ZEROS OF RANDOM TRIGONOMETRIC POLYNOMIALS WITH DEPENDENT COEFFICIENTS

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Abstract: It is well known that the expected number of real zeros of a random cosine polynomial (of degree n)

$$V_n(x) = \sum_{j=0}^{n} a_j \cos(jx), \quad x \in (0, 2\pi),$$

where the coefficients a_j are independent and identically distributed (i.i.d.) real-valued standard Gaussian random variables, is asymptotically $2n/\sqrt{3}$. To the best of our knowledge, the above asymptotic relation has always been the lower bound for the expected number of real zeros of V_n when the a_j employ different settings. However, this inequality is sharp for most of the cases that have been considered so far. Moreover, out of various ways to establish a set of dependent coefficients, one can sort out the coefficients in the blocks of the same length and then identify certain blocks. As one may expect, the expected number of real zeros of these polynomials is subject to how we identify the blocks, yet it might be independent of the size of the blocks. In this manuscript, we investigate four cases of random cosine polynomials where the blocks of the coefficients are identified in different fashions. The cases we study include the adjacent, palindromic, and periodic blocks as well as the case involving only two blocks, each of which possesses a different expected number of real zeros from one another.

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LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers: $1, 2, 3, \ldots$
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{T}$	The unit circle
$\mathbb{P}(A)$	The probability of an event A
$X \sim \mathcal{N}(\mu, \sigma^2)$	A random variable X distributed normally with mean μ and variance σ^2
$\mathbb{E}[X]$	The expectation of a random variable X
$\mathrm{Var}(X)$	The variance of a random variable X
$Z_S(f)$	The set of all zeros of f in the set S
$\#S$	The number of elements in a finite set S
$N(S)$	The number of zeros of a random function in a set S
$N_n(S)$	The number of zeros of a random function of degree n in a set S
$N_{n,K}(S)$	The number of K-level crossings of a random function of degree n in a set S
$f(n) = o(g(n))$	If $g(n) \neq 0$ and $f(n)/g(n) \to 0$ as $n \to \infty$
$f(n) = \mathcal{O}(g(n))$	If there exists $C > 0$ such that $\|f(n)\| \leq C\|g(n)\|$, for sufficiently large n
$f(n) \sim g(n)$	If $g(n) \neq 0$ and$f(n)/g(n) \to 1$ as $n \to \infty$
$f(n) \ll g(n)$	Vinogradov notation; $f(n) = \mathcal{O}(g(n))$
$\mathbb{1}_A$	The indicator function of a subset A of a set X
$\mathrm{sgn}(\cdot)$	The sign function; $\mathrm{sgn}(a) := \|a\| / a$, $a \neq 0$

CHAPTER I

INTRODUCTION

1.1 Plan of this dissertation

In this dissertation, we study zeros of random trigonometric polynomials with dependent coefficients. Chapter I contains a concise history of the subject.

The second chapter consists of the results from our published work [67]. The objective of the chapter is to introduce two different models of random trigonometric polynomials with dependent coefficients and to show the expected number of real zeros may remain intact or exceed that of the classical case with i.i.d. coefficients, depending on how one possibly sort the coefficients in different blocks.

Chapter III includes the results obtained in another published paper [68]. In that chapter, the expected number of real zeros of a random cosine polynomial with palindromic blocks of coefficients (of any fixed length $\ell \in \mathbb{N} \setminus \{1\}$) is considered. Our result generalizes the work of Farahmand and Li [35] on the expected number of real zeros of random cosine polynomials with palindromic coefficients, which corresponds to $\ell = 1$.

In the fourth and the last chapter, we study the same concept of the expected number of real zeros with quite a different setting. We determine the expected number of real zeros for algebraic, trigonometric, and cosine polynomials, where the coefficients are grouped in blocks of a fixed length recurring periodically.

1.2 A brief history of the study of random polynomials

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which we define complex-valued random variables $\eta_0, \eta_1, \ldots, \eta_n$. In addition, assume $S \subset \mathbb{C}$ and $f_0, f_1, \ldots, f_n : S \to \mathbb{C}$ are some functions. We consider a random function $F_n : S \to \mathbb{C}$ defined as

$$F_n(z) \equiv F_n(z, \omega) := \sum_{j=0}^{n} \eta_j(\omega) f_j(z), \tag{1.2.1}$$

which is indeed the linear combination of f_j's.

1.2.1 Random algebraic polynomials

Among all well-known random functions, of great importance are the random algebraic polynomials (or *Kac polynomials*) $P_n(z)$ plainly created by replacing the $f_j(z)$ in the above definition with the monomials z^j, that is,

$$P_n(z) := \eta_0 + \eta_1 z + \cdots + \eta_{n-1} z^{n-1} + \eta_n z^n, \tag{1.2.2}$$

where the η_j are complex-valued random variables and $\eta_n \neq 0$. Let $\mathbb{E}$ denote the mathematical expectation, $\mathbb{P}$ be the probability of an event, and $N_n(S)$ denote the number of zeros of P_n in the set S.

For the sake of clarity, we would like to reserve the coefficients η_j only for the case of complex-valued random (algebraic) polynomials and define real-valued ones as

$$P_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n, \tag{1.2.3}$$

where the a_j are real and chosen at random.

The work on the expected number of zeros of random polynomials initiated in the early 1930s. Bloch and Pólya [8] were the first to study the expected number of real zeros of a random polynomial whose coefficients are independent and identically distributed (i.i.d.) random variables. They assumed that $a_0 = 1$ almost surely (a.s.), namely $\mathbb{P}(\{\omega : a_0(\omega) = 1\}) = 1$, and that all other coefficients are uniformly chosen from the set $\{-1, 0, 1\}$, i.e.,

$$\mathbb{P}(\{\omega : a_j(\omega) = -1\}) = \mathbb{P}(\{\omega : a_j(\omega) = 0\}) = \mathbb{P}(\{\omega : a_j(\omega) = 1\}) = 1/3.$$

Bloch and Pólya showed that

$$\mathbb{E}[N_n(\mathbb{R})] = \mathcal{O}(\sqrt{n}), \quad \text{as } n \to \infty.$$

Throughout an almost-a-decade period, in a series of publications [58, 59, 60, 61, 62] Littlewood and Offord found and constantly modified upper and lower bounds for $N_n(\mathbb{R})$ of the random algebraic polynomial $P_n(x) = \sum_{j=0}^{n} a_j x^j$, where the coefficients a_j are real-valued random variables with either standard Gaussian or Bernoulli distribution, or uniformly distributed in $[-1, 1]$. More precisely, they proved that in such cases

$$\frac{\log n}{\log\log\log n} \ll N_n(R) \ll \log^2 n,$$

with probability $1 - o(1)$ as n tends to infinity.

When it comes to real-valued random polynomials P_n with i.i.d. standard Gaussian coefficients (with mean zero and unit variance), the expected number of real roots of these polynomials was the subject of the in-depth investigation by Kac [53]. He showed that

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{4}{\pi} \int_0^1 \frac{\sqrt{1 - h_n^2(x)}}{1 - x^2}\, dx,$$

where

$$h_n(x) = \frac{n x^{n-1}(1 - x^2)}{1 - x^{2n}}.$$

Using the above integral formula, Kac obtained the famed asymptotic relation

$$\mathbb{E}[N_n(\mathbb{R})] \sim \frac{2}{\pi} \log n, \tag{1.2.4}$$

and the estimate

$$\mathbb{E}[N_n(\mathbb{R})] \leqslant (2/\pi) \log n + 14/\pi, \quad n \in \mathbb{N} \setminus \{1\}.$$

In [54], using quite a different method, Kac also showed the same asymptotic as in (1.2.4) holds if the coefficients are i.i.d. random variables uniformly distributed in $[-1, 1]$, which was conjectured in his previous article [53]. The error term in asymptotic estimate

(1.2.4) established by Kac, which is $o(\log n)$, was later improved by many mathematicians. In fact, defining

$$A_0 := \lim_{n\to\infty} (\mathbb{E}[N_n(\mathbb{R})] - (2/\pi)\log(n+1)),$$

Jamrom [51, 52] and Wang [89] independently derived two integral representation forms of the constant A_0. More improved versions of Kac's asymptotic (1.2.4) may be found in the works of Hammersley [44], Yu [92], Edelman and Kostlan [18] and especially of Wilkins [90] who gives the asymptotic expansion

$$\mathbb{E}[N_n(\mathbb{R})] \sim \frac{2}{\pi}\log n + \sum_{k=0}^{\infty} A_k n^{-k},$$

where the coefficients A_0, A_2, A_4 were explicitly computed via integrals. Indeed, we approximately have $A_0 \approx 0.6257358$, $A_2 \approx -0.2426127$ and $A_4 \approx -0.0879406$ with $A_1 = A_3 = A_5 = 0$. It has still remained an open problem if $A_k = 0$ holds for the other odd k's.

Although the initial study of the number of real roots of random polynomials involved coefficients with discrete distribution, most of the following research focused on the case of the coefficients with normal distribution. However, an innovative method introduced by Erdős and Offord [21] showed that for a random polynomial with i.i.d. Bernoulli distribution, for sufficiently large n,

$$N_n(\mathbb{R}) = \frac{2}{\pi}\log n + o((\log n)^{2/3}\log\log n),$$

with probability $1 - o(1/\sqrt{\log\log n})$. By enhancing their method, Ibragimov and Maslova [46, 47, 48] showed that if the coefficients a_j are from the domain of attraction of the normal (proper) law, then

$$\mathbb{E}[N_n(\mathbb{R})] = \begin{cases} (2/\pi)\log n + o(\log n), & \text{if } \mathbb{E}(a_j) = 0, \\ (1/\pi)\log n + o(\log n), & \text{if } \mathbb{E}(a_j) \neq 0. \end{cases}$$

More recently, Nguyen, Nguyen and Vu [65] showed that the error term in the above estimate is literally bounded provided that the coefficients a_j have mean zero. The variance of the number of real zeros of random polynomials has also been of great interest. In 1974, Maslova

[64] showed that if the coefficients a_j are i.i.d. random variables distributed in such a way that $\mathbb{E}(a_j) = 0$ and $\mathbb{E}\big(|a_j|^{2+\delta}\big) < \infty$, $\delta > 0$, then

$$\mathrm{Var}[N_n(\mathbb{R})] \sim \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right)\log n.$$

While the number of real roots of the polynomial P_n is quite small, concentrating at ± 1, in the 1960s, Šparo and Šur [88] and Arnold [3] were among the first to show that most of the complex zeros concentrate on the unit circumference $\mathbb{T}$. Years later, Shepp and Vanderbei [86] showed that, for large enough n, about $n - (2/\pi)\log n$ of zeros accumulate on $\mathbb{T}$ whereas $(2/\pi)\log n$ of real roots gather at ± 1.

The fact of accumulation of complex zeros of polynomial P_n on the unit circumference may be viewed differently. Let $Z(P_n) = \{z_1, z_2, \ldots, z_n\}$ be the set of complex zeros of P_n (counted with multiplicity) and define the normalized zero counting measure

$$\tau_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j},$$

where δ_{z_j} is the unit point mass at z_j. The term *equidistributed* zeros refers to

$$\tau_n \xrightarrow{*} \mu_{\mathbb{T}}, \quad \text{as } n \to \infty.$$

Namely, τ_n converges (in the weak* topology sense, or weakly, in the language of probability theory) to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference $\mathbb{T}$ with probability 1, where $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$. More recent work on the limiting distribution of zeros of random polynomials are the papers of Ibragimov and Zeitouni [50], Hughes and Nikeghbali [45], Ibragimov and Zaporozhets [49], and Pritsker [69].

1.2.2 Random trigonometric polynomials

A random trigonometric polynomial of degree n is defined as

$$T_n(x) := \sum_{j=0}^{n} a_j \cos(jx) + b_j \sin(jx), \quad x \in (0, 2\pi), \tag{1.2.5}$$

where the coefficients a_j and b_j are chosen randomly. It is also convenient to tag the special case

$$V_n(x) := \sum_{j=0}^{n} a_j \cos(jx), \quad x \in (0, 2\pi), \tag{1.2.6}$$

as a random cosine polynomial if the coefficients a_j are random variables. We note that setting $z = e^{ix}$, (1.2.6) is literally depicted as

$$\begin{aligned} V_n(x) = \sum_{j=0}^{n} a_j \cos(jx) &= \frac{1}{2}\sum_{j=0}^{n} a_j(z^j + z^{-j}) \\ &= \frac{z^{-n}}{2}\sum_{k=-n}^{n} \eta_k z^{k+n} =: \frac{z^{-n}}{2} P_{2n}(z), \end{aligned} \tag{1.2.7}$$

where

$$\eta_j = \eta_{-j} = \begin{cases} 2a_0, & \text{if } j = 0, \\ a_j, & \text{if } 1 \leqslant j \leqslant n. \end{cases}$$

As (1.2.7) suggests, it is evident that the number of real zeros of V_n can not exceed $2n$.

The study of zeros of random trigonometric polynomials dates back to the 1960s when Dunnage [17] found the celebrated asymptotic for the expected number of real roots of random cosine polynomials. He showed that asymptotically about $1/\sqrt{3}$ of all zeros of V_n are real, in fact

$$\mathbb{E}[N_n(0, 2\pi)] \sim \frac{2n}{\sqrt{3}} \tag{1.2.8}$$

as long as $a_0 = 0$, and the a_j are i.i.d. random variables with standard normal distribution. Only two years after the work of Dunnage, as part of his Ph.D. thesis, Das [15] investigated random cosine polynomials of the form

$$V_{n,\delta}(x) := \sum_{j=0}^{n} j^{\delta} a_j \cos(jx), \quad a_j \sim \mathcal{N}(0,1),\ \delta > -3/2,$$

and showed that

$$\mathbb{E}[N_{n,\delta}(0, 2\pi)] = 2n\sqrt{\frac{1+2\delta}{3+2\delta}} + \mathcal{O}(n^{1/2-\delta}), \quad \text{as } n \to \infty.$$

As a matter of fact, the above asymptotic relation improves the error term $\mathcal{O}(n^{11/13}(\log n)^{3/13})$ obtained by Dunnage to $\mathcal{O}(\sqrt{n})$ only by setting $\delta = 0$. The error term $\mathcal{O}(1)$, the best known so far, appears in the work of Wilkins [91] where he proves that

$$\mathbb{E}[N_n(0,2\pi)] = \frac{2n+1}{\sqrt{3}} \sum_{r=0}^{3} \frac{D_r}{(2n+1)^r} + \mathcal{O}((2n+1)^{-3}), \quad \text{as } n \to \infty,$$

with $D_0 = 1$ and D_1, D_2, D_3 being explicitly computed.

There have also been developments in the study of K-level crossings, namely zeros of polynomials $V_n(x) = K$ with K being any constant which may depend on n but not on x, and the cases with coefficients having nonzero mean. For instance, Sambandham and Renganathan [76] showed that the expected number of real zeros of V_n remains invariant even if $\mu \neq 0$. Unlike random algebraic polynomials whose expectation of the K-level crossings decreases for relatively large values of K, more precisely the expected value of the K-level crossings in the interval $(-1,1)$ drops down from $(1/\pi)\log n$ to $(1/\pi)\log(n/K^2)$ as long as $K = o(\sqrt{n})$, the expected value of level crossings remains stable for the case of random cosine polynomials. The following characterization of the expected number of level crossings of random cosine polynomials [29, Theorem 4.1] given by Farahmand, which to some extent summarizes his own works [24, 26], reveals that, for sufficiently large n,

$$\mathbb{E}[N_{n,K}(0,2\pi)] = \begin{cases} \dfrac{2n}{\sqrt{3}} + \mathcal{O}(n^{3/4}), & \text{if } K = \mathcal{O}(n^{3/4}), \\ \dfrac{2n}{\sqrt{3}} + o(n), & \text{if } K = o(n), \end{cases} \tag{1.2.9}$$

where $N_{n,K}(0,2\pi)$ denotes the number of K-level crossings of the random cosine polynomial V_n whose coefficients a_j are i.i.d. and $a_j \sim \mathcal{N}(\mu,1)$. For larger values of K, up to this point, it is understood that the polynomial V_n, asymptotically and on average, is not to touch the level K if $K = \mathcal{O}(\sqrt{n\log n})$, for instance see [33].

Compared with the expected number of real zeros, our knowledge of the variance of real roots of random trigonometric polynomials was crude and just in the form of upper bounds. The first result on the variance of the number of real zeros was obtained by Qualls [73] in 1970 by using the theory of stationary processes. He considered a normalized version of random trigonometric polynomials defined as

$$X_n(x) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} a_j \cos(jx) + b_j \sin(jx), \quad x \in (0, 2\pi), \tag{1.2.10}$$

also known as Qualls' ensemble. Not only did he find that

$$\mathbb{E}[N_n(0, 2\pi)] = 2\sqrt{\frac{(2n+1)(n+1)}{6}},$$

but also showed that, for some positive constant C and sufficiently large enough n,

$$\Big| N_n(0, 2\pi) - \mathbb{E}[N_n(0, 2\pi)] \Big| \leqslant Cn^{3/4}$$

with probability $1 - o(1)$, provided that the coefficients a_j, b_j are i.i.d. random variables with standard normal distribution. Two fairly large upper bounds for the variance of real roots of polynomials V_n were established by Farahmand [25, 28] concluding that

$$\mathrm{Var}[N_n(0, 2\pi)] = \mathcal{O}(n^{3/4}), \quad \text{as } n \to \infty.$$

More recently, the asymptotic $\mathrm{Var}(N_n(0, 2\pi)) \sim cn$ conjectured by French physicists Bogomolny, Bohigas and Leboeuf [9] has been showed to be true (with $c = c_G \approx 0.5582$ being explicitly computed) by Granville and Wigman [43] for Qualls' ensemble with i.i.d. standard Gaussian coefficients. Moreover, they proved the Central Limit Theorem (CLT) by showing that the distribution

$$\frac{N_n(0, 2\pi) - \mathbb{E}[N_n(0, 2\pi)]}{\sqrt{c_G\, n}}$$

weakly converges to the standard normal distribution. In recent years, variance and CLT of the number of real roots of Qualls' ensemble has become an interesting direction of research, c.f. [87], [4], [6] and [16]. Do, Nguyen and Nguyen [16] found an asymptotic relation for the variance of the number of real roots for the Qualls' ensemble whose coefficients are i.i.d random variables belonging to a broad class of distributions (even discrete ones) satisfying the following (finite moments of large orders) criteria

$$\mathbb{E}[\xi] = 0, \ \mathrm{Var}(\xi) = 1 \text{ and } \mathbb{E}\big[\, |\xi|^{M_0} \,\big] < \infty, \quad \text{for large enough } M_0.$$

Moreover, the authors verified that the variance of the number of real roots, asymptotically, is linear in terms of the expectation by showing that

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Var}[N_n(0,2\pi)] = c_G + \frac{2}{15}\mathbb{E}[\xi^4 - 3].$$

It is then trivial to see that if the coefficients are uniformly chosen from $\{\pm 1\}$, also known as the Rademacher distribution, $\mathrm{Var}(N_n(0,2\pi)) \sim (c_G - 4/15)n$.

More on the history of the subject together with many additional references and further directions of work on a broad range of topics such as non-identical coefficients, sharp crossings, local extrema, points of inflection, and exceedance measure can be found in the books of Bharucha-Reid and Sambandham [7] or of Farahmand [29] and the references therein.

1.2.3 Dependent coefficients

Sambandham was one of the first mathematicians who studied the case of dependent coefficients for both algebraic and trigonometric polynomials. In a series of papers published in the 1970s, c.f. [83, 84], he investigated the expected number of real zeros of random algebraic polynomials with dependent coefficients of either constant or geometric correlations. Sambandham showed that for a random polynomial P_n with standard normal coefficients, the standard asymptotic relation (1.2.4) holds if the coefficients are of geometric correlation, that is,

$$\rho_{jk} := \mathbb{E}[a_j a_k] = \rho^{|j-k|}, \quad \rho \in (0, 1/2) \tag{1.2.11}$$

whereas $\mathbb{E}[N_n(\mathbb{R})]$ reduces asymptotically to $(1/\pi)\log n$ if the coefficients are of the constant correlation, i.e.,

$$\rho_{jk} = \begin{cases} 1, & \text{if } j = k, \\ \rho, & \text{if } j \neq k, \quad \rho \in (0,1). \end{cases} \tag{1.2.12}$$

The number of K-level crossings of P_n with coefficients being of the constant correlation was investigated by Farahmand in [23] showing that the number of K-level crossings reduces to half of that of in the case when the coefficients are independent–in contrast with the fact that the number of K-level crossings of P_n with coefficients being of the geometric correlation, defined in (1.2.11), still remains unsolved. The case of dependent coefficients of negative correlations was also discussed in the work of Farahmand and Nezakati [36] where they showed that the asymptotic (1.2.4) remains unaltered if the coefficients are of negative geometric correlation, namely $\rho_{jk} = -\rho^{|k-j|}$, $\rho \in (0, 1/3)$. Not always is $\mathcal{O}(\log n)$ the desired asymptotic. Farahmand and Nezakati [37] proved that the expected number of real zeros of P_n notably reduces to $\mathcal{O}(\sqrt{\log n})$ if the correlation between the coefficients satisfies $\rho_{jk} = 1 - |k-j|/n$. Moreover, in [38] they explored the expected number of real zeros of polynomials of the form

$$Q_n(x) = \sum_{j=0}^{n} \binom{n}{j}^{1/2} a_j x^j,$$

first introduced by Edelman and Kostlan [18], where $a_j \sim \mathcal{N}(0,1)$, and are of the constant correlation as defined in (1.2.12). Farahmand and Nezakati showed that $\mathbb{E}[N_n(\mathbb{R})] \sim \sqrt{n}/2$, which is indeed half of the value obtained in [18] for the case when the coefficients are independent.

The study of random trigonometric polynomials with dependent coefficients was first considered by Sambandham [82] where he showed that for the random cosine polynomials V_n with standard normal coefficients satisfying (1.2.12),

$$\mathbb{E}[N_n(0, 2\pi)] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{\varepsilon + 11/13}), \quad \text{as } n \to \infty,$$

with probability at least $1 - n^{-2\varepsilon}$, $\varepsilon \in (0, 1/13)$. Similarly, the case of random trigonometric polynomials with coefficients of geometric correlation was studied by Sambandham and Renganathan [77] confirming that $\mathbb{E}[N_n(0, 2\pi)] \sim 2n/\sqrt{3}$ is still valid. Analogous to (1.2.9), Farahmand [27] extended his own result to the case with the dependent coefficients of either constant or geometric correlation by showing that the number of K-level crossings stays unchanged as $2n/\sqrt{3}$ for large enough n. Angst, Dalmao and Poly [2] showed that the expected number of real zeros of Qualls' ensemble, as defined in (1.2.10), satisfies the universal asymptotic $2n/\sqrt{3}+o(n)$ as long as the coefficients are standard Gaussian random variables satisfying a general correlation function $\rho : \mathbb{N} \to \mathbb{R}$.

The common ground between the case with independent coefficients and those mentioned above with dependent coefficients satisfying some correlation properties is that $2n/\sqrt{3}$ is the universal expected value for the number of real roots (in one period) of a random trigonometric polynomial. Two examples of random trigonometric polynomials with *strongly* dependent Gaussian coefficients, i.e., the expected number of real roots are different from the standard one, are of great interest to us. The first example is the recent work of Pautrel [66] which considers the expected value of the number of real roots of Qualls' ensemble, where $a_j, b_j \sim \mathcal{N}(0,1)$ with $\mathbb{E}[a_j b_k] = 0$, and the correlation of the coefficients satisfies

$$\mathbb{E}[a_j a_k] = \mathbb{E}[b_j b_k] = \rho(|k-j|) := \cos(|k-j|\alpha), \quad \alpha \geqslant 0.$$

He shows that under this strong dependence condition, asymptotically, the expected number of real roots of X_n may significantly differ from the standard one. More precisely, he proves that for all $\varepsilon > 0$ and $l \in (\sqrt{2}, 2]$, there exist $\alpha = \alpha(l) \geq 0$, and infinitely many $n \in \mathbb{N}$, such that

$$\left| \frac{\mathbb{E}[N_n(0,2\pi)]}{n} - l \right| \leqslant \varepsilon.$$

The second example is the work of Farahmand and Li [35], where they extensively studied the expected number of real roots of polynomials T_n and V_n possessing palindromic coefficients, that is, $a_{n-j} = a_j$ and $b_{n-j} = b_j$. Unlike the algebraic polynomials with palindromic coefficients (studied in [32]) whose expected number of real roots still remains asymptotic to $(2/\pi)\log n$, a random cosine polynomial satisfying symmetry of the coefficients ends up with a non-universal result. In other words,

Theorem 1.2.1 (Farahmand & Li) *Let $n = 2m-1$, $m \in \mathbb{N}$, and $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$. We assume that the a_j, $0 \leqslant j \leqslant m-1$, are i.i.d. random variables with standard Gaussian distribution. If the a_j are palindromic, i.e., $a_j = a_{n-j}$, $0 \leqslant j \leqslant m-1$, then*

$$\mathbb{E}[N_n(0,2\pi)] = \frac{n}{\sqrt{3}} + \mathcal{O}(n^{3/4}), \quad \text{as } n \to \infty. \tag{1.2.13}$$

Let us reverse the order of the coefficients of V_n and define

$$\widetilde{V_n}(x) := \sum_{j=0}^{n} a_{n-j} \cos(jx), \quad x \in (0, 2\pi).$$

It then follows from Proposition 2.1 of [13] that

$$N_n(0, 2\pi) + \widetilde{N_n}(0, 2\pi) \geqslant 2n, \tag{1.2.14}$$

where $\widetilde{N_n}(0, 2\pi)$ refers to the number of zeros of $\widetilde{V_n}$ in one period, counted with multiplicity. This literally shows that if the coefficients of V_n are palindromic, n is the least average number of real zeros one can expect in $(0, 2\pi)$, which is obviously in direct contradiction with (1.2.13).

To resolve the issue, it is vital to point out that the asymptotic (1.2.13) solely counts the number of probabilistic real roots, and not those which are deterministic. To begin with, we observe that if the coefficients are palindromic, we can write

$$V_n(x) = \sum_{j=0}^{n} a_j \cos(jx) = \sum_{j=0}^{m-1} a_j \big[\cos(jx) + \cos(n-j)x\big] = 2\cos(nx/2)\, V_n^*(x),$$

where $m = (n+1)/2$, and

$$V_n^*(x) := \sum_{j=0}^{m-1} a_j \cos(n/2 - j)x.$$

Let us call $N_n(0, 2\pi)$ and $N_n^*(0, 2\pi)$ as the number of real zeros of V_n and V_n^* in $(0, 2\pi)$ respectively. Thus, with this notation, (1.2.13) should have been stated as

$$\mathbb{E}[N_n^*(0, 2\pi)] = \frac{n}{\sqrt{3}} + \mathcal{O}(n^{3/4}), \quad \text{as } n \to \infty. \tag{1.2.15}$$

Now, taking n distinct roots of $\cos(nx/2)$ into account, the expected value of the number of all real zeros of V_n is

$$\mathbb{E}[N_n(0, 2\pi)] = \mathbb{E}[n + N_n^*(0, 2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{3/4}), \quad \text{as } n \to \infty. \tag{1.2.16}$$

Remark 1.2.1 We note that Farahmand and Li [35] only considered the case where n is odd. They mentioned that the same result holds for even n's without making any further comments, which does not seem immediate at all. Therefore, we give a complete proof of the asymptotic relation (1.2.16) in the appendix to this manuscript, see Appendix, Theorem A.1, on p. 107.

1.2.4 Kac-Rice's formula

The Kac-Rice formula is our chief tool to study the number of real zeros of random functions' asymptotic behavior. The underlying formula on which the seminal Kac-Rice formula is built is called Kac's counting formula, see [53, Lemma 1], and is stated as follows.

Lemma 1.2.1 (Kac's Counting Formula I) *Let $F \in C^1[a,b]$, and assume that F' has finitely many zeros in (a,b). We define*

$$N^*(a,b) = N(a,b) + (\kappa(a) + \kappa(b))/2,$$

where $N(a,b)$ is the number of zeros of F in (a,b) and

$$\kappa(x) := \begin{cases} 1, & \text{if } F(x) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$N^*(a,b) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_a^b \mathbb{1}_{\{|F(x)|<\varepsilon\}}(x)\, |F'(x)|\, dx, \tag{1.2.17}$$

where multiple roots are counted only once.

We also note that $\psi_\varepsilon := (1/2\varepsilon)\mathbb{1}_{\{|F(x)|<\varepsilon\}}$ converges to Dirac's δ-measure (point mass measure at the origin) as $\varepsilon \to 0$, which implies that in each sufficiently small interval $I_k \subset (a,b)$ containing a zero of F, we have

$$\int_{I_k} \delta_0(F(x))\, |F'(x)|\, dx = 1.$$

Thus, summing over all the k, (1.2.17) may be written as

$$N^*(a,b) = \int_a^b \delta_0(F(x))\, |F'(x)|\, dx.$$

Remark 1.2.2 We can easily extend Kac's counting formula to find the number of K-level crossings of function F by replacing F with $F-K$ to obtain

$$
\begin{aligned}
N_K^*(a,b) &= \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon}\int_a^b \mathbb{1}_{\{|F(x)-K|<\varepsilon\}}(x)\,|F'(x)|\,dx \\
&= \int_a^b \delta_K(F(x))\,|F'(x)|\,dx,
\end{aligned}
\tag{1.2.18}
$$

where $N_K^*(a,b)$ is the number of K-level crossings and δ_K is the point mass at K.

Consider the Dirichlet integral

$$
\text{P.V. } \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha y)}{y}\,dy = \operatorname{sgn}(\alpha), \quad \alpha \neq 0,
$$

and let $a = x_0 < x_1 < \cdots < x_M < x_{M+1} = b$, where $x_1, \ldots, x_M$ are the roots of F'. It is clear that

$$
\begin{aligned}
&\text{P.V. } \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_a^b \cos(yF(x))\,|F'(x)|\,dx\,dy \\
&= \text{P.V. } \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=0}^{M}\int_{x_j}^{x_{j+1}} \cos(yF(x))\,|F'(x)|\,dx\,dy \\
&= \text{P.V. } \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=0}^{M}\pm\int_{x_j}^{x_{j+1}} \cos(yF(x))F'(x)\,dx\,dy \\
&= \frac{1}{2}\sum_{j=0}^{M}\pm\left\{\text{P.V. } \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(yF(x_{j+1}))-\sin(yF(x_j))}{y}\,dy\right\} \\
&= \frac{1}{2}\sum_{j=0}^{M}\pm\big[\operatorname{sgn}(F(x_{j+1}))-\operatorname{sgn}(F(x_j))\big] = N^*(a,b),
\end{aligned}
$$

which leads us to another version of Kac's counting formula:

Lemma 1.2.2 (Kac's Counting Formula II) *If $F \in C^1[a,b]$ and $F'(x)$ vanishes only at a finite number of points in (a,b), then*

$$
N^*(a,b) = \text{P.V. } \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_a^b \cos(yF(x))\,|F'(x)|\,dx\,dy. \tag{1.2.19}
$$

An essential contribution to this area was also made by Rice [78, 79], a computer scientist who is well-known in information theory, telecommunication, and signal processing, shortly after Kac published his pioneering paper having established the counting formula. To introduce the Rice formula, let us consider a real-valued stochastic process $\{F(x) : x \in I\}$ where I is an interval. The original Rice formula shows that the expected number of K-level crossings of a Gaussian centered stationary process with unite variance, in the interval I, is given by

$$\mathbb{E}[N_K^*(I)] = \frac{\sqrt{\lambda_2}\,|I|\,e^{-K^2/2}}{\pi},$$

where λ_2 is the second moment of the process. Roughly speaking, by taking expected value of both sides of (1.2.18), Kac-Rice's formula gives an explicit integral formula for the expectation of the number of K-level crossings, namely

$$\begin{aligned}
\mathbb{E}[N_K^*(I)] &= \mathbb{E}\left[\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_I \mathbb{1}_{\{|F(x)-K|<\varepsilon\}}(x)\,|F'(x)|\,dx\right] \\
&= \lim_{\varepsilon\to 0^+}\mathbb{E}\left[\frac{1}{2\varepsilon}\int_I \mathbb{1}_{\{|F(x)-K|<\varepsilon\}}(x)\,|F'(x)|\,dx\right] \\
&= \int_I \lim_{\varepsilon\to 0^+}\int_{K-\varepsilon}^{K+\varepsilon}\mathbb{E}\big[\,|F'(x)|\,\big|F(x)=y\big]\,p_{F(x)}(y)\,dy\,dx \\
&= \int_I \mathbb{E}\big[\,|F'(x)|\,\big|F(x)=K\big]\,p_{F(x)}(K)\,dx, \qquad (1.2.20)
\end{aligned}$$

where $p_{F(x)}$ denotes the probability density function for F. The above is just an upshot of a more general result known as the k-factorial moment of crossings, see [5, Theorem 3.2 & 3.4]. In particular, the expected number of K-level crossings of a real-valued Gaussian random function F is shown to be

$$\mathbb{E}[N_K(I)] = \int_I \int_{-\infty}^{\infty} |y|\; p_{F(x),F'(x)}(K,y)\,dy\,dx, \qquad (1.2.21)$$

where $p_{F(x),F'(x)}$ denotes the joint probability density for F and F', see [5, (3.20), p. 79] or [29, Theorem 2.1, p. 12].

Among different variants of Kac-Rice's formula, which are more or less equivalent, for instance see [1], Chapter 3 of [5], [14, p. 285], and [29, pp. 26–28], for the purpose of this text we stick to the one proved by Lubinsky, Pritsker and Xie [63, Proposition 1.1]. Using the counting formula (1.2.19), the authors showed that

Lemma 1.2.3 (Kac-Rice Formula) *Let* $[a,b] \subset \mathbb{R}$*, and consider real-valued functions* $f_j(x) \in C^1[a,b]$*,* $0 \leqslant j \leqslant n$*. Define the random function* $F_n(x) = \sum_{j=0}^{n} a_j f_j(x)$*, where the* a_j *are i.i.d. random variables with Gaussian distribution* $\mathcal{N}(0,\sigma^2)$*. Let*

$$A_n(x) := \sum_{j=0}^{n} (f_j(x))^2, \qquad B_n(x) := \sum_{j=0}^{n} f_j(x) f_j'(x), \qquad \text{and} \qquad C_n(x) := \sum_{j=0}^{n} (f_j'(x))^2.$$

If $A_n(x) > 0$ *on* $[a,b]$*, and there is* $M \in \mathbb{N}$ *such that* $F_n'(x)$ *has at most* M *zeros in* (a,b) *for all choices of coefficients, then the expected number of real zeros of* $F_n(x)$ *in the interval* (a,b) *is given by*

$$\mathbb{E}[N_n(a,b)] = \frac{1}{\pi} \int_a^b \frac{\sqrt{A_n(x)C_n(x) - B_n(x)^2}}{A_n(x)}\, dx. \tag{1.2.22}$$

Remark 1.2.3 It should be stressed that the original statement of the above lemma assumes that "$f_0(x)$ is a nonzero constant" in place of "$A_n > 0$ on $[a,b]$", a stronger hypothesis than stated. However, under this slightly weaker condition, $0 < \eta_n = \min_{x \in [a,b]} A_n(x)$ holds, and the proof is the same.

CHAPTER II

RANDOM TRIGONOMETRIC POLYNOMIALS WITH PAIRWISE EQUAL BLOCKS OF COEFFICIENTS

This chapter discusses the expected number of real zeros of some random trigonometric polynomials with pairwise equal blocks of coefficients. To begin with, we explore the polynomials T_n and V_n, defined in (1.2.5) and (1.2.6) respectively. We show that the expected number of real zeros of these polynomials in small intervals of length $\varepsilon > 0$ is negligible. Afterward, we discuss two cases of random trigonometric polynomials with pairwise equal blocks of coefficients, one with *adjacent* blocks, and the other with only two pairwise equal blocks.

2.1 Equidistribution and the expected number of zeros in negligible intervals

The study of the number of zeros of the polynomial $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx),\ x \in (0, 2\pi)$, with Gaussian coefficients, in a small interval dates back to the very first landmark in the subject by Dunnage [17, sec. 10, pp. 82–84] and extended shortly after by Das stating "the probability of $V_n(x)$ having an appreciable number of zeros in a small interval $t - \varepsilon < x < t + \varepsilon,\ t \in (0, 2\pi)$, is small", see [15, sec. 2, p. 721]. Thereafter, Jensen's inequality has been the essential tool to determine the expected number of real zeros of random trigonometric polynomials in negligible intervals; for instance see [26], [27], [30], [39], [34], and [35].

When a random trigonometric polynomial has i.i.d. Gaussian coefficients, one can effortlessly employ Flasche's result [40, Theorem 1, p. 3923] to show that, for any $\eta \in (0, 2\pi)$ and $\varepsilon > 0$,

$$\mathbb{E}[N_n(\eta, \eta + \varepsilon)] = \mathcal{O}(n\varepsilon), \quad \text{as } n \to \infty. \tag{2.1.1}$$

This section aims to find an analogous estimate for polynomials T_n and V_n without worrying whether the coefficients are independent or not.

As already observed in (1.2.7), real zeros of $V_n(x)$ are in fact the complex roots of an algebraic polynomial $P_{2n}(z)$ lying on the unit circumference $\mathbb{T}$. Hence in order to reach our desired result, it is natural to study complex zeros of random algebraic polynomials $P_n(z) = \sum_{j=0}^{n} \eta_j z^j$ in a very small annular sector containing the circular arc $e^{it},\ t \in (\eta, \eta+\varepsilon)$. As briefly discussed, see page 5, equidistribution of the zeros refers to

$$\tau_n \xrightarrow{*} \mu_{\mathbb{T}}, \quad \text{as } n \to \infty.$$

One can find more on the global limiting distribution of zeros of random polynomials, for instance, in the works of Ibragimov and Zeitouni [50], Hughes and Nikeghbali [45], and Ibragimov and Zaporozhets [49]. In particular, it was proved in [49] that for random algebraic polynomials P_n with the a_j being complex-valued i.i.d. random variables, $\mathbb{E}[\log^+ |a_0|] < \infty$ if and only if $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ almost surely. As we observe in all results addressed above, it is commonly assumed that the a_j are i.i.d. random variables. Pritsker [70] (see also [69]) showed that under assumption that the distribution function of the $|a_j|$ meets desirable growth conditions, $\tau_n \xrightarrow{*} \mu_{\mathbb{T}}$ a.s., where the a_j need not be identically distributed or even independent. One can study the deviation of τ_n from $\mu_{\mathbb{T}}$ through the discrepancy of these measures in the annular sectors

$$A_r(\alpha, \beta) = \{z \in \mathbb{C} : r < |z| < 1/r,\ \alpha \leqslant \arg z < \beta\}, \quad 0 < r < 1.$$

Pritsker and Sola [71, Theorem 3.7] considered a random polynomial $P_n(z)$, with not necessarily independent coefficients, and proved that the expected discrepancy of roots of P_n in the annular sector $A_r(\alpha, \beta)$, namely

$$\mathbb{E}\left[\left|\tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi}\right|\right]$$

decays like $\sqrt{\log n/n}$. We also note that in the case of deterministic polynomials, which was considered by Erdős and Turán [22], $\sqrt{\log n/n}$ is the optimal order one can obtain. Pritsker and Yeager [72] generalized the above asymptotic relation while removing many unnecessary restrictions. Herein, we quote one of their results on the rate of convergence for the expected discrepancy, which is vital to obtain our intended result on the number of real zeros of random trigonometric polynomials with dependent coefficients.

Lemma 2.1.1 (Pritsker & Yeager) *Let $P_n(z) = \sum_{j=0}^{n} a_{j,n} z^j$, $n \in \mathbb{N}$, be a sequence of random polynomials, where $a_{j,n}$ are complex-valued random variables. For a fixed $t \in (0,1]$, if*

$$M := \sup_{n\in\mathbb{N}} \left\{ \mathbb{E}[|a_{j,n}|^t] : 0 \leqslant j \leqslant n \right\} < \infty,$$

and

$$L := \inf_{n\in\mathbb{N}} \left\{ \mathbb{E}[\log |a_{j,n}|] : j = 0, n \right\} > -\infty,$$

then

$$\mathbb{E}\left[\left| \tau_n(A_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi} \right| \right] = \mathcal{O}\left(\sqrt{\frac{\log n}{n}} \right), \quad \text{as } n \to \infty, \tag{2.1.2}$$

where the implied constant (in big $\mathcal{O}$ notation) depends only on r, t, M and L but not on n.

So, we are in the position to claim that the expected number of real roots of polynomials V_n and T_n remains comparatively small in negligible intervals.

Lemma 2.1.2 *Let $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$ with the a_j being random variables, not necessarily independent, with Gaussian distribution $\mathcal{N}(0,\sigma^2)$. If $a \in (0, 1/2)$ is fixed, then for all $\delta \in [0, 2\pi)$, we have*

$$\mathbb{E}[N_n(\delta, \delta + n^{-a})] = \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty,$$

where the big $\mathcal{O}$ is uniform in δ.

Proof. Without loss of generality, let us set $\sigma = 1$. It follows from (1.2.7) that $V_n(x) = z^{-n} P_{2n}(z)/2$, where $z = e^{ix}$ and P_{2n} is a random polynomial of degree $2n$ with coefficients defined as $\eta_0 = 2a_0$ and $\eta_j = \eta_{-j} = a_j$, $1 \leqslant j \leqslant n$. Therefore, for $1 \leqslant j \leqslant n$,

$$\begin{aligned} \frac{\mathbb{E}[|\eta_0|]}{2} &= \mathbb{E}[|\eta_j|] = \mathbb{E}[|\eta_{-j}|] \\ &= \int_{-\infty}^{\infty} \frac{e^{-t^2/2}\, |t|\, dt}{\sqrt{2\pi}} = 2\int_{0}^{\infty} \frac{e^{-t^2/2}\, t\, dt}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}, \end{aligned}$$

and

$$\mathbb{E}\big[\log|\eta_{-n}|\big] = \mathbb{E}\big[\log|\eta_n|\big] = \int_{-\infty}^{\infty} \frac{e^{-t^2/2}\log|t|\,dt}{\sqrt{2\pi}}$$
$$= 2\int_0^{\infty} \frac{e^{-t^2/2}\log(t)\,dt}{\sqrt{2\pi}} = \frac{-\gamma - \log 2}{2} \approx -0.63518,$$

where $\gamma = -\int_0^\infty e^{-t}\log(t)\,dt$ is Euler-Mascheroni's constant. Now, applying Lemma 2.1.1 while setting $t = 1$, $r = 1/2$, $\alpha = \delta$ and $\beta = \delta + n^{-a}$, we have

$$\mathbb{E}\left[\left|\tau_{2n}(A_{1/2}(\delta, \delta + n^{-a})) - \frac{n^{-a}}{2\pi}\right|\right] = \mathcal{O}\left(\sqrt{\frac{\log 2n}{2n}}\right),$$

which implies that

$$\mathbb{E}\big[2n\,\tau_{2n}(A_{1/2}(\delta, \delta + n^{-a}))\big] = \mathcal{O}(\sqrt{n\log n}) + \mathcal{O}(n^{1-a}) = \mathcal{O}(n^{1-a}).$$

Let $N_n(\cdot)$ and $N^*_{2n}(\cdot)$ denote the number of zeros of V_n and P_{2n} respectively. Since $(\delta, \delta + n^{-a}) \subset A_{1/2}(\delta, \delta + n^{-a})$, it is immediate that

$$\mathbb{E}[N_n(\delta, \delta + n^{-a})] \leqslant \mathbb{E}\big[N^*_{2n}(A_{1/2}(\delta, \delta + n^{-a}))\big]$$
$$= \mathbb{E}\big[2n\,\tau_{2n}(A_{1/2}(\delta, \delta + n^{-a}))\big] = \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty,$$

where this estimate is uniform in δ since all r, t, M and L are fixed. ■

In a similar way, one can show that

Corollary 2.1.1 *Let $T_n(x) = \sum_{j=0}^n a_j \cos(jx) + b_j \sin(jx)$ with the a_j and b_j being random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. If $a \in (0, 1/2)$ is fixed, then for all $\delta \in [0, 2\pi)$, we have*

$$\mathbb{E}[N_n(\delta, \delta + n^{-a})] = \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty,$$

where the big $\mathcal{O}$ is uniform in δ.

Proof. Again, for simplicity let $\sigma = 1$. Similar to (1.2.7) and setting $z = e^{ix}$, we can write

$$T_n(x) = \frac{z^{-n}P_{2n}(z)}{2},$$

where $P_{2n}(z) := \sum_{k=-n}^{n} \eta_k z^{k+n}$ with

$$\eta_j = \overline{\eta_{-j}} = \begin{cases} 2a_0, & \text{if } j = 0, \\ a_j - ib_j, & \text{if } 1 \leqslant j \leqslant n. \end{cases}$$

It is then clear that $\mathbb{E}[|\eta_0|] = 2\sqrt{2/\pi}$, and for $1 \leqslant j \leqslant n$,

$$\mathbb{E}[|\eta_j|] = \mathbb{E}[|\eta_{-j}|] \leqslant \mathbb{E}[|a_j|] + \mathbb{E}[|b_j|] \leqslant 2\sqrt{\frac{2}{\pi}},$$

and

$$\mathbb{E}\big[\log|\eta_{-n}|\big] = \mathbb{E}\big[\log|\eta_n|\big] \geqslant \mathbb{E}\big[\log(\max\{|a_n|, |b_n|\})\big] = \frac{-\gamma - \log 2}{2},$$

which imply that $M < \infty$ and $L > -\infty$. The rest of the proof remains the same as in the proof of Lemma 2.1.2. ■

2.2 Random trigonometric polynomials with equal adjacent blocks, and only two equal blocks

Our primary motivation behind the study of roots of random trigonometric polynomials with pairwise equal blocks of coefficients is that the expected number of real zeros of a random cosine polynomial with palindromic coefficients is not universal any longer, i.e., it deviates from $2n/\sqrt{3}$. More precisely, as we see in (1.2.16), in such a case, one should expect about 36.6% more real zeros than in the classical case where the coefficients are i.i.d. Gaussian random variables that are centered with unit variance. This naturally raises the question of whether the expected value of the number of real roots remains universal if the coefficients are sorted and identified in different ways. In other words, we would like to know how strongly dependent the coefficients should be to force the expectation to become non-universal, how arranging the coefficients in blocks with a specific length changes the expected number of real zeros and what this length has to do with that expectation. As observed and expected by the author, a random cosine polynomial with adjacent coefficients, namely $a_{2j} = a_{2j+1}$, asymptotically has the same expected number of real zeros as the classical case. Since each coefficient by itself can be seen as a block of unit length, it is reasonable to investigate the number of real zeros of a random trigonometric polynomial whose coefficients are ordered in blocks instead.

Definition 2.2.1 An ℓ-tuple $(a_i, a_{i+1}, \ldots, a_{i+\ell-1})$ is called a block of coefficients of length ℓ.

The first model we consider is the one where successive blocks of coefficients are pairwise identified. In particular, we show no matter what the length of these adjacent blocks is, the expected value of the number of real zeros is universal as if the coefficients were independent in the first place. The construction goes as follows:
Fix $\ell \in \mathbb{N}$, and let $n = 2\ell m - 1 + r$, $m \in \mathbb{N}$ and $r \in \{0, 1, \ldots, 2\ell - 1\}$, namely r is the remainder and m is the quotient of dividing $n + 1$ by 2ℓ. We sort the coefficients in $2m$ blocks of length ℓ in the following fashion. Assume

$$A = (a_0, a_1, \ldots, a_n) = \bigcup_{i=0}^{2m-1} A_i \cup \tilde{A}_r,$$

where $A_i := (a_{\ell i}, a_{\ell i+1}, \ldots, a_{\ell i+\ell-1})$, and

$$\tilde{A}_r := \begin{cases} \emptyset, & \text{if } r = 0, \\ (a_{2\ell m}, \ldots, a_{2\ell m-1+r}), & \text{if } 1 \leqslant r \leqslant 2\ell - 1. \end{cases}$$

In other words, the set $\tilde{A}_r$ varies in size from empty to having $2\ell - 1$ elements and comes in the end of A. We further assume that the adjacent blocks are identified, i.e., $A_{2j+1} = A_{2j}$, $0 \leqslant j \leqslant m - 1$. In the following theorem, we prove that under these assumptions, the expected number of real zeros of V_n asymptotically stays unaltered as $2n/\sqrt{3}$ regardless of the size of the blocks.

Theorem 2.2.1 *Fix* $\ell \in \mathbb{N}$ *and let* $n = 2\ell m - 1 + r$, $m \in \mathbb{N}$, *and* $r \in \{0, 1, \ldots, 2\ell - 1\}$. *Assume* $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$, *and* $\bigcup_{j=0}^{m-1} A_{2j} \cup \tilde{A}_r$ *is a family of i.i.d. random* *variables with Gaussian distribution* $\mathcal{N}(0,\sigma^2)$. *For* $0 \leqslant j \leqslant m-1$ *and* $0 \leqslant k \leqslant \ell-1$, *we further assume* $a_{2\ell j+\ell+k} = a_{2\ell j+k}$, *in other words,* $A_{2j+1} = A_{2j}$. *Then*

$$\mathbb{E}[N_n(0,2\pi)] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{4/5}), \quad \text{as } n \to \infty,$$

where the implied constant (in big $\mathcal{O}$ *notation) only depends on* ℓ.

Similarly, let

$$B = (b_0, b_1, \dots, b_n) = \bigcup_{i=0}^{2m-1} B_i \cup \tilde{B}_r,$$

where $B_i := (b_{\ell i}, b_{\ell i+1}, \dots, b_{\ell i+\ell-1})$, and

$$\tilde{B}_r := \begin{cases} \emptyset, & \text{if } r = 0, \\ (b_{2\ell m}, \dots, b_{2\ell m-1+r}), & \text{if } 1 \leqslant r \leqslant 2\ell-1. \end{cases}$$

We can also show that the above result also holds for random trigonometric polynomials T_n. To put it in a nutshell, the expected value of the number of real zeros of a random trigonometric polynomial with adjacent blocks of coefficients is universal.

Theorem 2.2.2 *Fix* $\ell \in \mathbb{N}$ *and let* $n = 2\ell m - 1 + r$, $m \in \mathbb{N}$, *and* $r \in \{0, 1, \dots, 2\ell-1\}$. *Assume* $T_n(x) = \sum_{j=0}^{n} a_j \cos(jx) + b_j \sin(jx)$, $x \in (0, 2\pi)$, *and* $\bigcup_{j=0}^{m-1} (A_{2j} \cup B_{2j}) \cup (\tilde{A}_r \cup \tilde{B})$ *is a family of i.i.d. random variables with Gaussian distribution* $\mathcal{N}(0,\sigma^2)$. *For* $0 \leqslant j \leqslant m-1$, *we further assume* $A_{2j+1} = A_{2j}$ *and* $B_{2j+1} = B_{2j}$. *Then*

$$\mathbb{E}[N_n(0,2\pi)] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{4/5}), \quad \text{as } n \to \infty,$$

where the implied constant depends only on ℓ.

The second case to be studied is even more extreme because the set of random coefficients is composed of only two identical blocks. Let $A = (a_0, a_1, \dots, a_n) = A_0 \cup A_1 \cup \tilde{A}$, where

$$A_0 := (a_0, a_1, \dots, a_{\lceil n/2 \rceil - 1}), \quad A_1 := (a_{\lceil n/2 \rceil}, a_{\lceil n/2 \rceil + 1}, \dots, a_{2\lceil n/2 \rceil - 1}),$$

and

$$\tilde{A} := \begin{cases} \emptyset, & \text{if } n \text{ is odd,} \\ (a_n), & \text{if } n \text{ is even.} \end{cases}$$

We prove that if the condition $A_1 = A_0$ is imposed, the asymptotics of the expected number of real zeros of V_n is no more universal.

Theorem 2.2.3 *Let $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $n \in \mathbb{N}$, and $x \in (0, 2\pi)$. Assume $A_0 \cup \tilde{A}$ is a family of i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. For $0 \leqslant j \leqslant m := \lceil n/2 \rceil - 1$, we further assume $a_{j+\lceil n/2 \rceil} = a_j$, that is, $A_1 = A_0$. Then*

$$\mathbb{E}[N_n(0, 2\pi)] = \left(\frac{1}{2} + \frac{\sqrt{13}}{2\sqrt{3}}\right)n + \mathcal{O}(n^{4/5}), \quad \text{as } n \to \infty,$$

where the implied constant depends only on ℓ.

In a similar fashion, we define $B = (b_0, b_1, \ldots, b_n) = B_0 \cup B_1 \cup \tilde{B}$ with

$$B_0 := (b_0, b_1, \ldots, b_{\lceil n/2 \rceil - 1}), \quad B_1 := (b_{\lceil n/2 \rceil}, b_{\lceil n/2 \rceil + 1}, \ldots, b_{2\lceil n/2 \rceil - 1}),$$

along with

$$\tilde{B} := \begin{cases} \emptyset, & \text{if } n \text{ is odd,} \\ (b_n), & \text{if } n \text{ is even.} \end{cases}$$

We observe that the above result is also valid for polynomials T_n, i.e.,

Theorem 2.2.4 *Let $T_n(x) = \sum_{j=0}^{n} a_j \cos(jx) + b_j \sin(jx)$, $n \in \mathbb{N}$, and $x \in (0, 2\pi)$. Assume $A_0 \cup B_0 \cup \tilde{A} \cup \tilde{B}$ is a family of i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. If $A_1 = A_0$ and $B_1 = B_0$, then*

$$\mathbb{E}[N_n(0, 2\pi)] = \left(\frac{1}{2} + \frac{\sqrt{13}}{2\sqrt{3}}\right)n + \mathcal{O}(n^{4/5}), \quad \text{as } n \to \infty,$$

where the implied constant depends only on ℓ.

2.3 Proofs

In this section, we only give the proofs of Theorems 2.2.1 and 2.2.3 since they are more challenging. The proofs of Theorems 2.2.2 and 2.2.4 are much easier due to the appearance of many cancellations, and they employ precisely the same machinery as those of Theorem 2.2.1 and 2.2.3 and are therefore omitted. Before proving the first theorem, we need a handful of lemmas that are pretty useful for what comes.

Lemma 2.3.1 *Fix $\ell \in \mathbb{N} \setminus \{1\}$ and define $u_\ell(x) := \sin(\ell x)/\ell \sin(x)$, $x \in [0, \pi]$, then*

$$|u_\ell(x)| \leqslant 1, \quad x \in [0, \pi],$$

where the maximum is attained only at the endpoints $x = 0, \pi$.

Proof. Let $f(y) = T_\ell(y)$, $y \in [-1, 1]$, be the ℓ-th Chebyshev polynomial of the first kind, namely $T_\ell(y) := \cos(\ell \arccos(y))$ on $[-1, 1]$. It is clear that $f(y)$ is a polynomial of degree ℓ and $|f(y)| \leqslant 1$. It is also known that

$$f'(y) = \frac{d}{dy} T_\ell(y) = \ell\, U_{\ell-1}(y),$$

where $U_{\ell-1}(y)$ is the Chebyshev polynomial of the second kind defined as

$$U_{\ell-1}(y) := \frac{\sin(\ell \arccos(y))}{\sin(\arccos(y))}, \quad |y| \leqslant 1.$$

Now, f meets all the hypotheses of Markov's inequality for algebraic polynomials, see [74, Theorem 15.1.4, p. 567], that is, f is a polynomial of degree at most ℓ on $[-1, 1]$, and $|f(y)| \leqslant 1$. Hence Markov's inequality for algebraic polynomials implies that

$$|f'(y)| = \ell\, |U_{\ell-1}(y)| \leqslant \ell^2, \quad |y| \leqslant 1,$$

and the upper bound is achieved only at $y = \pm 1$. In other words,

$$|U_{\ell-1}(y)| \leqslant \ell, \quad |y| \leqslant 1,$$

and the upper bound is achieved only at $y = \pm 1$. This concludes the proof if we set $y = \cos(x)$. ■

The next lemma we wish to discuss here gives us a fairly nice estimate for the sum of the trigonometric functions and their derivatives.

Lemma 2.3.2 *Fix $a \in (0, 1/2)$ and $p \in \mathbb{N}$ and let $x \in E = [m^{-a}, \pi/p - m^{-a}]$, $m \in \mathbb{N}$. For $\lambda = 0, 1, 2,$ we define*

$$P_\lambda(p, m, x) := \sum_{j=0}^{m-1} j^\lambda \cos(2pj)x, \quad \textit{and} \quad Q_\lambda(p, m, x) := \sum_{j=0}^{m-1} j^\lambda \sin(2pj)x.$$

Then

$$P_\lambda(p, m, x),\ Q_\lambda(p, m, x) = \mathcal{O}(m^{\lambda+a}), \quad \text{as } n \to \infty,$$

where the implied constant depends only on p.

Proof. It follows from [42, 1.341(1,3), p. 29] that, for $p \neq 0$, $r \in \mathbb{N}$ and $x \in (0, \pi)$,

$$\sum_{j=0}^{r-1} \cos(2pj + q)x = \frac{\sin(rpx)\cos((r-1)p + q)x}{\sin(px)}, \tag{2.3.1}$$

and

$$\sum_{j=0}^{r-1} \sin(2pj + q)x = \frac{\sin(rpx)\sin((r-1)p + q)x}{\sin(px)}. \tag{2.3.2}$$

By (2.3.1), we have

$$P_0(p, m, x) = \sum_{j=0}^{m-1} \cos(2pj)x = \frac{\sin(mpx)\cos(m-1)px}{\sin(px)} = \mathcal{O}(m^a),$$

where the last estimate is reached by the fact that $\csc(px) = \mathcal{O}(m^a)$ on E. It is also easy to check, while employing (2.3.2), that

$$\begin{aligned} P_1(p, m, x) &= \frac{d}{dx}\left(\frac{Q_0(p, m, x)}{2p}\right) = \frac{d}{dx}\sum_{j=0}^{m-1} \frac{\sin(2pj)x}{2p} \\ &= \frac{d}{dx}\left(\frac{\sin(mpx)\sin(m-1)px}{2p\sin(px)}\right) \\ &= \frac{mp\cos(mpx)\sin(m-1)px}{2p\sin(px)} \\ &\quad + \frac{(m-1)p\sin(mpx)\cos(m-1)px}{2p\sin(px)} \\ &\quad - \frac{p\sin(mpx)\sin(m-1)px\ \cos(px)}{2p\sin^2(px)} \\ &= \mathcal{O}(m^{1+a}) + \mathcal{O}(m^{2a}) = \mathcal{O}(m^{1+a}). \end{aligned}$$

Similarly, by taking derivative of $Q_1(p,m,x)$, we can show that $P_2(p,m,x) = \mathcal{O}(m^{2+a})$, as m tends to infinity. The proof of estimates for the Q_λ is alike. ■

Corollary 2.3.1 *With the above assumptions, let*

$$R_\lambda(p,m,x) := \sum_{j=0}^{m-1} j^\lambda \cos(2pj+1)x, \quad \textit{and} \quad S_\lambda(p,m,x) := \sum_{j=0}^{m-1} j^\lambda \sin(2pj+1)x.$$

Then, from the preceding lemma, we have

$$R_\lambda(p,m,x),\ S_\lambda(p,m,x) = \mathcal{O}(m^{\lambda+a}), \quad \text{as } n \to \infty.$$

2.3.1 Proof of Theorem 2.2.1

Before we prove Theorem 2.2.1, we need a lemma. Keep in mind that in the following lemma, the implied constants (in big $\mathcal{O}$ notation) depend only on ℓ.

Lemma 2.3.3 *Fix $\ell \in \mathbb{N}$ and set $n = 2\ell m - 1$, $m \in \mathbb{N}$. Let us define*

$$V_n^*(x) := \sum_{j=0}^{m-1} \sum_{k=0}^{\ell-1} a_{2\ell j+k} \cos(2\ell j + \ell/2 + k)x,$$

where the $a_{2\ell j+k}$ are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0,\sigma^2)$. For a fixed $a \in (0,1/4)$, we define $E_0 = [0,\pi/\ell] \setminus F_0$ with

$$F_0 = [0,n^{-a}) \cup (\pi/2\ell - n^{-a}, \pi/2\ell + n^{-a}) \cup (\pi/\ell - n^{-a}, \pi/\ell].$$

Then

(1). $0 < A_n^*(x) = \dfrac{n}{4} + \mathcal{O}(n^a), \quad \text{as } n \to \infty \text{ and } x \in E_0,$

(2). $B_n^*(x) = \mathcal{O}(n^{1+a}), \quad \text{as } n \to \infty \text{ and } x \in E_0,$

(3). $C_n^*(x) = \dfrac{n^3}{12} + \mathcal{O}(n^{2+a}), \quad \text{as } n \to \infty \text{ and } x \in E_0,$

where $A_n^*(x)$, $B_n^*(x)$ and $C_n^*(x)$ are defined in a similar way as in Lemma 1.2.3.

Proof. It is trivial that

$$A_n^*(x) = \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} \cos^2(2\ell j + \ell/2 + k)x \geqslant \cos^2(\ell x/2) > 0, \quad x \in E_0.$$

We also observe that

$$\begin{aligned} A_n^*(x) &= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} \cos^2(2\ell j + \ell/2 + k)x \\ &= \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} \big[1 + \cos((4j+1)\ell + 2k)x\big] \\ &= \frac{\ell m}{2} + \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} \cos(4j+1)\ell x \, \cos(2kx) \\ &\quad - \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} \sin(4j+1)\ell x \, \sin(2kx) \\ &= \frac{n+1}{4} + \frac{R_0(2,m,\ell x)}{2}\sum_{k=0}^{\ell-1}\cos(2kx) - \frac{S_0(2,m,\ell x)}{2}\sum_{k=0}^{\ell-1}\sin(2kx). \end{aligned}$$

Hence the boundedness of $\sum_{k=0}^{\ell-1}\cos(2kx)$ and $\sum_{k=0}^{\ell-1}\sin(2kx)$ along with Corollary 2.3.1 gives that

$$A_n^*(x) = \frac{n}{4} + \mathcal{O}(n^a) = \frac{n\big(1 + \mathcal{O}(n^{-1+a})\big)}{4}, \quad \text{as } n \to \infty \text{ and } x \in E_0. \tag{2.3.3}$$

Proof of (2). We observe that

$$\begin{aligned} B_n^*(x) &= -\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j + \ell/2 + k)\cos(2\ell j + \ell/2 + k)x \, \sin(2\ell j + \ell/2 + k)x \\ &= -\frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j + \ell/2 + k)\sin((4j+1)\ell + 2k)x \\ &= -\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j + \ell/4 + k/2)\big[\sin(4j+1)\ell x \, \cos(2kx) + \cos(4j+1)\ell x \, \sin(2kx)\big] \\ &= -\ell\left(S_1(2,m,\ell x) + \frac{S_0(2,m,\ell x)}{4}\right)\sum_{k=0}^{\ell-1}\cos(2kx) - \frac{S_0(2,m,\ell x)}{2}\sum_{k=0}^{\ell-1}k\cos(2kx) \\ &\quad - \ell\left(R_1(2,m,\ell x) + \frac{R_0(2,m,\ell x)}{4}\right)\sum_{k=0}^{\ell-1}\sin(2kx) - \frac{R_0(2,m,\ell x)}{2}\sum_{k=0}^{\ell-1}k\sin(2kx). \end{aligned}$$

Thus, Corollary 2.3.1 implies that

$$B_n^*(x) = \mathcal{O}(n^{1+a}), \quad \text{as } n \to \infty \text{ and } x \in E_0.$$

Proof of (3). We see that

$$\begin{aligned} C_n^*(x) &= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2\sin^2(2\ell j+\ell/2+k)x \\ &= \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2\big[1-\cos((4j+1)\ell+2k)x\big] \\ &= \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2 \\ &\quad -\frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2\cos(4j+1)\ell x\,\cos(2kx) \\ &\quad +\frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2\sin(4j+1)\ell x\,\sin(2kx). \end{aligned}$$

It is clear that $\sum_{k=0}^{\ell-1}k^\lambda$ is bounded for $\lambda=1,2$, and $\sum_{j=0}^{m-1}j=\mathcal{O}(m^2)$. Thus, it is quite easy to check that

$$\begin{aligned} &\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2 \\ &\quad=\sum_{j=0}^{m-1}\left[\ell^3(4j^2+2j+1/4)+\frac{(\ell^3-\ell^2)(4j+1)}{2}+\frac{2\ell^3-3\ell^2+\ell}{6}\right] \\ &\quad=4\ell^3\sum_{j=0}^{m-1}j^2+\mathcal{O}(m^2)=\frac{4\ell^3(m-1)m(2m-1)}{6}+\mathcal{O}(m^2) \\ &\quad=\frac{4\ell^3m^3}{3}+\mathcal{O}(m^2)=\frac{n^3}{6}+\mathcal{O}(n^2). \end{aligned}$$

Next, we observe that

$$\begin{aligned} &\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j+\ell/2+k)^2\cos(4j+1)\ell x\,\cos(2kx) \\ &\quad=\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\big[(4j^2+2j+1/4)\ell^2+(4j+1)\ell k+k^2\big]\cos(4j+1)\ell x\,\cos(2kx) \\ &\quad=\ell^2\left[4R_2(2,m,\ell x)+2R_1(2,m,\ell x)+\frac{R_0(2,m,\ell x)}{4}\right]\sum_{k=0}^{\ell-1}\cos(2kx) \\ &\qquad+\ell(4R_1(2,m,\ell x)+R_0(2,m,\ell x))\sum_{k=0}^{\ell-1}k\cos(2kx)+R_0(2,m,\ell x)\sum_{k=0}^{\ell-1}k^2\cos(2kx) \\ &\quad=\mathcal{O}(m^{2+a})=\mathcal{O}(n^{2+a}), \end{aligned}$$

where the next to last equality is obtained by applying Corollary 2.3.1, and the fact that $\sum_{k=0}^{\ell-1} k^\lambda \cos(2kx)$ is bounded for $\lambda = 0, 1, 2$. Similarly, we have

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(2\ell j + \ell/2 + k)^2 \sin(4j+1)\ell x \, \sin(2kx) = \mathcal{O}(n^{2+a}).$$

Putting these estimates together, we have

$$C_n^*(x) = \frac{n^3}{12} + \mathcal{O}(n^{2+a}), \quad \text{as } n \to \infty \text{ and } x \in E_0,$$

as desired. ■

Proof of Theorem 2.2.1. We start with the simplest case when $r = 0$, that is, $n = 2\ell m - 1$. For $x \in (0, 2\pi)$, we see that

$$\begin{aligned} V_n(x) &= \sum_{j=0}^{n} a_j \cos(jx) \\ &= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} a_{2\ell j+k}\big[\cos(2\ell j + k)x + \cos(2\ell j + \ell + k)x\big] = 2\cos(\ell x/2)\, V_n^*(x), \end{aligned}$$

where

$$V_n^*(x) := \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} a_{2\ell j+k} \cos(2\ell j + \ell/2 + k)x.$$

Let us fix $a \in (0, 1/4)$ and let $N_n(\,\cdot\,)$ and $N_n^*(\,\cdot\,)$ be the number of real zeros of V_n and V_n^* respectively. It follows from Lemma 2.3.3 that

$$\begin{aligned} \Delta_n^*(x) &:= A_n^*(x)C_n^*(x) - B_n^*(x)^2 = \frac{n^4}{48} + \mathcal{O}(n^{3+a}) \\ &= \frac{n^4\big(1 + \mathcal{O}(n^{-1+a})\big)}{48}, \quad \text{as } n \to \infty \text{ and } x \in E_0. \end{aligned} \tag{2.3.4}$$

So, (2.3.3) and (2.3.4) as well as Lemma 1.2.3 (Kac-Rice's formula) give

$$\begin{aligned} \mathbb{E}[N_n^*(E_0)] &= \frac{1}{\pi}\int_{E_0} \frac{\sqrt{\Delta_n^*(x)}}{A_n^*(x)}\, dx \\ &= \frac{1}{\pi}\int_{E_0} \frac{n\big(1 + \mathcal{O}(n^{-1+a})\big)}{\sqrt{3}\big(1 + \mathcal{O}(n^{-1+a})\big)}\, dx \\ &= \frac{n + \mathcal{O}(n^a)}{\sqrt{3}\pi}\, |E_0| = \frac{\big(n + \mathcal{O}(n^a)\big)\big(\pi/\ell + \mathcal{O}(n^{-a})\big)}{\sqrt{3}\pi} \\ &= \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty. \end{aligned}$$

Let $E_k = E_0 + k\pi/\ell$ and $F_k = F_0 + k\pi/\ell,\ 0 \leqslant k \leqslant 2\ell - 1$. In the same way, we can show that

$$\mathbb{E}[N_n^*(E_k)] = \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Furthermore, Lemma 2.1.2 helps us write

$$\mathbb{E}[N_n^*(F_k)] \leqslant \mathbb{E}[N_n(F_k)] = \mathcal{O}(n^{1-a}).$$

Thus, for $0 \leqslant k \leqslant 2\ell - 1$,

$$\mathbb{E}\big[N_n^*[k\pi/\ell, (k+1)\pi/\ell]\big] = \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Therefore,

$$\mathbb{E}[N_n^*(0, 2\pi)] = \sum_{k=0}^{2\ell-1} \mathbb{E}\big[N_n^*[k\pi/\ell, (k+1)\pi/\ell]\big] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Now, taking ℓ distinct zeros of $\cos(\ell x/2)$ into account, we have

$$\mathbb{E}[N_n(0, 2\pi)] = \mathbb{E}[\ell + N_n^*(0, 2\pi)] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty. \tag{2.3.5}$$

Note that the above result holds for $n = 2\ell m - 1$. To generalize this result, we also need to study the expected number of real zeros of the polynomials $V_{n+1}, V_{n+2}, \dots, V_{n+2\ell-1}$, where $n = 2\ell m - 1$. For $x \in (0, 2\pi)$, we can write

$$V_{n+1}(x) = V_n(x) + a_{n+1} \cos(n+1)x = 2\cos(\ell x/2)\, V_n^*(x) + a_{n+1} \cos(n+1)x.$$

It is natural to estimate A_{n+1}, B_{n+1} and C_{n+1} in terms of A_n^*, B_n^* and C_n^* whose asymptotic estimations are already given in Lemma 2.3.3. In fact,

$$A_{n+1}(x) = 4\cos^2(\ell x/2) A_n^*(x) + \cos^2(n+1)x,$$

$$\begin{aligned} B_{n+1}(x) = &-2\ell \sin(\ell x/2)\cos(\ell x/2) A_n^*(x) \\ &+ 4\cos^2(\ell x/2) B_n^*(x) - (n+1)\sin(n+1)x\, \cos(n+1)x, \end{aligned}$$

and

$$C_{n+1}(x) = \ell^2 \sin^2(\ell x/2) A_n^*(x) + 4\cos^2(\ell x/2) C_n^*(x)$$
$$- 4\ell \sin(\ell x/2)\cos(\ell x/2) B_n^*(x) + (n+1)^2 \sin^2(n+1)x.$$

Thus,

$$\begin{aligned}\Delta_{n+1}(x) &:= A_{n+1}(x)C_{n+1}(x) - B_{n+1}(x)^2 = 16\cos^4(\ell x/2)\Delta_n^*(x)\\ &+ 4(n+1)^2\sin^2(n+1)x\ \cos^2(\ell x/2)A_n^*(x) + \ell^2\cos^2(n+1)x\ \sin^2(\ell x/2)A_n^*(x)\\ &+ 4\cos^2(n+1)x\ \cos^2(\ell x/2)C_n^*(x) - 2\ell\cos^2(n+1)x\ \sin(\ell x)B_n^*(x)\\ &- \ell(n+1)\sin(2n+2)x\ \sin(\ell x)A_n^*(x)\\ &+ 4(n+1)\sin(2n+2)x\ \cos^2(\ell x/2)B_n^*(x).\end{aligned}$$

Let $y = \pi/\ell - n^{-a}$. It is clear that

$$|\cos(\ell x/2)| \geqslant |\cos(\ell y/2)| = \sin(\ell n^{-a}/2) \geqslant \ell n^{-a}/\pi,\ x \in E_0,$$

which implies that $\sec(\ell x/2) = \mathcal{O}(n^a)$ on E_0. Now, this fact along with Lemma 2.3.3 and (2.3.4) helps us to write

$$\begin{aligned}\Delta_{n+1}(x) &= 16\cos^4(\ell x/2)\Delta_n^*(x) + \mathcal{O}(n^3)\\ &= \frac{n^4\cos^4(\ell x/2)\big(1+\mathcal{O}(n^{-1+a})\big)}{3} + \mathcal{O}(n^3)\\ &= \frac{n^4\cos^4(\ell x/2)\big(1+\mathcal{O}(n^{-1+a}) + \mathcal{O}(n^{-1+4a})\big)}{3}\\ &= \frac{n^4\cos^4(\ell x/2)\big(1+\mathcal{O}(n^{-1+4a})\big)}{3}, \quad \text{as } n\to\infty \text{ and } x \in E_0,\end{aligned}$$

Comparably, we see

$$\begin{aligned}A_{n+1}(x) &= 4\cos^2(\ell x/2)A_n^*(x) + \mathcal{O}(1) = n\cos^2(\ell x/2) + \mathcal{O}(n^a)\\ &= n\cos^2(\ell x/2)\big(1+\mathcal{O}(n^{-1+2a})\big), \quad \text{as } n\to\infty \text{ and } x\in E_0.\end{aligned}$$

Thereafter, applying Kac-Rice's formula gives us

$$\begin{aligned}
\mathbb{E}[N_{n+1}(E_0)] &= \frac{1}{\pi}\int_{E_0} \frac{\sqrt{\Delta_{n+1}(x)}}{A_{n+1}(x)}\,dx \\
&= \frac{1}{\pi}\int_{E_0} \frac{n\big(1+\mathcal{O}(n^{-1+4a})\big)}{\sqrt{3}\big(1+\mathcal{O}(n^{-1+2a})\big)}\,dx \\
&= \frac{n\big(1+\mathcal{O}(n^{-1+4a})\big)}{\sqrt{3}\pi}\,|E_0| \\
&= \frac{\big(n+\mathcal{O}(n^{4a})\big)\big(\pi/\ell+\mathcal{O}(n^{-a})\big)}{\sqrt{3}\pi} \\
&= \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{4a}) + \mathcal{O}(n^{1-a}), \quad \text{as } n\to\infty.
\end{aligned}$$

Now, we use Lemma 2.1.2 and observe that

$$\mathbb{E}[N_{n+1}[0,\pi/\ell]] = \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{4a}) + \mathcal{O}(n^{1-a}), \quad \text{as } n\to\infty.$$

Recall that $a\in(0,1/4)$. Thus, the most efficient estimate occurs at $a=1/5$, which implies that

$$\mathbb{E}[N_{n+1}[0,\pi/\ell]] = \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty.$$

Likewise, for $0\leqslant k\leqslant 2\ell-1$, we obtain that

$$\mathbb{E}[N_{n+1}[k\pi/\ell,(k+1)\pi/\ell]] = \frac{n}{\sqrt{3}\ell} + \mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty.$$

Hence taking sum over k's, we get the desired result

$$\mathbb{E}[N_{n+1}(0,2\pi)] = \sum_{k=0}^{2\ell-1}\mathbb{E}[N_{n+1}[k\pi/\ell,(k+1)\pi/\ell]] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty.$$

We can replicate this method to show that

$$\begin{aligned}
\mathbb{E}[N_{n+2\ell-1}(0,2\pi)] &= \cdots = \mathbb{E}[N_{n+2}(0,2\pi)] \\
&= \mathbb{E}[N_{n+1}(0,2\pi)] = \frac{2n}{\sqrt{3}} + \mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty.
\end{aligned}$$

Now, setting $a=1/5$ in (2.3.5) finishes the proof. ■

2.3.2 Proof of Theorem 2.2.3

To prove Theorem 2.2.3 we follow the same procedure as in the proof of Theorem 2.2.1, by estimating the auxiliary functions A_n^*, B_n^* and C_n^*.

Lemma 2.3.4 *Assume $n \in \mathbb{N}$ is odd, and set $m = (n-1)/2$. Let*

$$V_n^*(x) := \sum_{j=0}^{m} a_j \cos(j + (n+1)/4)x,$$

where the a_j are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. For a fixed $a \in (0, 1/4)$, we define $E = [0, \pi] \setminus F$ with $F = [0, n^{-a}) \cup (\pi - n^{-a}, \pi]$. Then

(1). $0 < A_n^*(x) = \dfrac{n}{4} + \mathcal{O}(n^a), \quad$ as $n \to \infty$ and $x \in E$,

(2). $B_n^*(x) = \mathcal{O}(n^{1+a}), \quad$ as $n \to \infty$ and $x \in E$,

(3). $C_n^*(x) = \dfrac{13n^3}{192} + \mathcal{O}(n^{2+a}), \quad$ as $n \to \infty$ and $x \in E$,

where $A_n^*(x)$, $B_n^*(x)$ and $C_n^*(x)$ are defined in a similar way as in Lemma 1.2.3.

Proof. We first show that $A_n^* > 0$ on E. It is clear that

$$A_n^*(x) = \sum_{j=0}^{m} \cos^2(j + (n+1)/4)x = \frac{m+1}{2} + \frac{1}{2}\sum_{j=0}^{m} \cos(2j + (n+1)/2)x.$$

We use (2.3.1) and observe that

$$\begin{aligned} A_n^*(x) &= \frac{m+1}{2} + \frac{\cos(nx)\sin(m+1)x}{2\sin(x)} \\ &= \frac{m+1}{2}\left(1 + \frac{\cos(nx)\sin(m+1)x}{(m+1)\sin(x)}\right) \\ &= \frac{(m+1)(1 + u_{m+1}(x)\cos(nx))}{2}, \end{aligned}$$

where

$$u_{m+1}(x) := \frac{\sin(m+1)x}{(m+1)\sin(x)}.$$

As a direct application of Lemma 2.3.1, it is immediate that $|u_{m+1}(x)| < 1$ on E which points out that $A_n^* > 0$ on E. It is achieved without great effort that

$$\begin{aligned} A_n^*(x) &= \sum_{j=0}^{m} \cos^2(j + (n+1)/4)x = \frac{1}{2} \sum_{j=0}^{m} \big[1 + \cos(2j + (n+1)/2)x\big] \\ &= \frac{m+1}{2} + \frac{\cos((n+1)x/2)}{2} \sum_{j=0}^{m} \cos(2jx) - \frac{\sin((n+1)x/2)}{2} \sum_{j=0}^{m} \sin(2jx) \\ &= \frac{n+1}{4} + \frac{\cos((n+1)x/2)\, P_0(1, m+1, x)}{2} - \frac{\sin((n+1)x/2)\, Q_0(1, m+1, x)}{2}. \end{aligned}$$

Once more Lemma 2.3.2 gives us

$$A_n^*(x) = \frac{n}{4} + \mathcal{O}(n^a) = \frac{n\big(1 + \mathcal{O}(n^{-1+a})\big)}{4}, \quad \text{as } n \to \infty \text{ and } x \in E. \tag{2.3.6}$$

Proof of (2). We write

$$\begin{aligned} B_n^*(x) &= -\sum_{j=0}^{m} \big[j + (n+1)/4\big] \sin(j + (n+1)/4)x \, \cos(j + (n+1)/4)x \\ &= -\frac{1}{2} \sum_{j=0}^{m} \big[j + (n+1)/4\big] \sin(2j + (n+1)/2)x \\ &= -\cos((n+1)x/2) \sum_{j=0}^{m} \big[j/2 + (n+1)/8\big] \sin(2jx) \\ &\quad - \sin((n+1)x/2) \sum_{j=0}^{m} \big[j/2 + (n+1)/8\big] \cos(2jx) \\ &= -\cos((n+1)x/2) \left(\frac{Q_1(1, m+1, x)}{2} + \frac{(n+1)\, Q_0(1, m+1, x)}{8} \right) \\ &\quad - \sin((n+1)x/2) \left(\frac{P_1(1, m+1, x)}{2} + \frac{(n+1)\, P_0(1, m+1, x)}{8} \right). \end{aligned}$$

Thus, using Lemma 2.3.2, we have the following estimate

$$B_n^*(x) = \mathcal{O}(n^{1+a}), \quad \text{as } n \to \infty \text{ and } x \in E.$$

Proof of (3). To estimate C_n^*, we see

$$\begin{aligned}
C_n^*(x) &= \sum_{j=0}^{m} \big[j + (n+1)/4\big]^2 \sin^2(j + (n+1)/4)x \\
&= \frac{1}{2}\sum_{j=0}^{m} \big[j + (n+1)/4\big]^2 (1 - \cos(2j + (n+1)/2)x) \\
&= \sum_{j=0}^{m} \big[j^2/2 + (n+1)j/4 + (n+1)^2/32\big] \\
&\quad - \cos((n+1)x/2) \sum_{j=0}^{m} \big[j^2/2 + (n+1)j/4 + (n+1)^2/32\big] \cos(2jx) \\
&\quad + \sin((n+1)x/2) \sum_{j=0}^{m} \big[j^2/2 + (n+1)j/4 + (n+1)^2/32\big] \sin(2jx).
\end{aligned}$$

A basic computation shows

$$\begin{aligned}
&\sum_{j=0}^{m} \big[j^2/2 + (n+1)j/4 + (n+1)^2/32\big] \\
&\qquad = \frac{m(m+1)(2m+1)}{12} + \frac{(n+1)m(m+1)}{8} + \frac{(n+1)^2(m+1)}{32} \\
&\qquad = \frac{13n^3}{192} + \mathcal{O}(n^2).
\end{aligned}$$

Moreover, applying Lemma 2.3.2 gives

$$\begin{aligned}
&\sum_{j=0}^{m} \big[j^2/2 + (n+1)j/4 + (n+1)^2/32\big] \cos(2jx) \\
&\quad = \frac{P_2(1, m+1, x)}{2} + \frac{(n+1)P_1(1, m+1, x)}{4} + \frac{(n+1)^2 P_0(1, m+1, x)}{32} \\
&\quad = \mathcal{O}(n^{2+a}).
\end{aligned}$$

Similar computations give us

$$\sum_{j=0}^{m} \big[j^2/2 + (n+1)j/4 + (n+1)^2/32\big] \sin(2jx) = \mathcal{O}(n^{2+a}).$$

Hence

$$C_n^*(x) = \frac{13n^3}{192} + \mathcal{O}(n^{2+a}), \quad \text{as } n \to \infty \text{ and } x \in E,$$

as required. ∎

Proof of Theorem 2.2.3. We provide the proof through two steps based on n being odd or even. As a matter of fact, the proof is straightforward for odd n due to the abundance of deterministic zeros.

First case: Assume n is odd and let $m = (n-1)/2$. For $x \in (0, 2\pi)$, observe that

$$\begin{aligned} V_n(x) &= \sum_{j=0}^{n} a_j \cos(jx) \\ &= \sum_{j=0}^{m} a_j \big[\cos(jx) + \cos(j + (n+1)/2)x\big] = 2\cos((n+1)x/4)\, V_n^*(x), \end{aligned}$$

where

$$V_n^*(x) := \sum_{j=0}^{m} a_j \cos(j + (n+1)/4)x.$$

Suppose $N_n(\,\cdot\,)$ and $N_n^*(\,\cdot\,)$ are the number of real zeros of V_n and V_n^* respectively. Let $a \in (0, 1/4)$ be arbitrary and fixed. We use the estimates given in Lemma 2.3.4, and observe that

$$\begin{aligned} \Delta_n^*(x) := A_n^*(x)C_n^*(x) - B_n^*(x)^2 &= \frac{13n^4}{768} + \mathcal{O}(n^{3+a}) \\ &= \frac{13n^4\big(1 + \mathcal{O}(n^{-1+a})\big)}{768}, \quad \text{as } n \to \infty \text{ and } x \in E. \end{aligned} \tag{2.3.7}$$

Therefore, (2.3.6) and (2.3.7) as well as Kac-Rice's formula (Lemma 1.2.3) suggest that

$$\begin{aligned} \mathbb{E}[N_n^*(E)] &= \frac{1}{\pi}\int_E \frac{\sqrt{\Delta_n^*(x)}}{A_n^*(x)}\,dx \\ &= \frac{1}{\pi}\int_E \frac{\sqrt{13}n\big(1 + \mathcal{O}(n^{-1+a})\big)}{4\sqrt{3}\big(1 + \mathcal{O}(n^{-1+a})\big)}\,dx \\ &= \frac{\sqrt{13}n\big(1 + \mathcal{O}(n^{-1+a})\big)}{4\sqrt{3}\pi}\,|E| \\ &= \frac{\big(\sqrt{13}n + \mathcal{O}(n^{a})\big)\big(\pi + \mathcal{O}(n^{-a})\big)}{4\sqrt{3}\pi} \\ &= \frac{\sqrt{13}n}{4\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty. \end{aligned}$$

Afterward, Lemma 2.1.2 guarantees that $\mathbb{E}[N_n^*(F)] = \mathcal{O}(n^{1-a})$, giving us

$$\mathbb{E}[N_n^*(0, 2\pi)] = 2\,\mathbb{E}[N_n^*(0, \pi)] = \frac{\sqrt{13}n}{2\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

In consequence, adding $(n+1)/2$ distinct zeros of $\cos((n+1)x/4)$ to the above estimate gives

$$\begin{aligned}\mathbb{E}[N_n(0,2\pi)] &= \mathbb{E}\left[\frac{n+1}{2} + N_n^*(0,2\pi)\right] \\ &= \left(\frac{1}{2} + \frac{\sqrt{13}}{2\sqrt{3}}\right)n + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty. \qquad (2.3.8)\end{aligned}$$

Second case: Assume n is even, and $a \in (0, 1/4)$ is fixed. To study the expected value of the number of real zeros of V_n, we start from the case of odd n and consider the polynomial V_{n+1} instead. For $x \in (0, 2\pi)$, we see that

$$\begin{aligned}V_{n+1}(x) &= V_n(x) + a_{n+1}\cos(n+1)x \\ &= 2\cos((n+1)x/4)\, V_n^*(x) + a_{n+1}\cos(n+1)x.\end{aligned}$$

It is then easy to compute A_{n+1}, B_{n+1} and C_{n+1} in terms of A_n^*, B_n^* and C_n^* whose asymptotics are already described in Lemma 2.3.4. In fact, we have

$$A_{n+1}(x) = 4\cos^2((n+1)x/4)A_n^*(x) + \cos^2(n+1)x,$$

$$\begin{aligned}B_{n+1}(x) = &-(n+1)\sin((n+1)x/4)\cos((n+1)x/4)A_n^*(x) \\ &+ 4\cos^2((n+1)x/4)B_n^*(x) - (n+1)\sin(n+1)x\ \cos(n+1)x,\end{aligned}$$

and

$$\begin{aligned}C_{n+1}(x) = &((n+1)/2)^2\sin^2((n+1)x/4)A_n^*(x) + 4\cos^2((n+1)x/4)C_n^*(x) \\ &- 2(n+1)\sin((n+1)x/4)\cos((n+1)x/4)B_n^*(x) + (n+1)^2\sin^2(n+1)x.\end{aligned}$$

Let

$$x_k^{n+1} := (4k+2)\pi/(n+1), \quad k \in \mathbb{N} \cup \{0\},$$

and Z be the set of all the x_k^{n+1}, which are in fact the roots of $\cos((n+1)x/4)$ lying in E. It is then easy to count and see that we possess $n/4 + \mathcal{O}(n^{1-a})$ of such *almost zeros*. We want to capture the reader's attention that, in this case, the integrand in Kac-Rice's formula obeys different asymptotics depending on the proximity of these so-called almost zeros. For each $x_k^{n+1} \in Z$, we define

$$J_k = \big[x_k^{n+1} - 2\pi/(n+1), x_k^{n+1} + 2\pi/(n+1)\big).$$

It is obvious that $x \in J_k$ can be written as $x = x_k^{n+1} + 4\pi u/(n+1)$, for some $u \in [-1/2, 1/2)$. This helps to express A_{n+1}, B_{n+1} and C_{n+1} explicitly as functions of u. In fact, for $x \in J_k$, we obtain

$$A_{n+1}(x) = 4\sin^2(\pi u)A_n^*(x) + \cos^2(4\pi u), \tag{2.3.9}$$

$$\begin{aligned} B_{n+1}(x) = {} & (n+1)\sin(\pi u)\cos(\pi u)A_n^*(x) \\ & + 4\sin^2(\pi u)B_n^*(x) - (n+1)\sin(4\pi u)\cos(4\pi u), \end{aligned}$$

and

$$\begin{aligned} C_{n+1}(x) = {} & ((n+1)/2)^2\cos^2(\pi u)A_n^*(x) + 4\sin^2(\pi u)C_n^*(x) \\ & + 2(n+1)\sin(\pi u)\cos(\pi u)B_n^*(x) + (n+1)^2\sin^2(4\pi u), \end{aligned}$$

which allow us to compute

$$\begin{aligned} \Delta_{n+1}(x) := {} & A_{n+1}(x)C_{n+1}(x) - B_{n+1}(x)^2 \\ = {} & 16\sin^4(\pi u)\Delta_n^*(x) + 4(n+1)^2\sin^2(\pi u)\sin^2(4\pi u)A_n^*(x) \\ & + ((n+1)/2)^2\cos^2(\pi u)\cos^2(4\pi u)A_n^*(x) \\ & + 4\sin^2(\pi u)\cos^2(4\pi u)C_n^*(x) + 2(n+1)\sin(\pi u)\cos(\pi u)\cos^2(4\pi u)B_n^*(x) \\ & + 2(n+1)^2\sin(\pi u)\cos(\pi u)\sin(4\pi u)\cos(4\pi u)A_n^*(x) \\ & + 8(n+1)\sin^2(\pi u)\sin(4\pi u)\cos(4\pi u)B_n^*(x). \end{aligned} \tag{2.3.10}$$

Our main objective is to prove that as n goes to infinity, on average, one root should be expected as we approach the x_k^{n+1}, namely u gets close to 0, which hence invites us to label these points as *almost zeros*.

As the first step, we approximate the number of real zeros of V_{n+1} while keeping enough distance from these almost zeros. To do so, we define

$$I'_k = \left[x_k^{n+1} - n^{-1-a}, x_k^{n+1} + n^{-1-a}\right], \quad x_k^{n+1} \in Z,$$

$$U = \bigcup_{x_k^{n+1} \in Z} I'_k, \quad \text{and} \quad G = E \setminus U.$$

Keep in mind that if $x \in G$,

$$|u| = \frac{\left|x - x_k^{n+1}\right| (n+1)}{4\pi} \geqslant \frac{n^{-1-a}(n+1)}{4\pi} \geqslant \frac{n^{-a}}{4\pi}.$$

Thus,

$$\frac{1}{|\sin(\pi u)|} \leqslant \frac{1}{2\,|u|} \leqslant 2\pi n^a,$$

which implies that $csc(\pi u) = \mathcal{O}(n^a)$ on G. With this fact in mind and using Lemma 2.3.4 and (2.3.7), we can estimate (2.3.10) as

$$\begin{aligned}
\Delta_{n+1}(x) &= 16 \sin^4(\pi u) \Delta_n^*(x) + \mathcal{O}(n^3) \\
&= \frac{13 n^4 \sin^4(\pi u)\big(1 + \mathcal{O}(n^{-1+a})\big)}{48} + \mathcal{O}(n^3) \\
&= \frac{13 n^4 \sin^4(\pi u)\big(1 + \mathcal{O}(n^{-1+a}) + \mathcal{O}(n^{-1+4a})\big)}{48} \\
&= \frac{13 n^4 \sin^4(\pi u)\big(1 + \mathcal{O}(n^{-1+4a})\big)}{48}, \quad \text{as } n \to \infty \text{ and } x \in G.
\end{aligned}$$

Comparably, we estimate

$$\begin{aligned}
A_{n+1}(x) &= 4 \sin^2(\pi u) A_n^*(x) + \mathcal{O}(1) \\
&= n \sin^2(\pi u)\big(1 + \mathcal{O}(n^{-1+2a})\big), \quad \text{as } n \to \infty \text{ and } x \in G.
\end{aligned}$$

Inserting these two estimates into the Kac-Rice formula gives

$$\begin{aligned}\mathbb{E}[N_{n+1}(G)] &= \frac{1}{\pi}\int_G \frac{\sqrt{\Delta_{n+1}(x)}}{A_{n+1}(x)}\,dx \\ &= \frac{1}{\pi}\int_G \frac{\sqrt{13}n\big(1+\mathcal{O}(n^{-1+4a})\big)}{4\sqrt{3}\big(1+\mathcal{O}(n^{-1+2a})\big)}\,dx \\ &= \frac{\sqrt{13}n\big(1+\mathcal{O}(n^{-1+4a})\big)}{4\sqrt{3}\pi}\,|G| \\ &= \frac{\big(\sqrt{13}n+\mathcal{O}(n^{4a})\big)\big(\pi+\mathcal{O}(n^{-a})\big)}{4\sqrt{3}\pi} \\ &= \frac{\sqrt{13}n}{4\sqrt{3}}+\mathcal{O}(n^{4a})+\mathcal{O}(n^{1-a}), \quad \text{as } n\to\infty.\end{aligned}$$

In addition, it follows from Lemma 2.1.2 that

$$\mathbb{E}[N_{n+1}(G\cup F)] = \frac{\sqrt{13}n}{4\sqrt{3}}+\mathcal{O}(n^{4a})+\mathcal{O}(n^{1-a}), \quad \text{as } n\to\infty.$$

The fact that $a\in(0,1/4)$ suggests that we may achieve the best possible estimate if we set $a=1/5$. Hence

$$\mathbb{E}[N_{n+1}(G\cup F)] = \frac{\sqrt{13}n}{4\sqrt{3}}+\mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty, \tag{2.3.11}$$

where $G=E\setminus U$ and $U=\cup_{x_k^{n+1}\in Z} I'_k$ with $I'_k=\big[x_k^{n+1}-n^{-6/5}, x_k^{n+1}+n^{-6/5}\big]$ are defined as above while setting $a=1/5$. In order to reach our desired result, we need to prove

$$\mathbb{E}[N_{n+1}(U)] = \frac{n}{4}+\mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty. \tag{2.3.12}$$

Recall that we have $n/4+\mathcal{O}(n^{4/5})$ of the intervals I'_k in $(0,\pi)$. Therefore, it suffices to show

$$\mathbb{E}[N_{n+1}(I'_k)] = 1+\mathcal{O}(n^{-1/5}), \quad \text{as } n\to\infty.$$

Equivalently, we intend to prove

$$\mathbb{E}[N_{n+1}(I_k)] = \frac{1}{2}+\mathcal{O}(n^{-1/5}), \quad \text{as } n\to\infty, \tag{2.3.13}$$

where

$$I_k=\big[x_k^{n+1}, x_k^{n+1}+n^{-6/5}\big],$$

and (2.3.13) does not depend on the choice of k. We divide the I_k into four subintervals

$$I_{k,1} = \big[x_k^{n+1} + n^{-5/4}, x_k^{n+1} + n^{-6/5}\big], \qquad I_{k,2} = \big[x_k^{n+1} + n^{-13/10}, x_k^{n+1} + n^{-5/4}\big],$$
$$I_{k,3} = \big[x_k^{n+1} + n^{-17/10}, x_k^{n+1} + n^{-13/10}\big], \quad \text{and} \quad I_{k,4} = \big[x_k^{n+1}, x_k^{n+1} + n^{-17/10}\big].$$

We study the expected number of real zeros in each subinterval separately. For $\varepsilon > 0$, we also define

$$I_{k,1,\varepsilon} = \big[x_k^{n+1} + n^{-5/4+\varepsilon}, x_k^{n+1} + n^{-6/5}\big], \quad \text{and} \quad I_{k,2,\varepsilon} = \big[x_k^{n+1} + n^{-13/10}, x_k^{n+1} + n^{-5/4-\varepsilon}\big].$$

First, we show that

$$\mathbb{E}[N_{n+1}(I_{k,1})] = \mathcal{O}(n^{-1/5}), \quad \text{as } n \to \infty. \tag{2.3.14}$$

If $x \in I_{k,1,\varepsilon}$, one can check that $\sin(\pi u) = \mathcal{O}(n^{-1/5})$ and $\csc(\pi u) = \mathcal{O}(n^{1/4-\varepsilon})$. Hence we can rewrite (2.3.9) as

$$\begin{aligned} A_{n+1}(x) &= 4\sin^2(\pi u) A_n^*(x) + \mathcal{O}(1) = n\sin^2(\pi u) + \mathcal{O}(1) \\ &= n\sin^2(\pi u)\big(1 + \mathcal{O}(n^{-1/2-2\varepsilon})\big), \quad \text{as } n \to \infty \text{ and } x \in I_{k,1,\varepsilon}. \end{aligned}$$

In a similar way, we estimate (2.3.10) and obtain

$$\begin{aligned} \Delta_{n+1}(x) &= 16\sin^4(\pi u)\Delta_n^*(x) + \mathcal{O}(n^3) = \frac{13n^4\sin^4(\pi u)}{48} + \mathcal{O}(n^3) \\ &= \frac{13n^4\sin^4(\pi u)\big(1 + \mathcal{O}(n^{-4\varepsilon})\big)}{48}, \quad \text{as } n \to \infty \text{ and } x \in I_{k,1,\varepsilon}. \end{aligned}$$

Therefore, Kac-Rice's formula gives us

$$\begin{aligned} \mathbb{E}[N_{n+1}(I_{k,1,\varepsilon})] &= \frac{1}{\pi}\int_{I_{k,1,\varepsilon}} \frac{\sqrt{\Delta_{n+1}(x)}}{A_{n+1}(x)}\,dx \\ &= \frac{1}{\pi}\int_{I_{k,1,\varepsilon}} \frac{\sqrt{13}n\big(1 + \mathcal{O}(n^{-4\varepsilon})\big)}{4\sqrt{3}\big(1 + \mathcal{O}(n^{-1/2-2\varepsilon})\big)}\,dx \\ &= \frac{\big(1 + \mathcal{O}(n^{-4\varepsilon})\big)}{\pi}\int_{I_{k,1,\varepsilon}} \frac{\sqrt{13}n}{4\sqrt{3}}\,dx \\ &= \big(1 + \mathcal{O}(n^{-4\varepsilon})\big)\mathcal{O}(n)\,\mathcal{O}(n^{-6/5}), \quad \text{as } n \to \infty. \end{aligned}$$

Now, (2.3.14) holds by letting $\varepsilon \to 0^+$.

Second step requires showing that

$$\mathbb{E}[N_{n+1}(I_{k,2})] = \mathcal{O}(n^{-1/5}), \quad \text{as } n \to \infty. \tag{2.3.15}$$

If $x \in I_{k,2,\varepsilon}$, we observe that $\sin(\pi u) = \mathcal{O}(n^{-1/4-\varepsilon})$ and $\csc(\pi u) = \mathcal{O}(n^{3/10})$. Plugging these estimates back into (2.3.9) and (2.3.10) gives us

$$\begin{aligned} A_{n+1}(x) &= 4\sin^2(\pi u) A_n^*(x) + \mathcal{O}(1) = n\sin^2(\pi u) + \mathcal{O}(1) \\ &= n\sin^2(\pi u)\big(1 + \mathcal{O}(n^{-2/5})\big), \quad \text{as } n \to \infty \text{ and } x \in I_{k,2,\varepsilon} \end{aligned}$$

and

$$\begin{aligned} \Delta_{n+1}(x) &= ((n+1)/2)^2 \cos^2(\pi u)\cos^2(4\pi u) A_n^*(x) + \mathcal{O}(n^{3-2\varepsilon}) \\ &= \frac{n^3 \cos^2(\pi u)\cos^2(4\pi u)}{16} + \mathcal{O}(n^{3-2\varepsilon}) \\ &= \frac{n^3 \cos^2(\pi u)\cos^2(4\pi u)\big(1 + \mathcal{O}(n^{-2\varepsilon})\big)}{16}, \quad \text{as } n \to \infty \text{ and } x \in I_{k,2,\varepsilon}. \end{aligned}$$

Therefore, by Lemma 1.2.3 (Kac-Rice's formula), we have

$$\begin{aligned} \mathbb{E}[N_{n+1}(I_{k,2,\varepsilon})] &= \frac{1}{\pi}\int_{I_{k,2,\varepsilon}} \frac{\sqrt{\Delta_{n+1}(x)}}{A_{n+1}(x)}\,dx \\ &= \frac{1}{\pi}\int_{I_{k,2,\varepsilon}} \frac{n^{3/2}\,|\cos(\pi u)|\,|\cos(4\pi u)|\,\big(1 + \mathcal{O}(n^{-2\varepsilon})\big)}{4n\sin^2(\pi u)\big(1 + \mathcal{O}(n^{-2/5})\big)}\,dx \\ &\leqslant \frac{1}{\pi}\int_{I_{k,2,\varepsilon}} \frac{n^{1/2}\big(1 + \mathcal{O}(n^{-2\varepsilon})\big)}{4\sin^2(\pi u)\big(1 + \mathcal{O}(n^{-2/5})\big)}\,dx \\ &= \big(1 + \mathcal{O}(n^{-2\varepsilon})\big)\,\mathcal{O}(n^{1/2})\int_{I_{k,2,\varepsilon}} \frac{dx}{\sin^2(\pi u)}, \quad \text{as } n \to \infty. \end{aligned}$$

Indeed, letting $\varepsilon \to 0^+$ implies

$$\mathbb{E}[N_{n+1}(I_{k,2})] = \mathcal{O}(n^{1/2})\int_{I_{k,2}} \frac{dx}{\sin^2(\pi u)}, \quad \text{as } n \to \infty.$$

Recall that $x = x_k^{n+1} + 4\pi u/(n+1)$, thus

$$\begin{aligned}
\int_{I_{k,2}} \frac{dx}{\sin^2(\pi u)} &= \frac{4\pi}{n+1} \int_{(n+1)n^{-13/10}/4\pi}^{(n+1)n^{-5/4}/4\pi} \frac{du}{\sin^2(\pi u)} \\
&\leqslant \frac{4\pi}{n+1} \int_{n^{-3/10}/4\pi}^{n^{-1/4}/2\pi} \frac{du}{\sin^2(\pi u)} \leqslant \frac{4\pi}{n+1} \int_{n^{-3/10}/4\pi}^{n^{-1/4}/2\pi} \frac{du}{4u^2} \\
&= \mathcal{O}(n^{-1})\, \mathcal{O}(n^{3/10}) = \mathcal{O}(n^{-7/10}), \quad \text{as } n \to \infty.
\end{aligned}$$

Now, (2.3.15) follows from the last two relations.

Next, we wish to show that

$$\mathbb{E}[N_{n+1}(I_{k,3})] = \frac{1}{2} + \mathcal{O}(n^{-1/5}), \quad \text{as } n \to \infty. \tag{2.3.16}$$

For $x \in I_{k,3}$, it is quite easy to see that $\sin(\pi u) = \mathcal{O}(n^{-3/10})$ and $\csc(\pi u) = \mathcal{O}(n^{7/10})$. Once more, applying these facts to (2.3.9) together with (2.3.6), while setting $a = 1/5$, gives us

$$\begin{aligned}
A_{n+1}(x) &= 4\sin^2(\pi u) A_n^*(x) + \cos^2(4\pi u) = n\sin^2(\pi u) + \cos^2(4\pi u) + \mathcal{O}(n^{-2/5}) \\
&= \big(n\sin^2(\pi u) + \cos^2(4\pi u)\big)\big(1 + \mathcal{O}(n^{-2/5})\big), \quad \text{as } n \to \infty \text{ and } x \in I_{k,3},
\end{aligned}$$

where the last equality is derived from the fact that $\cos^2(4\pi u) \geqslant 1/2$ for very small values of u, and

$$\frac{1}{n\sin^2(\pi u) + \cos^2(4\pi u)} \leqslant \frac{1}{\cos^2(4\pi u)} \leqslant 2.$$

We can also estimate (2.3.10) as

$$\begin{aligned}
\Delta_{n+1}(x) &= ((n+1)/2)^2 \cos^2(\pi u)\cos^2(4\pi u) A_n^*(x) + \mathcal{O}(n^{14/5}) \\
&= \frac{n^3\cos^2(\pi u)\cos^2(4\pi u)}{16} + \mathcal{O}(n^{14/5}) \\
&= \frac{n^3\cos^2(\pi u)\cos^2(4\pi u)\big(1 + \mathcal{O}(n^{-1/5})\big)}{16}, \quad \text{as } n \to \infty \text{ and } x \in I_{k,3}.
\end{aligned}$$

We also note that since $x \in I_{k,3}$, we have $u = \mathcal{O}(n^{-3/10})$ implying that both $\cos(\pi u)$ and

$\cos(4\pi u)$ are positive on $I_{k,3}$. Now, by Kac-Rice's formula

$$\begin{aligned}
\mathbb{E}[N_{n+1}(I_{k,3})] &= \frac{1}{\pi}\int_{I_{k,3}} \frac{\sqrt{\Delta_{n+1}(x)}}{A_{n+1}(x)}\,dx \\
&= \frac{1}{\pi}\int_{I_{k,3}} \frac{n^{3/2}\cos(\pi u)\cos(4\pi u)\big(1+\mathcal{O}(n^{-1/5})\big)}{4\big(n\sin^2(\pi u)+\cos^2(4\pi u)\big)\big(1+\mathcal{O}(n^{-2/5})\big)}\,dx \\
&= \frac{1+\mathcal{O}(n^{-1/5})}{4\pi}\int_{I_{k,3}} \frac{n^{3/2}\cos(\pi u)\cos(4\pi u)}{n\sin^2(\pi u)+\cos^2(4\pi u)}\,dx \\
&= \frac{1+\mathcal{O}(n^{-1/5})}{4\pi}\int_{n^{-1/5}}^{n^{1/5}} \frac{\cos(n^{-3/2}(n+1)t/4)\cos(n^{-3/2}(n+1)t)}{n\sin^2(n^{-3/2}(n+1)t/4)+\cos^2(n^{-3/2}(n+1)t)}\,dt, \qquad (2.3.17)
\end{aligned}$$

where the last equality directly follows from $x = x_k^{n+1} + 4\pi u/(n+1)$, and the change of variables $t = n^{3/2}(x - x_k^{n+1})$. Let us define

$$f_n(t) := \frac{\cos(n^{-3/2}(n+1)t/4)\cos(n^{-3/2}(n+1)t)}{n\sin^2(n^{-3/2}(n+1)t/4)+\cos^2(n^{-3/2}(n+1)t)} \cdot \mathbb{1}_{[n^{-1/5},n^{1/5}]}(t), \quad t\in\mathbb{R}^+.$$

It is easy then to check that

$$\sin(n^{-3/2}(n+1)t/4) \geqslant n^{-1/2}\,t/2\pi, \quad \text{and} \quad \cos(n^{-3/2}(n+1)t) \geqslant 1/2, \quad t\in[n^{-1/5}, n^{1/5}].$$

Therefore,

$$0 \leqslant f_n(t) \leqslant \frac{4\pi^2}{t^2+\pi^2} =: g(t), \quad \text{and} \quad g\in\mathbb{L}^1(\mathbb{R}^+).$$

Hence by Lebesgue's Dominated Convergence Theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{n^{-1/5}}^{n^{1/5}} &\frac{\cos(n^{-3/2}(n+1)t/4)\cos(n^{-3/2}(n+1)t)}{n\sin^2(n^{-3/2}(n+1)t/4)+\cos^2(n^{-3/2}(n+1)t)}\,dt \\
&= 16\int_0^\infty \frac{dt}{t^2+16} = 2\pi. \qquad (2.3.18)
\end{aligned}$$

If we implement the change of variables $x = n^{-1/2}\,t$, we proceed with

$$\begin{aligned}
0 &\leqslant \int_{n^{-1/5}}^{n^{1/5}} \frac{\cos(n^{-3/2}(n+1)t/4)\cos(n^{-3/2}(n+1)t)}{n\sin^2(n^{-3/2}(n+1)t/4)+\cos^2(n^{-3/2}(n+1)t)}\,dt \\
&\leqslant \int_{n^{-1/5}}^{n^{1/5}} \frac{dt}{n\sin^2(n^{-1/2}\,t/4)+\cos^2(2n^{-1/2}\,t)} \\
&= n^{1/2}\int_{n^{-7/10}}^{n^{-3/10}} \frac{dx}{n\sin^2(x/4)+\cos^2(2x)}.
\end{aligned}$$

Considering the convexity of the sine function on $[0, n^{-3/10}]$ we arrive at

$$\sin(x) \geqslant x n^{3/10} \sin(n^{-3/10}),$$

and as a result

$$\begin{aligned} n^{1/2} &\int_{n^{-7/10}}^{n^{-3/10}} \frac{dx}{n \sin^2(x/4) + \cos^2(2x)} \\ &\leqslant 16 n^{1/2} \int_{n^{-7/10}}^{n^{-3/10}} \frac{dx}{x^2 n^{8/5} \sin^2(n^{-3/10}) + 16 \cos^2(2x)} \\ &= 16 n^{1/2} \int_{n^{-7/10}}^{n^{-3/10}} \frac{dx}{x^2 n^{8/5} \sin^2(n^{-3/10}) + 16(1 - \sin^2(2x))} \\ &\leqslant 16 n^{1/2} \int_{n^{-7/10}}^{n^{-3/10}} \frac{dx}{x^2 n^{8/5} \sin^2(n^{-3/10}) + 16(1 - 4n^{-3/5})}, \end{aligned}$$

where the last inequality is derived from $\sin(2x) \leqslant 2x \leqslant 2n^{-3/10}$. Set

$$c_n = \frac{n^{4/5} \sin(n^{-3/10})}{4\sqrt{1 - 4n^{-3/5}}}.$$

Thus, the change of variables $t = c_n x$ gives that

$$\begin{aligned} 16 n^{1/2} &\int_{n^{-7/10}}^{n^{-3/10}} \frac{dx}{x^2 n^{8/5} \sin^2(n^{-3/10}) + 16(1 - 4n^{-3/5})} \\ &= \frac{4n^{-3/10}}{\sqrt{1 - 4n^{-3/5}} \sin(n^{-3/10})} \int_{c_n n^{-7/10}}^{c_n n^{-3/10}} \frac{dt}{1 + t^2} \\ &\leqslant \frac{4n^{-3/10}}{\sqrt{1 - 4n^{-3/5}} \sin(n^{-3/10})} \int_{c_n n^{-7/10}}^{\infty} \frac{dt}{1 + t^2} \\ &= d_n \big(2\pi - 4 \arctan(c_n n^{-7/10}) \big), \end{aligned}$$

where

$$d_n = \frac{n^{-3/10}}{\sqrt{1 - 4n^{-3/5}} \sin(n^{-3/10})}.$$

One may employ the estimate

$$\frac{x}{\sqrt{1 - 4x^2} \sin(x)} = 1 + \mathcal{O}(x^2), \quad \text{as } x \to 0,$$

to show that $d_n = 1 + \mathcal{O}(n^{-3/5})$. Moreover, the estimates

$$\frac{\sin(x)}{x\sqrt{1 - 4x^2}} = 1 + \mathcal{O}(x^2), \quad \text{and} \quad \arctan(x) = \mathcal{O}(x), \quad \text{as } x \to 0,$$

give

$$c_n = \frac{n^{1/2}\sin(n^{-3/10})}{4n^{-3/10}\sqrt{1-4n^{-3/5}}} = \mathcal{O}(n^{1/2})$$

as well as

$$\arctan(c_n n^{-7/10}) = \mathcal{O}(c_n n^{-7/10}) = \mathcal{O}(n^{-1/5}).$$

Thus, putting all the relations appeared after (2.3.18) together, we can claim that there exist $C > 0$ and $N \in \mathbb{N}$ such that for all $n \geqslant N$,

$$0 \leqslant \int_{n^{-1/5}}^{n^{1/5}} \frac{\cos(n^{-3/2}(n+1)t/4)\cos(n^{-3/2}(n+1)t)}{n\sin^2(n^{-3/2}(n+1)t/4) + \cos^2(n^{-3/2}(n+1)t)}\,dt \leqslant 2\pi + Cn^{-1/5}.$$

Now, this together with (2.3.18) guarantees that

$$\int_{n^{-1/5}}^{n^{1/5}} \frac{\cos(n^{-3/2}(n+1)t/4)\cos(n^{-3/2}(n+1)t)}{n\sin^2(n^{-3/2}(n+1)t/4) + \cos^2(n^{-3/2}(n+1)t)}\,dt = 2\pi + \mathcal{O}(n^{-1/5}). \tag{2.3.19}$$

Hence (2.3.16) follows from (2.3.17) and (2.3.19).

Finally, we prove that

$$\mathbb{E}[N_{n+1}(I_{k,4})] = \mathcal{O}(n^{-1/5}), \quad \text{as } n \to \infty. \tag{2.3.20}$$

If $x \in I_{k,4}$, we see that $\sin(\pi u) = \mathcal{O}(n^{-7/10})$. Plugging this into (2.3.9) and (2.3.10) assists us in writing

$$\begin{aligned}\Delta_{n+1}(x) &= ((n+1)/2)^2\cos^2(\pi u)\cos^2(4\pi u)A_n^*(x) + \mathcal{O}(n^{8/5})\\ &= \frac{n^3\cos^2(\pi u)\cos^2(4\pi u)}{16} + \mathcal{O}(n^{8/5})\\ &= \frac{n^3\cos^2(\pi u)\cos^2(4\pi u)\big(1+\mathcal{O}(n^{-7/5})\big)}{16}, \quad \text{as } n\to\infty \text{ and } x \in I_{k,4},\end{aligned}$$

and

$$\begin{aligned}A_{n+1}(x) &= 4\sin^2(\pi u)A_n^*(x) + \cos^2(4\pi u) = \cos^2(4\pi u) + \mathcal{O}(n^{-2/5})\\ &= \cos^2(4\pi u)\big(1+\mathcal{O}(n^{-2/5})\big), \quad \text{as } n\to\infty \text{ and } x\in I_{k,4}.\end{aligned}$$

It is worth mentioning that $x \in I_{k,4}$ gives $u = \mathcal{O}(n^{-7/10})$, hence both $\cos(\pi u)$ and $\cos(4\pi u)$ are positive on $I_{k,4}$. Now, Kac-Rice's formula (Lemma 1.2.3) gives that

$$\begin{aligned}\mathbb{E}[N_{n+1}(I_{k,4})] &= \frac{1}{\pi}\int_{I_{k,4}} \frac{\sqrt{\Delta_{n+1}(x)}}{A_{n+1}(x)}\,dx \\ &= \frac{1}{\pi}\int_{I_{k,4}} \frac{n^{3/2}\cos(\pi u)\big(1+\mathcal{O}(n^{-7/5})\big)}{4\cos(4\pi u)\big(1+\mathcal{O}(n^{-2/5})\big)}\,dx \\ &= \mathcal{O}(n^{3/2})\int_{I_{k,4}} \frac{dx}{\cos(4\pi u)}, \quad \text{as } n\to\infty.\end{aligned}$$

Recall that $x = x_k^{n+1} + 4\pi u/(n+1)$, hence the change of variables $t = 4\pi u$ implies that

$$\begin{aligned}\int_{I_{k,4}} \frac{dx}{\cos(4\pi u)} &= \frac{1}{n+1}\int_0^{(n+1)n^{-17/10}} \frac{dt}{\cos(t)} \leqslant \frac{1}{n}\int_0^{2n^{-7/10}} \frac{dt}{\cos(t)} \\ &= \frac{\log\big(\sec(2n^{-7/10}) + \tan(2n^{-7/10})\big)}{n} = \mathcal{O}(n^{-17/10}), \quad \text{as } n\to\infty,\end{aligned}$$

where the last equality is obtained by the estimate $\log(\sec(x)+\tan(x)) = \mathcal{O}(x)$ as $x \to 0$. Hence (2.3.20) holds.

We combine (2.3.14), (2.3.15), (2.3.16) and (2.3.20) and observe that (2.3.13) holds, so does (2.3.12). Thus, (2.3.11) and (2.3.12) lead us to the desired result, namely

$$\begin{aligned}\mathbb{E}[N_{n+1}(0,2\pi)] &= 2\,\mathbb{E}[N_{n+1}(0,\pi)] = 2\,\mathbb{E}[N_{n+1}(G\cup F\cup U)] \\ &= \left(\frac{1}{2} + \frac{\sqrt{13}}{2\sqrt{3}}\right)n + \mathcal{O}(n^{4/5}), \quad \text{as } n\to\infty. \qquad (2.3.21)\end{aligned}$$

At last, (2.3.21) along with setting $a = 1/5$ in (2.3.8) concludes the proof. ■

CHAPTER III

RANDOM TRIGONOMETRIC POLYNOMIALS WITH PALINDROMIC BLOCKS OF COEFFICIENTS

3.1 Self-reciprocal polynomials

Let

$$P_n(z) = \eta_0 + \eta_1 z + \cdots + \eta_{n-1} z^{n-1} + \eta_n z^n$$

be a polynomial of degree n with complex coefficients. We say P_n is palindromic (self-reciprocal) if $\eta_{n-j} = \eta_j$, $0 \leqslant j \leqslant n$, or equivalently $P_n(z) = z^n P_n(z^{-1})$. We also note that w is a zero of P_n if and only if w^{-1} is also a zero of P_n. Moreover, we note that if a polynomial $Q_n \in \mathbb{R}[z]$ has all its zeros on $\mathbb{T}$, then $Q_n(z) = \beta z^n Q_n(z^{-1})$ for some $\beta \in \mathbb{T}$. This could be shown by writing

$$Q_n(z) = a_n \prod_{i=1}^{n} (z - \alpha_i), \quad a_n \neq 0,\ |\alpha_i| = 1,\ 1 \leqslant i \leqslant n,$$

and noting that $Q_n(\alpha_i^{-1}) = Q_n(\bar{\alpha}_i) = 0$ since Q_n is a polynomial with real coefficients. Thus, we can rewrite Q_n as

$$Q_n(z) = a_n \prod_{i=1}^{n} \left(z - \alpha_i^{-1}\right) = (-1)^n \alpha^{-1} a_n \prod_{i=1}^{n} (1 - z\alpha_i),$$

where $\alpha = \prod_{i=1}^{n} \alpha_i$. It is now easy to check that

$$Q_n(z) = (-1)^n \alpha^{-1} z^n Q_n(z^{-1}).$$

The class of polynomials with palindromic coefficients and their zeros' behavior have attracted a lot of attention over recent years, and it turns out to have many applications in some areas of physics and mathematics.

There is extensive literature on the behavior and the location of the zeros of self-reciprocal polynomials with deterministic coefficients, for instance, focusing on the minimal conditions imposed on a self-reciprocal polynomial to make all its roots unimodular, that is, lying on the unit circle (see, e.g., [57],[55],[56] and [11]). There is also a direct connection between these polynomials and trigonometric ones. For instance, if we define

$$P_{2n}(z) := 2a_0 z^n + \sum_{j=1}^{n} a_j (z^{n-j} + z^{n+j})$$

with the coefficients being real, it is immediate that P_{2n} is a self-reciprocal polynomial of degree $2n$, and we can easily check that

$$e^{-inx} P_{2n}(e^{ix}) = 2V_n(x), \tag{3.1.1}$$

where V_n is a cosine polynomial as defined in (1.2.6). Another example is the trigonometric polynomial of the form

$$R_N(x) := \sum_{j=0}^{N-1} \big[a_{N-j}\cos(j+1/2)x + b_{N-j}\sin(j+1/2)x\big]. \tag{3.1.2}$$

The asymptotic expected value of the number of real zeros of (3.1.2) was surveyed by Farahmand, see [31] and [35, Case 3, p. 1884]. Note that (3.1.2) can be rephrased as

$$e^{-inx/2} P_n(e^{ix}) = 2R_N(x) + 2\cos(N+1/2)x,$$

where $n = 2N+1$, and $P_n(z) = \sum_{j=0}^{n} \eta_j z^j$ is a random polynomial with $\eta_j = a_j + ib_j$, $\eta_0 = \eta_n = 1$, and it is conjugate-reciprocal, i.e., $\eta_{n-j} = \overline{\eta_j}$.

The number of unimodular zeros of these polynomials has also been of interest to some mathematicians. More recently, Erdélyi [20] proved the following result: assume that $S \subset \mathbb{Z}$ is finite, and $P(z)$ is a palindromic polynomial whose coefficients lie in S. Then for a positive constant c, which only depends on $\varepsilon > 0$ and

$$M = M(S) := \max\{|z| : z \in S\},$$

the number of unimodular roots of P exceeds

$$c\big(\log\log\log|P(1)|\big)^{1-\varepsilon}-1,$$

which significantly improves the recent result of Sahasrabudhe [81] and his previous work [19]. This result combined with (3.1.1) establishes an explicit lower bound for the number of sign changes (in one period) of V_n with coefficients in S. More precisely, he proved that, for some $c>0$,

$$N_n^{\#}(0,2\pi)\geqslant\left(\frac{c}{1+\log M}\right)\frac{\log\log\log|V_n(0)|}{\log\log\log\log|V_n(0)|}-1,$$

where $N_n^{\#}(0,2\pi)$ is the number of sign changes of V_n in $(0,2\pi)$ and M as defined above, see [20, Corollary 2.3].

In the case of random polynomials with palindromic coefficients with the Gaussian distribution $\mathcal{N}(0,\sigma^2)$, it is understood from the work of Farahmand and Gao [32] that the expectation of the number of real roots still remains asymptotic to the universal value $(2/\pi)\log n$, see also [85] for the case where $\mathrm{Var}(a_j)=\sigma_j^2$.

3.2 Self-reciprocal random trigonometric polynomials

We would like to focus on random trigonometric polynomials with palindrome coefficients. Conrey et al. [13] proved that the number of real zeros of V_n with self-reciprocal i.i.d. coefficients exceeds n by showing that

$$N_n(0,2\pi)+\widetilde{N_n}(0,2\pi)\geqslant 2n,$$

where $\widetilde{N_n}$ denotes the number of zeros of $\widetilde{V_n}$ obtained by reversing the order of the coefficients of V_n, and slightly improves the result obtained by Borwein et al. [10, Theorem 2, p. 1152].

The palindromic polynomials V_n, T_n and R_n, as defined in (3.1.2), were considered by Farahmand and Li in [35], where they showed that the asymptotic expected number of real roots of T_n and R_n remains universal whereas it enlarges by 36.6% for a palindromic random cosine polynomial V_n, see (1.2.16) for our modification of their result as discussed in Chapter I. This poses the following research problem:

- Does the asymptotic expected number of real roots of a random cosine polynomial stay non-universal if we group the coefficients in palindromic blocks of a certain length?

- If so, what is the connection between the length of the blocks and the deviation from the universal asymptotics?

- What happens if one allows the size of the blocks to grow?

In this section, we attempt to answer some of these questions.

Similar to the first case discussed in Chapter II, fix $\ell \in \mathbb{N}\backslash\{1\}$, and let $A := (a_0, a_1, \ldots, a_n)$ be the set of all the coefficients of $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$. Set $n = 2\ell m - 1 + r$, $m \in \mathbb{N}$, where $r \in \{0, 1, \ldots, 2\ell - 1\}$. In other words, r is the remainder and m is the quotient of dividing $n+1$ by 2ℓ. We then sort out the coefficients into $2m$ blocks of a length ℓ as follows. Set

$$A = (a_0, a_1, \ldots, a_n) = \bigcup_{j=0}^{2m-1} A_j \cup \tilde{A}_r,$$

where

$$A_j := \begin{cases} (a_{\ell j}, a_{\ell j+1}, \ldots, a_{\ell(j+1)-1}), & \text{if } 0 \leqslant j \leqslant m-1, \\ (a_{\ell j+r}, a_{\ell j+1+r}, \ldots, a_{\ell(j+1)-1+r}), & \text{if } m \leqslant j \leqslant 2m-1, \end{cases}$$

and

$$\tilde{A}_r := \begin{cases} \emptyset, & \text{if } r = 0, \\ (a_{\ell m}, \ldots, a_{\ell m-1+r}), & \text{if } 1 \leqslant r \leqslant 2\ell - 1. \end{cases}$$

In other words, $\tilde{A}_r$ comes in the middle of A, and $\#\tilde{A}_r = r$, $0 \leqslant r \leqslant 2\ell - 1$.

The main theorem of this chapter, which is a generalization of Farahmand and Li's result, answers the first question raised above. In fact, we prove that the expected number of real zeros of a random cosine polynomial with palindromic blocks is non-universal.

Theorem 3.2.1 *Fix $\ell \in \mathbb{N}\backslash\{1\}$, and let $n = 2\ell m - 1 + r$, where $m \in \mathbb{N}$ and $r \in \{0, 1, \ldots, 2\ell - 1\}$. Assume $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$, and $\bigcup_{j=0}^{m-1} A_j \cup \tilde{A} = (a_0, a_1, \ldots, a_{\ell m-1+r})$*

is a family of i.i.d. random variables with Gaussian distribution $\mathcal{N}(0,\sigma^2)$. *For* $0 \leqslant j \leqslant m-1$ *and* $0 \leqslant k \leqslant \ell-1$, *we further assume* $a_{\ell(2m-1-j)+r+k} = a_{\ell j+k}$, *i.e.,* $A_{2m-1-j} = A_j$. *Let us define*

$$\mathrm{K}_\ell := \frac{1}{\pi^2}\int_0^\pi \int_0^{\pi/2} \sqrt{1+\frac{3(1-u_\ell^2(s))}{\big(1+u_\ell(s)\cos(t)\big)^2}}\,ds\,dt,$$

with

$$u_\ell(s) = \frac{\sin(\ell s)}{\ell \sin(s)}.$$

Then

$$\mathbb{E}[N_n(0,2\pi)] = \frac{2n}{\sqrt{3}}\,\mathrm{K}_\ell + \mathcal{O}(n^{3/4}), \quad \text{as } n\to\infty,$$

where the implied constant depends only on ℓ.

Remark 3.2.1 We note that the case $\ell = 1$ is indeed a random cosine polynomial with palindromic coefficients whose expected number of real zeros has already been discussed in (1.2.16), and in detail in Appendix, see pp. 101–107. Namely, if $\ell = 1$, then

$$\mathbb{E}[N_n(0,2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{3/4}), \quad \text{as } n\to\infty.$$

3.2.1 Properties of the K_ℓ

To answer the second question, our numerical computation suggests that $\{\mathrm{K}_\ell\}_{\ell=2}^\infty$ is decreasing, namely the smaller ℓ is, the more expected number of real roots deviates from the universal one. We also give a definitive answer to the last question by proving that K_ℓ converges to 1 as ℓ tends to infinity. This requires a quick lemma.

Lemma 3.2.1 *Let* $\varphi_n, \psi_n : X \to \mathbb{R}$, *where* $(X,\mathcal{A},\mu)$ *is a measure space. Assume that the* φ_n *are measurable, the* ψ_n *are integrable and* $|\varphi_n| \leqslant \psi_n$ *a.e. (almost everywhere) for all* n. *If* $\lim_{n\to\infty}\varphi_n = \varphi$ *a.e.,* $\lim_{n\to\infty}\psi_n = \psi$ *a.e.,* ψ *is integrable and* $\limsup_{n\to\infty}\int_X \psi_n = \int_X \psi$, *then* φ *is integrable and* $\lim_{n\to\infty}\int_X \varphi_n = \int_X \varphi$.

Proof. Our proof is based on that of Lebesgue's Dominated Convergence Theorem [41, Theorem 2.24]. That φ is measurable is immediate. Note that $|\varphi| \leqslant \psi$ a.e. and ψ is integrable, so is φ. It is also clear that $\psi_n + \varphi_n \geqslant 0$ a.e. and $\psi_n - \varphi_n \geqslant 0$ a.e., hence by Fatou's Lemma, we obtain

$$\begin{aligned}\int \psi + \int \varphi &\leqslant \liminf \int (\psi_n + \varphi_n) \\ &= \liminf \Big(\int \psi_n + \int \varphi_n \Big) + \liminf \Big(- \int \psi_n \Big) + \limsup \int \psi_n \\ &\leqslant \liminf \int \varphi_n + \limsup \int \psi_n = \liminf \int \varphi_n + \int \psi,\end{aligned}$$

and

$$\begin{aligned}\int \psi - \int \varphi &\leqslant \liminf \int (\psi_n - \varphi_n) \\ &= \liminf \Big(\int \psi_n - \int \varphi_n \Big) + \liminf \Big(- \int \psi_n \Big) + \limsup \int \psi_n \\ &\leqslant \liminf \Big(- \int \varphi_n \Big) + \limsup \int \psi_n = -\limsup \int \varphi_n + \int \psi.\end{aligned}$$

In other words, $\limsup \int \varphi_n \leqslant \int \varphi \leqslant \liminf \int \varphi_n$. Thus, by definition, $\int \varphi_n \to \int \varphi$. ■

Lemma 3.2.2 *For $\ell \in \mathbb{N} \setminus \{1\}$, let us define K_ℓ as in Theorem 3.2.1. Then*

(1). $1 < \mathrm{K}_\ell \leqslant (1+\sqrt{3})/2$.

(2). $\lim_{\ell\to\infty} \mathrm{K}_\ell = 1$.

Proof. For a fixed $\ell \in \mathbb{N} \setminus \{1\}$, we define

$$g_\ell(s,t) := \sqrt{1 + \frac{3(1-u_\ell^2(s))}{\big(1+u_\ell(s)\cos(t)\big)^2}}, \quad (s,t) \in \mathcal{R} = (0,\pi/2) \times (0,\pi),$$

and

$$f_\ell(s,t) := \frac{\sqrt{1-u_\ell^2(s)}}{1+u_\ell(s)\cos(t)}, \quad (s,t) \in \mathcal{R}.$$

In other words, we have $g_\ell(s,t) = \sqrt{1+3f_\ell^2(s,t)}$. Note that, by [42, 3.613(1), p. 366], we obtain

$$\int_0^\pi \frac{dt}{1+a\cos(t)} = \frac{\pi}{\sqrt{1-a^2}}, \quad |a| < 1. \tag{3.2.1}$$

This combined with the fact that $|u_\ell(s)| < 1$ on $(0, \pi/2)$, see Lemma 2.3.1, gives us

$$\frac{1}{\pi}\int_0^\pi f_\ell(s,t)\,dt = \frac{1}{\pi}\int_0^\pi \frac{\sqrt{1-u_\ell^2(s)}\,dt}{1+u_\ell(s)\cos(t)} = 1, \quad s \in (0,\pi/2). \tag{3.2.2}$$

It is a known fact that for any nonconstant $f \in \mathbb{L}^1((0,\pi))$ and a strictly convex function Φ on the real line, Jensen's inequality is strict (see Theorem 3.3 of [80]). That is,

$$\Phi\left(\frac{1}{\pi}\int_0^\pi f(t)\,dt\right) < \frac{1}{\pi}\int_0^\pi \Phi(f(t))\,dt.$$

Let $\Phi(y) := \sqrt{1+3y^2}$. It now follows from Fubini-Tonelli's Theorem, (3.2.2), and Jensen's inequality that

$$\begin{aligned} 1 &= \frac{1}{\pi^2}\int_0^{\pi/2} \pi\,\Phi\left(\frac{1}{\pi}\int_0^\pi f_\ell(s,t)\,dt\right)ds \\ &< \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^\pi \Phi(f_\ell(s,t))\,dt\,ds = \mathrm{K}_\ell \\ &\leqslant \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^\pi \big(1+\sqrt{3}f_\ell(s,t)\big)dt\,ds = \frac{1+\sqrt{3}}{2}. \end{aligned}$$

Proof of (2). In order to use Lemma 3.2.1, let $X = \mathbb{R}^2$, $\mathcal{R} = (0,\pi/2)\times(0,\pi)$, and define

$$\varphi_\ell(s,t) := \frac{g_\ell(s,t)}{\pi^2}\cdot \mathbb{1}_{\mathcal{R}}(s,t), \quad \text{and} \quad \psi_\ell(s,t) := \frac{1+\sqrt{3}f_\ell(s,t)}{\pi^2}\cdot \mathbb{1}_{\mathcal{R}}(s,t).$$

It is trial that $0 \leqslant \varphi_\ell(s,t) \leqslant \psi_\ell(s,t)$ on $\mathcal{R}$. Note that $\lim_{\ell\to\infty} u_\ell(s) = 0$ guarantees that

$$\lim_{\ell\to\infty}\varphi_\ell(s,t) = \frac{2}{\pi^2} =: \varphi(s,t), \quad \text{and} \quad \lim_{\ell\to\infty}\psi_\ell(s,t) = \frac{1+\sqrt{3}}{\pi^2} =: \psi(s,t).$$

Now, implementing (3.2.2), while using the Fubini-Tonelli Theorem, yields

$$\limsup_{\ell\to\infty}\int_{\mathcal{R}}\psi_\ell = \frac{1+\sqrt{3}}{2} = \int_{\mathcal{R}}\psi.$$

Therefore, it is immediate from Lemma 3.2.1 that

$$\lim_{\ell\to\infty}\mathrm{K}_\ell = \lim_{\ell\to\infty}\int_{\mathcal{R}}\varphi_\ell = \int_{\mathcal{R}}\varphi = 1.$$

■

3.3 Proof of Theorem 3.2.1

The proof of Theorem 3.2.1 is step-by-step through a sequence of lemmas. It is necessary to mention that in all of the following lemmas, the implied constants in each big $\mathcal{O}$ depend only on ℓ.

Lemma 3.3.1 *With the same assumptions as in Theorem 3.2.1, fix $a \in (0, 1/3)$ and let $E_\ell = (0, \pi) \setminus F_\ell$, where $F_\ell = \cup_{i=0}^{\ell} (i\pi/\ell - n^{-a}, i\pi/\ell + n^{-a})$. Then*

(1). $0 < A_n(x) = \dfrac{n\big(1 + u_\ell(x)\cos(nx)\big)}{2} + \mathcal{O}(n^a), \quad \text{as } n \to \infty \text{ and } x \in E_\ell,$

(2). $B_n(x) = -\dfrac{n^2 u_\ell(x)\sin(nx)}{4} + \mathcal{O}(n^{1+a}), \quad \text{as } n \to \infty \text{ and } x \in E_\ell,$

(3). $C_n(x) = \dfrac{n^3\big(2 - u_\ell(x)\cos(nx)\big)}{12} + \mathcal{O}(n^{2+a}), \quad \text{as } n \to \infty \text{ and } x \in E_\ell,$

where $A_n(x)$, $B_n(x)$ and $C_n(x)$ are defined in a similar way as in Lemma 1.2.3.

Proof. Setting $\tilde{J}_r = \{j : a_j \in \tilde{A}_r\}$ helps to write

$$\begin{aligned}
V_n(x) &= \sum_{j=0}^{n} a_j \cos(jx) \\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} a_{\ell j+k}\big[\cos(\ell j + k)x + \cos(\ell(2m-1-j) + r + k)x\big] + \sum_{j\in\tilde{J}_r} a_j \cos(jx) \\
&= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} a_{\ell j+k}\big[\cos(\ell j + k)x + \cos(n - \ell + 1 - \ell j + k)x\big] + \sum_{j\in\tilde{J}_r} a_j \cos(jx)
\end{aligned}$$

with the conventional notation of $\sum_{j\in\tilde{J}_r} a_j\cos(jx) = 0$ if $r = 0$. Let $x \in E_\ell$ and observe that

$$\begin{aligned}
A_n(x) &= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1} \big[\cos(\ell j + k)x + \cos(n - \ell + 1 - \ell j + k)x\big]^2 + \sum_{j\in\tilde{J}_r} \cos^2(jx) \\
&= 4\sum_{j=0}^{m-1} \cos^2\left(\frac{n-\ell+1-2\ell j}{2}\right)x \; \sum_{k=0}^{\ell-1} \cos^2\left(\frac{n-\ell+1+2k}{2}\right)x + \sum_{j\in\tilde{J}_r} \cos^2(jx) \\
&= \sum_{j=0}^{m-1} \big[1 + \cos(n-\ell+1-2\ell j)x\big] \sum_{k=0}^{\ell-1} \big[1 + \cos(n-\ell+1+2k)x\big] + \sum_{j\in\tilde{J}_r} \cos^2(jx) \\
&= \sum_{j=0}^{m-1} \big[1 + \cos(\ell+r+2\ell j)x\big] \sum_{k=0}^{\ell-1} \big[1 + \cos(n-\ell+1+2k)x\big] + \sum_{j\in\tilde{J}_r} \cos^2(jx),
\end{aligned}$$

where the last equality is reached by replacing j with $m-1-j$. Now, identity (2.3.1) simplifies $A_n(x)$ as

$$\begin{aligned} A_n(x) &= \left(m + \frac{\sin(m\ell x)\cos(m\ell+r)x}{\sin(\ell x)}\right)\left(\ell + \frac{\sin(\ell x)\cos(nx)}{\sin(x)}\right) + \sum_{j\in\tilde{J}_r}\cos^2(jx) \\ &= m\ell\big[1+u_m(\ell x)\cos(m\ell+r)x\big]\big[1+u_\ell(x)\cos(nx)\big] + \sum_{j\in\tilde{J}_r}\cos^2(jx). \end{aligned}$$

Note that, for $x \in E_\ell$, $|u_m(\ell x)| < 1$ and $|u_\ell(x)| < 1$, see Lemma 2.3.1. Thus, A_n is positive on E_ℓ. Moreover,

$$\begin{aligned} A_n(x) &= \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\big[\cos(\ell j+k)x + \cos(n-\ell+1-\ell j+k)x\big]^2 + \sum_{j\in\tilde{J}_r}\cos^2(jx) \\ &= \sum_{j=0}^{n}\cos^2(jx) + 2\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\cos(\ell j+k)x\,\cos(n-\ell+1-\ell j+k)x \\ &= \sum_{j=0}^{n}\cos^2(jx) + \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\cos(n-\ell+1+2k)x + \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\cos(n-\ell+1-2\ell j)x \\ &= \sum_{j=0}^{n}\cos^2(jx) + m\sum_{k=0}^{\ell-1}\cos(n-\ell+1+2k)x + \ell\sum_{j=0}^{m-1}\cos(\ell+r+2\ell j)x, \end{aligned}$$

where the last equality is obtained again by replacing j with $m-1-j$. It is clear from Lemma 2.3.2 that

$$\sum_{j=0}^{n}\cos^2(jx) = \frac{1}{2}\sum_{j=0}^{n}(1+\cos(2jx)) = \frac{n+1}{2} + \frac{P_0(1,n+1,x)}{2} = \frac{n}{2} + \mathcal{O}(n^a).$$

The fact that $\csc(x) = \mathcal{O}(n^a)$, $x \in E_\ell$, combined with (2.3.1) yields

$$\begin{aligned} m\sum_{k=0}^{\ell-1}\cos(n-\ell+1+2k)x &= \frac{(n+1-r)\sin(\ell x)\cos(nx)}{2\ell\sin(x)} \\ &= \frac{n\sin(\ell x)\cos(nx)}{2\ell\sin(x)} + \mathcal{O}(n^a) = \frac{nu_\ell(x)\cos(nx)}{2} + \mathcal{O}(n^a). \end{aligned}$$

Similarly, since $\csc(\ell x) = \mathcal{O}(n^a)$ on E_ℓ, we have

$$\sum_{j=0}^{m-1}\cos(\ell+r+2\ell j)x = \frac{\sin(m\ell x)\cos(m\ell+r)x}{\sin(\ell x)} = \mathcal{O}(n^a).$$

Putting all these estimates together gives the desired estimate

$$A_n(x) = \frac{n\big(1 + u_\ell(x)\cos(nx)\big)}{2} + \mathcal{O}(n^a), \quad \text{as } n \to \infty \text{ and } x \in E_\ell.$$

Proof of (2). Following the definition of B_n, we have

$$\begin{aligned} B_n(x) = &- \sum_{j\in\tilde{J}_r} j\sin(jx)\cos(jx) - \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\big[\cos(\ell j + k)x + \cos(n-\ell+1-\ell j+k)x\big] \\ &\times \big[(\ell j + k)\sin(\ell j+k)x + (n-\ell+1-\ell j+k)\sin(n-\ell+1-\ell j+k)x\big] \\ = &- \sum_{j=0}^{n} j\sin(jx)\cos(jx) - \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j+k)\sin(\ell j+k)x\,\cos(n-\ell+1-\ell j+k)x \\ &- \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1-\ell j+k)\sin(n-\ell+1-\ell j+k)x\,\cos(\ell j+k)x. \end{aligned}$$

We employ the identity $\sin(\alpha)\cos(\beta) = [\sin(\alpha+\beta)+\sin(\alpha-\beta)]/2$ to simplify the above as

$$\begin{aligned} B_n(x) = &-\sum_{j=0}^{n} j\sin(jx)\cos(jx) - \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1+2k)\sin(n-\ell+1+2k)x \\ &- \frac{1}{2}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1-2\ell j)\sin(n-\ell+1-2\ell j)x. \end{aligned}$$

By Lemma 2.3.2, we obtain

$$\sum_{j=0}^{n} j\sin(jx)\cos(jx) = \frac{1}{2}\sum_{j=0}^{n} j\sin(2jx) = \frac{Q_1(1,n+1,x)}{2} = \mathcal{O}(n^{1+a}).$$

Using the estimate $\sum_{k=0}^{\ell-1}(-\ell+1+2k)\sin(n-\ell+1+2k)x = \mathcal{O}(1)$ gives us

$$\begin{aligned} &\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1+2k)\sin(n-\ell+1+2k)x \\ &\quad = nm\sum_{k=0}^{\ell-1}\sin(n-\ell+1+2k)x + \mathcal{O}(m) \\ &\quad = \frac{n(n+1-r)\sin(\ell x)\sin(nx)}{2\ell\sin(x)} + \mathcal{O}(m) \\ &\quad = \frac{n^2 u_\ell(x)\sin(nx)}{2} + \mathcal{O}(n^{1+a}), \end{aligned}$$

where the penultimate equality is a direct application of identity (2.3.2). Note that

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1-2\ell j)\sin(2j+1)\ell x$$
$$=\ell(n-\ell+1)\sum_{j=0}^{m-1}\sin(2j+1)\ell x-2\ell^2\sum_{j=0}^{m-1}j\sin(2j+1)\ell x$$
$$=\ell(n-\ell+1)S_0(1,m,\ell x)-2\ell^2 S_1(1,m,\ell x)=\mathcal{O}(n^{1+a}).$$

Similarly, we have the following estimate

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1-2\ell j)\cos(2j+1)\ell x=\mathcal{O}(n^{1+a}).$$

Thus, combining the last two estimates gives

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(n-\ell+1-2\ell j)\sin(n-\ell+1-2\ell j)x=\mathcal{O}(n^{1+a}).$$

Hence

$$B_n(x)=-\frac{n^2u_\ell(x)\sin(nx)}{4}+\mathcal{O}(n^{1+a}),\quad \text{as } n\to\infty \text{ and } x\in E_\ell.$$

Proof of (3). We see that

$$\begin{aligned}C_n(x)&=\sum_{j\in\tilde{J}_r}j^2\sin^2(jx)\\&\quad+\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\big[(\ell j+k)\sin(\ell j+k)x+(n-\ell+1-\ell j+k)\sin(n-\ell+1-\ell j+k)x\big]^2\\&=\sum_{j=0}^{n}j^2\sin^2(jx)\\&\quad+\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j+k)(n-\ell+1-\ell j+k)\big[\cos(n-\ell+1-2\ell j)x-\cos(n-\ell+1+2k)x\big].\end{aligned}$$

Note that

$$\sum_{j=0}^{n}j^2\sin^2(jx)=\frac{1}{2}\sum_{j=0}^{n}j^2-\frac{1}{2}\sum_{j=0}^{n}j^2\cos(2jx)$$
$$=\frac{n(n+1)(2n+1)}{12}-\frac{P_2(1,n+1,x)}{2}=\frac{n^3}{6}+\mathcal{O}(n^{2+a}).$$

We use Lemma 2.3.2 and the boundedness of $\sum_{k=0}^{\ell-1} k^\lambda$, $\lambda = 1, 2$, to show that

$$\begin{aligned}
&\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j + k)(n-\ell+1-\ell j+k)\cos(2j+1)\ell x \\
&\quad = (n-\ell+1)\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j+k)\cos(2j+1)\ell x \\
&\quad + \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(k^2-\ell^2 j^2)\cos(2j+1)\ell x \\
&\quad = (n-\ell+1)\left(\ell^2 R_1(1,m,\ell x) + R_0(1,m,\ell x)\sum_{k=0}^{\ell-1} k\right) \\
&\qquad + R_0(1,m,\ell x)\sum_{k=0}^{\ell-1} k^2 - \ell^3 R_2(1,m,\ell x) = \mathcal{O}(n^{2+a}).
\end{aligned}$$

Likewise,

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j + k)(n-\ell+1-\ell j+k)\sin(2j+1)\ell x = \mathcal{O}(n^{2+a}).$$

The last two estimates combined with the following identity

$$\cos(n-\ell+1-2\ell j)x = \cos(n+1)x\,\cos(2j+1)\ell x + \sin(n+1)x\,\sin(2j+1)\ell x$$

yield

$$\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j + k)(n-\ell+1-\ell j+k)\cos(n-\ell+1-2\ell j)x = \mathcal{O}(n^{2+a}).$$

We employ identity (2.3.1), and observe that

$$\begin{aligned}
&\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}(\ell j + k)(n-\ell+1-\ell j+k)\cos(n-\ell+1+2k)x \\
&\quad = \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\left[(n-\ell+1)\ell j - \ell^2 j^2\right]\cos(n-\ell+1+2k)x \\
&\quad + \sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}\left[(n-\ell+1)k + k^2\right]\cos(n-\ell+1+2k)x \\
&\quad = \left[\frac{(n-\ell+1)\ell(m-1)m}{2} - \frac{\ell^2(m-1)m(2m-1)}{6}\right]\frac{\sin(\ell x)\cos(nx)}{\sin(x)} \\
&\quad + m\sum_{k=0}^{\ell-1}\left[(n-\ell+1)k + k^2\right]\cos(n-\ell+1+2k)x.
\end{aligned}$$

We already know that $\csc(x) = \mathcal{O}(n^a)$ on E_ℓ, and $\sum_{k=0}^{\ell-1} k^\lambda \cos(n-\ell+1+2k)x = \mathcal{O}(1)$, $\lambda = 1, 2$. Thus, the following is immediate

$$\begin{aligned}\sum_{j=0}^{m-1}\sum_{k=0}^{\ell-1}&(\ell j+k)(n-\ell+1-\ell j+k)\cos(n-\ell+1+2k)x\\ &= \left(\frac{n\ell^2m^2}{2}-\frac{\ell^3m^3}{3}\right)\frac{\sin(\ell x)\cos(nx)}{\ell\sin(x)}+\mathcal{O}(m^{2+a})\\ &= \frac{n^3u_\ell(x)\cos(nx)}{12}+\mathcal{O}(n^{2+a}).\end{aligned}$$

Putting all these estimates together, we observe that

$$C_n(x) = \frac{n^3\big(2-u_\ell(x)\cos(nx)\big)}{12}+\mathcal{O}(n^{2+a}), \quad \text{as } n\to\infty \text{ and } x\in E_\ell.$$

■

Lemma 3.3.2 *With the same assumptions as Lemma 3.3.1, we have*

$$\begin{aligned}&\frac{\sqrt{A_n(x)C_n(x)-B_n(x)^2}}{A_n(x)}\\ &\quad= \frac{n+\mathcal{O}(n^{3a})}{2\sqrt{3}}\sqrt{1+\frac{3(1-u_\ell^2(x))}{\big(1+u_\ell(x)\cos(nx)\big)^2}}, \quad \text{as } n\to\infty \text{ and } x\in E_\ell.\end{aligned}$$

Proof. Assume that $\ell \in \mathbb{N}\setminus\{1\}$ is fixed. Using A_n, B_n, and C_n as in Lemma 3.3.1 gives that

$$\begin{aligned}A_n(x)C_n(x)-B_n(x)^2 &= \frac{n^4\left[\big(1+u_\ell(x)\cos(nx)\big)^2+3(1-u_\ell^2(x))\right]}{48}+\mathcal{O}(n^{3+a})\\ &= \frac{n^4\left[\big(1+u_\ell(x)\cos(nx)\big)^2+3(1-u_\ell^2(x))\right]}{48}\\ &\quad\times\left(1+\frac{\mathcal{O}(n^{-1+a})}{\big(1+u_\ell(x)\cos(nx)\big)^2+3(1-u_\ell^2(x))}\right).\end{aligned}$$

It is clear from Lemma 2.3.1 that there exists a constant $\omega_\ell \in (0,1)$ such that

$$|u_\ell(x)| \leqslant \omega_\ell, \quad x\in[\pi/2\ell, \pi-\pi/2\ell], \tag{3.3.1}$$

Thus, for $x \in [\pi/2\ell, \pi-\pi/2\ell]$,

$$\begin{aligned}0 &< \frac{1}{\big(1+u_\ell(x)\cos(nx)\big)^2+3(1-u_\ell^2(x))} \leqslant \frac{1}{(1-\omega_\ell)^2+3(1-\omega_\ell^2)}\\ &= \frac{1}{(1-\omega_\ell)(4+2\omega_\ell)} =: c_\ell.\end{aligned}$$

So, for $x \in [\pi/2\ell, \pi - \pi/2\ell]$, we write

$$A_n(x)C_n(x) - B_n(x)^2 = \frac{n^4\big[\big(1+u_\ell(x)\cos(nx)\big)^2 + 3(1-u_\ell^2(x))\big]\big[1+\mathcal{O}(n^{-1+a})\big]}{48}.$$

Note that

$$0 \leqslant \cos(\ell x) \leqslant u_\ell(x) \leqslant \cos(x), \quad x \in [0, \pi/2\ell]. \tag{3.3.2}$$

This helps to write, for $x \in [n^{-a}, \pi/2\ell]$,

$$\begin{aligned} 0 &< \frac{1}{\big(1+u_\ell(x)\cos(nx)\big)^2 + 3(1-u_\ell^2(x))} \\ &\leqslant \frac{1}{\big(1-u_\ell(x)\big)^2 + 3(1-u_\ell(x)^2)} \leqslant \frac{1}{(1-u_\ell(x))(4+2u_\ell(x))} \\ &\leqslant \frac{1}{4(1-\cos(x))} = \frac{1}{8\sin^2(x/2)} \leqslant \frac{\pi^2}{8x^2} \leqslant \frac{\pi^2 n^{2a}}{8}. \end{aligned}$$

Therefore, for $x \in [n^{-a}, \pi/2\ell]$, we obtain

$$A_n(x)C_n(x) - B_n(x)^2 = \frac{n^4\big[\big(1+u_\ell(x)\cos(nx)\big)^2 + 3(1-u_\ell^2(x))\big]\big[1+\mathcal{O}(n^{-1+a})\big]}{48}.$$

Hence for any $x \in E_\ell$, we have

$$A_n(x)C_n(x) - B_n(x)^2 = \frac{n^4\big[\big(1+u_\ell(x)\cos(nx)\big)^2 + 3(1-u_\ell^2(x))\big]\big[1+\mathcal{O}(n^{-1+a})\big]}{48}.$$

Taking square root of both sides, while considering the fact that $a \in (0, 1/3)$, we see that, for large enough n and $x \in E_\ell$,

$$\sqrt{A_n(x)C_n(x) - B_n(x)^2} = \frac{n^2\big(1+\mathcal{O}(n^{-1+a})\big)\sqrt{\big(1+u_\ell(x)\cos(nx)\big)^2 + 3(1-u_\ell^2(x))}}{4\sqrt{3}}.$$

In a similar way, one can show that

$$\begin{aligned} A_n(x) &= \frac{n\big(1+u_\ell(x)\cos(nx)\big)}{2} + \mathcal{O}(n^a) \\ &= \frac{n\big(1+u_\ell(x)\cos(nx)\big)\big(1+\mathcal{O}(n^{-1+3a})\big)}{2}, \quad \text{as } n\to\infty \text{ and } x \in E_\ell. \end{aligned}$$

Now, since $A_n(x) > 0$ on E_ℓ, we see

$$\begin{aligned} \frac{\sqrt{A_n(x)C_n(x) - B_n(x)^2}}{A_n(x)} &= \frac{n\big(1+\mathcal{O}(n^{-1+3a})\big)}{2\sqrt{3}}\sqrt{1 + \frac{3(1-u_\ell^2(x))}{\big(1+u_\ell(x)\cos(nx)\big)^2}} \\ &= \frac{n+\mathcal{O}(n^{3a})}{2\sqrt{3}}\sqrt{1 + \frac{3(1-u_\ell^2(x))}{\big(1+u_\ell(x)\cos(nx)\big)^2}}, \quad \text{as } n\to\infty \text{ and } x \in E_\ell, \end{aligned}$$

as desired. ∎

Lemma 3.3.3 *With the same assumptions as Lemma 3.3.1, let $G_\ell = E_\ell \cap [0, \pi/2]$. Then, as n grows to infinity,*

$$\mathbb{E}[N_n(0,2\pi)] = \begin{cases} \dfrac{\left[n + \mathcal{O}(n^{3a})\right]\left[J_\ell^+(n) + J_\ell^-(n)\right]}{\sqrt{3}\pi} + \mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is even,} \\ \\ \dfrac{\left[2n + \mathcal{O}(n^{3a})\right] J_\ell^+(n)}{\sqrt{3}\pi} + \mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is odd,} \end{cases}$$

where

$$J_\ell^+(n) := \int_{G_\ell} g_n^+(x)\, dx, \quad \text{and} \quad J_\ell^-(n) := \int_{G_\ell} g_n^-(x)\, dx,$$

with

$$g_n^+(x) := \sqrt{1 + \frac{3(1-u_\ell^2(x))}{\left(1 + u_\ell(x)\cos(nx)\right)^2}}, \quad \text{and} \quad g_n^-(x) := \sqrt{1 + \frac{3(1-u_\ell^2(x))}{\left(1 - u_\ell(x)\cos(nx)\right)^2}}.$$

Proof. From Lemmas 2.1.2, we know that $\mathbb{E}[N_n(F_\ell)] = \mathcal{O}(n^{1-a})$, for large enough n. If we use Lemmas 3.3.2 and 1.2.3 (Kac-Rice's formula), we observe that

$$\begin{aligned} \mathbb{E}[N_n(0,2\pi)] &= 2\,\mathbb{E}[N_n(0,\pi)] = 2\,\mathbb{E}[N_n(E_\ell \cup F_\ell)] \\ &= 2\,\mathbb{E}[N_n(E_\ell)] + \mathcal{O}(n^{1-a}) = \frac{2}{\pi}\int_{E_\ell} \frac{\sqrt{A_n(x)C_n(x) - B_n(x)^2}}{A_n(x)}\, dx + \mathcal{O}(n^{1-a}) \\ &= \frac{n + \mathcal{O}(n^{3a})}{\sqrt{3}\pi} \int_{E_\ell} \sqrt{1 + \frac{3(1-u_\ell^2(x))}{\left(1 + u_\ell(x)\cos(nx)\right)^2}}\, dx + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty. \end{aligned} \tag{3.3.3}$$

It is also clear that, for $x \in [0, \pi/2]$,

$$u_\ell(\pi - x) = \begin{cases} -u_\ell(x), & \text{if } \ell \text{ is even,} \\ \\ u_\ell(x), & \text{if } \ell \text{ is odd,} \end{cases} \tag{3.3.4}$$

which gives that, for $x \in [0, \pi/2]$,

$$u_\ell(\pi - x)\cos(n(\pi - x)) = \begin{cases} -u_\ell(x)\cos(nx), & \text{if } n-\ell \text{ is even,} \\ \\ u_\ell(x)\cos(nx), & \text{if } n-\ell \text{ is odd.} \end{cases}$$

Therefore,

$$\int_{E_\ell} \sqrt{1+\frac{3(1-u_\ell^2(x))}{\big(1+u_\ell(x)\cos(nx)\big)^2}}\,dx = \begin{cases} J_\ell^+(n)+J_\ell^-(n), & \text{if } n-\ell \text{ is even,} \\ 2J_\ell^+(n), & \text{if } n-\ell \text{ is odd.} \end{cases}$$

Hence combining this with (3.3.3) yields the desired result. ■

The following lemma is also an essential step in proving Theorem 3.2.1.

Lemma 3.3.4 *With the same* g_n^- *as in Lemma 3.3.3, assume that* $\ell \in \mathbb{N}\setminus\{1\}$ *and* $c \in (0,1)$ *are fixed. Then*

$$\int_{\pi/2n}^{n^{-c}} g_n^-(x)\,dx = \mathcal{O}(n^{-c}), \quad \text{as } n \to \infty.$$

Proof. Recall that

$$g_n^-(x) = \sqrt{1+\frac{3(1-u_\ell^2(x))}{\big(1-u_\ell(x)\cos(nx)\big)^2}}.$$

For a fixed $\ell \in \mathbb{N}\setminus\{1\}$, let us choose sufficiently large $N \in \mathbb{N}$ so that $N^{-c} \leqslant \pi/2\ell$. This implies that for all $n \geqslant N$ and $x \in [\pi/2n, n^{-c}]$, $u_\ell(x) \geqslant 0$. For all $n \geqslant N$, we define

$$f_n^-(x) := \frac{\sqrt{1-u_\ell^2(x)}}{1-u_\ell(x)\cos(nx)}.$$

Since $1 \leqslant g_n^-(x) \leqslant 1+\sqrt{3}\, f_n^-(x)$, it suffices to show that

$$\int_{\pi/2n}^{n^{-c}} f_n^-(x)\,dx = \mathcal{O}(n^{-c}).$$

Let $n_1 := \lceil n^{1-c}/\pi \rceil$, and for $1 \leqslant k \leqslant n_1$ define

$$Q_k^+ = \big\{x \in [(k-1)\pi/n, k\pi/n] : \cos(nx) \geqslant 0\big\},$$

and

$$Q_k^- = \big\{x \in [(k-1)\pi/n, k\pi/n] : \cos(nx) \leqslant 0\big\}.$$

We know that $u_\ell(x) > 0$ on $[\pi/2n, n^{-c}]$ and $0 \leqslant f_n^-(x) \leqslant 1$ on Q_k^-, $1 \leqslant k \leqslant n_1$, hence

$$\int_{Q_k^-} f_n^-(x)\,dx \leqslant \frac{\pi}{2n}.$$

In order to obtain a fitting upper bound for our integral over the Q_k^+, we note that k needs to run from 2. Otherwise, f_n^- blows up as fast as $\mathcal{O}(x^{-2})$ over $Q_1^+ = [0, \pi/2n]$, which ends up with a divergent integral. Thus, for the rest of the proof, let $2 \leqslant k \leqslant n_1$. When $x \in Q_k^+$, $2 \leqslant k \leqslant n_1$, we observe that $x \geqslant k\pi/2n$ and consequently that $\sin(x) \geqslant 2x/\pi \geqslant k/n$. Thus,

$$0 \leqslant \cos(x) \leqslant \sqrt{1 - k^2/n^2} =: a_k < 1, \quad x \in Q_k^+,\ 2 \leqslant k \leqslant n_1.$$

Therefore, the above inequality combined with (3.3.2) gives us

$$\begin{aligned}\int_{Q_k^+} f_n^-(x)\,dx &= \int_{Q_k^+} \frac{\sqrt{1-u_\ell^2(x)}}{1-u_\ell(x)\cos(nx)}\,dx \leqslant \int_{Q_k^+} \frac{\sin(\ell x)\,dx}{1-\cos(x)\cos(nx)} \\ &\leqslant \frac{k\ell\pi}{n}\int_{Q_k^+} \frac{dx}{1-\cos(x)\cos(nx)} \leqslant \frac{k\ell\pi}{n}\int_{Q_k^+} \frac{dx}{1-a_k\cos(nx)} \\ &\leqslant \frac{k\ell\pi}{n}\int_{(k-1)\pi/n}^{k\pi/n} \frac{dx}{1-a_k\cos(nx)} = \frac{k\ell\pi}{n^2}\int_{(k-1)\pi}^{k\pi} \frac{dx}{1-a_k\cos(x)}.\end{aligned}$$

Applying the change of variables $u = x - (k-1)\pi$ for the odd k, similarly $u = k\pi - x$ if k is even, we obtain

$$\int_{(k-1)\pi}^{k\pi} \frac{dx}{1-a_k\cos(x)} = \int_0^{\pi} \frac{dx}{1-a_k\cos(x)} = \frac{\pi}{\sqrt{1-a_k^2}}, \tag{3.3.5}$$

where the latter equality is obtained by (3.2.1). It follows from the last two relations that for all $n \geqslant N$,

$$\int_{Q_k^+} f_n^-(x)\,dx \leqslant \frac{k\ell\pi^2}{n^2\sqrt{1-a_k^2}} = \frac{\ell\pi^2}{n}, \quad 2 \leqslant k \leqslant n_1.$$

Thus, there exists $M_\ell > 0$ such that, for all $n \geqslant N$,

$$\begin{aligned}\int_{\pi/2n}^{n^{-c}} f_n^-(x)\,dx &\leqslant \sum_{k=1}^{n_1}\int_{Q_k^-} f_n^-(x)\,dx + \sum_{k=2}^{n_1}\int_{Q_k^+} f_n^-(x)\,dx \\ &\leqslant n^{1-c}\left(\frac{\pi}{2n} + \frac{\ell\pi^2}{n}\right) = M_\ell\, n^{-c}.\end{aligned} \tag{3.3.6}$$

■

Lemma 3.3.5 *If $\ell \in \mathbb{N} \setminus \{1\}$ and $c \in (0,1)$ are fixed, then*

$$\int_0^{n^{-c}} g_n^+(x)\,dx = \mathcal{O}(n^{-c}), \quad \text{as } n \to \infty,$$

where g_n^+ is defined as in Lemma 3.3.3.

Proof. The proof of this lemma is almost identical to that of Lemma 3.3.4 . Let us define

$$f_n^+(x) := \frac{\sqrt{1-u_\ell^2(x)}}{1+u_\ell(x)\cos(nx)}.$$

Thus, we require showing that

$$\int_0^{n^{-c}} f_n^+(x)\,dx = \mathcal{O}(n^{-c}).$$

It is trivial that

$$\int_{Q_k^+} f_n^+(x)\,dx \leqslant \frac{\pi}{2n}, \quad 1 \leqslant k \leqslant n_1$$

and, for $1 \leqslant k \leqslant n_1$,

$$\begin{aligned}\int_{Q_k^-} f_n^+(x)\,dx &\leqslant \frac{k\ell\pi}{n}\int_{Q_k^-}\frac{dx}{1+\cos(x)\cos(nx)}\\ &\leqslant \frac{k\ell\pi}{n}\int_{Q_k^-}\frac{dx}{1+a_k\cos(nx)} \leqslant \frac{k\ell\pi}{n^2}\int_{(k-1)\pi}^{k\pi}\frac{dx}{1+a_k\cos(x)}.\end{aligned}$$

The same change of variable as in (3.3.5) gives

$$\int_{(k-1)\pi}^{k\pi}\frac{dx}{1+a_k\cos(x)} = \int_0^{\pi}\frac{dx}{1+a_k\cos(x)} = \frac{\pi}{\sqrt{1-a_k^2}},$$

which similarly implies that for all $n \geqslant N$ and $1 \leqslant k \leqslant n_1$,

$$\int_{Q_k^-} f_n^+(x)\,dx \leqslant \frac{\ell\pi^2}{n}.$$

Therefore, for all $n \geqslant N$, we have

$$\int_0^{n^{-c}} f_n^+(x)\,dx \leqslant \sum_{k=1}^{n_1}\int_{Q_k^+\cup Q_k^-} f_n^+(x)\,dx \leqslant n^{1-c}\left(\frac{\pi}{2n}+\frac{\ell\pi^2}{n}\right) = M_\ell\, n^{-c}.$$

■

Lemma 3.3.6 *With the same assumptions as in Theorem 3.2.1, fix $a \in (0, 1/3)$. Then, as n grows to infinity,*

$$\mathbb{E}[N_n(0,2\pi)] = \begin{cases} \dfrac{\left[n+\mathcal{O}(n^{3a})\right]\left[I_\ell^+(n)+I_\ell^-(n)\right]}{\sqrt{3}} + \mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is even,}\\[2ex] \dfrac{\left[2n+\mathcal{O}(n^{3a})\right]I_\ell^+(n)}{\sqrt{3}} + \mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is odd,}\end{cases}$$

where

$$I_\ell^+(n) := \frac{1}{\pi}\int_0^{\pi/2} g_n^+(x), \quad \text{and} \quad I_\ell^-(n) := \frac{1}{\pi}\int_{\pi/2n}^{\pi/2} g_n^-(x)\,dx.$$

Proof. Define

$$H_\ell = \bigcup_{i=1}^{[\ell/2]} [i\pi/\ell - n^{-a}, i\pi/\ell + n^{-a}].$$

It is known from (3.3.1) that $|u_\ell(x)| \leqslant \omega_\ell$, $x \in H_\ell$, which shows that

$$1 \leqslant g_n^+(x),\, g_n^-(x) \leqslant \sqrt{1+3/(1-\omega_\ell)}, \quad x \in H_\ell.$$

Therefore,

$$\int_{H_\ell} g_n^+(x)\,dx = \mathcal{O}(n^{-a}), \quad \text{and} \quad \int_{H_\ell} g_n^-(x)\,dx = \mathcal{O}(n^{-a}).$$

Thus, with the help of Lemma 3.3.4, we can write

$$\begin{aligned} J_\ell^-(n) &= \int_{G_\ell} g_n^-(x)\,dx \\ &= \int_{\pi/2n}^{\pi/2} g_n^-(x)\,dx - \int_{H_\ell} g_n^-(x)\,dx - \int_{\pi/2n}^{n^{-a}} g_n^-(x)\,dx = \pi I_\ell^-(n) + \mathcal{O}(n^{-a}). \end{aligned}$$

In a similar way, we have $J_\ell^+(n) = \pi I_\ell^+(n) + \mathcal{O}(n^{-a})$. Now, Lemma 3.3.3 helps to reach the desired result. ∎

Lemma 3.3.7 *Fix $\ell \in \mathbb{N}\setminus\{1\}$ and $a \in (0,1/3)$. For $1 \leqslant k \leqslant n' := [n/2]$ define*

$$\zeta_k := \begin{cases} k\pi, & \text{if } k \text{ is odd,} \\ (k-1)\pi, & \text{if } k \text{ is even.} \end{cases}$$

Then

$$I_\ell^-(n) = \frac{1}{n\pi}\sum_{k=1}^{n'} \int_{(k-1)\pi}^{k\pi} \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\,dt + \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty.$$

Proof. Note that that

$$\begin{aligned} I_\ell^-(n) &= \frac{1}{\pi}\int_{\pi/2n}^{\pi/2} g_n^-(x)\,dx \\ &= \frac{1}{\pi}\int_{\pi/2n}^{\pi/n} g_n^-(x)\,dx + \frac{1}{\pi}\int_{\pi/n}^{n'\pi/n} g_n^-(x)\,dx + \frac{1}{\pi}\int_{n'\pi/n}^{\pi/2} g_n^-(x)\,dx. \end{aligned}$$

When n is even, the last integral in the above vanishes, and it is as small as $\mathcal{O}(n^{-1})$ when n is odd. Since $0 \leqslant f_n^-(x) \leqslant 1$ on Q_1^-, we observe that

$$\frac{1}{\pi}\int_{\pi/2n}^{\pi/n} g_n^-(x)\,dx = \frac{1}{\pi}\int_{Q_1^-} g_n^-(x)\,dx \leqslant \frac{1}{\pi}\int_{Q_1^-}(1+\sqrt{3}\,f_n^-(x))dx \leqslant \frac{1+\sqrt{3}}{2n}.$$

Therefore,

$$I_\ell^-(n) = \frac{1}{\pi}\int_{\pi/n}^{n'\pi/n} g_n^-(x)\,dx + \mathcal{O}(n^{-1}).$$

We apply the change of variables $t = nx$, and observe that

$$\begin{aligned} I_\ell^-(n) &= \frac{1}{\pi}\int_{\pi/n}^{n'\pi/n} g_n^-(x)\,dx + \mathcal{O}(n^{-1}) \\ &= \frac{1}{n\pi}\sum_{k=2}^{n'}\int_{(k-1)\pi}^{k\pi}\sqrt{1+\frac{3(1-u_\ell^2(t/n))}{\big(1-u_\ell(t/n)\cos(t)\big)^2}}\,dt + \mathcal{O}(n^{-1}). \end{aligned}$$

By Lemma 2.3.1, we already know that $|u_\ell(\zeta_k/n)| < 1$, and in particular

$$|u_\ell(\zeta_1/n)| = |u_\ell(\pi/n)| < 1.$$

Therefore, (3.2.1) helps us write

$$\frac{1}{n\pi}\int_0^\pi \sqrt{1+\frac{3(1-u_\ell^2(\zeta_1/n))}{\big(1-u_\ell(\zeta_1/n)\cos(t)\big)^2}}\,dt \leqslant \frac{1}{n} + \frac{\sqrt{3}}{n\pi}\int_0^\pi \frac{\sqrt{1-u_\ell^2(\zeta_1/n)}\,dt}{1-u_\ell(\zeta_1/n)\cos(t)} = \frac{1+\sqrt{3}}{n}.$$

Considering the last two relations, to prove our desired result, we need to prove that

$$\begin{aligned} &\frac{1}{n\pi}\sum_{k=2}^{n'}\int_{(k-1)\pi}^{k\pi}\sqrt{1+\frac{3(1-u_\ell^2(t/n))}{\big(1-u_\ell(t/n)\cos(t)\big)^2}}\,dt \\ &\qquad = \frac{1}{n\pi}\sum_{k=2}^{n'}\int_{(k-1)\pi}^{k\pi}\sqrt{1+\frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\,dt + \mathcal{O}(n^{-a}), \end{aligned}$$

or equivalently, we desire to show that

$$\frac{1}{n\pi}\sum_{k=2}^{n'}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^-(t)\,dt = \mathcal{O}(n^{-a}), \quad \text{as } n\to\infty, \tag{3.3.7}$$

where, for $2 \leqslant k \leqslant n'$ and $t \in [(k-1)\pi, k\pi]$,

$$\Delta_{k,n}^-(t) := \sqrt{1+\frac{3(1-u_\ell^2(t/n))}{\big(1-u_\ell(t/n)\cos(t)\big)^2}} - \sqrt{1+\frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}.$$

Set $b_a := (1-a)/2$, where $a \in (0, 1/3)$ is fixed. With the same positive integer N coming from the proof of Lemma 3.3.4, let $n_1 := \lceil n^{1-b_a}/\pi \rceil$ and $n_2 := [n/2\ell]$, for all $n \geqslant N$. If $2 \leqslant k \leqslant n_1$, we replace a_k with $u_\ell(\zeta_k/n)$ in (3.3.5) to obtain

$$\begin{aligned}\frac{1}{n\pi} \int_{(k-1)\pi}^{k\pi} & \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\, dt \\ &\leqslant \frac{1}{n} + \frac{\sqrt{3}}{n\pi} \int_{(k-1)\pi}^{k\pi} \frac{\sqrt{1-u_\ell^2(\zeta_k/n)}\, dt}{1-u_\ell(\zeta_k/n)\cos(t)} \\ &= \frac{1}{n} + \frac{\sqrt{3}}{n\pi} \int_0^{\pi} \frac{\sqrt{1-u_\ell^2(\zeta_k/n)}\, dt}{1-u_\ell(\zeta_k/n)\cos(t)} = \frac{1+\sqrt{3}}{n}.\end{aligned}$$

It follows from $n_1 \leqslant n^{1-b_a}$ and the above inequality that

$$\frac{1}{n\pi} \sum_{k=2}^{n_1} \int_{(k-1)\pi}^{k\pi} \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\, dt \leqslant (1+\sqrt{3})n^{-b_a}.$$

Replacing c with b_a in (3.3.6) also gives us

$$\begin{aligned}\frac{1}{n} \sum_{k=2}^{n_1} & \int_{(k-1)\pi}^{k\pi} \sqrt{1 + \frac{3(1-u_\ell^2(t/n))}{\big(1-u_\ell(t/n)\cos(t)\big)^2}}\, dt \\ &= \sum_{k=2}^{n_1} \int_{(k-1)\pi/n}^{k\pi/n} \sqrt{1 + \frac{3(1-u_\ell^2(x))}{\big(1-u_\ell(x)\cos(nx)\big)^2}}\, dx \\ &= \sum_{k=2}^{n_1} \int_{(k-1)\pi/n}^{k\pi/n} g_n^-(x)\, dx \leqslant \pi n^{-b_a} + \sqrt{3} \sum_{k=2}^{n_1} \int_{(k-1)\pi/n}^{k\pi/n} f_n^-(x)\, dx \\ &\leqslant (\pi + \sqrt{3}M_\ell)n^{-b_a}.\end{aligned}$$

Hence putting the last two relations together and with the help of the triangle inequality we see that

$$\frac{1}{n\pi} \sum_{k=2}^{n_1} \int_{(k-1)\pi}^{k\pi} \Delta_{k,n}^-(t)\, dt = \mathcal{O}(n^{-b_a}). \tag{3.3.8}$$

On the other hand, if $n_1 + 1 \leqslant k \leqslant n'$, it is obvious that $\Delta_{k,n}^- \in C^1[(k-1)\pi, k\pi]$. Thus, by the Mean Value Theorem for Integrals, there exists $\alpha_k \in ((k-1)\pi, k\pi)$ such that

$$\left| \frac{1}{n\pi} \sum_{k=n_1+1}^{n'} \int_{(k-1)\pi}^{k\pi} \Delta_{k,n}^-(t)\, dt \right| = \left| \frac{1}{n} \sum_{k=n_1+1}^{n'} \Delta_{k,n}^-(\alpha_k) \right| \leqslant \frac{1}{n} \sum_{k=n_1+1}^{n'} \left| \Delta_{k,n}^-(\alpha_k) \right|.$$

For $n_1+1 \leqslant k \leqslant n'$, let us define

$$f_{k,n}^{-}(t) := \frac{\sqrt{1-u_\ell^2(t/n)}}{1-u_\ell(t/n)\cos(\alpha_k)}, \quad t \in [(k-1)\pi, k\pi].$$

Since $f_{k,n}^{-} > 0$ and $\sqrt{1+3y^2} \geqslant \sqrt{3}\,|y|$, we see that

$$\begin{aligned}
\left|\Delta_{k,n}^{-}(\alpha_k)\right| &= \left|\sqrt{1+3\big(f_{k,n}^{-}(\alpha_k)\big)^2} - \sqrt{1+3\big(f_{k,n}^{-}(\zeta_k)\big)^2}\right| \\
&= \frac{3\big[f_{k,n}^{-}(\alpha_k)+f_{k,n}^{-}(\zeta_k)\big]\left|f_{k,n}^{-}(\alpha_k)-f_{k,n}^{-}(\zeta_k)\right|}{\sqrt{1+3\big(f_{k,n}^{-}(\alpha_k)\big)^2}+\sqrt{1+3\big(f_{k,n}^{-}(\zeta_k)\big)^2}} \\
&\leqslant \sqrt{3}\left|f_{k,n}^{-}(\alpha_k)-f_{k,n}^{-}(\zeta_k)\right|.
\end{aligned}$$

We know that $0 \leqslant |u_\ell(t/n)| < 1$ on $[(k-1)\pi, k\pi]$, $n_1+1 \leqslant k \leqslant n'$, hence $0 < f_{k,n}^{-} \in C^1[(k-1)\pi, k\pi]$. Thus, by the Mean Value Theorem, there exists β_k lying between α_k and ζ_k such that

$$\left|f_{k,n}^{-}(\alpha_k)-f_{k,n}^{-}(\zeta_k)\right| = |\zeta_k-\alpha_k|\left|{f_{k,n}^{-}}'(\beta_k)\right| \leqslant \pi\left|{f_{k,n}^{-}}'(\beta_k)\right|.$$

So, we can write

$$\left|\frac{1}{n\pi}\sum_{k=n_1+1}^{n'}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^{-}(t)\,dt\right| \leqslant \frac{\sqrt{3}\pi}{n}\sum_{k=n_1+1}^{n'}\left|{f_{k,n}^{-}}'(\beta_k)\right|. \tag{3.3.9}$$

Let us define an auxiliary function

$$h_\ell(x) := \frac{u_\ell(x)\cos(x)-\cos(\ell x)}{\sin(x)\sqrt{1-u_\ell^2(x)}}, \quad x \in [0,\pi].$$

Some properties of h_ℓ are indeed quick and quite useful. It is trivial that, for $x \in [0, \pi/2]$,

$$h_\ell(\pi - x) = \begin{cases} h_\ell(x), & \text{if } \ell \text{ is even,} \\ -h_\ell(x), & \text{if } \ell \text{ is odd.} \end{cases} \tag{3.3.10}$$

Moreover, $h_\ell(x)$ could be simplified as

$$\begin{aligned}
h_\ell(x) &= \frac{\sin(\ell x)\cos(x)-\ell\sin(x)\cos(\ell x)}{\ell\sin^2(x)\sqrt{1-u_\ell^2(x)}} \\
&= \frac{(1-\ell)\sin(\ell+1)x+(1+\ell)\sin(\ell-1)x}{2\ell\sin^2(x)\sqrt{1-u_\ell^2(x)}}.
\end{aligned}$$

Note that $(1-\ell)\sin(\ell+1)x+(1+\ell)\sin(\ell-1)x$ vanishes at $x=0$. In addition, for $x \in [0, \pi/2\ell]$, we have

$$\frac{d}{dx}\big[(1-\ell)\sin(\ell+1)x+(1+\ell)\sin(\ell-1)x\big] = 2(\ell^2-1)\sin(\ell x)\sin(x) \geqslant 0.$$

This means that $h_\ell(x)$ is nonnegative on $[0, \pi/2\ell]$. Hence we apply (3.3.2), and see that

$$0 \leqslant h_\ell(x) \leqslant \frac{\cos^2(x)-\cos(\ell x)}{\sin^2(x)} =: \eta_\ell(x) \leqslant \frac{\ell^2-2}{2}, \quad x \in [0, \pi/2\ell], \tag{3.3.11}$$

where the last inequality follows from the facts that η_ℓ is a strictly decreasing function on $[0, \pi/2\ell]$, and

$$\lim_{x\to 0^+} \eta_\ell(x) = (\ell^2-2)/2.$$

With the help of (3.3.1), (3.3.10) and (3.3.11), one can show that h_ℓ is integrable on $[0, \pi]$. Namely,

$$\begin{aligned}\int_0^\pi |h_\ell(x)|\,dx &= 2\int_0^{\pi/2\ell} h_\ell(x)\,dx + \int_{\pi/2\ell}^{\pi-\pi/2\ell} |h_\ell(x)|\,dx \\ &\leqslant \frac{(\ell^2-2)\pi}{2\ell} + \int_{\pi/2\ell}^{\pi-\pi/2\ell} \frac{2\,dx}{\sin(x)\sqrt{1-\omega_\ell^2}} \\ &\leqslant \frac{(\ell^2-2)\pi}{2\ell} + \frac{2(\ell-1)\pi}{\sqrt{1-\omega_\ell}} < \infty. \end{aligned} \tag{3.3.12}$$

We also note that, using the triangle inequality, we obtain

$$\begin{aligned}
\left|{f^{-}_{k,n}}'(\beta_k)\right| &= \left| \frac{u_\ell^2(\beta_k/n)\cos(\beta_k/n) - u_\ell(\beta_k/n)\cos(\ell\beta_k/n)}{n\sin(\beta_k/n)\sqrt{1-u_\ell^2(\beta_k/n)}\big[1-u_\ell(\beta_k/n)\cos(\alpha_k)\big]} \right. \\
&\quad \left. - \frac{\sqrt{1-u_\ell^2(\beta_k/n)}\big[u_\ell(\beta_k/n)\cos(\beta_k/n)\cos(\alpha_k) - \cos(\ell\beta_k/n)\cos(\alpha_k)\big]}{n\sin(\beta_k/n)\big[1-u_\ell(\beta_k/n)\cos(\alpha_k)\big]^2} \right| \\
&= \frac{1}{n}\left| \frac{u_\ell(\beta_k/n)\cos(\beta_k/n) - \cos(\ell\beta_k/n)}{\sin(\beta_k/n)\sqrt{1-u_\ell^2(\beta_k/n)}} \right| \\
&\quad \times \left| \frac{u_\ell(\beta_k/n)}{1-u_\ell(\beta_k/n)\cos(\alpha_k)} - \frac{\big[1-u_\ell^2(\beta_k/n)\big]\cos(\alpha_k)}{\big[1-u_\ell(\beta_k/n)\cos(\alpha_k)\big]^2} \right| \\
&= \frac{|h_\ell(\beta_k/n)|}{n}\left| \frac{u_\ell(\beta_k/n)}{1-u_\ell(\beta_k/n)\cos(\alpha_k)} - \frac{\big[1-u_\ell^2(\beta_k/n)\big]\cos(\alpha_k)}{\big[1-u_\ell(\beta_k/n)\cos(\alpha_k)\big]^2} \right| \\
&\leqslant \frac{|h_\ell(\beta_k/n)|}{n}\left(\frac{|u_\ell(\beta_k/n)|}{1-|u_\ell(\beta_k/n)|} + \frac{1-u_\ell^2(\beta_k/n)}{\big[1-|u_\ell(\beta_k/n)|\big]^2} \right) \\
&= \frac{|h_\ell(\beta_k/n)|}{n} \times \frac{1+2\,|u_\ell(\beta_k/n)|}{1-|u_\ell(\beta_k/n)|} \leqslant \frac{3\,|h_\ell(\beta_k/n)|}{n\big[1-|u_\ell(\beta_k/n)|\big]}. \qquad (3.3.13)
\end{aligned}$$

First, let $n_1+1 \leqslant k \leqslant n_2$. It is obvious that $n^{-b_a} \leqslant \beta_k/n \leqslant \pi/2\ell$. It follows from (3.3.2), (3.3.11) and (3.3.13) that, for $n_1+1 \leqslant k \leqslant n_2$,

$$\begin{aligned}
\left|{f^{-}_{k,n}}'(\beta_k)\right| &\leqslant \frac{3\,|h_\ell(\beta_k/n)|}{n(1-|u_\ell(\beta_k/n)|)} \leqslant \frac{3(\ell^2-2)}{2n(1-u_\ell(\beta_k/n))} \\
&\leqslant \frac{3(\ell^2-2)}{2n(1-\cos(\beta_k/n))} = \frac{3(\ell^2-2)}{4n\sin^2(\beta_k/2n)} \\
&\leqslant \frac{3(\ell^2-2)\pi^2}{4n(\beta_k/n)^2} \leqslant \frac{3(\ell^2-2)\pi^2 n^{-1+2b_a}}{4} = \frac{3(\ell^2-2)\pi^2 n^{-a}}{4}.
\end{aligned}$$

Hence we use (3.3.9), and observe that

$$\begin{aligned}
\left| \frac{1}{n\pi}\sum_{k=n_1+1}^{n_2}\int_{(k-1)\pi}^{k\pi} \Delta^{-}_{k,n}(t)\,dt \right| &\leqslant \frac{\sqrt{3}\pi}{n}\sum_{k=n_1+1}^{n_2}\left|{f^{-}_{k,n}}'(\beta_k)\right| \\
&\leqslant \frac{3\sqrt{3}(\ell^2-2)\pi^3 n^{-a}}{8\ell} = C_\ell\, n^{-a}.
\end{aligned}$$

In other words,

$$\frac{1}{n\pi}\sum_{k=n_1+1}^{n_2}\int_{(k-1)\pi}^{k\pi} \Delta^{-}_{k,n}(t)\,dt = \mathcal{O}(n^{-a}). \qquad (3.3.14)$$

Next, let $n_2 + 1 \leqslant k \leqslant n'$. This means that $\pi/2\ell \leqslant \beta_k/n \leqslant \pi/2$. Therefore, (3.3.1) and (3.3.13) show that, for $n_2 + 1 \leqslant k \leqslant n'$,

$$\left| {f_{k,n}^{-}}'(\beta_k) \right| \leqslant \frac{3\,|h_\ell(\beta_k/n)|}{n(1-|u_\ell(\beta_k/n)|)}$$
$$\leqslant \frac{6}{n\sin(\beta_k/n)\sqrt{1-\omega_\ell^2}(1-\omega_\ell)} \leqslant \frac{6\ell}{n(1-\omega_\ell)^{3/2}},$$

where the last inequality comes from the fact that $\sin(\beta_k/n) \geqslant 2\beta_k/n\pi \geqslant 1/\ell$ for $n_2 + 1 \leqslant k \leqslant n'$. Thus, with the help of (3.3.9), we have

$$\left| \frac{1}{n\pi} \sum_{k=n_2+1}^{n'} \int_{(k-1)\pi}^{k\pi} \Delta_{k,n}^{-}(t)\,dt \right| \leqslant \frac{3\sqrt{3}\ell\pi}{n(1-\omega_\ell)^{3/2}} = D_\ell\, n^{-1}.$$

So, we have shown that

$$\frac{1}{n\pi} \sum_{k=n_2+1}^{n'} \int_{(k-1)\pi}^{k\pi} \Delta_{k,n}^{-}(t)\,dt = \mathcal{O}(n^{-1}). \tag{3.3.15}$$

Combining (3.3.8), (3.3.14) and (3.3.15) yields the desired result

$$\frac{1}{n\pi} \sum_{k=2}^{n'} \int_{(k-1)\pi}^{k\pi} \Delta_{k,n}^{-}(t)\,dt = \mathcal{O}(n^{-b_a}) + \mathcal{O}(n^{-a}) + \mathcal{O}(n^{-1}) = \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty$$

since $a < b_a = (1-a)/2$, $a \in (0, 1/3)$. Thus, (3.3.7) holds as required. ∎

Lemma 3.3.8 *With the same assumptions as Lemma 3.3.7, we have*

$$I_\ell^{+}(n) = \frac{1}{n\pi} \sum_{k=1}^{n'} \int_{(k-1)\pi}^{k\pi} \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\left(1+u_\ell(\zeta_k/n)\cos(t)\right)^2}}\,dt + \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty.$$

Proof. The proof of the lemma is quite similar to that of Lemma 3.3.7. For this reason, we just state the critical steps and leave the details to the reader. As in the proof of Lemma 3.3.7, we require showing that

$$\frac{1}{n\pi} \sum_{k=1}^{n'} \int_{(k-1)\pi}^{k\pi} \Delta_{k,n}^{+}(t)\,dt = \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty,$$

where, for $1 \leqslant k \leqslant n'$ and $t \in [(k-1)\pi, k\pi]$,

$$\Delta_{k,n}^{+}(t) := \sqrt{1 + \frac{3(1-u_\ell^2(t/n))}{\left(1+u_\ell(t/n)\cos(t)\right)^2}} - \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\left(1+u_\ell(\zeta_k/n)\cos(t)\right)^2}}.$$

It is quite easy to prove that

$$\frac{1}{n\pi}\sum_{k=1}^{n_1}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^{+}(t)\,dt=\mathcal{O}(n^{-b_a}). \tag{3.3.16}$$

For $n_1+1\leqslant k\leqslant n'$, let us define

$$f_{k,n}^{+}(t):=\frac{\sqrt{1-u_\ell^2(t/n)}}{1+u_\ell(t/n)\cos(\alpha_k)},\quad t\in[(k-1)\pi,k\pi].$$

The same argument that helped us obtain (3.3.9) gives that

$$\left|\frac{1}{n\pi}\sum_{k=n_1+1}^{n'}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^{+}(t)\,dt\right|\leqslant\frac{\sqrt{3}\pi}{n}\sum_{n_1+1}^{n'}\left|{f_{k,n}^{+}}'(\beta_k)\right|,$$

where

$$\begin{aligned}\left|{f_{k,n}^{+}}'(\beta_k)\right|&=\frac{|h_\ell(\beta_k/n)|}{n}\left|\frac{u_\ell(\beta_k/n)}{1+u_\ell(\beta_k/n)\cos(\alpha_k)}+\frac{\left[1-u_\ell^2(\beta_k/n)\right]\cos(\alpha_k)}{\left[1+u_\ell(\beta_k/n)\cos(\alpha_k)\right]^2}\right|\\&\leqslant\frac{3\,|h_\ell(\beta_k/n)|}{n\left[1-|u_\ell(\beta_k/n)|\right]}.\end{aligned}$$

We follow the same procedure as in Lemma 3.3.7, and observe that, for $n_1+1\leqslant k\leqslant n_2$,

$$\left|{f_{k,n}^{+}}'(\beta_k)\right|\leqslant\frac{3(\ell^2-2)\pi^2n^{-a}}{4},$$

and, for $n_2+1\leqslant k\leqslant n'$,

$$\left|{f_{k,n}^{+}}'(\beta_k)\right|\leqslant\frac{6\ell}{n(1-\omega_\ell)^{3/2}}.$$

Therefore,

$$\frac{1}{n\pi}\sum_{k=n_1+1}^{n_2}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^{+}(t)\,dt=\mathcal{O}(n^{-a}), \tag{3.3.17}$$

and

$$\frac{1}{n\pi}\sum_{k=n_2+1}^{n'}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^{+}(t)\,dt=\mathcal{O}(n^{-1}). \tag{3.3.18}$$

Now, (3.3.16)–(3.3.18) gives the desire result, that is

$$\frac{1}{n\pi}\sum_{k=1}^{n'}\int_{(k-1)\pi}^{k\pi}\Delta_{k,n}^{+}(t)\,dt=\mathcal{O}(n^{-a}),\quad\text{as } n\to\infty.$$

■

Lemma 3.3.9 *With the same assumptions as Lemma 3.3.7,*

$$I_\ell^-(n) = \mathrm{K}_\ell + \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty.$$

Proof. It follows from Lemma 3.3.7 that

$$I_\ell^-(n) = \frac{1}{n\pi} \sum_{k=1}^{n'} \int_{(k-1)\pi}^{k\pi} \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\, dt + \mathcal{O}(n^{-a}).$$

We use the same change of variables as in (3.3.5) and observe that

$$I_\ell^-(n) = \frac{1}{\pi^2} \int_0^\pi \frac{\pi}{n} \sum_{k=1}^{n'} \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\, dt + \mathcal{O}(n^{-a}). \tag{3.3.19}$$

We intend to show that

$$\begin{aligned} &\int_0^\pi \frac{\pi}{n} \sum_{k=1}^{n} \sqrt{1 + \frac{3(1-u_\ell^2(\zeta_k/n))}{\big(1-u_\ell(\zeta_k/n)\cos(t)\big)^2}}\, dt \\ &\qquad = \int_0^\pi \int_0^\pi \sqrt{1 + \frac{3(1-u_\ell^2(s))}{\big(1-u_\ell(s)\cos(t)\big)^2}}\, ds\, dt + \mathcal{O}\left(\frac{\log n}{n}\right). \end{aligned} \tag{3.3.20}$$

Let us define

$$g^-(s,t) := \sqrt{1 + \frac{3(1-u_\ell^2(s))}{\big(1-u_\ell(s)\cos(t)\big)^2}} \times \mathbb{1}_{(0,\pi)^2}(s,t), \quad (s,t) \in [0,\pi]^2.$$

For any fixed $t \in (0,\pi)$ and any $s \in [0,\pi]$, we set

$$g_t^-(s) := g^-(s,t), \quad \text{and} \quad I^-(t) := \int_0^\pi g_t^-(s)\, ds.$$

For the sake of simplicity of our computations, we suppose that n is even. Set

$$\begin{aligned} R_{n'}(g_t^-) := \frac{\pi}{n'} \sum_{k=1}^{n'} g_t^-\big((k-1/2)\pi/n'\big) &= \frac{2\pi}{n} \sum_{k=1}^{n'} g_t^-\big((2k-1)\pi/n\big) \\ &= \frac{\pi}{n} \sum_{k=1}^{n'} 2g_t^-\big((2k-1)\pi/n\big) = \frac{\pi}{n} \sum_{k=1}^{n} g_t^-(\zeta_k/n). \end{aligned}$$

Hence proving (3.3.20) is equivalent to showing that

$$\int_0^\pi R_{n'}(g_t^-)\, dt = \int_0^\pi I^-(t)\, dt + \mathcal{O}\left(\frac{\log n}{n}\right). \tag{3.3.21}$$

Define

$$f^-(s,t) := \frac{\sqrt{1-u_\ell^2(s)}}{1-u_\ell(s)\cos(t)} \times \mathbb{1}_{(0,\pi)^2}(s,t), \quad (s,t) \in [0,\pi]^2,$$

and for any fixed $t \in (0,\pi)$ let $f_t^-(s) := f^-(s,t)$, $s \in [0,\pi]$. It is clear that

$$0 < \sqrt{3} f_t^-(s) \leqslant g_t^-(s) \leqslant 1 + \sqrt{3} f_t^-(s),$$

and

$$\left|\frac{dg_t^-(s)}{ds}\right| = \frac{3 f_t^-(s)}{g_t^-(s)} \left|\frac{df_t^-(s)}{ds}\right| \leqslant \sqrt{3} \left|\frac{df_t^-(s)}{ds}\right|.$$

Therefore, proving that g_t^-, $t \in (0,\pi)$, is integrable and a function of bounded variation over $[0,\pi]$, simply requires showing that f_t^- is integrable and of bounded variation over the same interval. It is clear that

$$\begin{aligned}
\left|\frac{df_t^-(s)}{ds}\right| &= \left|\frac{u_\ell(s)\cos(s) - \cos(\ell s)}{\sin(s)\sqrt{1-u_\ell^2(s)}}\right| \left|\frac{u_\ell(s)}{1-u_\ell(s)\cos(t)} - \frac{\left[1-u_\ell^2(s)\right]\cos(t)}{\left[1-u_\ell(s)\cos(t)\right]^2}\right| \\
&= |h_\ell(s)| \left|\frac{u_\ell(s)}{1-u_\ell(s)\cos(t)} - \frac{\left[1-u_\ell^2(s)\right]\cos(t)}{\left[1-u_\ell(s)\cos(t)\right]^2}\right| \\
&\leqslant |h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)\cos(t)|} + \frac{\left[1-u_\ell^2(s)\right]|\cos(t)|}{\left[1-|u_\ell(s)\cos(t)|\right]^2}\right).
\end{aligned} \tag{3.3.22}$$

It is trivial that

$$0 \leqslant f_{\pi/2}^-(s) = \sqrt{1-u_\ell^2(s)} \leqslant 1,$$

which implies that $f_{\pi/2}^-$ is integrable over $[0,\pi]$. Moreover, (3.3.22) and (3.3.12) imply that

$$\int_0^\pi \left|\frac{df_{\pi/2}^-(s)}{ds}\right| ds = \int_0^\pi |h_\ell(s)|\,|u_\ell(s)|\,ds \leqslant \int_0^\pi |h_\ell(s)|\,ds < \infty.$$

Thus, $f_{\pi/2}^-$ is a function of bounded variation over $[0,\pi]$. Note that, by symmetry, we just need to prove the claims for $t \in (0,\pi/2)$. So, without loss of generality, fix $t \in (0,\pi/2)$. By (3.3.4), it is immediate that, for any $s \in [0,\pi/2]$,

$$f_t^-(\pi - s) = \begin{cases} f_t^+(s), & \text{if } \ell \text{ is even}, \\ f_t^-(s), & \text{if } \ell \text{ is odd}, \end{cases} \tag{3.3.23}$$

where

$$f_t^+(s) := \frac{\sqrt{1-u_\ell^2(s)}}{1+u_\ell(s)\cos(t)}.$$

This indicates that f_t^- is integrable and a function of bounded variation over $[0,\pi]$ if both f_t^- and f_t^+ are integrable and of bounded variation over $[0,\pi/2]$. It is obvious that

$$0 \leqslant f_t^+(s) \leqslant 1, \quad s \in [0, \pi/2\ell],$$

and

$$0 \leqslant f_t^-(s), f_t^+(s) \leqslant 1/(1-\omega_\ell), \quad s \in [\pi/2\ell, \pi/2].$$

Furthermore, (3.3.2) gives that

$$\begin{aligned}
\int_0^{\pi/2\ell} \left|f_t^-(s)\right| ds &= \int_0^{\pi/2\ell} \frac{\sqrt{1-u_\ell^2(s)}\, ds}{1-u_\ell(s)\cos(t)} \leqslant \int_0^{\pi/2\ell} \frac{\sin(\ell s)\, ds}{1-\cos(s)\cos(t)} \\
&\leqslant \frac{\ell\pi}{2}\int_0^{\pi/2\ell} \frac{\sin(s)\, ds}{1-\cos(s)\cos(t)} \\
&= \frac{\ell\pi}{2\cos(t)} \log\left(\frac{1-\cos(\pi/2\ell)\cos(t)}{1-\cos(t)}\right) < \infty.
\end{aligned}$$

So far, we showed that f_t^- and f_t^+ are integrable over $[0,\pi/2]$.

In order to see that f_t^- and f_t^+ are functions of bounded variation over $[0,\pi/2]$, similar to (3.3.22), we can write

$$\begin{aligned}
\left|\frac{df_t^+(s)}{ds}\right| &= |h_\ell(s)| \left|\frac{u_\ell(s)}{1+u_\ell(s)\cos(t)} + \frac{\left[1-u_\ell^2(s)\right]\cos(t)}{\left[1+u_\ell(s)\cos(t)\right]^2}\right| \\
&\leqslant |h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)\cos(t)|} + \frac{\left[1-u_\ell^2(s)\right]|\cos(t)|}{\left[1-|u_\ell(s)\cos(t)|\right]^2}\right). \qquad (3.3.24)
\end{aligned}$$

Up to this point, (3.3.22) and (3.3.24) state that, for a fixed $t \in (0,\pi/2)$ and any $s \in [0,\pi/2]$,

$$\left|\frac{df_t^-(s)}{ds}\right|, \left|\frac{df_t^+(s)}{ds}\right| \leqslant |h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{\left[1-u_\ell^2(s)\right]\cos(t)}{\left[1-|u_\ell(s)|\cos(t)\right]^2}\right). \qquad (3.3.25)$$

It is clear from (3.3.1) that, for $s \in [\pi/2\ell, \pi/2]$,

$$\begin{aligned}
&|h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{\left[1-u_\ell^2(s)\right]\cos(t)}{\left[1-|u_\ell(s)|\cos(t)\right]^2}\right) \\
&\leqslant |h_\ell(s)| \left(\frac{1}{1-\omega_\ell} + \frac{1-\omega_\ell^2}{(1-\omega_\ell)^2}\right) \leqslant \frac{2}{\sin(s)\sqrt{1-\omega_\ell^2}}\left(\frac{2+\omega_\ell}{1-\omega_\ell}\right) \\
&\leqslant \frac{6}{\sin(s)\sqrt{1-\omega_\ell^2}(1-\omega_\ell)} \leqslant \frac{6\ell}{(1-\omega_\ell)^{3/2}}, \qquad (3.3.26)
\end{aligned}$$

where the last inequality holds since $\sin(s) \geqslant 1/\ell$ on $[\pi/2\ell, \pi/2]$. Considering (3.3.25) and (3.3.26), we have showed that f_t^- and f_t^+ are of bounded variation over $[\pi/2\ell, \pi/2]$. It also remains to prove that f_t^- and f_t^+ are of bounded variation over $[0, \pi/2\ell]$. It follows from (3.3.11) and (3.3.2) that

$$\begin{aligned}
&\int_0^{\pi/2\ell} |h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{\big[1-u_\ell^2(s)\big]\cos(t)}{\big[1-|u_\ell(s)|\cos(t)\big]^2} \right) ds \\
&\quad \leqslant \frac{\ell^2-2}{2} \int_0^{\pi/2\ell} \left(\frac{1}{1-\cos(s)\cos(t)} + \frac{\sin^2(\ell s)\cos(t)}{\big[1-\cos(s)\cos(t)\big]^2} \right) ds \\
&\quad \leqslant \frac{\ell^2-2}{2} \int_0^{\pi/2\ell} \left(\frac{1}{1-\cos(s)\cos(t)} + \frac{\ell^2\pi^2\sin^2(s)\cos(t)}{4\big[1-\cos(s)\cos(t)\big]^2} \right) ds \\
&\quad \leqslant \frac{\ell^2-2}{2} \int_0^{\pi} \left(\frac{1}{1-\cos(s)\cos(t)} + \frac{\ell^2\pi^2\sin^2(s)\cos(t)}{4\big[1-\cos(s)\cos(t)\big]^2} \right) ds \\
&\quad = \frac{\ell^2-2}{2} \left(\frac{\pi}{\sin(t)} + \frac{\ell^2\pi^2}{4} \int_0^{\pi} \frac{\cos(s)\,ds}{1-\cos(s)\cos(t)} \right),
\end{aligned} \tag{3.3.27}$$

where the last equality is obtained by (3.2.1) and integration by parts. We employ [42, 3.613(1), p. 366], while setting $n = 1$, and observe that

$$\int_0^{\pi} \frac{\cos(s)\,ds}{1-\cos(s)\cos(t)} = \frac{\pi(1-\sin(t))}{\sin(t)\cos(t)}.$$

This helps to rewrite (3.3.27) as

$$\begin{aligned}
&\int_0^{\pi/2\ell} h_\ell(s) \left(\frac{u_\ell(s)}{1-u_\ell(s)\cos(t)} + \frac{\big[1-u_\ell^2(s)\big]\cos(t)}{\big[1-u_\ell(s)\cos(t)\big]^2} \right) ds \\
&\qquad\qquad \leqslant \frac{\ell^2-2}{2} \left(\frac{\pi}{\sin(t)} + \frac{\ell^2\pi^3(1-\sin(t))}{4\sin(t)\cos(t)} \right) < \infty.
\end{aligned}$$

This along with (3.3.25) implies that f_t^- and f_t^+ are of bounded variation over $[0, \pi/2\ell]$. To summarize, we have so far showed that g_t^-, $t \in (0, \pi)$, is integrable and a function of bounded variation over $[0, \pi]$.

Let us define

$$\nu_{n'}(s) := \pi\,\rho_{n'}(s) - n's, \quad s \in [0, \pi],$$

also known as the modified sawtooth function, and $\rho_{n'} := \sum_{k=1}^{n'} \mathbb{1}_{G_k}$ with $\mathbb{1}_{G_k}$ denoting the characteristic function on $G_k = [(2k-1)\pi/n, \pi]$. If we follow the technique stated in Section

2 of [12], then

$$I^-(t) - R_{n'}(g_t^-) = \frac{\pi}{n'}\int_0^\pi \nu_{n'}(s)\,dg_t^-(s) = \frac{2\pi}{n}\int_0^\pi \nu_{n'}(s)\,dg_t^-(s).$$

In particular, we have

$$\begin{aligned}\left|\int_0^\pi \big[I^-(t) - R_{n'}(g_t^-)\big]dt\right| &\leqslant \frac{2\sqrt{3}\pi}{n}\int_0^\pi\int_0^\pi |\nu_{n'}(s)|\left|\frac{df_t^-(s)}{ds}\right| ds\,dt \\ &\leqslant \frac{4\sqrt{3}\pi}{n}\int_0^\pi\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)|\left(\frac{|u_\ell(s)|}{1-|u_\ell(s)\cos(t)|} + \frac{\big[1-u_\ell^2(s)\big]\,|\cos(t)|}{\big[1-|u_\ell(s)\cos(t)|\big]^2}\right) ds\,dt,\end{aligned}$$

where the last inequality comes from (3.3.22)–(3.3.24), and the fact that $|\nu_{n'}|$ is symmetric about $s = \pi/2$. Note that

$$\begin{aligned}&\int_0^\pi \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)\cos(t)|} + \frac{\big[1-u_\ell^2(s)\big]\,|\cos(t)|}{\big[1-|u_\ell(s)\cos(t)|\big]^2}\right)dt \\ &\quad= \int_0^{\pi/2}\left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{\big[1-u_\ell^2(s)\big]\,|\cos(t)|}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right)dt \\ &\qquad+ \int_0^{\pi/2}\left(\frac{|u_\ell(s)|}{1+|u_\ell(s)|\cos(t)} + \frac{\big[1-u_\ell^2(s)\big]\,|\cos(t)|}{\big[1+|u_\ell(s)|\cos(t)\big]^2}\right)dt \\ &\quad\leqslant 2\int_0^{\pi/2}\left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right)dt.\end{aligned}$$

Therefore, we can write

$$\begin{aligned}&\left|\int_0^\pi \big[I^-(t) - R_{n'}(g_t^-)\big]dt\right| \\ &\qquad\leqslant \frac{8\sqrt{3}\pi}{n}\int_0^{\pi/2}\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)|\left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right)ds\,dt.\end{aligned} \tag{3.3.28}$$

It is quite easy to check that $|\nu_{n'}(s)| \leqslant \pi/2$, $s \in [0,\pi/2]$. Hence with the help of (3.3.26), we have

$$\int_{\pi/2\ell}^{\pi/2}\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)|\left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right)dt\,ds \leqslant \frac{3(\ell-1)\pi^3}{4(1-\omega_\ell)^{3/2}}. \tag{3.3.29}$$

Now, (3.3.2) helps to write

$$\begin{aligned}
&\int_0^{\pi/2\ell}\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right) dt\,ds \\
&\quad\leqslant \frac{\ell^2-2}{2}\int_0^{\pi/2\ell} |\nu_{n'}(s)| \int_0^{\pi/2} \left(\frac{1}{1-\cos(s)\cos(t)} + \frac{\ell^2\pi^2\sin^2(s)}{4\big[1-\cos(s)\cos(t)\big]^2}\right) dt\,ds \\
&\quad\leqslant \frac{\ell^2-2}{2}\int_0^{\pi/2\ell} |\nu_{n'}(s)| \int_0^{\pi} \left(\frac{1}{1-\cos(s)\cos(t)} + \frac{\ell^2\pi^2\sin^2(s)}{4\big[1-\cos(s)\cos(t)\big]^2}\right) dt\,ds \\
&\quad= \frac{\ell^2-2}{2}\int_0^{\pi/2\ell} |\nu_{n'}(s)| \left(\frac{\pi}{\sin(s)} + \frac{\ell^2\pi^2\sin^2(s)}{4}\int_0^{\pi} \frac{dt}{\big[1-\cos(s)\cos(t)\big]^2}\right) ds,
\end{aligned}$$

where the last equality is gained by (3.2.1). We apply [42, 2.554(3), p. 148] with $n=2$, $a=1$ and $b=-\cos(s)$, and obtain

$$\int_0^{\pi} \frac{dt}{\big(1-\cos(s)\cos(t)\big)^2} = \frac{1}{\sin^2(s)}\int_0^{\pi} \frac{dt}{1-\cos(s)\cos(t)} = \frac{\pi}{\sin^3(s)}.$$

Combining the last two relations, we can write

$$\begin{aligned}
&\int_0^{\pi/2\ell}\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right) dt\,ds \\
&\qquad\leqslant \frac{(\ell^2-2)(4\pi+\ell^2\pi^3)}{8}\int_0^{\pi/2\ell} \frac{|\nu_{n'}(s)|}{\sin(s)}\,ds \\
&\qquad\leqslant \frac{(\ell^2-2)(4\pi^2+\ell^2\pi^4)}{16}\left(\int_0^{\pi/n} \frac{|\nu_{n'}(s)|}{s}\,ds + \int_{\pi/n}^{\pi/2\ell} \frac{|\nu_{n'}(s)|}{s}\,ds\right),
\end{aligned}$$

with the last inequality obtained by $\sin(s)\geqslant 2s/\pi$. Note that $|\nu_{n'}(s)| = n's = ns/2$ on $[0,\pi/n]$, and $|\nu_{n'}(s)|\leqslant \pi/2$, $s\in[\pi/n,\pi/2\ell]$. Therefore,

$$\begin{aligned}
&\int_0^{\pi/2\ell}\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right) dt\,ds \\
&\qquad\qquad\leqslant \frac{(\ell^2-2)(4\pi^3+\ell^2\pi^5)(\log n-\log(2\ell)+1)}{32}. \qquad (3.3.30)
\end{aligned}$$

It follows from (3.3.29) and (3.3.30) that there exist $d_\ell>0$ and $N\in\mathbb{N}$ such that, for all $n\geqslant N$,

$$\int_0^{\pi/2}\int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)| \left(\frac{|u_\ell(s)|}{1-|u_\ell(s)|\cos(t)} + \frac{1-u_\ell^2(s)}{\big[1-|u_\ell(s)|\cos(t)\big]^2}\right) dt\,ds \leqslant d_\ell \log n.$$

Thus, this combined with (3.3.28) implies that, for all $n \geqslant N$, we have

$$\left|\int_0^\pi \left[I^-(t) - R_{n'}(g_t^-)\right] dt\right|$$
$$\leqslant \frac{8\sqrt{3}\pi}{n} \int_0^{\pi/2} \int_0^{\pi/2} |\nu_{n'}(s)|\,|h_\ell(s)| \left(\frac{|u_\ell(s)|}{1 - |u_\ell(s)|\cos(t)} + \frac{1 - u_\ell^2(s)}{\left[1 - |u_\ell(s)|\cos(t)\right]^2} \right) dt\, ds$$
$$\leqslant \frac{8\sqrt{3}\pi d_\ell \log n}{n},$$

where the interchange of integration order is justified by the Fubini-Tonelli Theorem. This shows that (3.3.21) holds, so does (3.3.20). Now, combining (3.3.20) with the fact that $g^-(s,t)$ is symmetric about the vertical line passing through the point $(\pi/2, \pi/2, 0)$ gives us

$$\int_0^\pi \frac{\pi}{n} \sum_{k=1}^{n'} \sqrt{1 + \frac{3(1 - u_\ell^2(\zeta_k/n))}{\left(1 - u_\ell(\zeta_k/n)\cos(t)\right)^2}}\, dt$$
$$= \int_0^\pi \int_0^{\pi/2} \sqrt{1 + \frac{3(1 - u_\ell^2(s))}{\left(1 - u_\ell(s)\cos(t)\right)^2}}\, ds\, dt + \mathcal{O}\left(\frac{\log n}{n}\right).$$

At last, this very last estimate and (3.3.19) allow us to write that

$$\begin{aligned}
I_\ell^-(n) &= \frac{1}{\pi^2} \int_0^\pi \frac{\pi}{n} \sum_{k=1}^{n'} \sqrt{1 + \frac{3(1 - u_\ell^2(\zeta_k/n))}{\left(1 - u_\ell(\zeta_k/n)\cos(t)\right)^2}}\, dt + \mathcal{O}(n^{-a}) \\
&= \frac{1}{\pi^2} \int_0^\pi \int_0^{\pi/2} \sqrt{1 + \frac{3(1 - u_\ell^2(s))}{\left(1 - u_\ell(s)\cos(t)\right)^2}}\, ds\, dt + \mathcal{O}\left(\frac{\log n}{n}\right) + \mathcal{O}(n^{-a}) \\
&= \frac{1}{\pi^2} \int_0^\pi \int_0^{\pi/2} \sqrt{1 + \frac{3(1 - u_\ell^2(s))}{\left(1 - u_\ell(s)\cos(t)\right)^2}}\, ds\, dt + \mathcal{O}(n^{-a}) \\
&= \mathrm{K}_\ell + \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty,
\end{aligned}$$

where the last equality comes from the fact that

$$\frac{1}{\pi^2} \int_0^\pi \int_0^{\pi/2} \sqrt{1 + \frac{3(1 - u_\ell^2(s))}{\left(1 - u_\ell(s)\cos(t)\right)^2}}\, ds\, dt$$
$$= \frac{1}{\pi^2} \int_0^\pi \int_0^{\pi/2} \sqrt{1 + \frac{3(1 - u_\ell^2(s))}{\left(1 + u_\ell(s)\cos(t)\right)^2}}\, ds\, dt = \mathrm{K}_\ell. \tag{3.3.31}$$

■

Similarly, we have the following lemma, whose proof is omitted since it follows the exact procedure as in the proof of Lemma 3.3.9.

Lemma 3.3.10 *With the same assumptions as Lemma 3.3.7,*

$$I_\ell^+(n) = \mathrm{K}_\ell + \mathcal{O}(n^{-a}), \quad \text{as } n \to \infty.$$

Finally, we reach the point to prove the main theorem of this chapter.

Proof of Theorem 3.2.1. Fix $a \in (0, 1/3)$. It follows from Lemma 3.3.6 that

$$\mathbb{E}[N_n(0,2\pi)] = \begin{cases} \dfrac{\left[n + \mathcal{O}(n^{3a})\right]\left[I_\ell^+(n) + I_\ell^-(n)\right]}{\sqrt{3}} + \mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is even,} \\ \\ \dfrac{\left[2n + \mathcal{O}(n^{3a})\right] I_\ell^+(n)}{\sqrt{3}} + \mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is odd.} \end{cases}$$

Therefore, with the help of Lemmas 3.3.9 and 3.3.10, we obtain that

$$\mathbb{E}[N_n(0,2\pi)] = \frac{2n}{\sqrt{3}}\,\mathrm{K}_\ell + \mathcal{O}(n^{3a}) + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

It is clear that the best estimate occurs when $a = 1/4$. Thus,

$$\mathbb{E}[N_n(0,2\pi)] = \frac{2n}{\sqrt{3}}\,\mathrm{K}_\ell + \mathcal{O}(n^{3/4}), \quad \text{as } n \to \infty,$$

as desired. ∎

CHAPTER IV

RANDOM TRIGONOMETRIC POLYNOMIALS WITH PERIODIC COEFFICIENTS

4.1 Trigonometric polynomials with periodic coefficients

This chapter focuses on the number of zeros of random algebraic and trigonometric (cosine) polynomials with periodic coefficients.

The motivation of studying the number of real zeros of random cosine (trigonometric) polynomials with periodic coefficients arises while studying a classical work of Szegő [75, p. 260] as stated below.

Theorem (Szegő) *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be a power series with only finitely many distinct coefficients. Then either $\mathbb{D}$ is the domain of holomorphy of f or f can be extended to a rational function $\hat{f}(z) = p(z)/(1 - z^k)$, where $p(z) \in \mathbb{C}[z]$ and $k \in \mathbb{N}$.*

To prove the above theorem, it suffices to show that if $\mathbb{D}$ is not the domain of holomorphy of f, then from some coefficient on all coefficients are periodic, i.e., there exist $\lambda, \mu \in \mathbb{N}$ with $\lambda < \mu$ such that $a_{\lambda+j} = a_{\mu+j}$, $j \in \mathbb{N} \cup \{0\}$. Therefore, defining $P(z) := \sum_{j=0}^{\lambda-1} a_j z^j$ and $Q(z) := \sum_{j=\lambda}^{\mu-1} a_j z^j$, $z \in \mathbb{D}$, we then write, while setting $k = \mu - \lambda$,

$$f(z) = P(z) + Q(z) + Q(z)z^k + Q(z)z^{2k} + \cdots = P(z) + \frac{Q(z)}{1 - z^k}, \quad z \in \mathbb{D}.$$

This naturally invites us to investigate the cosine series $V(z) := \sum_{j=0}^{\infty} a_j \cos(jz)$ with only finitely many distinct coefficients.

Proposition 4.1.1 *$V(z)$ with only finitely many distinct coefficients diverges in $\mathbb{C}$.*

Proof. Observe that

$$V(z) = \sum_{j=0}^{\infty} a_j \cos(jz) = \frac{1}{2}\left(\sum_{j=0}^{\infty} a_j e^{ijz} + \sum_{j=0}^{\infty} a_j e^{-ijz}\right) = \frac{1}{2}\left(\sum_{j=0}^{\infty} a_j w^j + \sum_{j=0}^{\infty} a_j w^{-j}\right),$$

where $w = e^{iz}$. Since we are dealing with only finitely many distinct coefficients, we then see that $\limsup_{j\to\infty} |a_j|^{1/j} = 1$. Thus, $\sum_{j=0}^{\infty} a_j w^j$ and $\sum_{j=0}^{\infty} a_j w^{-j}$ diverge in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and $\mathbb{D}$ respectively, which implies that $\sum_{j=0}^{\infty} a_j \cos(jz)$ diverges in $\mathbb{C} \setminus \mathbb{R}$.

Note that $V(x) = \sum_{j=0}^{\infty} a_j \cos(jx),\ x \in [0, 2\pi)$, with only finitely many distinct coefficients is also divergent since $\lim_{j\to\infty} a_j \cos(jx) \neq 0$. This could be shown by assuming to the contrary that $\lim_{j\to\infty} a_j \cos(jx_0) = 0$ for some $x_0 \in [0, 2\pi)$. Set $M := \min\{|a_j| : a_j \neq 0\}$ and let $\varepsilon \in (0, M/2)$ be arbitrary. We may find a large enough $J \in \mathbb{N}$ so that $|a_J \cos(Jx_0)| < \varepsilon$, for all $j \geqslant J$. It is then clear that $|\cos(2Jx_0)| < \varepsilon/M < 1/2$, and

$$|a_{2J} \cos(2Jx_0)| \geqslant M\,|\cos(2Jx_0)| = M(1 - 2\cos^2(Jx_0)) > M(1 - 2\varepsilon^2/M^2) > \varepsilon,$$

where the last inequality is derived from the fact that $2\varepsilon^2 + M\varepsilon - M^2 < 0$ for all $\varepsilon \in (0, M/2)$. Thus, $V(z) = \sum_{j=0}^{\infty} a_j \cos(jz)$ with finitely many distinct coefficients diverges in the entire complex plane. ■

Similarly, $T(z) := \sum_{j=0}^{\infty} a_j \cos(jz) + b_j \sin(jz)$ diverges under the same conditions. However, our objective is to study the number of real zeros of partial sums of these infinite series.

We begin with a model considering the coefficients being periodic from the trailing coefficient a_0 on, namely $a_k = a_{k+\ell}$, $k \in \mathbb{N} \cup \{0\}$, and a fixed $\ell \in \mathbb{N}$. Let $A = (a_0, a_1, \ldots, a_n)$ be the block of all coefficients and $n = \ell m - 1$, $m \in \mathbb{N}$, where ℓ is fixed. We split the coefficients into the blocks of length ℓ as the following. Set $A = (a_0, a_1, \ldots, a_n) = \bigcup_{j=0}^{m-1} A_j$, where

$$A_j := (a_{\ell j}, a_{\ell j+1}, \ldots, a_{\ell(j+1)-1}).$$

We further assume that the coefficients are periodic, i.e., $A_0 = A_1 = \cdots = A_{m-1}$. In what follows, we investigate the roots of polynomials P_n, T_n, and V_n with coefficients satisfying the above arrangement.

Lemma 4.1.1 *Fix $\ell \in \mathbb{N}$ and set $n = \ell m - 1$, $m \in \mathbb{N}$. Let $P_n(z) = \sum_{j=0}^{n} a_j z^j$, where $(a_0, a_1, \ldots, a_n) = \bigcup_{j=0}^{m-1} A_j$ with the A_j as above. We also assume that the coefficients are periodic, i.e., $a_{k+\ell j} = a_k$ for $0 \leqslant k \leqslant \ell - 1$ and $0 \leqslant j \leqslant m - 1$. Then*

$$P_n(z) = \frac{z^{\ell m} - 1}{z^{\ell} - 1} \sum_{k=0}^{\ell-1} a_k z^k.$$

Remark 4.1.1 From the above lemma, it is clear that P_n has at least $\ell(m-1) = n - \ell + 1$ zeros that are all unimodular. We also note that if $\ell = 1$, then all zeros are the m-th roots of unity other than 1.

Before stating the next lemma, let us define

$$B := (b_0, b_1, \ldots, b_n), \qquad \text{and} \qquad B_j := (b_{\ell j}, b_{\ell j+1}, \ldots, b_{\ell(j+1)-1}).$$

Theorem 4.1.1 *Fix $\ell \in \mathbb{N}$, and set $n = \ell m - 1$, $m \in \mathbb{N} \setminus \{1\}$. Let $T_n(x) = \sum_{j=0}^{n} a_j \cos(jx) + b_j \sin(jx)$, $x \in (0, 2\pi)$. Assume $A_0 \cup B_0$ is a family of i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. We also assume that the coefficients a_j and b_j are periodic, i.e., $a_{k+\ell j} = a_k$ and $b_{k+\ell j} = b_k$ for $0 \leqslant k \leqslant \ell - 1$ and $0 \leqslant j \leqslant m - 1$. Then*

$$\mathbb{E}[N_n(0, 2\pi)] = n + 1 - \ell + \sqrt{n^2 + \frac{\ell^2 - 1}{3}}.$$

Remark 4.1.2 If $\ell = 1$, then all the zeros happen to be real.

Theorem 4.1.2 *Fix $\ell \in \mathbb{N}$, and set $n = \ell m - 1$, $m \in \mathbb{N}$. Let $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$. Assume A_0 is a family of i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. We also assume that the coefficients a_j are periodic, i.e., $a_{k+\ell j} = a_k$ for $0 \leqslant k \leqslant \ell - 1$ and $0 \leqslant j \leqslant m - 1$. Then*

$$\mathbb{E}[N_n(0, 2\pi)] = 2n + \mathcal{O}(n^{2/3}) \quad \text{as } n \to \infty,$$

where the implied constant depends only on ℓ.

4.2 Proofs

Proof of Lemma 4.1.1. With the assumptions of Lemma 4.1.1, one can write

$$P_n(z) = \sum_{j=0}^{n} a_j z^j = \sum_{k=0}^{\ell-1} a_k \sum_{j=0}^{m-1} z^{k+\ell j} = \left(\sum_{j=0}^{m-1} z^{\ell j}\right)\left(\sum_{k=0}^{\ell-1} a_k z^k\right) = \frac{z^{\ell m}-1}{z^\ell - 1}\sum_{k=0}^{\ell-1} a_k z^k,$$

which gives us at least $\ell(m-1) = n - \ell + 1$ roots with modulus one. ■

Proof of Theorem 4.1.1. For $x \in (0, 2\pi)$, we apply (2.3.1) and (2.3.2) and observe that

$$\begin{aligned} T_n(x) &= \sum_{j=0}^{n} a_j \cos(jx) + b_j \sin(jx) \\ &= \sum_{k=0}^{\ell-1} a_k \sum_{j=0}^{m-1} \cos(k+\ell j)x + \sum_{k=0}^{\ell-1} b_k \sum_{j=0}^{m-1} \sin(k+\ell j)x \\ &= \frac{\sin(m\ell x/2)}{\sin(\ell x/2)} \sum_{k=0}^{\ell-1} \big[a_k \cos(k+(m-1)\ell/2)x + b_k \sin(k+(m-1)\ell/2)x\big] \\ &=: \frac{\sin(m\ell x/2)}{\sin(\ell x/2)}\, T_n^*(x). \end{aligned}$$

We first find the expected number of real zeros of T_n^*. We observe that, for $x \in (0, 2\pi)$,

$$A_n^*(x) = \sum_{k=0}^{\ell-1} \big[\cos^2(k+(m-1)\ell/2)x + \sin^2(k+(m-1)\ell/2)x\big] = \ell > 0, \tag{4.2.1}$$

$$\begin{aligned} B_n^*(x) = &-\sum_{k=0}^{\ell-1}[k+(m-1)\ell/2] \sin(k+(m-1)\ell/2)x \, \cos(k+(m-1)\ell/2)x \\ &+ \sum_{k=0}^{\ell-1}[k+(m-1)\ell/2] \sin(k+(m-1)\ell/2)x \, \cos(k+(m-1)\ell/2)x = 0, \end{aligned} \tag{4.2.2}$$

and

$$\begin{aligned}
C_n^*(x) &= \sum_{k=0}^{\ell-1} \big[k+(m-1)\ell/2\big]^2 \sin^2(k+(m-1)\ell/2)x \\
&\quad + \sum_{k=0}^{\ell-1} \big[k+(m-1)\ell/2\big]^2 \cos^2(k+(m-1)\ell/2)x \\
&= \sum_{k=0}^{\ell-1} \left(k+\frac{\ell(m-1)}{2}\right)^2 = \sum_{k=0}^{\ell-1} \left(k^2+\ell(m-1)k+\frac{\ell^2(m-1)^2}{4}\right) \\
&= \frac{(\ell-1)\ell(2\ell-1)}{6} + \frac{(\ell-1)\ell^2(m-1)}{2} + \frac{\ell^3(m-1)^2}{4} \\
&= \frac{\ell(3\ell^2m^2-6\ell m+\ell^2+2)}{12} = \frac{\ell\big[3(\ell m-1)^2+(\ell^2-1)\big]}{12} \\
&= \frac{\ell\big[3n^2+(\ell^2-1)\big]}{12}. \qquad (4.2.3)
\end{aligned}$$

It follows from (4.2.1)–(4.2.3) that

$$\frac{\sqrt{A_n^*(x)C_n^*(x)-B_n^*(x)^2}}{A_n^*(x)} = \frac{1}{2}\sqrt{n^2+\frac{\ell^2-1}{3}}.$$

Therefore, by Kac-Rice's formula (1.2.3), we obtain

$$\mathbb{E}[N_n^*(0,2\pi)] = \frac{1}{\pi}\int_0^{2\pi} \frac{\sqrt{A_n^*(x)C_n^*(x)-B_n^*(x)^2}}{A_n^*(x)}\,dx = \sqrt{n^2+\frac{\ell^2-1}{3}}. \qquad (4.2.4)$$

Let us define $\varphi_m(x) := \sin(m\ell x/2)/\sin(\ell x/2)$. We know that

$$Z(\varphi_m)\cap[0,2\pi/\ell] = \{2j\pi/m\ell : 1 \leqslant j \leqslant m-1\}.$$

Therefore, φ_m has $\ell(m-1) = n+1-\ell$ zeros in $[0,2\pi]$. Now, considering (4.2.4) and $n+1-\ell$ zeros of φ_m, we have

$$\mathbb{E}[N_n(0,2\pi)] = n+1-\ell+\mathbb{E}[N_n^*(0,2\pi)] = n+1-\ell+\sqrt{n^2+\frac{\ell^2-1}{3}},$$

as required. ■

Proof of Theorem 4.1.2. Fix $a \in (0,1/2)$, and define $E = [0,\pi]\setminus F$, where $F = [0,n^{-a})\cup(\pi-n^{-a},\pi]$.

For $x \in [0, \pi]$, we apply (2.3.1) and observe that

$$V_n(x) = \sum_{j=0}^{n} a_j \cos(jx) = \sum_{k=0}^{\ell-1} a_k \sum_{j=0}^{m-1} \cos(k + \ell j)x$$
$$= \frac{\sin(m\ell x/2)}{\sin(\ell x/2)} \sum_{k=0}^{\ell-1} a_k \cos(k + (m-1)\ell/2)x.$$

If we set $\ell = 1$, it is easy to check that V_n has exactly $2n$ zeros in $[0, 2\pi]$, so we assume that $\ell \in \mathbb{N} \setminus \{1\}$. Again, setting $\varphi_m(x) = \sin(m\ell x/2)/\sin(\ell x/2)$, we can write $V_n(x) = \varphi_m(x) V_n^*(x)$, where

$$V_n^*(x) := \sum_{k=0}^{\ell-1} a_k \cos(k + (m-1)\ell/2)x.$$

To discuss the expected number of real zeros of V_n^* in E, we compute A_n^*, B_n^* and C_n^*. First, for $x \in E$, we apply (2.3.1) and obtain

$$A_n^*(x) = \sum_{k=0}^{\ell-1} \cos^2(k + (m-1)\ell/2)x = \frac{1}{2} \sum_{k=0}^{\ell-1} \big[1 + \cos(2k + (m-1)\ell)x\big]$$
$$= \frac{1}{2}\left(\ell + \frac{\sin(\ell x)\cos(\ell m - 1)x}{\sin(x)}\right) = \frac{\ell[1 + u_\ell(x)\cos(nx)]}{2}, \tag{4.2.5}$$

where $u_\ell(x) := \sin(\ell x)/\ell \sin(x)$. Note that Markov's inequality (see Theorem 15.1.4 of [74]) guarantees that $|u_\ell(x)| < 1$ on E implying that $A_n^* > 0$ on E. Moreover,

$$B_n^*(x) = -\sum_{k=0}^{\ell-1} [k + (m-1)\ell/2] \sin(k + (m-1)\ell/2)x \, \cos(k + (m-1)\ell/2)x$$
$$= -\frac{1}{2} \sum_{k=0}^{\ell-1} [k + (m-1)\ell/2] \sin(2k + (m-1)\ell)x$$
$$= -\frac{\ell m}{4} \sum_{k=0}^{\ell-1} \sin(2k + (m-1)\ell)x - \frac{1}{2} \sum_{k=0}^{\ell-1} [k - \ell/2] \sin(2k + (m-1)\ell)x$$
$$= -\frac{\ell m}{4} \sum_{k=0}^{\ell-1} \sin(2k + (m-1)\ell)x + \mathcal{O}(1)$$
$$= -\frac{\ell m \sin(\ell x)\sin(\ell m - 1)x}{4\sin(x)} + \mathcal{O}(1) = -\frac{\ell^2 m u_\ell(x)\sin(nx)}{4} + \mathcal{O}(1),$$

where the last sum is obtained with the help of (2.3.2). Therefore,

$$B_n^*(x) = -\frac{\ell n u_\ell(x)\sin(nx)}{4} + \mathcal{O}(1), \quad \text{as } n \to \infty \text{ and } x \in E. \tag{4.2.6}$$

In addition, using (2.3.1), we obtain that

$$
\begin{aligned}
C_n^*(x) &= \sum_{k=0}^{\ell-1}\big[k+(m-1)\ell/2\big]^2\sin^2(k+(m-1)\ell/2)x \\
&= \frac{1}{2}\sum_{k=0}^{\ell-1}\big[k+(m-1)\ell/2\big]^2[1-\cos(2k+(m-1)\ell)x] \\
&= \frac{\ell^2m^2}{8}\left(\ell-\frac{\sin(\ell x)\cos(\ell m-1)x}{\sin(x)}\right)+\mathcal{O}(m) \\
&= \frac{\ell^3m^2(1-u_\ell(x)\cos(nx))}{8}+\mathcal{O}(m).
\end{aligned}
$$

Hence,

$$
C_n^*(x)=\frac{\ell n^2(1-u_\ell(x)\cos(nx))}{8}+\mathcal{O}(n),\quad \text{as } n\to\infty \text{ and } x\in E. \tag{4.2.7}
$$

Hence, (4.2.5)–(4.2.7) imply that

$$
\Delta_n^*(x):=A_n^*(x)C_n^*(x)-B_n^*(x)^2=\frac{\ell^2n^2(1-u_\ell^2(x))}{16}+\mathcal{O}(n),\quad \text{as } n\to\infty \text{ and } x\in E.
$$

As in the proof of Lemma 3.3.2, we can show that

$$
\Delta_n^*(x)=\frac{\ell^2n^2(1-u_\ell^2(x))\big[1+\mathcal{O}(n^{-1+2a})\big]}{16},\quad \text{as } n\to\infty \text{ and } x\in E.
$$

Therefore,

$$
\frac{\sqrt{\Delta_n^*(x)}}{A_n^*(x)}=\frac{n\big[1+\mathcal{O}(n^{-1+2a})\big]}{2}\times\frac{\sqrt{1-u_\ell^2(x)}}{1+u_\ell(x)\cos(nx)},\quad \text{as } n\to\infty \text{ and } x\in E.
$$

From this point on, the proof is very close to that of Theorem 3.2.1. Set $G=E\cap[0,\pi/2]=[n^{-a},\pi/2]$. As in the proof of Lemma 3.3.3, we can easily show that, as n tends to infinity,

$$
\mathbb{E}[N_n^*(0,2\pi)]=\begin{cases}\dfrac{\big[n+\mathcal{O}(n^{2a})\big]\big[\mathcal{J}_\ell^+(n)+\mathcal{J}_\ell^-(n)\big]}{\pi}+\mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is even},\\[3ex] \dfrac{\big[2n+\mathcal{O}(n^{2a})\big]\mathcal{J}_\ell^+(n)}{\pi}+\mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is odd},\end{cases}
$$

where

$$
\mathcal{J}_\ell^+(n):=\int_G f_n^+(x)\,dx,\quad \text{and}\quad \mathcal{J}_\ell^-(n):=\int_G f_n^-(x)\,dx,
$$

with

$$f_n^+(x) := \frac{\sqrt{1-u_\ell^2(x)}}{1+u_\ell(x)\cos(nx)}, \quad \text{and} \quad f_n^-(x) := \frac{\sqrt{1-u_\ell^2(x)}}{1-u_\ell(x)\cos(nx)}.$$

Note that while proving Lemmas 3.3.4 and 3.3.5, we showed

$$\int_0^{n^{-a}} f_n^+(x)\,dx = \mathcal{O}(n^{-a}), \quad \text{and} \quad \int_{\pi/2n}^{n^{-a}} f_n^-(x)\,dx = \mathcal{O}(n^{-a}).$$

Hence, likewise Lemma 3.3.6 we can write

$$\mathbb{E}[N_n^*(0,2\pi)] = \begin{cases} \left[n+\mathcal{O}(n^{2a})\right]\left[I_\ell^+(n)+I_\ell^-(n)\right]+\mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is even,} \\ \\ \left[2n+\mathcal{O}(n^{2a})\right]I_\ell^+(n)+\mathcal{O}(n^{1-a}), & \text{if } n-\ell \text{ is odd,} \end{cases} \tag{4.2.8}$$

where

$$\mathcal{I}_\ell^+(n) := \frac{1}{\pi}\int_0^{\pi/2} f_n^+(x), \quad \text{and} \quad \mathcal{I}_\ell^-(n) := \frac{1}{\pi}\int_{\pi/2n}^{\pi/2} f_n^-(x)\,dx.$$

We use the same method established through Lemmas 3.3.7–3.3.10 to show that

$$\begin{aligned} \mathcal{I}_\ell^-(n) &= \frac{1}{\pi}\int_{\pi/2n}^{\pi/2} \frac{\sqrt{1-u_\ell^2(x)}}{1-u_\ell(x)\cos(nx)}\,dx \\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi/2} \frac{\sqrt{1-u_\ell^2(s)}}{1-u_\ell(s)\cos(t)}\,ds\,dt + \mathcal{O}(n^{-a}) \\ &= \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi} \frac{\sqrt{1-u_\ell^2(s)}}{1+u_\ell(s)\cos(t)}\,dt\,ds + \mathcal{O}(n^{-a}), \end{aligned}$$

where the interchange of integration order is justified by the Fubini-Tonelli Theorem. Now, applying (3.2.1) yields

$$\int_0^{\pi} \frac{dt}{1-u_\ell(s)\cos(t)} = \frac{\pi}{\sqrt{1-u_\ell^2(s)}},$$

which gives us

$$\mathcal{I}_\ell^-(n) = \frac{1}{2} + \mathcal{O}(n^{-a}).$$

Similarly, we have

$$\mathcal{I}_\ell^+(n) = \frac{1}{2} + \mathcal{O}(n^{-a}).$$

Plugging these two into (4.2.8), we can write

$$\mathbb{E}[N_n^*(0, 2\pi)] = n + \mathcal{O}(n^{2a}) + \mathcal{O}(n^{1-a}).$$

It is clear that the best estimate in above occurs when $a = 1/3$. Thus,

$$\mathbb{E}[N_n^*(0, 2\pi)] = n + \mathcal{O}(n^{2/3}).$$

At last, considering $n + 1 - \ell$ distinct roots of φ_m, in $[0, 2\pi]$, we have

$$\mathbb{E}[N_n(0, 2\pi)] = n + 1 - \ell + \mathbb{E}[N_n^*(0, 2\pi)] = 2n + \mathcal{O}(n^{2/3}), \quad \text{as } n \to \infty,$$

as desired. ■

REFERENCES

[1] R. Adler and J. Taylor, *Random fields and geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR 2319516

[2] J. Angst, F. Dalmao, and G. Poly, *On the real zeros of random trigonometric polynomials with dependent coefficients*, Proc. Amer. Math. Soc. **147** (2019), no. 1, 205–214. MR 3876743

[3] L. Arnold, *Über die Nullstellenverteilung zufälliger Polynome*, Math. Z. **92** (1966), 12–18. MR 200966

[4] J. Azaïs and J. León, *CLT for crossings of random trigonometric polynomials*, Electron. J. Probab. **18** (2013), no. 68, 17. MR 3084654

[5] J. Azaïs and M. Wschebor, *Level sets and extrema of random processes and fields*, John Wiley & Sons, Inc., Hoboken, NJ, 2009. MR 2478201

[6] V. Bally, L. Caramellino, and G. Poly, *Non universality for the variance of the number of real roots of random trigonometric polynomials*, Probab. Theory Related Fields **174** (2019), no. 3-4, 887–927. MR 3980307

[7] A. Bharucha-Reid and M. Sambandham, *Random polynomials*, Probability and Mathematical Statistics, Academic Press, Inc., Orlando, FL, 1986. MR 856019

[8] A. Bloch and G. Pólya, *On the roots of certain algebraic equations*, Proc. London Math. Soc. (2) **33** (1931), no. 2, 102–114. MR 1576817

[9] E. Bogomolny, O. Bohigas, and P. Leboeuf, *Quantum chaotic dynamics and random polynomials*, J. Statist. Phys. **85** (1996), no. 5-6, 639–679. MR 1418808

[10] P. Borwein, T. Erdélyi, R. Ferguson, and R. Lockhart, *On the zeros of cosine polynomials: solution to a problem of Littlewood*, Ann. of Math. (2) **167** (2008), no. 3, 1109–1117. MR 2415396

[11] V. Botta, L. Marques, and M. Meneguette, *Palindromic and perturbed polynomials: zeros location*, Acta Math. Hungar. **143** (2014), no. 1, 81–87. MR 3215605

[12] C. Chui, *Concerning rates of convergence of Riemann sums*, J. Approx. Theory **4** (1971), 279–287. MR 289731

[13] J. Conrey, D. Farmer, and O. Imamoglu, *Palindromic random trigonometric polynomials*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1835–1839. MR 2470844

[14] H. Cramér and M. Leadbetter, *Stationary and related stochastic processes*, Dover Publications, Inc., Mineola, NY, 2004, Sample function properties and their applications, Reprint of the 1967 original. MR 2108670

[15] M. Das, *The average number of real zeros of a random trigonometric polynomial*, Proc. Cambridge Philos. Soc. **64** (1968), 721–729. MR 233398

[16] Y. Do, H. Nguyen, and O. Nguyen, *Random trigonometric polynomials: universality and non-universality of the variance for the number of real roots*, arXiv e-prints (2019), arXiv:1912.11901.

[17] J. Dunnage, *The number of real zeros of a random trigonometric polynomial*, Proc. London Math. Soc. (3) **16** (1966), 53–84. MR 0192532

[18] A. Edelman and E. Kostlan, *How many zeros of a random polynomial are real?*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), no. 1, 1–37. MR 1290398

[19] T. Erdélyi, *The number of unimodular zeros of self-reciprocal polynomials with coefficients in a finite set*, Acta Arith. **176** (2016), no. 2, 177–200. MR 3566640

[20] ———, *Improved lower bound for the number of unimodular zeros of self-reciprocal polynomials with coefficients in a finite set*, Acta Arith. **192** (2020), no. 2, 189–210. MR 4042412

[21] P. Erdős and A. Offord, *On the number of real roots of a random algebraic equation*, Proc. London Math. Soc. (3) **6** (1956), 139–160. MR 0073870

[22] P. Erdős and P. Turán, *On the distribution of roots of polynomials*, Ann. of Math. (2) **51** (1950), 105–119. MR 33372

[23] K. Farahmand, *On the average number of real roots of a random algebraic equation with dependent coefficients*, J. Indian Math. Soc. (N.S.) **50** (1986), no. 1-4, 49–58 (1988). MR 989014

[24] ———, *On the average number of level crossings of a random trigonometric polynomial*, Ann. Probab. **18** (1990), no. 3, 1403–1409. MR 1062074

[25] ———, *On the variance of the number of real roots of a random trigonometric polynomial*, J. Appl. Math. Stochastic Anal. **3** (1990), no. 4, 253–261. MR 1086972

[26] ———, *Level crossings of a random trigonometric polynomial*, Proc. Amer. Math. Soc. **111** (1991), no. 2, 551–557. MR 1015677

[27] ———, *Level crossings of a random trigonometric polynomial with dependent coefficients*, J. Austral. Math. Soc. Ser. A **58** (1995), no. 1, 39–46. MR 1313884

[28] ———, *On the variance of the number of real zeros of a random trigonometric polynomial*, J. Appl. Math. Stochastic Anal. **10** (1997), no. 1, 57–66. MR 1437951

[29] ———, *Topics in random polynomials*, Pitman Research Notes in Mathematics Series, vol. 393, Longman, Harlow, 1998. MR 1679392

[30] ———, *On random orthogonal polynomials*, J. Appl. Math. Stochastic Anal. **14** (2001), no. 3, 265–274. MR 1853083

[31] ———, *On zeros of self-reciprocal random algebraic polynomials*, J. Appl. Math. Stoch. Anal. (2007), Art. ID 43091, 7. MR 2373413

[32] K. Farahmand and J. Gao, *Algebraic polynomials with symmetric random coefficients*, Rocky Mountain J. Math. **44** (2014), no. 2, 521–529. MR 3240512

[33] K. Farahmand and P. Hannigan, *Large level crossings of random polynomials*, Stochastic Anal. Appl. **20** (2002), no. 2, 299–309. MR 1900362

[34] K. Farahmand and T. Li, *Random trigonometric polynomials with nonidentically distributed coefficients*, Int. J. Stoch. Anal. (2010), Art. ID 931565, 10. MR 2678921

[35] ———, *Real zeros of three different cases of polynomials with random coefficients*, Rocky Mountain J. Math. **42** (2012), no. 6, 1875–1892. MR 3028766

[36] K. Farahmand and A. Nezakati, *Algebraic polynomials with dependent random coefficients*, Commun. Appl. Anal. **9** (2005), no. 1, 95–104. MR 2194834

[37] ———, *Real zeros of algebraic polynomials with dependent random coefficients*, Stoch. Anal. Appl. **28** (2010), no. 3, 558–564. MR 2739575

[38] ———, *The expected number of real zeros of algebraic polynomials with dependent and non-identical random coefficients*, Stoch. Anal. Appl. **29** (2011), no. 3, 452–456. MR 2795556

[39] K. Farahmand and A. Shaposhnikov, *On the expected number of level crossings of random trigonometric polynomials*, Stoch. Anal. Appl. **23** (2005), no. 6, 1141–1147. MR 2177241

[40] H. Flasche, *Expected number of real roots of random trigonometric polynomials*, Stochastic Process. Appl. **127** (2017), no. 12, 3928–3942. MR 3718101

[41] G. Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999. MR 1681462

[42] I. Gradshteyn and I. Ryzhik, *Table of integrals, series, and products*, Academic Press, 1980. MR 582453

[43] A. Granville and I. Wigman, *The distribution of the zeros of random trigonometric polynomials*, Amer. J. Math. **133** (2011), no. 2, 295–357. MR 2797349

[44] J. Hammersley, *The zeros of a random polynomial*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II, University of California Press, Berkeley and Los Angeles, 1956, pp. 89–111. MR 0084888

[45] C. Hughes and A. Nikeghbali, *The zeros of random polynomials cluster uniformly near the unit circle*, Compos. Math. **144** (2008), no. 3, 734–746. MR 2422348

[46] I. Ibragimov and N. Maslova, *The average number of zeros of random polynomials*, Vestnik Leningrad. Univ. **23** (1968), no. 19, 171–172. MR 0238376

[47] ———, *The mean number of real zeros of random polynomials. I. Coefficients with zero mean*, Teor. Verojatnost. i Primenen. **16** (1971), 229–248. MR 0286157

[48] ———, *The mean number of real zeros of random polynomials. II. Coefficients with a nonzero mean*, Teor. Verojatnost. i Primenen. **16** (1971), 495–503. MR 0288824

[49] I. Ibragimov and D. Zaporozhets, *On distribution of zeros of random polynomials in complex plane*, Prokhorov and contemporary probability theory, Springer Proc. Math. Stat., vol. 33, Springer, Heidelberg, 2013, pp. 303–323. MR 3070481

[50] I. Ibragimov and O. Zeitouni, *On roots of random polynomials*, Trans. Amer. Math. Soc. **349** (1997), no. 6, 2427–2441. MR 1390040

[51] B. Jamrom, *On the average number of real roots of a random algebraic polynomial*, Vestnik Leningrad. Univ. Ser. Math. Meh. **4** (1971), no. 19, 152–156.

[52] ———, *The average number of real zeros of random polynomials*, Soviet Math. Dokl. **13** (1972), 1381–1383.

[53] M. Kac, *On the average number of real roots of a random algebraic equation*, Bull. Amer. Math. Soc. **49** (1943), 314–320. MR 7812

[54] ______, *On the average number of real roots of a random algebraic equation. II*, Proc. London Math. Soc. (2) **50** (1949), 390–408. MR 30713

[55] D. Kwon, *Reciprocal polynomials with all zeros on the unit circle*, Acta Math. Hungar. **131** (2011), no. 3, 285–294. MR 2786774

[56] ______, *Reciprocal polynomials with all but two zeros on the unit circle*, Acta Math. Hungar. **134** (2012), no. 4, 472–480. MR 2886220

[57] P. Lakatos, *On zeros of reciprocal polynomials*, Publ. Math. Debrecen **61** (2002), no. 3-4, 645–661. MR 1943722

[58] J. Littlewood and A. Offord, *On the number of real roots of a random algebraic equation*, J. London Math. Soc. **13** (1938), no. 4, 288–295. MR 1574980

[59] ______, *On the number of real roots of a random algebraic equation. II*, Proc. Camb. Philos. Soc. **35** (1939), no. 2, 133–148.

[60] ______, *On the number of real roots of a random algebraic equation. III*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 277–286. MR 0009656

[61] ______, *On the distribution of the zeros and a-values of a random integral function. I*, J. London Math. Soc. **20** (1945), 130–136. MR 19123

[62] ______, *On the distribution of zeros and a-values of a random integral function. II*, Ann. of Math. (2) **49** (1948), 885–952; errata 50, 990–991 (1949). MR 29981

[63] D. Lubinsky, I. Pritsker, and X. Xie, *Expected number of real zeros for random linear combinations of orthogonal polynomials*, Proc. Amer. Math. Soc. **144** (2016), no. 4, 1631–1642. MR 3451239

[64] N. Maslova, *The variance of the number of real roots of random polynomials*, Teor. Verojatnost. i Primenen. **19** (1974), 36–51. MR 0334327

[65] H. Nguyen, O. Nguyen, and V. Vu, *On the number of real roots of random polynomials*, Commun. Contemp. Math. **18** (2016), no. 4, 1550052. MR 3493213

[66] T. Pautrel, *New asymptotics for the mean number of zeros of randomtrigonometric polynomials with strongly dependentgaussian coefficients*, Electron. Commun. Probab. **25** (2020), 1–13.

[67] A. Pirhadi, *Real zeros of random trigonometric polynomials with pairwise equal blocks of coefficients*, Rocky Mountain J. Math. **50** (2020), no. 4, 1451–1471. MR 4154817

[68] ______, *Real zeros of random cosine polynomials with palindromic blocks of coefficients*, Anal. Math. **47** (2021), no. 1, 175–210. MR 4218584

[69] I. Pritsker, *Asymptotic zero distribution of random polynomials spanned by general bases*, Contemp. Math **661** (2016), 121–140. MR 3489554

[70] ______, *Zero distribution of random polynomials*, J. Anal. Math. **134** (2018), no. 2, 719–745. MR 3771497

[71] I. Pritsker and A. Sola, *Expected discrepancy for zeros of random algebraic polynomials*, Proc. Amer. Math. Soc. **142** (2014), no. 12, 4251–4263. MR 3266993

[72] I. Pritsker and A. Yeager, *Zeros of polynomials with random coefficients*, J. Approx. Theory **189** (2015), 88–100. MR 3280673

[73] C. Qualls, *On the number of zeros of a stationary Gaussian random trigonometric polynomial*, J. London Math. Soc. (2) **2** (1970), 216–220. MR 258110

[74] Q. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press, Oxford University Press, Oxford, 2002. MR 1954841

[75] R. Remmert, *Classical topics in complex function theory*, Graduate Texts in Mathematics, vol. 172, Springer-Verlag, New York, 1998, Translated from the German by Leslie Kay. MR 1483074

[76] N. Renganathan and M. Sambandham, *On the number of real zeros of a random trigonometric polynomial: coefficients with nonzero mean*, J. Indian Math. Soc. (N.S.) **45** (1981), no. 1-4, 193–203 (1984). MR 828871

[77] ______, *On the average number of real zeros of a random trigonometric polynomial with dependent coefficients. II*, Indian J. Pure Appl. Math. **15** (1984), no. 9, 951–956. MR 761283

[78] S. Rice, *Mathematical analysis of random noise*, Bell System Tech. J. **23** (1944), 282–332. MR 10932

[79] ______, *Mathematical analysis of random noise*, Bell System Tech. J. **24** (1945), 46–156. MR 11918

[80] W. Rudin, *Real and complex analysis*, second ed., McGraw-Hill, 1974. MR 0344043

[81] J. Sahasrabudhe, *Counting zeros of cosine polynomials: on a problem of Littlewood*, Adv. Math. **343** (2019), 495–521. MR 3883212

[82] M. Sambandham, *On the number of real zeros of a random trigonometric polynomial*, Trans. Amer. Math. Soc. **238** (1978), 57–70. MR 461648

[83] ______, *On the upper bound of the number of real roots of a random algebraic equation*, J. Indian Math. Soc. (N.S.) **42** (1978), no. 1-4, 15–26 (1979). MR 558980

[84] ______, *On the average number of real zeros of a class of random algebraic curves*, Pacific J. Math. **81** (1979), no. 1, 207–215. MR 543744

[85] S. Shemehsavar, *Expected number of real zeros of Gaussian self-reciprocal random algebraic polynomials*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2018), no. 1, 105–113. MR 3773130

[86] L. Shepp and R. Vanderbei, *The complex zeros of random polynomials*, Trans. Amer. Math. Soc. **347** (1995), no. 11, 4365–4384. MR 1308023

[87] Z. Su and Q. Shao, *Asymptotics of the variance of the number of real roots of random trigonometric polynomials*, Sci. China Math. **55** (2012), no. 11, 2347–2366. MR 2994124

[88] D. Šparo and M. Šur, *On the distribution of roots of random polynomials*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **1962** (1962), no. 3, 40–43. MR 0139199

[89] Y. Wang, *Bounds on the average number of real roots of a random algebraic equation*, Chinese Ann. Math. Ser. A **4** (1983), no. 5, 601–605, An English summary appears in Chinese Ann. Math. Ser. B **4** (1983), no. 4, 527. MR 742181

[90] J. Wilkins, *An upper bound for the expected number of real zeros of a random polynomial*, J. Math. Anal. Appl. **42** (1973), 569–577. MR 326842

[91] ———, *Mean number of real zeros of a random trigonometric polynomial*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 851–863. MR 1039266

[92] Z. Yu, *Bounds on the average number of real roots for a class of random algebraic equations*, J. Math. Res. Exposition **2** (1982), no. 2, 81–85. MR 669828

APPENDICES

Random cosine polynomials with palindromic coefficients

This appendix discusses the expected number of real zeros of random cosine polynomials with palindromic coefficients. Even though this subject was already studied by Farahmand and Li [35], we believe it is necessary to give a complete proof of the case with palindromic coefficients since the given result in [35] is not accurate. Besides, they only considered the case where n (the degree of V_n) is odd, see pp. 11–12 for our brief comments on their result.

We investigate the expected number of real zeros of random palindromic cosine polynomials through the following lemmas based on n being odd or even.

Lemma A.1 *Fix $a \in (0, 1/2)$ and let $n = 2m - 1$, $m \in \mathbb{N}$. Define $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$. We assume that the coefficients a_j, $0 \leqslant j \leqslant m-1$, are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. If the a_j are palindromic, i.e., $a_j = a_{n-j}$, $0 \leqslant j \leqslant m-1$, then*

$$\mathbb{E}[N_n(0, 2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Proof. Since the coefficients are palindromic, we observe that

$$\begin{aligned} V_n(x) &= \sum_{j=0}^{n} a_j \cos(jx) = \sum_{j=0}^{m-1} a_j \big[\cos(jx) + \cos(n-j)x\big] \\ &= 2\cos(nx/2) \sum_{j=0}^{m-1} a_j \cos(n/2 - j)x = 2\cos(nx/2)\, V_n^*(x), \end{aligned}$$

where $m = (n+1)/2$, and

$$V_n^*(x) := \sum_{j=0}^{m-1} a_j \cos(m - j - 1/2)x.$$

Let $N_n(\cdot)$ and $N_n^*(\cdot)$ be the number of real zeros of V_n and V_n^* respectively. We first find the expected number of real zeros of V_n^* in $(0, 2\pi)$. Therefore, we require computing $A_n^*(x)$, $B_n^*(x)$ and $C_n^*(x)$ for V_n^*, as they are defined in Lemma 1.2.3. Let $E = [0, \pi] \setminus F$, where $F = [0, n^{-a}) \cup (\pi - n^{-a}, \pi]$, and $a \in (0, 1/2)$ is fixed. We note that, for $x \in E$,

$$A_n^*(x) = \sum_{j=0}^{m-1} \cos^2(m - j - 1/2)x = \sum_{j=0}^{m-1} \cos^2(j + 1/2)x,$$

where the last equality is simply obtained by replacing j with $m - 1 - j$. It is trivial that

$$A_n^*(x) = \sum_{j=0}^{m-1} \cos^2(j + 1/2)x \geq \cos^2(x/2) > 0, \quad x \in E.$$

Moreover,

$$\begin{aligned} A_n^*(x) &= \sum_{j=0}^{m-1} \cos^2(j + 1/2)x = \frac{1}{2} \sum_{j=0}^{m-1} [1 + \cos(2j + 1)x] \\ &= \frac{m}{2} + \frac{1}{2} \sum_{j=0}^{m} \cos(2j + 1)x = \frac{m}{2} + \frac{R_0(1, m, x)}{2}. \end{aligned}$$

Now, Corollary 2.3.1 gives that

$$\begin{aligned} A_n^*(x) = \frac{m}{2} + \mathcal{O}(m^a) &= \frac{n+1}{4} + \mathcal{O}(^a) \\ &= \frac{n(1 + \mathcal{O}(n^{-1+a}))}{4}, \quad \text{as } n \to \infty \text{ and } x \in E. \end{aligned} \tag{A.1}$$

Note that

$$\begin{aligned} B_n^*(x) &= -\sum_{j=0}^{m-1} (m - j - 1/2) \sin(m - j - 1/2)x \, \sin(m - j - 1/2)x \\ &= -\sum_{j=0}^{m-1} (j + 1/2) \sin(j + 1/2)x \, \sin(j + 1/2)x \\ &= -\frac{1}{2} \sum_{j=0}^{m-1} (j + 1/2) \sin(2j + 1)x = -\frac{S_1(1, m, x)}{2} - \frac{S_0(1, m, x)}{4}, \end{aligned}$$

where the second equality is again reached by replacing j with $m - 1 - j$. Thus, Corollary 2.3.1 implies that

$$B_n^*(x) = \mathcal{O}(m^{1+a}) = \mathcal{O}(n^{1+a}), \quad \text{as } n \to \infty \text{ and } x \in E. \tag{A.2}$$

To estimate C_n^*, we see that

$$\begin{aligned}
C_n^*(x) &= \sum_{j=0}^{m-1}(m-j-1/2)^2\sin^2(m-j-1/2)x \\
&= \sum_{j=0}^{m-1}(j+1/2)^2\sin^2(j+1/2)x \\
&= \frac{1}{2}\sum_{j=0}^{m-1}(j^2+j+1/4)[1-\cos(2j+1)x].
\end{aligned}$$

Since $\sum_{j=0}^{m-1} j = \mathcal{O}(m^2)$, it is quite easy to check that

$$\begin{aligned}
\frac{1}{2}\sum_{j=0}^{m-1}(j^2+j+1/4) &= \frac{1}{2}\sum_{j=0}^{m-1} j^2 + \mathcal{O}(m^2) = \frac{(m-1)m(2m-1)}{12} + \mathcal{O}(m^2) \\
&= \frac{m^3}{6} + \mathcal{O}(m^2) = \frac{(n+1)^3}{48} + \mathcal{O}(n^2) = \frac{n^3}{48} + \mathcal{O}(n^2).
\end{aligned}$$

Next, with the help of Corollary 2.3.1, we observe that

$$\begin{aligned}
\sum_{j=0}^{m-1}(j^2+j+1/4)\cos(2j+1)x &= R_2(1,m,x) + R_1(1,m,x) + \frac{R_0(1,m,x)}{4} \\
&= \mathcal{O}(m^{2+a}) = \mathcal{O}(n^{2+a}).
\end{aligned}$$

Plugging the last two estimates into C_n^*, we have

$$C_n^*(x) = \frac{n^3}{48} + \mathcal{O}(n^{2+a}), \quad \text{as } n\to\infty \text{ and } x\in E. \tag{A.3}$$

It follows from (A.1)–(A.3) that

$$\begin{aligned}
\Delta_n^*(x) := A_n^*(x)C_n^*(x) - B_n^*(x)^2 &= \frac{n^4}{192} + \mathcal{O}(n^{3+a}) \\
&= \frac{n^4\big(1+\mathcal{O}(n^{-1+a})\big)}{192}, \quad \text{as } n\to\infty \text{ and } x\in E.
\end{aligned} \tag{A.4}$$

Thus, (A.1) and (A.4) as well as Lemma 1.2.3 (Kac-Rice's formula) give

$$\begin{aligned}
\mathbb{E}[N_n^*(E)] &= \frac{1}{\pi}\int_E \frac{\sqrt{\Delta_n^*(x)}}{A_n^*(x)}\,dx \\
&= \frac{1}{\pi}\int_E \frac{n\big(1+\mathcal{O}(n^{-1+a})\big)}{2\sqrt{3}\big(1+\mathcal{O}(n^{-1+a})\big)}\,dx \\
&= \frac{n+\mathcal{O}(n^a)}{2\sqrt{3}\pi}\,|E| = \frac{\big(n+\mathcal{O}(n^a)\big)\big(\pi+\mathcal{O}(n^{-a})\big)}{2\sqrt{3}\pi} \\
&= \frac{n}{2\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n\to\infty.
\end{aligned}$$

Furthermore, Lemma 2.1.2 implies that

$$\mathbb{E}[N_n^*(F)] \leqslant \mathbb{E}[N_n(F)] = \mathcal{O}(n^{1-a}).$$

Thus,

$$\mathbb{E}[N_n^*(0,2\pi)] = 2\,\mathbb{E}[N_n^*(0,\pi)] = 2\,\mathbb{E}[N_n^*(E \cup F)] = \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Now, taking n distinct roots of $\cos(nx/2)$ in $(0, 2\pi)$ into account, we have

$$\mathbb{E}[N_n(0,2\pi)] = \mathbb{E}[n + N_n^*(0,2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

■

At this point, we discuss the expected number of real zeros of random palindromic cosine polynomials of even degrees.

Lemma A.2 *Fix $a \in (0, 1/2)$ and let $n = 2m$, $m \in \mathbb{N}$. Define $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$, $x \in (0, 2\pi)$. We assume that the coefficients a_j, $0 \leqslant j \leqslant m$, are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0, \sigma^2)$. If the a_j are palindromic, i.e., $a_j = a_{n-j}$, $0 \leqslant j \leqslant m$, then*

$$\mathbb{E}[N_n(0,2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Proof. It is obvious that since n is even, the term $a_{n/2}\cos(nx/2) = a_m \cos(mx)$ remains unchanged in the middle. In other words,

$$\begin{aligned} V_n(x) = \sum_{j=0}^{n} a_j \cos(jx) &= a_m \cos(mx) + \sum_{j=0}^{m-1} a_j \big[\cos(jx) + \cos(n-j)x\big] \\ &= a_m \cos(mx) + 2\cos(nx/2) \sum_{j=0}^{m-1} a_j \cos(n/2 - j)x \\ &= 2\cos(nx/2)\, V_n^*(x), \end{aligned}$$

where $m = n/2$, and

$$V_n^*(x) := \frac{a_m}{2} + \sum_{j=0}^{m-1} a_j \cos(m-j)x.$$

Again, let $N_n(\cdot)$ and $N_n^*(\cdot)$ denote the number of real zeros of V_n and V_n^* respectively. We need to estimate the expected number of real zeros of V_n^* in $(0, 2\pi)$. Let us define $E = [0, \pi] \setminus F$, where $F = [0, n^{-a}) \cup (\pi - n^{-a}, \pi]$, and $a \in (0, 1/2)$ is fixed. We note that, for $x \in E$,

$$A_n^*(x) = \frac{1}{4} + \sum_{j=0}^{m-1} \cos^2(m-j)x > 0.$$

It is also obvious that

$$A_n^*(x) = \frac{1}{4} + \sum_{j=0}^{m-1} \cos^2(m-j)x = \frac{1}{4} + \sum_{j=0}^{m-1} \cos^2(j+1)x,$$

where the second equality is obtained by replacing j with $m-1-j$. It is clear that

$$\begin{aligned} A_n^*(x) &= \frac{1}{4} + \sum_{j=0}^{m-1} \cos^2(j+1)x = \frac{1}{4} + \frac{1}{2}\sum_{j=0}^{m-1}[1 + \cos(2j+2)x] \\ &= \frac{m}{2} + \frac{1}{4} + \frac{\cos(x)}{2}\sum_{j=0}^{m} \cos(2j+1)x - \frac{\sin(x)}{2}\sum_{j=0}^{m} \sin(2j+1)x \\ &= \frac{m}{2} + \frac{1}{4} + \frac{\cos(x)\, R_0(1,m,x)}{2} - \frac{\sin(x)\, S_0(1,m,x)}{2}. \end{aligned}$$

Therefore, it follows from Corollary 2.3.1 that

$$\begin{aligned} A_n^*(x) = \frac{m}{2} + \mathcal{O}(m^a) &= \frac{n}{4} + \mathcal{O}(n^a) \\ &= \frac{n(1 + \mathcal{O}(n^{-1+a}))}{4}, \quad \text{as } n \to \infty \text{ and } x \in E. \end{aligned} \tag{A.5}$$

We also observe that

$$\begin{aligned} B_n^*(x) &= -\sum_{j=0}^{m-1}(m-j)\sin(m-j)x\ \sin(m-j)x \\ &= -\sum_{j=0}^{m-1}(j+1)\sin(j+1)x\ \sin(j+1)x \\ &= -\frac{1}{2}\sum_{j=0}^{m-1}(j+1)\sin(2j+2)x = -\frac{\cos(x)\big[S_1(1,m,x) + S_0(1,m,x)\big]}{2} \\ &\quad - \frac{\sin(x)\big[R_1(1,m,x) + R_0(1,m,x)\big]}{2}, \end{aligned}$$

where the second equality is again reached by replacing j with $m-1-j$. Hence Corollary 2.3.1 implies that

$$B_n^*(x) = \mathcal{O}(m^{1+a}) = \mathcal{O}(n^{1+a}), \quad \text{as } n \to \infty \text{ and } x \in E. \tag{A.6}$$

Note that

$$\begin{aligned} C_n^*(x) &= \sum_{j=0}^{m-1} (m-j)^2 \sin^2(m-j)x \\ &= \sum_{j=0}^{m-1} (j+1)^2 \sin^2(j+1)x \\ &= \frac{1}{2} \sum_{j=0}^{m-1} (j^2+2j+1)[1-\cos(2j+2)x]. \end{aligned}$$

It follows from $\sum_{j=0}^{m-1} j = \mathcal{O}(m^2)$ that

$$\frac{1}{2}\sum_{j=0}^{m-1}(j^2+2j+1) = \frac{m^3}{6} + \mathcal{O}(m^2) = \frac{n^3}{48} + \mathcal{O}(n^2).$$

Moreover, with the help of Corollary 2.3.1, we have

$$\begin{aligned} \sum_{j=0}^{m-1}(j^2+2j+1)\cos(2j+2)x &= \cos(x)\big[R_2(1,m,x) + 2R_1(1,m,x) + R_0(1,m,x)\big] \\ &- \sin(x)\big[S_2(1,m,x) + 2S_1(1,m,x) + S_0(1,m,x)\big] = \mathcal{O}(m^{2+a}) = \mathcal{O}(n^{2+a}). \end{aligned}$$

Putting the last two estimates back into C_n^*, we obtain

$$C_n^*(x) = \frac{n^3}{48} + \mathcal{O}(n^{2+a}), \quad \text{as } n \to \infty \text{ and } x \in E. \tag{A.7}$$

From this point on, the proof is quite similar to that of Lemma A.1. Namely, we can show that

$$\mathbb{E}[N_n^*(0,2\pi)] = \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

Finally, considering n distinct roots of $\cos(nx/2)$ in $(0, 2\pi)$, we have

$$\mathbb{E}[N_n(0,2\pi)] = \mathbb{E}[n + N_n^*(0,2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty,$$

as required. ■

Now, Lemmas A.1 and A.2 lead us to the desired result, i.e.,

Theorem A.1 *Fix* $a \in (0, 1/2)$*, and let* $n = 2m-1+r$*, where* $m \in \mathbb{N}$ *and* $r \in \{0, 1\}$*. Define* $V_n(x) = \sum_{j=0}^{n} a_j \cos(jx)$*,* $x \in (0, 2\pi)$*. We assume that the coefficients* a_j*,* $0 \leqslant j \leqslant [n/2]$*, are i.i.d. random variables with Gaussian distribution* $\mathcal{N}(0, \sigma^2)$*. If the* a_j *are palindromic, i.e.,* $a_j = a_{n-j}$*,* $0 \leqslant j \leqslant [n/2]$*, then*

$$\mathbb{E}[N_n(0, 2\pi)] = n + \frac{n}{\sqrt{3}} + \mathcal{O}(n^{1-a}), \quad \text{as } n \to \infty.$$

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