r characters of $\mathrm{H}_1 \cong F^\times$ and we shall denote this decomposition by writing $\chi = (\chi_1, \ldots, \chi_r)$. Also we shall sometimes write "1" instead of "χ_0" for the trivial character, in order to avoid having subscripts on χ in the same formula meaning different things.

Proposition 1: *Let ω and ν be suitable characters of $\widetilde{\mathrm{Z}}(r)$. Then $\vartheta_{\gamma_0,\omega} \otimes \vartheta_{\gamma_0,\nu}$ is the direct sum of all characters $\chi = (\chi_1, \ldots, \chi_r)$ of H_r which satisfy $\chi_j^2 = 1$ for $j = 1, \ldots, r$ and, if r is odd, also satisfy $\prod_{j=1}^{r} \chi_j = \omega \cdot \nu$.*.

Proof: By Theorem 1(b) of section 3 we have

$$\vartheta_{\gamma_0,\omega} \cong (\chi_0)_{r,\omega} = \chi_0 \,\widetilde{\otimes}_\omega\, \chi_0 \,\widetilde{\otimes}\, \ldots \,\widetilde{\otimes}\, \chi_0$$

where there are r factors in the metaplectic tensor product. From the construction of metaplectic tensor products in section 1 it easily follows from this that

$$\dim(E_{\vartheta_{\gamma_0,\omega}}) = [F^\times : (F^\times)^2]^{\lfloor r/2 \rfloor}$$

and hence that

$$\dim(E_{\vartheta_{\gamma_0,\omega} \otimes \vartheta_{\gamma_0,\nu}}) = \begin{cases} [F^\times : (F^\times)^2]^r & \text{if } r \text{ is even} \\ [F^\times : (F^\times)^2]^{r-1} & \text{if } r \text{ is odd.} \end{cases} \qquad (1)$$

This dimension is equal to the number of characters of H_r which satisfy the conditions of the statement. Hence if we can show that all of these characters occur in the given representation then the proof will be complete.

Since H_r is abelian, $\vartheta_{\gamma_0,\omega} \otimes \vartheta_{\gamma_0,\nu}$ is a sum of characters. The group $\widetilde{\mathrm{H}}_r^2$ acts on both $\vartheta_{\gamma_0,\omega}$ and $\vartheta_{\gamma_0,\nu}$ by the same character and the square of this character is trivial, so H_r^2 acts on the tensor product trivially. Hence each of the characters in the sum must satisfy the first condition of the statement. The group $\widetilde{\mathrm{Z}}(r)$ acts on $\vartheta_{\gamma_0,\omega}$ via ω and on $\vartheta_{\gamma_0,\nu}$ by ν and thus $\mathrm{Z}(r)$ acts on the tensor product via the non-genuine character $\omega \cdot \nu$. If r is even, $\mathrm{Z}(r) = \mathrm{Z}^2(r)$ and this imposes no further condition, but if r is odd, $\mathrm{Z}(r) = \mathrm{Z}^1(r)$ and we obtain the second condition of the statement. It follows that every character occurring in the decomposition does satisfy the conditions of the statement.

Let us choose one character χ of H_r which occurs in the decomposition of the tensor product. Say that $\xi \neq 0$ in the space of the tensor product transforms by χ. Fix $a \in \widetilde{\mathrm{H}}_r$ and consider the vector

$$\xi_a = \big(\vartheta_{\gamma_0,\omega}(a) \otimes \vartheta_{\gamma_0,\nu}(e)\big)\xi\,.$$

For $h \in \mathrm{H}_r$ we have

$$\begin{aligned}
&\big(\vartheta_{\gamma_0,\omega} \otimes \vartheta_{\gamma_0,\nu}\big)(h)\xi_a \\
&= \big(\vartheta_{\gamma_0,\omega} \otimes \vartheta_{\gamma_0,\nu}\big)(h)\big(\vartheta_{\gamma_0,\omega}(a) \otimes \vartheta_{\gamma_0,\nu}(e)\big)\xi \\
&= \big(\vartheta_{\gamma_0,\omega}(\tilde{h}a) \otimes \vartheta_{\gamma_0,\nu}(\tilde{h})\big)\xi \qquad \text{where } p(\tilde{h}) = h \\
&= [\tilde{h}, a]\big(\vartheta_{\gamma_0,\omega}(a) \otimes \vartheta_{\gamma_0,\nu}(e)\big)\big(\vartheta_{\gamma_0,\omega}(\tilde{h}) \otimes \vartheta_{\gamma_0,\nu}(\tilde{h})\big)\xi
\end{aligned}$$

$$= [\tilde{h}, a]\chi(h)\xi_a$$

and so ξ_a transforms by the character $h \mapsto [\tilde{h}, a]\chi(h)$ where, as usual, $[\tilde{h}, a]$ denotes the commutator of $\tilde{h}$ and a. A direct calculation using formula (3) of Chapter 1, section 3 shows that if $h = \mathrm{diag}(h_1, \dots, h_r)$ and $p(a) = \mathrm{diag}(a_1, \dots, a_r)$ then

$$[\tilde{h}, a] = \big(\det(h), \det(a)\big) \cdot \prod_{j=1}^{r} (h_j, a_j)^{-1}.$$

From this it is easy to check that the kernel of $a \mapsto [\,\cdot\,, a]$ is $\widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2$ and so we have produced $[\widetilde{\mathrm{H}}_r : \widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2]$ distinct characters of H_r which occur in the tensor product. But it follows from (1) that

$$[\widetilde{\mathrm{H}}_r : \widetilde{\mathrm{Z}}(r) \cdot \widetilde{\mathrm{H}}_r^2] = \dim\big(E_{\vartheta_{\gamma_0,\omega} \otimes \vartheta_{\gamma_0,\nu}}\big)$$

and the proof is complete. □

That part of Bump's and Ginzburg's paper [BuG] on the symmetric square L-functions on GL(r) which deals with the local theory of these functions centers around two results, the "unramified computation" in section 4 and the local functional equation in section 5. The central point in the latter is to establish a certain representation-theoretic uniqueness statement (Theorem 5.1 of that paper) from which the local functional equation flows. The argument by which they establish this uniqueness statement appears to be susceptible to substantial refinement and we shall do this below. However, it is first necessary to point out two rather subtle errors in Bump's and Ginzburg's proof and to show how they may be overcome, so that we are refining a valid argument.

One of the errors in question involves a misuse of the properties of the functors $\Psi^{\pm}$ and $\Phi^{\pm}$ and their close relative $\hat{\Phi}^{+}$ and so it may be helpful to recall some of their properties and also to clarify some points which have not been adequately addressed in the literature. Here and throughout this work, we employ the normalized versions of the functors as defined in section 3 of [BZ2]. We shall assume below that the reader is familiar with this work and also with [BZ1]. As we have already remarked the basic properties of these functors (and that is all we shall need here) remain unchanged when we pass to the metaplectic setting. The next Proposition recalls the properties of these functors which will particularly concern us here and adds one which is not explicit in [BZ2]. Recall that $\mathcal{A}(G)$ denotes the category of algebraic representations of G for any ℓ-group G. We use $\hat{}$ to denote contragredient; this should not cause any confusion with its use in the symbol $\hat{\Phi}^{+}$.

If $\tau_1, \tau_2, \tau_3, \tau_4 \in \mathcal{A}(\widetilde{\mathrm{P}}(r))$ then for purely algebraic reasons we have a natural isomorphism

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau_1 \otimes \tau_2, \tau_3 \otimes \hat{\tau}_4) \cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau_1 \otimes \tau_4, \tau_3 \otimes \hat{\tau}_2)$$

but since $(\hat{\tau})\hat{}$ is not generally isomorphic to τ many of the other familiar isomorphisms involving contragredients do not hold on $\widetilde{\mathrm{P}}(r)$. For example, it is not generally true that

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau_1, \tau_2) \cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\hat{\tau}_2, \hat{\tau}_1).$$

Such properties can be recovered for admissible representations of $\widetilde{\mathrm{P}}(r)$, but these are comparatively rare – even if $\pi \in \mathcal{A}(\widetilde{\mathrm{G}}(r))$ is admissible (or even irreducible), $\pi|_{\widetilde{\mathrm{P}}(r)}$ will not be admissible in general.

Proposition 2: *Let* $\rho \in \mathcal{A}(\widetilde{\mathrm{G}}(r-1))$, $\tau \in \mathcal{A}(\widetilde{\mathrm{P}}(r))$ *and* $\kappa \in \mathcal{A}(\widetilde{\mathrm{P}}(r-1))$. *Then*

(i) $\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau, \Psi^+(\rho)) \cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-1)}(\Psi^-(\tau), \rho)$

(ii) $\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\Phi^+(\kappa), \tau) \cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-1)}(\kappa, \Phi^-(\tau))$

(iii) $\mathrm{Hom}_{\widetilde{\mathrm{P}}(r-1)}(\Phi^-(\tau), \kappa) \cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau, \hat{\Phi}^+(\kappa))$

(iv) $\widehat{\Psi^+(\rho)} \cong |\det|^{-1} \otimes \Psi^+(\hat{\rho})$

(v) $\widehat{\Phi^+(\kappa)} \cong |\det|^{-1} \otimes \hat{\Phi}^+(|\det| \otimes \hat{\kappa})$

(vi) $\widehat{\Phi^-(\tau)} \cong \Phi^-(\hat{\tau})$

Except for those in (v) and (vi), all the implied maps underlie natural transformations.

Proof: Statements (i)–(iii) are in [BZ2], Proposition 3.2 and (iv) and (v) are in [BZ2], Proposition 3.4 except for the naturality claim in (iv); however this follows immediately from the proof. The map underlying (v) in not natural, except in trivial situations, because it involves the choice of an element of $\widetilde{\mathrm{P}}(r)$ with which to conjugate $\bar{\theta}$ to θ. It could be made natural, if this was desired, by including the character θ explicity and using the conjugate character in the appropriate place. We now turn to (vi). We have

$$\begin{aligned}
\mathrm{Hom}_{\widetilde{\mathrm{P}}(r-1)}(\kappa, \Phi^-(\hat{\tau})) &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\Phi^+(\kappa), \hat{\tau}) && \text{by (ii)} \\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau, \widehat{\Phi^+(\kappa)}) \\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau, |\det|^{-1} \otimes \hat{\Phi}^+(|\det| \otimes \hat{\kappa})) && \text{by (v)} \\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-1)}(|\det| \otimes \Phi^-(\tau), |\det| \otimes \hat{\kappa}) && \text{by (iii)}
\end{aligned}$$

$$\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-1)}\big(\kappa, \widehat{\Phi^-(\tau)}\big)\,.$$

If we now take $\kappa = \Phi^-(\hat{\tau})$ and observe that at each stage of the above calculation an isomorphism in one Hom space is carried to an isomorphism in the next, we see that the identity map in the first Hom space gives rise to the required isomorphism in the last. $\square$

There doesn't seem to be a simple relationship like (vi) for the functor Ψ^-. However, if we assume that $\tau \in \mathcal{A}\big(\widetilde{\mathrm{P}}(r)\big)$ is such that $\Psi^-(\tau)$ is admissible (which would be true, for example, if τ were the restriction to $\widetilde{\mathrm{P}}(r)$ of a representation of finite length on $\widetilde{\mathrm{G}}\,(r)$) then we can prove the following.

Proposition 3: *Let $\tau \in \mathcal{A}\big(\widetilde{\mathrm{P}}(r)\big)$ be such that $\Psi^-(\tau)$ is admissible. Then there is a natural surjection*

$$\widehat{\Psi^-(\hat{\tau})} \twoheadrightarrow |\det| \otimes \Psi^-(\tau)\,.$$

Proof: For any $\rho \in \mathcal{A}\big(\widetilde{\mathrm{G}}\,(r-1)\big)$ we have

$$\begin{aligned}
\mathrm{Hom}_{\widetilde{\mathrm{G}}(r-1)}\big(\Psi^-(\tau), \hat{\rho}\big) &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(\tau, \Psi^+(\hat{\rho})\big) && \text{by Proposition 2(i)}\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(\tau, |\det| \otimes \widehat{\Psi^+(\rho)}\big) && \text{by Proposition 2(iv)}\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(|\det|^{-1} \otimes \Psi^+(\rho), \hat{\tau}\big)\,. && (2)
\end{aligned}$$

The short exact sequence

$$0 \to \Phi^+\Phi^-(\hat{\tau}) \to \hat{\tau} \to \Psi^+\Psi^-(\hat{\tau}) \to 0$$

gives rise to an exact sequence

$$\begin{aligned}\{0\} &\longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}(r)}}\big(|\det|^{-1}\otimes\Psi^+(\rho),\Phi^+\Phi^-(\hat\tau)\big)\\ &\longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}(r)}}\big(|\det|^{-1}\otimes\Psi^+(\rho),\hat\tau\big)\\ &\longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}(r)}}\big(|\det|^{-1}\otimes\Psi^+(\rho),\Psi^+\Psi^-(\hat\tau)\big)\end{aligned}\tag{3}$$

and since $\Phi^+\Phi^-(\hat\tau)$ embeds in $\hat\Phi^+\Phi^-(\hat\tau)$, the first possibly non-zero term in (3) embeds in

$$\begin{aligned}&\mathrm{Hom}_{\widetilde{\mathrm{P}(r)}}\big(|\det|^{-1}\otimes\Psi^+(\rho),\hat\Phi^+\Phi^-(\hat\tau)\big)\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}(r)}}\big(|\det|^{-1}\otimes\Phi^-\Psi^+(\rho),\Phi^-(\hat\tau)\big) \qquad \text{by Proposition 2(iii)}\\ &=\{0\}\end{aligned}$$

since $\Phi^-\Psi^+=0$. Using this in (3) we see that

$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathrm{G}(r-1)}}\big(\Psi^-(\tau),\hat\rho\big) \hookrightarrow\; &\mathrm{Hom}_{\widetilde{\mathrm{P}(r)}}\big(|\det|^{-1}\otimes\Psi^+(\rho),\Psi^+\Psi^-(\hat\tau)\big)\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}(r-1)}}\big(|\det|^{-1}\otimes\Psi^-\Psi^+(\rho),\Psi^-(\hat\tau)\big)\\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}(r-1)}}\big(|\det|^{-1}\otimes\rho,\Psi^-(\hat\tau)\big)\end{aligned}\tag{4}$$

since $\Psi^-\Psi^+\sim\mathrm{Id}$. Now if ρ is admissible (even though $\Psi^-(\hat\tau)$ need not be) then the last space in (4) is isomorphic to

$$\mathrm{Hom}_{\widetilde{\mathrm{G}(r-1)}}\big(|\det|^{-1}\otimes\widehat{\Psi^-(\hat\tau)},\hat\rho\big)$$

and so

$$\mathrm{Hom}_{\widetilde{\mathrm{G}(r-1)}}\big(\Psi^-(\tau),\hat\rho\big)\hookrightarrow \mathrm{Hom}_{\widetilde{\mathrm{G}(r-1)}}\big(|\det|^{-1}\otimes\widehat{\Psi^-(\hat\tau)},\hat\rho\big)$$

naturally for all admissible ρ. Taking $\rho = \widehat{\Psi^-(\tau)}$, which is admissible since $\Psi^-(\tau)$ is, and using naturality we obtain

$$|\det|^{-1} \otimes \widehat{\Psi^-(\hat{\tau})} \twoheadrightarrow \Psi^-(\tau)$$

or

$$\widehat{\Psi^-(\hat{\tau})} \twoheadrightarrow |\det| \otimes \Psi^-(\tau),$$

as required. $\square$

One adjoint functor is conspicuously absent from the list in Proposition 2, namely the right adjoint of Ψ^+. This functor arises so infrequently that it seems to have no standard name in the literature. We discuss it here because we shall need it subsequently. By analogy with Φ^+ and $\hat{\Phi}^+$ we propose to call it $\hat{\Psi}^-$. If $\tau \in \mathcal{A}(\widetilde{\mathrm{P}}(r))$ then $\widetilde{\mathrm{G}}\,(r-1)$ preserves the space of $\mathrm{N}^*((r-1,1))$-invariant vectors in the space of τ and restricting τ to this subspace yields a representation of $\widetilde{\mathrm{G}}\,(r-1)$. This representation twisted by $|\det|^{-1/2}$ will be $\hat{\Psi}^-(\tau)$. If $T \in \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau_1, \tau_2)$ then $\hat{\Psi}^-(T)$ will simply be the restriction of T to the space of $\hat{\Psi}^-(\tau_1)$. This gives us a functor from $\mathcal{A}(\widetilde{\mathrm{P}}(r))$ to $\mathcal{A}(\widetilde{\mathrm{G}}\,(r-1))$. It is clear that, for $\rho \in \mathcal{A}(\widetilde{\mathrm{G}}\,(r-1))$, the image of $S \in \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\Psi^+(\rho), \tau)$ lies in the space of $\hat{\Psi}^-(\tau)$. From this observation the existence of a natural isomorphism

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\Psi^+(\rho), \tau) \cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-1)}(\rho, \hat{\Psi}^-(\tau))$$

is immediate and thus $\hat{\Psi}^-$ is right adjoint to Ψ^+. We note that $\hat{\Psi}^-\Psi^+ = \mathrm{Id}$.

Proposition 4: *For $\tau \in \mathcal{A}(\widetilde{\mathrm{P}}(r))$ there is a natural isomorphism*

$$\widehat{\Psi^-(\tau)} \cong |\det| \otimes \hat{\Psi}^-(\hat{\tau}) .$$

Proof: Let $\rho \in \mathcal{A}(\widetilde{\mathrm{G}}(r-1))$. Then

$$\begin{aligned} \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-1)}(\rho, \hat{\Psi}^-(\hat{\tau})) &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\Psi^+(\rho), \hat{\tau}) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau, \widehat{\Psi^+(\rho)}) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(\tau, |\det|^{-1} \otimes \Psi^+(\hat{\rho})) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}(|\det| \otimes \tau, \Psi^+(\hat{\rho})) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-1)}(|\det| \otimes \Psi^-(\tau), \hat{\rho}) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-1)}(\rho, |\det|^{-1} \otimes \widehat{\Psi^-(\tau)}) . \end{aligned}$$

with naturality at every stage. It follows that $\hat{\Psi}^-(\hat{\tau}) \cong |\det|^{-1} \otimes \widehat{\Psi^-(\tau)}$ naturally; hence the result. $\square$

We now turn to explaining the errors made by Bump and Ginzburg in the proof of Theorem 5.1 of [BuG]. We assume that the reader has access to the paper. In comparing their work with what is done here it should be remembered that Bump and Ginzburg use the unnormalized versions of $\Psi^{\pm}$ and $\Phi^{\pm}$ (which they denote by $\Psi_{\pm}$ and $\Phi_{\pm}$) and that they do not refer explicitly to χ or ω when discussing $\vartheta_\gamma(\chi, \omega)$, so that twists are frequently absorbed into χ without mention. The preliminary material, on the first three pages of the proof, is all correct. In the last paragraph of the proof, in the course of completing the induction step, they write

$$\mathrm{Hom}_{\widetilde{P}_k}\left(\pi_k \otimes \delta_P^s \otimes \Psi_+\theta_{k-1}, \Phi_+\Phi_-\widehat{\theta}_k\right) \cong \mathrm{Hom}_{\widetilde{P}_k}\left(\Phi_+\Phi_-(\theta_k) \otimes \delta_P^s \otimes \Psi_+\theta_{k-1}, \widehat{\pi}_k\right).$$

They are using the duality property of contragredients, but as we have seen above,

$$\widehat{\Phi_+\Phi_-\theta_k} \cong \hat{\Phi}_+\big(\widehat{\Phi_-(\theta_k)}\big) \cong \hat{\Phi}_+\Phi_-\big(\widehat{\theta}_k\big)$$

and so the isomorphism is not obviously correct as it stands. This suggests that some other error in the argument might have led to $\Phi_+\Phi_-\widehat{\theta}_k$ standing in place of $\hat{\Phi}_+\Phi_-\widehat{\theta}_k$ in the second place in the first Hom space in the isomorphism. However this is not so; the representation $\Phi_+\Phi_-\widehat{\theta}_k$ got there because of its appearance in the short exact sequence

$$0 \to \Phi_+\Phi_-(\widehat{\theta}_k) \to \widehat{\theta}_k \to \Psi_+\Psi_-(\widehat{\theta}_k) \to 0$$

and there is no reason to believe that the natural map $\Phi_+\Phi_-(\widehat{\theta}_k) \to \widehat{\theta}_k$ extends to a map $\hat{\Phi}_+\Phi_-(\widehat{\theta}_k) \to \widehat{\theta}_k$. If the statement of Theorem 5.1 is modified to read " ... there exists at most one ... " instead of " ... there exists exactly one ... " then this problem can be avoided. Even though we shall point out a much more serious error in the proof of Theorem 5.1 below, it is worth explaining how to get around the difficulty because we shall use the same argument ourselves later on. The point is to observe that, since $\Phi_+\Phi_-\widehat{\theta}_k$ is a subrepresentation of $\hat{\Phi}_+\Phi_-\widehat{\theta}_k$, if we replace $\Phi_+\Phi_-\widehat{\theta}_k$ by $\hat{\Phi}_+\Phi_-\widehat{\theta}_k$ in the Hom space above then the dimension of the space can only increase and afterwards the inductive step can be completed.

The second error which Bump and Ginzburg commit in their proof of Theorem 5.1 involves a pun on the symbol $\widehat{\theta}_k$. If $\rho \in \mathcal{A}(\widetilde{\mathrm{G}}\,(r))$ then it is conventional to confuse ρ with $\rho|_{\widetilde{\mathrm{P}}(r)}$ whenever the Φ and Ψ functors are involved. This convention leads to an ambiguity when applied to the symbol $\hat{\rho}$; does it mean $(\hat{\rho})|_{\widetilde{\mathrm{P}}(r)}$ or $\widehat{(\rho|_{\widetilde{\mathrm{P}}(r)})}$? Unfortunately these objects are very different in general. For instance, if ρ has finite length then so does $(\hat{\rho})|_{\widetilde{\mathrm{P}}(r)}$ but not usually $\widehat{(\rho|_{\widetilde{\mathrm{P}}(r)})}$. We may give a more concrete example on $\mathrm{G}(2)$ where the situation is very well-understood. Let ρ be an irreducible cuspidal representation of $\mathrm{G}(2)$. Then, in the notation of [BZ1] (see §5.13ff), we have $\hat{\rho}|_{\widetilde{\mathrm{P}}(2)} \cong \tau_P^0$ and

$$\widehat{(\rho|_{\widetilde{\mathrm{P}}(2)})} \cong \widehat{\tau_P^0} \cong \tau_P\,.$$

Consideration of the restrictions of principal series representations of $\mathrm{G}(2)$ to $\mathrm{P}(2)$ (see [BZ1], §5.24) shows that τ_P/τ_P^0 (which is most naturally regarded as a representation of $\mathrm{G}(1)$) contains every character of $\mathrm{G}(1)$. In particular, $\Psi^-\big(\widehat{(\rho|_{\widetilde{\mathrm{P}}(2)})}\big)$ contains every character of $\mathrm{G}(1)$ and is certainly not of finite length.

Near the beginning of the main argument in the proof of Theorem 5.1 Bump and Ginzburg say

> " ... so this space is isomorphic to
>
> $$\mathrm{Hom}_{\widetilde{P}_r}(\pi_r \otimes \theta_r, \widehat{\theta}_{r-1} \otimes \delta_P^{1-s}) \cong \mathrm{Hom}_{\widetilde{P}_r}(\pi_r \otimes \delta_P^s \otimes \theta_{r-1}, \widehat{\theta}_r)\,."$$

In this congruence the symbol $\widehat{\theta}_{r-1}$ really means $\Psi_+(\widehat{\theta}_{r-1})$, where the contragredient is taken on $\widetilde{\mathrm{G}}\,(r-1)$, and this is isomorphic to $\widehat{\Psi_+(\theta_{r-1})}$. The symbol θ_r on the left means $\theta_r|_{\widetilde{\mathrm{P}}(r)}$ and so the symbol $\widehat{\theta}_r$ on the right means $\widehat{(\theta_r|_{\widetilde{\mathrm{P}}(r)})}$. The isomorphism is thus correct provided one changes δ_P^s to δ_P^{s-1} on the right — this change is not significant for their argument. Continuing with their analysis they eventually arrive (at the first inductive step with $k = r$) at the space

$$\mathrm{Hom}_{\widetilde{G}_{r-1}}\left(\Psi_-(\pi_r \otimes \delta_P^s \otimes \Psi_+\theta_{r-1}), \Psi_-\widehat{\theta}_r\right).$$

They correctly conclude that this is isomorphic to

$$\mathrm{Hom}_{\widetilde{G}_{r-1}}\left(\Psi_-(\pi_r) \otimes \delta_P^s \otimes \theta_{r-1}, \Psi_-\widehat{\theta}_r\right).$$

However they then behave as if the representation $\Psi_-\widehat{\theta}_r$ had finite length and analyze its composition factors in terms of their central characters. But we have seen that the symbol $\Psi_-\widehat{\theta}_r$ appearing on the right explicitly means $\Psi_-\left(\widehat{(\theta_r|_{\widetilde{P}_r})}\right)$ and there is no reason to believe that this has finite length. If it did then the above space would indeed be zero for general s as Bump and Ginzburg claim, but if not then we cannot draw this conclusion on general grounds alone. We commend to the reader's attention the following simple related example: If ρ is as at the end of the previous paragraph then the space

$$\mathrm{Hom}_{\mathrm{G}(1)}\left(|\cdot|^s, \Psi_-\left(\widehat{\rho|_{\mathrm{P}(2)}}\right)\right)$$

is non-zero for all $s \in \mathbb{C}$.

In light of these remarks I believe that Bump's and Ginzburg's proof of Theorem 5.1 of [BuG] must be rejected and with it their proof of the local functional equation for the symmetric square L-functions. Below I shall prove a result which implies the generic uniqueness part of Theorem 5.1 (though not the existence) and this will be enough to salvage the local functional equation. Bump and Ginzburg remark that their proof of Theorem 5.1 bears a close resemblance to the unfolding argument for their global Rankin-Selberg integral. However, this is not exactly true; the unfolding is "2-periodic" (that is, it takes two steps for the integrand to return to its original form) whereas their proof of Theorem 5.1 is 1-periodic. The proof of Theorem 1 below is 2-periodic, restoring the expected harmony between the local and global theories.

If $\pi \in \mathcal{A}(\widetilde{\mathrm{G}}(r))$ then we shall refer to $\pi^{(1)}, \ldots, \pi^{(r-1)}$ as the *intermediate derivatives* of π (that is, intermediate between $\pi^{(0)} = \pi$ and $\pi^{(r)}$).

Definition 1: *Let $\pi \in \mathcal{A}(\widetilde{\mathrm{G}}(r))$ and $s \in \mathbb{C}$. We say that π is general with respect to s if no non-zero subquotient of any of the odd intermediate derivatives $\pi^{(1)}, \pi^{(3)}, \ldots$ has central character, α, satisfying $\alpha^2 = |\det|^{-2s-1/2}$ and no non-zero subquotient of any of the even intermediate derivatives $\pi^{(2)}, \pi^{(4)}, \ldots$ has central character, α, satisfying $\alpha^2 = |\det|^{-2s+1/2}$.*

Theorem 1: *Suppose that F is not dyadic or that $r \leq 3$. Let ω and ν be suitable characters of $\widetilde{\mathrm{Z}}(r)$, $s \in \mathbb{C}$ and π a homogeneous, admissible representation of $\mathrm{G}(r)$ of finite length which is general with respect to s and whose central character, ω_π, satisfies $\omega_\pi|_{\mathrm{Z}(r)} = \omega \cdot \nu$. Then the dimension of the space of invariant functionals on the representation*

$$\vartheta_{r,\omega} \otimes \mathrm{ind}_{\widetilde{\mathrm{Q}}((r-1,1))}^{\widetilde{\mathrm{G}}(r)} \left(\vartheta_{(r-1,1),\nu} \otimes \delta^{s}_{\widetilde{\mathrm{Q}}((r-1,1))}\right) \otimes \pi$$

is at most the dimension of the space of Whittaker models of π.

Proof: We abbreviate $\mathrm{Q}((r-1,1))$ by Q. First suppose that r is odd. The space in question is then

$$\begin{aligned}
&\mathrm{Hom}_{\widetilde{\mathrm{G}}(r)}\big(\vartheta_{r,\omega}\otimes \mathrm{ind}_{\widetilde{\mathrm{Q}}}^{\widetilde{\mathrm{G}}(r)}(\vartheta_{(r-1,1),\nu}\otimes\delta_{\mathrm{Q}}^{s})\otimes\pi,1\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r)}\big(\vartheta_{r,\omega}\otimes\pi,\mathrm{ind}_{\widetilde{\mathrm{Q}}}^{\widetilde{\mathrm{G}}(r)}(\vartheta_{(r-1,1),\nu}\otimes\delta_{\mathrm{Q}}^{-s})\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{Q}}}\big(\vartheta_{r,\omega}\otimes\pi,\Psi^{+}(\vartheta_{(r-1,1),\nu})\otimes\delta_{\mathrm{Q}}^{-s}\big)\,. \qquad (5)
\end{aligned}$$

Since r is odd the representations in the first and second places in (5) have the same character under $\widetilde{\mathrm{Z}}^{1}(r)$ and, because $\widetilde{\mathrm{Q}}=\widetilde{\mathrm{Z}}^{1}(r)\cdot\widetilde{\mathrm{P}}(r)$, the space in (5) is isomorphic to

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(\vartheta_{r,\omega}\otimes\pi,\Psi^{+}(\vartheta_{(r-1,1),\nu}|_{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\mu_2})\otimes|\det|^{-s}\big)\,. \qquad (6)$$

But $(r-1)$ is even and so $\vartheta_{(r-1,1),\nu}|_{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\mu_2}\cong\vartheta_{r-1,\nu_{r-1}}$ where ν_{r-1} is the (unique) suitable character of $\widetilde{\mathrm{Z}}(r-1)$. Thus (6) is isomorphic to

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(\vartheta_{r,\omega}\otimes\pi,\Psi^{+}(\vartheta_{r-1,\nu_{r-1}})\otimes|\det|^{-s}\big)\,. \qquad (7)$$

If r is even then we must reach (7) by a slightly different route. In this case $(r-1)$ is odd and so

$$\vartheta_{(r-1,1),\nu}\cong \mathrm{ind}_{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}^{2}(1)}^{\widetilde{\mathrm{G}}(r-1)\,\widetilde{\times}\,\widetilde{\mathrm{G}}(1)}\big(\vartheta_{r-1,\nu_{r-1}}\otimes 1\big)$$

where ν_{r-1} is any suitable character of $\widetilde{\mathrm{Z}}(r-1)$. Thus if $\widetilde{\mathrm{Q}}'=\widetilde{\mathrm{P}}(r)\cdot\widetilde{\mathrm{Z}}^{2}(r)$ we have, by transitivity of induction, that

$$\mathrm{ind}_{\widetilde{\mathrm{Q}}}^{\widetilde{\mathrm{G}}(r)}\big(\vartheta_{(r-1,1),\nu}\otimes\delta_{\mathrm{Q}}^{s}\big)\cong \mathrm{ind}_{\widetilde{\mathrm{Q}}'}^{\widetilde{\mathrm{G}}(r)}\big(\vartheta_{r-1,\nu_{r-1}}\otimes 1\otimes\delta_{\mathrm{Q}}^{s}\big)$$

and repeating (5) with the right hand side of this isomorphism in place of the left we find that our space is isomorphic to

$$\mathrm{Hom}_{\widetilde{Q}'}\left(\vartheta_{r,\omega} \otimes \pi, \Psi^{+}(\vartheta_{r-1,\nu_{r-1}} \otimes 1) \otimes \delta_{Q}^{-s}\right)$$

and since the $\widetilde{Z}^2(r)$ characters in both places in this Hom space agree, this is isomorphic to (7).

From this point onwards the particular suitable characters with respect to which the exceptional representations are formed will not play a significant rôle. We shall thus allow ourselves to omit them from the notation and simply write ϑ_γ for any exceptional representation $\vartheta_{\gamma,\omega}$.

For $0 \leq k < r-1$ and $z \in \mathbb{C}$ we shall consider the spaces

$$\mathcal{H}_k(\pi, z) = \mathrm{Hom}_{\widetilde{P}(r-k)}\left(\vartheta_{r-k} \otimes (\Phi^{-})^{k}(\pi), \Psi^{+}(\vartheta_{r-k-1}) \otimes |\det|^{z}\right)$$

and

$$\mathcal{J}_k(\pi, z) = \mathrm{Hom}_{\widetilde{P}(r-k)}\left(\Psi^{+}(\vartheta_{r-k-1}) \otimes (\Phi^{-})^{k}(\pi), \vartheta_{r-k} \otimes |\det|^{z}\right).$$

We have a short exact sequence

$$0 \to \Phi^{+}\Phi^{-}(\vartheta_{r-k}) \to \vartheta_{r-k} \to \Psi^{+}\Psi^{-}(\vartheta_{r-k}) \to 0 \tag{8}$$

and since the tensor product yields an exact functor on the category of vector spaces we obtain from this an exact sequence

$$\begin{aligned} 0 &\longrightarrow \Phi^{+}\Phi^{-}(\vartheta_{r-k}) \otimes (\Phi^{-})^{k}(\pi) \\ &\longrightarrow \vartheta_{r-k} \otimes (\Phi^{-})^{k}(\pi) \\ &\longrightarrow \Psi^{+}\Psi^{-}(\vartheta_{r-k}) \otimes (\Phi^{-})^{k}(\pi) \longrightarrow 0. \end{aligned}$$

Using this sequence in the definition of $\mathcal{H}_k(\pi, z)$ we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\big(\Psi^+\Psi^-(\vartheta_{r-k}) \otimes (\Phi^-)^k(\pi), \Psi^+(\vartheta_{r-k-1}) \otimes |\det|^z\big)$$
$$\longrightarrow \mathcal{H}_k(\pi, z) \tag{9}$$
$$\longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\big(\Phi^+\Phi^-(\vartheta_{r-k}) \otimes (\Phi^-)^k(\pi), \Psi^+(\vartheta_{r-k-1}) \otimes |\det|^z\big).$$

Now

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\big(\Psi^+\Psi^-(\vartheta_{r-k}) \otimes (\Phi^-)^k(\pi), \Psi^+(\vartheta_{r-k-1}) \otimes |\det|^z\big)$$
$$\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\big(\Psi^-(\Psi^+\Psi^-(\vartheta_{r-k}) \otimes (\Phi^-)^k(\pi)), \vartheta_{r-k-1} \otimes |\det|^z\big)$$
$$\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\big(\Psi^-(\vartheta_{r-k}) \otimes \pi^{(k+1)} \otimes |\det|^{1/2}, \vartheta_{r-k-1} \otimes |\det|^z\big)$$
$$\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\big(\oplus\, \vartheta_{r-k-1} \otimes \pi^{(k+1)} \otimes |\det|^{1/4}, \vartheta_{r-k-1} \otimes |\det|^z\big)$$
$$\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\big(\oplus\, \vartheta_{r-k-1} \otimes \pi^{(k+1)}, \vartheta_{r-k-1} \otimes |\det|^{z-1/4}\big) \tag{10}$$

where $\oplus\vartheta_{r-k-1}$ denotes a finite direct sum of exceptional representations formed with respect to various suitable characters. In this calculation we used Proposition 2(i) from the first to the second line and Proposition 1 of section 4 from the third to the fourth. All the exceptional representations in (10) transform via a suitable character of $\widetilde{\mathrm{Z}}^2(r-k-1)$. The representation $\pi^{(k+1)}$ is of finite length and comparing $\widetilde{\mathrm{Z}}^2(r-k-1)$ characters in both entries in (10) we see that the space of homomorphisms is $\{0\}$ provided that no non-zero subquotient of $\pi^{(k+1)}$ has central character, α, satisfying $\alpha^2 = |\det|^{2z-1/2}$. If this is the case then (9) shows that $\mathcal{H}_k(\pi, z)$ may be regarded as a subspace of

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\big(\Phi^+\Phi^-(\vartheta_{r-k}) \otimes (\Phi^-)^k(\pi), \Psi^+(\vartheta_{r-k-1}) \otimes |\det|^z\big)$$

$$\cong \mathrm{Hom}_{\widetilde{P}(r-k)}\big(\Phi^+\Phi^-(\vartheta_{r-k}), \Psi^+(\vartheta_{r-k-1}) \otimes (\widehat{\Phi^-)^k(\pi}) \otimes |\det|^z\big)$$

$$\cong \mathrm{Hom}_{\widetilde{P}(r-k-1)}\big(\Phi^-(\vartheta_{r-k}), \Phi^-(\Psi^+(\vartheta_{r-k-1}) \otimes (\widehat{\Phi^-)^k(\pi}) \otimes |\det|^z)\big)$$

$$\cong \mathrm{Hom}_{\widetilde{P}(r-k-1)}\big(\Phi^-(\vartheta_{r-k}), \vartheta_{r-k-1} \otimes (\widehat{\Phi^-)^{k+1}}(\pi) \otimes |\det|^{z+1/2}\big)$$

$$\cong \mathrm{Hom}_{\widetilde{P}(r-k-1)}\big(\Phi^-(\vartheta_{r-k}) \otimes (\Phi^-)^{k+1}(\pi), \vartheta_{r-k-1} \otimes |\det|^{z+1/2}\big) \tag{11}$$

where we have used Proposition 2(ii) from line two to line three and Proposition 2(vi) from three to four.

Up until now we have not made any use of the hypothesis that F is not dyadic or else that $r \leq 3$. This hypothesis becomes necessary when we attempt to analyze (11) further. Applying the standard short exact sequence to the representation $\Phi^-(\vartheta_{r-k})$ we obtain

$$0 \to \Phi^+(\Phi^-)^2(\vartheta_{r-k}) \to \Phi^-(\vartheta_{r-k}) \to \Psi^+(\vartheta_{r-k}^{(2)}) \to 0\,. \tag{12}$$

Since $\vartheta_{r-k}^{(j)} = 0$ if $j \geq 3$, we see that all the proper derivatives of the representation $(\Phi^-)^2(\vartheta_{r-k})$ are equal to zero and arguing as in Proposition 9 of section 4 this implies that $(\Phi^-)^2(\vartheta_{r-k}) = 0$. Hence (12) implies that

$$\Phi^-(\vartheta_{r-k}) \cong \Psi^+(\vartheta_{r-k}^{(2)}) \cong |\det|^{-1/2} \otimes \Psi^+(\vartheta_{r-k-2})$$

by Proposition 2 of section 4.

Using this we see that (11) is isomorphic to

$$\mathrm{Hom}_{\widetilde{P}(r-k-1)}\big(|\det|^{-1/2} \otimes \Psi^+(\vartheta_{r-k-2}) \otimes (\Phi^-)^{k+1}(\pi), \vartheta_{r-k-1} \otimes |\det|^{z+1/2}\big)$$

$$\cong \mathrm{Hom}_{\widetilde{P}(r-k-1)}\big(\Psi^+(\vartheta_{r-k-2}) \otimes (\Phi^-)^{k+1}(\pi), \vartheta_{r-k-1} \otimes |\det|^{z+1}\big)$$

$$= \mathcal{J}_{k+1}(\pi, z+1)\,.$$

To summarize this part of the argument, we have shown that if $\pi^{(k+1)}$ has no non-zero subquotient with central character, α, satisfying $\alpha^2 = |\det|^{2z-1/2}$ then $\mathcal{H}_k(\pi, z)$ is a subspace of $\mathcal{J}_{k+1}(\pi, z+1)$.

We must now undertake a similar, but somewhat easier, analysis of $\mathcal{J}_k(\pi, z)$. Applying (8) in the second place in the definition of $\mathcal{J}_k(\pi, z)$ we obtain an exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\left(\Psi^+(\vartheta_{r-k-1}) \otimes (\Phi^-)^k(\pi), \Phi^+\Phi^-(\vartheta_{r-k}) \otimes |\det|^z\right) \\ &\longrightarrow \mathcal{J}_k(\pi, z) \qquad (13) \\ &\longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\left(\Psi^+(\vartheta_{r-k-1}) \otimes (\Phi^-)^k(\pi), \Psi^+\Psi^-(\vartheta_{r-k}) \otimes |\det|^z\right).\end{aligned}$$

The last term in this sequence is isomorphic to

$$\begin{aligned}&\mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\left(\vartheta_{r-k-1} \otimes \pi^{(k+1)} \otimes |\det|^{1/2}, \Psi^-(\vartheta_{r-k}) \otimes |\det|^z\right) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\left(\vartheta_{r-k-1} \otimes \pi^{(k+1)} \otimes |\det|^{1/2}, \oplus\vartheta_{r-k-1} \otimes |\det|^{z-1/4}\right) \\ &\cong \mathrm{Hom}_{\widetilde{\mathrm{G}}(r-k-1)}\left(\vartheta_{r-k-1} \otimes \pi^{(k+1)}, \oplus\vartheta_{r-k-1} \otimes |\det|^{z-3/4}\right) \qquad (14)\end{aligned}$$

where we have used Proposition 1 of section 4 from the first to the second line. As before, this space is $\{0\}$ if no non-zero subquotient of $\pi^{(k+1)}$ has central character, α, satisfying $\alpha^2 = |\det|^{2z-3/2}$. If this condition is satisfied then (13) shows that $\mathcal{J}_k(\pi, z)$ is isomorphic to

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\left(\Psi^+(\vartheta_{r-k-1}) \otimes (\Phi^-)^k(\pi), \Phi^+\Phi^-(\vartheta_{r-k}) \otimes |\det|^z\right)$$

which is a subspace of

$$\mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k)}\left(\Psi^+(\vartheta_{r-k-1}) \otimes (\Phi^-)^k(\pi), \hat{\Phi}^+\Phi^-(\vartheta_{r-k}) \otimes |\det|^z\right)$$

$$\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k-1)}\big(\vartheta_{r-k-1}\otimes(\Phi^-)^{k+1}(\pi)\otimes|\det|^{1/2}, \Phi^-(\vartheta_{r-k})\otimes|\det|^{z}\big)$$

$$\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k-1)}\big(\vartheta_{r-k-1}\otimes(\Phi^-)^{k+1}(\pi)\otimes|\det|^{1/2}, \Psi^+(\vartheta_{r-k-2})\otimes|\det|^{z-1/2}\big)$$

$$\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-k-1)}\big(\vartheta_{r-k-1}\otimes(\Phi^-)^{k+1}(\pi), \Psi^+(\vartheta_{r-k-2})\otimes|\det|^{z-1}\big)$$

$$= \mathcal{H}_{k+1}(\pi, z-1)\,.$$

To summarize, we have shown that if $\pi^{(k+1)}$ has no non-zero subquotient with central character, α, satisfying $\alpha^2 = |\det|^{2z-3/2}$ then $\mathcal{J}_k(\pi,z)$ may be regarded as a subspace of $\mathcal{H}_{k+1}(\pi, z-1)$.

With the analysis of $\mathcal{H}_k(\pi,z)$ and $\mathcal{J}_k(\pi,z)$ complete it is time to return to the space of invariant functionals whose dimension we are trying to estimate. In the first part of the proof we saw that this space is isomorphic to (7) and this is $\mathcal{H}_0(\pi,-s)$ by definition. In the diagram below we have indicated the sequence of inclusions which has been established. Each arrow is labelled with an abbreviated form of the condition under which it is valid:

$$\mathcal{H}_0(\pi,-s) \xhookrightarrow[\text{on } \pi^{(1)}]{} \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! _{\alpha^2\neq|\det|^{-2s-\frac{1}{2}}} \quad \mathcal{J}_1(\pi,1-s) \xhookrightarrow[\text{on } \pi^{(2)}]{} \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! _{\alpha^2\neq|\det|^{-2s+\frac{1}{2}}} \quad \mathcal{H}_2(\pi,-s) \xhookrightarrow[\text{on } \pi^{(3)}]{} \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! _{\alpha^2\neq|\det|^{-2s-\frac{1}{2}}}$$

Since we are assuming that π is general with respect to s, all these conditions are satisfied and we conclude that the space of invariant functionals is a subspace of either $\mathcal{H}_{r-1}(\pi,-s)$ or $\mathcal{J}_{r-1}(\pi,1-s)$ depending on the parity of r. Now

$$\mathcal{H}_{r-1}(\pi,-s) = \mathrm{Hom}_{\widetilde{\mathrm{P}}(1)}\big(\vartheta_1\otimes(\Phi^-)^{r-1}(\pi), \Psi^+(\vartheta_0)\otimes|\det|^{-s}\big) \tag{15}$$

and both entries in the Hom space are genuine representations of $\widetilde{\mathrm{P}}(1) = \mu_2$. Both the exceptional representations are one-dimensional and $(\Phi^-)^{r-1}(\pi)$ is realized on the same space as $\pi^{(r)}$. Hence (15) has the same dimension as the space of Whittaker models of π. The same conclusion holds for $\mathcal{J}_{r-1}(\pi, 1-s)$ by a similar argument. This finally completes the proof of the Theorem. □

Corollary 1: *Suppose that F is not dyadic or that $r \leq 3$. Let ω and ν be suitable characters of $\widetilde{\mathrm{Z}}(r)$ and π a homogeneous, admissible representation of $\mathrm{G}(r)$ of finite length which is general with respect to $1/4$. Then the dimension of the space of invariant linear functionals on $\vartheta_{r,\omega} \otimes \vartheta_{r,\nu} \otimes \pi$ is at most the dimension of the space of Whittaker models of π.*

Proof: If $\omega_\pi|_{\widetilde{\mathrm{Z}}(r)} \neq \omega \cdot \nu$ then the space has dimension zero, so we may assume that the central characters match. Combining parts (e) and (f) of Theorem 1 in section 3 we see that $\vartheta_{r,\nu}$ is isomorphic to a quotient of

$$\mathrm{ind}_{\widetilde{\mathrm{Q}}((r-1,1))}^{\widetilde{\mathrm{G}}(r)} \left(\vartheta_{(r-1,1),\nu} \otimes \delta_{\mathrm{Q}}^{1/4}\right)$$

and hence there is an injective map from the space of invariant linear functionals on $\vartheta_{r,\omega} \otimes \vartheta_{r,\nu} \otimes \pi$ to the space of invariant linear functionals on

$$\vartheta_{r,\omega} \otimes \mathrm{ind}_{\widetilde{\mathrm{Q}}((r-1,1))}^{\widetilde{\mathrm{G}}(r)} \left(\vartheta_{(r-1,1),\nu} \otimes \delta_{\mathrm{Q}}^{1/4}\right) \otimes \pi\,.$$

The result now follows from Theorem 1. □

Note that if π is a cuspidal representation of $\mathrm{G}(r)$ then $\pi^{(j)} = 0$ except for $j = 0$ and $j = r$ (see [BZ2], Theorem 4.4). Thus π is automatically general with respect to any $s \in \mathbb{C}$. If π is also irreducible then $\pi^{(r)} = 1$ and it follows from Corollary 1 that the space of invariant linear functionals on $\vartheta_{r,\omega} \otimes \vartheta_{r,\nu} \otimes \pi$ is always at most one-dimensional. In order to see that the hypothesis of generality in Corollary 1 is necessary let us take π to be the trivial representation of $\mathrm{G}(r)$ with $r \geq 2$. Then π has no Whittaker models, but the space of invariant linear functionals on $\vartheta_{r,\omega} \otimes \vartheta_{r,\omega} \otimes \pi$ is exactly one-dimensional because $\vartheta_{r,\omega}$ is irreducible and self-contragredient. The first derivative of π is the one-dimensional representation $|\det|^{-1/2}$ on $\mathrm{G}(r-1)$ and all higher derivatives of π are zero. Thus the condition of generality with respect to $1/4$ is violated in this case and this shows it to be a necessary assumption.

Definition 2: *If ω and ν are suitable characters and π is a representation of $\mathrm{G}(r)$ then we shall denote the space of invariant linear functionals on $\vartheta_{r,\omega} \otimes \vartheta_{r,\nu} \otimes \pi$ by $\mathcal{L}(\omega, \nu; \pi)$. If $\omega = \nu$ then we shall write $\mathcal{L}(\omega; \pi)$ in place of $\mathcal{L}(\omega, \omega; \pi)$.*

Theorem 2: *Suppose that F is not dyadic or that $r \leq 3$. Let ω and ν be suitable characters of $\widetilde{\mathrm{Z}}(r)$ and ρ a homogeneous admissible representation of $\mathrm{G}(r-1)$ of finite length. If r is odd then define a character α of $F^\times$ by*

$$\alpha(z) = \omega_\rho(zI_{r-1})^{-1} \cdot (\omega \cdot \nu)(zI_r) \tag{16}$$

where ω_ρ is the central character of ρ. If r is even then define α by (16) on $(F^\times)^2$ and extend it in any way to $F^\times$. Let

$$\pi = i_{(r-1,1),(r)}(\rho \otimes \alpha)\,.$$

If r is odd then there is an exact sequence

$$\{0\} \to \mathcal{L}(\eta; \rho) \to \mathcal{L}(\omega, \nu; \pi) \to \mathcal{L}(\omega', \nu'; \rho^{(1)})$$

where η is the unique suitable character of $\widetilde{\mathrm{Z}}(r-1)$ and ω' and ν' are suitable characters of $\widetilde{\mathrm{Z}}(r-2)$ related to ω and ν by

$$\omega = (\omega' \otimes 1_\psi)|_{\widetilde{\mathrm{Z}}(r)} \qquad \textit{and} \qquad \nu = (\nu' \otimes 1_\psi)|_{\widetilde{\mathrm{Z}}(r)}\,. \tag{17}$$

If r is even then $\omega = \nu$ and there is a space V which completes the following diagram

$$\begin{array}{ccccccc} & & & & \mathcal{L}(\omega;\pi) & & \\ & & & & \downarrow & & \\ \{0\} & \longrightarrow & \oplus_{\eta_1,\eta_2}\mathcal{L}(\eta_1,\eta_2;\rho) & \longrightarrow & V & \longrightarrow & \mathcal{L}(\omega';\rho^{(1)}) \end{array}$$

with the row exact. Here ω' is related to ω by (17) and the direct sum is over all pairs of suitable characters. The space V is independent of the extension of α from $(F^\times)^2$ to $F^\times$ initially chosen and if V is non-zero then $\mathcal{L}(\omega;\pi)$ is non-zero for at least one choice of extension.

Proof: We let $\mathrm{Q} = \mathrm{Q}((r-1,1))$ as before. The space $\mathcal{L}(\omega,\nu;\pi)$ is isomorphic to

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{G}(r)}\big(\vartheta_{r,\omega}\otimes\vartheta_{r,\nu}\otimes\pi, 1\big) \\ &\cong \mathrm{Hom}_{\mathrm{G}(r)}\big(\vartheta_{r,\omega}\otimes\vartheta_{r,\nu}, \mathrm{ind}_{\mathrm{Q}}^{\mathrm{G}(r)}(\hat\rho\otimes\alpha^{-1})\big) \\ &\cong \mathrm{Hom}_{\mathrm{Q}}\big(\vartheta_{r,\omega}\otimes\vartheta_{r,\nu}, \Psi^+(\hat\rho\otimes\alpha^{-1})\big)\,. \end{aligned} \tag{18}$$

We note explicitly that the $\hat\rho$ in (18) refers to the contragredient of ρ as a representation of $\mathrm{G}(r-1)$. The $\mathrm{Z}(r)$-character of the representations on the left and right of this Hom space have been arranged to match. Thus if r is odd we may drop the center to see that (18) is isomorphic to

$$\mathrm{Hom}_{\mathrm{P}(r)}\big(\vartheta_{r,\omega}\otimes\vartheta_{r,\nu}, \Psi^+(\hat\rho)\big)\,. \tag{19}$$

If r is even then (19) will be the space denoted by V in the statement. It evidently does not depend on the extension of α which was chosen and since the $\mathrm{P}(r)$-intertwining property required of elements of (19) is less restrictive than the Q-intertwining property required of elements of (18), $\mathcal{L}(\omega;\pi)$ is always a subspace of (19). Since $\mathrm{Z}^1(r)/\mathrm{Z}^2(r)$ is a finite abelian group, any element of (19) may be written as a sum of linear maps between the underlying spaces each of which is $\mathrm{P}(r)$-intertwining and transforms under $\mathrm{Z}^1(r)$ by one of the square-trivial characters. These summands give elements of the various $\mathcal{L}(\omega;\pi)$, where π is formed with one of the possible extensions of α. If the original map is non-zero then at least one of its summands must be non-zero and this shows that if $V \neq \{0\}$ then $\mathcal{L}(\omega;\pi) \neq \{0\}$ for at least one choice of α. This said, it remains to analyze (19).

We shall begin with the short exact sequence

$$0 \to \Phi^+\Phi^-(\vartheta_{r,\omega}) \otimes \vartheta_{r,\nu} \to \vartheta_{r,\omega} \otimes \vartheta_{r,\nu} \to \Psi^+\Psi^-(\vartheta_{r,\omega}) \otimes \vartheta_{r,\nu} \to 0$$

which yields an exact sequence

$$\begin{aligned} \{0\} &\longrightarrow \mathrm{Hom}_{\mathrm{P}(r)}\big(\Psi^+\Psi^-(\vartheta_{r,\omega}) \otimes \vartheta_{r,\nu}, \Psi^+(\hat{\rho})\big) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{P}(r)}\big(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu}, \Psi^+(\hat{\rho})\big) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{P}(r)}\big(\Phi^+\Phi^-(\vartheta_{r,\omega}) \otimes \vartheta_{r,\nu}, \Psi^+(\hat{\rho})\big)\,. \end{aligned} \tag{20}$$

The first term in (20) is easy to analyze. Indeed, by Proposition 2(i), it is isomorphic to

$$\mathrm{Hom}_{\mathrm{G}(r-1)}\big(|\det|^{1/2} \otimes \Psi^-(\vartheta_{r,\omega}) \otimes \Psi^-(\vartheta_{r,\nu}), \hat{\rho}\big)$$

$$\cong \mathrm{Hom}_{\mathrm{G}(r-1)}\big(|\det|^{1/2} \otimes \Psi^-(\vartheta_{r,\omega}) \otimes \Psi^-(\vartheta_{r,\nu}) \otimes \rho, 1\big)$$

and by Proposition 1 of section 4 this is isomorphic to $\mathcal{L}(\eta;\rho)$ if r is odd and to

$$\oplus_{\eta_1,\eta_2}\mathcal{L}(\eta_1,\eta_2;\rho)$$

if r is even. This gives the first part of the exact sequence in the statement.

The analysis of the third term in (20) is much more challenging and, in order to avoid a string of isomorphisms stretching for a whole page, we shall break it into shorter steps. First we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{P}(r)}\big(\Phi^+\Phi^-(\vartheta_{r,\omega}) \otimes \vartheta_{r,\nu}, \Psi^+(\hat{\rho})\big)\\
&\cong \mathrm{Hom}_{\mathrm{P}(r)}\big(\Phi^+\Phi^-(\vartheta_{r,\omega}) \otimes \vartheta_{r,\nu}, |\det| \otimes \widehat{\Psi^+(\rho)}\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(|\det|^{-1} \otimes \Psi^+(\rho) \otimes \vartheta_{r,\nu}, \Phi^+\widehat{\Phi^-(\vartheta_{r,\omega})}\big)
\end{aligned}$$

where we have used Proposition 2(iv) and the duality property of contragredients. By Proposition 2(v) this is isomorphic to

$$\begin{aligned}
&\mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(|\det|^{-1} \otimes \Psi^+(\rho) \otimes \vartheta_{r,\nu}, |\det|^{-1} \otimes \hat{\Phi}^+(|\det| \otimes \widehat{\Phi^-(\vartheta_{r,\omega})})\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r)}\big(\Psi^+(\rho) \otimes \vartheta_{r,\nu}, \hat{\Phi}^+(|\det| \otimes \widehat{\Phi^-(\vartheta_{r,\omega})})\big)\\
&\cong \mathrm{Hom}_{\widetilde{\mathrm{P}}(r-1)}\big(|\det|^{1/2} \otimes \rho \otimes \Phi^-(\vartheta_{r,\nu}), |\det| \otimes \widehat{\Phi^-(\vartheta_{r,\omega})}\big)\\
&\cong \mathrm{Hom}_{\mathrm{P}(r-1)}\big(|\det|^{-1/2} \otimes \rho \otimes \Phi^-(\vartheta_{r,\omega}) \otimes \Phi^-(\vartheta_{r,\nu}), 1\big) \qquad (21)
\end{aligned}$$

where we have used Proposition 2(iii) and the duality property of contragredients. The first hypothesis in the statement implies, as in the proof of Theorem 1, that

$$\Phi^-(\vartheta_{r,\omega}) \cong |\det|^{-1/2} \otimes \Psi^+(\vartheta_{r-2,\omega'})$$

and

$$\Phi^-(\vartheta_{r,\nu}) \cong |\det|^{-1/2} \otimes \Psi^+(\vartheta_{r-2,\nu'}) .$$

Hence (21) is isomorphic to

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{P}(r-1)}\big(|\det|^{-3/2} \otimes \rho \otimes \Psi^+(\vartheta_{r-2,\omega'}) \otimes \Psi^+(\vartheta_{r-2,\nu'}), 1\big) \\
&\cong \mathrm{Hom}_{\mathrm{P}(r-1)}\big(|\det|^{-3/2} \otimes \Psi^+(\vartheta_{r-2,\omega'}) \otimes \Psi^+(\vartheta_{r-2,\nu'}), \hat{\rho}\big) \\
&\cong \mathrm{Hom}_{\mathrm{P}(r-1)}\big(|\det|^{-1} \otimes \Psi^+(\vartheta_{r-2,\omega'} \otimes \vartheta_{r-2,\nu'}), \hat{\rho}\big)
\end{aligned}$$

where the symbol $\hat{\rho}$ now refers to the contragredient on $\mathrm{P}(r-1)$ of $\rho|_{\mathrm{P}(r-1)}$. By the discussion preceding Proposition 4, this is isomorphic to

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{G}(r-2)}\big(|\det|^{-1} \otimes \vartheta_{r-2,\omega'} \otimes \vartheta_{r-2,\nu'}, \hat{\Psi}^-(\hat{\rho})\big) \\
&\cong \mathrm{Hom}_{\mathrm{G}(r-2)}\big(|\det|^{-1} \otimes \vartheta_{r-2,\omega'} \otimes \vartheta_{r-2,\nu'}, |\det|^{-1} \otimes \widehat{\Psi^-(\rho)}\big) \\
&\cong \mathrm{Hom}_{\mathrm{G}(r-2)}\big(\vartheta_{r-2,\omega'} \otimes \vartheta_{r-2,\nu'}, \widehat{\Psi^-(\rho)}\big) \\
&\cong \mathrm{Hom}_{\mathrm{G}(r-2)}\big(\vartheta_{r-2,\omega'} \otimes \vartheta_{r-2,\nu'} \otimes \Psi^-(\rho), 1\big) \\
&= \mathcal{L}(\omega', \nu'; \rho^{(1)})
\end{aligned}$$

by Proposition 4 and the duality property of contragredients. This completes the proof. □

We shall now concentrate our attention on the principal series of $\mathrm{G}(r)$ and attempt to obtain further information about the spaces $\mathcal{L}(\omega, \nu; \pi)$ when π belongs to this series. It will be convenient to have a more compact notation for these representations than is presently available and so we shall write $\mathbb{I}(\chi)$ or $\mathbb{I}(\chi_1, \dots, \chi_r)$ for the representation obtained by normalized parabolic induction from the character $\chi = (\chi_1, \dots, \chi_r)$ of H_r.

Definition 3: *A character $\chi = (\chi_1, \dots, \chi_r)$ of H_r will be called balanced if there is an involution $j \mapsto j^*$ of the set $\{1, \dots, r\}$ such that, for all $1 \leq j \leq r$, $\chi_j^2 \chi_{j^*}^2 = 1$ if $j \neq j^*$ and $\chi_j^2 = 1$ if $j = j^*$.*

We note that if $r = 1$ then the character χ is balanced if and only if $\chi^2 = 1$ and if $r = 2$ then the character $\chi = (\chi_1, \chi_2)$ is balanced if and only if $\chi_1^2 \chi_2^2 = 1$. In general, if $\chi = (\chi_1, \dots, \chi_r)$ is balanced then $\prod_{j=1}^{r} \chi_j^2 = 1$, but additional restrictions are also being imposed when $r \geq 3$.

Theorem 3: *Suppose that F is not dyadic or that $r \leq 3$. If $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi)) \neq \{0\}$ for some choice of suitable characters ω and ν then χ is balanced.*

Proof: Since $\vartheta_{r,\omega} \otimes \vartheta_{r,\nu}$ transforms under $\mathrm{Z}(r)$ by $\omega \cdot \nu$, a first necessary condition for $\mathcal{L}(\omega, \nu; \pi) \neq \{0\}$ is that $\omega_\pi = \omega \cdot \nu$ on $\mathrm{Z}(r)$. (Note that $(\omega \cdot \nu) = (\omega \cdot \nu)^{-1}$.) In particular, ω_π must be square-trivial.

We shall establish the result by induction on r. If $r = 1$ or $r = 2$ then χ being balanced is equivalent to $\mathbb{I}(\chi)$ having square-trivial central character and hence the claim is true in either of these cases. Now suppose that $r \geq 3$ and that $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi)) \neq \{0\}$. By transitivity of induction we have

$$\mathbb{I}(\chi) \cong \mathrm{ind}_{\mathrm{Q}}^{\mathrm{G}(r)} \left(\mathbb{I}(\chi_1, \dots, \chi_{r-1}) \otimes \chi_r \right).$$

If we set $\rho = \mathbb{I}(\chi_1, \ldots, \chi_{r-1})$ then

$$\omega_\rho(zI_{r-1}) = \prod_{j=1}^{r-1} \chi_j(z)$$

and it follows from the remarks in the first paragraph that

$$\omega_\rho(zI_{r-1}) \cdot \chi_r(z) = (\omega \cdot \nu)(zI_r) \qquad \text{for } zI_r \in \mathrm{Z}(r).$$

Thus, regardless of the parity of r, $\alpha = \chi_r$ is one suitable choice in Theorem 2. We conclude from that Theorem that either $\mathcal{L}(\eta_1, \eta_2; \rho) \neq \{0\}$ for some suitable η_1 and η_2 or that $\mathcal{L}(\omega', \nu'; \rho^{(1)}) \neq \{0\}$. If the first possibility obtains then we conclude inductively that $(\chi_1, \ldots, \chi_{r-1})$ is balanced. In particular, $\prod_{j=1}^{r-1} \chi_j^2 = 1$ and hence $\chi_r^2 = 1$. If we extend the involution $j \mapsto j^*$ of $\{1, \ldots, r-1\}$ to $\{1, \ldots, r\}$ by setting $r^* = r$ then we obtain an involution which shows that χ is balanced.

Suppose now that the second possibility obtains. Using Corollary 4.6 of [BZ2] (the "Leibniz rule" for derivatives) we see that $\rho^{(1)}$ is glued from the representations

$$\rho_\ell = \mathbb{I}(\chi_1, \ldots, \widehat{\chi_\ell}, \ldots, \chi_{r-1})$$

where the hat denotes omission. Since $\mathcal{L}(\omega', \nu', \rho^{(1)}) \neq \{0\}$, it follows that we must have $\mathcal{L}(\omega', \nu'; \rho_\ell) \neq \{0\}$ for some ℓ. Then, by induction, $(\chi_1, \ldots, \widehat{\chi_\ell}, \ldots, \chi_{r-1})$ is balanced for that ℓ. In particular,

$$\chi_1^2 \cdot \chi_2^2 \cdot \ldots \cdot \chi_{\ell-1}^2 \cdot \chi_{\ell+1}^2 \cdot \ldots \cdot \chi_{r-1}^2 = 1$$

and since $\prod_{j=1}^{r} \chi_j^2 = 1$ it follows that $\chi_\ell^2 \chi_r^2 = 1$. Thus if we take the involution $j \mapsto j^*$ of $\{1, \ldots, \ell-1, \ell+1, \ldots, r-1\}$ corresponding to the induction datum of

ρ_ℓ being balanced and extend it to $\{1,\ldots,r\}$ by setting $\ell^* = r$ then we have an involution showing that χ is balanced. This completes the induction. □

As usual, a character $\chi = (\chi_1,\ldots,\chi_r)$ of H_r will be called *regular* if it is not fixed by any non-identity element of the Weyl group under the natural conjugation action; here it simply means that all the characters χ_j are distinct.

Theorem 4: *Suppose that F is not dyadic or that $r \leq 3$. Suppose that $\chi = (\chi_1,\ldots,\chi_r)$ is balanced and that χ^2 is regular. Then, for any suitable characters ω and ν,*

$$\dim_{\mathbb{C}}\left(\mathcal{L}(\omega,\nu;\mathbb{I}(\chi))\right) \leq 1\,. \tag{22}$$

Proof: We shall again use induction on r, beginning with $r = 1$ and $r = 2$. If $r = 1$ then $\mathbb{I}(\chi) = \chi_1$ is a square-trivial character and so $\dim_{\mathbb{C}}\left(\mathcal{L}(\omega,\nu;\mathbb{I}(\chi))\right)$ is one if $\chi_1 = \omega\cdot\nu$ and zero otherwise. Thus (22) holds. If $r = 2$ then we are dealing with an induction datum $\chi = (\chi_1,\chi_2)$ which satisfies $\chi_1^2\chi_2^2 = 1$ and $\chi_1^2 \neq \chi_2^2$; in particular, $\chi_1^2 \neq 1$. Thus $\mathcal{L}(\eta_1,\eta_2;\chi_1) = \{0\}$ for all suitable η_1 and η_2 and using Theorem 2 with $\rho = \chi_1$ and $\alpha = \chi_2$ we obtain an injection

$$\mathcal{L}\big(\omega;\mathbb{I}(\chi)\big) \hookrightarrow \mathcal{L}\big(\omega';\chi_1^{(1)}\big)\,.$$

But $\chi_1^{(1)}$ is the trivial representation of $\mathrm{G}(0)$ and hence $\mathcal{L}(\omega';\chi_1^{(1)}) \cong \mathbb{C}$. This gives (22) in this case.

Now suppose that $r \geq 3$. We shall apply Theorem 2 with $\rho = \mathbb{I}(\chi_1,\ldots,\chi_{r-1})$

and $\alpha = \chi_r$. First assume that $\chi_r^2 \neq 1$. Then $\omega_\rho^2 = \chi_r^{-2} \neq 1$ and so $\mathcal{L}(\eta_1, \eta_2; \rho) = \{0\}$ for all suitable η_1 and η_2. Theorem 2 then implies that there is an injection

$$\mathcal{L}\big(\omega, \nu; \mathbb{I}(\chi)\big) \hookrightarrow \mathcal{L}\big(\omega', \nu'; \rho^{(1)}\big)\,. \tag{23}$$

As in the proof of Theorem 3, $\rho^{(1)}$ is glued from the representations

$$\rho_\ell = \mathbb{I}(\chi_1, \ldots, \widehat{\chi_\ell}, \ldots, \chi_{r-1})$$

for $\ell = 1, \ldots, r-1$. If $\ell \neq \ell'$ but the central characters of ρ_ℓ and $\rho_{\ell'}$ have equal squares then we could conclude that $\chi_\ell^2 = \chi_{\ell'}^2$, contradicting the regularity assumption. Thus the squares of the central characters of the ρ_ℓ are all distinct. It follows that

$$\rho^{(1)} \cong \oplus_{\ell=1}^{r-1} \rho_\ell$$

and that at most one of the ρ_ℓ has square-trivial central character. Thus

$$\mathcal{L}\big(\omega', \nu'; \rho^{(1)}\big) \cong \oplus_{\ell=1}^{r-1} \mathcal{L}\big(\omega', \nu'; \rho_\ell\big) \tag{24}$$

and all but one of the summands on the right of (24) are zero on central character grounds. If one of them is non-zero then it is at most one-dimensional by the induction hypothesis. Hence the left hand side of (24) is at most one-dimensional and it follows from (23) that $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi))$ is at most one-dimensional. This completes the induction in this case.

Now assume that $\chi_r^2 = 1$. Then $\chi_j^2 \neq 1$ for all $j = 1, \ldots, r-1$ and so the involution $j \mapsto j^*$ of $\{1, \ldots, r\}$ which corresponds to χ being balanced must satisfy $r^* = r$. If r were even then the restriction of $j \mapsto j^*$ to $\{1, \ldots, r-1\}$

would necessarily have a fixed point and this would produce some $1 \leq j \leq r-1$ with $\chi_j^2 = 1$. This is impossible and so r must be odd. Hence we have an exact sequence

$$\{0\} \to \mathcal{L}(\eta, \rho) \to \mathcal{L}(\omega, \nu; \mathbb{I}(\chi)) \to \mathcal{L}(\omega', \nu'; \rho^{(1)}) . \tag{25}$$

The central character of ρ is square-trivial and thus if ρ_ℓ had square-trivial central character for some ℓ, we would conclude that $\chi_\ell^2 = 1$, contradicting regularity. Therefore $\mathcal{L}(\omega', \nu'; \rho_\ell) = \{0\}$ for all ℓ and so $\mathcal{L}(\omega', \nu'; \rho^{(1)}) = \{0\}$. Using this fact, (25) yields an isomorphism

$$\mathcal{L}(\omega, \nu; \mathbb{I}(\chi)) \cong \mathcal{L}(\eta; \rho)$$

and the space on the right is at most one-dimensional by the inductive hypothesis. We conclude that $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi))$ is at most one-dimensional and this completes the inductive step in this case also. □

We have two results available which allow us to estimate the dimension of the space $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi))$, namely Corollary 1 and Theorem 4. Their range of applicability is not the same and it seems that Corollary 1 should allow us to obtain an estimate for certain balanced characters χ with χ^2 irregular. Unfortunately, Corollary 1 is never applicable to $\mathbb{I}(\chi)$ with χ balanced once $r \geq 3$. We can, however, complete our results when $r = 2$. We need not even assume that F is not dyadic, since $2 \leq 3$.

Proposition 5: *Let* $\chi = (\chi_1, \chi_2)$ *be a character of* H_2 *satisfying* $\chi_1^2\chi_2^2 = 1$. *Then,* *if ω is the unique suitable character of $\widetilde{Z}(2)$, we have*

$$\dim_{\mathbb{C}} \left(\mathcal{L}(\omega; \mathbb{I}(\chi))\right) \leq 1\,.$$

Proof: The result follows from Theorem 4 unless χ^2 is irregular; that is, unless $\chi_1^2 = \chi_2^2$. If $\chi_1^2 = \chi_2^2$ then $\chi_1^4 = \chi_2^4 = 1$; let us assume that this is so. The only intermediate derivative of $\mathbb{I}(\chi)$ is $\mathbb{I}(\chi)^{(1)}$, which is glued from χ_1 and χ_2. We cannot have $\chi_1^2 = |\cdot|^{-1}$ or $\chi_2^2 = |\cdot|^{-1}$ since $|\cdot|^{-2} \neq 1$ and so $\mathbb{I}(\chi)$ is general with respect to 1/4 (see Definition 1). By Corollary 1 we know that $\dim_{\mathbb{C}} \left(\mathcal{L}(\omega; \mathbb{I}(\chi))\right)$ is at most equal to the dimension of the space of Whittaker models of $\mathbb{I}(\chi)$ and it is well-known that this is one. □

6. Tensor Products of Exceptional Representations II

In this section we shall continue with the investigation of the spaces $\mathcal{L}(\omega, \nu; \pi)$ which was begun in the previous section. Our focus here will be on existence results to complement the uniqueness results already obtained.

Proposition 1: *Let ω and ν be suitable characters and $\chi_1, \ldots, \chi_r$ be characters of $F^\times$ satisfying $\chi_j^2 = 1$ for all j and also $\prod_{j=1}^r \chi_j = \omega \cdot \nu$ if r is odd. Put $\chi = (\chi_1, \ldots, \chi_r)$. Then $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi)) \neq \{0\}$.*

Proof: Since $\mathbb{I}(\chi)^\frown \cong \mathbb{I}(\chi^{-1})$ we have

$$\mathcal{L}(\omega, \nu; \mathbb{I}(\chi)) = \mathrm{Hom}_{G(r)}\left(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu}, \mathbb{I}(\chi^{-1})\right)$$

$$\cong \operatorname{Hom}_{\mathrm{H}_r}\left(\varphi_{(r),\gamma_0}(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu}), \chi^{-1}\right)$$

by Frobenius reciprocity.

Next we wish to understand the relationship between $\varphi_{(r),\gamma_0}(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu})$ and

$$\varphi_{(r),\gamma_0}(\vartheta_{r,\omega}) \otimes \varphi_{(r),\gamma_0}(\vartheta_{r,\nu}),$$

which is slightly complicated by the normalizations of the Jacquet functors. Let $E = E_1 \otimes E_2$ where E_1 is the space of $\vartheta_{r,\omega}$ and E_2 that of $\vartheta_{r,\nu}$. We let $E(\mathrm{N}(\gamma_0))$ be the subspace of E spanned by the set

$$\left\{(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu})(n)\xi - \xi \mid n \in \mathrm{N}(\gamma_0), \xi \in E\right\}$$

and $E_{\mathrm{N}(\gamma_0)}$ be $E/E(\mathrm{N}(\gamma_0))$ as usual, with similar notation for the other spaces. For $n \in \mathrm{N}(\gamma_0)$, $\xi_1 \in E_1$ and $\xi_2 \in E_2$ we have

$$\begin{aligned}&(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu})(n)(\xi_1 \otimes \xi_2) - (\xi_1 \otimes \xi_2)\\ &= \left[\vartheta_{r,\omega}(n)\xi_1 - \xi_1\right] \otimes \vartheta_{r,\nu}(n)\xi_2 + \xi_1 \otimes \left[\vartheta_{r,\nu}(n)\xi_2 - \xi_2\right]\end{aligned}$$

it follows that $E(\mathrm{N}(\gamma_0)) \subseteq E_1(\mathrm{N}^*(\gamma_0)) \otimes E_2(\mathrm{N}^*(\gamma_0))$ and so the space of

$$\varphi_{(r),\gamma_0}(\vartheta_{r,\omega}) \otimes \varphi_{(r),\gamma_0}(\vartheta_{r,\nu})$$

is a quotient of the space of $\varphi_{(r),\gamma_0}(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu})$ and the resulting surjection intertwines the unnormalized H_r actions on these spaces. Taking into account the normalizations, we conclude that there is an intertwining map

$$\varphi_{(r),\gamma_0}\left(\vartheta_{r,\omega} \otimes \vartheta_{r,\nu}\right) \twoheadrightarrow \mu_{(r),\gamma_0}^{1/2} \otimes \varphi_{(r),\gamma_0}\left(\vartheta_{r,\omega}\right) \otimes \varphi_{(r),\gamma_0}\left(\vartheta_{r,\nu}\right).$$

Using Theorem 1(a) of section 3 this gives an intertwining map

$$\varphi_{(r),\gamma_0}\left(\vartheta_{r,\omega}\otimes\vartheta_{r,\nu}\right)\twoheadrightarrow\vartheta_{\gamma_0,\omega}\otimes\vartheta_{\gamma_0,\nu}$$

and hence a surjection

$$\mathcal{L}(\omega,\nu;\mathbb{I}(\chi))\to\operatorname{Hom}_{\mathrm{H}_r}\left(\vartheta_{\gamma_0,\omega}\otimes\vartheta_{\gamma_0,\nu},\chi^{-1}\right).$$

This last space is non-zero by Proposition 1 and the conclusion follows. □

We cannot expect to make much progress in determining the dimension of the space $\mathcal{L}(\omega,\nu;\mathbb{I}(\chi))$ for a general balanced χ by such simple methods as the above. What is needed is a systematic procedure for producing elements of $\mathcal{L}(\omega,\nu;\mathbb{I}(\chi))$. For most χ (those for which $\mathbb{I}(\chi)$ is irreducible) we may conjugate χ by an element of the Weyl group without altering $\mathbb{I}(\chi)$. Thus we shall largely be content to produce an element of $\mathcal{L}(\omega,\nu;\mathbb{I}(\chi))$ after replacing χ by some Weyl conjugate. We need several preliminary results.

Lemma 1: *Every balanced character $\chi=(\chi_1,\ldots,\chi_r)$ of H_r is conjugate to a character which is trivial on $\mathrm{T}_2^2(r)$. Conversely, a character of H_r which is trivial on $\mathrm{T}_2^2(r)$ is balanced.*

Proof: From the definition it follows that

$$\mathrm{T}_2(r)=\begin{cases}\{\operatorname{diag}(a_1,a_1,a_2,a_2,\ldots,a_q,a_q)\mid a_j\in F^\times\} & \text{if } r \text{ is even}\\ \{\operatorname{diag}(a_0,a_1,a_1,\ldots,a_q,a_q)\mid a_j\in F^\times\} & \text{if } r \text{ is odd}\end{cases}$$

where $q=\lfloor r/2\rfloor$ and to obtain $\mathrm{T}_2^2(r)$ we need only replace the condition $a_j\in F^\times$ by $a_j\in(F^\times)^2$. Every involution of $\{1,\ldots,r\}$ is a product of disjoint transpositions and the conjugacy class of the involution within $\mathfrak{S}_r$ is determined by the number of these which occur. Suppose that χ is balanced with respect to the involution $j \mapsto j^*$. After conjugating χ and hence the involution we may suppose that $j \mapsto j^*$ is equal to the product

$$(i, i+1)(i+2, i+3) \cdot \ldots \cdot (r-1, r)$$

in $\mathfrak{S}_r$ for some $i \geq 1$ with $i \equiv r-1 \pmod 2$. From the definition of balanced we then have $\chi_j^2 = 1$ for $j < i$ and $\chi_j^2 \chi_{j+1}^2 = 1$ for $j = i, i+2, \ldots$ and so χ is trivial on $\mathrm{T}_2^2(r)$. Conversely, if χ is trivial on $\mathrm{T}_2^2(r)$ then it is balanced with respect to the involution

$$(12)(34) \cdot \ldots \cdot (r-1, r)$$

if r is even and

$$(23)(45) \cdot \ldots \cdot (r-1, r)$$

if r is odd. $\square$

The next Lemma refines Proposition 6 of section 4 for semi-Whittaker functions of the second kind. The character η is defined immediately before that Proposition.

Lemma 2: *Let $\omega \in \Omega^2(r)$. Then there is a character η_ω of $\widetilde{\mathrm{T}}_2(r)$ agreeing with η on $\widetilde{\mathrm{T}}_2^2(r)$ such that*

$$\Xi_\xi^{2,\omega}(hg) = \eta_\omega(h) \mu_{(r),\gamma_0}^{1/4}(h)\, \Xi_\xi^{2,\omega}(g)$$

for all $h \in \widetilde{\mathrm{T}}_2(r)$, $\xi \in E_{\vartheta_{r,\omega}}$ *and* $g \in \widetilde{\mathrm{G}}\,(r)$.

Proof: Let λ be any semi-Whittaker functional of the second kind. We fix $h \in \widetilde{\mathrm{T}}_2(r)$ and consider the functional λ_h given by

$$\xi \mapsto \lambda\big(\vartheta_{r,\omega}(h^{-1})\xi\big)$$

on $E_{\vartheta_{r,\omega}}$. The group H_r acts on $\mathrm{N}^*(\gamma_0)$ by conjugation and the character θ^2 of $\mathrm{N}^*(\gamma_0)$ is fixed by $\widetilde{\mathrm{T}}_2(r)$ under this action. Thus

$$\lambda_h\big(\vartheta_{r,\omega}(n)\xi\big) = \theta^2(n)\lambda_h(\xi)$$

for all $n \in \mathrm{N}^*(\gamma_0)$ and so λ_h is a semi-Whittaker functional of the second kind. The map $h \mapsto (\lambda \mapsto \lambda_h)$ defines a representation of $\widetilde{\mathrm{T}}_2(r)$ on the space of semi-Whittaker functionals of the second kind. By Proposition 4 of section 4 this space is one-dimensional and hence there is a character κ of $\widetilde{\mathrm{T}}_2(r)$ such that $\lambda_h = \kappa(h^{-1})\lambda$ for all $h \in \widetilde{\mathrm{T}}_2(r)$. Thus if the semi-Whittaker functions are formed with respect to λ then

$$\begin{aligned}
\Xi_\xi^{2,\omega}(hg) &= \lambda\big(\vartheta_{r,\omega}(hg)\xi\big) \\
&= \lambda_{h^{-1}}\big(\vartheta_{r,\omega}(g)\xi\big) \\
&= \kappa(h)\lambda\big(\vartheta_{r,\omega}(g)\xi\big) \\
&= \kappa(h)\,\Xi_\xi^{2,\omega}(g)\,.
\end{aligned}$$

We know from Proposition 6 of section 4 that $\kappa(h) = \eta(h)\mu_{(r),\gamma_0}^{1/4}(h)$ for $h \in \widetilde{\mathrm{T}}_2^2(r)$ and the Lemma follows. $\square$

Using the Inductive Structure Theorem (Theorem 1 of section 4) it is possible to compute η_ω explicitly. The answer depends, as expected, on the choice of additive character ψ. We shall not need to record the result of this calculation here; we merely note that $\eta_\omega \cdot \eta_\omega$ is a character of $\mathrm{T}_2(r)$ trivial on $\mathrm{T}_2^2(r)$ and therefore, since $\mathrm{T}_2(r)\,/\mathrm{T}_2^2(r)$ is a finite group, the character η_ω is necessarily unitary.

Next we need to recall a particular case of a well-known result from the general measure theory of locally compact, Hausdorff topological groups. Unfortunately, it is difficult to give a reference for it in exactly the form we shall need. The reader unfamiliar with it may consult, for instance, section 2.6 of [Fol] (see also the discussion in the first three sections of chapter 6 of the same work), and will also find it discussed in most other books which deal with the theory of Haar measure on not necessarily abelian topological groups. Note, however, that Folland (and several other of the standard references) put the subgroup on the right in forming homogeneous spaces, whereas we shall always put it on the left. This leads to some inverses appearing in the formulæ when they are translated from Folland's notation to ours. A brief treatment of the result we want in close to the form we shall require is contained in sections 1.20 and 1.21 of [BZ1], but with hypotheses which are slightly too restrictive.

We suppose that L is a closed subgroup of H_r and consider the group $\mathrm{L}\cdot\mathrm{N}(\gamma_0)$. This is a closed subgroup of $\mathrm{H}_r \cdot \mathrm{N}(\gamma_0)$ and has the structure of a semi-direct product, with L acting by conjugation on $\mathrm{N}(\gamma_0)$. The module of this action is the restriction to L of $\mu_{(r),\gamma_0}$; this is the inverse of the modular character of $\mathrm{L} \cdot \mathrm{N}(\gamma_0)$

restricted to L (under the usual conventions).

We initially consider the space $\mathcal{D}_c$ of continuous complex-valued functions, f, on $\mathrm{G}(r)$ which satisfy the conditions

((Inv)) $\quad f(\ell n g) = \mu_{(r),\gamma_0}(\ell) f(g)$ for all $\ell \in \mathrm{L}$, $n \in \mathrm{N}(\gamma_0)$ and $g \in \mathrm{G}(r)$ $\quad$ and

((Supp)) $\quad \mathrm{supp}(f) \subseteq \mathrm{L} \cdot \mathrm{N}(\gamma_0) \cdot C$ for some compactum $C \subseteq \mathrm{G}(r)$.

The letter $\mathcal{D}$ is chosen to suggest the word "density", since a function satisfying (Inv) may be regarded as a section of the vector bundle of densities on $\mathrm{L} \cdot \mathrm{N}(\gamma_0) \backslash \mathrm{G}(r)$. The space $\mathcal{D}_c$ contains a cone of everywhere non-negative functions (note that $\mu_{(r),\gamma_0}$ is a positive real-valued character) and so it makes sense to speak of a functional on $\mathcal{D}_c$ as being positive. Also, $\mathcal{D}_c$ carries a natural topology as the inductive limit of the spaces $\mathcal{D}(C)$, which are defined in the same way as $\mathcal{D}_c$, but with the compactum C fixed, where we take the topology of uniform convergence on C on the space $\mathcal{D}(C)$. The first form of the result we shall need is that $\mathcal{D}_c$ carries a continuous, positive linear functional which is invariant under the action of $\mathrm{G}(r)$ on $\mathcal{D}_c$ given by $(g_0 \cdot f)(g) = f(g g_0)$. This functional is, in fact, unique up to positive scalar multiples. It corresponds, as in [Fol], to a Radon measure on $\mathrm{L} \cdot \mathrm{N}(\gamma_0) \backslash \mathrm{G}(r)$ which is "strongly quasi-invariant". It follows from this that the functional extends to several spaces related to $\mathcal{D}_c$. For instance, if $\mathcal{D}^+$ is the space of continuous positive real-valued functions on $\mathrm{G}(r)$ satisfying (Inv) then the functional extends to $\mathcal{D}^+$ as an $\overline{\mathbb{R}}$-valued functional. From here we may extend it to the space $\mathcal{D}$ consisting of those continuous complex-valued functions, f, satisfying (Inv) and such that the functional is finite on $\mathrm{Re}(f)^{\pm}$ and $\mathrm{Im}(f)^{\pm}$.

We shall follow an almost universal abuse of notation and write

$$\int\limits_{L\cdot N(\gamma_0)\backslash G(r)} f(g)\, dg \tag{1}$$

for the value of this functional (with a normalization fixed once and for all) on $f \in \mathcal{D}$ or $f \in \mathcal{D}^+$.

We shall also require a "convergence" criterion for pseudointegrals such as (1). Let us suppose that we have another closed subgroup L' of H_r and that $L \cdot L'$ is of finite index in H_r. Since $H_r \cdot N(\gamma_0)\backslash G(r)$ is compact we conclude from the construction of (1) that if $f \in \mathcal{D}^+$ then (1) is finite if and only if

$$\int\limits_{L\cap L'\backslash L'} \mu^{-1}_{(r),\gamma_0}(\ell')f(\ell')\, d\ell' < \infty\,. \tag{2}$$

Here $d\ell'$ is the Haar measure on $L \cap L'\backslash L'$, which exists since both L' and $L \cap L'$ are abelian and hence unimodular. Note that (Inv) implies that the integrand in (2) is $L \cap L'$-invariant on the left, so that the integral is well-defined. There is a similar absolute convergence criterion.

Definition 1: *Let χ be a character of H_r unitary on $T_2(r)$. We say that χ has positive real part, and write $\mathrm{Re}(\chi) > 0$, if $h \in H_r$ and $|h^\alpha| < 1$ for all $\alpha \in \Delta_2$ implies $|\chi(h)| < 1$.*

If χ is a character of H_r trivial on $T_2(r)$ then it is determined by its restriction to $T_1(r)$, which is necessarily trivial on $T_1(r) \cap T_2(r) = Z^1(r)$. The torus $Z^1(r)\backslash T_1(r)$ is isomorphic to $F^\times \times \cdots \times F^\times$ with $q = \lfloor r/2 \rfloor$ factors via the map $[h] \mapsto (h^\alpha)_{\alpha\in\Delta_2}$ (where we enumerate Δ_2 in the standard order). Every character,

χ, of H_r trivial on $\mathrm{T}_2(r)$ corresponds to a character of $F^\times \times \cdots \times F^\times$ (q factors) under this isomorphism, and we denote the resulting character by $\check{\chi}$. We may express $\check{\chi}$ as $(\check{\chi}_1, \ldots, \check{\chi}_q)$ where each $\check{\chi}_j$ is a character of $F^\times$. Note that χ has positive real part if and only if $|\check{\chi}_j(\varpi)| < 1$ for all $j = 1, \ldots, q$, where ϖ is any uniformizer of F.

We recall that the complex conjugate representation of $\vartheta_{r,\omega}$ is isomorphic to $\vartheta_{r,\omega^{-1}}$ since any suitable ω is, in particular, a unitary character. In the function space models of $\vartheta_{r,\omega}$ and $\vartheta_{r,\omega^{-1}}$ given in section 3 the map is literally complex conjugation of functions. Thus an element of $\mathcal{L}(\omega, \omega^{-1}; \pi)$ gives rise to a $\mathrm{G}(r)$-invariant *semi-Hermitian form* on $\vartheta_{r,\omega} \times \vartheta_{r,\omega} \times \pi$, by which I mean a form which is $\mathrm{G}(r)$-invariant, complex linear in its first and third arguments and complex anti-linear in its second. The converse is also true and it will be notationally more convenient to work with semi-Hermitian forms in what follows.

For the reader's convenience we add a few remarks on the relation between ω and ω^{-1} when ω is a suitable character. When r is even, so that ω is a character of $\widetilde{\mathrm{Z}}^2(r) \cong (F^\times)^2 \times \mu_2$ trivial on the first factor, we have $\omega = \omega^{-1}$ and hence $\mathcal{L}(\omega, \omega^{-1}; \pi) = \mathcal{L}(\omega; \pi)$. When $r \equiv 1 \pmod 4$ it follows from the discussion before Proposition 3 of section 3 that $\widetilde{\mathrm{Z}}^1(r) \cong F^\times \times \mu_2$ and ω is a genuine square-trivial character on this group. Thus again $\omega = \omega^{-1}$ and $\mathcal{L}(\omega, \omega^{-1}; \pi) = \mathcal{L}(\omega; \pi)$. In the last case, when $r \equiv 3 \pmod 4$, this is no longer necessarily true. In fact, a calculation using Proposition 3 of section 3 together with [Rao], Corollary A.5 (3) shows that

$$\omega^{-1}(\mathbf{s}(tI_r)) = \omega(\mathbf{s}(tI_r)) \cdot (-1, t)$$

for all suitable ω when $r \equiv 3 \pmod 4$. Thus $\mathcal{L}(\omega, \omega^{-1}; \pi)$ is not necessarily the same thing as $\mathcal{L}(\omega; \pi)$ in this case.

In what follows, if ω is a suitable character then we shall allow ourselves to write $\Xi_{\xi}^{2,\omega}$. This is legitimate since each suitable character extends uniquely to an element of $\Omega^2(r)$. Note that $\Xi_{\xi}^{1,\omega}$ would be ambiguous if r were even since in that case the unique suitable character extends to an element of $\Omega^1(r)$ in many ways.

Proposition 2: *Let χ be a character of H_r which is trivial on $\mathrm{T}_2(r)$ and has positive real part. For ω a suitable character, $\xi_1, \xi_2 \in E_{\vartheta_{r,\omega}}$ and $f \in E_{\mathbb{I}(\chi)}$ let*

$$\Upsilon(\xi_1, \xi_2, f) = \int_{\mathrm{T}_2(r)\cdot \mathrm{N}(\gamma_0)\backslash \mathrm{G}(r)} \Xi_{\xi_1}^{2,\omega}(g)\, \overline{\Xi_{\xi_2}^{2,\omega}}(g)\, f(g)\, dg\,. \tag{3}$$

Then Υ is a $\mathrm{G}(r)$-invariant semi-Hermitian form on $\vartheta_{r,\omega} \times \vartheta_{r,\omega} \times \mathbb{I}(\chi)$ and hence gives rise to an element of $\mathcal{L}(\omega, \omega^{-1}; \mathbb{I}(\chi))$.

Proof: The integral in (3) is meant in the sense of (1) and we must check that the integrand is a function of the correct kind. We note first that the product of the two (genuine) semi-Whittaker functions is being regarded as a non-genuine object in the usual way. We know from Lemma 2 that $\Xi_{\xi_1}^{2,\omega}$ and $\Xi_{\xi_2}^{2,\omega}$ both transform on the left under $\widetilde{\mathrm{T}}_2(r)$ by $\eta_\omega \cdot \mu_{(r),\gamma_0}^{1/4}$ and since η_ω is a unitary character it follows that

$$\Xi_{\xi_1}^{2,\omega}(hg)\, \overline{\Xi_{\xi_2}^{2,\omega}}(hg) = \mu_{(r),\gamma_0}^{1/2}(h)\, \Xi_{\xi_1}^{2,\omega}(g)\, \overline{\Xi_{\xi_2}^{2,\omega}}(g)$$

for all $h \in \mathrm{T}_2(r)$ and $g \in \mathrm{G}(r)$. Recalling that the induction is normalized, we have

$$f(hg) = \mu^{1/2}_{(r),\gamma_0}(h)\chi(h)f(g) = \mu^{1/2}_{(r),\gamma_0}(h)f(g)$$

for $h \in \mathrm{T}_2(r)$ and $g \in \mathrm{G}(r)$. Thus the integrand has the correct transformation law under $\mathrm{T}_2(r)$. Under left multiplication by $n \in \mathrm{N}(\gamma_0)$, the first factor transforms by $\theta^2(n)$, the second by $\overline{\theta^2(n)}$ and the third is invariant. Since θ^2 is a unitary character, the whole integrand is invariant on the left by $\mathrm{N}(\gamma_0)$. Every factor in the integrand is continuous (indeed locally constant) on $\mathrm{G}(r)$ and hence (3) is intelligible.

Next we must show that (3) converges, for which purpose we shall use the absolute convergence test mentioned above. We have $\mathrm{H}_r = \mathrm{T}_1(r) \cdot \mathrm{T}_2(r)$ and $\mathrm{T}_1(r) \cap \mathrm{T}_2(r) = \mathrm{Z}^1(r)$ and hence it suffices to establish the convergence of the integral

$$\begin{aligned} &\int\limits_{\mathrm{Z}^1(r)\backslash \mathrm{T}_1(r)} \mu^{-1}_{(r),\gamma_0}(h) \left| \Xi^{2,\omega}_{\xi_1}(h)\, \Xi^{2,\omega}_{\xi_2}(h)\, f(h) \right| dh \\ &= \int\limits_{\mathrm{Z}^1(r)\backslash \mathrm{T}_1(r)} \mu^{-1}_{(r),\gamma_0}(h) \left| \Xi^{2,\omega}_{\xi_1}(h)\, \Xi^{2,\omega}_{\xi_2}(h) \right| \mu^{1/2}_{(r),\gamma_0}(h) |\chi(h)|\, dh \cdot |f(e)| \\ &= \int\limits_{\mathrm{Z}^1(r)\backslash \mathrm{T}_1(r)} \left| \Xi^{2,\omega}_{\xi_1}(h)\, \Xi^{2,\omega}_{\xi_2}(h) \right| \mu^{-1/2}_{(r),\gamma_0}(h) |\chi(h)|\, dh \cdot |f(e)| . \end{aligned} \tag{4}$$

Combining Propositions 7 and 8 of section 4 we see that the function

$$\left| \Xi^{2,\omega}_{\xi_1}(h)\, \Xi^{2,\omega}_{\xi_2}(h) \right| \mu^{-1/2}_{(r),\gamma_0}(h)$$

is bounded on $\mathrm{T}_1(r)$ and vanishes whenever $|h^\alpha|$ is sufficiently large for some $\alpha \in \Delta_2$. Thus (4) is bounded by a constant times

$$\int\limits_{\{h \in \mathrm{Z}^1(r)\backslash \mathrm{T}_1(r) \text{ such that } |h^\alpha| \le C\ \forall\ \alpha \in \Delta_2\}} |\chi(h)|\, dh$$

$$= \int_{\{(a_1,\ldots,a_q)\in(F^\times)^q \text{ such that } |a_j|\leq C\}} \prod_{j=1}^{q} |\check{\chi}_j(a_j)| \cdot \prod_{j=1}^{q} d^\times a_j$$

which converges since $|\check{\chi}_j(\varpi)| < 1$ for all $j = 1, \ldots, q$. It follows that $\Upsilon(\xi_1, \xi_2, f)$ is well-defined and since it is clearly semi-Hermitian and $\mathrm{G}(r)$-invariant, the Proposition follows. □

The reader may have been surprised to see the hypothesis that χ is trivial on $\mathrm{T}_2(r)$ rather than that it is trivial on $\mathrm{T}_2^2(r)$ appearing in Proposition 2 and some explanation is called for. A little extra generality is possible; if we allowed two suitable characters instead of one in forming the integral (3) when r was odd it would be possible to require χ to be trivial only on those elements in $\mathrm{T}_2(r)$ for which the $(1,1)$-entry is a square. But this merely amounts to twisting both $\mathbb{I}(\chi)$ and one of the two exceptional representations by $\kappa(\det)$ where κ is a square trivial character. It can thus be deduced from Proposition 2. If the results of [Sav] are correct then, except for the representations covered by Proposition 1, of the irreducible $\mathbb{I}(\chi)$ with χ balanced, only those mentioned in Proposition 2 have $\mathcal{L}(\omega, \omega^{-1}; \mathbb{I}(\chi)) \neq \{0\}$ when $r = 3$. This suggests that the results of section 5 are not the whole truth and indeed the author does not at present see how to obtain elements of the other $\mathcal{L}(\omega, \nu; \mathbb{I}(\chi))$ spaces which the results of section 5 allow to be non-zero.

In order to prepare for the next result we must recall some further facts. If χ is a character of $F^\times$ then χ is said to be *unramified* if it is trivial on the units, $\mathcal{O}_F^\times$ of the ring of integers of F. If we fix a uniformizer ϖ of F then any character of $F^\times$ may be dcomposed as $\chi = \chi_u \cdot \chi_d$ where we define $\chi_d(x) = \chi(x\varpi^{-v(x)})$ and $\chi_u(x) = \chi(\varpi^{v(x)})$ Here $v : F^\times \to \mathbb{Z}$ is the normalized additive valuation on F. It is easy to check that both χ_u and χ_d are characters, that χ_u is unramified and that χ_d is "radial"; that is, $\chi_d(\varpi^m x) = \chi_d(x)$ for all $m \in \mathbb{Z}$. The character χ_d is determined by its restriction to $\mathcal{O}_F^\times$ and any character of $\mathcal{O}_F^\times$ may be extended to a radial character of $F^\times$. Now any unramified character of $F^\times$ has the form $|\cdot|^s$ for some $s \in \mathbb{C}/2\pi i \log(q)\mathbb{Z}$ where $q = |\varpi|^{-1}$ is the module of F. (The double use of q, once for $\lfloor r/2 \rfloor$ and once for the module of F should not cause confusion.) Thus the space of unramified characters of $F^\times$ has the structure of a complex manifold and this may be extended to the entire space of characters by giving the characters of $\mathcal{O}_F^\times$ the discrete topology (and corresponding unique zero dimensional complex manifold structure) and using the decomposition $\chi = \chi_u \cdot \chi_d$ above. Changing the choice of uniformizer produces a permutation of which connected components of the space of characters are labelled by which characters of $\mathcal{O}_F^\times$ and also a purely imaginary shift in the variable s on each connected component, but does not alter the complex structure. In particular, for a given uniformizer ϖ, the map $\chi \mapsto \chi(\varpi)$ is an analytic function on the space of characters. It gives a local coordinate around any point in the space of characters.

If we speak of a "Laurent polynomial" on the space of characters then this is to be understood with respect to the function $\chi \mapsto \chi(\varpi)$ for some (and hence any) uniformizer ϖ. That is, f a function on the space of characters is a Laurent polynomial if for each connected component there is a Laurent polynomial P

such that $f(\chi) = P(\chi(\varpi))$ on that component. (In the literature the phrase "Laurent polynomial in q^{-s}" often occurs; the notions are identical.) All of these considerations extend in an obvious way to the space of characters of $F^\times \times \cdots \times F^\times$.

We let $\mathrm{K}(r)$ be the natural maximal compact subgroup of $\mathrm{G}(r)$, namely $\mathrm{GL}(r, \mathcal{O}_F)$. It is well-known (see [Cas] for example) that as a representation of $\mathrm{K}(r)$, $\mathbb{I}(\chi)$ is isomorphic to

$$\mathrm{ind}_{\mathrm{K}(r)\cap \mathrm{H}_r}^{\mathrm{K}(r)} \left(\chi|_{\mathrm{K}(r)\cap\mathrm{H}_r}\right),$$

the isomorphism in one direction being simply restriction to $\mathrm{K}(r)$. Thus, once $\chi|_{\mathrm{K}(r)\cap\mathrm{H}_r}$ is fixed, all the induced representations may be realized on the same space. If $f \in E_{\mathbb{I}(\chi_*)}$ and $\chi|_{\mathrm{K}(r)\cap\mathrm{H}_r}$ and $\chi_*|_{\mathrm{K}(r)\cap\mathrm{H}_r}$ are equal then we shall denote by $[f]_\chi$ the element of $E_{\mathbb{I}(\chi)}$ whose restriction to $\mathrm{K}(r)$ is $f|_{\mathrm{K}(r)}$. Since we shall be dealing with $\mathbb{I}(\chi)$ only when χ is trivial on $\mathrm{T}_2(r)$ we shall also allow ourselves to write $[f]_{\breve{\chi}}$ with $\breve{\chi}$ a character of $(F^\times)^q$. The notation means $[f]_\chi$ where χ is the character of H_r trivial on $\mathrm{T}_2(r)$ corresponding to $\breve{\chi}$.

Proposition 3: *Fix a uniformizer, ϖ, of F. Let ω be a suitable character, $\xi_1, \xi_2 \in E_{\vartheta_{r,\omega}}$ and $f \in E_{\mathbb{I}(\chi_*)}$ where χ_* is a character of H_r trivial on $\mathrm{T}_2(r)$. Then the function*

$$\breve{\chi} \mapsto \prod_{j=1}^{q} (1 - \breve{\chi}_j(\varpi)^2) \cdot \Upsilon(\xi_1, \xi_2, [f]_{\breve{\chi}}) \tag{5}$$

on the space of characters $\breve{\chi}$ of $(F^\times)^q$ satisfying $|\breve{\chi}_j(\varpi)| < 1$ for $j = 1, \ldots, q$ and

$$\chi|_{\mathrm{K}(r)\cap\mathrm{H}_r} = \chi_*|_{\mathrm{K}(r)\cap\mathrm{H}_r} \tag{6}$$

is a Laurent polynomial.

Proof: We first note that (6) simply fixes the ramified part, χ_d of χ; thus the function we are discussing is defined on an open set (a poly-half-cylinder, to be precise) within a single connected component of the space of characters of $(F^\times)^q$. The condition $|\check{\chi}_j(\varpi)| < 1 \ \forall\ j$ is precisely what is required for $\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$ to be convergent.

Since all the functions in (3) are right $\mathrm{K}(r)$-finite and $\mathrm{T}_1^2(r)\backslash\mathrm{T}_1(r)$ is a finite group, $\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$ is a sum of finitely-many integrals of the form

$$\int\limits_{\mathrm{Z}^2(r)\backslash\mathrm{T}_1^2(r)} \mu_{(r),\gamma_0}^{-1}(h)\, \Xi_{\zeta_1}^{2,\omega}(h)\, \overline{\Xi_{\zeta_2}^{2,\omega}}(h) v^\chi(h)\, dh \tag{7}$$

where $\zeta_1, \zeta_2 \in E_{\vartheta_{r,\omega}}$ and $v^\chi \in E_{\mathbb{I}(\chi)}$ with $v^\chi|_{\mathrm{K}(r)}$ fixed as χ varies. It thus suffices to establish the claim with (7) in place of $\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$. Since $v^\chi(e)$ does not depend on χ, (7) is equal to a fixed multiple of

$$\int\limits_{\mathrm{Z}^2(r)\backslash\mathrm{T}_1^2(r)} \mu_{(r),\gamma_0}^{-1/2}(h)\, \Xi_{\zeta_1}^{2,\omega}(h)\, \overline{\Xi_{\zeta_2}^{2,\omega}}(h)\chi(h)\, dh\,. \tag{8}$$

We analyze (8) using Fubini's Theorem and the Inductive Structure Theorem (Theorem 1 of section 4). Firstly, it is easy to see that

$$\mathrm{Z}^2(r)\backslash\mathrm{T}_1^2(r) \cong \mathrm{Z}^2(r-2)\backslash\mathrm{T}_1^2(r-2) \times \mathrm{Z}^2(2)\backslash\mathrm{T}_1^2(2)$$

and with respect to this decomposition the Inductive Structure Theorem gives

$$\Xi_{\zeta_i}^{2,\omega}(h_1 h_2) = \mu_{(r),(r-2,2)}^{1/4}(h_1, h_2) \sum_{j=1}^{M_i} \Xi_{\zeta_{i,j}^{r-2}}^{2,\omega_{r-2}}(h_1)\, \Xi_{\zeta_{i,j}^2}^{2,\omega_2}(h_2)\,.$$

Since

$$\mu_{(r),\gamma_0} = \mu_{(r),(r-2,2)} \cdot \left(\mu_{(r-2),\gamma_0} \times \mu_{(2),\gamma_0}\right)$$

this and Fubini's Theorem express (8) as a sum of products of factors of the form

$$\int\limits_{Z^2(r-2)\backslash T_1^2(r-2)} \mu_{(r-2),\gamma_0}^{-1/2}(h_1)\, \Xi_{\zeta_{1,j}^{r-2}}^{2,\omega_{r-2}}(h_1)\, \overline{\Xi_{\zeta_{2,j}^{r-2}}^{2,\omega_{r-2}}}(h_1)\, \chi|_{T_1^2(r-2)}(h_1)\, dh_1 \tag{9}$$

and

$$\int\limits_{Z^2(2)\backslash T_1^2(2)} \mu_{(2),\gamma_0}^{-1/2}(h_2)\, \Xi_{\zeta_{1,j}^2}^{2,\omega_2}(h_2)\, \overline{\Xi_{\zeta_{2,j}^2}^{2,\omega_2}}(h_2)\, \chi|_{T_1^2(2)}(h_2)\, dh_2\,. \tag{10}$$

Inductively, the product of (9) with

$$\prod_{j=1}^{q-1}(1-\check{\chi}_j(\varpi)^2)$$

is a Laurent polynomial. Thus we are reduced to beginning the induction and completing the inductive step by showing that functions of the form

$$(1-\check{\chi}(\varpi)^2)\cdot \int\limits_{Z^2(2)\backslash T_1^2(2)} \mu_{(2),\gamma_0}^{-1/2}(h)\, \Xi_{\zeta_1}^{2,\omega_2}(h)\, \overline{\Xi_{\zeta_2}^{2,\omega_2}}(h)\, \chi(h)\, dh \tag{11}$$

are Laurent polynomials. Writing

$$\phi(t) = \mu_{(2),\gamma_0}^{-1/2}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \Xi_{\zeta_1}^{2,\omega_2}\left(\mathrm{s}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{\Xi_{\zeta_2}^{2,\omega_2}}\left(\mathrm{s}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right),$$

(11) becomes

$$(1-\check{\chi}(\varpi)^2)\cdot \int\limits_{t\in(F^\times)^2} \phi(t)\check{\chi}(t)\, d^\times t \tag{12}$$

and we know from section 4 that $\phi(t) = 0$ when $|t| \gg 1$, $\phi(t) = k$, a constant, when $|t| \ll 1$ and for intermediate $|t|$, $\phi(t)$ is a locally constant function. With the additional restriction $|t| > \epsilon$, the integral in (12) is itself a Laurent polynomial.

The integral over the range on which $\phi(t) = k$ may be evaluated directly; it is zero if $\check{\chi}^2$ is ramified and a constant times $\check{\chi}(\varpi)^{2m} \cdot (1 - \check{\chi}(\varpi)^2)^{-1}$ for some $m \in \mathbb{Z}$ if not. This completes the proof. $\square$

Proposition 3 shows that $\check{\chi} \mapsto \Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$ is a rational function of $\check{\chi}$ with at worst simple poles at the zeros of

$$\prod_{j=1}^{q} (1 - \check{\chi}_j(\varpi)^2) . \tag{13}$$

If $\check{\chi}_j^2$ is ramified then it is possible to choose the uniformizer ϖ so that $(1 - \check{\chi}_j(\varpi)^2)$ is non-vanishing and it follows that $\check{\chi} \mapsto \Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$ has no poles in the variable $\check{\chi}_j$. In any case Proposition 3 serves to analytically continue $\check{\chi} \mapsto \Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$ and hence give a meaning to $\Upsilon(\xi_1, \xi_2, f)$ with $f \in \mathbb{I}(\chi)$ provided that $\check{\chi}$ in not a zero of (13).

We shall also require the following estimate, which follows by a slight variation on the argument in Proposition 3.

Proposition 4: *Fix a uniformizer, ϖ, of F. Let ω be a suitable character, $\xi_1, \xi_2 \in E_{\vartheta_{r,\omega}}$ and χ_* a character of H_r trivial on $\mathrm{T}_2(r)$. Take $\delta > 0$ and let $\mathbf{X}_\delta$ be the set*

$$\{\check{\chi} \in \widehat{(F^\times)^q} \mid \chi|_{\mathrm{T}_2(r)} = 1, \chi|_{\mathrm{K}(r) \cap \mathrm{H}_r} = \chi_*|_{\mathrm{K}(r) \cap \mathrm{H}_r}, \delta < |\check{\chi}_j(\varpi)| < 1 \ \forall\ j\} .$$

Then there is a constant $C(\xi_1, \xi_2, \delta)$ such that

$$|\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})| \leq C(\xi_1, \xi_2, \delta) \cdot \prod_{j=1}^{q} |1 - \check{\chi}_j(\varpi)^2|^{-1} \cdot \|f\|_{\mathrm{K}(r)}$$

for all $\check{\chi} \in \mathbf{X}_\delta$ and $f \in E_{\mathbb{I}(\chi_*)}$, where $\|\cdot\|$ denotes the uniform norm.

Proof: We can estimate $|\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})|$ by

$$\int\limits_{\mathrm{T}_1(r)\cap\mathrm{K}(r)\backslash\mathrm{K}(r)} \left| \int\limits_{\mathrm{Z}^1(r)\backslash\mathrm{T}_1(r)} \mu_{(r),\gamma_0}^{-1/2}(h)\, \Xi_{\xi_1}^{2,\omega}(hk)\, \overline{\Xi_{\xi_2}^{2,\omega}}(hk)\, \chi(h)\, dh \right| |f(k)|\, dk\,. \qquad (14)$$

Since both ξ_1 and ξ_2 are $\widetilde{\mathrm{K}}(r)$-finite, the inner integral is a sum of finitely-many similar terms involving the semi-Whittaker functions of the $\widetilde{\mathrm{K}}$-translates of ξ_1 and ξ_2. Now any integral of the form

$$\int\limits_{\mathrm{Z}^1(r)\backslash\mathrm{T}_1(r)} \mu_{(r),\gamma_0}^{-1/2}(h)\, \Xi_{\zeta_1}^{2,\omega}(h)\, \overline{\Xi_{\zeta_2}^{2,\omega}}(h)\, \chi(h)\, dh$$

has been shown, in the course of the proof of Proposition 3, to equal

$$\prod_{j=1}^{q} (1 - \check{\chi}_j(\varpi)^2)^{-1} \cdot P_{\zeta_1,\zeta_2}(\check{\chi}_1(\varpi), \ldots, \check{\chi}_q(\varpi))$$

where P_{ζ_1,ζ_2} is a Laurent polynomial depending only on ζ_1 and ζ_2. On the set $\mathbf{X}_\delta$ any Laurent polynomial is bounded by a constant involving the absolute values of its coefficients, the multidegree of each term and δ. Thus the absolute value of the inner integral in (14) may be estimated by

$$C'(\xi_1, \xi_2, \delta) \cdot \prod_{j=1}^{q} |1 - \check{\chi}_j(\varpi)^2|^{-1}\,.$$

We conclude that

$$\begin{aligned} &|\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})| \\ &\leq C'(\xi_1, \xi_2, \delta) \cdot \mathrm{vol}(\mathrm{T}_1(r) \cap \mathrm{K}(r)\backslash\mathrm{K}(r)) \cdot \prod_{j=1}^{q} |1 - \check{\chi}_j(\varpi)^2|^{-1} \cdot \|f\|_{\mathrm{K}(r)}\,, \end{aligned}$$

as required. $\square$

Using Proposition 4 we can show that the $\mathrm{G}(r)$-invariance of $\Upsilon(\cdot,\cdot,\cdot)$ extends to the closure of the domain on which we presently have it, provided that we do not encounter a pole.

Proposition 5: *Fix a uniformizer, ϖ, of F. Let ω be a suitable character and χ_* a character of H_r trivial on $\mathrm{T}_2(r)$. Suppose that $|\breve{\chi}_{*,j}(\varpi)| \leq 1$ but $\breve{\chi}_{*,j}(\varpi) \neq \pm 1$ for all $j = 1,\ldots,q$. Then $\Upsilon(\cdot,\cdot,\cdot)$ is a $\mathrm{G}(r)$-invariant semi-Hermitian form on $E_{\vartheta_{r,\omega}} \times E_{\vartheta_{r,\omega}} \times E_{\mathbb{I}(\chi_*)}$.*

Proof: Let $\xi_1, \xi_2 \in E_{\vartheta_{r,\omega}}$ and $f \in E_{\mathbb{I}(\chi_*)}$. In order to save some notational clutter we shall write $g \cdot \xi_1$, $g \cdot f$ and so on for the action of g on vectors in the various representations. Fix

$$\delta < \min_j |\breve{\chi}_{*,j}(\varpi)|$$

and let $\mathbf{X}_\delta$ be as in the previous Proposition. Then we have

$$\Upsilon(\xi_1,\xi_2,f) = \lim_{\chi \to \chi_*, \breve{\chi} \in \mathbf{X}_\delta} \Upsilon(\xi_1,\xi_2,[f]_{\breve{\chi}}) \tag{15}$$

and

$$\begin{aligned} \Upsilon(g\cdot\xi_1, g\cdot\xi_2, g\cdot f) &= \lim_{\chi \to \chi_*, \breve{\chi} \in \mathbf{X}_\delta} \Upsilon(g\cdot\xi_1, g\cdot\xi_2, [g\cdot f]_{\breve{\chi}}) \\ &= \lim_{\chi \to \chi_*, \breve{\chi} \in \mathbf{X}_\delta} \Upsilon(\xi_1,\xi_2, g^{-1}\cdot[g\cdot f]_{\breve{\chi}}) \end{aligned} \tag{16}$$

by the $\mathrm{G}(r)$-invariance of Υ when $\breve{\chi} \in \mathbf{X}_\delta$. We must show that (15) and (16) are equal.

For $k \in \mathrm{K}(r)$, let $kg^{-1} = n(k)a(k)k'(k)$ be an expression for kg^{-1} with respect to the Iwasawa decomposition $\mathrm{G}(r) = \mathrm{N}(\gamma_0)\,\mathrm{H}_r\mathrm{K}(r)$. None of the factors in this expression is unique, but $a(k)$ is restricted to a compactum as k varies over $\mathrm{K}(r)$ and this will be sufficient. For $k \in \mathrm{K}(r)$ we have

$$\begin{aligned} g^{-1}\cdot[g\cdot f]_{\check{\chi}}(k) &= [g\cdot f]_{\check{\chi}}(kg^{-1}) \\ &= [g\cdot f]_{\check{\chi}}(n(k)a(k)k'(k)) \\ &= \mu_{(r),\gamma_0}^{1/2}(a(k))\chi(a(k))(g\cdot f)(k'(k)) \\ &= \mu_{(r),\gamma_0}^{1/2}(a(k))\chi(a(k))f(k'(k)g) \\ &= \mu_{(r),\gamma_0}^{1/2}(a(k))\chi(a(k))f(a(k)^{-1}n(k)^{-1}k) \\ &= \chi(a(k))\chi_*(a(k))^{-1}f(k) \end{aligned}$$

and so

$$\begin{aligned} &\|\,[f]_{\check{\chi}} - g^{-1}\cdot[g\cdot f]_{\check{\chi}}\,\|_{\mathrm{K}(r)} \\ &\le \|f\|_{\mathrm{K}(r)}\,\|1-\chi(a(k))\chi_*(a(k))^{-1}\|_{\mathrm{K}(r)} \\ &\le \|f\|_{\mathrm{K}(r)}\,\|\chi_*(a(k))\|_{\mathrm{K}(r)}^{-1}\,\|\chi_*(a(k))-\chi(a(k))\|_{\mathrm{K}(r)}\,. \end{aligned} \tag{17}$$

The topology on the space of characters is that of uniform convergence on compacta and so (17) implies that

$$\lim_{\chi\to\chi_*}\|\,[f]_{\check{\chi}} - g^{-1}\cdot[g\cdot f]_{\check{\chi}}\,\|_{\mathrm{K}(r)} = 0\,.$$

Thus

$$|\Upsilon(g\cdot\xi_1, g\cdot\xi_2, g\cdot f) - \Upsilon(\xi_1,\xi_2,f)|$$

$$= \lim_{\chi \to \chi_*, \check{\chi} \in \mathbf{X}_\delta} |\Upsilon(\xi_1, \xi_2, g^{-1} \cdot [g \cdot f]_{\check{\chi}} - [f]_{\check{\chi}})|$$

$$\leq C(\xi_1, \xi_2, \delta) \cdot \lim_{\chi \to \chi_*, \check{\chi} \in \mathbf{X}_\delta} \prod_{j=1}^{q} |1 - \check{\chi}_j(\varpi)^2|^{-1} \cdot \| g^{-1} \cdot [g \cdot f]_{\check{\chi}} - [f]_{\check{\chi}} \|_{\mathrm{K}(r)}$$

from Proposition 4

$$= C(\xi_1, \xi_2, \delta) \cdot \prod_{j=1}^{q} |1 - \check{\chi}_{*,j}(\varpi)^2|^{-1} \cdot 0 = 0 .$$

This shows that Υ is $\mathrm{G}(r)$-invariant, as required. □

If χ_* is a character of H_r trivial on $\mathrm{T}_2(r)$, $|\check{\chi}_{*,j}(\varpi)| \leq 1$ for $j = 1, \ldots, q$ and $\check{\chi}_{*,j}(\varpi)^2 = 1$ for $j \in J \subseteq \{1, \ldots, q\}$ then for a suitable $\delta > 0$ we may define

$$\Upsilon_{\mathrm{res},J}(\xi_1, \xi_2, f) =$$

$$\lim_{\chi \to \chi_*, \check{\chi} \in \mathbf{X}_\delta} \prod_{j \in J} (1 - \check{\chi}_{*,j}(\varpi)^2) \cdot \Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$$

to be the "residue" (more correctly, a multiple of a coefficient in the partial fractions expansion of) $\Upsilon(\xi_1, \xi_2, [f]_{\check{\chi}})$ at $\check{\chi}_*$. We know from Proposition 3 that this limit exists and by following the proof of Proposition 5 *mutatis mutandis* we find that $\Upsilon_{\mathrm{res},J}$ defines a $\mathrm{G}(r)$-invariant semi-Hermitian form on $E_{\vartheta_{r,\omega}} \times E_{\vartheta_{r,\omega}} \times E_{\mathbb{I}(\chi_*)}$.

In order to show that the apparatus so far developed is not vacuous we must demonstrate that Υ is at least sometimes non-zero. If this were our only aim then it could quickly be realized. Indeed, if we choose $\check{\chi} = (| \cdot |^{\sigma_1}, \ldots, | \cdot |^{\sigma_q})$ where $\sigma_1, \ldots, \sigma_q$ are positive real numbers then we may find $f \in E_{\mathbb{I}(\chi)}$ such that $f(g) > 0$ for all $g \in \mathrm{G}(r)$. Choosing a vector $\xi \in E_{\vartheta_{r,\omega}}$ we see that

$$\Upsilon(\xi, \xi, f) = \int_{\mathrm{T}_2(r) \cdot \mathrm{N}(\gamma_0) \backslash \mathrm{G}(r)} \left| \Xi_\xi^{2,\omega}(g) \right|^2 f(g) \, dg$$

and the integrand is everywhere non-negative. If $\Upsilon(\xi, \xi, f) = 0$ then it follows from $f(g) > 0$ and the local constancy of the semi-Whittaker function that $\Xi_\xi^{2,\omega}(g) = 0$ for all $g \in \widetilde{\mathrm{G}}\,(r)$ and hence $\xi = 0$. Thus $\Upsilon(\xi, \xi, f) > 0$ if $\xi \neq 0$. However, we shall learn a good deal more by computing $\Upsilon(\xi_1, \xi_2, f)$ explicitly when ξ_1, ξ_2 and f are "spherical".

We begin by reviewing a few known facts. Suppose that χ is an unramified character of H_r, which means that $\chi|_{\mathrm{K}(r)\cap\mathrm{H}_r} = 1$. Then we may define a vector $f_\circ \in E_{\mathbb{I}(\chi)}$ by setting

$$f_\circ(g) = \mu_{(r),\gamma_0}^{1/2}(a(g))\chi(a(g))$$

where $g = n(g)a(g)k(g)$ is an Iwasawa decomposition of g. This vector satisfies $k \cdot f_\circ = f_\circ$ for all $k \in \mathrm{K}(r)$ and is called the *(normalized) spherical vector* in $\mathbb{I}(\chi)$.

In the representations $\vartheta_{r,\omega}$ we cannot, of course, hope to find a vector fixed under $\widetilde{\mathrm{K}}(r)$ since $\widetilde{\mathrm{K}}(r) \supseteq \mu_2$ and the representations are genuine. Thus we must first enquire when the metaplectic double cover of $\mathrm{G}(r)$ is split over $\mathrm{K}(r)$. It turns out that this happens (for non-Archimedean ground fields, F) precisely when the residual characteristic of F is odd. This follows from the argument (though not, as implied in [KaP], the statement) of Lemma (11.3) of [Mo2]. *Thus in all our discussion of spherical vectors below we must assume that F is not dyadic and we shall do so from now until the end of this section.* The splitting of the metaplectic cover over $\mathrm{K}(r)$ is then unique and we shall suppose that $\mathbf{s} : \mathrm{K}(r) \to \mathrm{K}^*(r)$ has been chosen to be a homomorphism.

We note that under our new assumption on F the Hilbert symbol takes a particularly simple form. Indeed, by Proposition 8, §3, chapter XIV of [Ser] we have

$$(a,b) = \left[\overline{(-1)^{v(a)v(b)}\frac{a^{v(b)}}{b^{v(a)}}}\right]^{(q-1)/2}$$

where $v(\cdot)$ is the normalized additive valuation, q is the module of the field and the bar denotes reduction modulo the prime ideal in $\mathcal{O}_F$. In particular, $(a,b) = 1$ if a and b are both units and it is always possible to find a unit a so that $(a,\varpi) = -1$ for any given normalizer ϖ. This is arranged by choosing a so that its reduction modulo the prime ideal is a quadratic non-residue in $\mathbb{F}_q$.

It is still not always possible to find a $\mathrm{K}^*(r)$-fixed vector in the space of $\vartheta_{r,\omega}$, since the choice of suitable character ω might not allow it. Clearly for $\vartheta_{r,\omega}$ to have a non-zero $\mathrm{K}^*(r)$-fixed vector the character ω must be trivial on $\widetilde{\mathrm{Z}}(r) \cap \mathrm{K}^*(r)$. If r is even then there is only one choice of suitable character and it always satisfies this condition. If r is odd then

$$\widetilde{\mathrm{Z}}(r) \cap \mathrm{K}^*(r) = \widetilde{\mathrm{Z}}^1(r) \cap \mathrm{K}^*(r) \cong \mathcal{O}_F^\times\,,$$

as follows from the discussion preceding Proposition 3 in section 3 plus what we have just said about the Hilbert symbol. The character ω is already assumed to be trivial on $(\mathcal{O}_F^\times)^2$ and we are requiring that it should in fact be trivial on $\mathcal{O}_F^\times$. In the rather odd terminology of [KaP] such characters are referred to as "normalized" (their "unramified" characters must be trivial on $\mathrm{K}^*(r) \cap \widetilde{\mathrm{Z}}^2(r)$ but may be non-trivial on $\mathrm{K}^*(r) \cap \widetilde{\mathrm{Z}}^1(r)$ – a departure from the usual sense of this term). There are two suitable unramified normalized characters of $\widetilde{\mathrm{Z}}^1(r)$ when r is odd. They correspond (in the sense of Proposition 3 of section 3) to the trivial character and the "unramified signum" $\mathrm{sgn}_{\mathrm{nr}}(x) = (-1)^{v(x)}$ (the terminology is Casselman's). We shall simply call them *unramified suitable characters.* We note that if ω_1 and ω_2 are suitable characters then, since r is odd,

$$\vartheta_{r,\omega_1} \cong (\omega_2^{-1} \cdot \omega_1)(\det) \otimes \vartheta_{r,\omega_2}$$

and so

$$\vartheta_{r,\omega_1} \otimes \vartheta_{r,\omega_1} \cong \vartheta_{r,\omega_2} \otimes \vartheta_{r,\omega_2}$$

in any case.

If ω is an unramified suitable character then it follows from [KaP] Lemma I.1.3 that the induced representation $\pi_{(r)}(\chi_0, \omega)$ contains a unique normalized $\mathrm{K}^*(r)$-invariant vector and from [KaP] I.2.4 that this vector lies in the space of $\vartheta_{r,\omega}$ inside this induced representation. We shall denote this vector by $\xi_\circ$ (the normalization will not be important to us and so we shall not discuss it further).

Finally, in order to be able to compute the spherical semi-Whittaker functions, we shall have to assume that the additive character ψ is itself unramified; that is, it is trivial on $\mathcal{O}_F$ but non-trivial on $\varpi^{-1}\mathcal{O}_F$.

Lemma 3: *Let* $\Omega^2(2) = \{\omega\}$ *and put*

$$h = \mathbf{s}(\mathrm{diag}(t_1, t_2)) \in \widetilde{\mathrm{H}}_2 .$$

Then $\Xi_{\xi_\circ}^{2,\omega}(h) = 0$ *unless* $v(t_1) \geq v(t_2)$ *and* $v(t_1) \equiv v(t_2) \pmod 2$. *If these conditions are satisfied then*

$$\Xi_{\xi_\circ}^{2,\omega}(h) = \begin{cases} \mu_{(2),\gamma_0}^{1/4}(h) & \text{if } v(t_1) \text{ is even} \\ \mu_\psi(\varpi)\mu_{(2),\gamma_0}^{1/4}(h) & \text{if } v(t_1) \text{ is odd.} \end{cases}$$

Proof: That $\Xi^{2,\omega}_{\xi_\circ}(h) \neq 0$ implies that $v(t_1) \geq v(t_2)$ follows from the argument used to prove Proposition 7 of section 4 once we use the assumptions made on $\xi_\circ$ and ψ. The argument which shows that it also implies $v(t_1) \equiv v(t_2) \pmod 2$ will be made in generality in the proof of the next Proposition and need not be preempted here. Thus we are reduced to evaluating $\Xi^{2,\omega}_{\xi_\circ}(h)$ when $h \in \widetilde{\mathrm{G}}^2(2)$. In this case

$$\mathbf{s}(\varpi I_2)h = \mathbf{s}(\mathrm{diag}(\varpi t_1, \varpi t_2))$$

and since $\mathbf{s}(\varpi I_2) \in \widetilde{\mathrm{Z}}^1(2)$ and $\omega_2(\mathbf{s}(\varpi I_2)) = \mu_\psi(\varpi)$ (see Proposition 3 of section 3 and Lemma 2 of section 4) the claim when $v(t_1)$ is odd is reduced to the claim when it is even. But this value is computed (in a somewhat disguised form) in Proposition 4.4.2 of [GeP]. (Note the remark immediately after the statement.)

□

Observe that the conditions in the Lemma may be combined simply to say that $h^\alpha \in \mathcal{O}_F^2$ where $\Delta_2 = \{\alpha\}$.

Proposition 6: *Let ω be an unramified suitable character and put*

$$h = \mathbf{s}(\mathrm{diag}(t_1, t_2, \ldots, t_r)) \in \widetilde{\mathrm{H}}_r\,.$$

Then $\Xi^{2,\omega}_{\xi_\circ}(h) = 0$ unless

$$v(t_{r-2j+1}) \geq v(t_{r-2j+2}) \tag{18}$$

and

$$v(t_{r-2j+1}) \equiv v(t_{r-2j+2}) \pmod 2 \tag{19}$$

for all $j = 1, \ldots, q$. If these conditions are satisfied then

$$\Xi^{2,\omega}_{\xi_\circ}(h) = \mu_\psi(\varpi)^p \mu^{1/4}_{(r),\gamma_0}(h)$$

if r is even and

$$\Xi^{2,\omega}_{\xi_\circ}(h) = \eta_\omega(\mathbf{s}(\mathrm{diag}(t_1, 1, \ldots, 1)))\mu_\psi(\varpi)^p \mu^{1/4}_{(r),\gamma_0}(h)$$

if r is odd, where p is the number of pairs in (19) with odd valuation.

Proof: The map $\xi \mapsto [\xi]$ which occurs in the Inductive Structure Theorem (with $\ell = 1$) is a $\mathrm{K}^*(r-2) \widetilde{\times} \mathrm{K}^*(2)$-intertwining operator and it follows from the unicity of the spherical vector that $[\xi_\circ] = \xi_\circ^{r-2} \otimes \xi_\circ^2$. Thus we have

$$\Xi^{2,\omega}_{\xi_\circ}(h) = \mu^{1/4}_{(r),(r-2,2)}(h_1, h_2)\, \Xi^{2,\omega_{r-2}}_{\xi_\circ^{r-2}}(h_1)\, \Xi^{2,\omega_2}_{\xi_\circ^2}(h_2)\,, \tag{20}$$

where $h_1 = \mathbf{s}(\mathrm{diag}(t_1, \ldots, t_{r-2}))$ and $h_2 = \mathbf{s}(\mathrm{diag}(t_{r-1}, t_r))$, provided that $t_{r-1}t_r \in (F^\times)^2$.

From the previous Lemma we know that $\Xi^{2,\omega_2}_{\xi_\circ^2}(e) \neq 0$ and we conclude inductively using (20) that $\Xi^{2,\omega}_{\xi_\circ}(e) \neq 0$. If $k \in \widetilde{\mathrm{T}}_2(r) \cap \mathrm{K}^*(r)$ then k fixes $\xi_\circ$ and so

$$\begin{aligned}\Xi^{2,\omega}_{\xi_\circ}(e) &= \Xi^{2,\omega}_{\xi_\circ}(k) \\ &= \eta_\omega(k)\, \Xi^{2,\omega}_{\xi_\circ}(e)\end{aligned}$$

where η_ω is the character in Lemma 2. Thus $\eta_\omega(k) = 1$ and so $\Xi^{2,\omega}_{\xi_\circ}(kg) = \Xi^{2,\omega}_{\xi_\circ}(g)$ for all $k \in \widetilde{\mathrm{T}}_2(r) \cap \mathrm{K}^*(r)$ and $g \in \widetilde{\mathrm{G}}\,(r)$.

Let us choose $a \in \mathcal{O}_F^\times$ such that $(a, \varpi) = -1$ and set

$$k = \mathbf{s}(\mathrm{diag}(1, \ldots, 1, a, a^{-1})) \in \widetilde{\mathrm{T}}_2(r) \cap \mathrm{K}^*(r)\,.$$

A direct calculation using (3) of Chapter 1, section 3 shows that

$$hk = (a, t_{r-1}t_r^{-1})kh$$

and so

$$\begin{aligned}\Xi^{2,\omega}_{\xi_\circ}(h) &= \Xi^{2,\omega}_{\xi_\circ}(hk) \\ &= (a, t_{r-1}t_r^{-1})\, \Xi^{2,\omega}_{\xi_\circ}(kh) \\ &= (a, t_{r-1}t_r^{-1})\, \Xi^{2,\omega}_{\xi_\circ}(h)\,.\end{aligned}$$

It follows from this identity that $\Xi^{2,\omega}_{\xi_\circ}(h) = 0$ unless $t_{r-1}t_r \in (F^\times)^2$. Thus either $\Xi^{2,\omega}_{\xi_\circ}(h) = 0$ or (20) is applicable. We conclude inductively from this that $\Xi^{2,\omega}_{\xi_\circ}(h) = 0$ unless (19) is satisfied. The necessity of (18) for $j = 1$ now follows from Lemma 3 and then we can deduce it inductively for $j = 2, \ldots, q$.

Now suppose that (18) and (19) are satisfied. Assume for a moment that r is odd. Then we have

$$h = \mathbf{s}(\mathrm{diag}(t_1, 1, \ldots, 1))\mathbf{s}(\mathrm{diag}(1, t_2, \ldots, t_r))$$

and the first factor on the right hand side lies in $\widetilde{\mathrm{T}}_2(r)$. Thus

$$\Xi^{2,\omega}_{\xi_\circ}(h) = (\eta_\omega \cdot \mu^{1/4}_{(r),\gamma_0})\big(\mathbf{s}(\mathrm{diag}(t_1, 1, \ldots, 1))\big) \cdot \Xi^{2,\omega}_{\xi_\circ}(\mathbf{s}(\mathrm{diag}(1, t_2, \ldots, t_r)))$$

by Lemma 2 and, comparing this with the identity to be proved when r is odd, we see that we may henceforth suppose that $t_1 = 1$ when r is odd.

This said, (20) and Lemma 3 give inductively that

$$\begin{aligned}
&\Xi^{2,\omega}_{\xi_\circ}(h) \\
&= \mu_\psi(\varpi)^{p'}\mu_\psi(\varpi)^{p''}\mu^{1/4}_{(r),(r-2,2)}(h_1,h_2)\mu^{1/4}_{(r-2),\gamma_0}(h_1)\mu^{1/4}_{(2),\gamma_0}(h_2) \\
&= \mu_\psi(\varpi)^{p'+p''}\mu^{1/4}_{(r),\gamma_0}(h)
\end{aligned}$$

where p' is the number of pairs with odd valuation up to $r-2$ and p'' is 1 or 0 according as the pair t_{r-1}, t_r does or does not have odd valuation. Since $p = p' + p''$, the proof is complete. $\square$

Again, the conditions (18) and (19) may be combined to say that $h^\alpha \in \mathcal{O}_F^2$ for all $\alpha \in \Delta_2$.

Proposition 7: *Let ω be an unramified suitable character and χ be an unramified character of H_r trivial on $\mathrm{T}_2(r)$ and satisfying $|\check{\chi}_j(\varpi)| < 1$ for all $j = 1, \ldots, q$. Then*

$$\Upsilon(\xi_\circ;\xi_\circ,f_\circ) = C \cdot \prod_{j=1}^{q}(1-\check{\chi}_j(\varpi)^2)^{-1},$$

where C is a non-zero constant.

Proof: Since all the data are spherical we have

$$\Upsilon(\xi_\circ,\xi_\circ,f_\circ) = \int\limits_{\mathrm{Z}^1(r)\backslash\mathrm{T}_1(r)} \left|\Xi^{2,\omega}_{\xi_\circ}(h)\right|^2 \mu^{-1/2}_{(r),\gamma_0}(h)\chi(h)\,dh\,. \qquad (21)$$

Recalling that $h \mapsto (h^\alpha)_{\alpha \in \Delta_2}$ is an isomorphism from $\mathrm{Z}^1(r)\backslash \mathrm{T}_1(r)$ onto $(F^\times)^q$ and using Proposition 6 we see that (21) may be rewritten as

$$\Upsilon(\xi_\circ, \xi_\circ, f_\circ) = \int_{(\mathcal{O}_F^2)^q} \check{\chi}(t)\, d^\times t\,.$$

Since $\check{\chi}$ is unramified this has the stated value. □

We are now ready for the result which has been the principal goal of this section. In the light of Lemma 1, its range of applicability is larger than may appear from the hypotheses.

Theorem 1: *Let ω be an unramified suitable character and χ be an unramified character of H_r trivial on $\mathrm{T}_2(r)$. Then $\mathcal{L}(\omega, \omega^{-1}; \pi) \neq \{0\}$ for some constituent, π, of $\mathbb{I}(\chi)$.*

Proof: Without altering the constituents of $\mathbb{I}(\chi)$ we may conjugate χ by an element of the Weyl group in order to assume that $|\check{\chi}_j(\varpi)| \leq 1$ for $j = 1, \ldots, q$. Let us put

$$J = \{j \mid \check{\chi}_j(\varpi)^2 = 1\}$$

and define $\Upsilon_{\mathrm{res},J}$ as above (see the discussion after the proof of Proposition 5). Then $\Upsilon_{\mathrm{res},J}$ is a $\mathrm{G}(r)$-invariant semi-Hermitian form on $\vartheta_{r,\omega} \times \vartheta_{r,\omega} \times \mathbb{I}(\chi)$ and, from Proposition 7,

$$\Upsilon_{\mathrm{res},J}(\xi_\circ, \xi_\circ, f_\circ) = C \cdot \prod_{j \notin J} (1 - \check{\chi}_j(\varpi)^2)^{-1} \neq 0\,.$$

We know that $\overline{\vartheta_{r,\omega}} \cong \vartheta_{r,\omega^{-1}}$; let $\xi \mapsto \bar{\xi}$ be the map which realizes the isomorphism.

Then the uniqueness of the spherical vector implies that $\overline{\xi_{o,\omega}} = \xi_{o,\omega^{-1}}$. If we define

$$\langle \xi_1, \overline{\xi_2}, f \rangle = \Upsilon_{\mathrm{res},J}(\xi_1, \xi_2, f)$$

then we obtain a $\mathrm{G}(r)$-invariant trilinear form on

$$\vartheta_{r,\omega} \times \vartheta_{r,\omega^{-1}} \times \mathbb{I}(\chi)$$

with

$$\langle \xi_{o,\omega}, \xi_{o,\omega^{-1}}, f_o \rangle \neq 0 \,.$$

This gives rise to a non-zero element of $\mathcal{L}(\omega, \omega^{-1}; \mathbb{I}(\chi))$ and hence $\mathcal{L}(\omega, \omega^{-1}; \pi) \neq \{0\}$ for some constituent of $\mathbb{I}(\chi)$. This completes the proof. □

7. Addenda

Many natural questions remain open after the work of the previous section. One expects there to be numerous representations, π, of $\mathrm{G}(r)$ not belonging to the principal series for which

$$\mathcal{L}(\omega, \omega^{-1}; \pi) \neq \{0\}$$

and the methods of section 6 do not extend to these. Also, Theorem 1 of section 6 does not address the question of precisely which constituents, ρ, of a reducible spherical principal series representation satisfy

$$\mathcal{L}(\omega, \omega^{-1}; \rho) \neq \{0\} \,.$$

This section will be devoted to observations on these problems which seem worth making although the author has not as yet been able to achieve any general results by their use.

We first note that if $\omega \in \Omega^2(r)$ then $\omega^{-1} \in \Omega^1(r)$. This is clear if r is odd since then both $\Omega^1(r)$ and $\Omega^2(r)$ consist of all suitable characters. If r is even then $\Omega^2(r)$ is a singleton, but $\Omega^1(r)$ consists of all extensions of the unique suitable character to $\widetilde{\mathrm{Z}}^1(r)$ and the claim follows. We shall denote by $\mathbf{W}(r, \psi)$ the Gelfand-Graev representation of $\mathrm{G}(r)$ with trivial central character formed using the additive character ψ. That is,

$$\mathbf{W}(r, \psi) = \mathrm{Ind}_{\mathrm{Z}^1(r)\cdot \mathrm{N}(\gamma_0)}^{\mathrm{G}(r)} \big(1 \otimes \psi\big)$$

where the induction is smooth but with no restriction on supports. If $\omega \in \Omega^2(r)$ then we define

$$T : \vartheta_{r,\omega^{-1}} \otimes \vartheta_{r,\omega} \longrightarrow \mathbf{W}(r, \psi)$$

by

$$T(\xi_1 \otimes \xi_2)(g) = \Xi_{\xi_1}^{1,\omega^{-1}}(g)\, \Xi_{\xi_2}^{2,\omega}(g)\,, \tag{1}$$

where the product of genuine objects is being regarded as non-genuine in the usual way. Since the two semi-Whittaker functions transform on the left under $\widetilde{\mathrm{Z}}^1(r)$ by inverse characters, $T(\xi_1 \otimes \xi_2)$ transforms trivially on the left under $\mathrm{Z}^1(r)$. The fact that $\theta^1 \cdot \theta^2 = \psi$ shows similarly that $T(\xi_1 \otimes \xi_2)$ transforms correctly on the left under $\mathrm{N}(\gamma_0)$. Since both ξ_1 and ξ_2 are smooth, $T(\xi_1 \otimes \xi_2)$ is smooth and hence (1) is well-defined. Its very definition makes it clear that T is an intertwining operator.

We can actually be a little more precise about the range of T. If $W \in \mathbf{W}(r, \psi)$ then $|W(g)|^2$ is left invariant by both $\mathrm{Z}^1(r)$ and $\mathrm{N}(\gamma_0)$ and so it makes sense to consider the integral

$$\int_{Z^1(r)\cdot N(\gamma_0)\backslash G(r)} |W(g)|^2\, dg\,. \tag{2}$$

We shall denote by $\mathbf{W}^2(r,\psi)$ the subspace of $\mathbf{W}(r,\psi)$ consisting of functions for which (2) is convergent. (The Cauchy-Schwarz inequality shows as usual that it *is* a subspace.) Note that $\mathbf{W}^2(r,\psi)$ is a subrepresentation of $\mathbf{W}(r,\psi)$ and carries the structure of a pre-Hilbert space.

Proposition 1: *The range of T lies in $\mathbf{W}^2(r,\psi)$.*

Proof: The decomposition $G(r) = N(\gamma_0)\, H_r K(r)$ and the smoothness of the functions involved show that $W \in \mathbf{W}^2(r,\psi)$ if and only if

$$\int_{Z^1(r)\backslash H_r} |W(h)|^2 < \infty\,. \tag{3}$$

For any Whittaker function, W, we have $W(h) = 0$ if $|h^\alpha| \gg_W 1$ for some $\alpha \in \Delta$. Thus (3) is equivalent to

$$\int_{\{h\in Z^1(r)\backslash H_r \text{ such that } |h^\alpha|\leq 1\ \forall\ \alpha\in\Delta\}} |W(h)|^2\, dh < \infty\,. \tag{4}$$

If $\xi_1 \in E_{\vartheta_{r,\omega^{-1}}}$ and $\xi_2 \in E_{\vartheta_{r,\omega}}$ then we have

$$\left|\Xi^{1,\omega^{-1}}_{\xi_1}(h)\right| \ll \mu^{1/4}_{(r),\gamma_0}(h) \quad \text{and} \quad \left|\Xi^{2,\omega}_{\xi_2}(h)\right| \ll \mu^{1/4}_{(r),\gamma_0}(h)$$

by Proposition 8 of section 4. Thus

$$|T(\xi_1\otimes\xi_2)(h)| \ll \mu^{1/2}_{(r),\gamma_0}(h) \tag{5}$$

and we are reduced to the convergence of

$$\int_{\{h\in Z^1(r)\backslash H_r \text{ such that } |h^\alpha|\leq 1\ \forall\ \alpha\in\Delta\}} \mu_{(r),\gamma_0}(h)\, dh$$

which is well-known. $\square$

If π is an irreducible admissible generic representation of $\mathrm{G}(r)$ and $v \in E_\pi$ then we may define

$$\langle \xi_1, \xi_2, v \rangle = \int_{\mathrm{Z}^1(r)\cdot \mathrm{N}(\gamma_0)\backslash \mathrm{G}(r)} T(\xi_1 \otimes \xi_2)(g) W_v(g)\, dg\,, \tag{6}$$

for $\xi_1 \in E_{\vartheta_{r,\omega^{-1}}}$ and $\xi_2 \in E_{\vartheta_{r,\omega}}$, where $W_v \in \mathbf{W}(r, \overline{\psi})$ is the Whittaker function associated to v. By Proposition 1 we know that if $W_v \in \mathbf{W}^2(r, \overline{\psi})$ then (6) will be absolutely convergent (as (5) shows, this condition on W_v isn't sharp). When (6) is absolutely convergent for all $v \in E_\pi$, it defines an invariant trilinear form on

$$E_{\vartheta_{r,\omega^{-1}}} \times E_{\vartheta_{r,\omega}} \times E_\pi$$

which then extends to an element of $\mathcal{L}(\omega^{-1}, \omega; \pi)$. Thus we have a way of producing an element of $\mathcal{L}(\omega^{-1}, \omega; \pi)$ for any irreducible admissible generic representation, π, whose Whittaker model lies inside $\mathbf{W}^2(r, \overline{\psi})$. The difficult point is to show that the resulting functional is not identically zero. Of course, this cannot be true in general. There are many irreducible principal series representations whose induction datum is not balanced but whose Whittaker model lies in $\mathbf{W}^2(r, \overline{\psi})$. For these, (6) must be identically zero by Theorem 3 of section 5 (at least when F is not dyadic or $r \leq 3$).

We wish to close this section by showing that (6) gives rise to a non-zero element of $\mathcal{L}(\omega^{-1}, \omega; \pi)$ in at least one instance. We shall postpone a discussion of the significance of this result until after the rather lengthy calculation necessary to prove it. *From now on we assume that F is not dyadic.*

Let σ denote the *Steinberg representation* of G(2) (see [Cas], for example). That is, σ is the unique irreducible subrepresentation of the reducible principal series representation $\pi = \mathbb{I}(\mu^{1/2})$ (the symbol μ will henceforth stand for $\mu_{(2),(1,1)}$). The other constituent of π is the trivial representation and there is thus a (non-split) short exact sequence

$$0 \to \sigma \to \pi \to 1 \to 0. \tag{7}$$

The unique normalized spherical vector $v_\circ \in E_\pi$ has non-zero image in the trivial representation under the map in (7) and thus the representation σ is not spherical.

Let us introduce some notation which will be useful in the calculation to come. We set

$$\mathrm{K}_m = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{K}(2) \,\middle|\, c \in \varpi^m \mathcal{O} \right\}$$

for $m \geq 0$ and

$$\begin{aligned} t(x) &= \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, & a(x_1, x_2) &= \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \\ m(y) &= \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, & n(y) &= \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \\ w &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & r(x) &= \begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix} \end{aligned}$$

for $x_1, x_2, x \in F^\times$ and $y \in F$. The following identities connecting these matrices may easily be established by direct calculation:

$$a(x_1, x_2)m(y) = m(x_1^{-1}x_2 y)a(x_1, x_2) \tag{8}$$

$$a(x_1, x_2)n(y) = n(x_1 x_2^{-1} y)a(x_1, x_2) \tag{9}$$

$$m(x) = n(x^{-1})a(x^{-1}, x)r(x) \tag{10}$$

$$wm(y) = n(y)w \tag{11}$$

for $x_1, x_2, x \in F^\times$ and $y \in F$.

Lemma 1: *Let $v = v_\circ - \pi\big(t(\varpi^{-2})\big)v_\circ$. Then v is a non-zero vector in the space of σ which is fixed by* K_2.

Proof: In the function space model of π, $v_\circ$ is given by $v_\circ(nak) = \mu(a)$ for $n \in \mathrm{N}(\gamma_0)$, $a \in \mathrm{H}_2$ and $k \in \mathrm{K}(2)$. Thus $v(e) = 1 - q^2$ (where q is the module of F) and so $v \neq 0$. Clearly the image of v in the trivial representation in (7) is zero and so $v \in E_\sigma$. If $k_2 \in \mathrm{K}_2$ then

$$\begin{aligned}\sigma(k_2)v &= \pi(k_2)v_\circ - \pi\big(k_2 t(\varpi^{-2})\big)v_\circ \\ &= v_\circ - \pi\big(t(\varpi^{-2})\big)\pi\big(t(\varpi^2)k_2 t(\varpi^{-2})\big)v_\circ .\end{aligned} \tag{12}$$

If

$$k_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{K}_2$$

then

$$t(\varpi^2)k_2 t(\varpi^{-2}) = \begin{pmatrix} a & \varpi^2 b \\ \varpi^{-2}c & d \end{pmatrix} \in \mathrm{K}(2)$$

and so $\pi\big(t(\varpi^2)k_2 t(\varpi^{-2})\big)v_\circ = v_\circ$. From (12) we then obtain $\sigma(k_2)v = v$. $\square$

Lemma 2: *Let*

$$\mathcal{K}_2 = \{wm(x) \mid x \in [\varpi\mathcal{O}/\varpi^2\mathcal{O}]\} \cup \{m(x) \mid x \in [\mathcal{O}/\varpi^2\mathcal{O}]\}$$

where $[\mathcal{O}/\varpi^2\mathcal{O}]$ indicates a transversal for $\mathcal{O}/\varpi^2\mathcal{O}$ and so on. Then $\mathcal{K}_2$ is a transversal for $\mathrm{K}(2)/\mathrm{K}_2$.

Proof: This is a routine computation. □

The Steinberg representation is generic and we next wish to compute the Whittaker function of v. *We shall assume henceforth that the additive character ψ is trivial on $\mathcal{O}$ but non-trivial on $\varpi^{-1}\mathcal{O}$.* The computation can be reduced substantially by the following observations. The Whittaker function is given by the absolutely convergent integral

$$W_v(g) = \int_F v\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\overline{\psi(x)}\,dx\,. \tag{13}$$

Now the integral

$$W_{v_\circ}(g) = \int_F v_\circ\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\overline{\psi(x)}\,dx \tag{14}$$

is also absolutely convergent and, although it cannot strictly be referred to as the Whittaker function of $v_\circ$, its value may be computed by the same method which is used to compute the spherical Whittaker function for an irreducible spherical principal series representation. Indeed, the computation in [God], Theorem 11 remains valid for (14) and we obtain

$$W_{v_\circ}\begin{pmatrix}\varpi^m & 0\\ 0 & 1\end{pmatrix} = \begin{cases}0 & \text{if } m < 0\\ \frac{1-q^{-(m+1)}}{1-q^{-1}} & \text{if } m \geq 0.\end{cases} \tag{15}$$

The Whittaker function W_v satisfies

$$W_v(nzgk_2) = \psi(n)W_v(g)$$

for $n \in \mathrm{N}(\gamma_0)$, $z \in \mathrm{Z}^1(2)$ and $k_2 \in \mathrm{K}_2$ and so it suffices to compute its values on the set

$$\{t(\varpi^m)k \mid m \in \mathbb{Z}, k \in \mathcal{K}_2\}\,.$$

Lemma 3: *We have*

$$W_v\big(t(\varpi^m)\big) = \begin{cases} 0 & m \le -1 \\ 1 & m = 0 \\ q^{-(m-1)}(1+q^{-1}) & m \ge 1\,, \end{cases}$$

$$W_v\big(t(\varpi^m)m(x)\big) = \begin{cases} 0 & m \le -1 \\ 1-\psi(x^{-1}) & m = 0 \\ 0 & m \ge 1 \end{cases}$$

for $x \in \varpi\mathcal{O}^\times$,

$$W_v\big(t(\varpi^m)m(x)\big) = \begin{cases} 0 & m \le -3 \\ -\psi(\varpi^{-2}x^{-1}) & m = -2 \\ -\psi(\varpi^{-1}x^{-1})(1+q^{-1}) & m = -1 \\ -q^{-(m+1)}(1+q^{-1}) & m \ge 0 \end{cases}$$

for $x \in \mathcal{O}^\times$ *and*

$$W_v\big(t(\varpi^m)wm(x)\big) = \begin{cases} 0 & m \le -3 \\ -\psi(\varpi^{-2}x) & m = -2 \\ -q^{-(m+1)}(1+q^{-1}) & m \ge -1 \end{cases}$$

for $x \in \varpi\mathcal{O}$.

Proof: Initially imposing no restriction on $x \in \mathcal{O}$ we have

$$\begin{aligned} W_v\big(t(\varpi^m)m(x)\big) &= W_{v_\circ}\big(t(\varpi^m)m(x)\big) - W_{v_\circ}\big(t(\varpi^m)m(x)t(\varpi^{-2})\big) \\ &= W_{v_\circ}\big(t(\varpi^m)\big) - W_{v_\circ}\big(t(\varpi^{m-2})m(\varpi^{-2}x)\big)\,. \end{aligned} \tag{16}$$

If $x = 0$ then combining this with (15) we obtain the first displayed formula. Now suppose that x lies in $\mathcal{O}^\times$ or $\varpi\mathcal{O}^\times$. Then

$$m(\varpi^{-2}x) = n(\varpi^2x^{-1})a(\varpi^2x^{-1}, \varpi^{-2}x)r(\varpi^{-2}x),$$

from (10), and since $\varpi^2 x^{-1} \in \mathcal{O}$, $r(\varpi^{-2}x) \in \mathrm{K}(2)$. Thus (16) gives

$$\begin{aligned}
&W_v\big(t(\varpi^m)m(x)\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - W_{v_\circ}\big(t(\varpi^{m-2})n(\varpi^2x^{-1})a(\varpi^2x^{-1},\varpi^{-2}x)\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - W_{v_\circ}\big(n(\varpi^m x^{-1})t(\varpi^{m-2})a(\varpi^2x^{-1},\varpi^{-2}x)\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - \psi(\varpi^m x^{-1})W_{v_\circ}\big(t(\varpi^{m-2})a(\varpi^2x^{-1},\varpi^{-2}x)\big)\,. \qquad (17)
\end{aligned}$$

If $x \in \varpi\mathcal{O}^\times$ then $\varpi x^{-1} \in \mathcal{O}^\times$ and so

$$\begin{aligned}
W_{v_\circ}\big(t(\varpi^{m-2})a(\varpi^2x^{-1},\varpi^{-2}x)\big) &= W_{v_\circ}\big(t(\varpi^{m-2})a(\varpi,\varpi^{-1})a(\varpi x^{-1},\varpi^{-1}x)\big) \\
&= W_{v_\circ}\big(t(\varpi^m)a(\varpi^{-1},\varpi^{-1})\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big)\,,
\end{aligned}$$

which gives

$$W_v\big(t(\varpi^m)m(x)\big) = [1-\psi(\varpi^m x^{-1})]W_{v_\circ}\big(t(\varpi^m)\big)\,.$$

The second displayed formula follows from this and the fact that ψ is trivial on $\mathcal{O}$.

Now suppose that $x \in \mathcal{O}^\times$. Then (17) gives

$$\begin{aligned}
&W_v\big(t(\varpi^m)m(x)\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - \psi(\varpi^m x^{-1})W_{v_\circ}\big(t(\varpi^{m-2})a(\varpi^2,\varpi^{-2})\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - \psi(\varpi^m x^{-1})W_{v_\circ}\big(t(\varpi^{m+2})a(\varpi^{-2},\varpi^{-2})\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - \psi(\varpi^m x^{-1})W_{v_\circ}\big(t(\varpi^{m+2})\big)
\end{aligned}$$

from which the third displayed formula follows.

Finally, suppose that $x \in \varpi\mathcal{O}$. Then

$$
\begin{aligned}
& W_v\big(t(\varpi^m)wm(x)\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - W_{v_\circ}\big(t(\varpi^m)wm(x)t(\varpi^{-2})\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - W_{v_\circ}\big(t(\varpi^m)n(x)wt(\varpi^{-2})\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - W_{v_\circ}\big(n(\varpi^m x)t(\varpi^m)a(1,\varpi^{-2})w\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - \psi(\varpi^m x)W_{v_\circ}\big(t(\varpi^{m+2})a(\varpi^{-2},\varpi^{-2})\big) \\
&= W_{v_\circ}\big(t(\varpi^m)\big) - \psi(\varpi^m x)W_{v_\circ}\big(t(\varpi^{m+2})\big)
\end{aligned}
$$

which gives the last displayed formula. □

We shall need to compute in G(2). We let $\mathbf{s} : \mathrm{G}(2) \to \widetilde{\mathrm{G}}(2)$ be the section corresponding to the Kubota cocycle

$$\sigma(g_1, g_2) = \left(\frac{X(g_1g_2)}{X(g_1)}, \frac{X(g_1g_2)}{X(g_2)\det(g_1)}\right)$$

(see [KaP], page 41). Unfortunately this section is not a homomorphism over K(2) and we shall briefly require the section $\mathbf{t} : \mathrm{G}(2) \to \widetilde{\mathrm{G}}(2)$ which is. Let us define

$$\kappa(g) = \begin{cases} \big(c, d/\det(g)\big) & \text{if } 0 < |c| < 1 \\ 1 & \text{if } |c| = 0, 1 \end{cases}$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{K}(2)\,.$$

Then

$$\sigma(g_1, g_2) = \frac{\kappa(g_1g_2)}{\kappa(g_1)\kappa(g_2)}$$

for all $g_1, g_2 \in \mathrm{K}(2)$ (see [Kub] §3, Theorem 3) and hence if we set

$$\mathbf{t}(g) = \kappa(g)\mathbf{s}(g)$$

(with κ extended to $\mathrm{G}(2)$ in any way) then

$$\mathbf{t} : \mathrm{K}(2) \to \mathrm{K}^*(2)$$

is a homomorphism. It will be convenient to assume that κ is extended to be trivial on H_2. This is possible since κ is trivial on $\mathrm{H}_2 \cap \mathrm{K}(2)$ (because we are dealing with the unramified Hilbert symbol). It is easy to check that $\kappa(k) = 1$ for all $k \in \mathcal{K}_2$ and so $\mathbf{s} = \mathbf{t}$ on $\mathcal{K}_2$ and, with the assumption just made, $\mathbf{s} = \mathbf{t}$ on H_2. These facts will allow us largely to avoid the section $\mathbf{t}$ below. We shall write $\mathrm{K}_j^* = \mathbf{t}(\mathrm{K}_j)$ for $j \geq 0$.

Now let ω be the unique suitable character of $\widetilde{\mathrm{Z}}(2)$ and let $\xi_\circ$ denote the normalized spherical vector in the space of $\vartheta_{2,\omega}$ as in the previous section.

Lemma 4: *The vector* $\xi = \mathbf{s}\big(t(\varpi^{-2})\big) \cdot \xi_\circ$ *is fixed by* K_2^*.

Proof: Let

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{K}_2 .$$

Then

$$\mathbf{s}(k)\mathbf{s}\big(t(\varpi^{-2})\big) = \mathbf{s}\big(t(\varpi^{-2})\big)\mathbf{s}\begin{pmatrix} a & \varpi^2 b \\ \varpi^{-2}c & d \end{pmatrix}$$

after a simple Kubota cocycle calculation and the matrix

$$k' = \begin{pmatrix} a & \varpi^2 b \\ \varpi^{-2}c & d \end{pmatrix}$$

lies in K(2). We have

$$\kappa(k') = \begin{cases} (\varpi^{-2}c, d/\det(k')) & \text{if } c \notin \varpi^2\mathcal{O}^\times \cup \{0\} \\ 1 & \text{if } c \in \varpi^2\mathcal{O}^\times \cup \{0\} \end{cases}$$
$$= \begin{cases} (c, d/\det(k)) & \text{if } c \notin \varpi^2\mathcal{O}^\times \cup \{0\} \\ 1 & \text{if } c \in \varpi^2\mathcal{O}^\times \cup \{0\} \end{cases}$$

and

$$\kappa(k) = \begin{cases} (c, d/\det(k)) & \text{if } c \neq 0 \\ 1 & \text{if } c = 0 \,. \end{cases}$$

It follows that $\kappa(k) = \kappa(k')$ perhaps unless $c \in \varpi^2\mathcal{O}^\times$. But suppose that $c = \varpi^2 u \in \varpi^2\mathcal{O}^\times$. Then $d \in \mathcal{O}^\times$ and so

$$\begin{aligned}(c, d/\det(k)) &= (\varpi^2 u, d/\det(k)) \\ &= (u, d/\det(k)) \\ &= 1\end{aligned}$$

since the symbol is unramified. Thus $\kappa(k) = \kappa(k')$ in all cases and so

$$\mathbf{t}(k)\mathbf{s}(t(\varpi^{-2})) = \mathbf{s}(t(\varpi^{-2}))\mathbf{t}(k') \,.$$

The claim follows at once from this equation. □

We remark that the vector $\mathbf{s}(t(\varpi^{-1})) \cdot \xi_\circ$ is *not* fixed by K_1^*. This is why we are working at "level two" in this calculation.

From now on let ω be extended to be the unique element of $\Omega^2(2)$.

Lemma 5: *We have* $\Xi_{\xi_\circ}^{1,\omega^{-1}}(e) \neq 0$.

Proof: The set $\{e, a(\varpi, 1)\}$ is a complete set of representatives for the double coset space

$$\mathrm{Z}^1(2)\,\mathrm{H}_2^2\backslash \mathrm{H}_2/\mathrm{H}_2 \cap \mathrm{K}(2)$$

and it follows from the Iwasawa decomposition that

$$\mathrm{G}(2) = \mathrm{N}(\gamma_0)\,\mathrm{Z}^1(2)\,\mathrm{H}_2^2\mathrm{K}(2) \cup \mathrm{N}(\gamma_0)\,\mathrm{Z}^1(2)\,\mathrm{H}_2^2\,a(\varpi,1)\,\mathrm{K}(2)\,.$$

Hence

$$\widetilde{\mathrm{G}}\,(2) = \mathrm{N}^*(\gamma_0)\,\widetilde{\mathrm{Z}}^1(2)\,\widetilde{\mathrm{H}}_2^2\mathrm{K}^*(2) \cup \mathrm{N}^*(\gamma_0)\,\widetilde{\mathrm{Z}}^1(2)\,\widetilde{\mathrm{H}}_2^2\,\mathbf{s}\big(a(\varpi,1)\big)\,\mathrm{K}^*(2)\,.$$

If $n \in \mathrm{N}^*(\gamma_0)$, $z \in \widetilde{\mathrm{Z}}^1(2)$, $h \in \widetilde{\mathrm{H}}_2^2$ and $k \in \mathrm{K}^*(2)$ then

$$\Xi_{\xi_\circ}^{1,\omega^{-1}}(nzhgk) = \omega^{-1}(z)\eta(h)\mu^{1/4}(h)\,\Xi_{\xi_\circ}^{1,\omega^{-1}}(g)$$

for any $g \in \widetilde{\mathrm{G}}\,(2)$, by Proposition 6 of section 4 and the definition of a semi-Whittaker function of the first kind. Thus, for any $g \in \widetilde{\mathrm{G}}\,(2)$, $\Xi_{\xi_\circ}^{1,\omega^{-1}}(g)$ is a multiple of either $\Xi_{\xi_\circ}^{1,\omega^{-1}}(e)$ or $\Xi_{\xi_\circ}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi,1))\big)$. Since $\xi_\circ \neq 0$, $\Xi_{\xi_\circ}^{1,\omega^{-1}}(g) \neq 0$ for some $g \in \widetilde{\mathrm{G}}\,(2)$ and so to obtain the result it will suffice to show that

$$\Xi_{\xi_\circ}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi,1))\big) = 0\,.$$

Let $u \in \mathcal{O}^\times$ be such that $(u,\varpi) = -1$. Recalling that $\mathbf{t} = \mathbf{s}$ on H_2 we have

$$\begin{aligned}\Xi_{\xi_\circ}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi,1))\big) &= \Xi_{\xi_\circ}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi,1))\mathbf{s}(a(u,u))\big)\\ &= \Xi_{\xi_\circ}^{1,\omega^{-1}}\big((u,\varpi)\mathbf{s}(a(u,u))\mathbf{s}(a(\varpi,1))\big)\\ &= -\omega\big(\mathbf{s}(a(u,u))\big)\,\Xi_{\xi_\circ}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi,1))\big)\end{aligned}$$

and so the conclusion will follow if we can show that $\omega^{-1}\big(\mathbf{s}(a(u,u))\big) = 1$. We already know, from Lemma 3 of section 6, that $\Xi_{\xi_\circ}^{2,\omega}(e) \neq 0$. But

$$\begin{aligned}\Xi_{\xi_\circ}^{2,\omega}(e) &= \Xi_{\xi_\circ}^{2,\omega}\big(\mathbf{s}(a(u,u))\big)\\ &= \omega\big(\mathbf{s}(a(u,u))\big)\,\Xi_{\xi_\circ}^{2,\omega}(e)\end{aligned}$$

and so $\omega\big(\mathbf{s}(a(u,u))\big) = 1$, as required. $\square$

In the light of Lemma 5 we are free to normalize the semi-Whittaker functionals we are using in such a way that

$$\Xi_{\xi_\circ}^{1,\omega^{-1}}(e) = \Xi_{\xi_\circ}^{2,\omega}(e) = 1$$

and we shall do this in what follows.

Lemma 6: *We have*

$$\Xi_{\xi}^{1,\omega^{-1}}(e) = q^{1/2},$$

$$\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(m(x))\big) = \omega^{-1}\big(\mathbf{s}(\varpi^{-1}I_2)\big)(\varpi^{-1}x, \varpi)$$

for $x \in \varpi\mathcal{O}^\times$,

$$\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(m(x))\big) = q^{-1/2}$$

for $x \in \mathcal{O}^\times$ *and*

$$\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(wm(x))\big) = q^{-1/2}$$

for $x \in F$.

Proof: For the first equation,

$$\begin{aligned} \Xi_{\xi}^{1,\omega^{-1}}(e) &= \Xi_{\xi_\circ}^{1,\omega^{-1}}\big(\mathbf{s}(t(\varpi^{-2}))\big) \\ &= \mu^{1/4}\big(t(\varpi^{-2})\big)\,\Xi_{\xi_\circ}^{1,\omega^{-1}}(e) \\ &= q^{1/2}, \end{aligned}$$

as claimed. If $x \in F^\times$ then a routine Kubota cocycle calculation establishes the identity

$$\begin{aligned}&\mathbf{s}\big(m(x)\big)\mathbf{s}\big(t(\varpi^{-2})\big)\\&=\mathbf{s}\big(t(\varpi^{-2})\big)\mathbf{s}\big(n(\varpi^{2}x^{-1})\big)\mathbf{s}\big(a(\varpi^{2}x^{-1},\varpi^{-2}x)\big)\mathbf{s}\big(r(\varpi^{-2}x)\big)\,.\end{aligned}\tag{18}$$

Now suppose that $x\in\mathcal{O}^{\times}$ or $x\in\varpi\mathcal{O}^{\times}$. Then $\varpi^{2}x^{-1}\in\mathcal{O}$ and so $r(\varpi^{-2}x)\in\mathrm{K}(2)$. It is easy to check that $\kappa(r(\varpi^{-2}x))=1$ and so $\mathbf{s}\big(r(\varpi^{-2}x)\big)\in\mathrm{K}^{*}(2)$. Hence

$$\begin{aligned}\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(m(x))\big)&=\Xi_{\xi_{\circ}}^{1,\omega^{-1}}\big(\mathbf{s}(m(x))\mathbf{s}(t(\varpi^{-2}))\big)\\&=\mu^{1/4}\big(t(\varpi^{-2})\big)\,\Xi_{\xi_{\circ}}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi^{2}x^{-1},\varpi^{-2}x))\big)\\&=q^{1/2}\,\Xi_{\xi_{\circ}}^{1,\omega^{-1}}\big(\mathbf{s}(a(\varpi^{2}x^{-1},\varpi^{-2}x))\big)\,.\end{aligned}\tag{19}$$

Now suppose that $x\in\mathcal{O}^{\times}$. Then

$$\mathbf{s}\big(a(\varpi^{2}x^{-1},\varpi^{-2}x)\big)=\mathbf{s}\big(a(\varpi^{4},1)\big)\mathbf{s}\big(a(\varpi^{-2},\varpi^{-2})\big)\mathbf{s}\big(a(x^{-1},x)\big)$$

and since $\kappa(a(x^{-1},x))=1$, $\mathbf{s}(a(x^{-1},x))\in\mathrm{K}^{*}(2)$. Thus

$$\begin{aligned}\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(m(x))\big)&=q^{1/2}\mu^{1/4}\big(a(\varpi^{4},1)\big)\omega^{-1}\big(\mathbf{s}(a(\varpi^{-2},\varpi^{-2}))\big)\\&=q^{-1/2}\omega^{-1}\big(\mathbf{s}(a(\varpi^{-2},\varpi^{-2}))\big)\,.\end{aligned}$$

However, $\omega^{-1}\big(\mathbf{s}(a(\varpi^{-2},\varpi^{-2}))\big)=1$ since ω is suitable and the third equation follows.

To obtain the second equation, let us return to (19) with the assumption that $x\in\varpi\mathcal{O}^{\times}$. We have

$$\begin{aligned}&\mathbf{s}\big(a(\varpi^{2}x^{-1},\varpi^{-2}x)\big)\\&=(\varpi,\varpi^{-1}x)\,\mathbf{s}\big(a(\varpi,\varpi^{-1})\big)\,\mathbf{s}\big(a(\varpi x^{-1},\varpi^{-1}x)\big)\\&=(\varpi,\varpi^{-1}x)\,\mathbf{s}\big(t(\varpi^{2})\big)\,\mathbf{s}\big(a(\varpi^{-1},\varpi^{-1})\big)\,\mathbf{s}\big(a(\varpi x^{-1},\varpi^{-1}x)\big)\end{aligned}$$

and $\mathbf{s}\big(a(\varpi x^{-1}, \varpi^{-1}x)\big) \in \mathrm{K}^*(2)$ because $\kappa(a(\varpi x^{-1}, \varpi^{-1}x)) = 1$. Therefore

$$\begin{aligned}\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(m(x))\big) &= q^{1/2}(\varpi, \varpi^{-1}x)\mu^{1/4}\big(t(\varpi^2)\big)\omega^{-1}\big(\mathbf{s}(a(\varpi^{-1}, \varpi^{-1}))\big) \\ &= (\varpi, \varpi^{-1}x)\,\omega^{-1}\big(\mathbf{s}(\varpi^{-1}I_2)\big)\end{aligned}$$

giving the second equation.

Another routine Kubota cocycle calculation shows that

$$\mathbf{s}\big(wm(x)\big)\mathbf{s}\big(t(\varpi^{-2})\big) = \mathbf{s}\big(n(x)\big)\mathbf{s}\big(a(1, \varpi^{-2})\big)\mathbf{s}\big(w\big)$$

and $\kappa(w) = 1$ so that $\mathbf{s}\big(w\big) \in \mathrm{K}^*(2)$. Thus

$$\begin{aligned}\Xi_{\xi}^{1,\omega^{-1}}\big(\mathbf{s}(wm(x))\big) &= \Xi_{\xi_o}^{1,\omega^{-1}}\big(\mathbf{s}(wm(x))\mathbf{s}(t(\varpi^{-2}))\big) \\ &= \Xi_{\xi_o}^{1,\omega^{-1}}\big(\mathbf{s}(a(1, \varpi^{-2}))\big) \\ &= \mu^{1/4}\big(a(1, \varpi^{-2})\big) \\ &= q^{-1/2},\end{aligned}$$

giving the last equation. □

In the following Proposition the quantity

$$\mathbf{g} = -\omega^{-1}\big(\mathbf{s}(\varpi I_2)\big) \sum_{u \in [\mathcal{O}^\times/1+\varpi\mathcal{O}]} (u, \varpi)\,\overline{\psi}(\varpi^{-1}u^{-1}) \tag{20}$$

will occur and we wish to note that it is a very classical object in number theory. Under reduction modulo $\varpi\mathcal{O}$ the set $[\mathcal{O}^\times/1+\varpi\mathcal{O}]$ maps onto $\mathbb{F}_q^\times$, where $\mathbb{F}_q$ is the residue class field of F. The map $u \mapsto (u, \varpi)$ is then simply the Legendre symbol on $\mathbb{F}_q^\times$ and $u \mapsto \overline{\psi}(\varpi^{-1}u)$ is a non-trivial additive character on $\mathbb{F}_q$. Making the change of variable $u \mapsto u^{-1}$ in (20) we see that it is $-\omega^{-1}(\mathbf{s}(\varpi I_2))$ multiplied by a quadratic Gauss sum. Since ω^{-1} is a unitary character it follows that $|\mathbf{g}| = q^{1/2}$ from the classical theory of such sums. Although $\mathbf{g}$ could be evaluated explicitly, this observation will be enough for our purpose here.

Proposition 2: *We have*

$$\langle \xi, \xi_\circ, v \rangle = C\left(\mathbf{g} - q^{-1/2} \cdot \frac{1-q^{-1}}{1-q^{-3}}\right)$$

where C is a positive constant. In particular,

$$\langle \xi, \xi_\circ, v \rangle \neq 0 \, .$$

Proof: By definition

$$\langle \xi, \xi_\circ, v \rangle = \int\limits_{\mathrm{Z}^1(2)\cdot \mathrm{N}(\gamma_0)\backslash \mathrm{G}(2)} \Xi_\xi^{1,\omega^{-1}}(g)\, \Xi_{\xi_\circ}^{2,\omega}(g)\, W_v(g)\, dg$$

where W_v is formed with respect to $\overline{\psi}$. Every function in the integrand is K_2^*-invariant on the right and so we have

$$\begin{aligned}
&\langle \xi, \xi_\circ, v \rangle \\
&\sim \sum_{m\in\mathbb{Z}} \sum_{k\in\mathcal{K}_2} \Xi_\xi^{1,\omega^{-1}}\big(t(\varpi^m)k\big)\, \Xi_{\xi_\circ}^{2,\omega}\big(t(\varpi^m)k\big)\, W_v\big(t(\varpi^m)k\big) \\
&= \sum_{m\in\mathbb{Z}} \Xi_{\xi_\circ}^{2,\omega}\big(\mathbf{s}(t(\varpi^m))\big) \sum_{k\in\mathcal{K}_2} \Xi_\xi^{1,\omega^{-1}}\big(\mathbf{s}(t(\varpi^m))\mathbf{s}(k)\big)\, W_v\big(t(\varpi^m)k\big) \\
&= \sum_{m\in 2\mathbb{N}} \mu^{1/4}\big(t(\varpi^m)\big) \sum_{k\in\mathcal{K}_2} \Xi_\xi^{1,\omega^{-1}}\big(\mathbf{s}(t(\varpi^m))\mathbf{s}(k)\big)\, W_v\big(t(\varpi^m)k\big) \\
&= \sum_{m\in 2\mathbb{N}} \mu^{1/2}\big(t(\varpi^m)\big) \sum_{k\in\mathcal{K}_2} \Xi_\xi^{1,\omega^{-1}}\big(\mathbf{s}(k)\big)\, W_v\big(t(\varpi^m)k\big) \\
&= \sum_{m\in 2\mathbb{N}} q^{-m/2} \sum_{k\in\mathcal{K}_2} \Xi_\xi^{1,\omega^{-1}}\big(\mathbf{s}(k)\big)\, W_v\big(t(\varpi^m)k\big)
\end{aligned}$$

where $\sim$ denotes proportionalty by a positive constant (which depends on the choice of Haar measure with respect to which the integral is performed) and we have used Lemma 3 of section 5.

We now evaluate the inner sum by breaking it into four pieces, $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ and $\boxed{4}$. In the display below we indicate the range of summation for each of the pieces:

$$\boxed{1} \longleftrightarrow \{e\}$$

$$\boxed{2} \longleftrightarrow \{m(x) \mid x \in [\mathcal{O}/\varpi^2\mathcal{O}], x \in \mathcal{O}^\times\}$$

$$\boxed{3} \longleftrightarrow \{m(x) \mid x \in [\mathcal{O}/\varpi^2\mathcal{O}], x \in \varpi\mathcal{O}^\times\}$$

$$\boxed{4} \longleftrightarrow \{wm(x) \mid x \in [\varpi\mathcal{O}/\varpi^2\mathcal{O}]\}.$$

In the range $m \in 2\mathbb{N}$ the terms in the sums $\boxed{1}$, $\boxed{2}$ and $\boxed{4}$ do not depend on x and hence they may readily be evaluated using Lemmas 3 and 6. The results are

$$\boxed{1} = \begin{cases} q^{1/2} & m = 0 \\ q^{1/2}(1+q^{-1})q^{-(m-1)} & m \geq 2 \end{cases}$$

$$\boxed{2} = -(q^2 - q)q^{-1/2}(1+q^{-1})q^{-(m+1)}$$

$$\boxed{4} = -qq^{-1/2}(1+q^{-1})q^{-(m+1)}.$$

We note that

$$\boxed{2} + \boxed{4} = -q^{1/2}(1+q^{-1})q^{-m}.$$

For $\boxed{3}$ we obtain

$$\boxed{3} = \sum_{x \in [\mathcal{O}/\varpi^2\mathcal{O}], x \in \varpi\mathcal{O}^\times} \omega^{-1}\big(\mathbf{s}(\varpi^{-1}I_2)\big)\,(\varpi^{-1}x, \varpi)\,(1 - \overline{\psi}(x^{-1})) \tag{21}$$

if $m = 0$ and zero otherwise. Let us reindex (21) by setting $x = \varpi u$ where $u \in [\mathcal{O}^\times/1 + \varpi\mathcal{O}]$. Then (21) becomes

$$\begin{aligned}
&\omega^{-1}\big(\mathbf{s}(\varpi^{-1}I_2)\big) \sum_{u\in[\mathcal{O}^\times/1+\varpi\mathcal{O}]} (u,\varpi)\,(1-\overline{\psi}(\varpi^{-1}u^{-1}))\\
&= -\omega^{-1}\big(\mathbf{s}(\varpi^{-1}I_2)\big) \sum_{u\in[\mathcal{O}^\times/1+\varpi\mathcal{O}]} (u,\varpi)\,\overline{\psi}(\varpi^{-1}u^{-1})\\
&= \mathbf{g}
\end{aligned}$$

since $u \mapsto (u, \varpi)$ is a non-trivial character. Thus

$$\boxed{3} = \begin{cases} \mathbf{g} & m = 0 \\ 0 & m \geq 2\,. \end{cases}$$

Putting this all together we obtain

$$\begin{aligned}
\langle \xi, \xi_\circ, v\rangle &\sim \mathbf{g} + q^{1/2} - q^{1/2}(1+q^{-1})\\
&\quad + \sum_{m\in 2\mathbb{N}^+} q^{-m/2}[q^{1/2}(1+q^{-1})q^{-(m-1)} - q^{1/2}(1+q^{-1})q^{-m}]\\
&= \mathbf{g} - q^{-1/2} + \sum_{m\in 2\mathbb{N}^+} q^{-3m/2}q^{1/2}(1+q^{-1})(q-1)\\
&= \mathbf{g} - q^{-1/2} + \frac{q^{-3}}{1-q^{-3}}q^{1/2}(1+q^{-1})(q-1)\\
&= \mathbf{g} - q^{-1/2}\cdot\frac{1-q^{-1}}{1-q^{-3}}
\end{aligned}$$

after a little further algebra. This proves the first claim. For the second we note that, since $q > 1$, the reverse triangle inequality gives

$$|\langle \xi, \xi_\circ, v\rangle| \geq C\left(|\mathbf{g}| - q^{-1/2}\right) = C\left(q^{1/2} - q^{-1/2}\right)$$

and so $\langle \xi, \xi_\circ, v\rangle \neq 0$. $\square$

Theorem 1: *Suppose that F is not dyadic and let σ be the Steinberg representation of* $\mathrm{G}(2)$. *Then $\mathcal{L}(\omega;\sigma) \neq \{0\}$ for ω the unique suitable character.*

Proof: It is well-known (see [God] for example) that the Whittaker model of σ lies in $\mathbf{W}^2(2,\overline{\psi})$ and hence (6) defines an element of $\mathcal{L}(\omega;\sigma)$. (Recall that $\omega = \omega^{-1}$ in this case.) From Proposition 2 we know that this trilinear form is non-zero and the Theorem follows. $\square$

In order to understand the significance of this result, let us make some observations. First, Theorem 1 of section 6 assures us that some constituent, ρ, of $\pi = \mathbb{I}(\mu^{1/2})$ satisfies $\mathcal{L}(\omega;\rho) \neq \{0\}$ and Proposition 5 of section 5 tells us that, in any case, $\dim_{\mathbb{C}}(\mathcal{L}(\omega;\pi)) \leq 1$. Secondly, since $\vartheta_{2,\omega}$ is self-contragredient, $\mathcal{L}(\omega;1) \cong \mathbb{C}$ and any non-zero trilinear form in $\mathcal{L}(\omega;1)$ may be pulled back via the surjection $\pi \to 1$ to give a non-zero element of $\mathcal{L}(\omega;\pi)$ which is zero on $\vartheta_{2,\omega} \otimes \vartheta_{2,\omega} \otimes \sigma$. Thus $\mathcal{L}(\omega;\pi) \cong \mathbb{C}$ and we conclude that a non-zero element of $\mathcal{L}(\omega;\sigma)$ cannot extend to all of $\vartheta_{2,\omega} \otimes \vartheta_{2,\omega} \otimes \pi$. This is concordant with the fact that the integral (6) in this situation is generally divergent unless $v \in E_\sigma$.

We are led by this result to recognize that several constituents of a reducible spherical principal series representation may carry invariant trilinear forms. It would be interesting to determine whether any constituent which is neither spherical nor generic can carry such a form, but this will have to await more detailed investigation of GL(3) since such constituents do not arise on GL(2).

We note that Theorem 1 may be used to fill one of the gaps in Savin's results in [Sav]. If we let λ be a character of $F^\times$ such that $\lambda^2 = 1$ then Savin writes

$$\mathrm{St}_\lambda = \lambda(\det) \otimes \mathrm{St}$$

where $\mathrm{St} = \sigma$ is the Steinberg representation. On his list of constituents of principal series representations which may carry invariant trilinear forms are included the representations

$$i_{(2,1),(3)}(\mathrm{St}_\lambda \otimes 1)$$

for the various possible λ, but he does not decide whether or not they actually carry non-zero forms. Applying Theorem 2 of section 5 with $\rho = \mathrm{St}_\lambda$ (so that $\omega_\rho = 1$), ω any suitable character of $\widetilde{\mathrm{Z}}(3)$ and $\nu = \omega^{-1}$ we obtain an exact sequence

$$\{0\} \to \mathcal{L}(\eta; \mathrm{St}_\lambda) \to \mathcal{L}(\omega, \omega^{-1}; i_{(2,1),(3)}(\mathrm{St}_\lambda \otimes 1)) \to \mathcal{L}(\omega', (\omega^{-1})'; \mathrm{St}_\lambda^{(1)}) \qquad (22)$$

where η is the suitable character of $\widetilde{\mathrm{Z}}(2)$. Now $\mathrm{St}_\lambda^{(1)} \cong \lambda|\cdot|^{1/2}$ on $\mathrm{GL}(1)$ and since this is not square-trivial, $\mathcal{L}(\omega', (\omega^{-1})'; \mathrm{St}_\lambda^{(1)}) = \{0\}$ and (22) gives

$$\mathcal{L}(\omega, \omega^{-1}; i_{(2,1),(3)}(\mathrm{St}_\lambda \otimes 1)) \cong \mathcal{L}(\eta; \mathrm{St}_\lambda)\,.$$

If $\lambda^2 = 1$ then $\lambda \otimes \vartheta_{2,\eta} \cong \vartheta_{2,\eta}$ (by Proposition 4 of section 1, for instance) and so

$$\vartheta_{2,\eta} \otimes \vartheta_{2,\eta} \otimes \mathrm{St}_\lambda \cong \vartheta_{2,\eta} \otimes \vartheta_{2,\eta} \otimes \mathrm{St}$$

which gives

$$\mathcal{L}(\eta; \mathrm{St}_\lambda) \cong \mathcal{L}(\eta; \mathrm{St}) \cong \mathbb{C}$$

by Theorem 1 and Corollary 1 of section 5. We conclude that

$$\mathcal{L}(\omega, \omega^{-1}; i_{(2,1),(3)}(\mathrm{St}_\lambda \otimes 1)) \cong \mathbb{C}.$$

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APPENDIX

The purpose of this appendix is to point out a third error in [BuG]. We give some examples to show that an arithmetic claim made in that paper is false. Although this error is not as serious as that exposed in section 5 of Chapter 2, it does necessitate a modification of their local calculations and means that not all of their results can be relied on in detail. Furthermore, noticing this error restores the proper uniformity from the function field to the number field case; unfortunately it means that both cases are a little more complicated than Bump and Ginzburg allow. In order to explain this error let us recall some notation from §3 of [BuG]; in that section F is a global field, $\mathbb{A}$ its ring of adeles, $\mathbb{A}^\times$ the corresponding group of ideles and S a certain finite set of places of F including the Archimedean places. In the course of establishing the invariance of their proposed integrand under the center $\mathrm{Z}_{\mathbb{A}}$ of $\mathrm{GL}(r, \mathbb{A})$ when r is even and F is a number field, they make the following claim: "Now if F is a number field, a consequence of the strong approximation theorem is that $F^\times(\mathbb{A}^\times)^2(\prod_{v\notin S}\mathfrak{o}_v^\times) = \mathbb{A}^\times$, and so we have invariance under all of $\mathrm{Z}_{\mathbb{A}}$." They proceed to exclude the function field case when r is even in order to be able to make use of invariance under the center. Unfortunately, not only does the equality $F^\times(\mathbb{A}^\times)^2(\prod_{v\notin S}\mathfrak{o}_v^\times) = \mathbb{A}^\times$ not follow from the strong approximation theorem, but the equality itself is generally false, as we shall show below. This means that the integrand in Bump-Ginzburg's integral has not been show to be invariant under the center of $\mathrm{GL}(r, \mathbb{A})$ and their Rankin-Selberg integral has not been shown to be well-defined when r is even; in the absence of this, they gain nothing by restricting to the number field case.

Proposition 1: *Let F be a number field, $\mathbb{A}^\times$ its group of ideles, S some finite set of places of F including the Archimedean places and $Cl(F)$ the ideal class group of F. Then the finite group*

$$\mathbb{A}^\times/(\mathbb{A}^\times)^2 F^\times(\textstyle\prod_{v \notin S} \mathfrak{o}_v^\times)$$

maps onto the group $Cl(F)/(Cl(F))^2$.

Proof: Let S_∞ denote the set of Archimedean places of F. For any finite set $T \supseteq S_\infty$ put

$$\mathfrak{O}^\times(T) = \prod_{v \notin T} \mathfrak{o}_v^\times .$$

It is well-known that

$$\mathbb{A}^\times/F^\times(\textstyle\prod_{v \in S_\infty} F_v^\times)\mathfrak{O}^\times(S_\infty) \cong Cl(F)$$

and since $(\prod_{v \in S_\infty} F_v^\times)\mathfrak{O}^\times(S_\infty) \supseteq \mathfrak{O}^\times(S)$ it follows that the group

$$\mathbb{A}^\times/F^\times\mathfrak{O}^\times(S)$$

maps onto $Cl(F)$. Under this homomorphism the subgroup

$$(\mathbb{A}^\times)^2 \cdot (F^\times\mathfrak{O}^\times(S))/F^\times\mathfrak{O}^\times(S)$$

maps into the group of square classes and so the quotient by this subgroup maps onto $Cl(F)/(Cl(F))^2$. However, this quotient is isomorphic to

$$\mathbb{A}^\times / F^\times (\mathbb{A}^\times)^2 \mathfrak{O}^\times(S)$$

and the lemma follows. □

It follows from Proposition 1 that the equality $\mathbb{A}^\times = F^\times(\mathbb{A}^\times)^2(\prod_{v \notin S} \mathfrak{o}_v^\times)$ cannot hold for any number field F having even class number. It would be possible to reformulate the truth of the equality in question as a rather complicated condition involving only objects familiar from the classical (that is, non-adelic) description of the arithmetic of F. Rather than doing so we content ourselves with two further observations. The first is that it is easy to see that the equality does hold in the case $F = \mathbb{Q}$ for every choice of S. Hence any application of the details of Bump-Ginzburg's work in which the only ground field of interest was the rational numbers is not invalidated by the observations made here. The second is that it is not true that the only obstruction to Bump-Ginzburg's equality is 2-torsion in the class group of F. We make this claim precise in the following Proposition.

Proposition 2: *Let F be a number field of class number one and S a finite set of places of F containing the Archimedean places. Let*

$$\Sigma = \prod_{v \in S_{\text{real}}} \{\pm 1\}$$

where S_{real} denotes the set of real places of F and put

$$\Sigma_0 = \left\{ \left(\text{sgn}(\alpha_v)\right)_{v \in S_{\text{real}}} \,\middle|\, \alpha \in \mathcal{O}_F^\times \right\} \leq \Sigma \,.$$

Then the finite group

$$\mathbb{A}^\times/(\mathbb{A}^\times)^2 F^\times(\prod_{v\notin S}\mathfrak{o}_v^\times)$$

maps onto the group Σ/Σ_0.

Proof: We may regard Σ as a subgroup of $\mathbb{A}^\times$ and we do so in what follows. Since F has class number one we have

$$\mathbb{A}^\times = F^\times(\prod_{v\in S_\infty} F_v^\times)\mathfrak{O}^\times(S_\infty)$$

and hence

$$\begin{aligned}
&\mathbb{A}^\times/(\mathbb{A}^\times)^2 F^\times\mathfrak{O}^\times(S_\infty)\\
&= F^\times(\prod_{v\in S_\infty} F_v^\times)\mathfrak{O}^\times(S_\infty)/F^\times(\prod_{v\in S_\infty}(F_v^\times)^2)\mathfrak{O}^\times(S_\infty)\\
&= F^\times(\prod_{v\in S_\infty}(F_v^\times)^2)\,\Sigma\,\mathfrak{O}^\times(S_\infty)/F^\times(\prod_{v\in S_\infty}(F_v^\times)^2)\mathfrak{O}^\times(S_\infty)\\
&\cong \Sigma/\Sigma\cap\left(F^\times(\prod_{v\in S_\infty}(F_v^\times)^2)\mathfrak{O}^\times(S_\infty)\right).
\end{aligned}$$

Now let us suppose that an idele (ϵ_v) lies in $\Sigma\cap\left(F^\times(\prod_{v\in S_\infty}(F_v^\times)^2)\mathfrak{O}^\times(S_\infty)\right)$ and let us write it as $\alpha\cdot(\beta_v^2)_{v\in S_\infty}\cdot(\gamma_v)_{v\notin S_\infty}$ where $\alpha\in F$ and suppressed components of the ideles are all equal to 1. Comparing components we find that $\alpha_v=\gamma_v\in\mathfrak{o}_v^\times$ for all finite places v and consequently $\alpha\in\mathcal{O}_F^\times$, the group of units of the ring of integers of F. For $v\in S_{\text{real}}$ we have $\epsilon_v=\alpha_v\cdot\beta_v^2$ and since $\epsilon_v=\pm1$ this gives $\epsilon_v=\operatorname{sgn}(\alpha_v)$. It is easy to see that any $(\epsilon_v)\in\Sigma$ satisfying these two conditions is an element of the intersection. Thus

$$\Sigma\cap\left(F^\times(\prod_{v\in S_\infty}(F_v^\times)^2)\mathfrak{O}^\times(S_\infty)\right)=\Sigma_0$$

and it follows that

$$\mathbb{A}^\times/(\mathbb{A}^\times)^2 F^\times \mathfrak{O}^\times(S_\infty) \cong \Sigma/\Sigma_0 .$$

Since $\mathfrak{O}^\times(S_\infty) \supseteq \mathfrak{O}^\times(S)$, the Proposition follows from this. □

To show that Proposition 2 gives rise to obstructions to the Bump-Ginzburg equality beyond those arising from the class group we must give an example of a number field F with class number one but with the group Σ/Σ_0 non-trivial. This is easily done: consider for example the field $F = \mathbb{Q}(\sqrt{3})$. It is well-known that F has class number one and that $\mathcal{O}_F^\times = \{\pm 1\} \times \langle \epsilon \rangle$ where $\epsilon = 2 + \sqrt{3}$. The field F has two real places and so $\Sigma \cong \{\pm 1\} \times \{\pm 1\}$. Since ϵ is positive at both of these places we have $\Sigma_0 = \{(1,1), (-1,-1)\}$ and hence Σ/Σ_0 is cyclic of order two. So the Bump-Ginzburg equality fails for this field for any choice of S. More generally we might take F to be any real quadratic field with class number one and totally positive fundamental unit.

For completeness let us observe that the statements made about the function field case in [BuG] are also erroneous; generally the subgroup $F^\times(\mathbb{A}^\times)^2(\prod_{v \notin S} \mathfrak{o}_v^\times)$ of $\mathbb{A}^\times$ has index greater than two.

Proposition 3: *Let F be a function field, $\mathbb{A}^\times$ its group of ideles, S any finite set of places of F, $Cl(F)$ the set of divisor classes of F and $Cl^0(F)$ the set of divisor classes of degree zero. Then the finite group*

$$\mathbb{A}^\times/F^\times(\mathbb{A}^\times)^2(\textstyle\prod_{v \notin S} \mathfrak{o}_v^\times)$$

maps onto the group $\mathbb{Z}/2\mathbb{Z} \times Cl^0(F)/(Cl^0(F))^2$.

Proof: Using the same notation as above we have

$$\mathbb{A}^{\times}/F^{\times}\mathfrak{O}^{\times}(\emptyset) \cong Cl(F)$$

and if we let $\mathbb{A}^1$ denote the set of ideles whose idele norm is one then we have isomorphisms

$$\mathbb{A}^{\times}/F^{\times}\mathfrak{O}^{\times}(\emptyset) \cong \mathbb{Z} \times \mathbb{A}^1/F^{\times}\mathfrak{O}^{\times}(\emptyset)$$

and

$$\mathbb{A}^1/F^{\times}\mathfrak{O}^{\times}(\emptyset) \cong Cl^0(F)\,.$$

From this point on the proof proceeds exactly as in the number field case. □

It is not difficult to manufacture examples in which the group $Cl^0(F)$ has non-trivial 2-torsion. For instance, let $\mathbb{F}$ be a finite field of characteristic neither 2 nor 3 and let E be the elliptic curve over $\mathbb{F}$ with Weierstrass model $y^2 = x(x-a)(x-b)$ where a and b are distinct elements of $\mathbb{F}^{\times}$. This elliptic curve is its own Jacobian and its 2-torsion subgroup over $\mathbb{F}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If we let F be the field of functions of E then, as usual, this subgroup gives rise to a subgroup of $Cl^0(F)$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus the index of $(\mathbb{A}^{\times})^2F^{\times}(\prod_v \mathfrak{o}_v^{\times})$ in $\mathbb{A}^{\times}$ is at least eight for this field.

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VITA

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