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REOPENING THE CASE OF SCHEMA:
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DEPARTMENT OF MATHEMATICS

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DEDICATION

to

My husband,

Daniel Berger

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Abstract

Abstract mathematical topics, like topology, can initially be challenging for students. Further, there is a limited amount of literature available for how undergraduate students learn topology concepts. This work aims to contribute to the literature by investigating the nature of students' difficulties in topology. I employed Skemp's (1979) model of intelligence and schema theory to construct a framework for successful action in learning topology, which has the potential for application in a variety of areas of advanced mathematics. Possession of an appropriate schema (Knowing That), a path from the present state to the goal state (Knowing How), and adequate skill (Being Able) are all required for successful action to occur. When the action is physical, it is directed by a delta-one director system. When the action is mental and is changing the state of a delta-one director system, it is part of a delta-two director system, which is where intelligent learning occurs. When the quality of the schema is improved, the functioning of the director system is also improved.

A number of task-based interviews were conducted with a variety of participants: an undergraduate student, first-year graduate students, advanced graduate students, and a topologist. The framework was used to analyze the participants' activities when working with open sets and a basis for a topology. The results showed that undergraduate students can still be in the early stages of

basis schema development, even at the end of their topology course, but the first-year graduate students showed significant learning outcomes by the end of their course. When the participants had issues with completing the problems, their difficulties occurred all throughout the Knowing That, Knowing How, and Being Able stages. Unlike the topologist, the novice participants showed a lack of strength of conceptual connections, less organization of their schemas, and had issues in all three stages for successful action.

Chapter 1

Introduction

Advanced mathematics is often a challenge for undergraduate students (Weber, 2001), especially when students first take a topology course. Further, as we will see in Chapter 2, the research on topology education has been severely limited until recent years. Cheshire (2017) studies students' understanding of continuous functions and open sets through axiomatic definitions. A case study by Gallagher (2020) works with a student over the course of a semester while she was enrolled in a topology course and analyzed how her thinking on proofs was influenced by things like examples, gestures, and speech. In the limited amount of research on topology education, there is very little, if any, that focuses on students' understandings of a basis for a topology and how to utilize it in topology problems.

Multiple people have told me that if I want to learn more about a topic, then I should try to teach it to someone. In preparing to teach, something can be seen about the topic that was possibly not realized before. The strategy in the following study is similar. As a graduate student, there are at least two things that have been difficult to understand but I have wanted to learn more about: point-set topology and the construct of schema. In this work, I will carefully define schema, discuss the limited literature on topology education, create a framework based on schema that is applicable in learning topology, and investigate students' understanding of basic topology concepts.

When first learning about topology, a few hurdles to overcome are the definitions of a topology and basis, the abstractness of those definitions, and understanding what they are useful for. Additional ideas that are difficult to grasp at first are the purposes of topological spaces, knowing which topologies are understood to be assigned with which spaces, and knowing what it means to have a finer basis for a topology. Once some of these definitions and concepts finally made sense for me, I enjoyed thinking about them and could not understand why I had struggled so much with them or why it took so long. My struggle to grasp these abstract concepts is certainly not a unique one. I have also spoken with other peers and undergraduates who share a similar experience, namely that topology was difficult when they took their first class in it. My work aims to understand why a first course in topology is challenging and uncover some strategies to help alleviate that struggle.

The other object of study in this work is schema. In a sentence, schema is a mental organization of concepts which contains connections between ideas and is itself connected to other ideas or schemas. It is possible that you find this short definition too vague or lacking, which was my initial feeling when introduced to schema. Additionally, it seemed that the existing literature had various and inconsistent ways of thinking about it or explaining it. For example, the definition that is part of APOS theory (Arnon et al., 2014; Dubinsky & McDonald, 2001) is different from Skemp's (1962; 1979; 1987) perspective on schema, depending on if the schema is coherent or not. Therefore another goal of this work is to unravel what exactly a schema consists of and how it can be utilized.

1.1 Aims and Research Questions

My purpose statement follows Creswell's (2013) recommendations so that my purpose is not left implicit or unclear. The central purpose of this collective case study is to identify and describe the nature of undergraduate students' difficulties when learning topology concepts. Making sense of the schema construct can help with this and is therefore a second aim. As a result, the overarching goal is to construct a framework that provides details on what it means to possess an appropriate schema for topology concepts that is applicable to all levels of learners, both undergraduate and post-graduate.

My research questions are as follows, with the first question serving as my central question.

1. What is the nature of students' difficulties with topology? What are the particular difficulties within each stage of successful action?

2. How do topology students reach a goal state when they do not have a guaranteed Know How? What do students' successful director systems have in common?

3. What qualities of schema do undergraduate students demonstrate when they are learning about topology for the first time? What qualities are needed in order to begin learning about topology at the undergraduate level? What about at the graduate level? What impact does each quality have on the functioning of a director system in topology?

1.2 An Overview of the Dissertation

In this section, I will provide a brief description of each upcoming chapter. Chapter 2 reviews the previous literature on schema and related learning theories. In the last portion of that chapter, a summary of existing research on undergraduate topology education is given. The pilot studies are found in Chapter 3, which evaluated the level of undergraduate students' schema development for a basis and analyzed first-year graduate students' basis schema qualities. The theoretical framework and the theory it is primarily based from are presented in great detail in Chapter 4. This framework is used to understand three stages needed to reach goal states in topology and some forms of learning that can occur if the goal cannot be achieved yet. Chapter 5 describes the methods, participants, and data analysis used for this study. The first half of Chapter 6 reports what occurred in the interviews, which is then analyzed with the theoretical framework in the second half. These results are then discussed in Chapter 7 in order to address the research questions. Chapter 8 evaluates the contributions of this work and gives some pedagogical recommendations based on what was found.

Chapter 2

Literature Review

This chapter begins by reporting on the existing literature on schema, ranging from its origins to its common uses today, including Skemp's (1979; 1987) notion of schema. How a schema is created and updated is discussed, as well as some basic features of a schema. In the final section, the focus switches to topology education and current work in the area. The intention of this chapter is to not only present the established research, but also to illuminate gaps in understanding schema and topology, where my work can hope to contribute.

2.1 The History and Development of Schema

In Bartlett's (1932) time, a common way of thinking about remembering was that a memory was some 'trace' in the mind, and to actually recall it, your mind simply activated that trace. Bartlett argued that the past must work as an organized mass, or setting, rather than a group of individual traces. Munk (1890) regarded the brain as a "storehouse of images of movement", which Head and Rivers (1920) showed was not true. Instead, Head and Rivers (1920) came up with 'schema' and 'schemata', which Bartlett agreed with more, but still pointed out some difficulties with it.

Head and Rivers (1920) claimed that everything enters into consciousness already relating to previous experience and that "For this combined standard, against which all subsequent changes of posture are measured before they enter consciousness, we propose the word 'schema'" (p. 605-606). Here Head and Rivers defined schema to be the past experiences that have remained with an individual that they can then use to take in new experiences with. Throughout this work, I will use the term 'schemas' to be the plural of schema, since Head and Rivers had a different meaning in mind for the term 'schemata'. They defined schemata to be the organized models of ourselves that are formed from schemas (Head & Rivers, 1920). Based on these definitions, schemata and schema seem to be defined as mental constructs that exist mostly outside of consciousness whereas structures (in the sense of Piaget (1970)) are constructs that we consciously put onto things. Schema and schemata are something that are inherently in us that do have structure, but structures in general can live independently of human thought.

Bartlett (1932) used Head's ideas to form another definition for schema:

> 'Schema' refers to an active organisation of past reactions, or of past experiences, which must always be supposed to be operating in any well-adapted organic response. That is, whenever there is any order or regularity of behaviour, a particular response is possible only because it is related to other similar responses which have been serially organised, yet which operate, not simply as individual members coming one after another, but as a unitary mass. (p. 201)

This definition seems to combine Head's 'schema' and 'schemata' into one construct because everything is working together as one. This feels more natural because our past experiences are not just traces that are placed within organized models, but are organized within themselves as well. So it is redundant to distinguish between the two and to instead use 'schema' to describe the whole mass of organized, previous experience. Bartlett (1932) indicated that a function of a schema is that it works with incoming senses and makes a "specific adaptive reaction possible" (p. 207).

It would be unproductive if we went through the trouble to define schema so carefully but did not discuss the problem of its common use: the term is used too often to mean various things. Bartlett (1932) pointed this out in the context of psychology, but it is still true in modern mathematics education research and in the world in general. As we will see, there are many variations of how schema is defined and there are inconsistencies between them, therefore I will try to be clear in my use of the word.

In the world of philosophy, Johnson (1987) discussed what he calls an 'image schema'. He defined an image schema as a "recurring, dynamic pattern of our perceptual interactions and motor programs that given coherence and structure to our experience." Our image schemas are made of parts that come together to be unified wholes, which give our experiences meaning. He phrased the formal, rational, and intellectual component of the mind-body problem as "spontaneous organizing activities of our understanding" (p. xxvii). Additionally, he asserted that "Image schemata are abstract patterns in our experience and understanding that are not propositional in any of the standard senses of that term, and yet they are central to meaning and to the inferences we make" (p. 2).

Johnson (1987) then defined a schema as "a recurrent pattern, shape, and regularity in, or of, these ongoing ordering activites" (p. 29). (The activities being our actions, perceptions, and conceptions.) Although a schema can be built up from patterns, the idea of a schema being a pattern itself is not quite consistent with how I will use the term. Part of Johnson's definition that I will use is that they are "structures for organizing our experiences and comprehension" (p. 29). Furthermore, he identified schema as "a continuous structure of an organizing activity" (p. 29).

Johnson (1987) also summarized Kant's (1781) position on schemas. Kant's view of schemata was that they are "nonpropositional structures of imagination" (Johnson, 1987, p. 19), which feels quite vague. Johnson elaborates, "As we shall see, Kant's interpretation is somewhat limited by his peculiar view of concepts, but he does recognize the imaginative and nonpropositional nature of schemata" (p. 21). Later on Johnson elaborates:

> Kant went so far as to claim (in some passages, at least) that schemata are actually preconceptual structuring *processes* whose structures can "fit" general concepts and can generate particular images, thereby giving our experiences meaningful order and organization that we can understand. He also saw schemata as structures of imagination. ...for Kant, imagination is the very means by which we have *any* comprehensible structure in our experience. ...his stronger thesis that schemata are procedures for generating images that can fit concepts. (p. 29)

Johnson also discussed Neisser's (1976) definition of schema, which fits well with Bartlett's (1932) definition:

> A schema is that portion of the entire perceptual cycle which is inter-

> nal to the perceiver, modifiable by experience, and somehow specific to what is being perceived. The schema accepts information as it becomes available at sensory surfaces and is changed by that information; it directs movements and exploratory activities that make more information available, by which it is further modified. (Neisser, 1976, p. 54)

This definition emphasizes the malleability of schema. One function of schema is that it makes it "possible for us to recognize different kinds of things and events as being different *kinds*" (Johnson, 1987, p. 20). This is highly related to Skemp's (1979) object-concept, which he defines as "The very basic kind of concept by which we recognize an object as being (or not being) the same thing that we have seen before..." (p. 117). As we experience different types of the same thing or event, our schema for that thing or event may be modified to include the new experience. Johnson (1987) said that our structures are "altered in their application to particular situations" (p. 21).

Up to this point, the major characteristics of schema include that it operates as an organized mass that is structured and continuously active, existing either in or out of consciousness, and is based on our experiences. Because a schema takes in information through our senses, sight included, I should distinguish between it and a mental picture.

Schema and Mental Pictures

Because Johnson (1987) defined a schema as a recurrent pattern or regularity, it is possible to think of an example and come up with specific mental images, however, this is not what Johnson meant. "On the other hand, *image schemata* *are not rich, concrete images or mental pictures, either.* They are structures that organize our mental representations at a level more general and abstract than that at which we form particular mental images" (p. 23-24).

Johnson referenced Kant (1781) to talk about the distinction between schema and mental pictures. He emphasized that a mental picture may be too detailed and not broad enough to represent all kinds of that object. For example, a right triangle cannot represent all triangles, because there are triangles that do not have a right angle.

> No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought. (Kant, 1781, A141/B180)

One distinction between schema and mental pictures is what Anderson (1980) called "abstract analog". This is something that a schema has, but a mental picture does not. For example, "Anderson suggests the length of a line as an analog for a person's weight" (Johnson, 1987, p. 25). I can visualize a scale with someone's weight on it, but that visual remains fixed. The analog would be if I imagine a line with the same length in inches as the weight was in pounds. I am using my spatial schema to visualize a completely different mental picture and the original picture was not very helpful in transitioning to the line.

> Humans seem to have the ability to scan and transform mental arrays and image structures in a fashion analogous to the scanning and manipulation of physical objects. It is as though we have a "mental

> space" in which we perform image-schematic operations that may or may not involve visual rich images. (Johnson, 1987, p. 25)

Johnson (1987) also pointed out that schema can be influenced by general knowledge whereas mental pictures can not. "Schemata are more abstract and malleable than mental pictures" (p. 26). He then talks about a scenario where someone has to remember a mental picture. He makes a great point in that "If they only grasped the mental picture, and not some schematic representation, they should be able to reproduce the original drawing exactly" (p. 26). If they remember something similar to the needed picture, but some things about it are different, they likely remembered it through the use of a schema instead of rote memorization. With regards to math education, this may be where rote memorization is not necessarily a bad thing if you are trying to remember a formula, however, after remembering that formula accurately, you will likely need schematic remembering to make use of that formula.

2.1.1 Modern definitions

How is schema used in mathematics education today? Davis and Tall (2002) put it well: "It is fair to say that whilst the term scheme has been used in mathematics education (see, for example, Steffe, 1983, 1988; Davis, 1984; Dubinsky, 1992; Cottrill et al., 1996) there have not been many attempts to define more precisely what might constitute a scheme" (p. 141). For example, Harel and Tall (1991) defined three types of generalization in terms of schema, however, they did not themselves define schema.

In his book, Skemp (1979) gave his model of intelligence, which builds up to the idea of schema. I will explain this model in detail in Chapter 4 as part of building my theoretical framework. Olive and Steffe (2002) described Skemp's model and merged it with Piagetian Scheme Theory to make a complex model for an "interiorizing scheme" (p. 130), which they used to try to explain children's equi-partitioning schemes for fractions. It is important to note that in their study, "scheme" and "schema" are different. "Schemas are *representational* structures - they represent knowledge in the form of networks of connected concepts, whereas schemes are *action* structures" (Olive & Steffe, 2002, p. 120). In their writing, Olive and Steffe included three aspects of a schema that affect its quality. The type of concepts that the schema consists of will determine how abstract it is. The quality and quantity of the connections made between concepts also determines how complex a schema is.

Tall and Vinner (1981) described schema, but instead call it a *concept image*. "We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). They call the part of the schema that is activated by a stimulus the *evoked* concept image. The concept image may or may not include what they called the *concept definition*, which can be the formal definition in actuality, or it can be one's personal definition for the concept. Finally, one's concept image does not have to align with actuality. It can contain misunderstandings or potential for conflict if faced with a stimulus that contradicts the concept image.

A definition of schema that is commonly referenced among math educators is the 'S' of APOS Theory (Arnon et al., 2014; Dubinsky & McDonald, 2001). Actions, processes, and objects are used to define a Schema. Actions are external transformations of objects that become processes once internalized. After an individual becomes aware of a process and the transformations that can act on it, the process has become an object itself. Dubinsky and McDonald (2001) continue on to define Schema:

> Finally, a schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not. Because this theory considers that all mathematical entities can be represented in terms of actions, processes, objects, and schemas, the idea of schema is very similar to the concept image which Tall and Vinner introduce in "Concept image and concept definition in mathematics with particular reference to limits and continuity," Educational Studies in Mathematics, 12, 151-169 (1981). Our requirement of coherence, however, distinguishes the two notions. (p. 3)

Some caution is needed here since this idea of coherence makes the APOS notion of schema slightly different from our definitions of schema thus far. I do not typically capitalize 'schema', however, for the sake of this discussion, I will follow Trigueros's (2019) footsteps and use a capital S if referencing APOS theory. If your collections of actions, processes, and objects are scattered or disconnected then they are part of your concept image, but that concept image is not organized enough to be called a Schema according to the theory of APOS. Dubinsky and McDonald (2001) and Arnon et al. (2014) require the collection of actions, processes, and objects to make sense together with some sort of coherency in order to together be called a Schema. In contrast, Skemp (1979) alluded to inaccurate and incomplete schemas that are potentially the cause of a lack of understanding, while Dubinsky and McDonald would not consider these to be Schemas at all.

Arnon et al. (2014) describe schema, saying that “a Schema is a tool for understanding how knowledge is structured and its development through the learning process” (p. 111). When building up a schema, you must have some conceptual pieces in place before understanding can occur. These pieces make up your concept image and once coherent, become a Schema. I conjecture that this coherency that makes APOS’s Schema is what Skemp (1979) would call understanding. It is important to be aware of these distinctions since Dubinsky and McDonald’s (2001) definition is widely referenced with the use of APOS Theory. This work is primarily based on Skemp’s (1979) schema, which needs to be discussed in greater detail.

2.1.2 Skemp’s Schema

In an early paper, Skemp (1962) argued that psychological theories developed in laboratories instead of classrooms are not valid learning theories and that there was a need for such a theory:

> A theory is required which takes account (among other things) of the systematic development of an organised body of knowledge, which not only integrates what has been learnt, but is a major factor in new learning: as when a knowledge of arithmetic makes possible the learning of algebra, and when this knowledge of algebra is subsequently used for the understanding of analytical geometry. (p. 133)

In his 1962 study, Skemp (1962) defined schema as the "organised body of knowledge" that integrates existing knowledge and is a major factor for new learning (p. 133). Additionally, he compared schematic learning to rote learning (non-schematic learning). Unsurprisingly, he found that "Schematic learning...has a triple effect: more efficient current learning, preparation for future learning, and automatic revision of past learning" (p. 140). This is consistent with what Bartlett (1932) said about rote memorization: "In fact, if we consider evidence rather than presupposition, remembering appears to be far more decisively an affair of construction rather than one of mere reproduction" (p. 205). "All relatively low-level remembering tends, in fact, to be rote remembering, and rote memory is nothing but the repetition of a series of reactions in the order in which they originally occurred" (Bartlett, 1932, p. 203). Rote memory can be difficult because it's like re-exciting a trace, which has no relation to anything else. A schema has relationships between other concepts, making maintaining it easier and therefore making remembering easier. Skemp (1979) elaborated on this:

> It follows that if our mental models are to be of any use to us, they must represent, not singletons from among the infinite variety of actuality, but common properties of past experiences which we are able to recognize on future occasions. ... A major feature of intelligent learning is the discovery of these regularities, and the organizing of them into conceptual structures which are themselves orderly. (p. 116)

Skemp (1987) later gave a more detailed definition of schema in his chapter, "The Idea of a Schema". He described a system where concepts are embedded in a hierarchical structure of other concepts, these levels in the structure being classifications of concepts. For example, a train can be classified as a mode of transportation and can contribute to one's concept of transportation. There are not only single concepts, but when we pair concepts together, we can have a relation between them, for which a classification is also possible. The pairs

tracks, train; road, car; air, plane

have a relation between them by the idea of what object a particular mode of transportation travels on. We can also look at transformations of concepts, which can be combined to make other transformations. An example of a transformation that Skemp (1987) provided is sending a concept to its opposite:

good→bad hot→cold high→low

There is also the transformation of sending a concept to its extreme:

bad→worst cold→coldest low→lowest

These two transformations can be combined to make the following transformation:

good→worst hot→coldest high→lowest

Note that the train concept can be classified as a form of transportation, but it can also be classified as a toy that young children play with or as a character in a book. What makes this hierarchical structure of concepts, relations, and transformations so deep and complex is the fact that these classifications are not unique, giving way to multiple hierarchical structures, which can be interrelated. When components of these conceptual structures come together to make a structure that would not be realized by only looking at the individual components, we call this resulting structure a schema. For example, I know that a steam engine has cars that are linked together, wheels on each car, and coal fueling a fire in the front car. These components individually do not let me know how a steam engine can move forward, but together, they may provide some insight and a schema for the workings of a steam engine. Skemp (1987) claimed that a schema integrates existing knowledge, serves as a tool for future learning, and makes understanding possible.

Skemp (1987) incorporated Piaget's (1950) assimilation and accommodation in discussing schema. Assimilation occurs when a concept is built up within a schema, possibly forming connections to other concepts. If there is not an appropriate schema available for assimilation of a concept, then accommodation or reconstruction is needed. "...Not only are unsuitable schemas a major handicap to our future learning, but even schemas which have been of real value may cease to become so if new experience is encountered, new ideas need to be acquired, which cannot be fitted in to an existing schema" (Skemp, 1987, p. 27). Accommodation can be difficult and thus the need to modify a schema usually comes with resistance. For example, the mistake someone makes when writing $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ is a special case of "The Freshman's Dream". This mistake often presents itself in a College Algebra or Calculus class. I have stood at the front of my classroom and told my students explicitly that this is not true. I have them choose two (positive, nonzero) numbers to demonstrate that it is not true. Yet, I always have a handful of students who continue to try to use it for the remainder of the semester. There is much resistance to the modification of what I will call their "linearity" schema. As Skemp (1987) pointed out, assimilation can be a much more pleasant experience. "...Assimilation of a new experience to an existing schema gives a feeling of mastery and is usually enjoyed" (p. 28).

A question to now consider is how successful accommodation occurs in order to assimilate concepts. Clearly, showing only one counterexample to the Freshman's Dream is not enough to cause accommodation.

Eight years later, Skemp (1979) talked about how there is not a clear cut distinction between concepts and schema because they are the same, but viewed from different perspectives. He believed that "a schema is a highly abstract concept" (p. 167). There is a zooming feature, much like a camera, that Skemp called 'vari-focal'. Zoomed in, we may see a schema, but zoom out and you will see a general concept (possibly within a bigger schema). When you zoom in, however, schema brings greater detail than zooming in on a picture would. Instead of a blur, you will gain more information. This property Skemp called 'interiority', and together with vari-focality, they describe a schema's organization.

Skemp (1979) also distinguished between object-concepts and perceptual learning. Object-concepts are a specific type of concept that allows us to decide, when a new object is put in front of us, if that object is an example of the concept or not. For example, a small child may be shown a picture of a pig and then asked "Is this a cow?", to which they may respond "No! That is a pig!". The child has an object-concept of pigs that likely involves a pig being a pink color, having a squished, circular nose, and a curly tail, while their object-concept of cows involves animals that are larger, black and white, with longer faces and straight tails. The picture of the pig did not fit their object-concept of a cow, but it did fit their object-concept of a pig. Object-concepts can be developed fairly quickly and at young ages. Meanwhile, perceptual learning is learning that depends on 'sensory discrimination' and may need time to accomplish. By this, we mean that you are intentionally using your senses to discover something specific. You are teaching your senses to distinguish between things that you may have not been able to previously distinguish. Whenever I have gone to the doctor for an ultrasound, I have been amazed that the ultrasound technician can understand what they are seeing on the screen. I mostly see blobs, but they have trained their senses to be able to see what is on the screen and process it to make sense of what is shown. Similarly, Skemp used examples of doctors listening for characteristics of a heart beat, coffee tasters smelling and tasting varieties of coffee, and someone noticing differences in quality before and after switching to higher quality equipment. Perceptual learning in not instinctual like the kind of learning that builds object-concepts from regularities in our experiences.

A schema that is useful should have a vari-focal organizational quality so that it can be adaptable to various situations. Mathematics instructors usually desire for their students to have mathematical object-concepts, organized within a schema so they know when to use them for certain problems. We usually say that we want our students to understand the topics. This begs the question, with respect to schema, what does it mean to understand something and what occurs when a student does not understand?

2.2 Understanding

In his chapter on understanding, Skemp (1979) connected understanding and schema in a topological way. Imagine a graph with vertices and edges that connect some, but not necessarily all, of those vertices. An example of such a graph is shown in Figure 2.1. This graph can be thought of as a cognitive map with vertices being concepts and each edge representing a cognitive connection between two concepts. The overall conceptual structure is a schema, which may have schemas within it, meaning that the vertices may have some interiority to them, creating the hierarchical structure from before. For example, Figure 2.1 may be a schema for abstract algebra, and within that schema exists vertex Q, which may be a group theory schema, of which possibly holds a group actions schema. Schemas are nested within other schemas.

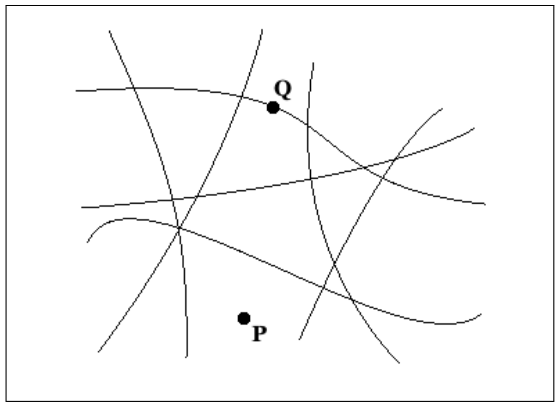


Figure 2.1: A visual for a concept map.

As previously stated, not all vertices are necessarily connected by edges. When a vertex is isolated, like point P is in Figure 2.1, this concept is mentally lost. According to Skemp (1979), “The achievement of understanding makes connections with an existing schema” (p. 145). So once an edge is created that connects point P with another point in the schema or map, understanding has occurred. This can be thought of as assimilation. If no understanding is taking place, there is a missing link within a person’s schema. In other words, that person does not have the existence of an appropriate schema to assimilate the concept into. With this relation between schema and understanding, Skemp (1979) presented five types of non-understanding:

1. Encountering an object or event that we cannot adequately categorize

2. Realizing an object or event, but being unable to connect it with an appro-

priate schema

3. Complete absence of an appropriate schema

4. Connecting an object or event incorrectly to an appropriate schema

5. Associating words or symbols with an inappropriate concept or being unable to associate them to a concept at all

The first type of non-understanding can be interpreted as seeing vertex P, but not knowing what it is because we have never encountered it before. Thus there is no way to make a connection between vertex P and a schema because vertex P is not known. An example of this could be an undiagnosed illness. There is no way to know a cure since the illness is unknown. The second type of non-understanding occurs when vertex P is known, the schema or goal state is known, but what is unknown is how to make a connection between them. The example here is an illness that has been diagnosed, health is the goal state, but a cure is not known. The third type is a complete absence of an appropriate schema. This can be thought of as ignorance, or a state in which you are not aware that you do not understand. Before becoming a mathematics major in my undergraduate work, I had never heard of topology. I easily did not understand topology because I was unaware of its existence. Even once I was aware of it, it still took much time and thought to understand concepts in the area. The fourth type of non-understanding can be thought of as a misunderstanding. This occurs when a concept is assimilated inappropriately. A person who lives in Oklahoma during a weather change may have sinus congestion that they believe to be seasonal allergies. They may wonder why their allergy medicine is not working if they misdiagnosed a case of the common cold to be allergies. Finally, the last type of non-understanding involves recognition of symbols. If a symbol is seen, but it is not associated with the right concept, confusion and frustration may follow. This particular misinterpretation of a symbol is too often demonstrated in undergraduates who are struggling in their math course.

Let us consider an example problem from a topology textbook:

> Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X. (Munkres, 1975, p. 83)

Some concepts that are needed in order to approach this problem include, but are not limited to, quantifiers, equivalence of sets, and the definitions of open, topology, and topological space. There are many places where non-understanding can take place. A student may have not yet seen the definition of a topological space or an open set yet (Skemp's first type of non-understanding). They could realize what open is, but not associate it with the topological definition, which is Skemp's second type of non-understanding. An example of the fourth type would be if a student tried to use the intersections part of the definition of topology. Finally, a student could incorrectly use symbols for the quantifiers or not know what they are (the fifth type). All of these non-understandings can prevent the student from finishing the exercise and need different types of interventions. The big question here is: if the student's attempt at the exercise is given to an instructor entirely on paper, can the instructor determine which type of non-understanding is occurring? If not, this makes interventions extremely challenging.

How these connections of understanding are formed and how schemas are reconstructed for assimilation is a difficult question. In terms of mathematics education, the ultimate role of a teacher is to help his or her students understand,

and anyone who has taught a math course before will tell you that this is not a trivial matter. Looking back at the Freshman's Dream example, in order to help students reconstruct their linearity schema, it would probably be most helpful to know what type of non-understanding is occurring. Do they not understand what the square root symbol means? Maybe they have an inappropriate schema that tells them that everything they know is linear. It is possible that they know that the Freshman's Dream is false, but they see no other way to make progress on a problem, so they decide to write it down since they have no where else to go. All three of these scenarios would require different interventions.

In order to successfully accommodate, I conjecture that any intervention would require an amount of the student's time working through algebra material, the desire to understand, and the belief that understanding is possible for them. Referring to Skemp's (1987) structural definition of schema, "The study of the structures themselves is an important part of mathematics, and the study of the ways in which they are built up and function is at the very core of the psychology of learning mathematics" (p. 23).

2.3 Schema Formation

Understanding occurs when connections are made to an existing schema, but how does a schema originate? "How are our active organised settings, our 'schemata', developed?" (Bartlett, 1932, p. 212). Piaget (1970) considered this by questioning if structures are formed or predetermined: "Have these composite wholes always been composed?...Or were they initially (and are they still) in *process* of composition? To put the question in a different way: Do structures call for *formation*, or is only some sort of eternal *preformation* compatible with them?" (p. 9). It is important to emphasize that a schema is a mental structure and therefore much of what Piaget says regarding structuralism can be applied to a schema. When it comes to an individual's schema, however, it becomes clear that this structure must be built up within the mind of the individual, even if the structure already exists in actuality. In this section, I will discuss various ways that a schema is formed for an individual and contributions to how a schema is formed.

Clark et al. (1997) provided a nice application of Piaget and Garcia's (1989) triad framework, Intra-, Inter-, and Trans-, to the chain rule in Calculus. This triad is a theory for schema development within the context of APOS. Arnon et al. (2014) explain the triad in great detail and give examples of schema development for functions, the derivative concept, and the chain rule. Additionally, Trigueros (2019) explains the triad framework in terms of relations: correspondence, transformation, and equivalence relations. Before a schema is coherent, it must go through these three stages. In the Intra- stage, an object is thought of in isolation from other actions, processes, or objects. Correspondence relations are used to compare and contrast structures. Once relationships are seen between the object and other actions, processes, objects, and schemas, the individual is in the Inter- stage, also known as a pre-schema. Here, "transformation relations are developed when they discover that some structures in the Schema can be related in terms of changes in the other" (Trigueros, 2019, p. 1056). In the Trans- stage, a coherent structure begins to underlie the relationships from the Inter- stage, and the original object in question can be identified as a schema. Equivalence relations are seen through conservation of properties and the underlying structure is understood. This triad can be used as a way of analyzing schema development. Trigueros (2019) creates a genetic decomposition for relations among the components of a Linear Algebra Schema and used the triad to analyze students' construction of the schema.

Since concept formation is the first step in schema formation (the 'second' step is making connections between concepts), it is important that we consider what goes into forming concepts. Bartlett (1932) emphasized, "All that goes to the building of a 'schema' has a chronological, as well as a qualitative, significance" (p. 208). Skemp (1979) also realized the importance of understanding schema formation. "Any progress we can make in identifying the factors which affect concept formation will have widespread value for setting up conditions favourable to intelligent learning" (Skemp, 1979, p. 121). Skemp identified contrast, symbols, and time as important factors to concept formation, but that list is certainly not exhaustive. Contrast refers to contributors and non-contributors, or in other words, examples and non-examples. Seeing a non-example can be just as helpful as seeing an example. Moreover, symbols and communication also contribute to forming a concept, which is what we typically experience in a mathematics lecture. Symbols can be more heavily used for forming secondary concepts. Skemp (1979) presented a very interesting question about how concepts are formed without a teacher:

> How can we account for concept formation in the absence of such help? In other words, how can someone group together the necessary contributors from which to form a new concept, if no one does it for him, and if he doesn't know himself what to group together because he hasn't the concept? The question is an important one, since this is how many or most of our concepts are formed, in the absence of a teacher, under the informal learning conditions of everyday life. (p. 122)

As for timing, Skemp (1979) said, "The shorter the time interval between similar experiences, the more likely they are to become contributors to a concept" (p. 121). Bartlett (1932) made a comment about the time factor and the past operating *almost* as a mass. He said that the latest additions to the schema have more of an influence on it than the older components and argued against Head and Rivers's (1920) idea that schemata are built up chronologically. Skemp's and Bartlett's ideas seem consistent with my experience in learning to play golf. When in a swing lesson, although you've practiced swinging a certain way for a while, the swing coach can say one small thing that completely fixes your swing. The most recent tweak is what was more effective than the older muscle memory.

When talking about schema formation in mathematics, it is important to distinguish between primary and secondary concepts. Primary concepts are "formed directly from sensory experience of actuality" (Skemp, 1979, p. 120). Secondary concepts can be formed through successive abstraction, which Skemp (1979) defined as "repeating the process of concept formation at a higher level; based now, not on regularities as experienced directly from actuality, but on the discovery of higher-order regularities among these regularities" (p. 119). Skemp (1979) defined abstraction as "The process by which certain qualities of actual objects and events are internalized as concepts, while other qualities are ignored" (p. 24).

Various perspectives on abstraction are found in undergraduate mathematics education literature, but there is agreement that being able to abstract is an important skill in mathematics (Hazzan, 1999). Hazzan (1999) presented three perspectives on abstraction and discusses how to reduce the level of abstraction. Reducing abstraction helps the learner give the concept to be learned some meaning that they can then build on. Hazzan's (1999) first perspective was *Abstraction level as the quality of the relationships between the object of thought* *and the thinking person* (Wilensky, 1991). This interpretation emphasizes that the level of abstraction is a property of the individual, not of the concept being learned. This is consistent with Skemp's (1979) perspective because it says that the more connections one has to a concept, the more concrete it feels. The example that Hazzan (1999) gave is in abstract algebra, where students would reduce the level of abstraction by connecting the concept of a group to number operations that they were very comfortable with.

The second perspective that Hazzan (1999) gave is *Abstraction level as reflection of the process-object duality* (Arnon et al., 2014; Dubinsky & McDonald, 2001). This interpretation is focused on the process and object components of APOS theory, where a concept as a process is less abstract than the concept as an object itself. This relies on Piaget's (1970) process of reflective abstraction, which is where one derives properties, not from things, but from acting on things. In other words, reflective abstraction is the mental activity to go from a process conception to an object conception.

The third perspective was *Abstraction level as the degree of complexity of the concept of thought*. Hazzan's (1999) assumption here was that "the more compound an entity is, the more abstract it is" (p. 82). In the above language of Skemp (1979), secondary concepts are more abstract than primary concepts. Replacing a secondary concept with a primary concept will reduce the level of abstraction. For example, a topology is a set where the elements are sets themselves. Reduced abstraction would be to focus on one or some of the sets within the topology.

The Intra-, Inter-, and Trans- phases can be used to measure schema development, which can be affected through things like contrast in examples, communication and symbols, and shorter increments of time between contributors. To develop secondary concepts in a schema, ideas must undergo successive abstraction, which can be made more accessible through reducing the level of abstraction. Because a schema is active, it is continually updated by our experiences, even after reaching the Trans- phase. Therefore the next section will be dedicated to ways a schema continually updates through reality testing.

2.4 Updating of a Schema

Once a schema is formed, it is not permanently fixed. We continue to take in experiences that either fit within our current understanding, or they do not. The schema may need to undergo accommodation in order to assimilate new information. Arnon et al. (2014) use the term 'equilibrium' to describe the fluidity of a schema. "Equilibrium is dynamic, so that through it the Schemas are constantly changing, although they maintain their identity" (p. 113). Bartlett (1932) described equilibrium as well, saying that it is "very essential to the whole notion, that the organised mass results of past changes of position and posture are actively doing something all the time; are, so to speak, carried along with us, complete, though developing, from moment to moment" (p. 201).

Before discussing the updating of schemas, we need to define some terms. Skemp (1979) clearly distinguished between actuality and reality. Actuality is what is happening in the world, independent of any individual, while reality is specific to an individual. For example, it exists in actuality that apples have color to them. However, the actual color that each individual sees changes depending on their eyesight. The reality of someone who is colorblind is different than the reality of someone who is not.

Knowledge, according to Skemp, is a concept or schema that is built from and tested in actuality. On the other hand, belief is a concept or schema that is considered as fact, but has not been tested or proven in actuality. Let's start with the assumption that a schema has been formed already. The knowledge and beliefs that make up the schema build one's reality. This reality then is tested in at least one of three ways: through an experiment, social discussion, or internal consistency. Experimentation tests reality with actuality in some concrete manner. Social discussion compares one's reality with others' realities, usually through discussion. Internal consistency is testing a specific reality with a more general reality.

For example, if my reality is that I am holding a purple apple, I can check to see if this reality is consistent with my reality, other's reality, and actuality. In checking with what I already know about apples, I may reason that this apple may not really be purple, since most apples vary from red to yellow to green. I can ask someone else who is nearby what color they think the apple is. Finally, if I have the right knowledge and equipment, I could perform a scientific experiment to test the contents of the apple. If no contradictions occur in reality testing, then the knowledge and beliefs are further solidified. If a contradiction occurs, then a change to the knowledge and beliefs occurs, thus updating them, which improves the accuracy and completeness of the schema.

Through testing reality, concepts and schemas are updated by the following processes: "realization, assimilation, expansion, differentiation, and re-construction" (Skemp, 1979, p. 125). There is a two-way relationship between actuality and our schema. What most call 'assimilation', Skemp broke down into realization and assimilation, each indicating the relational direction occurring. *Realization* maps actuality into an appropriate schema, while *assimilation* selectively takes an existing concept and views actuality through its lens. Both of these contribute to the formation of a schema, while the remaining three processes focus on accommodating an existing schema. Our experience can contribute to an existing concept, which is what Skemp called *expansion*. *Differentiation* is where new classifications are formed within a schema. The most frustrating form of accommodation is *reconstruction*, which is sometimes necessary. "Since this necessarily involves first taking it partially or completely to pieces, this is disruptive, unwelcome, and difficult: because while this is going on, we are unable to use our schemas effectively for directing our actions" (Skemp, 1979, p. 126).

Once a schema has been formulated, updated, and made coherent, what are some characteristics it should have? In other words, what makes a "good" schema? A select few characteristics will be discussed in the next section and in Section 4.2.3.

2.5 Features of a Schema

Piaget (1970) claimed that structures are made from three main ideas: wholeness, transformations, and self-regulation. Self-regulation is closely related to the idea of a schema being in equilibrium, which was discussed earlier. I will discuss the other two ideas here.

Wholeness of a Schema

Piaget (1970) began by considering structures versus the aggregates of a structure. A structure cannot be viewed entirely as a sum of its parts, nor can you completely ignore its parts. Each component brings something to contribute to the structure. If not, why would it be there? Furthermore, we cannot ignore the properties of components. Piaget used the integers as an example of a structure. The integers can be made of discrete sets of numbers, each with their own properties. We can look at even numbers, multiples of three, negative numbers, etc. Those individual properties, however, do not immediately reveal properties of integers. Many small children can count to the integer 2, but they do not yet know that there is such a number as -2 and that $(-2) + 2 = 0$. This property is not obvious from the number itself. You can also know of all of the integers, but you would still need to define addition before this property is revealed.

Alternatively, considering a structure as a whole and ignoring its components could overlook some properties of the components. Consider what Piaget (1970) said about substructures: "the laws of the substructure are not altered but conserved and the intervening change is an enrichment rather than an impoverishment" (p. 14). For example, consider the set of even integers unioned with zero, which is a group under addition, just like the integers are a group under addition. This subset still preserves properties of the integers, but has additional properties itself since it is also a group. Since we cannot view structures as only a whole or only by its aggregates, this leads to the compromise of operational structuralism: a relational perspective where the relations among elements are emphasized instead of the whole or the components.

Johnson (1987) also noticed this quality of wholeness. "The important point for our purposes is that schemata have a generality that raises them a level above the specificity of particular rich images" (p. 24). Skemp (1979) refered to the wholeness of a schema as its "organization", which has both vari-focal and interiority properties.

Transformations of a Schema

Transformations can be, but do not have to be, temporal processes. Piaget (1970) regarded transformations to be any change, temporal or not, to the structure. This brings up a question of distinction between transformations and formations. Formations are about how a structure was obtained, but structures are systems of transformations on its elements, so at what point is the formation done and are the transformations beginning? An answer to this question may not be as important as understanding that structures are not static, and that they undergo transformations frequently.

"Were it not for the idea of transformation, structures would lose all explanatory import, since they would collapse into static forms" (Piaget, 1970, p. 12). Instead of being static, structures are constantly working to evolve and be built up and are therefore always open to transformation. I currently have a schema for the definition of a topology, and therefore I am constantly testing my understanding of a topology with the world around me and the theorems I read. I am currently assuming I have a decent understanding of the definition, however, if I ever come across something that does not fit my current schema of a topology, my schema will have to transform and adjust to accommodate the new information. Bartlett (1932) discussed this idea in the context of memory:

> In any case, a storehouse is a place where things are put in the hope that they may be found again when they are wanted exactly as they were when first stored away. The schemata are, we are told, living, constantly developing, affected by every bit of incoming sensational experience of a given kind. The storehouse notion is as far removed from this as it well could be. (p. 200)

Every experience we have has an impact on how we view the world, so it is impossible for us to 'store' away a thought or a memory and then for us to recall it later exactly as we had left it. The storehouse notion is also inconsistent with the idea of incubation. Incubation is the mental, unconscious activity that occurs during a break in the conscious attempt of the activity (Segal, 2004; Wallas, 1926). When mathematicians are working on a problem and get stuck, they will most times take a break from it and move on to something else with the intention of coming back to their problem later. It is not uncommon for an idea regarding the problem to come to them during this break because the mind is still working, even if they are resting. This certainly does not match the storehouse notion. If I put a broken lawn mower in a storage shed and come back a week later, I will not find that the lawn mower has produced the part or tool necessary to fix it. It will either still be broken, or some other person will have had an impact on it. If this were how human minds worked, then the incubation effect would never occur.

At this point, the concept of schema, how it is formed, how it is updated, and some characteristics of it have been discussed at length. I will pause this discussion for now to review the literature on undergraduate topology education. Schema will return in Chapter 4, where it will be part of a larger model for learning topology concepts.

2.6 Topology for Undergraduates

Topology is an important course for advancement in mathematics. Narli (2010) pointed out that "topology allows mathematics to be generalized to all sets" and allows students to investigate problems from a "wider perspective" (p. 121). It is a course that is offered at the undergraduate level, and even if it is not a requirement for graduation, it is usually a requirement if students are to continue on into graduate school. It is typically a proofs-based course, which students are known to have trouble with, specifically with regards to proof construction (Weber, 2001). Students usually view topology as highly abstract and difficult. The abstractness perhaps cannot be avoided, however, research into the difficulties may be beneficial for many abstract mathematical topics. I argue that undergraduate topology has great potential for investigating students' transitions to formal, abstract mathematical thought and how students construct new schemas and modify existing ones. Topology is a topic that students do not typically encounter before their undergraduate work, so the concepts require schemas to be reorganized or built up in order to be assimilated into and for understanding to occur. Additionally, basic topological concepts can be ideal for tasked-based interviews since topics can be chosen to where not much background knowledge is necessary. Skemp (1987) used topology for the reason that "the relevant schema can be quickly built up, whereas most mathematical ones take longer" (p. 30). Although this study focuses on more advanced topology questions than Skemp used, I still believe that topology offers ideal topics to observe schema development. Most students do not encounter topology until late in their undergraduate work, meaning they can come into the course without any preconceived views or misunderstandings.

For these reasons, I wish to consider the research questions with respect to introductory topology. Unfortunately, there is a limited amount of research on how undergraduates first learn topology until recent years (Cheshire, 2017). Numerous searches for undergraduate topology education have turned up very little. For example, in a search through the abstracts for presentations in the Conference on Research in Undergraduate Mathematics Education for the years 2012-2020, ten presentations were made involving topology. Two involve algebraic topology (Norton, 2014; Stewart, Thompson, & Brady, 2018), which is typically not an undergraduate course. Kontorovich (2020) looks at a mathematician's epistemology of proof, but the study involves topology only because the participant was teaching it at the time. All the remaining research that focuses on undergraduate topology came from Berger and Stewart (2018, 2019, 2020), Cheshire (2015, 2016), and Gallagher and Infante (2019, 2020).

The studies by Berger and Stewart (2018, 2019) served as the pilot studies for this work and therefore can be found in detail in Chapter 3. The analysis of Pilot Study 1's data revealed that the majority of undergraduate students were in the beginning stages of schema development, even though they were completing a final examination at the end of their semester. Skemp (1979) listed out several qualities of schema and argues that "The qualities of our available schemas are crucial determinants of our success in action" (p. 191). Berger and Stewart (2019) examined students' development and qualities of their schemas and discussed why these qualities can be difficult for learners in a topology context. In that study (Pilot Study 2), the participants found it difficult to navigate the generality of a schema and the strength of its connections. In Berger and Stewart (2020), two advanced graduate student participants were interviewed and some preliminary data was discussed. One of the participants, Luke, demonstrated characteristics of what it means to *do* mathematics by trying to use lemmas, clever tricks, and pose additional questions. These indicated that Luke possessed a high-order schema, or a schema with a generality quality.

Cheshire (2016) explores forms of abstraction, instantiation, and representation when understanding the topological definition of a continuous function. Both of his CRUME presentations contributed to his dissertation (Cheshire, 2017), where he investigates in what ways undergraduate students' schemas undergo accommodation in order to understand the axiomatic structures of topology. He found that when considering open sets, students would rely on their previous mathematical experiences instead of the axiomatic definition. They were reluctant to say that $[a, b)$ is open in the lower limit topology because their experience is mostly associated with the standard topology.

Gallagher and Infante (2019) collected data through weekly study sessions with students in an undergraduate topology course and came up with the Topology Proving Framework. They explore how visual representations influenced a student in her proving process and helped her recognize the main idea of the proof. Gallagher and Infante (2020) discovered mismatches between students' concept images and concept definitions. Even though the definition of open and closed sets allows for a set to be both open and closed, one participant's concept image for open and closed did not allow for this and thus hindered her progress on problems. Finally, Gallagher (2020) looks at how structural examples and gestures were used by a student as part of her thinking on topology proofs.

In addition to the above, there are some studies that document how topology can be challenging for first-time learners. Some studies focus on set theory, which is a necessary foundation for topology. Narli (2010) found that there were many misunderstandings of a topology lesson due to issues with set theory and nearly all of the students demonstrated notational mistakes. Further, the tasks in the study asked students to recall the definition of a topology and then check to see if a given set qualified as a topology. The students struggled to apply the definition of a topology to the given set, which is significant given that over half of the students were taking the topology course for their second time and the tasks were relevant to the lesson they recently had. In an earlier study by Zazkis and Gunn (1997), complexities were found in preservice teachers' understanding when set elements were sets themselves, which is exactly what the definition of a topology requires. Additionally, they found that the participants assigned meanings to the concepts of set cardinality, set element, subset, and the empty set that were not consistent with mathematical conventions.

Although there is not a lot of research in this area, that does not mean it is not worth investigating. What research does exist shows that students have difficulties with fairly prerequisite concepts. Some basic introductory concepts that are the focus of this thesis is that of a topology and a basis.

Topology and a Basis

Some topological concepts that I am investigating in this thesis are the definition of a topology, a basis for a topology, and the connections between these concepts. Munkres (1975) gives the following definitions for a topology and a basis:

> A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:
>
> (1) $\emptyset$ and X are in $\mathcal{T}$.
>
> (2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
>
> (3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
>
> A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***. (p. 76)

> If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called ***basis elements***) such that
>
> (1) For each $x \in X$, there is at least one basis element B containing x.
>
> (2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.
>
> (p. 78)

A topology on a space determines which sets are open in it and a basis can generate a topology on a space. When an individual is first exposed to these definitions, they may find them difficult to cipher and likely will want to see some examples. Munkres (1975) even admited, "The definition finally settled on may seem a bit abstract, but as you work through the various ways of constructing topological spaces, you will get a better feeling for what the concept means" (p. 76).

Since these definitions are usually new to students, schemas need to be constructed. Consider what this schema development may look like using Piaget and Garcia's (1989) triad framework. Referring to the definition in order to check the properties of a topology for basic examples occurs in the Intra- stage of development. It is possible that one could consider this to also be in the Inter- stage if the examples are in a student's previous knowledge, but the action of checking the properties directly is very much an Intra- stage type action since it is isolated with the definition. Topology students likely have a previous knowledge of open intervals on the real number line. Connecting this previously built schema with the topological definition of open happens in the Inter- stage. As more connections are made in the Inter- stage, a coherency is cultivated and the schema goes into the Trans- stage. This coherency may take a bit of time and effort to be developed, however, since these definitions have not been met before.

In this chapter, I discussed the origins of the schema construct, compared and contrasted some modern definitions, and explained Skemp's (1979) contribution to schema theory. How schema is related to understanding, formed, and updated were also considered. The final portion of the chapter was dedicated to the subject of topology and what literature exists with respect to undergraduate education. Part of that literature included my two pilot studies, which are featured in the next chapter, where I utilized Piaget and Garcia's (1989) triad framework and Skemp's (1979) schema theories to analyze preliminary data.

Chapter 3

Pilot Studies

3.1 Pilot Study 1

3.1.1 Introduction

As seen in Chapter 2, research on student experiences with topology content are scarce. Since most of the literature is recent, this was especially true when I began this project. To embark on this journey, I wanted to first see whether or not my experience with topology was a unique one. Did other students also initially struggle with topology? Is this topic worth investigating? Therefore I sought out any data that could tell me what undergraduate topology students actually do and understand regarding the topic. I started this inquiry by reviewing undergraduate students' final exams for their introductory topology course, which served as my initial pilot data. The findings of this pilot study are described in this section and can also be found in Berger and Stewart (2018).

I employed the idea of schema and its development to gain more insight into the transition towards advanced mathematics, specifically towards topology. This pilot study drew on Skemp's (1962; 1987) schema and Piaget and Garcia's (1989) Intra-, Inter-, and Trans- triad framework. I also considered Dubinsky and McDonald's (2001) definition of schema, which is only seen in the Trans- stage of development since that is when a coherent structure appears. In comparison, Skemp's (1987) definition of schema is seen throughout all stages of the triad framework, which is how I refer to the schema construct. In my view, an idea does not have to be fully developed or correct in order to be a part of a schema.

The purpose of this pilot study was to understand what stages of schema development undergraduate topology students demonstrate at the end of their introductory topology course. At this point in my research, schema development was defined by Piaget and Garcia's (1989) triad framework and I was able to collect data regarding the basis for a topology. As such, my research question for this pilot study was, "With respect to the triad framework, how developed are introductory topology students' schemas for a basis for a topology?" (Berger & Stewart, 2018).

3.1.2 Method

This was a limited, instrumental case study (Creswell, 2013) into introductory topology students' thinking about a basis for a topology. Twelve final exams were collected and de-identified from a senior-level undergraduate topology class at the University of Oklahoma. This study focuses on the first of the nine exam questions, shown below.

1. (a) Let $(X, \mathcal{T}_X)$ and $Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.

   (b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$

is an open map.

This question was chosen for a couple of different reasons, as explained by Berger and Stewart (2018).

> First, it is structured such that students who are in-between the Intra and Inter stages of their schema development for a topology generated by a basis can still answer part a. Then part b requires students to be at least in the Inter stage of schema development. This question quickly reveals students whose schemas are still in the Intra stage.
>
> Compared to other questions on the exam, this problem is more consistent with content from a typical introductory topology class. It would be unusual if the product topology on $X \times Y$ and the use of a basis did not appear in a beginning topology course, and therefore this problem is one that can be considered for use in future expansions of this study. This problem was also the first on the exam and therefore all of the students made an attempt on it. (p. 1043)

The data was initially coded by identifying the types of errors made in each part of the problem (see Table 3.1). I then went through a second round of coding for consistency. The coding results can be seen in Table 3.2. The cells indicate whether the error was seen in part a or b of the problem. Student 8 was not considered in the results since they submitted a nearly blank exam. I then analyze the proofs and errors with the triad spectrum.

3.1.3 Results and Discussion

The product topology on $X \times Y$ can be defined using the collection $\mathcal{B} = \{U \times V | U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ as a basis. In other words, the product topology contains all

Table 3.1: Types of errors in students' proofs (Berger & Stewart, 2018).

Code	Description
B	Left blank or contributed no original thoughts
IN	Issues with notation
IL	Issue of beginning proof with conclusion/other incorrect logical statement
NB	No reference to a basis
LE	Lacking elegance in proof
LL	Lacking logical flow
LD	Lacking direction

Table 3.2: Students' codes in pilot study.

Student	B	IN	IL	NB	LE	LL	LD
1				a	b		
2		b					
3							
4			b	a			
5			b	a	b	b	a
6			b	a	b	b	
7	b			a			
8	a,b						
9		b	b		b		
10			b	a			
11		b			b		
12			a,b	a	b		

basis elements, as well as unions (both finite and infinite) of the basis elements. The proof for part b involves three main components:

A. Noting that all open sets can be written as a union of basis elements (this part may be considered part of the definition of a basis depending on how it was presented in class).

B. Verifying that the projection of a union is a union of projections.

C. Noting that the projection map is an open map for basis elements.

Each instructor has their own discretion as to how detailed students' proofs are expected to be, but these three components should at least be noted somehow in the proof. Berger and Stewart (2018) gives an overview of the proof schema, seen in Figure 3.1. The arrows in the figure indicate previous knowledge that is needed in order to complete parts of the problem.

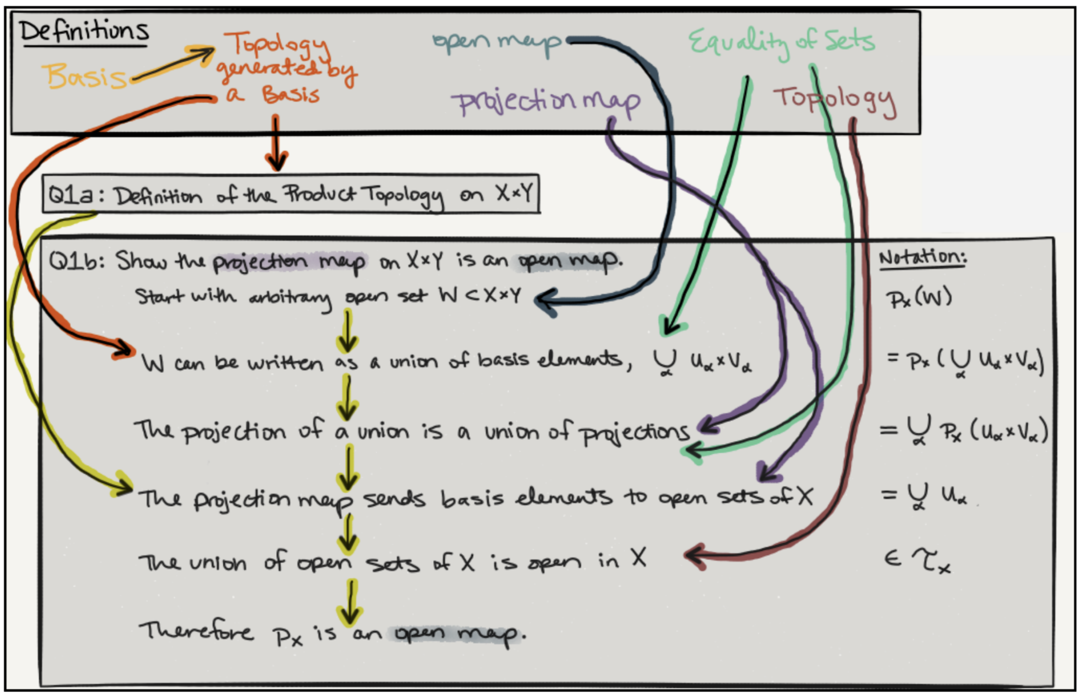
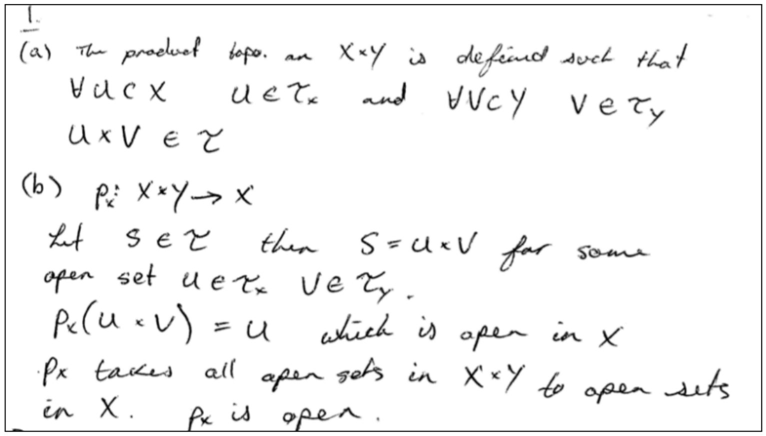


Figure 3.1: A proof schema for the problem (Berger & Stewart, 2018, p. 1044).

The Intra-basis stage

Seven of the eleven students did not use a basis to define the product topology on $X \times Y$ and six of those seven students claimed that the topology on $X \times Y$ is $\mathcal{T}_{X\times Y} = \{U \times V | U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$. A typical response of this type is shown in Figure 3.2.

The following argument from Berger and Stewart (2018) demonstrates why this response cannot be the topology on $X \times Y$ and why a basis is needed.


1.

(a) The product topo. on $X \times Y$ is defined such that
$\forall U \subset X \quad U \in \tau_X$ and $\forall V \subset Y \quad V \in \tau_Y$
$U \times V \in \tau$

(b) $p_x: X \times Y \to X$
Let $S \in \tau$ then $S = U \times V$ for some open set $U \in \tau_X$ $V \in \tau_Y$.
$p_x(U \times V) = U$ which is open in X
p_x takes all open sets in $X \times Y$ to open sets in X. p_x is open.

Figure 3.2: A response that does not utilize a basis (Berger & Stewart, 2018, p. 1045).

> Let $\mathcal{B} = \{U \times V | U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ be the basis for $\mathcal{T}_{X \times Y}$. $U_1 \times V_1$ and $U_2 \times V_2$ are both elements of $\mathcal{B}$ and therefore are also elements of $\mathcal{T}_{X \times Y}$. By the definition of a topology, $(U_1 \times V_1) \cup (U_2 \times V_2)$ is also an element of $\mathcal{T}_{X \times Y}$. Note, however, that the union is not of the same form as elements of $\mathcal{B}$ and cannot be in $\mathcal{B}$, as shown in [Figure 3.3]. So $\mathcal{B}$ cannot be the entire topology on $X \times Y$ [since not all elements are of the form $U \times V$]. (p. 1044)

Since the proof for part b depends on the use of a basis, the seven students who did not use a basis in part a were unable to write a complete proof for part b. They often showed component C of the proof but did not include components A or B. The students who had this type of response may not see the need for a basis, know when it is appropriate to use one, or know how to make use of it. If their basis schema was activated by this problem, they did not show it in their writing, suggesting that their basis schema was isolated from other actions, processes, and objects. Therefore these students' basis schemas are, at best, in the Intra-basis stage of schema development. They have not reached the Inter-

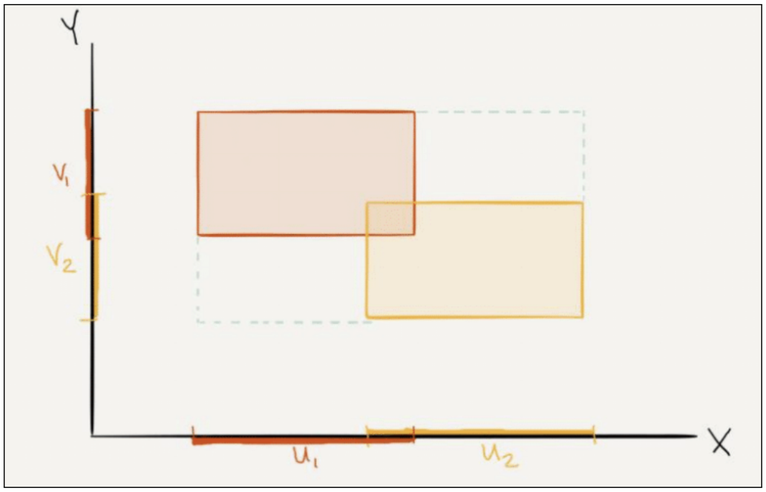


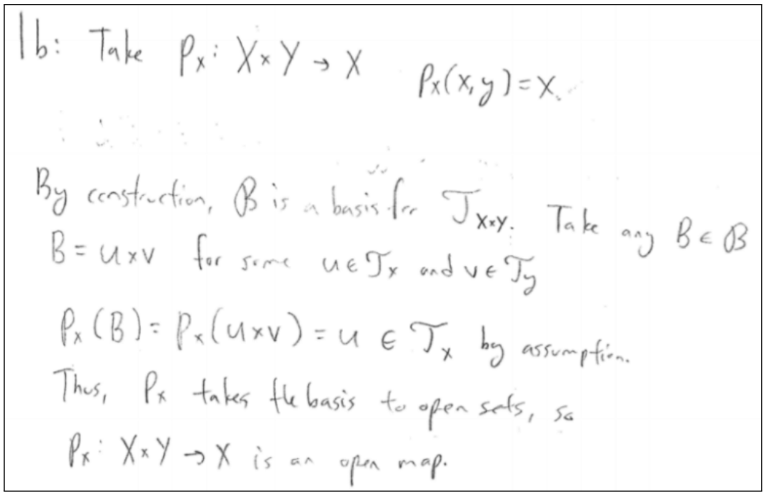
Figure 3.3: A visual representation of the need for a basis (Berger & Stewart, 2018, p. 1045).

basis stage since they are unable to connect a basis with other knowledge. These seven students are indicated in yellow in Table 3.2.

The Inter-basis stage

The four students who did make use of the basis to define the product topology had problems with incomplete proofs and notation in part b. One student (Student 3) wrote the proof for basis elements only and then immediately jumped to the conclusion of the proof without addressing components A or B of the proof. Their work is shown in Figure 3.4.

Whether or not the proof is considered to be correct depends on the instructor and the classroom norms. For this study, however, I am not as concerned about the validity of the proof as much as what it does (or in this case, does not) tell us about the student's basis schema. The use of the word "basis" might have been used as a substitute for component A of the proof, but we cannot assume that the student did or did not understand this. The same goes for component



1b: Take $p_x : X \times Y \to X$ $\quad p_x(x, y) = x$.

By construction, $\mathcal{B}$ is a basis for $\mathcal{T}_{X \times Y}$. Take any $B \in \mathcal{B}$

$B = U \times V$ for some $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$

$p_x(B) = p_x(U \times V) = U \in \mathcal{T}_X$ by assumption.

Thus, p_x takes the basis to open sets, so

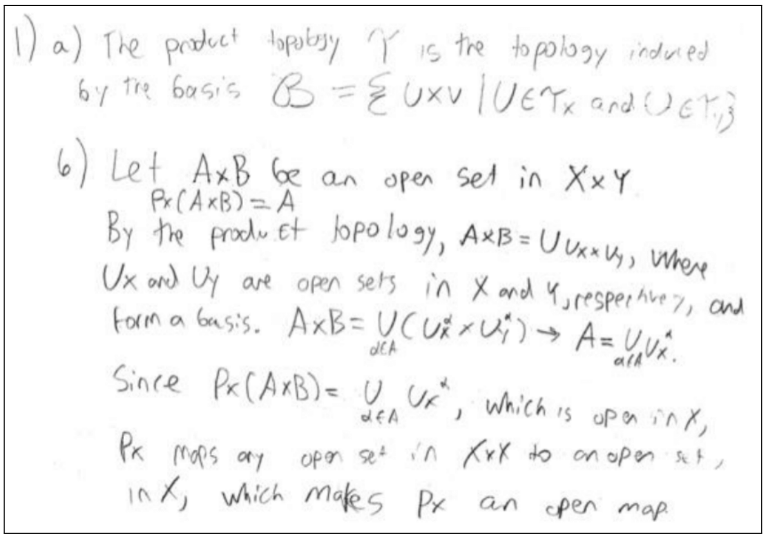
$p_x : X \times Y \to X$ is an open map.

Figure 3.4: Student 3's response, which has reached the Inter-basis stage (Berger & Stewart, 2018, p. 1046).

B, which may or may not have been considered trivial in the class. We can say that this student has reached the Inter-basis stage since they could relate a basis with the product space and send basis elements through the projection map. Due to the minimal amount of details in their proof, however, we cannot make any conclusions beyond this stage about their level of understanding. This student is indicated in blue in Table 3.2.

The Trans-basis stage

Students 2, 9, and 11 all had notational issues, even though they had the right idea for their proof. An interesting example of this is in Figure 3.5. Student 9 incorrectly used $A \times B$ as their arbitrary open set of $X \times Y$, yet still included component A of the proof by saying that $A \times B$ is a union of basis elements. This indicates that the student had an understanding of the need for component A in their proof schema, but they did not understand how to denote the arbitrary open set. The student has a coherent structure of a basis and its relationships to the product space, open sets, and unions, but their argument could be clearer with some corrections in notation. This student's response shows that they have reached the Trans-basis stage, but there are still some notational details needed to improve their overall schema. The students who reached the Trans-basis stage are indicated in Table 3.2 in green.


1) a) The product topology Υ is the topology induced by the basis $\mathcal{B} = \{U \times V \mid U \in \Upsilon_X \text{ and } V \in \Upsilon_Y\}$

b) Let $A \times B$ be an open set in $X \times Y$

$P_X(A \times B) = A$

By the product topology, $A \times B = \bigcup U_X \times U_Y$, where U_X and U_Y are open sets in X and Y, respectively, and form a basis. $A \times B = \bigcup_{\alpha \in A} (U_X^\alpha \times U_Y^\alpha) \rightarrow A = \bigcup_{\alpha \in A} U_X^\alpha$.

Since $P_X(A \times B) = \bigcup_{\alpha \in A} U_X^\alpha$, which is open in X, P_X maps any open set in $X \times X$ to an open set in X, which makes P_X an open map

Figure 3.5: Student 9's response, which has reached the Trans-basis stage (Berger & Stewart, 2018, p. 1046).

3.1.4 Concluding Remarks

The examples discussed in this study demonstrate three different places along the triad spectrum where student's schemas could be. Even though this problem came from a final exam at the end of the semester, a majority of the students surprisingly were still at the Intra-basis stage or lower in their schema development. I cannot comment on why this is since no data was collected regarding the class that these participants were in. This also means that what was considered to be trivial in the course is unknown, making it difficult to analyze student's responses that are similar to Figure 3.4. These students' schemas may or may not include components that were replaced with equivalent, but highly simplified, statements. It is also unknown how much the instructor emphasized the need for a basis for certain topologies or whether or not the students had seen this problem on a previous homework assignment, both of which would affect the students' schemas.

The other limitation to this study is that it is impossible to physically see the schema of another person, so at best conjectures can only be made about participants' schema development, especially since written proofs were analyzed. Therefore interactions with participants became a major requirement for data collection in the main study.

Other methods for the main study that were informed by this pilot study were that data collection should occur at both the beginning and the end of a semester, and interactions between participants are desirable, in either a partner or group setting. Participants should explain their schemas out loud so that I can observe progress in the development of their schemas over time. Additionally, the students in this pilot study were taking a final exam. They were likely stressed and they knew their instructor would be grading their proofs. Ideally, participants in the main study should explain their thought processes in a low-stress environment, away from mathematical authorities, like their instructor. This can help ensure that the data collected is more aligned with how students are actually thinking, not with statements they think their instructor wants to hear. Finally, a wider variety of questions over a basis for a topology will be asked to better understand what concepts and connections the basis schema actually consists of for students.

3.2 Pilot Study 2

3.2.1 Introduction

The section focuses on Pilot Study 2, which was reported by Berger and Stewart (2019). After Pilot Study 1, I continued to study and tried to make sense of Skemp's (1979) model of intelligence, of which schema is a central focus. A thorough explanation of this model will be given in Chapter 4, but I will give a short description here to make sense of Pilot Study 2.

A person or object has a present state that they are currently in, and a goal state that they would like to be in. Skemp (1979) defined the operand as "that which is changed from one state to another and kept there" (p. 41) and the operator is "that which actually does the work of changing the state of the operand" (p. 41). Finally, a director system is "that which directs the way in which the energy of the operator system is applied to the operand so as to take it to the required state and keep it there" (p. 41-42). Berger and Stewart (2019) summarized the example that Skemp gave to communicate these theoretical ideas.

> He referred to the temperature of an oven in many instances. Say an oven is at room temperature and needs to heat to 400 degrees Fahrenheit. The present state is the current temperature and the goal state is to reach 400 degrees Fahrenheit. The operand is the interior of the oven and the operator is the temperature of the oven. The thermostat in the oven is the director system. If there is a new goal state of 350 degrees Fahrenheit, the same director system (the thermostat) will be used.

Using the the idea of a director system, Skemp (1979) defined a schema as the highly abstract concepts that give a director system flexibility when states change. His work with director systems was also used by Olive and Steffe (2002) to build "a theoretical model of children's constructive activity in the context of learning about fractions" (p. 106) Skemp (1979) listed several qualities that help determine the strength of the schema, which were used to analyze this pilot data. The qualities of schema that were focused on and the definitions of certain words are shown in Table 3.3. In this pilot study, I examined students' development of their schemas and their qualities based on Skemp's (1979) model. The research question to guide this study was: "What qualities of schema do Topology students demonstrate?" (Berger & Stewart, 2019).

3.2.2 Method

The lessons learned from Pilot Study 1 informed the next round of data collection. The participants were first year graduate students who were enrolled in a graduate topology course at the University of Oklahoma. There were four participants, which were interviewed as two pairs. Each pair did two task-based interviews together, the first in the first month of the semester, and the second in the last month of the same semester. Pairs were used to try to get the participants to demonstrate their ideas and discuss how they think about the tasks to their partner, making their thoughts more observable. No data was collected regarding what took place in the classroom before, after, or between these interviews. The participants were given a definition sheet and a task sheet, found in Appendices A and B respectively. Each interview began with a period of time where the participants could look through and work on the tasks individually. After

Table 3.3: Some qualities of a schema (Berger & Stewart, 2019, p. 28).

Qualities of a schema	Definitions
(ii) "Relevance of content to the task in hand (rather obviously, but not always met)." (p. 190)	
(iii) "The extent of its domain." (p. 190)	Domain: "The set of states within which (and only within which) a director system can function, i.e., can take the operand to its goal state and keep it there, provided that the operators are capable." (p. 312)
(iv) "The accuracy with which it represents actuality." (p. 190)	
(v) "The completeness with which it represents actuality within this domain." (p. 190)	
(vi) "The quality of organization which makes it possible to use the concepts of lower or higher order as required, and to interchange concepts and schemas. (The vari-focal part of the model, linked with the idea of interiority.)" (p. 190)	Vari-focal: "A way of describing the different ways in which the same concept or schema can be viewed, from a simple entity to a complex and detailed structure." (p. 316)
(vii) "By a high-order schema, we mean one containing high-order concepts…This determines its generality…" (p. 190)	
(viii) "The strength of the connections." (p. 190)	
(ix) "The quality of the connections, whether associative or conceptual." (p. 190)	
(x) "The content of ready-to-hand plans…" (p. 191)	Plan: "A path from a present state to a goal state, together with a way of applying the energies available to the operators in such a way as to take the operand along this path. A plan is thus one essential part of the director system." (p. 314)

that, they worked on the tasks as a pair, explaining their thoughts to each other and coming to a consensus for each problem. In the final part of each interview, the pairs were asked follow up questions about what they thought was needed to complete each task. They were also asked about their background with Topology. In the interviews that occurred during finals week, they were additionally given their work from the early interview and asked to discuss their progress between then and the late interview. These prompts can be found in Appendix C. Each interview was video recorded and then transcribed. If the participants utilized the white boards, their written work was also transcribed. In the transcriptions, a scribble indicates that the pair erased something on the board.

This pilot study focused on the same task as Pilot Study 1. This choice was made because it is consistent with Pilot Study 1, but additionally, this problem requires a higher-order schema for a basis. Because of this, I hypothesized that the data would be more informative about certain qualities of schema needed. After the data was transcribed, some themes were established from Skemp's (1979) model based on the qualities of schema. Further, the conceptual framework shown in Table 3.4 was created. After creating an ideal proof with ideal qualities, the data was examined against it.

The step by step proof of part b has been illustrated in Table 3.4. The lower-order concepts needed for this proof, as well as the schema qualities ideal for completing each portion of the proof, are shown. More qualities of schema can be applied in each step, however, the listed qualities were focused on based on my experience with this problem and the data set. After being transcribed, the data was divided up by what portion of the proof it aligned with. Each portion was analyzed in terms of Skemp's (1979) qualities of schema. Ideally, learners' schemas become more structured as they go throughout the course, but this study was not solely focused on comparing the early and late interviews. Instead, I looked for changes in students' schemas and what qualities were involved in those changes.

Table 3.4: Qualities of each portion of the task in Pilot Study 2 (Berger & Stewart, 2019, p. 30).

Portion of proof	Proof for part (b)	Explanation of Proof Step	Lower-order Concepts Needed	Qualities of Schema
b1	Consider $p_X(W)$ where $W \subset X \times Y$ is an arbitrary open set of $X \times Y$.	Start with an arbitrary open set of $X \times Y$ and see where p_X sends it.	-Open map	-Plan -Domain -Relevance
b2	Now $p_X(W) = p_X(\cup_\alpha (U_\alpha \times V_\alpha))$ where $U_\alpha \times V_\alpha \in \mathcal{T}$ are basis elements.	By the definition of a basis, W can be written as a union of basis elements.	-Topology generated by a basis -Equality of sets	-Strength -Quality -Domain
b3	Note $p_X(\cup_\alpha (U_\alpha \times V_\alpha)) = \cup_\alpha p_X(U_\alpha \times V_\alpha)$.	The projection of a union is a union of projections.	-Projection map -Equality of sets	-Generality -Domain -Accuracy -Strength -Quality
b4	Now $\cup_\alpha p_X(U_\alpha \times V_\alpha) = \cup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{T}_X$.	The projection map sends basis elements to open sets of X.	-Projection map -Definition of the product topology on $X \times Y$	-Accuracy -Completeness
b5	Since $\cup_\alpha U_\alpha \in \mathcal{T}_X$, $p_X(W) \in \mathcal{T}_X$ and p_X is an open map.	The union of open sets of X is also open in X.	-Topology -Open map	-Strength -Vari-focal

3.2.3 Results and Discussion

For the purposes of this pilot study, only one of the pairs of participants were considered: Brandon and Kyle (pseudonyms). Brandon had not completed a Topology course before the initial interview and Kyle had previously taken an introductory Topology course, so their experiences with the subject matter were limited. Additionally, this pair was more interactive with each other and took their time discussing each task out loud.

Early Interview

In their early interview, Brandon and Kyle had a proof similar to the Intra-basis stage proofs in Pilot Study 1. The pair incorrectly stated that the product topology consisted of all sets of the form $U \times V$, not the unions of such sets, and they did not mention a basis at all. For b1, they took their arbitrary open set to be $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Their director system functioned appropriately, but their mistake in part a led them to start part b at the wrong present state. From there, they followed the definition of the projection map (part of b4) and immediately got that the output would be U, which they said was open in X. Since this step was a fairly straightforward computation for the pair, their ideas were accurate and complete, but only relative to their incorrect definition in part a. Because they began part b from a poor present state, they did not provide any information about their topology schema for b2, b3, and part of b4. With regards to their schemas for a basis, no claims were made since their work here provided no evidence regarding a basis.

Late Interview

In the late interview, Brandon and Kyle correctly defined the product topology by generating it with a basis. For b1, Brandon quickly wrote "W open in $X \times Y$" on the board, and neither Brandon nor Kyle discussed it. Brandon had an available plan and began executing it without discussing it with Kyle. Mathematically, this was relevant and fit within the domain of the problem.

For b2, Brandon wrote W as a union of basis elements, but the notation was not quite clear, so Kyle tried to improve it. He came up with a notation that a lot of B's and β's in it, which he still did not like. Brandon was less concerned about the notation, so they moved on. They both knew how to do b2, but struggled to denote it in a satisfactory way. This was supported when Kyle said "W is the union of elements of...some arbitrary union of elements of that B thing [referring to the basis they wrote in part a]." This statement demonstrates a strong conceptual connection between open sets and a basis. Again, this was appropriate within the domain of the problem.

Brandon and Kyle started off b3 by quickly saying that the projection gives you a union of open sets, but did not give a reason why. Kyle started thinking about if "weirder things" could occur, which sent the pair into a discussion about whether the projection map was open or closed, trying to check their initial idea. They eventually agreed that their conclusion was valid, but did not add anything to their written proof on the board. They discussed what exactly the projection did (b4), which prompted Kyle to suddenly write up the proof on the board (see Figure 3.6). They did not explicitly discuss b5.

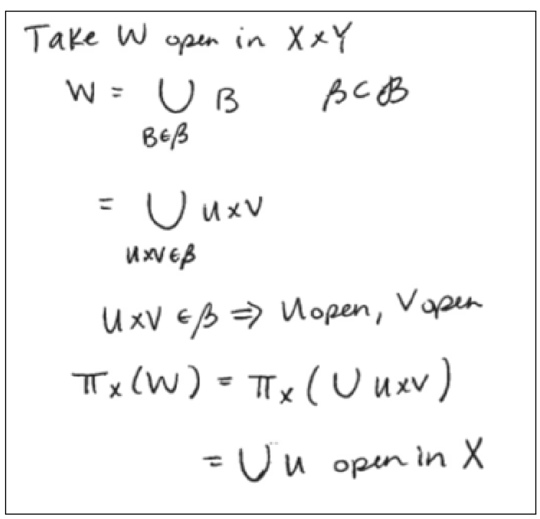


Figure 3.6: Brandon and Kyle's proof (Berger & Stewart, 2019, p. 32).

The pair got off topic during their b3 discussion, which demonstrates a loss of relevance to the task and not as strong of connections between the projection map and unions. At the beginning and end of b3, however, they did demonstrate accuracy in their statements about what the projection map does. They combined b4 and b5 of the proof and did not make any conclusion statements for their proof. Their final statements are accurate, but do not give evidence for complete ideas, leaving me unable to say anything conclusive about what qualities they demonstrated for b4 or b5.

Towards the end of the late interview, the pair was asked to reflect on their work from the initial interview. After reviewing their previous work, Brandon and Kyle quickly confirmed that they had not considered a basis in the first interview. When asked about what could have aided in correcting that mistake, Brandon responded that what corrected it for him was getting feedback on his homework "...with that specific thing being torn apart on it." Kyle admitted that he didn't "get the basis topology stuff at all when [they] were going over it in class" but later realized the importance when reading through the textbook. These testimonies support the idea that understanding the basis concept was a long-term process for them. Brandon also commented on their earlier work, saying that "it just seems natural to just go to one thing instead of considering the most general thing." This reflection suggested that the generality and relevance of a schema are not necessarily intuitive.

3.2.4 Concluding Remarks

In this preliminary case study, Berger and Stewart (2019) summarized the results.

> We saw that accuracy and completeness are not typically a difficulty,

> but rather the generality of a schema, strength of connections, and the relevance of a schema can be difficult to navigate. Both Brandon and Kyle demonstrated strong, conceptual connections between lower-order concepts and a basis in the final interview, but not in the early interview. In the final interview, the pair demonstrated a weaker connection between the projection map and unions.

These results are highly specific to these participants and task, meaning that additional data needs to be considered if I want the results to be generalizable in any way. This data set, along with Brandon and Kyle's other tasks and other participants, is explored in further detail in Chapter 6 by utilizing the framework developed in Chapter 4.

Chapter 4

Theoretical Framework

A theoretical framework or model is a necessary part of research. Kieran (1998) distinguished between a model and a theory in mathematics education research and discusses how the development of a model can be a research result itself:

> So, we are beginning to see more and more often that the reporting of research results is not simply the enumeration of the observed empirical facts but also the description of a model that has been developed (or an existing model that has been further refined) to explain what has been identified; that is, that models are as much the results of doing research as are the empirical observations. (p. 213-214)

A model is not intended to explain completely, but instead is meant to represent or simplify a situation and is meant to evolve over time. "[Models] tend to be situated within the context of a more global, overarching theory" (Kieran, 1998, p. 218). A theory, on the other hand, is a conceptual system that should explain and predict experiences. Dubinsky and McDonald (2001) offered six features of models and theories. They can "support prediction, have explanatory power, be applicable to a broad range of phenomena, help organize one's thinking about complex, interrelated phenomena, serve as a tool for analyzing data, and provide a language for communication of ideas about learning that go beyond superficial descriptions" (p. 275).

In the first half of this chapter, I will explain Skemp's model for director systems. This will then be used along with schema theory to develop a framework for students' learning of topology, which is described in the second half of this chapter.

4.1 Director Systems

One of the significant roles of a schema is to take in new information, organize it, and then let it inform our thoughts and actions (Skemp, 1979). Bartlett (1932) pointed out that incoming information does not direct our actions without interacting with a schema first.

> A new incoming impulse must become not merely a cue setting up a series of reactions all carried out in a fixed temporal order, but a stimulus which enables us to go direct to that portion of the organised setting of past responses which is most relevant to the needs of the moment...An organism has somehow to acquire the capacity to turn round upon its own 'schemata' and to construct them afresh. (p. 206)

The theoretical framework that I am presenting here is based on ideas found in Skemp's (1979) book, *Intelligence, Learning, and Action*. In this book, he discussed director systems in great detail (Figure 4.1). Director systems are responsible for directing all of our actions, both physical and mental actions.

This includes problem-solving, proof-writing, and other ways of thinking about mathematics, but I will start off with a more basic, physical example. In every action, we have a present state and a goal state, that is, a state at which we are and then would like to be, respectively. These are represented in Figure 4.1 with the letters P and G. The operand is the object, possibly ourselves, whose state is being changed, and the operators are what is changing the state of the operand.

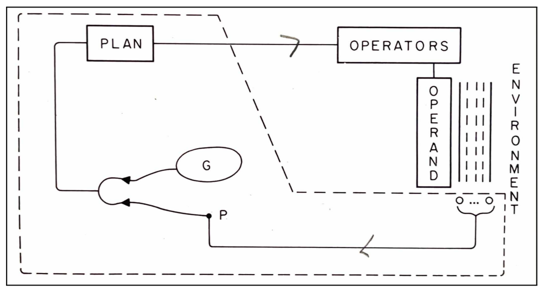


Figure 4.1: Skemp's (1979) director system (p. 49).

Consider an example with golf. Suppose that I am starting a round on the first hole. I tee up my golf ball on the tee box and am getting ready to hit. My goal state is to get the ball into the first hole, which is likely hundreds of yards away. My present state is that the ball is on the tee on the first tee box. The operand in this example is the golf ball and the operators are myself and my golf clubs, swinging to hit the ball. So I step up and hit my first shot using my driver. Let's suppose the ball lands in the middle of the fairway and stops about 125 yards away from the hole. The empty circles in Figure 4.1 represent sensors of some kind that can determine the present state of the operand. In this example, my senses, primarily sight, are the sensors that can see where the ball ended up after that first shot. I can also compare where the ball currently is and where I want it to be, which in this case, is a physical distance. I serve as the "comparator" in this example, which is what is able to compare the present and goal states. The comparator is represented in Figure 4.1 with the C-like arc connecting P and G.

Once I have identified the ball's updated present state and confirmed that it has not reached the goal state, I develop a plan for trying to make it reach the goal state. In this example, the plan is to swing my 8-iron so that I land the ball on the green, ideally to roll into the hole. Then the cycle of the director system continues: The operators (myself and my 8-iron) act on the operand (the ball), and my sensors (my eyes) tell me that the new present state of the ball is that it is on the green, about five feet away from the hole. The updated plan is to take my putter and putt it into the hole. If I make it, then the ball has reached the goal state and the director system is done. If I miss the putt, then the director system cycles around again, continually comparing where the ball is relative to the hole and updating the plan accordingly.

Now that was an example of a director system for a specific hole on a specific course, which we would classify as a Δ_1 director system, due to its physical operand. When the operand of a director system is a Δ_1 director system itself, Skemp called that a Δ_2 director system. For example, let's suppose that I did not perform so well on that first hole. Let's say that my drive landed to the right of the fairway and there was a tree between my ball and the green. I would want to improve my performance; specifically, I would want to improve my swing of the driver acting on the ball. The present state of my Δ_1 director system is that the shots keep fading to the right and the goal state is that the shots go straight, towards where I aim. Improving my golf swing (the functioning of this Δ_1 system) would be a Δ_2 goal. My sensors would be my experience and confidence level as I practice. The plan is to continue practicing and the operators would be my muscle memory. Note that this example is what Skemp would call a *teachable* system, as opposed to an innate director system. As Skemp put it, "learning...involves the action of a delta-two system on a teachable delta-one system" (p. 75).

It should be noted that the outcomes of our director systems are not disjoint from our emotions. Usually failure of a Δ_1 director system is associated with negative feelings (disappointment, anger, frustration, and any other emotions you feel when you hit a bad golf shot). However, failure of the Δ_1 system can still contribute positively towards a Δ_2 goal (through things like learning from mistakes) and this can bring about positive emotions. This is why people, especially younger children and athletes, can enjoy repetitive practice as much as they do. Through practice, making mistakes, and trying again, there is progress towards achieving the Δ_2 goal, which brings about positive feelings such as satisfaction, relief, confidence, or security.

To relate this to mathematics, which is a bit more mental and abstract, we can still distinguish between a Δ_1 and Δ_2 director system. In a Δ_1 director system, the mathematical operand may be physical, or possibly able to be represented physically, through things like symbols. For example, consider solving a linear equation. The linear equation can be written on a chalkboard for all to see and the goal of solving the equation is either done successfully, or it isn't. A mathematical Δ_1 director system, like this, looks more like mathematics being performed and mathematical facts being recalled and utilized. On the other hand, if this Δ_1 director system becomes the operand, then we would be talking about a Δ_2 director system whose goal state is to be able to solve any linear equation quickly, correctly, and without difficulty. This director system is less about performing specific mathematics than it is about thinking mathematically in general. Instead of specific mathematical facts being recalled, mathematical skills would come into play here.

In the golfing example, how did I know what club to use next and in what way to swing it? In other words, how did I know what plan to make and how to update it? Those strategies were informed by my *schema* for golfing. Skemp defined a schema as a structure of connected concepts that determines the effectiveness of our director systems. Not only is it composed of many concepts connected together, but a schema can also be regarded as an abstract concept itself. Skemp gave the visual in Figure 4.2 as a schema, where P and G are concepts that are connected together in many ways. He also gives the following explanation.

> The idea of a cognitive map is a useful introduction, a simple particular example of a schema at one level of abstraction only, having concepts with little or no interiority, and representing actuality as it has been experienced. A schema in its general form contains many levels of abstraction, concepts with interiority, and represents possible states (conceivable states) as well as actual states. (p. 190)

My golfing schema at one level of abstraction might be applied on a specific golf course on a specific day, but then there is another level of abstraction where I can talk about the general actions of hitting with woods and irons, chipping, and putting. I have possible states on an actual golf course, but then I also have conceivable goal states that I would like to achieve in general, like making a hole-in-one, for example, regardless of what the course is.

My golfing schema contains everything I have learned over my lifetime about golf, including how to hit a golf ball, how to decide which club to hit for certain yardages, what effect the type of grass may have on a putting green, etc. This schema is what informs the plans in my director systems and it gives my

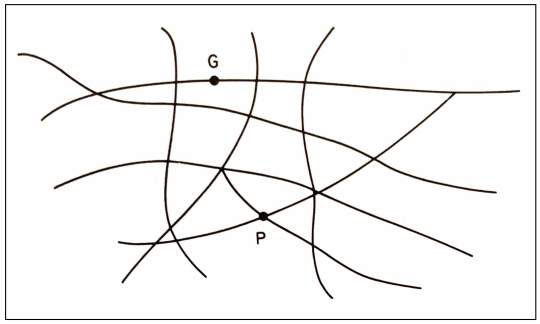


Figure 4.2: Skemp's (1979) visual for schema at one level of abstraction (p. 144).

director systems flexibility when states change (like if the wind changes, for example). Skemp said, "The greatest adaptability of behavior is made possible by the possession of an appropriate schema, from which a great variety of paths can be derived, connecting any particular present location to any required goal location" (p. 169). To go back to the mathematics example, I have what we will call a "solving" schema, which is what informs my plan for solving linear equations. Let's say I need to solve $3x - 29 = 19$. My schema contains information like "adding the same value to both sides of an equation only changes it superficially" and that, among other concepts, will be utilized in the plan for solving this. Now if I change the problem to be $3x - 292 = 19$, the exact same strategy for the first problem (adding 29 to both sides) will not be applicable to this new problem. Having well-connected, available concepts in my solving schema is what will give me the flexibility to solve this problem also.

It is important to discuss what Skemp called the domain of a director system (Figure 4.3). The domain consists of all present states such that there exists a plan (or path) (f) from the present state (P) to the goal state (G) and the director system is able to take the operand along the path. In other words, the director system can function successfully if and only if the present state is within its domain. If a state is outside of the domain, then there is no path currently available to take the operand to the goal state. The boundary of the domain was referred to by Skemp as the 'frontier zone'.

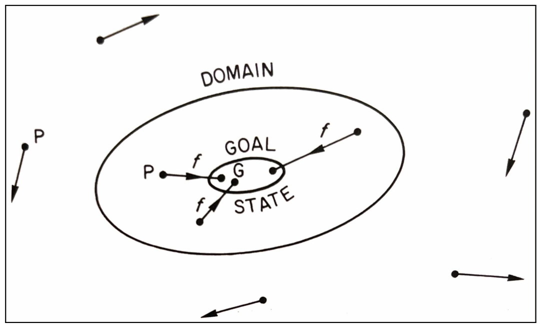


Figure 4.3: Skemp's (1979) domain of a director system (p. 61).

Consider a student solving a linear equation similar to what was presented above. If the student cannot solve the equation at all, then their current state is outside of the domain of their 'solving' director system. If they can solve the equation confidently, then their present state is within the domain. On the other hand, there is a third case where the student can make an attempt at solving the equation with some level of confidence, but not full confidence. This is an example of when their present state is in the frontier zone. When the student has practiced enough to reach full confidence in what they are doing, the frontier zone is transformed into domain. This should give us great relief that the domain and frontier zone are not fixed. Unlike what some students try to convince us of, they don't have to stay in the frontier zone and never be confident in what they are doing. With the right amounts of practice, questioning, discussion, and feedback, they can move the states that are in the frontier zone into established domain and can then extend their frontier zone to new concepts to work on.

Sometimes a student might practice a type of problem so much that there is only one path they can go on in their domain. If we're thinking of the map in Figure 4.2 as possible paths from a present state, P, to a goal state, G, this would be as if only one path was activated and all other paths connecting P and G don't exist at all. This is what most would call a 'habit'. This plan can be over practiced and can become fixed if the student is too dependent on it, which would leave the student unable to adapt if presented with a similar problem that needs a slight adjustment in the plan. In thinking about problem solving, Skemp (1979) said that "one requirement is the ability to set aside existing pre-formed habits and make equally available for use the whole of an appropriate schema" (p. 174). Having an appropriate schema available in the domain gives multiple paths between the present state and goal state and allows flexibility when presented with a new problem. It also allows for what Skemp called intelligent learning. This learning comes from someone being able to reflect upon their process (or director system) and transform their frontier zone into domain. This is opposite of surface learning, which is rote memorization.

4.2 Three Stages for Successful Action

Skemp (1979) admited that "an appropriate schema is necessary, but not sufficient, for successful action" (p. 167). He distinguished between what he calls Knowing That, Knowing How, and Being Able, which are all necessary for successful functioning of a director system.

4.2.1 Knowing That

Knowing That is the possession of an appropriate schema and activation of the necessary concepts within the schema. It might be helpful to reference Figure 4.2

again for a visual of a schema, where P and G are concepts within that schema. When faced with a task, goal state, or information in general, certain concepts come to mind, or are activated. This activation of concepts in a schema is part of Knowing That.

Because concepts within a schema are connected, activating one concept can activate neighboring concepts at the same time. Or, if they are not fully activated, then at least the threshold for activating them has been lowered so that they are more easily activated. It is possible that an entire path is activated at once, which we call intuitive path finding, or informally, “following your nose”. It is possible that the entire path isn’t activated at once, but maybe just the first half of it is activated and the second half activates once the halfway point on the path has been reached. This could still be considered intuitive path finding because the path is naturally being activated without too much conscious planning.

4.2.2 Knowing How

On the other hand, what if activating the present state does not lead to an intuitive path? Deriving a path, whether intuitively or not, is what Skemp called Knowing How. If a path cannot easily be activated intuitively, then a plan needs to be made consciously. This can be difficult the first time, but the next time a similar situation occurs, then it might be easier to remember the path that was derived before and it can be applied now. This is how routine functioning is developed. Something is done often enough, it becomes routine.

For example, when golfing, consider the goal state of having your ball on the green (or even better, in the hole) and an unfortunate present state of your ball being in a sand trap. If you have never golfed before, consider what your plan might be for getting your ball out of the sand trap. If you have golfed before, try to remember the first time you tried to hit out of a sand trap. Having never encountered this present state before, there is likely not an intuitive path for this situation. So maybe you come up with the plan to try to hit the ball just the same as you would from the fairway. From experience, I can say that you will likely find that strategy problematic, especially if you are in a trap with fine sand. With some practice and coaching, you will eventually learn that a strategy to get the ball out is to hit the sand right behind the ball, so that the power behind the sand, not the club, is mostly lifting the ball out of the trap. Encountering that state frequently, whether in practice or on the course, will give you enough opportunity to make that plan routine. Each time the plan is activated, it is easier to strategize because it has been done many times before. This is one example of reflective planning. Note that reflective planning is not the same thing as a habit, because reflective planning allows the plan to be modified for specific circumstances, while a habit remains fixed. (Maybe the sand trap is on a course in rural Oklahoma and so the sand isn't fine at all. This means you would want to modify your strategy and hit less sand and more ball.)

Now let's think about this in terms of a mathematical example. For some first-year students, solving a linear equation does not come intuitively for them. Their path is not activated when they see the problem statement. We, as instructors, hope that with enough instruction, practice, and reflection on their process, they get to a point where they see that solving linear equations can occur from roughly the same plan every time. By the end of a semester, they develop some routine, yet flexible, strategy for that type of problem.

4.2.3 Qualities of a Schema

Deriving a plan depends on what connections exist between concepts in our schema and the quality of those concepts and connections. Skemp (1979) listed twelve qualities of a schema, saying that they influence the Knowing How and "collectively determine the effectiveness of the director system" (p. 190). Therefore it will be important to discuss these qualities before moving on to the Being Able stage. Although Skemp (1979) listed twelve qualities (p. 190-191), I will focus on eight of them here. I will use these qualities as a sort of measure for the effectiveness of Knowing That and Knowing How.

(i) **Relevance** of the schema to the director system. If the activated concepts are not relevant, a successful plan cannot be devised. As an instructor of first-year mathematics, I have seen this before on exams when I ask students to find the domain of a function and they instead solve for the zeros of the function. When seeing the function, their 'set equal to zero and solve' schema was activated instead of their 'domain' schema.

(ii) **Accuracy** of the schema in actuality. Skemp distinguished between reality and actuality by using 'actuality' in reference to the physical world that we and others experience, while we all individually have our own unique 'reality' that we live in. When we reference the accuracy of a schema, it is accurate (or not) in comparison to actuality. In mathematics, this is usually the quality that assessments are focused on.

(iii) **Completeness** of the schema in actuality. Does the schema contain all of the necessary concepts, or is some information unknown? Children learn to count before they learn about the existence of negative numbers or fractions.

Their schema of the real numbers is incomplete at first, even though negative numbers and fractions exist in actuality.

(iv) **Organization** of concepts. Skemp used the term 'vari-focal' to describe the conceptual hierarchy of a schema and 'interiority' to convey how a concept could have an interior to 'zoom in' on and view without losing detail. For example, consider a schema for playing golf. This schema could have three concepts in it: long distance hitting, short distance hitting, and putting. However, in zooming in on putting, we find more structure, like a putter, putting distances, and how to read the slopes of the green. On the other hand, if we zoom out of the playing golf schema, we find that it is one component of a broader golf schema, which also includes concepts like golf courses, golf merchandise, and even a social concept that contains things like etiquette. At each level, the concepts can have interiority and be a schema in their own right, or they can be viewed as a simple entity that is part of a larger structure. Skemp said that the quality of this organization "makes it possible to use concepts of lower or higher order as required, and to interchange concepts and schemas" (p. 190).

(v) **Generality** of a schema. A schema that contains higher-order concepts in its organization is more general. This quality is certainly applicable in abstract mathematical topics, like topology, where lemmas and theorems are applicable to a wide variety of spaces and applications.

(vi) **Strength** of connections within a schema. Activating one concept can activate neighboring concepts through the connections between them. If the neighborhood around an activated concept is large, that is, if it contains many other concepts that also become activated, we consider those connections to be 'stronger'. However, the connections are weaker if activating one concept does not also activate the concepts that are connected. The strength of connections determines whether or not an entire path can be found intuitively or not.

(vii) **Type** of connections within a schema. (Skemp called this the 'quality' of connections, but I would like to avoid confusion with this overall list of 'qualities'.) Skemp categorized connections as either associative or conceptual. Associative connections relate two concepts through rote memorization, like a connection between a person and their phone number. Conceptual connections are relational links through patterns, rules, logic, or some other kind of action, and can be applied to multiple scenarios. For example, a conceptual connection between Eastern Daylight Time (EDT) and Central Daylight Time (CDT) is that CDT is one hour behind EDT. This connection can be applied at all times, so a person in Savannah, Georgia, with this connection can always determine the current time in Norman, Oklahoma. Skemp points out that once a conceptual connection is realized, you can "put a name to it, communicate it, and make it an object for reflective intelligence" (p. 188). A conceptual connection can be applied to a wide variety of situations, whereas an associative connection only has the one application. Because of this, a schema is considered to be 'better' or 'improved' if the ratio of conceptual connections to associative connections increases.

(viii) **Plans** that result from routine functioning. Skemp called some of these *ready-to-hand* plans because they are available to use from previous experiences. The content of these plans stay intact with the schema and are flexible if necessary. The more ready-to-hand plans available in a schema, the more adaptable and effective it can be. If a routine plan is routinized so much that it weakens the connections to neighboring concepts, this routine plan becomes a *fixed plan* and adaptability is lost. Dogan (2019) recently looked at linear algebra students' schemas and found that they primarily contained three specific routinized plans, some of which were activated based on the task at hand. "Some participants displayed instances where ready-to-hand plans were activated, but only on a few occasions" (p. 1180). Dogan (2019) also gives some open problems with the quality of plans: What are the implications on the growth of one's schema functioning with a limited number of plans? What factors contribute to the activation of ready-to-hand plans? What are the implications of applying the same ready-to-hand plan with different representations or underlying ideas?

4.2.4 Being Able

The possession of a schema (Knowing That) helps in deriving a plan (Knowing How) for the director system. The qualities of the schema affect the quality of the plan for the director system. Even if the plan is one that should lead to the goal state, it is still possible that the directory system is not successful. Being Able is having the ability to put the plan into action, and could very well rely on the operators of the system. A physical example is easiest to demonstrate this distinction. A golfer might know how to hit a golf ball, but if they have some kind of physical injury, they might not be able to actually swing their club. The "operator" that acts on their operand is broken. A basic example that we have all experienced at some point is when we have the right Knowing That and Knowing

How for a mathematics problem, but we make some kind of miscalculation in the middle of actually working it out. This miscalculation, although it is able to be corrected, temporarily keeps us from successfully answering the problem. This is why instructors usually encourage their students to include checking their answers in their plans.

There is a spectrum of quality with Being Able that should be discussed which Skemp called "skill". A director system can function inefficiently and slowly, or it can run smoothly and quickly. There is successful action at both ends of the spectrum, but the more skillful end is usually desirable. In fact, the more skillful end of the spectrum is usually the goal state of a Δ_2 director system.

4.3 A Framework for Successful Action in Learning Topology

Now that I have discussed some physical examples and lower-level mathematics examples, I am going to switch to focusing on higher-level mathematics, specifically in topology. This is the focus of the framework in Table 4.1, which gives examples of what the three stages (Knowing That, Knowing How, and Being Able) might look like in director systems in topology.

4.3.1 Delta One

A Δ_1 director system in topology might look like someone is taking in facts and lemmas and applying them in a basic way to their problem. The concepts needed in order to Know That are often prerequisite topics or first-order concepts, like set theory or the definition of a topology. These concepts are already assimilated

Table 4.1: A Framework for Successful Action in Learning Topology.

	Δ_1 Director System	Δ_2 Director System
Attributes	○ Physical operand ○ Consciousness of the physical operand ○ Performing mathematics ○ Mathematical facts ○ Ability or inability to take in facts and apply them	○ Mental operand ○ Consciousness of the Δ 1 operand ○ Thinking mathematically ○ Mathematical skills ○ Ability or inability to improve skill
Knowing That (KT) Possession of a Schema Activation of Concepts **Knowing How (KH)** Influenced by Qualities of Schema Connections of Concepts Path Finding **Being Able (BA)** Ability of Operators Ability to Follow Plan	**Knowing That** ○ Definition of topology ○ Subspace topology ○ Absolute values ○ Endpoints ○ Interior point ○ Basis ○ Basic open sets ○ Set theory ○ Sequences **Knowing How** ○ Following definitions ○ Following given functions ○ Using basic theorems ○ WLOG ○ Basic proof strategies ○ Initial proof setup **Being Able to** ○ Work with technical details, such as notation ○ Calculate	**Knowing That** ○ Combination of KT, KH, and BA from Δ 1 with strong Qualities of Schema ○ Contributing Examples ○ Context and Motivation for Concepts **Knowing How** ○ Proof-writing ○ Switching between notation and graphical representation ○ Utilizing definitions ○ Utilizing other mathematical results/propositions ○ Alternative definitions/techniques ○ Working with cases **Being Able to** ○ Abstract ○ Recognize patterns ○ Sketch for problem solving ○ Visualize
Domain States and paths where director system functions successfully **Frontier Zone** Boundary of domain **Learning** Occurs when frontier zone is transformed into domain	**Short-term learning** ○ Rote memorization and recall	**Intelligent learning** Reflective Intelligence ○ Reflective extrapolation: ability to reflect on existing concepts and apply them elsewhere ○ Reflective planning: ability to reflect on previous experience and derive a plan ○ Reflective problem-solving: ability to construct conceptual bridge previously not in existence

into one's schema through a successful Δ_2 director system. In topology, knowing basic examples of a concept contributes to the strength of Knowing That.

The Know How for a Δ_1 director system often involves basic proof strategies that might be introduced in a first proof-writing course, or the path might even be intuitive. There are many proof-writing strategies that might be ready-to-hand plans for undergraduates, like direct proof, proof by contradiction, proof by induction, etc. Another example might be if someone was to write a proof framework (Selden & Selden, 1995) to get started, and then writing that framework helped activate a path for completing the proof.

In a topology Δ_1 director system, Being Able is a little bit more difficult to see because there is not a large distinction between Knowing How and Being Able, due to the abstract nature of doing topology. If a path is found through activated concepts, then one's ability is usually not limited. This is because the operator, when performing mathematics, is the mind, which is also where the schema lives, which informs the plan. However, here is a simple example where there is a small distinction between the two. Perhaps I have the Know How for a proof I am working on and it involves intersecting an unknown number of sets (possibly an infinite number of sets). Then when I try to write down the notation for what I am thinking, I get stuck about how to denote the intersection. So I have the plan for the proof, but my ability to follow the plan is limited by a notational issue. From there, I might modify my plan so that I do not have to denote that intersection technically, or perhaps I seek some help from the textbook to clarify the issue.

4.3.2 Delta Two

Now let us consider Knowing That, Knowing How, and Being Able in the context of a Δ_2 director system. Because the operand of this system is a Δ_1 system itself, the operand is likely not physical, especially in the context of topology.

The successful functioning of a Δ_2 director system is more about thinking mathematically and improving mathematical skills rather than recalling mathematical facts or completing a mathematical task. In order to improve mathematical skills, the Knowing That could include the entirety of the Δ_1 system's Knowing That, Knowing How, and Being Able. For example, basic proof strategies are part of Knowing How in a Δ_1 system, but these strategies can be a concept themselves within a schema, making them a Know That for a Δ_2 system.

Examples play a major role in learning mathematics. When we see several examples of the same kind, they help build up a concept in our schema. Basic examples play a part in understanding the concepts in a Δ_1 Knowing That, but when we are able to take the examples, identify the commonalities, and apply them for use in other areas, they are part of a higher-order structure, making them a Δ_2 Knowing That.

Another Δ_2 Knowing That is a context for concepts to assimilate into. At some point in time, we have all learned a concept just because our instructor taught it, but we were not sure exactly what that concept would be used for or where the instructor was going with it. Without that context, it is difficult to connect that concept with other concepts in a schema, leaving it isolated and weak. If the concept is introduced with a context, however, then more connections are made when the concept is assimilated into the schema and those connections are available for future use. A context is classified as a Δ_2 because it is contributing to a stronger, well-connected schema, but it may not be necessary for some Δ_1 systems that utilize that concept.

The Know Hows for a Δ_2 director system involve paths and connections that are usually tied to norms for what it means to do mathematics. For example, switching between different representations is something that mathematicians typically do to help make sense of their scenario and it can improve their argument or be used to check their logic. Proving something in more than one way is another example of a Knowing How that helps improve a mathematician's functioning of their Δ_1 director system.

Being Able to improve the functioning of a Δ_1 director system can be the most challenging part of doing higher-level mathematics. Even with appropriate concepts activated and more sophisticated strategies available to use, sometimes students (and mathematicians) still are unable to prove something. Maybe the quality of their schema and Know How is not sufficient yet to be able to put everything together. Something that is necessary for topology specifically is abstraction. Being able to abstract information and recognize patterns can greatly improve a student's cognitive skills in proving, and likewise, not being able to can make the subject difficult to grasp (Hazzan, 1999).

The examples given in Table 4.1 focus specifically on topology, however, the framework is applicable to many other areas of advanced mathematics. To apply this framework to another subject, consider what the prerequisite topics are and include them in the Δ_1 Knowing That instead of the given topology prerequisites.

4.3.3 Domain and Learning

The bottom row of Table 4.1 includes the ideas of Domain, Frontier Zone, and Learning. These are included because Knowing That, Knowing How, and Being Able must be part of or accessible from the present state in order for the director system to operate successfully. In other words, these three stages can indicate whether the present state is in the domain or not. If it is not, then some kind of learning needs to occur before the goal state can be reached.

Recall that learning occurs when the frontier zone is transformed into established domain. Skemp (1979) talked about short-term learning, which is rote memorization, and then intelligent learning. Short-term learning is not adaptable. If you forget that three times nine is twenty-seven, then it is simply forgotten. If you have a conceptual understanding of multiplication, however, three times nine can be derived in another way. For example, maybe you remember that three times ten is thirty and that three times nine will be one less set of three from thirty, therefore it must be twenty-seven. This way of thinking comes from intelligent learning, which Skemp (1979) defined as "an advanced kind of learning, by which director systems are built, tested, up-dated, and improved in their internal organization" (p. 313). He also says that "intelligent learning can be explained in terms of a second-order director system, delta-two, acting on a teachable first-order director system, delta-one" (p. 81). This is why I have put intelligent learning in the Δ_2 column of Table 4.1.

In addition to intelligent learning, Skemp (1979) also discussed Piaget's (1950) reflective intelligence. By reflective, Skemp meant that "in which consciousness is centered in delta-two; the objects of consciousness being concepts, schemas, plans, or activities, in delta-one" (p. 315). Reflective intelligence is a significant type of intelligent learning that is necessary in mathematics. This framework will focus on three reflective actions, even though there certainly are others: reflective extrapolation, reflective planning, and reflective problem-solving.

Any mathematics instructor can attest that students have difficulty with problem-solving. This is why we typically like to rely on reflective extrapolation, which can be more accessible for students. Another word for reflective extrapolation is "generalization". In both of his books, Skemp (1979; 1987) explained the generalization process in three stages. First, several examples of one type formulate a schema or method, which can be applied to other basic examples of the same type. For example, a mathematics instructor might give examples for how to find the domain of rational functions and expect their students to be able to find the domain of basic rational functions afterwards. The method here is to set the denominator not equal to zero to find valid input values. The second stage occurs when the students reflect on their schema and identify what features were important for the method to be applied. In this example, they would focus on the fact that there is a fraction to consider and the denominator of that fraction cannot be zero. The final stage is to apply this method to examples of a new type. For example, even though $f(x) = \frac{1}{\ln x}$ is not a rational function, the same schema would apply to find its domain. Our method tells us that the denominator cannot be zero, so $\ln x$ cannot be zero, meaning that x cannot be one. This overall generalization process can be found in Figure 4.4.

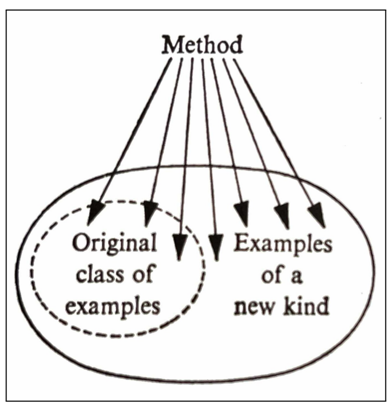


Figure 4.4: Skemp's (1987) process of mathematical generalization (p. 40).

Skemp (1987) described why this process is both sophisticated and powerful:

> Sophisticated, because it involves reflecting on the *form* of the method while temporarily ignoring its content. Powerful, because it makes possible conscious, controlled and accurate reconstruction of one's

> existing schemas - not only in response to the demands for assimilation of new situations as they are encountered *but ahead of these demands*, seeking or creating new examples to fit the enlarged concept.

Reflective planning was discussed at length earlier as part of Knowing How. This was about deriving a plan based on previous experience and ranges from ready-to-hand plans to plans that are not routine yet. If a path from present state to goal state is not yet developed, then reflective problem-solving might be necessary. Skemp (1979) said that the path can only be completed by a "conceptual bridge" that might not exist yet and that bridge is constructed through reflective intelligence, or in this case, it is called reflective problem-solving. In other words, this is where concepts and plans are utilized in Δ_1 to formulate new ideas to reach a goal state that has not been reached before.

Reflective extrapolation and planning relies on previous knowledge or experience to derive new ideas, but reflective problem-solving does not have as much previous experience to lean on. With problem-solving, conceptual connections that did not exist previously are being constructed, which is why it can be difficult. To make problem-solving more accessible, Skemp (1979) pointed to collaboration as a tool. Skemp also said that there are two kinds of problems that require problem-solving. The first is when the goal state is known and can be imagined, but a path does not yet exist for reaching the goal state. For many students, this is what they experience when they come across a mathematics problem of a type they have not seen before. Creating a successful path for the Δ_1 director system can be useful for not just that one problem, but perhaps many problems after that. The other type of problem is when we experience some kind of actuality that cannot be explained with our current schema. For example, a student emailed me the other week saying that they saw a convincing proof that $0.\overline{9} = 1$, but they knew something had to be wrong in it because they *knew* that $0.\overline{9} \neq 1$. The argument that the student read was actually correct, but that did not align with their current (incorrect) understanding. This student's schema is going to have to accommodate new knowledge in order to make sense of what they read.

Reflective extrapolation, planning, and problem-solving, are all examples of reflective intelligence, which allows opportunity for learning for a Δ_2 director system.

In summary, the success of our actions depends on the qualities of our schema (Knowing That), plans (Knowing How), and skills (Being Able). The type of director system, Δ_1 or Δ_2, determines the needed level of organization of Knowing That (interiority or vari-focal), what kind of plans and connections are necessary for the Knowing How (fixed or ready-to-hand), and whether Being Able requires physical skills or cognitive skills. If the present state is within the domain of the director system, then the goal state can be reached. If the present state is in the frontier zone, the goal state can be reached with some support. When outside of the domain and frontier zone, a form of learning must occur before the director system can function successfully. This Framework for Successful Action in Learning Topology will be used in Chapter 6 to analyze the results from the data.

Chapter 5

Methodology

In order to identify the nature of students' difficulties with topology, I wanted data from a variety of levels of expertise, but was also limited to a small number of potential participants. I also knew I was working with an abstract theoretical perspective and that analyzing large amounts of data would not be feasible. Therefore I knew I would want a good quality, in-depth look at what data I could obtain. In this chapter, I will explain the methodological choices I made, who my participants were, and how their data was obtained and analyzed.

In his book, Creswell (2013) compares and contrasts five possible methodologies for qualitative research: narrative research, phenomenology, grounded theory, ethnography, and a case study. Narrative research is best suited for individual experiences that need to be told and phenomenology is used to describe the "essence of a lived phenomenon" by studying multiple participants with a shared experience (Creswell, 2013, p. 104). Grounded theory develops a theory that comes from the view of the participants. Ethnography studies a culture and case studies analyze a defined case or cases. The aim of this study is to understand the nature of students' difficulties in the subject of topology. I am not focusing on the student experience in a topology course, so narrative research and phenomenology were not appropriate to use here. Although there certainly is a culture surrounding higher-level mathematics, the culture itself is not my focus, so ethnography was not appropriate here either. Creswell (2013) says that "a case study is a good approach when the inquirer has clearly identifiable cases with boundaries and seeks to provide an in-depth understanding of the cases or a comparison of several cases" (p. 100). This study's identifiable cases are students studying topology, which can be compared to each other, but also have the potential to provide data that can be analyzed deeply. By analyzing these cases thoroughly, I hope to show instructors any complexities involved in the learning process that they may have not considered previously, and I also wish to give some recommendations for students who are wanting to learn an abstract topic, like topology.

5.1 Case Studies

A case study involves an in-depth look at a real-life scenario within some defined, bounded system, also known as a case (Creswell, 2013; Yin, 2009). Creswell (2013) gives a detailed description.

> Case study research is a qualitative approach in which the investigator explores a real-life, contemporary bounded system (a case) or multiple bounded systems (cases) over time, through detailed, in-depth data collection involving multiple sources of information (e.g., observations, interviews, audiovisual material, and documents and reports), and reports a case description and case themes. (p. 97)

Creswell also identifies three kinds of case studies: the single instrumental case study, the collective case study, and the intrinsic case study. The first and last of these focus on a single case, whereas the collective case study selects one issue to study, but does so through multiple cases. There are two types of intent for the study: intrinsic and instrumental (Creswell, 2013; Yin, 2009). A case study that illustrates an interesting case is intrinsic and a case study that tries to understand a specific problem is instrumental.

One of the strategies for addressing my research question is to interview participants with varying levels of experience with learning topology. Therefore this is an instrumental, collective case study where the cases consist of several individual learners at the University of Oklahoma whose common activity is that they were studying in the area of topology to some degree at the time of their interview. The cases range from a novice learner to an experienced researcher and one case involves two learners paired together. Where possible, the logic of replication (Yin, 2009) is used across cases during data collection. Within each case, the participants discuss the same four tasks in interviews, where both auditory and visual data are studied, and related follow-up questions are asked.

5.2 Participants and Settings

There are five participants discussed in this study, all at various stages in their mathematical careers. All participants were from the University of Oklahoma and are identified in this study using pseudonyms. Any identifying information that was given during the interview was generalized during transcription. Dr. T was a tenured professor who does research in geometry and topology. Jordan was a graduate student who had passed his qualifying exams, chosen an advisor, and had starting studying in the area of geometry and topology. Brandon and Kyle were first-year graduate students who were enrolled in the first semester qualifying course for topology. They did two interviews, what we are calling the “early” interview and “late” interview. The early interview was conducted around the fourth week of the semester and the late interview occurred the week following finals week of that same semester. At the time of the early interview, Brandon had not completed a topology course and Kyle had previously taken an introductory topology course, so their experience with the subject matter was limited. The final participant is an undergraduate student named Phil. At the time of his interview, Phil was in the final weeks of the undergraduate topology course, which was the first course in topology that he had taken.

An introductory topology course at the University of Oklahoma covers metric spaces and topological spaces, as well as topics like continuity, connectedness, and compactness. It is intended for undergraduate students to take in one of their final semesters of their mathematics program. There are also two qualifying topology courses which are part of a two-semester sequence that serve to prepare graduate students for their qualifying exam in topology. The first semester focuses on point-set topology while the second semester is on algebraic topology. Even though the first qualifying topology course is more accelerated than the introductory topology course, graduate students do not have to take the introductory course since the same topics are also covered in the first qualifying course. The qualifying course is intended for all first-year graduate students, even if they have not taken a course in topology before. Topology is one of three areas that mathematics Ph.D. candidates are required to pass a qualifying exam for. For graduate students who have passed their topology qualifying exam and are interested in the subject, there is also a two-semester sequence of courses on algebraic topology available to take.

There were four tasks used in this study, which are found in Appendix B. These tasks are considered to be introductory point-set topology tasks for undergraduate or beginning graduate students. Task One focuses on whether or not sets are open, depending on what topological space is being considered. This task specifically focuses on the standard topology on $\mathbb{R}$ and the subspace topology on $[-1, 1]$. Task Two asks to show that two sets are each a basis for a topology on $\mathbb{R}$. This task utilizes the definition of a basis and is dependent on set theory. The product topology on $X \times Y$ is defined using a basis in Task Three and then it wants a proof that the projection map in $X \times Y$ is an open map. This task is highly dependent on an understanding of a basis for a topology. Finally, Task Four requires more abstract thought about a basis and the topology generated by it. Task Three was the first question on the exam analyzed in the pilot study (see Chapter 3), which is how I first became interested in how students understood the concept of a basis for a topology. Tasks Two and Four were selected because they involved the basis concept, each at a different level of abstraction and difficulty. Task One does not require the basis concept and thus provided opportunity for participants to demonstrate their understanding of set theory and a topology, even if they did not know what a basis was.

5.3 Data Collection

Initially, I wanted to select participants from students enrolled in either the undergraduate or graduate introductory topology courses since the aim of this research involves schema construction for abstract topology topics. In these courses, students will see the definition of a topology and a basis for a topology, likely for the first time. Participants from these courses will be ideal for observing schema construction since the topics are new to them. Even within the graduate course, there is typically a mix of students who have seen the definition of a topology before and some that have not. The students who have not will fall into the same category as the undergraduate students. The students who have seen the definition prior to taking the course can be as informative in interviews if they still need to accommodate their schema so they can assimilate new ideas regarding topology.

I began by collecting some data from first-year graduate students with the sampling strategy of collecting typical cases of students learning topology (Creswell, 2013). After doing so, it became clear that it would be beneficial to the creation of my framework to collect data on the same tasks from experts in the area of topology. I started with advanced graduate students whose subject of study was related to topology and then later also interviewed a topologist. I additionally had an opportunity to interview an undergraduate enrolled in a topology course towards the end of my data collection. My resulting sampling strategy was to get maximum variation in order to document similarities and differences in the various levels of expertise in the subject (Creswell, 2013). Because this data was collected over the course of a couple of years, there are slight variations in each phase of the data collection, so I will discuss the phases separately.

For all phases, IRB approval was received before beginning recruitment and collection. When recruiting, the potential participants were given a brief description of the study, the anticipated time commitment for each interview, when in the semester they would be conducted, and why they were potential participants. There was no coercion or influence during any of the the recruitment processes. The interviews were only conducted with those who chose to contact me based on the recruitment prompts and who chose to consent after receiving the consent forms. I did not contact those who did not respond to the recruitment prompt.

5.3.1 Phase 1: First-Year Graduate Students

I met with the instructor on record for the graduate course to provide information about this study and to ask permission to recruit during 5-10 minutes of class time. Four first-year graduate students responded to the prompt and were divided into two pairs to conduct their interviews together. One of these pairs was Brandon and Kyle and the other pair remains unnamed since they are not included in the results of this study. Pairs were used to try to get the participants to demonstrate their ideas more explicitly than they would on their own and to discuss out loud how they think about the tasks to their partner. Additionally, if one participant reached a point where they would not be able to make any further progress on their own, there was a chance that their partner could do or say something to help move the pair forward in their ideas and work, which would give me a better understanding of how they were thinking. It is worth noting that no data was collected regarding what took place in the classroom before, after, or between these interviews.

For each pair of participants, two sets of semi-structured, task-based interviews were conducted: one in the first month of the semester and the second in the last month of the semester. The interviewer prompts can be found in Appendix C. In both of these interviews, students were given the definition sheet found in Appendix A and then were asked to individually consider the four tasks found in Appendix B. After working individually for a short time, they then discussed their thoughts with their partner, with the goal of coming to a consensus on each task. In the final part of each interview, the pairs were asked follow-up questions about what they thought was needed to complete each task. They were also asked about their background with topology.

In the late interview, after working through the tasks, the participants reflected on what changed for them in how they think about topology problems since the beginning of the semester. They were given their work from earlier in the semester and asked to discuss their progress between then and now.

These interviews were video recorded via two different recording devices, both of which included audio. One device was meant to focus on the participants when working at the table and the other device was intended to focus on when the participants used the whiteboard. They both served as a back up recording device in case the other malfunctioned, which was necessary once. The interviews were then transcribed and de-identified. The recordings, transcriptions, and scans of written work were all stored on my computer and were password protected.

5.3.2 Phase 2: Advanced Graduate Students

An email was sent out to graduate students who were in the research phase of their program in the Mathematics Department, meaning that they had already passed all three of their qualifying exams, including the topology exam. Additionally, their selected research area was in geometry and topology. Three graduate students responded and participated in their own individual interviews. One of the participants was Jordan, one was Luke, and the third remains unnamed. Luke's results did not make it into the current study, but they were discussed by Berger and Stewart (2020). These interviews were conducted with individuals rather than pairs because these participants were considered to be experts on the basic topology concepts that were being studied and therefore did not need to rely on collaboration to discuss the tasks. Additionally, the prompt asked them to explain their work as if they were in a teaching or tutoring setting, which is more appropriate if they are interviewed separately.

The semi-structured, task-based interviews were conducted in the first month of the semester, although that timing is not as relevant for the advanced graduate students as it was for the first-year graduate students. The interviewer prompts can be found in Appendix D and the participants were also given the definition sheet found in Appendix A. In these interviews, participants worked on the tasks and explained their thought process as if they were teaching to a group of students. They also reflected on what prerequisite concepts, skills, or techniques they thought were necessary for undergraduate students to possess when working through the tasks. Because the participants were experienced with conducting research related to topology, I interviewed them with the perspective that they are experts in the field.

The interviews were video recorded via a Zoom room camera that was able to capture both the participants and their work on the whiteboard. The recordings were later transcribed and de-identified. The recordings and transcriptions were all stored on my computer and were password protected.

5.3.3 Phase 3: The Undergraduate Student and the Topologist

For undergraduate recruitment, I emailed the instructor on record for the undergraduate introductory course to provide information about this study and to ask permission to recruit from his class via email. One undergraduate student, Phil, responded to the email prompt. It should be noted that no data was collected regarding what took place in the introductory topology course.

A semi-structured, task-based interview was conducted in the last two weeks of the semester. The interviewer prompts can be found in Appendix E and the participant was given access to the definition sheet found in Appendix A. The participant was asked to consider and try to work through the four tasks found in Appendix B. If they finished their work and something was incorrect or incomplete, I intervened and gave them another opportunity to progress further in the task. This method allowed me to understand what they would have completed if working on their own, but also gave them an opportunity to show more of their understanding of the content that previously was hidden behind a mistake or misunderstanding. In the final part of the interview, the participant was asked about their experience with topology and what they have found helpful in their learning experience.

The topologist, Dr. T, was recruited via email and interviewed after the semester had ended. The interviewer prompts can be found in Appendix F. Dr. T was also given the same tasks and definitions as the undergraduate participant, Phil. In this interview, the participant worked on the tasks and explained his thought process. He also reflected on what prerequisite concepts, skills, or techniques were necessary for undergraduate students to possess when working through the tasks. Because the participant was highly experienced with the content and the teaching of the topology courses, this interview was less structured so that his contribution was not restricted. He was given the space to talk about any aspect of teaching and learning topology that he felt compelled to.

Both of these interviews were conducted via Zoom meetings where I had the definitions and tasks open in a OneNote file and then shared my screen with the participant. I gave the participants remote control of the Zoom meeting so that they could scroll between the definitions and tasks and write their work on the shared screen. The Zoom meetings were recorded to a secure cloud server that is password protected. They were also downloaded and stored on my computer. The interviews were transcribed and de-identified. The recordings, transcriptions, and screenshots of written work were all stored on my computer and were password protected.

For all phases and interviews, there was a portion of the interview that was less structured. Mainly, the point in time after a participant had finished a task, but had not yet moved on to the next prompt or task. In this bit of time, I would follow up with the participant on what they just did, asking some open ended questions suggested by Hunting (1997) to give more opportunity for discussion and reveal more of their thought processes. Some of Hunting's (1997) questions that I used are listed below.

- *Can you tell me how you worked that out?*
- *Do you know a way to check whether you are right?*
- *Pretend you are the teacher. Could you explain what you think to a student?*

5.4 Data Analysis

Clement (2000) discussed clinical interviews from the perspective that their primary goal of analysis is to construct "a model of hidden mental structures and processes that are grounded in detailed observations from protocols" (p. 549). Creswell (2013) describes qualitative data analysis in three broad stages: organizing the data, reducing the data into codes and themes, and representing the data through discussion, figures, or tables. He also summarizes three perspectives that all fit within his stages, one of which includes a traditional approach by Wolcott (1994), where patterned regularities are identified and then contextualized to relate to the framework from the literature. Creswell (2013) presents his conceptualization of data analysis in his “Data Analysis Spiral”, which emphasizes a process of cycling through phases instead of following a linear approach. When multiple cases are being analyzed, Yin (2009) recommends a cross-case synthesis of the data. This can be made easier with helpful tables or figures that display the findings and compare the cases. Creswell (2013) describes two kinds of analyses for collective case studies. “When multiple cases are chosen, a typical format is to provide first a detailed description of each case and themes within the case, called a within-case analysis, followed by a thematic analysis across the cases, called a cross-case analysis, as well as assertions or an interpretation of the meaning of the case” (p. 101).

Braun and Clarke (2006) gave six phases of thematic analysis, which are shown in Table 5.1. They defined thematic analysis as “a method for identifying, analysing and reporting patterns (themes) within data” (p. 79). These themes, or categories, could be pre-existing from the literature, or they can be emergent (Creswell, 2013). Stake (1995) called this search for meaningful themes and patterns “categorical aggregation”. Maguire and Delahunt (2017) give a detailed example of using these six phases and agree with Braun and Clarke and Creswell that these phases are not linear. Instead, these phases are a guide to a recursive process that develops over time.

The above sources all informed how I approached my data analysis, which can be summarized as both within-case and cross-case analyses with open, emergent coding of themes. I have collated Creswell’s (2013) data analysis spiral with

Table 5.1: Braun and Clarke's (2006) phases of thematic analysis.

Phase	Description of the process
1. Familiarizing yourself with your data	Transcribing data (if necessary), reading and re-reading the data, noting down initial ideas.
2. Generating initial codes	Coding interesting features of the data in a systematic fashion across the entire data set, collating data relevant to each code.
3. Searching for themes	Collating codes into potential themes, gathering all data relevant to each potential theme.
4. Reviewing themes	Checking if the themes work in relation to the coded extracts (Level 1) and the entire data set (Level 2), generating a thematic 'map' of the analysis.
5. Defining and naming themes	Ongoing analysis to refine the specifics of each theme, and the overall story the analysis tells, generating clear definitions and names for each theme.
6. Producing the report	The final opportunity for analysis. Selection of vivid, compelling extract examples, final analysis of selected extracts, relating back of the analysis to the research question and literature, producing a scholarly report of the analysis.

Braun and Clarke's (2006) phases in Figure 5.1 to give an overview of the data analysis process that I used. The figure is organized by Creswell's (2013) three stages of organizing the data, coding and finding themes, and representing the findings. Note that Braun and Clarke's (2006) first phase is included in the organizing stage and their sixth phase is part of the representing stage. The four phases between those two all occur in the coding and themes stage.

I began with several rounds of reading through the transcribed data and watching the recordings. I highlighted key concepts, took brief notes, and reported my initial ideas to my research advisor. I then used open coding (Strauss

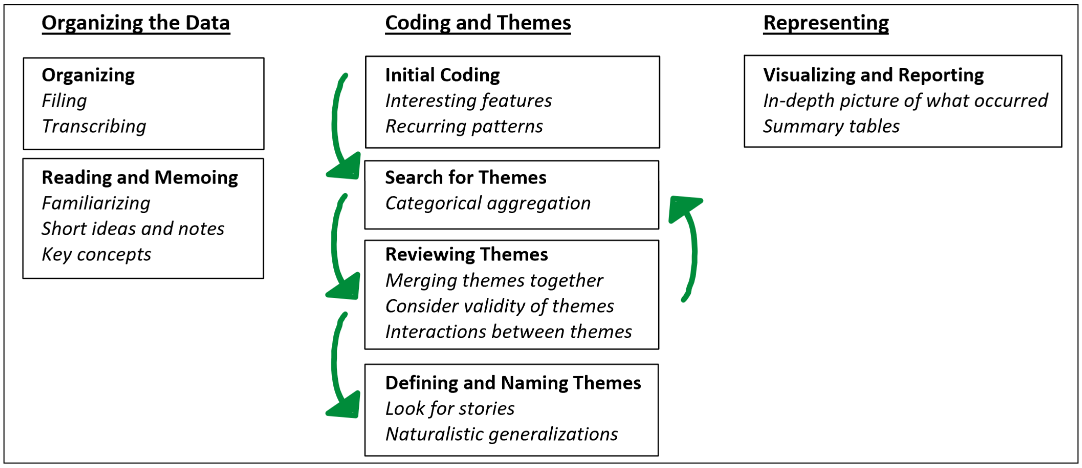


Figure 5.1: The data analysis procedures for this study.

& Corbin, 1998) for interesting, recurring patterns I saw in the data, which were not dependent on the theoretical ideas I was considering. Creswell (2013) says that reliability in qualitative research "often refers to the stability of responses to multiple coders of data sets" (p. 253). In order to have reliability in this study, my advisor also coded the data and we met weekly to compare codes and come to a consensus.

The codes were classified into a smaller number of broad themes, which were then considered with the literature to begin building my theoretical framework. The themes and codes can be found in Table 5.2 and select examples are given in Table 5.3. These codes were not mutually exclusive. For example, the quote from Dr. T in the Topology Concepts theme was also coded under the Mathematical Norms theme for other mathematical results. Several rounds occurred where I continued to search for prominent themes, reviewed them with the proposed framework, revised the framework, and searched again.

Table 5.2: Themes and codes.

Topology Concepts	**Proving Basics**	**Mathematical Norms**	**Cognitive Skills**
- Absolute values - Basis - Choice of a topology - Endpoints - Interior points - Basic open sets - Sequences - Set theory - Subspace topology	- Understanding definitions - WLOG - Initial proof setup	- Proof-writing - Switching representations - Utilizing definitions - Other math results - Utilizing propositions - Alternative definitions - Alternative techniques - Cases	- Abstraction - Pattern recognition - Sketching for problem solving - Visualization

Table 5.3: Some examples of the codes.

Topology Concepts: basic open sets	"You show that the open rectangles are a basis and then you have this basic fact that says that if you have a basis for a topology then a set is open if it's a union of basis elements." Dr. T
Proving Basics: WLOG	"Since they have negative infinity, there's really kind of only one case to consider. If it's contained in two, then it's contained in whichever one's smaller." Phil
Mathematical Norms: switching representations	"Here is a visual presentation of it and here's like this open interval presentation. They have to be able to go back and forth between these things." Jordan
Cognitive Skills: abstraction	"Well, it is when you start off with point-set topology, like following Munkres, the level of the abstraction is really, I think, hard to get used to." Dr. T

Chapter 6

Results

The participants in this study presented their ideas in a variety of ways, primarily through written and verbal communication. Typically what the participants wrote did not make sense without the verbal explanation, and likewise, they wrote down symbols and pictures to better convey what they were saying.

In this chapter, I will analyze the data using my theoretical framework. This chapter is organized by the four tasks, each beginning with the problem statement and a possible solution, and then followed up with an account for each participant who attempted that task. If a participant is not included, then they did not have enough time during their interview to consider that task.

6.1 Task One

Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \left\{x \mid \tfrac{1}{2} < |x| < 1\right.$$

$$B = \left\{x \mid \tfrac{1}{2} < |x| \leq 1\right.$$

$$C = \left\{x \mid \tfrac{1}{2} \leq |x| < 1\right.$$

$$D = \left\{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbb{Z}_+\right.$$

6.1.1 Solution

It is natural to want to know what these sets look like visually before getting started. The first three sets are easily drawn (see Figure 6.1), while drawing set D is less straightforward.

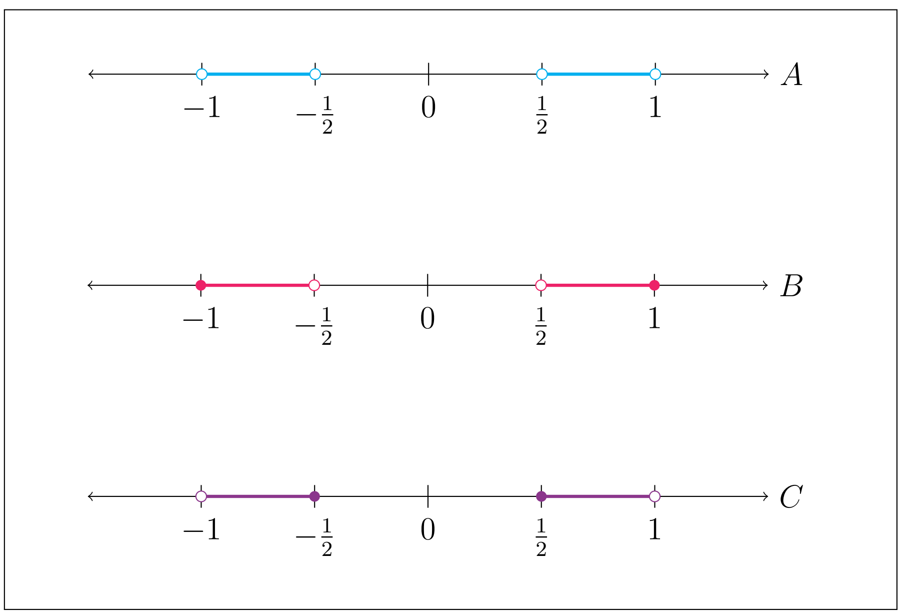


Figure 6.1: The first three sets of Task One.

For set D, start with the open interval $(-1, 0) \cup (0, 1)$. Then there is the condition that for all $x \in D$, $\frac{1}{x} \notin \mathbb{Z}_+$. After thinking about this condition for a moment, and even maybe playing around with a few example values of x, you realize that this condition means rational numbers of the form $\frac{1}{n}$ need to be removed, where n is any positive integer. Thus, remove $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., and continue infinitely (see Figure 6.2).

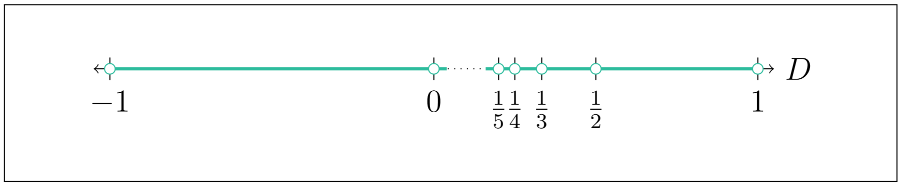


Figure 6.2: The fourth set of Task One.

From these visuals, we can write each set using interval notation:

$$A = (-1, -0.5) \cup (0.5, 1)$$

$$B = [-1, -0.5) \cup (0.5, 1]$$

$$C = (-1, -0.5] \cup [0.5, 1)$$

$$D = (-1, 0) \cup ... \cup \left(\tfrac{1}{5}, \tfrac{1}{4}\right) \cup \left(\tfrac{1}{4}, \tfrac{1}{3}\right) \cup \left(\tfrac{1}{3}, \tfrac{1}{2}\right) \cup \left(\tfrac{1}{2}, 1\right)$$

The definition of the standard topology gives that basic open sets of $\mathbb{R}$ are of the form (a, b), where a and b are real numbers. It also gives that unions (even an infinite amount of unions) of basic open sets are open in the standard topology. Since we can write A and D as unions of basic open sets, those two sets are open in $\mathbb{R}$. Because we cannot write B and C in this way, those two are not open in $\mathbb{R}$.

Now consider whether or not each set is open in the subspace $Y = [-1, 1]$. The subspace topology consists of all intersections of open sets of $\mathbb{R}$ with Y. In other words, in order for a set to be open in the subspace, it must be able to be written as some open set intersected with $[-1, 1]$. Consider set A, which is open in $\mathbb{R}$. Because $A \cap Y = A$, A is open in Y. From this, it is not much of a stretch to see why any set within Y that is open in $\mathbb{R}$ will also be open in Y. So the same reasoning applies to set D and it is also open in Y.

For set B, we have to search a tiny bit to find an open set in $\mathbb{R}$ that will satisfy the subspace topology definition. If you extend B past -1 and 1 a little bit with an open endpoint, you'll find a set that works. For example, the set $(-1.5, -0.5) \cup (0.5, 1.5)$ is open in $\mathbb{R}$ and intersects with Y to give B. So B is open in Y even though it is not open in $\mathbb{R}$.

For set C, you could try to find such an open set, but you will eventually realize that no such set exists. What is more challenging is to technically prove that no such set exists. The argument involves the closed ends at -0.5 and 0.5 and how those two values cannot be part of an open set of $\mathbb{R}$ without including values closer to 0 in the intersection. So C is neither open in Y nor $\mathbb{R}$.

6.1.2 The professor, Dr. T

With Task One, Dr. T quickly decided which sets are open in $\mathbb{R}$ and then almost immediately after that made conclusions about whether or not they were open in the subspace. Very little explanation was given and he viewed the question as fairly technical. He did elaborate about the boundary points with set C when prompted:

> I look at that and I know right away the issue is just with minus one and one, the endpoints. And if you have something like one-half is a boundary point in C, but it's not minus one or one ... It's just a matter of the only difference is going to be what happens at minus one and one.

When Dr. T was asked about what students need to understand in order to complete this task, the first thing he said was that they "have to be adept at understanding the set notation" and "describing these sets." He also discussed the two aspects of this task:

> You know, the two parts, whether it's open in $\mathbb{R}$ or open in Y are sort of two separate things. I mean, presumably in the class you're going to initially spend time on just identifying open sets in $\mathbb{R}$. The sets that are open in the subspace topology; that's a little bit of a more technical thing that might come later.

He is emphasizing how the task should be approached in two stages. First, determining if the set is open in $\mathbb{R}$, then secondarily determining if the set is open in the subspace. At the end of discussing Task One, Dr. T had a final comment regarding how he did this problem compared to how his students probably do it. "Well the thing is, I have so much experience with working with those things, that...the way I think about it is probably pretty far removed from the way students think about it."

6.1.3 The doctoral student, Jordan

Immediately when reading the task prompt, Jordan confirmed with the interviewer that the topology on $\mathbb{R}$ is understood to be the standard topology, demonstrating that he knew there was a connection between open sets and a topology on a space. He then said that "we should try to understand what these sets look like." For each set he considered, he went to the right side of the white board, drew a number line, and sketched the intervals for the set he was currently thinking about. Once he had the visual, he decided whether or not the set was open in $\mathbb{R}$ and Y and recorded his decision on the left side of the board (see Figure 6.3).

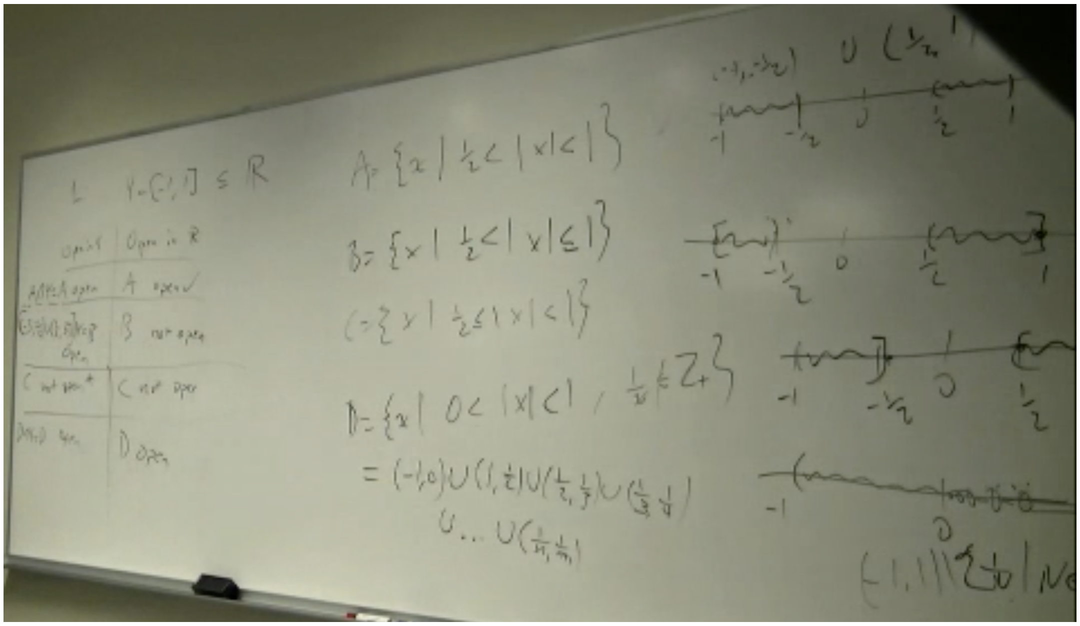


Figure 6.3: Jordan's board work during Task One.

Jordan arrived to the correct conclusions for set A quickly. Once he had the visual representation and interval notation on the right side of the board, he explained, "First of all, this set is open at $\mathbb{R}$ and so by the definition of the subspace topology, that means it is open in Y because it is contained in Y." He then recorded this on the left side of the board. He continued in this cycle for the remainder of the sets where he went to the right side to sketch and brainstorm and then wrote his results on the left side.

Before sketching set C, Jordan (half-jokingly) made a quick conjecture that C might be open in $\mathbb{R}$, but not Y. "So, A is open in both. B is open in just Y. I wonder if C is open in just $\mathbb{R}$. [Smiles mischievously.] Actually, no, it shouldn't be." It is likely that he quickly retracted this conjecture because of what he said earlier about why set A was open in Y. Any set that is open in $\mathbb{R}$ and contained within Y will also be open in Y, so it is not possible for any of these four sets to be open in $\mathbb{R}$ only. A few moments later, while Jordan was sketching set C, he realized exactly what the result would be: "Oh, this will be open in neither." He very quickly had an intuition that C would not be open in either, but acknowledged that technically proving it might be complicated. "Now I guess you have to show that C is not the intersection of any open set and [Y]...oh, maybe that's not 100% obvious." He then mentioned this set not being open in Y in "other results", indicating that there was something else in his background knowledge that he was utilizing in reaching this result. Jordan then wrote a star next to his set C result, commenting that there might be more he needs to show there.

Moving on to set D, Jordan re-read over the given set and explained that you start with this open set from negative one to one and you are "going to be taking away points from this set." He wrote $\{\frac{1}{N} \mid N \in \mathbb{Z}_+$ as the values that will need to be excluded from the set and drew open holes on his number line. He then rewrote D in a (self-proclaimed) nonsensical order of interval notation as $(-1,0) \cup (1, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{4}) \cup ... \cup (\frac{1}{M}, \frac{1}{M+1}) \cup ....$ He then explained, "Because we realize D as the infinite union of a bunch of open intervals, D is open [in $\mathbb{R}$]. So, since D intersect Y is precisely D, D is open [in Y]."

Once Jordan finished the task, he was asked to reflect on the task and explain what he believed undergraduate students would need to know in order to successfully complete the same task. Below is some of his response.

> *Jordan:* Here is like a visual presentation of it and here's like this open-interval presentation. They have to be able to go back and

forth between these things. So, that's one thing. You need to know the definition of the topology and the subspace topology and then you need to...so, like there's this notion of like an interior point to a set. So, a point's in the interior if there's like some open set around it. So like an alternative definition for openness is that every point's interior. So, you might find that helpful to make some of these arguments like this one. And I think that's what you need to do here [set C] although it doesn't seem as easy to me.

Interviewer: Yeah.

Jordan: Oh, I can finish that then [set C]. Yeah. I think that's exactly what you need. So, set theory, interior points, and then, of course you need to know the subspace topology. You need to know stuff about bases. [Writing on board.] Anything else I used?

Interviewer: Where did you use a basis?

Jordan: You need to know what the basic open sets in $\mathbb{R}$ look like to maybe argue stuff about interior points.

Interviewer: Okay. Yeah, that's true.

Jordan: Maybe something about sequences. I don't think this [set D] is like a kind of a normal set.

$\vdots$

Jordan: Really what was going on here is using my set theory to go back and forth between different pictures. I don't know if

> people...okay, I don't do this. Maybe some people do. I'm not going to be able to tell you whether that's open or not unless I have some sort of visualization of that.

He listed off prerequisite topics like set theory, topology, subspace topology, interior points, bases, and sequences. When he was talking about interior points, he realized that was what he needed to finish his argument for why set C is not open in Y. When he mentioned knowing things about bases, he was asked where exactly he used a basis in this task and he replied that they were needed to understand the basic open sets of $\mathbb{R}$. Jordan also emphasized more than once the method of switching back and forth between representations of the same set and needing to have a visual presentation for this task. Overall, Jordan worked through the first task fairly efficiently, while still showing all his work and explaining his reasoning. He was able to discuss in detail what ideas he pulled from in order to do this task.

6.1.4 The first-year graduate students, Brandon and Kyle

The Early Interview

Kyle first confirmed with Brandon that $\mathbb{Z}_+$ (in set D) meant the positive integers and then suggested that nothing else would be excluded from $(-1, 0) \cup (0, 1)$ because one was already excluded. Brandon said that sounded good to him. Kyle then tried to confirm that D would be open in both $\mathbb{R}$ and Y, but Brandon wanted to clarify what topology they were using. They quickly agreed that it was understood to be the standard topology on $\mathbb{R}$.

Kyle described to Brandon how he had started these on his own. He had broken up these sets into the two pieces, the negative piece and the positive piece, and figured out what points were included or not. With words and hand gestures, he described his visual for the sets. The pair never drew their pictures on paper nor on the whiteboard for any of the four sets during this interview.

In considering set B, Kyle pointed out that the closed bracket around the negative one and one would keep B from being open in $\mathbb{R}$. This thought was then interrupted by the pair moving to work on the whiteboard. Once they got to the whiteboard, they rewrote A in interval notation and agreed that A was open in both $\mathbb{R}$ and Y. Kyle did elaborate as to why A was open in Y, but accidentally said "union" instead of "intersection" several times, even though he reached the conclusion he would have if he had said "intersection".

For set B, the pair immediately wrote the interval notation on the board. They came up with the open set $(-3, -\frac{1}{2}) \cup (\frac{1}{2}, 3)$ to intersect with Y to prove that B was open in Y, but not in $\mathbb{R}$. Kyle continued to say "union", but at this points, Brandon interrupted him to correct him with "intersection".

Kyle started off discussing C by declaring that it would be open in neither Y nor $\mathbb{R}$. They started to write the interval notation, but didn't write out the endpoints or the union, so it just looked like ( ] [ ). They both started saying that C would not be open in Y due to the closed brackets being at $-\frac{1}{2}$ and $\frac{1}{2}$ instead of at -1 and 1.

> *Kyle:* There's nothing that you're going to intersect with that's an open set that's going to give you
>
> *Brandon:* both of those [referring to the brackets on the board].
>
> *Kyle:* Yeah.

They then went back to discussing set D and again concluded (incorrectly) that D is just $(-1, 0) \cup (0, 1)$. Kyle guessed that the notation was there just as something to think about, even though it didn't exclude any additional points from the set. They then agreed that this would mean that D is open in both $\mathbb{R}$ and Y.

The Late Interview

Brandon and Kyle worked together at the whiteboard, with Kyle doing the writing at first. He tried to draw a visual for set A, but mistakenly drew it as if x was a radius and the space they were considering was $\mathbb{R}^2$ (see Figure 6.4). Brandon immediately pointed out that they were just in $\mathbb{R}$ and Kyle corrected his drawing (see Figure 6.5). They agreed that A was a union of open balls and was therefore open. They did not clarify if they mean open in $\mathbb{R}$ or open in Y.

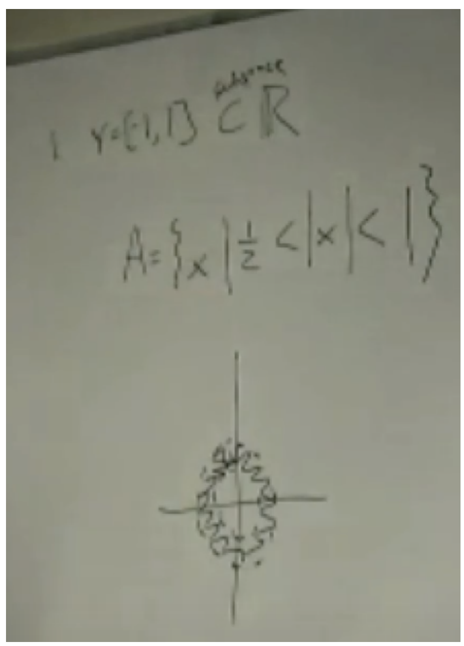


Figure 6.4: Kyle's initial set A.

They then drew a picture for B and concluded that it would not be open, again not clarifying whether in $\mathbb{R}$ or Y. They tried to give a reason for B not being open, but it was difficult to follow. They were attempting to use the argument that a set is open in $\mathbb{R}$ if for every point in the set, you can find an open ball around it that is still contained within the set, but for set B, the end point of 1 will not be able to satisfy this condition. They then said the same ideas applied to C, so it was not open as well.

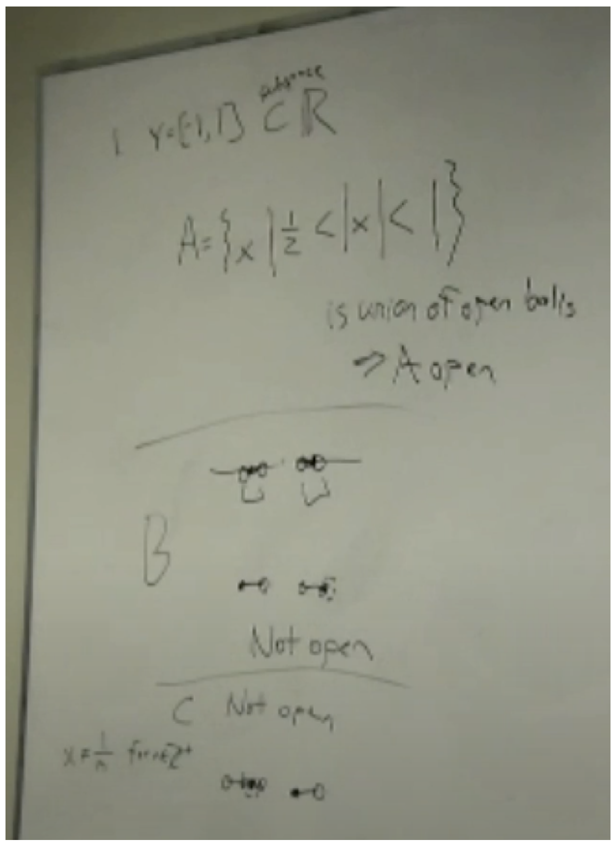


Figure 6.5: Brandon and Kyle's board work for Task One.

When they got to D, they took a moment to read it and figure out what it looked like. They agreed that $x \neq \frac{1}{n}$ for $n \in \mathbb{Z}^+$ (as seen in Figure 6.5). Kyle hypothesized that D would still be open but took a bit of time to formalize why. (Brandon started to write Task Two on the board and jumped back to what Kyle was doing once Kyle started talking again.) Kyle's board work is typed in

Figure 6.6

$$\underbrace{\frac{1}{n+1} < x}_{\delta_a}\underbrace{< \frac{1}{n}}_{\delta_b}$$

Figure 6.6: Kyle's board work for set D.

Kyle: However close your point is to zero...it's going to have to be between some one over n and one over n plus one, right? ... And so the idea is there's gonna be some distance from here to here, some distance from here to here. Call this δ_a, δ_b. You set delta equal to the min[imum] of these two, right? Then the ball of radius delta will be...

Brandon: contained in there.

Kyle: So yeah, for any point you can find a ball around it which is contained within it [set D], so it's, it's open.

Brandon: Okay.

At this point, Brandon and Kyle moved on to Task Two. They did not ever consider whether or not these sets were open in Y. Later in the interview, they were asked if they could identify any misunderstandings they may have had in their early interview by looking over their handwritten work and they realized their mistake in this interview of not considering Y.

Kyle: I mean, this one thing I wrote here does not immediately make sense. So that's questionable.

Interviewer: Which bit?

Kyle: "Not open in either."

Brandon: In either?

Kyle: Yeah, I don't know what the like. We were only talking about $\mathbb{R}$. Right? So...

Brandon: Oh no, it says which sets are open in Y and then which sets are open in $\mathbb{R}$.

Kyle: Oh, we didn't do that other part, did we?

Brandon: Nope.

Kyle: Oh wait.

Brandon: Nope.

Kyle: No, we, yeah.

Brandon: Wow.

Kyle: Wow. Well, you know.

Interviewer: So just to clarify, you only did $\mathbb{R}$ just now?

Kyle: Yeah we only did $\mathbb{R}$ just now.

Brandon: Yeah.

Kyle: Yeah. We're good at reading and stuff.

Due to time constraints, they were not able to go back and finish the problem. Even though they did not successfully complete the task in this interview, this transcript shows that they should not be discredited. They acknowledged the other half of the problem, so we know that they have more understanding than what was demonstrated in the data. We cannot make any assumptions about what they would have done past understanding that there was a second half to the problem.

6.1.5 The undergraduate student, Phil

Phil started Task One by assuming he was working with the standard topology and made a statement of "The subspace topology is still the standard topology, I think." He said A was open because "there's no strictly equal sign" and for similar reasons (mainly that there was an equality on an endpoint), concluded that B and C were not open. He then spent a couple of minutes thinking to himself, making sure he understood set D and what it looked like. He concluded that D was open in Y and said that "every set that's open in Y is open in $\mathbb{R}$, I think." He justified his conclusion that D was open by correctly describing the rational values that are excluded from D and then also tried to draw the set (see Figure 6.7). It should be noted that Phil faced some technological challenges in drawing on the screen, so the interviewer aided in drawing the tick marks at the rational values after Phil explained what should be there. The yellow highlighter is what Phil said set D was.

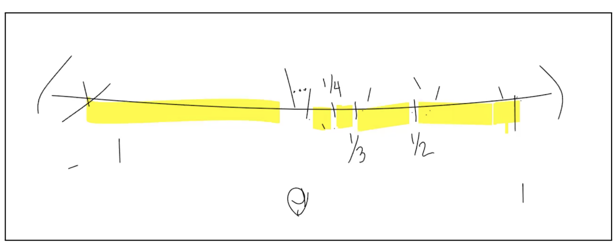


Figure 6.7: Phil's sketch for Task One, set D.

After Phil was done with the sketch, the interviewer clarified what he was concluding with set D.

> *Interviewer:* You said your final conclusion for set D is what?
>
> *Phil:* That it is open.
>
> *Interviewer:* Okay. And it's open in $\mathbb{R}$?
>
> *Phil:* Yes.
>
> *Interviewer:* Are you saying anything different about being open in Y?
>
> *Phil:* Not to my knowledge.

If he was not being interviewed, that is where Phil would have stopped working on the problem. He had only made one conclusion for each of the four sets, equating what it meant to be open in $\mathbb{R}$ and open in Y.

Phil was asked if he had anything else to say about whether the sets were open in Y and he said he didn't think so. The interviewer then pointed out the definition of a subspace topology and how open in Y is a different statement than open in $\mathbb{R}$. We could hear Phil's "lightbulb" moment as he said "Oh, I see. Yeah. Huh. That's cool." Phil was then given the opportunity to finish the task.

He hypothesized at first that all four sets would be open in Y. When asked why he thought that, he again cited a misremembered lemma about open sets of Y being open in $\mathbb{R}$. It is possible that he was thinking of the following lemma from Munkres (1975).

> ***Lemma*** Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Note that this lemma does not apply to this task since Y is not open in $\mathbb{R}$. The other possibility is that Phil was not thinking of this lemma, but instead had his logic backwards or stated what he was thinking in the reverse order. He then said that A and D were open since they were open in $\mathbb{R}$, but he did not correctly say why at this point. He then thought about B and C.

> I think B and C are open because they are at least half open, and then that means there's an open interval which if you intersect with Y, as long as it goes beyond one... [pause] Wait a minute. Hold on. [pause] Hmm... [long pause] Let me change that. C I think is op-. No, wait, hold on. Wrong. I'm fairly sure B is open in Y. I'm starting to not think so about C.

Phil was then prompted to draw a picture of B to explain what he was thinking and he correctly reasoned about why B is open in Y. He was then asked again to explain why A and D are open in Y and he finally came up with a valid reason, saying "I mean, I suppose you can think of them as, like...my reasoning is mostly because they're open in $\mathbb{R}$, but also, like, if you intersect Y with A and D, you'll get A and D. And A and D are open in $\mathbb{R}$."

All that was left unresolved at this point was set C. Phil said he was struggling to find an open set that would intersect with Y to give C, but couldn't come up with a reasoning for it quickly. Due to time constraints, the interviewer needed to move on to the next task, so they helped explain why Phil couldn't find such a set and he agreed that their explanation made sense.

6.1.6 Task One Summary

On Task One, Dr. T rattled off the answers immediately. In comparison, the other participants took a substantial amount of time to think about and answer the multiple components of the problem. For example, the argument needed for set C took some time for Jordan, Brandon, and Kyle, while Phil was unable to come up with it during the interview. In order to say that a set is open in the subspace topology, you must find some set that intersects with the subspace to become that set (in this case, set C). If set C were open in the subspace, this wouldn't necessarily take a large amount of time because you only have to find one set. However, set C is not open in the subspace, so it takes some time to realize that no such set exists and then come up with the details to prove why. Brandon and Kyle reached the correct conclusion in their early interview, but did not give reasoning beyond stating that no such set exists. Phil took a significant amount of time to reach this conclusion and did not have enough time to come up with a justification. Jordan reached the right conclusion and had an intuition about why, but didn't come up with a formal reasoning until later in the interview when reflecting on Task One. In comparison, Dr. T was able to declare that set C was not open in the subspace relatively immediately and explained why.

Jordan presented his thoughts as if he were explaining them to a group of students and I could follow everything he was saying. He knew what he was doing, but at the same time, wrote everything out as if he needed to verify it all for himself. All of the other participants with Task One were guilty at some point of stating their conclusions, without actually explaining how they came to those conclusions. Phil was the participant who needed the most prompting to state his reasoning out loud. However, Jordan drew number lines for all four sets, sketched the intervals on them, and even wrote out an alternative notation for set D. He also made a table to summarize his conclusions and filled it in only if he had confirmed and explained the result. Out of all of the participants, Jordan came the closest to laying out his schema on the white board during this task.

For Task One, Brandon and Kyle did well in their early interview. They were able to visualize and reason successfully through the prompts for sets A, B, and C. For set D, however, they did not read the set correctly and instead claimed that set D was $(-1,0)\cup(0,1)$. When considering the condition that $\frac{1}{x}$ cannot be in the positive integers, they seemed to only consider x values that were integers and it did not occur to them to consider rational x values. For the late interview, Brandon and Kyle correctly identified what set D looked like, however, they neglected the technical half of the task where they consider the subspace Y. They only said whether each set is open in $\mathbb{R}$ and did so using metric space arguments. It was not until the end of the interview when they were asked to reflect on their early interview that they realized their mistake in this task.

Phil would have not gotten Task One completely correct in his first try if he were not in an interview setting. He claimed that open in Y and open in $\mathbb{R}$ meant the same thing and therefore only did half of the task. (The same half that Brandon and Kyle did in their late interview.) Once done, the interviewer pointed out the differences in the subspace topology and the standard topology and Phil was given the opportunity to complete the task. After some prompting from the interviewer to explain his reasoning out loud, Phil had to reconsider some misunderstandings he had. Once he did so, he was able to reason correctly through sets A, B, and D. In the case of sets A and D, he was even asked multiple times to explain his reasoning for those because he continued to give reasoning that was not entirely correct or clear. The interviewer continued to give Phil opportunities to reason through the mathematics on his own. For set C, Phil reached the correct conclusion eventually, but never gave a reasoning on his own. Phil needed several opportunities before he was completely done with the task with correct conclusions and valid justifications.

Dr. T and Jordan demonstrated great mathematical understanding of this task and their responses can somewhat be thought of as an ‘answer key’. Although there certainly were some commendable moments mathematically with Brandon, Kyle, and Phil, it was very clear when they showed gaps in their understanding or were slower to reach their conclusions. For instance, the way in which Brandon and Kyle discussed set theory during the early interview makes it obvious that they are more of a novice status than Jordan was, demonstrated by things like misunderstanding what set D was and when Kyle continued to say “union” instead of “intersection” several times.

6.2 Task Two

Show that each collection of subsets of $\mathbb{R}$ is a basis for a topology on $\mathbb{R}$.

$$\mathcal{B}_1 = \{(a, b] \mid a < b\}, \text{ where } (a, b] = \{x \mid a < x \leq b\}$$

$$\mathcal{B}_2 = \{(-\infty, a) \mid a \in \mathbb{R}\}, \text{ where } (-\infty, a) = \{x \mid x < a\}$$

6.2.1 Solution

The definition of a basis gives two conditions that need to be checked in order for a collection to be a basis. We will check both of these for each of our given collections.

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

For $\mathcal{B}_1$, consider any real number x. Note that $x \in (x-1, x]$ and $(x -$

$1, x] \in \mathcal{B}_1$, which satisfies the first condition. For the second condition, consider if $x \in (a, b] \cap (c, d]$. Without loss of generality, there are two cases for what this intersection looks like. Either the intersection is the empty set, or the intersection looks like an element of $\mathcal{B}_1$, which immediately satisfies the condition. Therefore, $\mathcal{B}_1$ is a basis for a topology on $\mathbb{R}$.

For $\mathcal{B}_2$, consider any real number x. Note that $x \in (-\infty, x+1)$ and $(-\infty, x+1) \in \mathcal{B}_2$, which satisfies the first condition. For the second condition, consider if $x \in (-\infty, a) \cap (-\infty, b)$. Note that $(-\infty, a) \cap (-\infty, b) = (-\infty, \min\{a, b\})$, which is also an element of $\mathcal{B}_2$. So the second condition is also satisfied. Therefore, $\mathcal{B}_2$ is a basis for a topology on $\mathbb{R}$.

6.2.2 The professor, Dr. T

Dr. T completed this task in an efficient manner. He focused on the second condition of the definition of a basis because he said it was "fairly clear" that "every real number is in one of these sets" (the first condition). In considering the second condition, Dr. T explained that you do not have to find a smaller subset in the intersections of the basis elements:

> That's actually [true] for both I think. That when you intersect two intervals of the form $(-\infty, a)$, the intersection will be $(-\infty$, whichever the minimum of the two values of a is). So if I'm thinking of this in terms of these examples, trying to explain what a basis is for a student, these are somewhat limited in usefulness. . . the issue of having to find a smaller set that's contained in the intersection doesn't arise with these examples....I mean, they're examples that work and then you will get some sort of interesting topologies associated with them, but

it doesn't quite capture what the full idea of a basis is in some sense.

He gave an example to demonstrate his point. If you consider $\mathbb{R}^2$ and a basis consisting of open circles, intersecting two open circles could give something like a elongated football shape (the yellow region in Figure 6.8), which is not itself a circle. So you have to explain why you are able to find an open circle (containing the element x) within that football shape. In the sets given in Task Two, however, when you intersect two basis elements, you get exactly another basis element, so there's no explanation, searching, nor creating needed for these examples.

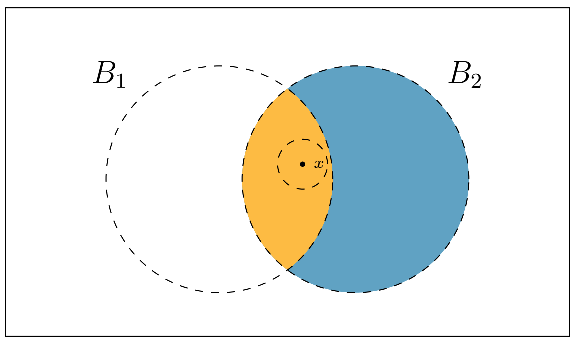


Figure 6.8: An example in $\mathbb{R}^2$.

Dr. T said (multiple times) that the key thing needed for this task is that students need to be able to analyze intersections of sets. Since these two bases are not as complicated, Dr. T said that he would want to see that the student would be able to check the definition given to them. He pointed out that there's no trick or other method to doing this task, but still some students have trouble with definition checking:

> Some students have trouble adapting to the concept of definitions and theorems...It's always challenging because...it's really hard to get across for some students the point of, the definition is where you start. You don't look for some...there isn't some secret thing that you don't

> know about of why something is true. So, these things are good as far as just emphasizing you have to work from what the definition says.

More than once, Dr. T mentioned that these two bases will generate interesting and unnatural topologies. He never elaborated about what those topologies looked like specifically or why he was thinking about that.

6.2.3 The doctoral student, Jordan

To start off this task, Jordan wrote out the two bases, as well as the definition of a basis (see Figure 6.9). Before writing the definition of a basis down, he was strategizing, "So, I don't guess you gave me any propositions on here. Maybe we'll use the definition. I may need to prove a proposition." After writing the definition, he explained what the second condition was asking for. "So, what this is saying is if you have two basic open sets, the intersection may not be basic, but at least you can find a subset of that intersection around x that is basic." He then reasoned through the first condition of the definition of a basis in the following way.

> *Jordan:* So, we just have to show that these collections satisfy this definition. So, one is going to be verified for both of these. . . . The first condition, you can show pretty easily it applies to both of these. So, for the first one, if you take an arbitrary point x, then you could take this half open interval and you could put x as your right endpoint. Then, for the next one, just do minus infinity to $x + 1$, whatever x is.

$\vdots$

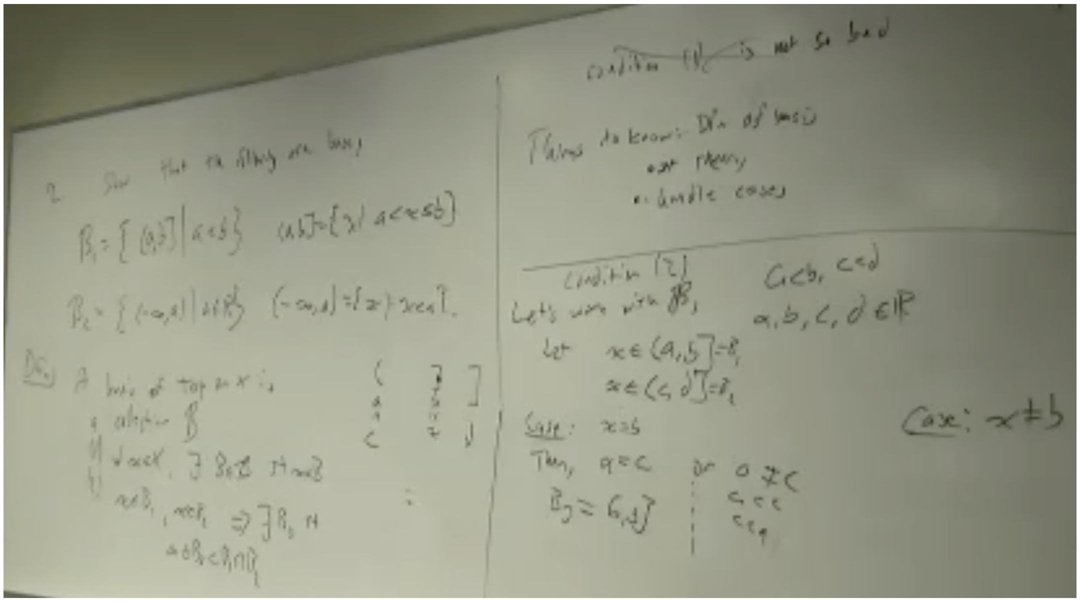


Figure 6.9: Jordan's board work during Task Two.

> *Jordan:* We don't need the proposition. Condition one is not so bad. [Writes "Condition (1) is not so bad."] So, I'll leave some space, and then condition two. Okay. It's less obvious to me. So, let's work with just $\mathcal{B}_1$ first. We're going to try to prove that condition two is true.

Jordan fairly quickly determined that "there's going to be lots of cases" for the intersection of basis elements that contain x and those cases could "make this weird." For example, "maybe x is the end point [referring to the closed end of one of the basis elements]". When exploring the first case, Jordan wrote his solution using symbols, but then also drew a visual representation to help show what he was saying.

Once the first case for $\mathcal{B}_1$ was done, the interviewer asked Jordan to refrain from completing the remaining cases due to time, but did ask that he walk through exactly what cases he was going to do. He had written the two basis elements that contain x as $(a, b]$ and $(c, d]$ where $a < b$ and $c < d$. His main cases were whether or not x is one of the closed endpoints. So without loss of generality, he chose $x = b$ and within that, he considered if $a = c$, $a < c$, or $c < a$. Then he said he should have the same three sub-cases for if $x \neq b$.

Jordan also clarified what he meant when he mentioned a proposition earlier:

> So, you need to know this definition and there is a proposition about bases. There's some easy condition to prove something's a basis, which I don't remember now, but I know it's slightly easier than the definition. I was going to see if I can prove it, but I think I can do it straight from the definition because it's not so bad for these collections.

When thinking about this mystery proposition, Jordan had another guess for what he was trying to think of.

> It's something like if you, like if you take an arbitrary open set... oh. You already have to know the topology on this, I think. I think it's something if you take an open set, each point you can find a basic open set around it. Which, that might be more helpful to compare two topologies, but we don't know what topologies these things generate so I don't think that proposition would be helpful. That was just something I remembered and was thinking maybe that's helpful. Maybe that would be helpful here.

Jordan was asked about what concepts students need to be able to do this task and in terms of prerequisite topics, his list was short: they need to know the definition of a basis, they need to be very familiar with set theory, and they need to be comfortable breaking things down into cases.

6.2.4 The first-year graduate students, Brandon and Kyle

The Early Interview

Brandon and Kyle took a few minutes at first to review the task and the given definition of a basis. For the first condition of $\mathcal{B}_1$, Brandon suggested the basis element $(x-1, x+1]$ to contain $x \in \mathbb{R}$. For the second condition, they chose the notation $B_1 = (a, b]$ and $B_2 = (c, d]$ and then considered if $x \in (a, b] \cap (c, d]$. At first, Kyle concluded that the intersection must be $(c, b]$, but then soon realized that other cases are possible, such as the case when the intersection is empty. The other cases they considered were when $(a, b] \subset (c, d]$ or $(c, d] \subset (a, b]$, and when $a < c < b < d$ or $c < a < d < b$. They verbally reasoned through these cases, but did not write their reasoning down formally.

When they considered $\mathcal{B}_2$, they defined $B = (-\infty, x+1)$ to satisfy the first condition. For the second condition, they chose $B_1 = (-\infty, a)$ and $B_2 = (-\infty, b)$ and immediately identified that $B_1 \cap B_2 = (-\infty, \min\{a, b\})$. Kyle started to define d as the distance from x to $\min\{a, b\}$, but then changed it so that d is the average between x and $\min\{a, b\}$. He did this because d will certainly be between x and $\min\{a, b\}$, so he defined $B_3 = (-\infty, d)$. In reviewing their B_3, they checked to make sure that x couldn't equal the minimum of a and b. Brandon pointed to the open right endpoints as the reason, while Kyle mentioned an open ball argument around x. They then got off on a tangent that caused them to modify their definition of B_3.

> *Brandon:* Now what if x is the minimum?
>
> *Kyle:* Well, x can't be the minimum if it's in this open set, right? Because, yeah, that has to be either a or b.

Brandon: Yeah, x can't be, can't be either one of those. Because then it wouldn't be in one of [those, pointing to B_1 and B_2].

Kyle: Because there has to be some open ball around x or…

Brandon: You keep saying metric space stuff.

Kyle: Right, I'm thinking metric space stuff and that's not true, but okay, if x is in this interval, right? Then we know that x can't be b...x can't be equal to the minimum of a, b, right? Because that point isn't included in this interval.

Brandon: Gotcha.

Kyle: Okay? So that means there has to be some distance between x and min a b. So what we could do, we could do min a b minus the distance between x and min a b divided by two. And you could set that distance equal to d and then set this equal to negative infinity to min a b minus d, right? Does that make sense?

Brandon: Kind of.

Kyle: Okay, so, so if we instead said...So d is just the distance between x and min a b, right? So x minus min a b. Okay. And then so x has to be in negative infinity to min a b minus... min a b minus d divided by two, right? [Writes typo on board.] Oh no, sorry. A swing and a miss. [Erases typo.] Min a b minus d divided by two. There we go. Does that make sense? So this right here, this is the distance between. So you have x is falling somewhere on your number line, right? And then you have min a b, right? So we take the distance between these, d, right? We cut it in half. And so now x has to be in this interval, right? And then this point is just min a b minus d over two. So, that, which is kind of ugly looking. Yeah.

To summarize what they did with notation, they defined $d = |\min\{a, b\} - x|$ and then set $B_3 = (-\infty, \min\{a, b\} - \frac{d}{2})$.

The Late Interview

When Brandon and Kyle considered $\mathcal{B}_1$ in this interview, they went about it in a more efficient way in comparison to the early interview. For the first condition of $\mathcal{B}_1$, they created the basis element $(x - 1, x]$. For the second condition, they did not go down a road of cases. Instead, they said suppose $x \in (a, b] \cap (c, d]$ where $a < b$ and $c < d$ and then said that $x \in (\max\{a, c\}, \min\{b, d\}] \subset B_1 \cap B_2$.

For the basis $\mathcal{B}_2$, they showed the first condition in exactly the same way as they did in the early interview. They were much more efficient for showing the second condition however. Kyle reasoned, "Alright, well, either B_1 is a subset of B_2, or B_2 is a subset of B_1. So, just take that [subset] to be your element in the intersection." Brandon agreed and wrote out $x \in B_3 = (-\infty, \min(a, b)) \subset B_1 \cap B_2$.

6.2.5 The undergraduate student, Phil

Phil immediately said that Task Two "seems like this is just kind of a definition thing." For $\mathcal{B}_1$, he verbally proved the first condition by saying, "Any a comma x where a is less than x contains x, so you've met that requirement." For the second condition, he considered two basis elements that contained x and discussed an idea of having x be the right endpoint for a third basis element, with the left endpoint being "less than or equal to the closest thing to x". The interviewer followed up on that idea by drawing two scenarios and asked Phil what basis elements he would choose (see Figure 6.10). Phil then gave the intervals pictured on the right of each scenario. Interestingly, he gave basis elements that were not the same as the idea he had described verbally. His earlier idea was to choose the basis element $(c, x]$ for both of these scenarios. After these scenarios, Phil did not discuss $\mathcal{B}_1$ any further.

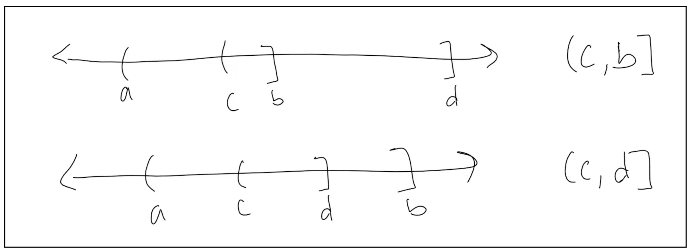


Figure 6.10: Phil's scenarios for Task Two.

Phil moved on to $\mathcal{B}_2$ and quickly addressed the first condition (in a similar way to how he did for $\mathcal{B}_1$). "Well, you can pick any x and then that's contained in $(-\infty, x+1)$." For the second condition, he pointed out that there's really only one case to consider because you consider the basis element with the smaller of the two right endpoints. In Figure 6.11, he's referring to $(-\infty, a)$. He was not sure if he was allowed to just use $(-\infty, a)$ as his basis element, but said it didn't matter because if he had to find another basis element within that, it would not be difficult because he could just pick any number between x and a since this was on the real number line. The interviewer suggested to pick a specific value, like the halfway point, and Phil agreed that that would work as the right endpoint.

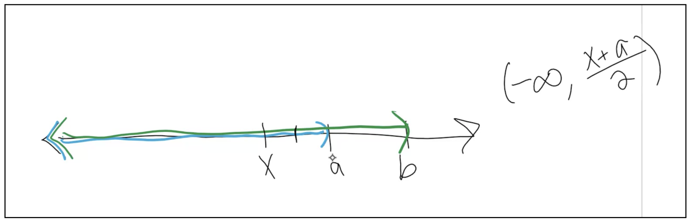


Figure 6.11: Phil explains the second condition of $\mathcal{B}_2$.

6.2.6 Task Two Summary

In their early interview, Brandon and Kyle broke their proof for $\mathcal{B}_1$ down into cases verbally, but did not write out every detail of those cases. Interestingly, they went into great detail to prove that $\mathcal{B}_2$ was a basis, while the other participants did not see a need to. They interpreted the subset notation as if the third basis element could not be equal to the intersection of the two basis elements. Therefore they worked hard to find a proper subset of the intersection, which was unnecessary for the problem. In the late interview, they came up with a basis element of $\mathcal{B}_1$ that was unlike anything the other participants created and did not require cases. For $\mathcal{B}_2$, they were no longer concerned about creating a proper subset and came up with the same basis element that Dr. T did.

Jordan not only broke his proof down into cases for $\mathcal{B}_1$, but he also tried to recall a proposition that he thought might be helpful instead of following the definition. After explaining the first condition for both bases, he realized that a proposition would not be necessary.

Jordan, Brandon, and Kyle both tried to break this problem down into cases for $\mathcal{B}_1$. They ended up thinking about cases resulting from labeling their basis elements and it became quite detailed, but Dr. T just noted how the intersections look exactly like the basis elements and was done. Dr. T emphasized that these two bases were good for seeing if students could check definitions, but they were not great for conveying what a basis is to students.

Phil quickly identified this task as a definition checking task. He had an interesting idea for the second condition of $\mathcal{B}_1$ that was somewhat similar to Brandon and Kyle's idea and did not require cases, but when asked to clarify, Phil abandoned his idea. He did not mention cases explicitly, but when the interviewer followed up with scenarios to clarify what Phil meant, he responded as if he had gone the cases route. For $\mathcal{B}_2$, Phil was not sure if he needed a proper subset or not, but explained that it did not matter since he could make a proper subset if necessary.

6.3 Task Three

(a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.

(b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$ is an open map.

6.3.1 Solution

(a) The product topology on $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$. In other words, the basis elements are open rectangles, but then open sets can be written as unions of these open rectangles.

To show that $\mathcal{B}$ is a basis for a topology on $X \times Y$, you have to check the two conditions from the definition of a basis (given in Task Two's solution).

Note that X is open in X and Y is open in Y, so $X \times Y \in \mathcal{B}$. Therefore the first condition is trivial. For the second condition, consider the intersection of two open rectangles. This intersection is either empty or another open rectangle, so the second condition will hold true.

(b) The projection map will be an open map if every open set of $X \times Y$ maps to an open set of X. So start by considering an arbitrary open set of $X \times Y$, which by part (a), looks like a (possibly infinite) union of basis elements.

$$(U_1 \times V_1) \cup (U_2 \times V_2) \cup ... \cup (U_n \times V_n)$$

Now consider what the projection map sends this set to. It takes a little bit of checking that the projection of a union is the same as the union of corresponding projections. Most can check this intuitively, but I will also give a technical proof here. Note that $x \in X$ will be in the projection

$$p_X\left((U_1 \times V_1) \cup (U_2 \times V_2) \cup ... \cup (U_n \times V_n)\right)$$

if and only if there exists a $y \in Y$ such that

$$(x, y) \in (U_1 \times V_1) \cup (U_2 \times V_2) \cup ... \cup (U_n \times V_n).$$

By definition of a union, this occurs if and only if (x, y) is an element of at least one of the open rectangles, which means that x is an element of one of the U_i's. Therefore $x \in U_1 \cup U_2 \cup ... \cup U_n$. Since $U_i = p_X(U_i \times V_i)$, we have

$$x \in p_X(U_1 \times V_1) \cup p_X(U_2 \times V_2) \cup ... \cup p_X(U_n \times B_n).$$

Now there is only one comment that needs to be made for the other direction. If you suppose $x \in p_X(U_1 \times V_1) \cup p_X(U_2 \times V_2) \cup ... \cup p_X(U_n \times B_n)$, then there is at least one $p_X(U_i \times V_i)$ that x is in, which means there exists a $y \in V_i$ such that $(x, y) \in U_i \times V_i$. So (x, y) is an element of at least one of the open rectangles, and the remaining steps were given before as if and only if statements. So we have that the projection of unions is equal to the union of corresponding projections.

Now that this has been verified, we can see that our set maps to $U_1 \cup U_2 \cup ... \cup U_n$. Since each U_i was open in X by definition, we can see that this union is also open in X. Therefore the projection map is open.

6.3.2 The professor, Dr. T

Dr. T correctly defined the product topology and then concluded that it would not be that difficult to show that the open rectangles satisfy the definition of a basis. He did not go into the details of showing this once he realized it would not be hard to do. After the basis was established, the key portion of the proof that Dr. T considered and explained was what the projection of a union of sets would look like. He said that you cannot get around needing to "describe what the image of the union of sets is on your function" because "an open set in X cross Y is going to be a union of open rectangles." He did not lay out every detail for this task. Instead, he just gave the highlights of what would need to be shown for the two parts.

The interviewer followed up with Dr. T about showing that the set in part (a) is a basis, asking him if that part was necessary. He said that it was necessary for the task, or you could define the topology without referencing a basis, which would mean you define it as all unions of open rectangles. After the interview, I realized that this alternative route would require that you prove that the set you came up with was actually a topology. Realizing this parallel logic made it easier for me to understand why you have to prove that the set in part a is a basis. In considering this alternative way of defining the topology, Dr. T explained what he would prefer pedagogically.

> I would want to encourage them to be thinking on this in terms of a basis. You show that the open rectangles are a basis and then you have this basic fact that says that if you have a basis for a topology then a set is open if it's a union of basis elements. And then for part b, it's clear that the projection of an open rectangle will be an open set.

Finally, Dr. T had two different views regarding this problem, depending on his audience. He said that "the graduate students in topology absolutely need to be totally comfortable with doing this," while he might not emphasize this kind of task to undergraduates as much, since their focus is likely different from the graduate students' needs.

6.3.3 The first-year graduate students, Brandon and Kyle

Note that this portion of the data was the focus of Pilot Study 2, however, a different unit of analysis was used than what was presented in Chapter 4. Further, the results here are described in greater detail than in the pilot study.

The Early Interview

In their initial interview, Brandon and Kyle defined the product topology without using a basis and then had a short proof based on their incorrect definition. This was the typical mistake that was found in Pilot Study 1. Specifically, in defining the product topology in part a, the pair incorrectly stated that the product topology consisted of all sets of the form $U \times V$, not the unions of such sets. Therefore, they set themselves up for a fairly trivial proof for part b. They both took some time wrapping their minds around the problem, with Kyle doing so in a more verbal manner. For part b, they took their arbitrary open set to be exactly what is expected based on their response to part a, which is $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ (see Figure 6.12). From there, they followed the definition of the projection map and immediately got what they needed in order to show that the map is open (see Figure 6.13).

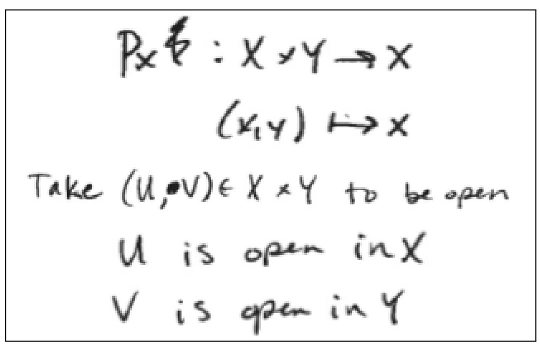


Figure 6.12: Brandon and Kyle's attempt early in the semester.

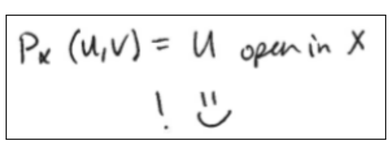


Figure 6.13: The end of Brandon and Kyle's proof.

The Late Interview

Brandon and Kyle did significantly better on this task in their late interview. For part a, they correctly defined the product topology by generating it with a basis. For part b, Brandon quickly wrote an arbitrary open set, W, on the board (see Figure 6.14).

W open in $X \times Y$

Figure 6.14: The start of Brandon and Kyle's proof late in the semester.

Then Brandon began writing W as a union of basis elements, but Kyle seemed unsatisfied and wanted to make the notation clearer. Brandon asked how they wanted to proceed in showing the union. Kyle took over writing on the board, erased the union that Brandon had written, and came up with the beta notation shown in Figure 6.15. After this, he was still unsatisfied with his notation and the usage of too many b's, but Brandon said it was fine, so they moved on. Note that Kyle immediately knew that they were wanting to write a union of basis elements, but the pair spent their time here struggling to denote it. Kyle stated "W is the union of elements of...some arbitrary union of elements of that B thing [referring to the basis they wrote in part a]."

$$\Rightarrow W = \cancel{\bigcup(U \times V)}$$

$$\bigcup_{B \in \beta} B \quad \text{for some } \beta \subset \mathcal{B}$$

Figure 6.15: Where Brandon and Kyle had notational difficulties.

In moving forward in their proof, they started off quickly saying that the projection gives you a union of open sets, but then Kyle started thinking about if they could get "weirder things". This launched the pair into a discussion about what they showed in class regarding the projection map. Brandon finally said something that took them back to what they originally (and correctly) said, which prompted Kyle to read the task again and agree that they had been on the right track. Although their discussion may have been helpful for them in checking their ideas, it ended up not affecting the proof that they wrote down on the board, as the discussion was entirely verbal and ended with the same conclusions that they began with. They then discussed what exactly the projection does, and this prompted Kyle to suddenly write up the proof seen in Figure 6.16. They acknowledged that they get a union of open sets of X from the basis elements and use this as the end of their proof.

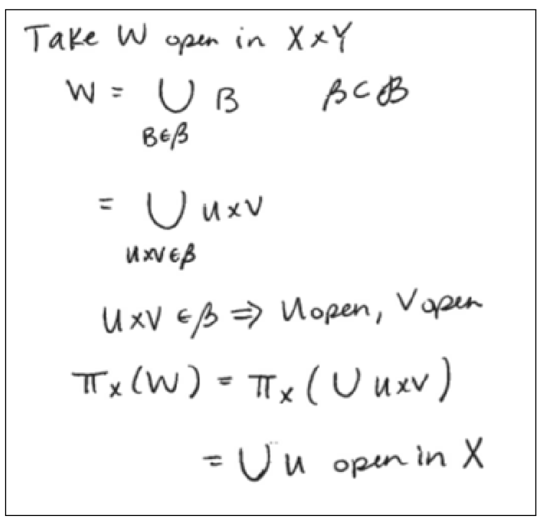


Figure 6.16: Brandon and Kyle's proof for Task Three.

Towards the end of the late interview, Brandon and Kyle watched the recording of them working on Task Three in their early interview and were asked to see if they could identify any misunderstandings they may have had before. Before the video was done playing, Brandon interrupted, "Oh, I already know what we did wrong. Yeah, we didn't consider an arbitrary open [set]. We just considered a basis element, didn't we?" When asked about ideas for anything that could have helped in correcting that mistake earlier, Brandon said, "I know what *did* correct that mistake is getting a homework back, with that specific thing being torn apart on it. For not considering arbitrary elements and just products of open sets and not unions of them. So that got me. So I don't forget that anymore." Kyle chimed in, admitting that he didn't "get the basis topology stuff at all when [they] were going over it in class" but later realized the importance when reading through the textbook. Brandon commented on their earlier work on the problem with,

> In terms of when we first did this, I guess, um, it's easy to just jump straight into just choosing U or something because of the way we defined the basis of just being an open set cross an open set where each of those are coming from the individual...so...it just seems natural to just go to one thing instead of considering the most general thing, which is a union of those things.

6.3.4 The undergraduate student, Phil

Phil immediately defined the product topology on $X \times Y$ by the basis $\{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$. He indicated that he was working entirely off of his memory from the topology course he was taking for part a. He then read the prompt for part b, re-read the definition of an open map, and then gave his proof for part b entirely verbally.

> So if you have some open set in the product topology, then, um, all basis...okay. I found this online, so I'm hoping it's true, but I

> remember reading something like, all open sets are unions of basis elements. So it's unions of open sets in X cross...open sets in Y, and unions of open sets are open. So when you project it back to X, it's still just a bunch of open sets in X. And that's my reasoning for b.

The interviewer then asked Phil to try to write or type that reasoning up as a proof, which is shown in Figure 6.17.

> Let A be an open set in $X \times Y$. By definition of a basis, $A = B_1 \cup B_2 \cup \ldots \cup B_n$ for some $B_n \in X \times Y$. All $B_k = U_k \times V_k$ for some $U_k \in \mathcal{T}_X$, $V_k \in \mathcal{T}_Y$. Then $p(A) = p(B_1 \cup B_2 \cup \ldots \cup B_n) = U_1 \cup U_2 \cup \ldots \cup U_n$ (by $U_k \times V_k = B_k$). By union of open sets open, we conclude $p(A)$ open.

Figure 6.17: Phil's proof for Task Three part b.

It is fair to say that if Phil was not in an interview setting, he would have been done with this task by this point. He had no further comments to make. The interviewer wanted to see if he could reason through why the projection of unions is a union of projections, so they gave him the visual in Figure 6.18 and asked how he knew he would get $U_1 \cup U_2$. Phil's response is below.

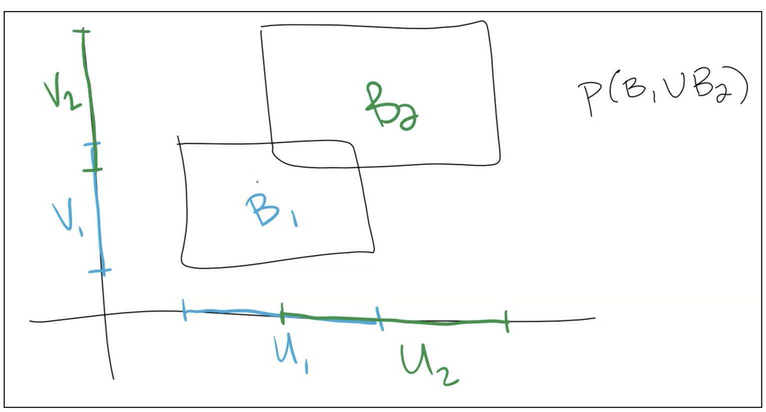


Figure 6.18: Interviewer's picture for Phil.

> You can pick an (x, y) in...either one I suppose, but it's in...so if you pick an (x, y), it's in U_1, U_2 and it sends the point (x, y) to x. So you know that the x is in U_1, U_2 cross...well it's U_1, V_1 union U_2, V_2 and you can rearrange stuff to be U_1 union U_2, cross V_1 union V_2, I think. And then you can say, oh well then x has to be in U_1, U_2 because that's the X part of the thing.

Phil did not seem confident about rearranging stuff to get the needed format for the projection map. It is actually good that he was reserved about it, because his logic in the second half of his statement is not correct regarding the rearrangement. He said that you could take $(U_1 \times V_1) \cup (U_2 \times V_2)$ and rearrange things to equivalently have $(U_1 \cup U_2) \times (V_1 \cup V_2)$, which is not true. The first is what is pictured in Figure 6.18 and the second would be an open rectangle that contains the first.

6.3.5 Task Three Summary

In their early interview, Brandon and Kyle did not consider a basis in this task at all, leading them towards a trivial proof for part b. For their late interview, they did much better with the task. They not only remembered the basis, but they then used an arbitrary open set of $X \times Y$, wrote it as a union of basis elements, and took the time to consider what the projection of that union could look like. In reflecting on the attempt during the early interview, Brandon admitted that his professor pointed out this misunderstanding on his homework, which is why he was able to do it correctly in the late interview. Similarly, Kyle said that it took him reading through the textbook again after class to understand more about using a basis.

Phil did a commendable job on this task considering what Brandon and Kyle did in their early interview. Phil utilized a basis, wrote an arbitrary open set as a union of basis elements, and had the rough outline of the proof. He glossed over why the projection of a union is the union of projections. When prompted to explain that further, he came up with an explanation that started off with the right idea, but then took a turn with a false equivalency.

Dr. T gave an abbreviated version of this task, with his main focus being on what the projection of unions of open sets looked like in part b. In part a, however, he mentioned needing to show that the defined basis actually satisfied the definition of a basis. This is something that none of the other participants discussed or considered. Note that Munkres (1975) checks that the open rectangles form a basis after defining the product topology on $X \times Y$, so the context for this task would be important as to whether or not an instructor is expecting their students to do this. If part a is something that was discussed in class and part b was homework, then it would probably be reasonable for students to not show this again. On the other hand, if Task Three were being asked on an exam, the instructor might want to see students consider this.

6.4 Task Four

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

6.4.1 Solution

Let $\mathcal{T}_{\mathcal{A}}$ be the topology generated by $\mathcal{A}$ and let $\mathcal{T}$ be the intersection of all topologies on X that contain $\mathcal{A}$. Because $\mathcal{T}_{\mathcal{A}}$ is a topology on X that contains

$\mathcal{A}$, it is one of the topologies that was intersected to get $\mathcal{T}$. Since intersections create *smaller* sets, we can see that $\mathcal{T} \subset \mathcal{T}_{\mathcal{A}}$.

For the other direction, it is helpful to know the following lemma from Munkres (1975).

> ***Lemma*** Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Using our notation for Task Four, this lemma means that $\mathcal{T}_{\mathcal{A}}$ contains $\mathcal{A}$, all unions of elements of $\mathcal{A}$, and nothing more. Now consider all topologies containing $\mathcal{A}$ and note that each one must contain all unions of elements of $\mathcal{A}$ by definition of a topology. Therefore, all unions of elements of $\mathcal{A}$ are also contained in the intersection, $\mathcal{T}$, so $\mathcal{T}_{\mathcal{A}} \subset \mathcal{T}$.

6.4.2 The professor, Dr. T

As you can see from the solution, there are two directions for this task: (1) $\mathcal{T} \subset \mathcal{T}_{\mathcal{A}}$ and (2) $\mathcal{T}_{\mathcal{A}} \subset \mathcal{T}$. Direction (1) was considered trivial by Dr. T and was hardly discussed. For direction (2), Dr. T says, "If you have a topology containing $\mathcal{A}$, you know that the union of...sets in $\mathcal{A}$ has to be in that topology by definition of topology. And every element in the topology generated by $\mathcal{A}$ is the union of elements of $\mathcal{A}$." (Therefore the topology generated by $\mathcal{A}$ is a subset of every topology containing $\mathcal{A}$, and thus the intersection also.)

Notation can be a huge stumbling block for this task. If you do not want to create notation for these topologies, then you have to describe them every time you reference them, which can easily confuse your audience, or even yourself. As for Dr. T, he explained this proof with only words, and with a minimal amount of them at that.

Dr. T liked this problem because it "requires them [students] to understand that every open set in the topology generated by $\mathcal{A}$ is a union of basic open sets." He considered whether or not there was another way to go about this proof, but did not come up with anything else during the interview. He predicted a wide variety in responses by undergraduates to this task and could imagine a lot of students who would not know where to start with this task. He hypothesized that "the students that have a good sense of what's going on will probably know how to do this very quickly. The students that don't may struggle with this a lot just because it makes it sound like they need to understand all of the topologies on X that contain $\mathcal{A}$." He thought students may miss the abstractness of the task, which he also said is important to be comfortable with in order to do this task. He spoke about how you have to be thinking in general terms in order to do this task. The problem is abstract, so you cannot get too specific about it. Additionally, students need to be comfortable with the idea of many possible topologies containing $\mathcal{A}$. For the students who would struggle, he proposed breaking the problem down into a part a and b. He suggested a part a that asks students to show that if a topology on X contains $\mathcal{A}$, then it contains the topology generated by $\mathcal{A}$. This scaffolding structure is likely showing us how his solution is organized in his schema.

Dr. T provided his perspective about why students struggle with this topic at first. "The whole thing about a lot of these definitions in topology is that it's all very abstract standards. You have to get comfortable...with a certain level of abstractness." He said this due to the abstract nature of Task Four, however, he quickly switched to talking in this same way about topology in general.

6.4.3 The first-year graduate students, Brandon and Kyle

The Early Interview

Brandon started off at the board with this task and wrote the following on the board.

$$\mathcal{A} = \text{Basis for Top on } X$$

$$\text{Then } \mathcal{A} = \bigcap \mathcal{T}_x \text{ s.t. } \mathcal{T}_x \supseteq \mathcal{A}$$

The notation was immediately difficult to follow because Brandon used $\mathcal{A}$ to denote both the basis and the topology generated by the basis. Kyle tried to verbally say what they were going to show, but was struggling with it when Brandon chimed in with "So we're going to show double containment?" Kyle sighed and said "Yeah, I think that's probably the easiest way." Brandon then wrote "open" on each side of his equality statement, to which Kyle pointed out that $\mathcal{A}$ is not a topology, so open in $\mathcal{A}$ doesn't quite make sense. Kyle then suggested that they change one of their $\mathcal{A}$'s to be $\mathcal{T}_\mathcal{A}$ to distinguish between the basis and the topology generated by the basis and they erased Brandon's "open" comments.

Kyle pointed out that $\mathcal{A} \subset \mathcal{T}_\mathcal{A}$ and confirmed with Brandon that they did not need to show that $\mathcal{T}_\mathcal{A}$ was actually a topology. They took a moment to think about it, but then Kyle thought of one direction. He wrote $\bigcap \mathcal{T} \subset \mathcal{T}_\mathcal{A}$ on the board and reasoned in the following way.

> *Kyle:* So this is a topology that contains $\mathcal{A}$, right? [Points at $\mathcal{T}_\mathcal{A}$ on board.] So, that means that the intersections of $\mathcal{T}$, such that $\mathcal{A}$ is contained in $\mathcal{T}$ ... So this [the intersection] has to be a subset

of $\mathcal{T}_{\mathcal{A}}$, right?

[Long pause]

Kyle: So $\mathcal{A}$ is contained in $\mathcal{T}_{\mathcal{A}}$, right? So then the intersection of all topologies containing $\mathcal{A}$...this [$\mathcal{T}_{\mathcal{A}}$] is one of the things being intersected, right? So then, this [the intersection] has to be a subset of this [$\mathcal{T}_{\mathcal{A}}$], right? Because you're taking the intersection of a bunch of things, this [$\mathcal{T}_{\mathcal{A}}$] is one of them, right?

Brandon: I think we're going down a different path.

Kyle: You think we're going down a different path. Okay. So what are you thinking?

Brandon: Just looking at the definition real quick.

Brandon was referencing Definition 3 of the given definition sheet (see Appendix A). He then went to the left side of the board and tried to write out this definition using notation:

$$\mathcal{T}_{\mathcal{A}} = \{U \subset X \mid \text{ if } x \in U, \exists A \in \mathcal{A} \text{ s.t. } x \in A \subset U\}$$

Brandon's idea was to choose an arbitrary $U \in \mathcal{T}_{\mathcal{A}}$ and show that U is in the intersection and then vice versa. Kyle did not agree that that strategy would prove equality. He said that that only shows that their intersection is nontrivial. There seemed to be multiple kinds of misunderstandings at this point between what each of them meant and what their strategies were. Brandon asked Kyle to clarify what the "thing on the left" was [$\mathcal{T}_{\mathcal{A}}$] and Kyle said that it was a collection of sets. Then Brandon clarified that he was choosing an *arbitrary* set with his method, which caused them to be in agreement again that that method could work, but they still were not on the same page about which direction they were working on and what notation they were using.

Brandon: So what is the thing on the left?

Kyle: The thing on the left is a collection of sets; subsets of X.

Brandon: Okay, so if we show that every set in that is also in the intersection of all those things...

Kyle: Okay. So you're showing that *every* set in this...

Brandon: Yeah and that starts by just choosing an arbitrary thing.

Kyle: ...of this. Okay.

Brandon: I'm not just showing one thing. I'm showing this, choosing something we can show is in there.

Kyle: Okay. So you're showing that an arbitrary set... Yeah. Right. ... You're right. And yeah, you're showing an arbitrary thing from this, has to be in this, which means that this is a subset of this, right?

Brandon: Yeah. And then if we show the other way, then they have to be equal.

Kyle: Right. Well what I was doing here was showing the other way.

Brandon: Okay, so you're just... So we're given this thing [points to $\mathcal{T}_x \supseteq \mathcal{A}$].

Kyle: Right. So basically I just wrote this down differently.

Brandon: Okay.

Brandon and Kyle then clarified their notation and Kyle added to what he had written earlier, so that it now looked like $\bigcap_{\mathcal{A}\subset\mathcal{T}} \mathcal{T} \subset \mathcal{T}_{\mathcal{A}}$. Kyle then explained again the direction he had explained earlier, this time with Brandon following along and giving verbal confirmation that he was understanding what Kyle was saying. They agreed that this direction was done, but they needed to figure out how to prove the other direction. It was at this point that they had to stop working because they were out of time for the interview. So they did not figure out in time during the early interview how to prove the $\mathcal{T}_{\mathcal{A}} \subset \bigcap \mathcal{T}$ direction.

The Late Interview

To discuss these results, I will again refer to the two directions from the solution of the task where $\mathcal{T} \subset \mathcal{T}_{\mathcal{A}}$ is direction (1) and $\mathcal{T}_{\mathcal{A}} \subset \mathcal{T}$ is direction (2).

For the late interview, Brandon started writing the conclusion, using $\mathcal{T}_{\mathcal{A}}$ to denote the topology generated by $\mathcal{A}$ and $\bigcap \mathcal{T}_x$ such that $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{T}_x$ to denote the intersection of all topologies containing $\mathcal{A}$. (see the top line of Figure 6.19.)

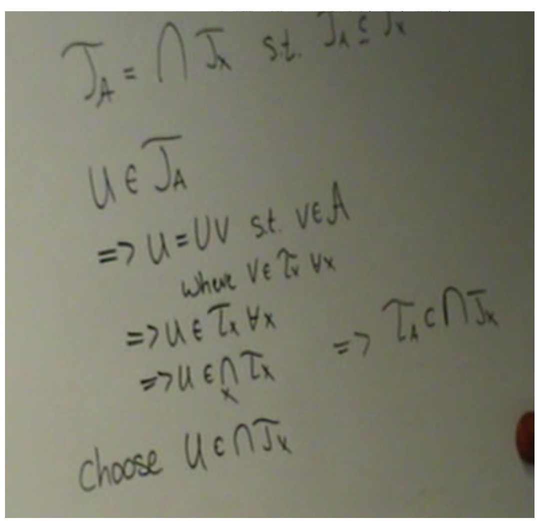


Figure 6.19: Brandon's writing for Task Four.

Verbally, Kyle reasoned that "For starters, $\mathcal{T}_\mathcal{A}$ is going to be a subset of the intersection since it's contained in all of the elements, right? So now we just have to do the other way." He then wrote $\mathcal{T}_\mathcal{A} \subset \bigcap \mathcal{T}_x$ (direction (2)) on the board. Brandon then tried to reason why that would be true, saying that he would choose an open set in $\mathcal{T}_\mathcal{A}$ and would need to show why it has to be in every $\mathcal{T}_x$. It is not clear whether Kyle wrote down the direction he thought was obvious or if he wrote down the direction he still needed to prove and had misspoken earlier. At this point, it is also not clear if Brandon followed Kyle's previous statement.

They both thought silently for a moment, and then Kyle approached the whiteboard. Something Kyle said prompted Brandon to clarify out loud that, "$\mathcal{A}$ is a basis and $\mathcal{T}_\mathcal{A}$ is a union of elements of $\mathcal{A}$." This shows that Brandon knows and is using the lemma that makes this task easier to think about. Then Brandon pointed to the board and said, "We want to choose something over here now [pointing to the intersection] and show it's over here [pointing to $\mathcal{T}_\mathcal{A}$]." This indicates that Brandon is wanting to show direction (1). They then pause for a few more moments and then go back to direction (2).

Kyle: This [indicates $\mathcal{T}_\mathcal{A}$] is a subset of the intersection of all of these things...

Brandon: Yeah.

Kyle: ...because...

Brandon: ...it's contained in every single one of them.

Kyle: Well...

Brandon: So it's contained in the intersection.

Kyle: Well $\mathcal{T}_\mathcal{A}$ is not...it's not given to us that $\mathcal{T}_\mathcal{A}$ is contained in every one of them. But we have that $\mathcal{A}$ is contained in every one of them, right...

Brandon: Oh, I get what you're saying.

Kyle: ...and so you take unions of $\mathcal{A}$ that has to be contained in your topology.

Brandon: Yeah.

Kyle: So that has to be contained in $\mathcal{T}_{\mathcal{A}}$. So maybe we should give a little more justification there because...

Brandon: I get what you're saying.

Kyle: I kind of overlooked that at first, but there's a little more to say, right?

This discussion prompted Kyle to try to write this down, but he struggled with the notation. Brandon tried to help verbally, but Kyle backed away so that Brandon could write what he was envisioning. Brandon erased Kyle's attempted notation and quickly wrote out the bulk of the work shown in Figure 6.19. They both worked together to explain the details while Brandon wrote. They said again that every open set in the topology generated by $\mathcal{A}$ is a union of basis elements and every one of those basis elements is in all of the topologies that contain $\mathcal{A}$. At this point, they finally had clear, concrete reasoning for direction (2).

For direction (1), Brandon started to write on the board to use the same notational strategy as direction (2), but Kyle was explaining why this direction was easy, so Brandon handed over the marker to Kyle, which is when Kyle wrote what we see in Figure 6.20.

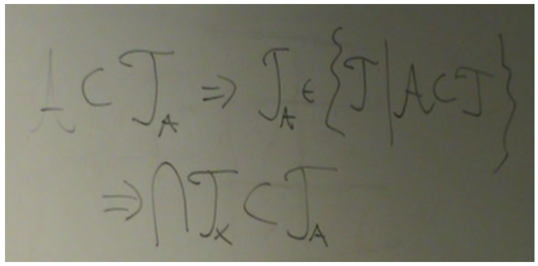


Figure 6.20: Kyle's writing for Task Four.

6.4.4 The undergraduate student, Phil

For this task, Phil said that it seemed like it would be a "subset problem" or an "element chasing thing". He was referring to the method of taking an arbitrary element of one set and showing that it is in the other set, and then vice versa. He thought out loud for a moment, making it seem as if one direction was clear. When asked to elaborate, he said it was clear that "$\mathcal{A}$ is a subset of the intersection of all topologies that contain $\mathcal{A}$ and the reason that, to me, it's clear is that...the requirement of all $\mathcal{T}_i$ is that they contain $\mathcal{A}$, so they...if all $\mathcal{T}_i$ contain $\mathcal{A}$ then their intersection must contain $\mathcal{A}$." Here we can see that Phil has not distinguished between $\mathcal{A}$ and the topology generated by $\mathcal{A}$. While it is true that $\mathcal{A} \subset \bigcap \mathcal{T}_i$, that is not productive for what the problem was asking.

In considering the other direction, Phil asked, "Are we allowed to count the topology generated by $\mathcal{A}$ as a thing that contains $\mathcal{A}$? Because if so, then the other argument is a lot easier." It was at this point that Phil finally addressed the topology generated by $\mathcal{A}$ and denoted it by $\mathcal{T}_\mathcal{A}$. Even though he is right that this direction is easier, he did not reason through it correctly. He still indicated as if he were trying to show $\bigcap \mathcal{T}_i \subset \mathcal{A}$, even though he had acknowledged $\mathcal{T}_\mathcal{A}$ in his reasoning. It was then suggested that Phil type his thoughts on the screen so that nothing was unclear, which is shown in Figure 6.21. He typed (ii) first and struggled with (i). He admitted that he might need help with (i).

(i) Let $x \in \mathcal{T}_{\mathcal{A}}$.
(ii) Let $x \in \bigcap \mathcal{T}_i$. Then as $\mathcal{A} \subset \mathcal{T}_{\mathcal{A}}$, $\mathcal{T}_{\mathcal{A}} \in \{\mathcal{T}_i\}$, so $x \in \mathcal{T}_{\mathcal{A}}$.

Figure 6.21: Phil's proof for Task Four.

After long moments of silence, the interviewer asked Phil if he was trying anything in his head that he could spitball out loud. Phil said that he was trying to think about things logically, but couldn't give specifics. It was at this point that the interviewer asked if he wanted help and he said yes, so they walked him through the (i) direction. In doing this, they drew out a picture to help Phil follow what they were saying. Once the visual was there, Phil was on the same page and said that it made more sense.

6.4.5 Task Four Summary

Dr. T considered the first direction of this proof to be trivial. For the nontrivial direction, he relied on the lemma that was given in the solution. Dr. T was completely verbal in his explanation and did not write anything on the screen. He pointed to the abstract nature of this task as the main issue for students. He also predicted that there would be a wide variety in responses for this task from undergraduates and that many of them might not know where to start. He then suggested a scaffolding for the problem to help out those students who are stuck from the beginning.

Brandon and Kyle spent a good amount of time wrapping their minds around this task in their early interview. They spent time reading the definition sheet, denoting the different sets of the task in various ways, and were not always on the same page with each other. This is the task where there was a bit of friction in their early interview. They thought of this proof in ways that were different from the other and they were even thinking about different directions of the proof at one point. They had a lot of discussion and some debate in order to show the direction that Dr. T considered to be trivial and did not have enough time to successfully show the other direction.

For their late interview, Brandon and Kyle still stumbled around at first with notation and details, but quickly were on the same page and showed the nontrivial direction first. They seemed to have a much better intuition for what the topology generated by $\mathcal{A}$ was and never referred to the definition sheet to look at the formal definition of it, which helped them not get bogged down by the details. They successfully finished both directions of the proof in their late interview.

Phil was unable to complete this proof in the interview without help. He showed the trivial direction, but in a way that was different from the other participants. He considered an element x of the intersection and showed that it had to be in the topology generated by $\mathcal{A}$. This strategy was more granular than the other participants' strategies, which remained on the subset level.

The participants' responses to this task were consistent with what stages they were at in their mathematical careers. Dr. T explained this proof efficiently, quickly, and verbally. Meanwhile, Brandon and Kyle were able to prove this with a small amount of time and discussion in their late interview. In their early interview, they debated heavily with each other and were only able to show the trivial direction. As for Phil, he also was only able to do the trivial direction, and he did so with some unconventional notation.

6.5 Reflections from the Participants

Additional comments from Dr. T

Dr. T was the only participant who was asked the following question at the end of their interview: Do you have any general thoughts or feelings about students learning topology for the first time? Why do you think that first experience can be challenging and how do you think it could be alleviated? In his responses, he pointed to the abstractness of topology as a primary reason students struggle initially. Additionally, there could be little to no context or motivation mathematically, which is something he tries to address when he teaches it.

> Well, it is when you start off with point-set topology, like following Munkres, the level of the abstraction is really, I think, hard to get used to. I mean, because you're just, you're just going through these definitions, but you don't have a context for...what this is all about [or] why you would want to do that. So I mean, sometimes in mathematics things are like that where...something looks really abstract. But if someone says it's really important to know this, then we should just sort of jump in and try to see where it leads. But again I think what's hard is that there is no context about, well, why did, why do we need to do this level of abstraction?

Whenever Dr. T mentioned students, he was either talking about the struggles he has witnessed students have in topology, or he was talking about their background and where they are going once they graduate. He acknowledged that most of them are not continuing on to graduate school, which has a significant impact on how he chooses to teach the course. He notes that other instructors will teach the course rigorously, but he finds that that does very little in terms of motivating the students or making the subject interesting. Instead, his approach to teaching the course involves using topology concepts to prove theorems students learned in Calculus, as well as giving some fundamentals in order to understand where topology research is at today.

In this final discussion, Dr. T also lists some prerequisite knowledge that is necessary for first learning topology.

> Getting started in point-set topology, the students have to be really, really comfortable with dealing with sets and functions and unions and intersections. And taking...as well as the concept of cardinality, uncountable sets and stuff. So, so there is that aspect of topology that you need to be really grounded in basic set theory. And sometimes the students that are having trouble with that are the ones where the subject is going to be more difficult.

Additional comments from Brandon and Kyle

During their late interview, Brandon and Kyle were asked to reflect on when they did these tasks earlier in the semester compared to what they just did at the end of the semester. The first thing Brandon pointed out was that the definitions were not as big of a deal anymore. Kyle agreed and said that they were more automatic now. Overall, they did not need to reference the definition sheet that they were given in their late interview. They also said that they had seen those problems and concepts a lot more now that it was the end of the semester. When they did their initial interview, only the ideas in Task One were remotely familiar, but overall everything was new definitions and ideas to them. They had not heard of a basis for a topology before their initial interview. Brandon said the following about the familiarity of the material.

> It's definitely not as foreign. I remember before this, I never had really seen topology and now after a semester, it's not as intimidating, I guess. A lot of it seemed pretty weird. I'm comfortable with it, I guess. Not that it's easy or anything, but it's definitely approachable now, for sure.

When asked about general things that helped them learn topology, Brandon said "The examples helped a lot. The examples give you a lot of ways to see the weirder side of things." Kyle chimed in that "understanding the examples and counterexamples" helped him. In looking back on his first topology course, Brandon says he wished he paid more attention to his proving process and gives his view of topology as a subject.

> There were just things that I should've worried about in terms of like, the proof process, that I didn't exactly give emphasis on that I should have. That I had to compensate for about a fourth of the way through, because, those first couple of homeworks, not that they were terrible, but they just...the things that he [the instructor] had complained about were things that I thought I didn't need to say. And then that's just...it becomes habit after that that you just have to, you have to say a lot of stuff that you wouldn't normally want to. But that's just nature of it...I feel like it's supposed to be something that's a little uncomfortable at first. As a subject. Like it's, it's different. It's definitely different than everything I've seen before.

> But it still has a logical structure and everything makes sense if you give it enough thought.

Additional comments from Phil

Phil was asked about his experience in his first topology course, and he immediately identified topology as the most abstract class he has taken. He indicated that he had a nice balance between struggling and understanding the content.

> It's really easy to feel like you don't understand anything, because the concept is very, like, at least for a math class I've taken, I feel like topology is the most abstract of all the abstract classes. It's very, like, wonky. But I feel like I must be doing something right, because I am able to do the homework and the exams for the most part. I can't say it's been struggle-free, but I also wouldn't say, like, there's been a point where I was like, okay I really don't understand this. I guess my experience with the content has been good, although like, if you ask me how much I think I know about topology, I'll tell you very little. But I can tell you what most things are.

Phil was also asked about what prerequisite topics he believed were needed for taking a first topology course. He identified set theory like the other participants did, but he also mentioned an ability to draw spaces and visualize.

> You need a lot of set theory and, maybe this isn't a prerequisite but a co-requisite, I think, it might be because I'm a visual learner, but I think drawing stuff out, you need a teacher, or whoever is teaching you needs to draw stuff out... Like all math, you should be pretty

familiar with definitions. But I guess in terms of actual prerequisites, I would say if you have a great understanding of set theory, that's all you would need.

6.6 Theoretical Analysis

In this section, I will analyze the previous results through the framework given in Table 4.1 and the qualities discussed in Section 4.2.3. The qualities help show whether or not someone demonstrates a certain component from the framework. The section will be divided up by the three stages for successful action (Knowing That, Knowing How, and Being Able), with both Δ_1 and Δ_2 director systems discussed within them. Because Task Two was entirely within the domain of all of the participants' director systems, I will discuss that task and overall successful action at the end of this chapter.

For Dr. T and Jordan, the tasks were entirely within the domains of their director systems. For the other participants, there were opportunities for learning to occur. Because Knowing That, Knowing How, and Being Able help identify when learning has occurred, the results about learning are discussed throughout those three sections.

In Table 6.1, I have summarized each participant's Δ_1 success with the tasks. If they were successful, I indicate what level of skill they were successful with. If they did not complete a task successfully, then the components that were primarily responsible for preventing success are listed. Note that I am not indicating Δ_2 success because not all participants had a specific Δ_2 goal that was observable, making "success" difficult to measure here.

Table 6.1: Participants' Δ_1 Success.

	Task One	**Task Two**	**Task Three**	**Task Four**
Dr. T The Professor	Goal state reached: High level of skill and speed	Goal state reached: High level of skill and speed	Goal state reached: High level of skill and speed	Goal state reached: High level of skill and speed
Jordan The doctoral student	Goal state reached: High level of skill	Goal state reached: Less efficient	Did not attempt	Did not attempt
Brandon and Kyle Late Interview The first-year graduate students	**Did not reach goal state** ! BA	Goal state reached: Highly efficient	Goal state reached: Adequate level of skill	Goal state reached: Adequate level of skill
Brandon and Kyle Early Interview The first-year graduate students	Goal state reached for A-C: Adequate level of skill **Did not reach goal state for D** ! KT	Goal state reached: Less efficient	**Did not reach goal state** ! KT	**Did not reach goal state** ! KT ! KH ! BA
Phil The undergraduate	**Did not reach goal state** ! KT	Goal state reached: Lower level of skill	**Did not reach goal state** ! KH ! BA	**Did not reach goal state** ! KT ! KH ! BA

6.6.1 Knowing That: Possession of a Schema

Δ_1 *Knowing That - Task One*

In this study, Dr. T and Jordan served as the experts, with Dr. T being the more experienced expert. As such, they were asked about what topics they felt were necessary for students to understand in order to be able to do the tasks, and they provided many prerequisite topics. Because Jordan was thorough about explaining what he was doing while doing it, we can also find many of these topics mentioned throughout his results while he was working. These topics are listed by task in Table 6.2.

Table 6.2: Δ_1 Prerequisite Topics for the Four Tasks.

Task One	Task Two	Task Three	Task Four
- Absolute values - Basis - Definition of a topology - Endpoints - Interior points - Open sets - Sequences - Set notation - Set theory - Standard topology - Subspace topology	- Definition of a basis - Set theory - Subset notation	- Basis - Open map - Open sets in $X \times Y$ - Product topology - Projection map - Unions of basis elements	- Basis - Definition of a topology - Set theory - Topology generated by a basis - Topology that contains a basis - Unions of basis elements

These prerequisite topics were easy for Dr. T and Jordan to identify because of the quality of their schemas in topology. In considering all of the results from Dr. T and Jordan, we can argue that they provided evidence of all of the qualities at some point, but I will just mention a few here. Because of their experience compared to the other participants, their schemas in topology are more complete, organized, and general. They contain more difficult and abstract topics than needed for these four tasks, but they were able to "zoom in" on the necessary ideas.

Since Dr. T and Jordan demonstrated strong Δ_1 director systems, it was easy to see when Brandon, Kyle, and Phil had gaps or weaknesses in theirs. Probably the most obvious quality to see throughout the data is the accuracy (or inaccuracy) of someone's schema. In both of their interviews, Brandon and Kyle had issues with reading and understanding the set notation on Task One, which in the case of the early interview, caused them to be unsuccessful in considering set D. Kyle first interpreted set D's notation to mean $(-1, 0) \cup (0, 1)$, Brandon agreed with him, and then they addressed it again later in the same interview and arrived again at the same incorrect conclusion. This was not just a simple mistake or typo. Multiple times they agreed on this, showing that the set theory part of their Δ_1 Knowing That was inaccurate at the time of the early interview. In moving forward with this incorrect set, they did apply a valid Knowing How and Being Able sufficiently, so this serves as a good example of how Knowing That is a required component for successful action. By the late interview, some kind of learning had occurred because they had satisfactory knowledge of set theory available to understand set D correctly. This likely can be categorized as Intelligent Learning because they were applying what they knew about set notation to the sets in Task One. In other words, they had not memorized what these particular four sets looked like in order to do this problem. This means that their set theory schema was available for a variety of problems and is not only a Δ_1 Knowing That, but also part of a Δ_2 Knowing That.

A major inaccuracy in Phil's Δ_1 director system for Task One was when he equated the subspace and standard topologies. The inaccuracy of this connection implies that it is a type of associative connection in his topology schema, or if not, that there were inaccuracies littered throughout his schema. Phil was fairly insistent on this idea because he referenced it no less than three times before the interviewer's intervention. Even after the intervention, he tried to use a related connection, saying that open sets of the subspace topology are open in the standard topology. This inaccurate Knowing That reduced the complexity of the problem, meaning that Phil had an unproductive Knowing How as a result. Phil's response to this task was hindered by his Knowing That.

Δ_1 ***Knowing That - Task Three***

For Task Three, Dr. T emphasized relying on a basis and also focused on the fact that open sets look like unions of basis elements. After defining the product topology, he said that it was necessary to prove that the defined basis actually qualified as a basis. This detail shows us that Dr. T's schema for this Δ_1 system is complete with respect to this task. Brandon and Kyle in their late interview, as well as Phil, all defined the product topology using a basis, but they did not ever consider checking that what they defined satisfied the definition of a basis. Because of the setting of the interviews, it is possible that they thought they did not need to check this. Both Phil and Brandon and Kyle (in the late interview) indicated that they were regurgitating the definition of the product topology, showing evidence of associative connections in their schemas. In their early interview, Brandon and Kyle did not define the product topology correctly, so the fact that they did in the late interview shows some short-term learning occurred between the two interviews. For Brandon and Kyle in the early interview, the missing basis concept was the primary thing that prevented them from completing this task successfully.

Another associative connection that Phil had was that all open sets in $X \times Y$ are unions of basis elements. He admitted that this was rote memorization because he said that he had read it online and that he hoped it was true. It is worth noting that I will later discuss this same connection between open sets and unions of basis elements as a Δ_1 Knowing How, but Phil utilized this connection as a Δ_1 Knowing That. This connection in Task Three was specific to the product topology on $X \times Y$, but a more general version of this connection is needed in Task Four.

Δ_1 ***Knowing That - Task Four***

The prerequisite topics listed in Table 6.2 need to be assimilated into one's schema in order to successfully complete these four tasks. However, recall that schemas do not consist of concepts only, but also the connections between those concepts. Dr. T gave some necessary Δ_1 connections while discussing Task Four. One of them was a conceptual connection that "every open set in the topology generated by $\mathcal{A}$ is a union of basic open sets." This was given as a lemma in my solution, but when used often, it is not so much a strategy as it is a connection within a schema. It is important that this connection is in one's schema for Task Four. If it is not, then you end up making the problem more challenging than it should be, which is what Brandon was doing in the early interview by trying to follow the original definition for a topology generated by a basis. This connection was not mentioned in their early interview at all, but it was mentioned in their late interview more than once. This Knowing That was initially missing for Brandon and Kyle, but was assimilated into their schemas by the late interview. Additionally, I would say this connection was even a strong one in the late interview because it was activated when the topology generated by $\mathcal{A}$ was activated, demonstrated by when Brandon clarified that "$\mathcal{A}$ is a basis and $\mathcal{T}_{\mathcal{A}}$ is a union of elements of $\mathcal{A}$." Over the course of the semester, Brandon and Kyle had learned to think of the topology generated by a basis in this way and therefore this problem moved from outside of their domain in the early interview, to the frontier zone at some point during the semester, and then finally was transformed into established domain by the time they got to the late interview.

Another Δ_1 connection given by Dr. T demonstrates the generality of his schema. He said that students have to be comfortable with the concept of many topologies containing $\mathcal{A}$, while not actually understanding what any of them look like (because no details were given about what $\mathcal{A}$ is). Task Four asked participants to prove a statement that was not about a specific basis, but any basis. This means you have to rely on higher-order concepts for this task. The generality of Dr. T's schema helped in activating the needed concepts quickly. None of the participants tried to get specific about this problem, but they did struggle because of these high-order concepts.

In their late interview, Brandon and Kyle were able to utilize their set theory to reason, but they began their work with an incorrect assumption. When Brandon wrote the problem, he wrote $\mathcal{T}_{\mathcal{A}} \subseteq \mathcal{T}_x$ instead of $\mathcal{A} \subset \mathcal{T}_x$, where $\mathcal{T}_x$ is representing a topology that contains $\mathcal{A}$. Even if this was just a typo, Kyle continued with it to say that $\mathcal{T}_{\mathcal{A}}$ would be in the intersection since it was in every $\mathcal{T}_x$. This mistake make the more difficult direction quite trivial. In trying to continue with the problem, they eventually realized their assumption was incorrect and modified their argument accordingly. Once they fixed their mistake, their reasoning and work was clear. This chain of events demonstrates the generality and organization of Brandon and Kyle's schemas because they were able to work with these higher-order ideas and they were able to clearly distinguish between a basis and the topology generated by a basis.

Brandon and Kyle also demonstrated strength of conceptual connections in the late interview because they never referred to the definition sheet and were able to work more from their intuition for the topology generated by $\mathcal{A}$. In comparison, in their early interview they spent time reading the definition sheet multiple times and did not have a grasp on what they were trying to do. They had to clarify that $\mathcal{A}$ was not a topology, it was a subset of the topology generated by it, and that the topology generated by $\mathcal{A}$ was one of the topologies being intersected, which are all basic facts necessary for this task. In reflecting on their interviews, they said that the definitions were more automatic by the late interview because in the early interview, the ideas were mostly new to them. They had not heard of a basis for a topology by the early interview. These critical concepts being weak or completely absent from their schemas in the early interview played a major part in them being unsuccessful for Task Four.

Phil also had these same issues in Task Four. He interchanged $\mathcal{A}$ and the topology generated by $\mathcal{A}$ at first and concluded that $\mathcal{A} \subset \bigcap \mathcal{T}_i$, which was not relevant to the goal state. He was not sure at first if the topology generated by $\mathcal{A}$ contained $\mathcal{A}$. These show that there were gaps in his schema involving things like the topology generated by a basis and the topology that contains a basis.

In summary, Δ_1 Knowing That played a role in unsuccessful action for Brandon and Kyle in their early interview, as well as for Phil. It was the primary cause for issues in Task One and then also Task Three for Brandon and Kyle. Gaps in schemas certainly caused problems during Task Four, but as we will see, Knowing That was not the only cause of difficulties with that task.

Δ_2 *Knowing That*

Identifying Δ_2 Knowing That's can be less obvious because they are usually first identified as a Knowing That, Knowing How, or Being Able in a Δ_1 director system. I will give some examples from the data to demonstrate this, followed up by results that are more explicitly Δ_2 in nature.

When reflecting back on the semester, Kyle talked about how he did not understand "the basis topology stuff" at first, but then he read through the textbook more to understand. Here Kyle is referring to a schema that contains Δ_1 Knowing That's, like topology and basis, but the general schema itself is part of his Δ_2 Knowing That. Similarly, Dr. T indicated in Task Three that the Δ_1 Knowing How of showing that open rectangles satisfy the definition of a basis is part of his Δ_2 Knowing That. Understanding that checking the definition is necessary shows some of the completeness of Dr. T's schema for Task Three.

Ready-to-hand plans can serve as a Δ_1 Knowing How, but because they are "ready to go" and are a result of reflective planning, they are also a concept themselves in a Δ_2 Knowing That. Jordan worked through Task One set-by-set, first identifying if the set was open in $\mathbb{R}$, then if it was open in Y. By the time he needed to think about set C, he was able to quickly look ahead and say that C will be open in neither before writing anything down. Here Jordan was able to reflect on his Δ_1 Knowing How for sets A and B and made a similar plan for C. His strategy for checking if a set is open $\mathbb{R}$ and the subspace became part of a higher-order schema that could be applied to any given set in the subspace.

In reflecting on the four tasks at the end of their late interview, Brandon and Kyle said that they had seen these tasks and had worked with these ideas throughout the semester, so they were more familiar with them by the late interview. They also discussed learning from their mistakes on their homework and how they do not forget those ideas anymore. Because of these, they had reflective planning helping them in the late interview. Some of their strategies were easier to come up with and use because they had thought about it before. Phil also mentioned a specific ready-to-hand plan of Task Four being an "element chasing thing" or a "subset problem". All of these plans are part of the participants' Δ_2

systems and schemas.

Brandon gave a little bit of his perspective on topology as a subject and showed some organization to his topology schema when he said that topology still "has a logical structure and everything makes sense if you give it enough thought." Since this is about topology as a subject and he was speaking in the context of learning as a goal state itself, this logical structure is part of his Δ_2 Knowing That.

Recall that Dr. T admitted that he likely thinks about topology concepts in a way that is fairly different from how students think about them and he pointed to his experience as a reason for this. His experience is what informs his ready-to-hand plans, which was how he was able to give solutions so quickly and accurately. The concepts, strategies, and details of these tasks were all part of his Δ_2 Knowing That coming into the interview. His experience not only gave him available plans, but also a well-structured organization of concepts and high-order generality of his schema for those plans to assimilate into.

Something unexpected that Dr. T discussed throughout his interview was the context around topology concepts. When teaching the undergraduate topology course, he likes to have students prove Calculus theorems by using topology in order to give them some context for the material. However, he admits that when you are getting started in learning topology, you do not have that context and that makes it challenging. If we consider a lone topology concept with no context, then there are little or no connections between it and outside concepts. However, if we are able to assimilate the concept into our schema with a context, then connections are made to other schemas, giving us more organization and a more complete schema. For these reasons, a context classifies as a Knowing That. We know that it informs Δ_2 director systems because the operand affected by a context is usually a problem itself, which needs its own Δ_1 director system.

One final reflection from Brandon and Kyle involves examples in topology. They discussed how examples and counterexamples helped them to learn topology ideas. Because of the nature of the four tasks in this study, we did not see a lot of evidence of reflective extrapolation, but Brandon and Kyle made sure to mention how helpful examples were to them. Because examples in topology can include a large range of things, from specific topological spaces to writing particular kinds of proofs, they can assimilate into one's schema as their own concept, ready to be utilized for other Δ_1 systems, making them a Δ_2 Knowing That.

6.6.2 Knowing How: Forming a Plan

Knowing How - Task One

Task One asks the reader to categorize the four sets as being open or not in two spaces: $\mathbb{R}$ and Y. Dr. T talked about this task asking two separate things, meaning that there are two separate paths to follow: identifying open sets in $\mathbb{R}$, and then more technically, identifying open sets in Y. Because of the way the task is phrased, a novice might read the task and think of these two categorizations as related, or even the same, which is what we saw in Phil's attempt. However, Dr. T emphasized that they are separate activities, showing that his schema is well-organized and he used two separate Δ_1 Knowing Hows for this task. The first relied on utilizing the definition of the standard topology, which is an intuitive path for most, and the second relied on following the definition of a subspace topology, which is a very technical path.

Dr. T also made this comment about multiple paths in general, without a specific task in mind. He said that he thinks about topology concepts differently than the students most likely. He is saying that his Δ_2 Know How is likely different from the students' and maybe the two different paths are not even near each other in his schema. A more experienced schema, like Dr. T's, has more ready-to-hand plans, conceptual connections, and stronger connections when compared to a novice. Therefore, the path for thinking about something in Dr. T's schema might be pretty far off from the path that a novice takes.

Jordan's Task One will be discussed in greater detail later in this chapter, but it should be noted that he also followed the same definitions as Dr. T, displayed intuitive path finding, and used reflective planning in his Δ_1 system for this task. He referred to set C not being open in other mathematical results, referenced an alternative definition for a set being open, and wrote an alternative notation for set D, which are all Δ_2 techniques that could have improved the efficiency of his Δ_1 system. Another Δ_2 strategy that he relied on heavily was switching back and forth between representations of the four sets and he pointed to his set theory schema for what allowed him to do that.

Brandon and Kyle in their early interview were able to successfully complete the task for all sets except for D. We can see the same Δ_1 actions from them as we see with Dr. T and Jordan, mainly obtaining a visual for the sets, rewriting the set using interval notation, and following their definitions. For set C, Kyle demonstrated some reflective planning by stating the conclusion for C at the very start of considering it. As for their late interview, because they did not consider what it meant to be open in Y at all, there were not as many Δ_1 Knowing Hows to observe. They still tried to get a visual for the sets and followed the definition of the standard topology, however, Kyle was very focused on the technical details of proving this. All other participants used their intuition for whether or not a set was open in $\mathbb{R}$, but Kyle used a detailed metric space argument for showing that D was open. This showed his connections between topological spaces and metric spaces, making his schema more complete.

Phil explored many paths during Task One, some of which were Δ_1 and some Δ_2. He first reasoned that A was open in $\mathbb{R}$ because the inequality did not include "or equal to", which might be how he assimilated the standard topology into his schema. He was the only participant to use this reasoning in Task One. When he got to set D, he tried to understand what it looked like and described the rational values that were excluded from the set. At no point did Phil follow the definition of the subspace topology on his own. That was not part of his plan for the problem. After the intervention from the interviewer, he did not immediately reason correctly as to why A and D were open in Y. He tried citing a lemma that was not relevant to the task. He started to go down an inaccurate path saying that B and C were open in Y because they were half open, but that reasoning was only going to work for B. At this point, we are not seeing any of the qualities that we were seeing with Dr. T. Phil did not have a ready-to-hand plan for determining if a set is open in the subspace topology, he did not show accurate nor conceptual connections between the standard and subspace topologies, and this task certainly was not intuitive for him, meaning the connections he did have were not strong. However, he continued to think about it and we actually could hear him changing his mind about set C. Phil serves as a good example in this task of what it looks like to be in the frontier zone. This task was not in his domain at first, but with an intervention and some more reflection on the definition of the subspace topology, he was able to make valid arguments for sets A, B, and D. He did not have enough time in the interview to figure out an argument for set C, but he did say that he was trying to think of an open set that intersected with Y to give C, which is the first step on the right path.

One final comment about Phil and Task One was that we saw a lot of hypothesizing and guessing with him. Even though some of these were not accurate, they were what his intuition was telling him. All of the participants at some point in Task One demonstrated intuitive path finding by trying to visualize the sets and used intuitive reasoning for why a set is open or not open in $\mathbb{R}$. Brandon and Kyle had the intuition to say that C was not open because no set could be found to intersect with Y, but they did not go beyond that in terms of proving why not.

Knowing How - Task Three

For part a of Task Three, most participants simply stated what they thought the product topology was. Dr. T was the only one to consider whether or not what he stated actually qualified as a basis for a topology. Reciting the product topology was a Δ_1 Knowing That, but the Δ_1 Knowing How of checking the conditions of a basis is also necessary in part a. Dr. T also discussed an alternative way of defining the product topology, but said that it would be more preferable for students to be thinking about the product topology in terms of a basis. By using a basis, you have the lemma from the Task Four solution in Section 6.4.1 available to use. Alternative definitions and utilizing lemmas demonstrate a more complete and better organized schema available for use in Δ_2 systems. Brandon and Kyle demonstrated this as well in their late interview when they utilized this lemma in part b to write their arbitrary open set as a union of basis elements. In their early interview, they had not done this because they did not use a basis in part a. However, they had made this same mistake in a homework during the semester and they said that their feedback on that homework caused them to do this task correctly in their late interview. Therefore we can classify the part of their plan when they write their set as a union of basis elements as a result of reflective planning, making it a Δ_2 Knowing How. Because they learned to work in terms of an arbitrary open set instead of just a basis element, the generality of their schemas also improved over the semester.

For part b, Dr. T's main focus was on what the projection of a union of sets would be. Following the projection map is the vital Δ_1 Knowing How for Task Three. Brandon and Kyle almost took this step for granted in their late interview, but they paused and discussed if the projection of the union is actually equal to the union of projections. It was clear that checking their assumptions and ideas had become part of their Δ_2 Knowing How from past experiences with "weirder things" (as Kyle puts it). In their early interview, Brandon and Kyle had followed the projection map correctly, so their Δ_1 Knowing How was not the component that was responsible for their director system being unsuccessful that time. Phil followed the projection map as well and had that in his plan, but he was unable to justify why he got open sets out of the map. He used an incorrect Δ_1 strategy of rearranging unions and products to get what he wanted, which was not mathematically accurate. This rearrangement was a path that took him further away from successfully completing the task.

All of the participants started their proof for part b by considering the projection of an open set. They knew the definition of an open map and utilized it to inform the beginning of their proof. This initial proof setup was a Δ_1 strategy that no one had difficulties with.

Dr. T gave the main outline for both parts of Task Three and did not focus on proving every detail. Because he could consider this "zoomed out" view of the problem, he showed an organizational quality of his schema that helped his Δ_1 director system. This quality also shows us that his Δ_2 proof-writing also has this organizational ability.

Knowing How - Task Four

A needed Δ_1 connection for Task Four is that unions of sets in $\mathcal{A}$ are also in a topology containing $\mathcal{A}$ because of the definition of a topology. You also need the lemma given in the solution in Section 6.4.1, as well as the basic proof strategy the equality can be shown through double containment. Dr. T utilized all three of these in the briefest of ways, while Brandon and Kyle used them in a less skillful, but still commendable, way in their late interview. Brandon and Kyle figured out the more difficult direction of the proof first, so when they started to think about the trivial direction, Kyle understood it immediately, demonstrating an intuitive path. Before Kyle explained his reasoning, Brandon had begun to write on the board the initial assumption for the trivial direction, which was a ready-to-hand plan from having already done the non-trivial direction. He did not get to continue with this plan, however, because Kyle saw how to do it by the time Brandon was finished writing his proof setup. Through these Δ_1 Knowing How's, we see evidence of the relevance and generality qualities of schema, as well as evidence of conceptual connections. These participants' thinking remained relevant to the task at hand, which is not always the case with abstract problems like Task Four. Further, these ideas are needing to be considered in the most general sense, making them higher-order concepts. Finally, these participants discussed the needed lemma, not as if it were a lemma, but rather as if it were their fundamental understanding of a topology generated by a basis. Recall that the definition sheet (Appendix A) has a technical definition of a topology generated by a basis, but that is not what Dr. T nor Brandon and Kyle used here. This was a strong, conceptual Δ_1 connection for them.

Recall that Brandon and Kyle in their early interview, as well as Phil, did not have success in completing this task on their own. They did acknowledge double containment as a proof strategy, with Phil saying it could be that or an "element chasing thing". This was the one task where Brandon and Kyle somewhat argued with each other and were starting down different paths from each other. Kyle started off explaining the trivial direction, but Brandon was focused on utilizing the definition of a topology generated by a basis. He then tried to explain to Kyle how he was going to show the double containment, but Kyle did not catch that Brandon was choosing an arbitrary set contained in the topologies, so he was confused further about what Brandon was doing. Once they ironed out what they meant by showing double containment, they then confused each other about what direction they were working on. By the time they were back to working together, they were only able to prove the trivial direction. In this early interview, Brandon and Kyle did not have the two necessary connections described above. They only had the idea of double containment. With this interview, we do not see conceptual connections at all because of the heavy reference to the definition sheet.

Phil also did not give any evidence of the two needed connections. He said that he was trying to think about things logically, but he was unable to make significant progress in the time of the interview. He was trying to rely on the organization of his schema, but it seems that many connections were missing, leaving the needed organizational structure unsupported.

Dr. T spent a part of his interview predicting what Task Four would look like with undergraduate students. He imagined a wide variety of attempts, which seems to be confirmed by Brandon and Kyle's early interview. He also could see students who would not know where to get started at all with this task, meaning that they would not be able to formulate a plan for their Δ_1 director system, so they would remain stuck at their present state of having made no progress on the problem. Because of this prediction, Dr. T suggested a scaffolding to the problem, which demonstrated his mental organization of this proof. This proof template is being classified as a Δ_2 Knowing How because of its ability to improve the efficiency of the Δ_1 director system for Task Four.

In summary, the paths for Task One were fairly intuitive for most participants and Phil was the only one to struggle with finding a successful path. An inaccurate Knowing How was a primary issue for Phil during Task Three, while everyone else followed the definition of the projection map well. In Task Four, there were two Δ_1 Knowing How's missing for Brandon and Kyle in their early interview, as well as for Phil. Any Δ_2 Knowing How's included mathematical norms and what the participant thought might be helpful in completing their task. In many cases, these norms were productive for the participant, like how Jordan switched representations in Task One, or Brandon and Kyle checked their ideas in Task Three. However, there were also some cases where the norms were a distraction to the director system, like how Phil persistently tried to use another mathematical result in Task One instead of going to the definition of the subspace topology, or when Dr. T described an alternative technique for defining the product topology in Task Three that he said was not his preferred way for students to conceptualize the product topology.

6.6.3 Being Able: Level of Skill

Being Able - Task Four

In most tasks, we saw improved skill if the participant was more experienced and less of a novice. Task Four is a perfect example of this. Phil used a confusing and unconventional notation and also interchanged $\mathcal{A}$ and the topology generated by $\mathcal{A}$. In their early interview, Brandon also interchanged $\mathcal{A}$ and the topology generated by $\mathcal{A}$, but Kyle corrected it after a short amount of time. They also had discrepancies about their notation and what direction they were working on, but they eventually improved their notation and were working together by the end. They went through all this debate to prove only the trivial direction. Phil, Brandon, and Kyle (in their early interview) all had Δ_1 Being Able as a road block that kept their director systems from running successfully. Brandon and Kyle then showed improved skill by their late interview because their struggle with the notation was short-lived. Even though they still needed some time, they were able to come up with correct reasoning, making them successful on this task.

Quite obviously, Dr. T serves as the most skilled participant for all tasks. In Task Four, he was able to explain his reasoning with only a verbal representation and did not need to elaborate on details in order to do so. He considered one direction trivial while all other participants still needed to reason through it. He did this task very quickly and expected that students who "have a good sense of what's going on" will also be able to work quickly. However, Brandon and Kyle in their late interview did not work nearly as quickly as Dr. T, even though they were successful with the problem.

This emphasizes that your present state may be in the domain of your Δ_1

director system, but your Δ_1 Being Able determines how efficiently your director system operates. For Dr. T, he operated as quickly as possible, while Brandon and Kyle were unable to reason as quickly. This is probably because Task Four was not in their domain during the early interview, but it was by their late interview, so they had recently been in the frontier zone between their interviews. The domain they were in by the late interview was recently established domain, and being in the domain does not mean you are instantly transported to the goal state. If we think of the paths to the goal state as having speeds assigned to them, some paths will be faster than others. The better established your Knowing That, Knowing How, and Being Able are, and the stronger your connections are, the faster you will go along your path and reach your goal state.

Being Able - Task Three

For Task Three, the major necessary detail involves articulating why the projection of a union is equal to the union of projections. This is the portion of the proof Dr. T emphasized as being unavoidable and what graduate students need to be comfortable doing. Brandon and Kyle in their early interview wrote a trivial proof based on their incorrect Δ_1 Knowing That, so they did not have the opportunity to prove this. They did have the opportunity in their late interview, where they discussed these ideas, but ended up not writing the details down. I would say that Brandon and Kyle successfully completed Task Three in this interview, but their lack of details shows room for improvement. At first glance, Phil seemed like he was entirely able to complete Task Three, but when prompted to explain why the projection of a union was a union of open sets in X, he was lacking a concrete and correct reasoning. Because this was the portion of the proof that Dr. T emphasized, I am considering Phil to have been unsuccessful with Task Three as a result of his Δ_1 Being Able.

The other detail for Task Three is about notation. Phil did not have an issue with notation on this problem, but Brandon and Kyle stumbled a bit with it in their late interview. They struggled to denote a (possibly infinite) union of basis elements. They tried using a beta notation that had too many kinds of B's and they were unsatisfied with it. They never really improved their notation, but it was not bad enough to prevent them from completing the proof.

Being Able - Task One

Task One was the task where Brandon and Kyle did *not* show improved skill from the early interview to the late interview. In the early interview, they were generally successful for sets A, B, and C, however, some of the details of their work showed less skill. Kyle continued to say "union" even though he meant "intersection". Neither of them elaborated as to why you cannot find a set to intersect with Y and get C. They also did not draw out visuals for these sets on the whiteboard, which if they had, could have helped them realize that they were not reading set D correctly. A Δ_1 skill that helped them, however, was their ability to write out the interval notation for the sets and coming up with a set that showed why B was open in Y.

In the late interview, the major detail they overlooked was that the problem was asking which sets were open in *both* $\mathbb{R}$ and Y. Separate from this mistake, Brandon and Kyle also showed less skill when explaining why B was not open because their verbal reasoning was not coherent enough for an outsider to follow along. The one place where we can point to an improvement in this task is that they did ready set D correctly in the late interview. Even so, they still had to take some time to think about what it looked like.

Phil struggled to show details in Task One without being prompted first. He could not come up with the details needed to explain why C was not open in Y and needed several opportunities to give correct reasoning for the other sets. Overall, he took a significant amount of time for Task One; much more than any of the other participants took for this task. This demonstrates his lower skill-level compared to the other participants. Meanwhile, Jordan showed much more skill in how quickly he could think about and explain this task, in how he came up with a formal reasoning for C and in how he showed his "calculations", like $D \cap Y = D$. Unsurprisingly, Dr. T had the best skillset for this task, as evidenced by how he reached the correct conclusions for each set almost immediately.

In summary, Δ_1 Being Able prevented successful action for Brandon and Kyle's Task One in their late interview, as well as for Phil's Task Three. Difficulties with notation had an impact in Task Four for Phil and Brandon and Kyle in their early interview.

Δ_2 *Being Able*

Dr. T pointed to abstract standards as something that topology students have difficulty with and usually is the primary difficulty when starting a topology course. He said they need to get comfortable with working in general terms, like what was needed in Task Four. Phil's reflections on his first topology course were consistent with what Dr. T said. Phil felt like topology was "the most abstract of all the abstract classes". Making leaps of abstraction is not something that necessarily helps with a single problem; it helps with improving mathematical skills and proving in general. For that reason, being able to work with abstract standards is being qualified as a Δ_2 Being Able. We saw that Brandon and Kyle were able to think in general terms in their late interview for Task Four, but they were not able to as easily in their early interview. Between the interviews, they were working through a first-year graduate topology course that undoubtedly helped them get better accustomed to abstracting.

Another needed skill for topology that Phil identified in his reflections was an ability to visualize and maybe even sketch the topological spaces you are working with. Jordan emphasized visualization as well during Task One. Sketching a visual can be a Δ_1 detail for a specific director system, but this overall ability is an important Δ_2 Being Able that helps improve mathematical skill. Although Jordan was working on Task One, he was generally saying that he will not be able to tell you if a set is open or not if he does not have a visualization of it.

6.6.4 Successful Action

In this section, I will discuss a few cases of successful action in this study. These cases demonstrated all three of Knowing That, Knowing How, and Being Able and were therefore able to complete their task. First, I will discuss Jordan during Task One and then I will discuss all participants during Task Two.

Task One - Jordan

When Jordan worked through the first task, he never said explicitly what his overall strategy would be. He started off by trying to wrap his mind around the four sets by sketching them on number lines. He would not formulate a plan for the task until he had a visual for each set. He used his knowledge of set theory to do this.

> So I guess something you need to know [in order] to know what this set is, is to know that the absolute value of x describes the distance

> from x to the origin. So the distance from x for any point in the set needs to be bigger than one-half and less than one. So that'll be the open set over here. [Draws open interval from one-half to one.] So one-half to one and then also the other open set over here. [Draws open interval from negative one to negative one-half.]

Jordan's set theory schema was part of his Δ_1 Know That and this plan of visualizing the sets was the first step in his Δ_1 Know How.

Then once he knew what the four sets looked like, he was able to discuss whether or not they were open in $\mathbb{R}$ and Y. Because set D was not as easy to think about, he also tried to write it in interval notation to see that it was a union of an infinite number of open sets. For set A, he quickly explained that A is open in $\mathbb{R}$ and therefore A can be used in the definition of a subspace topology to show that it is also open in Y. He had the definition of a subspace topology as part of his Δ_1 Knowing That and utilizing it was part of his Δ_1 Knowing How.

Through his process and how he explained everything out loud, Jordan demonstrated intuitive path finding in a Δ_1 director system. I could see how the set notation in the problem activated the concept of sketching on a number line, which activated interval notation, and so on. He followed what was in front of him and organically came to his conclusions about these four sets. He also had the necessary Know That's along this intuitive path so that everything activated as needed. For the first two sets, he did not activate all needed concepts at the same time since he didn't announce his strategy at any point, nor did he reach a correct conclusion before showing the reasoning. By the time he thought about set C, however, he was able to activate more concepts along the path at one time (because he had already done this process with the first two sets), which was why he was able to know his result before writing out the details. This reflective planning came from Jordan seeing the pattern in the process with Task One.

For more difficult sets like set C, Jordan's initial plan started off the same and he had to adjust it slightly when he ran into stumbling blocks with the details. He had the visual of set C and knew that it was not open in $\mathbb{R}$. He also knew that C would not be open in Y, but wasn't immediately sure that he knew how to prove that. He knew that he would have to show that C is *not* the result of an intersection. Later in the interview, his concept of interior points was activated, which gave him the idea for how to finish proving that set C was not open in Y. His Knowing That was activated and helped him formulate a plan for achieving his goal state of completing the proof for set C. Part of the plan for this was to utilize a proof strategy that was not needed for the other sets and to utilize another mathematical concept, but what strategy and what concept were not immediately clear. Once Jordan had the concept though, he knew how to implement it because he was experienced enough to have these mathematical norms available in his schema of doing mathematics and proving. These norms were part of his Δ_2 Knowing How and he was able to achieve his Δ_1 goal state once all the pieces came together at the same time. Overall, Jordan completed this task in a very logical sequence of events and efficiently, which demonstrates his skill. In this case, we see all three components contributing to the success of his Δ_1 director system.

After having finished a task, Jordan then responded to my question about what an undergraduate student would need in order to be able to do this problem. His responses were consistent with those of Dr. T for the prerequisite concepts, but he brings up some interesting points about switching representations and utilizing other mathematical results. I think Jordan's responses revolve around what it means to do mathematics and he mentions several norms within the mathematical community. I think Dr. T and Jordan together are telling us that topology can be challenging for students because they have to learn these concepts on top of navigating some new ways of thinking that they maybe didn't need in previous classes.

Because of the nature of Jordan's interview, we can identify components for both Δ_1 and Δ_2 director systems (see Table 6.3). He not only completed the tasks, but he also was able to reflect on his processes and explain them to me. A Δ_1 system, for example, is how he completed Task One. I then asked him to reflect on what undergraduate students need in order to be able to do something like Task One. He said, "they need to be comfortable with set theory first of all...So, this is one presentation of this set. Here is a visual presentation of it and here's like this open-interval presentation. They have to be able to go back and forth between these things." This helps us understand his process in achieving his Δ_1 goal. His Δ_1 director system was the operand for a Δ_2 system. The Δ_2 operator, which changed the state of his progress on Task One, was his ability to go back and forth between the visual presentation and set notation. Because of his mathematical experience, Jordan's Δ_1 system was able to operate through intuitive-path finding.

Task Two - Dr. T

Task Two was unique because it was the only task where all participants were successful. Interestingly, even though they were all able to complete the task by following the definition of a basis, the sets they created to satisfy the definition's conditions all looked different from each other's.

Dr. T serves as the most efficient example of what Task Two could look like.

Table 6.3: Jordan's Successful Action in Task One.

Δ_1 Director System	Δ_2 Director System

Knowing That
○ Set theory
○ Absolute values
○ Subspace topology
○ Interior point
○ Basic open set
○ Sequences
Knowing How
○ Following definitions
○ Intuitive path:
  ○ Visualize sets
  ○ Write interval notation
Being Able to
○ Find open set to intersect with Y
	Knowing That
	○ All KT, KH, and BA from Δ 1 with strong Qualities of Schema:
	  ○ Relevance
	  ○ Accuracy
	  ○ Strength of connections
	  ○ Plans
	Knowing How
	○ Switching between notation and graphical representation
	○ Utilizing definitions
	○ Utilizing other mathematical results/propositions
	Being Able to
	○ Visualize
	Intelligent Learning
	○ Reflective planning

His Δ_1 Knowing That contained the needed prerequisite concepts, like the definition of a basis and set theory. Further, his schema for a basis contained a Δ_1 Knowing How of finding a smaller basis element that's contained in an intersection, making this strategy a Δ_2 Knowing That. This is an excellent example in topology of vari-focality: the strategy of finding a smaller basis element is in this basis schema, which is in his Knowing That for the task. Even though Dr. T possessed this Δ_1 Knowing How, it was not necessary for this task. Instead, a satisfactory Δ_1 Knowing How is working from what the definition gives you, which was also a Δ_2 Knowing That for Dr. T. Because he was working from the definition, he emphasized analyzing intersections of sets for this task. This is part of his basis schema, which is a Δ_2 Knowing That, but for this specific task, analyzing intersections is a needed Δ_1 Being Able. In analyzing the intersections, he noted that every intersection is in the form of a basis element, so the second condition of the definition was satisfied immediately. He did not spend much time on the first condition of the definition because he said it was “fairly clear”.

Overall, Dr. T displayed several qualities in this task. We saw his organization through his vari-focal schema. He displayed generality when he was able to analyze the intersections from a “zoomed out” perspective to see that they will be basis elements themselves. By following the definition of a basis, he showed relevance of his schema to the task. His experience also made following the definition a ready-to-hand plan. Dr. T’s Knowing That, Knowing How, and Being Able are summarized in Table 6.4.

Before moving on to other participants, it should be noted that Dr. T knew that equality was acceptable when the second condition of the definition needed $B_3 \subset B_1 \cap B_2$, which was part of what made this task so simple. We will see that this was not obvious to all of the participants.

Table 6.4: Dr. T’s Successful Action in Task Two.

Δ_1 **Director System**	Δ_2 **Director System**
Knowing That ○ Definition of a Basis ○ Set theory **Knowing How** ○ Following the definition of a basis ○ Finding a smaller basis element within an intersection of basis elements **Being Able to** ○ Analyzing intersections of sets	**Knowing That** ○ All KT, KH, and BA from $\Delta 1$ with strong Qualities of Schema: ○ Relevance ○ Organization ○ Generality ○ Plans

Task Two - Jordan

Jordan knew he had options for paths with this task, saying that he might use the definition or that he might need a proposition. This flexibility came from his Δ_2 Knowing How. He ended up just using the definition like Dr. T did, which was relevant to the director system. He also explained what the second condition of the definition was, demonstrating that he had the same Δ_1 Knowing How as Dr. T. When Jordan considered the first condition of the definition of a basis, he created sets that would satisfy it, showing a Δ_1 Being Able that Dr. T could do, but didn't explicitly give in the interview. When considering the second condition of the definition, Jordan took a more granular approach than Dr. T did. He said that it was less obvious to him, which indicates that he was not able to analyze the intersections with the generality that Dr. T did. Instead, Jordan considered six cases and his ability to work through them was part of his Δ_1 Being Able. This norm of considering cases was part of his Δ_2 Knowing How and was influenced by the completeness of his schema. When considering the first case, he also drew a visual representation of it, pulling from a Δ_2 Being Able. As we can see in Table 6.5, we saw more approaches from Jordan because proving the second condition was not as automatic for him, and not all of these approaches and ideas were utilized for his final proof. Additionally, Jordan might have considered other paths and shown more drawings because he was trying to do the task as if he were teaching it to a class. He was trying to lay everything out carefully.

Table 6.5: Jordan's Successful Action in Task Two.

Δ_1 **Director System**	Δ_2 **Director System**
Knowing That ○ Definition of a Basis ○ Set theory **Knowing How** ○ Following the definition of a basis ○ Finding a smaller basis element within an intersection of basis elements **Being Able to** ○ Find a specific basis element that contains x ○ Analyzing intersections of sets (via cases)	**Knowing That** ○ All KT, KH, and BA from Δ 1 with strong Qualities of Schema: ○ Relevance ○ Completeness ○ Organization ○ Plans **Knowing How** ○ Utilizing definitions ○ Utilizing other mathematical results/propositions ○ Working with cases **Being Able to** ○ Visualize

Task Two - Brandon and Kyle

In their late interview, Brandon and Kyle were very efficient in proving Task Two. They knew the definition of a basis and were comfortable enough with basic set theory to have the necessary Δ_1 Knowing That's. They then followed each part of the definition for each basis in the task. They were able to think about the intersections without going into cases, but they also were not thinking as generally as Dr. T was. Instead, they drew visuals, which helped them use maximums and minimums to create the basis elements that satisfied the needed condition. Ironically, both of the B_3's that they created were equal to the intersection of B_1 and B_2, but at no point did they notice this out loud. Following the definition was a ready-to-hand plan that they played out with decent skill and was relevant to the task. Their schemas also had a completeness quality because they did not leave any detail unaddressed. Their Knowing That, Knowing How, and Being Able is shown in Table 6.6.

Table 6.6: Brandon and Kyle's Successful Action in Task Two (late interview).

Δ_1 Director System	Δ_2 Director System

Knowing That
○ Definition of a Basis
○ Set theory
Knowing How
○ Following the definition of a basis
Being Able to
○ Find a specific basis element that contains x
○ Analyzing intersections of sets
	Knowing That
	○ All KT, KH, and BA from Δ 1 with strong Qualities of Schema:
	  ○ Relevance
	  ○ Completeness
	  ○ Plans
	Being Able to
	○ Visualize

In their early interview, Brandon and Kyle reviewed the definition of a basis from their definition sheet and they were not sure whether the notation $B_3 \subset B_1 \cap B_2$ allowed for equality or not. They assumed that an equality would not satisfy the subset, so they did a lot more work than necessary because of this Δ_1 misunderstanding. For $\mathcal{B}_2$, they correctly identified what the intersection was, but then spent a significant amount of time creating a basis element that was within, but not equal to, the intersection. For $\mathcal{B}_1$, Kyle started to say the intersection was a specific set, but then he realized that other cases were possible. Therefore Brandon and Kyle spent time on five cases, which were more difficult to follow than Jordan's cases were. Like the other participants, they followed the definition as their Δ_1 Knowing How and were easily able to find specific basis elements that satisfied the first condition of the definition. These participants on this task are not demonstrating the relevance quality since they spent the bulk of their work on something that was not necessary, but they did show completeness and worked from their available plan of following a definition. Table 6.7 shows their components, but with one new feature. There is an exclamation point next to "Notation" in Being Able. This is indicating that there was some kind of misunderstanding or issue with this Being Able. We can see that Brandon and Kyle were still successful in completing their task, so an exclamation item does not immediately indicate a lack of success, but rather perhaps there were just some road blocks on their path or lesser skill shown on the way to their goal state.

Table 6.7: Brandon and Kyle's Successful Action in Task Two (early interview).

Δ_1 **Director System**	Δ_2 **Director System**
Knowing That ○ Definition of a Basis ○ Set theory ○ Metric space **Knowing How** ○ Following the definition of a basis ○ Finding a smaller basis element within an intersection of basis elements **Being Able to** ! Notation ○ Find a specific basis element that contains x ○ Analyzing intersections of sets (via cases)	**Knowing That** ○ All KT, KH, and BA from Δ 1 with strong Qualities of Schema: ○ Completeness ○ Plans **Knowing How** ○ Working with cases

Task Two - Phil

Phil immediately started out by identifying the Δ_1 Knowing How of following the definition for this task. We know he had the needed Δ_1 concepts because he was able to create sets that satisfied the conditions of the definition. However, he was not consistent with what sets he was utilizing. He started out $\mathcal{B}_1$ with a creative idea to use x as the right endpoint, which none of the other participants did. After prompting for further details, however, he changed his set to be more like what Brandon and Kyle created in their late interview. I would not say his schema was complete because he did not consider all possible cases, nor did he ever reason in general terms. He left $\mathcal{B}_1$ slightly unfinished, but had described more than one successful route.

For $\mathcal{B}_2$, Phil noticed that there was only one case to consider, which showed that he had a Δ_1 Knowing How that showed some generality of his schema. At this point, he questioned whether or not his B_3 could equal the intersection, but realized that it did not matter since he could just choose a number between x and a like what Brandon and Kyle did in their early interview. A summary of Phil's Knowing That, Knowing How, and Being Able is in Table 6.8.

Table 6.8: Phil's Successful Action in Task Two.

Δ_1 **Director System**	Δ_2 **Director System**

Knowing That
○ Definition of a Basis
○ Set theory
Knowing How
○ Following the definition of a basis
○ WLOG (B_2 only)
○ Finding a smaller basis element within an intersection of basis elements
Being Able to
○ Find a specific basis element that contains x
○ Analyzing intersections of sets
! Notation
	Knowing That
	○ All KT, KH, and BA from $\Delta 1$ with strong Qualities of Schema:
	    ○ Relevance
	    ○ Generality (B_2 only)
	    ○ Plans
	Knowing How
	○ Working with cases

Chapter 7

Discussion

In this chapter, I will address some highlights from the analysis of the data, address the research questions, and back up some of my claims by referring to the literature.

For the purposes of this discussion, I will refer to Brandon and Kyle in their early interview, along with Phil, as the "novice" participants. Dr. T and Jordan are considered to be the experts, while Brandon and Kyle in their late interview are somewhere between novice and expert. As seen in Table 6.1, the majority of participants who did not reach their goal state for a task were novices, and the number of stages that prevented successful action increased as the tasks increased in abstractness. Similar to what was seen in Chapter 3, the analysis of the data revealed that students new to the subject of topology are in the early stages of schema development, even after having been in a course for an entire semester. This provides evidence that difficulties do exist when first learning topology, and brings me to my central research question.

What is the nature of students' difficulties with topology?

7.1 Students' Difficulties with Topology

The analysis of the data revealed that the novice participants had issues that prevented them from correctly finishing three of the four tasks, while Task Two posed less difficulty. This was because Task Two could be done by following the definition of a basis and the bases selected for the task had elements that intersected to give another basis element. Even if a participant did not recognize that the intersection was another basis element, they were able to analyze it enough to show the second condition of the definition. Further, the definition of a basis (Knowing That) was given to the participants on the definition sheet. Even though all participants were successful to some extent, the skill with which they operated their director systems for Task Two varied. Unsurprisingly, Dr. T was the most efficient participant and the undergraduate student, Phil, demonstrated the lowest level of skill. Brandon and Kyle were successful in both interviews, but took a much longer route in the early interview than they did in the late interview, which provided evidence of Intelligent Learning over the course of their semester. No one struggled with understanding the definition of a basis and following it, nor did they struggle with analyzing intersections of the sets given.

What happened in the other three tasks to cause difficulty? It should be noted that in the analysis of the results, we saw issues during all three stages of successful action (Knowing That, Knowing How, and Being Able) and we did not see substantially more issues at one stage than another. For Tasks One and Three, Brandon, Kyle, and Phil all were missing something at a stage that prevented them from successfully completing the tasks. Whether they were completely unsuccessful, or maybe just missing some important details, we can isolate their issues within one of the stages.

Task Three goes a step further than Task Two and requires an awareness of a basis and how it can be used to define the product topology (Knowing That). Brandon and Kyle were missing this in their early interview, even though they had successfully worked with bases in Task Two. Task Four requires a conceptual understanding of basis and the ability to work with abstract ideas instead of specific sets. We can see that this task was the most difficult one because the novice participants did not just have issues with one of the stages, but rather, they had issues with all three of them.

Because issues were seen at some point in all three stages, it will make sense to address the following research question for each stage over the next three sections.

What are the particular difficulties within each stage of successful action?

7.1.1 Students' Gaps in Their Schema (Knowing That)

The topics that were relevant to these four tasks were given in Table 6.2. Although I chose the basis concept to be a focus of this study, set theory emerged as a recurring theme. In his final reflections, Dr. T gave several concepts that are needed when beginning a topology course, with most of them being part of set theory. This is consistent with Narli (2010), who found that misunderstandings of a first topology lesson were caused by issues with set theory. We saw Brandon and Kyle struggle with reading set notation for set D of Task One in their early interview, and we saw that Phil had an issue with set theory during Task Three. Therefore, set theory may be something to emphasize with undergraduate students and graduate students in their first topology course, even if it is considered to be a prerequisite topic.

In many of the cases where a task was not completed correctly, it was due to an incomplete schema. The participants simply did not have the necessary concepts or connections needed to think about the task. We saw this with Brandon and Kyle because their early interview took place before they had been introduced to a basis for a topology in their course. They were only familiar with the ideas in Task One, and for all other tasks, they were relying on their definition sheet and each other, showing that they were missing conceptual connections that would have been helpful. Therefore it comes as no surprise that their lack of knowledge (Knowing That) stood in the way of them completing Tasks Three and Four. In Task Three, they were missing the basis concept entirely and in Task Four, they were missing the connection between open sets and a basis given from the Lemma in the Task Four solution (see Section 6.4.1). They relied heavily on their definition sheet in the early interview, compared to their late interview, in which they hardly needed it, showing the existence of conceptual connections that helped them reach their goal state.

It is worth noting that possession of the definition sheet was not sufficient for the participants to "Know That". Brandon and Kyle showed evidence of a significantly more complete schema for all tasks in their late interview, meaning that the needed concepts were already assimilated. For the early interview, however, reading the definition sheet did not give them sufficient time and information in order to assimilate the concepts to a level of quality that could be utilized successfully for the tasks. This supports the Δ_2 portion of the framework, in that one not only need possession of a schema, but one needs strong qualities of schema in order to have the Knowing That that contributes to a successful Δ_2 director system.

What may be interesting to consider is how Phil showed gaps in his schema, even though his interview took place at the end of the semester during the week when he was studying for his final exam. In Task Four, he was missing information about the topology generated by a basis and a topology that contains a basis. In Task One, however, Phil showed inaccuracies more than gaps. There existed some kind of concept in his schema for the standard topology and another for the subspace topology, but he continued to incorrectly say that they were the same because he was trying to remember a lemma that made him think they were.

Dr. T spent a significant amount of time in his interview laying out how he likes to teach a first topology course. Some of his goals involved teaching enough content that students could prove some major Calculus theorems by using topology proofs. Theoretically, this would give some context for the students and this new area they have learned about. I identified contexts like this as a Δ_2 Knowing That in my analysis because a concept can be assimilated into a context, creating more connections than the concept would have when isolated. Although there was little data collected involving contexts, Dr. T testified to a lack of context being an issue for students.

Knowing That centers around possessing a schema that contains all needed concepts and connections for the task at hand. When there are concepts and connections that are missing, they need to be assimilated into the schema. On the other hand, another possibility is that the concepts and connections exist, but they contain inaccuracies. In this case, the schema needs to undergo reconstruction, or accommodation, to correct these issues, which can be difficult (Skemp, 1979, 1987). We saw this with Phil in Task One when he still tried to say the standard and subspace topologies were the same, even after the intervention and his "lightbulb" moment. The results of this study are consistent with the idea that assimilation and accommodation are sufficient interventions for addressing the difficulties participants had with Knowing That. There were no cognitive gaps or misunderstandings that could not be resolved through some form of assimilation or accommodation.

Hazzan (1999) suggests reducing the level of abstraction for easier assimilation of new, abstract concepts.

> Given the abstraction level in which abstract algebra concepts are usually presented to students in lectures, and the lack of time for activities which may help students grasp these concepts, many of the students fail in constructing mental objects for the new ideas and in assimilating them with their existing knowledge. The mental mechanism of reducing the level of abstraction enables students to base their understanding on their current knowledge, and to proceed towards mental construction of mathematical concepts conceived on higher level of abstraction. (p. 84)

This statement was made in the context of abstract algebra, however, topology requires a similar level of abstractive abilities. Reducing the level of abstraction can help with assimilation of topology concepts into a student's schema.

7.1.2 Students' Difficulties with Planning (Knowing How)

It can be difficult to distinguish between Knowing That and Knowing How when thinking about connections in a schema. It depends on who possesses the schema and in what way they are utilizing the connection. The connection can be something that is necessary to understand before working towards the goal state (Knowing That), or the same connection can be thought of as a path that needs to be traveled on the way to the goal state (Knowing How). This is demonstrated in Task Three with the connection that open sets are unions of basis elements.

Phil recalled this as a fact that he read on the internet and treated the union as the input of the projection map, but Brandon and Kyle in their late interview discussed it as if writing the union of basis elements was a step they needed to perform before sending through the map. A question to consider in the future is, what does this difference in use of the connection indicate regarding successful action of the director system?

In completing Tasks One and Two, no participants showed difficulties with Knowing How to complete them. Their paths were very intuitive for Task One and the first path that everyone tried on Task Two eventually led them to the goal state. Tasks Three and Four are less concrete and therefore have more opportunities for students to get lost or travel down a bad path. Even so, none of the participants had problems with the initial proof setup for Task Three, where they let the definition of an open map guide them. Phil started down the correct path, following the definition of an open map and writing his open set as a union of basis elements, but when it came time to consider what the projection map does to a union, he inaccurately rearranged unions and products. This comes back to an issue of accuracy with set theory within an application in a two-dimensional space. Considering the union of rectangles in a product space and what the projection map sends that union to is the portion of this proof that Dr. T emphasized the most in Task Three, but Phil tried to "hand wave" through.

Phil had memorized for the product topology that all open sets are unions of basis elements. For Task Four, however, he needed another version of this connection that said these unions of basis elements make up the topology generated by the basis (also known as the lemma from Section 6.4.1). Phil's rote memorization of this connection and Brandon and Kyle's complete lack of it in Task

Three helps explain why they were unable to apply it during Task Four. The other Knowing How that the novices did not show during Task Four was that the topologies containing $\mathcal{A}$ were also going to contain unions of elements of $\mathcal{A}$. This part of the path is created from using the definition of a topology. These missing connections for the novices are not directly tied to set theory. Instead, what they have in common is that they are some of the new concepts in a first topology course. It is understandable that Brandon and Kyle were lacking these paths since they did their early interview in the beginning of their semester, however, Phil's interview took place towards the end of his semester. Phil's connections between a topology and basis were not very strong, which suggests that a topology schema and a basis schema can be difficult and slow to develop for students new to the subject. This is consistent with what Narli (2010) observed when students who were retaking a topology course struggled with the concepts in the first lesson of the semester.

The proof strategy of showing double containment was not a challenge for most participants. Brandon and Kyle debated about it for a moment, which was due to them misunderstanding each other, not a misunderstanding about the strategy itself. The paths that were more difficult for the novice students to travel were the ones created from set theory and new connections between topology concepts, not the ones from proof-writing strategies. It should be noted that this study had a small number of participants, so I am *not* generalizing this to say that proof-writing is simple. In fact, the literature says the opposite. Weber (2001) found that the primary issue with undergraduate abstract algebra students' proof construction was their lack of strategic knowledge. This knowledge included things like proof techniques, knowing which theorems are important and useful, and knowing when to use (or not use) a syntactic strategy (one that does not require conceptual understanding, but instead pushes symbols). In Weber's (2001) study, the undergraduate students had gaps in this strategic knowledge while the graduate students did not. The majority of participants in my study were graduate students or higher, with the exception of Phil. Even though Phil is an undergraduate student, he was in the top of his class, so the fact that he did not have difficulties with proving techniques is not entirely surprising.

7.1.3 Students' Skill Levels (Being Able)

The difficulties discussed in the previous two sections centered around the need for a schema with strong qualities, available for use in a Δ_1 director system. In identifying what occurred during the Being Able stage, we saw a wider variety of difficulties arise, some of which can occur even if an appropriate schema is available.

Part of Brandon and Kyle's Being Able during Task Three in their late interview was that they discussed the details of how the projection of a union equals the union of corresponding projections and they could not think of "weird" cases where that was not true. In comparison, Phil did not consider these details at all and tried to brush over them in his initial attempt of the task. It was not until the interviewer prompted him to explain more that he came up with an incorrect reasoning. These details stem from needed set theory "calculations".

Recall that the participants' skill on Task Four was improved if they were more of an expert. The novice participants confused the basis and the topology generated by the basis, and also struggled with notation. This is likely due to how the task was phrased since the only notation given to the participants was $\mathcal{A}$, meaning they had to select their own notations for the topology generated by $\mathcal{A}$ and the topologies that contain $\mathcal{A}$. Further, these topologies were general concepts, not specific topologies, so the novice participants were less comfortable creating notation for abstract ideas. Phil chose a notation that was fairly unconventional, using x as an element of a topology. The elements in a topology are sets, but mathematicians typically use x to represent an element of a set, not a set itself. Phil's notational detail here is consistent with results from Zazkis and Gunn (1997), which showed that students experienced issues when set elements are sets themselves.

The details of a Δ_1 Being Able can keep someone from their goal state, even if they have done the problem successfully before. Brandon and Kyle were the only participants who had early and late interviews, so they were able to demonstrate this with their first task. They successfully determined if sets A, B, and C were open in $\mathbb{R}$ and in Y during their early interview, but then completely overlooked the instruction about open in Y in the late interview. It may seem unfortunate to label Brandon and Kyle as unsuccessful because of a small Δ_1 detail, however, in mathematics, one small mistake can lead you completely astray and we saw this in their case.

The difficulties we saw in the Being Able stage included set theory calculations involving unions of products, creating notation for abstract ideas, and keeping up with the small details. In Task Two, we saw that the participants did *not* have issues with calculating basic intersections of sets.

7.2 In Search of a Path

In mathematics, the Knowing How stage has a significant impact on our success. If only I knew how to prove this theorem, I would just do it! If I do not quickly find a path, I can experience feelings of frustration or anxiety. As an instructor, I like to ask my students what paths they have tried so far because it offers me insight about how they are thinking about the concepts and gives me clues as to how to best help them. Therefore, it could be helpful to instructors to address the following research question about what students' attempts look like in topology when they do not have a ready-to-hand plan available.

How do topology students reach a goal state when they do not have a guaranteed Know How?

In Task One, we saw lots of intuitive path finding through the participants visualizing the four sets, rewriting some of those sets in interval notation, and following the definition of the subspace topology. Phil did try to visualize set D, but he also tried some other paths that we only saw with him. To determine whether a set was open in $\mathbb{R}$, he focused on the inequalities in the notation given in the task and looked for inequalities that included "or equal to". Even after the intervention, he did this again and considered whether or not the set was at least half open to be open in Y.

The path that Phil traveled for the longest time was trying to cite a lemma that he was not remembering correctly. He tried to apply this Δ_2 Knowing How, which is certainly not a bad thing to try, but it did not get him any closer to the goal state in this case. Sometimes Δ_2 Knowing How's are helpful for improving the efficiency of our Δ_1 director system, but they can also be a distraction if a student is just looking for a shortcut. Dr. T also pointed this out during Task Three when he gave an alternate route for defining the product topology. Even though using an alternate definition can demonstrate flexibility of a schema, it can also indicate avoidance of a certain concept, like a basis.

Even though all participants were successful with Task Two, they did not necessarily have a ready-to-hand plan for all of their paths. Jordan expressed this explicitly by saying he might need to follow the definition of a basis or use a proposition and that showing the second condition of the definition was not immediately obvious to him. He followed the definition route first and it led him to his goal state, so he did not need to explore the proposition route further. Once he reached the second condition, he broke the problem down into six cases and drew a visual representation in considering the first case. The novice participants also considered various cases in order to make progress with Task Two. Note that there existed a highly abstract and general path for completing Task Two, which was demonstrated by Dr. T. Hazzan (1999) pointed out that "even in cases where students have a relatively advanced conception of mathematical notions, sometimes they tend to use a (less abstract) canonical procedure at the cost of many calculations" (p. 82). This certainly applies to Jordan in this case. He was highly capable of completing the task in a more abstract way, but working with the cases reduced the level of abstraction, making the content easier to work with. Recall that Jordan took the interview prompt seriously and worked as if he were teaching to a group of students. Reducing the level of abstraction here could have been a pedagogical decision.

All of the participants followed the definition of an open map to begin working through Task Three. Brandon and Kyle (in both interviews) were able to follow the projection map, but this is where Phil had difficulties. After being prompted to explain in detail why he got the image he claimed to get, he tried a path that, if written on physical paper, would have appeared to be symbolic manipulation. This path was not an accurate one and therefore took him further from the goal state.

We know that the novice participants did not have a plan ready for Task Four because they were not successful with it and they had difficulties at all three stages of their director system. The one plan (Knowing How) that they did try that was productive was the strategy of showing double containment in order to prove equality. Additionally, Phil mentioned "element chasing" as something that he might try. Phil contemplated this task quietly for a while, so we can never know the extent of paths he tested in his mind, but he eventually did say that he was trying to think about the problem logically. By this, I believe he was saying that he was trying to think of the concepts and any connections to them to see if any connections were activated intuitively. This did not work for him in the interview because he did not possess strong enough connections that were needed for the task.

In their early interview, Brandon and Kyle wanted to travel down different paths from each other when they did not know how to move forward. Kyle was thinking about a set theory argument to explain the trivial direction, but while he was explaining it, Brandon was trying to follow the definition of a topology generated by a basis. Their paths were so far apart from each other's that there was a moment where they were arguing with one another.

Some of the attempts that were not productive for the participants were when they tried to use a Δ_2 Know How before following a definition, incorrect symbolic manipulation, and "element chasing" in topologies, where the elements are sets (Zazkis & Gunn, 1997). However, there were also many of these attempts that were productive and moved the participant closer to the goal state. Identifying these paths (Knowing How) will help address a subquestion of the second research question.

What do students' successful director systems have in common?

The most helpful thing that the participants tried was following the definitions relevant to the problem. This was the main strategy in Task Two, but they also found their paths in Task One by following the subspace topology definition and in Task Three by following what it means to be an open map and what the projection map does. Breaking the problem down into cases, a form of reducing abstraction (Hazzan, 1999), also helped the more novice participants in Task Two and considering double containment helped with proving equality in Task Four. If we consider Brandon and Kyle in their late interview, they were successful in Task Four because they possessed stronger connections about topologies and a basis than they did in the early interview.

7.3 Qualities of a Schema for Topology

Analyzing the qualities from Section 4.2.3 has helped with understanding schema better. Specifically, what qualities are desired for students taking their first topology course, and what qualities are they demonstrating? The following series of research questions were asked in order to address this.

What qualities of schema do undergraduate students demonstrate when they are learning about topology for the first time? What qualities are needed in order to begin learning about topology at the undergraduate level? What about at the graduate level? What impact does each quality have on the functioning of a director system in topology?

In Table 7.1, I have summarized which participants demonstrated certain qualities. There are four indicators in each cell that represent the four tasks, in order from left to right. Jordan only has two indicators since he only had time for Tasks One and Two. For example, consider Brandon and Kyle in their early interview. They demonstrated a more complete schema for the ideas in Task Two and provided some evidence during Task Three that their schema for that task was not as complete as it needed to be. There was not any evidence from them during Tasks One and Four that was noteworthy, which is represented by the dashes. Note that I am not claiming that this table is complete. You can imagine that Dr. T's schema probably had all of these qualities during all four of the tasks, but he does not have checkmarks in every spot. I only put a ✓or a × if there was significant evidence regarding that quality, which was discussed in Section 6.6.

Table 7.1: Evidence of schema qualities noted during the interviews.

	Dr. T	Jordan	Brandon and Kyle (late)	Brandon and Kyle (early)	Phil
Relevance	- ✓ - -	- ✓	- ✓ - ✓	- - - -	- ✓ - ×
Accuracy	✓✓✓✓	✓ -	✓ - - -	× - - -	× - × -
Completeness	- - ✓ -	- ✓	✓✓× -	- ✓× -	- - × ×
Organization	✓✓✓✓	- ✓	- - ✓✓	- - - -	- - - ×
Generality	- ✓ - ✓	✓×	- - ✓✓	- × - -	- ✓ - -
Strength	✓ - - -	✓ -	- - - ✓	- - - ×	× - - ×
Type (Conceptual)	✓ - - ✓	- -	- - - ✓	- - - ×	× - - -
Plans	✓✓✓✓	✓✓	- ✓ - ✓	- ✓ - -	× ✓ - ✓

✓Evidence of possession of quality found
- No evidence regarding quality found
× Evidence of lacking in quality found

In reviewing Table 7.1, we see a stark contrast between the experts and novices. Not only do the novice participants not have many checkmarks, but they demonstrated a lower quality of schema more often than I expected. The quality that all participants demonstrated was having ready-to-hand plans available. These plans were part of their Δ_2 Knowing That and Knowing How, ready to use as needed. The plans that were used the most were proof strategies (Knowing How), like following a definition and showing double containment.

None of the participants showed that they were lacking generality in Tasks Three and Four, however, when reflecting on Task Three, Brandon pointed out that working with abstract concepts is not intuitive.

> In terms of when we first did this, I guess, um, it's easy to just jump straight into just choosing U or something because of the way we defined the basis of just being an open set cross an open set where each of those are coming from the individual...so...it just seems natural to just go to one thing instead of considering the most general thing, which is a union of those things.

Additionally, in Dr. T's views, students have to be comfortable with working in general terms. Working with higher-order concepts is an innate part of what it means to be able to think about higher-level mathematics, thus it is important for students to be able to think in general terms when learning about topology. Because these participants were advanced, they already had gone through some prerequisite courses where they had to train themselves to think in more general terms, so they did not show great deficiency in this quality. However, Dr. T thought of a way to scaffold the problem for students because he expects some students to struggle with generality at the undergraduate level. Additionally, we did see a lower level of skill from participants who lacked generality during Task Two when they worked from cases instead of seeing the general form of the intersection.

A complete schema is one that contains all necessary concepts and connections needed to reach the goal state. An incomplete schema was usually indicated by gaps where an expert might have given what the necessary concepts and connections were. In Table 7.1, we see a mixture of complete and incomplete schemas, even for the same participant, confirming that the completeness quality is highly dependent on what the goal state is. Many participants were considered to have an incomplete schema for Task Three because they did not know they had to show that their basis satisfied the definition of a basis. Then on Task Two, they showed completeness by thoroughly considering all cases necessary. Not only is completeness a quality of a schema, but we can also envision a Δ_2 director system where a more complete schema is a goal state in itself. In this system, the present state at the beginning of the semester for most students is that they have a highly incomplete topology schema and their goal state is to have the completeness quality by the end of the semester by learning about all of the topics selected for the course.

Four of the qualities show a pattern in Table 7.1 where the expert participants showed evidence of the quality and the novice participants showed evidence that they lacked the quality. These four qualities are the strength of connections, the type of connections, organization, and accuracy. These qualities were specifically on a lower level or perhaps even absent with less experienced students. Because the expert participants demonstrated these consistently, we can say that these qualities are certainly desirable for anyone studying advanced mathematical topics, which certainly includes graduate students.

To address the first research question in the series, we can look to the novice participants. In this series of tasks, they demonstrated relevance to the problem, having ready-to-hand plans from past mathematics classes, and being able to consider higher-order concepts. Additionally, they showed the quality of an *incomplete* schema. If we compare and contrast the participants' tables for Task Two in Section 6.6.4, we see that the most efficient director systems have fewer items listed in the three stages because they took a direct path with no detours. However, the most efficient director systems also have more qualities demonstrated. Students are likely not entering a first topology course with rich qualities of schema, but our goal as instructors is to help them build up these qualities throughout the semester so they can run efficient director systems for a wide variety of problems they encounter.

Although this study is focused on gaining a better understanding of learning in undergraduate topology courses, there are reasons to briefly discuss first-year graduate courses as well. A fascinating phenomenon that is shown in Table 7.1 is what happened with Brandon and Kyle. In their early interview, they did not demonstrate many qualities at all, but by their late interview, they demonstrated all of them. The monstrous amount of learning that occurred in the weeks between their interviews is likely credited to them having been enrolled in a graduate qualifying course in topology. This should be highly encouraging for students in advanced mathematics because Brandon and Kyle here have shown that you do not have to begin a course with several rich qualities of your schema in order to be successful. The data points to generality and some ready-to-hand plans as qualities that are desired in students starting out, but Brandon and Kyle show that these are not strict prerequisites for successful learning.

We saw schemas with richer qualities in the more experienced participants, so that is where we should look to address the final research question.

What impact does each quality have on the functioning of a director system in topology?

From the experienced participants, we saw director systems reach their goal state, usually with a high level of skill, speed, or efficiency. These participants were flexible in their approaches, like when Jordan said that he could follow the definition or use a proposition in Task Two, or when Dr. T gave two approaches for defining the product topology. We also saw more variety in the responses, like in Task Two where everyone came up with different sets to satisfy the second condition of the definition. Additionally, the experienced participants showed that their schemas were organized and contained strong, conceptual connections, making them more likely to find intuitive paths with which to reach their goal state.

7.4 Summary

There were several notable results discussed in this chapter. Students like Phil and the students from Pilot Study 1 could spend a semester learning about topology and still be in the early stages of schema development at the end, showing that understanding these concepts takes significant time for students new to the subject. This study showed this in particular for the topology and basis schemas. Even in some cases where an appropriate schema is available or the system saw success previously, a director system may be unable to reach the goal state because of issues with the Being Able stage. This demonstrates that there are various opportunities for difficulties to occur in an advanced topic like topology.

The Framework for Successful Action in Learning Topology helped diagnose where difficulties stemmed from for the participants, most of which occurred with the novice participants. For Task Four, unsuccessful action was a result of issues in all three stages of Knowing That, Knowing How, and Being Able. These cases demonstrated that novice students had more serious issues when the problem was more abstract. Given the scarce amount of literature on the learning of topology concepts, especially with regards to the basis concept, this work serves as an "existence proof" that these ideas are challenging for students to navigate at first.

These particular participants did *not* struggle with basic proof-writing ideas like breaking a situation down into cases, double containment, and following technical definitions, but that is not to say that future participants will not struggle with these. Some difficulties that were demonstrated in this study included set theory calculations, creating notation for abstract ideas, and missing connections between concepts like open sets and a basis. The Δ_2 Knowing How's usefulness for a director system depended on the participant and goal state. These strategies could have been productive towards the goal state, like how Jordan switched representations in Task One, or they can completely derail the director system, like when Phil tried to force a lemma into his work on Task One.

Strong qualities of schema were demonstrated by the expert participants and can serve as a goal state in their own right for novices. In particular, all participants had ready-to-hand plans available to use, but the novice participants showed a lack of strength of connections, types of connections, organization, and accuracy. There was a lower level of skill seen in participants who lacked generality and the completeness quality was dependent on the goal state for each participant. The most encouraging result was that these qualities are not strict prerequisites for learning topology concepts, as demonstrated by Brandon and Kyle.

Chapter 8

Concluding Remarks

In this chapter, I will briefly summarize the major contributions of this work and assess the usefulness and validity of the theoretical framework. I will then discuss some limitations of this work, ideas for future work, and offer some pedagogical suggestions.

8.1 Contributions

A significant contribution of this work is the development of the Framework for Successful Action in Learning Topology. The idea of schema can be elusive and difficult to fully comprehend. The framework made this less so by situating schema and its qualities in a director system model, bringing Skemp's (1979) notions to life, but at the same time, preserving all of the elements of his theories. This study exemplified some of Skemp's ideas by applying them to describe students' understanding of topology. Additionally, it offers examples of what kinds of introductory ideas are more challenging for topology students, specifically with respect to a basis schema for topology. This is especially informative given the limited amount of literature on undergraduate topology education.

8.2 The Framework for Successful Action in Learning Topology

One of the aims of this study was to develop a framework that helped make sense of the schema construct, while also shedding light on students' ways of thinking and dealing with topology concepts. Ideally, a framework should be useful across multiple contexts and should "give us a sense of understanding by providing satisfying explanations about hidden processes underlying the phenomena in an area" (Clement, 2000, p. 559). Did the Framework for Successful Action in Learning Topology accomplish this? In this section, the viability of this framework will be evaluated through considering plausibility, triangulation, explanatory adequacy, empirical support, external coherence, extendability, and predicability (Clement, 2000).

In this study, participants' thought processes were explained from the perspective of the theoretical framework and there were no anomalies to report. The framework was applicable across all levels of expertise, and in turn, was further supported by the data. The participants with successful Δ_1 director systems had evidence showing their ability to navigate all three stages, while the participants with difficulties had gaps or misconceptions in at least one stage. In some cases, the participants were even explicit about what was causing them trouble, like when Kyle was not satisfied with his basis notation and Phil admitted that he recalled something from rote memory. These types of observations make stronger connections to the framework and thus we can confidently triangulate from the data to the framework. All of the above shows plausibility of the framework through explanatory adequacy and empirical support.

There is potential for external coherence of the framework. How coherent is it with other learning theories accepted in the Research in Undergraduate Mathematics Education (RUME) community? APOS (Arnon et al., 2014; Dubinsky & McDonald, 2001) and Piaget and Garcia's (1989) triad framework explained how a schema, or Knowing That, is built. Hazzan's (1999) reduction of the level of abstraction can help with the Δ_2 Being Able stage.

What is the external viability of the Framework for Successful Action in Learning Topology? The components in Knowing That, Knowing How, and Being Able were written with introductory topology tasks in mind. Additionally, they were informed by expert accounts of these tasks. The framework shows extendability in that it can be modified for other advanced mathematical topics, especially in consultation with an expert in the topic. The stability of the framework depends on the level of expert consulted. The more they can say about what is needed for success in their topic, the more the framework can be filled out, which makes it easier to identify what concepts, connections, and details are missing for the novice.

The framework also has some predictive power from the perspective of the learner. A learner can identify their goal state and then use the three stages as a checklist of sorts to gain a sense of the likelihood of success in reaching their goal state. The framework provides an improved sense of what is cognitively needed, as well as what road blocks potentially stand in the way for learners. Predictiveness from the perspective of an instructor is possible when deciding on questions for assessments. Instructors can identify what is needed from the three stages for successful action (Knowing That, Knowing How, and Being Able) for a question and use that to judge whether or not the question is appropriate for their assessment. The framework can also be useful for introductory topology instructors in identifying their students' gaps in understanding, which can inform what intervention is needed.

When I first attempted to understand what a schema was and how it worked, I was generally left feeling like it was still too abstract to truly understand anything about it. This feeling was not improved until I tried to think about schema in the context of a director system. The director system is more relatable, where concrete present states, goal states, and plans can be thought of. In a sense, assimilating 'schema' into the context of how a director system reaches its goal state reduced the level of abstraction of schema, making schema theory more accessible. Additionally, incorporating qualities of schema also improved understanding by providing a set of standards by which schema can be approximately measured. Finally, the framework clarifies what cognitive structures, processes, and skills are necessary for learning beginning topology concepts, a goal that is not always achieved easily.

8.3 Limitations and Future Work

Because this was a collective case study, there were a limited number of participants. Therefore the study results do not have as much generalizability to other students in topology. However, the framework can be applied to more studies in topology in the future, which will help with understanding the current status of topology education. Along these lines, there is a need for a larger scale study with a wider variety of introductory topics so that the main issues for students can be identified in a general setting.

The design of the study created Δ_1 goals for the participants through the four tasks, but identifiable Δ_2 goals were in short supply because much of the data was collected in a one-time interview. The exception to this was Brandon and Kyle, which is where we saw evidence of learning, however, little data was collected about their experience during the semester and *how* they came to learn about these concepts. In an attempt to collect data that tells us more about Δ_2 director systems, future work would ideally involve long-term data collection, a variety of data sources (such as student journals, observations of the class sessions, interviews, etc.), and prompts that are goal oriented and directly involve learning and skills objectives for the students. This would help improve empirical support for the Δ_2 portion of the Framework for Successful Action in Learning Topology.

The four tasks were chosen to focus on the basis concept, however, other choices could have answered lingering questions. For example, Task One focused on the standard topology on $\mathbb{R}$ and the subspace topology. Did Phil struggle with the subspace topology because it was different than the standard topology or did was it something about the subspace topology itself that caused issues? If the task had involved multiple other topologies in addition to the standard and subspace topologies, this question possibly could have been addressed. Recall that Phil tried to remember a lemma when thinking about this task. A suggestion for future work on the definition of a topology would be to use or create a topology that is not found in a standard textbook so that participants have to build up the necessary concepts and connections to do the task. In this case, Phil would likely not have tried to use a lemma.

Participants did well with following definitions, but following the definitions was not a complex process for Tasks One and Two. An interesting future study would be to investigate students' Δ_1 director systems when faced with a task where following the definition is not straight forward.

8.4 Some Pedagogical Recommendations

Based on this work, some teaching recommendations could include considering ways to reduce the level of abstraction for students so that the time needed for assimilation is shortened. There is a conflict between the amount of content instructors need to get through in a semester and the amount of time students need to digest these topics. Additionally, emphasis can be made on conceptual connections between concepts in order to assimilate the concept into a schema. To help students improve their qualities of schema, an instructor can be more explicit about theirs. For example, the instructor has the organizational quality for their topology schema and could point out places of interiority of concepts. An instructor could emphasize scenarios where arbitrary objects are necessary in higher-order topics to help students with the generality of their schemas.

The results support that set theory should be emphasized as a prerequisite for a topology course. If an instructor is not convinced their students have this prerequisite, then they might want to consider some support with set theory concepts in the first few weeks of the semester.

During the interview, Dr. T gave a bit of his pedagogical perspective for the introductory topology course. Although I was able to make some recommendations above based on the results and theoretical framework, it is also gratifying to get recommendations from an expert who has taught the course many times before. One major theme in Dr. T's recommendations is that students need to be motivated for the subject because they do not know very much about it coming in. He tries to teach the content in a way that students can prove a couple of theorems from calculus through topology proofs. He relates these new, abstract ideas to concepts that students are more comfortable with. This recommendation is consistent with the theory because having a context can allow for a larger number of connections to form as a concept is assimilated into one's schema. Additionally, Dr. T acknowledges that rigor in proof is not the goal with introductory students. In the quote below, he describes how he approaches an introductory topology course.

> I don't really view that, when I teach it, I don't view that course as being a course that's trying to prepare, to rigorously prepare, students for topology. Because when I've taught it, they've been, you know, oftentimes the class has been large, like maybe thirty students or so, which is large for that class, I think. Most of the students aren't, when I've taught it, most of the students aren't students that are really thinking about going on to graduate level study in mathematics. So I view it, if you're going to phrase it in terms of the undergraduate course, I think about that course a little bit differently. It's sort of not necessarily trying to get students rigorously able to work on topology questions, but to instead to expose them to the basic ideas and give them some idea of the extent of the field of topology.

8.5 Summary

To summarize, students need an appropriate schema in place in order to assimilate or understand new concepts. Schemas centered around physical activity, like the game of golf, can be easier to construct. As the topic becomes less concrete and more abstract, like topology, the more difficult schema formation becomes.

Understanding how schemas are developed and used in action can positively inform the teaching of advanced mathematics. Possession of an appropriate schema is required for Knowing That, and if the schema contains strong qualities, Knowing How to reach the goal state can be intuitive, like what was seen in Tasks One and Two. Even if both the Knowing That and Knowing How are understood, the operators of the director system still have to Be Able in order for successful action to take place. A direct result of learning is the improved functioning of a Δ_1 director system, therefore learning is a major component of a Δ_2 director system, which needs further study. Two participants, Brandon and Kyle, showed substantial intelligent learning in the results, saying that they were slow to understand the basis concept earlier in the semester, but demonstrated strong, organized, conceptual connections in their basis schema by the end of the semester. This supports Skemp's (1962) theory that learning will be much slower at the stage when a new schema is being formed than later when it has become available for use as a tool of further learning.

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Appendix A

Definitions given to participants

All definitions are from Munkres (1975).

Definition 1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\emptyset$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**. We say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Definition 2. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that

(1) For each $x \in X$, there is at least one basis element B containing x.

(2) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Definition 3. If $\mathcal{B}$ is a basis for a topology on X, the **topology $\mathcal{T}$ generated by $\mathcal{B}$** is described as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Definition 4. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X; its open sets consist of all intersections of open sets of X with Y.

Definition 5. A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y.

Appendix B

The Four Tasks

All tasks are from Munkres (1975).

Task One

Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \{x \mid \tfrac{1}{2} < |x| < 1$$

$$B = \{x \mid \tfrac{1}{2} < |x| \leq 1$$

$$C = \{x \mid \tfrac{1}{2} \leq |x| < 1$$

$$D = \{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbb{Z}_+$$

Task Two

Show that each collection of subsets of $\mathbb{R}$ is a basis for a topology on $\mathbb{R}$.

$$\mathcal{B}_1 = \{(a, b] \mid a < b\}, \text{ where } (a, b] = \{x \mid a < x \leq b\}$$

$$\mathcal{B}_2 = \{(-\infty, a) \mid a \in \mathbb{R}\}, \text{ where } (-\infty, a) = \{x \mid x < a\}$$

Task Three

(a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.

(b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$ is an open map.

Task Four

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

Appendix C

Questions for Partner Interviews

1. I have four tasks for you to work on individually and I have a definition sheet that you are welcome to use if you find it helpful. I will let you work individually for about ten minutes and then I will have you collaborate with your partner on them.

2. We'll now move on to work with your partner. Go through each task and discuss your attempts with one another, trying to come to some sort of consensus.

   (i) Try to explain your ideas to your partner.

   (ii) Try to come to some agreement for each task together.

   (iii) (If a task was successfully completed) What do you feel was necessary for you to know in order to accomplish this task?

   (iv) (If a task was left incomplete) What do you think you may be missing in order to complete this task?

   (v) Have you taken a topology course/seminar/experience before this semester?

3. (For second round of interviews only. Hand participants a copy of their work from the first interview.) Here is your work on these same problems from our first meeting. Take a moment to read through them.

   (i) How much progress in these tasks, if any, do you feel you have made between then and now?

   (ii) Can you identify any misunderstandings you may have had in that first attempt?

   (iii) Do you have any ideas for anything that could or did aid you in correcting these misunderstandings you had?

Appendix D

Questions for Advanced Graduate Student Interviews

1. Go through each task and discuss your thoughts about a solution to each task. Feel free to act as though you are teaching or explaining the ideas to me and that I do not have much background knowledge. Also feel free to use the white board.

   (a) What do you feel is necessary for introductory Topology students to know in order to accomplish this task?

2. Can you talk for a moment about your background in Topology? What is your research and how is it related to Topology?

Appendix E

Questions for the Undergraduate Interview

1. I have a definition sheet here for your use. If you have notes on these concepts, you are welcome to use them as well. I have four tasks for you to consider in any order you want. Read each task and try to work through them and explain what you are doing as if I am your peer.

   (i) If a task was not successfully completed, intervene as if in a tutoring session.

   (ii) Any questions about this problem?

   (iii) (If a task was successfully completed) What do you feel was necessary for you to know in order to accomplish this task?

2. What is your experience with topology? What classes have you taken and how have you felt about them?

3. Has there been anything that you have found helpful in learning topology?

Appendix F

Questions for the Topologist Interview

1. Go through each task and discuss your thoughts about a solution to each task. Feel free to act as though you are explaining the ideas to me and that I do not have much background knowledge. Also feel free to use the white board.

2. What do you feel is necessary for introductory Topology students to know in order to accomplish this task?

3. Can you talk for a moment about your background in Topology? What is your research and how is it related to Topology?