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TOPOLOGICAL CONVEXITY STRUCTURES AND

THE KREIN-MILMAN THEOREM

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TOPOLOGICAL CONVEXITY STRUCTURES AND THE KREIN-MILMAN THEOREM

CHAPTER I

INTRODUCTION

The setting for this dissertation is a set X and a family of subsets of X, called convex sets, satisfying given convexity properties. In this setting the convex hull operator can be used to define the analogues of segments and lines. Basic relationships between the hull operator and what are called segment and line operators are derived, leading to Kuratowski-like theorems for each. A set of alignment axioms is considered, which allows a line operator to be developed from a segment operator, where the lines involved are shown to possess a linear order.

Properties of convex-preserving functions and convexity isomorphisms are derived, including conditions for continuity and the preservation of Carathéodory, Helly and Radon numbers.

A topology is then introduced which leads to several results which parallel fundamental properties in the theory of linear topological spaces, culminating in a condition analogous to local convexity and the proof of the Krein-Milman theorem applied to this setting. Thus is obtained, among other classical results, a Krein-Milman theorem for a more general class of spaces than that normally considered. In particular, it is shown that no underlying algebraic structure is needed.

The usual set-theoretic symbols will be used throughout without definition. In particular, the notation established in Valentine [12] will be employed.

CHAPTER II

RELATIONSHIPS BETWEEN CONVEXITY STRUCTURES AND SEGMENT OPERATORS

2.1. Definition. Let $\mathcal{C}$ be a family of subsets of a set X. $\mathcal{C}$ is called a convexity structure if

(i) $\bigcap \mathcal{J} \equiv \bigcap \{C : C \in \mathcal{J}\} \in \mathcal{C}$ for $\mathcal{J} \subset \mathcal{C}$,

(ii) ϕ and $X \in \mathcal{C}$.

Further if $\mathcal{C}$ satisfies

(iii) $\{x\} \in \mathcal{C}$ for each $x \in X$

then $\mathcal{C}$ is said to be T_1.

2.2. Definition. The hull operator H associated with $\mathcal{C}$ is defined by

$$H(S) = \bigcap\{C \in \mathcal{C} : C \supset S\}, \qquad S \subset X.$$

2.3. Theorem. Let $\mathcal{C}$ be a convexity structure and H the corresponding hull operator. Then H satisfies the following:

(a) $A \subset H(A)$ for any $A \subset X$.

(b) $H(A) \subset H(B)$ if $A \subset B$.

(c) $H \circ H = H$.

(d) $A \in \mathcal{C}$ iff $H(A) = A$.

Conversely let H be a map from P(X), the power set of X, to P(X) such that $H(\phi) = \phi$.

Also suppose that H satisfies (a), (b), (c) and $\mathcal{C}$ is defined as in

(d) ; then $\mathcal{C}$ is a convexity structure, and the corresponding hull operator H' coincides with H.

Proof. Axioms (a), (b), and (c) can be demonstrated by direct applications of the definition of H. To prove (d) consider $A \in \mathcal{C}$. Then since $A \in \mathcal{C}$ and $A \supset A$, the definition of H implies $H(A) \subset A$. By (a) $A \subset H(A)$ so $A = H(A)$. Also if $A = H(A) \in \mathcal{C}$ then $A \in \mathcal{C}$.

Conversely let $\mathcal{C} = \{A : H(A) = A\}$ where H is an operator from $P(X)$ to $P(X)$ such that $H(\phi) = \phi$ and H satisfies (a), (b) and (c). Now $\phi \in \mathcal{C}$, and $X \in \mathcal{C}$ by applying (a), so it must be shown that $\mathcal{C}$ is closed under arbitrary intersections. Let $\mathcal{J} \subset \mathcal{C}$ and note that $\bigcap\mathcal{J} \subset C$, where C is any member of $\mathcal{J}$. So by (b) and the definition of $\mathcal{C}$, $H(\bigcap\mathcal{J}) \subset H(C) = C$. herefore, $H(\bigcap\mathcal{J}) \subset \bigcap\{C : C \in \mathcal{J}\} = \bigcap\mathcal{J}$. By (a) it must be that $\bigcap\mathcal{J} \subset H(\bigcap\mathcal{J})$. Therefore $H(\bigcap\mathcal{J}) = \bigcap\mathcal{J}$, and $\bigcap\mathcal{J} \in \mathcal{C}$.

Let H' be the corresponding hull operator associated with $\mathcal{C}$. If $A \subset X$ then $H'(A) = \bigcap\{C \in \mathcal{C} : C \supset A\}$, so $H'(A) \in \mathcal{C}$ and $A \subset H'(A)$. By (b) and the definition of $\mathcal{C}$, $H(A) \subset H(H'(A)) = H'(A)$. But by (c) $H(H(A)) = H(A)$, so $H(A) \in \mathcal{C}$. Since $H(A) \in \mathcal{C}$ and $A \subset H(A)$, by definition of H',

$$H'(A) = \bigcap\{C \in \mathcal{C} : C \supset A\} \subset H(A).$$

Therefore $H'(A) = H(A)$ for each subset $A \subset X$, completing the proof.

2.4. Definition. Any mapping $\sigma : X \times X \to P(X)$ such that $\sigma(x,y) = \sigma(y,x)$, $\sigma(x,x) = x$, and $x \in \sigma(x,y)$ for all $(x,y) \in X \times X$ is called a segment operator on $X \times X$. For any segment operator σ, a corresponding join operator J_n on all n-tuples of X, $n \in N$, is defined inductively by $J_1(x_1) = \{x_1\}$, $J_2(x_1,x_2) = \sigma(x_1,x_2)$ and, if J_{n-1} is defined, $J_n(x_1,\cdots,x_n) = \bigcup\{\sigma(x_n,u) : u \in J_{n-1}(x_1, \cdots, x_{n-1}\}$. We call J_n commutative iff for any n-tuple $(x_1,\cdots,x_n)$ and any permutation $(x_{k_1},\cdots,x_{k_n})$ it is true that $J_n(x_1,\cdots,x_n) = J_n(x_{k_1},\cdots,x_{k_n})$. When J_n is commutative and $B = \{x_1,\cdots,x_n\}$ then we write simply $J_n(B)$ for $J_n\{x_1,\cdots,x_n\}$.

2.5. Definition. The join operator on a point $x \in X$ and set $A \subset X$ is defined by a mapping $J:X \times P(X) \to P(X)$, where for $x \in X$ and $A \subset X$

$$J(x,A) = \bigcup\{\sigma(x,a):a \in A\}.$$

Thus the operator J_n above may be defined inductively by $J_n(x_1,\cdots,x_n) = J(x_n,J_{n-1}(x_1,\cdots,x_{n-1}))$.

If H is a hull operator (satisfying axioms (a), (b) and (c) of Theorem 2.3) and if, in addition, $H(x) = \{x\}$ for $x \in X$, then a natural segment operator is that defined by $\sigma(x,y) = H\{x,y\} \equiv H(x,y)$ for each $(x,y) \in X \times X$. In this case the join operator $J:X \times P(x) \to P(x)$ becomes

$$J(x,A) = \bigcup\{H(x,a):\ a \in A\}.$$

In classical convexity, if xA denotes the join of x and A in the usual sense we have

$$\text{conv } xA = x \text{ conv } A \text{ for } x \in X \text{ and } A \subset X.$$

In abstract settings this property becomes

$$H(J(x,A)) = J(x,H(A)) \qquad (*)$$

2.6. Definition. Property (*) is called join-hull commutativity for either $\mathcal{C}$ or H. If (*) is required only for finite sets A, then we say that $\mathcal{C}$ or H is finitely join-hull commutative.

Another important property in classical convexity is the following, due to Carathéodory: If $x \in$ conv A there is a finite set $B \subset A$ such that $x \in$ conv B. In abstract settings this property may be assumed as an axiom.

2.7. Definition. If for each subset A of X, $x \in H(A)$ implies there exists a finite subset B of A such that $x \in H(B)$ then either $\mathcal{C}$ or H is said to be domain finite.

Kay and Womble [8] have established the first two following lemmas.

2.8. Lemma. A convexity structure $\mathcal{C}$ is [finitely] join-hull commutative iff for each $x \in X$ and [finite] subset $A \subset X$ it is true that $H(x \cup A) \subset J(x,H(A))$.

2.9. Remark. Let $\mathcal{C}$ be a convexity structure on a set X. It can easily be shown that if $x \in X$ and $A \subset X$ then $H(x \cup A) = H(J(x,A))$.

2.10. Lemma. If $\mathcal{C}$ is a convexity structure that is finitely join-hull commutative and domain finite then $\mathcal{C}$ is join-hull commutative.

2.11. Lemma. If the convexity structure $\mathcal{C}$ is finitely join-hull commutative, then for any finite set $B = \{x_1,\cdots,x_n\}$, $J_n(x_1,\cdots,x_n\} = H(B)$. Consequently J_n is commutative for all $n \geq 2$.

Proof. The proof is by induction on n. Now $J_2(x_1,x_2) = H(x_1,x_2)$ so the assertion is true for $n = 2$. Let $B = \{x_1,\cdots,x_n\}$ and $A = B \setminus \{x_n\}$; then by the induction hypotheses, finite join-hull commutativity and Remark 2.9 we have $J_n(x_1,\cdots,x_n) = J(x_n,J_{n-1}(A)) = J(x_n,H(A)) = H(J(x_n,A)) = H(x_n \cup A) = H(B)$. Hence if $x_{k_1},\cdots,x_{k_n}$ is any permutation of $x_1,\cdots,x_n$ then $J_n(x_{k_1},\cdots,x_{k_n}) = H(x_{k_1}\cdots,x_{k_n}) = H(x_1,\cdots,x_n) = J_n(x_1,\cdots,x_n)$.

It is often useful to develop a convexity structure from a segment operator and vice versa. The following theorem gives the relationship which arises when basic properties of each are desired.

2.12. Theorem. Let $\mathcal{C}$ be a convexity structure on X, whose corresponding hull operator H satisfies the axioms:

(a) H is T_1.

(b) H is domain finite.

(c) H is finitely join-hull commutative.

(d) If $z \in H(x,y)$ then $H(x,y = H(x,z) \cup H(z,y)$.

Let a segment operator on X be defined by $\sigma(x,y) = H(x,y)$ for each $(x,y) \in X \times X$, with <u>σ-sets</u> defined as those sets C with the property that $\sigma(x,y) \subset C$ for x and y in C. Then σ will obey the following:

(a') If x, y, z are in X, $u \in \sigma(x,y)$ and $v \in \sigma(u,z)$ then there is a $w \in \sigma(y,z)$ such that $v \in \sigma(x,w)$.

(b') $z \in \sigma(x,y)$ implies $\sigma(x,y) = \sigma(x,z) \cup \sigma(z,y)$.

(c') $\mathcal{C}$ is precisely the family of σ-sets.

Conversely, if σ is a segment operator on X satisfying axioms (a') and (b'), and a convexity structure $\mathcal{C}$ is defined as in (c'), then the corresponding hull operator H will obey axioms (a), (b), (c) and (d) and the segment operator σ' defined from $\mathcal{C}$ will coincide with the original operator σ.

<u>Proof</u>.

(a') Let x, y, $z \in X$, $u \in \sigma(x,y)$ and $v \in \sigma(u,z)$. By definition $v \in J(z,\sigma(x,y) = J_3(z,x,y)$. Lemma 2.11 implies that $J_3(z,x,y) = J_3(x,y,z)$. Hence, $v \in J_3(x,y,z) = J(x,\sigma(y,z))$ so there exists $w \in \sigma(y,z)$ such that $v \in \sigma(x,w)$.

(b') This follows directly from (d).

(c') This is a result proved by Kay and Womble [8]

For the converse, let σ satisfy (a') and (b') and let $\mathcal{C}$ be the set of σ-sets, with H the corresponding hull operator. $\mathcal{C}$ is obviously closed under intersections, since any intersection of σ-sets is a σ-set. By the definition of segment operators, a singleton set $\{x\}$ is a σ-set since $\sigma(x,x) = \{x\}$, so (a) follows.

(b) Set $C = \bigcup \{H(B)\colon B$ is a finite subset of $A\}$. We show that $C \in \mathcal{C}$ and $A \subset C \subset H(A)$, thereby proving that $H(A) = C$ and establishing domain finiteness. If $x,y \in C$ then there exists finite subsets B_1, B_2 of A such that $x \in H(B_1)$ and $y \in H(B_2)$. But $B_1 \cup B_2$ is a finite subset of A, and since $H(B_1 \cup B_2)$ is a member of $\mathcal{C}$ and thus is a σ - set, $\sigma(x,y) \subset H(B_1 \cup B_2) \subset C$ and so $C \in \mathcal{C}$. It is obvious that $A \subset C$; if $x \in C$ then $x \in H(B)$ for some finite set $B \subset A$, and hence $x \in H(B) \subset H(A)$.

(c) We first show that for any $C \in \mathcal{C}$, $J(x,C) \in \mathcal{C}$. Let $y', z' \in J(x,C)$ and $v \in \sigma(y',z')$. Then there exists y and z in C such that $y' \in \sigma(x,y)$ and $z' \in \sigma(x,z)$. Consider z, x, y' in X, $z' \in \sigma(z,x)$, and $v \in \sigma(z',y')$. By (a') there is $u \in \sigma(x,y')$ such that $v \in \sigma(z,u)$. But $\sigma(x,y') \cup \sigma(y',y) = \sigma(x,y)$ by (b'), so $u \in \sigma(x,y)$. Consider x, y, z in X with $u \in \sigma(x,y)$, $v \in \sigma(u,z)$; by (a') there is $w \in \sigma(y,z)$ such that $v \in \sigma(x,w)$. Since C is a σ-set with both $y \in C$ and $z \in C$, $\sigma(y,z) \subset C$. Hence, $w \in C$. Then $v \in \bigcup\{\sigma(x,w)\colon w \in C\} = J(x,C)$, so $\sigma(y',z') \subset J(x,C)$. Therefore $J(x,C)$ is a σ-set, so it belongs to $\mathcal{C}$. Now for any $A \subset X$ $H(A) \in \mathcal{C}$; therefore since $x \cup A \subset J(x,A) \subset J(x,H(A))$ we have $H(x \cup A) \subset H(J(x,H(A)) = J(x,H(A))$. By Lemma 2.8 $\mathcal{C}$ is join-hull commutative.

(d) We must show that the segment operator given by $\sigma'(x,y) = H(x,y)$ coincides with σ, that is, that $\sigma(x,y) = H(x,y)$. If $u,v \in \sigma(x,y)$ then by (b') $v \in \sigma(x,y) = \sigma(x,u) \cup \sigma(u,y)$, so $v \in \sigma(x,u)$ or $v \in \sigma(u,y)$. In the first case, $\sigma(x,v) \cup \sigma(v,u) = \sigma(x,u) \subset \sigma(x,y)$, and in the second, $\sigma(u,v) \cup \sigma(v,y) = \sigma(u,y) \subset \sigma(x,y)$. Since $\sigma(u,v) = \sigma(v,u)$ either case implies that $\sigma(u,v) \subset \sigma(x,y)$. Therefore $\sigma(x,y)$ is a σ-set, and

$\sigma(x,y) \in \mathcal{G}$, and $\{x,y\} \subset \sigma(x,y)$ implies $H(x,y) \subset \sigma(x,y)$. Also $\{x,y\} \subset H(x,y)$, so since $H(x,y)$ is a σ-set, $\sigma(x,y) \subset H(x,y)$ implying that $\sigma(x,y) = H(x,y)$.

CHAPTER III

THE AXIOMATIC DEVELOPMENT OF LINES FROM SEGMENT OPERATORS

In this chapter a system of axioms is considered which essentially allows us to build lines from properties of segments. These axioms serve as a replacement for axiom systems for lines yielding the usual alignment properties as normally encountered in foundations of geometry [11].

Let σ be a segment operator on $X \times X$ which for arbitrary v, w, x, y, z in X satisfies the following alignment axioms:

(a) If $z \in \sigma(x,y)$ then $\sigma(x,y) = \sigma(x,z) \cup \sigma(z,y)$.

(b) If $\sigma(x,y) = \sigma(x,z)$ then $y = z$.

(c) $\sigma(x,y)$ contains at least three elements of X.

(d) If $y \in \sigma(x,z) \cap \sigma(x,w)$ and $y \neq x$ then there exists $v \in X$ such that both z and w belong to $\sigma(x,v) \sim \{v\}$.

(e) If $x \in \sigma(w,y)$ and $y \in \sigma(x,z)$ then $x \in \sigma(w,z)$.

For brevity xy is written for $\sigma(x,y)$. The notation (xyz) means $y \in \sigma(x,z) \sim \{x,z\}$, and further (wxyz) means (wxy), (wxz), (wyz) and (xyz).

3.1. Theorem. If (xyz) then not (xzy) nor (yxz).

Proof. If (xyz) and (xzy) then by (a) $xz = xy \cup yz$ and $xy = xz \cup zy$. Therefore $xy \subset xy \cup yz = xz$ and $xz \subset xz \cup zy = xy$, or $xy = xz$. Axiom (b) then implies that $y = z$, which is a contradiction. A similar argument implies not both (xyz) and (yxz).

<u>3.2. Theorem</u>. If (wxy) and (wyz), then (wxyz).

<u>Proof</u>. By definition it need only be shown that (wxz) and (xyz) hold. Now (wxy) implies that $w \neq x$, and if $x = z$ then (wxy) and (wyz) become (wxy) and (wyx), which is impossible by Theorem 3.1. By hypothesis and axiom (a) $x \in wy \subset wy \cup yz = wz$; therefore (wxz). Again by hypotheses and axiom (a) $x \neq y$, $y \neq z$, and $x \in wy \subset wy \cup yz = wz$, hence $y \in wz = wx \cup xz$. The proof will be complete by showing $y \notin wx$. But $y \neq w$, $y \neq x$, so $y \in wx \to$ (wyx), which together with (wxy), is a contradiction.

The following theorem follows directly from axiom (c).

<u>3.3. Theorem</u>. If $x,y \in X$ and $x \neq y$ there exists $z \in X$ such that (xzy).

<u>3.4. Theorem</u>. If $x,y \in X$ and $x \neq y$ then there is a point $z \in X$ such that (xyz).

<u>Proof</u>. By Theorem 3.3 there is $w \in X$ such that (xwy), so applying axiom (d) to $w \in \sigma(x,y) \cap \sigma(x,y)$ there is $z \in X$ such that $y \in \sigma(x,z) \sim \{z\}$. But $y \neq x$ so (xyz).

<u>3.5. Theorem</u>. If (wxy) and (xyz) then (wxyz).

<u>Proof</u>. By axiom (e) $x \in \sigma(w,z)$; but $x \neq w$ and $x \neq z$ so (wxz). Hence Theorem 3.2 implies that (wxyz).

<u>3.6. Theorem</u>. If $x \neq y$, (wxz) and (wyz) then (wxyz) or (wyxz).

<u>Proof</u>. The hypotheses and axiom (a) imply $x \in wz = wy \cup yz$. If $x \in wy$ then $x \neq w$ and $x \neq y$ imply (wxy), and if $x \in yz$ then $x \neq y$ and $x \neq z$ imply (yxz). But by Theorem 3.2 (wxy) and (wyz) imply (wxyz), while (yxz) and (wyz) imply (wyxz).

<u>3.7. Theorem</u>. If (wxy) and (wxz) then $y = z$, (wxyz), or (wxzy).

<u>Proof</u>. By axiom (d) there exists $v \in X$ such that (wyv) and (wzv).

If $y \neq z$ Theorem 3.6 implies (wzyv) or (wyzv). Now (wzyv) implies (wzy) which with (wxz) implies (wxzy). Also (wyzv) implies (wyz) which with (wxy) implies (wxyz).

3.8. Definition. If $x,y \in X$, and $x \neq y$ then a ray $R(x,y)$ from x in the direction of y is defined as $R(x,y) \equiv \{z: z = x,\ z = y,\ (xzy \text{ or } (xyz)\}$.

3.9. Theorem. If $z \in R(x,y)$ and $z \neq x$ then $R(x,y) \subset R(x,z)$.

Proof. Assuming $z \neq y$ then either (xzy) or (xyz). Suppose (xzy) and that $v \in R(x,y)$; we prove $v \in R(x,z)$. If $v = y$ then (xzy) implies (xzv) and $v \in R(x,z)$. The result is obvious for $v = x$ and $v = z$. If (xvy) then (xzy) and Theorem 3.6 imply either (xvzy) or (xzvy). Thus, either (xvz) or (xzv), either of which implies that $v \in R(x,z)$. If (xyv) then (xzy) and Theorem 3.2 imply that (xzyv), which implies (xzv) so again, $v \in R(x,v)$.

If (xyz) then again let $v \in R(x,y)$. If $v = x$, y, or z then $v \in R(x,z)$ as above. If (xvy) then Theorem 3.2 and (xyz) imply (xvyz) which implies (xvz) and $v \in R(x,z)$. If (xyv) then (xyz) and Theorem 3.7 imply (xyvz) or (xyzv), so (xvz) or (xzv), and in each case this implies $v \in R(x,z)$. Therefore $R(x,y) \subset R(x,z)$.

An obvious corollary is the following.

3.10. Corollary. If $z \in R(x,y)$ and $z \neq x$ then $R(x,y) = R(x,z)$.

3.11. Theorem. If $z \in \mathcal{C}(x,y) \sim \{y\}$ then $R(z,y) \subset R(x,y)$.

Proof. The case $z = x$ clearly satisfies the theorem, so assume $z \neq x$, then (xzy). If $v \in R(z,y)$ either $v = z$, $v = y$, (zvy) or (zyv). If $v = z$ or $v = y$ then v is clearly in $R(x,y)$. If (zvy) then (xzy) and Theorem 3.2 imply (xzvy) which implies (xvy); hence $v \in R(x,y)$. If (zyv) then (xzy) and Theorem 3.5 imply (xzyv), which implies (xyv) and again

$v \in R(x,y)$ proving the theorem.

3.12. Corollary. If (zxy) then $R(x,y) \subset R(z,x)$.

3.13. Definition. The line through the points x and y $(x \neq y)$ is defined by

$$L(x,y) = R(x,y) \cup R(y,x).$$

3.14. Theorem. If $z \in L(x,y)$ and $z \neq x$ then $L(x,y) = L(x,z)$.

Proof. The case $z = y$ is obvious. If $z \neq y$ then either (zxy), (xzy) or (xyz) must hold.

In case (zxy) then by Corollary 3.12 $R(x,y) \subset R(z,x) \subset L(x,z)$. But (zxy) also means (yxz) so by Corollary 3.12 $R(x,z) \subset R(y,x) \subset L(x,y)$. Let $v \in R(y,x)$. If $v = y$, then (zxy) implies $v \in R(z,x) \subset L(x,z)$. If $v = x$, $v \in R(x,z)$. If (yvx) then (zxy) and Theorem 3.2 imply $(zxvy)$, which implies (zxv), so $v \in R(z,x) \subset L(x,z)$. If (yxv) then (zxy) (which is the same as (yxz)) and Theorem 3.7 implies either $v = z$, $(yxvz)$, or $(yxzv)$. If $v = z$ then $v \in L(x,z)$. If $(yxvz)$ then (xvz), which implies $v \in R(x,z) \subset L(x,z)$. If $(yxzv)$ then (xzv), which implies $v \in R(x,z) \subset L(x,z)$. Therefore $R(y,x) \subset L(x,z)$, so $L(x,y) \subset L(x,z)$. It can also be shown that $R(z,x) \subset L(x,y)$ so that $L(x,z) \subset L(x,y)$. The other two cases (xzy) or (xyz) are left to the reader.

3.15. Theorem If $v,z \in L(x,y)$ and $v \neq z$, then $L(x,y) = L(v,z)$. That is, two points lie on a unique line.

Proof. Assume v and z are not x and y in some order, and unless $v \neq x$, $v \neq y$, $z \neq x$, and $z \neq y$ Theorem 3.14 implies the result. Otherwise $v \in L(x,y)$ and $v \neq x$; so $L(x,y) = L(x,v) = L(v,x)$ by Theorem 3.14. But $z \in L(x,y) = L(v,x)$ and $v \neq z$; so again by Theorem 3.14 $L(v,x) = L(v,z)$. Therefore $L(x,y) = L(v,z)$.

3.16. Definition. A relation $<$ on $L(x,y)$ is defined by $u < v$ where $u,v \in L(xy)$ iff $R(u,v) \cap R(x,y) = R(z,w)$ for some $z \neq w$.

3.17. Theorem. The above relation on the line $L(x,y)$ is a linear order.

Proof. If $u \neq v$ and $u,v \in L(x,y)$ then it can be shown that $u < v$ or $v < u$. A typical case is when (uxy) and (vxy); then since $u \neq v$, Theorem 3.7 implies $(uvxy)$ or $(vuxy)$. But $(uvxy)$ implies $R(u,v) \cap R(x,y) = R(u,v)$, so $u < v$. The relation $(vuxy)$ implies $R(v,u) \cap R(x,y) = R(v,u)$, so $v < u$. The relation $<$ is also antisymmetric, nonreflexive and transitive. The proofs involve several cases each of which is easily analyzed.

It can be shown that if $\sigma'(x,y)$ is the set of points on $L(x,y)$ between x and y in the above linear order then $\sigma' = \sigma$, the original segment operator. So starting from a segment operator σ a "line operator" L has been constructed whose "restriction" is σ.

If $\mathcal{Q}$ is the family of all flats (translates of subspaces) in a vector space, one can consider the corresponding (affine) hull operator as before, since $\mathcal{Q}$ is closed under intersections. A member F of $\mathcal{Q}$ is characterized by the property; $x \in F$ and $y \in F$ iff the line $L(x,y)$ determined by x and y is contained in F. Hence, we might consider the question whether there is a theorem for an affine hull operator analogous to 2.12, pertaining to the convex hull operator. The following theorem is the analogue we seek.

3.18. Theorem. Let $\mathcal{Q}$ be a convexity structure on X, whose corresponding hull operator A satisfies the axioms:

(a) A is T_1.

(b) A is domain finite.

(c) A is finitely join-hull commutative.

(d) If $u,v \in A(x,y)$ then $A(u,v) = A(x,y)$.

Let a mapping from $X \times X$ to $P(X)$ be defined by $L(x,y) = A(x,y)$ for each $x,y \in X$, with L-sets defined as those sets F with the property that $L(x,y) \subset F$ if x,y are in F. Then L will obey the following:

(a') $L(x,x) = x$.

(b') If x, y, z are in X and $u \in L(x,y)$, $v \in L(u,z)$, then there is $w \in L(y,z)$ such that $v \in L(x,w)$.

(c') $u,v \in L(x,y)$ implies $L(x,y) = L(u,v)$.

(d') $\mathcal{Q}$ is precisely the family of L-sets.

Conversely if L is a map from $X \times X$ into $P(X)$ satisfying (a'), (b'), (c') and a convexity structure $\mathcal{Q}$ is defined as in (d'), then the corresponding hull operator A will obey (a), (b), (c) and (d) and the mapping L' defined by $L'(x,y) = A(x,y)$ will coincide with L.

Proof. The arguments used in Theorem 2.12 apply.

If $X = R^2$ and A is the usual affine hull operator on X then $L(x,y) = A(x,y)$ gives lines in R^2. Note that in this case A is not finitely join-hull commutative, since the join of a line L and a point $x \notin L$ does not contain the line through x parallel to L. However, this is not the case in the projective plane since every pair of lines must intersect. Therefore, the projective plane X, with $\mathcal{Q} \equiv$ the family of all flats, would be an example of a system satisfying the hypothesis of Theorem 3.18. Note also that the projective plane with the line at infinity removed is isomorphic to the Euclidean plane. Hence, the properties of line operators could be studied in the projective plane, then transferred to the Euclidean plane by simply removing some line.

The above idea admits the following generalization: Let (X,F) be a vector space over F. Introduce the equivalence relation $\equiv$ on X by

$$x \equiv \lambda x \quad \text{iff} \quad \lambda \neq 0.$$

<u>3.19. Definition</u>. Let $X^* = \{\bar{x}: x \in X,\ x \neq 0\}$ then X^* is called a <u>projective space</u> over F of dimension equal to dim X - 1 if dim X< ∞, and infinite otherwise. Let H be a hyperplane of X then $H^* = \{\bar{x}: x \in H,\ x \neq 0\}$ is called a <u>projective hyperplane</u>. We state without proof

<u>3.20. Theorem</u>. If H is any hyperplane in a vector space X, then $X^* \sim H^*$ is a vector space over F that is isomorphic to H.

Thus, as in the 2-dimensional case, properties of line operators for X^* have implications for vector spaces by lifting their properties from $X^* \sim H^*$.

CHAPTER IV

PROPERTIES OF CONVEX AND PRECONVEX FUNCTIONS

4.1. Definition. If $\mathcal{C}$ is a convexity structure on the set X then the pair $(X, \mathcal{C})$ is called a convexity space.

4.2. Definition. Consider the convexity spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ and a mapping f from X to Y. The map f is said to be convex (with respect to $\mathcal{C}$ and $\mathcal{D}$) if $f(C) \in \mathcal{D}$ when $C \in \mathcal{C}$. Further, f is said to be preconvex (with respect to $\mathcal{C}$ and $\mathcal{D}$) if $f^{-1}(D) \in \mathcal{C}$ when $D \in \mathcal{D}$. The function f is a convexity isomorphism if f is one to one, onto, convex and preconvex. In such a case the convexity structures are called isomorphic.

Recall that a map f from R^n to R is called convex (in the classical sense) if for each x and y in R^n

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \text{for } 0 \leq \lambda \leq 1.$$

Any such function is continuous and so preserves connectedness. Let $\mathcal{C}^n$ and $\mathcal{C}$ denote the usual convexity structure on R^n and R respectively. Then $C \in \mathcal{C}^n$ implies $f(C) \in \mathcal{C}$, because f(C) is connected and therefore it is an interval. So, every classically convex function is convex with respect to $\mathcal{C}^n$ and $\mathcal{C}$.

Meyer and Kay [10] have established the following result, which shows a close relationship between classically linear functions and functions which are convex in our sense.

4.3. Theorem. If V and W are real vector spaces with dim V > 1 and

$f: V \to W$ is a one to one mapping which preserves the usual convex sets then if $f(0) = 0$, f is a linear map.

The following question, referred to as "the linearization problem" is of interest: Find a set of conditions on a given convexity space $(X, \mathscr{C})$ involving only the members of $\mathscr{C}$ which will imply the existence of a vector space structure (X,F) over an ordered field F such that the usual convex sets of (X,F) become the elements of $\mathscr{C}$. In a recent paper[7] Kay has answered this question in the affirmative.

The following theorem allows the linearization problem to be stated in terms of the existence of a certain convexity isomorphism.

<u>4.4. Theorem</u>. Let $(X, \mathscr{C})$ and $(Y, \mathscr{D})$ be convexity spaces, where Y is a vector space over an ordered field F and $\mathscr{D}$ is the usual convexity structure; that is, $D \in \mathscr{D}$ iff $x,y \in D$ implies that $\alpha x + (1-\alpha)y \in D$ for $\alpha \in F$ and $0 \leq \alpha \leq 1$. If $(X, \mathscr{C})$ is isomorphic to $(Y, \mathscr{D})$, then there exists a vector space structure on X over F such that $\mathscr{C}$ is the usual convexity structure.

<u>Proof</u>. Let f be the convexity isomorphism in the hypotheses. For any $x,y \in X$ and $\alpha \in F$ define $x + y$ and $\alpha \cdot x$ to be the elements of X whose images are $f(x) + f(y)$ and $\alpha f(x)$ respectively; then $(X,+,\cdot,F)$ is a vector space. By definition of $+,\cdot$

$$\begin{aligned} f[\alpha\cdot(x+y)] = \alpha f(x+y) = \alpha[f(x) + f(y)] &= \alpha f(x) + \alpha f(y) \\ &= f(\alpha\cdot x) + f(\alpha\cdot y) \\ &= f[\alpha\cdot x + \alpha\cdot y] \end{aligned}$$

Since f is $1 - 1$ this implies $\alpha\cdot(x + y) = \alpha\cdot x + \alpha\cdot y$. The demonstration of the other vector space properties is similar. Let $C \in \mathscr{C}$ and suppose $x,y \in C$. Then if $0 \leq \alpha \leq 1$,

$f[\alpha x + (1-\alpha)y] = \alpha f(x) + (1-\alpha)f(y)$. Since f is convex $f(C) \in \mathfrak{D}$ so that $f(x), f(y) \in f(C)$ implies $\alpha f(x) + (1-\alpha)f(y) \in f(C)$; therefore, $\alpha x + (1-\alpha)y \in C$, and C is convex in the usual sense.

Conversely let C be any set satisfying the property that if $u,v \in C$ $\alpha u + (1-\alpha)v \in C$ for $0 \leq \alpha \leq 1$. Then f(C) has the property that for any pair $f(x), f(y) \in f(C)$, $\alpha f(x) + (1-\alpha)f(y) = f[\alpha x + (1-\alpha)y] \in f(C)$ so that $f(C) \in \mathfrak{D}$. But since f is preconvex $C \in \mathcal{C}$. Therefore, $\mathcal{C}$ is the usual convexity structure on (X,F).

If in the above theorem one replaces the usual convexity structure $\mathfrak{D}$ by the family of all flats in (Y,F), then an argument similar to the one above shows that $\mathcal{C}$ would be the usual flats on the vector space (X,F). In this case it could be said that f is linear and prelinear since the image of a line is a line, and conversely. More generally, one can use this terminology for a mapping f from a "linear" space $(X, \mathcal{Q})$ into $(Y, \mathcal{B})$, where $\mathcal{Q}$ and $\mathcal{B}$ are thought of as affine hull operators, instead of convex hull operators.

We now show that a number of properties which have been considered by other authors are isomorphism invariants in our terminology.

4.5. Definition. A convexity structure $\mathcal{C}$ is said to have Carathéodory number c iff c is the smallest positive integer for which it is true that the $\mathcal{C}$-hull of any set $S \subset X$ is the union of the $\mathcal{C}$-hulls of those subsets of S of cardinality less than or equal to c.

4.6. Definition. A convexity structure has Helly number h iff h is the smallest positive integer for which it is true that a finite subfamily $\mathcal{J}$ of sets in $\mathcal{C}$ has nonempty intersection provided each h members of $\mathcal{J}$ have nonempty intersection.

4.7. Definition. A convexity structure $\mathcal{C}$ has Radon number r iff r is the smallest positive integer for which it is true that any set S with cardinality greater than or equal to r has a Radon partition, that is, S may be partitioned into two nonempty subsets (S_1, S_2) such that $H(S_1) \cap H(S_2) \neq \phi$.

4.8. Theorem. If $(x, \mathcal{C})$ is isomorphic to $(Y, \mathcal{D})$ then $f[H(S)] = H(f[S])$ for $S \subset X$.

Proof. Since $S \subset H(S)$ and f is convex, $f[S] \subset f[H(S)] \in \mathcal{D}$; therefore, by the definition of H, $H(f[S]) \subset f[H(S)]$.

Conversely, if $D \in \mathcal{D}$ and $f(S) \subset D$ then $S \subset f^{-1}(D)$ and $f^{-1}(D) \in \mathcal{C}$ since f is preconvex. Therefore

$$H(S) \equiv \bigcap_{\substack{C \supset S \\ C \in \mathcal{C}}} C \subset \bigcap_{\substack{D \in \mathcal{D} \\ D \supset f(S)}} f^{-1}(D) = f^{-1}\left(\bigcap_{\substack{D \in D \\ D \supset f(S)}} D\right) = f^{-1}[H(f[S]],$$

so $f[H(S)] \subset H(f[S])$.

4.9. Corollary. If $(X, \mathcal{C})$ is isomorphic to $(Y, \mathcal{D})$, then $(X, \mathcal{C})$ has Carathéodory number c iff $(Y, \mathcal{D})$ has Carathéodory number c.

Proof. Let $\mathcal{C}$ have Carathéodory number c. Then if $D \in \mathcal{D}$

$$f^{-1}[H(D)] = H(f^{-1}[D]) = \bigcup\{H(T): T \subset f^{-1}[D] \text{ and } |T| \leq c\},$$

where $|T|$ denotes the cardinality of T. Hence $H(D) = f[\bigcup H(T): T \subset f^{-1}[D], |T| \leq c\}] = \bigcup\{f[H(T)]: T \subset f^{-1}(D), |T| \leq c\} = \bigcup\{H(f[T]) : f(T) \subset D \text{ and } |f(T)| \leq c\}$, so $\mathcal{D}$ has Carathéodory number less than or equal to c. The argument is symmetric so the Carathéodory numbers are equal.

4.10. Corollary. Domain finiteness and join-hull commutativity are preserved by isomorphisms.

4.11. Theorem. If f maps X onto Y and f is preconvex with respect to $\mathcal{C}$ and $\mathcal{D}$ then if $(X,\mathcal{C})$ has Helly number h, $(Y,\mathcal{D})$ has Helly number $\leq h$.

Proof. Let $\mathcal{F} = \{D_k : k = 1,2,\cdots,n\ D_k \in \mathcal{D}\}$ and suppose that the intersection of each h members of $\mathcal{F}$ have a nonempty intersection. Let $C_k = f^{-1}(D_k)$ and $H = \{C_k : k = 1,\cdots,n\}$; since f is preconvex, $C_k \in \mathcal{C}$ for each k. Let $C_{k_1}, C_{k_2}, \cdots, C_{k_h}$ be any h elements of H. Then

$\bigcap_{i=1}^{h} C_{k_i} = \bigcap_{i=1}^{h} f^{-1}(D_{k_i}) = f^{-1}(\bigcap_{i=1}^{h} D_{k_i})$. But $\bigcap_{i=1}^{h} D_{k_i} \neq \phi$ and since f is onto this implies $\bigcap_{i=1}^{h} C_{k_i} \neq \phi$. But $(X,\mathcal{C})$ has Helly number h so $\bigcap_{k=1}^{n} C_k \neq \phi$. Since

$$\bigcap_{k=1}^{n} C_k = \bigcap^{n} f^{-1}(D_k) = f^{-1}(\bigcap D_k),$$

$\bigcap_{k=1}^{n} D_k \neq \phi$. Therefore, $(Y,\mathcal{D})$ has Helly number less than or equal to h.

4.12. Corollary. If $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ are isomorphic, then $(X,\mathcal{C})$ has Helly number h iff $(Y,\mathcal{D})$ has Helly number h.

4.13. Theorem. Let $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ be convexity spaces and f a one to one map from X onto Y which is preconvex. If $(X,\mathcal{C})$ has finite Radon number r then the Radon number of $(Y,\mathcal{D})$ is $\leq r$.

Proof. Let D be any set in Y with cardinality greater than or equal to the number h. Since f is onto, $|C| \geq h$ where $C \equiv f^{-1}(D)$, so there exists a Radon partition (C_1,C_2) of C. Since f is one to one, $D_1 \equiv f(C_1)$ and $D_2 \equiv f(C_2)$ partition D. But $f^{-1}[H(D_i)] \supset f^{-1}(D_i) \supset C_i$, $i = 1,2$, and since f is preconvex $f^{-1}[H(D_i)] \in \mathcal{C}$ so $H(C_i) \subset f^{-1}[H(D_i)]$. Therefore,

$\phi \neq H(C_1) \cap H(C_2) \subset f^{-1}[H(D_1) \cap H(D_2)]$ so $H(D_1) \cap H(D_2) \neq \phi$.

4.14. Corollary. If $(X,\mathscr{C})$ and $(Y,\mathscr{D})$ are isomorphic, then $(X,\mathscr{C})$ has Radon number r iff $(Y,\mathscr{D})$ has Radon number r.

The composition of convex isomorphisms is again a convex isomorphism and in fact, under the operation of composition the set of convex isomorphisms from a convexity space $(X,\mathscr{C})$ to itself forms a group. The next result, which is used later, is the analogue of a classic theorem in linear topological spaces.

4.15. Definition. A set S is said to be starshaped at a point x iff for each $y \in S$ it is true that $H(x,y) \subset S$. The notation $(X,\mathscr{C},\mathscr{T})$ is used for a set X with convexity structure $\mathscr{C}$ and topology $\mathscr{T}$. It is said that $(X,\mathscr{C},\mathscr{T})$ is locally starshaped at x if for each neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and V is starshaped with respect to x. If the above holds for each $x \in X$ then $(X,\mathscr{C},\mathscr{T})$ is called locally starshaped.

4.16 Theorem. If $(X,\mathscr{C},\mathscr{T})$ is locally starshaped and f is a convex function from $(X,\mathscr{C})$ to the real numbers (with the usual convexity structure) then f is continuous iff $f^{-1}(\alpha)$ is closed for each real number α.

Proof. If f is continuous then the preimage of a point must be closed.

Conversely, suppose $f^{-1}(\alpha)$ is closed for each $\alpha \in R$ and that there is a point $x \in X$ at which f is not continuous. Let $\varepsilon > 0$ be such that every neighborhood U of x contains at least one point y such that $|f(y) - f(x)| > \varepsilon$. Let $K = f^{-1}(f(x) - \varepsilon/2) \cup f^{-1}(f(x) + \varepsilon/2)$; then K is closed and $x \notin K$. So there is a neighborhood U of x such that $U \cap K = \phi$. Since $(X,\mathscr{C},\mathscr{T})$ is locally starshaped, let V be a starshaped neighborhood of x such that $V \subset U$; then $V \cap K = \phi$. But there is a point $y \in V$ such that $|f(y) - f(x)| > \varepsilon$. Now $H(x,y) \subset V$ and $f[H(x,y)]$ must be an interval containing the points $f(x)$ and $f(y)$. Hence, $|f(y) - f(x)| > \varepsilon$ implies some point $z \in H(x,y)$ exists whose image is $f(x) + \varepsilon/2$ or $f(x) - \varepsilon/2$. Hence $z \in H(x,y) \cap K$, implying that $V \cap K \neq \phi$, which is a contradiction.

CHAPTER V

ANALOGUES OF THE KREIN-MILMAN THEOREM

M. C. Gemignani [5], and P. C. Hammer [6] have considered relationships between convexity and topology in nonclassical settings. In the classical setting continuity of vector addition and scalar multiplication provide a connecting link between convexity and topology. The convex hull and join operators provide this link in abstract settings, as we shall see.

5.1. Definition. Recall that in a topological space $(X,\mathcal{J})$ a sequence of sets S_n converges to the set S, denoted $S_n \to S$, iff $\lim\sup S_n = \lim\inf S_n = S$. Thus a topology known as the Hausdorff topology is determined on $P(X)$, the power set of X.

Let $(X,\mathcal{C})$ be a convexity space and $\mathcal{J}$ a topology on X. The hull operator H associated with $\mathcal{C}$ is a set function from $P(X)$ to $P(X)$. A natural property would be the requirement that H be continuous relative to the Hausdorff topology of $P(X)$. But if X is the plane R^2 with the usual topology and convexity structure then H is not continuous.

5.2. Example. Let $S_n = \{(-\frac{1}{n},y) : y \in R\} \cup \{(1,\frac{1}{n})\}$ and $S = \{(0,y) : y \in R\} \cup \{(1,0)\}$ then $S_n \to S$ but $H(S_n) \to \{(x,y) \quad 0 \leq x \leq 1\}$ and $H(S) = \{(x,y) : 0 \leq x < 1\} \cup \{(1,0)\}$. A concept which has been found to generalize the idea of a linear topological space is contained in the following definition.

5.3. Definition. The triple $(X, \mathcal{C}, \mathcal{T})$ is called a topological convexity space (T.C.S.) if $\mathcal{C}$ is a convexity structure for X and $\mathcal{T}$ is a topology on X for which:

(a) $\mathcal{T}$ is first countable and Hausdorff

(b) $U \in \mathcal{T}$ implies for $x \in X$, $J(x,U) \sim \{x\} \in \mathcal{T}$

(c) If $\{S_n\}$ is a sequence of p-element sets contained in a compact set, where p is a positive integer, then $S_n \to S$ implies $H(S_n) \to H(S)$.

The following are examples of topological convexity spaces.

5.4. Example. Let $(X, \mathcal{T})$ be any linear topological space and $\mathcal{C}$ the usual convexity structure on X. If $\mathcal{T}$ is first countable and Hausdorff (therefore, metrizable) then $(X, \mathcal{C}, \mathcal{T})$ is a T.C.S. The proof involves continuity of addition, scalar multiplication and Carathéodory's theorem.

5.5. Example. Let $X = R^2$ and $\mathcal{C}$ the usual convexity structure on R^2. Let a basis for $\mathcal{T}$ be given by sets of the form $\{(x,y) : a < x \leq b,\ c < y \leq d\}$ where a,b,c, and d are real numbers such that $a < b$ and $c < d$.

5.6. Example. Let $X = R^2$ and $\mathcal{T}$ the usual topology on R^2. Let $\mathcal{C}$ be generated (see Theorem 2.12) by the segment operator σ defined below. Every pair of points $u,v \in X$ either lie on a vertical line or determine a parabola of the form $y = x^2 + bx + c$. In either case let $\sigma(x,y)$ be the closed curve between x and y.

5.7. Example. Let $X = R^n$, $\mathcal{T}$ the usual topology on R^n and let $\mathcal{C}$ be any one of the following families of subsets:

(a) the closed subsets of X.

(b) the closed convex subsets of X.

(c) X and all bounded, convex subsets of X.

Then in each case $(X, \mathcal{C}, \mathcal{J})$ is a T.C.S.

5.8. Example. Let X consist of n-dimensional hyperbolic or elliptic geometry with the usual segment operator $\sigma(x,y) \equiv$ the points x and y, together with the points between x and y. Take $\mathcal{C}$ to be the family of σ-sets (see Theorem 2.12) and $\mathcal{J}$ the topology of X as a metric space. Then $(X, \mathcal{C}, \mathcal{J})$ is a T.C.S.

5.9 Theorem. Let $(X, \mathcal{C}, \mathcal{J})$ be a T.C.S. and $S \subset X$ which is starshaped at a point x. Then cl S, the closure of S, is also starshaped at x.

Proof. Let $x_o \in$ cl S and $\{x_n\}$ a sequence in S converging to x_o. Let $S_n = \{x_n, x\}$ and $S_o = \{x_o, x\}$ then S_n is a 2-element sequence and $\bigcup_{n=1}^{\infty} S_n \subset \{x, x_o, x_1, \cdots, x_n, \cdots\}$ which is a compact set. Further $S_n \to S_o$ so $H(S_n) \to H(S)$. But S is starshaped at x so $H(S_n) = H(x_n, x) \subset S$ for each n. Therefore, by the definition of convergence, $H(x, x_o) \subset$ cl S.

5.10 Remark. If $\mathcal{C}$ is domain finite then $S \subset X$ is $\mathcal{C}$-convex iff for any finite set $\{x_1, \cdots, x_n\}$ contained in S it is true that $H\{x_1, \cdots, x_n\} \subset S$.

5.11 Theorem. If S is a $\mathcal{C}$-convex set in a domain finite T.C.S. $(X, \mathcal{C}, \mathcal{J})$ then cl S is also $\mathcal{C}$-convex.

Proof. Let $\{x_1, \cdots, x_p\}$ be any finite subset of cl S; by the above remark it is enough to show that $H(x_1, \cdots, x_p) \subset$ cl S. For each i, i=1,2, $\cdots$,p, let $\{x_n^i\}$ be a sequence in S converging to x_i. Define $S_n = \{x_n^1, \cdots x_n^p\}$ and $S_o = \{x_1, \cdots, x_p\}$; then the sequence of p-element sets $\{S_n\}$ converges to S_o, and is contained in the compact set $(\bigcup_{n=1}^{\infty} S_n) \cup S_o$. Therefore, $H(S_n) \to H(S_o)$ and since $S_n \subset S$ and S is $\mathcal{C}$-convex, $H(S_n) \subset S$. Hence, it follows that $H(S_o) \subset$ cl S. That is, $H(x_1, \cdots, x_p) \subset$ cl S and cl S is $\mathcal{C}$-convex.

A modification of Example 5.7 (c) shows that the assumption of domain finiteness is needed in Theorem 5.11: Simply include one nonclosed, unbounded, convex subset of X in the family $\mathcal{C}$.

5.12. Theorem. Let $(X,\mathcal{T},C)$ be a domain finite and join-hull commutative T.C.S. then for any $A \subset X$, $H(\text{int } A) \subset \text{int } H(A)$.

Proof. Let $x \in H(\text{int } A)$. Then domain finiteness implies there is a minimal integer n such that $x \in H(a_1,\cdots,a_n)$ for $a_i \in \text{int } A$ $(1 \leq i \leq n)$. But by join-hull commutativity $x \in J(a_1, H(a_2,\cdots,a_n))$. Since n was minimal, $x \notin H(a_2,\cdots,a_n)$, so $x \in H(a_1,b) \sim \{b\} \subset J(b,U) \sim \{b\}$, where b is some point in $H(a_2,\cdots,a_n)$ and U is an open set such that $a_1 \in U \subset A$. Now $b \in H(A)$ and $U \subset A \subset H(A)$, so $x \in G \equiv J(b,U) \sim \{b\} \subset J(b,U) = H(b \cup U) \subset H(A)$. Since G is open, $x \in \text{int } H(A)$.

Two results follow immediately (valid in any domain finite and join-hull commutative T.C.S.)

5.13. Corollary. If $S \in \mathcal{T}$ then $H(S) \in \mathcal{T}$.

5.14. Corollary. The interior of a $\mathcal{C}$-convex set is $\mathcal{C}$-convex.

5.15. Theorem. If $\mathcal{C}$ has Carathéodory number c and $(X,\mathcal{C},\mathcal{T})$ is a T.C.S., then for each compact set K, H(K) is closed.

Proof. Let z be a limit point of H(K); then there is a sequence $\{z_n\} \subset H(K)$ such that $z_n \to z$. Note that if y is any point of H(K), then since $\mathcal{C}$ has Carathéodory number c there are points $y_1, y_2, \cdots, y_m$ in K such that $y \in H(y_1,\cdots,y_m)$, where $m \leq c$. If $m < c$ choose points $y_{m+1} = y_{m+2} = \cdots = y_c = y_m$ in K so that $y \in H(y_1,y_2,\cdots,y_c)$. Therefore, we may assume that $z_n \in H(x_n^1,\cdots,x_n^c)$ where $x_n^i \in K$, $1 \leq i \leq c$. Let $\{x_{n_k}^1\}$ be a subsequence of $\{x_n^1\}$ converging to say $x_1 \in K$; then $z_{n_k} \in H(x_{n_k}^1,\cdots,x_{n_k}^c)$. Repeat the process for the sequence $\{x_{n_k}^2\}$, and continue inductively. By change of notation it may be assumed that $z_n \in H(x_n^1,\cdots,x_n^c)$ where $x_n^i \to x_i$ and

$x_i \in K$, $1 \leq i \leq c$. Let $S_n \equiv \{x_n^1, \cdots, x_n^c\}$ and $S = \{x_1, \cdots x_c\}$. Then $\{S_n\}$ is a c-element sequence contained in K so $H(S_n) \to H(S)$. Since $S \subset K$, we have $H(S) \subset H(K)$. But $z_n \to z$ and $z_n \in H(S_n)$ so $z \in \lim\sup H(S_n) = H(S) \subset H(K)$.

It will be assumed that $\mathcal{C}$ is domain finite and join-hull commutative in all that follows.

5.16 Definition. Let $(X, \mathcal{C}, \mathcal{J})$ be a T.C.S. If $C, D \in \mathcal{C}$ then C and D are said to be complementary $\mathcal{C}$-half spaces if C and D are nonempty, $C \cup D = X$, and $C \cap D = \phi$.

Ellis [3] has proved the following.

5.17. Theorem. Let $(X, \mathcal{C})$ be a domain finite and finitely join-hull commutative convexity space. If $u \in H(x,y)$ and $v \in H(x,z)$ imply that $H(u,z) \cap H(v,y) \neq \phi$ then for any two disjoint sets $A, B \in \mathcal{C}$ there exist complementary $\mathcal{C}$ half spaces C and D such that $C \supset A$ and $D \supset B$.

5.18. Definition. Let C and D be complementary $\mathcal{C}$ half spaces and let $H \equiv \text{cl } C \cap \text{cl } D$. If $H \neq \phi$ and $H \neq X$ then H is called the closed hyperplane determined by C and D.

5.19. Remark. Since $D = X \sim C$ and $C = X \sim D$, the set H just defined is actually the boundary of both C and D, since $\text{bd } C = \text{cl } C \cap \text{cl}(X \sim C) = \text{cl } (X \sim D) \cap \text{cl } D = \text{bd } D$. Thus if $H = \phi$ then $\text{bd } C = \phi$ so $C = \text{int } C$, and $D = \text{int } D$. Thus, if X is connected and either $\text{int } C \neq \phi$ or $\text{int } D \neq \phi$, then both $H \neq \phi$ and $H \neq X$, and H is a closed hyperplane.

5.20. Theorem. If H is a closed hyperplane determined by C and D then either $H \supset C$, $H \supset D$, or the sets int C, int D, and H are each nonempty members of $\mathcal{C}$ and partition X, with $\text{int } C = C \sim H$ and $\text{int } D = D \sim H$.

Proof. Suppose $H \not\supset C$ and $H \not\supset D$. Then $\text{int } C \neq \phi$, $\text{int } D \neq \phi$ and $H \neq \phi$. Since $C, D \in \mathcal{C}$ Theorem 5.14 implies that int C and int D are in $\mathcal{C}$.

Also by Theorem 5.11 cl C and cl D are in $\mathcal{C}$; hence $H \in \mathcal{C}$. Suppose $x \in X \sim (H \cup \text{int } C \cup \text{int } D)$. Since $x \in X = \text{cl } C \cup \text{cl } D$ and $x \notin \text{cl } C \cap \text{cl } D$ it may be assumed that $x \notin \text{cl } D$. Thus, $x \in X \sim \text{cl } D \subset X \sim D = C$, so $x \in \text{int } C$. This contradiction implies that $X = H \cup \text{int } C \cup \text{int } D$, and since int C, int D and H are disjoint they partition X. It is a result from topology that $\text{int } C = C \sim H$ and $\text{int } D = D \sim H$, so the proof is complete.

The theory necessary to derive the analogue of the Krein-Milman theorem will now be introduced.

5.21. Definition. If $K \in \mathcal{C}$ and $x \in K$, then x *is an extreme point of* K if $y, z \in K$ and $x \in H(y,z)$ imply that $x = y$ or $x = z$. A nonempty set M contained in a set $K \subset X$, is called an *extremal subset of* K if $x, y \in K$ and $M \cap H(x,y) \sim \{x,y\} \neq \phi$ imply $\{x,y\} \subset M$.

The next theorem requires an extensive hypothesis, which is, of course satisfied in the usual setting of locally convex linear topological spaces in which the theorem of Krein-Milman normally applies. However, it is not difficult to see that this hypothesis is satisfied also by a few of our previous examples (5.6 and 5.8). Following the proof, we show how to obtain simiiar hypotheses in a class of spaces studied by Cantwell [6], providing further examples. Parts (a) and (b) of the hypothesis are general requirements pertaining to the space X (in particular, to the convexity structure $\mathcal{C}$), while part (c) concerns properties demanded of closed hyperplanes (involving both $\mathcal{C}$ and $\mathcal{J}$).

5.22. Definition. For convenience, let us call a closed hyperplane H *flat* if whenever the set $H(x,y) \cap H$ contains at least two points then $H(x,y) \subset H$. Let H be called *regular* if there exists a continuous convex function f from X onto R (with usual convexity structure and topology on R) such that for some real α, $H = f^{-1}(\alpha)$. A regular closed hyperplane $H = \{x : f(x) = \alpha\}$ is called translatable iff $f^{-1}(\beta)$ is a closed hyperplane for all real β.

5.23. Definition. A T.C.S. $(X, \mathcal{C}, \mathcal{J})$ is said to be locally convex at a point x if for each neighborhood U of x there exists a neighborhood V of x such that $V \in \mathcal{C}$ and $V \subset U$. If $(X, \mathcal{C}, \mathcal{J})$ is locally convex at each point then it is said to be locally convex.

5.24. Theorem. (Krein-Milman) Let $(X, \mathcal{C}, \mathcal{J})$ be a connected locally convex T.C.S. satisfying:

(a) $\mathcal{C}$ is domain finite and finitely join-hull commutative.

(b) If $x, y, z \in X$ and $u \in H(x,y)$, $v \in H(x,z)$ then $H(u,z) \cap H(v,y) \neq \phi$.

(c) Each closed hyperplane is flat, regular, and translatable.

If A is any compact element of $\mathcal{C}$ then the above properties imply that $A = cl(H(ext\ A))$, where ext A is the set of all extreme points of A.

The proof will be given in a sequence of lemmas.

5.25. Lemma. If $\mathcal{F}$ is a family of extremal subsets of a set K, then a nonempty intersection of any subfamily of $\mathcal{F}$ is an extremal subset of K.

Proof. Let $F_i \in \mathcal{F}$, $i \in A$. Choose $x \in K$, $y \in K$. If $(\bigcap_{i \in A} F_i) \cap H(xy) \sim \{x,y\} \neq \phi$ then $F_i \cap H(xy) \sim \{x,y\} \neq \phi$ so that $\{x,y\} \subset F_i$, $i \in A$. Therefore $\{x,y\} \subset \bigcap_{i \in A} F_i$.

5.26. Lemma. Let H be a regular closed hyperplane determined by complementary sets C and D. Then there exists a continuous convex function f from X onto R and $\alpha \in R$, such that $H = \{x: f(x) = \alpha\}$, and $int\ C = \{x: f(x) < \alpha\}$ and $int\ D = \{x \mid f(c) > \alpha\}$.

Proof. There exists a continuous convex function f_o from X onto R and

$\alpha \in R$, such that $H = f^{-1}(\alpha)$. There is $x_0 \notin H$, so either $x_0 \in C$ or $x_0 \in D$, say $x_0 \in C$. Then $x_0 \in C \sim H = \text{int } C$. If $f_0(x_0) < \alpha$ take $f = f_0$; if $f_0(x_0) > \alpha$ define f by $f(x) = 2\alpha - f_0(x)$ for $x \in X$. Then f is also a continuous convex function from X onto R, and $f(x_0) < \alpha$ in either case. Let y_0 be any point in X such that $f(y_0) > \alpha$; we show that $y_0 \in D$. If not then $y_0 \in C$ and hence $y_0 \in C \sim H = \text{int } C$. By the convexity of f it follows there is a point $z_0 \in H(x_0, y_0)$ such that $f(z_0) = \alpha$. Thus, $z_0 \in \text{int } C$ (since $\text{int } C \in \mathcal{C}$) and $z_0 \in H = \text{bd } C$, an impossibility. Therefore, $y_0 \in D \sim H = \text{int } D$. The same argument proves $\{x | f(x) > \alpha\} \subset \text{int } D$; similarly, $\{x | f(x) < \alpha\} \subset \text{int } C$. Now we have $H \not\supset C$ and $H \not\supset D$ so by Theorem 5.20 the sets int C, int D, H partition X. Since also the sets $\{x: f(x) < \alpha\}$, $\{x: f(x) = \alpha\} = H$, and $\{x: f(x) > \alpha\}$ partition X, the assertion follows.

5.27. Lemma. With hypotheses as in Theorem 5.24, a compact set A has at least one extreme point.

Proof. Let $\mathcal{F}$ be the set of all compact extremal subsets of A. Since $A \in \mathcal{F}$ the family $\mathcal{F} \neq \phi$. An application of Zorn's lemma implies that $\mathcal{F}$ has a minimal element F. The goal is to prove that F is a single point. (Hence F is an extreme point of A.) Suppose x, y are distinct points of F. Since $(X, \mathcal{C}, \mathcal{F})$ is a locally convex space and is Hausdorff, there exist disjoint $\mathcal{C}$-convex neighborhoods A, B of x and y respectively. Theorem 5.17 implies there exists complementary $\mathcal{C}$-half spaces C and D such that $A \subset C$ and $B \subset D$. Since $H = \text{cl } C \cap \text{cl } D$ is a closed hyperplane, Theorem 5.26 implies there exists a continuous convex onto function $f: X \to R$ such that $H = \{x: f(x) = \alpha\}$ and $f(x) < \alpha < f(y)$. Since f is continuous and F is compact

$$\beta = \sup_{z \in F}\{f(z)\} = f(z_o)$$

exists for some $z_o \in F$. Also by hypothesis $H_1 = \{x: f(x) = \beta\} \neq \phi$ is a closed hyperplane. If $E = F \cap H$, then E is compact, $z_o \in E$, and $x \notin E$ so E is a proper subset of F. We show that E is an extremal subset of A. If $u,v \in A$ with $w \in [H(u,v) \sim \{u,v\}] \cap E$ then $u,v \in F$ because $w \in E \subset F$ and F is an extremal subset of A. But $u,v \in F$ implies $f(u) \leq \beta$ and $f(v) \leq \beta$. If both $f(u) < \beta$ and $f(v) < \beta$ then $u,v \in \{x: f(x) < \beta\} \in \mathcal{C}$, implying $w \in H(u,v) \subset \{x: f(x) < \beta\}$. This is impossible since $w \in H_1$. If $f(u) < \beta$ and $f(v) = \beta$ then $w,v \in H(u,v) \cap H_1$ so by hypothesis $H(u,v) \subset H_1$, implying that $u,v \in E$. The above proves that E is an extremal subset of A. Hence, $E \in \mathfrak{F}$, contradicting the minimality of F.

Proof of Theorem 5.24. Let $B \equiv cl(H(\text{ext } A))$. Since ext $A \subset A$ it is true that

$$H(\text{ext } A) \subset H(A) = A \text{ and } cl(H(\text{ext } A) \subset cl\ A = A.$$

It remains to prove that $A \subset B$. Suppose $x \in A \sim B$. Since B is closed $X \sim B$ is a neighborhood of x. From the local convexity of $(X, \mathcal{C}, \mathfrak{F})$ there is a $\mathcal{C}$-convex neighborhood U of x such that $x \in U \subset X \sim B$. From Theorem 5.10, B is $\mathcal{C}$-convex so $U,B \in \mathcal{C}$ and $U \cap B = \phi$. Again by Theorem 5.17 there is a closed hyperplane H determined by C and D such that $U \subset C$ and $B \subset D$; thus $U \subset$ int C. By hypothesis H is regular, which with Theorem 5.26 implies the existence of a continuous convex function f and $\alpha \in R$, such that $H \in f^{-1}(\alpha)$ and $f(\text{int } C) < \alpha < f(\text{int } D)$. Since f is continuous and A is compact, define $\beta = \inf_{y \in A}\{f(y)\}$. (Note that $\beta < \alpha$ since $x \in A$ and $f(x) < \alpha$). Let $H_1 = \{y | f(y) = \beta\}$; then $E \equiv H_1 \cap A \neq \phi$ (f takes its minimal value at an element of A). Now E is compact, so by Lemma 5.27

there is a point $w \in \text{ext } E$. It will be shown that $w \in \text{ext } A$. If $u \in A$, $v \in A$ and $w \in H(u,v) \sim \{u,v\}$, then from the definition of β, $f(u) \geq \beta$ and $f(v) \geq \beta$. If both $f(u) > \beta$ and $f(v) > \beta$, then $u,v \in \{x:f(x) > \beta\} \in \mathcal{C}$ so $w \in H(u,v) \subset \{x:f(x) > \beta\}$. But this contradicts the fact that $f(w) = \beta$, since $w \in E \subset H_1$. If $f(u) = \beta$ and $f(v) > \beta$ then $u,w \in H(u,v) \cap H_1$ and hence, $H(u,v) \subset H_1$. Therefore $f(u) = f(v) = \beta$ and $u,v \in E$. But this contradicts the fact that $w \in \text{ext } E$, since $w \in H(u,v) \sim (u,v)$ where $u,v \in E$. Hence $w \in \text{ext } A$. But then $w \in \text{ext } A \subset B \subset D$ implies $f(w) \geq \alpha$, a contradiction since $f(w) = \beta < \alpha$. Therefore $A = B$.

We now develop the theory for the Krein-Milman theorem where the convexity structure and topology involved are related to lines instead of segments, which yields added properties sufficient to develop a general hyperplane.

5.28. Definition. Consider a set X and a family $\mathcal{L}$ of subsets called lines. We assume every line has a given total ordering. If $x,y \in L \in \mathcal{L}$, $x \neq y$ let (x,y) denote the set of points on L strictly between x and y. Similarly define $[x,y] = (x,y) \cup \{x,y\}$, $[x,y), = (xy) \cup \{x\}$ and $(x,y] = (x,y) \cup \{y\}$. The pair $(X,\mathcal{L})$ is called a generalized linear space if $\mathcal{L}$ satisfies the following axioms:

a) Every line is order isomorphic to the real numbers.

b) Each pair of points in X belongs to a unique member of $\mathcal{L}$.

c) If $x,y,z \in X$, $u \in (x,y)$, and $v \in (u,z)$ then there is $w \in (y,z)$ such that $v \in (x,w)$.

The family $\mathcal{L}$ determines a convexity structure $(X, \mathcal{C})$ where

$\mathcal{C} = \{C: x,y \in C \text{ implies } [x,y] \subset C\}$.

5.29. Definition. If $C \subset X$ then lin $C = \{x \in X: \text{there is } y \in C \text{ with } [y,x) \subset C\}$

and <u>core</u> $C = \{x \in C\colon$ for each $y \neq x$ there is $z \in (x,y)$ such that $(x,z) \subset C\}$.

<u>5.30. Definition</u>. A subset $F \subset X$ is called a <u>flat</u> if $x,y \in F$ implies $L(x,y) \subset F$, where $L(x,y)$ is the line in $\mathcal{L}$ which, by axiom (b), is determined by x and y. The <u>affine hull</u> of a set $S \subset X$ is defined as $\mathrm{fl}(S) \equiv \bigcap\{F\colon F \supset S, F$ a flat$\}$. A flat H <u>separates</u> X <u>into</u> (A,B) if $X \sim H = A \cup B$ with $A,B \in \mathcal{C}$, where A and B are nonempty disjoint and $x \in A$, $y \in B$ implies $(x,y) \cap H \neq \phi$. A flat H is a <u>hyperplane</u> iff it separates X. A flat is of <u>deficiency</u> 1 if there exists $x \notin H$ such that $\mathrm{fl}(x,H) = X$.

Cantwell [2], who has worked extensively with generalized linear spaces, has proved the following results:

<u>5.31. Lemma</u>. H is a hyperplane iff it is of deficiency 1, and further, a hyperplane is a maximal proper flat.

<u>5.32. Lemma</u>. If $A,B \in \mathcal{C}$ and $A \cap B = \phi$ there exist complementary $\mathcal{C}$-half spaces C,D such that $A \subset C$ and $B \subset D$.

<u>5.33. Lemma</u>. If C and D are complementary $\mathcal{C}$ half-spaces, $H = \mathrm{lin}\, C \cap \mathrm{lin}\, D$, and $H \neq X$, then H is a hyperplane.

<u>5.34. Definition</u>. The space X is said to be <u>linearly decomposable</u> if for any hyperplane $H \subset X$ there exist a line L and a family of hyperplanes $\{H_x\colon x \in L\}$ such that

(a) The family of sets $\{H_x\colon x \in L\}$ partitions X.

(b) $H_x \cap L = \{x\}$.

(c) For some $x \in L$, $H_x = H$.

<u>5.35. Theorem</u>. If X is linearly decomposable then for each hyperplane H there exists a convex and preconvex function f from $(X, \mathcal{C})$ onto the reals, where $\mathcal{C}$ is the convexity structure determined by $\mathcal{L}$. Further there is $\alpha \in R$ such that $H = \{x : f(x) = \alpha\}$.

Proof. Let L and $\{H_x : x \in L\}$ be the line and family of hyperplanes partitioning X as determined by H. Define $f: X \to L$ by $f(x) = y$ iff $x \in H_y$. Let $C \in \mathcal{C}$ and $z \in [x,y]$, where $x,y \in f(C)$. Suppose $u,v \in C$ and $f(u) = x$, $f(v) = y$; then $[u,v] \subset C$. Suppose $H_z \cap [u,v] = \phi$. Since H_z is a hyperplane it separates X into (A,B), and if $u \in A$, $v \in B$ then $(u,v) \cap H_z \neq \phi$, which is not the case. Therefore suppose $u,v \in A$. Then $[u,v] \subset A$. If $x \in B$ then $[u_o x] \cap H_z \neq \phi$, but $[u,x] \subset H_x$ and $H_x \cap H_z = \phi$, a contradiction. Therefore, $x \in A$. Similarly $y \in A$, so $[x,y] \subset A$ implies $H_z \cap [x,y] \neq \phi$, a contradiction. Hence we conclude $H_z \cap [u,v] = \{p\}$, so that $f(p) = z$. Hence $z \in f(C)$ and $f(C)$ is convex.

Similar reasoning shows that f is preconvex. Since L is order isomorphic to the reals, the existence of the desired function is now apparent.

5.36. Remark. Since $\{H_x : x \in L\}$ partitions X, $f^{-1}(x) = H_x$ for any $x \in L$. Hence the preimage of a point is a hyperplane, and each hyperplane is translatable.

5.37. Definition. If $(X, \mathcal{L})$ is a generalized linear space, then $(X, \mathcal{L}, \mathcal{T})$ is said to be a generalized linear topological space (G.L.T.S.) if:

(a) $\mathcal{T}$ is a first countable Hausdorff topology for X.

(b) If $x \in U \in \mathcal{T}$ then $x \in$ core U.

(c) Whenever $\{x_n\}$, $\{y_n\}$ are sequences which converge to x and y respectively, then $L(s_n, y_n) \to L(x,y)$ (Hausdorff limit understood).

5.38. Theorem. If F is a flat then cl F is a flat.

Proof. If $x,y \in$ cl F let $\{x_n\},\{y_n\}$ be sequences in F converging to x and y respectively; then $L(x_n, y_n) \to L(x,y)$. Since F is a flat

$L(x_n, y_n) \subset F$, so $L(x,y) \subset \mathrm{cl}\ F$.

5.39. Theorem. If H is a hyperplane, then either $\mathrm{cl}\ H = H$ or $\mathrm{cl}\ H = X$.

Proof. Lemma 5.38 implies that $\mathrm{cl}\ H$ is a flat. From Lemma 5.31 a hyperplane is a maximal proper flat, hence $\mathrm{cl}\ H = H$ or $\mathrm{cl}\ H = X$.

5.40. Theorem. If $C \in \mathcal{C}$ then $\mathrm{lin}\ C \subset \mathrm{cl}\ C$.

Proof. Let $x \in \mathrm{lin}\ C$; then there is $y \in C$ such that $[y,x) \subset C$. If U is a neighborhood of x since $(X, \mathcal{L}, \mathcal{J})$ is a G.L.T.S. there is $z \in (y,x)$ such that $(z,x) \subset U$. Therefore $U \cap C \neq \phi$.

5.41. Theorem (Krein-Milman) Suppose $(X, \mathcal{L}, \mathcal{J})$ is a locally convex, linearly decomposable G.L.T.S. then if $A \in \mathcal{C}$ is compact

$$A = \mathrm{cl}(H(\mathrm{ext}\ A)).$$

The following lemma is proved using the hypotheses of Theorem 5.41.

5.42. Lemma. If U and V are disjoint open convex sets there exists a hyperplane $H = \{x : f(x) = \alpha\}$ such that $f(U) < \alpha < f(V)$, and further, f is continuous.

Proof. By Lemma 5.32 there exists complementary $\mathcal{C}$-half spaces C and D such that $U \subset C$ and $V \subset D$. Let $H \equiv \mathrm{lin}\ C \cap \mathrm{lin}\ D$; then by Theorem 5.40, $H \subset \mathrm{cl}\ C \cap \mathrm{cl}\ D \neq X$. By Lemma 5.33 H is a hyperplane. Now $\mathrm{cl}\ H \subset \mathrm{cl}\ C \cap \mathrm{cl}\ D \neq X$, so by Theorem 5.39 $\mathrm{cl}\ H = H$ and H is closed. By Theorem 5.35 there is a convex and preconvex function f from X to R such that $H = \{x : f(x) = \alpha\}$. As in Theorem 5.26 it can be shown that $f(U) < \alpha < f(V)$. If $\beta < \alpha$ then $f^{-1}(\beta) \cap V = \phi$. From the construction of f, $f^{-1}(\beta)$ is a hyperplane, so $f^{-1}(\beta) \cap V = \phi$ implies $f^{-1}(\beta)$ is closed. Similarly, if $\beta > \alpha$ then $f^{-1}(\beta)$ is closed. Therefore, by Theorem 4.16 f is continuous.

We now have sufficient power to prove Theorem 5.41 by a method parallel to the proof of Theorem 5.24. Thus, we find that the class of linearly decomposable generalized linear topological spaces is a class of spaces in which a Krein-Milman type theorem holds.

We remark that in the presence of local convexity it can be shown that a generalized linear topological space $(X,\mathcal{L},\mathcal{J})$ is a topological convexity space, $(X,\mathcal{C},\mathcal{J})$, where $\mathcal{C}$ is the convexity structure determined by $\mathcal{L}$. The rather lengthy proof involves the following property: If $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences converging to x, and z respectively, where for each n, x_n, y_n and z_n lie on some line L_n and $y \in (x,z)$ then for all n sufficiently large $y_n \in (x_n, z_n)$.

5.43. Example. Let $X = R^2$ and $\mathcal{J}$ the usual topology on R^2. Let $\mathcal{L}$ consist of:

1) all ordinary Euclidean lines of zero, negative or infinite slope

2) all broken lines of positive slope consisting of two half-lines meeting at the x axis with the slope of the upper half-line twice the slope of the lower half-line.

One can easily show that $(X,\mathcal{J})$ is not equivalent to an open subset of R^2, since this is the classic example of a non-Desarguesian affine plane.

This construction can be applied to any finite dimension, with the usual topology of R^n.

More generally, Cantwell's axiomatic system in finite dimensions satisfies the linear decomposition property, and so provides a class of spaces more general than the classical setting for which the analogue of the Krein-Milman theorem holds.

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