MAHLER MEASURE AND ITS BEHAVIOR UNDER ITERATION

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Abstract: For an algebraic number α we denote by $M(\alpha)$ the Mahler measure of α. Mahler measure is a height function on polynomials with integer coefficients. Moreover, as $M(\alpha)$ is again an algebraic number (indeed, an algebraic integer), $M(\cdot)$ is a self-map on $\mathbb{A}$(Sometimes denoted $\overline{\mathbb{Q}}$), and therefore defines a dynamical system. The *orbit size* of α, denoted $\#\mathcal{O}_M(\alpha)$, is the cardinality of the forward orbit of α under M. In this thesis, we will start by introducing the background of Mahler measure as a height and a dynamical system, we will review previous results on the orbit sizes of lower degree algebraic integers and lower degree number fields, then we discuss results on the orbit sizes of algebraic integers with degrees at least 3 and non-unit norm. After that, we will turn our focus to the behavior of algebraic units, which are of interest in Lehmer's problem. We will prove the results regarding algebraic units of degree 4 and discuss that if α is an algebraic unit of degree $d \geq 5$ such that the Galois group of the Galois closure of $\mathbb{Q}(\alpha)$ contains A_d, then the orbit size must be 1, 2 or ∞. Furthermore, we will show that there exist units with orbit sizes larger than 2. We will also show a few experimental results on the behavior of $(\log(M^n(\alpha)))$. In chapter five, we will prove partial results on the classification of number fields based on the existence of wandering point.

TABLE OF CONTENTS

Chapter Page

I. INTRODUCTION . . . 1

1.1 Background . . . 1

1.2 Mahler measure, Lehmer's problem and partial results . . . 4

1.2.1 Lehmer's Problem . . . 7

1.2.2 Partial results towards Lehmer's problem . . . 8

1.3 Dynamical systems, the Northcott theorem . . . 12

1.4 Properties of M as a dynamical system . . . 16

1.5 Main results and conjectures . . . 18

II. LOWER DEGREE CASES . . . 21

2.1 Orbit sizes of lower degree algebraic integers . . . 21

2.1.1 Degree 1 . . . 21

2.1.2 Degree 2 . . . 21

2.1.3 Degree 3 . . . 22

2.2 Number fields with degree less than 4 . . . 24

2.3 An example of degree 4 unit . . . 24

III. NON-UNITS ALGEBRAIC INTEGERS . . . 26

3.1 A few isolated cases . . . 26

3.2 A generalization of Dubickas's result on non-units . . . 29

IV. UNITS OF HIGHER DEGREES . . . 36

4.1 Orbit sizes of degree 4 units . . . 36

Chapter Page

4.1.1 The case $M^{(2)}(\alpha) = \pm\frac{\alpha_1}{\alpha_4}$ 39

4.1.2 The case $M^{(2)}(\alpha) = \pm\frac{\alpha_1}{\alpha_3}$ 40

4.1.3 The case $M^{(2)}(\alpha) = \pm\frac{\alpha_2}{\alpha_4}$ 40

4.2 Units with higher degree with restrictions 41

4.3 The existence of units with orbit sizes greater than 2 49

V. CLASSIFICATION OF NUMBER FIELDS BY ORBIT SIZE 51

5.1 Fields of degree four 51

5.1.1 Totally imaginary extensions 51

5.1.2 Biquadratic extensions 52

5.1.3 Totally real extension of degree 4 55

5.2 Fields of degree five 58

5.3 Abelian extensions 65

5.3.1 Multiquadratic fields 65

5.3.2 Abelian extensions of degree six 68

5.3.3 Abelian extensions of degree nine 69

5.3.4 Cyclic extensions of odd degree ≥ 5 70

5.3.5 Classification of Abelian extensions 82

REFERENCES 84

LIST OF TABLES

Table | Page

1 Classification of Abelian extensions 83

LIST OF FIGURES

Figure		Page
1	Roots of $P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.	8

CHAPTER I

INTRODUCTION

1.1 Background

In his 1933 paper [16], Lehmer tried to find large primes among the prime factors of Pierce numbers $\Delta_n = \prod_{i=1}^{d}(\alpha_i^n - 1)$, where the α_i's are the Galois conjugates of an algebraic integer α. Since $\lim_{n\to\infty} |\Delta_{n+1}/\Delta_n|$ is equal to the Mahler measure of α, denoted $M(\alpha)$, Lehmer argued that the Mahler measure can be used to measure the rate of increase of the Pierce numbers. Moreover, since it helped his search to choose algebraic integers such that Δ_n does not increase rapidly, Lehmer wanted to find Mahler measures that are very close to 1.

Inspired by this, D.H. Lehmer asked in 1933 if the Mahler measure for an algebraic number which is not a root of unity can be arbitrarily close to 1. This question became known as *Lehmer's problem*. It is often suggested that the minimal value of Mahler measure that is greater than 1 is a Salem number, namely $\tau = 1.17\ldots$, which is the largest real root of the polynomial $f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, discovered by Lehmer in his paper [16].

Although there has been much computational work performed in order to find irreducible polynomials of small Mahler measure, remarkably, no polynomial of smaller nontrivial Mahler measure has been found since Lehmer's original 1933 work. Since that time, the best asymptotic bound towards Lehmer's problem was discovered by Dobrowolski [9]. It is clear that in considering the problem, one can reduce to considering the Mahler measure of algebraic units. Smyth [29] found a lower bound for Mahler measures of non-reciprocal units. In another direction, Borwein, Dobrowolski and Mossinghoff proved the Lehmer conjecture for polynomials with only odd coefficients [3].

One direction we can take to explore Lehmer's problem is to investigate Mahler measure as a dynamical system. The study of iteration of the Mahler measure began with questions about which algebraic numbers are themselves Mahler measures. Adler and Marcus [1] proved that every Mahler measure is a Perron number and asked if the Perron numbers given by the positive roots of $x^n - x - 1$ are also values of the Mahler measure for any $n > 3$. Recall that α is a Perron number if and only if $\alpha > 1$ is a real algebraic integer such that all conjugates of α over $\mathbb{Q}$ have absolute value $< \alpha$. This notion of 'Perron number' was introduced by Lind [17] who also proved several properties of the class of Perron numbers in [18], including that they are closed under addition and multiplication and are dense in the real interval $[1, \infty)$. Boyd [4] proved that the positive roots of $x^n - x - 1$ for $n > 3$ were not values of the Mahler measure, but Dubickas [11] showed that for every Perron number β, there exists a natural number n such that $n\beta$ is a value of the Mahler measure. Dixon and Dubickas [7] and Dubickas [13] established further results on which numbers are in the value set of M. However, the question whether a given number is a Mahler measure of an algebraic number is very hard to answer in general. For instance, it is an open question of Schinzel [24] whether or not $\sqrt{17} + 1$ is the Mahler measure of an algebraic number.

Dubickas [10] appears to have been the first to pose questions on the Mahler measure as a dynamical system, introducing the concept of the *stopping time* of an algebraic number under M, defined as the number of iterations required to reach a fixed point. We note that the stopping time is one less than the cardinality of the forward orbit of the number under iteration of M, which we will call the *orbit size*. Specifically, we set $M^{(0)}(\alpha) = \alpha$ and let $M^{(n)}(\alpha) = M \circ \cdots \circ M(\alpha)$ denote the nth iteration of M. We define the *orbit of α under M* to be the set:

$$\mathcal{O}_M(\alpha) = \{M^n(\alpha) : n \geq 0\}. \tag{1.1.1}$$

Then the orbit size of α is $\#\mathcal{O}_M(\alpha)$, while the stopping time is $\#\mathcal{O}_M(\alpha) - 1$. It is easy to see that for any algebraic number α, $M(\alpha) \leq M^{(2)}(\alpha)$, so M is nondecreasing after at least one iteration, and thus, the Mahler measure either grows, or is fixed.

In fact, by Northcott's theorem, it is easy to see that if α is a wandering point of M, then $M^{(n)}(\alpha) \to \infty$, as the degree of $M^{(n)}(\alpha)$ can never be larger than the degree of the Galois closure of the field $\mathbb{Q}(\alpha)$. In particular, there are no cycles of length greater than 1; each number α either wanders (that is, the orbit under M is infinite), or it is preperiodic and ends in a fixed point of M. Dubickas claimed in [10] that 'generically' $M^{(n)}(\alpha) \to \infty$, however, he did not give an example or a proof of this. In my master's thesis, I showed that if $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 3$, then $\#\mathcal{O}_M(\alpha) < \infty$, and we will present an algebraic number α of degree 4 with minimal polynomial $x^4 + 5x^2 + x - 1$ such that $M^{(2n)}(\alpha) = M^{(2)}(\alpha)^{2^{n-1}}$, proving that $M^{(n)}(\alpha) \to \infty$ for this example.

Further, it is trivial to see that the fixed points of M correspond to natural numbers, Pisot-Vijayaraghavan numbers, and Salem numbers. This raises several natural questions: for example, can one show that the Lehmer problem could be reduced to the study of fixed points of M? The answer to such a question might help establish the long held folklore conjecture that Salem numbers are indeed minimal for Lehmer's problem. The fixed points for the dynamical system induced by the multiplicative Weil-height have recently been classified by Dill [6].

Dubickas posed several questions in [10], including whether one could classify all numbers of stopping time 1 (that is, numbers which are not fixed by M, but for which $M(\alpha)$ is fixed), and whether algebraic numbers of arbitrary stopping time existed. In a later paper [11], he established, among other things, that for every $k \in \mathbb{N}$, there exists a cubic algebraic integer of norm 2 with stopping time k.

After the publishing of the paper by Paul Fili, Lukas Pottmeyer and me [15], Dubickas posed the question that whether one could find all number fields K that do not contain algebraic numbers with infinite orbit(private communication). It is clear that such fields include all numbers fields K satisfying $[K : Q] \leq 3$, but are there any other K with this property? It appears that if the corresponding Galois group is big enough, then there will be a wandering unit in the extension, but what if the Galois group is small? We will explore these questions and give partial results in chapter 5.

Part of this thesis is taken from the paper by Paul Fili, Lukas Pottmeyer and me [15], and my master's thesis. In particular, part of Chapter I is from [15] and my master's thesis; Part of Chapter II is from my master's thesis; Chapter III is from [15]; Part of Chapter IV is from [15].

1.2 Mahler measure, Lehmer's problem and partial results

The Mahler measure was defined by Kurt Mahler [19] in 1962, but appeared earlier in a paper of Lehmer [16] in an alternative form.

Definition 1 *Let* $P(z) = a_0 z^d + ... + a_d = a_0 \prod_{i=1}^{d} (z - \alpha_i)$ *be a non constant polynomial with integer cofficients. The* Mahler measure of P *is defined to be*

$$M(P) = \exp\left(\int_0^1 \log |P(e^{2\pi i t})| \, dt\right)$$

which is the geometric mean of $|P(z)|$ *for* z *on the unit circle. We refer to* $m(P) = \log M(P)$ *as the* logarithmic Mahler measure.

We first give a simple method to compute the Mahler measure, based on Jensen's formula:

Lemma 1.2.1 *We have*

$$M(P) = |a_0| \prod_{i=1}^{d} \max\{1, |\alpha_i|\} = |a_0| \prod_{|\alpha_i| \geq 1} |\alpha_i| = \pm a_0 \prod_{\substack{i=1 \\ |\alpha_i|>1}}^{d} \alpha_i,$$

where $\alpha_1, \ldots, \alpha_n$ *are the roots of* $P(z)$*, that is,* $P(z) = a_0 \prod_{i=1}^{d} (z - \alpha_i) \in \mathbb{Z}[x]$.

Proof. First we will show that we can assume that $P(0) \neq 0$ so that we can apply Jensen's formula. Since $P(z) = a_0 \prod_{i=1}^{d} (z - \alpha_i)$, notice that if one of the roots $\alpha_i = 0$, then $P(z) = a_0 (z)(z - \beta_1)(z - \beta_2) ... (z - \beta_{d-1})$, where β_i are the rest of $d - 1$ roots.

Define $P_1(z) = a_0(z-\beta_1)(z-\beta_2)\ldots(z-\beta_{d-1})$. observe that $M(P) = M(P_1)$ by either of the definitions. So we can avoid all the cases of $P(0) = 0$. Now P is an analytic function on the complex plane, $P(0) \neq 0$, so by Jensen's Formula,

$$\log|P(0)| = \log|a_0\alpha_1\cdots\alpha_d| = \sum_{|\alpha_i|<1}\log|\alpha_i| + \int_0^1 \log|P(e^{2\pi it})|\,dt$$

Thus,

$$\begin{aligned}
m(P) &= \log\left(\exp\left(\int_0^1 \log\left|P\left(e^{2\pi it}\right)\right|dt\right)\right)\\
&= \log|a_0\alpha_1\ldots\alpha_d| - \sum_{|\alpha_i|<1}\log|\alpha_i|\\
&= \log|a_0| + \log|\alpha_1| + \ldots \log|\alpha_d| - \sum_{|\alpha_i|<1}\log|\alpha_i|\\
&= \log|a_0| + \sum_{|\alpha_i|\geq 1}\log|\alpha_i|\\
&= \log\left(|a_0|\prod_{|\alpha_i|\geq 1}|\alpha_i|\right).
\end{aligned}$$

Since log is one-to-one on $\mathbb{R}$,

$$\exp\left(\int_0^1 \log|P(e^{2\pi it})|\,dt\right) = |a_0|\prod_{|\alpha_i|\geq 1}|\alpha_i| = \pm a_0\prod_{\substack{i=1\\|\alpha_i|>1}}^{d}\alpha_i.$$

The last equality is because, if α_i with $|\alpha_i| > 1$ is non-real, then some $\alpha_j = \overline{\alpha_i}$ is its conjugate. Note that this gives $|\alpha_j| = |\alpha_i| > 1$, and $\alpha_i\alpha_j$ is a real number. Therefore, the absolute value signs can be dropped. ■

Definition 2 *If $\alpha \in \mathbb{C}$ is a root of a polynomial $f(x) = a_dz^d + a_{d-1}z^{d-1} + \ldots + a_0 \in \mathbb{Z}[z]$, then α is called an* algebraic number.

We let $\mathbb{A} \subset \mathbb{C}$ denote the set of all algebraic numbers. It is well-known that $\mathbb{A}$ forms a subfield of $\mathbb{C}$.

Definition 3 *For $\alpha \in \mathbb{A}$ we define $M(\alpha)$ to be the Mahler measure of the minimal polynomial P_α of α, that is, where P_α is a generator (unique up to sign) of the ideal:*

$$P_\alpha = \{f(x) \in \mathbb{Z}[x] : f(\alpha) = 0\} \subset \mathbb{Z}[x].$$

We note in passing that $\deg P_\alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Note that by construction, $M(\sigma\alpha) = M(\alpha)$ for any σ in the Galois group of the Galois closure of $\mathbb{Q}(\alpha)$.

Observe that, with the convention of Definition 3, M in fact defines a function

$$M : \mathbb{A} \to \mathbb{A}$$

In addition, the Mahler measure is actually a height function on polynomials with integer coefficients because there are only a finite number of such polynomials of bounded degree and bounded Mahler measure. In fact, by Mahler's formula, the Weil height is related to Mahler measure by the following equation:

$$h(\alpha) = \frac{1}{\deg P_\alpha} \log M(\alpha)$$

The study of heights has led to the introduction of potential theoretic techniques in number theory and led to the resolution of classical problems like the Bogomolov conjecture, and may help with many more related questions.

In this paper we explore the behavior of the Mahler measure as a dynamical system on the set of algebraic numbers $\mathbb{A}$. The inspiration for this study comes from the observation, proven in Theorem 17, that the fixed points of the Mahler measure contain a class of algebraic numbers which in light of experimental evidence(we refer the reader to M. Mossinghoff's website [20] for the latest tables of known polynomials, as well as the papers by Mossinghoff [21] and Mossinghoff, Rhin, and Wu [22]) are widely believed to be minimal for the Mahler measure, namely, Salem numbers. Inspired by the analogy with the definition of the canonical height for rational maps, one might hope that a method might be found for bounding the Mahler measure of points which are wandering. This result, together with a better understanding of preperiodic points, may one day lead to a reduction of the Lehmer problem to the class of Salem numbers. We will now introduce the basic background regarding Lehmer's problem which motivates this study. In Section 1.3 we will introduce dynamical systems and the particular questions which we will address in Chapter II. Finally in Section 1.5 we will state the main results of this thesis and the conjectures formed from our study of the subject.

Lehmer sought large primes amongst the so-called *Pierce numbers*, given by

$$\Delta_n(F) = \prod_{i=1}^{d} \left(\alpha_i^n - 1\right),$$

where α_i are the roots of a monic integral polynomial $F(x)$. He proved that $\Delta_n(F)$ is more likely to produce primes if it does not grow too rapidly, and measured the rate of growth by $\frac{\Delta_{n+1}(F)}{\Delta_n(F)}$, which is where the Mahler measure comes into play.

Lemma 1.2.2 *For a monic $F \in \mathbb{Z}[x]$ with no roots on the unit circle,*

$$\lim_{n\to\infty} \left| \frac{\Delta_{n+1}(F)}{\Delta_n(F)} \right| = \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

Proof. We can treat each term in the product separately:

$$\lim_{n\to\infty} \left| \frac{\alpha_i^{n+1} - 1}{\alpha_i^n - 1} \right| = \begin{cases} |\alpha_i| & \text{if } |\alpha_i| > 1 \\ 1 & \text{if } |\alpha_i| < 1 \end{cases}$$

The case when $|\alpha_i| = 1$ is excluded by the assumption that F has no roots on the unit circle. ■

It follows that for such $F \in \mathbb{Z}[x]$,

$$\lim_{n\to\infty} \left| \frac{\Delta_{n+1}(F)}{\Delta_n(F)} \right| = M(F),$$

so that Lehmer's search for prime numbers led naturally to the question of finding monic polynomials with integer coefficients with Mahler measure close to (but not equal to) 1. This question has since become known as *Lehmer's problem.*

1.2.1 Lehmer's Problem

Among those monic integer coefficients polynomials with $M(P) > 1$, could the polynomials be chosen with $M(P)$ arbitrarily close to 1? Today it is widely believed that this is impossible and that the values are bounded away from 1. This statement is commonly called *Lehmer's conjecture*. The smallest known value of $M(P) > 1$ was actually found by Lehmer in his 1933 paper [16] and is $M(P) \approx 1.17638...$, where $P(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$.

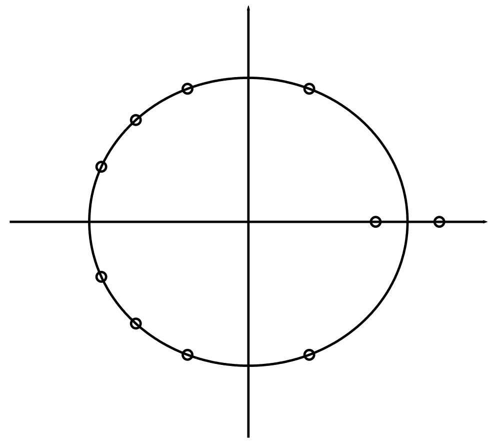

Figure 1: Roots of $P(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$.

As shown in Figure 1, among the roots of $P(z)$, 8 of the 10 lie on the unit circle, which is very special. Lehmer's conjecture seems to be a very deep problem and remains unsolved till now, but there are versions of this problem for certain classes of polynomials have been solved during these years.

1.2.2 Partial results towards Lehmer's problem

There are several partial results towards Lehmer's conjecture. Schinzel's theorem gives the lower bound of the Mahler measure of polynomials with all real roots. Smyth proved that Lehmer's conjecture is true for all polynomials that are not reciprocal, and in 1979, Dobrowolski gave essentially the best known unconditional result towards the conjecture. We will now review these results.

Now, if our setting is for polynomials with integer coefficients, then $|a_0| \geq 1$, so $M(P) \geq 1$. In Lehmer's conjecture, we only care about the case when $M(P) > 1$, so when does $M(P) = 1$? The answer is when $P(z)$ is a power of z times a product of cyclotomic polynomial, which will follow from a lemma of Kronecker:

Lemma 1.2.3 (Kronecker's lemma) *Suppose $\alpha_1 \neq 0$ is an algebraic integer $|\alpha_1| \leq 1$ and the algebraic conjugates $\alpha_2, \alpha_3, ...\alpha_{d-1}$ of α all have modulus $|\alpha_i| \leq 1$ then α_1 is a root of unity.*

Proof. Consider the polynomial $P_n(X) = \prod_{i=1}^{d}(X - \alpha_i^n)$, then P_1 is the minimal polynomial for α_1.

The coefficients of P_n are elementary symmetric functions of the nth powers of the roots $\alpha_1, \alpha_2, ...\alpha_d$, so they are rational integers. Each of the coefficients is uniformly bounded for every n by a combinatorial constant depending only on the degree d, since $|\alpha_i| \leq 1$ for all $1 \leq i \leq d$. So the collection of polynomials $\{P_n\}_{n\in\mathbb{N}}$ is finite. Hence there exist positive integers $n_1 < n_2$ such that $P_{n_1} = P_{n_2}$. Therefore

$$\{\alpha_1^{n_1}, \alpha_2^{n_1}, ..., \alpha_d^{n_1}\} = \{\alpha_1^{n_2}, \alpha_2^{n_2}, ..., \alpha_d^{n_2}\}.$$

Consider the permutation group S_d, define $\sigma \in S_d$ to be the action on the set of roots such that

$$\alpha_i^{n_1} = \alpha_{\sigma(i)}^{n_2}.$$

Which gives

$$(\alpha_i^{n_1})^{n_1} = (\alpha_{\sigma(i)}^{n_1})^{n_2}.$$

But since

$$\alpha_{\sigma(i)}^{n_1} = \alpha_{\sigma^2(i)}^{n_2}$$

$$\alpha_i^{n_1^2} = \alpha_{\sigma^2(i)}^{n_2^2}$$

Similarly,

$$\alpha_i^{n_1^3} = \alpha_{\sigma^3(i)}^{n_2^3}$$

$$\vdots$$

So if σ has order r in S_d,

$$\alpha_i^{n_1^r} = \alpha_i^{n_2^r}$$

and therefore $\alpha_i^{n_1^r}\left(\alpha_i^{n_2^r - n_1^r} - 1\right) = 0$, since $\alpha_i \neq 0$, So $\alpha_i^{n_2^r - n_1^r} = 1$, α_i is a root of unity. ∎

We now classify when the Mahler measure is trivial:

Theorem 4 *Suppose $F \in \mathbb{Z}[x]$ is non-zero, non-constant, and that the coefficients of F have no common factor other than 1. Then $M(F) = 1$ if and only if all the zeroes of F are roots of unity or 0. In particular, $M(F) = 1$ if and only if there exist integers $r \geq 1$, $t \in \{0, 1, ...\}$ and cyclotomic polynomials $c_1, \ldots, c_r$ such that $f(z) = \pm z^t$ or $f(z) = \pm z^t \prod_{i=1}^{r} c_i(z)$.*

Proof. Suppose all the zeros of F are roots of unity or 0, then F has a factor x^t where $t \in \{0, 1....\}$, and $F \left| x^t \left(x^N - 1 \right) \right.$ for some $N \geq 1$. Hence the leading coefficient of F must be ± 1, so it follows that $M(F) = 1$ by definition.

Now suppose that $M(F) = 1$. It follows that F must be a polynomial with leading coefficient ± 1, so all of the zeroes are algebraic integers, and satisfy $|\alpha_i| \leq 1$ for all $1 \leq i \leq n$. By the previous lemma, all the roots of F are roots of unity or zero.

For the second part of the theorem, we know from the argument above that if $M(F) = 1$, then all zeroes of f are roots of unity or 0. So the leading coefficient of f must be ± 1, and F has a factor z^t where $t \in \{0, 1, ...\}$. Now if α is a root of unity, then an irreducible polynomial $c(z)$ is a factor of f for which α is a root, and $c(z)$ is a cyclotomic polynomial. This is because all the roots of a non-zero irreducible integral polynomial are Galois conjugates, since $\alpha^n = 1$ then $\sigma^n(\alpha) = \sigma(\alpha^n) = \sigma(1) = 1$. So all the conjugates (the other roots of the irreducible polynomial) are roots of unity. A irreducible integral polynomial with roots that are all roots of unity is a cyclotomic polynomial.

Hence each root of f (that is root of unity) will fix a irreducible factor of f, which is a cyclotomic polynomial, and the product of these cyclotomic polynomials is a factor of f, the only other factor is $\pm z^t \, (t \in \{0, 1....\})$. The converse direction is immediate from the definition of M. ■

We now state some of the partial results towards Lehmer's conjecture that motivate our study.

Theorem 5 (Schinzel [25]) *Suppose that $F \in Z[X]$ is monic with degree d, $F(-1)\,F(1) \neq$*

0 *and* $F(0) = \pm 1$. *If the zeroes of* F *are all real then* $M(F) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}$ *with equality if and only if* F *is a product of a power of* $x^2 - x - 1$ *and a power of* $1 - x - x^2$.

Before introducing Smyth's theorem, we need the definition of reciprocal polynomials.

Definition 6 *Suppose* $F \in \mathbb{C}[x]$ *has degree* d*; write* $F^*(x) = x^d F(x^{-1})$. *Then* F *is reciprocal if* $F = F^*$, *and is non-reciprocal otherwise.*

As an example, the polynomial $x^2 - x + 1$ is reciprocal. On the hand, if we let

$$f(x) = x^3 - x - 1.$$

Then

$$x^3 \left(f\left(x^{-1}\right)\right) = \left(x^{-3} - x^{-1} - 1\right) x^3 = 1 - x^2 - x^3$$

So $f(x)$ is not reciprocal.

Theorem 7 (Smyth [29]) *If* $F(x) \in \mathbb{Z}[x]$ *is a non-reciprocal polynomial, and* $F(0) F(1) \neq 0$ *then* $m(F) \geq m(x^3 - x - 1) = \log(1.324\ldots) = 0.281\ldots$

Note that $F(0) \neq 0 \Rightarrow F$ is not divisible by $x - 1$. If F is reciprocal , then given F is of degree d,

$$x^d \left(F\left(x^{-1}\right)\right) = F(x)$$

Now define $G(x) = F(x) F(x-1)$ then G is of degree $(d+1)$.

$$\begin{aligned} x^{d+1}\left(G\left(x^{-1}\right)\right) &= x^{d+1} F\left(x^{-1}\right)\left(x^{-1} - 1\right) \\ &= x^d F\left(x^{-1}\right) - x^{d+1} F\left(x^{-1}\right) \\ &= F(x) - x F(x) \\ &= F(x)(1 - x) \\ &= -G(x) \end{aligned}$$

So $G(x)$ is not reciprocal. Hence condition about divisibility by $x - 1$ is required.

Lastly, we state a result of Dobrowolski, which gives essentially the best known asymptotic lower bound.

Theorem 8 (Dobrowolski [8]) *Let α be algebraic number of degree d, then for $d \geq 2$*

$$M(\alpha) > 1 + \frac{1}{1200}\left(\frac{\log\log d}{\log d}\right)^3.$$

We note that the constant of 1/1200 in the above theorem has since been improved to 1/4 by Voutier [30].

Classical results for Mahler measure and partial results towards Lehmer's problem are extensively surveyed in Smyth's paper [27].

1.3 Dynamical systems, the Northcott theorem

We mentioned at the beginning that the Mahler measure function maps algebraic numbers into itself. This is a specific example of a *dynamical system*:

Definition 9 *For a set X, let f be a function which maps X to itself, that is, $f : X \to X$. We call the pair (X, f) a* dynamical system*.*

Loosely speaking, dynamics refers to the study of the behavior of the points in X under iteration of the map f. We write

$$f^n = \underbrace{f \circ f \circ f \circ ...f}_{\text{n iterations}}$$

A primary goal in the study of dynamic is to classify the points of X by the behavior of their orbits, $O_f(\alpha)$, where $O_f(\alpha) = \{\alpha, f(\alpha), f^2(\alpha), f^3(\alpha), \ldots\}$.

We will give some definitions to classify what the orbit of a point in X looks like under iteration of f.

Definition 10 *A point $x \in X$ is called a* periodic point *for the dynamical system (X, f) if there exists $n > 0$ such that $f^n(x) = f(f(\cdots f(x)) = x$.*

Definition 11 *A point $x \in X$ is called a* preperiodic point *for the dynamical system (X, f) if there exists $n > m \geq 0$ such that $f^n(x) = f^m(x)$.*

We note that a number is preperiodic if and only if the orbit is finite:

Lemma 1.3.1 *Let (X, f) be a dynamical system, $x \in X$. Then the orbit $O_f(x)$ is finite if and only if x is preperiodic.*

Proof. If x is preperiodic, then $f^n(x) = f^m(x)$ for some $n > m > 0$. If $n - m = d$, $f^{m+1}(x) = f^{n+1}(x), f^{m+2}(x) = f^{n+2}(x), \cdots f^n(x) = f^{n+d}(x) = f^m(x)$ and as the iteration continue, the numbers repeat, so the orbit $O_f(x)$ has no more than $m + d$ elements.
If $O_f(x)$ is finite, then, for some $i \geq 1$, $f^i(x) = f^j(x)$ for some $j < i$, otherwise we will get infinitely many different outputs for the iteration, and $O_f(x)$ cannot be finite. ■

Definition 12 *A point $x \in X$ is a* wandering point *for a dynamical system (X, f) if it is not preperiodic (equivalently, if the orbit is infinite).*

The most basic questions we might ask about a dynamical system are the following:

1. What points are fixed by f?

2. What points are periodic?

3. What points are preperiodic?

Note that fixed points are periodic (of period $n = 1$), and periodic points are also preperiodic, but that none of the converses necessarily hold.

We wish to study $M : \mathbb{A} \to \mathbb{A}$ as a dynamical system. To study problem (1), we begin by prove that certain algebraic numbers are fixed points for M.

Proposition 13 *Every $n \in \mathbb{N} \subset \mathbb{A}$ is a fixed point of M.*

Proof. The minimal polynomial of n is $F(x) = x - n$. By definition, $M(F) = |a_0| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$, so $M(n) = M(x - n) = n$ for all $n \in \mathbb{N}$ ■

As a convention throughout this section, for any $\alpha \in \mathbb{A}$, we let $P_\alpha(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α, and denote the roots of P_α, that is, the Galois conjugates of α, by $\alpha = \alpha_1, \ldots, \alpha_n$ where $n = \deg P_\alpha$.

Definition 14 *We say that $\alpha \in \mathbb{A}$ is a* Pisot-Vijayaraghavan number *if:*

1. *α is an algebraic integer.*

2. *$\alpha \in \mathbb{R}$ and $\alpha > 1$,*

3. *All Galois conjugates of $\alpha, \alpha_1, ..., \alpha_n, \alpha_i \neq \alpha$, satisfy $|\alpha_i| < 1$.*

We note that vacuously, the set of natural numbers are Pisot-Vijayaraghavan.

Definition 15 *We say that $\alpha \in \mathbb{A}$ is a* Salem number *if*

1. *α is an algebraic integer.*

2. *$\alpha \in \mathbb{R}$ and $\alpha > 1$.*

3. *All Galois conjugates of $\alpha, \alpha_1, ..., \alpha_n, \alpha_i \neq \alpha$ satisfy $|\alpha_i| \leq 1$ and at least one conjugate α_m has $|\alpha_m| = 1$.*

Notice that a Salem number must have degree at least 2 over $\mathbb{Q}$.

Proposition 16 *If α is Salem, then for any conjugate α_i, the number α_i^{-1} is also a Galois conjugate of α, that is, $\alpha_i^{-1} = \alpha_j$ for some $1 \leq j \leq n$. In particular, the minimal polynomial P_α is reciprocal.*

Proof. If α is Salem, then there is one conjugate α_i such that $|\alpha_i| = 1$ then $\overline{\alpha_i}$ is also a conjugate of α, and $\alpha_i\overline{\alpha_i} = 1$. Now, $\alpha_j = \sigma(\alpha_i)$ for some $\sigma \in Gal(\mathbb{Q}(\alpha_1, ..., \alpha_d)/\mathbb{Q})$. Now $\sigma(\alpha_i)\sigma(\overline{\alpha_i} = \sigma(\alpha_i\overline{\alpha_i}) = \sigma(1) = 1$.
So $\alpha_j \cdot \sigma(\overline{\alpha_i}) = 1$; $\sigma(\overline{\alpha_i} = \alpha_j^{-1}$, and it is also a conjugate. That P_α is reciprocal now follows immediately, as the roots of P_α^* are precisely the inverses of the roots of P_α. ■

We now classify the fixed points of M:

Theorem 17 *Let $\alpha \in \mathbb{A} \setminus \{0, 1\}$. Then $M(\alpha) = \alpha$ if and only if α is a Pisot number or a Salem number.*

Proof. Let the minimal polynomial of α be

$$\begin{aligned} f(z) =& a_0 z^d + a_1 z^{d-1} + \cdots + a_d \\ =& a_0 \prod_{i=1}^{d}(z-\alpha_i) = \prod_{i=1}^{d}(z-a_i) \end{aligned} \tag{1.3.1}$$

Then $M(f) = \prod_{n=1}^{d} \max\{1, |\alpha_i|\}$. If α is a Pisot number, then $M(f) = |\alpha| = \alpha$, since all other roots have modulus less than 1. The proof is similar if α is a Salem number.

On the other hand , if $M(\alpha) = \alpha$, then

$$M(\alpha) = |a_0||\alpha| = \alpha > 1$$

So α must be a real number that is greater than 1, and $a_0 = \pm 1$, that is, α is an algebra integer.

α has no other conjugates with $|\alpha_i| > 1$, hence $|\alpha_i| \leq 1$ for all conjugates α_i other than α, Therefore by definitions, α is either an Pisot number or a Salem number.

■

It is a folklore conjecture that Lehmer's conjecture can be reduced to consideration of Salem numbers (Pisot-Vijayaraghavan numbers are non-reciprocal and hence have Mahler measure bounded away from 1 by Smyth's theorem), and the above result suggests that this may be true because of the special nature of such numbers under the iteration of M. For more information on recent results and applications of this important class of numbers, the readers may want to read Smyth's survey article [28].

In regard to classifying the periodic points and the preperiodic points, we will now proceed show the proof that, for degree no larger than 3 polynomials, the iteration of $M : \mathbb{A} \to \mathbb{A}$ will give the output that stabilizes eventually, those are examples of $\alpha \in \mathbb{A}$ that are preperiodic. There are many other questions that we are interested in, for example:

If we increase the degree of $f(x)$, which is the initial input of the iteration, does the "tail" length before stabilization increase? If we have the initial algebraic unit to be degree 6, will it take more than 1 application of M to get to the stabilized output? Is it true that if the degree of the starting algebraic unit is $d \geq 4$, then its orbit size is bounded by $d-1$? Can we find algebraic units with arbitrarily long but finite orbits in large enough degree extensions? And how large is "large enough" degree?

Remark 1.3.1 *Note that $M^{n+1}(\alpha) \geq M^n(\alpha)$ for all algebraic number α, the iterates of Mahler measure is non-decreasing(after one iteration), it is clear that we cannot have cycles of length greater than 1, such as $\alpha \to M(\alpha) \to M^2(\alpha) \to \alpha$.*

As we will see later, it is possible to have either a wandering point, or a preperiodic point with a long "tail" before reaching a fixed point.

One key theorem for the background of Mahler measure as a dynamical system is Northcott Theorem.

Theorem 18 (Northcott [23]) *For any $T, D > 0$, the set*

$$\{\alpha \in \mathbb{A} : M(\alpha) \leq T \text{ and degree of } \alpha \leq D\}$$

is finite.

If the orbit of α under M, $O_M(\alpha) = \{M^n(\alpha) : n \geq 0\}$ is infinite, then either the degree of the algebraic number $M^n(\alpha)$ tends to infinity as $n \to \infty$ or the number $M^n(\alpha) \to \infty$ as $n \to \infty$ by Northcott's theorem. In fact, it must be $M^n(\alpha) \to \infty$ as $n \to \infty$. All values of $M^n(\alpha)$ live in the field: $Q(\alpha_1, ..., \alpha_n, |\alpha_1|, ..., |\alpha_n|)$, where $\alpha_1, \ldots, \alpha_n$ are all the Galois conjugates of α, this is of finite degree over $\mathbb{Q}$.

In the next section, we will introduce some important properties of M as a dynamical system.

1.4 Properties of M as a dynamical system

In [7], Dixon and Dubickas gave their proofs for the properties presented in this section, and I will rephrase their proofs.

Definition 19 *We say that $\alpha \in \overline{\mathbb{Q}}^{\times}$ is torsion-free(See [12] and [14]) if for any $\sigma \in$ $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, if $\alpha \neq \sigma(\alpha)$, then $\frac{\alpha}{\sigma(\alpha)} \notin \mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$.*

Proposition 20 *Suppose $\alpha \in \overline{\mathbb{Z}}$, then $M(\alpha^{-1}) = M(\alpha)$.*

Proof. This is immediate by Definition 1. ■

Proposition 21 *Suppose $\alpha \in \overline{\mathbb{Z}}$, let the minimal polynomial of α be f, let G_α be the Galois group of f. Suppose that $-\alpha = \sigma(\alpha)$ for some $\sigma \in G_\alpha$ then $M(\alpha^2) = M(\alpha)$.*

Proof. Soppose there exists a Galois conjugate β such that $\beta \neq -\alpha$ and $\beta \neq \alpha$. Then there exists $\tau \in G_\alpha$ such that $\tau(\alpha) = \beta$, but then $\tau(-\alpha) = \tau\sigma(\alpha) = -\tau(\alpha) = -\beta$. Hence $-\beta$ is also a Galois conjugate of α. Now, let the complete set of conjugates of α be $\{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, ..., \alpha_n, -\alpha_n\}$. Since the conjugates of α^2 are the squares of the conjugates of α, $M(\alpha^2) = \prod\limits_{|\alpha_i| \geq 1} |\alpha_i^2| = M(\alpha) = \prod\limits_{|\alpha_i| \geq 1} |\alpha_i|\,|-\alpha_i|$. ■

Proposition 22 *Soppose $\alpha \in \overline{\mathbb{Z}}$ and $\beta = M(\alpha)$. let the minimal polynomial of β be f, let G_β be the Galois group of f. Then β is torsion-free and $M(\beta^n) = M(\beta)^n$ for any $n \in \mathbb{N}$.*

Proof. We know that $M^{(n)}(\alpha)$ is a Perron number [7], hence the conjugates of β are all less than β in absolute value. Therefore, for any $\sigma \in G_\beta$, if $\beta \neq \sigma(\beta)$ then $\frac{\beta}{\sigma(\beta)} \notin \mathrm{Tor}(\overline{\mathbb{Q}}^{\times})$, thus β is torsion-free. Now, this implies that $[\mathbb{Q}(\beta^n) : \mathbb{Q}] = [\mathbb{Q}(\beta) : \mathbb{Q}]$. Let the set of conjugates of β be $\{\beta_1, ..., \beta_m\}$, then the set of conjugates of β^n is $\{\beta_1^n, ..., \beta_m^n\}$, hence $M(\beta^n) = \prod\limits_{|\beta_i^n| \geq 1} |\beta_i^n| = \prod\limits_{|\beta_i| \geq 1} |\beta_i|^n = (\prod\limits_{|\beta_i| \geq 1} |\beta_i|)^n = (M(\beta))^n$. ■

Theorem 23 *If α is torsion-free and $M(\alpha) = \alpha^n$ for some integer $n > 1$, then α is a wandering point for M. In particular, if α is a Perron number, then it satisfies this conclusion as well.*

Proof. This is immediate from Theorem 22. ■

1.5 Main results and conjectures

In this thesis, we will prove several other results regarding the stopping time of algebraic numbers. Our first result is a direct generalization of Dubickas's result:

Theorem 24 *For any $d \geq 3$, $l \in \mathbb{Z} \setminus \{\pm 1, 0\}$ and $k \in \mathbb{N}$ there is an algebraic integer α of degree d, $N(\alpha) = l$ and $\#\mathcal{O}_M(\alpha) = k$.*

The proof of Theorem 24 will be given in §III below. To study the possible behaviour of algebraic units under iteration of M is more delicate. It is clear that $\#\mathcal{O}_M(\alpha) \leq 2$ for all algebraic units of degree at most 3, and this result is (non-trivially) also true if the degree is 4:

Theorem 25 *Let α be an algebraic unit of degree 4. Then either $\#\mathcal{O}_M(\alpha) \leq 2$ or $\#\mathcal{O}_M(\alpha) = \infty$. Moreover, if $\#\mathcal{O}_M(\alpha) = \infty$, then $M^{(3)}(\alpha) = M(\alpha)^2$.*

The first algebraic unit α with $\#\mathcal{O}_M(\alpha) \geq 3$ we found has degree 6 and orbit size 5. It is given by any root of $x^6 - x^5 - 4x^4 - 2x^2 - 4x - 1$. Despite an extensive search, we did not find any unit of degree 5 of orbit size ≥ 3, nor a unit of degree 6 of finite orbit size ≥ 6.

It will follow from the proof of Theorem 25 that we have the following corollary:

Corollary 26 *If α is an algebraic unit of degree 4, then the sequence $(\log M^{(n)}(\alpha))_{n \in \mathbb{N}}$ satisfies a linear homogeneous recursion.*

The proofs of Theorem 25 and Corollary 26 are given in §IV. We note that, in the example of a degree 4 wandering point given by Zhang [31], the sequence $(\log M^{(n)}(\alpha))_{n \in \mathbb{N}}$ satisfied the recursion relation $x_n = 2x_{n-2}$ for $n \geq 3$. Based on the above corollary and further experimental data, we make the following conjecture:

Conjecture 1 *For every algebraic unit α, there exists a constant k such that the sequence $(\log(M^{(n)}(\alpha)))_{n \geq k}$ satisfies a linear homogeneous recursion.*

We note that, in the case of a large Galois group, the behavior of units is particularly simple. We prove that, if the Galois group contains the alternating group, then the orbit of a unit must either stop after at most one iteration, or the unit wanders. Specifically, we prove in §IV the following theorem:

Theorem 27 *If α is an algebraic unit of degree d such that the Galois group of the Galois closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ contains the alternating group A_d, then $\#\mathcal{O}_M(\alpha) \in \{1, 2, \infty\}$.*

More precisely, if α is as above, of degree ≥ 5, and such that none of $\pm\alpha^{\pm 1}$ is conjugate to a Pisot number, then $\#\mathcal{O}_M(\alpha) = \infty$.

One might be led by Theorems 25 and 27 to suspect that, in fact, algebraic units cannot have arbitrarily large but finite orbits under M. However, we prove that this is not the case.

Theorem 28 *Let $S \in \mathbb{N}$ be arbitrary, and let $d \geq 12$ be divisible by 4. Then there exist algebraic units of degree d whose orbit size is finite but greater than S.*

The proof is given in Section IV. It would be interesting to know whether there are large finite orbits of algebraic units in any degree less than 12.

We know that all numbers fields K satisfying $[K : Q] \leq 3$ do not contain algebraic numbers a with infinite orbit. But are they all such fields? No. We know that there are quartic K that don't have a wandering point, as Theorem 29 below shows.

Theorem 29 *Let $K/\mathbb{Q}$ be of degree 4. If K is totally real, then it contains a wandering point under iteration of M if and only if K is not biquadratic. If K is totally imaginary, then there are no wandering points in K.*

Note that the classification of the extensions of signature $(2, 1)$ is still partially open.

In extensions of degree 5, there always exists an algebraic unit which is wandering under iteration of M.

Theorem 30 *Let $K/\mathbb{Q}$ be an extension of degree five. Then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Furthermore, in chapter 5, we will discuss the complete classification of all the Abelian extensions that don't contain a wandering point:

Theorem 31 *Let $K/\mathbb{Q}$ be an Abelian extension. Then K does not contain a wandering point under iteration of M if and only if the maximal real subfield of K has Galois group isomorphic to C_1, C_2, $C_2 \times C_2$, or C_3.*

CHAPTER II

LOWER DEGREE CASES

Part of this chapter is from my master's thesis.

2.1 Orbit sizes of lower degree algebraic integers

Let α be an algebraic integer, that is, an algebraic number with minimal polynomial that is monic, irreducible and with integer coefficients. The main result of this section is the following:

Theorem 32 *If the degree of α is at most 3, then the orbit of α under M is eventually fixed (i.e., it* stabilizes*).*

We break the proof down by degree.

2.1.1 Degree 1

Suppose that α is of degree 1. Then the minimal polynomial $f(x) = x + b$ with $b \in \mathbb{Z}$. If $b = 0$, then $M(\alpha) = \prod\limits_{|\alpha_i|\geq 1} \max\{1, |\alpha_i|\} = 1$, with the minimal case being $f(x) = x - 1$, and $M(M(\alpha)) = 1$, stabilized. If $b \neq 0$, then $M(\alpha) = \prod\limits_{|\alpha_i|\geq 1} |\alpha_i| = |b| = \pm b$ (with minimal polynomial $f(x) = x \mp b$). Then $M(M(\alpha)) = M(|b|) = |b| = \pm b$ and is stabilized.

2.1.2 Degree 2

Now suppose α is of degree 2. Let the minimal polynomial be $x^2 + ax + b \in \mathbb{Z}[x]$. Then $x^2 + ax + b = (x - \alpha)(x - \overline{\alpha})$. If $\alpha \notin \mathbb{R}$, then $|\alpha| = |\overline{\alpha}| \geq 1$. Then $|b| = |\alpha||\overline{\alpha}| \geq 1$. So in this case, $M(\alpha) = \prod\limits_{|\alpha_i|\geq 1} |\alpha_i| = |\alpha||\overline{\alpha}| = |b|$, so $M(M(\alpha)) = M(|b|) = |b|$, stabilized. If $\alpha \in \mathbb{R}$, then we can't have both $|\alpha|, |\overline{\alpha}| < 1$, so at least one value is ≥ 1. If both $|\alpha|, |\overline{\alpha}| \geq 1$, then again, $M(\alpha) = |\alpha \cdot \overline{\alpha}| = |b|$ and $M(M(\alpha)) = M(|b|) = |b|$.

If $|\alpha| \geq 1$, but $|\overline{\alpha}| < 1$. Without loss of generality, $M(\alpha) = |\alpha| = \pm\alpha$ and $M(M(\alpha)) = M(\pm\alpha) = |\alpha|$, so the Mahler measure stabilizes.

2.1.3 Degree 3

Now we suppose $\alpha = \alpha_1$ is of degree 3,

$$f(x) = x^3 + ax^2 + bx + c = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \text{ where } |c| \geq 1.$$

Case 1: There is one real root and two complex roots. Without loss of generality, let $\alpha_1 \in \mathbb{R}$, $\alpha_3 = \overline{\alpha_2}$ are complex numbers ($\alpha_3 = \overline{\alpha_2}$ is from complex conjugate root theorem).

So $\alpha_2\alpha_3 \in \mathbb{R}$. Now, $|\alpha_2\alpha_3|$ and $|\alpha_1|$ cannot be both less than 1, so at least one of them is ≥ 1.

1. If both ≥ 1, then $|\alpha_2\alpha_3| = |\alpha_2| \cdot |\alpha_3| \geq 1$. Since $\alpha_3 = \overline{\alpha_2}$, $|\alpha_2| = |\alpha_3|$. Hence $|\alpha_2| = |\alpha_3| \geq 1$ and $|\alpha_1| \geq 1$. Therefore,
   $M(\alpha) = \prod_{|\alpha_i| \geq 1} |\alpha_i| = |\alpha_1||\alpha_2||\alpha_3| = |c|$, So $M^2(\alpha) = M(|c|) = |c|$. Thus stabilized, minimal $f(x) = x - |c|$.

2. If $|\alpha_2\alpha_3| \geq 1$ but $|\alpha_1| < 1$ then since $|\alpha_2| = |\alpha_3|$, $|\alpha_2| \geq 1$, $|\alpha_3| \geq 1$. Hence $M(\alpha) = |\alpha_2||\alpha_3| = \alpha_2\alpha_3$. Since $\alpha_1\alpha_2\alpha_3 = -c$, $\alpha_2\alpha_3 = \dfrac{-c}{\alpha_1} \in \mathbb{Q}(\alpha_1)$. So the minimal polynomial of $\alpha_2\alpha_3$ is of degree 3. degree $[\mathbb{Q}(\alpha_2, \alpha_3) : \mathbb{Q}] = 3$

   $\tau : \alpha_1 \to \alpha_3 \to \alpha_2 \to \alpha_1, \tau(\alpha_2\alpha_3) = \alpha_1\alpha_2, \tau(\alpha_1\alpha_2) = \alpha_3\alpha_1$.

   So the conjugates are: $\alpha_3\alpha_1$ and $\alpha_1\alpha_2$. Now if $|\alpha_1\alpha_2| < 1$ then $|\alpha_3\alpha_1| = |\alpha_1\alpha_2| < 1$. Then $M(\alpha_2\alpha_3) = \alpha_2\alpha_3$, Stabilized.

If $|\alpha_1\alpha_2| \geq 1$, then both $|\alpha_3\alpha_1|$ and $|\alpha_2\alpha_3| \geq 1$. $M(\alpha_2\alpha_3) = |\alpha_1\alpha_2| \cdot |\alpha_2\alpha_3| \cdot |\alpha_3\alpha_1| = (\alpha_1\alpha_2\alpha_3)^2 = c^2$. So Stabilized.

3. If $|\alpha_2\alpha_3| < 1$ but $|\alpha_1| \geq 1$ then $|\alpha_2| < 1, |\alpha_3| < 1$, but $|\alpha_1| \geq 1$. $M(\alpha_1) = |\alpha_1|$ and $M(|\alpha_1|) = |\alpha_1|$, Stabilized.

Case 2: All 3 roots, $\alpha_1, \alpha_2, \alpha_3$ are real.

Then $|\alpha_1|, |\alpha_2|, |\alpha_3|$ cannot all be less than 1. Without loss of generality, we have three cases:

1. $|\alpha_1|, |\alpha_2|, |\alpha_3|$ are all ≥ 1. Then $M(\alpha) = \prod\limits_{|\alpha_i| \geq 1} |\alpha_i| = |\alpha_1||\alpha_2||\alpha_3| = |c|$. $M^2(\alpha) = M(|c|) = |c|$. Stabilized.

2. $|\alpha_1| \geq 1$ and $|\alpha_2| < 1, |\alpha_3| < 1$. Then $M(\alpha) = \prod\limits_{|\alpha_i| \geq 1} |\alpha_i| = |\alpha_1|$ and $M^2(\alpha) = |\alpha_1|$, stabilized.

3. If $|\alpha_1| \geq 1, |\alpha_2| \geq 1$, and $|\alpha_3| < 1$, then $M(\alpha) = \prod\limits_{|\alpha_i| \geq 1} |\alpha_i| = |\alpha_1||\alpha_2| = |\alpha_1\alpha_2|$. $\sigma(\alpha_1\alpha_2) = \alpha_2\alpha_3, \tau(\alpha_1\alpha_2) = \alpha_1\alpha_3$. So if $|\alpha_2\alpha_3| < 1, |\alpha_1\alpha_3| < 1$, then $M(|\alpha_1\alpha_2|) = |\alpha_1\alpha_2|$. Stabilized.

   If $|\alpha_2\alpha_3| \geq 1, |\alpha_1\alpha_3| \geq 1$, then $M(|\alpha_1\alpha_2|) = c^2$, stabilized.

   If $|\alpha_2\alpha_3| \geq 1$, $|\alpha_1\alpha_3| < 1$, then $M(|\alpha_1\alpha_2|) = |\alpha_1\alpha_2^2\alpha_3| = |c\alpha_2|$

   Conjugates of $|c\alpha_2| = \epsilon \cdot c\alpha_2$, where $\epsilon \in \{\pm 1\}$ are $\epsilon c\alpha_1, \epsilon c\alpha_3$.

   Now, $|\epsilon c\alpha_1| \geq 1, |\epsilon c\alpha 2| \geq 1$. Now if $|c\alpha_3| < 1$, then $M(|c\alpha_2|) = \pm c^2\alpha_1\alpha_2 = |c^2\alpha_1\alpha_2| = \epsilon c^2\alpha_1\alpha_2$. If $\epsilon c^2\alpha_1\alpha_3 < 1$, then $M(\epsilon c^2\alpha_1\alpha_2) = |c^4 c\alpha_2|$. Once again, $|c^4 c\alpha_1| \geq 1, |c^4 c\alpha_2| \geq 1$. Now, if $|c^5\alpha_3| < 1$, then continue the process until the power of c big enough to make $|c^m\alpha_3| \geq 1$, and the output of M will be stabilized then;

   Otherwise, if $|c^5\alpha_3| \geq 1$, then

$$\begin{aligned} M(|c^4 c\alpha_2|) &= |c^4 c\alpha_2| \cdot |c^4 c\alpha_1| \cdot |c^4 c\alpha_3| \\ &= |c^{15}\alpha_1\alpha_2\alpha_3| = |c^{16}| \end{aligned} \tag{2.1.1}$$

and $M(|c^{16}|) = |c^{16}|$, so stabilized. Similarly, if $|\alpha_1\alpha_3| \geq 1$, $|\alpha_2\alpha_3| < 1$, the output of M will stabilize eventually.

2.2 Number fields with degree less than 4

Theorem 33 *Let $K/\mathbb{Q}$ be a number field satisfying $[K : Q] \leq 3$, then K does not contain algebraic numbers with infinite orbit.*

Proof. Suppose that α is an element in K, then α is an algebraic number with degree $n \leq 3$, and $\alpha_1, \cdots, \alpha_n$ are the distinct Galois conjugates of α. Note that if c is a positive integer, then the Galois conjugates of $c\alpha$ are among $c\alpha_1, \cdots, c\alpha_n$. This, and a similar argument as before, gives that $\#\mathcal{O}_M(\alpha) < \infty$. ■

2.3 An example of degree 4 unit

Proposition 34 *Let α be a root of $f(x) = x^4 + 5x^2 + x - 1$. Then $\{M^n(\alpha) : n \in \mathbb{N}\}$ is infinite.*

Proof. Let the Galois conjugates of α be denoted $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where α_1 and α_2 are real, α_3 and α_4 are complex, $|\alpha_1| \approx 0.5393$, $|\alpha_2| \approx 0.3547$, $|\alpha_3| = |\alpha_4| \approx 2.2859$. Then $M(\alpha_1) = |\alpha_3||\alpha_4|$ since only α_3 and α_4 are outside or on the unit circle. Now consider $M(\alpha_3\alpha_4)$. Note that $f(x)$ factors into irreducibles as $(x+1)(x^3+x^2+1) \pmod 2$, and f is irreducible mod 5. Therefore the Galois group of f contains a 3-cycle and a 4-cycle. This means that the Galois group has order at least 12, and has an odd permutation, which implies that it is S_4. Those are all the permutations for 4 elements. So there are $\binom{4}{2} = 6$ combinations, $\alpha_3\alpha_4$ has degree 6. When we check in Mathematica, only 3 roots of the minimal polynomial of $\alpha_3\alpha_4$ have absolute value ≥ 1. Again, if we check in Mathematica, those roots are $\alpha_1\alpha_3, \alpha_1\alpha_4$, and $\alpha_3\alpha_4$.

Hence, $M^2(\alpha_1) = M(\alpha_3\alpha_4) = |\alpha_1^2\alpha_3^2\alpha_4^2| = \left|\dfrac{1}{\alpha_2^2}\right|$.

Conjugates of $\alpha_1\alpha_3\alpha_4$ with absolute value ≥ 1 are $\alpha_1\alpha_3\alpha_4$ and $\alpha_2\alpha_3\alpha_4$. Reciprocals of α_i are conjugates.

In fact, $M^3(\alpha_1) = M(|1/\alpha_2^2|) = M(|\alpha_2^2|) = M(|\alpha_1^2|) = |\alpha_3^2\alpha_4^2|$. Then $M^4(\alpha_1) = M(\alpha_3^2\alpha_4^2) = |\alpha_1^4\alpha_3^4\alpha_4^4|$ since conjugates of $\alpha_3^2\alpha_4^2$ that has modulus ≥ 1 are $\alpha_1^2\alpha_4^2, \alpha_1^2\alpha_3^2, \alpha_3^2\alpha_4^2$. $|\alpha_1^4\alpha_3^4\alpha_4^4| = \left|\dfrac{1}{\alpha_2^4}\right|$.

Again, $M^5(\alpha_1) = M(|\frac{1}{\alpha_2^4}|) = M(|\alpha_2^4|) = M(|\alpha_1^4|) = |\alpha_3^4\alpha_4^4|$. Then $M^6(\alpha_1) = M(|\alpha_3^4\alpha_4^4|) = |\alpha_1^8\alpha_3^8\alpha_4^8| = \left|\dfrac{1}{\alpha_2^8}\right|$, and we can continue the process and $M^n(\alpha_1)$ will be increasing(goes to infinity) as $n \to \infty$, since $|\alpha_2| < 1$. We can prove by induction that $M^{2n-1}(\alpha_1) = M(\alpha_1)^{2^{n-1}}$ and $M^{2n}(\alpha_1) = M^2(\alpha_1)^{2^{n-1}}$.

We will prove $M^{2n-1}(\alpha_1) = M(\alpha_1)^{2^{n-1}}$. When $n = 1$, $M(\alpha_1) = |\alpha_3\alpha_4|, M(\alpha_1)^{2^{n-1}} = |\alpha_3\alpha_4|$. So $M^{2n-1}(\alpha_1) = M(\alpha_1)^{2^{n-1}}$ holds for case $n = 1$. Suppose the case of $n-1$ holds. Then

$$\begin{aligned}
&M^{2(n-1)-1}(\alpha_1) = M(\alpha_1)^{2^{(n-1)-1}} \\
&M^{2n-3}(\alpha_1) = M(\alpha_1)^{2^{n-2}} \text{ then} \\
&M^{2n-2}(\alpha_1) = M((M(\alpha_1))^{2^{n-2}}) \\
&\qquad\qquad\quad = M(|\alpha_3^{2^{n-2}}\alpha_4^{2^{n-2}}|)
\end{aligned} \tag{2.3.1}$$

Conjugates of $\alpha_3^{2^{n-2}}\alpha_4^{2^{n-2}}$ that have absolute value ≥ 1 are $\alpha_1^{2^{n-2}}\alpha_4^{2^{n-2}}, \alpha_1^{2^{n-2}}\alpha_3^{2^{n-2}}$ and $\alpha_3^{2^{n-2}}\alpha_4^{2^{n-2}}$.

So, $M(|\alpha_3^{2^{n-2}}\alpha_4^{2^{n-1}}|) = |\alpha_1^{2^{n-1}}\alpha_3^{2^{n-1}}\alpha_4^{2^{n-1}}| = \left|\dfrac{1}{\alpha_2^{2^{n-1}}}\right|$.

Now,

$$\begin{aligned}
M^{2n-1}(\alpha_1) &= M(|1/\alpha_2^{2^{n-1}}|) \\
&M(|\alpha_2^{2^{n-1}}|) = M(|\alpha_1^{2^{n-1}}|) \\
&= |\alpha_3^{2^{n-1}}\alpha_4^{2^{n-1}}| = |\alpha_3\alpha_4|^{2^{n-1}} = M(\alpha_1)^{2^{n-1}}
\end{aligned} \tag{2.3.2}$$

Hence, the result holds for the case of n. Similarly, we can prove that $M^{2n}(\alpha_1) = M^2(\alpha_1)^{2^{n-1}}$ and thus the orbit of α under the Mahler measure tends to infinity (doubly exponentially). ∎

CHAPTER III

NON-UNITS ALGEBRAIC INTEGERS

This chapter is from [15].

3.1 A few isolated cases

In [11], Dubickas proved the case $d = 3$ and $l = 2$ (and k arbitrary). In order to prove Theorem 24, we will start with a few examples.

Example 3.1.1 *Since there are Pisot-Vijayaraghavan numbers of any degree and norm, we know that for any $d \in \mathbb{N}$ and any $l \in \mathbb{Z} \setminus \{\pm 1, 0\}$ there are algebraic numbers α of degree d, norm l and orbit size 1. By Perron's criterion, we may take the largest root of $x^d + l^2 x^{d-1} + l$.*

Similarly, the polynomial $x^d + l^d x + l$ has precisely one root β inside the unit circle and all other roots are of absolute value $> |l|$. Hence, the polynomial is irreducible. Let α be the largest root of this polynomial. Then $M(\alpha) = |\frac{l}{\beta}|$, which is a Pisot number. Thus, α has norm l, degree d and orbit size 2.

Example 3.1.2 *For any $l \in \mathbb{Z} \setminus \{\pm 1, 0\}$ we consider $f(x) = x^3 - l^2 x + l$. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of f ordered such that $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3|$.*

If $l \geq 2$ we have

$$\begin{aligned} f(-l-1) &= -2l^2 - 2l - 1 < 0 \qquad & f(-l) &= l > 0 \\ f(l-1) &= -2l^2 + 4l - 1 < 0 \qquad & f(l) &= l > 0 \\ f(1) &= 1 - l^2 + l < 0 \qquad & f(\tfrac{1}{l}) &= \tfrac{1}{l^3} > 0 \end{aligned}$$

Hence, the three roots are real and none of them is an integer. If f is reducible, then one of the factors must be linear, this is a contradiction since f is monic. Hence, f is *irreducible and it follows* $\alpha_1 \in (-l-1,-l)$, $\alpha_2 \in (l-1,l)$ *and* $\alpha_3 \in (\frac{1}{l},1)$. *Therefore we find* $M^{(0)}(\alpha_1) = \alpha_1$, $M^{(1)}(\alpha_1) = -\alpha_1\alpha_2 = \frac{l}{\alpha_3}$, $M^{(2)}(\alpha_1) = M(\frac{l}{\alpha_3}) = \frac{l^2}{\alpha_2\alpha_3} = -\alpha_1 l$, $M^{(3)}(\alpha_1) = M(-\alpha_1 l) = \alpha_1 l\alpha_2 l\alpha_3 l = l^4 \in \mathbb{Z}$. *These are all elements in the orbit of* α_1 *under iteration of* M. *Hence,* α_1 *is an algebraic integer of degree* 3, $N(\alpha_1) = l$ *and* $\#\mathcal{O}_M(\alpha_1) = 4$. *Moreover* $-\alpha_1$ *is an algebraic integer of degree* 3, $N(-\alpha_1) = -l$ *and* $\#\mathcal{O}_M(-\alpha_1) = 4$.

In the same fashion one can prove that any root of the polynomial $x^3 + lx^2 - l$ *is of degree* 3, *norm* $-l$ *and orbit size* 3.

Example 3.1.3 *Again let* $l \in \mathbb{Z}\setminus\{\pm 1, 0\}$ *be arbitrary and consider* $f(x) = x^4 - l^2x^2 + (l^2 - l)x + l$. *The four roots of* f *are ordered as* $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3| \geq |\alpha_4|$. *A direct computation shows that* f *is irreducible and* $\#\mathcal{O}_M(\alpha_1) = 4$ *if* $l \in \{-3,-2,-4\}$. *If* $l \notin \{-3,-2-1,0,1,2\}$, *then we show as in the last example that*

$$\alpha_1 \in (-l-1,-l), \quad \alpha_2 \in (l-1,l), \quad \alpha_3 \in (1,2), \quad \alpha_4 \in (-1,-\frac{1}{l^2})$$

if $l > 0$, *and*

$$\alpha_1 \in (-l-1,-l), \quad \alpha_2 \in (l-1,l), \quad \alpha_3 \in (1,2), \quad \alpha_4 \in (1,\frac{1}{l^2})$$

if $l < 0$. *Obviously* f *has no linear factor. Moreover,* α_4 *and* α_1 *must be Galois conjugates, since the norm of* α_1 *has to be a divisor of* l. *Hence, if* f *is not irreducible it factors into* $g(x) = (x-\alpha_1)(x-\alpha_4)$ *and* $h(x) = (x-\alpha_2)(x-\alpha_3)$. *This can only occur if* g *and* h *are in* $\mathbb{Z}[x]$. *Comparing the size of the roots, the only possibilities are* $g(x) = x^2 + (l+1)x + 1$ *and* $h(x) = x^2 - (l+1)x + l$. *However, multiplying these two polynomials does not give* f. *Hence,* f *is irreducible.*

Now we calculate the orbit size of α_1. *We have* $M(\alpha_1) = -\frac{l}{\alpha_4}$, $M^{(2)}(\alpha_1) = \pm l^2\alpha_1$, $M^{(3)}(\alpha_1) = \pm l^9$, *and hence* $\#\mathcal{O}_M(\alpha_1) = 4$. *We have shown, that any root* α *of* f *is an algebraic integer of degree* 4, *norm* l *and orbit size* 4.

Example 3.1.4 *One can show with similar methods as above, that any root of* $x^d - l^{d-2}x + l$

has orbit size 3*, for all* $d \geq 4$ *and* $l \in \mathbb{Z} \setminus \{\pm 1, 0\}$*: To this end, we note*

$$|-l^{d-2}z| = |l|^{d-2} > |l| + 1 \geq |z^d + l| \quad \forall\, z \in \mathbb{C},\ |z| = 1, \tag{3.1.1}$$

and

$$|z^d| = |l|^d > |l|^{d-1} + |l| \geq |-l^{d-2}z + l| \quad \forall\, z \in \mathbb{C},\ |z| = |l|. \tag{3.1.2}$$

Now we apply Rouché's theorem. Then (3.1.1) *tells us that* $x^d - l^{d-2}x + l$ *has precisely one root* α_d *inside the unit circle, and* (3.1.2) *tells us that all roots* $\alpha_1, \ldots, \alpha_d$ *of* $x^d - l^{d-2}x + l$ *have absolute value* $< |l|$*.*

Before we proceed with calculating the orbit size of one of these roots, we need to show that $x^d - l^{d-2}x + l$ *is irreducible. This is obviously the case if* $|l|$ *is a prime number. So in particular, we can assume that* $|l| \geq 4$*. Using this assumption and* $d \geq 4$*, the same calculation as in* (3.1.1) *proves that there is precisely one root of* $x^d - l^{d-2}x + l$ *of absolute value* $\leq \sqrt{|l|}$ *(necessarily* α_d*).*

It follows that no product of two or more of the elements $\alpha_1, \ldots, \alpha_{d-1}$ *can be a divisor of* l*. Hence, the only possibility for* $x^d - l^{d-2}x + l$ *to be reducible is, if it has a root* $a \in \mathbb{Z}$*. This* a *must be a divisor of* $|l|$ *and it must satisfy* $a^d = l^{d-2}a - l$*. Hence,* $a^{d-1} \mid l$ *which implies* $|a|^{d-1} \leq |l|$*. This is not possible, as we have just seen that* $|a| \geq \sqrt{|l|}$*. It follows that* $x^d - l^{d-2}x + l$ *is indeed irreducible, and* α_1 *is an algebraic integer of degree* d*, and norm* l*.*

We then have:

- $M^{(1)}(\alpha_1) = \alpha_1 \cdots \alpha_{d-1} = \frac{l}{|\alpha_d|} \notin \mathbb{Z}$*,*
- $M^{(2)}(\alpha_1) = M(\pm\frac{l}{\alpha_d}) = \pm\prod_{i=1}^{d} \frac{l}{\alpha_i} \in \mathbb{Z}$*, and*
- $M^{(n)}(\alpha_1) = M^{(2)}(\alpha_1)$ *for all* $n \geq 2$*.*

Hence α_1 *has orbit size* 3*.*

3.2 A generalization of Dubickas's result on non-units

Proposition 35 *Let $d \geq 3$ be an integer and let $\alpha_1, \ldots, \alpha_d$ be a full set of Galois conjugates of an algebraic integer α. Assume the following conditions:*

i. $|\alpha_1| > |\alpha_2| \geq \ldots \geq |\alpha_{d-1}| > 1 > |\alpha_d|$,

ii. $|\alpha_i| \leq |N(\alpha)|$ *for all* $i \in \{2, \ldots, d\}$,

Then α is a pre-periodic point of M. More precisely, if we let

$$c(\alpha) = \min\{\min\{k \in \mathbb{N} : 2 \mid k \text{ and } |\alpha_d \cdot N(\alpha)^{b_k}| > 1\}, \\ \min\{k \in \mathbb{N} : 2 \nmid k \text{ and } |\alpha_1| < |N(\alpha)^{b_k}|\}\},$$

where we define $b_1 = 1$, and $b_n = b_{n-1} \cdot (d-1) + (-1)^{n-1}$ for all $n \geq 2$, then $\#\mathcal{O}_M(\alpha) = c(\alpha) + 2$.

Proof. First we note, that α cannot be an algebraic unit. Hence, $|N(\alpha)| \geq 2$ and $b_k \geq 1$ for all k. We claim that $b_k \to \infty$. To see this, notice that $b_1 = 1, b_2 = d - 2 \geq 1$, and we want to show that for $n \geq 3$, $b_n \geq (d-2)(d-1)^{n-2} + 1$. Now, this is true for $n = 3$, since $b_3 = (d-2)(d-1) + 1$. By induction, suppose $b_{n-1} \geq (d-2)(d-1)^{n-3} + 1$, then $b_n \geq ((d-2)(d-1)^{n-3} + 1)(d-1) + (-1)^{n-1} = (d-2)(d-1)^{n-2} + (d-1) + (-1)^{n-1} \geq (d-2)(d-1)^{n-2} + 1$, as desired. Therefore, $b_n \geq 1$ for all n, and $b_n \to \infty$.

So the integer $c := c(\alpha)$ does indeed exist. We claim that for all $k \leq c$ we have

$$M^{(k)}(\alpha) = \begin{cases} \pm\frac{N(\alpha)^{b_k}}{\alpha_d} & \text{if } 2 \nmid k \\ \pm N(\alpha)^{b_k} \cdot \alpha_1 & \text{if } 2 \mid k \end{cases} \tag{3.2.1}$$

Note that $\alpha_1, \alpha_d \in \mathbb{R}$, since there is no other conjugate of the same absolute value. Therefore, the sign in (3.2.1) has to be chosen such that the value is positive. We prove the claim by induction.

For $k = 1$, we calculate $M^{(1)}(\alpha) = M(\alpha) = \pm\alpha_1 \cdot \ldots \cdot \alpha_{d-1} = \pm\frac{N(\alpha)}{\alpha_d} = \pm\frac{N(\alpha)^{b_1}}{\alpha_d}$, by assumption (i). Now assume, that (3.2.1) is correct for a fixed $k < c$. If k is even, then by assumption (i) we have

$$\begin{aligned} M^{(k+1)}(\alpha) &= M(\pm N(\alpha)^{b_k} \cdot \alpha_1) = \pm N(\alpha)^{b_k \cdot (d-1)} \cdot \alpha_1 \cdot \ldots \cdot \alpha_{d-1} \\ &= \pm\frac{N(\alpha)^{b_k \cdot (d-1)+1}}{\alpha_d} = \pm\frac{N(\alpha)^{b_{k+1}}}{\alpha_d}. \end{aligned}$$

Here we have used that $k < c$ and hence $|N(\alpha)^{b_k} \cdot \alpha_d| < 1$.

If k is odd, then by assumption (ii) we have

$$\begin{aligned} M^{(k+1)}(\alpha) &= M(\pm\frac{N(\alpha)^{b_k}}{\alpha_d}) = \pm\frac{N(\alpha)^{b_k}}{\alpha_d} \cdot \frac{N(\alpha)^{b_k}}{\alpha_{d-1}} \cdot \ldots \cdot \frac{N(\alpha)^{b_k}}{\alpha_2} \\ &= \pm\frac{N(\alpha)^{b_k \cdot (d-1)}}{\alpha_2 \cdot \ldots \cdot \alpha_{d-1}} = \pm N(\alpha)^{b_k \cdot (d-1)-1} \cdot \alpha_1 = \pm N(\alpha)^{b_{k+1}} \cdot \alpha_1. \end{aligned}$$

Here we have used that $k < c$ and hence $|\frac{N(\alpha)^{b_k}}{\alpha_1}| < 1$. This proves the claim. Moreover, the proof of the claim shows that $M^{(k+1)}(\alpha) > M^{(k)}(\alpha_1)$ for all $k \in \{0, \ldots, c-1\}$.

Now, we calculate $M^{(c+1)}(\alpha)$. By definition of c, every conjugate of $M^c(\alpha)$ is greater than 1 in absolute value. Therefore, $M^{c+1}(\alpha) \in \mathbb{N}$. It follows, that $M^{(c+2)}(\alpha) = M^{(c+1)}(\alpha)$. Hence, $\#\mathcal{O}_M(\alpha_1) = c + 2$ as claimed. ∎

It remains to prove the existence of an algebraic number of degree d satisfying the assumptions of Proposition 35 for an arbitrary c.

The strategy is as the following: We will prove the locations of the roots of a class of irreducible polynomials satisfying assumptions (i) and (ii) from Proposition 35, then by Proposition 35, show that any root of one of the polynomials in the class will have desired degree, norm and orbit size.

We fix for the rest of this section arbitrary integers $d \geq 3$, $c \geq 2$ and $l \in \mathbb{Z} \setminus \{\pm 1, 0\}$. Moreover, we define

$$f_n(x) = x \cdot (x^{d-2} - 2) \cdot (x - n) + l$$

and denote the roots of f_n by $\alpha_1^{(n)}, \ldots, \alpha_d^{(n)}$ ordered such that

$$|\alpha_1^{(n)}| \geq |\alpha_2^{(n)}| \geq \ldots \geq |\alpha_d^{(n)}|.$$

Lemma 3.2.1 *Let $n \geq |l|+3$ be an integer. With the notation from above we have $\alpha_1^{(n)} \in (n-\frac{1}{n}, n+\frac{1}{n})$, $\alpha_d^{(n)} \in (-\frac{|l|}{n}, -\frac{1}{2n}) \cup (\frac{1}{2n}, \frac{|l|}{n})$, and $|\alpha_i^{(n)}| \in (1, \sqrt[d-2]{3-\frac{1}{d}})$ for all $i \in \{2, \ldots, d-1\}$. Moreover, $\alpha_d^{(n)}$ is negative if and only if $\alpha_1^{(n)} < n$.*

Proof. We apply Rouché's theorem and first prove the location of $\alpha_1^{(n)}$. Let z be any complex number with $|z| = n + \frac{1}{n}$. Then

$$\begin{aligned}
&|z \cdot (z^{d-2} - 2) \cdot (z - n)| \\
&\geq \left|n + \frac{1}{n}\right| \cdot \left|(n + \frac{1}{n})^{d-2} - 2\right| \cdot \frac{1}{n} \\
&= \left|1 + \frac{1}{n^2}\right| \cdot \left|(n + \frac{1}{n})^{d-2} - 2\right| \\
&> |l|
\end{aligned}$$

Hence by Rouché's theorem, f_n has exactly as many roots of absolute value $< n + \frac{1}{n}$ as $x \cdot (x^{d-2} - 2) \cdot (x - n)$, so f_n has d roots of absolute value $< n + \frac{1}{n}$. Now, let z be any complex number with $|z| = n - \frac{1}{n}$, suppose that $n = |l| + m$ where $m \geq 3$. Then

$$\begin{aligned}
&\left|z \cdot (z^{d-2} - 2) \cdot (z - n)\right| \\
&\geq \left|n - \frac{1}{n}\right| \cdot \left|(n - \frac{1}{n})^{d-2} - 2\right| \cdot \frac{1}{n} \\
&\geq \left|n - \frac{1}{n}\right| \cdot \left|(n - \frac{1}{n}) - 2\right| \cdot \frac{1}{n} \\
&= (|l| + m - \frac{1}{|l|+m})(|l| + m - \frac{1}{|l|+m} - 2) \cdot \frac{1}{|l|+m} \\
&= (1 - \frac{1}{(|l|+m)^2})(|l| - \frac{1}{|l|+m} + m - 2) \\
&= |l| - \frac{|l|}{(|l|+m)^2} - \frac{1}{|l|+m} + (m-2) + \frac{1}{(|l|+m)^3} - \frac{m}{(|l|+m)^2} + \frac{2}{(|l|+m)^2} > |l|,
\end{aligned}$$

since $m \geq 3$. Again by Rouché's theorem, f_n has $d-1$ roots of absolute value $< n - \frac{1}{n}$. Since f_n has no roots on the circle $|z| = n - \frac{1}{n}$, f_n has a single root in $(-n-\frac{1}{n}, -n+\frac{1}{n}) \cup (n-\frac{1}{n}, n+\frac{1}{n})$.

Now,

$$\begin{aligned}
&\left|(-n-\frac{1}{n})((-n-\frac{1}{n})^{d-2}-2)(-2n-\frac{1}{n})\right| \\
&\geq (|l|+2)\left|(n+\frac{1}{n})^{d-2}-2\right|(2n+\frac{1}{n}) \\
&\geq (|l|+2)||l|(2(|l|+2)) \\
&\geq (|l|+2)||l|(2|l|+4) \\
&\geq |l|^2 > |l|.
\end{aligned}$$

Similarly,

$$\left|(-n+\frac{1}{n})((-n+\frac{1}{n})^{d-2}-2)(-2n+\frac{1}{n})\right| \geq 2|l|^2 > |l|.$$

Since

$$(-n-\frac{1}{n})((-n-\frac{1}{n})^{d-2}-2)(-2n-\frac{1}{n})$$

has the same sign as

$$(-n+\frac{1}{n})((-n+\frac{1}{n})^{d-2}-2)(-2n+\frac{1}{n}),$$

$f_n(-n+\frac{1}{n})$ has the same sign as $f_n(-n-\frac{1}{n})$. Therefore, since there is only one root in the annulus $|z| \in (n-\frac{1}{n}, n+\frac{1}{n})$, which is necessarily real, f_n cannot have any root in the interval $(-n-\frac{1}{n}, -n+\frac{1}{n})$, thus f_n has a single root in the interval $(n-\frac{1}{n}, n+\frac{1}{n})$.

To prove the location of $\alpha_d^{(n)}$, let z be any complex number with $|z| = \frac{|l|}{n}$. Then,

$$\begin{aligned}
|z\cdot(z^{d-2}-2)\cdot(z-n)| &\geq \frac{|l|}{n}\cdot(2-\frac{|l|}{n})\cdot(n-\frac{|l|}{n}) \\
=&2|l|-2\frac{|l|^2}{n^2}-\frac{|l|^2}{n}+\frac{|l|^3}{n^3} > 2|l|-2\frac{|l|^2}{n^2}-\frac{|l|^2}{n} \\
\geq&2|l|-|l|\frac{|l|^2+4|l|}{(|l|+2)^2} > |l|.
\end{aligned}$$

By Rouché's theorem, f_n has exactly as many roots of absolute value $< \frac{|l|}{n}$ as the polynomial $x\cdot(x^{d-2}-2)\cdot(x-n)$. This is, f_n has exactly one root of absolute value $< \frac{|l|}{n}$. This root is necessarily real. A straightforward computation shows that $f_n(\pm\frac{1}{2n})$ have the same sign as $f_n(0)$. Hence f_n cannot have any root in the interval $(-\frac{1}{2n}, \frac{1}{2n})$.

To show the location of $\alpha_i^{(n)}$ for all $i \in \{2, \ldots, d-1\}$, let z be any complex number with $|z| = 1$. Then,

$$
\begin{aligned}
&|z \cdot (z^{d-2} - 2) \cdot (z - n)| \\
&= |z^{d-2} - 2| \cdot |z - n| \\
&\geq n - 1 > |l|,
\end{aligned}
$$

so f_n has a single root of absolute value < 1. The argument above also shows that f_n has no roots on the circle $|z| = 1$. Now, let z be any complex number with $|z| = \sqrt[d-2]{3 - \frac{1}{d}}$. Then,

$$
\begin{aligned}
&|z \cdot (z^{d-2} - 2) \cdot (z - n)| \\
&\geq (3 - \frac{1}{d})^{\frac{1}{d-2}} \cdot (1 - \frac{1}{d}) \cdot (n - (3 - \frac{1}{d})^{\frac{1}{d-2}}).
\end{aligned}
$$

Notice that since $n \geq |l| + 3$, $n - (3 - \frac{1}{d})^{\frac{1}{d-2}} > |l|$, hence it suffices to show that $(3 - \frac{1}{d})^{\frac{1}{d-2}} \cdot (1 - \frac{1}{d}) > 1$. Indeed, by elementary calculus, $(3 - \frac{1}{d})(1 - \frac{1}{d})^{d-2} > 1$ for all $d \geq 3$, which gives $|z \cdot (z^{d-2} - 2) \cdot (z - n)| > |l|$, hence by Rouché's theorem, f_n has $d - 1$ roots of absolute value less than $\sqrt[d-2]{3 - \frac{1}{d}}$. Therefore, f_n has exactly $d - 2$ roots with absolute values in the interval $(1, \sqrt[d-2]{3 - \frac{1}{d}})$.

The last part of the lemma is obvious, since $x \cdot (x^{d-2} - 2) \cdot (x - n)$ changes the sign at 0 and at n in the same way. ■

Lemma 3.2.2 *Let $n \geq |l| + 3$. Then f_n is irreducible in $\mathbb{Q}[x]$ whenever l is odd.*

Proof. From Lemma 3.2.1 we know $\alpha_1^{(n)} > |l|$. Hence, $\alpha_1^{(n)}$ must be a conjugate of the only root of f_n which is less than 1 in absolute value. If f_n would be reducible, then some product of the elements $\alpha_2^{(n)}, \ldots, \alpha_{d-1}^{(n)}$ must be a divisor of l. But every such product lies strictly between 1 and 3. Since 2 is no divisor of l by assumption, f_n is necessarily irreducible. ■

Lemma 3.2.3 *Let p be a prime and let $f = x^d + a_{d-1}x^{d-1} + \ldots + a_2x^2 + a_1x + a_0 \in \mathbb{Z}[x]$ such that $p \mid a_i$ for all $i \in \{0, \ldots, d-1\}$ and $p^2 \nmid a_2$. Then either f has a divisor of degree ≤ 2 or f is irreducible.*

Proof. This follows exactly as the classical Eisenstein criterion. Assume, that $f = g \cdot h$ where

$$g(x) = x^r + g_{r-1}x^{r-1} + \ldots + g_0 \quad \text{and} \quad h(x) = x^s + h_{s-1}x^{s-1} + \ldots + h_0 \in \mathbb{Z}[x]$$

with $r, s \geq 3$. Since the reduction of $g \cdot h$ modulo p is equal to $x^d \in \mathbb{Z}p\mathbb{Z}[x]$ and $\mathbb{Z}p\mathbb{Z}[x]$ is an integral domain, we know that each coefficient of g and h is divisible by p. It follows $p^2 \mid g_0h_2 + g_1h_1 + g_2h_0 = a_2$, which is a contradiction. ■

Lemma 3.2.4 *Let $n \geq |l| + 3$ and $|l|$ both be even. Then f_n is irreducible.*

Proof. We first note that f_n does not have a factor of degree 1. Otherwise, some divisor a of l would be a root of f_n. But $|a(a-n)| \geq n - 1 \geq |l| + 1$. Hence, in particular, $f_n(a) \neq 0$ for all $a \mid l$. It follows, that f_n is irreducible for $d = 3$. From now on we assume $d \geq 4$.

If l and n are even, then $f_n(x) = x(x^{d-2} - 2)(x - n) + l = x^d - nx^{d-1} - 2x^2 + 2nx + l$ is – by Lemma 3.2.3 – irreducible if it does not have a factor of degree 2.

Since $\alpha_1^{(n)}$ is larger than $|l|$ (which is the absolute value of product of all roots of f_n), it must be conjugate to $\alpha_d^{(n)}$ which is the only root of absolute value ≤ 1. If $\alpha_d^{(n)}$ would be the only conjugate of $\alpha_1^{(n)}$, then $\alpha_1^{(n)} + \alpha_d^{(n)} \in \mathbb{Z}$. This is not possible by Lemma 3.2.1. This means, that there is no factor of degree 2, having $\alpha_1^{(n)}$ or $\alpha_d^{(n)}$ as a root. This proves that f_n is irreducible for $d = 4$. For $d \geq 5$ the only possibility of a divisor of degree 2 is $x^2 - (\alpha_i^{(n)} + \alpha_j^{(n)})x + \alpha_i^{(n)}\alpha_j^{(n)}$, for $i \neq j \in \{2, \ldots, d-1\}$. By Lemma 3.2.1, we have $|\alpha_i^{(n)}\alpha_j^{(n)}| > 1$ and $|\alpha_i^{(n)}\alpha_j^{(n)}| < \sqrt[d-2]{3 - \frac{1}{d}}^{\,2} < 2$. Hence, such polynomial is not in $\mathbb{Z}[x]$. We conclude that f_n does not have a factor of degree ≤ 2 and therefore f_n is irreducible. ■

Theorem 36 *Let $d \geq 3$ and $l \in \mathbb{Z} \setminus \{\pm 1, 0\}$ such that $(d, l) \notin \{(3, 2), (3, -2)\}$. Moreover, let $b_1, b_2, \ldots$ be the sequence from Proposition 35 and $c \geq 2$ be an integer with $c \neq 2$ if $d \in \{3, 4\}$. Then any root α of $f_{|l|^{b_c - 1}}(x) = x(x^{d-2} - 2)(x - |l|^{b_c - 1}) + l$ is an algebraic integer of degree d, norm l, and orbit size $c + 2$.*

Proof. The cases we have to exclude, are those which violate assumption (ii) in Proposition 35 or satisfy $|l^{b_c - 1}| < |l| + 3$.

In Lemmas 3.2.2 and 3.2.4, we proved that α has degree d. Moreover, by Lemma 3.2.1, α satisfies assumptions (i) and (ii) from Proposition 35. As usual we denote with $\alpha_1, \ldots, \alpha_d$ the full set of conjugates of α. Then by Lemma 3.2.1, we achieve $|\alpha_d l^{b_c}| > \frac{|l|}{2} \geq 1$ and $|\alpha_1| < |l^{b_c - 1}| + 1 \leq |l^{b_c}|$.

Furthermore, we know $|\alpha_1| > |l|^{b_c - 1} - 1 \geq |l|^{b_{c-1}}$ and $|\alpha_d l^{b_{c-1}}| < \frac{|l|^{b_{c-1}+1}}{|l|^{b_c - 1}} < 1$. Again from Lemma 3.2.1 we also have $|\alpha_d l^{b_{c-2}}| < 1$ and $|\alpha_1| > l^{b_{c-2}}$, if $c \geq 3$.

What we have shown is that in the notation from Proposition 35, we have $c(\alpha) = c$, and hence $\#\mathcal{O}_M(\alpha) = c + 2$. ■

A closed formula for the recursion $b_1, b_2, \ldots$ is $b_n = \frac{1}{d}((d-1)^n + (-1)^{n-1})$. So Theorem 36 is fairly effective.

Corollary 37 *For any triple (d, l, k) of integers, with $d \geq 3$, $l \notin \{\pm 1, 0\}$, and $1 \leq k$, there are algebraic integers α with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d$, $N(\alpha) = l$ and $\#\mathcal{O}_M(\alpha) = k$.*

Proof. For $(3, 2, k)$ and $(3, -2, k)$ this is due to Dubickas [11] (note that he states the case $N(\alpha) = 2$, but then $-\alpha$ does the job in the case of negative norm). Together with Theorem 36 and the examples in 3.1, we conclude the corollary. ■

CHAPTER IV

UNITS OF HIGHER DEGREES

Part of this chapter is from [15].

4.1 Orbit sizes of degree 4 units

In light of Theorem 24, one might ask if arbitrarily long but finite orbits occur for algebraic units. In this section we will prove Theorem 25, which states that the orbit size of an algebraic unit of degree 4 must be 1, 2, or ∞.

Let α be an algebraic unit of degree 4. If α is a root of unity, a Pisot number, a Salem number or an inverse of such number we surely have $\#\mathcal{O}_M(\alpha) \leq 2$. Hence, we may and will assume for the rest of this section that the conjugates of α satisfy

$$|\alpha_1| \geq |\alpha_2| > 1 > |\alpha_3| \geq |\alpha_4|.$$

Denote the Galois group of $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)/\mathbb{Q}$ by G_α. For any $\beta \in \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ we denote the Galois orbit of β by $G_\alpha \cdot \beta$.

Then $M(\alpha) = \pm\alpha_1\alpha_2$ and

$$G_\alpha \cdot (\alpha_1\alpha_2) \subseteq \{\alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_1\alpha_4, \alpha_2\alpha_3, \alpha_2\alpha_4, \alpha_3\alpha_4\}.$$

Lemma 4.1.1 *If $|\alpha_1\alpha_4| = 1$ or $|\alpha_1\alpha_3| = 1$, then we have either $\#\mathcal{O}_M(\alpha) = 2$ or $\#\mathcal{O}_M(\alpha) = \infty$.*

Proof. If $|\alpha_1\alpha_4| = 1$, then also $|\alpha_2\alpha_3| = 1$, and if $|\alpha_1\alpha_3| = 1$, then also $|\alpha_2\alpha_4| = 1$. In both cases we see

$$|\alpha_1| = |\alpha_2| \iff |\alpha_3| = |\alpha_4|. \tag{4.1.1}$$

We first assume that $\alpha_1 \notin \mathbb{R}$. Then $\alpha_2 = \overline{\alpha_1}$ and hence $|\alpha_1| = |\alpha_2|$. Obviously it is $M(\alpha_1) = \alpha_1\alpha_2$. By our assumptions and (4.1.1), all values $|\alpha_1\alpha_3|$, $|\alpha_1\alpha_4|$, $|\alpha_2\alpha_3|$, $|\alpha_2\alpha_4|$, $|\alpha_3\alpha_4|$ are less or equal to 1. Hence $M^{(2)}(\alpha_1) = M(\alpha_1\alpha_2) = \alpha_1\alpha_2$. Therefore, $\#\mathcal{O}_M(\alpha_1) = 2$.

If $\alpha_1 \in \mathbb{R}$ and $|\alpha_1| = |\alpha_2|$, then $\alpha_2 = -\alpha_1$ and $\alpha_4 = -\alpha_3$. Hence, the only non-trivial Galois conjugate of $M(\alpha_1) = \alpha_1^2$ is α_3^2 and lies inside the unit circle. Therefore, $M^{(2)}(\alpha_1) = \alpha_1^2$ and $\#\mathcal{O}_M(\alpha_1) = 2$.

From now on we assume that $|\alpha_1| \neq |\alpha_2|$. Then, by (4.1.1), we have

$$|\alpha_1| > |\alpha_2| > 1 > |\alpha_3| > |\alpha_4|$$

and α_1 must be totally real. Moreover, we see

$$\alpha_1^n, \alpha_2^n, \alpha_3^n, \alpha_4^n \quad \text{are pairwise distinct for all } n \in \mathbb{N}, \tag{4.1.2}$$

and

$$(\alpha_1\alpha_2)^n, (\alpha_3\alpha_4)^n, (\alpha_1\alpha_3)^n, (\alpha_2\alpha_4)^n \quad \text{are pairwise distinct for all } n \in \mathbb{N}. \tag{4.1.3}$$

We notice, that in this situation it is not possible that $|\alpha_1\alpha_3| = 1$, since otherwise $|\alpha_2\alpha_4| < 1$ which contradicts $1 = |\alpha_1\alpha_2\alpha_3\alpha_4|$. Therefore, $|\alpha_1\alpha_4| = 1$, and $\alpha_4 = \pm\alpha_1^{-1}$. It follows that also $\alpha_3 = \pm\alpha_2^{-1}$. This gives natural constraints on the Galois group G_α, namely

$$G_\alpha \subseteq \{\mathrm{id}, (12)(34), (13)(24), (14)(23), (14), (23), (1342), (1243)\} \subseteq S_4.$$

In particular, since G_α is a transitive subgroup of S_4 with order divisible by 4,

$$G_\alpha = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\} \text{ or } \{\mathrm{id}, (1342), (14)(23), (1243)\} \subseteq G_\alpha.$$

In the first case, $G_\alpha \cdot (\alpha_1\alpha_2) = \{\alpha_1\alpha_2, \alpha_3\alpha_4\}$, which implies that $\alpha_1\alpha_2$ is a quadratic unit. Hence $\#\mathcal{O}_M(\alpha) = \#\mathcal{O}_M(\alpha_1\alpha_2) + 1 = 2$.

In the second case, $G_\alpha \cdot (\alpha_1\alpha_2) = \{\alpha_1\alpha_2, \alpha_3\alpha_4, \alpha_1\alpha_3, \alpha_2\alpha_4\}$. Note that $\alpha_1\alpha_2$ is still of degree 4 by (4.1.3). Hence $M^{(2)}(\alpha_1) = M(\alpha_1\alpha_2) = \pm\alpha_1^2\alpha_2\alpha_3 = \alpha_1^2$. By (4.1.2) it follows $M^{(3)}(\alpha) = M(\alpha_1^2) = (\alpha_1\alpha_2)^2 = M(\alpha)^2$. Now, by induction and (4.1.3) and (4.1.2), it follows $M^{(n)}(\alpha_1) = \alpha_1^{2^n}$ for all even $n \in \mathbb{N}$. Hence $\#\mathcal{O}_M(\alpha_1) = \infty$. ■

From now on, we assume:

$$|\alpha_1\alpha_4| \neq 1 \neq |\alpha_1\alpha_3|. \tag{4.1.4}$$

Lemma 4.1.2 *Assuming* (4.1.4), *if* $\alpha_1^n = \alpha_2^n$ *or* $\alpha_3^n = \alpha_4^n$ *for some* $n \in \mathbb{N}$, *then* $\#\mathcal{O}_M(\alpha_1) = 2$.

Proof. Let $\alpha_1^n = \alpha_2^n$ for some $n \in \mathbb{N}$. Then $\frac{\alpha_1}{\alpha_2}$ is a root of unity. Since none of the elements $\frac{\alpha_1}{\alpha_3}, \frac{\alpha_1}{\alpha_4}, \frac{\alpha_2}{\alpha_3}, \frac{\alpha_2}{\alpha_4}, \frac{\alpha_3}{\alpha_1}, \frac{\alpha_3}{\alpha_2}, \frac{\alpha_4}{\alpha_1}, \frac{\alpha_4}{\alpha_2}$ lies on the unit circle, we have $G_\alpha \cdot (\frac{\alpha_1}{\alpha_2}) \subseteq \{\frac{\alpha_1}{\alpha_2}, \frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_4}, \frac{\alpha_4}{\alpha_3}\}$. Hence

$$G_\alpha \subseteq \{\mathrm{id}, (12), (12)(34), (13)(24), (14)(23), (1324), (1423)\}.$$

This implies $M^{(2)}(\alpha_1) = M(\pm\alpha_1\alpha_2) = \pm\alpha_1\alpha_2 = M(\alpha_1)$, and hence $\#\mathcal{O}_M(\alpha_1) = 2$. The same proof applies if $\alpha_3^n = \alpha_4^n$. ■

Lemma 4.1.3 *Assuming* (4.1.4) *and* $\#\mathcal{O}_M(\alpha_1) > 2$, *then*

a. $|\alpha_1\alpha_2| > 1$, $|\alpha_1\alpha_3| > 1$.

b. $|\alpha_3\alpha_4| < 1$, $|\alpha_2\alpha_4| < 1$.

c. one of the values $|\alpha_1\alpha_4|$ *and* $|\alpha_2\alpha_3|$ *is* < 1 *and the other is* > 1.

d. α_1^n, α_2^n, α_3^n, α_4^n *are pairwise distinct for all* $n \in \mathbb{N}$.

e. $(\alpha_1\alpha_2)^n$, $(\alpha_3\alpha_4)^n$, $(\alpha_1\alpha_3)^n$, $(\alpha_2\alpha_4)^n$ *are pairwise distinct for all* $n \in \mathbb{N}$.

Proof. Obviously $|\alpha_1\alpha_2| > 1$ and $|\alpha_3\alpha_4| < 1$. Moreover, $1 \neq |\alpha_1\alpha_3| \geq |\alpha_2\alpha_4|$ and $|\alpha_1\alpha_3| \cdot |\alpha_2\alpha_4| = 1$. This means $|\alpha_1\alpha_3| > 1$ and $|\alpha_2\alpha_4| < 1$, proving parts (a) and (b).

Since $|\alpha_1\alpha_4| \cdot |\alpha_2\alpha_3| = 1$ and $|\alpha_1\alpha_4| \neq 1$, part (c) follows.

The elements α_1 and α_2 lie outside the unit circle, and α_3 and α_4 lie inside or on the unit circle. Hence, the only possibilities for (d) to fail are $\alpha_1^n = \alpha_2^n$ or $\alpha_3^n = \alpha_4^n$ for some $n \in \mathbb{N}$. By the previous lemma, both implies $\#\mathcal{O}_M(\alpha_1) = 2$, which is excluded by our assumptions.

Part (e) follows immediately from (a), (b) and (d). ■

Lemma 4.1.4 *If $M^{(3)}(\alpha_1) = M(\alpha_1)^2$ and $\#\mathcal{O}_M(\alpha_1) > 2$, then $\#\mathcal{O}_M(\alpha_1) = \infty$.*

Proof. This is true if assumption (4.1.4) is not satisfied, by Lemma 4.1.1. If we assume (4.1.4), then by Lemma 4.1.3 (d) and (e), we are in the same situation as at the end of the proof of Lemma 4.1.1. Hence, an easy induction proves the claim. ■

We now complete the proof of the statement that $\#\mathcal{O}_M(\alpha_1) \in \{1, 2, \infty\}$. It suffices to prove this under the assumption (4.1.4). From now on we assume $\#\mathcal{O}_M(\alpha) > 2$ and show that this implies $\#\mathcal{O}_M(\alpha) = \infty$. By Lemma 4.1.3, we have

$$\begin{aligned} M^{(2)}(\alpha) &\in \{\pm\alpha_1^2\alpha_2\alpha_3, \pm\alpha_1^3\alpha_2\alpha_3\alpha_4, \pm\alpha_1^2\alpha_2^2\alpha_3^2, \pm\alpha_1^2\alpha_2\alpha_4, \pm\alpha_1\alpha_2^2\alpha_3\} \\ &= \{\pm\frac{\alpha_1}{\alpha_4}, \pm\alpha_1^2, \pm\frac{1}{\alpha_4^2}, \pm\frac{\alpha_1}{\alpha_3}, \pm\frac{\alpha_2}{\alpha_4}\} \end{aligned} \tag{4.1.5}$$

In two of these cases the orbit of α can be determined immediately:

- If $M^{(2)}(\alpha) = \pm\alpha_1^2$, then (since we have $\#\mathcal{O}_M(\alpha) > 2$) it is $\alpha_1^n \neq \alpha_2^n$ for all $n \in \mathbb{N}$. Hence $M^{(3)}(\alpha) = M(\alpha)^2$ which implies $\#\mathcal{O}_M(\alpha) = \infty$.

- Similarly, if $M^{(2)}(\alpha) = \pm\frac{1}{\alpha_4^2}$, then (since $\#\mathcal{O}_M(\alpha) > 2$) it is $\alpha_3^n \neq \alpha_4^n$ for all $n \in \mathbb{N}$. Hence $M^{(3)}(\alpha) = M(\alpha_4^2) = M(\alpha)^2$ and again $\#\mathcal{O}_M(\alpha) = \infty$.

We now study the other three cases.

4.1.1 The case $M^{(2)}(\alpha) = \pm\frac{\alpha_1}{\alpha_4}$

This case occurs if $\alpha_1\alpha_3 \in G_\alpha \cdot (\alpha_1\alpha_2)$, and

- $|\alpha_1\alpha_4| > 1$ but $\alpha_1\alpha_4 \notin G_\alpha \cdot (\alpha_1\alpha_2)$, or

- $|\alpha_2\alpha_3| > 1$ but $\alpha_2\alpha_3 \notin G_\alpha \cdot (\alpha_1\alpha_2)$.

In both cases the only possibilities for G_α are the following copies of the cyclic group C_4 and the dihedral group D_8:

(I) $C_4 = \{\mathrm{id}, (1342), (14)(23), (1243)\}$, or

(II) $D_8 = \{\mathrm{id}, (1243), (14)(23), (1342), (12)(34), (13)(24), (14), (23)\}$.

In both cases a full set of conjugates of $\frac{\alpha_1}{\alpha_4}$ is $\{\frac{\alpha_1}{\alpha_4}, \frac{\alpha_3}{\alpha_2}, \frac{\alpha_4}{\alpha_1}, \frac{\alpha_2}{\alpha_3}\}$. It follows

$$M^{(3)}(\alpha) = M(\frac{\alpha_1}{\alpha_4}) = \pm\frac{\alpha_1}{\alpha_4} \cdot \frac{\alpha_2}{\alpha_3} = (\alpha_1\alpha_2)^2 = M(\alpha)^2$$

Hence, by Lemma 4.1.4 we have $\#\mathcal{O}_M(\alpha) = \infty$.

4.1.2 The case $M^{(2)}(\alpha) = \pm\frac{\alpha_1}{\alpha_3}$

This case occurs if $\alpha_1\alpha_3 \notin G_\alpha \cdot (\alpha_1\alpha_2)$, and $|\alpha_1\alpha_4| > 1$, and $\alpha_1\alpha_4 \in G_\alpha \cdot (\alpha_1\alpha_2)$.

Hence, the only possibilities for G_α are the following copies of the cyclic group C_4 and the dihedral group D_8:

(I) $C_4 = \{\mathrm{id}, (1234), (13)(24), (1432)\}$, or

(II) $D_8 = \{\mathrm{id}, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$.

In both cases a full set of conjugates of $\frac{\alpha_1}{\alpha_3}$ is $\{\frac{\alpha_1}{\alpha_3}, \frac{\alpha_2}{\alpha_4}, \frac{\alpha_3}{\alpha_1}, \frac{\alpha_4}{\alpha_2}\}$. It follows

$$M^{(3)}(\alpha) = M(\frac{\alpha_1}{\alpha_3}) = \pm\frac{\alpha_1}{\alpha_3} \cdot \frac{\alpha_2}{\alpha_4} = (\alpha_1\alpha_2)^2 = M(\alpha)^2$$

Hence, again we have $\#\mathcal{O}_M(\alpha) = \infty$ by Lemma 4.1.4.

4.1.3 The case $M^{(2)}(\alpha) = \pm\frac{\alpha_2}{\alpha_4}$

This case occurs if $\alpha_1\alpha_3 \notin G_\alpha \cdot (\alpha_1\alpha_2)$, and $|\alpha_2\alpha_3| > 1$, and $\alpha_2\alpha_3 \in G_\alpha \cdot (\alpha_1\alpha_2)$.

Hence, the only possibilities for G_α are the following copies of the cyclic group C_4 and the dihedral group D_8:

(I) $C_4 = \{\mathrm{id}, (1234), (13)(24), (1432)\}$, or

(II) $D_8 = \{\mathrm{id}, (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$.

In both cases a full set of conjugates of $\frac{\alpha_2}{\alpha_4}$ is $\{\frac{\alpha_2}{\alpha_4}, \frac{\alpha_3}{\alpha_1}, \frac{\alpha_4}{\alpha_2}, \frac{\alpha_1}{\alpha_3}\}$. It follows

$$M^{(3)}(\alpha) = M(\frac{\alpha_2}{\alpha_4}) = \pm\frac{\alpha_2}{\alpha_4} \cdot \frac{\alpha_1}{\alpha_3} = \pm(\alpha_1\alpha_2)^2 = M(\alpha)^2$$

Hence, also in this case we have $\#\mathcal{O}_M(\alpha) = \infty$.

This concludes the proof of Theorem 25. We now prove Corollary 26:

Proof of Corollary 26. Let α be an algebraic unit of degree 4. We set

$$a_n = \log(M^{(n)}(\alpha))$$

for all $n \in \mathbb{N}$. If $\#\mathcal{O}_M(\alpha) \leq 2$, then $a_{n+1} = a_n$ for all $n \in \mathbb{N}$. If $\#\mathcal{O}_M(\alpha) = \infty$, then Theorem 25 tells us $a_3 = 2a_1$. Moreover, $M^{(4)}(\alpha) = M(M^{(3)}(\alpha)) = M(M(\alpha)^2) = M(M(\alpha))^2 = M^{(2)}(\alpha)^2$. Hence, $a_4 = 2a_2$, and by induction we find $a_{n+1} = 2a_{n-1}$, proving the claim. ∎

4.2 Units with higher degree with restrictions

In this section we will prove Theorem 27. We know that $\#\mathcal{O}_M(\alpha) \in \{1, 2, \infty\}$ whenever α is an algebraic unit of degree ≤ 4. (We note in passing that the orbit size for units of degree less than 4 is trivially 1 or 2.) So we assume from now on that α is an algebraic unit with $[\mathbb{Q}(\alpha) : \mathbb{Q}] = d \geq 5$. Denote by G_α the Galois group of the Galois closure of $\mathbb{Q}(\alpha)$. We assume that G_α contains a subgroup isomorphic to A_d, so G_α is either the full symmetric group or the alternating group. Every self-reciprocal polynomial admits natural restrictions on which permutations of the zeros are given by field automorphisms. Hence, α cannot be conjugated to $\pm$ a Salem number (see [5] for more precise statements on the structure of the Galois group G_α, when α is a Salem number). If one of $\pm\alpha^{\pm 1}$ is conjugated to a Pisot number, then surely $\#\mathcal{O}_M(\alpha) \in \{1, 2\}$. Hence, we assume from now on that none of $\pm\alpha^{\pm 1}$ is conjugated to a Pisot number.

Hence, if we denote by $\alpha_1, \ldots, \alpha_d$ the Galois conjugates of α, we assume

$$|\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_r| > 1 \geq |\alpha_{r+1}| \geq \ldots \geq |\alpha_d|, \tag{4.2.1}$$

where $r \in \{2, \ldots, d-2\}$ and $1 > |\alpha_{d-1}|$.

We identify G_α with a subgroup of S_d, by the action on the indices of $\alpha_1, \ldots, \alpha_d$. In particular, for any $\sigma \in A_d$ and any $f_1, \ldots, f_d \in \mathbb{Z}$ the element

$$\sigma \cdot (\alpha_1^{f_1} \cdot \ldots \cdot \alpha_d^{f_d}) := \alpha_{\sigma(1)}^{f_1} \cdot \ldots \cdot \alpha_{\sigma(d)}^{f_d}$$

is a Galois conjugate of $\alpha_1^{f_1} \cdot \ldots \cdot \alpha_d^{f_d}$.

Lemma 4.2.1 *Let $i, j, k, l \in \{1, \ldots, d\}$ be pairwise distinct, and let $f_1, \ldots, f_d \in \mathbb{Z}$. Then*

a. $(i, j, k) \cdot (\alpha_1^{f_1} \cdots \alpha_d^{f_d}) = \alpha_1^{f_1} \cdots \alpha_d^{f_d} \iff f_i = f_j = f_k.$

b. $(i, j)(k, l) \cdot (\alpha_1^{f_1} \cdots \alpha_d^{f_d}) = \alpha_1^{f_1} \cdots \alpha_d^{f_d} \iff f_i = f_j$ *and* $f_k = f_l.$

Proof. In both statements, the implication $\Longleftarrow$ is trivial. Lets start with the other implication in (a). It is

$$(i, j, k) \cdot (\alpha_1^{f_1} \cdots \alpha_d^{f_d}) = \alpha_1^{f_1} \cdots \alpha_d^{f_d} \implies \alpha_j^{f_i - f_j} \cdot \alpha_k^{f_j - f_k} \cdot \alpha_i^{f_k - f_i} = 1$$

Since $d \geq 5$, we may choose two conjugates of α not among $\alpha_i, \alpha_j, \alpha_k$ – say α_p and α_q. Since G_α contains A_d, the elements $(i, j)(p, q)$, $(i, k)(p, q)$, $(j, k)(p, q)$, (i, j, k), and (i, k, j) are all contained in G_α. Applying these automorphisms to $\alpha_j^{f_i - f_j} \cdot \alpha_k^{f_j - f_k} \cdot \alpha_i^{f_k - f_i} = 1$, yields

$$\begin{aligned}
\alpha_j^{f_i - f_j} \cdot \alpha_k^{f_j - f_k} \cdot \alpha_i^{f_k - f_i} &= 1 = \alpha_j^{f_i - f_j} \cdot \alpha_i^{f_j - f_k} \cdot \alpha_k^{f_k - f_i} \\
\alpha_i^{f_i - f_j} \cdot \alpha_j^{f_j - f_k} \cdot \alpha_k^{f_k - f_i} &= 1 = \alpha_k^{f_i - f_j} \cdot \alpha_j^{f_j - f_k} \cdot \alpha_i^{f_k - f_i} \\
\alpha_k^{f_i - f_j} \cdot \alpha_i^{f_j - f_k} \cdot \alpha_j^{f_k - f_i} &= 1 = \alpha_i^{f_i - f_j} \cdot \alpha_k^{f_j - f_k} \cdot \alpha_j^{f_k - f_i}.
\end{aligned}$$

Hence

$$\left(\frac{\alpha_i}{\alpha_k}\right)^{2f_k - f_i - f_j} = 1, \qquad \left(\frac{\alpha_i}{\alpha_k}\right)^{2f_i - f_j - f_k} = 1, \quad \text{and} \quad \left(\frac{\alpha_i}{\alpha_k}\right)^{2f_j - f_k - f_i} = 1.$$

But $\frac{\alpha_i}{\alpha_k}$ is no root of unity, since it is a Galois conjugate of $\frac{\alpha_1}{\alpha_d}$, which lies outside the unit circle. It follows $2f_k - f_i - f_j = 2f_i - f_j - f_k = 2f_j - f_k - f_i = 0$, and hence $f_i = f_j = f_k$. This proves part (a).

Part (b) follows similarly: $(i,j)(k,l)\cdot(\alpha_1^{f_1}\cdots\alpha_d^{f_d}) = \alpha_1^{f_1}\cdots\alpha_d^{f_d}$ implies

$$\alpha_i^{f_j}\cdot\alpha_j^{f_i}\cdot\alpha_k^{f_l}\cdot\alpha_l^{f_k} = \alpha_i^{f_i}\cdot\alpha_j^{f_j}\cdot\alpha_k^{f_k}\cdot\alpha_l^{f_l}.$$

Without loss of generality, we assume $f_j \geq f_i$ and $f_k \geq f_l$. Using that $(i,l)(j,k)$ is an element of G_α, we get

$$\left(\frac{\alpha_j}{\alpha_i}\right)^{f_j-f_i} = \left(\frac{\alpha_k}{\alpha_l}\right)^{f_k-f_l} \quad \text{and} \quad \left(\frac{\alpha_k}{\alpha_l}\right)^{f_j-f_i} = \left(\frac{\alpha_j}{\alpha_i}\right)^{f_k-f_l}.$$

Multiplying both equations yields

$$\left(\frac{\alpha_j}{\alpha_i}\right)^{(f_j-f_i)+(f_k-f_l)} = \left(\frac{\alpha_k}{\alpha_l}\right)^{(f_j-f_i)+(f_k-f_l)},$$

and hence

$$\left(\frac{\alpha_j\cdot\alpha_l}{\alpha_i\cdot\alpha_k}\right)^{(f_j-f_i)+(f_k-f_l)} = 1.$$

Again, $\frac{\alpha_j\cdot\alpha_l}{\alpha_i\cdot\alpha_k}$ is a Galois conjugate of $\frac{\alpha_1\cdot\alpha_2}{\alpha_{d-1}\cdot\alpha_d}$, which lies outside the unit circle, and hence is not a root of unity. Therefore $(f_j-f_i)+(f_k-f_l)=0$. Since $f_j \geq f_i$ und $f_k \geq f_l$, it follows $f_j = f_i$ and $f_k = f_l$, proving the lemma. ∎

Lemma 4.2.2 *Let $f_1,\ldots,f_d$ be pairwise distinct integers. Then $[\mathbb{Q}(\alpha_1^{f_1}\cdots\alpha_d^{f_d}):\mathbb{Q}] = \#G_\alpha$.*

Proof. The proof is essentially the same as the proof of part (1) in Theorem 1.1 from [2]. Assume there is a $\sigma^{-1}\in G_\alpha\subseteq S_d$ such that $\alpha_1^{f_1}\cdots\alpha_d^{f_d} = \sigma^{-1}\cdot(\alpha_1^{f_1}\cdots\alpha_d^{f_d})$. Then

$$1 = \alpha_1^{f_1-f_{\sigma(1)}}\cdots\alpha_d^{f_d-f_{\sigma(d)}}. \tag{4.2.2}$$

If σ is an odd permutation, then $G_\alpha = S_d$, then it was already proven by Smyth (see Lemma 1 of [26]) that $f_i = f_{\sigma(i)}$ for all i, hence that $\sigma = \mathrm{id}$. If σ is an even permutation, then by repeated application of Lemma 4.2.1 to equation (4.2.2) above, this is only possible if $f_i - f_{\sigma(i)}$ is the same integer for all $i\in\{1,\ldots,d\}$, say $f_i - f_{\sigma(i)} = k$.

Since $\sigma^{d!} = \mathrm{id}$, it follows

$$f_1 = k + f_{\sigma(1)} = 2k + f_{\sigma^2(1)} = \ldots = d!\cdot k + f_{\sigma^{d!}(1)} = d!\cdot k + f_1,$$

and hence $k = 0$. Therefore we have $f_i = f_{\sigma(i)}$ for all $i \in \{1, \ldots, d\}$. But by assumption the integers $f_1, \ldots, f_d$ are pairwise distinct, hence, we must again have $\sigma = \mathrm{id}$. Since in either case, $\sigma = \mathrm{id}$, this means that the images of $\alpha_1^{f_1} \cdots \alpha_d^{f_d}$ are distinct under each non-identity element of G_α, so $[\mathbb{Q}(\alpha_1^{f_1} \cdots \alpha_d^{f_d}) : \mathbb{Q}] = \#G_\alpha$. ■

Proposition 38 *Let $M^{(n)}(\alpha) = \alpha_1^{e_1} \cdot \ldots \cdot e_d^{e_d}$ such that the exponents $e_1, \ldots, e_d$ are pairwise distinct. Then $M^{(n+1)}(\alpha) > M^{(n)}(\alpha)$.*

Proof. We denote by Z_3 the set of 3-cycles in $G_\alpha \subseteq S_d$. For any $k \in \{1, \ldots, d\}$, the number of 3-cycles which fix k is equal to $\frac{(d-1)(d-2)(d-3)}{3}$. For any pair $k \neq k' \in \{1, \ldots, d\}$, the number of 3-cycles sending k to k' is $(d-2)$. Therefore,

$$\left|\prod_{\tau \in Z_3} \tau \cdot M^{(n)}(\alpha)\right| = \left|\alpha_1^{\frac{(d-1)(d-2)(d-3)}{3} e_1 + (d-2)\sum_{k \neq 1} e_k} \cdot \ldots \cdot \alpha_d^{\frac{(d-1)(d-2)(d-3)}{3} e_d + (d-2)\sum_{k \neq d} e_k}\right|. \quad (4.2.3)$$

Since α is an algebraic unit, we have $\prod_{j=1}^d \alpha_j^{\sum_{k=1}^d e_k} = \pm 1$. Hence, the value in (4.2.3) is equal to

$$\left|\prod_{j=1}^d \alpha_j^{\left(\frac{(d-1)(d-2)(d-3)}{3} - (d-2)\right)e_j}\right| = M^{(n)}(\alpha)^{\frac{(d-1)(d-2)(d-3)}{3} - (d-2)} > M^{(n)}(\alpha).$$

The last inequality follows from our general hypothesis that $d \geq 5$. Since $e_1, \ldots, e_d$ are assumed to be pairwise distinct, it follows from Lemma 4.2.2 that the factors $\tau \cdot M^{(n)}(\alpha)$ in (4.2.3) are also pairwise distinct conjugates of $M^{(n)}(\alpha)$. In particular

$$M^{(n+1)}(\alpha) = M(M^n(\alpha)) \geq \left|\prod_{\tau \in Z_3} \tau \cdot M^{(n)}(\alpha)\right| > M^{(n)}(\alpha)$$

which is what we needed to prove. ■

Lemma 4.2.3 *Let $n \in \mathbb{N}$ and let $M^{(n)}(\alpha) = \alpha_1^{e_1} \cdots \alpha_d^{e_d}$. Then we have:*

1. *$e_i \geq e_{i+1}$ for all but at most one $i \in \{1, \ldots, d-1\}$.*

2. *If $e_i < e_{i+1}$ for some $i \in \{2, \ldots, d-1\}$, then $e_{i-1} > e_{i+1}$.*

3. If $e_i < e_{i+1}$ for some $i \in \{1, \ldots, d-2\}$, then $e_i > e_{i+2}$.

4. If $e_i < e_{i+1}$ for some $i \in \{1, \ldots, d-1\}$, then

$$e_1 > e_2 > \cdots > e_{i-1} > e_{i+1} > e_i > e_{i+2} > e_{i+3} > \cdots > e_d.$$

Proof. It is known that $M^{(n)}(\alpha)$ is a Perron number, which means that $M^{(n)}(\alpha)$ does not have a Galois conjugate of the same or larger modulus (cf. [11] for this and other properties of values of the Mahler measure). This fact will be used several times in the following proof.

To prove (1), we have two cases: there are three distinct elements $1 \leq i < j < k \leq d$ such that $e_i < e_j < e_k$, or else there exist $1 \leq i < j < k < l \leq d$ such that $e_i < e_j$ and $e_k < e_l$. Assume first that there are three distinct elements $1 \leq i < j < k \leq d$ such that $e_i < e_j < e_k$. Recall that by definition we have $|\alpha_i| \geq |\alpha_j| \geq |\alpha_k|$. Therefore $|\alpha_k|^{e_k - e_i} \leq |\alpha_j|^{e_k - e_i}$, which implies

$$\begin{aligned}
&\underbrace{|\alpha_i|^{e_i - e_j}}_{\leq |\alpha_j|^{e_i - e_j}} \cdot |\alpha_j|^{e_j - e_k} \cdot |\alpha_k|^{e_k - e_i} \leq |\alpha_j|^{e_i - e_k} \cdot |\alpha_k|^{e_k - e_i} \leq 1 \\
\implies\ & |\alpha_i^{e_i} \cdot \alpha_j^{e_j} \cdot \alpha_k^{e_k}| \leq |\alpha_i^{e_j} \cdot \alpha_j^{e_k} \cdot \alpha_k^{e_i}| \\
\implies\ & |M^{(n)}(\alpha)| \leq |(i, k, j) \cdot M^{(n)}(\alpha)|.
\end{aligned}$$

By Lemma 4.2.1, $(i, j, k) \cdot M^{(n)}(\alpha) \neq M^{(n)}(\alpha)$ is a Galois conjugate of $M^{(n)}(\alpha)$. This contradicts the fact that $M^{(n)}(\alpha)$ is a Perron number. In particular, it is not possible that $e_i > e_{i+1} > e_{i+2}$ for any $i \in \{1, \ldots, d-2\}$.

Now assume that we have $1 \leq i < j < k < l \leq d$ such that $e_i < e_j$ and $e_k < e_l$. Then $|\alpha_i| \geq |\alpha_j|$ and $|\alpha_k| \geq |\alpha_l|$ imply

$$|\alpha_i|^{e_j - e_i} \cdot |\alpha_k|^{e_l - e_k} \geq |\alpha_j|^{e_j - e_i} \cdot |\alpha_l|^{e_l - e_k},$$

and hence

$$|\alpha_i^{e_i} \cdot \alpha_j^{e_j} \cdot \alpha_k^{e_k} \cdot \alpha_l^{e_l}| \leq |\alpha_i^{e_j} \cdot \alpha_j^{e_i} \cdot \alpha_k^{e_l} \cdot \alpha_l^{e_k}|.$$

This, however, is equivalent to $|M^{(n)}(\alpha)| \leq |(i,j)(k,l) \cdot M^{(n)}(\alpha)|$, which is not possible by Lemma 4.2.1, since $M^{(n)}(\alpha)$ is a Perron number. This proves part (1) of the lemma.

In order to prove part (2), we assume for the sake of contradiction that $e_i < e_{i+1}$ but $e_{i-1} \leq e_{i+1}$ for some $i \in \{2, \dots, d-1\}$. By part (1), since we already have $e_i < e_{i+1}$, we know that $e_{i-1} \geq e_i$. We have

$$\begin{aligned}
&(i-1, i, i+1) \cdot |\alpha_{i-1}|^{e_{i-1}} |\alpha_i|^{e_i} |\alpha_{i+1}|^{e_{i+1}} \\
&= |\alpha_i|^{e_{i-1}} |\alpha_{i+1}|^{e_i} |\alpha_{i-1}|^{e_{i+1}}
\end{aligned}$$

Now,

$$\begin{aligned}
&|\alpha_{i-1}|^{e_{i-1}-e_{i+1}} |\alpha_i|^{e_i-e_{i-1}} |\alpha_{i+1}|^{e_{i+1}-e_i} \\
&= |\alpha_{i-1}|^{e_{i-1}-e_{i+1}} |\alpha_i|^{e_i-e_{i-1}} |\alpha_{i+1}|^{e_{i+1}-e_{i-1}} |\alpha_{i+1}|^{e_{i-1}-e_i} \\
&\leq |\alpha_{i-1}|^{e_{i-1}-e_{i+1}} |\alpha_i|^{e_i-e_{i-1}} |\alpha_{i-1}|^{e_{i+1}-e_{i-1}} |\alpha_i|^{e_{i-1}-e_i} \\
&= 1
\end{aligned}$$

Therefore,

$$|M^{(n)}(\alpha)| \leq |(i-1, i, i+1) \cdot M^{(n)}(\alpha)|,$$

giving a contradiction to $M^{(n)}(\alpha)$ being a Perron number.

Similarly, if $e_i < e_{i+1}$ and $e_i \leq e_{i+2}$, then we know by (a) that $e_{i+1} \geq e_{i+2}$. This implies that $|M^{(n)}(\alpha)| \leq |(i, i+2, i+1) \cdot M^{(n)}(\alpha)|$. This proves part (3).

So far we have proven that if $e_i < e_{i+1}$ for some $i \in \{1, \dots, d-1\}$, then we have

$$e_1 \geq e_2 \geq \ldots \geq e_{i-1} > e_{i+1} > e_i > e_{i+2} \geq e_{i+3} \geq \ldots \geq e_d.$$

We need to show that all of the above inequalities are strict. Assume that this is not the case, and that $e_k = e_{k+1}$. Then $k, k+1, i, i+1$ must be pairwise distinct. It follows, that $|(i, i+1)(k, k+1) \cdot M^{(n)}(\alpha)| = |(i, i+1) \cdot M^{(n)}(\alpha)| > |M^{(n)}(\alpha)|$, which is a contradiction. The last inequality just follows from the fact that $|\alpha_i|^{e_{i+1}} \cdot |\alpha_{i+1}|^{e_i} > |\alpha_i|^{e_i} \cdot |\alpha_{i+1}|^{e_{i+1}}$. ■

Lemma 4.2.4 *Let $f_1 \geq f_2 \geq \ldots \geq f_k \geq 0$ be integers, with $f_1 \geq 1$, and let $a_1 \geq a_2 \geq \ldots \geq a_d > 0$ be real numbers such that $\prod_{i=1}^k a_i > 1$. Then $\prod_{i=1}^k a_i^{f_i} > 1$.*

Proof. We prove the statement by induction on k, where the base case $k = 1$ is trivial. Now assume that the statement is true for k and that there are real numbers $a_1 \geq \ldots \geq a_{k+1} > 0$, with $\prod_{i=1}^{k+1} a_i > 1$, and integers $f_1 \geq \ldots \geq f_{k+1} \geq 0$, with $f_1 \geq 1$. If $f_1 = f_{k+1}$, then the claim follows immediately. Hence, we assume $f_1 > f_{k+1}$. Set $f_i' = f_i - f_{k+1}$ for all $i \in \{1, \ldots, k+1\}$. Then

$$f_1' \geq f_2' \geq \ldots f_k' \geq f_{k+1}' = 0 \text{ and } f_1' \geq 1.$$

Moreover, $\prod_{i=1}^k a_i$ is either greater than or equal to $\prod_{i=1}^{k+1} a_i > 1$ (if $a_{k+1} \leq 1$), or it is a product of real numbers > 1. Hence, our induction hypothesis states $\prod_{i=1}^k a_i^{f_i'} > 1$. This implies

$$\prod_{i=1}^{k+1} a_i^{f_i} = \underbrace{\left(\prod_{i=1}^{k+1} a_i\right)^{f_{k+1}}}_{\geq 1} \cdot \left(\prod_{i=1}^{k} a_i^{f_i'}\right) > 1,$$

proving the lemma. ■

Proposition 39 *Let $M^{(n)}(\alpha) = \alpha_1^{e_1} \cdots \alpha_d^{e_d}$. If $e_{i+1} \leq e_i$ for all $i \in \{1, \ldots, d-1\}$, then $M^{(n+1)}(\alpha) > M^{(n)}(\alpha)$.*

Proof. We show that $M^{(n)}(\alpha)$ has a non-trivial Galois conjugate outside the unit circle. This immediately implies the claim.

Since α is an algebraic unit, we may assume that $e_d = 0$. Note however, that this uses our assumption $e_{i+1} \leq e_i$ for all i. We set

$$s := \max\{i \in \{1, \ldots, d\} | e_i \neq 0\}.$$

By Proposition 38 we may assume that we have $e_i = e_{i+1}$ for some $i \in \{1, \ldots, d-1\}$. This i is not equal to s, since $e_s \neq 0 = e_{s+1}$ by definition. If $i \notin \{s-1, s+1\}$, then $(i, i+1)(s, s+1) \cdot M^{(n)}(\alpha) = \alpha_1^{e_1} \cdots \alpha_{s-1}^{e_{s-1}} \alpha_s^{e_{s+1}} \alpha_{s+1}^{e_s}$. If $i = s-1$, then $(s-1, s+1, s) \cdot$

$M^{(n)}(\alpha) = \alpha_1^{e_1} \cdots \alpha_{s-2}^{e_{s-2}} \alpha_{s-1}^{e_s} \alpha_s^{e_{s+1}} \alpha_{s+1}^{e_{s-1}} = \alpha_1^{e_1} \cdots \alpha_{s-1}^{e_{s-1}} \alpha_s^{e_{s+1}} \alpha_{s+1}^{e_s}$. If finally $i = s+1$, then $(s, s+1, s+2) \cdot M^{(n)}(\alpha) = \alpha_1^{e_1} \cdots \alpha_{s-1}^{e_{s-1}} \alpha_s^{e_{s+2}} \alpha_{s+1}^{e_s} \alpha_{s+2}^{e_{s+1}} = \alpha_1^{e_1} \cdots \alpha_{s-1}^{e_{s-1}} \alpha_s^{e_{s+1}} \alpha_{s+1}^{e_s}$.

Since $e_{s+1} = 0$, we see that in any case

$$\alpha_1^{e_1} \cdots \alpha_{s-1}^{e_{s-1}} \alpha_{s+1}^{e_s} \text{ is a non-trivial Galois conjugate of } M^{(n)}(\alpha). \tag{4.2.4}$$

We will prove that this Galois conjugate lies outside the unit circle. Again we distinguish several cases.

If $s \leq r-1$, then all of the elements $\alpha_1, \ldots, \alpha_{s+1}$ lie outside the unit circle. Hence $|\alpha_1 \cdots \alpha_{s-1} \alpha_{s+1}| > 1$.

If $s \geq r+1$, then $|\alpha_1 \cdots \alpha_{s-1} \alpha_{s+1}| = |\alpha_s \alpha_{s+2} \cdots \alpha_d|^{-1} > 1$, since all of $\alpha_s, \ldots, \alpha_d$ lie inside the closed unit disc and $|\alpha_d| < 1$.

Lastly, we consider the case $2 \leq s = r \leq d-2$. Then surely $|\alpha_1 \cdots \alpha_{r-1}| \geq |\alpha_r|$ and $|\alpha_{r+1}| \geq |\alpha_{r+2} \cdots \alpha_d|$, where the first inequality is strict whenever $r \neq 2$, and the second inequality is strict whenever $r \neq d-2$. By our general assumption it is $d \geq 5$ and hence $|\alpha_1 \cdots \alpha_{r-1} \alpha_{r+1}| > |\alpha_r \alpha_{r+2} \cdots \alpha_d|$. Since the product of all α_i is ± 1, it follows $|\alpha_1 \cdots \alpha_{s-1} \alpha_{s+1}| > 1$.

Hence, in any case we have $|\alpha_1| \cdots |\alpha_{s-1}| \cdot |\alpha_{s+1}| > 1$. From our assumption $e_1 \geq \ldots \geq e_d$ it follows by Lemma 4.2.4 that $|\alpha_1^{e_1} \cdots \alpha_{s-1}^{e_{s-1}} \alpha_{s+1}^{e_s}| > 1$. Therefore, $M^{(n)}(\alpha)$ has a non-trivial Galois conjugate outside the unit circle (see (4.2.4)). Hence $M^{(n+1)}(\alpha) = M(M^{(n)}(\alpha)) > M^{(n)}(\alpha)$. ■

We are now ready to prove Theorem 27.

Proof of Theorem 27. As stated at the beginning of this section, we may assume that $d \geq 5$, and that the elements $\pm\alpha^{\pm 1}$ are neither conjugates of a Pisot, nor a Salem number. Hence, we may assume that the hypothesis (4.2.1) is met. Let $n \in \mathbb{N}$ be arbitrary. Then for some $e_1, \ldots, e_d \in \mathbb{N}_0$, we have $M^{(n)}(\alpha) = \alpha_1^{e_1} \cdots \alpha_d^{e_d}$. We have seen in Lemma 4.2.3, that one of the following statements applies:

1. $e_1 \geq e_2 \geq \ldots \geq e_d$, or

2. the integers $e_1, \ldots, e_d$ are pairwise distinct.

In case (i), we have $M^{(n+1)}(\alpha) > M^{(n)}(\alpha)$ by Proposition 39. In case (ii), we have $M^{(n+1)}(\alpha) > M^{(n)}(\alpha)$ by Proposition 38. Hence $\#\mathcal{O}_M(\alpha) = \infty$. ■

4.3 The existence of units with orbit sizes greater than 2

Let $d = 4k$, with an integer $k \geq 3$. Now, we will show that there exist algebraic units of degree d with arbitrarily large orbit size, proving Theorem 28.

Proof of Theorem 28. Let α_1, β_1 be positive real algebraic units satisfying:

1. $[\mathbb{Q}(\beta_1) : \mathbb{Q}] = 2$, $\beta_1 > 1$,

2. α_1 is a Salem number of degree $2k$.

3. The fields $\mathbb{Q}(\alpha_1)$ and $\mathbb{Q}(\beta_1)$ are linearly disjoint.

For any $k \geq 3$ we can indeed find such α_1 and β_1. Since there are Salem numbers of any even degree ≥ 4 we find an appropriate α_1. Now, we take any prime p which is unrammified in $\mathbb{Q}(\alpha_1)$, and let $\beta_1 > 1$ be an algebraic unit in $\mathbb{Q}(\sqrt{p})$. Note that if the above conditions are met by α_1 and β_1, then they are met by α_1^ℓ and $\beta_1^{\ell'}$, for any $\ell, \ell' \in \mathbb{N}$.

We denote the conjugates of α_1 by $\alpha_2, \cdots, \alpha_{2k}$, with $\alpha_{2k} = \alpha_1^{-1}$, and the conjugate of β_1 is $\beta_2 = \beta_1^{-1}$. Note that $\alpha_2, \cdots, \alpha_{2k-1}$ all lie on the unit circle. By assumption (3) the element $\alpha_1\beta_1$ has degree $4k$ and a full set of Galois conjugates of $\alpha_1\beta_1$ is given by

$$\{\alpha_i\beta_j : (i,j) \in \{1, \ldots, 2k\} \times \{1, 2\}\}.$$

There are two cases. First, if $\beta_1 > \alpha_1$, then $|\alpha_i\beta_1| > 1$ for all $i \in \{1, \cdots, 2k\}$ and $|\alpha_i\beta_2| < |\alpha_i\alpha_6| \leq 1$ for all $i \in \{1, \cdots, 2k\}$, hence,

$$M(\alpha_1\beta_1) = \left|\prod_{n=1}^{2k} \alpha_i\beta_1\right| = \beta_1^{2k} \tag{4.3.1}$$

For the second case, if $\beta_1 < \alpha_1$, then

$$|\alpha_i\beta_1| > 1 \iff i \in \{1, \cdots, 2k-1\}, \text{ and } |\alpha_i\beta_2| > 1 \iff i = 1.$$

Therefore

$$M(\alpha_1\beta_1) = |\alpha_1\beta_1| \cdot \left|\prod_{n=2}^{2k-1} \alpha_i\beta_1\right| \cdot |\alpha_1\beta_2| = \alpha_1^2\beta_1^{2k-2}. \tag{4.3.2}$$

We now construct an algebraic unit of degree $4k$ of finite orbit size $> S$. Let $\ell \in \mathbb{N}$ be such that $(\alpha_1^\ell)^{2^S} > \beta_1^{(2k-2)^S}$. Then by (4.3.2), we have $M(\alpha_1^\ell\beta_1) = (\alpha_1^\ell)^2\beta_1^{2k-2}$, $M^{(2)}(\alpha_1^\ell\beta_1) = M((\alpha_1^\ell)^2)(\beta_1^{2k-2})) = (\alpha_1^\ell)^{2^2}\beta_1^{(2k-2)^2}, \cdots, M^{(S)}(\alpha_1^\ell\beta_1) = (\alpha_1^\ell)^{2^S}\beta_1^{(2k-2)^S}$. Hence, the orbit size of $\alpha_1^\ell\beta_1$ is greater than S. However, there exists $S' > S$ such that $(\alpha_1^\ell)^{2^{S'}} < \beta_1^{(2k-2)^{S'}}$. Assume that S' is minimal with this property. Then we have

$$M^{(S'+1)}(\alpha_1^\ell\beta_1) = (\alpha_1^\ell)^{2^{S'}}\beta_1^{(2k-2)^{S'}} \overset{(4.3.1)}{=} (\beta_1^{4^{S'}})^{2k},$$

which is of degree 2. Therefore, the orbit size of $\alpha_1^\ell\beta_1$ is $S' + 2 > S$. ■

CHAPTER V

CLASSIFICATION OF NUMBER FIELDS BY ORBIT SIZE

5.1 Fields of degree four

In this section, we will prove Theorem 29 and Proposition 42.

5.1.1 Totally imaginary extensions

We will show that

Proposition 40 *If $K/\mathbb{Q}$ is a totally complex number field of degree 4, then all elements in K are preperiodic under iteration of the Mahler measure.*

Let $\alpha \in \overline{\mathbb{Q}}$ be of degree 4 and totally imaginary. Moreover, let $a \in \mathbb{N}$ be the leading coefficient of its minimal polynomial (note, that we do not assume that α is an algebraic integer). Denote the Galois conjugates of α by α_1, α_2, α_3, α_4, such that $|\alpha_1| = |\alpha_2| > |\alpha_3| = |\alpha_4|$. Then the Mahler measure of α is one of the elements a, $a\alpha_1\alpha_2$, $a\alpha_1\alpha_2\alpha_3\alpha_4$.

Since a and $a\alpha_1\alpha_2\alpha_3\alpha_4 =: e$ are in $\mathbb{Z}$, our α is preperiodic, whenever $M(\alpha) \neq a\alpha_1\alpha_2$. Hence, from now on we assume

$$|\alpha_1| = |\alpha_2| > 1 > |\alpha_3| = |\alpha_4|.$$

The Galois conjugates of $a\alpha_1\alpha_2$ all lie in the set

$$\{a\alpha_1\alpha_2, a\alpha_1\alpha_3, a\alpha_1\alpha_4, a\alpha_2\alpha_3, a\alpha_3\alpha_4\}.$$

Before we proceed, we remark that $a\alpha_1\alpha_2$ is a Perron integer.

Lemma 5.1.1 *If $\alpha_1\alpha_3 = \alpha_2\alpha_4$ (resp. $\alpha_1\alpha_4 = \alpha_2\alpha_3$), then $\alpha_1\alpha_3$ (resp. $\alpha_1\alpha_4$) is not a Galois conjugate of $\alpha_1\alpha_2$.*

Proof. Assume $\alpha_1\alpha_3 = \alpha_2\alpha_4$, then $(\alpha_1\alpha_3)^2 = \alpha_1\alpha_2\alpha_3\alpha_4 \in \mathbb{Q}$. Therefore, $\alpha_1\alpha_3$ cannot be a Galois conjugate of the Perron number $\alpha_1\alpha_2$. The same argument applies if $\alpha_1\alpha_4 = \alpha_2\alpha_3$. ■

We proceed to prove Proposition 40:

Proof for Proposition 40. If all Galois conjugates of $a\alpha_1\alpha_2$ have absolute value ≥ 1, then $a\alpha_1\alpha_2$ – and hence α – is surely preperiodic. So we assume that $a\alpha_3\alpha_4 < 1$.

Since in any case we have $\alpha_1\alpha_3 = \overline{\alpha_2\alpha_4}$, it follows,

$$\begin{aligned} M^{(2)}(\alpha) &= M(a\alpha_1\alpha_2) \\ &\in \{a\alpha_1\alpha_2, a^3\alpha_1\alpha_2\alpha_1\alpha_3\alpha_2\alpha_4, a^5\alpha_1\alpha_2\alpha_1\alpha_3\alpha_1\alpha_4\alpha_2\alpha_4\alpha_2\alpha_3\} \\ &= \{a\alpha_1\alpha_2, a^2e\alpha_1\alpha_2, a^3e^2\alpha_1\alpha_2\} \end{aligned}$$

If $|ae| = 1$, which is precisely the case if α is an algebraic unit, then $M^{(2)}(\alpha) = M(\alpha)$ and α is preperiodic. Assuming that $|ae| > 1$, and $M^{(2)}(\alpha) \neq M(\alpha)$, then the possible Galois conjugates of $M^{(2)}(\alpha)$ are

$$\{a^ie^j\alpha_1\alpha_2, a^ie^j\alpha_1\alpha_3, a^ie^j\alpha_1\alpha_4, a^ie^j\alpha_2\alpha_3, a^ie^j\alpha_3\alpha_4\},$$

with $(i,j) \in \{(2,1),(3,2)\}$. As before, either $|a^ie^j\alpha_3\alpha_4| > 1$, and we are done, or $M^{(3)}(\alpha)$ is an element from

$$\{a^ie^j\alpha_1\alpha_2, a^{3i-1}e^{3j+1}\alpha_1\alpha_2, a^{5i-2}e^{5j+2}\alpha_1\alpha_2\}.$$

Since each iteration increases the power of the whole number in front of the $\alpha_1\alpha_2$, after finitely many iterations, all Galois conjugates of some $M^{(k)}(\alpha)$ lie outside the unit circle, and it is fixed. Hence, α is preperiodic in all cases as claimed. ■

5.1.2 Biquadratic extensions

Let $p, q \in \mathbb{Z}$ be squarefree integers. We will prove, that all elements in $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ are preperiodic under iteration of the Mahler measure. (By the last section, we can then assume that p and q are positive).

Let $\alpha \in K$ be arbitrary. Since $K/\mathbb{Q}$ is Galois, $M(\alpha)$ is still in K. Moreover, $M(\alpha)$ is an algebraic integer. Hence, we may assume without loss of generality, that α is an algebraic integer, with Galois conjugates

$$|\alpha_1| > |\alpha_2| \geq |\alpha_3| \geq |\alpha_4|.$$

If precisely one, or precisely four, conjugates of α lie outside the unit circle, then α is obviously preperiodic. We assume first, that precisely three conjugates of α lie outside the unit circle. Then $M(\alpha) = |\alpha_1\alpha_2\alpha_3| = \left|\frac{N(\alpha)}{\alpha_4}\right|$. Again, if one or four conjugates of $\frac{N(\alpha)}{\alpha_4}$ lie outside the unit circle, then α is obviously preperiodic. But if precisely three conjugates of $\frac{N(\alpha)}{\alpha_4}$ lie outside the unit circle, then α satisfies the assumptions of Proposition 35, and we can conclude that α is preperiodic.

Hence, we are left with the case that precisely two conjugates of α or $M(\alpha)$ lie outside the unit circle. Since, as before, $M(\alpha)$ is also an element from K, we may assume without loss of generality that

$$|\alpha_1| > |\alpha_2| > 1 \geq |\alpha_3| \geq |\alpha_4|.$$

Since $\alpha = \alpha_1 \in K$, there are $a, b, c, d \in \mathbb{Q}$, such that $\alpha = a + b\sqrt{p} + c\sqrt{q} + d\sqrt{pq}$. Then α_2 must be one of the elements

$$a - b\sqrt{p} + c\sqrt{q} - d\sqrt{pq} \quad , \quad a + b\sqrt{p} - c\sqrt{q} - d\sqrt{pq} \quad , \quad a - b\sqrt{p} - c\sqrt{q} + d\sqrt{pq}.$$

In any case, $M(\alpha) = \pm\alpha_1\alpha_2$ is a quadratic integer, and hence α is preperiodic. This gives the following proposition

Proposition 41 *If $K/\mathbb{Q}$ is biquadratic, then all elements in K are preperiodic under iteration of the Mahler measure.*

Some extensions of signature (2,1) contain a wandering point:

Proposition 42 *If $a, b, d \in \mathbb{Z}$ are such that $\sqrt{a+b\sqrt{d}}$ generates a real number field K of signature $(2,1)$, then every element in K is preperiodic under iteration of the Mahler measure.*

Proof. Let $d \in \mathbb{N}$, and $a, b \in \mathbb{Z}$, such that $a + b\sqrt{d} > 0$ and $a - b\sqrt{d} < 0$. Set $K = \mathbb{Q}(\sqrt{a+b\sqrt{d}})$. Note, that the case $a = 0$, $b = 1$ gives a radical extension $K = \mathbb{Q}(\sqrt[4]{d})$. Moreover, our assumptions guarantee that K has signature $(2,1)$. We aim to prove that all elements in K are preperiodic under iteration of M.

Let $\alpha \in K$ be of degree four. Then there are $r, s, t, u \in \mathbb{Q}$ such that

$$\alpha = \alpha_1 = r + s\sqrt{a+b\sqrt{d}} + t(a+b\sqrt{d}) + u\left(\sqrt{a+b\sqrt{d}}\right)^3.$$

Since the degree of α is four, the Galois conjugates of α are

$$\begin{aligned}
\alpha_2 &= r - s\sqrt{a+b\sqrt{d}} + t(a+b\sqrt{d}) - u\left(\sqrt{a+b\sqrt{d}}\right)^3 \\
\alpha_3 &= r + s\sqrt{a-b\sqrt{d}} + t(a-b\sqrt{d}) + u\left(\sqrt{a-b\sqrt{d}}\right)^3 \\
\alpha_4 &= r - s\sqrt{a-b\sqrt{d}} + t(a-b\sqrt{d}) - u\left(\sqrt{a-b\sqrt{d}}\right)^3.
\end{aligned}$$

Let $a \in \mathbb{N}$ denote the leading coefficient of the minimal polynomial of α. Then

$$M(\alpha) \in \{a\alpha_1, a\alpha_2, a\alpha_1\alpha_2, a\alpha_1\alpha_3\alpha_4, a\alpha_2\alpha_3\alpha_4, a\alpha_3\alpha_4, a\alpha_1\alpha_2\alpha_3\alpha_4\}.$$

Note, that by assumption $\alpha_3 = \overline{\alpha_4}$. Moreover, $\alpha_3\alpha_4, \alpha_1\alpha_2 \in \mathbb{Q}(\sqrt{d})$, and $K = \mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_2)$. Hence, $M(\alpha) \in K$.

Therefore, we may assume that $\alpha = \alpha_1$ is an algebraic integer in K of absolute value > 1. If $\alpha_2, \alpha_3, \alpha_4$ lie all outside (or inside) the closed unit circle, α is surely preperiodic. So we assume that one of $\alpha_2, \alpha_3, \alpha_4$ lies inside, and another lies outside the unit circle. Then

$$M(\alpha) \in \{\alpha_1\alpha_2, \alpha_1\alpha_3\alpha_4\}.$$

If $M(\alpha) = \alpha_1\alpha_2 \in \mathbb{Q}(\sqrt{d})$, we are done, since then α is preperiodic. So we proceed with $M(\alpha) = \alpha_1\alpha_3\alpha_4 = \frac{N(\alpha)}{\alpha_2} \in K$. As before we see, that if the number of Galois conjugates of

$\frac{N(\alpha)}{\alpha_2}$ outside the unit circle is equal to 1, 2 or 4, then α is preperiodic. But if this number is equal to 3, then α satisfies the assumption of Proposition 35, and hence it is preperiodic in this case as well. This proves Proposition 42. ■

The assumption on the signature is crucial, as the following example shows: The element $\alpha = \frac{1}{4}\left(-1-\sqrt{17}-\sqrt{2(17+\sqrt{17})}\right)$ is an algebraic unit in the totally real field $\mathbb{Q}\left(\sqrt{2(17+\sqrt{17})}\right)$. This unit has precisely two conjugates outside the unit circle. The first is $\alpha_1 = \alpha$ and the second is $\alpha_2 = \frac{1}{4}\left(-1+\sqrt{17}+\sqrt{2(17-\sqrt{17})}\right)$. Hence,

$$M(\alpha) = \alpha_1\alpha_2 = \frac{(-\sqrt{17}-5)\sqrt{34-2\sqrt{17}}}{16} - \frac{\sqrt{17}}{2} - 1.$$

Again, $M(\alpha)$ has precisely two conjugates outside the unit circle. Hence, $M^{(2)}(\alpha) \neq M(\alpha)$. It follows, that the orbit of α contains at least three elements. Hence, by Theorem 25, α is a wandering point.

5.1.3 Totally real extension of degree 4

Lemma 5.1.2 *Let K be a totally real number field of degree 4. Suppose that K is embedded in $\mathbb{R}$, so $K \subset \mathbb{R}$, and denote by $\sigma_1, \sigma_2, \sigma_3$ the remaining non-trivial embeddings of K into $\mathbb{R}$. Then there exists an algebraic unit $\alpha \in K$, such that*

i. $|\alpha| > |\sigma_1(\alpha)| > 1$,

ii. $|\sigma_2(\alpha)|, |\sigma_3(\alpha)| < 1$, *and*

iii. $|\alpha\sigma_i(\alpha)| \neq 1$ *for all* $i \in \{1,2,3\}$.

Proof. Let $\mathcal{O}_K^*$ be the unit group of K. For $\alpha \in \mathcal{O}_K^*$ we define

$$L(\alpha) = (\log|\alpha|, \log|\sigma_1(\alpha)|, \log|\sigma_2(\alpha)|, \log|\sigma_3(\alpha)|).$$

By Dirichlet's unit theorem $L(\mathcal{O}_K^*)$ is a lattice of rank 3 in the hyperplane

$$\{(x_1, x_2, x_3, -x_1-x_2-x_3) | x_1, x_2, x_3 \in \mathbb{R}\} \subseteq \mathbb{R}^4.$$

Let $B \in \mathbb{R}$ be larger than any vector spanning a fixed fundamental domain of the lattice $L(\mathcal{O}_K^*)$. Then there exists an element $(x_1, x_2, x_3, x_4) \in L(\mathcal{O}_K^*)$ such that $x_1 > 2B$, $0 < x_2 < B$, and $-B > x_3 > -2B$. It follows that $x_4 = -x_1 - x_2 - x_3 < -2B + 2B = 0$ and $x_1 + x_3 \neq 0 \neq x_1 + x_4$. This means that any $\alpha \in \mathcal{O}_K^*$ such that $L(\alpha) = (x_1, x_2, x_3, x_4)$ satisfies the statements (i), (ii), (iii) from the lemma. ■

Proposition 43 *Let $K/\mathbb{Q}$ be totally real of degree four and let L be the Galois closure of K over $\mathbb{Q}$. If there is a $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that all embeddings of K are given by* id*, $\sigma|_K$, $\sigma^2|_K$, and $\sigma^3|_K$, then there is an algebraic unit in K, which is a wandering point under iteration of the Mahler measure.*

Proof. By Lemma 5.1.2 there is an algebraic unit $\alpha \in K$ such that the Galois conjugates of α satisfy

$$|\alpha| > |\sigma(\alpha)| > 1 \quad \text{and} \quad |\sigma^2(\alpha)|, |\sigma^3(\alpha)| < 1 \quad \text{and} \quad |\alpha\sigma^3(\alpha)| \neq 1. \tag{5.1.1}$$

Now, the Mahler measure of α is $M(\alpha) = \pm\alpha\sigma(\alpha)$. The elements

$$\alpha\sigma(\alpha) \quad , \quad \sigma(\alpha)\sigma^2(\alpha) \quad , \quad \sigma^2(\alpha)\sigma^3(\alpha) \quad , \quad \sigma^3(\alpha)\alpha$$

are pairwise distinct Galois conjugates of $\alpha\sigma(\alpha)$. Moreover, since α is an algebraic unit, we have

$$|\sigma(\alpha)\sigma^2(\alpha)| \cdot |\sigma^3(\alpha)\alpha| = 1,$$

and by (5.1.1) $|\sigma^3(\alpha)\alpha| \neq 1$. It follows, that precisely one of $\sigma^3(\alpha)\alpha$ and $\sigma(\alpha)\sigma^2(\alpha)$ lies outside the unit circle. Hence $M^{(2)}(\alpha) = M(\alpha\sigma(\alpha)) > M(\alpha)$, and the orbit of α under iteration of M contains at least three elements. It follows, that α is a wandering point. ■

Corollary 44 *Let $K/\mathbb{Q}$ be totally real of degree four, and let L be the Galois closure of K over $\mathbb{Q}$. If $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the cyclic group with four elements C_4 or to the dihydral group D_4, then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. If $\mathrm{Gal}(L/\mathbb{Q}) \cong C_4$, then $L = K$ and the assumptions from Proposition 43 are met. Hence, there is a wandering unit in this case. Now assume that $\mathrm{Gal}(L/\mathbb{Q}) \cong D_4 = \langle \sigma, \tau \rangle$, with σ of order 4 and τ of order 2. In particular $K/\mathbb{Q}$ is not Galois in this case!

If $\sigma|_K$ or $\sigma^3|_K$ were the identity map, then K would be fixed by the group $\langle \sigma \rangle$ of order 4, and hence $[K : \mathbb{Q}] \leq 2$, which is not possible. If $\sigma^2|_K$ were the identity map, then K would be the fixed field of $\langle \sigma^2 \rangle$. But $\langle \sigma^2 \rangle$ is normal in D_4, and hence $K/\mathbb{Q}$ would be normal, which is not the case.

It follows, that $\mathrm{id}|_K$, $\sigma|_K$, $\sigma^2|_K$, and $\sigma^3|_K$ are pairwise distinct embeddings of K. So again the assumptions from Proposition 43 are met, and K contains a wandering unit. ∎

Proposition 45 *Let $K/\mathbb{Q}$ be totally real of degree four, and let L be the Galois closure of K over $\mathbb{Q}$. If $\mathrm{Gal}(L/\mathbb{Q})$ contains the alternating group A_4, then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. By Lemma 5.1.2 there is an algebraic unit α in K such that the Galois conjugates $\alpha = \alpha_1$, α_2, α_3, α_4 satisfy

$$|\alpha_1| > |\alpha_2| > 1 > |\alpha_3| \geq |\alpha_4| \quad .$$

Again, $M(\alpha) = \pm\alpha_1\alpha_2$. Since $\mathrm{Gal}(L/\mathbb{Q})$ contains A_4, the Galois conjugates of $\alpha_1\alpha_2$ are precisely

$$\alpha_1\alpha_2 \quad , \quad \alpha_1\alpha_3 \quad , \quad \alpha_1\alpha_4 \quad , \quad \alpha_2\alpha_3 \quad , \quad \alpha_2\alpha_4 \quad , \quad \alpha_3\alpha_4 .$$

(Consider the action of the group on the indexes). It is $|\alpha_1\alpha_3| > |\alpha_2\alpha_4|$ and $|\alpha_1\alpha_2\alpha_3\alpha_4| = 1$. Hence $\alpha_1\alpha_3$ is a Galois conjugate of $\alpha_1\alpha_2$ outside the unit circle. As before it follows that the orbit of α under iteration of the Mahler measure contains at least three elements, which implies that α is a wandering point. ∎

At last, we have the following theorem:

Theorem 46 *Let $K/\mathbb{Q}$ be totally real of degree four. Then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M, if and only if K is not biquadratic.*

Proof. If K is biquadratic, then there are no wandering points by Proposition 41. So let K not be biquadratic. This means that the Galois group of the Galois closure L of K over $\mathbb{Q}$ is not the Klein four-group. Since $\mathrm{Gal}(L/\mathbb{Q})$ must be a transitive subgroup of S_4 with order divisible by 4, it must be isomorphic to C_4, D_4, A_4 or S_4. In all these cases there exists a wandering unit by Corollary 44 and Proposition 45. ■

Discussion in Section 5.1 gives us Theorem 29.

5.2 Fields of degree five

We will now show that in extensions of degree 5, there always exists an algebraic unit which is wandering under iteration of M.

Lemma 5.2.1 *Let K be a totally real number field of degree 5, which we assume is embedded in $\mathbb{R}$, hence $K \subset \mathbb{R}$. Denote by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ the remaining embeddings of K into $\mathbb{R}$. Then there exists an algebraic unit $\alpha \in K$, such that*

i. $|\alpha| > |\sigma_1(\alpha)| > |\sigma_2(\alpha)| > 1 > |\sigma_3(\alpha)| > |\sigma_4(\alpha)|$,

ii. $|\alpha\sigma_4(\alpha)| < 1, |\sigma_2(\alpha)\sigma_3(\alpha)| < 1$, *and*

iii. $|\alpha\sigma_4(\alpha)| > |\sigma_2(\alpha)\sigma_3(\alpha)|$.

Proof. Let $\mathcal{O}_K^*$ be the unit group of K. For $\alpha \in \mathcal{O}_K^*$ we define

$$L(\alpha) = (\log|\alpha|, \log|\sigma_1(\alpha)|, \log|\sigma_2(\alpha)|, \log|\sigma_3(\alpha)|), \log|\sigma_4(\alpha)|).$$

By Dirichlet's unit theorem $L(\mathcal{O}_K^*)$ is a lattice of rank 4 in the hyperplane

$$\{(x_1, x_2, x_3, x_4, -x_1 - x_2 - x_3 - x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\} \subseteq \mathbb{R}^5.$$

Let $B \in \mathbb{R}$ be larger than any vector spanning a fixed fundamental domain of the lattice $L(\mathcal{O}_K^*)$. Then there exists an element $(x_1, x_2, x_3, x_4, x_5) \in L(\mathcal{O}_K^*)$ such that $9B < x_1 < 10B$, $8B < x_2 < 9B$, $0 < x_3 < B$, $-7B < x_4 < -6B$. therefore, $-14B < x_5 = -x_1 - x_2 - x_3 - x_4 <$

$-10B$, and so $x_1+x_5<0$, $x_3+x_4<0$, $x_1+x_5>x_3+x_4$. This gives that any $\alpha\in\mathcal{O}_K^*$ such that $L(\alpha)=(x_1,x_2,x_3,x_4,x_5)$ satisfies the statements (i), (ii), (iii) from the lemma. ∎

Lemma 5.2.2 *Let K be an extension of degree 5 with signature (3,1) that is embedded in $\mathbb{R}$. Denote by σ_1,σ_2 the remaining embeddings of K into $\mathbb{R}$, and by σ_3,σ_4 the complex conjugate pair of embeddings into $\mathbb{C}$. Then there exists an algebraic unit $\alpha\in K$, such that*

i. $|\alpha|>|\sigma_1(\alpha)|>|\sigma_2(\alpha)|>1>|\sigma_3(\alpha)|=|\sigma_4(\alpha)|$,

ii. $|\alpha\sigma_4(\alpha)|<1, |\sigma_2(\alpha)\sigma_3(\alpha)|<1$, *and*

iii. $|\alpha\sigma_4(\alpha)|>|\sigma_2(\alpha)\sigma_3(\alpha)|$.

Proof. Let $\mathcal{O}_K^*$ be the unit group of K. For $\alpha\in\mathcal{O}_K^*$ we define

$$L(\alpha)=(\log|\alpha|,\log|\sigma_1(\alpha)|,\log|\sigma_2(\alpha)|,2\log|\sigma_3(\alpha)|).$$

By Dirichlet's unit theorem $L(\mathcal{O}_K^*)$ is a lattice of rank 3 in the hyperplane

$$\{(-x_2-x_3-2x_4,x_2,x_3,2x_4)|x_2,x_3,x_4\in\mathbb{R}\}\subseteq\mathbb{R}^4.$$

Let $B\in\mathbb{R}$ be larger than any vector spanning a fixed fundamental domain of the lattice $L(\mathcal{O}_K^*)$. Then there exists an element $(x_1,x_2,x_3,2x_4)\in L(\mathcal{O}_K^*)$ such that $9B<x_2<10B$, $7B<x_3<8B$, $-15B<x_4<-14B$, therefore, $10B<x_1=-x_2-x_3-2x_4<14B$. We see that $x_1+x_4<0$, $x_3+x_4<0$, $x_1+x_4>x_3+x_4$. This gives that any $\alpha\in\mathcal{O}_K^*$ such that $L(\alpha)=(x_1,x_2,x_3,2x_4)$ satisfies the statements (i)-(iii) from the lemma. ∎

Lemma 5.2.3 *Let K be an extension of degree 5 with signature (1,2) and assume that K is embedded into $\mathbb{R}$. Denote by σ_1,σ_2, and σ_3,σ_4 the complex conjugate pairs of embeddings of K into $\mathbb{C}$. Then there exists an algebraic unit $\alpha\in K$, such that*

i. $|\alpha|>|\sigma_1(\alpha)|=|\sigma_2(\alpha)|>1>|\sigma_3(\alpha)|=|\sigma_4(\alpha)|$,

ii. $|\alpha\sigma_4(\alpha)|<1, |\sigma_2(\alpha)\sigma_3(\alpha)|<1$, *and* *iii.* $|\alpha\sigma_4(\alpha)| > |\sigma_2(\alpha)\sigma_3(\alpha)|$.

Proof. Let $\mathcal{O}_K^*$ be the unit group of K. For $\alpha \in \mathcal{O}_K^*$ we define

$$L(\alpha) = (\log|\alpha|, 2\log|\sigma_1(\alpha)|, 2\log|\sigma_3(\alpha)|).$$

By Dirichlet's unit theorem $L(\mathcal{O}_K^*)$ is a lattice of rank 2 in the hyperplane

$$\{(-2x_2 - 2x_4, 2x_2, 2x_4)|x_2, x_4 \in \mathbb{R}\} = \{(-x-y, x, y)|x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3.$$

Let $B \in \mathbb{R}$ be larger than any vector spanning a fixed fundamental domain of the lattice $L(\mathcal{O}_K^*)$. Then there exists an element $(x_1, 2x_2, 2x_4) \in L(\mathcal{O}_K^*)$ such that $9B < x_2 < 10B$, $-17B < x_4 < -16B$, therefore, $12B < x_1 = -2x_2 - 2x_4 < 16B$. We see that $x_1 + x_4 < 0$, $x_2 + x_4 < 0$, $x_1 + x_4 > x_2 + x_4$. This gives that any $\alpha \in \mathcal{O}_K^*$ such that $L(\alpha) = (x_1, 2x_2, 2x_4)$ satisfies the statements (i)-(iii) from the lemma. ■

Proposition 47 *Let $K/\mathbb{Q}$ be totally real of degree five and let L be the Galois closure of K over $\mathbb{Q}$. If there is a $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that all embeddings of K are given by* id*, $\sigma|_K$, $\sigma^2|_K$, $\sigma^3|_K$,and $\sigma^4|_K$, then there is an algebraic unit in K, which is a wandering point under iteration of the Mahler measure.*

Proof. By Lemma 5.2.1, there is an algebraic unit $\alpha \in K$ such that the Galois conjugates of α satisfy

1. $|\alpha| > |\sigma(\alpha)| > |\sigma^2(\alpha)| > 1 > |\sigma^3(\alpha)| > |\sigma^4(\alpha)|$,

2. $|\sigma^2(\alpha)\sigma^3(\alpha)| < |\alpha\sigma^4(\alpha)| < 1$.

Now, $M(\alpha) = \pm\alpha\sigma(\alpha)\sigma^2(\alpha)$ and

$$|\sigma(\alpha)\sigma^2(\alpha)\sigma^3(\alpha)| = \left|\frac{1}{\alpha\sigma^4(\alpha)}\right| > 1, |\sigma^2(\alpha)\sigma^3(\alpha)\sigma^4(\alpha)| = \left|\frac{1}{\sigma(\alpha)\alpha}\right| < 1,$$

$$|\sigma^3(\alpha)\sigma^4(\alpha)\alpha| = \left|\frac{1}{\sigma(\alpha)\sigma^2(\alpha)}\right| < 1, |\sigma^4(\alpha)\alpha\sigma(\alpha)| = \left|\frac{1}{\sigma^2(\alpha)\sigma^3(\alpha)}\right| > 1,$$

and

$$|\alpha\sigma(\alpha)\sigma^2(\alpha)| = \left|\frac{1}{\sigma^3(\alpha)\sigma^4(\alpha)}\right| > 1$$

are pairwise distinct Galois conjugates of $|\alpha\sigma(\alpha)\sigma^2(\alpha)|$.Therefore,

$$M^{(2)}(\alpha) = \pm\alpha^2\sigma(\alpha)^3\sigma^2(\alpha)^2\sigma^3(\alpha)\sigma^4(\alpha) = \pm\alpha\sigma(\alpha)^2\sigma^2(\alpha).$$

Now,

$$|\alpha\sigma(\alpha)^2\sigma^2(\alpha)| = \left|\frac{\sigma(\alpha)}{\sigma^3(\alpha)\sigma^4(\alpha)}\right| > 1, |\sigma(\alpha)\sigma^2(\alpha)^2\sigma^3(\alpha)| = \left|\frac{\sigma^2(\alpha)}{\sigma^4(\alpha)\alpha}\right| > 1,$$
$$|\sigma^2(\alpha)\sigma^3(\alpha)^2\sigma^4(\alpha)| = \left|\frac{\sigma^3(\alpha)}{\alpha\sigma(\alpha)}\right| < 1, |\sigma^3(\alpha)\alpha\sigma^4(\alpha)^2| = \left|\frac{\sigma^4(\alpha)}{\sigma(\alpha)\sigma^2(\alpha)}\right| < 1,$$

and

$$|\alpha^2\sigma^4(\alpha)\sigma(\alpha)| = \left|\frac{\alpha}{\sigma^2(\alpha)\sigma^3(\alpha)}\right| > 1$$

are pairwise distinct Galois conjugates of $M^{(2)}(\alpha)$.

We will show that when $n \geq 2$, $M^{(n)}(\alpha) = \alpha^i\sigma(\alpha)^j\sigma^2(\alpha)^i$ for some $i < j$ such that $j - i \leq i$, and that $M^{(n)}(\alpha)$ has five distinct Galois conjugates, among which three conjugates are outside of the unit circle while the other two lie inside of the unit circle, which gives that α is a wandering point.

We will show this by induction. The case for $n = 1$ and $n = 2$ are as above. Suppose that $M^{(n-1)}(\alpha) = \alpha^i\sigma(\alpha)^j\sigma^2(\alpha)^i$ for some $i < j$, then

$$|\alpha^i\sigma(\alpha)^j\sigma^2(\alpha)^i| = \left|\frac{\sigma(\alpha)^{j-i}}{\sigma^3(\alpha)^i\sigma^4(\alpha)^i}\right| > 1,$$

$$|\sigma(\alpha)^i\sigma^2(\alpha)^j\sigma^3(\alpha)^i| = \left|\frac{\sigma^2(\alpha)^{j-i}}{\sigma^4(\alpha)^i\alpha^i}\right| > 1,$$

$$|\sigma^2(\alpha)^i\sigma^3(\alpha)^j\sigma^4(\alpha)^i| = \left|\frac{\sigma^3(\alpha)^{j-i}}{\alpha^i\sigma(\alpha)^i}\right| < 1,$$

$$|\sigma^3(\alpha)^i\alpha^i\sigma^4(\alpha)^j| = \left|\frac{\sigma^4(\alpha)^{j-i}}{\sigma(\alpha)^i\sigma^2(\alpha)^i}\right| < 1,$$

and

$$|\alpha^j\sigma^4(\alpha)^i\sigma(\alpha)^i| = \left|\frac{\alpha^{j-i}}{\sigma^2(\alpha)^i\sigma^3(\alpha)^i}\right| > 1$$

are pairwise distinct Galois conjugates of $M^{(2)}(\alpha)$.

Since

$$\sigma^3(\alpha)^{j-i}\sigma(\alpha)^i\sigma^2(\alpha)^i < \sigma^4(\alpha)^{j-i}\alpha^i\sigma(\alpha)^i$$

and

$$\sigma(\alpha)^{j-i}\sigma^2(\alpha)^i\sigma^3(\alpha)^i > \alpha^{j-i}\sigma^3(\alpha)^i\sigma^4(\alpha)^i,$$

the five absolute values above are pairwise distinct Galois conjugates of $M^{(n-1)}(\alpha)$.

Therefore, $M^{(n)}(\alpha) = \alpha^{j+i}\sigma(\alpha)^{i+j+i}\sigma^2(\alpha)^{i+j}\sigma^3(\alpha)^i\sigma^4(\alpha)^i = \alpha^j\sigma(\alpha)^{i+j}\sigma^2(\alpha)^j$.

By the same proof as above, $M^{(n)}(\alpha)$ has five distinct Galois conjugates, three of them are strictly outside of the unit circle while the other two lie strictly inside of the unit circle. This proves that α is a wandering point. ■

Corollary 48 *Let $K/\mathbb{Q}$ be totally real of degree five, and let L be the Galois closure of K over $\mathbb{Q}$. If $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the cyclic group with five elements C_5 or to the dihedral group D_5 or to the semidirect product $\mathbb{Z}_4 \ltimes \mathbb{Z}_5$, then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. If $\mathrm{Gal}(L/\mathbb{Q}) \cong C_5$, then $L = K$ and the assumptions from Proposition 47 are satisfied. Therefore, there is a wandering unit in this case. Now assume instead that $\mathrm{Gal}(L/\mathbb{Q}) \cong D_5 = \langle \sigma, \tau \rangle$, with σ of order 5 and τ of order 2. Note that $K/\mathbb{Q}$ is not Galois in this case.

If $\sigma|_K$ or $\sigma^2|_K$ or $\sigma^3|_K$ or $\sigma^4|_K$ were the identity map, then K would be fixed by the group $\langle \sigma \rangle$ of order 5, and hence $[K : \mathbb{Q}] \leq 2$, which is not possible.

Therefore, $\mathrm{id}|_K$, $\sigma|_K$, $\sigma^2|_K$, $\sigma^3|_K$ and $\sigma^4|_K$ are pairwise distinct embeddings of K. So in this case the assumptions from Proposition 47 are met, and K contains a wandering unit.

Now, assume that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5 = \langle \sigma, \tau \rangle$, where σ is of degree 5 and τ is of degree 4. Notice that in this case $K/\mathbb{Q}$ is again not Galois. By the same argument as above, $\mathrm{id}|_K$, $\sigma|_K$, $\sigma^2|_K$, $\sigma^3|_K$ and $\sigma^4|_K$ are pairwise distinct embeddings of K. Hence, once again, the assumptions from Proposition 47 are met and K contains a wandering unit. ■

Proposition 49 *Let $K/\mathbb{Q}$ be totally real of degree five, and let L be the Galois closure of K over $\mathbb{Q}$. If* $\mathrm{Gal}(L/\mathbb{Q})$ *contains the alternating group A_5, then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. By Lemma 5.2.1, there is an algebraic unit α in K such that the Galois conjugates $\alpha = \alpha_1$, α_2, α_3, α_4, α_5 satisfy

$$|\alpha_1| > |\alpha_2| > |\alpha_3| > 1 > |\alpha_4| > |\alpha_5| \quad .$$

Therefore, $M(\alpha) = \pm\alpha_1\alpha_2\alpha_3$. Since $\mathrm{Gal}(L/\mathbb{Q})$ contains A_5, the Galois conjugates of $\alpha_1\alpha_2\alpha_3$ are precisely

$$\alpha_1\alpha_2\alpha_3 \quad , \quad \alpha_2\alpha_4\alpha_5 \quad , \quad \alpha_2\alpha_3\alpha_5 \quad , \quad \alpha_3\alpha_4\alpha_5 \quad , \quad \alpha_2\alpha_3\alpha_4 \quad ,$$

$$\alpha_1\alpha_4\alpha_5 \quad , \quad \alpha_1\alpha_3\alpha_5 \quad , \quad \alpha_1\alpha_3\alpha_4 \quad , \quad \alpha_1\alpha_2\alpha_5 \quad , \quad \alpha_1\alpha_2\alpha_4 \quad .$$

Notice that $|\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5| = 1$ and $|\alpha_1\alpha_2\alpha_4| > |\alpha_3\alpha_5|$, which gives $|\alpha_1\alpha_2\alpha_4| > 1$.

Therefore the orbit of α under iteration of M contains at least 3 elements. By Theorem 27, α is a wandering unit. ■

To summarize, we get the following theorem:

Theorem 50 *Let $K/\mathbb{Q}$ be totally real of degree five. Then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. Since $\mathrm{Gal}(L/\mathbb{Q})$ must be a transitive subgroup of S_5 with order divisible by 5, it must be isomorphic to C_5, D_5, $\mathbb{Z}_4 \ltimes \mathbb{Z}_5$, A_5 or S_5. In all these cases there exists a wandering unit by Corollary 48 and Proposition 49. ■

The proofs for signature (3,1) and (1,2) are similar.

Proposition 51 *Let $K/\mathbb{Q}$ be of degree five with signature (3,1) and let L be the Galois closure of K over $\mathbb{Q}$. If there is a $\sigma \in$* $\mathrm{Gal}(L/\mathbb{Q})$ *such that all embeddings of K are given by* id, *$\sigma|_K$, $\sigma^2|_K$, $\sigma^3|_K$, and $\sigma^4|_K$, then there is an algebraic unit in K, which is a wandering point under iteration of the Mahler measure.*

Proof. By Lemma 5.2.2, there is an algebraic unit $\alpha \in K$ such that the Galois conjugates of α satisfy

1. $|\alpha| > |\sigma(\alpha)| > |\sigma^2(\alpha)| > 1 > |\sigma^3(\alpha)| = |\sigma^4(\alpha)|$,

2. $|\sigma^2(\alpha)\sigma^3(\alpha)| < |\alpha\sigma^4(\alpha)| < 1$.

Then, by the same proof as above for the totally real case, α is a wandering point. ■

Corollary 52 *Let $K/\mathbb{Q}$ be of degree five with signature (3,1), and let L be the Galois closure of K over $\mathbb{Q}$. If $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to the dihedral group D_5 or to the semidirect product $\mathbb{Z}_4 \ltimes \mathbb{Z}_5$, then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. Assume that $\mathrm{Gal}(L/\mathbb{Q}) \cong D_5 = \langle \sigma, \tau \rangle$, where σ is of order 5 and τ order 2. By the same proof as in Corollary 48, there is a wandering unit in K in this case.

Now assume that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_5 = \langle \sigma, \tau \rangle$ where σ is of order 5 and τ is of order 4. In this case, K also contains a wandering point by the same argument as in Corollary 48. ■

Proposition 53 *Let $K/\mathbb{Q}$ be of degree five with signature (3,1), and let L be the Galois closure of K over $\mathbb{Q}$. If $\mathrm{Gal}(L/\mathbb{Q})$ contains the alternating group A_5, then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. By the same argument as in the proof for Proposition 49, in this case there exists an algebraic unit which is wandering under iteration of M. ■

To summarize, we have

Theorem 54 *Let $K/\mathbb{Q}$ be of degree five with signature (3,1). Then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. Since $\mathrm{Gal}(L/\mathbb{Q})$ must be a transitive subgroup of S_5 with order divisible by 5, it must be isomorphic to C_5, D_5, $\mathbb{Z}_4 \ltimes \mathbb{Z}_5$, A_5 or S_5. $\mathrm{Gal}(L/\mathbb{Q})$ cannot be isomorphic to C_5 since

K is not Galois over $\mathbb{Q}$. In all the rest of these cases there exists a wandering unit by the corollary and proposition above. ■

Theorem 55 *Let $K/\mathbb{Q}$ be of degree five with signature (1,2). Then there exists an algebraic unit in K which is wandering under iteration of the Mahler measure M.*

Proof. By Lemma 5.2.3, there is an algebraic unit $\alpha \in K$ such that the Galois conjugates of α satisfy

1. $|\alpha| > |\sigma(\alpha)| = |\sigma^2(\alpha)| > 1 > |\sigma^3(\alpha)| = |\sigma^4(\alpha)|$,

2. $|\sigma^2(\alpha)\sigma^3(\alpha)| < |\alpha\sigma^4(\alpha)| < 1$.

Then, once again, the rest of the proof is essentially the same as the proof for the totally real case. ■

Finally, altogether, we have Theorem 30.

5.3 Abelian extensions

5.3.1 Multiquadratic fields

For any $r \in \mathbb{N}$ we call a field extension $K/\mathbb{Q}$ *r-quadratic*, if there are $d_1, \ldots, d_r \in \mathbb{Z}$ such that $K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_r})$ and $[K : \mathbb{Q}] = 2^r$.

Let K be r-quadratic, then $K/\mathbb{Q}$ is Galois and hence K is either totally real or totally complex. Moreover, any proper subfield of K is r'-quadratic with $r' < r$.

We first consider totally real 3-quadratic fields $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$. For each $i \in \{1, 2, 3\}$ choose a fundamental unit $\beta_i \in \mathbb{Q}(\sqrt{d_i})$ with $|\beta_i| > 1$. Moreover, let n_1, n_2, n_3 be positive integers, and set $\alpha_i = \beta_i^{n_i}$. Denote with σ_i the nontrivial element in the Galois group of $\mathbb{Q}(\sqrt{d_i})/\mathbb{Q}$. Then $|\alpha_i\sigma_i(\alpha_i)| = 1$ for all $i \in \{1, 2, 3\}$. The Galois conjugates of $\alpha = \alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}$ are

$$\begin{array}{cccc} \alpha_1\alpha_2\alpha_3 & \sigma_1(\alpha_1)\alpha_2\alpha_3 & \alpha_1\sigma_2(\alpha_2)\sigma_3 & \alpha_1\alpha_2\sigma_3(\alpha_3) \\ \sigma_1(\alpha_1)\sigma_2(\alpha_2)\alpha_3 & \sigma_1(\alpha_1)\alpha_2\sigma_3(\alpha_3) & \alpha_1\sigma_2(\alpha_2)\sigma_3(\sigma_3) & \sigma_1(\alpha_1)\sigma_2(\alpha_2)\sigma_3(\alpha_3) \end{array}$$

By choosing the exponents n_i's such that $|\beta_1^{n_1}| \approx |\beta_2^{n_2}| \approx |\beta_3^{n_3}|$, we can ensure that $|\beta_1^{n_1}| \geq 2$, and $|\beta_2^{n_2}|$,$|\beta_3^{n_3}| \in \left(|\beta_1^{n_1}| - \frac{1}{2}, |\beta_1^{n_1}| + \frac{1}{2}\right)$, and we have

- $|\alpha_1\alpha_2\alpha_3|, |\alpha_1|^3 > 1$,
- $|\sigma_1(\alpha_1)\alpha_2\alpha_3|, |\alpha_1\sigma_2(\alpha_2)\sigma_3|, |\alpha_1\alpha_2\sigma_3(\alpha_3)|, |\alpha_1| > 1$,
- $|\sigma_1(\alpha_1)\sigma_2(\alpha_2)\alpha_3|, |\sigma_1(\alpha_1)\alpha_2\sigma_3(\alpha_3)|, |\alpha_1\sigma_2(\alpha_2)\sigma_3(\sigma_3)|, |\alpha_1|^{-1} < 1$,
- $|\sigma_1(\alpha_1)\sigma_2(\alpha_2)\sigma_3(\alpha_3)|, |\alpha_1|^{-3} < 1$.

In particular, α is a Perron number (and hence torsion-free) of degree 8, and the Mahler measure of α is

$$M(\alpha) = |\alpha_1\alpha_2\alpha_3\sigma_1(\alpha_1)\alpha_2\alpha_3\alpha_1\sigma_2(\alpha_2)\sigma_3\alpha_1\alpha_2\sigma_3(\alpha_3)| = |\alpha_1^2\alpha_2^2\alpha_3^2| = \alpha^2.$$

Since α is torsion-free, it is $M^{(2)}(\alpha) = M(\alpha^2) = M(\alpha)^2 = \alpha^4$. More generally, for any $n \geq 1$ it is $M^{(n)}(\alpha) = \alpha^{2^n}$, which immediately implies that α is wandering under iteration of M. We have just seen:

Proposition 56 *Let K be a totally real 3-quadratic field. Then there are wandering units under iteration of the Mahler measure.*

This gives:

Corollary 57 *Let K be a r-quadratic field. If K is totally real and $r \in \{1,2\}$, or if K is totally imaginary and $r \in \{1,2,3\}$, all elements in K are preperiodic under iteration of the Mahler measure. In all other cases there are wandering units under iteration of the Mahler measure.*

Proof. Assume first that K is totally real. By Proposition 41, all elements of K are preperiodic, if $r \in \{1,2\}$. If $r \geq 3$, then it contains a 3-quadratic subfield. By Proposition 56, K contains a wandering unit.

Now, let K be totally imaginary. If $r \in \{1, 2, 3\}$, then for any $\alpha \in K$ it is $M(\alpha) \in K \cap \mathbb{R}$. This is $M(\alpha)$ lies in a totally real $(r-1)$-quadratic subfield of K (where a 0-quadratic field is just $\mathbb{Q}$). By the first part of the proof, we conclude that α, and hence all elements in K are preperiodic. If $r \geq 4$, then $K \cap \mathbb{R}$ is a totally real $(r-1)$-quadratic subfield of K. Again by the first part of the proof, there is a wandering unit in K. ■

A totally imaginary 3-quadratic field gives an example of a field of degree 8 in which all elements are preperiodic. Similarly, any totally imaginary Galois extension of degree 6 has this property: The Mahler measure of any element in such a field is real and must live in a proper subfield. This implies that the Mahler measure of an element in our field is of degree 1, 2, or 3. Hence, any element is preperiodic.

We are now ready to prove the following theorem:

Theorem 58 *Let $K/\mathbb{Q}$ be an abelian 2-extension. Then K does not contain an element wandering under iteration of M, if and only if the maximal real subfield of K has Galois group isomorphic to C_1, C_2 or $C_2 \times C_2$. In all other cases, K contains a wandering unit.*

Proof. Let $\alpha \in K$ be arbitrary. Then $M(\alpha)$ is a real algebraic number in K (since $K/\mathbb{Q}$ is Galois). By assumption $K/\mathbb{Q}$ is abelian and hence all subfields are Galois over $\mathbb{Q}$ as well. It follows that $M(\alpha)$ lies in the maximal real subfield of K. If this maximal real subfield is a trivial or a quadratic extension of $\mathbb{Q}$, then surely there are no wandering elements in K. If it is a $C_2 \times C_2$ extension, it does not contain a wandering point by Proposition 41.

In all other cases, K contains a totally real field of Galois group isomorphic to C_4 or isomorphic to $C_2 \times C_2 \times C_2$. Both yields the existence of wandering units by Corollary 44 and Proposition 56. ■

From now on we will only consider abelian extensions. We aim to classify all the Abelian extensions that do not contain a wandering point. Let us start with the smallest remaining degree.

5.3.2 Abelian extensions of degree six

Proposition 59 *Let $K/\mathbb{Q}$ be an abelian extension of degree* 6*. Then there exists an element in K that is wandering under iteration of M if and only if K is totally real.*

Proof. If $K/\mathbb{Q}$ is totally imaginary, then $M(\alpha)$ – as a real element – must lie in a proper subfield of K for all $\alpha \in K$. This means that $M(\alpha)$ is of degree 1, 2, or 3. In all cases α is a preperiodic point. Hence for the rest of this proof we assume that K is totally real. We will show that K contains a wandering unit.

For any $\alpha \in K$, we set $\alpha_1 = \alpha$ and $\alpha_{i+1} = \sigma(\alpha_i)$ for all $i \in \mathbb{N}$, where σ is a generator of $\mathrm{Gal}(K/\mathbb{Q})$.

By Dirichlet's unit theorem there is an algebraic unit $\alpha \in K$, such that

$$|\alpha_1| > |\alpha_6| > |\alpha_2| > 1 > |\alpha_5| > |\alpha_3| > |\alpha_4|. \tag{5.3.1}$$

Then $M(\alpha) = |\alpha_1\alpha_6\alpha_2|$. In particular, we get

$$\begin{aligned} |\beta_1| &= |\alpha_1\alpha_6\alpha_2|, \quad |\beta_2| = |\alpha_2\alpha_1\alpha_3|, \quad |\beta_3| = |\alpha_3\alpha_2\alpha_4| \\ |\beta_4| &= |\alpha_4\alpha_3\alpha_5|, \quad |\beta_5| = |\alpha_5\alpha_4\alpha_6|, \quad |\beta_6| = |\alpha_6\alpha_5\alpha_1|. \end{aligned}$$

We will see in a moment that β satisfies the same distribution of Galois conjagates as α in (5.3.1). Then, inductively, it follows that α is a wandering point. By using the table above and (5.3.1) we immediately get

$$|\beta_1| > |\beta_6|, \quad |\beta_6| > |\beta_2|, \quad |\beta_2| > |\beta_5|, \quad |\beta_5| > |\beta_3|, \quad |\beta_3| > |\beta_4|.$$

Since α is an algebraic unit, we have $|\beta_2| \cdot |\beta_5| = |\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6| = 1$. We have already noticed, that $|\beta_2| > |\beta_5|$. Hence, we have $|\beta_2| > 1 > |\beta_5|$. We have shown that $M(\alpha) = \beta$ has the same distribution of Galois conjugates as α. As noticed above, this implies that α is a wandering point. ∎

5.3.3 Abelian extensions of degree nine

We proceed with studying abelian extensions, and whether they contain wandering points or not.

Proposition 60 *Let $K/\mathbb{Q}$ be an abelian extension with Galois group isomorphic to $C_3 \times C_3$. Then there are wandering units in K.*

Proof. By Galois theory, we can write K as the compositum of two linearly disjoint fields K_1 and K_2, both of degree 3. Since K is Galois and of odd degree, it must be totally real. In particular K_1 and K_2 are totally real.

Let α be a Pisot unit in K_1 and let β be a Pisot unit in K_2 (recall that any real number field of degree greater than 1 contains a Pisot unit). Denote by α_1, α_2, α_3 the Galois conjugates of α, and by β_1, β_2, β_3 the Galois conjugates of β, ordered such that

$$|\alpha_1| > 1 > |\alpha_2| > |\alpha_3| \qquad \text{and} \qquad |\beta_1| > 1 > |\beta_2| > |\beta_3|.$$

Since K_1 and K_2 are linearly disjoint, the Galois conjugates of $\alpha_1\beta_1$ are precisely

$$\alpha_1\beta_1, \quad \alpha_1\beta_2, \quad \alpha_1\beta_3, \quad \alpha_2\beta_1, \quad \alpha_2\beta_2, \quad \alpha_2\beta_3, \quad \alpha_3\beta_1, \quad \alpha_3\beta_2, \quad \alpha_3\beta_3.$$

Since powers of α and β are also Pisot units, we may raise α and β to some power to assume that α and β satisfy

$$|\alpha_1\beta_3| > 1 \qquad \text{and} \qquad |\beta_1\alpha_3| > 1.$$

Then we find

$$M(\alpha_1\beta_1) = |(\alpha_1\beta_1)(\alpha_1\beta_2)(\alpha_1\beta_3)(\alpha_2\beta_1)(\alpha_3\beta_1)| = |\alpha_1^2\beta_1^2| = (\alpha_1\beta_1)^2.$$

By construction $\alpha_1\beta_1$ is a Perron number, and hence torsion free. It follows from the last displayed formula, that $M^n(\alpha_1\beta_1) = (\alpha_1\beta_1)^{2^n}$ for all $n \in \mathbb{N}$. Hence $\alpha_1\beta_1 \in K$ is a wandering unit. ∎

5.3.4 Cyclic extensions of odd degree ≥ 5

We define for all integers a the integer $\overline{a}$, with $\overline{a} \equiv a \mod n$ and $\overline{a} \in \{1, \ldots, n\}$. Let $n \geq 5$ be an odd integer and we fix any Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong C_n$. Let σ be a fixed generator of this Galois group. For any element $\alpha \in K$, we set $\alpha^{(i)} = \sigma^{i-1}(\alpha)$ for all $i \in \{2, \ldots, n\}$. This implies

$$\sigma(\alpha^{(i)}) = \alpha^{(i+1)} \quad \text{and} \quad \sigma(\alpha^{(n)}) = \alpha^{(1)} = \alpha.$$

Note that K is necessarily totally real, as a Galois extension of odd degree.

We break such cyclic extensions into two cases: $n \equiv 3 \mod 4$ and $n \equiv 1 \mod 4$.

Assume first that $n \equiv 3 \mod 4$.

Lemma 5.3.1 *Let $\alpha \in K$ be such that*

$$\left|\alpha^{(1)}\right| > \left|\alpha^{(n)}\right| > \left|\alpha^{(2)}\right| > \left|\alpha^{(n-1)}\right| > \left|\alpha^{(3)}\right| > \left|\alpha^{(n-2)}\right| > \left|\alpha^{(4)}\right| > \ldots \tag{5.3.2}$$

In other words, we assume that

a. $\left|\alpha^{(i)}\right| > \left|\alpha^{(n+1-i)}\right| \ \forall\, i \in \{1, \ldots, \frac{n-1}{2}\}$, *and*

b. $\left|\alpha^{(\overline{n+2-i})}\right| > \left|\alpha^{(i)}\right| \ \forall\, i \in \{2, \ldots, \frac{n+1}{2}\}$.

Assume that precisely $\frac{n+1}{2}$ conjugates of α lie outside the unit circle. Then the conjugates of $\beta = M(\alpha)$ are precisely distributed as the following

$$\left|\beta^{(1)}\right| > \left|\beta^{(2)}\right| > \left|\beta^{(n)}\right| > \left|\beta^{(3)}\right| > \left|\beta^{(n-1)}\right| > \left|\beta^{(4)}\right| > \left|\beta^{(n-2)}\right| > \ldots, \tag{5.3.3}$$

that is,

c. $\left|\beta^{(i)}\right| > \left|\beta^{(n+2-i)}\right| \ \forall\, i \in \{2, \ldots, \frac{n+1}{2}\}$, *and*

d. $\left|\beta^{(\overline{n+2-i})}\right| > \left|\beta^{(i+1)}\right| \ \forall\, i \in \{1, \ldots, \frac{n-1}{2}\}$.

and $M(\alpha)$ has either precisely $\frac{n+1}{2}$ or precisely $\frac{n-1}{2}$ conjugates outside the unit circle.

Proof. We have

$$\beta = M(\alpha) = \left|\alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(\frac{n+1}{4})}\right| \cdot \left|\alpha^{(n)} \cdot \alpha^{(n-1)} \cdots \alpha^{(\frac{3n+3}{4})}\right|$$
$$= \left|\alpha^{(\overline{\frac{3n+3}{4}})} \cdot \alpha^{(\overline{\frac{3n+3}{4}+1})} \cdot \alpha^{(\overline{\frac{3n+3}{4}+2})} \cdots \alpha^{(\overline{\frac{3n+3}{4}+\frac{n-1}{2}})}\right|.$$

Note that β is a Perron number, hence we $\beta^{(1)} = \beta$ is the largest conjugate of β. Now, define for all $i \in \{0, \ldots, \frac{n-1}{2}\}$,

$$\gamma_i = \frac{\left|\alpha^{(\overline{\frac{3n+3}{4}-i})} \cdot \alpha^{(\overline{\frac{3n+3}{4}+1-i})} \cdots \alpha^{(\overline{\frac{3n+3}{4}+\frac{n-1}{2}-i})}\right|}{\left|\alpha^{(\overline{\frac{3n+3}{4}+i})} \cdot \alpha^{(\overline{\frac{3n+3}{4}+1+i})} \cdots \alpha^{(\overline{\frac{3n+3}{4}+\frac{n-1}{2}+i})}\right|} = \frac{\left|\beta^{(\overline{1-i})}\right|}{\left|\beta^{(\overline{1+i})}\right|}.$$

Notice that we have $\gamma_0 = 1$. Further, we have

$$\gamma_i = \gamma_{i-1} \cdot \frac{\left|\alpha^{(\overline{\frac{3n+3}{4}-i})} \cdot \alpha^{(\overline{\frac{3n+3}{4}+(i-1)})}\right|}{\left|\alpha^{(\overline{\frac{3n+3}{4}+\frac{n-1}{2}+i})} \cdot \alpha^{(\overline{\frac{3n+3}{4}+\frac{n-1}{2}-(i-1)})}\right|} = \gamma_{i-1} \cdot \frac{\left|\alpha^{(\overline{\frac{3n+3}{4}-i})} \cdot \alpha^{(\overline{\frac{3n-1}{4}+i})}\right|}{\left|\alpha^{(\overline{\frac{n+1}{4}+i})} \cdot \alpha^{(\overline{\frac{n+5}{4}-i})}\right|} \tag{5.3.4}$$

for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. By assumption (a) on the $\alpha^{(j)}$'s we find that for all $i \in \{1, \ldots, \frac{n-3}{4}\}$ we have

$$\left|\alpha^{(\overline{\frac{3n+3}{4}-i})}\right| < \left|\alpha^{(\overline{\frac{n+1}{4}+i})}\right| \quad \text{and} \quad \left|\alpha^{(\overline{\frac{3n-1}{4}+i})}\right| < \left|\alpha^{(\overline{\frac{n+5}{4}-i})}\right|.$$

This shows that $1 = \gamma_0 > \gamma_1 > \ldots > \gamma_{\frac{n-3}{4}}$. For all $i \in \{\frac{n+5}{4}, \ldots, \frac{n-1}{2}\}$, we have

$$\left|\alpha^{(\overline{\frac{n+1}{4}+i})}\right| < \left|\alpha^{(\overline{\frac{3n+3}{4}-i})}\right| \quad \text{and} \quad \left|\alpha^{(\overline{\frac{n+5}{4}-i})}\right| < \left|\alpha^{(\overline{\frac{3n-1}{4}+i})}\right|.$$

Hence, $\gamma_{\frac{n+5}{4}} < \gamma_{\frac{n+9}{4}} < \ldots < \gamma_{\frac{n-1}{2}}$. For $i = \frac{n+1}{4}$, we have

$$\left|\alpha^{(\overline{\frac{n+1}{4}+i})}\right| = \left|\alpha^{(\overline{\frac{3n+3}{4}-i})}\right| \quad \text{and} \quad \left|\alpha^{(\overline{\frac{n+5}{4}-i})}\right| = \left|\alpha^{(\overline{\frac{3n-1}{4}+i})}\right|.$$

Therefore, $\gamma_{\frac{n-3}{4}} = \gamma_{\frac{n+1}{4}}$.

Since we have

$$\gamma_{\frac{n-1}{2}} = \frac{\left|\alpha^{(\frac{n+5}{4})} \cdots \alpha^{(\frac{3n+3}{4})}\right|}{\left|\alpha^{(\frac{n+1}{4})} \cdots \alpha^{(\frac{3n-1}{4})}\right|} = \frac{\left|\alpha^{(\frac{3n+3}{4})}\right|}{\left|\alpha^{(\frac{n+1}{4})}\right|} \overset{(a)}{<} 1,$$

we have that $\gamma_i < 1$ for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. This means

$$\left|\beta^{(\overline{1+i})}\right| > \left|\beta^{(\overline{1-i})}\right| = \left|\beta^{(\overline{n+1-i})}\right| = \left|\beta^{(\overline{n+2-(1+i)})}\right| \quad \forall\, i \in \left\{1, \ldots, \frac{n-1}{2}\right\}.$$

Hence,

$$\left|\beta^{\overline{(i)}}\right| > \left|\beta^{\overline{(n+2-i)}}\right| \qquad \forall\ i \in \left\{2, \ldots, \frac{n+1}{2}\right\}.$$

This proves assumption (c) for the $\beta^{(i)}$'s. Now we prove that $\beta^{(i)}$'s satisfy the condition (d).

Define for all $i \in \{1, \ldots, \frac{n-1}{2}\}$,

$$\mu_i = \frac{\left|\alpha^{\left(\frac{3n+3}{4}-(i-1)\right)} \cdot \alpha^{\left(\frac{3n+3}{4}+1-(i-1)\right)} \cdots \alpha^{\left(\frac{3n+3}{4}+\frac{n-1}{2}-(i-1)\right)}\right|}{\left|\alpha^{\left(\frac{3n+3}{4}+i\right)} \cdot \alpha^{\left(\frac{3n+3}{4}+1+i\right)} \cdots \alpha^{\left(\frac{3n+3}{4}+\frac{n-1}{2}+i\right)}\right|} = \frac{\left|\beta^{\overline{(2-i)}}\right|}{\left|\beta^{\overline{(i+1)}}\right|}.$$

Note that $\mu_1 = \frac{\left|\beta^{(1)}\right|}{\left|\beta^{(2)}\right|} > 1$. Now,

$$\mu_{i+1} = \mu_i \cdot \frac{\left|\alpha^{\left(\frac{3n+3}{4}-i\right)}\right| \cdot \left|\alpha^{\left(\frac{3n+3}{4}+i\right)}\right|}{\left|\alpha^{\left(\frac{3n+3}{4}+\frac{n-1}{2}+(i-1)\right)}\right| \cdot \left|\alpha^{\left(\frac{3n+3}{4}+\frac{n-1}{2}-(i-1)\right)}\right|} \tag{5.3.5}$$

$$= \mu_i \cdot \frac{\left|\alpha^{\left(\frac{3n+3}{4}-i\right)}\right| \cdot \left|\alpha^{\left(\frac{3n+3}{4}+i\right)}\right|}{\left|\alpha^{\left(\frac{5n+5}{4}+i\right)}\right| \cdot \left|\alpha^{\left(\frac{5n+5}{4}-i\right)}\right|} \tag{5.3.6}$$

$$= \mu_i \cdot \frac{\left|\alpha^{\left(\frac{3n+3}{4}-i\right)}\right| \cdot \left|\alpha^{\left(\frac{3n+3}{4}+i\right)}\right|}{\left|\alpha^{\left(\frac{n+5}{4}+i\right)}\right| \cdot \left|\alpha^{\left(\frac{n+5}{4}-i\right)}\right|}, \tag{5.3.7}$$

for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. By assumption (b) on the $\alpha^{(j)}$'s, for all $i \in \{1, \ldots, \frac{n-3}{4}\}$, we have

$$\left|\alpha^{\left(\frac{3n+3}{4}-i\right)}\right| > \left|\alpha^{\left(\frac{n+5}{4}+i\right)}\right|, \quad \text{and} \quad \left|\alpha^{\left(\frac{3n+3}{4}+i\right)}\right| > \left|\alpha^{\left(\frac{n+5}{4}-i\right)}\right|.$$

This implies that $1 < \mu_1 < \mu_2 < \ldots < \mu_{\frac{n+1}{4}}$. For all $i \in \{\frac{n+5}{4}, \ldots, \frac{n-1}{2}\}$,

$$\left|\alpha^{\left(\frac{n+5}{4}+i\right)}\right| > \left|\alpha^{\left(\frac{3n+3}{4}-i\right)}\right|, \quad \text{and} \quad \left|\alpha^{\left(\frac{n+5}{4}-i\right)}\right| > \left|\alpha^{\left(\frac{3n+3}{4}+i\right)}\right|.$$

Hence, $\mu_{\frac{n+5}{4}} > \ldots > \mu_{\frac{n-1}{2}}$. Since

$$\mu_{\frac{n-1}{2}} = \frac{\left|\alpha^{\left(\frac{n+9}{4}\right)} \cdots \alpha^{\left(\frac{3n+7}{4}\right)}\right|}{\left|\alpha^{\left(\frac{n+1}{4}\right)} \cdots \alpha^{\left(\frac{3n-1}{4}\right)}\right|} = \frac{\left|\alpha^{\left(\frac{3n+3}{4}\right)} \cdot \alpha^{\left(\frac{3n+7}{4}\right)}\right|}{\left|\alpha^{\left(\frac{n+1}{4}\right)} \cdot \alpha^{\left(\frac{n+5}{4}\right)}\right|} \overset{(b)}{>} 1.$$

We then have that $\mu_i > 1$ for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. This means

$$\left|\beta^{(\overline{n+2-i})}\right| = \left|\beta^{(\overline{2-i})}\right| > \left|\beta^{(\overline{i+1})}\right|,$$

for all $i \in \{1, \cdots, \frac{n-1}{2}\}$. This completes the proof of assumption (d) for the $\beta^{(i)}$'s.

By the distribution of the conjugates of β, we have

$$\underbrace{\cdots}_{\frac{n-3}{2}\text{-roots}} > \left|\beta^{(\frac{3n+7}{4})}\right| > \left|\beta^{(\frac{n+5}{4})}\right| > \left|\beta^{(\frac{3n+3}{4})}\right| > \underbrace{\cdots}_{\frac{n-3}{2}\text{-roots}}.$$

Hence, we need to show that we have $\left|\beta^{(\frac{3n+7}{4})}\right| > 1$ and $\left|\beta^{(\frac{3n+3}{4})}\right| < 1$.

We have $\left|\beta^{(\frac{3n+7}{4})}\right| = \left|\alpha^{(\frac{n+3}{2})} \cdot \alpha^{(\frac{n+5}{2})} \cdots \alpha^{(1)}\right|$. Assume that this is ≤ 1. Then by (b)

$$\left|\alpha^{(2)}\right| \cdot \left|\alpha^{(\frac{n+1}{2})} \cdot \alpha^{(\frac{n-1}{2})} \cdot \alpha^{(\frac{n-3}{2})} \cdots \alpha^{(2)}\right| < 1.$$

However,

$$\left|\alpha^{(2)}\right| \cdot \underbrace{\left|\alpha^{(\frac{n+1}{2})} \cdot \alpha^{(\frac{n-1}{2})} \cdot \alpha^{(\frac{n-3}{2})} \cdots \alpha^{(2)}\right| \cdot \left|\alpha^{(\frac{n+3}{2})} \cdot \alpha^{(\frac{n+5}{2})} \cdots \alpha^{(1)}\right|}_{=|N(\alpha)|=1} < 1,$$

which is a contradiction. Hence, $\left|\beta^{(\frac{3n+7}{4})}\right| > 1$. We argue similarly for $\left|\beta^{(\frac{3n+3}{4})}\right| < 1$. Assume

$$\left|\beta^{(\frac{3n+3}{4})}\right| = \left|\alpha^{(\frac{n+1}{2})} \cdot \alpha^{(\frac{n+3}{2})} \cdots \alpha^{(n)}\right| \geq 1.$$

Now by (a),

$$\left|\alpha^{(\overline{\frac{n+1}{2}})}\right| \cdot \left|\alpha^{(1)}\alpha^{(2)} \cdots \alpha^{(\frac{n-1}{2})}\right| > 1,$$

which implies that,

$$\left|\alpha^{(\overline{\frac{n+1}{2}})}\right| \cdot \left|\alpha^{(1)}\alpha^{(2)} \cdots \alpha^{(\frac{n-1}{2})}\right| \cdot \left|\alpha^{(\frac{n+1}{2})} \cdot \alpha^{(\frac{n+3}{2})} \cdots \alpha^{(n)}\right| > 1.$$

But this implies that $\left|\alpha^{(\overline{\frac{n+1}{2}})}\right| > 1$, a contradiction. This proves the lemma. ■

Lemma 5.3.2 *Let $\alpha \in K$ be such that*

$$\left|\alpha^{(1)}\right| > \left|\alpha^{(2)}\right| > \left|\alpha^{(n)}\right| > \left|\alpha^{(3)}\right| > \left|\alpha^{(n-1)}\right| > \left|\alpha^{(4)}\right| > \left|\alpha^{(n-2)}\right| > \ldots, \qquad (5.3.8)$$

that is,

a. $\left|\alpha^{(\overline{n+2-i})}\right| > \left|\alpha^{(i+1)}\right| \;\forall\; i \in \{1, \ldots, \frac{n-1}{2}\}$.

b. $\left|\alpha^{(i)}\right| > \left|\alpha^{(n+2-i)}\right| \;\forall\; i \in \{2, \ldots, \frac{n+1}{2}\}$.

Assume that precisely $\frac{n+1}{2}$ conjugates of α lie outside of the unit circle. Then the conjugates of $\beta = M(\alpha)$ are precisely distributed as the following

$$\left|\beta^{(1)}\right| > \left|\beta^{(n)}\right| > \left|\beta^{(2)}\right| > \left|\beta^{(n-1)}\right| > \left|\beta^{(3)}\right| > \left|\beta^{(n-2)}\right| > \left|\beta^{(4)}\right| > \ldots \tag{5.3.9}$$

That is,

c. $\left|\beta^{(i)}\right| > \left|\beta^{(n+1-i)}\right| \;\forall\; i \in \{1, \ldots, \frac{n-1}{2}\}$, *and*

d. $\left|\beta^{(\overline{n+2-i})}\right| > \left|\beta^{(i)}\right| \;\forall\; i \in \{2, \ldots, \frac{n+1}{2}\}$,

and $M(\alpha)$ has either precisely $\frac{n+1}{2}$ or precisely $\frac{n-1}{2}$ conjugates outside the unit circle.

Proof. We have

$$\begin{aligned}\beta = M(\alpha) &= \left|\alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(\frac{n+5}{4})}\right| \cdot \left|\alpha^{(n)} \cdot \alpha^{(n-1)} \cdots \alpha^{(\frac{3n+7}{4})}\right| \\ &= \left|\alpha^{(\overline{\frac{3n+7}{4}})} \cdot \alpha^{(\overline{\frac{3n+7}{4}+1})} \cdot \alpha^{(\overline{\frac{3n+7}{4}+2})} \cdots \alpha^{(\overline{\frac{3n+7}{4}+\frac{n-1}{2}})}\right|.\end{aligned}$$

Note that β is a Perron number, so $\beta^{(1)} = \beta$ is the largest conjugate of β. Define for all $i \in \{0, \ldots, \frac{n-1}{2}\}$,

$$\gamma_i = \frac{\left|\alpha^{(\overline{\frac{3n+7}{4}+i})} \cdot \alpha^{(\overline{\frac{3n+7}{4}+1+i})} \cdots \alpha^{(\overline{\frac{3n+7}{4}+\frac{n-1}{2}+i})}\right|}{\left|\alpha^{(\overline{\frac{3n+7}{4}-i})} \cdot \alpha^{(\overline{\frac{3n+7}{4}+1-i})} \cdots \alpha^{(\overline{\frac{3n+7}{4}+\frac{n-1}{2}-i})}\right|} = \frac{\left|\beta^{(\overline{1+i})}\right|}{\left|\beta^{(\overline{1-i})}\right|}.$$

Notice that we have $\gamma_0 = 1$. Now, we have

$$\gamma_i = \gamma_{i-1} \cdot \frac{\left|\alpha^{(\overline{\frac{3n+7}{4}+\frac{n-1}{2}-(i-1)})} \cdot \alpha^{(\overline{\frac{3n+7}{4}+\frac{n-1}{2}+i})}\right|}{\left|\alpha^{(\overline{\frac{3n+7}{4}+(i-1)})} \cdot \alpha^{(\overline{\frac{3n+7}{4}-i})}\right|} = \gamma_{i-1} \cdot \frac{\left|\alpha^{(\overline{\frac{n+9}{4}-i})} \cdot \alpha^{(\overline{\frac{n+5}{4}+i})}\right|}{\left|\alpha^{(\overline{\frac{3n+3}{4}+i})} \cdot \alpha^{(\overline{\frac{3n+7}{4}-i})}\right|} \tag{5.3.10}$$

for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. By assumption (a) on the $\alpha^{(j)}$'s we see that for all $i \in \{1, \ldots, \frac{n-3}{4}\}$ we have

$$\left|\alpha^{(\overline{\frac{n+9}{4}-i})}\right| < \left|\alpha^{(\overline{\frac{3n+3}{4}+i})}\right| \quad \text{and} \quad \left|\alpha^{(\overline{\frac{n+5}{4}+i})}\right| < \left|\alpha^{(\overline{\frac{3n+7}{4}-i})}\right|.$$

This shows that $1 = \gamma_0 > \gamma_1 > \ldots > \gamma_{\frac{n-3}{4}}$. For all $i \in \{\frac{n+5}{4}, \ldots, \frac{n-1}{2}\}$, we have

$$\left|\alpha^{\overline{(\frac{3n+3}{4}+i)}}\right| < \left|\alpha^{\overline{(\frac{n+9}{4}-i)}}\right| \quad \text{and} \quad \left|\alpha^{\overline{(\frac{3n+7}{4}-i)}}\right| < \left|\alpha^{\overline{(\frac{n+5}{4}+i)}}\right|.$$

Hence, $\gamma_{\frac{n+1}{4}} < \gamma_{\frac{n+5}{4}} < \ldots < \gamma_{\frac{n-1}{2}}$.

Now, we have

$$\gamma_{\frac{n-1}{2}} = \frac{\left|\alpha^{(\frac{n+5}{4})} \cdot \alpha^{(\frac{n+9}{4})} \cdots \alpha^{(\frac{3n+3}{4})}\right|}{\left|\alpha^{(\frac{n+9}{4})} \cdot \alpha^{(\frac{n+13}{4})} \cdots \alpha^{(\frac{3n+7}{4})}\right|} = \frac{\left|\alpha^{(\frac{n+5}{4})}\right|}{\left|\alpha^{(\frac{3n+7}{4})}\right|} \overset{(a)}{<} 1,$$

we have that $\gamma_i < 1$ for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. This means

$$\left|\beta^{\overline{(1+i)}}\right| < \left|\beta^{\overline{(1-i)}}\right| = \left|\beta^{\overline{(n+1-i)}}\right| \quad \forall\, i \in \left\{1, \ldots, \frac{n-1}{2}\right\}.$$

Hence,

$$\left|\beta^{\overline{(i)}}\right| < \left|\beta^{\overline{(n+2-i)}}\right| \quad \forall\, i \in \left\{2, \ldots, \frac{n+1}{2}\right\}.$$

This proves assumption (d) for the $\beta^{(i)}$'s. Now we prove that $\beta^{(i)}$'s satisfy the condition (c). Define for all $i \in \{1, \ldots, \frac{n-1}{2}\}$,

$$\mu_i = \frac{\left|\alpha^{\left(\overline{\frac{3n+7}{4}+(i-1)}\right)} \cdot \alpha^{\left(\overline{\frac{3n+7}{4}+1+(i-1)}\right)} \cdots \alpha^{\left(\overline{\frac{3n+7}{4}+\frac{n-1}{2}+(i-1)}\right)}\right|}{\left|\alpha^{\left(\overline{\frac{3n+7}{4}-i}\right)} \cdot \alpha^{\left(\overline{\frac{3n+7}{4}+1-i}\right)} \cdots \alpha^{\left(\overline{\frac{3n+7}{4}+\frac{n-1}{2}-i}\right)}\right|} = \frac{\left|\beta^{(\bar{i})}\right|}{\left|\beta^{(\overline{1-i})}\right|}.$$

Note that $\mu_1 = \frac{\left|\beta^{(1)}\right|}{\left|\beta^{(n)}\right|} > 1$. Now,

$$\mu_{i+1} = \mu_i \cdot \frac{\left|\alpha^{\left(\overline{\frac{3n+7}{4}+\frac{n-1}{2}-i}\right)}\right| \cdot \left|\alpha^{\left(\overline{\frac{3n+7}{4}+\frac{n-1}{2}+i}\right)}\right|}{\left|\alpha^{\left(\overline{\frac{3n+7}{4}+(i-1)}\right)}\right| \cdot \left|\alpha^{\left(\overline{\frac{3n+7}{4}-(i+1)}\right)}\right|} \tag{5.3.11}$$

$$= \mu_i \cdot \frac{\left|\alpha^{\left(\overline{\frac{n+5}{4}-i}\right)}\right| \cdot \left|\alpha^{\left(\overline{\frac{n+5}{4}+i}\right)}\right|}{\left|\alpha^{\left(\overline{\frac{3n+3}{4}+i}\right)}\right| \cdot \left|\alpha^{\left(\overline{\frac{3n+3}{4}-i}\right)}\right|}, \tag{5.3.12}$$

for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. By assumption (b) on the $\alpha^{(j)}$'s, we see that for all $i \in \{1, \ldots, \frac{n-3}{4}\}$, we have

$$\left|\alpha^{\left(\overline{\frac{n+5}{4}-i}\right)}\right| > \left|\alpha^{\left(\overline{\frac{3n+3}{4}+i}\right)}\right|, \quad \text{and} \quad \left|\alpha^{\left(\overline{\frac{n+5}{4}+i}\right)}\right| > \left|\alpha^{\left(\overline{\frac{3n+3}{4}-i}\right)}\right|.$$

This implies that $1 < \mu_1 < \mu_2 < \ldots < \mu_{\frac{n+1}{4}}$. For all $i \in \{\frac{n+5}{4}, \ldots, \frac{n-1}{2}\}$,

$$\left|\alpha^{\left(\frac{n+5}{4}-i\right)}\right| < \left|\alpha^{\left(\frac{3n+3}{4}+i\right)}\right|, \quad \text{and} \quad \left|\alpha^{\left(\frac{n+5}{4}+i\right)}\right| < \left|\alpha^{\left(\frac{3n+3}{4}-i\right)}\right|.$$

Hence, $\mu_{\frac{n+5}{4}} > \mu_{\frac{n+9}{4}} > \ldots > \mu_{\frac{n-1}{2}}$. Since

$$\mu_{\frac{n-1}{2}} = \frac{\left|\alpha^{\left(\frac{n+1}{4}\right)} \cdot \alpha^{\left(\frac{n+5}{4}\right)} \cdots \alpha^{\left(\frac{3n-1}{4}\right)}\right|}{\left|\alpha^{\left(\frac{n+9}{4}\right)} \cdot \alpha^{\left(\frac{n+13}{4}\right)} \cdots \alpha^{\left(\frac{3n+7}{4}\right)}\right|} = \frac{\left|\alpha^{\left(\frac{n+1}{4}\right)} \cdot \alpha^{\left(\frac{n+5}{4}\right)}\right|}{\left|\alpha^{\left(\frac{3n+3}{4}\right)} \cdot \alpha^{\left(\frac{3n+7}{4}\right)}\right|} \overset{(b)}{>} 1.$$

We then have that $\mu_i > 1$ for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. This means

$$\left|\beta^{(\bar{i})}\right| > \left|\beta^{(\overline{1-i})}\right| = \left|\beta^{(\overline{n+1-i})}\right|,$$

for all $i \in \{1, \cdots, \frac{n-1}{2}\}$. This completes the proof of assumption (c) for the $\beta^{(i)}$'s.

Recall the distribution of the conjugates of $\beta^{(i)}$'s, we have

$$\underbrace{\cdots}_{\frac{n-3}{2}\text{-roots}} > \left|\beta^{(\frac{n+1}{4})}\right| > \left|\beta^{(\frac{3n+3}{4})}\right| > \left|\beta^{(\frac{n+5}{4})}\right| > \underbrace{\cdots}_{\frac{n-3}{2}\text{-roots}}.$$

Hence, we need to show that we have $\left|\beta^{(\frac{n+1}{4})}\right| > 1$ and $\left|\beta^{(\frac{n+5}{4})}\right| < 1$.

We have $\left|\beta^{(\frac{n+1}{4})}\right| = \left|\alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(\frac{n+1}{2})}\right|$. Assume that this is ≤ 1, then by (b),

$$\left|\alpha^{(2)}\right| \cdot \left|\alpha^{(n+2-2)} \cdot \alpha^{(n+2-3)} \cdot \cdots \cdot \alpha^{(n+2-\frac{n+1}{2})}\right| = \left|\alpha^{(2)}\right| \cdot \left|\alpha^{(n)} \cdot \alpha^{(n-1)} \cdot \cdots \cdot \alpha^{(\frac{n+3}{2})}\right| < 1.$$

This gives,

$$\left|\alpha^{(2)}\right| \cdot \underbrace{\left|\alpha^{(n)} \cdot \alpha^{(n-1)} \cdots \alpha^{(\frac{n+3}{2})}\right| \cdot \left|\alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(\frac{n+1}{2})}\right|}_{=|N(\alpha)|=1} < 1.$$

A contradiction. Hence, $\left|\beta^{(\frac{n+1}{4})}\right| > 1$. We now argue similarly for $\left|\beta^{(\frac{n+5}{4})}\right|$. Assume

$$\left|\beta^{(\frac{n+5}{4})}\right| = \left|\alpha^{(2)} \cdot \alpha^{(3)} \cdots \alpha^{(\frac{n+3}{2})}\right| \geq 1.$$

Now by (a),

$$\left|\alpha^{(n+2-1)}\alpha^{(n+2-2)} \cdots \alpha^{(n+2-\frac{n-1}{2})}\right| \cdot \left|\alpha^{(\overline{\frac{n+1}{2}})}\right| = \left|\alpha^{(1)}\alpha^{(n)} \cdots \alpha^{(\frac{n+5}{2})}\right| \cdot \left|\alpha^{(\overline{\frac{n+1}{2}})}\right| > 1,$$

But then,

$$\left|\alpha^{(\overline{\frac{n+1}{2}})}\right| \cdot \left|\alpha^{(1)}\alpha^{(n)}\cdots\alpha^{(\frac{n+5}{2})}\right| \cdot \left|\alpha^{(2)}\cdot\alpha^{(3)}\cdots\alpha^{(\frac{n+3}{2})}\right| > 1.$$

Since $\left|\alpha^{(\overline{\frac{n+1}{2}})}\right| < 1$, this is a contradiction. This proves $\left|\beta^{(\frac{n+5}{4})}\right| < 1$, so the lemma is proved. ■

Proposition 61 *Let $n \geq 7$ be such that $n \equiv 3 \mod 4$. Moreover, let $K/\mathbb{Q}$ be a Galois extension with Galois group isomorphic to the cyclic group C_n. Then K contains a wandering unit under iteration of the Mahler measure.*

Proof. We fix a generator σ of $\mathrm{Gal}(K/\mathbb{Q})$ and use the notations as above. By Dirichlet's unit theorem, we find an algebraic unit $\alpha \in K$ such that α satisfies the assumptions from Lemma 5.3.1. If $\beta = M(\alpha)$ has $\frac{n-1}{2}$ conjugates outside the unit circle, then β^{-1} has precisely $\frac{n+1}{2}$ conjugates outside the unit circle. These are distributed as

$$(\beta^{(\frac{n+3}{2})})^{-1} > (\beta^{(\frac{n+1}{2})})^{-1} > (\beta^{(\frac{n+5}{2})})^{-1} > (\beta^{(\frac{n-1}{2})})^{-1}\ldots$$

Let $(\beta^{(\frac{n+3}{2})})^{-1} = \gamma$, then we have

$$\gamma^{(1)} > \gamma^{(n)} > \gamma^{(2)} > \gamma^{(n-1)} > \gamma^{(3)} > \ldots$$

Now, γ satisfies the assumptions of Lemma 5.3.1, and we have $M^2(\alpha) = M(\beta) = M(\gamma)$. Suppose that $M(\gamma)$ has precisely $\frac{n-1}{2}$ conjugates outside the unit circle. Then, as before, some conjugate of $M(\gamma)^{-1}$ satisfies the assumptions of Lemma 5.3.1, and $M^3(\alpha) = M(M(\gamma)) = M(M(\gamma)^{-1})$. Now, either $M^3(\alpha)$ satifies the assumptions of Lemma 5.3.1 or Lemma 5.3.2, or some conjugate of $M^3(\alpha)^{-1}$ satisfies Lemma 5.3.1 or Lemma 5.3.2, and the argument continues. To summarize, for any iterate $M^j(\alpha)$, either it satifies the assumptions of Lemma 5.3.1 or Lemma 5.3.2, or some conjugate of $M^j(\alpha)^{-1}$ satisfies Lemma 5.3.1 or Lemma 5.3.2. This implies that α is a wandering unit. ■

Now, instead, assume $n \equiv 1 \mod 4$. As a reminder, we define for all integers a the integer

$\overline{a}$, with $\overline{a} \equiv a \mod n$ and $\overline{a} \in \{1, \ldots, n\}$.

Lemma 5.3.3 *Let $\alpha \in K$ be such that*

$$\left|\alpha^{(1)}\right| > \left|\alpha^{(2)}\right| > \left|\alpha^{(n)}\right| > \left|\alpha^{(3)}\right| > \left|\alpha^{(n-1)}\right| > \left|\alpha^{(4)}\right| > \left|\alpha^{(n-2)}\right| > \ldots \tag{5.3.13}$$

Then we have

a. $\left|\alpha^{(i)}\right| > \left|\alpha^{(n+2-i)}\right| \ \forall\, i \in \{2, \ldots, \frac{n+1}{2}\}$, *and*

b. $\left|\alpha^{(\overline{n+2-i})}\right| > \left|\alpha^{(i+1)}\right| \ \forall\, i \in \{1, \ldots, \frac{n-1}{2}\}$.

Assume further that precisely $\frac{n+1}{2}$ conjugates of α lie outside the unit circle. Then the conjugates of $\beta = M(\alpha)$ are precisely distributed as in (5.3.13) *and $M(\alpha)$ has either precisely $\frac{n+1}{2}$ or precisely $\frac{n-1}{2}$ conjugates outside the unit circle.*

Proof. Our assumptions guarantee that we have

$$\begin{aligned}\beta = M(\alpha) &= \left|\alpha^{(1)} \cdot \alpha^{(2)} \cdots \alpha^{(\frac{n+3}{4})}\right| \cdot \left|\alpha^{(n)} \cdot \alpha^{(n-1)} \cdots \alpha^{(n-\frac{n-5}{4})}\right| \\ &= \left|\alpha^{(\overline{\frac{3n+1}{4}+1})} \cdot \alpha^{(\overline{\frac{3n+1}{4}+2})} \cdots \alpha^{(\overline{\frac{3n+1}{4}+\frac{n+1}{4}})}\right|.\end{aligned}$$

As a Mahler measure, β is a Perron number, and hence we know that $\beta^{(1)} = \beta$ is the largest conjugate of β. In order to prove that the $\beta^{(i)}$ satisfy the assumption (a) from the lemma, we define for all $i \in \{1, \ldots, \frac{n-1}{2}\}$

$$\gamma_i = \frac{\left|\alpha^{(\overline{\frac{3n+5}{4}+i})} \cdot \alpha^{(\overline{\frac{3n+5}{4}+1+i})} \cdots \alpha^{(\overline{\frac{3n+5}{4}+\frac{n-1}{2}+i})}\right|}{\left|\alpha^{(\overline{\frac{3n+5}{4}-i})} \cdot \alpha^{(\overline{\frac{3n+5}{4}+1-i})} \cdots \alpha^{(\overline{\frac{3n+5}{4}+\frac{n-1}{2}-i})}\right|} = \frac{\left|\beta^{(\overline{i+1})}\right|}{\left|\beta^{(\overline{1-i})}\right|}.$$

Obviously, we have $\gamma_0 = 1$. Moreover, we see

$$\gamma_i = \gamma_{i-1} \cdot \frac{\left|\alpha^{(\overline{\frac{3n+5}{4}+\frac{n-1}{2}+i})} \cdot \alpha^{(\overline{\frac{3n+5}{4}+\frac{n-1}{2}-(i-1)})}\right|}{\left|\alpha^{(\overline{\frac{3n+5}{4}+(i-1)})} \cdot \alpha^{(\overline{\frac{3n+5}{4}-i})}\right|} = \gamma_{i-1} \cdot \frac{\left|\alpha^{(\overline{\frac{n+3}{4}+i})} \cdot \alpha^{(\overline{\frac{n+7}{4}-i})}\right|}{\left|\alpha^{(\overline{\frac{3n+1}{4}+i})} \cdot \alpha^{(\overline{\frac{3n+5}{4}-i})}\right|} \tag{5.3.14}$$

for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. By assumption (a) on the $\alpha^{(j)}$'s we find that for all $i \in \{1, \ldots, \frac{n-1}{4}\}$, we have

$$\left|\alpha^{(\overline{\frac{n+3}{4}+i})}\right| > \left|\alpha^{(\overline{\frac{3n+5}{4}-i})}\right| \quad \text{and} \quad \left|\alpha^{(\overline{\frac{n+7}{4}-i})}\right| > \left|\alpha^{(\overline{\frac{3n+1}{4}+i})}\right|.$$

This proves $1 = \gamma_0 < \gamma_1 < \ldots < \gamma_{\frac{n-1}{4}}$. For all $i \in \{\frac{n+3}{4}, \ldots, \frac{n-1}{2}\}$ assumption (a) yields

$$\left|\alpha^{\overline{(\frac{3n+1}{4}+i)}}\right| > \left|\alpha^{\overline{(\frac{n+7}{4}-i)}}\right| \quad \text{and} \quad \left|\alpha^{\overline{(\frac{3n+5}{4}-i)}}\right| > \left|\alpha^{\overline{(\frac{n+3}{4}+i)}}\right|.$$

Hence, we have $\gamma_{\frac{n+3}{4}} > \gamma_{\frac{n+7}{4}} > \ldots > \gamma_{\frac{n-1}{2}}$. Since we have

$$\gamma_{\frac{n-1}{2}} = \frac{\left|\alpha^{(\frac{n+3}{4})} \cdots \alpha^{(\frac{3n+1}{4})}\right|}{\left|\alpha^{(\frac{n+7}{4})} \cdots \alpha^{(\frac{3n+5}{4})}\right|} = \frac{\left|\alpha^{(\frac{n+3}{4})}\right|}{\left|\alpha^{(\frac{3n+5}{4})}\right|} \overset{(A)}{>} 1,$$

we find $\gamma_i > 1$ for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. This means

$$\left|\beta^{\overline{(1+i)}}\right| > \left|\beta^{\overline{(1-i)}}\right| = \left|\beta^{\overline{n+2-(1+i)}}\right| \quad \forall\, i \in \left\{1, \ldots, \frac{n-1}{2}\right\}$$

or equivalently

$$\left|\beta^{\overline{(i)}}\right| > \left|\beta^{\overline{(n+2-i)}}\right| \quad \forall\, i \in \left\{2, \ldots, \frac{n+1}{2}\right\}.$$

This proves assumption (a) for the $\beta^{(j)}$'s. Assumption (b) can be proven similarly: For all $i \in \{1, \ldots, \frac{n-1}{2}\}$, define

$$\mu_i = \frac{\left|\alpha^{\overline{\left(\frac{3n+9}{4}-i\right)}} \cdot \alpha^{\overline{\left(\frac{3n+9}{4}+1-i\right)}} \cdots \alpha^{\overline{\left(\frac{3n+9}{4}+\frac{n-1}{2}-i\right)}}\right|}{\left|\alpha^{\overline{\left(\frac{3n+5}{4}+i\right)}} \cdot \alpha^{\overline{\left(\frac{3n+5}{4}+1+i\right)}} \cdots \alpha^{\overline{\left(\frac{3n+5}{4}+\frac{n-1}{2}+i\right)}}\right|} = \frac{\left|\beta^{\overline{(2-i)}}\right|}{\left|\beta^{\overline{(i+1)}}\right|}.$$

Note that $\mu_1 = \frac{\left|\beta^{(1)}\right|}{\left|\beta^{(2)}\right|} > 1$. We have

$$\mu_{i+1} = \mu_i \cdot \frac{\left|\alpha^{\overline{\left(\frac{3n+9}{4}-i-1\right)}}\right| \cdot \left|\alpha^{\overline{\left(\frac{3n+5}{4}+i\right)}}\right|}{\left|\alpha^{\overline{\left(\frac{3n+9}{4}+\frac{n-1}{2}-i\right)}}\right| \cdot \left|\alpha^{\overline{\left(\frac{3n+5}{4}+\frac{n-1}{2}+i+1\right)}}\right|} \tag{5.3.15}$$

$$= \mu_i \cdot \frac{\left|\alpha^{\overline{\left(\frac{3n+5}{4}-i\right)}}\right| \cdot \left|\alpha^{\overline{\left(\frac{3n+5}{4}+i\right)}}\right|}{\left|\alpha^{\overline{\left(\frac{5n+7}{4}-i\right)}}\right| \cdot \left|\alpha^{\overline{\left(\frac{5n+7}{4}+i\right)}}\right|} \tag{5.3.16}$$

$$= \mu_i \cdot \frac{\left|\alpha^{\overline{\left(\frac{3n+5}{4}-i\right)}}\right| \cdot \left|\alpha^{\overline{\left(\frac{3n+5}{4}+i\right)}}\right|}{\left|\alpha^{\overline{\left(\frac{n+7}{4}-i\right)}}\right| \cdot \left|\alpha^{\overline{\left(\frac{n+7}{4}+i\right)}}\right|}, \tag{5.3.17}$$

for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. By assumption b on the $\alpha^{(j)}$'s, for all $i \in \{1, \ldots, \frac{n-5}{4}\}$,

$$\left|\alpha^{\left(\frac{3n+5}{4}+i\right)}\right| > \left|\alpha^{\left(\frac{n+7}{4}-i\right)}\right|, \quad \text{and} \quad \left|\alpha^{\left(\frac{3n+5}{4}-i\right)}\right| > \left|\alpha^{\left(\frac{n+7}{4}+i\right)}\right|.$$

For $i = \frac{n-1}{4}$,

$$\left|\alpha^{\left(\frac{3n+5}{4}+i\right)}\right| > \left|\alpha^{\left(\frac{n+7}{4}-i\right)}\right|, \quad \text{but} \quad \left|\alpha^{\left(\frac{3n+5}{4}-i\right)}\right| = \left|\alpha^{\left(\frac{n+7}{4}+i\right)}\right|.$$

Therefore, above gives $1 < \mu_1 < \mu_2 < \ldots < \mu_{\frac{n+3}{4}}$. For all $i \in \{\frac{n+3}{4}, \ldots, \frac{n-1}{2}\}$ assumption (b) gives

$$\left|\alpha^{\left(\frac{n+7}{4}-i\right)}\right| > \left|\alpha^{\left(\frac{3n+5}{4}+i\right)}\right|, \quad \text{and} \quad \left|\alpha^{\left(\frac{n+7}{4}+i\right)}\right| > \left|\alpha^{\left(\frac{3n+5}{4}-i\right)}\right|,$$

which implies that $\mu_{\frac{n+3}{4}} > \mu_{\frac{n+7}{4}} > \ldots > \mu_{\frac{n-1}{2}}$. Now, by (b),

$$\mu_{\frac{n-1}{2}} = \frac{\left|\alpha^{\left(\frac{n+11}{4}\right)} \cdots \alpha^{\left(\frac{3n+9}{4}\right)}\right|}{\left|\alpha^{\left(\frac{n+3}{4}\right)} \cdots \alpha^{\left(\frac{3n+1}{4}\right)}\right|} = \frac{\left|\alpha^{\left(\frac{3n+9}{4}\right)} \cdot \alpha^{\left(\frac{3n+5}{4}\right)}\right|}{\left|\alpha^{\left(\frac{n+3}{4}\right)} \cdot \alpha^{\left(\frac{n+7}{4}\right)}\right|} > 1.$$

Hence, $\mu_i > 1$ for all $i \in \{1, \ldots, \frac{n-1}{2}\}$. We conclude that

$$\left|\beta^{\left(\overline{n+2-i}\right)}\right| = \left|\beta^{\left(\overline{2-i}\right)}\right| > \left|\beta^{\left(\overline{i+1}\right)}\right|,$$

for all $i \in \{1, \cdots, \frac{n-1}{2}\}$. This completes the proof of assumption (b) for the β's.

By the distribution of the conjugates of β we get

$$\underbrace{\cdots}_{\frac{n-3}{2}\text{-many}} > \left|\beta^{(\frac{n+3}{4})}\right| > \left|\beta^{(\frac{3n+5}{4})}\right| > \left|\beta^{(\frac{n+7}{4})}\right| > \underbrace{\cdots}_{\frac{n-3}{2}\text{-many}}.$$

Hence, in order to prove the lemma, we have to show that we have $\left|\beta^{(\frac{n+3}{4})}\right| > 1$ and $\left|\beta^{(\frac{n+7}{4})}\right| < 1$. But this is almost obvious once we have written the $\beta^{(i)}$'s in terms of the $\alpha^{(i)}$'s.

We have $\left|\beta^{(\frac{n+3}{4})}\right| = \left|\alpha^{(1)} \cdots \alpha^{(\frac{n+1}{2})}\right|$. Assume that this is ≤ 1. Then by (a) also

$$\begin{aligned} &\left|\alpha^{(2)}\right| \cdot \left|\alpha^{(n+2-2)}\alpha^{(n+2-3)} \cdots \alpha^{(n+2-\frac{n+1}{2})}\right| \\ &= \left|\alpha^{(2)}\right| \cdot \left|\alpha^{(n)}\alpha^{(n-1)} \cdots \alpha^{(\frac{n+3}{2})}\right| < 1. \end{aligned}$$

Multiplying both terms ≤ 1 gives

$$\left|\alpha^{(2)}\right| \cdot \underbrace{\left|\alpha^{(n)}\alpha^{(n-1)}\cdots\alpha^{(\frac{n+3}{2})}\right| \cdot \left|\alpha^{(1)}\cdots\alpha^{(\frac{n+1}{2})}\right|}_{=|N(\alpha)|=1} \leq 1.$$

This is a contradiction. Hence, $\left|\beta^{(\frac{n+3}{4})}\right| > 1$. To show that $\left|\beta^{(\frac{n+7}{4})}\right| < 1$, we argue in the same way. Assume

$$\left|\beta^{(\frac{n+7}{4})}\right| = \left|\alpha^{(2)}\cdots\alpha^{(\frac{n+3}{2})}\right| \geq 1.$$

Then by (b) (and (a) for the last estimate) we also have

$$\left|\alpha^{(\overline{n+2-1})}\alpha^{(\overline{n+2-2})}\cdots\alpha^{(\overline{n+2-\frac{n-1}{2}})}\right| \cdot \left|\alpha^{(\overline{\frac{n+1}{2}})}\right| = \left|\alpha^{(1)}\alpha^{(n)}\cdots\alpha^{(\frac{n+5}{2})}\right| \cdot \left|\alpha^{(\overline{\frac{n+1}{2}})}\right| > 1.$$

However, multiplying both terms and applying that the norm of α has absolute value one, we get $\left|\alpha^{(\overline{\frac{n+1}{2}})}\right| > 1$, which is nonsense. This proves the lemma. ■

Lemma 5.3.4 *Let $\alpha \in K$ be such that*

$$\left|\alpha^{(1)}\right| > \left|\alpha^{(n)}\right| > \left|\alpha^{(2)}\right| > \left|\alpha^{(n-1)}\right| > \left|\alpha^{(3)}\right| > \left|\alpha^{(n-2)}\right| > \left|\alpha^{(4)}\right| > \ldots \qquad (5.3.18)$$

In other words,

a. $\left|\alpha^{(i)}\right| < \left|\alpha^{(n+2-i)}\right|\ \forall\ i \in \{2, \ldots, \frac{n+1}{2}\}$ *and*

b. $\left|\alpha^{(\overline{n+1-i})}\right| < \left|\alpha^{(1)}\right|\ \forall\ i \in \{1, \ldots, \frac{n-1}{2}\}$.

Assume further that precisely $\frac{n+1}{2}$ conjugates of α lie outside the unit circle. Then the conjugates of $\beta = M(\alpha)$ are precisely distributed as in (5.3.18) *and $M(\alpha)$ has either precisely $\frac{n+1}{2}$ or precisely $\frac{n-1}{2}$ conjugates outside the unit circle.*

Proof. This can be proved by an argument similar to those in the proofs of Lemma 5.3.1, 5.3.2 and 5.3.3. Note that all we have done is to replace i by $\overline{2+n-i}$ for all $i \in \{1, \ldots, n\}$ in Lemma 5.3.3. This does not affect the result. ■

Proposition 62 *Let $n \geq 5$ be such that $n \equiv 1 \mod 4$. Moreover, let $K/\mathbb{Q}$ be a Galois extension with Galois group isomorphic to the cyclic group C_n. Then K contains a wandering unit under iteration of the Mahler measure.*

Proof. We fix once and for all a generator σ of $\mathrm{Gal}(K/\mathbb{Q})$ and use the notation as above. By Dirichlet's unit theorem, we find an algebraic unit $\alpha \in K$ such that α satisfies the assumptions from Lemma 5.3.3. If $\beta = M(\alpha)$ has $\frac{n-1}{2}$ conjugates outside the unit circle, then β^{-1} has precisely $\frac{n+1}{2}$ conjugates outside the unit circle. These are distributed as

$$(\beta^{(\frac{n+3}{2})})^{-1} > (\beta^{(\frac{n+1}{2})})^{-1} > (\beta^{(\frac{n+5}{2})})^{-1} > \dots$$

Setting $(\beta^{(\frac{n+3}{2})})^{-1} = \gamma$, then this is nothing but

$$\gamma^{(1)} > \gamma^{(n)} > \gamma^{(2)} > \gamma^{(n-1)} > \gamma^{(3)} > \dots$$

In other words, γ satisfies the assumptions of Lemma 5.3.4. Moreover, we have $M^2(\alpha) = M(\beta) = M(\gamma)$. Assume that $M(\gamma)$ has precisely $\frac{n-1}{2}$ conjugates outside the unit circle. Then, as before, some conjugate of $M(\gamma)^{-1}$ satisfies the assumptions of Lemma 5.3.3, and $M(M(\gamma)) = M(M(\gamma)^{-1})$. We conclude, that any iterate $M^k(\alpha)$ either satisfies the assumptions of Lemma 5.3.3 or of Lemma 5.3.4. In particular, no iterate of α is a fixed point and hence α is a wandering unit. ■

Altogether, we have the following proposition:

Proposition 63 *Let $K/\mathbb{Q}$ be a cyclic extension of odd degree $n \geq 5$ with Galois group isomorphic to C_n. Then there are wandering units in K.*

5.3.5 Classification of Abelian extensions

Table 1 on the next page shows the results so far on Abelian extensions.

In conclusion, we have Theorem 31.

Proof for Theorem 31. Suppose that the maximal real subfield of K has Galois group isomorphic to C_1, C_2, $C_2 \times C_2$, or C_3, then $M(\alpha)$ as a real number must lie in a totally real subfield of K with Galois group isomorphic to C_1, C_2, $C_2 \times C_2$, or C_3, but we know that all elements in such fields are preperiodic. We assume now that the maximal real subfield of

Galois group of maximal real subfield	Contains Wandering point
C_1, C_2, or $C_2 \times C_2$	No
C_3	No
Contains C_4 or $C_2 \times C_2 \times C_2$	Yes
Contains C_n, where $n \geq 5$ is odd	Yes
Contains C_6	Yes
Contains $C_3 \times C_3$	Yes

Table 1: Classification of Abelian extensions

K has Galois group that is not isomorphic to one of C_1, C_2, $C_2 \times C_2$, or C_3, then its Galois group has to contain one of the following:

(a) C_4

(b) $C_2 \times C_2 \times C_2$

(c) C_n, where $n \geq 5$ is odd

(d) C_6

(e) $C_3 \times C_3$

Then, K has a totally real subfield with Galois group isomorphic to one of (a)-(e). By the results we proved in this chapter, such a subfield contains a wandering unit, this proves the forward direction. ■

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