SPECTRAL CHARACTERIZATION OF MULTI-INPUT DYNAMIC SYSTEMS

By

S. SRINATHKUMAR

Bachelor of Engineering
University of Mysore
Bangalore, India
1960

Master of Science
University of Hawaii
Honolulu, Hawaii
1973

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
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DYNAMIC SYSTEMS

Thesis Approved:

Ronald P. Rhoten

Thesis Adviser

Rao Yarlagadda

Robert J. Mulholland

Larry D. Zirkle

N N Durham

Dean of the Graduate College

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I dedicate this dissertation to the loving memory of my mother.

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NOMENCLATURE

Matrices	- Upper case alphabet
Vectors	- Lower case alphabet
Scalars	- Lower case Greek alphabet
I_n	- n x n identity matrix
Diag[•]	- Diagonal matrix
A^T	- Transpose of a matrix
A^{-1}	- Inverse of a matrix
$A^{\dagger}$	- Pseudo-inverse of a matrix
det[•]	- Determinant of a matrix
$\|\|\cdot\|\|$	- Matrix or vector norm
$\|\cdot\|$	- Absolute value
$\{\cdot\}$	- Set
ε	- Element of a set
ϕ	- Order of
Π	- Continued product
$\neq$	- Equality does not hold
$\leq$	- Less than or equal to
$>>$	- Far greater than
$\approx$	- Approximately equal to

CHAPTER I

INTRODUCTION

1.1 Multivariable System Synthesis

Once a physical system has been modelled by an appropriate mathematical structure, various techniques of analysis can be applied to characterize the behavior of the system relative to solution representation, stability, boundedness, etc. The synthesis problem, on the other hand, is to modify the system response to a more desirable form.

If the modelling structure is a linear time invariant vector differential equation, two appropriate synthesis procedures are the linear quadratic optimal control solution and pole-placing techniques.

The optimal control procedure is particularly well suited to those systems which have desired response time histories as performance specifications, and which allow off-line computation of the feedback gain matrix. It is often the case, however, that real time computations are desired, entailing updates in the feedback gains as improved estimates of system parameters are generated. In such cases the computationally complex procedures of the quadratic regulator solution require extensive on-line computer facilities. However, if stabilization of the system is the primary design objective, an alternative synthesis procedure based on Lyapunov's functions could be employed [1].

Another difficulty which may be encountered in the formulation of the optimal control problem is that of performance specification.

Whether through necessity or utility, many system performance specifications are formulated in the frequency domain rather than the time domain [2,3]. Such criteria are commonly presented as desirable locations for the closed-loop system poles (eigenvalues in the state variable formulation), and pole-placement techniques thus become appropriate synthesis procedure.

Although a wide variety of multivariable system synthesis algorithms based on pole-placement concepts have been developed, none have proved to be practical design procedures. The common difficulty is not the fault of the algorithms, but is really due to the inherent inability of pole location specifications to characterize the actual variable responses. For multi-input systems, the feedback control law assigning a specified set of eigenvalues is not unique. That is, an infinite number of control laws will yield the same pole locations but different eigenvectors and thus lead to radically different output responses.

In the existing algorithms this nonuniqueness is a liability in that it either substantially complicates the selection of certain transformations or is restricted at the outset, without knowledge of how such a restriction will affect variable responses. In essence, existing techniques solve the problem as posed, but from a design perspective, solve the wrong problem. Since a system eigenvector determines the influence of its associated eigenvalue on each state variable response, control of the closed-loop modal matrix (matrix of eigenvectors) is as necessary as control of pole locations if acceptable dynamic behavior is to be achieved.

The key concept of rationally utilizing the freedom in control law selection to satisfy eigenvector as well as eigenvalue specifications forms the basis for the development of the synthesis procedure reported in this dissertation.

1.2 Review of Current Techniques and Problem Statement

The classical pole-placement problem may be stated as follows: given the linear time-invariant system

$$\dot{x} = Ax + Bu \tag{1.1}$$

where x is the state n-vector, u is the control m-vector and A and B are constant matrices of appropriate dimensions, find a control law of the form $u = Kx + v$, where K is an m x n matrix of constants and v is an m-reference input vector, such that the closed-loop coefficient matrix $\hat{A} = A + BK$ has arbitrarily assigned eigenvalues. It is assumed that (A,B) is a controllable pair so that the matrix $[B, AB, A^2B, \cdots, A^{n-1}B]$ has rank n.

Existing algorithms for the solution of the problem stated above can be broadly classified as pole-shifting methods or direct methods. The former methods [4], classically known as modal control theory, rely on the knowledge of the eigenvalues and eigenvectors of the open-loop system in generating the control law. The obvious computational burden associated with such schemes make them less attractive in an on-line implementation framework. Further, they do not possess features to significantly utilize the multi-input design freedom mentioned earlier, and hence will not be considered further.

Anderson and Luenberger [5] have proposed a direct method which reduces (1.1) to a special canonical form in which the system matrix has a block triangular structure, with the diagonal blocks in companion form. This canonical form is viewed as a set of uni-directionally coupled subsystems, so that pole-assignment of the first block is unaffected by the second, and so on. The arbitrariness in the realization of $\hat{A}$ arises because the transformation matrix is not unique, and further, after arriving at the canonical form, feedback gains to achieve pole-placement are non-unique. Thus nothing can be inferred about the dynamics of the resulting system for some initial restriction of arbitrariness. In addition, as the authors point out, the main computational difficulty in the scheme is the determination of the linear dependence among a set of vectors needed in the generation of the transformation matrix.

Fallside and Seraji [6] propose a scheme which reduces the multi-input system to an equivalent single input system by imposing a dyadic structure on the feedback matrix so that $K = qf^T$, with f^T indicating the transpose of f. Here, q is an arbitrary m-vector chosen so that (A,Bq) is a controllable pair, and f is an n-vector. With this structure, $\hat{A} = A + Bqf^T$, and the algorithm utilizes the characteristic polynomial of A to complete the synthesis. Using this technique the authors have been able to assign part of the numerator dynamics of the closed-loop transfer function matrix arbitrarily.

More recently, Chidambara, et al. [7] essentially construct a closed-loop system matrix which is similar to a block triangular matrix whose eigenvalues are the ones to be assigned to the system. The procedure still requires determination of the linear dependence of vectors as in [5], but with vectors of reduced dimensions, thus alleviating to a certain extent the computational problems. Again there is no direct correlation between system response and the arbitrary design parameters involved in the algorithm. However the technique has been used to arbitrarily assign some residues connected with the dominant modes of the closed-loop transfer function matrix [8].

If these algorithms are appraised in terms of the effective utilization of the free choices in the feedback matrix K, it is apparent that either no meaningful interpretation can be given to the arbitrariness arising in the algorithms [5,7], or the design freedom is unduly curtailed, apriori, by assigning a structure to the feedback matrix [6]. Furthermore, in all these methods, if a nominal design fails to yield a satisfactory system, the design procedures give no guidance as to the means of achieving a system with improved response. This failing is crucial, for even if an off-line solution is all that is required, the design process reduces to a random search procedure. The problem arises from the fact that there is no link between the control parameters and the system dynamical behavior.

These discussions aid in developing the problem statement for the synthesis procedure to be presented in Chapter II. In essence, a linear feedback control law is to be selected for the system of (1.1) such that a combination of eigenvalue and eigenvector constraints are satisfied. In addition the procedure developed will ensure a direct relation between dynamical response characteristics and control gain values. While a more detailed problem statement would perhaps seem appropriate, later chapters will show that the wide variety of performance specifications, design alternatives and additional applications would be unduly restricted by a premature attempt to structure the research objectives.

Finally, it is appropriate to mention that the multivariable synthesis problem can also be analyzed from a frequency domain perspective [2,3]. Horowitz [3] provides an interesting comparison between the relative merits of state space oriented design and transfer function synthesis procedures. The discussions in later chapters also bring out the role played by the closed-loop modal matrix in relating time/frequency domain attributes of the feedback system.

1.3 Organization

The dissertation is presented in the following format. Chapter II introduces a new formulation of the problem which maps the nonuniqueness in the feedback gain matrix K to an equivalent freedom in the selection of the modal matrix entries. This in essence leads to the spectral characterization of all possible closed-loop modal structures for a given plant and its specified closed-loop poles (eigenvalues). Chapter III outlines a practical multivariable synthesis procedure based on the spectral characterization formulation introduced in Chapter II, by synthesizing a hover controller for a helicopter.

Chapter IV discusses the utility of the new formulation in the design of asymptotic state estimators and in the synthesis of systems insensitive to plant parameter perturbations. The feature of the algorithm developed in Chapter II when used as an on-line adaptive controller is also highlighted. Finally, Chapter V presents a summary of the results and indicates areas of future research.

CHAPTER II

SPECTRAL CHARACTERIZATION

2.1 Introduction

The need to control simultaneously the modes and the associated modal structure in order to ensure acceptable dynamic responses of the output variables was established in Chapter I. Unfortunately the process of coupling the effect of the individual modes to the output variables through the entries of the eigenvectors is nonlinear, either through the construction of the total solution via reciprocal basis vectors and constituent solutions or the modal matrix and its inverse. However it may be possible to identify certain desired closed-loop system structures and thus the associated modal structures. For example it is often the case that higher order systems may be considered as a coupling of lower order subsystems, each with its own specifications of acceptable performance. In such a case, the eigenvectors should be selected so that the eigenvalues appropriate to one set of response variables do not unduly influence the other responses. Similarly it may be desirable to segregate short time constant variables from long time constant modes, or ensure that systems with both real and complex pair poles have eigenvectors selected so that minimal oscillatory behavior will arise in those responses associated with real eigenvalues.

The eigenvectors may of course be modified without disturbing pole locations through the non-uniqueness of the modal control process.

Equally clearly, there is not sufficient freedom to arbitrarily select these eigenvectors, except in the pathological case of an n-state, n-input system. This makes a precise problem statement somewhat difficult to formulate, but in general the approach will be to structure the control design process so that maximum capabilities are achieved for satisfying whatever eigenvalue/eigenvector specifications exist. The following sections will present these results.

1. The pole-placement problem will be reformulated, and it will be shown that n eigenvalues and a maximum of $n \cdot m$ eigenvector entries can be arbitrarily specified.
2. In general, no more than m entries of any one eigenvector can be chosen arbitrarily.
3. While n eigenvalue specifications can be achieved exactly (for a controllable system), a superior design algorithm results if the specification is relaxed to allow the closed-loop eigenvalue to be arbitrarily close to that desired.

2.2 A New Formulation

The pole-placement problem may be reformulated as an eigenvalue/eigenvector selection problem as follows.

Given the controllable system

$$\dot{x} = Ax + Bu \quad , \tag{2.1}$$

find the state feedback law

$$u = Kx \tag{2.2}$$

such that the closed-loop system matrix

$$\hat{A} = [A + BK] \tag{2.3}$$

satisfies

$$\hat{A}U = U\Lambda \quad , \tag{2.4}$$

where Λ is the diagonal matrix of desired eigenvalues and U is the modal matrix satisfying some given constraints. Note that in the more general case of multiple root assignment, Λ becomes the appropriate Jordon canonical form. The case of multiple root assignment will be considered in section 2.4.4.

To see the freedom which exists in the choice of U, partition the matrices in (2.3) and (2.4) as

$$A = \left[\begin{array}{c|c} D & E \\ \hline T & R \end{array}\right] \quad ,$$

$$B = \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix} \quad ,$$

$$K = [K_1 : K_2] \quad ,$$

$$\Lambda = \left[\begin{array}{c|c} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{array}\right] \quad ,$$

and

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] ,$$

where D, B_1, K_1, Λ_1 and U_{11} are m x m, and the other matrices are compatibly dimensioned. It is also assumed that B_1 is nonsingular. The invertibility of B_1 can be obtained by at most a reordering of the state variables.

For simplicity of analysis assume that the system (2.1) is already in rank reduced form with B_1 nonsingular and B_2 identically zero. This can always be achieved by a coordinate transformation [7] of the state vector by

$$L = \left[\begin{array}{c|c} I_m & 0 \\ \hline B_2B_1^{-1} & I_{n-m} \end{array}\right] \tag{2.5}$$

with I_k the identity matrix of order k.

With this transformation, (2.4) can be written in partitioned form as

$$\left[\begin{array}{c|c} D+B_1K_1 & E+B_1K_2 \\ \hline T & R \end{array}\right]\left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right]\left[\begin{array}{c|c} \Lambda_1 & 0 \\ \hline 0 & \Lambda_2 \end{array}\right] . \tag{2.6}$$

Completing the multiplication of the partitioned matrices in (2.6) yields

$$[D + B_1K_1]U_{11} + [E + B_1K_2]U_{21} = U_{11}\Lambda_1 \tag{2.7}$$

$$[D + B_1K_1]U_{12} + [E + B_1K_2]U_{22} = U_{12}\Lambda_2 \tag{2.8}$$

$$TU_{11} = U_{21}\Lambda_1 - RU_{21} \tag{2.9}$$

$$TU_{12} = U_{22}\Lambda_2 = RU_{22} \quad . \tag{2.10}$$

Equations (2.9) and (2.10) can also be expressed as a set of linear constraints on the individual eigenvector entries of the following form.

(1) For real eigenvalues

$$[\lambda_i I_{n-m} - R]w_i = Tz_i, \ i = 1,2,\cdots,n_1 \quad . \tag{2.11}$$

(2) For complex pairs in quasidiagonal form

$$\rho_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}, \ j = n_1+1, n_1+3, \cdots, n-1 \ ,$$

and

$$\left[\begin{array}{c|c} \alpha_j I_{n-m} - R & -\beta_j I_{n-m} \\ \hline \beta_j I_{n-m} & \alpha_j I_{n-m} - R \end{array}\right] \begin{bmatrix} w_j \\ \cdots\cdot \\ w_{j+1} \end{bmatrix} = \left[\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right] \begin{bmatrix} z_j \\ \cdots\cdot \\ z_{j+1} \end{bmatrix} \tag{2.12}$$

where

$$\begin{bmatrix} z_i \\ \cdots\cdot \\ w_i \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{c|c} z_j & z_{j+1} \\ \hline w_j & w_{j+1} \end{array}\right]$$

are the real eigenvectors associated with eigenvalues λ_i and $\alpha_j \pm i\beta_j$, respectively, z's are m-vectors and w's are (n - m) vectors. (For a derivation of the relation between complex eigenvectors and equivalent real eigenvectors see Appendix A.)

If the system is not in rank reduced form (2.11) and (2.12) take the form

(1) for real eigenvalues

$$(\lambda_i I_{n-m} - F)w_i = (G + \lambda_i H)z_i, \; i = 1,2,\cdots,n_1 \quad ,$$

and

(2) for complex pairs

$$\left[\begin{array}{c|c} \alpha_j I_{n-m} - F & -\beta_j I_{n-m} \\ \hline \beta_j I_{n-m} & \alpha_j I_{n-m} - F \end{array}\right] \begin{bmatrix} w_j \\ \cdots \\ w_{j+1} \end{bmatrix} = \left[\begin{array}{c|c} G+\alpha_j H & -\beta_j H \\ \hline \beta_j H & G+\alpha_j H \end{array}\right] \begin{bmatrix} z_j \\ \cdots \\ z_{j+1} \end{bmatrix} \quad ,$$

$$j = n_1+1, n_1+3, \cdots, n-1$$

where

$$H = B_2 B_1^{-1} \quad ,$$

$$G = T - HD \quad ,$$

$$F = R - HE \quad .$$

Equations (2.11) and (2.12) constitute a set of under-determined linear homogeneous equations, and even if all the eigenvalues are fixed, m entries in each eigenvector are arbitrary. Thus a total of n • m entries in the modal matrix can be arbitrarily selected subject only to the constraint that U be nonsingular.

It is also crucial to note that (2.11) and (2.12) show the complete relationship between the eigenvalues and associated eigenvectors. This property not only illustrates what can be achieved by the controller, but also may immediately point out that preconceived design objectives may be impossible to achieve. Later examples will further illustrate this point and emphasize that the design approach is complete in the sense that when it fails to satisfy specifications, it does so by showing no control law could satisfy them.

It is interesting to note that the free $n \cdot m$ elements of U exactly correspond to the $n \cdot m$ arbitrary elements of the feedback matrix K. This establishes the parametric equivalence between the nonunique feedback matrix K and the arbitrary modal entries.

With a nonsingular U chosen to meet the restrictions of (2.11) and (2.12), the required feedback matrix K may then be easily evaluated by

$$K_1 = B_1^{-1}[\hat{A}_{11} - D] \tag{2.13}$$

$$K_2 = B_1^{-1}[\hat{A}_{12} - E] \tag{2.14}$$

where

$$U\Lambda U^{-1} = \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] , \tag{2.15}$$

and with $[K_1 \vdots K_2]$ selected as in (2.13) and (2.14), relations (2.7) and (2.8) are identically satisfied.

The validity of the above analysis of course depends on the guaranteed generation of the nonsingular transformation U in (2.15). Appendix B develops a constructive proof which assures the realization of such a transformation, thus establishing the solution of the pole-placement problem posed in (2.1-2.4). However it is also apparent, from

Appendix B, that the process of mathematical validation required to meet exact pole-placement has unduly restricted the structure of the transformation matrix U to be block triangular. From a practical synthesis standpoint this is truely undesirable. It is also interesting to note that a relaxation of exact pole specifications results in significant gains in flexibility of selection of a nonsingular modal structure and consequent better control of system response. This design philosophy has lead to the development of the algorithms described in section 2.4. These algorithms have the desirable feature of imposing minimal restrictions on the structure of U at the cost of constructing, in some pathological cases, an n-dimensional eigenspace for a matrix $(\Lambda + \delta\Lambda)$ arbitrarily close to the desired matrix Λ of (2.4). It is also emphasized that this is in no way a serious limitation since pole specifications are rarely intended to be exact.

While the above flexible modal structural representation may be used for a variety of goals, the most readily apparent is that of mode decoupling. A performance specification might fix (or at least bound) pole locations and require minimal interaction of modes, implying that each state variable response be dominated by only one corresponding eigenvalue. From the solution representation

$$x(t) = U \exp (\Lambda t) U^{-1} x(0) \quad , \qquad (2.16)$$

such performance will be achieved if both U and its inverse are dominated by the main diagonal elements. While the entries of U^{-1} are nonlinear functions of the entires of U, it is easy to show that if the off-diagonal elements of a matrix are of order ε compared to the main diagonal elements, the same is true of the inverse.

While the diagonally dominant modal structure may be a desirable design specification, whether or not it can be synthesized for a plant description needs to be ascertained. This is vividly displayed in the relations (2.11) and (2.12). For the present, assume that the modes λ_i and ρ_i to be assigned do not coincide with those of matrix R. Then (2.12) can be written as

$$w_i = C_i z_i \tag{2.17}$$

with

$$C_i = (\lambda_i I_{n-m} - R)^{-1} T \quad . \tag{2.18}$$

Similarly (2.13) can be expressed as

$$\begin{bmatrix} w_j \\ \cdots \\ w_{j+1} \end{bmatrix} = C_j \begin{bmatrix} z_j \\ \cdots \\ z_{j+1} \end{bmatrix} \tag{2.19}$$

with

$$C_j = \left[\begin{array}{c|c} \alpha_j I_{n-m} - R & -\beta_j I_{n-m} \\ \hline \beta_j I_{n-m} & \alpha_j I_{n-m} - R \end{array}\right] \left[\begin{array}{c|c} T & 0 \\ \hline 0 & T \end{array}\right] \quad . \tag{2.20}$$

The matrices C_k define the couplings that exist among the eigenvector entries, and give an apriori indication of inevitable mode coupling that may result for a given selection of z_k. Alternatively this representation also points out how z_k could be selected to suppress certain modes from influencing selected response variables. Further, since $C_k = C_k(\lambda_k)$, selection of modes takes on a new significance. In practice

Λ is not fixed but is required to lie in a subset $\Lambda\varepsilon\Omega$ of the stable complex plane. Then it becomes practical to search in Ω for a $C_k(\lambda_k)$ which gives the closest desirable structure for the corresponding eigenvector. This makes pole-assignment more meaningful and the λ_k's become additional design parameters available for manipulation to meet the performance specification.

Before proceeding with the development of algorithms for the solution of the reformulated pole-placement problem, it is worthwhile to clarify the concepts introduced so far with two simple numerical examples.

2.3 Numerical Examples

2.3.1 A First Tutorial Example

The following problem is presented to describe the synthesis procedure and indicate the tradeoffs between design specifications and achievable results which may be required. Suppose the plant dynamics are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{2.21}$$

and

$$\begin{bmatrix} \mu_1 \\ \cdots\cdot \\ \mu_2 \end{bmatrix} = Kx$$

is to be selected so that:

(1) pole locations are at approximately -1, -10, and -100;

(2) x_1, x_2 and x_3 should exhibit respectively the short, intermediate and long time constant transients, namely 0.01, 0.1, 1 seconds, respectively; and

(3) the responses should be decoupled in the sense that while x_3 may exhibit short and intermediate transients (e^{-t} will rapidly dominate other terms), x_2 should not have e^{-t} terms and x_1 should exhibit neither e^{-t} nor e^{-10t} transients.

Note that this is not a trivial problem since the open-loop eigenvalues are at 0, 1, and 2 and the crucial long time constant is to be associated with x_3, the variable not directly influenced by the input.

The time solution for the closed loop system in terms of the modal matrix U can be expressed as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \cdot e^{\lambda_1 t} \\ \alpha_2 \cdot e^{\lambda_2 t} \\ \alpha_3 \cdot e^{\lambda_3 t} \end{bmatrix} \tag{2.22}$$

where the α_i's depend on U^{-1} and the initial conditions, and $\lambda_1 \approx -100$, $\lambda_2 \approx -10$ and $\lambda_3 \approx -1$. While the entries of the modal matrix influence the α_i's in a nonlinear fashion, dominance arguments show that the design specifications will be satisfied if U is of the form

$$U = \begin{bmatrix} u_{11} & \varepsilon_1 & \varepsilon_2 \\ \delta_1 & u_{22} & \varepsilon_3 \\ \delta_2 & \delta_3 & u_{33} \end{bmatrix} , \tag{2.23}$$

where the ε's and δ's should be small compared to u_{ii} (i = 1,2,3). This will yield minimal cross coupling and preserve the diagonal dominance character of U^{-1}.

To begin the synthesis, denote each column of U by u_j, and

$$u_j = \begin{bmatrix} z_{1j} \\ z_{2j} \\ \cdots \\ w_{1j} \end{bmatrix} .$$

For the system (2.21) the respective matrices of (2.11) and (2.12) are

$$R = -1, \; T = (1 \quad 1) .$$

Requiring $\Lambda \approx \mathrm{Diag}(-100,-10,-1)$, and starting with $j = 3$ ($\lambda_3 \approx -1$ is the most crucial eigenvalue), the procedure is to select z_{13}, z_{23} and λ_3 and evaluate w_{13} from

$$w_{13} = \frac{z_{13} + z_{23}}{\lambda_3 + 1} .$$

This is seen to be a "best case" problem (at least thus far), since λ_3 may be selected as exactly -1, both z_{13} and z_{23} to be zero and $w_{13} = 1$. Thus

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Now for $j = 2$,

$$w_{12} = \frac{z_{12} + z_{22}}{\lambda_2 + 1} ,$$

where z_{12} and w_{12} (corresponding to ε_1 and δ_3 in (2.23)) should be small compared to z_{22}, and λ_2 should be about -10. For numerical simplicity select $\lambda_2 = -11$, $z_{22} = 1$, $z_{12} = 0$ and w_{12} is then found to be -0.1. Note that this selection of eigenvector entries is not "best case", since w_{12} and z_{12} cannot be both zero if z_{22} is to be non-zero. This implies there must be some coupling of system modes. Continuing

$$u_2 = \begin{bmatrix} 0 \\ 1 \\ -0.1 \end{bmatrix} ,$$

and while a higher dimension problem might require the systematic procedures to be developed in section 2.4 to ensure linear independence of u_3 and u_2, it is easily seen, by inspection, that for this case the two eigenvectors are independent.

Now for j = 1,

$$w_{11} = \frac{z_{11} + z_{12}}{\lambda_1 + 1} ,$$

and selecting $\lambda_1 = -101$, it is desired that z_{11} dominate both w_{11} and z_{12}. Clearly both conditions cannot be satisfied (again implying some coupling is unavoidable), but by selecting $z_{11} = 1$ and $z_{12} = -1$, w_{11} will be zero and the inverse of U will preserve the diagonally dominant structure. Again,

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} ,$$

is clearly linearly independent of u_2 and u_3 by inspection. The modal matrix U has now been found to be

$$U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -0.1 & 1 \end{bmatrix},$$

with inverse

$$U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0.1 & 0.1 & 1 \end{bmatrix}.$$

The closed-loop system matrix is found to be

$$\hat{A} = U\Lambda U^{-1}$$

$$= \begin{bmatrix} -101 & 0 & 0 \\ 90 & -11 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

and the required feedback gains are easily calculated from (2.13) and (2.14) as

$$K = \begin{bmatrix} -102 & -1 & 1 \\ 90 & -14 & 2 \end{bmatrix}.$$

The time response of the system is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = U \exp [\Lambda t] \, U^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} .$$

Thus

$$x_1(t) = x_{10} \, e^{-101t}$$

$$x_2(t) = -x_{10} \, e^{-101t} + (x_{10} + x_{20})e^{-11t}$$

$$x_3(t) = -0.1(x_{10} + x_{20})e^{-11t} + (0.1x_{10} + 0.1x_{20} + x_{30})e^{-t} .$$

It is seen that the performance specifications have been achieved, and that the only coupling is from transients which rapidly become dominated by the desired time constant solutions. It is again emphasized that the example is "best case", since the ability to select four identically zero entries in the modal matrix ensured the total unidirectional coupling. To now consider a "worst case" example, suppose the problem were the same except that

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 1 & 1 & 19 \end{bmatrix} .$$

The first step of the solution is to again select $\lambda_3 \approx -1$, with

$$w_{13} = \frac{z_{13} + z_{23}}{\lambda_3 + 19}$$

and $|w_{13}| >> |z_{13}|$, $|w_{13}| >> |z_{23}|$. These conditions certainly cannot even be approximately satisfied, indicating that any solution will exhibit considerable mode coupling.

One particular compromise solution yielded a modal matrix

$$U = \begin{bmatrix} -1 & 0 & -10 \\ 0 & -1 & 0 \\ 0.05 & 0.033 & -0.1 \end{bmatrix}$$

with Λ = Diag[-1,-11,-81], and the corresponding state variable responses

$$x_1(t) = (-0.167x_{10} + 0.555x_{20} + 16.667x_{30})e^{-t}$$

$$+ 10(0.083x_{10} + 0.056x_{20} + 1.667x_{30})e^{-81t}$$

$$x_2(t) = x_{20}\ e^{-11t}$$

$$x_3(t) = 0.05(-0.167x_{10} + 0.555x_{20} + 16.667x_{30})e^{-t}$$

$$- 0.03x_{20}\ e^{-11t} + 0.1(0.083x_{10} + 0.056x_{20} + 1.667x_{30})e^{-81t} \quad .$$

The following observations may be made about this particular solution.

(1) The solution for $x_1(t)$ is the only one which does not meet specifications.

(2) If $x_1(t)$ were a more important variable than $x_2(t)$, selection of modal matrix entries could be made to have $x_1(t)$ free of cross coupled transients (e^{-t}, e^{-10t}), but this would introduce e^{-t} transient in the $x_2(t)$ response.

(3) If relative magnitudes of initial conditions are known, further improvement could be easily obtained.

To summarize, the proposed technique yields impressive results in "best case" examples, and for "worst case" problems, at least forewarns of the type of mode couplings that will occur. Thus in these less tractable problems the displaying of the system structure shows that no other solution will yield the desirable response. These discussions also demonstrate the inevitable trade off involved in any synthesis procedure, and highlight the need to understand fully the physical constraints of the plant, as exhibited by the $C_i(\lambda_i)$ matrices, to evolve acceptable design goals.

2.3.2 A Second Illustrative Example

A simple numerical example will now be presented to highlight the final basis for the establishment of the design algorithm to be presented in section 2.4. It will illustrate that certain procedural idiosyncrasies, as simple as selecting the sequence of synthesizing eigenvectors, may induce problems which could be resolved by inspection for lower order systems, but do require an algorithmic process for more complex systems.

Suppose the plant dynamics are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} .$$

Suppose further that design specifications are $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -1$ and an eigenvector structure

$$U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ w_1 & w_2 & w_3 \end{bmatrix} ,$$

with the w's computed to meet the desired pole-specifications.

To complete the design, Equation (2.11) can be used immediately (where R = 0 and T = [-1 -1]) to yield

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} .$$

It is immediately obvious that the resulting modal matrix u will be singular, since u_2 and u_3 are identical. For a 3rd order problem such as this, Equation (2.11) could be examined directly for each eigenvalue, and a sound decision made as how to relax which specification to give a system performance very close to that desired. Of course, for substantially higher order systems, or for an automated on-line controller, a precisely defined algorithm is required. Such an algorithm will now be presented, and the above problem reexamined as an illustration of its utility.

2.4 An Algorithm for Eigenvalue/Eigenvector Assignment

As the discussions in the previous sections revealed, the central question in the new formulation of the pole-placement problem is the guaranteed generation of the nonsingular matrix U satisfying (2.11) and (2.12). An examination of the eignvector constraints shows that there is an m-dimensional subspace associated with each eigenvalue. Thus the problem reduces to selecting a nonsingular set of n eigenvectors with one vector included from a subspace associated with each eigenvalue. System controllability assures the existence of such a set, and indeed in the multi-input case it is an infinite set. However this is an inefficient way to synthesize since the designer loses direct control of the selection of arbitrary elements of the modal matrix (z-vectors). Also, any algorithm would become computationally intractable since it would involve pairing n vectors from a set of n • m vectors until a nonsingular set resulted.

Alternatively, if the z-vectors are allowed to be chosen arbitrarily, then it is important to keep track of the linear independence of the eigenvectors as they are sequentially generated. It would appear that an easy way to accomplish this, would be to construct the projector of the subspace spanning the eigenvectors already synthesized in the sequence as

$$P^{(k-1)} = N[N^T N]^{-1} N^T \tag{2.24}$$

where $N = [u_1 : u_2 : \cdots u_{k-1}]$ are the first (k-1) linearly independent eigenvectors, and selecting u_k such that

$$P^{(k-1)} u_k \neq u_k \quad . \tag{2.25}$$

Unfortunately this procedure is potentially susceptible to the generation of a singular set of eigenvectors since it is quite likely that no closed-loop matrix with the precise set of eigenvalues/eigenvectors as selected would exist for any choice of control law. This is exactly what occurred in the example of section 2.3.2. However, it is possible to detect the occurrence of such a situation, and by slightly relaxing the specification of exact assignment of the modes and/or the corresponding design vectors (z), it is always possible to generate a nonsingular modal structure arbitrarily close to the desired one. It is emphasized again that the advantage of maintaining modal structural flexibility more than off-sets the disadvantage of not attaining exact pole assignment.

An algorithm will now be presented which incorporates the condition

(2.25) without having to explicitly compute the projection matrices $P^{(k)}$ of (2.24) and allows maximal flexibility in the selection of z-vectors.

Again for simplicity of presentation the system (2.1) is assumed to be in rank reduced form. In this form (R,T) is a controllable pair [7]. For clarity, the algorithm will be presented for real eigenvalue assignment and the extension to complex pairs discussed later.

2.4.1 An Algorithm to Ensure $\det[U] \neq 0$

The following notations will be used throughout the algorithm presentation.

(1) e_r is an n-vector with rth entry equal to unity and all other entries zero.

(2) $u_k = \begin{bmatrix} z_k \\ \cdots \\ w_k \end{bmatrix}$ is the kth eigenvector, where z_k is an m-vector, specified in advance.

(3) $u_k^{(k)} = Q^{(k-1)} u_k$ where $Q^{(0)} = I_n$ and $Q^{(i)}$, $i \neq 0$ will be defined in (6) below.

(4) $u_{rk}^{(k)}$ is the rth entry of $u_k^{(k)}$ and $u_{rk}^{(k)} = \sigma_k$ provides notation compatible with Appendix C.

(5) $M^{(k)}$ is an n x n matrix of the form

$$M^{(k)} = \left[\begin{array}{c|c|c} I_{n-r} & m_a^{(k)} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & m_b^{(k)} & I_{r-1} \end{array}\right] \tag{2.26}$$

where the vectors $m_a^{(k)}$ and $m_b^{(k)}$ are defined by (Appendix C without involving the permutation)

$$M^{(k)}u_k^{(k)} = \sigma_k e_r \; ; \quad r\varepsilon\{\Delta^{(k)}\} \qquad , \tag{2.27}$$

where $\Delta^{(k)}$ is a subset of integers $\{1,2,\cdots,n\}$ containing the indices not already used in the construction of the matrices $M^{(1)},M^{(2)},\cdots,M^{(k-1)}$, and $\Delta^{(1)}$ is the complete set $\{1,2,\cdots,n\}$. Note $M^{(k)}$ can be constructed if and only if $\sigma_k \neq 0$.

(6)
$$Q^{(k-1)} = M^{(k-1)}M^{(k-2)} \cdots M^{(1)} \qquad . \tag{2.28}$$

The algorithm now proceeds as follows.

Step 1: For $k = 1,2,\cdots,n$ do Steps 2-5.

Step 2: For $\lambda = \lambda_k$ compute $\det[\lambda_k I_{n-m} - R]$.

(a) If det = 0 perturb λ_k to $(\lambda_k + \delta\lambda_k)$ and repeat Step 2.

(b) If det $\neq$ 0, go to Step 3.

Step 3: For $\lambda = \lambda_k$ compute C_k (Equation (2.18)).

Step 4: For some $r\varepsilon\{\Delta^{(k)}\}$

(a) compute

$$[g_r^{(k)}]^T = f_r^{(k-1)} + h_r^{(k-1)}C_k \tag{2.29}$$

where $(f_r^{(k-1)} : h_r^{(k-1)})$ is the rth row of the transformation $Q^{(k-1)}$ (Equation (2.28)), $g_r^{(k)}$ is a m-vector and $h_r^{(k-1)}$ is a (n - m)-row vector.

(b) Compute

$$\sigma_k = [g_r^{(k)}]^T z_k \tag{2.30}$$

where z_k is the arbitrarily specified m design vector.

(i) If $\sigma_k \neq 0$, compute w_k (Equation (2.17)) and $M^{(k)}$ (Equation (2.27)) and go to Step 1.

(ii) If $\sigma_k = 0$, select another $r\varepsilon\{\Delta^{(k)}\}$ and return to Step 4a.

(iii) If $\sigma_k = 0$ for all $r\varepsilon\{\Delta^{(k)}\}$, go to Step 5.

Step 5: For some $r\varepsilon\{\Delta^{(k)}\}$

(a) If $g_r^{(k)} \neq 0$, perturb z_k to $(z_k + \delta z_k)$ to make $\sigma_k \neq 0$ (Equation (2.30)), compute w_k and $M^{(k)}$ and go to Step 1.

(b) If $g_r^{(k)} = 0$, select another $r\varepsilon\{\Delta^{(k)}\}$ and repeat Step 5a.

(c) If $g_r^{(k)} = 0$, all $r\varepsilon\{\Delta^{(k)}\}$, perturb λ_k to $(\lambda_k + \delta\lambda_k)$ and go to Step 2.

Step 6: Compute the feedback gains using (2.13-2.15).

Step 7: Stop.

In order to clearly see that the kth linearly independent eigenvector u_k can be synthesized provided $\sigma_k \neq 0$, assume, without loss of generality, that the first (k - 1) eigenvectors are generated with indices $r = 1,2,\cdots,(k-1)$. Then these vectors are transformed into a canonical form under $Q^{(k-1)}$ as

$$Q^{(k-1)}[u_1 \vdots u_2 \vdots \cdots u_{k-1}] = \begin{bmatrix} \mathrm{Diag}[\sigma_1,\sigma_2,\cdots,\sigma_{k-1}] \\ \hline 0 \end{bmatrix} , \qquad (2.31)$$

with

$$\sigma_i \neq 0 \ ; \quad i = 1,2,\cdots,k-1$$

and the projector spanning the subspace of these transformed eigenvectors has the simple form

$$P^{(k-1)} = \left[\begin{array}{c|c} I_{k-1} & 0 \\ \hline 0 & 0 \end{array}\right] \quad . \tag{2.32}$$

Now choosing the kth eigenvector so that its transformed vector $u_k^{(k)} = Q^{(k-1)}u_k$, causes

$$\sigma_k \neq 0 \ ; \quad r\varepsilon\{k,k+1,\cdots,n\} \quad , \tag{2.33}$$

and ensures the linear independence of u_k since the constraint (2.25) is clearly satisfied.

The following observations can be made regarding the algorithm outlined above.

1. The algorithm can be directly extended to assign complex pairs in quasi-diagonal form by noting that two eigenvectors are synthesized in one iteration and Equation (2.19) is used instead of (2.17). Further to ensure mutual independence between the two eigenvectors corresponding to ρ_j, the test condition $\sigma_k \neq 0$ in Step 4.b.i is modified to testing the nonsingularity of a 2 x 2 matrix Σ_j. This matrix is constructed by selecting two rows of the transformation matrix $Q^{(j-1)}$ obtained in the previous iteration and developing the condition similar to (2.30). The transformation matrix $M^{(j)}$ for the complex pair is obtained as a product of two transformations similar to (2.27) corresponding to the real eigenvectors $[u_j : u_{j+1}]$ associated with ρ_j.

2. Explicit evaluation of the eigenvalues of matrix R is not needed to detect coincident mode assignment. It is sufficient to determine the

appropriate determinant in Step 2, with this determinant available as a by product in the synthesis of the kth eigenvector when C_k is evaluated. If a mode is coincident with the spectrum of R then a perturbation of the mode is required to ensure $[\lambda_k I_{n-m} - R]$ is nonsingular and hence that (2.11) has a solution for any arbitrary z_k. The degree of perturbation needed depends on the numerical tolerance set on the evaluation of the determinant. Also since the eigenvalue shift in Step 2 is a designer's choice, system stability is always assured.

3. The iterative procedure in Step 4.b.ii attempts to meet exact eigenvalue/eigenvector specifications. In Step 5.b an attempt is made to meet exact eigenvalue specifications with slightly relaxed eigenvector specifications (z-vector). The test in Step 5.c indicates that the eigenvalue specification implies that the corresponding eigenvector will lie in the eigensubspace already synthesized, thus requiring a perturbation in eigenvalue specification.

4. Since the matrices $M^{(k)}$ have only one nontrivial column, coordinate transformations in (2.28) reduce to simple vector multiplications. Further, the inverse of U required to evaluate the feedback gains in (2.13) and (2.14) is easily evaluated by noting that $Q^{(n)}U$ has the general form

$$Q^{(n)}U = \mathrm{Diag}[\sigma_1, \sigma_2, \cdots, \sigma_n]S \qquad (2.34)$$

where S is an elementary permutation dependent on the sequence of generating the indices $r \varepsilon \{\Delta^{(k)}\}$ in Steps 4 and 5, the σ_k are the nonzero pivotal elements in (2.27) and

$$U^{-1} = S^{-1}\ \mathrm{Diag}[\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_n}]Q^{(n)} \quad . \qquad (2.35)$$

Also notice that

$$|\det[U]| = \left| \prod_{i=1}^{n} \sigma_i \right| \quad , \qquad (2.36)$$

since $|\det[Q^{(n)}]|$ and $|\det[S]|$ are unity.

Thus the numbers σ_i provide a good measure of the linear independence between the eigenvectors provided the eigenvector entries are scaled to a standard basis, the largest element of each eigenvector being normalized to 1 for example.

The computational advantages discussed above become apparent when the algorithm is mechanized for on-line applications. In Chapter IV these aspects will be explored in greater detail when the applicability of the algorithm as an on-line adaptive controller is investigated.

5. A noteworthy feature of the algorithm is that the eigenvectors do not explicitly undergo any change in the sequence of transformations (2.28). This keeps the mode coupling characteristics of C_k transparent during synthesis, a very desirable feature for an off-line synthesis problem.

2.4.2 A Numerical Example

The example given in section 2.3.2 will now be used to highlight the features of the algorithm described in section 2.4.1. Applying the algorithm step by step to the system in section 2.3.2 yields the following synthesis sequence.

First Mode: Let

$$\lambda_1 = -1 \ ; \quad z_1^T = (1 \quad 0) \quad .$$

$$\Delta^{(1)} = \{1,2,3\} \ ; \quad \text{Choose } r = 1 \quad .$$

Then

$$C_1 = (1 \quad 1) \ ; \quad g_1^{(1)} = (1 \quad 0)^T \quad \text{and} \quad \sigma_1 = 1 \quad ;$$

since $\sigma_1 \neq 0$ the 1st eigenvector can be synthesized as $u_1 = (1 \quad 0 \quad 1)^T$; and the transformation matrix $Q^{(1)}$ becomes

$$Q^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad .$$

Second Mode: Let

$$\lambda_2 = -2 \ ; \quad z_2^T = (-1 \quad 1) \quad .$$

$$\Delta^{(2)} = \{2,3\} \ ; \quad \text{Choose } r = 2 \quad .$$

Then

$$C_2 = (\tfrac{1}{2} \quad \tfrac{1}{2}) \ ; \quad g_2^{(2)} = (0 \quad 1)^T \quad \text{and} \quad \sigma_2 = 1 \quad ;$$

since $\sigma_2 \neq 0$, the second eigenvector can be synthesized as $u_2 = (-1 \quad 1 \quad 0)^T$; $u_2^{(2)} = (-1 \quad 1 \quad 1)^T$ and the transformation $Q^{(2)}$ becomes

$$Q^{(2)} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad .$$

Third Mode: Let

$$\lambda_3 = -1 \ ; \quad z_3^T = (-1 \quad 1) \quad .$$

$$\Delta^{(3)} = \{3\} \ ; \quad \text{thus } r = 3 \quad .$$

Then

$$C_3 = (1 \quad 1) \ ; \quad g_3^{(3)} = [0 \quad 0]^T \quad \text{and} \quad \sigma_3 = 0 \quad ;$$

since $\sigma_3 = 0$, the third eigenvector cannot be synthesized. Further since $g_3^{(3)} = [0 \quad 0]^T$, the eigenvalue specification cannot be met. This is obvious since the basis vectors spanning the 2-dimensional subspace corresponding to $\lambda_1 = -1$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and $\{u_1, u_2\}$ already span this subspace. Thus a second eigenvector associated with $\lambda = -1$ cannot be synthesized as revealed by the null vector $g_3^{(3)}$. This implies λ_3 must be perturbed slightly. Let $\lambda_3 = (-1-\varepsilon)$; $\varepsilon > 0$. Then

$$C_3 = (\frac{1}{1+\varepsilon} \quad \frac{1}{1+\varepsilon}) \ ; \quad g_3^{(3)} = (\frac{-\varepsilon}{1+\varepsilon} \quad \frac{-\varepsilon}{1+\varepsilon})^T \quad \text{and again} \quad \sigma_3 = 0 \quad ,$$

implying the third eigenvector still cannot be synthesized. In this case, since $g_3^{(3)}$ is not the null vector, the eigenvalue specification can be met but the eigenvector specification requires slight perturbation. Let $z_3^T = (-1 \quad 1+\delta)$; $\delta \neq 0$. Then

$$\sigma_3 = \frac{-\varepsilon\delta}{1 + \varepsilon} \neq 0 \quad ;$$

and u_3 can now be synthesized as

$$u_3 = (-1 \quad 1+\delta \quad \frac{\delta}{1+\varepsilon})^T \quad .$$

The closed-loop modal matrix U is

$$U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1+\delta \\ 1 & 0 & \frac{\delta}{1+\varepsilon} \end{bmatrix}$$

and $\Lambda = \mathrm{Diag}[-1 \quad -2 \quad -1-\varepsilon]$. Further

$$|\det[U]| = |\sigma_3| = \frac{\varepsilon\delta}{1 + \varepsilon}$$

since $\sigma_1 = \sigma_2 = 1$. Thus, in this case, setting a tolerance on the value of σ_3 would directly control the numerical illconditioning of U and consequently influence the choice of the perturbations ε and δ.

It is also interesting to note that if the sequence of assignment of modes were changed to $\lambda_1 = -1$, $\lambda_2 = -1$ and $\lambda_3 = -2$, then both eigenvectors corresponding to $\lambda_1 = -1$ could be synthesized as

$$[u_1 : u_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad ,$$

and $\lambda_3 = -2$ could still be assigned without perturbation since $g_3^{(3)} = (-\frac{1}{2} \quad -\frac{1}{2})^T$. However the eigenvector specification could not be met since $\sigma_3 = 0$. Thus a perturbation in z_3^T would allow completion of the synthesis.

2.4.3 Algorithm to Generate Special Modal Structures

For the mode decoupling designs introduced in section 2.2 it is possible to identify λ_i and z_i which yield minimal interacting eigenvector structures. For example if z_i is chosen to be in the null space of T, (2.11) has an attractive solution with w_i the null vector for all λ_i as

$$u_i = \begin{bmatrix} z_i \\ \cdots \\ 0 \end{bmatrix} \quad . \tag{2.37}$$

In the algorithm presented in section 2.4.1, if an assignable mode λ_i happened to coincide with the spectrum of R, a perturbation in its assignment was needed to ensure (2.11) had a solution for any arbitrary z_i. However for the coincident spectrum case, (2.11) also has a solution for z_i identically zero as

$$u_i = \begin{bmatrix} 0 \\ \cdots \\ w_i \end{bmatrix} \tag{2.38}$$

where w_i is the eigenvector of matrix R corresponding to the mode λ_i. Since u_i is completely defined in (2.38) a deflation technique suggested in [7] can be employed not only to assign the coincident mode λ_i but also reduce the dimension of the pole-placement problem for the remaining mode assignments as follows. It is assumed, in the discussion to follow that the eigenvalues to be assigned are distinct.

Case 1: Coincident real eigenvalues

$$\lambda_i (i = 1,2,\cdots,k_1) \; ; \quad k_1 \leq n - m \qquad .$$

Let λ_1 be a coincident eigenvalue. Then the corresponding eigenvector takes the form

$$u_1^{(0)} = \begin{bmatrix} 0 \\ \cdots\cdot \\ w_1 \end{bmatrix} \tag{2.39}$$

where w_1 satisfies

$$Rw_1 = \lambda_1 w_1 \qquad . \tag{2.40}$$

It can be proved (Appendix C) that given w_1 there exists a matrix F_1 such that

$$F_1 R F_1^{-1} = \left[\begin{array}{c|c} R_1 & 0 \\ \hline r_1 & \lambda_1 \end{array}\right] \qquad . \tag{2.41}$$

Let

$$x = L_0 \hat{x}$$

where

$$L_0 = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & F_1 \end{array}\right] \qquad ,$$

so that (2.1) under this coordinate transformation is given by

$$\dot{\hat{x}} = \left[\begin{array}{c|c} D & E\,F_1^{-1} \\ \hline F_1 T & \left[\begin{array}{c|c} R_1 & 0 \\ \hline r_1 & \lambda_1 \end{array}\right] \end{array}\right] \hat{x} + \begin{bmatrix} B_1 \\ \cdots\cdots \\ 0 \end{bmatrix} u \,. \qquad (2.43)$$

Further $u_1^{(0)}$ is transformed to

$$[\hat{u}_1^{(0)}]^T = (0 \quad 0 \quad \cdots \quad \sigma_1)^T \quad \text{with} \quad \sigma_1 \neq 0 \quad .$$

Equation (2.43) can now be written in detail as

$$\dot{\hat{x}} = \left[\begin{array}{c|c|c} D & E_1 & f_1 \\ \hline T_1 & R_1 & 0 \\ \hline t_1 & r_1 & \lambda_1 \end{array}\right] \hat{x} + \begin{bmatrix} B_1 \\ \cdots\cdots \\ 0 \\ \cdots\cdots \\ 0 \end{bmatrix} u \qquad (2.44)$$

where

$$EF_1^{-1} = [E_1 \; : \; f_1] \qquad ,$$

$$F_1 T = \begin{bmatrix} T_1 \\ \cdots\cdots \\ t_1 \end{bmatrix} \qquad ,$$

with f_1 and t_1 being m x 1 and 1 x m vectors, respectively. At this stage the first mode λ_1 and its eigenvector have been synthesized. Thus we only need to consider the (n - 1)th order subsystem for the remaining assignments as

$$\dot{x}_1 = \left[\begin{array}{c|c} D & E_1 \\ \hline T_1 & R_1 \end{array}\right] x_1 + \left[\begin{array}{c} B_1 \\ \cdots\cdot \\ 0 \end{array}\right] u \qquad (2.45)$$

where x_1 is an $(n - 1)$-vector, and $[B_1 : 0]$ is $(n - 1) \times m$. Since it can be proved [8] that (T_1,R_1) is a controllable pair, the deflation technique can be reapplied to assign the second coincident eigenvalue λ_2 to the subsystem (2.45) by the transformation

$$x_1 = L_1\hat{x}_1 \qquad , \qquad (2.46)$$

where

$$L_1 = \left[\begin{array}{c|c} I_m & 0 \\ \hline 0 & F_2 \end{array}\right] \qquad ,$$

is a $(n - 1) \times (n - 1)$ matrix and F_2 is such that

$$F_2R_1F_2^{-1} = \left[\begin{array}{c|c} R_2 & 0 \\ \hline r_2 & \lambda_2 \end{array}\right] \qquad . \qquad (2.47)$$

In general after k_1 deflation and reduction transformations (2.45) has the form

$$\dot{x}_{k_1} = \left[\begin{array}{c|c} D & E_{k_1} \\ \hline T_{k_1} & R_{k_1} \end{array}\right] x_{k_1} + \left[\begin{array}{c} B_1 \\ \cdots\cdot \\ 0 \end{array}\right] u \quad , \qquad (2.48)$$

with (T_k, R_{k_1}) forming a controllable pair.

Now the remaining $(n - k_1)$ noncoincident eigenvalues can be assigned by the algorithm of section 2.4.1 for the system (2.48). Let $U^{(k_1)}$ be the $(n - k_1)$ nonsingular modal matrix corresponding to the pole assignment in (2.48). Then the n dimensional modal matrix corresponding to (2.4) can now be constructed by the following sequence of expansion/inflation transformations.

The eigenvectors of (2.48) have the form

$$u_i^{(k_1)} = \begin{bmatrix} z_i \\ \cdots \\ w_i^{(k_1)} \end{bmatrix}, \quad i = 1,2,\cdots,(n-k_1) \tag{2.49}$$

where $w_i^{(k_1)}$ is a $(n - m - k_1)$-vector.

Step 1: Expansion

Construct

$$\hat{u}_i^{(k_1-1)} = \begin{bmatrix} u_i^{(k_1)} \\ \cdots \\ \eta_i^{(k_1)} \end{bmatrix}; \quad i = 1,2,\cdots,(n-k_1) \tag{2.50}$$

where $\eta_i^{(k_1)}$ is computed from the relation

$$-r^{(k_1)} w_i^{(k_1)} + (\lambda_i - \lambda_{k_1})\eta_i^{(k_1)} = t^{(k_1)} z_i \tag{2.51}$$

where $r^{(k_1)}$ and $t^{(k_1)}$ are the first rows of R_{k_1} and T_{k_1}, respectively. Also note $\lambda_i \neq \lambda_{k_1}$ $(i = 1,2,\cdots,(n-k_1)$. The expanded $(n - k_1 + 1) \times (n - k_1 + 1)$ modal matrix has the form

$$\hat{U}^{(k_1-1)} = \left[\begin{array}{c|c} u_i^{(k_1)} & 0 \\ \hline y^{(k_1)} & \sigma_{k_1} \end{array}\right] , \qquad (2.52)$$

where $y^{(k_1)} = (\eta_1^{(k_1)}, \eta_2^{(k_1)}, \cdots, \eta_{n-k_1}^{(k_1)})$ and $\sigma_{k_1} \neq 0$.

Step 2: Inflation

Compute

$$U^{(k_1-1)} = [L_{k_1-1}]^{-1} \hat{U}^{(k_1-1)} . \qquad (2.53)$$

Step 3: Repeat Steps 1 and 2 until $U^{(0)}$ is constructed, which is the required nonsingular modal matrix.

Case 2: Coincident complex pairs

$$\rho_j (j = 1,2,\cdots,k_2) .$$

Let ρ_1 be a coincident mode. The corresponding real eigenvectors have the form

$$u_{1,2} = \begin{bmatrix} 0 \\ \cdots\cdot \\ w_1 : w_2 \end{bmatrix} , \qquad (2.54)$$

where $[w_1 : w_2]$ satisfy

$$\left[\begin{array}{c|c} \alpha_1 I_{n-m} - R & -\beta_1 I_{n-m} \\ \hline \beta_1 I_{n-m} & \alpha_1 I_{n-m} - R \end{array}\right] \begin{bmatrix} w_1 \\ \cdots\cdot \\ w_2 \end{bmatrix} = 0 \qquad (2.55)$$

It is again possible to prove (Appendix C) that there exists a matrix F_1 such that

$$F_1 R F_1^{-1} = \left[\begin{array}{c|c} R_1 & 0 \\ \hline R_2 & \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix} \end{array}\right] , \qquad (2.56)$$

and thus ρ_1 has been deflated. Procedures similar to the real coincident eigenvalues case can now be developed to assign these complex modes.

2.4.4 Assignment of Multiple Real Eigenvalues Non-Coincident With Spectrum of R

Multiple root assignment is generally of academic interest since, as emphasized earlier, exact assignment of modes is not needed. However the following observations can be made regarding the possible assignment of multiple roots.

As noted earlier, (2.11) has m-linearly independent solutions. Thus if the algebraic multiplicity r of a repeated eigenvalue λ_r is less than or equal to the geometric multiplicity ($r \leq m$), we can indeed generate a diagonal canonical form for the closed-loop system. If $r > m$ the canonical form degenerates to the Jordon form with m Jordon chains as

$$\Lambda_r = \left[\begin{array}{c|c|c|c} J_1 & 0 & \cdots & 0 \\ \hline 0 & J_2 & & \vdots \\ \vdots & & \ddots & 0 \\ \hline 0 & \cdots & \cdots & J_m \end{array}\right] \qquad (2.57)$$

where each Jordon block has the form

$$\begin{bmatrix} \lambda_r & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \lambda_r & 1 & 0 & & \cdot \\ \cdot & 0 & \cdot & & & \cdot \\ \cdot & & & \cdot & & 0 \\ \cdot & & & & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & \lambda_r \end{bmatrix} \tag{2.58}$$

of size $(\eta_i \times \eta_i)$. The size of each Jordon block is arbitrary except

$$\sum_{i=1}^{m} \eta_i = r \quad . \tag{2.59}$$

For each J_i(2.12) has the structure

$$[w_1 \cdots w_{\eta_i}]J_i - R[w_1 \cdots w_{\eta_i}] = T[z_1 \cdots z_{\eta_i}] \; . \tag{2.60}$$

The first vector

$$\begin{bmatrix} z_1 \\ \cdots\cdot \\ w_1 \end{bmatrix}$$

is an eigenvector and is synthesized directly by (2.11). The remaining generalized vectors of (2.60) can now be recursively synthesized using the algorithm in section 2.4.1, and in order to ensure all the vectors generated are linearly independent, it may be necessary to synthesize the multiple roots first. The problem becomes more complex if many sets of multiple eigenvalues are to be assigned since there is no way to guarantee the linear independence of all the eigenvectors. Indeed the example in section 2.4.2 demonstrates graphically this fact. Thus from a practical synthesis point of view it may not be worthwhile to attempt synthesizing Jordon forms of the type (2.57), and it is better to synthesize an equivalent diagonal matrix which has roots arbitrarily close to λ_r.

2.5 Summary

In this chapter a new formulation of the pole-placement problem was introduced and solved. By establishing the parametric equivalence between the nonunique feedback matrix K and the arbitrary closed-loop modal entries (z), it was possible to directly interpret the effect of each design choice on the resultant closed-loop response of the system. This becomes possible since in essence (2.11) and (2.12) provide the complete spectral characterization of all the closed-loop eigenvector structures for the given triple (A,B,Λ). This characterization has an important property that if the design process fails to satisfy a given eigenvalue/eigenvector specification, it does so by showing no control law could meet that specification. This completely eliminates unnecessary synthesis effort being expended on an unattainable design objective.

It also became evident from the analysis that meeting exact pole-specifications resulted in significant curtailment in the choice of eigenvector forms and consequently limited the flexibility in shaping the closed-loop response. However, for practical systems, pole-specifications are rarely intended to be exact but are usually required to lie in a subset of the stable complex plane. This observation lead to the development of the algorithm in section 2.4.1 which allows maximal flexibility in the choice of closed-loop modal entries and hence the dynamic response, while assuring the generation of an n-dimensional eigenspace required for the solution. The simplicity of the algorithm makes it admirably suited for computer implementation as a powerful iterative/interactive design tool for multivariable system synthesis.

The utility of the new formulation as a practical synthesis procedure is demonstrated in the next chapter by designing a hover controller for a helicopter. The potentialities of the algorithm developed in section 2.4.1 as an on-line adaptive controller will be explored in Chapter IV.

CHAPTER III

APPLICATION OF SPECTRAL SYNTHESIS TO THE DESIGN OF A HELICOPTER HOVER CONTROLLER

3.1 Introduction

The design of fixed gain feedback controllers to improve the response of a system by simultaneous control of modes and the associated modal structure was introduced in Chapter II. The application of the technique to the design of a hover controller for a helicopter will be illustrated in this chapter. The hover controller problem will be considered for three reasons. First, helicopter dynamics, by virtue of substantial longitudinal/lateral mode coupling and cross axis coupling, provide an excellent case study to illustrate the application of modal control to axis decoupling, stabilization and the minimization of mode interaction. Second, the 9th order linear model of the helicopter certainly represents a non-trivial problem of significant synthesis complexity. Finally, helicopter hover dynamics have been used as a representative model in the literature to illustrate various synthesis techniques based on optimal control theory [9,10] and classical modal control theory [11].

3.2 A Helicopter Hover Controller

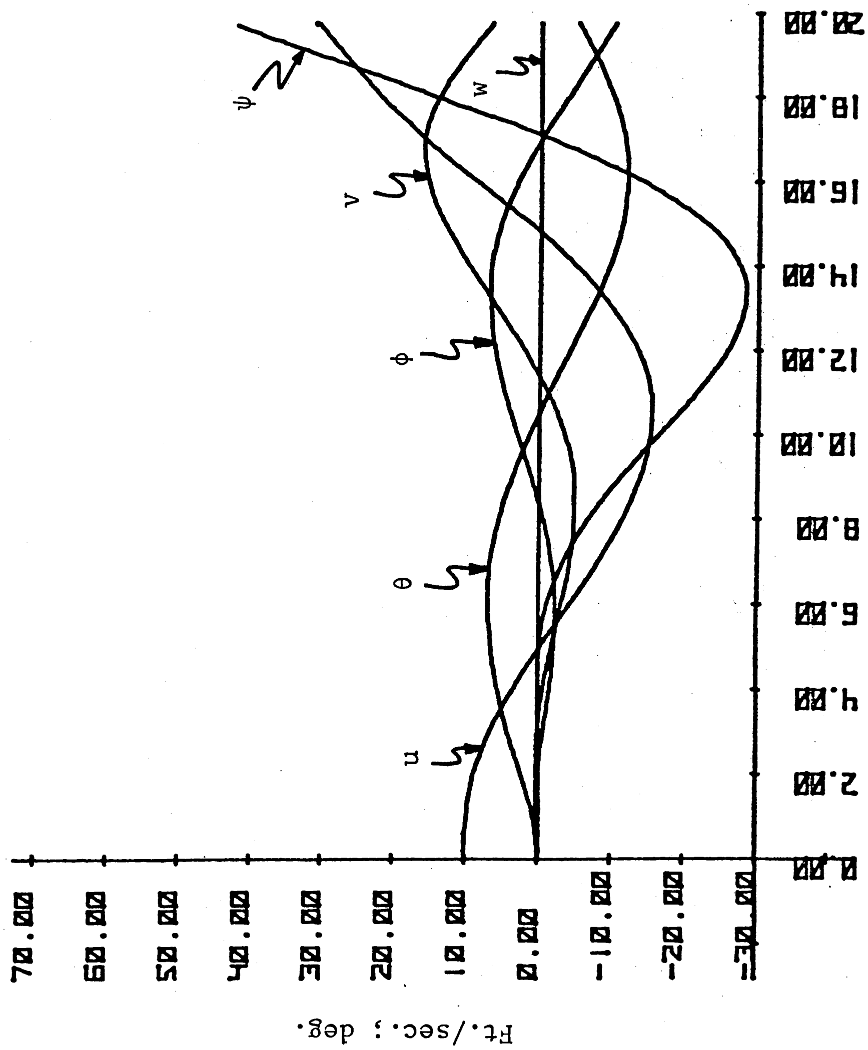
The design of controllers for helicopters is more difficult than for fixed-wing aircraft, since the lateral and longitudinal motions are highly interactive. This coupling not only causes unstable responses to affect all variables, but substantially distorts the intended non-interactive functions of the pilot controls. A typical initial condition response is presented in Figure 1, in terms of variables to be defined later. Even if stabilizing controllers are designed (pole-placement only), the pilot work load is still high due to the cross-coupling which follows application of any one input. This example, therefore, illustrates the use of eigenvalue/eigenvector modification to both stabilize responses and decouple lateral and longitudinal motions.

The 9th order linear perturbation model of the Sikorsky SH-3D helicopter in a hover mode has state variables of longitudinal (u), lateral (v), and vertical (w) velocities in feet/second; pitch (q), roll (p) and yaw (r) rates in degrees/second; and pitch (θ), roll (ϕ) and yaw (ψ) angles in degrees. The inputs are main rotor collective pitch (u_C), tail rotor collective pitch (u_T), longitudinal cyclic pitch (u_P) and lateral cyclic pitch (u_R), all in degrees.

The open-loop dynamics may then be structured as

$$\begin{bmatrix} \dot{x}_1 \\ \cdots \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} + \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \begin{bmatrix} u_1 \\ \cdots \\ u_2 \end{bmatrix} , \tag{3.1}$$

where $x_1 = [u \; w \; q \; \theta]^T$ and $u_1 = [u_P \; u_C]^T$ are the longitudinal variables and controls, and $x_2 = [v \; p \; \phi \; r \; \psi]^T$ and $u_2 = [u_R \; u_T]^T$ are the lateral variables and controls. The normalized system matrices [11], scaled by a rotor tip speed of 680 ft/sec, are given in Table I, along with the open-loop eigenvalues.



Time (sec.)

Figure 1. Open-Loop Response; u(0) = 10 ft./sec.

TABLE I

OPEN-LOOP DYNAMICS

$$
A = \left[\begin{array}{cccc|ccccc}
-0.016 & 0 & 0.0297 & -0.5934 & -0.0047 & -0.0012 & 0 & 0 & 0 \\
0 & -0.3242 & 0.0021 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.1666 & 0.0843 & -0.5420 & 0 & 0.062 & 0.548 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0.0047 & -0.0007 & -0.0285 & 0 & -0.033 & -0.0297 & 0.5934 & 0.0107 & 0 \\
0.2199 & -0.0137 & -1.94 & 0 & -0.6109 & -1.96 & 0 & 0.01 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0.0013 & -0.0163 & -0.0083 & 0 & 0.471 & -0.0043 & 0 & -0.303 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

$$
B^T = \left[\begin{array}{cccc|ccccc}
60.7609 & -5.0318 & -2.13 & 0 & 0.1187 & 0.69 & 0 & 0.005 & 0 \\
0.5934 & 0 & -6.15 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0.5934 & 21.81 & 0 & 0.174 & 0 \\
0 & 0 & 0 & 0 & 0.2611 & 0.3475 & 0 & -7.48 & 0
\end{array}\right]
$$

Open-loop Eigenvalues: $0, -0.305, -0.324, 0.08 \pm j0.313, -1.31 \pm j0.65, -0.047 \pm j0.414$.

The problem of synthesizing systems wherein independent single inputs influence only specified single outputs has been well studied as the classical decoupling problem [12,13]. A combination of feedforward and feedback control law of the form

$$u = Kx + Gv \quad , \qquad (3.2)$$

where G is a m x m nonsingular feedforward matrix, is required to accomplish this decoupling. This is because the coupling in the system arises both due to input mixing through the distribution matrix B and the modal matrix which controls the interaction between the response variables. In systems where this complete input/output decoupling can be achieved, some degree of design freedom is generally lost rendering in some cases even arbitrary assignment of all system poles impossible [12]. An examination of Table I reveals that for the helicopter system the input matrix B does not substantially contribute to cross-input coupling. Consequently if the closed-loop modal structure is chosen to be diagonally dominant, good non-interacting control will result even without resorting to complex control laws of the form (3.2), while fully retaining the multi-input design freedom to shape the dynamics of the feedback system.

To pose the specifications, a control $u = Kx$ is desired which uncouples lateral and longitudinal motions and also assigns eigenvalues selected to meet desired handling criteria. In the discussions to follow a response variable is assumed to be not influenced by a mode if its entry in the associated eigenvector is less than 1% of the dominant entry.

At the outset it is seen that if a modal matrix $U = I_n$ were synthesized, in addition to eigenvalue assignment, the specifications defined earlier would be completely met. This is possible only in the trivial case when the system has n-inputs. Thus in the present case a judicious choice of the modes (λ_i) and the free design entries of the eigenvectors (z_i) must be made to evolve a closed-loop modal matrix which is at least block diagonally dominant. The role played by the coupling matrices $C_i(\lambda_i)$ in making these selections was discussed in section 2.3.1. Using these design aids it was possible to associate the following eigenvalues with the indicated variables.

Eigenvalue	Response Variable
-4.5	u
-0.324	w
$-1.5 \pm j1$	q, θ
-0.3	v
$-1.5 \pm j1$	p, ϕ
$-1.5 \pm j1$	r, ψ

The vertical velocity mode was specially chosen to yield the coincident spectrum case discussed in section 2.4.3. This choice gave an exceptionally good non-interacting structure for the associated eigenvector, with the vertical velocity variable having the dominant entry. It was also found impossible to decouple roll dynamics from lateral velocity response and vice-versa, without introducing coupling into the longitudinal and heading (yaw) response variables. For example if the free parameters (z-vector) corresponding to the lateral velocity mode are selected as

$$z^T = (z_q z_v z_r z_u)^T = (0 \quad 1 \quad 0 \quad 0)^T \qquad ,$$

where the subscripts indicate the respective response variable eigenvector entries, then the remaining eigenvector entries

$$w^T = (w_w, w_p, w_\phi, w_\theta, w_\psi)^T$$

can be evaluated for each mode selection using (2.11). Figure 2 shows the variation of these coupling coefficients for the range of assignable modes. The best compromise between response time and low roll coupling appears to be achieved for a lateral velocity mode of -0.3. Notice also the large variation of the vertical velocity coefficient (w_w) near the mode -0.324, which corresponds to the coincident eigenvalue of matrix R of (2.11). A similar analysis was adopted to select the roll mode as $-1.5 \pm j1$ to minimize the influence of this mode on the lateral velocity variable. It is also evident from the physics of the process that this roll/lateral velocity coupling is acceptable since helicopters achieve lateral motions by rolling in the intended direction.

With these initial selection of modes (λ_i) and design vectors (z_i) the algorithm of Chapter II was used to generate a nonsingular modal matrix and to compute the feedback gain matrix K. Table II gives the resulting control and closed-loop state matrices. Table III gives the final modal matrix and the diagonal matrix of eigenvalues. The block diagonally dominant structure of the resultant modal matrix is apparent with the reduction in the coupling between the longitudinal and lateral dynamics best measured in terms of a matrix norm as shown in Table IV. It is important to note that in addition to longitudinal/lateral interaction minimization, excellent decoupling between the motions in the three rotational axes has also been achieved. The initial condition responses in Figures 3-8 illustrate that excellent mode decoupling has

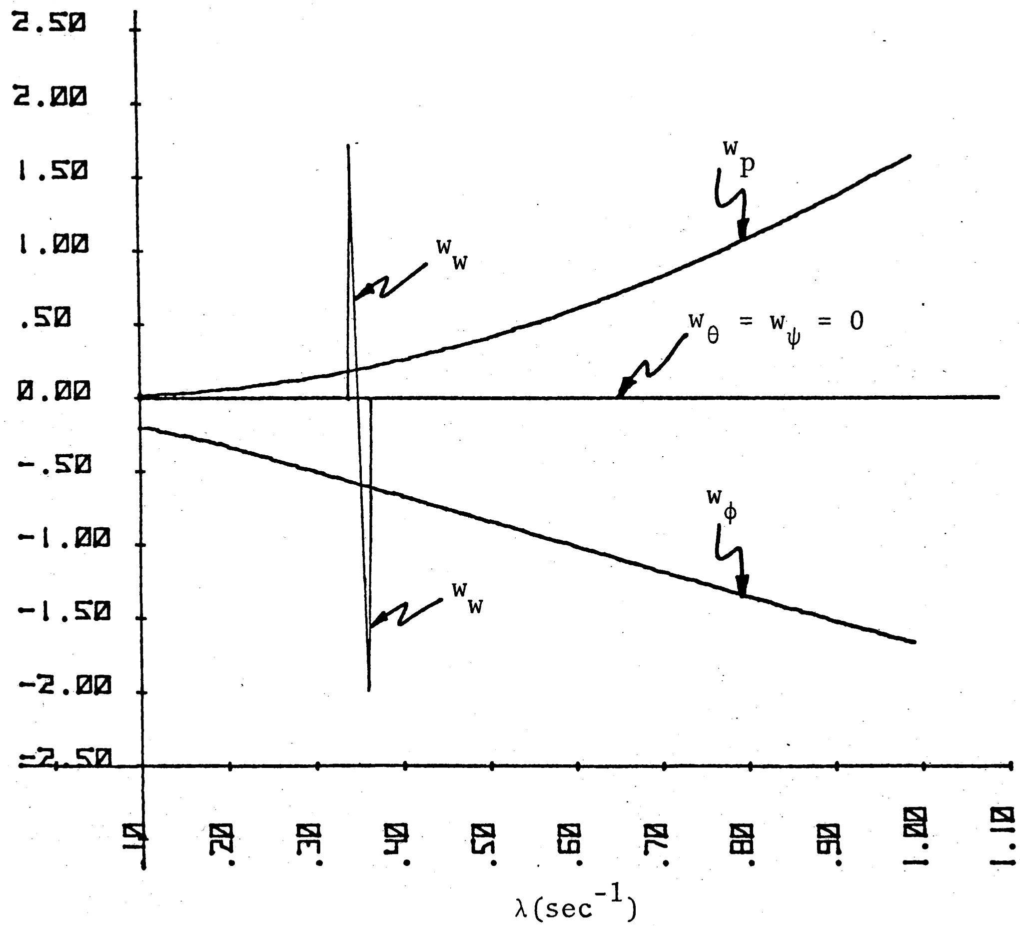


Figure 2. Variation of Mode Coupling Coefficients

TABLE II

CLOSED-LOOP DYNAMICS

$$\hat{A} = \begin{bmatrix} -4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3739 & -0.3235 & 0.0243 & -0.0243 & 0.0001 & 0.0044 & 0 & 0 & 0 \\ -0.0001 & -0.0011 & -3 & -3.25 & 0.0001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0086 & -0.0008 & 0.0241 & 0.0015 & -0.0455 & -0.0655 & 0.4805 & 0.0996 & 0.1093 \\ 0.0036 & 0.0045 & -0.0003 & 0.0386 & -1.6405 & -3.2545 & -4.1095 & -0.0552 & 0.0004 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -3.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.0743 & -0.0001 & -0.0044 & 0.0046 & 0 & -0.0009 & 0 & 0 & 0 \\ 0.0527 & 0.0139 & 0.4012 & 0.5268 & 0.0101 & 0.0894 & 0 & 0 & 0 \\ -0.0076 & 0.0009 & 0.0891 & 0.0016 & -0.0482 & -0.0593 & -0.1884 & -0.0087 & -0.0069 \\ 0 & 0.0022 & 0.001 & 0 & 0.0619 & -0.002 & -0.0044 & 0.3604 & 0.4343 \end{bmatrix}$$

TABLE III

CLOSED-LOOP MODAL STRUCTURE

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0895 & 12.9761 & 0.0052 & -0.0254 & 0.0225 & -0.0012 & -0.0029 & 0 & 0 \\ 0 & 0.002 & 1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0062 & -0.3077 & -0.5385 & 0 & 0 & 0 & 0 & 0 \\ 0.0019 & 0.0353 & -0.0074 & -0.0128 & 0.6550 & 0.006 & 0.2045 & -0.0337 & -0.0169 \\ 0 & 0 & 0 & 0 & 0.1 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3333 & -0.3077 & -0.5385 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3077 & -0.5385 \end{bmatrix}$$

$$\Lambda = \mathrm{Diag}[-4.5 \quad -0.324 \quad -1.5 \pm j1 \quad -0.3 \quad -1.5 \pm j1 \quad -1.5 \pm j1 \,]$$

TABLE IV

NORMED LONGITUDINAL/LATERAL CROSS COUPLING

	$\frac{\|\|A_{12}\|\|}{\|\|A_{11}\|\|}$	$\frac{\|\|A_{21}\|\|}{\|\|A_{11}\|\|}$	$\frac{\|\|A_{12}\|\|}{\|\|A_{22}\|\|}$	$\frac{\|\|A_{21}\|\|}{\|\|A_{22}\|\|}$
OPEN-LOOP	0.2233	0.8099	0.1022	0.3706
CLOSED-LOOP	0.0004	0.0066	0.0003	0.0045

$$||A|| = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$$

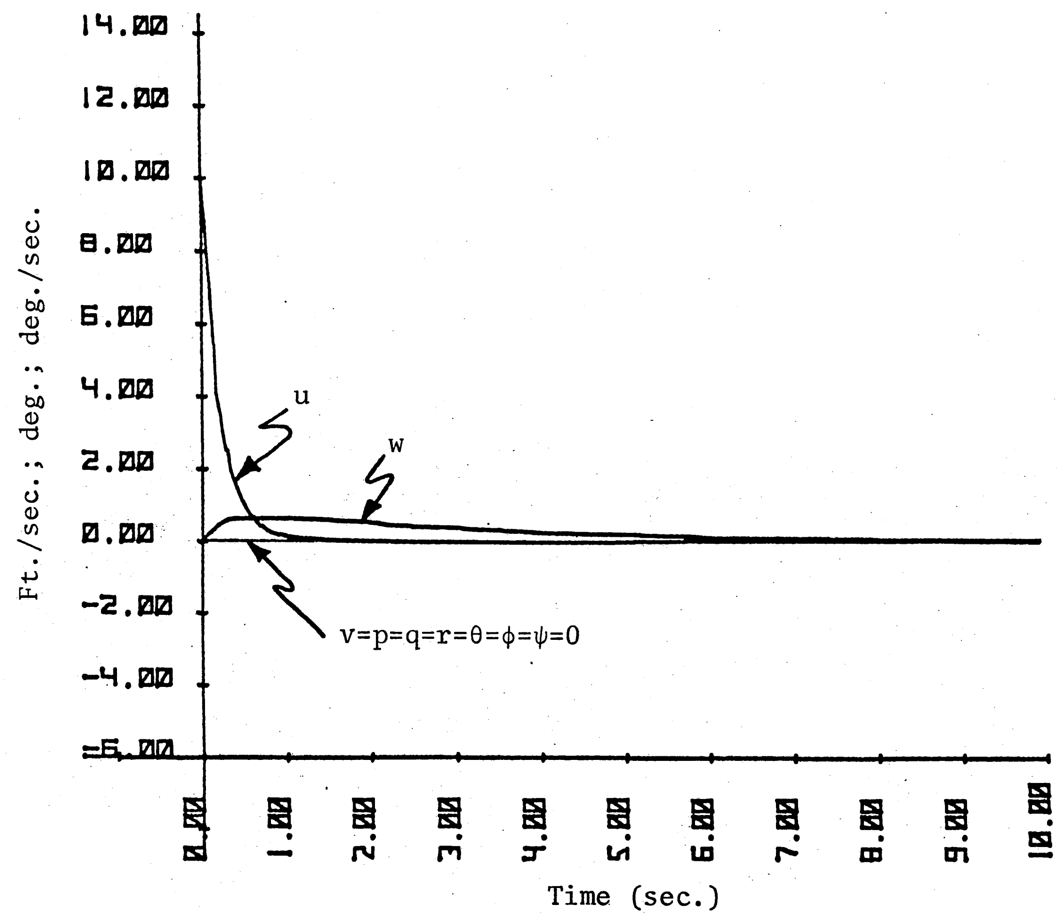


Figure 3. Closed-Loop Response; u(0) = 10 ft./sec.

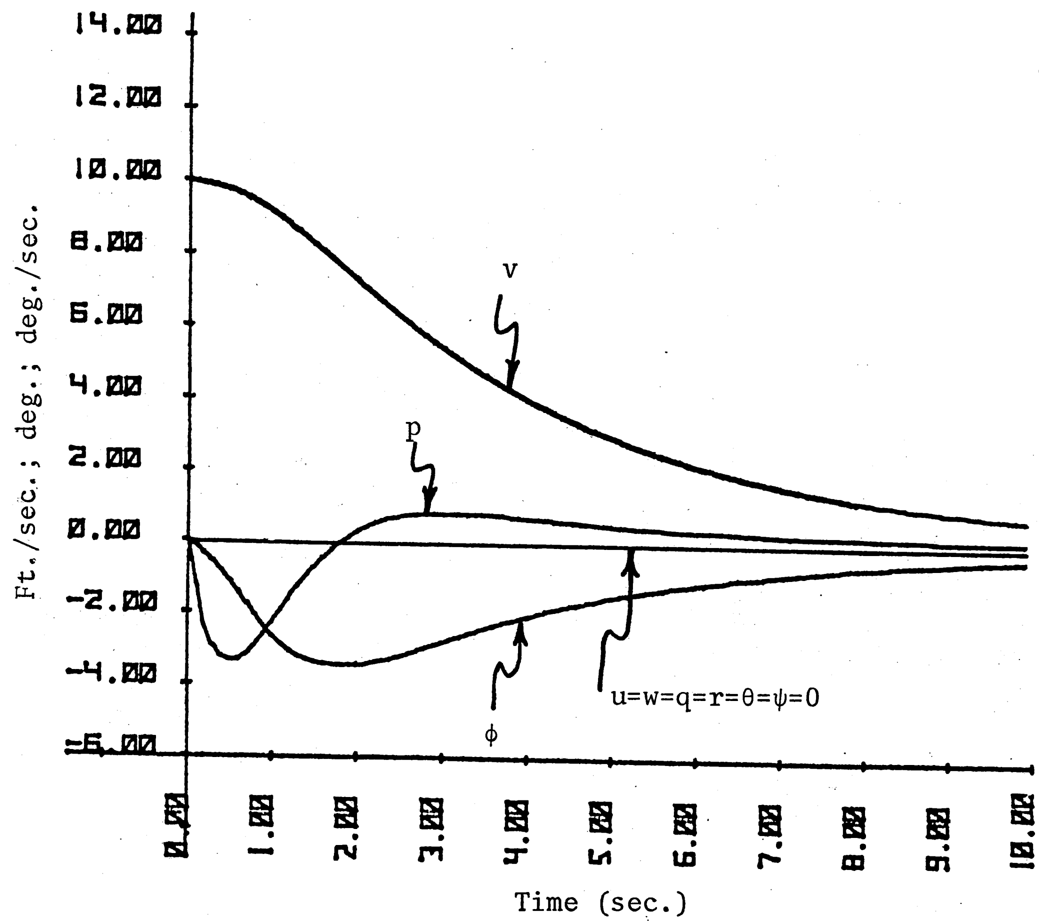


Figure 4. Closed-Loop Response; v(0) = 10 ft./sec.

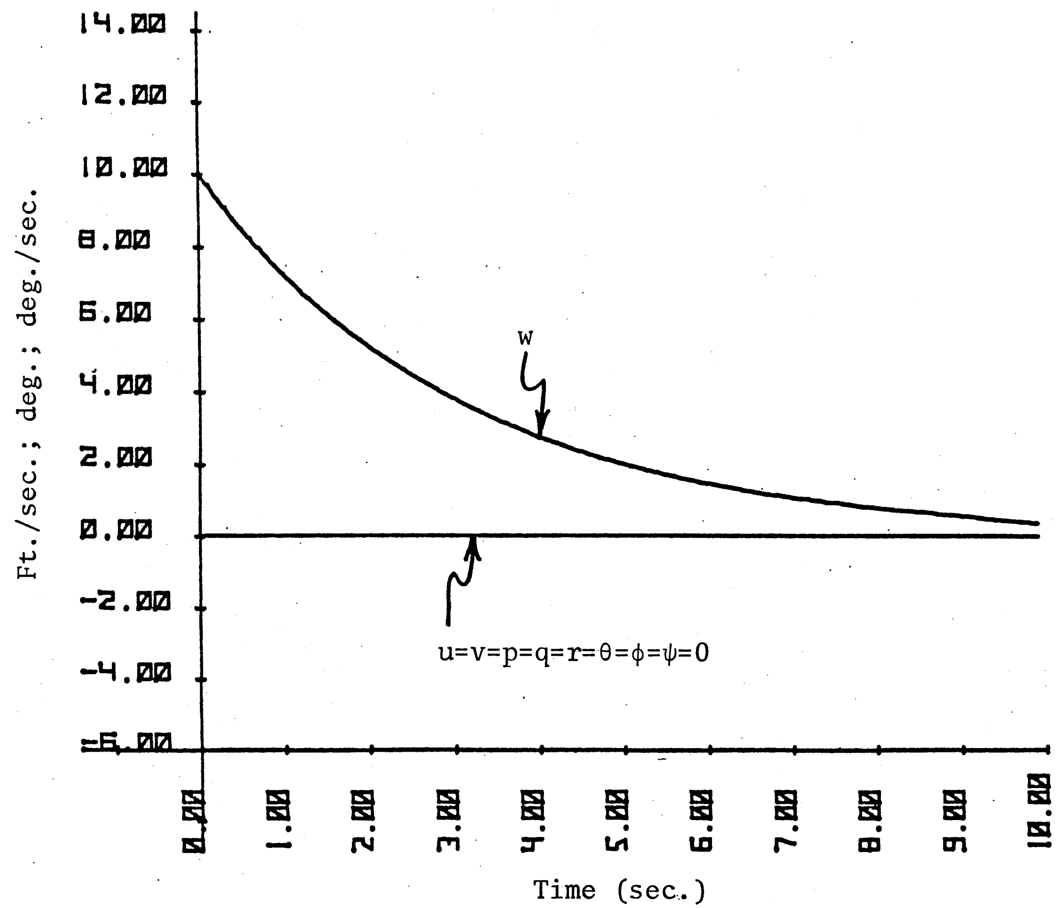


Figure 5. Closed-Loop Response; w(0) = 10 ft./sec.

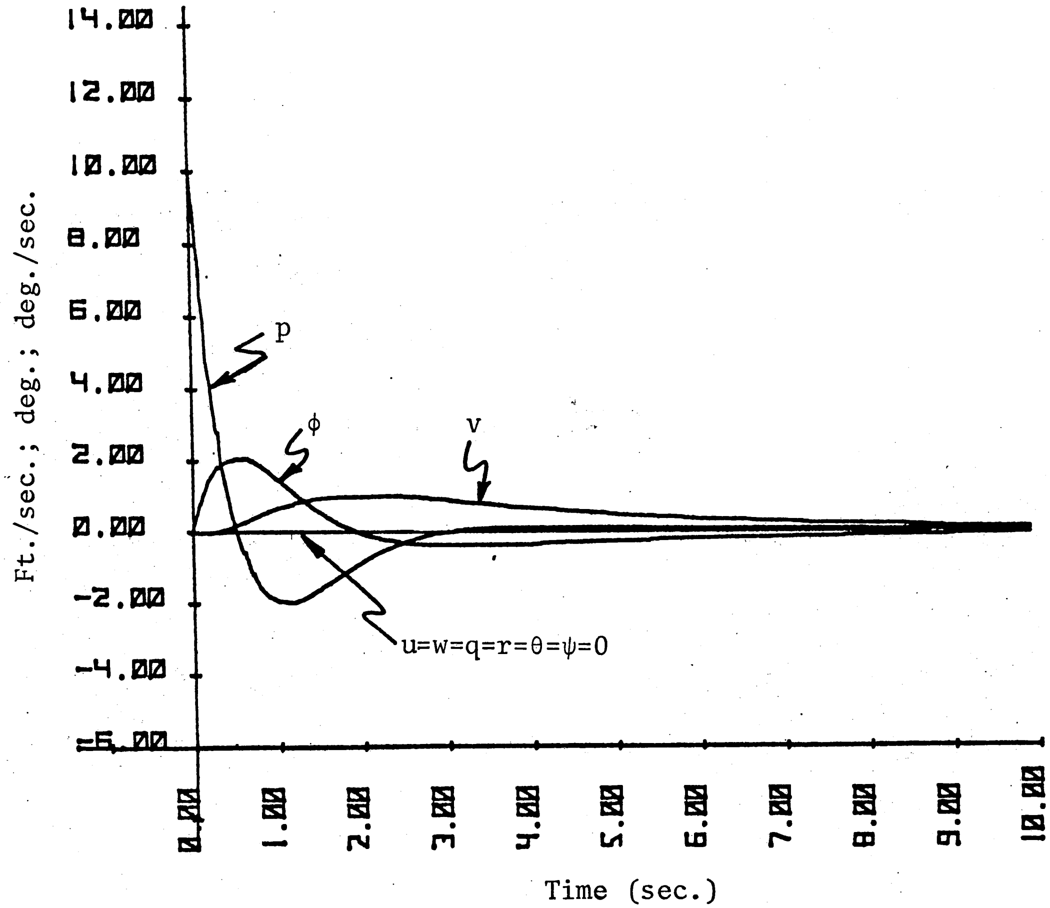


Figure 6. Closed-Loop Response; p(0) = 10 deg./sec.

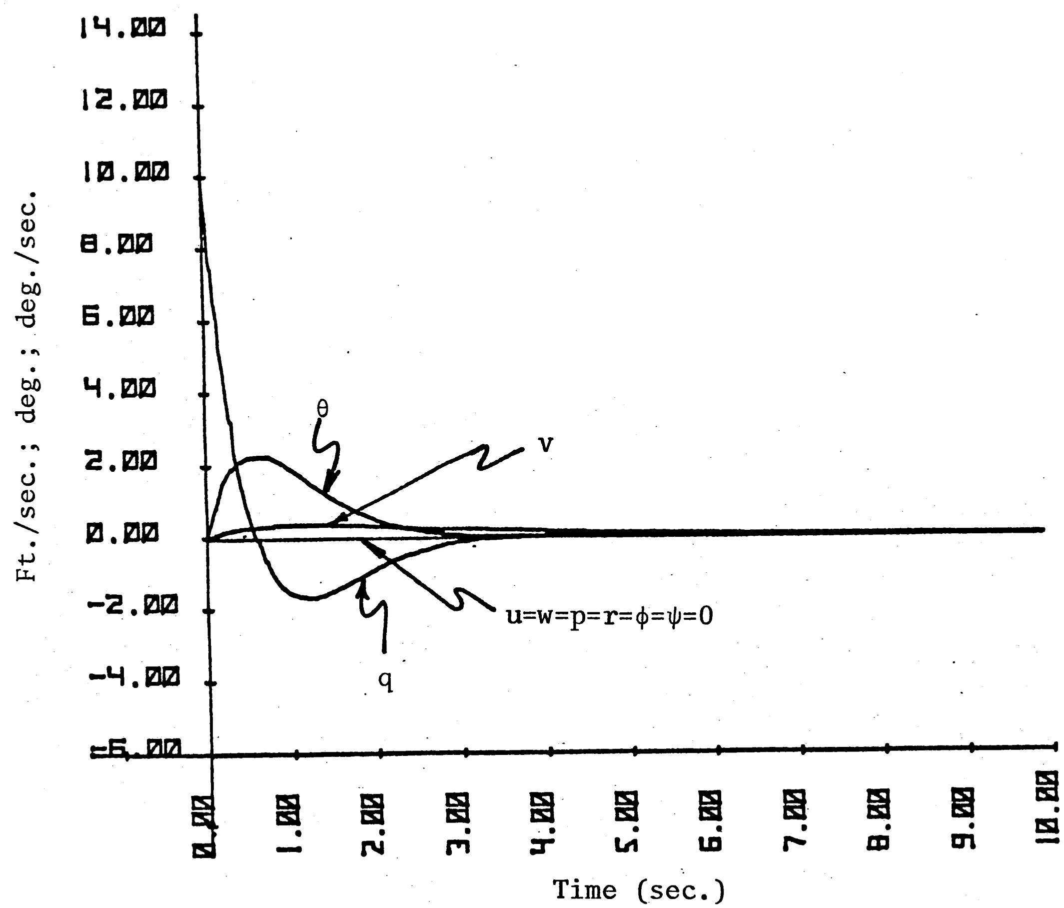


Figure 7. Closed-Loop Response; q(0) = 10 deg./sec.

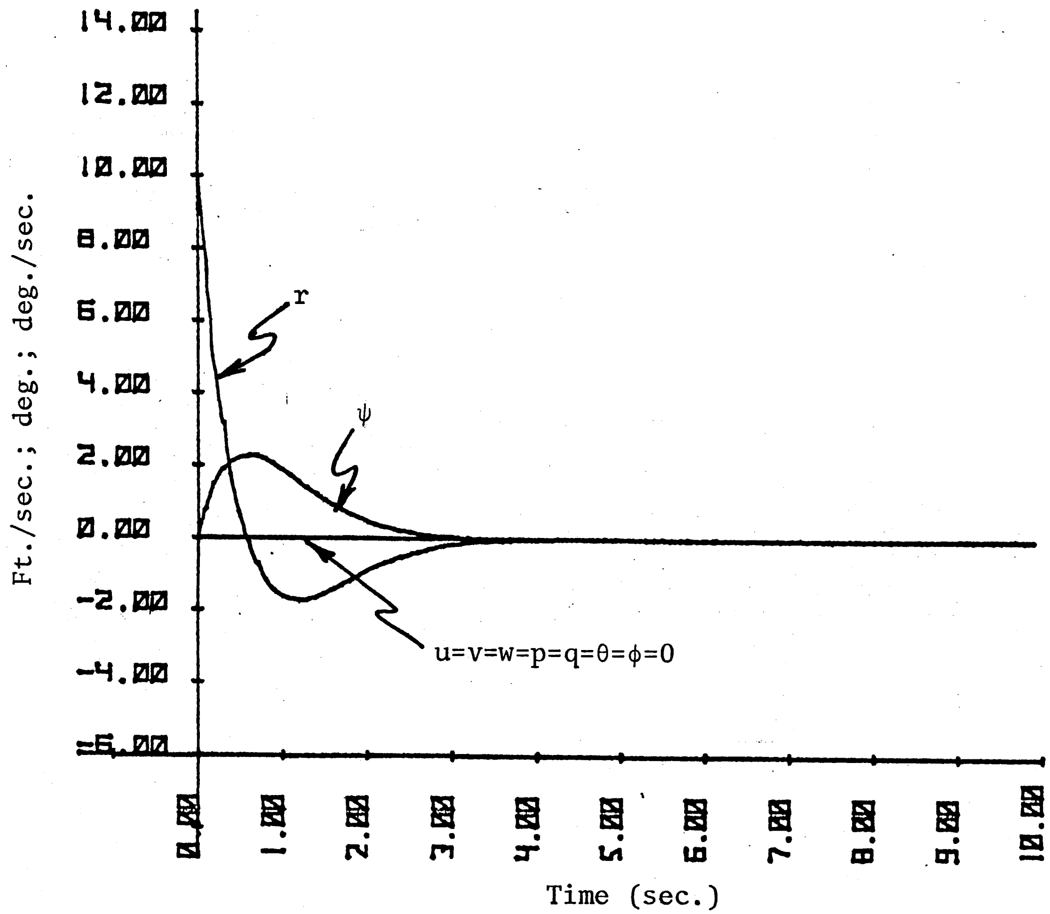


Figure 8. Closed-Loop Response; r(0) = 10 deg./sec.

resulted in all cases except for mode interactions between lateral velocity and roll variables, coupling required by the physics of the process. The closed-loop system was also subjected to step control inputs and the responses are summarized in Figures 9-12. The responses confirm that good non-interacting behavior, as predicted, has been achieved with intended variables independently excited by corresponding inputs. Notice that substantial transients in rate variables (q,p,r) are physically required for large steady state angular deflections (θ,ϕ,ψ), and similarly large transients are required in variables p and ϕ to induce large steady state values in v. These response results clearly show that exceptional performance has been achieved.

3.3 Summary

In this chapter the synthesis techniques of Chapter II were used to design a helicopter hover controller. This example highlights the mode-decoupling design concepts introduced in Chapter II. By utilizing the spectral characterization it was possible to establish that lateral motions could not be made non-interacting with roll dynamics without introducing unwanted coupling into longitudinal/heading motions. This demonstrates the clear insight the new characterization provides in visualizing attainable closed-loop modal structures. The synthesis procedure discussed in this chapter is purely an off-line effort requiring interaction between the designer and the computer. The features of the algorithm of Chapter II when used for on-line applications will be studied in the next chapter.

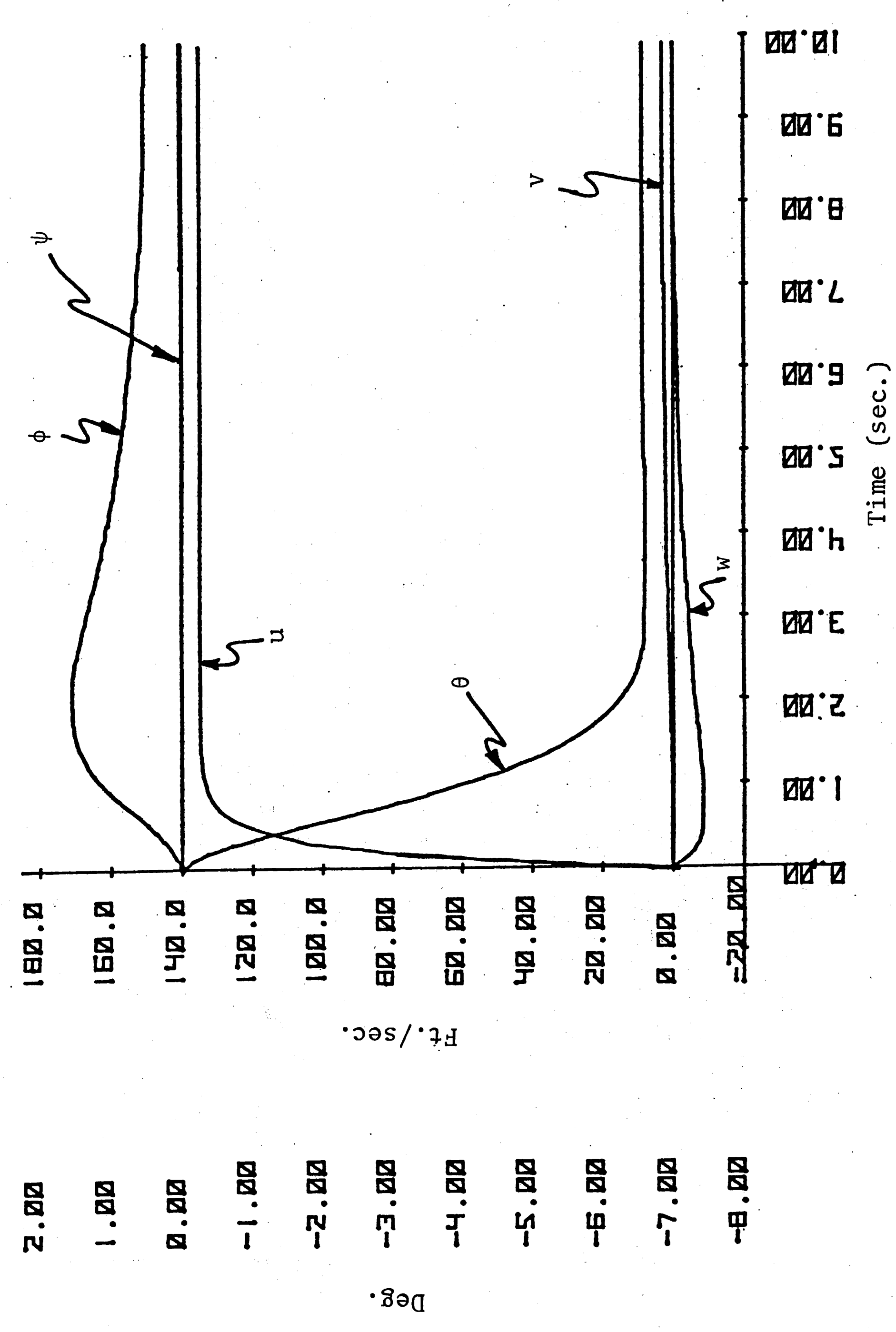


Figure 9. Step Response; u_p = 10 deg.

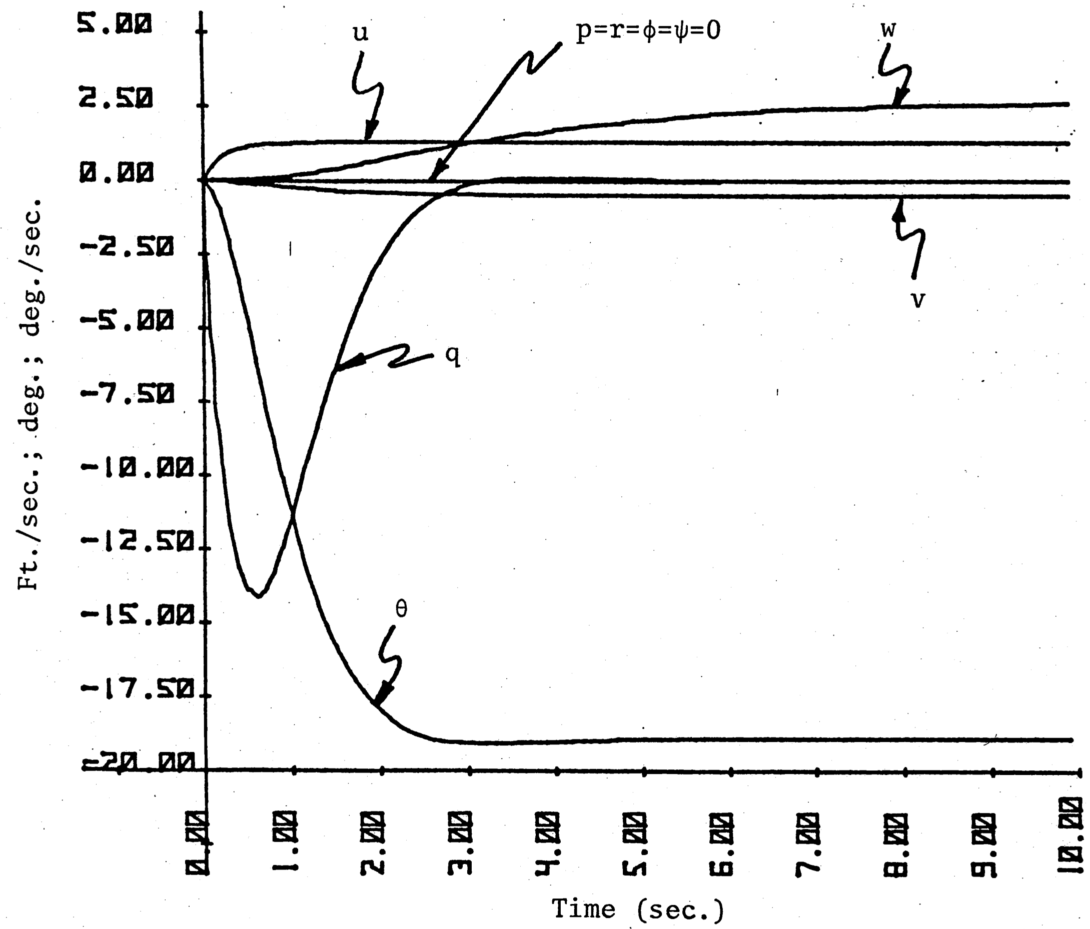


Figure 10. Step Response; u_C = 10 deg.

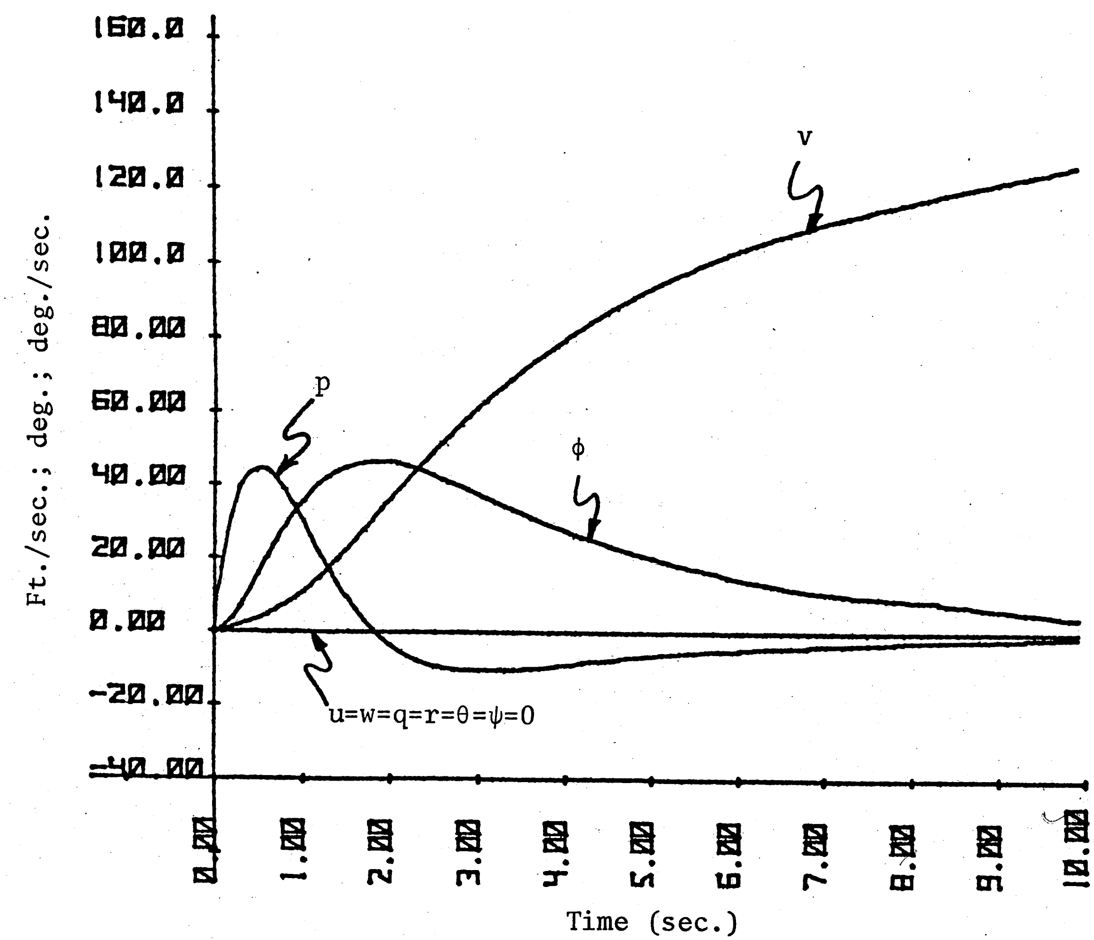


Figure 11. Step Response; u_R = 10 deg.

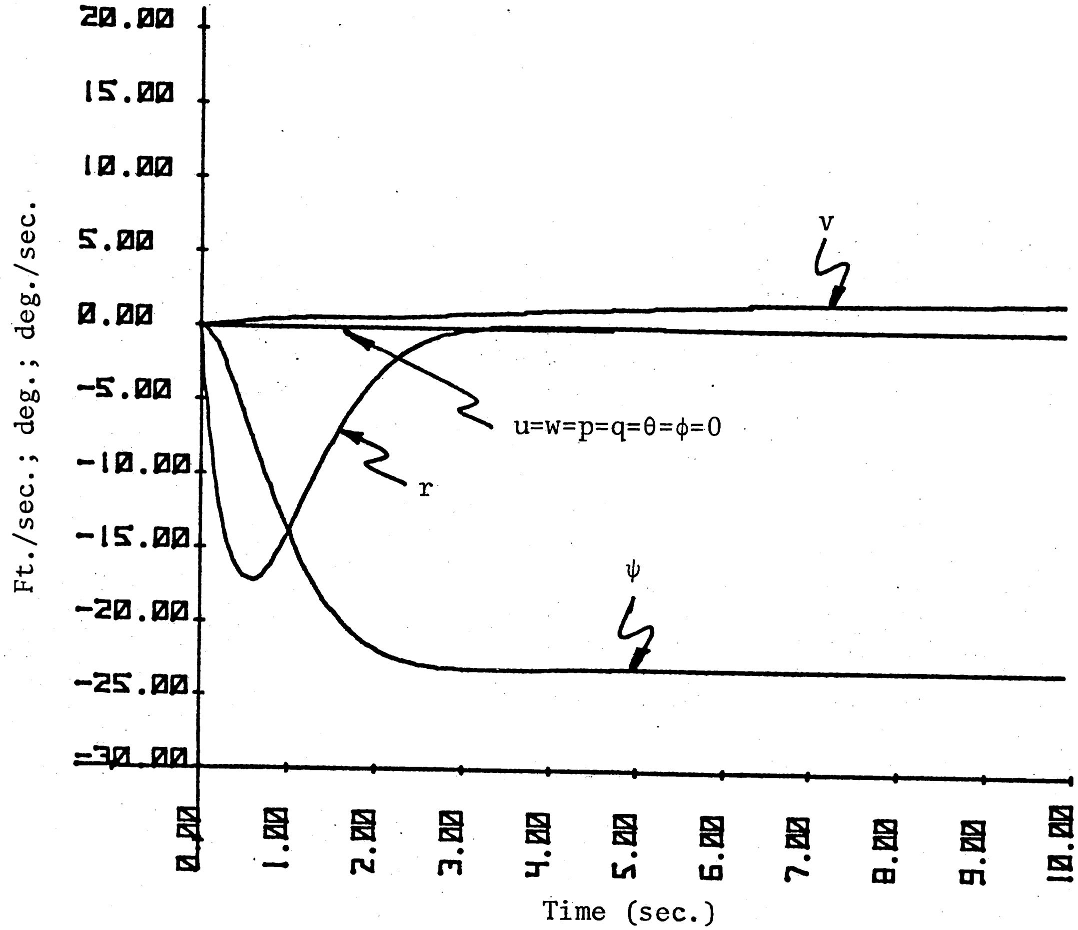


Figure 12. Step Response; u_T = 10 deg.

CHAPTER IV

RELATED TOPICS

4.1 Introduction

The analysis presented in Chapter II can be extended to many related multivariable synthesis problems since the closed-loop modal matrix plays such a focal role in characterizing varied attributes of the system being synthesized. While an extensive treatment of all these topics is beyond the scope of this dissertation, two important problems, observer design and minimum sensitivity solutions, will be analyzed in detail in sections 4.2 and 4.3 with spectral characterization techniques emphasized. Chapter V will provide some perspective into possible extensions to other associated problems.

4.2 The Problem of State Estimation

In order to generate the feedback law to assign the closed-loop eigenvalues, the technique of Chapter II relied on knowledge of the complete state vector. In practical situations it is often the case that not all state variables can be measured, and thus some form of estimating the inaccessible states becomes mandatory. A classical solution to this problem is the deterministic state observer originated by Luenberger [14]. He established that given an nth order completely observable system with r outputs it is possible to construct a dynamical observer of order (n - r) which asymptotically estimates the state vector. Many algorithms have been proposed [7,15], again with varied degree of computational complexity, to design these observers. Some variants of the basic Luenberger design have also been proposed by converting the state estimation problem to an equivalent pole-placement problem [16,17]. Of course, observer eigenvalue specifications are again found to leave considerable room for improvement by considering the associated nonunique transformation matrices. Specifically a decoupled structure, or at least one with required coupling carefully controlled in view of time constant magnitudes, would minimize large estimate errors in one variable exciting errors in other variable estimates. Hence it is apparent that the algorithms of Chapter II are directly applicable to the synthesis of state estimators, and indeed all the previous observations made regarding the preservation of maximum flexibility in the design structure apply to this dual problem.

Section 4.2.1 will now specifically characterize the solution of the observer synthesis, while section 4.2.2 will investigate the more general problem of combined observer and control dynamics.

4.2.1 Observer Problem for Multi-Output Systems

Consider the dynamic system

$$\dot{x} = Ax + Bu \tag{4.1}$$

$$y = Cx \tag{4.2}$$

where x is the state n-vector, u is the control m-vector, y is the output r-vector ($r > 1$) and A, B and C are constant matrices of appropriate dimensions. Assume that C is of full rank and that (A,C) is an observable pair (the matrix $[C^T, A^TC^T, (A^2)^TC^T, \cdots, (A^{n-1})^TC^T]$ is of rank n). The problem then is to design an (n - r) order dynamic observer which asymptotically estimates the state vector x. For simplicity of presentation assume that the output matrix C is in rank reduced identity form $C = [I_r : 0]$. This structure can always be obtained by a coordinate transformation T_0 of (4.1-4.2), where

$$T_0 = \left[\begin{array}{c|c} C_1^{-1} & C_1^{-1}C_2 \\ \hline 0 & I_{n-r} \end{array}\right] , \tag{4.3}$$

and the nonsingularity of the r x r matrix C_1 is obtained by at most a reordering of the state variables. Under this transformation (4.1) and (4.2) can be written in partitioned form as

$$\dot{x} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ \cdots \\ B_2 \end{bmatrix} u \tag{4.4}$$

$$y = [I_r : 0] \begin{bmatrix} x_1 \\ \cdots \\ x_2 \end{bmatrix} \tag{4.5}$$

where A_{11} is r x r, B_1 is r x m and x_1 is a r-vector, and the other sub-matrices are compatably dimensioned. Notice also that in the special case of incomplete state feedback (4.4-4.5) can be obtained by at most a reordering of state variables. Since in this new structure the output contains complete knowledge of x_1 it only remains to construct an observer to estimate x_2 using the knowledge of x_1 and u.

The following derivation of the Luenberger observer closely follows the method of Macfarlane [18].

Let $\hat{x}$ be an estimate of x and define

$$\hat{x} = \left[\begin{array}{c|c} I_r & 0 \\ \hline L_1 & L_2 \end{array}\right]^{-1} \left[\begin{array}{c} x_1 \\ \cdots\cdot \\ \theta \end{array}\right] \tag{4.6}$$

where $L = [L_1 \; : \; L_2]$ is yet to be determined and

$$\theta = Lx \tag{4.7}$$

is an (n - r) observation vector. Since x_2 can only be estimated, we can only generate an estimate $\hat{\theta}$ of the observation vector θ. Let e be the associated error in estimate defined by

$$e = \hat{\theta} - \theta \quad , \tag{4.8}$$

and suppose $\hat{\theta}$ satisfies the differential equation

$$\dot{\hat{\theta}} = F\hat{\theta} + ECx + Hu \tag{4.9}$$

where F, E and H are constant matrices to be selected and C is as in (4.5). Then

$$\dot{e} = Fe + [FL + EC - LA]x + [H - LB]u \quad , \tag{4.10}$$

and if F, E and H (which define the observer [14]) are chosen such that

$$LA - FL = EC \tag{4.11}$$

$$H = LB \quad , \tag{4.12}$$

the error dynamical model reduces to

$$\dot{e} = Fe \quad . \tag{4.13}$$

Clearly if F is a stable matrix e will decay exponentially with time. If $\hat{\theta}$ is properly initialized in (4.9) as

$$\hat{\theta}(0) = Lx(0)$$

(if, for example, the initial condition of even the unobserved states are somehow known), the unobservable state values are tracked with zero error for all $t > 0$. Otherwise a biased estimate results, which asymptotically approaches zero at a rate determined by the real parts of the eigenvalues of F. The estimate of the state vector is available from (4.6) provided the indicated inverse exists. Thus the observer design involves the construction of the matrix L so that (4.10) is satisfied and the transformation (4.6) is nonsingular. The following analysis adapts the algorithms presented in Chapter II for the design of observers.

Writing (4.10) in partitioned form yields

$$[L_1 \vdots L_2]\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] - F[L_1 \vdots L_2] = [E \vdots 0] \quad . \tag{4.14}$$

Completing the multiplication in (4.14) gives

$$L_1A_{11} + L_2A_{21} - FL_1 = E \tag{4.15}$$

$$L_1A_{12} + L_2A_{22} - FL_2 = 0 \tag{4.16}$$

Thus L_1 and L_2 must be chosen to satisfy (4.16) for a specified F and at the same time assure the transformation in (4.6) nonsingular. E is then computed using (4.15) and H is evaluated using (4.12).

Transposing (4.16) and choosing F as a diagonal matrix with desired observer eigenvalues yields

$$WF - A_{22}^{T}W = A_{12}^{T}Z \tag{4.17}$$

where $W = L_2^T$, $Z = L_1^T$. Equation (4.17) can also be expressed in terms of individual vectors w_i and z_i $(i = 1,2,\cdots,n-r)$, similar to (2.11) and (2.12) of Chapter II, and each vector equation constitutes a set of $(n - r)$ linear equations in n unknowns. The similarity to the eigenvector assignment problem of Chapter II is easily established by formulating (4.17) as the synthesis of $(n - r)$ linearly independent vectors $[z_i : w_i]^T$, so that they do not lie in the subspace already generated by C^T, namely $[I_r : 0]^T$, and the algorithms of Chapter II are now directly applicable.

4.2.2 Observers in a State Feedback Control System

The analysis in section 4.2.1 noted that the main problem arising in the design of observers is due to imperfect initialization of the observation vector in (4.8). Since the initial values of the state vector are not always known, the observer must then apparently be designed so that the real part of its eigenvalues are far larger than those of the plant, in order to rapidly stabilize the biased estimate errors. This increases the bandwidth of the observation channels and consequently increases the noise sensitivity. In general the selection of eigenvalues is done by actual simulation of different configurations under predicted noisy conditions [19]. However attempts have also been made to optimally locate observer eigenvalues [20] by minimizing a performance index related to the bias error. The objective of course is to find that compromise which provides fairly rapid decay of estimate errors yet also achieve acceptable noise rejection properties.

An alternative approach is suggested for those combined observer/control problems where the observed values are used to generate some desired control law. In these cases, the observer could be designed so that the biased estimate errors minimally influence the system closed-loop dynamics. Quite simply, if one of the variables to be estimated influences the response of some output variables substantially, its biased estimate errors should be effectively supressed in the corresponding error input channels. The design freedom in choosing the observation matrix L clearly plays a vital role in such a synthesis approach. This can be visualized by deriving the augmented feedback system and observer as follows [18].

Suppose a feedback control law is derived based on state feedback of Chapter II as

$$u = Kx \tag{4.18}$$

and is expressed in terms of available output y and the observation vector θ as

$$u = G_1 y + G_2 \theta = G_1 Cx + G_2 Lx \tag{4.19}$$

so that

$$K = G_1 C + G_2 L \quad . \tag{4.20}$$

If a feedback is applied using only the estimate of θ as

$$\hat{u} = G_1 y + G_2 \hat{\theta} \quad , \tag{4.21}$$

it is possible from (4.7) to express the combined closed-loop and observer error dynamics as

$$\begin{bmatrix} \dot{x} \\ \cdots \\ \dot{e} \end{bmatrix} \quad \left[\begin{array}{c|c} A+BK & BG_2 \\ \hline 0 & F \end{array}\right] \begin{bmatrix} x \\ \cdots \\ e \end{bmatrix} \tag{4.22}$$

From (4.22) it is clear that the eigenvalues of the closed-loop system are as they would be had the originally desired state feedback been directly implemented. The observer merely adds its own eigenvalues to those of the state feedback system [18]. Further, the effect of the biased estimate errors can be interpreted as a disturbance input to the closed-loop system through the distribution matrix $[BG_2]$. This immediately suggests the possibility of choosing the observer matrix L to suppress and localize the estimate error inputs. In order to see the relation between the matrices L and $[BG_2]$, write K in (4.18) in partitioned form as

$$K = [K_1 \; : \; K_2] \tag{4.23}$$

where K_1 is m x r. Then (4.20) can be written as

$$K_1 = G_1 + G_2 L_1 \tag{4.24}$$

$$K_2 = G_2 L_2 \tag{4.25}$$

and

$$BG_2 L_2 = BK_2 \quad . \tag{4.26}$$

Thus for a given K_2 (state feedback law), L_2 must be chosen so that $[BG_2]$ possesses the desirable structure. Alternatively if the relative effect of the biased error on the closed-loop dynamics is to be minimized, then it is possible to specify a suitable matrix $[BG_2]$ of rank $(n - r)$ such that

$$||BG_2|| << ||A + BK||$$

and minimize

$$||L_2 - [BK_2][BG_2]^{\dagger}|| \quad , \qquad (4.27)$$

where $[BG_2]^{\dagger}$ is the generalized inverse given by [43]

$$[BG_2]^{\dagger} = \{[BG_2]^T[BG_2]\}^{-1}[BG_2]^T \quad .$$

Equation (4.27) together with (4.17) can now be formulated as a constrained minimization problem in terms of individual vectors w_i of L_2^T as follows. Let

$$[BG_2]^{\dagger}[BK_2] = M \qquad (4.28)$$

and

$$M = \begin{bmatrix} m_1 \\ \cdots\cdot \\ m_2 \\ \cdots\cdot \\ \vdots \\ m_{n-r} \end{bmatrix} \qquad (4.29)$$

where m_i^T are $(n - r)$-vectors. Then minimizing $||w_i - m_i{}^T||$ subject to the observer constraints of (4.17), completes the observer synthesis.

It should be pointed out, however, that the above analysis presupposes that $\hat{\theta}(0)$ has been initialized, based on the statistics of x(0), to minimize the initial biased error e(0). That is, the suggested approach will provide improved response only if the initial conditions of the observation vector $\hat{\theta}$ can be estimated more accurately than the normally assumed zero mean. A numerical example will now be presented to illustrate the features of a combined observer/controller design.

4.2.3 Controllers for the Lateral Dynamics of Aircraft

The linear perturbation dynamics for the lateral motions of an aircraft can be modelled as (4.1-4.2) where x is the state vector of roll rate (ρ), yaw rate (γ), sideslip (β) and bank angle (ϕ), respectively, and u is the control vector of aileron (δ_a) and rudder (δ_r) angular deflections. The measured outputs are roll rate and yaw rate. Thus it is required to design a feedback controller which includes an observer to estimate the inaccessible states side slip and bank angle.

The system of equations modelling an F8-C aircraft at an altitude of 50,000 ft, a mach number of 1·1 and angle of attack of 8.6 degrees as (4.1-4.2) is shown in Table V. Initially a state feedback controller was designed to meet the following performance specifications [21]. The fourth order system can be considered as two second order subsystems, with side slip and yaw rate exhibiting a coupled damped dutch roll (a pair of complex conjugate roots), and roll rate/bank angle constituting the second subsystem. Experience indicates that a long time constant is desirable for the bank angle, indicating a pole close to the origin.

TABLE V

LATERAL DYNAMICS OF AIRCRAFT

$$A = \begin{bmatrix} -1.38 & 0.223 & -33.1 & 0 \\ -0.0037 & -0.196 & 6.71 & 0 \\ 0.115 & -1 & -0.107 & 0.032 \\ 0.989 & 0.149 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 11.6 & 0.209 & -0.0014 & 0 \\ 4.43 & -1.76 & 0.017 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

These specifications lead to the following mode/variable assignment.

Eigenvalues	Response Variables
$1.5 \pm j2$	γ, β
-3	ρ
-0.01	ϕ

It is again emphasized that the direct association of modes with output variables as indicated is valid only if the resulting modal matrix has the desirable decoupled structure.

The method of Chapter II yields a closed-loop system as shown in Table VI. If all four variables were available for control law implementation, the response of the system to a 10 degrees per second initial condition for the roll rate would appear as in Figures 13-16, Case A, displaying the desired minimal interaction of the side slip and yaw rate responses.

Since β and ϕ are inaccessible states an observer must be augmented to the feedback system. Following the analysis of sections 4.2 and 4.2.2, in order to make the observer fast as compared to the system itself the eigenvalues are chosen to be -30 and -25. A nominal design meeting the constraint equations (4.17) resulted in the respective observer matrices given in Table VII. The response of the composite system (4.22) for an initial condition of 10 degrees per second roll rate and an initial bias error of the same magnitude ($e_1(0) = e_2(0) = 10$) is shown in Figures 13-16 as Case B. There has been considerable coupled response in both yaw rate and side slip indicating significant performance degradation. The cause of this behavior is apparent by noting the entries in the error distribution matrix BG_2 in Table VII. The norm minimizing technique discussed in section 4.2.2 was next used to redesign

TABLE VI

MODAL AND CONTROL MATRICES FOR STATE FEEDBACK DESIGN

$$U = \begin{bmatrix} 1 & -0.1 & -0.1 & -0.015 \\ 0 & 1 & 1 & 0.03 \\ -0.03 & -0.085 & 0.486 & -0.0034 \\ -0.329 & 0.004 & -0.028 & 1.037 \end{bmatrix}$$

$$\Lambda = \mathrm{Diag}[-3, -1.5 \pm j2, -0.01]$$

$$K = \begin{bmatrix} -0.083 & -0.576 & 2.733 & 0.019 \\ -0.156 & 1.456 & 0.157 & -0.056 \end{bmatrix}$$

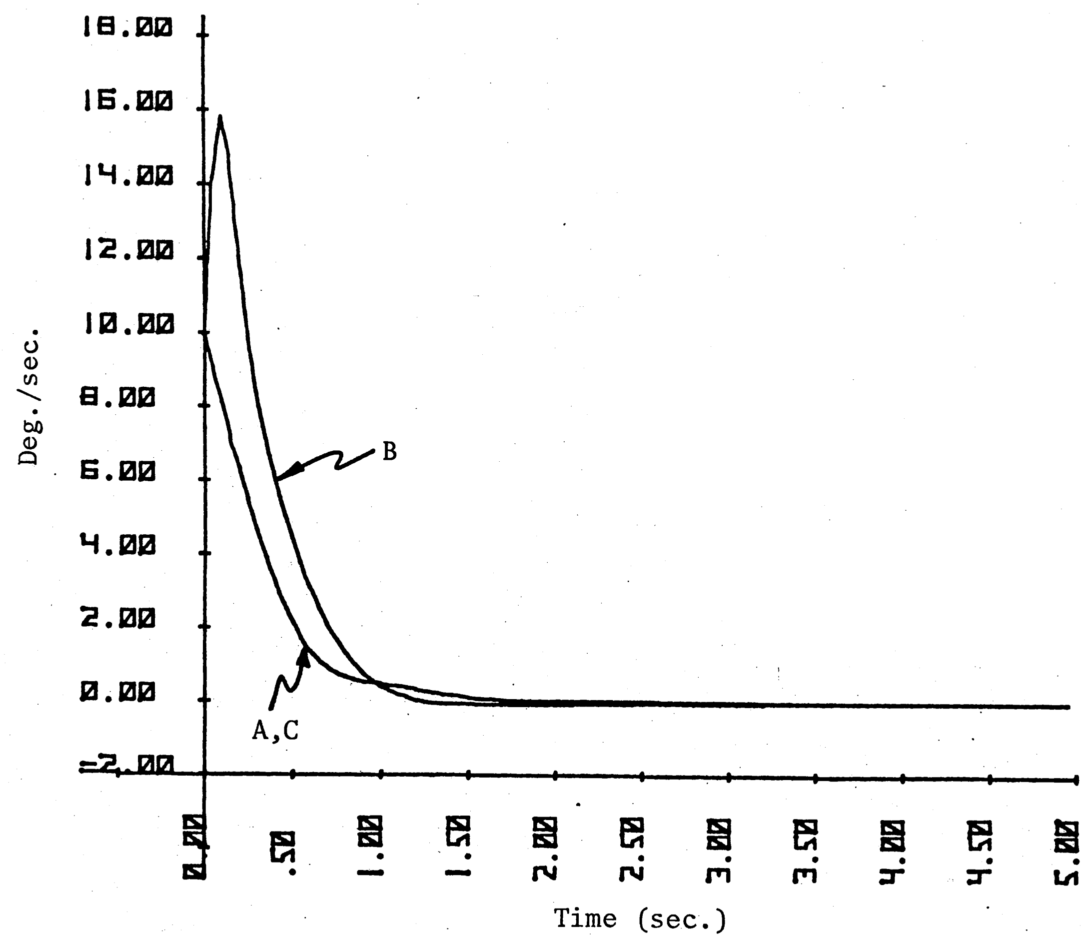


Figure 13. Roll Rate Response; $\rho(0)$ = 10 deg./sec.

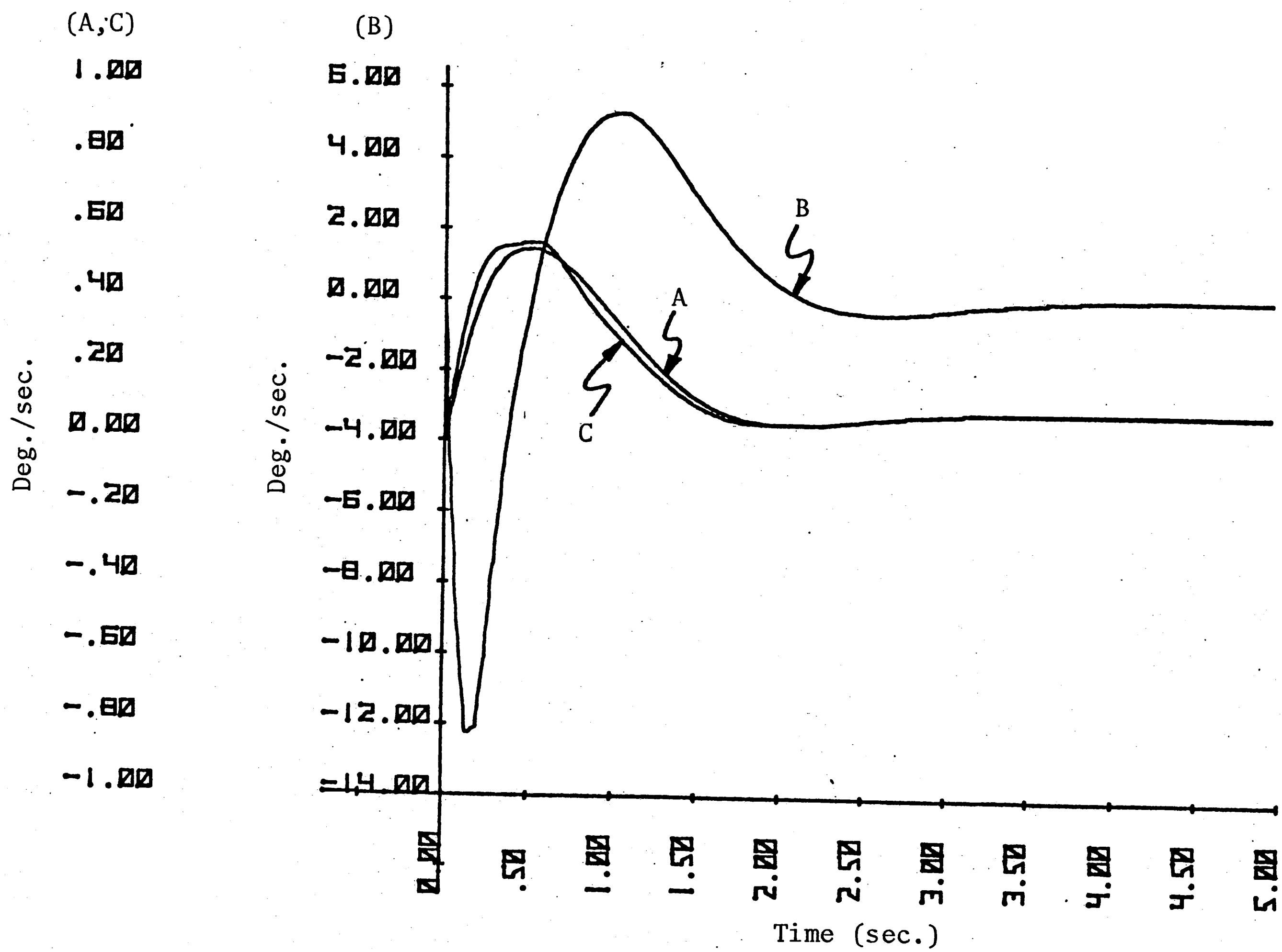


Figure 14. Yaw Rate Response; $\rho(0)$ = 10 deg./sec.

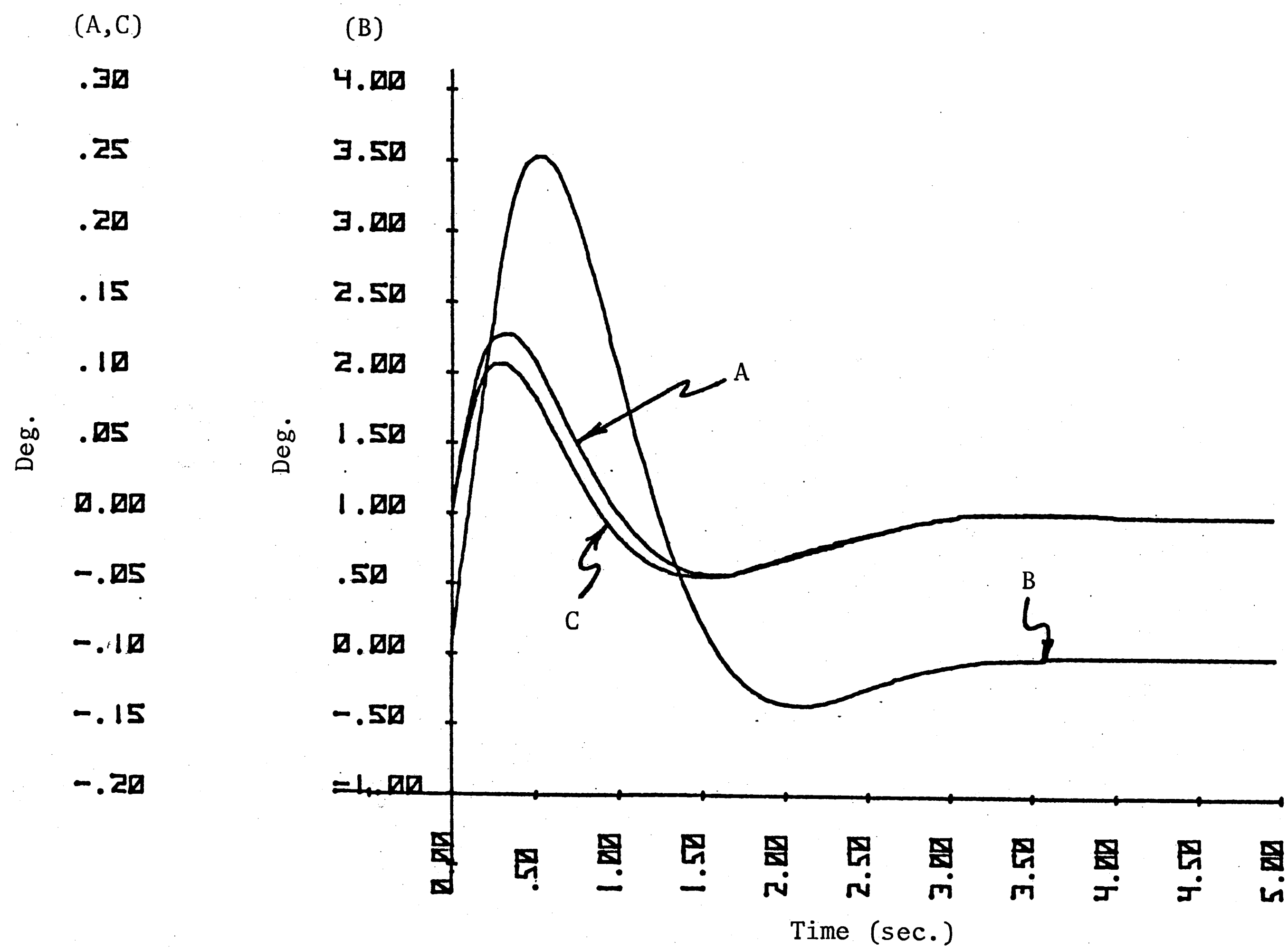


Figure 15. Sideslip Response; $\rho(0)$ = 10 deg./sec.

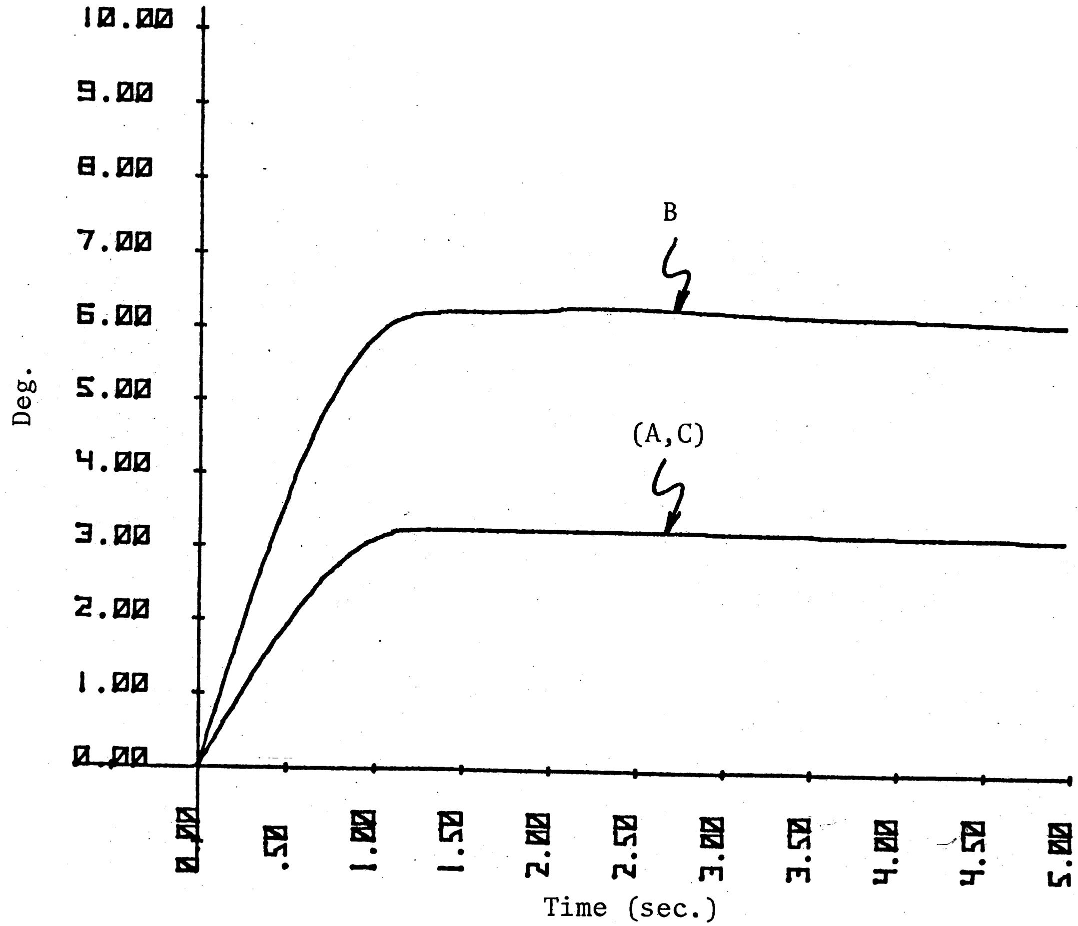


Figure 16. Bank Angle Response; $\rho(0)$ = 10 deg./sec.

TABLE VII

OBSERVER; NOMINAL DESIGN

$$F = \mathrm{Diag}[-30,-20]$$

$$L = \begin{bmatrix} 0.9 & 0 & 1 & -0.001 \\ 0.68 & 0 & 1 & -0.0013 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 26.838 & -24.021 \\ -66.282 & 65.2918 \end{bmatrix}$$

$$BG_2 = \begin{bmatrix} 17.695 & 10.59 \\ 122.2658 & -119.933 \\ -1.164 & 1.143 \\ 0 & 0 \end{bmatrix}$$

the observer. Because of the special structure of the constraint equations (4.17) for this problem it was possible to directly express L_2^{-1} as a function of observer eigenvalues and the resulting matrices for the observer are listed in Table VIII. The response of the composite system to the same initial conditions in Case B is shown in Figures 13-16 as Case C. Indeed, in this case the feedback controller with the observer behaves practically like an equivalent state feedback controller even though subjected to significant bias errors.

4.2.4 Summary

In this section the algorithms developed for the synthesis of multivariable systems were shown to be directly adaptable to the design of observers. While the selection of observer poles faster than the plant dynamics alone may be adequate to track the open-loop dynamics, sufficient caution should be exersized in the selection of the transformation matrix L to provide good closed-loop tracking in presence of large bias errors. The flexibility inherent in the selection of the matrix L can thus be effectively used to limit/suppress bias errors from unduly influencing the closed-loop dynamics. The ability to characterize all the realizable L matrices given the triple (A,C,F) finds application in the synthesis of special observer structures like zero sensitivity observers to plant variations [22].

4.3 Modal Sensitivity Reduction to Parameter Variations

An important goal in feedback design is to guarantee achievement of specified tolerances on system response over specified bounds of plant

TABLE VIII

OBSERVER; MINIMUM NORM DESIGN

$$F = \mathrm{Diag}[-30,-25]$$

$$L = \begin{bmatrix} -135 & 0 & -150 & 0.150 \\ -113 & 0 & -150 & 0.182 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} -0.46 & 0.45 \\ 1.23 & -1.22 \end{bmatrix}$$

$$BG_2 = \begin{bmatrix} -0.0108 & -0.199 \\ -2.265 & 2.249 \\ 0.0215 & -0.0215 \\ 0 & 0 \end{bmatrix}$$

parameter values [23]. The significant contributions towards this sensitivity theory are well documented in [24]. The primary concern here is the modal sensitivity of closed-loop systems to parameter perturbations.

Consider the nominal closed-loop system

$$A_C = A + BK \tag{4.30}$$

with desired eigenvalues. The first order differential change dA_C due to perturbations in A, B and K can be expressed as

$$dA_C = dA + dB \cdot K + B \cdot dK \quad . \tag{4.31}$$

Assuming A_C has been assigned distinct eigenvalues, the corresponding first order perturbations in the eigenvalues of A_C can be expressed [25] as

$$d\lambda_i = v_i^T [dA_C] \, u_i \tag{4.32}$$

where u_i and v_i are the eigenvectors and reciprocal basis vectors of A_C with

$$u_i^T u_i = u_i^T v_i = 1 \quad . \tag{4.33}$$

The corresponding eigenvector change is given by

$$du_i = \sum_{j=1}^{n-1} \alpha_{ij} u_j \qquad (j \neq i) \tag{4.34}$$

with

$$\alpha_{ij} = \frac{v_j^T (dA_C) u_i}{(\lambda_i - \lambda_j)} \qquad (j \neq i) \quad . \tag{4.35}$$

Similar relations can be developed for complex conjugate pairs (Appendix A).

There are primarily two approaches to reduce the differential eigensensitivity coefficients formulated in (4.32) and (4.34), for a specified plant variation. One may either seek a closed-loop system which is inherently insensitive to the given perturbation, or design a corrective feedback controller which zeros the eigensensitivity coefficients.

At the outset an examination of (4.32) and (4.35) reveals that the closed-loop eigenvectors directly influence the sensitivity characteristics of the system, and the utility of direct control of their structure is quite obvious. Seraji [26] attempts to relate the design freedom available in a unity rank feedback system to control indirectly the closed-loop modal structure and hence evolves a procedure based on the first approach mentioned above. Tzafesta [27] develops a corrective controller which nulls only the first order eigenvalue sensitivity coefficients defined in (4.32). The corrective feedback scheme has three major limitations.

1. It is not applicable to large parameter variations ($|\delta a_{ij}| > 1$).

2. System stability is not assured since eigenvalues undergo uncontrolled, though limited, shifts during feedback update.

3. Even if a stable system results there is no guarantee that system response is held within bounds since eigenvector sensitivity coefficients are not nulled.

However it is possible to generalize the result of Tzafesta [27] to null as many of the eigenvalues/eigenvector sensitivity coefficients in (4.32) and (4.34) as the multi-input design freedom permits, using the following analysis.

Consider the sensitivity matrix

$$S = V^T dA_C U \quad , \qquad (4.36)$$

where U is the closed-loop modal matrix, V^T is the matrix of reciprocal basis vectors. Each entry of S can be expressed as

$$s_{ij} = v_i^T dA_C u_j \ ; \quad i,j = 1,2,\cdots,n \quad . \tag{4.37}$$

From (4.32) and (4.35) it follows that the diagonal entries of S correspond to first order eigenvalue sensitivity coefficients, the off diagonal entries are eigenvector sensitivity coefficients normalized by the factor

$$\frac{1}{(\lambda_i - \lambda_j)} \quad ,$$

and s_{ij} $(i \neq j)$ determines the influence of the jth eigenvector on the sensitivity of the ith eigenvector.

Substituting for dA_C from (4.31) in (4.36) gives

$$S = V^T(dA + dB \cdot K)U + V^T BdKU \tag{4.38}$$

where K is the nominal feedback in (4.30). Let $\hat{B} = V^T B$ and

$$P = V^T(dA + dB \cdot K)U \quad .$$

Then (4.38) can be written in elemental form as

$$s_{ij} - p_{ij} = \hat{b}_i^T dKu_j \tag{4.39}$$

where $\hat{b}_i^T$ is the ith row of $\hat{B}$, u_j is the jth column of U and dK is the corrective feedback gain matrix.

Define the column string

$$k_s = (dk_{11}, dk_{12}, \cdots, dk_{1m}, dk_{12}, dk_{22}, \cdots, dk_{m2}, dk_{1n}, \cdots, dk_{mn}) \tag{4.40}$$

Thus we can write (4.39) as a linear constraint in the $n \cdot m$ elements of k_s as

$$s_{ij} - p_{ij} = c_i^T k_s \qquad (4.41)$$

where

$$c_i^T = (\hat{b}_i^T * u_j)$$

and the operation (*) is defined by

$$\hat{b}_i^T * u_j = (u_{1j} \cdot \hat{b}_i^T : u_{2j} \cdot \hat{b}_i^T : \cdots : u_{nj}\hat{b}_i^T) \quad , \qquad (4.42)$$

where $\hat{b}_i^T$ is 1 x m, u_j is n x 1 and c_i^T is 1 x mn. Thus the upper bound on the n^2 first order sensitivity coefficients of S that can be set to zero is mn, which is the design freedom inherent in pole-placement, provided the corresponding system of equations

$$Ck_s = y \quad , \qquad (4.43)$$

derived from the set (4.41) is consistent, where $y_i = -p_{ij}$ is an mn-vector obtained by setting the corresponding $s_{ij} = 0$ in (4.41). Since for practical systems $mn < n^2$, the best choice is to set $s_{ii} = 0$ ($i = 1,2,\cdots,n$) and use the remaining freedom to limit the eigenvector perturbations corresponding to the dominant modes of the system. If the rank of C in (4.43) is less than mn, a generalized minimum norm least square solution can be obtained.

Selection of a closed loop modal structure insensitive to a specified plant variation is attractive when only a few isolated parameters in the plant undergo perturbation. In such cases dA_C will usually be sparse and singular and the appropriate zerosensitivity eigenvectors are the ones that span the null space of dA_C provided these eigenvectors can be synthesized under the pole-placement constraints. Indeed it is also possible to identify certain perturbation structures that completely allow restoration of the nominal modal structure of (4.30) by a simple feedback update procedure. This is immediately apparent if we consider the plant in rank reduced form of (2.1). If R and T do not undergo any variations, then the required feedback correction matrix is

$$dK = -B_1^{-1}[D : E] \quad . \tag{4.44}$$

In the framework of the above discussions it becomes clear that the synthesis algorithm of Chapter II possesses all the desired attributes for evolving minimum sensitivity designs with the added advantage of assuring closed-loop stability. Thus in the next section the features of an adaptive controller based on the spectral synthesis algorithm will be examined.

4.3.1 Real Time Adaptive Controller Design

With the advent of inexpensive high performance digital computers, it appears realistic to implement control algorithms on a real time basis to update the feedback law as revised estimates of plant parameters become available. In these on-line applications the computational complexity of the algorithms naturally take paramount importance. Optimal control techniques [28] have been considered for such applications.

The corrective feedback controllers of section 4.3 and the synthesis algorithm of Chapter II are also potential candidates for real time implementation; thus it is appropriate to compare these algorithms in terms of the two computational performance indices, operation count and storage requirement. Table IX provides such a comparison. For ease of representation it is assumed that the number of inputs $m = \frac{n}{2}$. The optimal control algorithm is based on Kleinman's [29,30] method, where

TABLE IX

COMPUTATIONAL REQUIREMENTS FOR ADAPTIVE CONTROLLERS

Algorithm	Approximate Multiplications	Approximate Storage
Optimal Control	$(\frac{n^6}{8} + \frac{3}{8}n^5 + \frac{5}{8}n^4 + \frac{11}{4}n^3)$[1]	$n^4 + 2n^3 + 7n^2$
Eigensensitivity[2]	$n^6 + 0.33\ n^3$	$0.5\ n^4 + 1.5\ n^2$
Spectral Synthesis (Chapter II)	$k_1\ n^4 + (k_2 + 4)n^3$ $k_1 \leq 3,\ k_2 \leq 7$	$15\ n^2 + 12n$

[1]Multiplications per iteration.

[2]Minimum norm least squares solution.

the solution to the nonlinear algebraic Riccati equation is obtained as an iterative solution of a set of linear equations. The number of iterations depends on the system and stipulated accuracy. Narendra, et al. [28] consider this algorithm for their adaptive controller. The eigensensitivity algorithm discussed in section 4.3 has been included to illustrate that it does not enjoy any computational advantage.

For computer implementation of the synthesis algorithm of Chapter II, it was advantageous to use the following representation for complex pairs as suggested in [7] instead of Equation (2.19):

$$\begin{bmatrix} w_j \\ \cdots \\ w_{j+1} \end{bmatrix} = \left[\begin{array}{c|c} P_j T & Q_j T \\ \hline -Q_j T & P_j T \end{array}\right] \begin{bmatrix} z_j \\ \cdots \\ z_{j+1} \end{bmatrix} \tag{4.45}$$

where

$$\hat{R} = [R^2 - 2\alpha_j R + (\alpha_j^2 + \beta_j^2) I_{n-m}]^{-1} \quad ,$$

$$P_j = \hat{R}[\alpha_j I_{n-m} - R] \quad ,$$

and

$$Q_j = \hat{R}\beta_j \quad .$$

The upper bound on the multiplicative constants k_1 and k_2 in Table IX are based on worst case situations wherein repetitive execution of steps 4 and 5 of the algorithm in section 2.4.2 are required for every eigenvector synthesis, which is not generally the case in practice. However even in such cases the operation counts are significantly less than the optimal control solution.

Reduced computation and storage requirements for the new algorithm are to be expected since the solution of linear equations in general requires $\phi(d^3/3)$ multiplications and $\phi(d^2)$ storage locations, where d is the dimension of the system and d is $n(n + 1)/2$ for the optimal control solution, $n^2/2$ for the eigensensitivity algorithm and $n/2$ for the spectral synthesis algorithm.

A numerical example will be presented in the next section to illustrate how the spectral synthesis algorithm could be used to hold response deviations within close bounds, even under large plant parameter variations.

4.3.2 Adaptive Controllers for Helicopter

It is well recognized that the use of linear perturbation models about a nominal flight condition to evolve feedback control laws are inadequate for highly responsive aircraft systems. Thus some form of on-line identification/adaptive scheme becomes essential to continuously monitor the system and update the feedback gains in order to hold the closed-loop response within acceptable bounds. Narendra, et al. [28] have discussed an adaptive scheme based on optimal control techniques for the control of the longitudinal dynamics of a helicopter. The same model will be used here to design an adaptive controller using the algorithm of Chapter II. The dynamics of the helicopter can again be modelled as (2.1) with state vector $x = [u,v,q,\theta]^T$ and control vector $u = [u_p,u_C]^T$, where the notations of Chapter III are used for the variables. Table X gives the nominal A and B matrices corresponding to an airspeed of 135 knots. As the airspeed changes large perturbations occur in the elements a_{32}, a_{34} and b_{21}. The bounds of these parameters over the operating

TABLE X

HELICOPTER LONGITUDINAL DYNAMICS

$$A = \begin{bmatrix} -0.03 & 0.027 & 0.018 & -0.45 \\ 0.04 & -1.01 & 0.0024 & -4.02 \\ 0.1 & 0.368^{\dagger} & -0.707 & 1.42^{\dagger} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[B^T] = \begin{bmatrix} 0.442 & 3.545^{\dagger} & -5.52 & 0 \\ 0.176 & -7.59 & 4.49 & 0 \end{bmatrix}$$

†Parameters undergoing change. All other parameters are assumed to be constant. The bounds on the variable parameters for the speed range 60-170 knots are (i) $0.06 \leq a_{32} \leq 0.505$, (ii) $0.119 \leq a_{34} \leq 2.52$, and (iii) $0.977 \leq b_{21} \leq 5.11$.

speed range are also included in Table X. It is thus required to design an adaptive controller which updates the feedback gains to keep the deviation in the closed-loop modal structure a minimum for the complete speed range. The modes of the system were selected to yield as far as possible eigenvectors which would undergo minimal deviation under update feedback for the complete speed range. The needed information is readily available by examining the coupling matrices $C_i(\lambda_i)$ of (2.18) and (2.20) corresponding to the two range limit speeds (60 knots and 170 knots) and the nominal speed (135 knots). This gave the desired mode/variable assignments as

Mode	Variable
-0.2	u
-0.5	v
$-1.5 \pm j1$	q, θ

Table XI lists the resulting modal structure and the feedback gains obtained for the three air speeds. The free design parameters corresponding to the first two rows of the modal matrix were held the same for all speeds, and the change in modal structure due to plant variation is revealed in the last two rows of the modal matrix. For comparison purposes Table XI also includes the case of a fixed gain controller based on the nominal 135 knots design. The significant mode coupling introduced at speeds of 60 and 170 knots using the fixed controller is very apparent. The consequent response deviations are graphically illustrated in Figures 17-19, where responses labelled A and C show the dynamics of the fixed gain system at 170 and 60 knots, respectively. Responses labelled B show the nominal 135 knots responses and the adaptive controller responses for the speed range 60-170 knots, indicating negligible deviation in dynamics

TABLE XI

MODAL MATRICES FOR DIFFERENT AIR SPEEDS

	Speed	60 Knots				135 Knots				170 Knots			
Adaptive Controller	λ	-0.2	-0.5	-1.5 ± j1		-0.2	-0.5	-1.5 ± j1		-0.2	-0.5	-1.5 ± j1	
	U	1	0	0.1	0.1	1	0	0.1	1	1	0	0.1	0.1
		0	1	0	0	0	1	0	0	0	1	0	0
		-0.04	0.004	-0.54	0.06	-0.05	-0.018	-0.54	-0.03	-0.07	-0.06	-0.54	-0.19
		0.23	0.009	0.26	0.13	0.26	0.036	0.24	0.18	0.38	0.12	0.19	0.25
	K	-0.124	-0.043	0.486	0.30	-0.21	-0.007	0.685	0.808	-0.43	-0.03	0.88	1.52
		-0.009	-0.073	0.063	-0.49	-0.09	-0.07	0.32	-0.15	-0.28	-0.09	0.59	0.49
Fixed Gain Controller (135 knots design)	λ	-0.16	-0.44	-1.53 ±j1.41		As Above				-0.25	-0.35	-1.05 ±j2.04	
	U	0.75	-0.118	-0.13	-0.16					-0.7	0.29	0.36	0.198
		0.65	0.989	0.63	0.3					-0.59	0.91	0.06	0.29
		-0.02	0.03	1	0					0.09	-0.09	-0.67	-0.84
		0.109	-0.07	-0.35	-0.32					-0.38	0.27	0.63	0.41

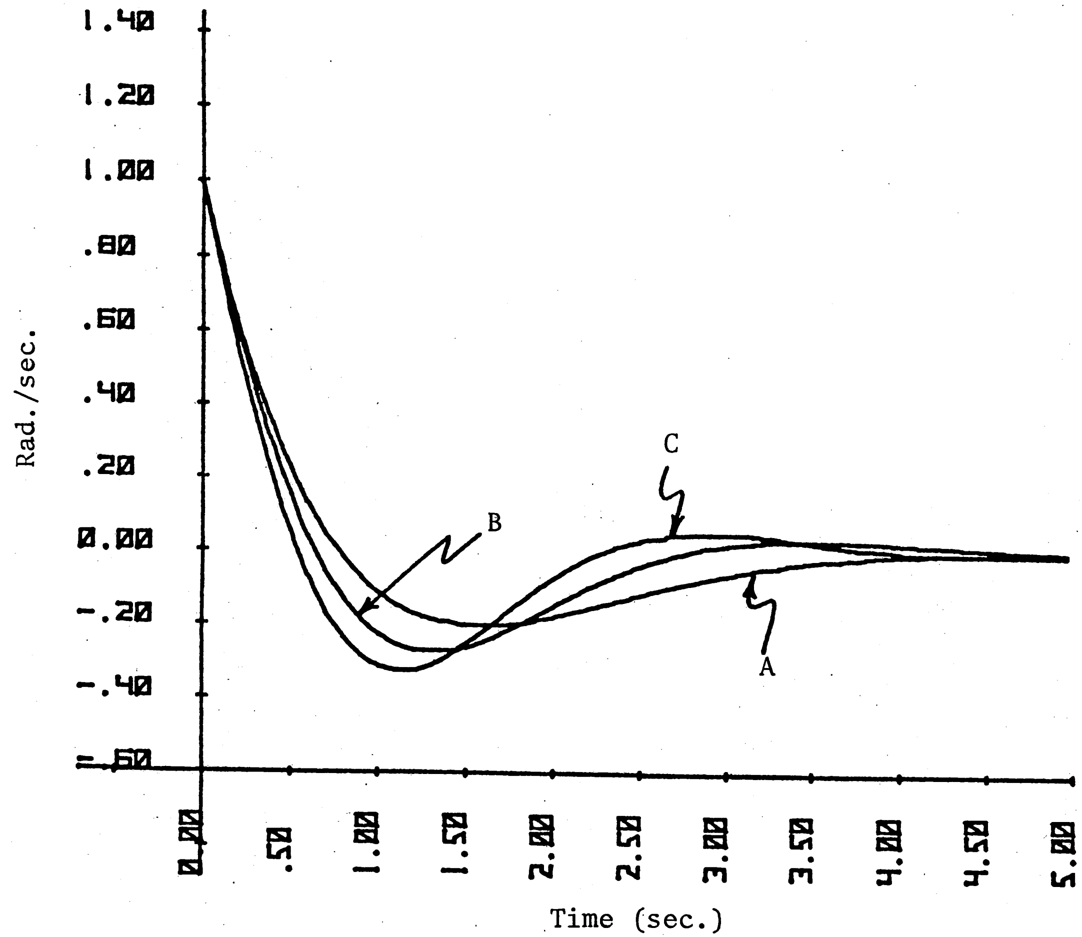


Figure 17. Pitch Rate Response; q(0) = 1 rad./sec.

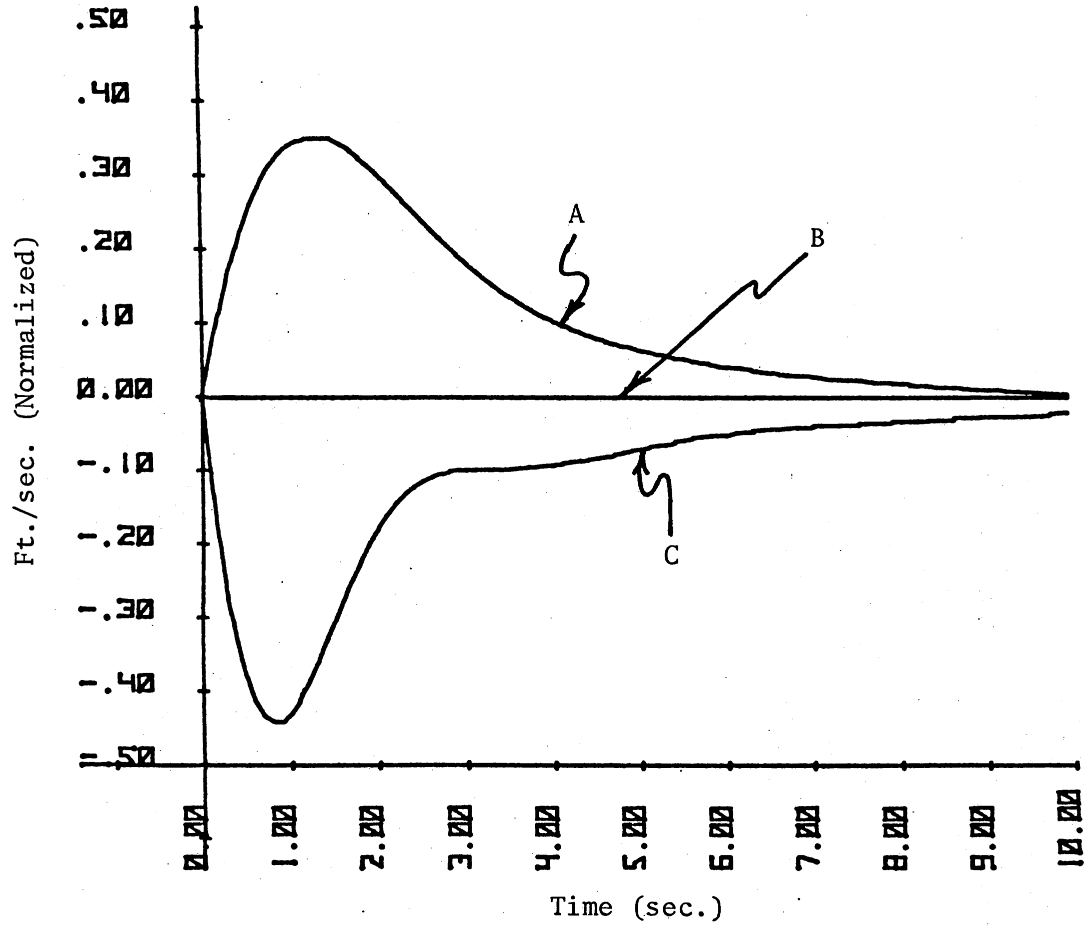


Figure 18. Vertical Velocity Response; q(0) = 1 rad./sec.

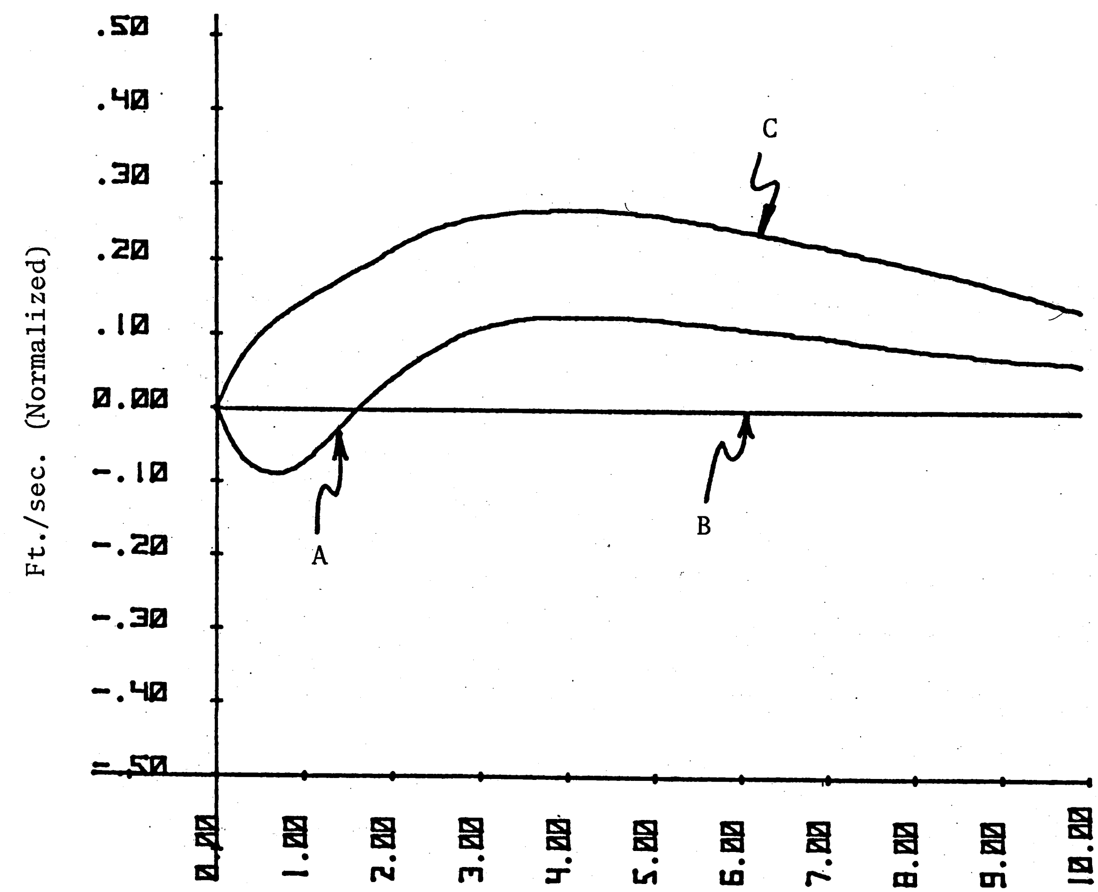


Figure 19. Vertical Velocity Response;
u(0) = 1 ft./sec. (Normalized)

from the nominal has occurred.

4.3.3 Summary

The modal sensitivity of systems subject to plant variations has been examined. Freedback update procedures, based on the differential eigenvalue sensitivity of Tzafesta [27], were extended to include eigenvector sensitivity coefficients. From system stability considerations, all such sensitivity designs are less attractive for on-line applications. It was also noted that despite solving the non-linear Riccati equations as an iterative set of linear equations, the storage and operational count for optimal control schemes were found to be greater than for the algorithm of Chapter II. Two advantages of spectral synthesis, identification of modes leading to modal structures minimally affected by apriori specified plant variations and subsequent rapid feedback update to synthesize these structures, were illustrated in a numerical example.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary

In this dissertation a new approach to the synthesis of multivariable systems was presented. The design process originated from the observation that sufficient freedom exists in a multi-input state feedback system to simultaneously assign eigenvalues and part of the closed-loop modal structure arbitrarily. Since the eigenvectors determine the influence of the associated modes on the response variables, selection of the modal entries directly control the dynamic characteristics of the feedback system. This key concept lead to the spectral characterization analysis developed in Chapter II. It was shown that the maximum number of entries in each eigenvector that can be arbitrarily selected is identically equal to the number of inputs. While this design freedom could be effectively used to meet a variety of design objectives, the most readily apparent, mode decoupled structure synthesis, was analyzed in this chapter. Further, in order to retain maximum flexibility in the realization of these closed-loop eigenvector forms, a new algorithm was developed for the eigenvalue/eigenvector assignment problem, which guarantees generation of a non-singular modal matrix required for the solution.

In the algorithm developed, it is quite possible that a particular set of eigenvalue/eigenvector specifications may lead to a singular modal matrix, indicating that no control law exists to meet that specification. In such an event it becomes necessary to relax eigenvalue and/or eigenvector specifications to meet the nonsingular modal matrix constraint. This is in no way a serious limiation since pole specifications are rarely intended to be exact. However the algorithm detects the occurrence of a possible singular assignment and allows for a designer controlled corrective perturbation to be made in eigenvalue/eigenvector specifications. This assures the synthesis of a stable feedback system with its dynamical behavior arbitrarily close to the desired specification.

In Chapter III a practical application of the new design procedure was illustrated by synthesizing a hover controller for a helicopter. This example highlights the features of a solution arising from a combination of frequency and time-domain specifications for the desired responses. The analysis in this chapter also revealed that for plants characterized by minimum input mixing through the distribution matrix B, it is possible to achieve non-interacting input/output behavior by state feedback alone, provided the corresponding closed-loop modal matrix can be synthesized to have a block diagonally dominant structure.

The direct applicability of the synthesis algorithm to the design of state observers was established in Chapter IV. Again it was noted that for a multi-output system significant flexibility exists in the synthesis of observer structures. An application of this freedom in the synthesis of combined observer/feedback control system insensititive to initial bias errors was also presented. Thus, in essence, the algorithm presented in Chapter II could be used to synthesize closed-loop systems with acceptable dynamic response even when all the states are not available for measurement.

Finally the utility of the new formulation was exemplified in the design of controllers insensitive to plant parameter variations. The central role the closed-loop modal structure plays in evolving minimum sensitivity designs was clearly demonstrated. The potentialities of the synthesis algorithm as an on-line adaptive controller was also investigated in this chapter and found to be computationally superior to existing methods.

5.2 Conclusions

Synthesis of multivariable systems based on pole-placement techniques are potentially attractive since there is adequate freedom in the design to modify and control the closed-loop dynamics through the non-unique feedback law meeting desired pole-specifications. Since the synthesis algorithms reported in the literature are unable to relate this design freedom in terms of an attribute of the system being synthesized they have not emerged as a practical design tool. The new formulation presented in this dissertation has significantly eliminated this limitation by characterizing the non-unique feedback law in terms of the closed-loop dynamics through eigenvalue/eigenvector assignment. By shifting the focus on the closed-loop spectral characteristics (eigenvalues/eigenvectors) as contrasted with the open-loop spectral characterization of the classical modal control theory, it has been possible to rationally relate the multi-input design freedom to shaping the feedback system response, while retaining all the intuitively satisfying concepts of mode oriented design. Thus, spectral characterization theory as developed in this dissertation can be successfully used as a complementary design methodology along with classical optimal control procedures.

The inherently simple synthesis algorithm is admirably suited for both off/on-line applications. In particular in an iterative off-line dynamic response optimization synthesis effort, additional savings in computation results since the time response histories can be efficiently computed using (2.16) instead of resorting to numerical integration of the system differential equations.

With the advent of small inexpensive digital computers considerable attention is being focused on the development of computer based on-line adaptive controllers. These controllers attempt to hold the response of the plant within tight bounds even under widely varying operating conditions. The computational complexity of the algorithms naturally take paramount importance in such applications, and for highly responsive systems, such as aircraft, these requirements become more critical. Conventional designs based on quadratic regulator procedures, in view of their extensive computations associated with the solution of non-linear Riccati equations, have found limited applications in this area. Thus the new synthesis algorithm, by virtue of dual advantage of computational simplicity and ability to synthesize minimum sensitivity systems, shows promise in meeting the rigid requirements of these real-time controllers.

5.3 Topics for Further Research

The pivotal role the modal structure plays in characterizing varied system attributes makes the new formulation potentially attractive to many related multivariable synthesis problems. Output feedback control, transfer function synthesis, dynamic compensators [31], disturbance localization [32], and inverse-optimal control problem [33], are typical examples. Some recommendations of future study would be as follows.

5.3.1 Output Feedback Problem

Consider the controllable/observable system (4.1-4.2). It is required to find a control law

$$u = Ky \tag{5.1}$$

which arbitrarily assigns eigenvalues to the closed-loop system

$$\hat{A} = A + BKC \quad . \tag{5.2}$$

This problem is as yet unsolved and numerous attempts [34,35,36,37] have been made to provide partial solutions to this problem. In general the number of poles that can be "almost arbitrarily assigned" [35], close to the desired values has been $\min(n, m + r - 1)$. More recently [38] an iterative technique has been suggested which improves the number of attainable poles to $\min(n, m \cdot r)$ provided a solution exists. In the framework of the formulation of Chapter II, it is immediately clear that at most $\max(m, r)$ eigenvalues can be assigned arbitrarily with $m \cdot r$ modal entries freely chosen. Indeed Shaw, et al. [39], have tried to extend the formulation suggested in Chapter II [40], for the output feedback case and have pointed out the complexity of the resulting analysis. Kimura [36] attempts to characterize the output feedback solutions in terms of the closed-loop eigenspace. These efforts point towards the hope that the closed-loop eigenvector structures may still hold the key to the solution and thus warrants some more investigation. In particular it is of interest to see if an iterative scheme could be developed which, in addition to partial pole placement, at least guarantees closed-loop stability.

5.3.2 Transfer Function Synthesis

From a frequency domain viewpoint the transfer function matrix with all its poles and zeros specified characterizes the dynamic response of the system, and attempts have been made [41,42] to use the multi-input freedom to specify arbitrary zeros in addition to pole-placement. Chen [41] provides a partial solution to this pole/zero assignment problem from a frequency domain analysis approach. Interestingly, the closed-loop modal matrix is still the link between the frequency/time-domain formulations since the transfer function for the closed-loop system (4.1-4.2) can be expressed as

$$T(s) = CU[sI - \Lambda]^{-1}VB \tag{5.3}$$

where U is the modal matrix, Λ is the matrix of eigenvalues, $V^T = [U^{-1}]^T$ is the matrix of reciprocal basis vectors and s is the Laplace operator. By expanding (5.3) in terms of the powers of s (Leverrier's algorithm [41]), it is possible to derive a set of linear constraints on the reciprocal vectors v_i^T, (when C is of rank n), for each zero specification. Unfortunately the pole-placement specifications generate constraints on the eigenvectors and thus in general it is not possible to provide closed form solutions to this problem. However it is still important to investigate the possibility of approximate transfer function synthesis using the pseudo-inverse matrices [43] $B^{\dagger}$ and $C^{\dagger}$.

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APPENDIX A

COMPLEX CONJUGATE PAIR IN QAUSI-DIAGONAL FORM

In order to allow for real arithmetic computation in the algorithms developed in Chapter II the following well known transformation relating the complex pair of eigenvalues/eigenvectors to an equivalent real pair is established for reference.

Consider the eigenvalue/eigenvector relation

$$A(u + jv) = (\alpha + j\beta)(u + jv) \qquad (A.1)$$

where $\alpha + j\beta$ is a complex eigenvalue of A and $(u + jv)$ the associated eigenvector. (A.1) can be solved for real and imaginary parts as

$$Au = \alpha u - \beta v \qquad (A.2)$$

$$Av = \beta u + \alpha v \qquad (A.3)$$

with the same relationship holding for the conjugate eigenvalue/eigenvector. (A.2) and (A.3) can also be written as

$$A(u : v) = (u : v)\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad (A.4)$$

where u and v are real vectors corresponding to the real and imaginary parts of the complex eigenvector and using the standard quasidiagonal representation of the complex pair of eigenvalues.

It is also possible to derive the first order eigensensitivity relations of section (4.3) for complex pairs in terms of real eigenvectors as follows.

Let u_r, u_c be the real eigenvector pair corresponding to

$$\rho = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

and let v_r and v_c be the reciprocal vectors such that

$$\begin{bmatrix} v_r^T \\ \cdots \\ v_c^T \end{bmatrix} (u_r : u_c) = I_2 \quad . \qquad (A.5)$$

Then the complex eigenvector and reciprocal vector corresponding to $\alpha + j\beta$ are

$$u_1 = u_r + ju_c \qquad (A.6)$$

$$v_1 = v_r - jv_c \qquad (A.7)$$

with the scale factor $(1/\sqrt{2})$ suppressed. Then

$$d\lambda = v_1^T dA_c u_1 \quad , \qquad (A.8)$$

and for $d\lambda = 0$, by equating real and imaginary parts to zero,

$$v_r^T dA_c u_r = -v_c^T dA_c u_c \qquad (A.9)$$

$$v_r^T dA_c u_c = v_c^T dA_c u_r \quad . \qquad (A.10)$$

(A.9) and (A.10) can now be used in (4.32) to develop two relations corresponding to the complex pair eigenvalues.

APPENDIX B

CONSTRUCTION OF THE NONSINGULAR MODAL MATRIX

The procedure outlined is a constructive guarantee of the realization of a nonsingular U, and closely follows the development given in [7]. The notation of [7] is followed for clarity, and the matrix partitions are not necessarily those of Chapter II. The matrix M is constructed as an intermediate step in the realization of the required transformation U. The matrix of eigenvalues will be denoted here by J so that partitions J_i will not be confused with earlier partitions Λ_i:

Rewrite (2.1) as

$$\dot{x} = Fx + Gu \tag{B.1}$$

with matrix G such that

$$G = \begin{bmatrix} G_1 \\ \cdots\cdot \\ G_2 \end{bmatrix}$$

where G_1 is a p x p nonsingular matrix.

Step 1: Let z = Tx where

$$T = \left[\begin{array}{c|c} I_p & 0 \\ \hline -G_2G_1^{-1} & I_{n-p} \end{array}\right]$$

and I_m indicates a m x m identity matrix. This reduces (B.1) to

$$\dot{z} = \hat{F}z + \hat{G}u$$

where $\hat{F} = TFT^{-1}$

$$\hat{G} = \begin{bmatrix} G_1 \\ \cdots\cdot \\ 0 \end{bmatrix} .$$

Express $\hat{F}$ in partitioned form as

$$\hat{F} = \left[\begin{array}{c|c|c} F_{11} & F_{12} & F_{13} \\ \hline F_{21} & \Delta_1 & \Delta_2 \\ \hline \hat{B}_1 & \hat{B}_2 & A \end{array}\right] \tag{B.2}$$

where A is m x m (with m = n - p), $\hat{B}_2$ is m x r, $\hat{B}_1$ is mx(n - m - r) and r is such that $[\hat{B}_2 : A]$ spans an m dimensional subspace. This is achieved by permuting the columns of F [7].

Step 2: Find a nonsingular M such that

$$MJM^{-1} = F + \begin{bmatrix} G_1 \\ \cdots\cdot \\ 0 \end{bmatrix} (K_1 : K_2) \tag{B.3}$$

where J is the diagonal matrix of desired eigenvalues. Then

$$J = \left[\begin{array}{c|c|c} J_1 & 0 & 0 \\ \hline 0 & J_2 & 0 \\ \hline 0 & 0 & J_3 \end{array}\right]$$

and

$$M = \left[\begin{array}{c|c|c} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & M_{33} \end{array}\right] .$$

The partitions of M and J are compatible with the partitioning in (B.2). Now choosing M_{12}, M_{13}, M_{21} as null matrices and using the method outlined in Chapter II, (B.3) reduces to the following equations

$$M_{31}J_1 + AM_{31} = \hat{B}_1M_{11} \tag{B.4}$$

$$M_{32}J_2 + AM_{32} = \hat{B}_2M_{22} \tag{B.5}$$

$$M_{33}J_3 + AM_{33} = \hat{B}_2M_{23} \quad . \tag{B.6}$$

(B.5) and (B.6) can be solved using Theorem 1 of [7] to yield

$$\left[\begin{array}{c|c} M_{22} & M_{23} \\ \hline M_{32} & M_{33} \end{array}\right]$$

nonsingular. Now choosing M_{11} nonsingular results in M invertible. The required transformation U of Chapter II is $U = T^{-1}M$.

APPENDIX C

EIGENVALUE DEFLATION TECHNIQUES

The following well known result in matrix theory [43] is proved for reference in order to emphasize the computational aspects. The notations used in the analysis to follow are not completely compatible with those used in Chapter II.

Theorem: Let x be an eigenvector of a matrix A corresponding to the eigenvalue λ. Let $Q = MP$ be a nonsingular matrix such that $Qx = \mu e_n$. Then

$$QAQ^{-1} = \left[\begin{array}{c|c} C & 0 \\ \hline h^T & \lambda \end{array}\right] , \qquad (C.1)$$

where e_n is a vector whose nth component is unity and other components zero, P is an elementary permutation of the form

$$
P = \begin{array}{cc}
 & \begin{array}{ccccc} & i & & j & \end{array} \\
\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} &
\left[\begin{array}{ccccc}
1 \ddots 1 & & & \vdots & \\
\cdots & 0 & \cdots & 1 & \cdots \\
 & \vdots & & \vdots & \\
\cdots & 1 & \cdots & 0 & \cdots \\
 & \vdots & & \vdots & 1 \ddots
\end{array}\right]
\end{array}
\qquad (C.2)
$$

and M is an elementary upper triangular matrix of order n and index 1 of the form

$$M = I_n = me_n^T \qquad (C.3)$$

with

$$e_n^T m = 0 \quad . \qquad (C.4)$$

Proof by construction. Initially let

$$P = I_n$$

and

$$M = \left[\begin{array}{c|c} I_{n-1} & m \\ \hline 0 & 1 \end{array}\right] \qquad (C.5)$$

with

$$m = (-\mu_1, -\mu_2, \cdots, -\mu_{n-1})^T \quad . \qquad (C.6)$$

Construct M such that

$$Mx = \sigma e_n \quad . \tag{C.7}$$

(C.7) implies

$$\begin{bmatrix} \zeta_1 - \mu_1\zeta_n \\ \cdot \\ \cdot \\ \cdot \\ \zeta_{n-1} - \mu_{n-1}\zeta_n \\ \zeta_n \end{bmatrix} = \sigma e_n$$

where

$$x = (\zeta_1, \zeta_2, \cdots, \zeta_n)^T \quad .$$

This yields

$$\sigma = \zeta_n \tag{C.8}$$

and

$$\mu_i = \left(\frac{\zeta_i}{\zeta_n}\right) \qquad (i = 1,2,\cdots,n-1) \quad . \tag{C.9}$$

Since P only permutes the elements of x the best choice of P from a computational viewpoint is one that prevents numerical overflow in (C.9). Hence choose P such that x is permuted to have

$$|\zeta_n| \geq |\zeta_i| \qquad (i = 1,2,\cdots,n-1) \quad .$$

Now

$$MAM^{-1} = \left[\begin{array}{c|c} I_{n-1} & m \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} A_{11} & a_1 \\ \hline a_2 & a_{nn} \end{array}\right]\left[\begin{array}{c|c} I_{n-1} & -m \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11}+ma_2 & -A_{11}m-ma_2m+a_1+ma_{nn} \\ \hline a_2 & -a_2m+a_{nn} \end{array}\right] \quad . \tag{C.10}$$

Since x is an eigenvector

$$\left[\begin{array}{c|c} A_{11} & a_1 \\ \hline a_2 & a_{nn} \end{array}\right]\left[\begin{array}{c} x_1 \\ \cdots\cdot \\ \zeta_n \end{array}\right] = \lambda \left[\begin{array}{c} x_1 \\ \cdots\cdot \\ \zeta_n \end{array}\right] \tag{C.11}$$

Completing the multiplication in (C.11) gives

$$A_{11}x_1 + a_1\zeta_n = \lambda x_1 \tag{C.12}$$

$$a_2x_1 + a_{nn}\zeta_n = \lambda\zeta_n \quad . \tag{C.13}$$

From (C.9), (C.12) and (C.13)

$$x_1 = -\zeta_n \cdot m \tag{C.14}$$

$$-a_2m + a_{nn} = \lambda \tag{C.15}$$

$$m(a_{nn} - a_2m) + a_1 - A_{11}m = 0 \quad . \tag{C.16}$$

Thus (C.11) can be written as

$$MAM^{-1} = \left[\begin{array}{c|c} A_{11}+ma_2 & 0 \\ \hline a_2 & \lambda \end{array}\right] \tag{C.17}$$

which is in the form (C.1).

VITA 2

S. Srinathkumar

Candidate for the Degree of

Doctor of Philosophy

Thesis: SPECTRAL CHARACTERIZATION OF MULTI-INPUT DYNAMIC SYSTEMS

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Mysore, India, December 28, 1939, the son of Mr. and Mrs. S. N. Srikantia.

Education: Received the Bachelor of Engineering degree in Electrical Engineering from University of Mysore, India in October, 1960; received the Master of Science degree in Electrical Engineering from the University of Hawaii in December, 1973; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1976.

Professional Experience: Scientist, National Aeronautical Laboratory, Bangalore, India, 1961-1971; East-West Scholar, University of Hawaii, Honolulu, Hawaii, 1971-1973; Graduate Assistant, Electrical Engineering, Oklahoma State University, Stillwater, Oklahoma, 1974-present.

Professional Organizations: Member of the Institute of Electrical and Electronic Engineers and Phi Kappa Phi.