ANALYSIS OF BINARY DATA IN DESIGNED EXPERIMENTS

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CHAPTER I

INTRODUCTION AND LITERATURE REVIEW

Statement of the Problem

The statistician is often faced with the data from an experiment in which the individual responses are of a binary (quantal) or categorical nature. Such data can arise in experiments from many fields including psychology, pharmacology, bacteriology, and sample survey designs. It is very essential to have some methods of analyzing the data from these experiments.

The area of the analysis of binary or categorical data is still fertile and an open field for the researchers. This is also indicated by Light and Margolin (1971, p. 534) in the following statement:

> A common problem confronting researchers concerns devising useful methods for analyzing categorical, or nominal scale, data. Researchers familiar with analysis of variance have well-developed techniques for quantitative variables, but must switch to a completely different set of varied techniques when they deal with categorical data.

The objective of this study is to develop a test procedure, hopefully analogous to the analysis of variance, for the analysis of binary data occurring either in one-way classification experiments or in balanced two-way classification experiments with one or more observations per cell.

The concentration will be mainly on testing various kinds of hypotheses of interest. Factorial arrangements of treatments will be given attention. Interactions which occur in the classical treatment of the above designs will also be considered.

An attempt will be made to keep the procedures computationally simple and, hence, the use of transformations will be avoided. The new techniques will be compared with some of their existing competitors, whenever possible.

Literature Review for the Analysis of Binary and Categorical Responses in One-Way Classification

Let t denote the number of experimental groups (treatments) and c denote the number of response categories. Let n_{ij} represent the number of responses in category i for group j, $i = 1,\ldots,c$ and $j = 1,\ldots,t$. The number of responses, or sample size, from group j is $n_{\cdot j} = \sum_{i=1}^{c} n_{ij}$. Similarly, the numbering of responses in the i^{th} category is $n_{i\cdot} = \sum_{j=1}^{t} n_{ij}$. Thus the total number of responses in the study is:

$$N = \sum_{j=1}^{t} n_{\cdot j} = \sum_{i=1}^{c} n_{i\cdot} = \sum_{i=1}^{c} \sum_{j=1}^{t} n_{ij}.$$

An alternative way of viewing this data is via a c x t contingency table where n_{ij} is the count in the $(i,j)^{th}$ cell.

Let p_{ij} be the probability that any experimental unit from group j will yield a response in category i, and $\sum_{i=1}^{c} p_{ij} = 1$ for all j.

A common model for the one-way classification, assuming the responses within groups and from group to group to be stochastically independent and following a multinomial model, can be written as:

$$\Pr\{(n_{11},\ldots,n_{c1},\ldots,n_{1t},\ldots,n_{ct})\}$$

$$= \prod_{j=1}^{t} \left[\binom{n_{\cdot j}}{n_{ij},\ldots,n_{cj}} \prod_{i=1}^{c} (p_{ij})^{n_{ij}} \right] , \qquad (1.1)$$

where $n_{\cdot j} > 0$ and $\sum_{i=1}^{c} p_{ij} = 1$ for all j.

The standard null hypothesis of interest for a one-way classification based on categorical data is that the t samples are from the same population, i.e.,

$$H_0 : p_{ij} = p_i \text{ for all i and j} \qquad (1.2)$$

against

$$H_A: \text{ Not } H_0.$$

Several techniques exist in the literature for testing the hypothesis given by (1.2). One group of techniques uses the data in their original form while the second group of techniques transforms the data in such a fashion that it can be treated by the existing methods for quantitative data. Pearson's chi-square test (1900), the likelihood ratio test due to Wilks (1935) and the CATANOVA procedure of Light and Margolin (1971) are some of the examples for the first group of techniques while the logit transformation proposed by Cox (1969), Winsor (1948), Dyke and Patterson (1952), and Cart and Zweifel (1967) are some of the examples for the second group of techniques.

Pearson's chi-square is most commonly used among researchers. The χ^2 statistic is given by

$$\chi^2 = \sum_{i=1}^{c} \sum_{j=1}^{t} \left[\frac{\left(n_{ij} - \frac{n_{i.} n_{.j}}{N} \right)^2}{\frac{n_{i.} n_{.j}}{N}} \right] . \qquad (1.3)$$

The asymptotic null distribution of this χ^2 statistic is chi-square with (c-1)(t-1) degrees of freedom.

This procedure will be referred here as a χ^2 test procedure with the understanding that it is a χ^2 test procedure for the one-way classification and not for the independence. The χ^2 test statistic in the one-way classification for testing the hypothesis of a common population based on several samples of grouped data is computationally equivalent to that of testing the hypothesis of independence in a two-dimensional contingency table, but due to different experimental situations and sampling procedures involved, they give two different tests.

Wilks (1935) has presented likelihood ratios for several situations in contingency tables. For each case, -2(natural logarithm of the likelihood ratio) is approximately distributed as a χ^2. Wilks showed in his paper that the χ^2 method for testing several hypotheses in a contingency table has no greater theoretical validity than that of the likelihood ratio method. For testing the usual null hypothesis given by (1.2) in the general c x t table under the model in (1.1), the likelihood ratio is:

$$L = \left(\prod_{i=1}^{c} n_{i.}^{\,n_{i.}}\right)\left(\prod_{j=1}^{t} n_{.j}^{\,n_{.j}}\right)\Bigg/\left(N^N \prod_{i=1}^{c}\prod_{j=1}^{t} n_{ij}^{\,n_{ij}}\right).$$

The statistic $-2(\ln L)$ is approximately distributed as χ^2 with $(c-1)(t-1)$ degrees of freedom.

For the one-way classification, $-2(\ln L)$ is the same as $2\hat{I}$, the Kullback's (1962) minimum discrimination information statistic:

$$2\hat{I} = 2\left[N(\ln N) + \sum_{j=1}^{t}\sum_{i=1}^{c} n_{ij}(\ln n_{ij}) - \sum_{i=1}^{c} n_{i.}(\ln n_{i.}) - \sum_{j=1}^{t} n_{.j}(\ln n_{.j})\right],$$

with the asymptotic null distribution of χ^2 with $(c-1)(t-1)$ degrees of freedom. Here $0(\ln 0)$ is defined to be 0.

Obviously, one can use the methods described above and the CATANOVA procedure of Light and Margolin (to be discussed at the end of this section) for the analysis of binary responses in one-way classification just by letting $c = 2$. Below is a technique proposed by Brown and Mood (1948) which can be used indirectly in this situation.

Let n_{1j} denote the number of 1's under group (treatment) j and let n_{2j} denote the number of 0's under group j for

$j = 1,\ldots,t$. Note that $n_{2j} = n_{.j} - n_{1j}$, and $n_{2.} = N - n_{1.}$. Then it is of the interest to test the usual hypothesis of the equality of treatment effects given by (1.2), i.e.,

$$H_0: \; p_1 = \cdots = p_t \; (=p \text{ say})$$

against $\quad H_A: \;$ Not H_0 ,

where p_j is the probability of success under treatment j.

Under the null hypothesis, the probability distribution of the random variables, $n_{11},\ldots,n_{1t}$, is

$$f(n_{11},\ldots,n_{1t}) = \frac{\prod_{j=1}^{t} \binom{n_{.j}}{n_{1j}} p^{n_{1j}} (1-p)^{n_{.j}-n_{1j}}}{\binom{N}{n_{1.}} p^{n_{1.}} (1-p)^{N-n_{1.}}} = \frac{\prod_{j=1}^{t} \binom{n_{.j}}{n_{1j}}}{\binom{N}{n_{1.}}} \quad , \tag{1.4}$$

which is a multivariate hypergeometric point probability.

The observed significance level for a given set of data can be calculated by summing the point probabilities for all the data sets as extreme or more so with the same row and column marginal frequencies. The null hypothesis is rejected when this sum is smaller than the desired significance level. Unfortunately, with large values of $n_{.j}$ or t, calculations become tedious and time consuming and as a result, this test is rarely carried out by this procedure. Fortunately, a fairly good approximation to (1.4) is available when $n \geq 20$

and all $n_{.j} \geq 5$. The test statistic under this criterion is:

$$T = \frac{N^2}{n_{1.}n_{2.}} \sum_{j=1}^{t} \frac{\left(n_{1j} - \frac{n_{1.}n_{.j}}{N}\right)^2}{n_{.j}}$$

or

$$T = \frac{N^2}{n_{1.}n_{2.}} \sum_{j=1}^{t} \frac{\left(n_{2j} - \frac{n_{2.}n_{.j}}{N}\right)^2}{n_{.j}} .$$

One can show that the above test statistic T turns out to be a special case of the χ^2 statistic defined by (1.3) for c = 2 as follows:

$$\chi^2 = \frac{N}{n_{1.}} \sum_{j=1}^{t} \left[\frac{\left(n_{1j} - \frac{n_{1.}n_{.j}}{N}\right)^2}{n_{.j}}\right] + \frac{N}{n_{2.}} \sum_{j=1}^{t} \left[\frac{\left(n_{2j} - \frac{n_{2.}n_{.j}}{N}\right)^2}{n_{.j}}\right]$$

$$= \left(\frac{N}{n_{1.}} + \frac{N}{n_{2.}}\right) \sum_{j=1}^{t} \left[\frac{\left(n_{1j} - \frac{n_{1.}n_{.j}}{N}\right)^2}{n_{.j}}\right]$$

$$= \frac{N^2}{n_{1.}n_{2.}} \sum_{j=1}^{t} \left[\frac{\left(n_{1j} - \frac{n_{1.}n_{.j}}{N}\right)^2}{n_{.j}}\right]$$

$$= T .$$

When H_0 is true, the distribution of T is approximately chi-square with (t-1) degrees of freedom. According to Mood (1950), this approximation can be improved by multiplying the statistic T by $(\frac{N-1}{N})$, obtaining

$$T' = \left(\frac{N-1}{N}\right) T = \left(\frac{N-1}{N}\right) \chi^2$$

For further discussion of the Brown-Mood Median Test, refer to Bradley (1968), Gibbons (1971), and Mood (1950). Gibbons (1950) has discussed some tests of the equality of independent samples in Chapter II. The Kruskal-Wallis one-way ANOVA test discussed in this chapter is not quite appropriate for the analysis of binary data but sometimes is used in practice.

Gabriel (1963) has given some F tests, estimates, and confidence bounds for dichotomous data.

Cox (1969), Winsor (1948), Dyke and Patterson (1952), Gart and Zweifel (1967) have proposed logit transformations which allow treating the data by standard analysis of variance techniques.

In the book by Cox (1969), various usual situations of the design of experiments and regression involving binary data are treated by considering the models in which the logistic transform of the probability of success is a linear combination of unknown parameters. These linear logistic models play about the same role as do the normal theory models in the analysis of continuously distributed data. Mainly the test statistics are based on the sufficient statistics and the maximum likelihood procedure. Some of the exact tests presented in this book can be laborious and time consuming.

Winsor (1948) in his presentation of factorial analysis of a multiple dichotomy has indicated a method which, where applicable, provides the standard errors of the estimates, together with significance tests for effects which have been assumed non-existent. The method proposed is essentially that suggested by Yates (1934).

Dyke and Patterson (1952) have provided the logit transformation to transform the observations in the new scale of measurements which can reasonably be represented as linear functions of a number of parameters. Maximum likelihood estimates of these parameters are then found.

Gart and Zweifel (1967) have investigated the bias of several logit estimators and their corresponding estimators in small samples.

For the special case of dichotomous data, it is Cochran's (1950) suggestion to assign the values of 0 or 1 to represent the responses in the two categories and then to use the analysis of variance technique for analyzing the data.

CATANOVA Procedure

Light and Margolin (1971) have proposed "An Analysis of Variance" for categorical data, referred to as CATANOVA.

In the case of continuous data, the coefficient of multiple determination provides a measure of association between response and predictor variables. This measure depends upon the ratio of two appropriate sums of squares of the response variable. It can be interpreted as the proportion of total variation observed in the response variable that is attributed to (or explained by) the predictor or classification variables. Various measures of association for categorical data have appeared in the literature; however, none can be given the above "proportion of explained variation" interpretation since the concept of partitioning variation was never applied to categorical data. Goodman and Kruskal (1954, 1959, 1963, 1972) have a series of research papers on measures of association. A bibliography on the measures of association can be found in a book by Lancaster (1969).

Light and Margolin (1971) credit C. W. Gini (1912) with having noted that the sum of squares of deviations from the mean for the quantitative measurements can be expressed solely as a function of the squares of the pairwise differences for all $\binom{n}{2}$ pairs. Specifically, if $X_1,\ldots,X_n$ denote the measurements, then

$$\sum_{i=1}^{n} (X_i - \overline{X})^2 = \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^2 ,$$

where $\overline{X} = \sum_{i=1}^{n} X_i/n$ and $d_{ij} = X_i - X_j$.

Reasoning by analogy, Gini later developed in his 1938 Variabilità E Concentrazione a measure of variation a measure of variation for categorical data. Assume that each of the responses $X_1,\ldots,X_n$ names one and only one of c possible categories and define $d(X_i, X_j) = d_{ij}$ as:

$$d_{ij} = \begin{cases} 1 \text{ if } X_i \text{ and } X_j \text{ name different categories} \\ 0 \text{ if } X_i \text{ and } X_j \text{ name the same category.} \end{cases}$$

Then,

Definition: The variation for categorical responses $X_1,\ldots,X_n$ is

$$\frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 = \frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij},$$

where each response names one and only one of c possible categories and d_{ij} is defined as above.

If n_i is the number of responses naming the i^{th} category for $i = 1,\ldots,c$, then $\sum_{i=1}^{c} n_i = n$ and the variation of $X_1,\ldots,X_n$ becomes:

$$\frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} = \frac{1}{2n}\left[\sum_{i=1}^{n} n_i \, (n - n_i)\right]$$

$$= \frac{n}{2} - \frac{1}{2n}\sum_{i=1}^{c} n_i^2.$$

Gini's measure of variation possesses the following two desirable properties:

(1) The variation is minimized to zero if and only if all n categorical responses name the same category.

(2) The variation is maximized when the responses are distributed among the available categories as evenly as possible.

In the spirit of Gini's measure of variation for categorical data, the total variation observed in the response variable, or "total sum of squares" is equal to:

$$TSS = \frac{N}{2} - \frac{1}{2N} \sum_{i=1}^{c} n_{i.}^2 \ . \tag{1.5}$$

The total within-group variation or "within group sum of squares" is obtained by applying Gini's definition within each group and then summing over the t groups to give:

$$\begin{aligned} WSS &= \sum_{j=1}^{t} \left(\frac{n_{.j}}{2} - \frac{1}{2n_{.j}} \sum_{i=1}^{c} n_{ij}^2 \right) \\ &= \frac{N}{2} - \frac{1}{2} \sum_{j=1}^{t} \frac{1}{n_{.j}} \sum_{i=1}^{c} n_{ij}^2 \ . \end{aligned} \tag{1.6}$$

The between-group variation or sum of squares is equal to:

$$BSS = TSS - WSS = \frac{1}{2} \sum_{j=1}^{t} \frac{1}{n_{.j}} \sum_{i=1}^{c} n_{ij}^2 - \frac{1}{2N} \sum_{i=1}^{c} n_{i.}^2 \ .$$

Then Light and Margolin (1971) proposed an "analysis of variance" technique to test the standard null hypothesis for a one-way classification based on categorical data which is given by (1.2). They named their proposed test statistic, C, as the CATANOVA (categorical analysis of variation) statistic.

$$C = (N-1)(c-1) \cdot \frac{BSS}{TSS} \ .$$

Under this common multinomial model, given by (1.1), the null distribution of the CATANOVA statistic, C, is asymptotically chi-square with (c-1)(t-1) degrees of freedom. Their simulation study indicated that even for small group sizes, the statistic C is approximated quite well by chi-square with (c-1)(t-1) degrees of freedom under H_0.

They found that asymptotically with large $n_{.j}$, TSS and BSS are independent under H_0 which is just the opposite of the situation in the standard analysis of variance. This fact made them depart from the standard AOV theory.

Light and Margolin (1971) have failed to note that even though the χ^2 statistic for testing the hypothesis of independence is computationally equivalent to the χ^2 statistic for testing the hypothesis of common population in several samples of grouped data, they give different tests due to differences in the sampling procedures.

They have proposed a measure of association between the grouping and response variables which may be given a "proportion of variation explained" interpretation. This measure is defined as:

$$R^2 = \frac{BSS}{TSS} = \frac{\left(\sum_{j=1}^{t} \frac{1}{n_{.j}} \sum_{i=1}^{c} n_{ij}^2\right) - \frac{1}{N} \sum_{i=1}^{c} n_{i.}^2}{N - \frac{1}{N} \sum_{i=1}^{c} n_{i.}^2} \; .$$

The R^2 defined above has some nice properties as one would expect of a measure of association. As they have noted, $R^2 = 0$ if $\frac{n_{ij}}{n_{.j}} = f_i$, $i = 1,\ldots,c$; $j = 1,\ldots,t$, i.e., if there is no association--no effect of group on distribution of category. $R^2 = 1$ if for each j, $j = 1,\ldots,t$, there exists an i such that $n_{ij} = n_{.j}$, i.e., if there is perfect predictability. Otherwise, $0 < R^2 < 1$.

Further, R^2 is the proportion of total variation in the response variable which is accounted for by the knowledge of the grouping variable. Multiplying all entries in a contingency table by any positive constant leaves R^2 unchanged.

In their papers (1971, 1974), Light and Margolin did not reach the general analytic results for the c x t tables which can make the comparison of CATANOVA and χ^2 methods an easier task. However, they have done simulation studies on the computer for 3x2 tables under some selected alternative hypotheses. In their study of 3x2 tables, they generated 1,000 samples for each of the ten table structures under the multinomial model. For both the groups, $n_{.j}$ was fixed to be 100.

They observed that if one group's probabilities are held at (1/3, 1/3, 1/3) and for the other group if one category has a high response probability and the other two have low probabilities, then the power of CATANOVA statistic is higher than that of a chi-square statistic. On the other hand, if one group's probabilities are held at (1/3, 1/3, 1/3) and for the other group if one category has a low response probability and the other two have high probabilities, then the power of the chi-square statistic is higher than that of the CATANOVA statistic. They have noted that under both H_0 and H_A, the two techniques give orderings of data sets that are highly correlated.

They also have concluded that the MANOVA (one-way multivariate analysis of variance) test statistic is a monotonically increasing function of the CATANOVA test statistic.

Light and Margolin (1974) presented some empirical evidence that in small samples the distribution of C is somewhat better approximated by $\chi^2_{[(c-1)(t-1)]}$ than is the distribution of the χ^2 statistic, and both are considerably better approximated by $\chi^2_{[(c-1)(t-1)]}$ than is the null distribution of $2\hat{I}$. For further discussion concerning this subject matter, refer to the papers by Light and Margolin (1971, 1974).

Literature Review for the Analysis of Binary and Categorical Responses in Two-Way Classification

Now consider an experiment conducted as a two-way classification with an equal number of binary observations that are sampled from each population corresponding to each cell. As some of the assumptions underlying the analysis of variance are violated, it may not be appropriate to use the ANOVA technique in this situation. Some of the rank tests may also not be appropriate here but are often used because of their simplicity or the lack of more appropriate techniques. Cochran's Q test (1950) and Friedman's rank test (1937) are examples of this and will be discussed next.

Cochran's Q Test

Now consider the situation where there are b rows (blocks) and t columns (treatments) with one binary observation per cell. Let x_{ij} denote the observation in the i^{th} row corresponding to the j^{th} column. Cochran (1950) suggested a Q statistic to test the equality of column effects under such a situation. However, he assumed the row totals to be fixed. He showed that the asymptotic null distribution of the quantity Q,

$$Q = \frac{t(t-1) \sum_{i=1}^{t} \left(T_i - \bar{T}\right)^2}{t\left(\sum_{j=1}^{b} u_j\right) - \left(\sum_{j=1}^{b} u_j^2\right)}$$

as b increases, is the chi-square with (t-1) degrees of freedom, where T_i is the number of 1's in the i^{th} column, $\bar{T}$ is the mean of the T_i's, and u_j the number of 1's in the j^{th} row.

The requirement of large b enables one to assume the joint distribution of column totals to be multivariate normal which is necessary in the derivation of Q.

At first, it is not clear how one should interpret or justify the assumption of the fixed row totals. This assumption makes sense if one is willing to rank the observations within each row (block) and use the mid-ranks technique for handling ties. If one does this, then the way of treating the data becomes the same as with Friedman's rank test which is to be discussed briefly below.

The Friedman's rank test (1937) considers the same situation as that of Cochran's Q test but instead of binary data, it assumes the data is measured on an ordinal scale within each row (block). So theoretically, it assumes no ties in the ranks within each row, while that will not be the case in Cochran's Q test if the number of treatments is greater than two. This will make Friedman's rank test an inappropriate one to use for the present situation with binary data. However, it should be pointed out that it is used incorrectly many times in practice.

Friedman's test statistic to test the hypothesis of the equality of treatment effects is given by

$$\chi_b^2 = \frac{12}{bt(t+1)} \sum_{j=1}^{t} \left[R_{.j} - \frac{b(t+1)}{2} \right]^2 ,$$

where $R(X_{ij})$ is the rank of X_{ij} and

$$R_{.j} = \sum_{i=1}^{b} R(X_{ij}) .$$

The statistic χ_b^2 is a special case of the form of Sen's (1968) statistic S_n which for the present situation reduces to

$$S_n = \frac{t-1}{t} \frac{\sum_{j=1}^{t} \left[\bar{R}_{.j} - E(\bar{R}_{.j}) \right]^2}{V(\bar{R}_{.j})} , \quad (1.7)$$

where $\bar{R}_{.j} = R_{.j} / b$.

Sen (1968) showed that the general statistic S_n has the asymptotic null distribution of central chi-square with (t-1) degrees of freedom and hence the distribution of χ_b^2 is the same as that of S_n.

The following structure due to Brown and Mood, discussed by Claypool (1975), is of interest as it deals with binary data under a specified structure and has a satisfactory theoretical base behind it. Under the Brown-Mood structure, each of the b observers assigns a value of 1 for the $k(<t)$ most preferred treatments (out of t) and 0 to the remaining (t-k) less preferred treatments. Under this situation, one will have the fixed and equal row totals. As is mentioned by Claypool (1975), Sen's statistic S_n can be used here to test the treatments. The Brown-Mood structure can be thought of as a special case of Cochran's Q test.

One can show that the Cochran's Q statistic is of the form S_n defined by (1.5), as was the Friedman's rank statistic, χ_b^2, and hence its asymptotic null distribution is chi-square with (t-1) degrees of freedom. Brownlee (1965) has shown Cochran's Q statistic to be a special case of Friedman's statistic; however, he credits Nancy D. Bailey and William H. Kruskal with demonstrating this proof to him.

The asymptotic null distribution of the statistic Q will be derived in Chapter II using a different approach.

In his paper, Cochran (1950) presented the comparison of the Q test with the ordinary χ^2 test for one-way classification which is valid when the samples are independent.

These two tests coincide when the probability of success does not change from row to row. The appropriate χ^2 statistic is

$$\chi^2_{col} = \frac{\sum_{i=1}^{t}\left(T_i - \bar{T}\right)^2}{b\,\frac{\bar{u}}{t}\left(1 - \frac{\bar{u}}{t}\right)}, \text{ where } \bar{u} = \frac{\sum_{j=1}^{b} u_j}{b} .$$

Similarly, the appropriate χ^2 statistic to test the row effects is

$$\chi^2_{row} = \frac{\sum_{j=1}^{b}\left(u_j - \bar{u}\right)^2}{t\,\frac{\bar{u}}{t}\left(1 - \frac{\bar{u}}{t}\right)} .$$

Cochran (1950) concluded that the Q test gives more significant results when χ^2_{row} exceeds its expectation, and fewer significant results when χ^2_{row} is below expectation.

Cochran (1950, p. 262) mentions:

> If the data had been measured variables that appeared normally distributed, instead of a collection of 1's and 0's, the F-test would be almost automatically applied as the appropriate method. Without having looked into the matter, I had once or twice suggested to research workers that the F-test might serve as an approximation even when the table consists of 1's and 0's. As a testimony to the modern teaching of statistics, the suggestion was received with incredulity, the objection being made that the F-test requires normality, and that a mixture of 1's and 0's could not by any stretch of the imagination be regarded as normally distributed. The same workers raised no objection to a χ^2 test, not having realized that both tests require to some extent an assumption of normality, and that it is not obvious whether F or χ^2 is more sensitive to the assumption. Inclusion of the F-test is also worthwhile in view of the widespread interest in the application of the analysis of variance to non normal data.

So later in his paper, Cochran considered the F test as an alternative to his Q test and compared them. He concluded that the use of the χ^2 approximation for the Q statistic is preferable to the F statistic after correction for continuity since it is easier to calculate. Neither method is free from bias. Both the methods are "close enough" for routine decisions.

Tate and Brown (1970) have done an extensive study on the distribution of Q in small samples and have given a rule of thumb which aids in judging when the chi-square approximation to Q is satisfactory for practical purposes. The rule of thumb as given by them is as follows:

Delete each row containing only 1's or only 0's. Let r denote the number of rows remaining. If $rt \geq 24$ and $r \geq 4$ then the approximation is generally satisfactory. Otherwise the tables given by Tate and Brown (1964) should be used or the exact distribution constructed. The range of errors, however, suggests the results be interpreted cautiously when the chi-square probability turns out to be near a critical value.

Recently, Patil (1975) proposed a relatively simple method for computing the exact probability distribution of the Q statistic and extended the tables of Tate and Brown (1964).

Tallis (1964) has suggested a method of analyzing the similar situation with one or more observations per cell. His development is based on the model similar to that of the standard two-way analysis of variance with usual constraints. Maximum likelihood estimation for the parameters in the model is also discussed.

As discussed earlier, Cochran's procedure assumes row totals to be fixed and since it is usually not possible to specify row (block) totals before the data is collected, Cochran's Q test is not explicitly appropriate for this situation. A more general procedure which does not require the assumption of fixed row totals will be proposed in this dissertation. A case with more than one observation per cell will also be considered with possible interactions and factorial arrangements of treatments.

Homogeneity of Two-Way Tables

Now consider the same situation with more than one <u>binary</u> observation (say n) per treatment x block cell. Then the data can be arranged in a three-way contingency table of size t x b x 2. This t x b x 2 table can be thought of as being a set of t independent two-way tables of size b x 2, each table corresponding to each treatment with fixed total. It will be of interest to test for the homogeneity of these b x 2 tables which is the same as testing the equality of the treatment effects. Similarly, it will also be interesting to test the equality of block effects and possible interactions.

The two-way tables are said to be homogeneous if the probabilities associated with corresponding cells are homogeneous. So in order to test the treatments (i.e., homogeneity of t tables of size b x 2), the null hypothesis can be formulated as:

$$H_0: \ p_{ijk} = p_{.jk},$$

where p_{ijk} is the probability of the observation under i^{th} treatment and j^{th} block to fall in k^{th} category, $i = 1,\ldots,t$; $j = 1,\ldots,b$; $k = 1,2$; and $\sum_{k=1}^{2} p_{ijk} = 1$ for fixed i and j.

As is apparent, treatment totals and block totals (within each treatment) are fixed for this situation.

In the literature, not much work can be found in this direction. The work done by the author in this direction will be presented in later chapters. One might be able to find a way to analyze this situation using the exponential model. Some discussion regarding this will also be given in later chapters.

Kullback (1959) has worked on a similar problem but he has assumed only the treatment totals to be fixed and not the block totals under each treatment or each b x 2 table. Notice that for his case, p_{ijk} denotes the probability of the observation under the i^{th} treatment to fall in j^{th} block and k^{th} category and hence $\sum_{k=1}^{2} \sum_{j=1}^{b} p_{.jk} = 1$. For this situation Kullback (1959) has given a test statistic based on information theory to test treatments, which is:

$$2\hat{I} = 2 \sum_{k=1}^{2} \sum_{j=1}^{b} \sum_{i=1}^{t} n_{ijk} \ \ln\left(\frac{N \cdot n_{ijk}}{n_{i..} n_{.jk}}\right)$$

where n_{ijk} is the frequency of occurrence in the i^{th} treatment, j^{th} block, and k^{th} category. $\sum_{k=1}^{2} \sum_{j=1}^{b} \sum_{i=1}^{t} n_{ijk} = N$, $n_{i..} = \sum_{k=1}^{2} \sum_{j=1}^{b} n_{ijk}$ and $n_{.jk} = \sum_{i=1}^{t} n_{ijk}$.

The asymptotic null distribution of the statistic $2\hat{I}$ is central chi-square with $(t-1)(bk-1)$ degrees of freedom. He has also shown that the statistic to test the independence of treatment classification with (block, category) classification is also of the same form, as above.

In the literature, the majority of the tests which are given for the analysis of multi-dimensional contingency tables are for testing the various types of interactions rather than directly for testing the main effects.

Hoyt, Krishnaih and Torrance (1959) have given the derivation of maximum likelihood estimates of probabilities that are used for testing certain hypotheses regarding interactions in contingency tables. They also have given a four-dimensional illustrative contingency table and have demonstrated how to apply their procedure for testing various hypotheses of independence.

Darroch (1962) has compared interactions in contingency tables with interactions in the analysis of variance. He pointed out that the interactions in contingency tables possess only a few of the fortuitously simple properties of interactions in the analysis of variance.

In three-way and multi-way contingency tables, Birch (1963) has considered interactions as certain linear combinations of the logarithms of the expected frequencies. Maximum likelihood estimation is also presented in this paper for multi-way tables.

Roy and Mitra (1956) have discussed the analysis of p-variate responses arranged in a q-way classification.

Lewis (1962) has presented a very general review of the important methods of analysis in multi-way contingency tables, along with a selection of procedures which are computationally the simplest available, and which may be adapted for use with different sampling schemes and/or with theoretical rather than estimated parameters.

Ku, Varner, and Kullback (1971) have described the principle of minimum discrimination information estimation and have used it to generate estimates for tests of hypotheses regarding various interactions and effects in the analysis of multi-dimensional contingency tables. According to them, with this principle, when certain marginals are fixed, all classical hypotheses for contingency tables can be generated.

CHAPTER II

ANALYSIS OF BINARY DATA IN ONE-WAY CLASSIFICATION SITUATIONS WITH EQUAL NUMBER OF OBSERVATIONS PER TREATMENT

The BIANOVA Technique

In this chapter and onwards, slightly different notations will be employed compared with that in the previous chapter.

Consider an experiment conducted as a one-way classification with t treatments and n binary observations per treatment. Let X_{ij} denote the binary response of the j^{th} subject under the i^{th} treatment for $i = 1, \ldots, t$ and $j = 1, \ldots, n$. Let n_i be the total number of 1's ("successes") under the i^{th} treatment. Then the data appear as follows:

treatment number →	1	2	...	t
	X_{11}	X_{21}	...	X_{t1}
	⋮	⋮	...	⋮
	X_{1n}	X_{2n}		X_{tn}
treatment total →	n_1	n_2	...	n_t

All the X_{ij} are assumed to be independent of each other. Then testing of the following hypothesis is of interest:

H_0: The treatments are equally effective.

H_A: At least one treatment is different in effectiveness from at least one other. (2.1)

Suppose that p_i is the true probability of "sucess" under treatment i. Then the above hypothesis may be restated in mathematical terms as follows:

H_0: $p_1 = p_2 = \ldots = p_t$ (= p say)

H_A: At least one p_i is different from at least one other.

The following test statistic, B, is proposed to test the above hypothesis:

$$B = \frac{nt \sum_{i=1}^{t} \left(\hat{p}_i - \hat{\bar{p}}\right)^2}{\sum_{i=1}^{t} \hat{p}_i \hat{q}_i}$$

where $\hat{p}_i = \frac{n_i}{n}$, $\hat{q}_i = 1-\hat{p}_i$, and $\hat{\bar{p}} = \sum_{i=1}^{t} \hat{p}_i/t = \sum_{i=1}^{t} n_i/nt$.

The following form of B is more suitable for computational and accuracy purposes:

$$B = \frac{nt \sum_{i=1}^{t} n_i^2 - n\left(\sum_{i=1}^{t} n_i\right)^2}{n \sum_{i=1}^{t} n_i - \sum_{i=1}^{t} n_i^2}$$

Note that

$$\sum_{j=1}^{n} \sum_{i=1}^{t} \left(\hat{p}_i - \hat{\bar{p}}\right)^2 = \sum_{j=1}^{n} \left[\sum_{i=1}^{t} \hat{p}_i{}^2 - t\hat{\bar{p}}\right]^2$$

$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{t} \frac{n_i^2}{n} - \frac{t \left(\sum_{i=1}^{t} n_i\right)^2}{n^2 t^2} \right]$$

$$= \sum_{i=1}^{t} \frac{n_i^2}{n} - \frac{\left(\sum_{i=1}^{t} n_i\right)^2}{nt}$$

and

$$\sum_{j=1}^{n} \sum_{i=1}^{t} \hat{p}_i \cdot \hat{q}_i = \sum_{j=1}^{n} \sum_{i=1}^{t} \left(\hat{p}_i - \hat{p}_i{}^2\right)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{t} \left(\frac{n_i}{n} - \frac{n_i^2}{n^2}\right)$$

$$= \sum_{i=1}^{t} \left(n_i - \frac{1}{n} n_i{}^2\right)$$

$$= \sum_{i=1}^{t} \left[\sum_{j=1}^{n} x_{ij}{}^2 - \frac{1}{n} \left(\sum_{j=1}^{n} x_{ij}\right)^2\right] .$$

Hence, the statistic B can be written as:

$$B = \frac{t \sum_{j=1}^{n} \sum_{i=1}^{t} \left(p_i - \hat{\bar{p}}\right)^2}{\sum_{j=1}^{n} \sum_{i=1}^{t} \hat{p}_i \; \hat{q}_i / n}$$

$$B = nt \frac{\left[\sum_{i=1}^{t} \frac{n_i^2}{n} - \frac{1}{nt} \times \left(\sum_{i=1}^{t} n_i\right)^2\right]}{\sum_{i=1}^{t}\left[\sum_{j=1}^{n} x_{ij}{}^2 - \frac{1}{n}\left(\sum_{j=1}^{n} x_{ij}\right)^2\right]}$$

The statistic B can be expressed as follows in terms of the components from the standard analysis of variance table:

$$B = nt \, \frac{\text{between group SS}}{\text{within group SS}}$$

$$= \frac{n(t-1)}{n-1} \, \frac{\text{between group MS}}{\text{within group MS}}$$

$$= \frac{n(t-1)}{n-1} \, F\,[(t-1),\ t(n-1)] \quad !!!$$

As the statistic B is made up of some of the components from the regular analysis of variance table, it is named as BIANOVA statistic (binary analysis of variation statistic) and this technique as the BIANOVA technique.

Notice that for fixed n and t, the usual ANOVA test statistic, F, is a monotonically increasing function of the BIANOVA test statistic, B.

Under this situation of binary responses, it turns out that Gini's definition of categorical variation is equivalent to the "usual" definition of variation in ANOVA. This can be demonstrated as follows:

Total SS in Gini's sense (repeating (1.5)) is:

$$\text{Total SS} = \frac{N}{2} - \frac{1}{2N} \sum_{i=1}^{c} n_{i.}^2$$

which can be written as follows in the notation used in this chapter:

$$\text{Total SS} = \frac{nt}{2} - \frac{1}{2nt}\left[\left(\sum_{i=1}^{t} n_i\right)^2 + \left(nt - \sum_{i=1}^{t} n_i\right)^2\right]$$

$$= \frac{nt}{2} - \frac{1}{2nt}\left[\left(\sum_{i=1}^{t} n_i\right)^2 + n^2t^2 + \left(\sum_{i=1}^{t} n_i\right)^2 - 2nt\left(\sum_{i=1}^{t} n_i\right)\right]$$

$$= \sum_{i=1}^{t} n_i - \frac{1}{nt}\left(\sum_{i=1}^{t} n_i\right)^2$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{n} X_{ij}^2 - \frac{1}{nt}\left(\sum_{i=1}^{t}\sum_{j=1}^{n} X_{ij}\right)^2$$

= "usual" Total SS in ANOVA.

Within SS in Gini's sense (repeating (1.6)) is:

$$\text{Within SS} = \sum_{j=1}^{t}\left(\frac{n_{.j}}{2} - \frac{1}{2n_{.j}}\sum_{i=1}^{c} n_{ij}^2\right)$$

which can be written as follows in the notation used in this chapter:

$$\text{Within SS} = \sum_{i=1}^{t}\left[\frac{n}{2} - \frac{1}{2n}\left\{n_i^2 + (n - n_i)^2\right\}\right]$$

$$= \sum_{i=1}^{t}\left[\frac{n}{2} - \frac{1}{2n}\left(n_i^2 + n^2 + n_i^2 - 2nn_i\right)\right]$$

$$= \sum_{i=1}^{t}\left[n_i - \frac{1}{n}\left(n_i^2\right)\right]$$

$$\text{Within SS} = \sum_{i=1}^{t}\left[\sum_{j=1}^{n} x_{ij}^{2} - \frac{1}{n}\left(\sum_{j=1}^{n} x_{ij}\right)^{2}\right]$$

= "usual" Within SS in ANOVA.

Hence for binary data, the CATANOVA statistic becomes:

$$C = (nt-1)\,\frac{\text{"usual" between group SS}}{\text{"usual" total SS}}$$

Distributional Derivation of the BIANOVA Statistic

Now it will be shown that under H_0, the asymptotic distribution of B is central chi-square with (t-1) degrees of freedom (same as that of C!) and under H_A, the approximate asymptotic distribution of B is non-central chi-square with (t-1) degrees of freedom and the non-centrality parameter is

$$\lambda = \frac{nt \sum_{i=1}^{t} \left(p_i - \bar{p}\right)^2}{2 \sum_{i=1}^{t} p_i q_i} > 0 .$$

The loss of 1 degree of freedom might be explained by the fact that $\sum_{i=1}^{t} \left(\hat{p}_i - \hat{\bar{p}}\right) = 0$.

<u>Derivation</u>

The test statistic B can be written as B = N/D, where

$$N = \sum_{i=1}^{t} \frac{\left(\hat{p}_i - \hat{\bar{p}}\right)^2}{pq/n} \quad \text{and} \quad D = \sum_{i=1}^{t} \frac{\hat{p}_i\,\hat{q}_i}{nt} \Big/ \frac{pq}{n} .$$

Notice that N can be written in a quadratic form as nY'AY/pq where $Y' = \left(\hat{p}_1, \ldots, \hat{p}_t\right)$ and $A = \left(I_t - \frac{1}{t} J_t^t\right)$. I_t denotes the txt identity matrix and J_t^t is the txt matrix with all the elements equal to 1. Observe that A is a symmetric idempotent matrix of rank (t-1).

It can be seen that n_i is distributed as binomial with the parameters n and p_i, i = 1,...,t. The mean and variance of n_i are np_i and np_iq_i, respectively. Asymptotically, n_i can be said to be distributed as a normal random variable with mean np_i and variance np_iq_i. This implies that $\hat{p}_i$ is asymptotically normal with mean p_i and variance p_iq_i/n.

Under H_0, $\hat{p}_i \overset{a.d.}{\sim} N(p, pq/n)$, i = 1,...,t. (Note that $\overset{a.d.}{\sim}$ denotes "asymptotically distributed as".) Due to the independence of the X_{ij}, $Y \overset{a.d.}{\sim} N_t\left(\mu = pJ_1^t,\ \Sigma = (pq/n)\ I_t\right)$. Then by Theorems 1 and 3 (in Appendix)

$$N = nY'AY/pq \overset{a.d.}{\sim} \chi'^2(t-1, 0) \text{ under } H_0.$$

From Theorem 5, it is known that $\hat{p}_i$ converges in probability to p_i. So under H_0, by Theorem 6,

$$D = \sum_{i=1}^{t} \frac{\hat{p}_i \hat{q}_i}{nt} \bigg/ \frac{pq}{n} \xrightarrow{\text{Prob}} 1 .$$

Hence by Theorem 4, test statistic B converges in distribution to a central chi-square with (t-1) degrees of freedom under H_0.

The asymptotic distribution of B cannot be F, as N and D are asymptotically dependent. Now it remains to find the asymptotic alternative distribution of the statistic B.

Under H_A, $\hat{p}_i \overset{a.d.}{\sim} N\left(p_i,\ p_iq_i/n\right)$.

Some attempts were made to find the closed form for the distribution of the statistic B under H_A, but none were successful due to the inequality of the variances of $\hat{p}_i$'s. However, it was decided to approximate the asymptotic alternative distribution of $\hat{p}_i$ by a normal distribution with mean p_i and variance c, where c is a constant such that $\sum_{i=1}^{t}\left(\frac{p_i q_i}{n} - c\right)^2$ is minimum. Such c is given by $\sum_{i=1}^{t} \frac{p_i q_i}{nt}$. Notice the error $= e_i = |(p_i q_i/n) - c| \to 0$ with increasing n, and even for small n, $e_i << 0.25$.

In later sections, the same sort of approximation is used several times. In each case, this results in approximate asymptotic distributions. The approximate asymptotic distribution will be denoted by the symbol $\overset{a.d.}{\underset{\cdot}{\sim}}$.

Consider $\hat{p}_i \overset{a.d.}{\underset{\cdot}{\sim}} N\left(p_i, \sum_{i=1}^{t} p_i q_i/nt\right)$.

B can be written as $B = N'/D'$ where

$$N' = \sum_{i=1}^{t} \left(\hat{p}_i - \hat{\bar{p}}\right)^2 \Big/ \sum_{i=1}^{t} \frac{p_i q_i}{nt} = \frac{Y'AY}{\sum_{i=1}^{t} p_i q_i/nt}$$

and

$$D' = \sum_{i=1}^{t} \frac{\hat{p}_i \hat{q}_i}{nt} \Bigg/ \sum_{i=1}^{t}\left(\frac{p_i q_i}{nt}\right) .$$

After an adjustment,

$$Y \overset{a.d.}{\underset{\cdot}{\sim}} N\left[\mu' = (p_1, \ldots, p_t), \Sigma = \left(\sum_{i=1}^{t} \frac{p_i q_i}{nt}\right) I_t \right].$$

By Theorems 1 and 3, $N' \overset{a.d.}{\sim} \chi'^2(t-1, \lambda)$ where

$$\lambda = nt \sum_{i=1}^{t} \left(p_i - \bar{p}\right)^2 / \, 2 \sum_{i=1}^{t} p_i q_i \quad .$$

By Theorems 5 and 6, $D' \xrightarrow{\text{Prob}} 1$.

Hence by Theorem 4, the test statistic B converges approximately in distribution to $\chi'^2(t-1),\lambda)$ under H_A.

<u>Note 1:</u> In the above derivation, under H_A, another possible c, say $c' = \bar{p}\bar{q}/n$ was also considered. This c' gives another test statistic, say B', where

$$B' = n \sum_{i=1}^{t} \left(\hat{p}_i - \hat{\bar{p}}\right)^2 \, / \, \hat{\bar{p}}\hat{\bar{q}} \quad .$$

By following the same reasoning as in the above derivation, one can easily show that the asymptotic null distribution of B' is $\chi^2(t-1)$ while its approximate asymptotic alternative distribution is $\chi'^2(t-1,\lambda')$ where

$$\lambda' = n \sum_{i=1}^{t} \left(p_i - \bar{p}\right)^2 / 2 \, \bar{p}\bar{q} \; .$$

It will be shown that $\lambda > \lambda'$, which means that in testing the hypotheses given by (2.1), B will yield uniformly higher power than B'. Then the idea of using c' and hence B' was dropped.

The following arguments show that $\lambda > \lambda'$ and hence the power of B is uniformly higher than that of B'.

Obviously, $\sum_{i=1}^{t} \left(p_i - \bar{p}\right)^2 \geq 0$.

Under H_A, $\sum_{i=1}^{t} \left(p_i - \bar{p}\right)^2 > 0$,

i.e. $\sum_{i=1}^{t} p_i^2 > \left(\sum_{i=1}^{t} p_i\right)^2/t$,

i.e. $\sum_{i=1}^{t} p_i - \sum_{i=1}^{t} p_i^2 < \sum_{i=1}^{t} p_i - \left[\left(\sum_{i=1}^{t} p_i\right)^2/t\right]$,

i.e. $\sum_{i=1}^{t} p_i q_i/t < \bar{p}\bar{q}$,

$\Longrightarrow \lambda > \lambda'$.

Now observe that B and B' have the same asymptotic null distribution and hence their critical points are also the same. Now as mentioned by Johnson and Kotz (1970, p. 141), if a normal distribution is fitted to the non-central chi-square distribution with d degrees of freedom and the non-centrality parameter ℓ, then

$$F(x;d,\ell) \doteq \Phi\left[\frac{x - d - \ell}{\{2\ (d+2\ell)\}^{1/2}}\right]$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{y} e^{-u^2/2}\, du.$$

It can be seen that $F(x;d,\ell)$ is a decreasing function of ℓ and this will simply imply that the power of B is uniformly higher than the power of B'.

Note 2: The asymptotic distribution of N' might be "better" approximated by the distribution of the sum of the non-central chi-square random variables.

<u>Note 3:</u> The distributional derivation of B is based on the assumption that $n_i \overset{a.d.}{\sim} N\left(np_i, np_iq_i\right)$. One might wonder or question about how large n should be in order to satisfy the above assumption. There is no unique answer to this in the literature. Kempthorne and Folks (1971, p. 103) suggest that the normal approximation to the binomial is better for p near 1/2 since the binomial distribution is in that case symmetric. The approximation also improves as n increases. According to Brownlee (1965, p. 140), the approximation is satisfactory if $npq > 9$. Thus if $p = 1/2$, an n of 36 is large enough, but if $p = 1/10$, n needs to be ≥ 100. As a rule of thumb, Remington and Schork (1970, p. 138) suggest having both np and nq greater than 5 for an adequate approximation. According to Mendenhall and Reinmuth (1974, p. 148-149), the approximation will be reasonably good if the interval $np \pm 2\sqrt{npq}$ lies within the binomial bounds, 0 and n.

Putting $c = 2$ and $n_{\cdot j} = n$ in the CATANOVA statistic of Light and Margolin and the χ^2 statistic of Pearson mentioned earlier, one can compare the BIANOVA test with CATANOVA and Pearson's chi-square tests. In practice, sometimes some people do use ANOVA technique for this situation even though it is not appropriate because of the violation of some of the assumptions underlying analysis of variance, so it will be worthwhile to include the F-test for comparison also. This will be done in the next section.

Comparisons of BIANOVA Test With CATANOVA, Chi-Square, and F Tests

For the situation described in this chapter, statistics C and χ^2 can be reduced to the following after doing appropriate substitutions and the necessary algebra:

$$C = \frac{n\sum_{i=1}^{t}\left(\hat{p}_i - \hat{\bar{p}}\right)^2}{\hat{\bar{p}}\ \hat{\bar{q}}} \cdot \frac{nt - 1}{nt}$$

$$\chi^2 = \frac{n\sum_{i=1}^{t}\left(\hat{p}_i - \hat{\bar{p}}\right)^2}{\hat{\bar{p}}\ \hat{\bar{q}}} .$$

The above form of χ^2 looks very familiar. It is the same as B' mentioned in Note 1 of the previous section which summarizes the null and alternative distributions of B' and hence χ^2! The derivation for the null distribution of χ^2 presented in the previous section through Note 1 is simple compared to those given in the literature. The alternative distribution of χ^2 is not widely known.

Notice that:

(i) B evaluated at $\left(\hat{p}_1,\ldots,\hat{p}_t\right)$ is the same as B evaluated at $\left(\hat{q}_1,\ldots,\hat{q}_t\right)$.

(ii) B evaluated at $\left(\hat{p}_1,\ldots,\hat{p}_t\right)$ is the same as B evaluated at (any permutation of $\hat{p}_1,\ldots,\hat{p}_t$).

(iii) From (i) and (ii) above, it can be seen that B evaluated at $\left(\hat{p}_1,\ldots,\hat{p}_t\right)$, (any permutation of $\hat{p}_i,\ldots,\hat{p}_t\Big)$, $\Big(\hat{q}_1,\ldots,\hat{q}_t\Big)$ and at $\Big($any permutation of $\hat{q}_1,\ldots,\hat{q}_t\Big)$ will be the same. For example, if $t = 2$, then this will imply the power for the B test under the following four situations will be the same:

(a) $p_1 = p_{10}$, $p_2 = p_{20}$. (p_{10} and p_{20} are fixed constants and their range is from 0 to 1.)

(b) $p_1 = p_{20}$, $p_2 = p_{10}$.

(c) $p_1 = q_{10}$, $p_2 = q_{20}$.

(d) $p_1 = q_{20}$, $p_2 = q_{10}$.

(iv) The above three properties hold for the C, χ^2, and F statistics, also.

(v) $C < \chi^2 < B$ and $F < B$ for fixed $\hat{p}_1,\ldots,\hat{p}_t$.

(vi) From the two expressions for C and χ^2 given above, it is obvious that the difference between the calculated C and χ^2 statistics becomes negligible with the increasing value of the product $n \cdot t$.

In standard text books, e.g., by Walpole and Myers (1972), by Snedecor and Cochran (1972), and by others, for $t = 2$, the Z test is given to test H_0: $p_1 = p_2$. The test statistic Z is given by

$$Z = \frac{\left(\hat{p}_1 - \hat{p}_2\right)}{\sqrt{\hat{\hat{p}}\,\hat{\hat{q}}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where $\hat{\hat{p}} = \dfrac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$, which for the present case ($n_1 = n_2 = n$) reduces to

$$Z' = \frac{\left(\hat{p}_1 - \hat{p}_2\right)}{\sqrt{\hat{\hat{p}}\,\hat{\hat{q}}\,(2/n)}} .$$

Under H_0, the approximate distribution of Z is N(0,1). For Z' (in general for Z also), it can be shown that

$$(Z')^2 = \frac{2\sum_{i=1}^{n}\left(\hat{p}_i - \hat{\hat{p}}\right)^2}{\hat{\hat{p}}\,\hat{\hat{q}}\quad(2/n)} = B' = \chi^2 .$$

Hence, the so-called Z test is identical to Pearson's chi-square test! In Note 1 of the previous section, it is already pointed out that the BIANOVA statistic, B, gives higher power than B', and hence χ^2.

Because of observations (iv) and (v) previously, it was decided to compare the BIANOVA test with the chi-square and F tests only. It is easy to observe that the chi-square test will yield higher power than C in the case of binary responses. For empirical comparisons, t = 2 and n = 10 were selected as the simplest case. Tables I through VII give the empirical power of BIANOVA (or B test), chi-square and F tests at various α levels and for various values of p_1 and p_2. To arrive at these tables, 2000 data sets were generated for each selected configuration of p_1 and p_2. p_1 and p_2 take values from 0 to 1 with the increments of 0.1. Figures 1 through 11 give the power curves based on the information from these Tables I through VII. Due to symmetry, Figures 9, 8, 7, and 6 are just the mirror images of Figures 1, 2, 3, and 4, respectively.

TABLE I

EMPIRICAL PROBABILITY OF REJECTING Ho: p1=p2 FOR BIANOVA AND CHI-SQUARE TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.05 AND 0.07, RESPECTIVELY

P2	TEST & LEVEL	P1 VALUES 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B 0.05	0.0460	0.1220		0.4690	0.6580	0.8025		0.9715	1.0000
	CHI-SQ 0.07	0.0460	0.1220		0.4690	0.6580	0.8025		0.9715	1.0000
0.2	B 0.05	0.1220	0.1075	0.1455		0.3675	0.5850	0.7245		0.9715
	CHI-SQ 0.07	0.1220	0.1075	0.1455		0.3675	0.5850	0.7245		0.9715
0.3	B 0.05		0.1455	0.0895	0.1135		0.3535		0.7245	
	CHI-SQ 0.07		0.1455	0.0895	0.1135		0.3535		0.7245	
0.4	B 0.05	0.4690		0.1135 *	0.0930	0.1264		0.3535	0.5850	0.8025
	CHI-SQ 0.07	0.4690		0.1135	0.0930	0.1264		0.3535	0.5850	0.8025
0.5	B 0.05	0.6580	0.3675 *		0.1264 *	0.0800 *	0.1264		0.3675	0.6580
	CHI-SQ 0.07	0.6580	0.3675		0.1264	0.0800	0.1264		0.3675	0.6580
0.6	B 0.05	0.8025 *	0.5850 *	0.3535		0.1264	0.0930 *	0.1135		0.4690
	CHI-SQ 0.07	0.8025	0.5850	0.3535		0.1264	0.0930	0.1135		0.4690
0.7	B 0.05		0.7245		0.3535 *		0.1135	0.0895 *	0.1455	
	CHI-SQ 0.07		0.7245		0.3535		0.1135	0.0895	0.1455	
0.8	B 0.05	0.9715		0.7245 *	0.5850	0.3675		0.1455 *	0.1075 *	0.1220
	CHI-SQ 0.07	0.9715		0.7245	0.5850	0.3675		0.1455	0.1075	0.1220
0.9	B 0.05	1.0000 *	0.9715 *		0.8025	0.6580 *	0.4690 *		0.1220 *	0.0460 *
	CHI-SQ 0.07	1.0000	0.9715		0.8025	0.6580	0.4690		0.1220	0.0460

* indicates the values of p1 and p2 under which the empirical study was carried out. The remaining cells of this table were completed using the results summarized on page 36. In the remaining tables, * should be located in the same cells indicating the same as here.

TABLE II

EMPIRICAL PROBABILITY OF REJECTING Ho: p1=p2 FOR BIANOVA, CHI-SQUARE AND F TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.01, 0.05 AND 0.05, RESPECTIVELY

P2	TEST & LEVEL	P1 VALUES								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B 0.01	0.0090	0.0400		0.2935	0.4700	0.6620		0.9420	0.9960
	CHI-SQ 0.05	0.0090	0.0400		0.2935	0.4700	0.6620		0.9420	0.9960
	F 0.05	0.0090	0.0400		0.2935	0.4700	0.6620		0.9420	0.9960
0.2	B 0.01	0.0400	0.0305	0.0605		0.2340	0.4590	0.6145		0.9420
	CHI-SQ 0.05	0.0400	0.0305	0.0605		0.2340	0.4590	0.6145		0.9420
	F 0.05	0.0400	0.0305	0.0605		0.2340	0.4590	0.6145		0.9420
0.3	B 0.01		0.0605	0.0320	0.0630		0.2440		0.6145	
	CHI-SQ 0.05		0.0605	0.0320	0.0630		0.2440		0.6145	
	F 0.05		0.0605	0.0320	0.0630		0.2440		0.6145	
0.4	B 0.01	0.2935		0.0630	0.0430	0.0584		0.2440	0.4590	0.6620
	CHI-SQ 0.05	0.2935		0.0630	0.0430	0.0584		0.2440	0.4590	0.6620
	F 0.05	0.2935		0.0630	0.0430	0.0584		0.2440	0.4590	0.6620
0.5	B 0.01	0.4700	0.2340		0.0584	0.0430	0.0584		0.2340	0.4700
	CHI-SQ 0.05	0.4700	0.2340		0.0584	0.0430	0.0584		0.2340	0.4700
	F 0.05	0.4700	0.2340		0.0584	0.0430	0.0584		0.2340	0.4700
0.6	B 0.01	0.6620	0.4590	0.2440		0.0584	0.0430	0.0630		0.2935
	CHI-SQ 0.05	0.6620	0.4590	0.2440		0.0584	0.0430	0.0630		0.2935
	F 0.05	0.6620	0.4590	0.2440		0.0584	0.0430	0.0630		0.2935
0.7	B 0.01		0.6145		0.2440		0.0630	0.0320	0.0605	
	CHI-SQ 0.05		0.6145		0.2440		0.0630	0.0320	0.0605	
	F 0.05		0.6145		0.2440		0.0630	0.0320	0.0605	
0.8	B 0.01	0.9420		0.6145	0.4590	0.2340		0.0605	0.0305	0.0400
	CHI-SQ 0.05	0.9420		0.6145	0.4590	0.2340		0.0605	0.0305	0.0400
	F 0.05	0.9420		0.6145	0.4590	0.2340		0.0605	0.0305	0.0400
0.9	B 0.01	0.9960	0.9420		0.6620	0.4700	0.2935		0.0400	0.0090
	CHI-SQ 0.05	0.9960	0.9420		0.6620	0.4700	0.2935		0.0400	0.0090
	F 0.05	0.9960	0.9420		0.6620	0.4700	0.2935		0.0400	0.0090

TABLE III

EMPIRICAL PROBABILITY OF REJECTING Ho: p1=p2 FOR BIANOVA AND CHI-SQUARE TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.03 AND 0.0512, RESPECTIVELY

P2	TEST & LEVEL	P1 VALUES 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B 0.03	0.0100	0.0500		0.3705	0.5740	0.7420		0.9545	1.0000
	CHI-SQ 0.0512	0.0100	0.0500		0.3705	0.5740	0.7420		0.9545	1.0000
0.2	B 0.03	0.0500	0.0505	0.0990		0.2990	0.5060	0.6445		0.9545
	CHI-SQ 0.0512	0.0500	0.0505	0.0990		0.2990	0.5060	0.6445		0.9545
0.3	B 0.03		0.0990	0.0535	0.0855		0.2745		0.6445	
	CHI-SQ 0.0512		0.0990	0.0535	0.0855		0.2745		0.6445	
0.4	B 0.03	0.3705		0.0855	0.0590	0.0736		0.2745	0.5060	0.7420
	CHI-SQ 0.0512	0.3705		0.0855	0.0590	0.0736		0.2745	0.5060	0.7420
0.5	B 0.03	0.5740	0.2990		0.0736	0.0490	0.0736		0.2990	0.5740
	CHI-SQ 0.0512	0.5740	0.2990		0.0736	0.0490	0.0736		0.2990	0.5740
0.6	B 0.03	0.7420	0.5060	0.2745		0.0736	0.0590	0.0855		0.3705
	CHI-SQ 0.0512	0.7420	0.5060	0.2745		0.0736	0.0590	0.0855		0.3705
0.7	B 0.03		0.6445		0.2745		0.0855	0.0535	0.0990	
	CHI-SQ 0.0512		0.6445		0.2745		0.0855	0.0535	0.0990	
0.8	B 0.03	0.9545		0.6445	0.5060	0.2990		0.0990	0.0505	0.0500
	CHI-SQ 0.0512	0.9545		0.6445	0.5060	0.2990		0.0990	0.0505	0.0500
0.9	B 0.03	1.0000	0.9545		0.7420	0.5740	0.3705		0.0500	0.0100
	CHI-SQ 0.0512	1.0000	0.9545		0.7420	0.5740	0.3705		0.0500	0.0100

TABLE IV

EMPIRICAL PROBABILITY OF REJECTING Ho:p1=p2 FOR BIANOVA, CHI-SQUARE AND F TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.005, 0.01 AND 0.01, RESPECTIVELY

P2	TEST & LEVEL	P1 VALUES 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B 0.005	0.0010	0.0110		0.1405	0.2920	0.4750		0.8435	0.9400
	CHI-SQ 0.01	0.0010	0.0110		0.1405	0.2920	0.4750		0.8435	0.9400
	F 0.01	0.0010	0.0110		0.1405	0.2920	0.4750		0.8435	0.9400
0.2	B 0.005	0.0110	0.0105	0.0220		0.1305	0.2860	0.4255		0.8435
	CHI-SQ 0.01	0.0110	0.0105	0.0220		0.1305	0.2860	0.4255		0.8435
	F 0.01	0.0110	0.0105	0.0220		0.1305	0.2860	0.4255		0.8435
0.3	B 0.005		0.0220	0.0110	0.0225		0.1175		0.4255	
	CHI-SQ 0.01		0.0220	0.0110	0.0225		0.1175		0.4255	
	F 0.01		0.0220	0.0110	0.0225		0.1175		0.4255	
0.4	B 0.005	0.1405		0.0225	0.0140	0.0224		0.1175	0.2860	0.4750
	CHI-SQ 0.01	0.1405		0.0225	0.0140	0.0224		0.1175	0.2860	0.4750
	F 0.01	0.1405		0.0225	0.0140	0.0224		0.1175	0.2860	0.4750
0.5	B 0.005	0.2920	0.1305		0.0224	0.0120	0.0224		0.1305	0.2920
	CHI-SQ 0.01	0.2920	0.1305		0.0224	0.0120	0.0224		0.1305	0.2920
	F 0.01	0.2920	0.1305		0.0224	0.0120	0.0224		0.1305	0.2920
0.6	B 0.005	0.4750	0.2860	0.1175		0.0224	0.0140	0.0225		0.1405
	CHI-SQ 0.01	0.4750	0.2860	0.1175		0.0224	0.0140	0.0225		0.1405
	F 0.01	0.4750	0.2860	0.1175		0.0224	0.0140	0.0225		0.1405
0.7	B 0.005		0.4255		0.1175		0.0225	0.0110	0.0220	
	CHI-SQ 0.01		0.4255		0.1175		0.0225	0.0110	0.0220	
	F 0.01		0.4255		0.1175		0.0225	0.0110	0.0220	
0.8	B 0.005	0.8435		0.4255	0.2860	0.1305		0.0220	0.0105	0.0110
	CHI-SQ 0.01	0.8435		0.4255	0.2860	0.1305		0.0220	0.0105	0.0110
	F 0.01	0.8435		0.4255	0.2860	0.1305		0.0220	0.0105	0.0110
0.9	B 0.005	0.9400	0.8435		0.4750	0.2920	0.1405		0.0110	0.0010
	CHI-SQ 0.01	0.9400	0.8435		0.4750	0.2920	0.1405		0.0110	0.0010
	F 0.01	0.9400	0.8435		0.4750	0.2920	0.1405		0.0110	0.0010

TABLE V

EMPIRICAL PROBABILITY OF REJECTING Ho: p1=p2 FOR BIANOVA, CHI-SQUARE AND F TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.03, 0.03 AND 0.03, RESPECTIVELY

P2	TEST&LEVEL	P1 VALUES 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B 0.03	0.0100	0.0500		0.3705	0.5740	0.7420		0.9545	1.0000
	CHI-SQ 0.03	0.0090	0.0400		0.2935	0.4700	0.6620		0.9420	0.9960
	F 0.03	0.0090	0.0400		0.2935	0.4700	0.6620		0.9420	0.9960
0.2	B 0.03	0.0500	0.0505	0.0990		0.2990	0.5060	0.6445		0.9545
	CHI-SQ 0.03	0.0400	0.0350	0.0605		0.2340	0.4590	0.6145		0.9420
	F 0.03	0.0400	0.0350	0.0605		0.2340	0.4590	0.6145		0.9420
0.3	B 0.03		0.0990	0.0535	0.0855		0.2745		0.6445	
	CHI-SQ 0.03		0.0605	0.0320	0.0630		0.2440		0.6145	
	F 0.03		0.0605	0.0320	0.0630		0.2440		0.6145	
0.4	B 0.03	0.3705		0.0855	0.0590	0.0736		0.2745	0.5060	0.7420
	CHI-SQ 0.03	0.2935		0.0630	0.0430	0.0584		0.2440	0.4590	0.6620
	F 0.03	0.2935		0.0630	0.0430	0.0584		0.2440	0.4590	0.6620
0.5	B 0.03	0.5740	0.2990		0.0736	0.0490	0.0736		0.2990	0.5740
	CHI-SQ 0.03	0.4700	0.2340		0.0584	0.0430	0.0584		0.2340	0.4700
	F 0.03	0.4700	0.2340		0.0584	0.0430	0.0584		0.2340	0.4700
0.6	B 0.03	0.7420	0.5060	0.2745		0.0736	0.0590	0.0855		0.3705
	CHI-SQ 0.03	0.6620	0.4590	0.2440		0.0584	0.0430	0.0630		0.2935
	F 0.03	0.6620	0.4590	0.2440		0.0584	0.0430	0.0630		0.2935
0.7	B 0.03		0.6445		0.2745		0.0855	0.0535	0.0990	
	CHI-SQ 0.03		0.6145		0.2440		0.0630	0.0320	0.0605	
	F 0.03		0.6145		0.2440		0.0630	0.0320	0.0605	
0.8	B 0.03	0.9545		0.6445	0.5060	0.2990		0.0990	0.0505	0.0500
	CHI-SQ 0.03	0.9420		0.6145	0.4590	0.2340		0.0605	0.0305	0.0400
	F 0.03	0.9420		0.6145	0.4590	0.2340		0.0605	0.0305	0.0400
0.9	B 0.03	1.0000	0.9545		0.7420	0.5740	0.3705		0.0500	0.0100
	CHI-SQ 0.03	0.9660	0.9420		0.6620	0.4700	0.2935		0.0400	0.0090
	F 0.03	0.9660	0.9420		0.6620	0.4700	0.2935		0.0400	0.0090

TABLE VI

EMPIRICAL PROBABILITY OF REJECTING Ho: p1=p2 FOR BIANOVA, CHI-SQUARE AND F TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.005, 0.005 AND 0.005, RESPECTIVELY

			P1 VALUES							
P2	TEST & LEVEL		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B	0.005	0.0010	0.0110		0.1405	0.2920	0.4750		0.8435	0.9400
	CHI-SQ	0.005	0.0000	0.0005		0.0620	0.1550	0.2860		0.6880	0.8680
	F	0.005	0.0000	0.0005		0.0740	0.2000	0.3755		0.7775	0.9400
0.2	B	0.005	0.0110	0.0105	0.0220		0.1305	0.2860	0.4255		0.8435
	CHI-SQ	0.005	0.0005	0.0020	0.0055		0.0575	0.1370	0.2550		0.6880
	F	0.005	0.0005	0.0030	0.0085		0.0935	0.2190	0.3385		0.7775
0.3	B	0.005		0.0220	0.0110	0.0225		0.1175		0.4255	
	CHI-SQ	0.005		0.0055	0.0040	0.0055		0.0535		0.2550	
	F	0.005		0.0085	0.0055	0.0120		0.0825		0.3385	
0.4	B	0.005	0.1405		0.0225	0.0140	0.0224		0.1175	0.2860	0.4750
	CHI-SQ	0.005	0.0620		0.0055	0.0055	0.0040		0.0535	0.1370	0.2860
	F	0.005	0.0740		0.0120	0.0100	0.0080		0.0825	0.2190	0.3755
0.5	B	0.005	0.2920	0.1305		0.0224	0.0120	0.0224		0.1305	0.2920
	CHI-SQ	0.005	0.1550	0.0575		0.0040	0.0030	0.0040		0.0575	0.1550
	F	0.005	0.2000	0.0935		0.0080	0.0080	0.0080		0.0935	0.2000
0.6	B	0.005	0.4750	0.2860	0.1175		0.0224	0.0140	0.0225		0.1405
	CHI-SQ	0.005	0.2860	0.1370	0.0535		0.0040	0.0055	0.0055		0.0620
	F	0.005	0.3755	0.2190	0.0825		0.0080	0.0100	0.0120		0.0740
0.7	B	0.005		0.4255		0.1175		0.0225	0.0110	0.0220	
	CHI-SQ	0.005		0.2550		0.0535		0.0055	0.0040	0.0055	
	F	0.005		0.3385		0.0825		0.0120	0.0055	0.0085	
0.8	B	0.005	0.8435		0.4255	0.2860	0.1305		0.0220	0.0105	0.0110
	CHI-SQ	0.005	0.6880		0.2550	0.1370	0.0575		0.0055	0.0020	0.0005
	F	0.005	0.7775		0.3385	0.2190	0.0935		0.0085	0.0030	0.0005
0.9	B	0.005	0.9400	0.8435		0.4750	0.2920	0.1405		0.0110	0.0010
	CHI-SQ	0.005	0.8680	0.6880		0.2860	0.1550	0.0620		0.0005	0.0000
	F	0.005	0.9400	0.7775		0.3755	0.2000	0.0740		0.0005	0.0000

TABLE VII

EMPIRICAL PROBABILITY OF REJECTING Ho: p1=p2 FOR BIANOVA, CHI-SQUARE AND F TESTS AT VARIOUS SELECTED COMBINATIONS OF p1 AND p2, FOR THE CASE OF t=2 AND n=10, WITH α=0.07, 0.07 AND 0.07, RESPECTIVELY

P2	TEST & LEVEL	P1 VALUES 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	B 0.07	0.0460	0.1220		0.4695	0.6650	0.8150		0.9825	1.0000
	CHI-SQ 0.07	0.0460	0.1220		0.4690	0.6580	0.8025		0.9715	1.0000
	F 0.07	0.0460	0.1178		0.4475	0.6140	0.7585		0.9550	1.0000
0.2	B 0.07	0.1220	0.1080	0.1485		0.3875	0.6240	0.7805		0.9825
	CHI-SQ 0.07	0.1220	0.1075	0.1455		0.3675	0.5850	0.7245		0.9715
	F 0.07	0.1178	0.1010	0.1325		0.3080	0.5110	0.6460		0.9550
0.3	B 0.07		0.1485	0.0940	0.1284		0.4095		0.7805	
	CHI-SQ 0.07		0.1455	0.0895	0.1135		0.3535		0.7245	
	F 0.07		0.1325	0.0675	0.0900		0.2750		0.6460	
0.4	B 0.07	0.4695		0.1284	0.1105	0.1528		0.4095	0.6240	0.8150
	CHI-SQ 0.07	0.4690		0.1135	0.093	0.1264		0.3535	0.5850	0.8025
	F 0.07	0.4475		0.0900	0.0605	0.0835		0.2750	0.5110	0.7585
0.5	B 0.07	0.6650	0.3875		0.1528	0.1100	0.1528		0.3875	0.6650
	CHI-SQ 0.07	0.6580	0.3675		0.1264	0.0800	0.1264		0.3675	0.6580
	F 0.07	0.6140	0.3080		0.0835	0.0490	0.0835		0.3080	0.6140
0.6	B 0.07	0.8150	0.6240	0.4095		0.1528	0.1105	0.1284		0.4695
	CHI-SQ 0.07	0.8025	0.5850	0.3535		0.1264	0.0930	0.1135		0.4690
	F 0.07	0.7585	0.5110	0.2750		0.0835	0.0605	0.0900		0.4475
0.7	B 0.07		0.7805		0.4095		0.1284	0.0940	0.1485	
	CHI-SQ 0.07		0.7245		0.3535		0.1135	0.0895	0.1455	
	F 0.07		0.6460		0.2750		0.0900	0.0675	0.1325	
0.8	B 0.07	0.9825		0.7805	0.6240	0.3875		0.1485	0.1080	0.1220
	CHI-SQ 0.07	0.9715		0.7245	0.5850	0.3675		0.1455	0.1075	0.1220
	F 0.07	0.9550		0.6460	0.5110	0.3080		0.4475	0.1010	0.1178
0.9	B 0.07	1.0000	0.9825		0.8150	0.6650	0.4695		0.1220	0.0460
	CHI-SQ 0.07	1.0000	0.9715		0.8025	0.6580	0.4690		0.1220	0.0460
	F 0.07	1.0000	0.9550		0.7585	0.6140	0.4475		0.1178	0.0460

TABLE VIII

MONOTONE RELATION OF CHI-SQUARE AND BIANOVA STATISTICS FOR t=2 AND n=10

OBS	P1	P2	CHSQ	B	RANKCHSQ	RANKB
1	.1	0.2	0.3922	0.4000	7.5	7.5
2	.1	0.3	1.2500	1.3333	15.5	15.5
3	.1	0.4	2.4000	2.7273	22.5	22.5
4	.1	0.5	3.8095	4.7059	28.5	28.5
5	.1	0.6	5.4945	7.5758	33.5	33.5
6	.1	0.7	7.5000	12.0000	37.5	37.5
7	.1	0.8	9.8990	19.6000	40.5	40.5
8	.1	0.9	12.8000	35.5556	43.0	43.0
9	.1	1.0	16.3636	90.0000	45.0	45.0
10	.2	0.3	0.2667	0.2703	5.5	5.5
11	.2	0.4	0.9524	1.0000	12.5	12.5
12	.2	0.5	1.9780	2.1951	19.5	19.5
13	.2	0.6	3.3333	4.0000	25.5	25.5
14	.2	0.7	5.0505	6.7568	31.5	31.5
15	.2	0.8	7.2000	11.2500	36.0	36.0
16	.2	0.9	9.8990	19.6000	40.5	40.5
17	.2	1.0	13.3333	40.0000	44.0	44.0
18	.3	0.4	0.2198	0.2222	3.5	3.5
19	.3	0.5	0.8333	0.8696	10.5	10.5
20	.3	0.6	1.8182	2.0000	17.5	17.5
21	.3	0.7	3.2000	3.8095	24.0	24.0
22	.3	0.8	5.0505	6.7568	31.5	31.5
23	.3	0.9	7.5000	12.0000	37.5	37.5
24	.3	1.0	10.7692	23.3333	42.0	42.0
25	.4	0.5	0.2020	0.2041	1.5	1.5
26	.4	0.6	0.8000	0.8333	9.0	9.0
27	.4	0.7	1.8182	2.0000	17.5	17.5
28	.4	0.8	3.3333	4.0000	25.5	25.5
29	.4	0.9	5.4945	7.5758	33.5	33.5
30	.4	1.0	8.5714	15.0000	39.0	39.0
31	.5	0.6	0.2020	0.2041	1.5	1.5
32	.5	0.7	0.8333	0.8696	10.5	10.5
33	.5	0.8	1.9780	2.1951	19.5	19.5
34	.5	0.9	3.8095	4.7059	28.5	28.5
35	.5	1.0	6.6667	10.0000	35.0	35.0
36	.6	0.7	0.2198	0.2222	3.5	3.5
37	.6	0.8	0.9524	1.0000	12.5	12.5
38	.6	0.9	2.4000	2.7273	22.5	22.5
39	.6	1.0	5.0000	6.6667	30.0	30.0
40	.7	0.8	0.2667	0.2703	5.5	5.5
41	.7	0.9	1.2500	1.3333	15.5	15.5
42	.7	1.0	3.5294	4.2857	27.0	27.0
43	.8	0.9	0.3922	0.4000	7.5	7.5
44	.8	1.0	2.2222	2.5000	21.0	21.0
45	.9	1.0	1.0526	1.1111	14.0	14.0

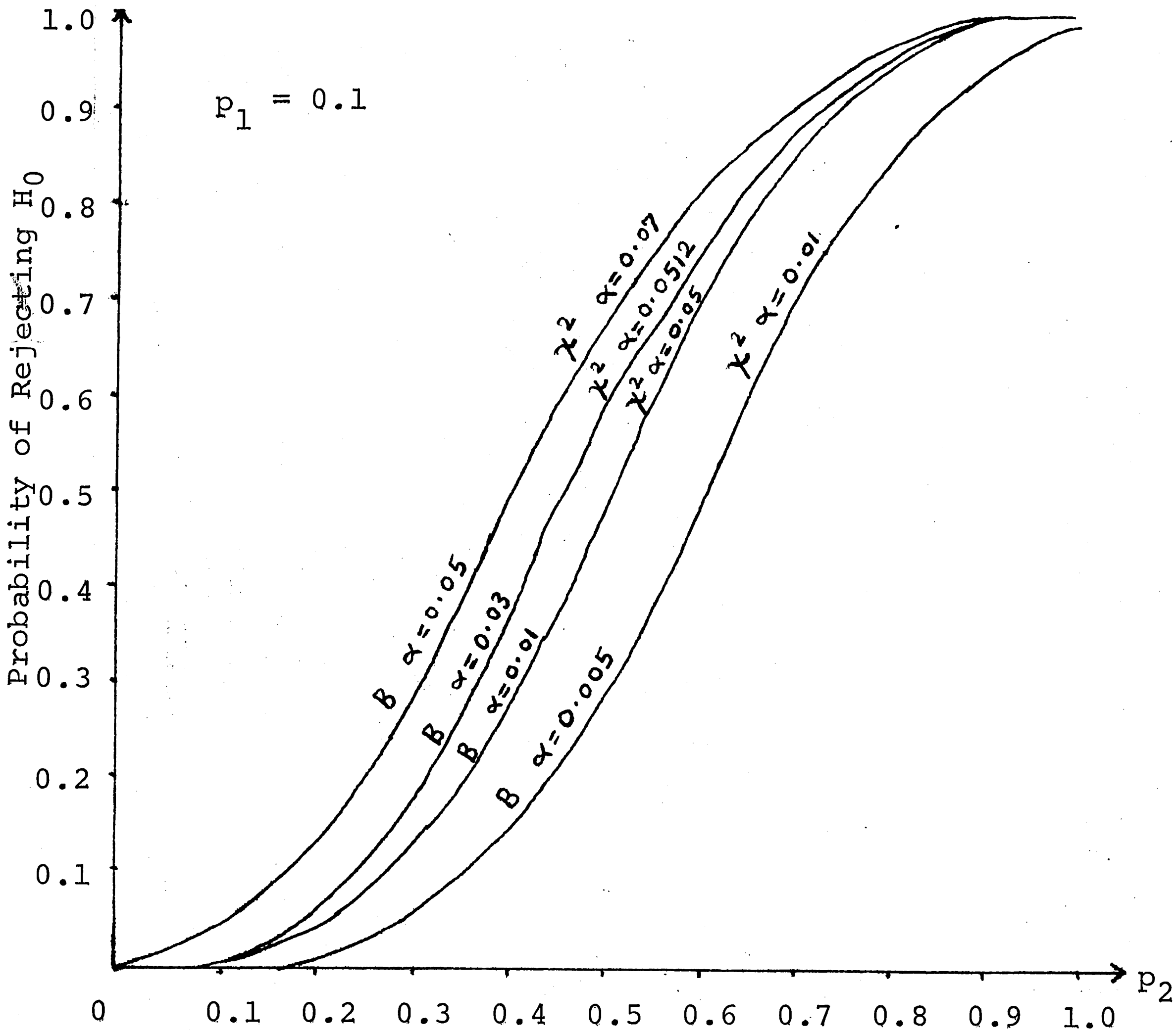

Figure 1. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.1$ and p_2 as a Variable.

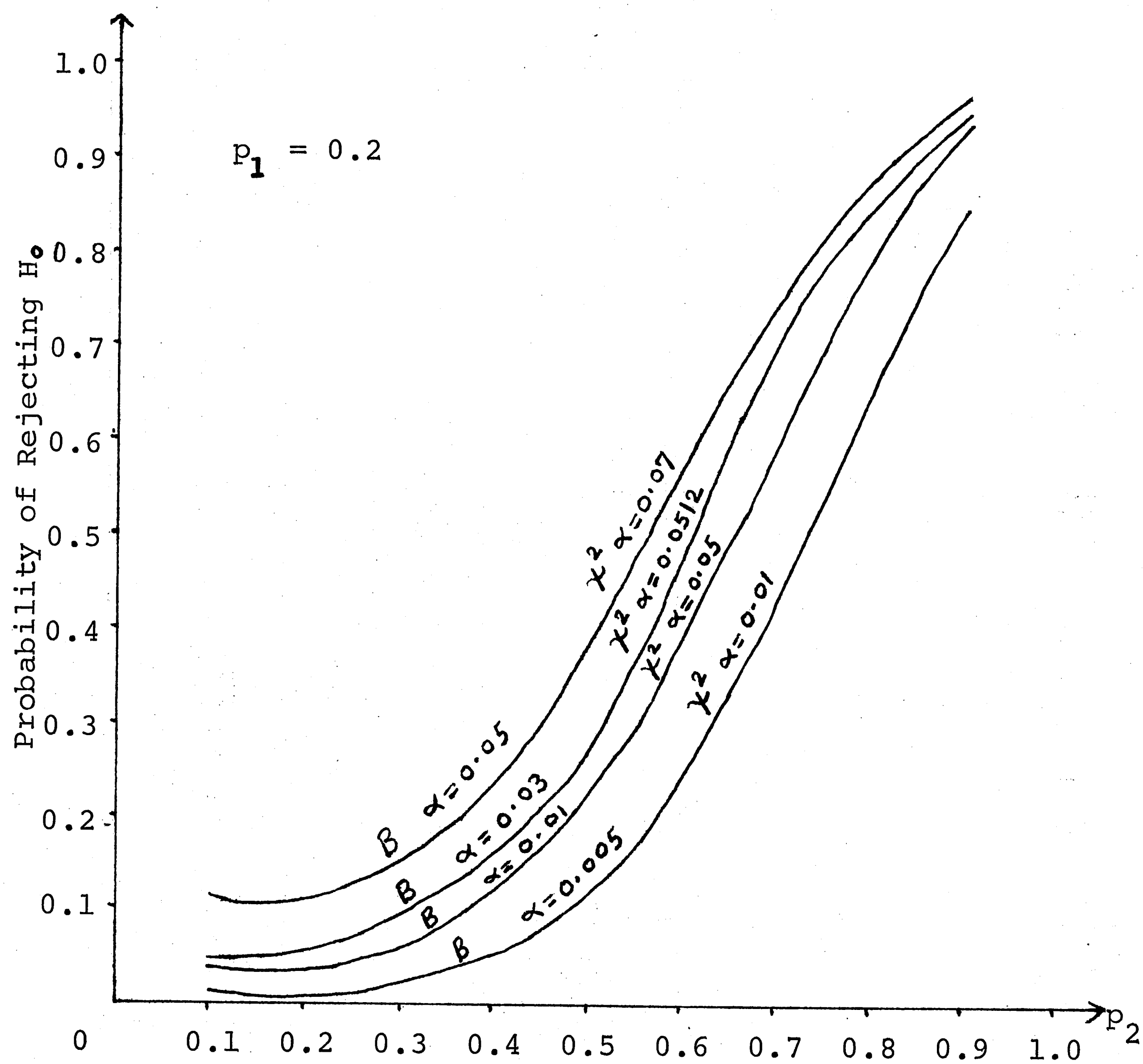

Figure 2. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.2$ and p_2 as a Variable.

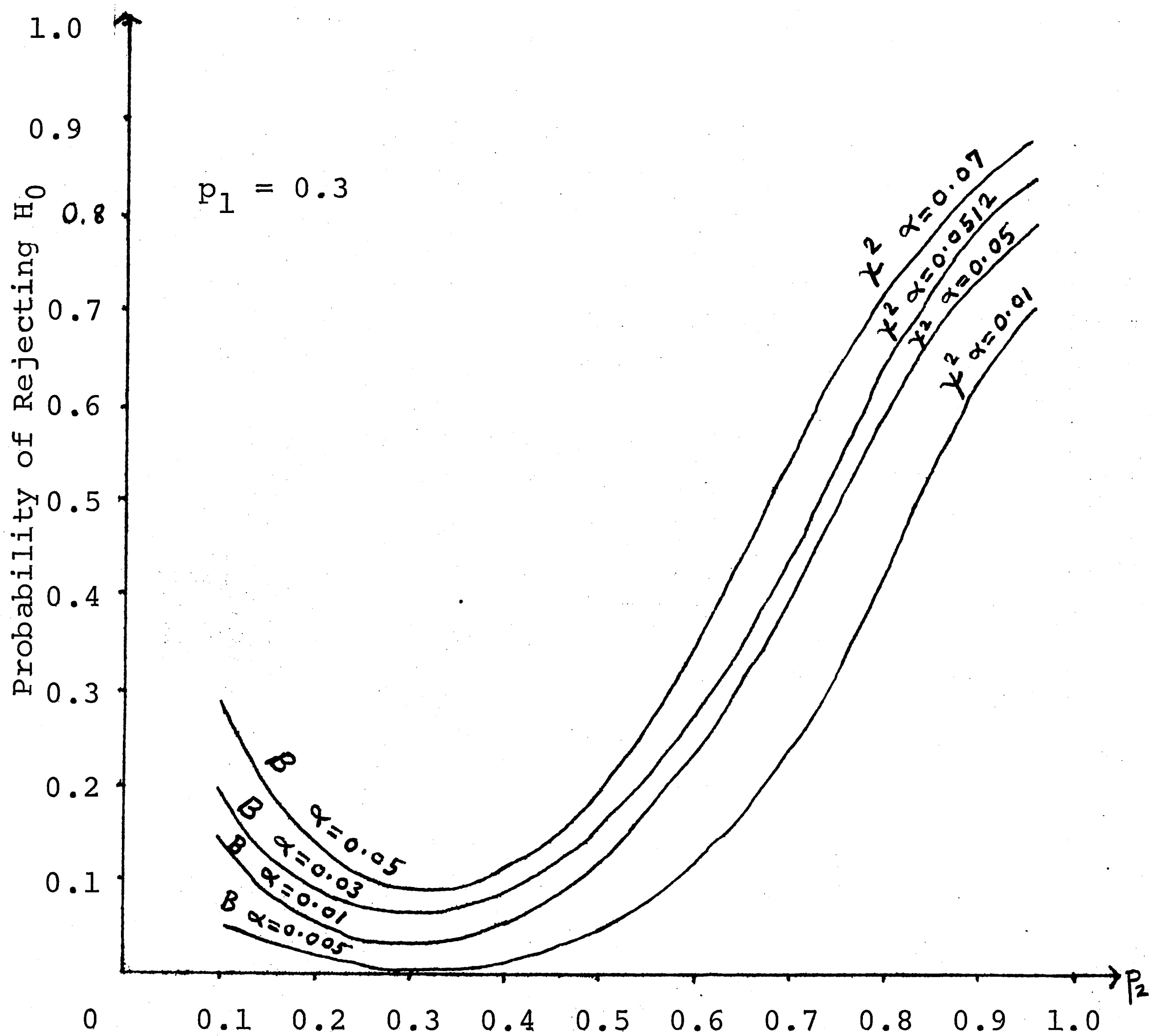

Figure 3. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.3$ and p_2 as a Variable.

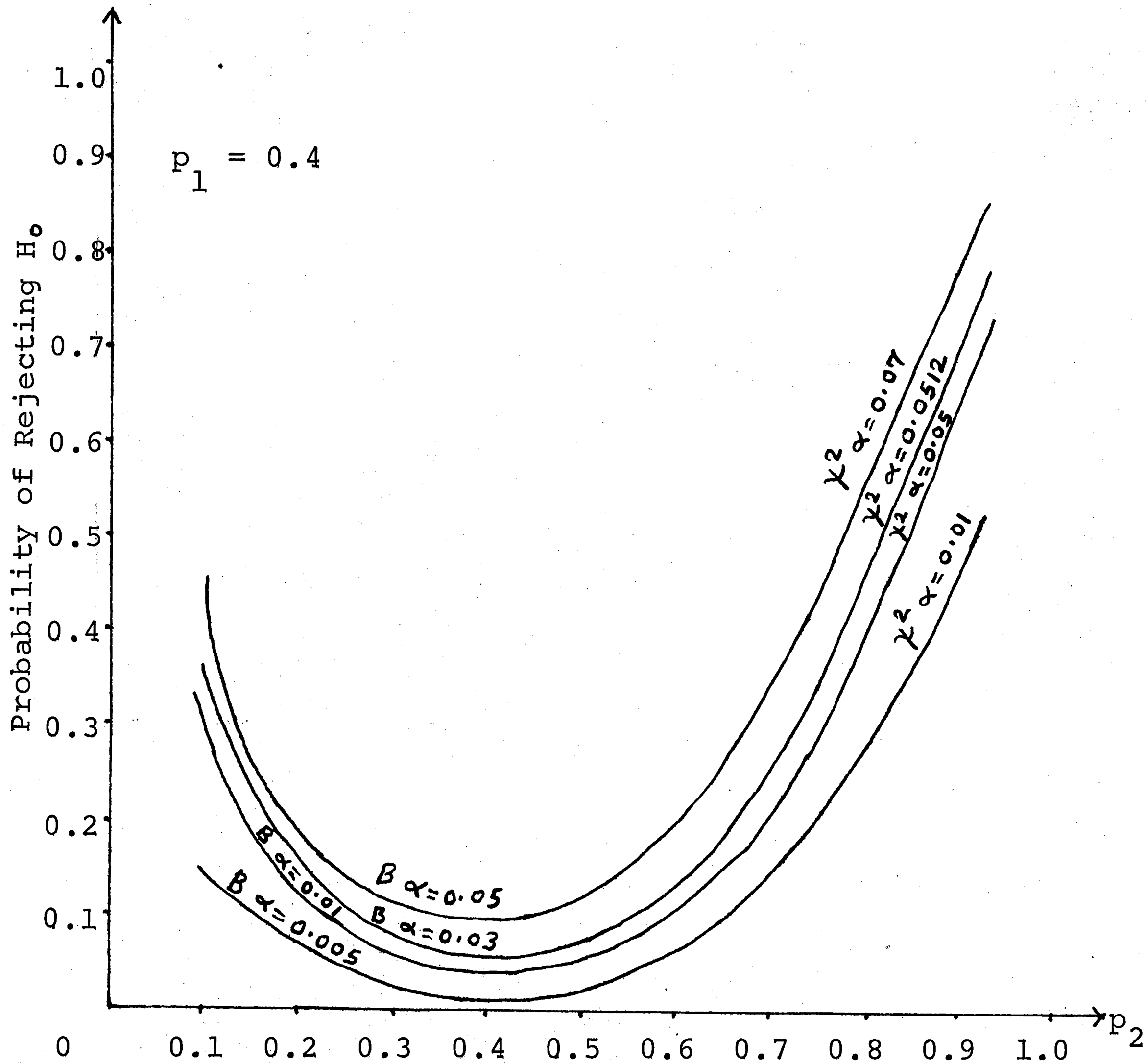

Figure 4. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.4$ and p_2 as a Variable.

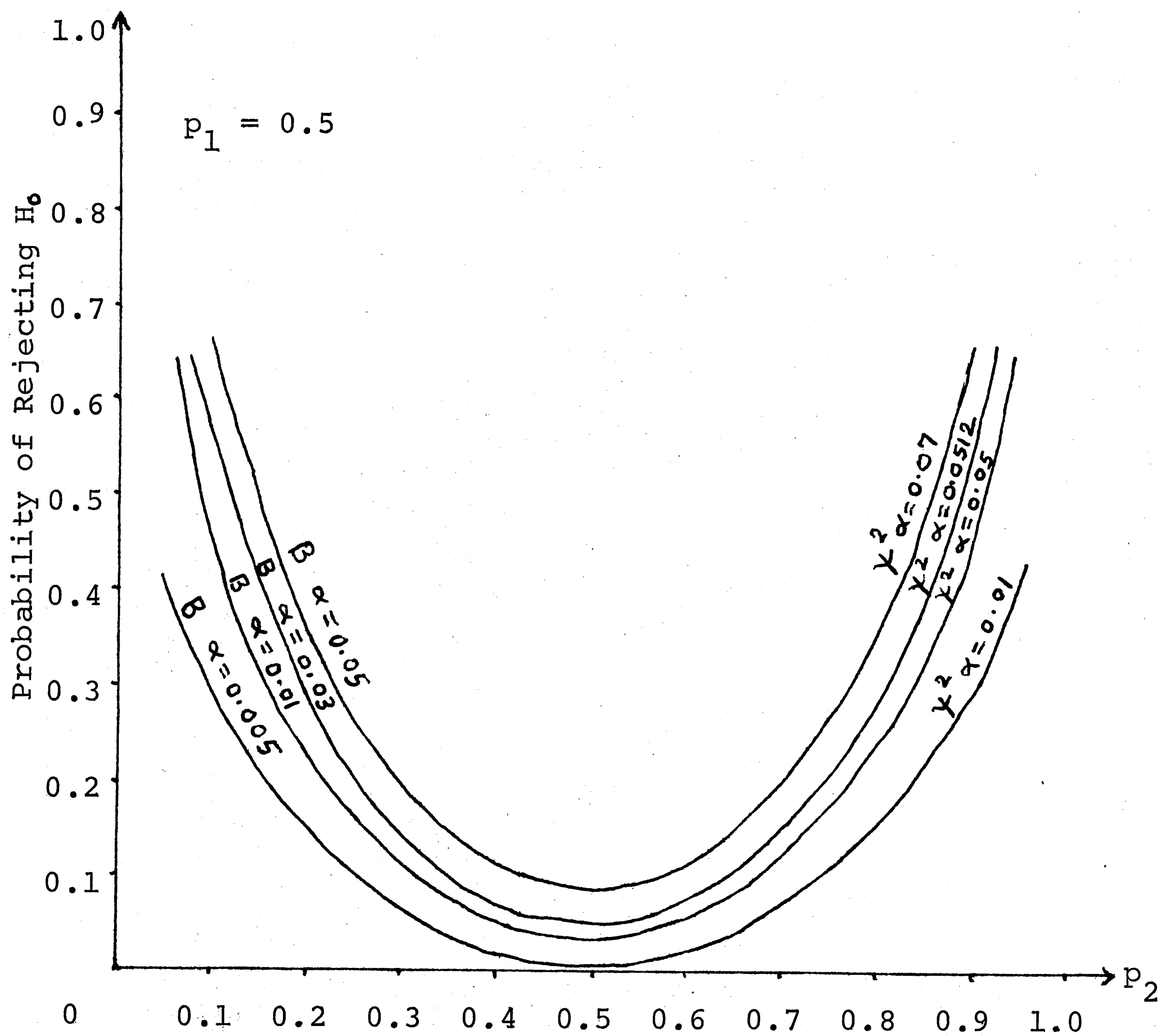

Figure 5. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.5$ and p_2 as a Variable.

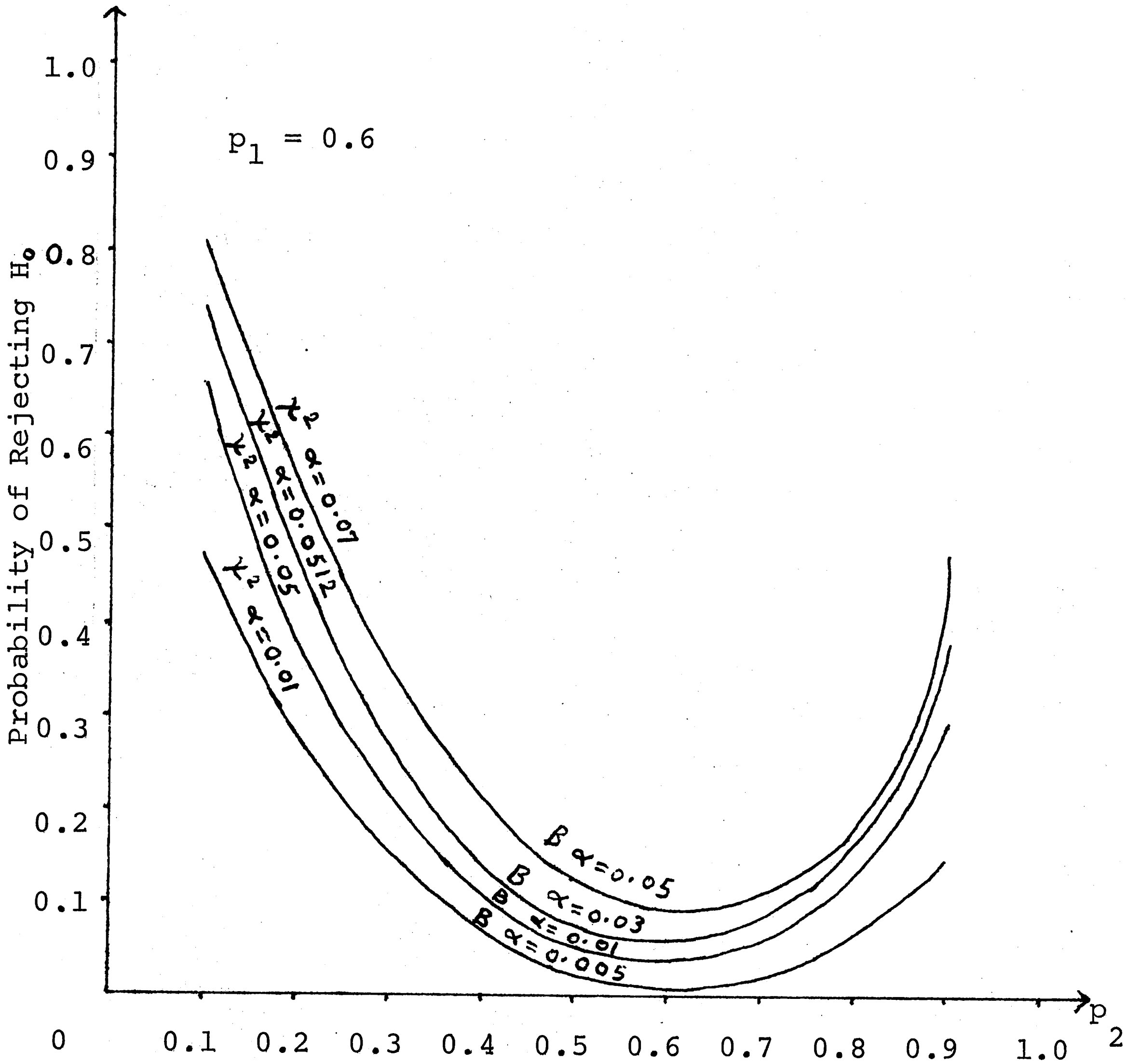

Figure 6. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.6$ and p_2 as a Variable.

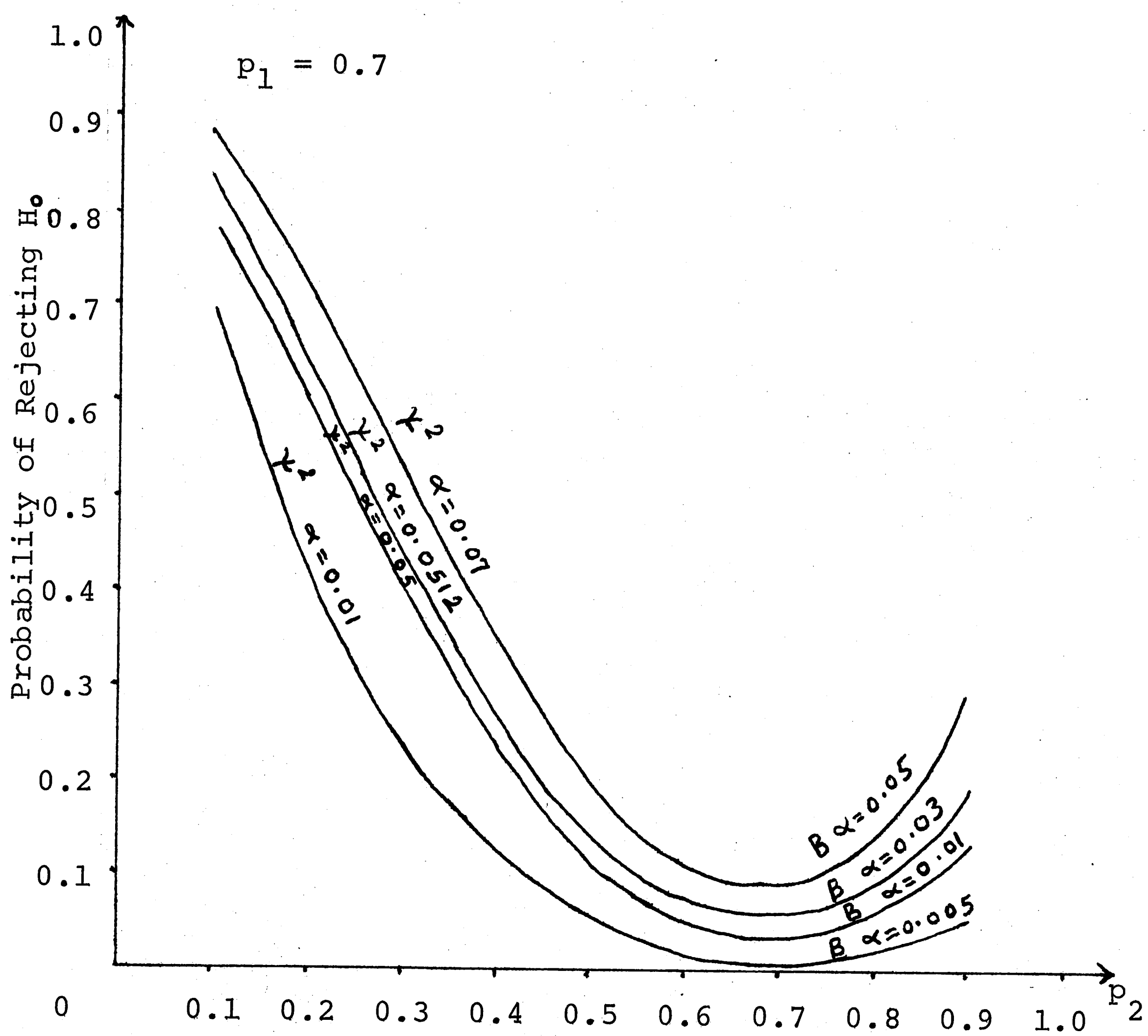

Figure 7. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.7$ and p_2 as a Variable.

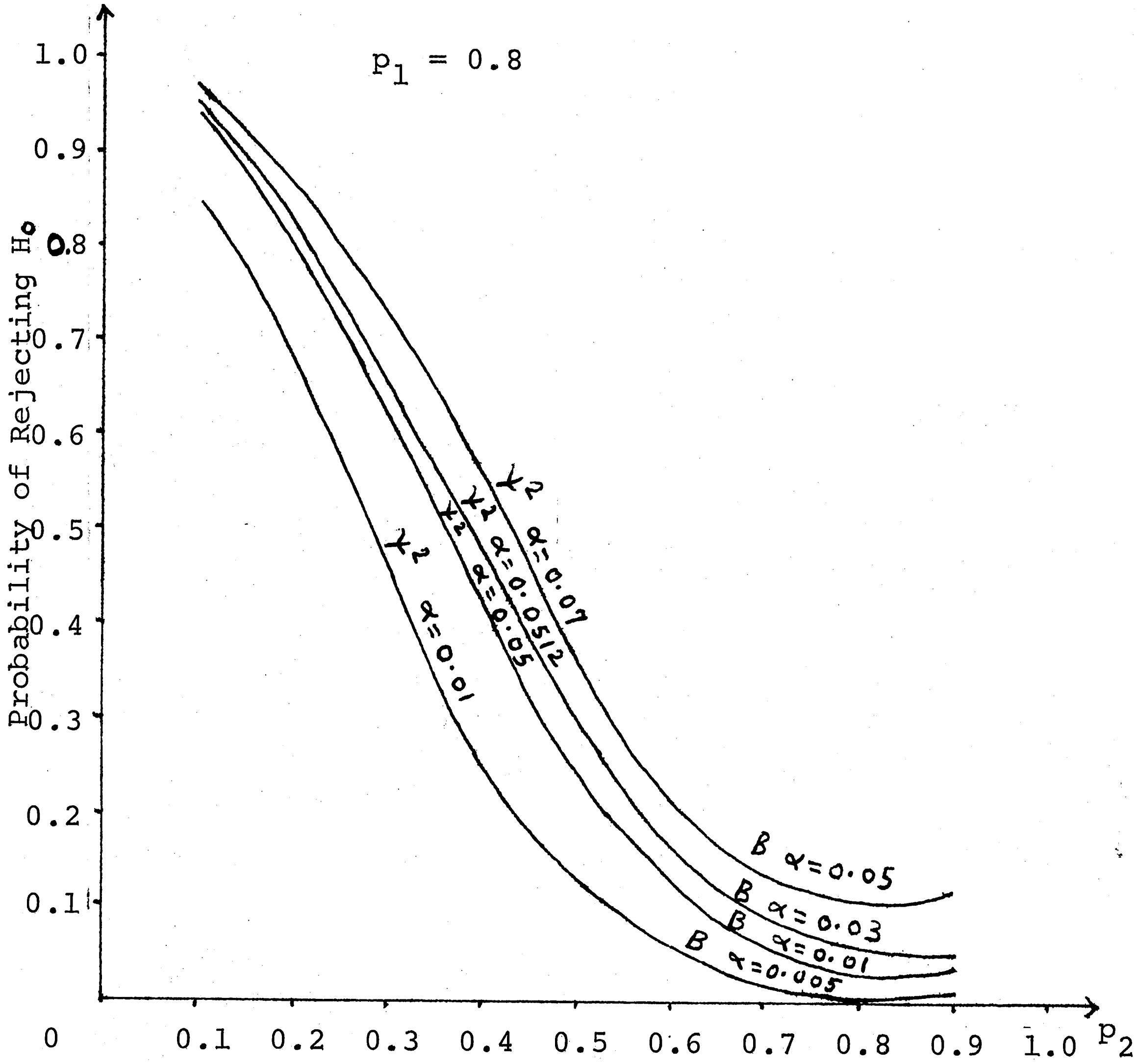

Figure 8. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.8$ and p_2 as a Variable.

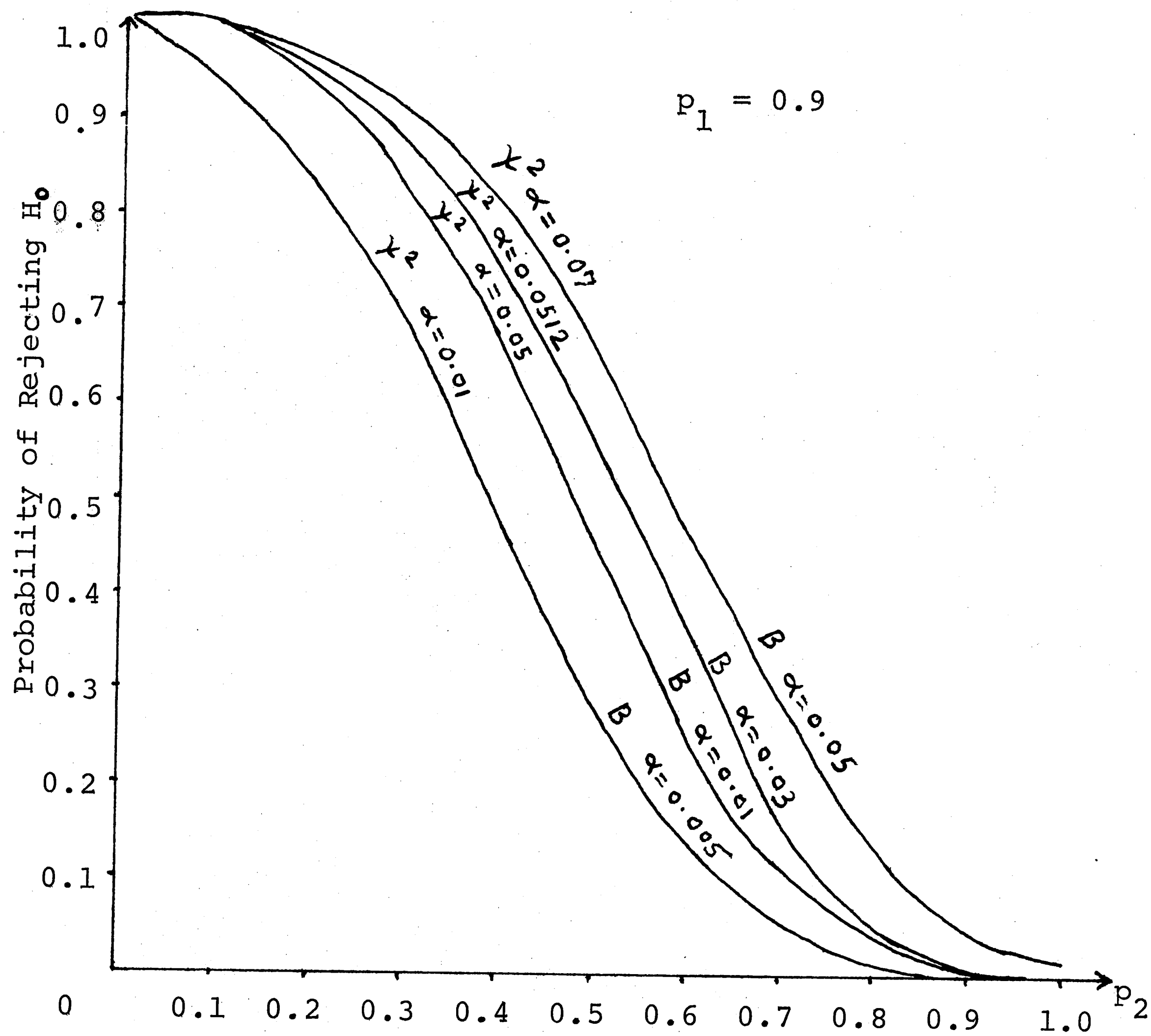

Figure 9. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests at Various α Levels Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.9$ and p_2 as a Variable.

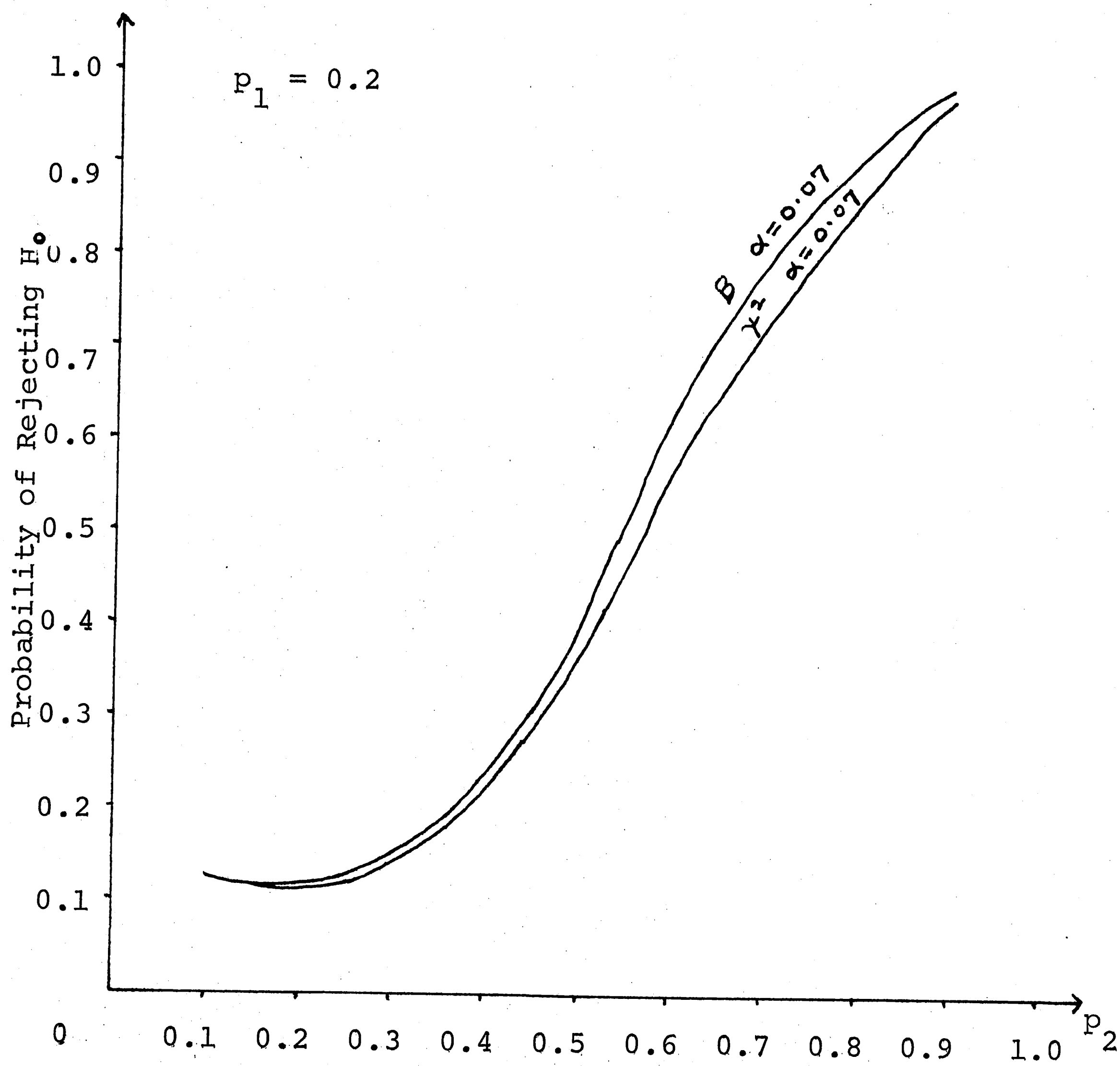

Figure 10. Empirical Power Curves for BIANOVA and CHI-SQUARE Tests, both at 0.07 Level, Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.2$ and p_2 as a Variable

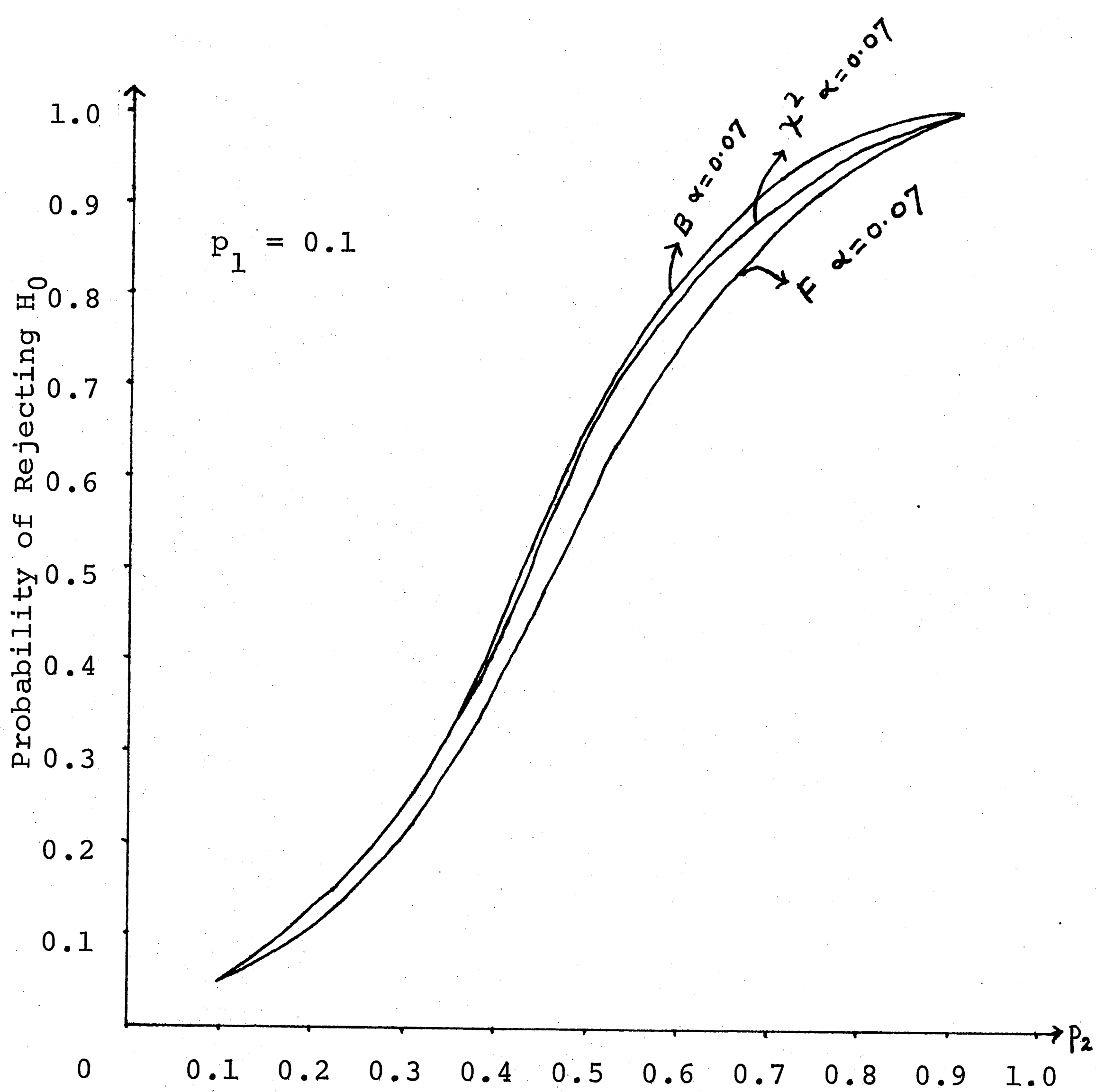

Figure 11. Empirical Power Curves for BIANOVA, CHI-SQUARE and F Tests, All Three at 0.07 Level, Under the Situations with $t = 2$, $n = 10$, $p_1 = 0.1$ and p_2 as a Variable.

From these figures and tables, notice the following points:

(i) The B test is uniformly more powerful than the chi-square and F tests.

(ii) Power curves for the B test at levels of 0.05, 0.03, 0.01, and 0.005 coincide with the power curves for the chi-square test at the levels 0.07, 0.0512, 0.05, and 0.01, respectively.

(iii) Power curves for the chi-square test and the F test coincide with each other at levels 0.05, 0.03, and 0.01. However, they do not coincide at levels 0.07 and 0.005.

(iv) From (ii) and (iii) above, it can be concluded that the power curves for B test at 0.01 level, chi-square test at 0.05 level, and F test at 0.05 level coincide with each other. Also, the power curves for B tests at 0.005 level, chi-square test at 0.01 level, and F test at 0.01 level coincide with each other.

(v) The chi-square test yields uniformly higher power than F test at the level of 0.07; however, the opposite is true at the level of 0.005.

(vi) Because of the small sample size, the test statistics become discontinuous (discrete) and as a result, chi-square (and F test, also) gives the same power at the different levels of 0.03 and 0.05. For some reason, this is not true for the B test!

(vii) Under H_0, for small sample sizes (such as n = 10), B, chi-square, and F tests do not attain the desired fixed α levels.

(viii) Frequently, under H_0, the level reached by the B test is somewhat higher than that reached by the chi-square and F tests. This makes the B test somewhat more liberal than the chi-square and F tests.

The incidence of coinciding power curves indicates that all three tests are ordering the data sets in a similar manner. Of course, for the different values of n and t, different types of relationship will hold between the sets of fixed α levels in order to achieve the coinciding power curves. One can investigate this interesting relationship for some cases, with enough time and computing money. However, it would be difficult to arrive at general conclusions. Also for a given value of t, one can investigate the minimum value for n for which all three tests will attain the fixed desired α levels under H_0.

From the above observations, it becomes difficult to say which test is "superior". Probably, it will depend very much upon an individual's taste for hypothesis testing and also upon the experimental situation. However, the situation in Figures 10 and 11 make the B test look somewhat more "attractive" than chi-square and F tests.

In Figure 10, p_1 is fixed at 0.2, and p_2 varies. Here, the desired α levels for both the tests are set at 0.07. It can be seen that under H_0, both the tests attain approximately the same level of 0.108 (different from the desired 0.07 level) and yet under alternative hypotheses, the B test is uniformly more powerful than the χ^2 test. With further study, it is possible to find different α levels (such as 0.07) for which both tests attain the same level (may be different from the desired level) under H_0 and yet under alternative hypotheses, the B test is more powerful than the chi-square test.

In Figure 11, p_1 is fixed at 0.01 level, and p_2 varies. Here, the desired α levels for all the three tests (B, chi-square, and F) are set at 0.07. It can be observed that under H_0, all three tests attain the same level of 0.046 (different from the desired 0.07 level) and yet under the alternative hypotheses, the B test gives higher power than both the chi-square and F tests, and the chi-square test gives higher power than the F test. With further study one can find different α levels (such as 0.07) for which all the three tests attain the same level (may be different from the desired level) under H_0 and yet under alternative hypotheses, the B test is more powerful than the chi-square and F tests.

Ordering of the Collection of Data Sets by B, Chi-Square, and F Tests

An empirical study for $n = 10$, $t = 2$, and $t = 3$, was done which indicated that the B, chi-square, and F tests order the collection of data sets in a similar fashion. Since it was previously indicated that for a fixed n and t, the F statistic is a monotonically increasing function of the BIANOVA test statistic B, then it is sufficient to compare the BIANOVA test statistic B with the chi-square test statistic, thereby providing a comparison of the BIANOVA test statistic B, the chi-square statistic, and the F statistic.

For $t = 2$ and $n = 10$, various values of $\hat{p}_1$ and $\hat{p}_2$, with the increments of 0.1 were considered and the corresponding values of B and χ^2 were found. These values are given in Table VIII. Also, for $t = 3$ and $n = 10$, various values of $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$, with the increments of 0.1, were considered and the corresponding values of B and χ^2 were found, which are not included here because of the length involved. From Table VIII, it can be seen that statistic B is a monotonically increasing function of the χ^2 statistic, and hence that all three tests (B, chi-square, and F) order (rank) the data sets in a similar fashion. However, in order to generalize this statement for all values of n and t, some more work needs to be done.

This problem of same ordering can be stated as follows:

$$\text{Suppose } \chi^2\left(\hat{p}_1,\ldots,\hat{p}_t\right) = \sum_{i=1}^{t} n\left(\hat{p}_i - \hat{\bar{p}}\right)^2/\hat{\bar{p}}\,\hat{\bar{q}}$$

$$\text{and } B\left(\hat{p}_1,\ldots,\hat{p}_t\right) = \sum_{i=1}^{t} n\left(\hat{p}_i - \hat{\bar{p}}\right)^2/\sum_{i=1}^{t}\left(\hat{p}_i\,\hat{q}_i/\,t\right)$$

are two t-dimensional functions. It is known that $\chi^2 < B$ at any fixed point in t-dimensions. Assume that there exists a point, say, $\left(\hat{p}_1',\ldots,\hat{p}_t'\right)$ such that $\chi^2\left(\hat{p}_1',\ldots,\hat{p}_t'\right) > \chi^2\left(\hat{p}_1,\ldots,\hat{p}_t\right)$. Does this imply $B\left(\hat{p}_1',\ldots,\hat{p}_t'\right) > B\left(\hat{p}_1,\ldots,\hat{p}_t\right)$, and vice versa?

It is difficult to solve this problem only with algebra. Some of the results achieved in trying to work this problem geometrically which might be of help are as follows:

$$\frac{\partial\chi^2}{\partial\hat{p}_i} = \left[\left\{t\sum_{j=1}^{t}\hat{p}_j - \left(\sum_{j=1}^{t}\hat{p}_j\right)^2\right\}\left\{2nt\left(t\hat{p}_i - \sum_{j=1}^{t}\hat{p}_j\right)\right\}\right.$$

$$\left.-\left\{n\sum_{j=1}^{t}\left(t\hat{p}_j - \sum_{j=1}^{t}\hat{p}_j\right)^2\right\}\left\{t-2\left(\sum_{j=1}^{t}\hat{p}_j\right)\right\}\right] \div$$

$$\left[\left(\sum_{j=1}^{t}\hat{p}_j\right)^2\left(t - \sum_{j=1}^{t}\hat{p}_j\right)^2\right]$$

$$= t^2\left[2n\left(\sum_{j=1}^{t}\hat{p}_j - \sum_{j=1}^{t}\hat{p}_j^{\,2}\right)\left(t\hat{p}_i - \sum_{j=1}^{t}\hat{p}_j\right) - \frac{n}{t}\sum_{j=1}^{t}\left(t\hat{p}_j\right.\right.$$

$$\left.\left.-\sum_{j=1}^{t}\hat{p}_j\right)^2\left(1 - 2\hat{p}_i\right)\right] \quad \div \quad \left[\left(\sum_{j=1}^{t}\hat{p}_j\right)^2\left(t - \sum_{j=1}^{t}\hat{p}_j\right)^2\right]$$

$$\frac{\partial B}{\partial \hat{p}_i} = \left[2n\left(\sum_{j=1}^{t}\hat{p}_j - \sum_{j=1}^{t}\hat{p}_j^{\,2}\right)\left(t\hat{p}_i - \sum_{j=1}^{t}\hat{p}_j\right) - \frac{n}{t}\sum_{j=1}^{t}\left(t\hat{p}_j - \sum_{j=1}^{t}\hat{p}_j\right)^2 \left(1 - 2\hat{p}_i\right)\right] \div \left[\left(\sum_{j=1}^{t}\hat{p}_j - \sum_{j=1}^{t}\hat{p}_j^{\,2}\right)^2\right]$$

Hence,

$$\nabla\,(\chi^2) = C^{+}_{\{\hat{p}_j\}} \cdot \nabla(B)\,,$$

where

$$C^{+}_{\{\hat{p}_j\}} = \frac{t^2\left(\sum_{j=1}^{t}\hat{p}_j - \sum_{j=1}^{t}\hat{p}_j^{\,2}\right)^2}{\left(\sum_{j=1}^{t}\hat{p}_j\right)^2\left(t - \sum_{j=1}^{t}\hat{p}_j\right)^2} > 0 \;.$$

Now

$$D_{\vec{u}}(\chi^2) = \nabla(\chi^2)\,\frac{\vec{u}}{|\vec{u}|}$$

$$= \left[C^{+}_{\{\hat{p}_j\}}\;\nabla(B)\right]\frac{\vec{u}}{|\vec{u}|}$$

$$= C^{+}_{\{\hat{p}_j\}}\;D_{\vec{u}}(B)\;.$$

(Definitions 4 and 5 in the Appendix define the gradient, ∇, and the directional derivative, respectively.)

The above result simply implies that along some path (and not for any two random points) of $\left(\hat{p}_1,\ldots,\hat{p}_t\right)$ in t-dimensions, an increase (a decrease) occurs in the χ^2 statistic if and only if an increase (a decrease) occurs in the statistic B, but that is not exactly what we want in order to make a general conclusion regarding the ordering of data sets by the B, chi-square, and F tests. This can be seen from the following figure:

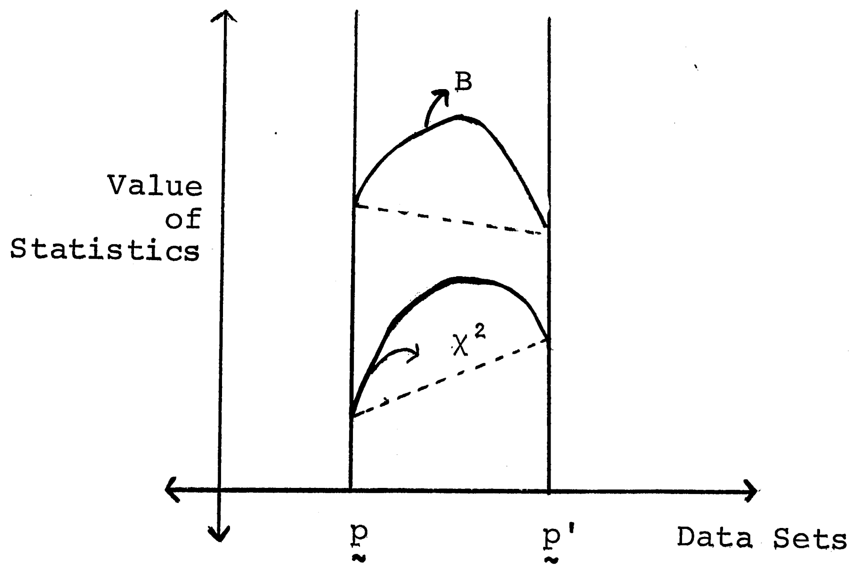

Figure 12. Ordering of the Data Sets by B and Chi-Square Tests

In Figure 12, the B curve increases (and decreases) with the χ^2 curve along some path, and yet both the tests order the two data sets, $\underset{\sim}{p}$ and $\underset{\sim}{p}'$, differently. The B test declares data set $\underset{\sim}{p}$ to be more significant, while the chi-square test declares data set $\underset{\sim}{p}'$ to be more significant. However, if one can find a continuous path from $\left(\hat{p}_1, \ldots, \hat{p}_t\right)$ to $\left(\hat{p}_1', \ldots, \hat{p}_t'\right)$ such that the χ^2 curve (or B curve) is monotonically increasing, then this problem is essentially solved because the monotonicity implies that $D_{\vec{u}}(\chi^2) > 0$. Thus, from the above result, it follows that $D_{\vec{u}}(B) > 0$. But to find a path from $\left(\hat{p}_1, \ldots, \hat{p}_t\right)$ to $\left(\hat{p}_1', \ldots, \hat{p}_t'\right)$ is a difficult or tricky task. This is a problem of topology, or more specifically, of Morse theory. Milnor (1963) has discussed Morse theory, which might be of some help in solving this problem.

CHAPTER III

TWO-WAY CLASSIFICATION WITH EQUAL NUMBER OF BINARY OBSERVATIONS PER CELL

One Binary Observation Per Cell

Consider an experiment conducted as a two-way classification with one binary observation per cell. Cochran's Q test is most frequently used for the analysis under this situation.

As before, assume that there are t treatments and b blocks. Let X_{ij} denote the observation corresponding to the i^{th} treatment and j^{th} block; $n_{i.}$, the i^{th} treatment total, and $n_{.j}$, the j^{th} block total, $i = 1,...,t$ and $j = 1,...,b$. Then the data appear as follows:

Block ↓	Treatment 1	2	. . .	t	Block Total
1	X_{11}	X_{21}	. . .	X_{2t}	$n_{.1}$
.	.	.		.	.
.	.	.		.	.
.	.	.		.	.
b	X_{1b}	X_{2b}	. . .	X_{tb}	$n_{.b}$
Trt Total	$n_{1.}$	$n_{2.}$	. . .	$n_{t.}$	$n_{..}$

It is assumed that all X_{ij} are independent, and that the blocks are randomly selected from the population of all possible blocks. Under these assumptions, it would be of interest to test the equality of treatment effects within each block. If p_{ij} denotes the true probability of "success" under treatment i and block j, then the null hypothesis may be restated in mathematical terms as follows:

H_0: $p_{1j} = p_{2j} = \ldots = p_{tj}$ ($= p_{.j}$ say) for each j from 1 to b

H_A: $p_{ij} \neq p_{kj}$ for some $i \neq k$, and for some j.

The following test statistic, B_1, is proposed to test the above hypotheses:

$$B_1 = \frac{b^2 \sum_{i=1}^{t} \left(\hat{p}_{i.} - \hat{\hat{p}}\right)^2}{\sum_{j=1}^{b} \hat{p}_{.j} \hat{q}_{.j}} ,$$

where $\hat{p}_{i.} = \frac{n_{i.}}{b}$, $\hat{p}_{.j} = \frac{n_{.j}}{t}$ and $\hat{\hat{p}} = \sum_{i=1}^{t} n_{i.}/tb = \sum_{j=1}^{b} n_{.j}/tb$.

The following form of B_1 is more suitable for computational and accuracy purposes.

$$B_1 = \frac{t^2 \sum_{i=1}^{t} n_{i.}^2 - t\left(\sum_{i=1}^{t} n_{i.}\right)^2}{t\left(\sum_{j=1}^{b} n_{.j}\right) - \sum_{j=1}^{b} n_{.j}^2} .$$

Note that

$$\sum_{j=1}^{b} \sum_{i=1}^{t} \left(\hat{p}_{i.} - \hat{\hat{p}}\right)^2 = \sum_{j=1}^{b} \left(\sum_{i=1}^{t} \hat{p}_{i.}^2 - t\,\hat{\hat{p}}^2\right)$$

$$= \sum_{j=1}^{b} \left[\sum_{i=1}^{t} \frac{n_{i.}^2}{b^2} - \frac{t\left(\sum_{i=1}^{t} n_{i.}\right)^2}{b^2\, t^2} \right]$$

$$= \sum_{i=1}^{t} \frac{n_{i.}^2}{b} - \frac{\left(\sum_{i=1}^{t} n_{i.}\right)^2}{bt}$$

and

$$\sum_{j=1}^{b} \sum_{i=1}^{t} \hat{p}_{.j}\, \hat{q}_{.j} = \sum_{j=1}^{b} \sum_{i=1}^{t} \left(\hat{p}_{.j} - \hat{p}_{.j}^{\,2}\right)$$

$$= \sum_{j=1}^{b} \sum_{i=1}^{t} \left(\frac{n_{.j}}{t} - \frac{n_{.j}^2}{t^2}\right)$$

$$= \sum_{j=1}^{b} \left(n_{.j} - \frac{n_{.j}^2}{t}\right)$$

$$= \sum_{j=1}^{b} \left[\sum_{i=1}^{t} x_{ij}^2 - \frac{1}{t}\left(\sum_{i=1}^{t} x_{ij}\right)^2 \right] .$$

Hence, statistic B_1 can be written as

$$B_1 = \frac{b \sum_{j=1}^{b} \sum_{i=1}^{t} \left(\hat{p}_{i.} - \hat{\bar{p}}\right)^2}{\sum_{j=1}^{b} \sum_{i=1}^{t} \hat{p}_{.j}\, \hat{q}_{.j}/t}$$

$$= \frac{b\left[\frac{1}{b} \sum_{j=1}^{t} n_{i.}^2 - \frac{1}{bt}\left(\sum_{i=1}^{t} n_{i.}\right)^2\right]}{\frac{1}{t}\left[\sum_{j=1}^{b}\left\{\sum_{i=1}^{t} x_{ij}^2 - \frac{1}{t}\left(\sum_{i=1}^{t} x_{ij}\right)^2\right\}\right]} .$$

The statistic B_1 can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_1 = \frac{b \cdot \text{treatment SS}}{\frac{1}{t}\text{ within block SS}}$$

$$= t\,\frac{\text{treatment MS}}{\text{within block MS}}$$

$$= t \cdot F\left[(t-1),\ b(t-1)\right] \quad !!!$$

The above result makes B_1 computationally simple. One can just run ANOVA on binary data and then by putting the appropriate components together, one can get numerical value of the B_1 statistic.

Also, $B_1 = \left(\frac{t}{t-1}\right)Q$, where Q is Cochran's Q statistic.

Now it will be shown that the asymptotic null distribution of B_1 is a central chi-square with (t-1) degrees of freedom. A distribution of B_1 under H_A is difficult to obtain.

<u>Derivation:</u>

$$\hat{p}_{i.} = \frac{1}{b}\sum_{j=1}^{b} X_{ij}$$

under H_0,

$$\hat{p}_{i.} \overset{a.d.}{\sim} N\left[\frac{1}{b}\sum_{j=1}^{b} p_{.j},\ \frac{1}{b^2}\sum_{j=1}^{b} p_{.j}\,q_{.j}\right],$$

$$i = 1,\ldots,t.$$

The statistic B_1 can be written as $B_1 = N/D$, where

$$N = \frac{\sum_{i=1}^{t} \left(\hat{p}_{i.} - \hat{\bar{p}}\right)^2}{\sum_{j=1}^{b} p_{.j}\, q_{.j}/b^2} \quad \text{and} \quad D = \frac{\sum_{j=1}^{b} \hat{p}_{.j}\, \hat{q}_{.j}/b^2}{\sum_{j=1}^{b} p_{.j}\, q_{.j}/b^2} .$$

Notice that N can be expressed in a quadratic form as

$$N = \frac{Y'AY}{\sum_{j=1}^{b} \left(p_{.j}\ q_{.j}\right)/b^2} ,$$

where $Y' = \left(\hat{p}_{1.}, \ldots, \hat{p}_{t.}\right)$ and $A = \left(I_t - \frac{1}{t} J_t^t\right)$. Note that A is symmetric idempotent matrix of rank (t-1).

It can be seen that, under H_0,

$$Y \overset{a.d.}{\sim} N_t \left[\left(\frac{1}{b} \sum_{j=1}^{b} p_{.j}\right) J_1^t, \left(\frac{1}{b^2} \sum_{j=1}^{b} p_{.j}\ \ q_{.j}\right) I_t\right] .$$

Then by Theorems 1 and 3 (in Appendix), $N \overset{a.d.}{\sim} \chi^2(t-1)$. By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Hence by Theorem 4, $B_1 = \frac{N}{D} \xrightarrow{dist} \chi^2(t-1)$ under H_0.

Note: In the above derivation, an assumption of fixed block totals is not required. Multiplying B_1 by $\left(\frac{t-1}{t}\right)$ might improve the approximation and if that is the case, then one will get Cochran's Q statistic. By making appropriate changes in the statistic B_1, one can test the equality of block effects under each treatment. In this case, testing of the treatments averaged over blocks is not possible as the estimate for $Var\left(\hat{p}_{i.}\right)$ is not available.

More Than One Binary Observation Per Cell

Now instead of just one binary observation per cell, suppose that there are n binary observations per cell. Let x_{ijk} denote the k^{th} response corresponding to i^{th} treatment and j^{th} block, $i = 1,\ldots,t;\ j = 1,\ldots,b$ and $k = 1,\ldots,n$. Then the data appears as follows:

Block ↓	Treatment 1	2	. . .	t	Block Total
1	x_{111} ⋮ x_{11n}	x_{211} ⋮ x_{21n}	. . . . . .	x_{t11} ⋮ x_{t1n}	$n_{.1}$
⋮	.	.	. . .	.	⋮
b	x_{1b1} ⋮ x_{1bn}	x_{2b1} ⋮ x_{2bn}	. . . . . .	x_{tb1} ⋮ x_{tbn}	$n_{.b}$
Trt. Totals	$n_{1.}$	$n_{2.}$		$n_{t.}$	$n_{..}$

Testing the Treatments Block-Wise

It is assumed that the responses in one cell are independent of the responses in other cells. Under this assumption, it is of interest to test the equality of treatment effects block-wise, i.e.

$$H_0: \; p_{1j} = p_{2j} = \cdots = p_{tj} \; (= p_{.j} \text{ say}) \text{ for } j = 1,\ldots,b,$$

where p_{ij} is the true probability of success under treatment i and block j.

This hypothesis is equivalent to that of testing for treatment effects and for treatment x block interaction effect simultaneously in the regular analysis of variance structure.

The following test statistic B_2 is proposed to test the above hypothesis:

$$B_2 = \frac{nbt \sum_{j=1}^{b} \sum_{i=1}^{t} \left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2}{\sum_{j=1}^{b} \sum_{i=1}^{t} \hat{p}_{ij} \, \hat{q}_{ij}}$$

Note that

$$\sum_{i=1}^{t} \sum_{j=1}^{b} \sum_{k=1}^{n} \left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2 = \sum_{j=1}^{b} \sum_{k=1}^{n} \left(\sum_{i=1}^{t} \hat{p}_{ij}^{\,2} - t\,\hat{p}_{.j}^{\,2}\right)$$

$$= \sum_{j=1}^{b} \left[\sum_{i=1}^{t} \frac{n_{ij}^{\,2}}{n} - \frac{\left(\sum_{i=1}^{t} n_{ij}\right)^2}{nt} \right]$$

and

$$\sum_{i=1}^{t} \sum_{j=1}^{b} \sum_{k=1}^{n} \hat{p}_{ij} \, \hat{q}_{ij} = \sum_{i=1}^{t} \sum_{j=1}^{b} \sum_{k=1}^{n} \left(\frac{n_{ij}}{n} - \frac{n_{ij}^{\,2}}{n^2}\right)$$

$$= \sum_{i=1}^{t} \sum_{j=1}^{b} \left(n_{ij} - \frac{n_{ij}^2}{n} \right)$$

$$= \sum_{i=1}^{t} \sum_{j=1}^{b} \left[\sum_{k=1}^{n} X_{ijk}^2 - \frac{\left(\sum_{k=1}^{n} X_{ij}\right)^2}{n} \right].$$

Hence, the statistic B_2 can be written as

$$B_2 = \frac{tb \sum_{i=1}^{t} \sum_{j=1}^{b} \sum_{k=1}^{n} \left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2}{\sum_{k=1}^{n} \sum_{j=1}^{b} \sum_{i=1}^{t} \hat{p}_{ij} \hat{q}_{ij} / n}$$

$$= \frac{nbt \sum_{j=1}^{b} \left[\sum_{i=1}^{t} \frac{n_{ij}^2}{n} - \frac{\left(\sum_{i=1}^{t} n_{ij}\right)^2}{nt} \right]}{\sum_{i=1}^{t} \sum_{j=1}^{b} \left[\sum_{k=1}^{n} X_{ijk}^2 - \left(\sum_{k=1}^{n} X_{ij}\right)^2 / n \right]} .$$

The statistic B_2 can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_2 = ntb \; \frac{\text{treatments within blocks SS}}{\text{experimental error SS}}$$

$$= \frac{nb(t-1)}{n-1} \; \frac{\text{treatments within blocks MS}}{\text{experimental error MS}}$$

$$= \frac{nb(t-1)}{n-1} \; F \left[b(t-1),\ tb\,(n-1) \right] .$$

The approximate asymptotic null distribution of B_2 is a central chi-square with $b(t-1)$ degrees of freedom and its approximate asymptotic alternative distribution is non-central chi-square with b(t-1) degrees of freedom and the non-centrality parameter is

$$\lambda_2 = \frac{nbt \sum_{i=1}^{t} \sum_{j=1}^{b} \left(p_{ij} - p_{.j}\right)^2}{2 \sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}\, q_{ij}} .$$

Derivation: A distributional derivation for a very general statistic, $B_{1.(g)}$ given by (4.1), is given in Chapter IV, of which this becomes a particular case. A substitution of $r = t$ and $s = 1$ in that derivation will provide the required distributional derivation of the statistic B_2, and hence is not presented here to avoid the duplication.

Note: A substitution of $b = 1$ in B_2 will give back the BIANOVA statistic, B, for the one-way classification. Hence the statistic B_2 is a generalization of the statistic B, for testing the treatments block-wise. Note that a substitution of $n = 1$ will make both B_2 and B undefined.

Another Approach

Another test statistic to test the same null hypothesis of the equality of treatment effects block-wise is developed and its asymptotic null distribution is found. Here n is assumed to be greater than 1.

This new statistic is

$$B_2' = \frac{nb \sum_{j=1}^{b} \sum_{i=1}^{t} \left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2}{\sum_{j=1}^{b} \hat{p}_{.j}\, \hat{q}_{.j}} .$$

Note that

$$\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n} \hat{p}_{.j}\,\hat{q}_{.j} = \sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\frac{n_{.j}}{nt} - \frac{n_{.j}^{2}}{n^{2}t^{2}}\right)$$

$$= \sum_{j=1}^{b}\left(n_{.j} - \frac{n_{.j}^{2}}{nt}\right)$$

$$= \sum_{j=1}^{b}\left[\sum_{i=1}^{t}\sum_{k=1}^{n} X_{ijk}^{2} - \frac{\left(\sum_{i=1}^{t}\sum_{k=1}^{n} X_{ijk}\right)^{2}}{nt}\right].$$

Hence, the statistic B_2' can be written as

$$B_2' = \frac{b\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\hat{p}_{ij} - \hat{p}_{.j}\right)^{2}}{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n}\hat{p}_{.j}\,\hat{q}_{.j} \,/\, nt}$$

$$= ntb\,\frac{\sum_{j=1}^{b}\left[\sum_{i=1}^{t}\frac{n_{ij}^{2}}{n} - \frac{\left(\sum_{i=1}^{t} n_{ij}\right)^{2}}{nt}\right]}{\sum_{j=1}^{b}\left[\sum_{i=1}^{t}\sum_{k=1}^{n} X_{ijk}^{2} - \frac{\left(\sum_{i=1}^{t}\sum_{k=1}^{n} X_{ijk}\right)^{2}}{nt}\right]}\,.$$

The statistic B_2' can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_2' = ntb \, \frac{\text{treatments within blocks SS}}{\text{within blocks SS}}$$

$$= \frac{ntb(t-1)}{nt-1} \, \frac{\text{treatments within blocks MS}}{\text{within blocks MS}}$$

$$= \frac{ntb(t-1)}{nt-1} \; F\left[b(t-1),\, b(nt-1)\right] .$$

The approximate asymptotic null distribution of B_2' is central chi-square with b(t-1) degrees of freedom. Its distribution under H_A is difficult to obtain. The derivation for the asymptotic null distribution of B_2' is given below:

Derivation:

$$\text{Under } H_0,\ \hat{p}_{ij} \overset{a.d.}{\sim} N\left[p_{.j}\,, p_{.j}q_{.j} / n\right] .$$

Observe the inequality of the variances of $\hat{p}_{ij}$'s. To overcome this problem, without knowing how critical it is, it was decided to approximate the distribution of $\hat{p}_{ij}$ by a normal distribution with mean $p_{.j}$ and variance c, where c is such that $\sum_{j=1}^{b}\left(\frac{p_{.j}q_{.j}}{n} - c\right)^2$ is minimum. This implies

$$c = \sum_{j=1}^{b} p_{.j}q_{.j}/nb.$$

Now B_2' can be expressed as $B_2' = N/D$, where

$$N = \frac{\sum_{i=1}^{t}\sum_{j=1}^{b}\left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2}{\sum_{j=1}^{b} p_{.j}\, q_{.j} / nb} \quad \text{and} \quad D = \frac{\sum_{j=1}^{b} \hat{p}_{.j}\, \hat{q}_{.j} / nb}{\sum_{j=1}^{b} p_{.j}\, q_{.j} / nb} .$$

Now, N can be written in quadratic form as

$$N = \frac{nb(Y'AY)}{\sum_{j=1}^{b} p_{.j}\, q_{.j}} ,$$

where $Y' = \left(\hat{p}_{11},\ldots,\hat{p}_{t1},\ldots,\hat{p}_{1b},\ldots,\hat{p}_{tb}\right)$ and

$$A = \begin{pmatrix} I_t - \frac{1}{t} J_t^t & \phi & \cdots & \phi \\ \phi & I_t - \frac{1}{t} J_t^t & \cdots & \phi \\ \vdots & \vdots & \ddots & \vdots \\ \phi & \phi & \cdots & I_t - \frac{1}{t} J_t^t \end{pmatrix}.$$

One can show that A is a symmetric idempotent matrix of rank $b(t-1)$.

After pooling the variances of $\hat{p}_{ij}$'s under H_0,

$$Y \overset{a.d.}{\sim} N\left[\left(p_{.1},\ldots,p_{.1},\ldots,p_{.b},\ldots,p_{.b}\right), \left(\sum_{j=1}^{b} p_{.j} q_{.j} / nb\right) I_{bt}\right].$$

By Theorems 1 and 3 (in Appendix), $N \overset{a.d.}{\sim} \chi^2\,[b(t-1)]$. By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Hence by Theorem 4, $B_2' = \frac{N}{D} \xrightarrow{dist} \chi^2\,[b(t-1)]$, approximately, under H_0.

<u>Notes:</u> When $n = 1$, treatments within block SS becomes identical to within blocks SS; and hence $B_2' = tb$, regardless of observations. This is the reason why n is restricted to being greater than 1 in this test procedure. This is the main reason why the author used $\sum_{i=1}^{t} (\hat{p}_{i.} - \hat{\hat{p}})^2$ as the test criterion rather than $\sum_{j=1}^{b} \sum_{i=1}^{t} (\hat{p}_{ij} - \hat{p}_{.j})^2$ in the construction of the statistic B_1. The above discussion also applies to Cochran's Q statistic.

Since under H_0, an unbiased estimator of the $\text{var}\left(\hat{p}_{ij}\right)$ is $\left(\frac{nt}{nt-1}\right)\left[\frac{\hat{p}_{.j}\hat{q}_{.j}}{n}\right]$, multiplying B_2' by $\left(\frac{nt-1}{nt}\right)$ might improve the approximation and if that is the case, one will have a new statistic, $B_2'' = \left(\frac{nt-1}{nt}\right) B_2'$, whose asymptotic null distribution is the same as that of B_2'. Notice that a substitution of $b = 1$ in B_2' and B_2'' will give back the Pearson's χ^2 statistic and the CATANOVA statistic of Light and Margolin for binary data in one-way classification, respectively. Hence, the statistics, B_2' and B_2'', are the generalizations of the statistics, χ^2 and C for binary data in one-way classification, respectively, to test the treatments block-wise.

Now it becomes a question whether to use B_2 or B_2' or B_2'' in order to test the equality of treatments block-wise. One can show that $B_2 > B_2' > B2''$, for a fixed set of data, and hence B_2 will yield uniformly higher power than B_2' and B_2''. The behavior of B_2, B_2', and $B2''$ under H_0 has not been studied yet. For $b = 1$, some evidence in favor of B_2 over B_2' is given in Chapter II through Figures 10 and 11. From this, it seems that through empirical search, some type of evidence in favor of B_2 over B_2' and B_2'', for $b > 1$, can also be found.

It is obvious that $B_2' > B_2''$ for a fixed set of data. The following steps demonstrate the $B_2 > B_2'$, for a fixed set of data.

Clearly, $\sum_{i=1}^{t} \left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2 \geq 0$. Under H_A, $\sum_{i=1}^{t} \left(\hat{p}_{ij} - \hat{p}_{.j}\right)^2 > 0$, for all j.

i.e., $\sum_{i=1}^{t} \hat{p}_{ij}^{\,2} > \left(\sum_{i=1}^{t} \hat{p}_{ij}\right)^2 / t$,

i.e., $\sum_{i=1}^{t} \hat{p}_{ij} - \sum_{i=1}^{t} \hat{p}_{ij}^{\,2} < \sum_{i=1}^{t} \hat{p}_{ij} - \left\{\left(\sum_{i=1}^{t} \hat{p}_{ij}\right)^2 / t\right\}$,

i.e., $\sum_{i=1}^{t} \hat{p}_{ij}\, \hat{q}_{ij} / t < \hat{p}_{.j}\, \hat{q}_{.j}$, for all j,

i.e., $\sum_{i=1}^{t} \sum_{j=1}^{b} \hat{p}_{ij}\, \hat{q}_{ij} / ntb < \sum_{j=1}^{b} \hat{p}_{.j}\, \hat{q}_{.j} / nb$,

$\Rightarrow \quad B_2 > B_2'$.

By making appropriate changes in the statistics, B_2, B_2', and B_2'', one can test the hypothesis of the equality of block effects treatment-wise.

Testing the Treatments Average-wise

Now consider the problem of testing for the treatment effects averaged over blocks, i.e.,

$$H_0:\ p_{1.} = p_{2.} = \dots = p_{t.} \ (= p \text{ say}).$$

This hypothesis is equivalent to that of testing for treatment effects in the regular analysis of variance structure.

The following test statistic, B_3, is proposed to test the above hypothesis:

$$B_3 = \frac{ntb^2 \sum_{i=1}^{t} \left(\hat{p}_{i.} - \hat{\bar{p}}\right)^2}{\sum_{i=1}^{t} \sum_{j=1}^{b} \hat{p}_{ij}\, \hat{q}_{ij}} .$$

Note that

$$\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\hat{p}_{i.} - \hat{\bar{p}}\right)^2 = \sum_{j=1}^{b}\sum_{k=1}^{n}\left(\sum_{i=1}^{t}\hat{p}_{i.}^{\,2} - t\hat{\bar{p}}^2\right)$$

$$= \sum_{j=1}^{b}\sum_{k=1}^{n}\left[\sum_{i=1}^{t}\frac{n_{i.}^{\,2}}{n^2b^2} - t\frac{\left(\sum_{i=1}^{t} n_{i.}\right)^2}{n^2b^2t^2}\right]$$

$$= \sum_{i=1}^{t}\frac{n_{i.}^{\,2}}{nb} - \frac{\left(\sum_{i=1}^{t} n_{i.}\right)^2}{nbt} .$$

Now the statistic B_3 can be written as

$$B_3 = \frac{tb\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(\hat{p}_{i\cdot} - \hat{\bar{p}}\right)^2}{\sum_{i=1}^{t}\sum_{j=1}^{b}\sum_{k=1}^{n}\hat{p}_{ij}\,\hat{q}_{ij} / n} =$$

$$nbt\,\frac{\left[\sum_{i=1}^{t}\frac{n_i^{\,2}}{nb} - \frac{\left(\sum_{i=1}^{t} n_{i.}\right)^2}{nbt}\right]}{\sum_{i=1}^{t}\sum_{j=1}^{b}\left[\sum_{k=1}^{n} X_{ijk}^{\,2} - \frac{1}{t}\left(\sum_{k=1}^{n} X_{ijk}\right)^2\right]} .$$

The statistic B_3 can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_3 = ntb\,\frac{\text{treatment SS}}{\text{experimental error SS}}$$

$$= \frac{n(t-1)}{n-1}\;\frac{\text{treatment MS}}{\text{experimental error MS}}$$

$$= \frac{n(t-1)}{n-1}\; F\left[(t-1),\ tb(n-1)\right] .$$

Hence as the other BIANOVA statistics, B_3 is also made up of the components from the regular analysis of variance table. Its approximate asymptotic null distribution is central chi-square with (t-1) degrees of freedom, while its approximate asymptotic alternative distribution is non-central chi-square with (t-1) degrees of freedom and non-centrality parameter is

$$\lambda_3 = \frac{ntb^2 \sum_{i=1}^{t} \left(p_{i.} - \bar{p}\right)^2}{2 \sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}\, q_{ij}} .$$

Derivation: A distributional derivation for a very general statistic, $B_{1.(.)}$ given by (4.2), is given in Chapter IV, of which this becomes a particular case. A substitution of $r = t$ and $s = 1$ in that derivation will provide the required distributional derivation of the statistic B_3, and hence is not presented here to avoid duplication.

Note: A substitution of $b = 1$ in B_3 will give back the BIANOVA statistic, B, for one-way classification (as was the case with B_2). The obvious reason for this is that when $b = 1$, testing the treatments block-wise is the same as testing the treatments average-wise. Note that the statistic B_3 is a generalization of the statistic B, for testing the treatments average-wise.

By making appropriate changes in the statistic B_3, one can test the hypothesis of the equality of block effects average-wise (i.e., averaged over treatments).

Test for Block x Treatment Interaction

Now consider the problem of testing for a block x treatment interaction. The hypothesis of no block x treatment interaction in mathematical terms is stated as:

$$H_0\colon\ p_{ij} - p_{i.} - p_{.j} + \bar{p} = 0 \text{ for all } i \text{ and } j.$$

The following test statistic, B_{bxt}, is proposed to test the above hypothesis:

$$B_{bxt} = \frac{\sum_{i=1}^{t} \sum_{j=1}^{b} \left(\hat{p}_{ij} - \hat{p}_{i.} - \hat{p}_{.j} + \hat{\bar{p}}\right)^2}{\sum_{i=1}^{t} \sum_{j=1}^{b} \hat{p}_{ij}\,\hat{q}_{ij} \,/\, nbt} \;.$$

The statistic B_{bxt} can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_{bxt} = \frac{nbt \sum_{i=1}^{t} \sum_{j=1}^{b} \sum_{k=1}^{n} \left(\hat{p}_{ij} - \hat{p}_{i.} - \hat{p}_{.j} + \hat{\bar{p}}\right)^2}{\sum_{i=1}^{t} \sum_{j=1}^{b} \sum_{k=1}^{n} \hat{p}_{ij}\,\hat{q}_{ij}}$$

$$= \frac{nbt\ (\text{block x treatment interaction SS})}{\text{experimental error SS}}$$

$$= \frac{(b-1)(t-1)n}{n-1}\ \frac{\text{block x treatment interaction MS}}{\text{experimental error MS}}$$

$$= \frac{(b-1)(t-1)n}{n-1}\ F\left[(b-1)(t-1),\ bt(n-1)\right] .$$

The approximate asymptotic null distribution of B_{bxt} is central chi-square with $(b-1)(t-1)$ degrees of freedom and its approximate asymptotic alternative distribution is non-central chi-square with (b-1)(t-1) degrees of freedom and non-centrality parameter is

$$\lambda_{bxt} = \frac{nbt \sum_{i=1}^{t} \sum_{j=1}^{b} \left(p_{ij} - p_{i.} - p_{.j} + \bar{p}\right)^2}{2 \sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}\, q_{ij}} .$$

Derivation: Under H_0 (or H_A), $\hat{p}_{ij} \overset{a.d.}{\sim} N\left(p_{ij}, \frac{p_{ij}q_{ij}}{n}\right)$.

Notice the inequality of variance of the $\hat{p}_{ij}$'s. This fact causes problems. However, it was decided to approximate the asymptotic distribution of $\hat{p}_{ij}$ by $\hat{p}_{ij} \overset{a.d.}{\sim} N\left[p_{ij}, c\right]$, where c is such that $\sum_{i=1}^{t} \sum_{j=1}^{b} \left(\frac{p_{ij}q_{ij}}{n} - c\right)^2$ is minimum. Choose $c = \sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}q_{ij}/nbt$. Here, in other words, variances are pooled in order to get a "common variance". This is open for criticisms provided there is a better way out.

The statistic B_{bxt} can be expressed as $B_{bxt} = N/D$, where

$$N = \frac{\sum_{i=1}^{t} \sum_{j=1}^{b} \left(\hat{p}_{ij} - \hat{p}_{i.} - \hat{p}_{.j} + \hat{\bar{p}}\right)^2}{\sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}\, q_{ij} \,/\, nbt} \quad \text{and} \quad D = \frac{\sum_{i=1}^{t} \sum_{j=1}^{b} \hat{p}_{ij}\hat{q}_{ij}/nbt}{\sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}q_{ij}/nbt} .$$

Note that N can be expressed in a quadratic form as

$$N = \frac{Y'AY}{\sum_{i=1}^{t} \sum_{j=1}^{b} p_{ij}\, q_{ij} \,/\, nbt} ,$$

where $Y' = \left(\hat{p}_{11}, \ldots, \hat{p}_{1b}, \ldots, \hat{p}_{t1}, \ldots, \hat{p}_{tb}\right)$

and

$$A = I_{bt} - \frac{1}{b}\begin{pmatrix} J_b^b & & \phi \\ & \ddots & \\ \phi & & J_b^b \end{pmatrix} - \frac{1}{t}\begin{pmatrix} I_b & \cdots & I_b \\ \vdots & \vdots & \vdots \\ I_b & \cdots & I_b \end{pmatrix} + \frac{1}{bt} J_{bt}^{bt} .$$

It can be shown that A is a symmetric idempotent matrix of rank $bt - t - b + 1 = (b-1)(t-1)$.

It can be seen that under H_0 (or H_A),

$$Y_{btx1} \overset{a.d.}{\underset{\sim}{\cdot}} N_{bt}\left[\mu' = \left(p_{11},\ldots,p_{1b},\ldots,p_{t1},\ldots,p_{tb}\right),\right.$$

$$\left.\left(\sum_{i=1}^{t}\sum_{j=1}^{b} \frac{p_{ij}q_{ij}}{nbt}\right) I_{bt}\right] .$$

Now by Theorems 1 and 3 (in Appendix), $N \overset{a.d.}{\underset{\sim}{\cdot}} \chi^2\left[(b-1)(t-1)\right]$. By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Then by Theorem 4, $B_{bxt} = \frac{N}{D} \xrightarrow{dist} \chi^2\left[(b-1)(t-1)\right]$ approximately, under H_0.

It remains to find the distribution of B_{bxt} under H_A. Under H_A, by Theorems 1 and 3, $N \overset{a.d.}{\underset{\sim}{\cdot}} \chi'^2\left[(b-1)(t-1), \lambda_{bxt}\right]$ where

$$\lambda_{bxt} = \frac{\sum_{i=1}^{t}\sum_{j=1}^{b}\left(p_{ij} - p_{i.} - p_{.j} + \bar{p}\right)^2}{2\sum_{i=1}^{t}\sum_{j=1}^{b} p_{ij}\, q_{ij} / nbt} .$$

By Theorems 5 and 6, $D \xrightarrow{prob} 1$, as before. Hence by Theorem 4,

$$B_{bxt} = \frac{N}{D} \xrightarrow{dist} \chi'^2\left[(b-1)(t-1),\ \lambda_{bxt}\right]$$

approximately, under H_A.

Problem: Assume a situation with b = 2, t = 2, and n = 100. Suppose $p_{11} = 0.2$, $p_{21} = 0.8$, and $p_{12} = 0.7$. What should be a reasonable value of p_{22} in order to conclude no block x treatment interaction?

It seems that there does not exist a reasonable value of p_{22} which will help in concluding that there is no block x treatment interaction, at least computationally, by following the present definition. Practically, if p_{22} takes its maximum value of 1, then it becomes a question of opinion whether the block x treatment interaction is present or not.

CHAPTER IV

FACTORIAL ARRANGEMENTS OF TREATMENTS

Two Factors

Consider an experiment having t treatments, b blocks, and n(>1) binary observations per each treatment x block cell. Suppose that the treatments are factorial. To start with, assume that there are two factors, A and B, at r and s levels, respectively. Let $\pi_{ij(g)}$ be the true probability of success under the i^{th} level of factor A and the j^{th} level of factor B for block g, $i = 1, \ldots, r$; $j = 1, \ldots, s$; $g = 1, \ldots, b$. Let $\pi_{ij(.)}$ be the true probability of success under the i^{th} level of factor A and the j^{th} level of factor B, averaged over blocks. Note that

$$\hat{\pi}_{i.(g)} = \sum_{j=1}^{s} \hat{\pi}_{ij(g)}/s = \sum_{j=1}^{s} n_{ij(g)}/ns \ ,$$

$$\hat{\pi}_{.j(g)} = \sum_{i=1}^{r} \hat{\pi}_{ij(g)}/r = \sum_{i=1}^{r} n_{ij(g)}/nr \ ,$$

$$\hat{\pi}_{i.(.)} = \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}/sb = \sum_{j=1}^{s} \sum_{g=1}^{b} n_{ij(g)}/nsb \ ,$$

$$\hat{\pi}_{.j(.)} = \sum_{i=1}^{r} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}/rb = \sum_{i=1}^{r} \sum_{g=1}^{b} n_{ij(g)}/nrb \ ,$$

$$\hat{\pi}_{..(g)} = \sum_{i=1}^{r} \sum_{j=1}^{s} \hat{\pi}_{ij(g)}/rs = \sum_{i=1}^{r} \sum_{j=1}^{s} n_{ij(g)}/nrs ,$$

and

$$\hat{\bar{\pi}} = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}/rsb = \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} n_{ij(g)}/nrsb .$$

The following layout of probabilities would help in visualizing the situation:

	Treatment Combination			
Block ↓	(1,1),...,(1,s)	...	(r,1),...,(r,s)	Block Prob. ↓
1	$\hat{\pi}_{1.(1)}$	...	$\hat{\pi}_{r.(1)}$	$\hat{\pi}_{..(1)}$
2	$\hat{\pi}_{1.(2)}$	...	$\hat{\pi}_{r.(2)}$	$\hat{\pi}_{..(2)}$
⋮	⋮	⋮	⋮	⋮
b	$\hat{\pi}_{1.(b)}$	...	$\hat{\pi}_{r.(b)}$	$\hat{\pi}_{..(b)}$
Trt. Prob.	$\hat{\bar{\pi}}_{1.(.)}$	...	$\hat{\bar{\pi}}_{r.(.)}$	

It is assumed that the response in one (treatment combination x block) cell is independent of the responses in other (treatment combination x block) cells.

Testing the Levels of Factor A Block-wise

Under this situation, it would be of interest to test the homogeneity of levels of factor A (averaged over levels of factor B) within each block, i.e.

$$H_0: \pi_{1.(g)} = \pi_{2.(g)} = \cdots = \pi_{r.(g)} \left(= \pi_{..(g)} \text{ say}\right)$$

for all g.

This hypothesis is equivalent to that of testing whether both the main effect of factor A and the A x block interaction effect are zero in the regular analysis of variance structure.

The following test statistic, $B_{1.(g)}$, is proposed to test the above hypothesis:

$$B_{1.(g)} = \frac{nbrs^2 \sum_{g=1}^{b} \sum_{i=1}^{r} \left(\hat{\pi}_{i.(g)} - \hat{\pi}_{..(g)}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right)} . \quad (4.1)$$

Note that,

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{i.(g)} - \hat{\pi}_{..(g)}\right)^2$$

$$= \sum_{j=1}^{s} \sum_{m=1}^{n} \left(\sum_{g=1}^{b} \sum_{i=1}^{r} \hat{\pi}^2_{i.(g)} - r \sum_{g=1}^{b} \hat{\pi}^2_{..(g)} \right)$$

$$= \sum_{g=1}^{b} \left[\sum_{i=1}^{r} \frac{n^2_{i.(g)}}{ns} - \frac{n^2_{..(g)}}{nrs} \right]$$

and

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right)$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\frac{n_{ij(g)}}{n} - \frac{n^2_{ij(g)}}{n^2} \right)$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} n_{ij(g)} - \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \frac{n^2_{ij(g)}}{n} .$$

The statistic $B_{1.(g)}$ can be expressed as follows in terms of the regular analysis of variance table:

$$B_{1.(g)} = \frac{brs \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{i.(g)} - \hat{\pi}_{..(g)}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ij(g)} \left[1 - \hat{\pi}_{ij(g)}\right] / n}$$

$$= nbrs \frac{\text{factor A within blocks SS}}{\text{experimental error SS}}$$

$$= \frac{nb(r-1)}{n-1} \frac{\text{factor A within blocks MS}}{\text{experimental error MS}}$$

$$= \frac{nb(r-1)}{n-1} F\left[b(r-1), rsb(n-1)\right] .$$

The approximate asymptotic null distribution of $B_{1.(g)}$ is central chi-square with $b(r-1)$ degrees of freedom, while its approximate asymptotic alternative distribution is non-central chi-square with $b(r-1)$ degrees of freedom, and the non-centrality parameter is

$$\lambda_{B_{1.(g)}} = \frac{nbrs^2 \sum_{g=1}^{b} \sum_{i=1}^{r} \left(\pi_{i.(g)} - \pi_{..(g)}\right)^2}{2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)} \left(1 - \pi_{ij(g)}\right)} .$$

<u>Derivation:</u>

Under H_0, $\hat{\pi}_{i.(g)} \overset{a.d.}{\sim} N\left[\pi_{..(g)}, \frac{1}{ns^2} \sum_{j=1}^{s} \pi_{ij(g)}\left\{1 - \pi_{ij(g)}\right\}\right].$

Notice the inequality of the variances of $\hat{\pi}_{i.(g)}$'s. To overcome the problem, without knowing how critical it is, it was decided to approximate the asymptotic distribution of $\hat{\pi}_{i.(g)}$ by

$$\hat{\pi}_{i.(g)} \overset{a.d.}{\sim} N\left[\pi_{..(g)}, c\right], \text{ where c is}$$

such that

$$\sum_{i=1}^{r} \sum_{g=1}^{b} \left[\frac{1}{ns^2} \sum_{j=1}^{s} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right) - c\right]^2$$

is minimum. Choose

$$c = \frac{1}{nbrs^2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right) .$$

Now $B_{1.(g)}$ can be expressed as $B_{1.(g)} = N/D$, where

$$N = \frac{nbrs^2 \sum_{g=1}^{b} \sum_{i=1}^{r} \left(\hat{\pi}_{i.(g)} - \hat{\pi}_{..(g)}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)}$$

and

$$D = \frac{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}\left(1 - \hat{\pi}_{ij(g)}\right)/nbrs^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)/nbrs^2} .$$

Notice that N can be expressed in a quadratic form as

$$N = \frac{nbrs^2\ Y'AY}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)} ,$$

where

$$Y = \left(\hat{\pi}_{1.(1)}, \ldots, \hat{\pi}_{r.(1)}, \ldots, \hat{\pi}_{1.(b)}, \ldots, \hat{\pi}_{r.(b)}\right)$$

and

$$A_{rbxrb} = \begin{pmatrix} I_r - \frac{1}{r}J_r^r & & \phi \\ & \ddots & \\ \phi & & I_r - \frac{1}{r}J_r^r \end{pmatrix} .$$

Observe that A is a symmetric idempotent matrix of rank $b(r-1)$.

$$Y \overset{a.d.}{\sim} N\left[\mu' = \left(\pi_{..(1)}, \ldots, \pi_{..(1)}, \ldots, \pi_{..(b)}, \ldots, \pi_{..(b)}\right), \right.$$

$$\left. \Sigma = \left\{\frac{1}{nbrs^2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)\right\} I_{rb}\right] .$$

Now, by Theorems 1 and 3 (in Appendix), $N \overset{a.d.}{\sim} \chi^2\left[b(r-1)\right]$. By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Hence by Theorem 4, $B_{1.(g)} = \frac{N}{D} \xrightarrow{dist} \chi^2\left[b(r-1)\right]$ approximately, under H_0.

Under H_A,

$$Y \overset{a.d.}{\sim} N\left[\mu' = \left(\pi_{1.(1)}, \ldots, \pi_{r.(1)}, \ldots, \pi_{1.(b)}, \ldots, \pi_{r.(b)}\right), \right.$$

$$\left. \Sigma = \left\{\frac{1}{nbrs^2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)\right\} I_{rb}\right] .$$

Then by Theorems 1 and 3, $N \overset{a.d.}{\sim} \chi'^2\left[b(r-1), \lambda_{B_{1.(g)}}\right]$, where

$$\lambda_{B_{1.(g)}} = \frac{nbrs^2 \sum_{g=1}^{b} \sum_{i=1}^{r} \left(\pi_{i.(g)} - \pi_{..(g)}\right)^2}{2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)} .$$

$D \xrightarrow{prob} 1$, by Theorems 5 and 6. Hence by Theorem 4, $B_{1.(g)} = \frac{N}{D} \xrightarrow{dist} \chi'^2 \left[b(r-1), \lambda_{B_{1.(g)}}\right]$ approximately.

Notes: Notice that substitution of $s = 1$ in the statistic $B_{1.(g)}$ will give the statistic B_2 proposed in Chapter III to test the equality of treatment effects block-wise in the usual two-way classification with n binary observations per cell.

By making appropriate changes in the statistic $B_{1.(g)}$ and in its derived approximate asymptotic distributions, one can obtain the statistic $B_{.2(g)}$ and its approximate asymptotic distributions, to test the levels of factor B, averaged over the levels of factor A, within each block. To obtain the statistic $B_{.2(g)}$, one has to change i to j, r to s, s to r, and $\hat{\pi}_{i.(g)}$ to $\hat{\pi}_{.j(g)}$, in the numerator of the statistic $B_{1.(g)}$ given by (4.1).

Testing the Levels of Factor A
Average-wise

Consider the problem of testing for the levels of factor A, averaged over blocks and the levels of factor B, i.e.

H_0: $\pi_{1.(.)} = \pi_{2.(.)} = \cdots = \pi_{r.(.)} = (\pi \text{ say})$.

The following test statistic, $B_{1.(.)}$ is proposed to test the above hypothesis:

$$B_{1.(.)} = \frac{nrs^2b^2 \sum_{i=1}^{r} \left(\hat{\pi}_{i.(.)} - \hat{\hat{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right)} . \quad (4.2)$$

The statistic $B_{1.(.)}$ can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_{1.(.)} = \frac{rsb \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{i.(.)} - \hat{\hat{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right) \Big/ n}$$

$$= nrsb \frac{\text{factor A SS}}{\text{experimental error SS}}$$

$$= \frac{(r-1)n}{n-1} \frac{\text{factor A MS}}{\text{experimental error MS}}$$

$$\frac{(r-1)n}{n-1} \; F\left[(r-1),\; rsb(n-1)\right] .$$

The approximate asymptotic null distribution of $B_{1.(.)}$ is central chi-square with (r-1) degrees of freedom, while its approximate asymptotic alternative distribution is non-central chi-square with (r-1) degrees of freedom and non-centrality parameter is

$$\lambda_{B_{1.(.)}} = \frac{nrs^2b^2 \sum_{i=1}^{r} \left(\pi_{i.(.)} - \bar{\pi}\right)^2}{2\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)} \quad .$$

The derivation is given below:

Derivation:

$$\text{var}\left(\hat{\pi}_{i.(.)}\right) = \text{var}\left(\sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}/sb\right)$$

$$= \frac{1}{nb^2s^2} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}\left(1 - \hat{\pi}_{ij(g)}\right) .$$

Then under H_0,

$$\hat{\pi}_{i.(.)} \overset{a.d.}{\sim} N\left[\pi = \pi_{i.(.)} = \sum_{g=1}^{b} \pi_{i.(g)}/b \right.,$$

$$\left. \frac{1}{nb^2s^2} \sum_{g=1}^{b} \sum_{j=1}^{s} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)\right] .$$

Notice the inequality of the variances of $\hat{\pi}_{i.(.)}$'s, $i = 1, \ldots, r$. From this, it can be observed that the construction of a "legitimate test" is difficult or may not be possible. However, it was decided to approximate the asymptotic distribution of $\hat{\pi}_{i.(.)}$ by

$$\hat{\pi}_{i.(.)} \overset{a.d.}{\approx} N\left[\pi_{i.(.)}, c\right] \quad \text{where c is such that}$$

$$\sum_{i=1}^{r} \left[\frac{1}{nb^2s^2} \sum_{g=1}^{b} \sum_{j=1}^{s} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right) - c\right]^2$$

is minimum.

Choose

$$c = \frac{1}{nrb^2s^2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right) .$$

By using c, it can be seen that the variance of $\hat{\pi}_{i.(.)}$'s are being pooled in order to get a common variance.

Now, $B_{1.(.)}$ can be written as $B_{1.(.)} = N/D$, where

$$N = \frac{nrs^2b^2 \sum_{i=1}^{r} \left(\hat{\pi}_{i.(.)} - \hat{\hat{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)}$$

$$\text{and } D = \frac{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)}\left(1 - \hat{\pi}_{ij(g)}\right)/nrs^2b^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)/nrs^2b^2} .$$

Notice that N can be expressed in a quadratic form as

$$nrs^2b^2\ Y'AY \Big/ \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right), \text{ where}$$

$$Y' = \left(\hat{\pi}_{1.(.)}, \ldots, \hat{\pi}_{r.(.)}\right) \text{ and } A = \left(I_r - \frac{1}{r}J_r^r\right) .$$

Observe that A is a symmetric idempotent matrix of rank (r-1).

After pooling the variances of $\hat{\pi}_{i.(.)}$'s under H_0,

$$Y \overset{a.d.}{\sim} N\left[\pi J_1^r, \left\{\frac{1}{nrs^2b^2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)\right\} I_r\right]$$

Now by Theorems 1 and 3 (in Appendix), $N \overset{a.d.}{\sim} \chi^2(r-1)$. By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Hence, by Theorem 4, $B_{1.(.)} = \frac{N}{D} \xrightarrow{dist} \chi^2(r-1)$ approximately, under H_0.

Under H_A,

$$\underset{\sim}{Y} \overset{a.d.}{\sim} N\left[\left(\pi_{1.(.)},\ldots,\pi_{r.(.)}\right)',\left\{\frac{1}{nrs^2b^2}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{g=1}^{b}\pi_{ij(g)}\times\right.\right.$$

$$\left.\left.\left(1-\pi_{ij(g)}\right)\right\}I_r\right] .$$

Then by Theorems 1 and 3, $N \overset{a.d.}{\sim} \chi'^2\left[(r-1),\lambda_{B_{1.(.)}}\right]$,

where

$$\lambda_{B_{1.(.)}} = \frac{nrs^2b^2\sum_{i=1}^{r}\left(\pi_{i.(.)}-\bar{\pi}\right)^2}{2\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{g=1}^{b}\pi_{ij(g)}\left(1-\pi_{ij(g)}\right)} .$$

As before, $D \xrightarrow{prob} 1$, by Theorems 5 and 6. Hence, by Theorem 4, $B_{1.(.)} = \frac{N}{D} \xrightarrow{dist} \chi'^2\left[(r-1),\lambda_{B_{1.(.)}}\right]$, approximately.

<u>Notes:</u>

A substitution of $b = 1$ in this test will give a test for testing the levels of factor A average-wise in one-way classification with n binary observations per treatment, which in this case is equivalent to testing the levels of factor A block-wise. Notice that for $s = 1$, the statistic $B_{1.(.)}$ becomes identical to the statistic B_3 proposed in Chapter III to test the treatments average-wise.

By making appropriate changes in the statistic $B_{1.(.)}$ and in its derived approximate asymptotic distributions, one can obtain the statistic $B_{.2(.)}$ and its approximate asymptotic distributions to test the levels of factor B, averaged over blocks and the levels of factor A. To obtain the statistic $B_{.2(.)}$, one has to change i to j, r to s, s to r, and $\hat{\pi}_{i.(.)}$ to $\hat{\pi}_{.j(.)}$, in the numerator of the statistic $B_{1.(.)}$ given by (4.2).

Test for Factor A x Factor B Interaction

Consider a problem of testing for factor A x factor B interaction (averaged over blocks). The hypothesis of no factor A x factor B interaction can be expressed in mathematical terms as follows:

H_0: $\pi_{ij(.)} - \pi_{i.(.)} - \pi_{.j(.)} + \bar{\pi} = 0$ for all i and j.

The following test statistic, $B_{12(.)}$, is proposed to test the above hypothesis:

$$B_{12(.)} = \frac{nrsb^2 \sum_{j=1}^{s} \sum_{i=1}^{r} \left(\hat{\pi}_{ij(.)} - \hat{\pi}_{i.(.)} - \hat{\pi}_{.j(.)} + \hat{\pi}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right)} . \tag{4.3}$$

The statistic $B_{12(.)}$ can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_{12(.)} = \frac{rsb \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{ij(.)} - \hat{\pi}_{i.(.)} - \hat{\pi}_{.j(.)} + \hat{\pi}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right)/n}$$

$$= nrsb \cdot \frac{\text{factor A x factor B interaction SS}}{\text{experimental error SS}}$$

$$= \frac{n(r-1)(s-1)}{n-1} \; \frac{\text{factor A x factor B interaction MS}}{\text{experimental error MS}}$$

$$= \frac{n(r-1)(s-1)}{n-1} \; F\left[(r-1)(s-1),\ (n-1)rsb\right] .$$

The approximate asymptotic null distribution of $B_{12(.)}$ is central chi-square with (r-1)(s-1) degrees of freedom and its approximate asymptotic alternative distribution is non-central chi-square with (r-1)(s-1) degrees of freedom and the non-centrality parameter is

$$\lambda_{B_{12(.)}} = \frac{nrsb^2 \sum_{j=1}^{s} \sum_{i=1}^{r} \left(\pi_{ij(.)} - \pi_{i.(.)} - \pi_{.j(.)} + \bar{\pi}\right)^2}{2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right)} .$$

The derivation is given below:

<u>Derivation:</u>

Under H_0 (or H_A), $\hat{\pi}_{ij(i)} \overset{a.d.}{\sim} N\left[\pi_{ij(.)}, \frac{1}{nb^2} \sum_{g=1}^{b} \pi_{ij(g)} \times \left(1 - \pi_{ij(g)}\right)\right] .$

Notice the inequality of variances of $\hat{\pi}_{ij(.)}$'s. This fact causes problems. In order to avoid some problems, it was decided to approximate the asymptotic distribution of $\hat{\pi}_{ij(.)}$ by

$$\hat{\pi}_{ij(.)} \overset{a.d.}{\sim} N\left[\pi_{ij(.)}, c\right] ,$$ where c is such that

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \left[\frac{1}{nb^2} \sum_{g=1}^{b} \pi_{ij(g)}\left(1 - \pi_{ij(g)}\right) - c\right]^2$$ is minimum.

Such c is given by

$$c = \frac{1}{nrsb^2} \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)} \left(1 - \pi_{ij(g)}\right) .$$

Here, the pooling of variances of $\hat{\pi}_{ij(.)}$'s is subject to criticism.

The statistic $B_{12(.)}$ can be expressed as $B_{12(.)} = N/D$, where

$$N = \frac{nrsb^2 \sum_{j=1}^{s} \sum_{i=1}^{r} \left(\hat{\pi}_{ij(.)} - \hat{\pi}_{i.(.)} - \hat{\pi}_{.j(.)} + \hat{\bar{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)} \left(1 - \pi_{ij(g)}\right)}$$

and

$$D = \frac{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \hat{\pi}_{ij(g)} \left(1 - \hat{\pi}_{ij(g)}\right) / nrsb^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)} \left(1 - \pi_{ij(g)}\right) / nrsb^2} .$$

Notice that N can be written in a quadratic form as

$$N = \frac{nrsb^2 Y'AY}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{g=1}^{b} \pi_{ij(g)} \left(1 - \pi_{ij(g)}\right)} , \text{ where}$$

$$Y' = (\hat{\pi}_{11(.)}, \ldots, \hat{\pi}_{1s(.)}, \ldots, \hat{\pi}_{r1(.)}, \ldots, \hat{\pi}_{rs(.)})$$

and

$$A = I_{rs} - \frac{1}{s} \begin{pmatrix} J_s^s & & \phi \\ & \ddots & \\ \phi & & J_s^s \end{pmatrix} - \frac{1}{r} \begin{pmatrix} I_s & \cdots & I_s \\ \vdots & \ddots & \vdots \\ I_s & \cdots & I_s \end{pmatrix} + \frac{1}{rs} J_{rs}^{rs} .$$

One can show that A is a symmetric idempotent matrix of rank $(r-1)(s-1)$.

After pooling the variances of $\hat{\pi}_{ij(\cdot)}$'s, under H_0 (or H_A),

$$Y_{rs\times 1} \overset{a.d.}{\sim} N\left[\mu' = \left(\hat{\pi}_{11(.)},\ldots,\hat{\pi}_{1s(.)},\ldots\hat{\pi}_{r1(\cdot)},\ldots,\hat{\pi}_{rs(.)}\right),\right.$$

$$\left.\Sigma = \left\{\frac{1}{nrsb^2}\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{g=1}^{b}\pi_{ij(g)}\left(1-\pi_{ij(g)}\right)\right\} I_{rs}\right].$$

Now by Theorems 1 and 3, (in Appendix), $N \overset{a.d.}{\sim} \chi^2\left[(r-1)(s-1)\right]$. By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Hence by Theorem 4, $B_{12(.)} = \frac{N}{D} \xrightarrow{dist} \left[\chi^2(r-1)(s-1)\right]$ approximately, under H_0. By Theorems 1 and 3 under H_A,

$$N \overset{a.d.}{\sim} \chi'^2\left[(r-1)(s-1),\lambda_{B_{12(.)}}\right], \text{ where}$$

$$\lambda_{B_{12(\cdot)}} = \frac{nrsb^2\left(\sum_{j=1}^{s}\sum_{i=1}^{r}\pi_{ij(.)} - \pi_{i.(.)} - \pi_{\cdot j(.)} - \bar{\pi}\right)^2}{2\sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{g=1}^{b}\pi_{ij(g)}\left(1-\pi_{ij(g)}\right)}.$$

By Theorems 5 and 6, $D \xrightarrow{prob} 1$. Then by Theorem 4, $B_{12(.)} = \frac{N}{D} \xrightarrow{dist} \chi'^2\left[(r-1)(s-1),\lambda_{B_{12(.)}}\right]$ approximately, under H_A.

Note: By substituting b = 1 and then considering the levels of factor B as blocks, the statistic $B_{12(.)}$ reduces to the statistic B_{bxt} of Chapter III.

Three Factors

Now consider 3 factors, A, B, and C, at r, s, and ℓ levels, respectively. The notations used for 2 factors will be extended here for 3 factors in an obvious manner. The following layout of probabilities would help in understanding the notations and in visualizing the situation.

Block ↓	Treatment Combinations (1,1,1),...,(1,s,ℓ)	...	(r,1,1),...,(r,s,ℓ)	Block Prob.
1	$\hat{\pi}_{1..(.)}$	...	$\hat{\pi}_{r..(1)}$	$\hat{\pi}_{...(1)}$
2	$\hat{\pi}_{1..(2)}$	...	$\hat{\pi}_{r..(2)}$	$\hat{\pi}_{...(2)}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
b	$\hat{\pi}_{1..(b)}$	...	$\hat{\pi}_{r..(b)}$	$\hat{\pi}_{...(b)}$
Trt. Prob.	$\hat{\pi}_{1..(.)}$	...	$\hat{\pi}_{r..(.)}$	$\hat{\bar{\pi}}$

It is assumed that the response in one (treatment combination x block) cell is independent of the responses in other (treatment combination x block) cells.

Testing the Levels of a Given Factor

Block-wise

Under the present situation, it might be of interest to test for the levels of a given factor (say A), averaged over levels of other two factors (B and C), within each block, i.e.

H_0: $\pi_{1..(g)} = \pi_{2..(g)} = \cdots = \pi_{r..(g)}$ $\left(= \pi_{...(g)} \text{ say}\right)$ for all g. This hypothesis is equivalent to that of testing the main effect of factor A and the A x block interaction effect, both to be zero, in the regular analysis of variance structure.

The following test statistic, $B_{1..(g)}$, is proposed to test this hypothesis.

$$B_{1..(g)} = \frac{nbrs^2\ell^2 \sum_{g=1}^{b} \sum_{i=1}^{r} \left(\hat{\pi}_{i..(g)} - \hat{\pi}_{...(g)}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right)} \quad (4.4)$$

The statistic $B_{1..(g)}$ can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_{1..(g)} = \frac{brs\ell \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{i..(g)} - \hat{\pi}_{...(g)}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right)/n}$$

$$= nbrs\ell \frac{\text{factor A within blocks SS}}{\text{experimental error SS}}$$

$$= \frac{bn(r-1)}{n-1} \; \frac{\text{factor A within blocks MS}}{\text{experimental error MS}}$$

$$= \frac{bn(r-1)}{n-1} \; F\left[b(r-1),\ (n-1)rs\ell b\right] \quad .$$

The approximate asymptotic null distribution of the statistic $B_{1..(g)}$ is central chi-square with b(r-1) degrees of freedom, while its approximate asymptotic alternative distribution is non-central chi-square with b(r-1) degrees of freedom and the non-centrality parameter is

$$\lambda_{B_{1..(g)}} = \frac{nbrs^2\ell^2 \sum_{g=1}^{b} \sum_{i=1}^{r} \left(\pi_{i..(g)} - \pi_{...(g)}\right)^2}{2\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \pi_{ijk(g)}\left(1 - \pi_{ijk(g)}\right)} .$$

Derivation:

The distributional derivation of the statistic $B_{1..(g)}$ can be obtained from that of the statistic $B_{1.(g)}$ by its straight forward extension.

Notes: A substitution of $\ell = 1$ in the statistic $B_{1..(g)}$ will give back the statistic $B_{1.(g)}$.

By making appropriate changes in the statistic $B_{1..(g)}$ and in its derived asymptotic distributions, one can obtain the statistic $B_{.2.(g)}$ (or $B_{..3(g)}$) and its asymptotic distributions to test the levels of factor B (or C), averaged over the levels of factors A and C (or A and B), within each block. In order to obtain the statistic $B_{.2.(g)}$, one has to change i to j, r to s, s to r, and $\hat{\pi}_{i..(g)}$ to $\hat{\pi}_{.j.(g)}$, in the numerator of the statistic $B_{1..(g)}$ given by (4.4). To obtain the statistic $B_{..3(g)}$, one has to change i to k, r to ℓ, ℓ to r and $\hat{\pi}_{i..(g)}$ to $\hat{\pi}_{..k(g)}$, in the numerator of the statistic $B_{1..(g)}$ given by (4.4).

Testing the Levels of a Given Factor

Average-wise

Consider the problem of testing for the levels of a given factor (say A), averaged over blocks and the levels of other two factors (B and C), i.e.,

$$H_0: \pi_{1..(.)} = \pi_{2..(.)} = \cdots = \pi_{r..(.)} \left(= \pi \text{ say}\right).$$

The following test statistic $B_{1..(.)}$, is proposed to test the above hypothesis:

$$B_{1..(.)} = \frac{nrb^2 s^2 \ell^2 \sum_{i=1}^{r} \left(\hat{\pi}_{i..(.)} - \hat{\bar{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right)} \quad . \quad (4.5)$$

The statistic $B_{1..(.)}$ can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_{1..(.)} = \frac{rs\ell b \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{i...(.)} - \hat{\bar{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right) / n}$$

$$= nrs\ell b \, \frac{\text{factor A SS}}{\text{experimental error SS}}$$

$$= \frac{n(r-1)}{n-1} \, \frac{\text{factor A MS}}{\text{experimental error MS}}$$

$$= \frac{n(r-1)}{n-1} \; F\left[(r-1,\ (n-1)rs\ell b\right] .$$

The approximate asymptotic null distribution of the statistic $B_{1..(.)}$ is central chi-square with (r-1) degrees of freedom while its approximate asymptotic alternative distribution is non-central chi-square with (r-1) degrees of freedom and non-centrality parameter is

$$\lambda_{B_{1..(.)}} = \frac{nrb^2 s^2 \ell^2 \sum_{i=1}^{r} \left(\pi_{i..(.)} - \bar{\pi}\right)^2}{2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \pi_{ijk(g)} \left(1 - \pi_{ijk(g)}\right)} .$$

Derivation:

The distributional derivation of the statistic $B_{1..(.)}$ can be obtained from that of $B_{1.(.)}$ by a straightforward extension.

Notes: A substitution of $\ell = 1$ in the statistic $B_{1..(.)}$ will give back the statistic $B_{1.(.)}$.

By making appropriate changes in the statistic $B_{1..(.)}$ and in its derived asymptotic distributions, one can obtain the statistic $B_{.2.(.)}$ (or $B_{..3(.)}$) and its asymptotic distributions to test the levels of factor B (or C), averaged over blocks and the levels of factors A and C (or A and B). In order to obtain the statistic $B_{.2.(.)}$, one has to change i to j, r to s, s to r, and $\hat{\pi}_{i..(.)}$ to $\hat{\pi}_{.j.(.)}$, in the numerator of the statistic $B_{1..(.)}$ given by (4.5). Similarly, to obtain the statistic $B_{..3(.)}$, one has to change i to k, r to ℓ, ℓ to r, and $\hat{\pi}_{i..(.)}$ to $\hat{\pi}_{..k(.)}$, in the numerator of the statistic $B_{..1(.)}$ given by (4.5).

Test for Two-Factor Interaction

Consider testing for a two-factor interaction (averaged over third factor and blocks). To start with, say one wants to test for factor A x factor B interaction (averaged over factor C and blocks). This hypothesis in mathematical terms can be stated as

$$H_0: \pi_{ij.(.)} - \pi_{i..(.)} - \pi_{.j.(.)} + \bar{\pi} = 0 \text{ for all } i \text{ and } j.$$

The following test statistic, $B_{12.(.)}$, is proposed to test the above hypothesis:

$$B_{12.(.)} = \frac{nrs\ell^2 b^2 \sum_{i=1}^{r} \sum_{j=1}^{s} \left(\hat{\pi}_{ij.(.)} - \hat{\pi}_{i..(.)} - \hat{\pi}_{.j.(.)} + \hat{\bar{\pi}}\right)^2}{\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right)} . \tag{4.6}$$

The statistic $B_{12.(.)}$ can be expressed as follows in terms of the components from the regular analysis of variance table:

$$B_{12.(.)} = \left[rs\ell b \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \sum_{m=1}^{n} \left(\hat{\pi}_{ij.(.)} - \hat{\pi}_{i..(.)} - \pi_{.j.(.)} + \hat{\bar{\pi}}\right)^2\right] \div \left[\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \sum_{m=1}^{n} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right)/n\right]$$

$$= nrs\ell b \, \frac{\text{factor A x factor B interaction SS}}{\text{experimental error SS}}$$

$$= \frac{n(r-1)(s-1)}{n-1} \quad \frac{\text{factor A x factor B interaction MS}}{\text{experimental error MS}}$$

$$= \frac{n(r-1)(s-1)}{n-1} \quad F\left[(r-1)(s-1),\ (n-1)rs\ell b\right] \ .$$

The approximate asymptotic null distribution of the statistic $B_{12.(.)}$ is central chi-square with (r-1)(s-1) degrees of freedom, while its approximate asymptotic alternative distribution is non-central chi-square with (r-1)(s-1) degrees of freedom and the non-centrality parameter is

$$\lambda_{B_{12.(.)}} = \frac{nrs\ell^2 b^2 \sum_{i=1}^{r} \sum_{j=1}^{s} \left(\pi_{ij.(.)} - \pi_{i..(.)} - \pi_{.j.(.)} + \bar{\pi}\right)^2}{2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \pi_{ijk(g)} \left(1 - \pi_{ijk(g)}\right)} \ .$$

Derivation:

The distributional derivation of the statistic $B_{12.(.)}$ can be obtained from that of $B_{12(.)}$ by its straight forward extension.

Notes: A substitution of $\ell = 1$ in the statistic $B_{12.(.)}$ will give back the statistic $B_{12(.)}$.

By making appropriate changes in the statistic $B_{12.(.)}$ and in its derived asymptotic distributions, one can obtain the statistic $B_{1.3(.)}$ (or $B_{.23(.)}$) and its asymptotic distributions to test factor A x factor C interaction (or factor B x factor C interaction). In order to obtain the statistic $B_{1.3(.)}$, one has to change j to k, s to ℓ, ℓ to s, $\hat{\pi}_{ij.(.)}$ to $\hat{\pi}_{i.k(.)}$, and $\hat{\pi}_{.j.(.)}$ to $\hat{\pi}_{..k(.)}$, in the numerator of the statistic $B_{12.(.)}$ given by (4.6). Similarly, to get $B_{.23(.)}$, one has to change i to k, r to ℓ, ℓ to r, $\hat{\pi}_{ij.(.)}$ to $\hat{\pi}_{.jk(.)}$, and $\hat{\pi}_{i..(.)}$ to $\hat{\pi}_{..k(.)}$, in the numerator of the statistic $B_{12.(.)}$ given by (4.6).

Test for Three Factor Interaction

Now consider testing for the three factor interaction (averaged over blocks). In mathematical terms, this hypothesis can be stated as

$$H_0: \pi_{ijk(.)} - \pi_{ij.(.)} - \pi_{i.k(.)} - \pi_{.jk(.)} + \pi_{i..(.)} + \pi_{.j.(.)} + \pi_{..k(.)} - \bar{\pi} \left(= L \text{ say}\right) = 0 \text{ for all } i,\ j, \text{ and } k.$$

The following test statistic, $B_{123(.)}$, is proposed to test the above hypothesis:

$$B_{123(.)} = nrs\ell b^2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \left(\hat{\pi}_{ijk(.)} - \hat{\pi}_{ij.(.)} - \hat{\pi}_{i.k(.)} - \hat{\pi}_{.jk(.)} + \hat{\pi}_{i..(.)} + \hat{\pi}_{.j.(.)} + \hat{\pi}_{..k(.)} - \hat{\bar{\pi}}\right)^2 \div \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \hat{\pi}_{ijk(g)} \left(1 - \hat{\pi}_{ijk(g)}\right) . \quad (4.7)$$

If one is to construct an analysis of variance table for this situation, then $B_{123(.)}$ can be expressed in terms of the components from analysis of variance as follows:

$$B_{123(.)} = nrs\ell b \, \frac{\text{factors A x B x C interaction SS}}{\text{experimental error SS}}$$

$$= \frac{n(r-1)(s-1)(\ell-1)}{n-1} \, \frac{\text{factors A x B x C interaction MS}}{\text{experimental error MS}}$$

$$= \frac{n(r-1)(s-1)(\ell-1)}{n-1} F\left[(r-1)(s-1)(\ell-1),(n-1)\ rs\ell b\right].$$

The approximate asymptotic null distribution of the statistic $B_{123(.)}$ is central chi-square with $(r-1)(s-1)(\ell-1)$ degrees of freedom, while its approximate asymptotic alternative distribution is non-central chi-square with $(r-1)(s-1)$ $(\ell-1)$ degrees of freedom and non-centrality parameter is

$$\lambda_{B_{123(.)}} = \frac{nrs\ell b^2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} (L)^2}{2 \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \sum_{g=1}^{b} \pi_{ijk(g)}\left(1 - \pi_{ijk(g)}\right)},$$

where L is as defined in the above H_0.

Derivation: The distributional derivation of the statistic $B_{123(.)}$ can be obtained from that of $B_{12(.)}$ by its straight forward generalization. The following information will be very useful in order to do such a generalization.

Consider Y to be a vector of the order $rs\ell x1$ such that

$$Y' = \left(\pi_{111(.)},\ldots,\pi_{11\ell(.)},\pi_{121(.)},\ldots,\pi_{12\ell(.)},\ldots,\pi_{1s1(.)},\ldots,\pi_{1s\ell(.)},\ldots,\pi_{r11(.)},\ldots,\pi_{r1\ell(.)},\pi_{r21(.)},\ldots,\pi_{r2\ell(.)},\ldots,\pi_{rs1(.)},\ldots,\pi_{rs\ell(.)}\right)_{1 \times rs\ell}.$$

Then

$$\sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{\ell} \left(\hat{\pi}_{ijk(.)} - \hat{\pi}_{ij.(.)} - \hat{\pi}_{i.k(.)} - \hat{\pi}_{.jk(.)} + \hat{\pi}_{i..(.)} + \hat{\pi}_{.j.(.)} + \hat{\pi}_{..k(.)} - \hat{\bar{\pi}}\right)^2$$

can be expressed in a quadratic as Y'AY where

$$A = M_T - M_{AB} - M_{AC} - M_{BC} + M_A + M_B + M_C - M_{C.F}\text{, where}$$

$$M_T = I_{rs\ell},\ M_{C.F} = \frac{1}{rs\ell}\ J_{rs\ell}^{rs\ell}\ ,$$

$$M_A = \frac{1}{s\ell}\begin{pmatrix} J_{s\ell}^{s\ell} & & \phi \\ & \ddots & \\ \phi & & J_{s\ell}^{s\ell} \end{pmatrix},\quad M_C = \frac{1}{rs}\begin{pmatrix} I_\ell & \cdots & I_\ell \\ \vdots & \ddots & \vdots \\ I_\ell & \cdots & I_\ell \end{pmatrix},$$

$$M_{BC} = \frac{1}{r}\begin{pmatrix} I_{s\ell} & \cdots & I_{s\ell} \\ \vdots & \ddots & \vdots \\ I_{s\ell} & & I_{s\ell} \end{pmatrix},\quad M_{AB} = \frac{1}{\ell}\begin{pmatrix} J_\ell^\ell & & \phi \\ & \ddots & \\ \phi & & J_\ell^\ell \end{pmatrix},$$

$$M_B = \frac{1}{r\ell}\left(\begin{array}{ccc|c|ccc} J_\ell^\ell & & \phi & & J_\ell^\ell & & \phi \\ & \ddots & & \cdots & & \ddots & \\ \phi & & J_\ell^\ell & & \phi & & J_\ell^\ell \\ \hline & \vdots & & \vdots & & \vdots & \\ \hline J_\ell^\ell & & \phi & & J_\ell^\ell & & \phi \\ & \ddots & & \cdots & & \ddots & \\ \phi & & J_\ell^\ell & & \phi & & J_\ell^\ell \end{array}\right)_{s\ell \times rs\ell},$$

and

$$
M_{AC} = \frac{1}{s}\left(\begin{array}{ccccc|c|ccccc}
I_\ell & \cdots & I_\ell & & & & & & & & \\
\vdots & \ddots & \vdots & & & \phi & & & \phi & & \\
I_\ell & \cdots & I_\ell & & & & & & & & \\
\hline
& \phi & & & & \ddots & & & \phi & & \\
\hline
& & & & & & I_\ell & \cdots & I_\ell & & \\
& \phi & & & & \phi & \vdots & \ddots & \vdots & & \\
& & & & & & I_\ell & \cdots & I_\ell & &
\end{array}\right)_{s\ell \times rs\ell} .
$$

One can show that A is a symmetric idempotent matrix of rank $(r-1)(s-1)(\ell-1)$.

<u>Notes:</u> One can easily generalize the techniques proposed in this chapter for any number of factors at any number of levels.

<u>Comments:</u> A substitution of $b = 1$ in the test procedures proposed in this chapter will give the corresponding test procedures for a balanced one-way classification with binary responses. In this case $(b = 1)$, it is obvious that the test procedure for testing the levels of a given factor block-wise is equivalent to that of testing the levels of that factor average-wise (averaged over blocks).

CHAPTER V

POSSIBLE EXTENSIONS

In the previous chapters, some transformation-free test procedures for testing the various kinds of hypotheses in the balanced one-way and two-way classifications (including factorial arrangements of treatments and possible interactions) with binary responses are proposed. Following are some possible extensions to this work which demand further attention:

1. Note that in order to arrive at the approximate asymptotic null and alternative distributions of some of the test statistics, pooling of the unequal variances was employed with the criterion of a minimum error, to get a constant variance. This may or may not give a "satisfactory" approximation. Hence the immediate thing which needs to be done is to check out the consequences of such pooling, at least under the null hypotheses. It seems that the general analytic conclusions will probably be hard to reach and one will have to do some simulation studies considering only particular cases. As a result, the

conclusions will be restricted to those particular cases only.

2. Since the present work is limited for the balanced one-way and two-way classifications only, it needs to be generalized for the balanced multi-way classifications. It is the author's opinion that this is a very simple and straight forward generalization which does not require a great deal of work.

3. Since all of the test procedures proposed up to this stage are restricted to the balanced cases, it certainly would be of interest to expand them gradually for the unbalanced multi-way classifications. It would be very nice if this were possible without the use of transformations. If not, then the following procedure might be of help.

Assume a two-way classification with N_{ij} observations per cell; $i = 1,\ldots,t$ and $j = 1,\ldots,b$. Then consider,

$$\ln\left(\frac{p_{ij}}{1-p_{ij}}\right) = p + \alpha_i + \beta_i + \gamma_{ij} \ ,$$

where

$$\sum_{i=1}^{t} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{t} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0 \ .$$

Then the parameter model is

$$p_{ij} = \frac{e^{p+\alpha_i+\beta_j+\gamma_{ij}}}{1+e^{p+\alpha_i+\beta_j+\gamma_{ij}}}, \left(\Rightarrow q_{ij} = \frac{1}{1+e^{p+\alpha_i+\beta_j+\gamma_{ij}}}\right).$$

Now, the full probability model (likelihood equation), L, can be shown to be a member of the exponential family in the following manner:

$$L = \prod_{i=1}^{t} \prod_{j=1}^{b} p_{ij}^{n_{ij}} \left(1 - p_{ij}\right)^{N_{ij} - n_{ij}}$$

$$= \prod_{i=1}^{t} \prod_{j=1}^{b} \left(\frac{p_{ij}}{1-p_{ij}}\right)^{n_{ij}} \left(1 - p_{ij}\right)^{N_{ij}}$$

$$= \prod_{i=1}^{t} \prod_{j=1}^{b} \left(1 - p_{ij}\right)^{N_{ij}} \left\{ e^{\sum_{i=1}^{t} \sum_{j=1}^{b} \left(p+\alpha_i+\beta_j+\gamma_{ij}\right) n_{ij}} \right\}$$

$$= \left\{ \prod_{i=1}^{t} \prod_{j=1}^{b} \left(1 - p_{ij}\right)^{N_{ij}} \right\}$$

$$\times \left\{ e^{p n_{..} + \sum_{i=1}^{t} \alpha_i n_{i.} + \sum_{j=1}^{b} \beta_j n_{.j} + \sum_{i=1}^{t} \sum_{j=1}^{b} n_{ij} \gamma_{ij}} \right\},$$

with the canonical parameters p, α_i, β_j, and γ_{ij} for i = 1,...,t, and j = 1,...,b. A minimal sufficient statistic for these parameters is n_{ij} for all i and j.

In the absence of block x treatment interaction, testing $\alpha_i = 0$ for all i will imply no treatment effects and similarly, testing $\beta_j = 0$ for all j will imply no block effects and so on.

4. Throughout the work presented in this dissertation, responses are assumed to be of a binary nature and hence it would be a step ahead if the present work is gradually generalized for the balanced and finally for the unbalanced multi-way classifications with categorical data.
5. A simulation study for the comparison of the BIANOVA test with chi-square and F tests, done in Chapter II, is not quite sufficient. This study can certainly be continued for different choices of n, t, and probability structure. Also, for a given probability structure and a value of t, one can investigate the minimum value of n for which all three tests, BIANOVA, chi-square, and F, attain the fixed desired α levels under H_0. However, from this study, general conclusions will be hard to make.
6. Some ideas have been sketched in Chapter II which might be of great help in proving that the orderings of a collection of data sets given by the B and χ^2 statistics are the same. Further study in this direction can be continued also.

In the literature, analysis of two-way or multi-way classifications with binary or categorical data is not given as much attention as that given to the multi-way contingency tables. The work presented in this dissertation represents a beginning in this direction of the analysis of unbalanced multi-way classifications with binary or categorical observations. It is the author's plan to observe this work more critically and then to work with possible extensions.

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APPENDIX

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Theorems

Theorem 1: If Y is distributed $N(\mu, \sigma^2 I)$, then $Y'AY/\sigma^2$ is distributed as $\chi'^2(k,\lambda)$, where $\lambda = \mu' A\mu/2\sigma^2$, if and only if A is an idempotent matrix of rank k.

Theorem 2: If A is idempotent of rank r, then trace (A) = r.

Theorem 3. If g is a continuous function and $x_n \overset{dist}{\rightarrow} X$, then $g(X_n) \overset{dist}{\rightarrow} g(X)$.

Theorem 4: (Cramer's Theorem)

Suppose $X_n \overset{dist}{\rightarrow} X$ and $Y_n \overset{prob}{\rightarrow} c$ (constant), then

(1) $X_n + Y_n \overset{dist}{\rightarrow} X + c$.

(2) $X_n/Y_n \overset{dist}{\rightarrow} X/c$ if $c \neq 0$.

(3) $X_n Y_n \overset{dist}{\rightarrow} cX$.

Theorem 5: (Tchebycheff's Theorem)

Let $\xi_1, \xi_2, \ldots,$ be random variables, and let m_n and σ_m denote the mean and the s.d. of ξ_n. If $\sigma_n \rightarrow 0$ as $n \rightarrow \infty$, then $\xi_n - m_n$ converges in probability to zero.

Theorem 6: If $\xi_n, \eta_n, \ldots, \rho_n$ are random variables converging in probability to the constants x,y,...r, respectively, any rational function $R(\xi_n, \eta_n, \ldots, \rho_n)$ converges in probability to the constant R(x,y ...,r), provided that the latter is finite. It follows that any power $R^k(\xi_n, \eta_n, \ldots, \rho_n)$ with $k > 0$ converges in probability to $R^k(x, y, \ldots, r)$.

Note: These theorems are taken from the texts by Rao (1973), Graybill (1961), and Cramer (1966).

Definitions

Definition 1: Convergence in Probability

Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables. We say that ξ_n converges in probability to a constant c, if for any $\varepsilon > 0$, the probability of the relation $|\xi_n - c| > \varepsilon$ tends to zero as $n \to \infty$.

Definition 2: Convergence in Distribution

A sequence of random variables $Z_1, Z_2, \ldots$ converges in distribution to the random variable with distribution function F whenever $\lim_{n\to\infty} F_n(x) = F(x)$ for all points of continuity of F, where F_n is the distribution function of Z_n.

Definition 3: Rational Function

A rational function f(x) is any function that can be expressed as the quotient of two polynomials, i.e. $f(x) = g(x)/h(x)$ where g(x) and h(x) are polynomials, and $h(x) \neq 0$.

Definition 4: If f is a differentiable real-valued function, $R^n \overset{f}{\to} R$, then the function $\nabla f(x)$ is defined by

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n} \right).$$

<u>Definition 5:</u> The directional derivative of f with respect to a vector u is $D_{\vec{u}}(f(x)) = f(x).\vec{u}/|u|$, where $|u|$ is a norm of the vector u.

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