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A PENALTY FUNCTION APPROACH TO GLOBAL EXTREMA FOR CERTAIN CONTROL PROBLEMS

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A PENALTY FUNCTION APPROACH TO GLOBAL

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FOR CERTAIN CONTROL PROBLEMS

CHAPTER I

INTRODUCTION

This paper is primarily concerned with optimal control problems to be described in Section 2.2 of the next chapter. We consider "solving an optimal control problem" to mean

(1) determining a function u_0, in a certain set U of admissible controls, that is a candidate for furnishing the desired infimum of a given cost function (criterion for optimality) $J(\cdot)$ over U,

and

(2) demonstrating that the candidate u_0 does indeed furnish the desired infimum.

Actually, in practice it is seldom possible to carry out steps (1) and (2) above to completion but in many cases the problem can be at least partially solved.

It is well known that solving or partially solving an optimal control problem can be extremely difficult. Frequently, it leads to the solution of a nonlinear two-point boundary value problem or to the solution of a large system of algebraic or transcendental equations. If either of these latter situations occurs then a further attempt to obtain a solution or partial solution is usually by recourse to numerical methods.

The need for such numerical methods has long been recognized as indicated by the 1909 paper of Ritz [21]. However, in recent years, the effort to develop numerical methods has greatly increased.

The numerical methods have been generally classified as either indirect or direct numerical methods. Descriptively stated, indirect methods use the Euler or other necessary conditions and seek, by various iterative procedures to satisfy these conditions, whereas, direct numerical methods use the cost function and the side and end conditions and attempt to solve the problem without resorting to the Multiplier Rule or other necessary conditions [24, p. 2].

An indirect numerical method was suggested as early as 1949 by Hestenes in [11]. A direct computational procedure was presented in 1960 by Kelley in [14, pp. 205-254]. Since these times several new or modified methods have been added to each general class.

Some difficulties are associated with these numerical methods. There is a general lack of criteria for selecting a method for a particular problem. In case an indirect method is used, then the terminal conditions are extremely sensitive to variations in the initial Lagrange multipliers, [9, p. 295]. Also, sometimes the solution of the dynamical equations is required at each step, which may be prohibitive in terms of both computer time and storage. On the other hand, the direct numerical methods have the inherent disadvantage of very slow convergence in the neighborhood of the optimal solution, [23].

There seems to be a basic misconception in numerous published papers that if one follows a direct computational technique associated with necessary conditions such as Pontryagin's Maximal Principle, Bellman's dynamic programming, etc., and if this yields a sequence $\{u_n\}$, in the set U of admissible controls, that appears to converge to some u_0 in U, then

$$(1.1) \qquad J(u_0) = \inf\{J(u) : u \in U\}.$$

Of the many recent examples, [15] and [20] are typical. It may be the case that $J(u_0)$ is only a local minimum, that is,

$$(1.2) \qquad J(u_0) < J(u)$$

for $u \in U$ and u sufficiently near u_0 in one sense or another. Another possibility is that u_0 fails to satisfy some necessary condition for even a local minimum. There usually are not enough criteria given to rule out these unwanted possibilities.

Given a problem

$$J(u) = \underline{\text{global}}\ \underline{\text{minimum}}\ \underline{\text{on}}\ U,$$

let u_α denote a computed function of which it is hoped that

$$(1.3) \qquad J(u_\alpha) < \inf\{J(u) : u \in U\} + \delta, \ \delta > 0.$$

There is a paucity of criteria that will say with certainty, or even high probability, that (1.3) holds for an explicit small δ.

Recently, the Calculus of Variation, which dates back to the seventeenth century, has been formulated in the language and notation of control systems. However, in spite of this long history, the potential usefulness in engineering and other human affairs continues to be only marginally realized because of the previously mentioned computational difficulties.

There is such a variety of problems, $J(u)$ = global minimum on U, depending on the nature of the admissible controls $u \in U$ and the kind of side and end conditions that it is vain to hope that one theory or procedure can encompass them all. We shall restrict attention to certain control problems with suitable convexity properties. It is noteworthy that many of the illustrative examples on which numerical methods have been tried out in the literature are of the type to be considered, for example, see [16, p. 210], [19, pp. 402-406], [20, pp. 344-347], and [22, p. 236].

The approach used here is the penalty function method as treated by Balakrishnan in [2]. We are indebted to his suggestive work. However, his paper has some seemingly vague or missing details that we will attempt to clarify or supply.

The class of problems presented in Section 2.2 is somewhat different from that of Balakrishnan in [2]. However, following his approach, the original control problem is replaced by the so-called auxiliary problem which is described in Section 2.3. Then Balakrishnan's major results of [2] are obtained, namely, the existence of solutions of the auxiliary problems and the use of these solutions to approximate the infimum of the original problem. Also, with additional assumptions, several new results involving existence and uniqueness of a solution for the original control problem are presented in Sections 2.6 and 2.7 of Chapter II.

Some of the ideas in the paper [6] by Budak, Berkovich and Solov'eva are used in Chapter III to prove that the infimum of the continuous auxiliary problem is approximated by the infimum of the corresponding discrete auxiliary problem. Then, with reference to some of the work in Chapter II, it is concluded that the infimum of the original control problem can be approximated by the infimum of a corresponding discrete auxiliary problem.

CHAPTER II

A PENALTY FUNCTION TECHNIQUE

2.1 Introduction

The use of penalty functions in minimization problems with equality constraints seems to go back to the 1945 work of R. Courant in [7, pp. 270-280]. He replaced the ordinary constrained minimization problem, $F(x) = \underline{\text{minimum}}$ subject to the constraint $G(x) = 0$, by a sequence of free problems, that is,

$$F_n(x) \equiv F(x) + n|G(x)|^2 = \underline{\text{minimum}},$$

for each positive integer n, where $|\cdot|$ in the <u>penalty-function</u> $n|G(x)|^2$ denotes the euclidean norm. Observe that unless $|G(x)|^2$ is suitably near zero for an x value that minimizes F_n, then it seems likely that F_n becomes large as n becomes large. Hence it is plausible to hope that, as $n \to \infty$, a minimizing x_n for $F_n(x)$ will converge to a limit x_0 such that $G(x_0) = 0$ and such that x_0 solves the original minimum problem. Courant proved that indeed this is so under certain conditions including lower semicontinuity of F and G.

Recently, Balakrishnan has discussed in [2], [3], [4], and [5], a penalty function method for solving certain control problems for dynamical systems. Although the method is similar in concept to that of Courant's work, Balakrishnan credits J. L. Lions with the suggestion.

The technique used by Balakrishnan avoids explicit solution of the dynamical equations, which seems to give it an inherent computational advantage. Also, this approach has certain advantages in proving the existence and uniqueness of a minimizing pair for the control problem.

2.2 The Control Problem

Let $\mathcal{P}$ be the class of all pairs (x,u) of functions $x = (x^1,\cdots,x^n)$ and $u = (u^1,\cdots,u^p)$ from the fixed interval [0,T] to R^n and R^p, respectively, satisfying the conditions that

(2.1)

(i) x *is absolutely continuous* (AC) *on* [0,T],

(ii) $\dot{x} = f[t,x,u(t)]$ *almost everywhere* (a.e.) *on* [0,T], *where* f *is a vector-valued function defined on* $[0,T] \times R^n \times R^p$,

(iii) $x(0) = a_0$, *a constant vector in* R^n,

(iv) u *is Lebesgue measurable on* [0,T] *such that*

$$|u^j(t)| \leq b_1^j < \infty,\ 0 \leq t \leq T, \text{ and } j = 1,\cdots,p.$$

Define a functional J: $\mathcal{P} \to R$, the real numbers, by the statement that

$$J(x,u) \equiv \int_0^T g[t,x(t),u(t)]dt, \tag{2.2}$$

where T is a positive constant and g is a scalar function defined on $[0,T] \times R^n \times R^p$.

Additional assumptions are:

(2.3) f(t,x,u), g(t,x,u) *and all first order partials with respect to components of* x *and* u *are continuous on* $[0,T] \times R^n \times R^p$,

and

$$J(x,u) \geq 0, \text{ for all } (x,u) \in \mathcal{P}. \tag{2.4}$$

In order to use the appropriate existence and uniqueness theorem [18, pp. 342-346] for the initial value problem (2.1) (ii), (iii) we require that

(2.5) <u>there</u> <u>exist</u> <u>a</u> <u>constant</u> $b_2 > 0$ <u>such</u> <u>that</u>

$$|f(t,x,u) - f(t,y,u)| \le b_2|x - y|, \quad x,y \in R^n,$$

<u>for</u> $t \in [0,T]$ <u>and</u> u <u>satisfying</u> (2.1) (iv),

and

(2.6) $|f(t,x,u)| \le \mu(t)[c + |x|]$, <u>for</u> <u>all</u> $t \in [0,T]$ <u>and</u> <u>all</u> u <u>satisfying</u> (2.1) (iv), <u>where</u> μ <u>is</u> <u>integrable</u> <u>on</u> $[0,T]$ <u>and</u> c <u>is</u> <u>a</u> <u>positive</u> <u>constant</u>.

The problem, herein referred to as the <u>control</u> <u>problem</u>, is to investigate the existence, the properties, and the possible approximation by numerical methods of a pair $(x_0,u_0) \in \mathcal{O}$ such that $J(x_0,u_0)$ is the infimum of $J(x,u)$ on $\mathcal{O}$ subject to conditions (2.3) through (2.6).

Let $x(\cdot;u)$ denote the response x from (2.1)(ii) corresponding to a given control u. It is shown in [13, pp. 74-78] that conditions (2.5), (2.6) and the continuity of f given in (2.3) are sufficient for $|x(t;u)|$ to be bounded, that is, there exist positive constants b_3^i, $i = 1,2,\cdots,n$, such that

(2.7) $|x^i(t;u)| \le b_3^i$, <u>for</u> <u>all</u> $t \in [0,T]$, <u>for</u> <u>all</u> u <u>such</u> <u>that</u> $(x,u) \in \mathcal{O}$, <u>and</u> $i = 1,2,\cdots,n$.

As a consequence of (2.7), the differential equation (2.1)(ii), the boundedness (2.1)(iv) of u and the continuity of f, the derivative $\dot{x}$ of every x such that $(x,u) \in \mathcal{O}$ is bounded on $[0,T]$ except for a subset of measure zero.

2.3 The Auxiliary Problem

Let $\mathcal{P}^+$ be the class of pairs (x,u), $x = (x^1,\cdots,x^n)$, $u = (u^1,\cdots,u^p)$ satisfying the conditions that

$$
(2.8)\quad
\begin{aligned}
&\text{(i)} && x \text{ is AC on } [0,T] \text{ and } \dot{x} \in L_2([0,T]), \\
&\text{(ii)} && x(0) = a_0, \text{ a constant vector in } R^n, \\
&\text{(iii)} && |x^i(t)| \le b_3^i, \text{ for all } t \in [0,T] \text{ and } i = 1,\cdots,n, \\
&\text{(iv)} && u \text{ is Lebesgue measurable on } [0,T] \text{ and such that} \\
& && \qquad |u^j(t)| \le b_1^j < \infty, \quad 0 \le t \le T \text{ and } j = 1,\cdots,p.
\end{aligned}
$$

Every $(x,u) \in \mathcal{P}^+$ (or $\mathcal{P}$) will be called an <u>admissible pair for the original problem or the auxiliary problem</u> as the case may be, with such qualification omitted if it is clear from the context which problem is involved. The x and u of an admissible pair will be called an <u>admissible response</u> and an <u>admissible control</u>, respectively.

Every x such that $(x,u) \in \mathcal{P}$ is an x satisfying the given differential equation (2.1)(ii) for some admissible control u and hence every such x is an $x(\cdot;u)$ satisfying (2.7). Although the symbol $x(\cdot;u)$ does not make sense in connection with the class $\mathcal{P}^+$ because this class is free of the requirement (2.1)(ii), it remains true of $\mathcal{P}$ that every x such that $(x,u) \in \mathcal{P}$ satisfies the condition (2.7) that

$$|x^i(t)| \le b_3^i, \text{ for all } t \in [0,T] \text{ and } i = 1,2,\cdots,n.$$

It is clear from the boundedness of $\dot{x}$ implied by (2.7) that every such x is such that $\dot{x} \in L_2([0,T])$. These considerations together with the comparison of conditions (2.1) and (2.8) show that every pair $(x,u) \in \mathcal{P}$ is also a pair (x,u) in $\mathcal{P}^+$ but not conversely. Thus, $\mathcal{P} \subset \mathcal{P}^+$ and $\mathcal{P} \neq \mathcal{P}^+$.

We now introduce with Balakrishnan an <u>auxiliary problem</u> also called the ε-<u>problem</u>. Let conditions (2.3) through (2.6) hold with (2.4) henceforth strengthened by the replacement of $\mathcal{O}$ by $\mathcal{O}^+$ and denoted by $(2.4)^+$. For an arbitrary but fixed $\varepsilon > 0$, we wish to minimize

$$J_\varepsilon(x,u) \equiv \int_0^T \{g[t,x(t),u(t)] + \frac{1}{2\varepsilon}\,|\dot{x}(t) - f[t,x(t),u(t)]|^2\}dt \tag{2.9}$$

over the class $\mathcal{O}^+$.

Recall that $J(x,u) = \int_0^T g[t,x(t),u(t)]dt$. The following condition will be a hypothesis in several of the theorems that follow:

$$\liminf_{\nu \to \infty} J(x_\nu,u_\nu) \geq J(x_0,u_0), \tag{2.10}$$

<u>for an arbitrary sequence</u> $\{(x_\nu,u_\nu)\}$ <u>in</u> $\mathcal{O}^+$ <u>such that</u> x_ν <u>converges uniformly to a limit</u> x_0 <u>and</u> u_ν <u>converges weakly to a limit</u> u_0.

By definition [10, p. 270] a sequence $\{u_\nu \in L_2([0,T])\}$ converges weakly to $u_0 \in L_2([0,T])$ if

$$\int_0^T (u_\nu - u_0)\phi \to 0 \text{ \underline{with} } 1/\nu \text{ \underline{for all} } \phi \in L_2([0,T]).$$

2.4 <u>Existence of a Solution of the Auxiliary Problem</u>

This section is devoted to showing that a solution for the ε-problem exists, that is, there is a minimizing pair $[x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon)]$ in $\mathcal{O}^+$ for the problem $J_\varepsilon(x,u)$ = <u>global minimum</u> on $\mathcal{O}^+$.

Balakrishnan has no condition corresponding to $(2.4)^+$ in his paper [2]. However, in a related paper, [3, p. 373] he included the condition

$$g(t,x,u) \geq 0, \text{ \underline{for all} } (t,x,u) \in [0,T] \times R^n \times R^p, \tag{2.11}$$

which is certainly sufficient for $(2.4)^+$. Either (2.11), $(2.4)^+$ or some other similar condition seems to be essential to ensure the existence of a minimizing sequence for the auxiliary problem.

It will be notationally convenient at times to suppress the fixed ε, that is, $x(\cdot,\varepsilon)$, $\dot{x}(\cdot,\varepsilon)$ and $u(\cdot,\varepsilon)$ may be written as $x(\cdot)$, $\dot{x}(\cdot)$, and $u(\cdot)$, respectively.

Let $\{\varepsilon_k\}$ denote a strictly decreasing sequence of real numbers which converges to zero. Let ε_ℓ be a fixed member of $\{\varepsilon_k\}$ and set

$$h(\varepsilon_\ell) \equiv \inf\{J_{\varepsilon_\ell}(x,u) : (x,u) \in \mathcal{O}^+\}.$$

It follows from $(2.4)^+$ and the definition of $J_\varepsilon(x,u)$ in (2.9) that $h(\varepsilon_\ell)$ is nonnegative. The form of $J_{\varepsilon_\ell}(x,u)$ and the continuity of f and g make it clear that there exists a pair (y,v) in $\mathcal{O}^+$ such that $J_{\varepsilon_\ell}(y,v)$ is finite so there necessarily exists a <u>minimizing sequence</u> $\{(x_\nu,u_\nu)\}$, that is, a sequence such that

$$\lim_{\nu\to\infty} J_{\varepsilon_\ell}(x_\nu,u_\nu) = h(\varepsilon_\ell). \tag{2.12}$$

LEMMA 2.1. <u>Let</u> $\{(x_\nu,u_\nu)\}$ <u>denote a minimizing sequence for the problem</u> $J_{\varepsilon_\ell}(x,u)$ = <u>global minimum on</u> $\mathcal{O}^+$ <u>and let</u> $\{u_\nu\}$ <u>be the corresponding sequence of admissible controls. Then there exists a subsequence of</u> $\{u_\nu\}$ <u>that converges weakly to an admissible control</u> u_0.

<u>Proof</u>. Let j be a fixed integer in the set $\{1,2,\cdots,p\}$. By condition (2.8)(iv) we have that

$$\|u_\nu^j\| \equiv \left(\int_0^T [u_\nu^j(t)]^2 dt\right)^{1/2} \le b_1^j\, T^{1/2},$$

where $\|\cdot\|$ denotes the L_2 norm. The weak compactness [10, p. 275] of a closed ball in L_2 ensures that there is a subsequence of $\{u_\nu^j\}$, which we again denote by $\{u_\nu^j\}$, and which converges weakly to a limit v_0^j in the given ball. Let E_+ and E_- denote subsets of $[0,T]$ consisting of points t such that

$$v_0^j(t) > b_1^j \quad \underline{\text{and}} \quad v_0^j(t) < -b_1^j,$$

respectively. If $\{u_\nu^j\}$ converges weakly to a limit v_0^j then by definition, [10, p. 270], we have that

$$\lim_{\nu \to \infty} \int_0^T u_\nu^j \phi = \int_0^T v_0^j \phi, \quad \underline{\text{for}} \quad \underline{\text{all}} \quad \phi \in L_2([0,T]).$$

Choose ϕ as the characteristic function χ_{E_+}. Then

$$\int_0^T u_\nu^j \chi_{E_+} = \int_{E_+} u_\nu^j,$$

$$\int_0^T v_0^j \chi_{E_+} = \int_{E_+} v_0^j,$$

and

$$\lim_{\nu \to \infty} \int_{E_+} u_\nu^j = \int_{E_+} v_0^j. \tag{2.13}$$

If E_+ has positive Lebesgue measure, $\lambda(E_+)$, condition (2.8)(iv) requires that the left member of (2.13) be less than or equal to $b_1^j \lambda(E_+)$, while by our definition of E_+ the right member is greater than $b_1^j \lambda(E_+)$. This is a contradiction unless $\lambda(E_+) = 0$. By a similar argument it is necessary that $\lambda(E_-) = 0$. We now define

$$u_0^j(t) \equiv \begin{cases} v_0^j(t) & \text{if} \quad |v_0^j(t)| \le b_1^j, \\ 0 & \text{if} \quad |v_0^j(t)| > b_1^j. \end{cases}$$

Then $u_0^j(t)$ is equivalent to $v_0^j(t)$, has property (2.8)(iv) and is also the weak limit of the sequence $\{u_\nu^j\}$.

The integer j was an arbitrary but fixed member of the set $\{1,2,\cdots,p\}$ so the above argument could be applied with $j = 1$ to obtain a first subsequence of $\{(x_\nu,u_\nu)\}$ (retain the same notation) such that $\{u_\nu^1\}$ converges weakly to u_0^1. Then the argument could be applied again to obtain a subsequence of the preceding subsequence (again retaining the same notation) such that $\{u_\nu^2\}$ converges to u_0^2. Repeating this procedure we finally obtain a p^{th} subsequence which provides weak convergence of all components of u_ν to those of u_0 and hence by definition the weak convergence of u_ν to $u_0 = (u_0^1,\cdots,u_0^p)$.

THEOREM 2.1. _Let the following condition hold: if_ $(y,w) \in \mathcal{D}^+$ _then_

$$\lim_{\rho\to\infty}\inf \int_0^T |\dot{x}_\rho(t) - f[t,x_\rho(t),u_\rho(t)]|^2 dt \geq \int_0^T |\dot{y}(t) - f[t,y(t),w(t)]|^2 dt, \tag{2.14}$$

for every sequence $\{(x_\rho,u_\rho)\}$ _of admissible pairs in_ $\mathcal{D}^+$ _such that the corresponding sequence_ $\{x_\rho\}$ _of admissible responses converges uniformly to_ y _on_ $[0,T]$ _and the corresponding sequence_ $\{u_\rho\}$ _of admissible controls converges weakly to_ w _on_ $[0,T]$_. If, in addition, condition_ (2.10) _holds then there exists an optimizing pair for the_ ε_ℓ_-problem,_ $J_{\varepsilon_\ell}(x,u) =$ _global minimum on_ $\mathcal{D}^+$.

Proof. Let $\{(x_\nu,u_\nu)\}$, a minimizing sequence for the ε_ℓ-problem, be chosen so that the corresponding sequence $\{u_\nu\}$ of admissible controls converges weakly to an admissible control u_0 as in Lemma 2.1. Let i be a fixed integer in the set $\{1,2,\cdots,n\}$, and let $\{x_\nu^i\}$ denote the sequence of i^{th} components of the admissible responses in the minimizing sequence $\{(x_\nu,u_\nu)\}$.

Define

$$\dot{z}_\nu \equiv \dot{x}_\nu(t) - f[t,x_\nu(t),u_\nu(t)].$$

Then, from (2.9) and (2.12) we have that

$$\lim_{\nu\to\infty} \left\{\int_0^T g[t,x_\nu(t),u_\nu(t)]dt + \frac{1}{2\varepsilon_\ell}\int_0^T |\dot{z}_\nu(t)|^2 dt\right\} = h(\varepsilon_\ell). \tag{2.15}$$

Since $h(\varepsilon_\ell)$ is finite and the first integral in (2.15) is nonnegative by (2.4), we can suppose the sequence $\{(x_\nu,u_\nu)\}$ and hence the sequence $\{\dot{z}_\nu\}$ to have been so chosen that the second term in (2.15) also converges. Hence $\int_0^T |\dot{z}_\nu(t)|^2 dt$ is uniformly bounded. Conditions (2.8)(iii) and (2.8)(iv) require that $|x_\nu(t)|$ and $|u_\nu(t)|$ be uniformly bounded independently of ν and t. These conditions, the continuity of f and the fact that

$$|\dot{x}_\nu^i(t)| = |\dot{z}_\nu^i(t) + f^i[t,x_\nu(t),u_\nu(t)]| \le |\dot{z}_\nu^i(t)| + |f^i[t,x_\nu(t),u_\nu(t)]|,$$

can be used together with the Cauchy-Buniakowski-Schwarz (CBS) inequality to obtain the existence of a constant, say b_4^i, such that,

$$\int_0^T [\dot{x}_\nu^i(t)]^2 dt \le b_4^i. \tag{2.16}$$

Now, for t_1 and t_2 in $[0,T]$, with $t_1 < t_2$,

$$|x_\nu^i(t_2) - x_\nu^i(t_1)|^2 = \left|\int_{t_1}^{t_2} \dot{x}_\nu^i(t)dt\right|^2 \le (t_2 - t_1)\int_0^T |\dot{x}_\nu^i(t)|^2 dt,$$

where the last inequality follows from the CBS inequality. Thus, the x_ν^i's are equicontinuous and equally bounded and, by the Arzelà-Ascoli theorem, there exists a subsequence of $\{x_\nu^i(t)\}$ (denote the subsequence by the same symbol $\{x_\nu^i(t)\}$) that converges uniformly to a limit, say $x_0^i(t,\varepsilon_\ell)$. It is shown in [1, pp. 133-134] that $x_0^i(t,\varepsilon_\ell)$ is AC on $[0,T]$. From the weak compactness theorem [10, p. 275] and with reference to (2.16), a further subsequence can be so chosen, for which we again retain the same notation, so that the sequence $\{\dot{x}_\nu^i(t)\}$ of derivatives converges weakly to some limit, which we denote by $\dot{y}^i(t)$. Then, the equalities

$$x_0^i(t,\varepsilon_\ell) - x_0^i(0,\varepsilon_\ell) = x_0^i(t,\varepsilon_\ell) - a_0^i = \lim_{\nu\to\infty}\int_0^t \dot{x}_\nu^i(s,\varepsilon_\ell)ds = \int_0^t \dot{y}^i(s,\varepsilon_\ell)ds,$$

imply via a standard theorem [10, p. 241] that

$$\dot{x}_0^i(t,\varepsilon_\ell) = \dot{y}^i(t,\varepsilon_\ell) \quad \text{a.e. } \underline{\text{on}}\ [0,T].$$

The above argument holds for every choice of i and hence the selection of appropriate subsequences can be repeated for each of the n components. Observe that $x_0 \equiv (x_0^1,\cdots,x_0^n)$ is AC and also satisfies (2.8)(iii). Moreover, Theorem 8.1 of [17, p. 612] can be applied in order to conclude that the integral $\int_0^T |\dot{x}_\nu^i(t)|^2 dt$ is lower semicontinuous on the set $\{x : x \text{ is AC, } x(0) = a_0,\ \dot{x} \in L_2([0,T])\}$, hence

$$\int_0^T |\dot{x}_0^i(t)|^2 dt \le \liminf_{\nu\to\infty}\int_0^T |\dot{x}_\nu^i(t)|^2 dt \le b_4^i,$$

so that $\dot{x}_0 \in L_2([0,T])$ as required of admissible responses. Thus x_0 is an admissible response.

The definition of $h(\varepsilon)$ and hypotheses (2.10) and (2.14) can now be used to obtain that

$$\text{(2.17)}\quad h(\varepsilon_\ell) = \lim_{\nu\to\infty}\{\int_0^T g[t,x_\nu(t),u_\nu(t)]dt + \frac{1}{2\varepsilon_\ell}\int_0^T|\dot{x}_\nu(t) - f[t,x_\nu(t),u_\nu(t)]|^2 dt\} =$$

$$\liminf_{\nu\to\infty}\{\int_0^T g[t,x_\nu(t),u_\nu(t)]dt + \frac{1}{2\varepsilon_\ell}\int_0^T|\dot{x}_\nu(t) - f[t,x_\nu(t),u_\nu(t)]|^2 dt\} \ge$$

$$\int_0^T g[t,x_0(t),u_0(t)]dt + \frac{1}{2\varepsilon_\ell}\int_0^T |\dot{x}_0(t) - f[t,x_0(t),u_0(t)]|^2 dt \ge h(\varepsilon_\ell).$$

It is clear that equality must hold throughout in the preceding. Therefore, (x_0,u_0) or $[x_0(\cdot,\varepsilon_\ell),u_0(\cdot,\varepsilon_\ell)]$ in the more complete notation is a minimizing pair for the ε_ℓ-problem and the proof of the theorem is complete.

The hypotheses (2.10) and (2.14) are very strong so the important question remains, for what functions f and g can we be assured of these semicontinuity properties.

Observe that the admissible controls u play the role of derivatives in the classical formulation of variational problems. Since each admissible u is in $L_2([0,T])$, it is in $L_1([0,T])$, so we can define

$$(2.18) \qquad v(t) \equiv \int_0^t u(\tau)d\tau, \quad \underline{\text{for}} \quad t \in [0,T].$$

Then, $\dot{v}(t) = u(t)$ a.e. on $[0,T]$ by a standard theorem. Moreover,

$$(2.19) \qquad v^j(t_2) - v^j(t_1) = \int_{t_1}^{t_2} u^j(\tau)d\tau,$$

and from the boundedness (2.8)(iv) of the admissible u's we have that

$$(2.20) \qquad |v^j(t_2) - v^j(t_1)| \leq b_1^j|t_2 - t_1| \quad \underline{\text{for}} \quad j = 1,\cdots,p,$$

and

$$(2.21) \qquad |v(t_2) - v(t_1)| \leq B|t_2 - t_1| \quad \underline{\text{where}} \quad B = [\sum_{j=1}^{p} (b_1^j)^2]^{1/2}.$$

Thus our integral (2.2) can now be written $J(x,v) = \int_0^T g[t,x(t),\dot{v}(t)]dt$ and the penalty term can be written

$$\frac{1}{2\varepsilon_\ell}\int_0^T |\dot{x}(t) - f[t,x(t),\dot{v}(t)]|^2 dt.$$

In this notation $J_{\varepsilon_\ell}(x,v)$ can now be regarded as a nonparametric integral in $(n+p+1)$-space with an integrand in $(t,x,v,\dot{x},\dot{v})$ that is free of v.

We have already shown that successive refinements of a given minimizing sequence $\{(x_\nu,u_\nu)\}$ can be made to obtain a sequence such that x_ν converges uniformly to an admissible x_0 and u_ν converges weakly to an admissible u_0. Let v be the integral (2.18) of u_ν. Then all the functions v_ν are equilipschitzian by (2.21), hence they are equally bounded and equicontinuous and the Arzelà-Ascoli theorem assures a subsequence converging uniformly to a limit v_0. For lower semicontinuity in terms of uniform convergence of pairs (x_ν,v_ν) to a limit pair (x_0,v_0) in a class of vector-valued functions (x,v) of equally bounded total variation, convexity of $g(t,x,\dot{v})$ in $\dot{v}$ suffices for (2.10) and likewise convexity of $|\dot{x} - f(t,x,\dot{v})|^2$ in $(\dot{x},\dot{v})$ suffices for (2.14) as shown by McShane in [17, p. 612]. That all components of admissible responses x_ν appearing in a minimizing sequence $\{(x_\nu,u_\nu)\}$ have appropriately bounded total variation is asserted by (2.16). Similarly, for the components of u_ν, we have $\dot{v}_\nu = u_\nu$ from above and the u_ν's are bounded by (2.8)(iv).

Thus the conclusion of Theorem 2.1 applies at least to all integrals J_{ε_ℓ} such that g is convex in $\dot{v}$ and such that f is linear in $\dot{v} = u$, that is,

(2.22) $$f^i(t,x,u) = r^i(t,x) + s^{ij}(t,x)u^j, \text{ \underline{where} } i = 1,\cdots,n$$
$$\text{\underline{and summation is on} } j \text{ \underline{for} } j = 1,\cdots,p.$$

We call attention at this juncture to two things. We have made a concerted effort to verify the conclusion stated by Balakrishnan at the bottom of his page 169 in [2]. This would establish the existence of a solution to the ε_ℓ-problem under weaker conditions than (2.10) and (2.14). However, the difficulty expressed in the present notation is that $\dot{x}_\nu$ is known only to converge weakly on [0,T] to $\dot{x}_0$. Balakrishnan in a later reference, namely [4, p. 366] assumes a condition which encompases our conditions (2.10) and (2.14). Also, in order to prove existence of a minimizing pair for a similar ε-problem, S. De Julio, a former student of Balakrishnan, assumes conditions very similar to (2.10) and (2.14) on pages 11 and 15 of his 1968 Ph.D. Dissertation, <u>Study of a New Computing Technique for Distributed Parameter Systems</u>.

We remark secondly that even with restriction to functions f that are linear in u we have verified the major conclusions of Balakrishnan for a larger class of functions f than those treated in Balakrishnan's Section 2 of [2] where f is linear in both x and u.

2.5 Approximate Solution of the Control Problem

Again let $\{\varepsilon_k\}$ be a strictly decreasing sequence of positive reals that converges to zero and recall the definition of $h(\varepsilon_k)$ preceding Lemma 2.1. If solutions of the ε_k-problems exist for $k = 1,2,\cdots$, then the following theorem shows that these solutions can be used to approximate the infimum of $J(x,u)$ on $\mathcal{P}$.

THEOREM 2.2 *Let $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$, $k = 1,2,3,\cdots$, denote a sequence of minimizing pairs in $\mathcal{P}^+$ for the ε_k-problems. Let $\{\hat{x}(\cdot,\varepsilon_k)\}$ be the sequence of unique solutions on the full interval $[0,T]$ of $\dot{x} = f[t,x,u_0(t,\varepsilon_k)]$, satisfying $x(0) = a_0$ (this sequence is ensured by a standard existence theorem [18, pp. 342-346]). Then*

$$\text{(2.23)}\qquad \lim_{k\to\infty} h(\varepsilon_k) = \inf\{J(x,u) : (x,u) \in \mathcal{P}\} =$$

$$\lim_{k\to\infty}\int_0^T g[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt =$$

$$\lim_{k\to\infty}\int_0^T g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt.$$

Proof. Select ε_ℓ and ε_m from $\{\varepsilon_k\}$ such that $0 < \varepsilon_m < \varepsilon_\ell$, but which are otherwise arbitrary. Let

$$\dot{z}_0(t,\varepsilon_k) \equiv \dot{x}_0(t,\varepsilon_k) - f[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)],$$

$$F(\varepsilon_k) \equiv \int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt,$$

and

$$G(\varepsilon_k) \equiv \int_0^T g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt.$$

By hypothesis, $[x_0(\cdot,\varepsilon_m),u_0(\cdot,\varepsilon_m)]$ denotes a minimizing pair for the ε_m-problem, $J_{\varepsilon_m}(x,u)$ = <u>global minimum on</u> $\mathcal{P}^+$. Thus, we have that

$$J_{\varepsilon_m}[x_0(\cdot,\varepsilon_m),u_0(\cdot,\varepsilon_m)] \le J_{\varepsilon_m}[x_0(\cdot,\varepsilon_\ell),u_0(\cdot,\varepsilon_\ell)].$$

This latter inequality can be written as

$$\text{(2.24)} \qquad \frac{1}{2\varepsilon_m} F(\varepsilon_m) + G(\varepsilon_m) \le \frac{1}{2\varepsilon_m} F(\varepsilon_\ell) + G(\varepsilon_\ell).$$

Similarly,

$$J_{\varepsilon_\ell}[x_0(\cdot,\varepsilon_\ell),u_0(\cdot,\varepsilon_\ell)] \le J_{\varepsilon_\ell}[x_0(\cdot,\varepsilon_m),u_0(\cdot,\varepsilon_m)],$$

that is,

$$\text{(2.25)} \qquad \frac{1}{2\varepsilon_\ell} F(\varepsilon_\ell) + G(\varepsilon_\ell) \le \frac{1}{2\varepsilon_\ell} F(\varepsilon_m) + G(\varepsilon_m).$$

Since $0 < \varepsilon_m < \varepsilon_\ell$ it follows from inequalities (2.24) and (2.25) that

$$\text{(2.26)} \qquad F(\varepsilon_m) \le F(\varepsilon_\ell)$$

and that

$$\text{(2.27)} \qquad G(\varepsilon_m) \ge G(\varepsilon_\ell).$$

Alternately stated, $\{F(\varepsilon_k)\}$ and $\{G(\varepsilon_k)\}$ are both monotonic sequences. Hence from the definitions of F and G we conclude that

$$\lim_{k\to\infty}\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2dt \quad \underline{\text{and}} \quad \lim_{k\to\infty}\int_0^T g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt$$

both exist.

Observe next that, since $\mathcal{P}^+ \supset \mathcal{P}$, then

(2.28) $$\inf\{J_{\varepsilon_k}(x,u) : (x,u) \in \mathcal{P}^+\} \le \inf\{J_{\varepsilon_k}(x,u) : (x,u) \in \mathcal{P}\}.$$

But $\dot{x} = f[t,x,u(t)]$ a.e. on $[0,T]$ for every pair (x,u) in $\mathcal{P}$, so that, by definition (2.9) of J_ε, $J_{\varepsilon_k}(x,u) = J(x,u)$ and hence

(2.29) $$\inf\{J_{\varepsilon_k}(x,u) : (x,u) \in \mathcal{P}\} = \inf\{J(x,u) : (x,u) \in \mathcal{P}\}.$$

It follows from (2.28) and (2.29) that

(2.30) $$\inf\{J_{\varepsilon_k}(x,u) : (x,u) \in \mathcal{P}^+\} \le \inf\{J(x,u) : (x,u) \in \mathcal{P}\}.$$

In the proof of Theorem 2.1 it was shown that there is an admissible pair $[x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]$ such that

$$\inf\{J_{\varepsilon_k}(x,u) : (x,u) \in \mathcal{P}^+\} = J[x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)] + \frac{1}{2\varepsilon_k}\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt,$$

so we have that

(2.31) $$J[x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)] + \frac{1}{2\varepsilon_k}\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt \le \inf\{J(x,u) : (x,u) \in \mathcal{P}\}.$$

The right-hand member of (2.31) is free of k and is necessarily finite. Let $k \to \infty$ in the left-hand member of (2.31) and use the monotoneity of $G(\varepsilon_k)$. It follows that

(2.32) $$\lim_{k\to\infty} J[x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)] \le \inf\{J(x,u) : (x,u) \in \mathcal{P}\}.$$

If $\lim\limits_{k\to\infty}\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt$ were positive, then $\lim\limits_{k\to\infty}\frac{1}{2\varepsilon_k}\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt = \infty$

and we have a contradiction. It must be inferred that

(2.33) $$\lim_{k\to\infty}\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt = 0.$$

Recall that $\hat{x}(\cdot,\varepsilon_k)$ is the unique solution on $[0,T]$ of the initial value problem (2.1)(ii), (iii) with $u(t) = u_0(t,\varepsilon_k)$. By the mean value theorem there exists a $\theta(t,\varepsilon_k)$ in the open interval $(0,1)$ such that

$$g[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)] - g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)] =$$

$$g_{x^i}[t,x_0(t,\varepsilon_k) + \theta(t,\varepsilon_k)\overline{\hat{x}(t,\varepsilon_k) - x_0(t,\varepsilon_k)},u_0(t,\varepsilon_k)]\cdot[\hat{x}^i(t,\varepsilon_k) - x_0^i(t,\varepsilon_k)],$$

with summation on i from 1 to n. Since each $u_0(t,\varepsilon_k)$ satisfies (2.8)(iv) it follows that $|u_0(t,\varepsilon_k)|$ is uniformly bounded independently of t and k. For each ε_k we have that $[\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)] \in \mathcal{P}$ so that from (2.7) and the discussion immediately preceding (2.7) we can conclude that $|\hat{x}(t,\varepsilon_k)|$ is bounded independently of t and k. These boundedness conditions and the continuity of the g_{x^i}'s are sufficient for the existence of a constant M_1 such that

(2.34) $$|g[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)] - g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]| \le M_1|\hat{x}(t,\varepsilon_k) - x_0(t,\varepsilon_k)|,$$

on $[0,T]$. Next, let

$$y(t,\varepsilon_k) \equiv \hat{x}(t,\varepsilon_k) - x_0(t,\varepsilon_k).$$

Then, in a manner somewhat similar to the preceding, it can be shown that

$$\dot{y}(t,\varepsilon_k) = f[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)] - f[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)] - \dot{z}_0(t,\varepsilon_k)$$

can be written as

(2.35) $$\dot{y}(t,\varepsilon_k) = M(t)y(t,\varepsilon_k) - \dot{z}_0(t,\varepsilon_k)$$

where $M(t)$ is uniformly bounded on $[0,T]$ independently of ε_k. Solving the first order linear differential equation (2.35) by the familiar formula leads to the existence of a constant M_2 such that

$$(2.36) \qquad |y(t,\varepsilon_k)| \le M_2 \left(\int_0^T |\dot{z}_0(t,\varepsilon_k)|^2 dt\right)^{1/2},$$

on $[0,T]$. Inequality (2.36) together with (2.33) implies that

$$\lim_{k\to\infty} |y(t,\varepsilon_k)| = 0,$$

so it follows from (2.34) and the definition of $y(t,\varepsilon_k)$ that

$$(2.37) \quad \lim_{k\to\infty} \int_0^T |g[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)] - g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]|dt = 0.$$

Hence,

$$(2.38) \quad \lim_{k\to\infty} \int_0^T g[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt = \lim_{k\to\infty} \int_0^T g[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt.$$

Since $[\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)]$ is a pair in $\mathcal{P}$,

$$(2.39) \qquad \inf\{J(x,u) : (x,u) \in \mathcal{P}\} \le J[\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)].$$

Taking the limit as $k \to \infty$ in (2.39) we obtain that

$$(2.40) \quad \inf\{J(x,u) : (x,u) \in \mathcal{P}\} \le \lim_{k\to\infty} J[\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)] =$$

$$\lim_{k\to\infty} \int_0^T g[t,\hat{x}(t,\varepsilon_k),u_0(t,\varepsilon_k)]dt,$$

and the conclusion of the theorem now follows from (2.32), (2.38) and (2.40).

2.6 Existence of a Minimizing Pair for the Control Problem

Theorem 2.2 is probably adequate for potential applications since the desired infimum can be approximated with any desired accuracy for k sufficiently large. However, one would like to know at this point whether or not there is a minimizing pair for the original control problem. The remainder of this chapter extends the work of Balakrishnan by establishing

(1) the existence of a minimizing pair for the original control problem,

and, under certain convexity conditions, showing both

(2) the uniqueness of such a pair,

and that

(3) for an arbitrary strictly decreasing sequence $\{\varepsilon_k\}$ of positive reals converging to zero the corresponding sequence of minimizing pairs $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$ for the ε_k-problems "converges" to the unique minimizing pair for the original control problem (see Theorem 2.5).

THEOREM 2.3 *Let* $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$ *denote a sequence of minimizing pairs for the* ε_k*-problems*, $k = 1,2,3,\cdots$, *where* $\{\varepsilon_k\}$ *is a strictly decreasing sequence of positive reals converging to zero. If conditions* (2.10) *and* (2.14) *hold then*

(i) *there exists a subsequence of* $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$, *call it* $\{[x_0(\cdot,\varepsilon_{k_\nu}),u_0(\cdot,\varepsilon_{k_\nu})]\}$, *such that the corresponding sequence* $\{x_0(\cdot,\varepsilon_{k_\nu})\}$ *of admissible responses converges uniformly to a limit* x_*,

(ii) *there exists a subsequence of* $\{[x_0(\cdot,\varepsilon_{k_\nu}),u_0(\cdot,\varepsilon_{k_\nu})]\}$, *for which we retain the same notation, such that the corresponding sequence* $\{u_0(\cdot,\varepsilon_{k_\nu})\}$ *of admissible controls converges weakly to a limit* u_*,

(iii) $(x_*,u_*) \in \mathcal{P}$

(iv) $J(x_*,u_*) = \inf\{J(x,u) : (x,u) \in \mathcal{O}\}$.

Proof. For each ε_k it was shown in Theorem 2.1 that a minimizing pair $[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]$ exists and belongs to $\mathcal{O}^+$. Recall that in the proof of Theorem 2.2 we defined

$$F(\varepsilon_k) = \int_0^T |\dot{x}_0(t,\varepsilon_k) - f[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]|^2 dt$$

and it was shown that $F(\varepsilon_k)$ is non-increasing as $k \to \infty$. Hence,

$$F(\varepsilon_k) \le F(\varepsilon_1), \text{ for all } k.$$

It follows, for components of $\dot{x}_0(t,\varepsilon_k)$ and $f[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]$, that

$$\int_0^T \{\dot{x}_0^i(t,\varepsilon_k) - f^i[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]\}^2 dt \le F(\varepsilon_1);$$

hence that

$$\|\dot{x}_0^i(t,\varepsilon_k) - f^i[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]\| \le \sqrt{F(\varepsilon_1)}\ ,$$

and by the triangle property of the L_2-norm that

$$\|\dot{x}_0^i(t,\varepsilon_k)\| \le \sqrt{F(\varepsilon_1)} + \|f^i[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]\|. \tag{2.41}$$

Since $x_0(t,\varepsilon_k)$ and $u_0(t,\varepsilon_k)$ are bounded independently of k by conditions (2.8)(iii) and (2.8)(iv) and f^i is continuous by condition (2.3) we have that $\|f^i[t,x_0(t,\varepsilon_k),u_0(t,\varepsilon_k)]\|$ is bounded independently of k. We can now conclude from (2.41) that $\int_0^T [\dot{x}_0^i(t,\varepsilon_k)]^2 dt$ is bounded independently of k.

Let t_1 and t_2 in $[0,T]$ satisfy the inequality $t_1 < t_2$. Then

$$|x_0^i(t_2,\varepsilon_k) - x_0^i(t_1,\varepsilon_k)|^2 = \left|\int_{t_1}^{t_2} \dot{x}_0^i(t,\varepsilon_k)dt\right|^2 \le (t_2 - t_1)\int_0^T |\dot{x}_0^i(t,\varepsilon_k)|^2 dt,$$

where the last inequality follows by the CBS inequality. Thus, the $x_0^i(\cdot,\varepsilon_k)$'s are Hölder continuous of order 1/2, hence equicontinuous and equally bounded and by the Arzelà-Ascoli theorem, there exists a subsequence of $\{x_0^i(\cdot,\varepsilon_k)\}$ (relabel it $\{x_0^i(\cdot,\varepsilon_k)\}$) that converges uniformly to a limit, say x_*^i. The function x_*^i is AC on [0,T] as shown in [1, pp. 133-134]. Since each $x_0^i(\cdot,\varepsilon_k)$ satisfies (2.8)(iii) we have that

$$b_3^i \geq \lim_{k\to\infty} |x_0^i(t,\varepsilon_k)| = |x_*^i(t)|, \ \underline{\text{for}}\ \underline{\text{all}}\ t \in [0,T].$$

Thus, x_*^i satisfies (2.8)(iii). By applying the above procedure for $i = 1,2,\cdots,n$ and by the successive selection of subsequences as in the proof of Lemma 2.1, one obtains finally a subsequence $\{x_0(\cdot,\varepsilon_{k_\nu})\}$ of the original sequence such that every component $x_0^i(\cdot,\varepsilon_{k_\nu})$ converges uniformly to an AC limit x_*^i, $i = 1,2,\cdots,n$, and we denote the vector function with these components by x_*.

Let $\{[x_0(\cdot,\varepsilon_{k_\nu}),u_0(\cdot,\varepsilon_{k_\nu})]\}$ denote the sequence of pairs such that $\{x_0(\cdot,\varepsilon_{k_\nu})\}$ converges uniformly to x_*. Recall that $[x_0(\cdot,\varepsilon_{k_\nu}),u_0(\cdot,\varepsilon_{k_\nu})] \in \mathcal{O}^+$, which requires under (2.8)(iv) that $|u_0(t,\varepsilon_{k_\nu})| \leq b_1^j$. Thus, for any fixed integer j in the set $\{1,2,\cdots,p\}$ we have that

$$\|u_0^j(t,\varepsilon_{k_\nu})\| \equiv \left(\int_0^T |u_0^j(t,\varepsilon_{k_\nu})|^2 dt\right)^{1/2} \leq b_1^j T^{1/2}.$$

Hence, each component $u_0^j(\cdot,\varepsilon_{k_\nu})$ of $u_0(\cdot,\varepsilon_{k_\nu})$ is in the closed ball $B(b_1^j T^{1/2},\theta)$, with center at the zero element θ of $L_2([0,T])$ and radius $b_1^j T^{1/2}$, and so, by the weak compactness theorem [10, p. 275] for such balls, there exists a subsequence (relabel it $\{u_0^j,(\cdot,\varepsilon_{k_\nu})\}$) which converges weakly to a limit, say u_*^j. Repeating the above procedure of selecting appropriate subsequences for each of the p components and retaining the same notation each time we can conclude that $\{u_0(\cdot,\varepsilon_{k_\nu})\}$ converges weakly to u_*. It follows as in Lemma 2.1 that u_* satisfies (2.8)(iv).

It has been shown that x_* is AC, x_* satisfies (2.8)(iii) and that u_* satisfies (2.8)(iv). It is readily verifiable that $\dot{x}_* \in L_2([0,T])$ and that $x_*(0) = a_0$. Thus, $(x_*,u_*) \in \mathcal{P}^+$. We want to show that $(x_*,u_*) \in \mathcal{P}$ so it remains to establish that the differential equation (2.1)(ii) is satisfied by the pair (x_*,u_*), that is, that

$$\dot{x}_* = f[t,x_*,u_*(t)] \quad \text{a.e. on } [0,T].$$

From (2.33) and the definition in the proof of Theorem 2.2 of $\dot{z}_0(t,\varepsilon_{k_\nu})$ we have that

$$\lim_{\nu\to\infty} \int_0^T |\dot{x}_0(t,\varepsilon_{k_\nu}) - f[t,x_0(t,\varepsilon_{k_\nu}),u_0(t,\varepsilon_{k_\nu})]|^2 dt = 0. \tag{2.42}$$

By condition (2.14) we have that

$$\liminf_{\nu\to\infty} \int_0^T |\dot{x}_0(t,\varepsilon_{k_\nu}) - f[t,x_0(t,\varepsilon_{k_\nu}),u_0(t,\varepsilon_{k_\nu})]|^2 dt \geq \int_0^T |\dot{x}_*(t) - f[t,x_*(t),u_*(t)]|^2 dt. \tag{2.43}$$

Clearly, the right-hand member of (2.43) is nonnegative so it follows from (2.42) and (2.43) that

$$\int_0^T |\dot{x}_*(t) - f[t,x_*(t),u_*(t)]|^2 dt = 0. \tag{2.44}$$

Hence $\dot{x}_*(t) = f[t,x_*(t),u_*(t)]$ a.e. on $[0,T]$. Thus condition (2.1)(ii) is satisfied by (x_*,u_*) and we conclude that the pair (x_*,u_*) is in $\mathcal{P}$.

2.7 A Convex Control Problem

Theorem 2.3 states the existence of a solution for the original control problem of Section 2.2. A uniqueness theorem is given in this section for the control problem of Section 2.2 with certain additional convexity assumptions.

The functional J is said to be *strictly convex at* (x,u) *in* $\mathcal{P}$ *relative to* $\mathcal{P}$ if corresponding to (x,u) and to every $(y,v) \in \mathcal{P}$, $(y,v) \neq (x,u)$ there is a positive number $e(x,u,y,v) \leq 1$ such that $[x + \tau(y-x), u + \tau(v-u)] \in \mathcal{P}$ and

$$(2.45) \qquad J(x,u) + \tau[J(y,v) - J(x,u)] > J[x + \tau(y-x),\ u + \tau(v-u)]$$

whenever τ is in the open interval $(0,e(x,u,y,v))$. Observe that the special case in which $J(x,u)$ is convex in (x,u) in the ordinary sense occurs if $e(x,u,y,v) = 1$ for all quadruples (x,u,y,v).

THEOREM 2.4. *Let the conditions in the formulations of the control problem and the auxiliary problem of Sections* 2.2 *and* 2.3 *hold. Let conditions* (2.10) *and* (2.14) *hold. Let* $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$, *a sequence of minimizing pairs for the* ε_k*-problems*, $k = 1,2,\cdots$, *be chosen so that it converges in the sense of Theorem* 2.3 *to a minimizing pair* (x_*,u_*) *for* J *on* $\mathcal{P}$. *If* J *is strictly convex at* (x_*,u_*) *relative to* $\mathcal{P}$ *then* (x_*,u_*) *is unique*.

Proof. From Theorem 2.3 we have that

$$(2.46) \qquad \inf\{J(x,u) : (x,u) \in \mathcal{P}\} = J(x_*,u_*).$$

Suppose that (x_{**},u_{**}) is a pair in $\mathcal{P}$ distinct from (x_*,u_*) such that

$$(2.47) \qquad \inf\{J(x,u) : (x,u) \in \mathcal{P}\} = J(x_{**},u_{**}).$$

The strict convexity of J at (x_*,u_*) yields that

$$(2.48)\qquad J(x_*,u_*) + \tau[J(x_{**},u_{**}) - J(x_*,u_*)] >$$

$$J[x_* + \tau(x_{**} - x_*),u_* + \tau(u_{**} - u_*)],$$

provided that $0 < \tau < e(x_*,u_*,x_{**},u_{**})$. By (2.46) and (2.47) we have that

$$J(x_*,u_*) = J(x_{**},u_{**})$$

so that (2.48) can be written as

$$(2.49)\qquad J(x_*,u_*) > J[x_* + \tau(x_{**} - x_*),u_* + \tau(u_{**} - u_*)],$$

provided that $0 < \tau < e(x_*,u_*,x_{**},u_{**})$. However, (2.49) now contradicts the hypothesis (2.46) that (x_*,u_*) minimizes $J(x,u)$ on $\mathcal{P}$, and we must conclude that (x_*,u_*) is unique.

The proof of Theorem 2.3 makes substantial use of subsequences of sequences of minimizing pairs for the ε_k-problems and this makes it difficult if not impossible to use this theorem in formulating a numerical procedure for approximating x_* and u_* on a grid of t values. However, with the added convexity hypothesis of Theorem 2.4, the following theorem shows that to obtain the optimal pair (x_*,u_*) it is sufficient to consider any sequence of minimizing pairs for the ε_k-problems, $k = 1,2,\cdots$, where $\{\varepsilon_k\}$ is an arbitrary strictly decreasing sequence of positive reals converging to zero.

THEOREM 2.5. <u>Let the conditions in the formulations of the control problem and the auxiliary problem of Sections 2.2 and 2.3 hold. Let $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$ denote a sequence of minimizing pairs for the problems $J_{\varepsilon_k}(x,u)$ = global minimum on $\mathcal{P}^+$, $k = 1,2,\cdots$, with $\{x_0(\cdot,\varepsilon_k)\}$ and $\{u_0(\cdot,\varepsilon_k)\}$ denoting the corresponding sequences of admissible responses and admissible controls, respectively. Further, let conditions</u>

(2.10) *and* (2.14) *hold and let* (x_*,u_*) *denote a minimizing pair for* J *on* $\mathcal{P}$. *If* J *is strictly convex at* (x_*,u_*) *relative to* $\mathcal{P}$ *then*

(i) *the sequence* $\{x_0(\cdot,\varepsilon_k)\}$ *converges uniformly to* x_*,

(ii) *the sequence* $\{u_0(\cdot,\varepsilon_k)\}$ *converges weakly to* u_*.

Proof. All of the hypotheses of Theorem 2.4 hold so that (x_*,u_*) is the unique minimizing pair of J on $\mathcal{P}$.

Suppose that it is necessary to select a proper subsequence of $\{[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]\}$ in order to have the convergence of (i) and (ii) in accord with Theorem 2.3. Then we can consider the possibilities in two cases. In the first case we assume that the original sequence $\{x_0(\cdot,\varepsilon_k)\}$ does not converge uniformly to x_*. Then there necessarily exists a subsequence $\{\varepsilon_{k_\ell}\}$ of $\{\varepsilon_k\}$ so that $\{x_0(\cdot,\varepsilon_{k_\ell})\}$ does not have $x_*(\cdot)$ as an accumulation point. We can apply Theorem 2.3 to the sequence $\{[x_0(\cdot,\varepsilon_{k_\ell}),u_0(\cdot,\varepsilon_{k_\ell})]\}$ of minimizing pairs for the ε_{k_ℓ}-problems to obtain another optimal pair for J on $\mathcal{P}$ which contradicts the uniqueness of (x_*,u_*).

In the second case we assume that the original sequence $\{x_0(\cdot,\varepsilon_k)\}$ converges uniformly to x_* but $\{u_0(\cdot,\varepsilon_k)\}$ does not converge weakly to u_*. Then there necessarily exists a subsequence $\{\varepsilon_{k_m}\}$ of $\{\varepsilon_k\}$ so that $\{u_0(\cdot,\varepsilon_{k_m})\}$ does not have u_* as a weak accumulation point. We can then apply Theorem 2.3 to the sequence $\{[x_0(\cdot,\varepsilon_{k_m}),u_0(\cdot,\varepsilon_{k_m})]\}$ of minimizing pairs for the ε_{k_m}-problems to obtain another optimal pair for J on $\mathcal{P}$ which contradicts the uniqueness of (x_*,u_*). Therefore, we conclude that it is unnecessary to consider subsequences of $[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]$ in order to obtain the pair (x_*,u_*).

CHAPTER III

DISCRETE APPROXIMATIONS FOR THE CONTINUOUS AUXILIARY PROBLEM

3.1 Introduction

One must discretize in some way if he hopes to approximate an optimal solution numerically. The auxiliary problem, $J_\varepsilon(x,u)$ = global minimum on $\mathcal{S}^+$, which was described in Section 2.3, is a continuous problem. In this chapter we consider the corresponding discrete auxiliary problem, i.e., the problem obtained by partitioning the fixed interval [0,T] (details described in the next section) and then restricting the domains of the functions of the continuous auxiliary problem to the partition points.

The literature relating a solution of the associated discrete problems to a solution of the original continuous control problem is limited and, in general, it seems merely to be assumed that a desirable relationship exists [8, p. 33]. Balakrishnan's papers [2], [3], [4], and [5] appear to be consistent with this remark since none of them includes any explicit results relating the continuous problem to an associated discrete problem.

The objective of this chapter is to demonstrate that under suitable hypotheses the infimum of an associated discrete auxiliary problem approximates the infimum of the corresponding continuous auxiliary problem.

Some of the ideas for the work in this chapter were derived from [6].

3.2 A Discrete Auxiliary Problem

Let $m = 2^r$ for some fixed positive integer r and let

$$Q_m \equiv \{t_{m0}, t_{m1}, \cdots, t_{mk}, t_{m(k+1)}, \cdots, t_{mm}\},$$

denote a partition of $[0,T]$ such that

$$t_{mk} = Tk/m, \qquad k = 0,1,\cdots,m.$$

The common length, T/m, of each subinterval will be denoted by τ_m.

Let $(x,u) \in \mathcal{P}^+$, the class of pairs defined by (2.8), and let

$$(x_{mk}, u_{mk}) \equiv [x(t_{mk}), u(t_{mk})], \qquad k = 0,1,\cdots,m. \tag{3.1}$$

It follows from (2.8) and (3.1) that

(3.2)

(i) $x_{m0} = a_0$, a constant vector in R^n,

(ii) $|x^i_{mk}| \le b^i_3$, $i = 1,2,\cdots,n$, and $k = 0,1,\cdots,m$,

(iii) $|u^j_{mk}| \le b^j_1$, $j = 1,2,\cdots,p$, and $k = 0,1,\cdots,m$.

Let $x_m = (x^1_m,\cdots,x^n_m)$ and $u_m = (u^1_m,\cdots,u^p_m)$ denote the vector-valued functions of the argument k, taking on values x_{mk} and u_{mk}, respectively, for $k = 0,1,\cdots,m$. Let $\mathcal{P}^+_m$ denote the class of all pairs (x_m,u_m) such that

(3.3)

(i) (x_m,u_m) is the restriction of a pair $(x,u) \in \mathcal{P}^+$ to the discrete domain Q_m via (3.1),

(ii) for all values of m the first component x_m of (x_m,u_m) satisfies the condition that

$$\sum_{k=0}^{m-1} [(x^i_{m(k+1)} - x^i_{mk})/\tau_m]^2 \tau_m \le b^i < \infty, \quad i = 1,\cdots,m,$$

for some set of positive constants $b^1,\cdots,b^m$.

This is a discrete analog of the condition that a class of functions $x: [0,T] \to R^n$ have derivatives in $L_2([0,T])$, namely that

$$\|\dot{x}^i\|^2 = \int_0^T (\dot{x}^i)^2 dt$$

be bounded.

The <u>discrete</u> ε-<u>problem</u> is: for a fixed partition Q_m and a fixed $\varepsilon > 0$, minimize

(3.4) $$J_{\varepsilon,m}(x_m,u_m) \equiv$$

$$\sum_{k=0}^{m-1} [g(t_{mk},x_{mk},u_{mk}) + \frac{1}{2\varepsilon} \left| \frac{x_{m(k+1)} - x_{mk}}{\tau_m} - f(t_{mk},x_{mk},u_{mk}) \right|^2]\tau_m,$$

on $\mathcal{P}_m^+$.

It should be noted that $J_{\varepsilon,m}$ is a function of $m(n+p)$ real variables on a compact subset of $R^{m(n+p)}$ determined by the inequalities (3.2)(ii), (iii). The definition (3.4) of $J_{\varepsilon,m}$ together with the continuity of f and g implies that $J_{\varepsilon,m}$ is continuous on this compact set. Thus, there exists a pair (x_m^*,u_m^*) in $\mathcal{P}_m^+$ such that

$$J_{\varepsilon,m}(x_m^*,u_m^*) = \inf\{J_{\varepsilon,m}(x_m,u_m) : (x_m,u_m) \in \mathcal{P}_m^+\}.$$

3.3 <u>Restriction of J_ε to a Domain Obtained by Extensions of Pairs in $\mathcal{P}_m^+$</u>

Let (x_m,u_m) be a pair in $\mathcal{P}_m^+$. Define a <u>piecewise-linear extension</u> $\tilde{x}_m$ (abbreviated PWLE) of the function x_m from the discrete domain Q_m to $[0,T]$ by setting

$$\tilde{x}_m(t) \equiv x_{mk} + \frac{x_{m(k+1)} - x_{mk}}{\tau_m}(t - t_{mk}), \quad t_{mk} \le t \le t_{m(k+1)},$$

for $k = 0,1,\cdots,m-1$. Also, define a piecewise-constant extension $\tilde{u}_m$ (abbreviated PWCE) of the function u_m by the statement that

$$\tilde{u}_m(t) \equiv u_{mk}, \qquad t_{mk} \le t < t_{m(k+1)},$$

for $k = 0,1,\cdots,m-1$, and $\tilde{u}_m(T) = u_{mm}$. Let

$$\tilde{\mathcal{O}}_m^+ \equiv \{(\tilde{x}_m,\tilde{u}_m) : (x_m,u_m) \in \mathcal{O}_m^+\}.$$

Each component $\tilde{x}_m^i$ of each $\tilde{x}_m$ is a function from $[0,T]$ to R with a broken line graph and each component $\tilde{u}_m^j$ of each $\tilde{u}_m$ is a step function from $[0,T]$ to R.

It is clear from the definition of the PWLE $\tilde{x}_m$ that $\tilde{x}_m$ is AC. Directly from the definition of $\tilde{x}_m$ and conditions (3.2)(i),(ii) we have that $\tilde{x}_m$ satisfies (2.8)(ii),(iii). The definition of $\tilde{x}_m$ and differentiation yield that

$$\dot{\tilde{x}}_m^i(t) = \frac{x^i_{m(k+1)} - x^i_{mk}}{\tau_m} \quad \text{for } t_{mk} < t < t_{m(k+1)},$$

whence,

$$\int_0^T |\dot{\tilde{x}}_m^i(t)|^2 dt = \sum_{k=0}^{m-1} \int_{t_{mk}}^{t_{m(k+1)}} \left| \frac{x^i_{m(k+1)} - x^i_{mk}}{\tau_m} \right|^2 dt \le b^i,$$

where the last inequality follows from (3.3)(ii). Indeed (3.3)(ii) was imposed to ensure this inequality. Consequently, $\dot{\tilde{x}}_m^i(t) \in L_2([0,T])$. Also, note that $\tilde{u}_m$ is Lebesgue measurable and satisfies the inequality of (2.8)(iv). Thus, $\tilde{\mathcal{O}}_m^+ \subset \mathcal{O}^+$ and conditions similar to the hypotheses of Lemma 2.1 and Theorem 2.1 hold with $\mathcal{O}^+$ in each of them replaced now by $\tilde{\mathcal{O}}_m^+$. Hence there exists $(y_0,v_0) \in \mathcal{O}^+$ such that

$$J_\varepsilon(y_0,v_0) = \inf\{J_\varepsilon(\tilde{x}_m,\tilde{u}_m) : (\tilde{x}_m,\tilde{u}_m) \in \tilde{\mathcal{O}}_m^+\},$$

where J_ε is the functional defined by (2.9). Observe that, since $\tilde{\mathcal{O}}_m^+ \subset \mathcal{O}^+$,

$$(3.5) \qquad \inf\{J_\varepsilon(x,u) : (x,u) \in \mathcal{O}^+\} \leq \inf\{J_\varepsilon(\tilde{x}_m,\tilde{u}_m) : (\tilde{x}_m,\tilde{u}_m) \in \tilde{\mathcal{O}}_m^+\}.$$

3.4 <u>A Lemma for Theorem</u> 3.1

For every pair (x,u), (y,v) of pairs in $\mathcal{O}^+$ define a norm

$$(3.6) \qquad N[(x,u),(y,v)] \equiv \|x-y\| + \|\dot{x}-\dot{y}\| + \|u-v\|,$$

where $\|\cdot\|$ again denotes the $L_2([0,T])$ norm.

LEMMA 3.1 <u>If $\|\dot{x}\|$ and $\|\dot{y}\|$ are bounded then the functional J_ε is continuous in the norm (3.6), that is, for every $\alpha > 0$ there exists a number β_α such that if</u>

$$N[(x,u),(y,v)] < \beta_\alpha, \quad \underline{\text{and}} \quad (x,u),(y,v) \in \mathcal{O}^+,$$

<u>then</u>

$$|J_\varepsilon(x,u) - J_\varepsilon(y,v)| < \alpha.$$

<u>Proof</u>. Let (x,u) and (y,v) be in $\mathcal{O}^+$. It follows from definition (2.9) of J_ε that

$$(3.7) \qquad J_\varepsilon(x,u) - J_\varepsilon(y,v) = \int_0^T [g(t,x,u) - g(t,y,v)]dt +$$

$$\frac{1}{2\varepsilon}\int_0^T [|\dot{x}-f(t,x,u)|-|\dot{y}-f(t,y,v)|][|\dot{x}-f(t,x,u)| + |\dot{y}-f(t,y,v)|]dt.$$

This last equation together with the triangle inequality for the euclidean norm yields that

$$(3.8)\quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le \int_0^T |g(t,x,u) - g(t,y,v)|dt +$$

$$\frac{1}{2\varepsilon}\int_0^T |\dot{x}-\dot{y}|[|\dot{x}| + |f(t,x,u)| + |\dot{y}| + |f(t,y,v)|]dt +$$

$$\frac{1}{2\varepsilon}\int_0^T |f(t,x,u) - f(t,y,v)|[|\dot{x}| + |f(t,x,u)| + |\dot{y}| + |f(t,y,v)|]dt.$$

The vector functions x and y are bounded as indicated by (2.8)(iii) whereas u and v are bounded as in (2.8)(iv). The functions $\dot{x}$ and $\dot{y}$ are bounded in the integral sense as given by the hypotheses of the lemma. Application of the CBS inequality to each of the last two integrals in (3.8) yields that

$$(3.9)\quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le \int_0^T |g(t,x,u) - g(t,y,v)|dt +$$

$$\frac{1}{2\varepsilon}\|\dot{x}-\dot{y}\|(\|\dot{x}\| + \|f(t,x,u)\| + \|\dot{y}\| + \|f(t,y,v)\|) +$$

$$\frac{1}{2\varepsilon}\|f(t,x,u) - f(t,y,v)\|(\|\dot{x}\| + \|f(t,x,u)\| + \|\dot{y}\| + \|f(t,y,v)\|).$$

We can conclude from (3.9), the continuity of f and the boundedness (2.8)(iii),(iv) that there exists a positive constant b_5 such that

$$(3.10)\quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le$$

$$\int_0^T |g(t,x,u) - g(t,y,v)|dt + \frac{b_5}{2\varepsilon}(\|\dot{x}-\dot{y}\| + \|f(t,x,u) - f(t,y,v)\|).$$

The mean value theorem can be applied to both the difference in the f terms and the difference in the g terms of this last inequality to obtain that

$$(3.11)\quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le \int_0^T |g_{x^i}(t,z_1,w_1)\cdot(x^i-y^i) + g_{u^j}(t,z_1,w_1)\cdot(u^j-v^j)|dt$$

$$+ \frac{b_5}{2\varepsilon}(\|\dot{x}-\dot{y}\| + \|f_{x^i}(t,z_2,w_2)\cdot(x^i-y^i) + f_{u^j}(t,z_2,w_2)\cdot(u^j-v^j)\|),$$

where summation is on i from 1 to n and on j from 1 to p, and

$$z_k = x + \theta_k(x-y), \ \underline{\text{for}}\ \underline{\text{some}}\ \theta_k \in (0,1),\ k = 1,2,$$

$$w_\ell = u + \phi_\ell(u-v), \ \underline{\text{for}}\ \underline{\text{some}}\ \phi_\ell \in (0,1),\ \ell = 1,2.$$

By application of Minkowski's inequality to the last term in (3.11), we have that

$$(3.12)\quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le \int_0^T |g_{x^i}(t,z_1,w_1)\cdot(x^i-y^i) + g_{u^j}(t,z_1,w_1)\cdot(u^j-v^j)|dt +$$

$$\frac{b_5}{2\varepsilon}(\|\dot{x}-\dot{y}\| + \|f_{x^i}(t,z_2,w_2)\cdot(x^i-y^i)\| + \|f_{u^j}(t,z_2,w_2)\cdot(u^j-v^j)\|).$$

The continuity of each f_{x^i}, $i = 1,2,\cdots,n$, given in (2.3), plus the boundedness of the arguments t, z_2 and w_2 ensure the existence of positive constants b_6^i, $i = 1,\cdots,n$, such that

$$(3.13)\qquad |f_{x^i}(t,z_2,w_2)| \le b_6^i,\quad i = 1,\cdots,n.$$

Similarly, there exist positive constants b_7^j, $j = 1,\cdots,p$, such that

$$(3.14)\qquad |f_{u^j}(t,z_2,w_2)| \le b_7^j,\quad j = 1,\cdots,p.$$

Let $b_8 = \max\{b_6^1,\cdots,b_6^n,b_7^1,\cdots,b_7^p\}$. Then from (3.12), (3.13) and (3.14) we can conclude that

$$(3.15)\quad |J_\varepsilon(x,u)-J_\varepsilon(y,v)| \le \int_0^T |g_{x^i}(t,z_1,w_1)\cdot(x^i-y^i) + g_{u^j}(t,z_1,w_1)\cdot(u^j-v^j)|dt$$

$$+ \frac{b_5}{2\varepsilon}(\|\dot{x}-\dot{y}\| + b_8\sum_{i=1}^{n}\|x^i-y^i\| + b_8\sum_{j=1}^{p}\|u^j-v^j\|).$$

Clearly,

$$\|x^i - y^i\| \le \|x-y\|,\quad i = 1,\cdots,n,$$

and

$$\|u^j - v^j\| \le \|u-v\|,\quad j = 1,\cdots,p.$$

These inequalities together with (3.15) imply that

$$(3.16) \quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le \int_0^T |g_{x^i}(t,z_1,w_1)\cdot(x^i-y^i) + g_{u^j}(t,z_1,w_1)\cdot(u^j-v^j)|\,dt$$

$$+ \frac{b_5}{2\varepsilon}(\|\dot{x} - \dot{y}\| + nb_8\|x - y\| + pb_8\|u - v\|).$$

A similar argument applies to g_{x^i} and g_{u^j} to ensure the existence of a positive constant b_9 so that from (3.16) we can conclude that

$$(3.17) \quad |J_\varepsilon(x,u) - J_\varepsilon(y,v)| \le (nb_9 + \frac{nb_5b_8}{2\varepsilon})\|x - y\| + \frac{b_5}{2\varepsilon}\|\dot{x} - \dot{y}\| +$$

$$(pb_9 + \frac{pb_5b_8}{2\varepsilon})\|u - v\|.$$

Let $\mu = \max\{n(b_9 + \frac{b_5b_8}{2\varepsilon}), \frac{b_5}{2\varepsilon}, p(b_9 + \frac{b_5b_8}{2\varepsilon})\}$. Then if

$$N[(x,u),(y,v)] = \|x - y\| + \|\dot{x} - \dot{y}\| + \|u - v\| < \beta_\alpha = \frac{\alpha}{3\mu},$$

we have from (3.17) that

$$|J_\varepsilon(x,u) - J_\varepsilon(y,v)| < \alpha,$$

which completes the proof of the theorem.

3.5 A Discrete Approximation for the Continuous Problem

Let e be an arbitrary positive number and let $\mathcal{P}_e^+$ denote the class of pairs (x_e,u_e), $x_e = (x_e^1,\cdots,x_e^n)$, $u = (u_e^1,\cdots,u_e^p)$, satisfying the conditions that

(3.18)

(i) x_e is AC on $[0,T]$ and $\dot{x}_e \in L_2([0,T])$,

(ii) $x_e(0) = a_0$, a constant vector in R^n,

(iii) $|x_e^i(t)| \le b_3^i + e$, for all $t \in [0,T]$ and $i = 1,\cdots,n$,

(iv) u_e is Lebesgue measurable on $[0,T]$ and such that

$$|u_e^j(t)| < b_1^j + e, \text{ \underline{for all} } t \in [0,T] \text{ \underline{and} } j = 1,\cdots,p.$$

We proceed as in Sections 3.2 and 3.3 and obtain $\tilde{\mathcal{O}}^+_{m,e}$, a class of piecewise extended pairs $(\tilde{x}_{m,e},\tilde{u}_{m,e})$, where now in lieu of (3.2) we have that

$$\begin{array}{lll} & \text{(i)} & x_{m0,e} = a_0, \text{ \underline{a constant vector in} } R^n, \\ (3.19) & \text{(ii)} & |x^i_{mk,e}| \le b^i_3 + e,\ i = 1,2,\cdots,n \text{ \underline{and} } k = 0,1,\cdots,m, \\ & \text{(iii)} & |u^j_{mk,e}| \le b^j_1 + e,\ j = 1,2,\cdots,p \text{ \underline{and} } k = 0,1,\cdots,m. \end{array}$$

We now compare the pairs $(\tilde{x}_{m,e},\tilde{u}_{m,e})$ in $\tilde{\mathcal{O}}^+_{m,e}$ with pairs $(\tilde{x}_m,\tilde{u}_m)$ in $\tilde{\mathcal{O}}^+_m$. Some of the PWLE components $\tilde{x}^i_{m,e}$ will be such that $|\tilde{x}^i_{m,e}(t)| > b^i_3$ for some values of $t \in [0,T]$ and some of the PWCE components $\tilde{u}^j_{m,e}$ will be such that $|\tilde{u}^j_{m,e}(t)| > b^j_1$ for some values of $t \in [0,T]$. Also, some of the absolute slopes $|\dot{\tilde{x}}^i_{m,e}(t)|$ will exceed $|\dot{\tilde{x}}^i_m(t)|$ for $t \in (0,T)$. Thus, we have that $\tilde{\mathcal{O}}^+_m \subset \tilde{\mathcal{O}}^+_{m,e}$ and $\tilde{\mathcal{O}}^+_m \neq \tilde{\mathcal{O}}^+_{m,e}$.

LEMMA 3.2. <u>Let</u> $\varepsilon > 0$ <u>and</u> m, <u>a natural number, be fixed. Let</u>

$$\tilde{\gamma}^+_{\varepsilon,m} \equiv \inf\{J_\varepsilon(\tilde{x}_m,\tilde{u}_m) : (\tilde{x}_m,\tilde{u}_m) \in \tilde{\mathcal{O}}^+_m\}, \tag{3.20}$$

and

$$\tilde{\gamma}^+_{\varepsilon,m,e} \equiv \inf\{J_\varepsilon(\tilde{x}_{m,e},\tilde{u}_{m,e}) : (\tilde{x}_{m,e},\tilde{u}_{m,e}) \in \tilde{\mathcal{O}}^+_{m,e}\}. \tag{3.21}$$

Then,

$$\lim_{e \to 0} \tilde{\gamma}^+_{\varepsilon,m,e} = \tilde{\gamma}^+_{\varepsilon,m}. \tag{3.22}$$

<u>Proof</u>. There exists a minimizing sequence $\{(\tilde{x}_{m,e,\nu},\tilde{u}_{m,e,\nu})\}$ in $\tilde{\mathcal{O}}^+_{m,e}$ for J_ε such that

$$J_\varepsilon(\tilde{x}_{m,e,\nu},\tilde{u}_{m,e,\nu}) < \tilde{\gamma}^+_{\varepsilon,m,e} + 1/\nu, \quad \nu = 1,2,\cdots. \tag{3.23}$$

Let α be the arbitrary positive number of Lemma 3.1. Then we can suppose e to have been chosen so small that corresponding to each pair in the minimizing sequence there is a pair $(\tilde{x}_{m,\nu},\tilde{u}_{m,\nu})$ in $\tilde{\mathcal{P}}^{+}_{m}$ such that

$$\|\tilde{x}_{m,e,\nu} - \tilde{x}_{m,\nu}\| + \|\dot{\tilde{x}}_{m,e,\nu} - \dot{\tilde{x}}_{m,\nu}\| + \|\tilde{u}_{m,e,\nu} - \tilde{u}_{m,\nu}\|$$

is less than the positive number β_α of Lemma 3.1 for all values $\nu = 1,2,\cdots$. Hence we have by Lemma 3.1 that

$$J_\varepsilon(\tilde{x}_{m,e,\nu},\tilde{u}_{m,e,\nu}) > J_\varepsilon(\tilde{x}_{m,\nu},\tilde{u}_{m,\nu}) - \alpha. \tag{3.24}$$

From (3.20), (3.21), (3.23) and (3.24) it follows that

$$\tilde{\gamma}^{+}_{\varepsilon,m} \leq J_\varepsilon(\tilde{x}_{m,\nu},\tilde{u}_{m,\nu}) < \tilde{\gamma}^{+}_{\varepsilon,m,e} + 1/\nu + \alpha,$$

hence that

$$\tilde{\gamma}^{+}_{\varepsilon,m} < \tilde{\gamma}^{+}_{\varepsilon,m,e} + 1/\nu + \alpha.$$

The natural number ν can be chosen at pleasure and the positive number α can be as near zero as desired provided that e (and hence β_α) is sufficiently small. Thus,

$$\tilde{\gamma}^{+}_{\varepsilon,m} \leq \liminf_{e \to 0} \tilde{\gamma}^{+}_{\varepsilon,m,e}. \tag{3.25}$$

Since $\tilde{\mathcal{P}}^{+}_{m} \subset \tilde{\mathcal{P}}^{+}_{m,e}$, we have that

$$\tilde{\gamma}^{+}_{\varepsilon,m,e} \leq \tilde{\gamma}^{+}_{\varepsilon,m} \quad \underline{\text{for}}\ \underline{\text{all}}\ e > 0,$$

hence

$$\limsup_{e \to 0} \tilde{\gamma}^{+}_{\varepsilon,m,e} \leq \tilde{\gamma}^{+}_{\varepsilon,m}. \tag{3.26}$$

The desired conclusion (3.22) now follows from (3.25) and (3.26).

Recall that Theorem 2.1 ensures the existence of an optimal pair $[x_0(\cdot,\varepsilon),u_0(\cdot,\varepsilon)]$ for the problem $J_\varepsilon(x,u)$ = <u>global minimum on</u> $\mathcal{P}^{+}$.

THEOREM 3.1. <u>Let</u> $\tilde{\gamma}^{+}_{\varepsilon,m}$ <u>denote the infimum of</u> J_ε <u>on</u> $\tilde{\mathcal{P}}^{+}_{m}$ <u>and let</u> (x_0,u_0)

<u>be an optimal pair for</u> J_ε <u>in</u> $\mathcal{O}^+$. <u>Then, for each choice of</u> ε,

$$\lim_{m\to\infty} \gamma^{+}_{\varepsilon,m} = J_\varepsilon(x_0,u_0). \tag{3.27}$$

<u>Proof.</u> Let i be a fixed integer in the set $\{1,\cdots,n\}$. By use of (2.16) it was shown in the proof of Theorem 2.1 that

$$\|\dot{x}_0^i(t)\| \leq (b_4^i)^{1/2}.$$

Thus, $\dot{x}_0^i$ is in the closed ball $B((b_4^i)^{1/2},\theta)$ in $L_2([0,T])$. Using the separability of L_2, let X be a countable set of continuous functions dense in $B((b_4^i)^{1/2},\theta)$. For an example of such a set see [10, p. 270]. Let δ_4^i be an arbitrary positive number. Then there exists a continuous function in X, which can be regarded as the derivative $\dot{y}$ of some y, such that

$$\|\dot{x}_0^i - \dot{y}^i\| < \delta_4^i. \tag{3.28}$$

By definition of the L_2-norm we have that

$$\|\dot{x}_0 - \dot{y}\| = \left(\int_0^T |\dot{x}_0 - \dot{y}|^2 dt\right)^{1/2} = \left[\int_0^T \sum_{i=1}^n (\dot{x}_0^i - \dot{y}^i)^2 dt\right]^{1/2} \leq \tag{3.29}$$

$$\sum_{i=1}^n \left[\int_0^T (\dot{x}_0^i - \dot{y}^i)^2 dt\right]^{1/2}.$$

We can conclude from (3.28) and (3.29) that

$$\|\dot{x}_0 - \dot{y}\| < \sum_{i=1}^n \delta_4^i. \tag{3.30}$$

Inequality (3.28) together with the CBS inequality yields that

$$|x_0^i(t) - y^i(t)| = \left|\int_0^t [\dot{x}_0^i(s) - \dot{y}^i(s)]ds\right| \leq \tag{3.31}$$

$$\int_0^T |\dot{x}_0^i(t) - \dot{y}^i(t)|dt \leq T^{1/2}\|\dot{x}_0^i - \dot{y}^i\| < T^{1/2}\delta_4^i.$$

From this last inequality it follows that

$$\|x_0^i - y^i\| < T\delta_4^i, \tag{3.32}$$

and hence that

$$\|x_0 - y\| < T \sum_{i=1}^{n} \delta_4^i. \tag{3.33}$$

Let j be a fixed positive number in the set $\{1,2,\cdots,p\}$. From condition (2.8)(iv) and the admissibility of u_0 we have that u_0^j is a measurable function on $[0,T]$ such that

$$|u_0^j(t)| \le b_1^j, \quad \underline{\text{for}} \quad t \in [0,T].$$

By Theorem 9.15 in [10, p. 268] there exists a real-valued continuous function v^j on $[0,T]$ such that

$$|v^j(t)| \le b_1^j, \quad \text{for} \quad t \in [0,T], \tag{3.34}$$

and

$$\|u_0^j - v^j\| < \delta_1^j$$

for an arbitrary positive number δ_1^j. It readily follows that

$$\|u_0 - v\| < [\sum_{j=1}^{p} (\delta_1^j)^2]^{1/2}. \tag{3.35}$$

As remarked at the beginning of the present proof, $\|\dot{x}_0^i\|$ is bounded by $(b_4^i)^{1/2}$, hence, $|\dot{x}_0|$ is bounded and from (3.30) we can conclude that $\|\dot{y}\|$ is bounded. Thus, the hypotheses of Lemma 3.1 hold. We have from (3.30), (3.33) and (3.35) that

$$N[(x_0,u_0),(y,v)] = \|x_0 - y\| + \|\dot{x}_0 - \dot{y}\| + \|u_0 - v\| < (1+T)\sum_{i=1}^{n} \delta_4^i + [\sum_{j=1}^{p} (\delta_1^j)^2]^{1/2}.$$

Now, for an arbitrary preassigned positive number $\alpha/2$ and for any choice of the δ's such that

$$(1 + T) \sum_{i=1}^{n} \delta_4^i + [\sum_{j=1}^{p} (\delta_1^j)^2]^{1/2} \le \beta_{\alpha/2},$$

we can conclude from Lemma 3.1 that

$$(3.36) \qquad |J_\varepsilon(x_0,u_0) - J_\varepsilon(y,v)| < \alpha/2.$$

The <u>oscillation of a function</u> h <u>on an interval</u> I is the supremum of all numbers $|h(s) - h(s')|$, with $s,s' \in I$. Let τ_m be so small and hence m so large that the oscillation of the continuous function $\dot{y}^i \in X$ on each of the subintervals $[t_{mk}, t_{m(k+1)}]$, $k = 0,1,\cdots,m-1$, is less than the positive number δ_4^i introduced in (3.28). Inequality (3.31) gives the existence of an $e = T^{1/2}\, \delta_4^i$ such that we can define the functions $\tilde{x}^i_{m,e}$, $i = 1,\cdots,n$, on [0,T] by

$$(3.37) \quad \tilde{x}^i_{m,e}(t) \equiv y^i(t_{mk}) + \frac{y^i(t_{m(k+1)}) - y^i(t_{mk})}{\tau_m}(t - t_{mk}),\ t_{mk} \le t \le t_{m(k+1)},$$

$k = 0,1,\cdots,m-1$. Observe that $\tilde{x}^i_{m,e}(t)$ approximates $y^i(t)$ uniformly on [0,T] with accuracy within δ_4^i and hence it follows that

$$(3.38) \qquad \|y - \tilde{x}_{m,e}\| < T^{1/2}[\sum_{i=1}^{n} (\delta_4^i)^2]^{1/2}.$$

From (3.37) we have that

$$(3.39) \qquad \dot{\tilde{x}}^i_{m,e}(t) = \frac{y^i(t_{m(k+1)}) - y^i(t_{mk})}{\tau_m},\ t_{mk} < t < t_{m(k+1)},$$

$k = 0,1,\cdots,m-1$. The continuity of $\dot{y}^i$ on [0,T] together with the mean value theorem implies the existence of c_k in $(t_{mk}, t_{m(k+1)})$ such that

$$(3.40) \qquad \dot{y}^i(c_k) = \frac{y^i(t_{m(k+1)}) - y^i(t_{mk})}{\tau_m},$$

for $k = 0,1,\cdots,m-1$. Thus, from (3.39) and (3.40) we conclude that

$$(3.41) \qquad \dot{\tilde{x}}^i_{m,e}(t) = \dot{y}^i(c_k),\ \underline{\text{for}}\ \ t_{mk} < t < t_{m(k+1)},$$

$k = 0,1,\cdots,m-1$. Since the oscillation of $\dot{y}^i(t)$ on $[t_{mk},t_{m(k+1)}]$ is less than δ_4^i it now follows from (3.41) that $\dot{\tilde{x}}_{m,e}(t)$ approximates $\dot{y}(t)$ uniformly on $[0,T]$ with accuracy within $[\sum_{i=1}^{n} (\delta_4^i)^2]^{1/2}$, and we have that

$$(3.42) \qquad \|\dot{y} - \dot{\tilde{x}}_{m,e}\| < T^{1/2}[\sum_{i=1}^{n} (\delta_4^i)^2]^{1/2}.$$

Recall that a continuous function v^j was introduced in (3.34). Let τ_m be so small that the oscillation of the continuous function v^j on the subintervals $[t_{mk},t_{m(k+1)}]$, $k = 0,1,\cdots,m-1$, is less than the positive number δ_1^j. Then from (3.34) it is seen that we can define functions $\tilde{u}^j_{m,e}$, $j = 1,\cdots,p$, on $[0,T]$ by

$$u^j_{m,e}(t) \equiv v^j(t_{mk}), \quad t_{mk} \le t < t_{m(k+1)},$$

for $k = 0,1,\cdots,m-1$, and $\tilde{u}^j_{m,e}(T) \equiv v^j(T)$. It follows that

$$(3.43) \qquad \|v - \tilde{u}_{m,e}\| < [\sum_{j=1}^{p} (\delta_1^j)^2]^{1/2}.$$

From (3.38), (3.42) and (3.43) we have that

$$(3.44) \qquad N[(y,v),(\tilde{x}_{m,e},\tilde{u}_{m,e})] = \|y - \tilde{x}_{m,e}\| + \|\dot{y} - \dot{\tilde{x}}_{m,e}\| + \|v - \tilde{u}_{m,e}\| <$$

$$2T^{1/2}[\sum_{i=1}^{n} (\delta_4^i)^2]^{1/2} + [\sum_{j=1}^{p} (\delta_1^j)^2]^{1/2}.$$

Thus, if the δ's are sufficiently small such that

$$2T^{1/2}[\sum_{i=1}^{n} (\delta_4^i)^2]^{1/2} + [\sum_{j=1}^{p} (\delta_1^j)^2]^{1/2} \le \beta_{\alpha/2},$$

we can conclude from Lemma 3.1 that

$$(3.45) \qquad |J_\varepsilon(y,v) - J_\varepsilon(\tilde{x}_{m,e},\tilde{u}_{m,e})| < \alpha/2.$$

By adding (3.36) and (3.45) we have that

$$(3.46) \qquad |J_\varepsilon(x_0,u_0) - J_\varepsilon(\tilde{x}_{m,e},\tilde{u}_{m,e})| < \alpha,$$

provided that $\sum_{i=1}^{n}(\delta_4^i)^2$ and $\sum_{j=1}^{p}(\delta_1^j)^2$ are sufficiently small. From inequality (3.46) and (3.21) we have that

$$\tilde{\gamma}^+_{\varepsilon,m,e} \le J_\varepsilon(\tilde{x}_{m,e},\tilde{u}_{m,e}) \le J_\varepsilon(x_0,u_0) + \alpha.$$

Now from this last inequality and Lemma 3.2 we have that

$$(3.47) \qquad \tilde{\gamma}^+_{\varepsilon,m} = \lim_{e\to 0}\tilde{\gamma}^+_{\varepsilon,m,e} \le J_\varepsilon(x_0,u_0) + \alpha.$$

Finally, from inequality (3.47), the definition of (x_0,u_0) as an optimal pair for J_ε on $\mathcal{P}^+$, the definition (3.20) of $\tilde{\gamma}^+_{\varepsilon,m}$ as the infimum of J_ε on $\tilde{\mathcal{P}}^+_m$, and from the fact that $\tilde{\mathcal{P}}^+_m \subset \mathcal{P}^+$, we have that

$$(3.48) \qquad J_\varepsilon(x_0,u_0) \le \tilde{\gamma}^+_{\varepsilon,m} \le J_\varepsilon(x_0,u_0) + \alpha.$$

Since (3.45) and hence (3.46) and (3.47) hold for a given α provided m is sufficiently large we see that in (3.48) as $\alpha \to 0$ then $m \to \infty$ and the conclusion of the theorem is obtained.

LEMMA 3.3. <u>Given</u> $x_m = \{x_{m0},x_{m1},\cdots,x_{mm}\}$, <u>the first component of a pair</u> $(x_m,u_m) \in \mathcal{P}^+_m$, <u>let</u> $\tilde{x}_m$ <u>be the first component of the corresponding pair in</u> $\tilde{\mathcal{P}}^+_m$. <u>Then</u>

$$\lim_{m\to\infty}|\tilde{x}_{mk} - x_{mk}| = \lim_{m\to\infty}\left|\frac{x_{m(k+1)} - x_{mk}}{\tau_m}(t - t_{mk})\right| = 0,$$

<u>for any</u> $k \in \{0,1,\cdots,m-1\}$ <u>and</u> $t \in [t_{mk},t_{m(k+1)}]$.

<u>Proof</u>. For $t \in [t_{mk},t_{m(k+1)}]$ it follows from the definition of τ_m that

$$|(t - t_{mk})/\tau_m| \le 1,$$

hence that

$$(3.49) \qquad |(x_{m(k+1)} - x_{mk})(t - t_{mk})/\tau_m| \le |x_{m(k+1)} - x_{mk}|.$$

Let i be any fixed integer in the set $\{1,2,\cdots,n\}$. Choose any dyadic rational multiple of T in $[0,1]$ as the fixed left endpoint of an interval $\left[\frac{k_m}{m}T, \frac{(k_m+1)}{m}T\right]$. Then, by the definition of Q_m in Section 3.2, the left-hand endpoint of this interval is in the partitions $Q_m, Q_{m+1}, \cdots$. Assume that $x^i_{m(k_m+1)}$ does not converge to $x^i_{mk_m}$, that is,

$$|x^i_{m(k_m+1)} - x^i_{mk_m}| \not\to 0 \text{ as } m \to \infty.$$

Then, by definition, there exists an $\alpha > 0$ such that for every natural number λ there exists a natural number $\nu \ge \lambda$ such that

$$|x^i_{\nu(k_m+1)} - x^i_{\nu k_m}| \ge \alpha.$$

Let $\lambda_1 = 1$. From the above there exists a natural number $\nu_1 \ge 1$ such that

$$|x^i_{\nu_1(k_m+1)} - x^i_{\nu_1 k_m}| \ge \alpha.$$

Next, let $\lambda_2 = \nu_1 + 1$ and we are assured of the existence of a natural number $\nu_2 \ge \lambda_2$ such that

$$|x^i_{\nu_2(k_m+1)} - x^i_{\nu_1 k_m}| \ge \alpha.$$

Repeating the above procedure indefinitely (letting $\lambda_{\mu+1} = \nu_\mu + 1$, $\mu = 1,2,3,\cdots$) we obtain a strictly increasing sequence $\{\nu_\beta\}$ such that

$$(3.50) \qquad |x^i_{\nu_\beta(k_m+1)} - x^i_{\nu_\beta k_m}| \ge \alpha, \quad \beta = 1,2,3,\cdots.$$

It follows that

$$(3.51) \qquad |x^i_{\nu_\beta(k_m+1)} - x^i_{\nu_\beta k_m}|^2/\tau_{\nu_\beta} \geq \alpha^2/\tau_{\nu_\beta} = \alpha^2\nu_\beta/T,$$

where the latter equality of (3.51) follows from the definition of $\tau_{\nu_\beta} = T/\nu_\beta$. Since $\{\nu_\beta\}$ is a strictly increasing sequence there necessarily exists a ν_β sufficiently large so that (3.51) implies that

$$(3.52) \quad \sum_{k=0}^{\nu_\beta-1} \left|(x^i_{\nu_\beta(k+1)} - x^i_{\nu_\beta k})/\tau_{\nu_\beta}\right|^2 \tau_{\nu_\beta} \geq (x^i_{\nu_\beta(k_m+1)} - x^i_{\nu_\beta k_m})^2/\tau_{\nu_\beta} > b^i,$$

which contradicts (3.3). Therefore, we conclude that

$$|x^i_{m(k_m+1)} - x^i_{mk_m}| \to 0 \quad \text{as} \quad m \to \infty.$$

Since the above argument holds for $i = 1,2,\cdots,n$, it follows that

$$|x_{m(k_m+1)} - x_{mk_m}| \to 0 \quad \text{as} \quad m \to \infty,$$

and the conclusion of the lemma now follows from (3.49).

We now define

$$K \equiv \{(t,x,u): t \in [0,T],\ |x^i| \leq b^i_3 < \infty,\ i = 1,\cdots,n, \text{ and } |u^j| \leq b^j_1 < \infty,\ j = 1,\cdots,p\}.$$

Observe that K is a "box" in R^{1+n+p} including all of its boundary points. Thus, K is a bounded and closed, hence compact subset of R^{1+n+p}. Since the function g was given to be continuous on $[0,T] \times R^n \times R^p$ and hence on K by condition (2.3) then it is uniformly continuous on K in all $1 + n + p$ variables, hence uniformly continuous in $(t,x^1,\cdots,x^n)$ for each fixed $u = (u^1,\cdots,u^p)$. Similarly, the function f is uniformly continuous in $x = (x^1,\cdots,x^n)$ for each fixed $(t,u^1,\cdots,u^p)$.

THEOREM 3.2. <u>If (x_m,u_m) is a pair in $\mathcal{P}_m^+$ and $(\tilde{x}_m,\tilde{u}_m)$ is the corresponding extended pair in $\tilde{\mathcal{P}}_m^+$ then</u>

$$(3.53)\qquad |J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - J_{\varepsilon,m}(x_m,u_m)| \to 0 \quad \text{as} \quad m \to \infty.$$

<u>Proof.</u> From definition (2.9) of J_ε and definition (3.4) of $J_{\varepsilon,m}$ we have that

$$(3.54)\qquad |J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - J_{\varepsilon,m}(x_m,u_m)| =$$

$$\Big|\int_0^T g(t,\tilde{x}_m,\tilde{u}_m)dt - \sum_{k=0}^{m-1} g(t_{mk},x_{mk},u_{mk})\tau_m +$$

$$\frac{1}{2\varepsilon}\int_0^T |\dot{\tilde{x}}_m - f(t,\tilde{x}_m,\tilde{u}_m)|^2 dt - \frac{1}{2\varepsilon}\sum_{k=0}^{m-1}\left|\frac{x_{m(k+1)}-x_{mk}}{\tau_m} - f(t_{mk},x_{mk},u_{mk})\right|^2\tau_m\Big|.$$

Direct use in (3.54) of the definitions from Section 3.3 of the PWLE $\tilde{x}_m$ and the PWCE $\tilde{u}_m$ gives that

$$(3.55)\qquad |J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - J_{\varepsilon,m}(x_m,u_m)| = \Big|\sum_{k=0}^{m-1}\int_{t_{mk}}^{t_{m(k+1)}} [g(t,x_{mk} + \frac{x_{m(k+1)}-x_{mk}}{\tau_m}(t-t_{mk}),u_{mk})$$

$$- g(t_{mk},x_{mk},u_{mk}) + \frac{1}{2\varepsilon}\left|\frac{x_{m(k+1)}-x_{mk}}{\tau_m} - f(t_{mk},x_{mk}+\frac{x_{m(k+1)}-x_{mk}}{\tau_m}(t-t_{mk}),u_{mk})\right|^2$$

$$- \frac{1}{2\varepsilon}\left|\frac{x_{m(k+1)}-x_{mk}}{\tau_m} - f(t_{mk},x_{mk},u_{mk})\right|^2]dt\Big|.$$

By appropriate use of some elementary triangle inequality properties for the euclidean norm $|\cdot|$ we can conclude from (3.55) that

$$(3.56)\qquad |J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - J_{\varepsilon,m}(x_m,u_m)| \le$$

$$\sum_{k=0}^{m-1}\int_{t_{mk}}^{t_{m(k+1)}} [|g(t,x_{mk} + \frac{x_{m(k+1)}-x_{mk}}{\tau_m}(t-t_{mk}),u_{mk}) - g(t_{mk},x_{mk},u_{mk})|$$

$$+ \frac{1}{2\varepsilon}|f(t_{mk},x_{mk} + \frac{x_{m(k+1)}-x_{mk}}{\tau_m}(t-t_{mk}),u_{mk}) - f(t_{mk},x_{mk},u_{mk})|]dt.$$

Lemma 3.3 states that

$$\left|x_{mk} + \frac{x_{m(k+1)} - x_{mk}}{\tau_m} (t - t_{mk}) - x_{mk}\right| \to 0 \text{ as } m \to \infty,$$

for each $k \in \{0,1,\cdots,m-1\}$. Recall from the paragraph preceding the present theorem that g is uniformly continuous in x and t on a certain compact set K. Hence there exists an m sufficiently large so that

$$(3.57) \qquad \left|g(t, x_{mk} + \frac{x_{m(k+1)} - x_{mk}}{\tau_m} (t - t_{mk}), u_{mk}) - g(t_{mk}, x_{mk}, u_{mk})\right| < \alpha/2T,$$

for some preassigned small positive number α. Similarly, from the uniform continuity of f in x on K, as remarked preceding this theorem, we have that there exists an m sufficiently large such that

$$(3.58) \qquad \left|f(t_{mk}, x_{mk} + \frac{x_{m(k+1)} - x_{mk}}{\tau_m} (t - t_{mk}), u_{mk}) - f(t_{mk}, x_{mk}, u_{mk})\right| < \frac{\alpha\varepsilon}{T}.$$

Then for m sufficiently large so that both (3.57) and (3.58) hold we can obtain the conclusion of the theorem from (3.56), (3.57) and (3.58).

THEOREM 3.3. <u>If</u>

$$\tilde{\gamma}^+_{\varepsilon,m} = \inf \{J_\varepsilon(\tilde{x}_m, \tilde{u}_m) : (\tilde{x}_m, \tilde{u}_m) \in \tilde{\mathcal{O}}^+_m\},$$

<u>and</u>

$$J_{\varepsilon,m}(x^*_m, u^*_m) = \inf \{J_{\varepsilon,m}(x_m, u_m) : (x_m, u_m) \in \mathcal{O}^+_m\},$$

<u>then</u>

$$(3.59) \qquad |\tilde{\gamma}^+_{\varepsilon,m} - J_{\varepsilon,m}(x^*_m, u^*_m)| \to 0 \ \underline{\text{as}} \ m \to \infty.$$

<u>Proof</u>. It was shown immediately preceding Section 3.3 that there exists a pair (x^*_m, u^*_m) in $\mathcal{O}^+_m$ with the property stated in the hypotheses of this theorem. We can apply (3.53), the conclusion of Theorem 3.2, to conclude that for the corresponding pairs $(\tilde{x}^*_m, \tilde{u}^*_m)$, (x^*_m, u^*_m) in $\tilde{\mathcal{O}}^+_m$, $\mathcal{O}^+_m$

that

$$(3.60) \qquad |J_\varepsilon(\tilde{x}_m^*,\tilde{u}_m^*) - J_{\varepsilon,m}(x_m^*,u_m^*)| \to 0 \text{ as } m \to \infty.$$

From the definition of convergence, (3.60) means that for an arbitrary but fixed $e > 0$, there exists an $M_e^{(1)}$ such that $m \geq M_e^{(1)}$ implies that

$$(3.61) \qquad |J_\varepsilon(\tilde{x}_m^*,\tilde{u}_m^*) - J_{\varepsilon,m}(x_m^*,u_m^*)| < e.$$

From (3.61) and the definition of $\tilde{\gamma}_{\varepsilon,m}^+$ we have that

$$(3.62) \qquad \tilde{\gamma}_{\varepsilon,m}^+ - J_{\varepsilon,m}(x_m^*,u_m^*) \leq J_\varepsilon(\tilde{x}_m^*,\tilde{u}_m^*) - J_{\varepsilon,m}(x_m^*,u_m^*) < e,$$

provided that $m \geq M_e^{(1)}$.

For $e/2$ and for every m there necessarily exists a pair $(\tilde{x}_m,\tilde{u}_m) \in \tilde{\mathcal{O}}_m^+$ such that

$$(3.63) \qquad J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - \tilde{\gamma}_{\varepsilon,m}^+ = |J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - \tilde{\gamma}_{\varepsilon,m}^+| < e/2.$$

Suppose that m in the preceding has been chosen sufficiently large, say $m \geq M_{e/2}^{(2)}$, such that we can conclude from Theorem 3.2 that

$$(3.64) \qquad |J_\varepsilon(\tilde{x}_m,\tilde{u}_m) - J_{\varepsilon,m}(x_m,u_m)| < e/2.$$

Then from (3.63) and (3.64) we have that

$$(3.65) \qquad |J_{\varepsilon,m}(x_m,u_m) - \tilde{\gamma}_{\varepsilon,m}^+| < e,$$

provided that $m \geq M_{e/2}^{(2)}$. Hence we have that

$$(3.66) \qquad -e < \tilde{\gamma}_{\varepsilon,m}^+ - J_{\varepsilon,m}(x_m,u_m) \leq \tilde{\gamma}_{\varepsilon,m}^+ - J_{\varepsilon,m}(x_m^*,u_m^*),$$

provided that $m \geq M_{e/2}^{(2)}$. Let $M_e = \max\{M_e^{(1)},M_{e/2}^{(2)}\}$ and we can conclude from (3.62) and (3.66) that

$$(3.67) \qquad |\tilde{\gamma}_{\varepsilon,m}^+ - J_{\varepsilon,m}(x_m^*,u_m^*)| < e,$$

provided that $m \geq M_e$. Since e was an arbitrary positive number (3.67) is the desired conclusion.

THEOREM 3.4. If

$$J_{\varepsilon,m}(x_m^*,u_m^*) = \inf\{J_{\varepsilon,m}(x_m,u_m) : (x_m,u_m) \in \mathcal{P}_m^+\},$$

then

$$\lim_{m\to\infty} J_{\varepsilon,m}(x_m^*,u_m^*) = J_\varepsilon(x_0,u_0) = \inf\{J_\varepsilon(x,u) : (x,u) \in \mathcal{P}^+\}.$$

Proof. For an arbitrary but fixed $\alpha > 0$ the conclusion (3.27) of Theorem 3.1 implies that

$$|\tilde{\gamma}_{\varepsilon,m}^+ - J_\varepsilon(x_0,u_0)| < \alpha/2 \tag{3.68}$$

for m sufficiently large. Now from Theorem 3.3 we can conclude that

$$|\tilde{\gamma}_{\varepsilon,m}^+ - J_{\varepsilon,m}(x_m^*,u_m^*)| < \alpha/2 \tag{3.69}$$

for m sufficiently large. Hence if m is sufficiently large for both (3.68) and (3.69) to hold we have that

$$|J_{\varepsilon,m}(x_m^*,u_m^*) - J_\varepsilon(x_0,u_0)| < \alpha. \tag{3.70}$$

Since α was an arbitrary positive number the conclusion of the theorem holds.

Recall that $\{\varepsilon_k\}$ denotes a strictly decreasing sequence of positive reals converging to zero, and that

$$J_{\varepsilon_k,m}[x_m^*(\cdot,\varepsilon_k),u_m^*(\cdot,\varepsilon_k)] = \inf\{J_{\varepsilon_k,m}(x_m,u_m) : (x_m,u_m) \in \mathcal{P}_m^+\},$$

and

$$J_{\varepsilon_k}[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)] = \inf\{J_{\varepsilon_k}(x,u) : (x,u) \in \mathcal{P}^+\}.$$

THEOREM 3.5. If the conditions in the formulation of the continuous and discrete auxiliary problems of Sections 2.3 and 3.2 hold, then

$$\lim_{k\to\infty}\lim_{m\to\infty} J_{\varepsilon_k,m}[x_m^*(\cdot,\varepsilon_k),u_m^*(\cdot,\varepsilon_k)] = \lim_{k\to\infty} J_{\varepsilon_k}[x_0(\cdot,\varepsilon_k),u_0(\cdot,\varepsilon_k)]$$

$$= J(x_*,u_*) = \inf\{J(x,u):(x,u) \in \mathcal{P}\}.$$

Proof. This theorem is obtained as a direct consequence of Theorems 2.2 and 3.4.

From Theorem 3.5 we see that obtaining an approximate solution of the original control problem of Section 2.2 essentially reduces to solving the corresponding discrete auxiliary problems for small positive ε_k. Various mathematical programming techniques exist to solve the discrete auxiliary problem.

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