A DEVELOPMENT OF MATERIALS TO BE USED IN A LABORATORY APPROACH TO A MATHEMATICS CONTENT COURSE FOR PRE-SERVICE ELEMENTARY TEACHERS AND THE EFFECTS OF THIS APPROACH ON ACHIEVEMENT AND ATTITUDE

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CHAPTER I

INTRODUCTION

Background of the Study

The Influence of Jean Piaget on Learning Theory

How does man learn? In spite of the "knowledge explosion" of our century, this question has yet to be definitively answered. Theories of learning abound, however, and one which has currently received much attention is based on the studies of the Swiss epistemologist, Jean Piaget. Piaget claims the essence of knowledge is operation. "To know an object, to know an event, is not simply to look at it and make a mental copy or image of it. To know an object is to act on it" (23, p.176). Operation must be reversible; that is, it must go in both directions as, for example, joining and separating, adding and subtracting. Furthermore, an operation does not exist in isolation, but is linked to other operations to constitute a structure. According to Piaget, it is these operational structures that form the basis of knowledge.

Piagetian research indicates that the development of these structures in the normal person proceeds in four distinct stages or levels. These stages always occur in the same order but continue for varying lengths of time in different individuals. They are the sensory-motor level, pre-operational level, concrete operational level and formal operational level.

The sensory-motor level is a pre-verbal stage which begins at birth and continues for approximately eighteen months. During this stage the child learns very basic things about his physical environment: for example, an object does not cease to exist because it is out of sight; if an object is pushed, it will move. This practical knowledge forms the basis for the representational knowledge of later stages.

In the pre-operational stage, which usually lasts from eighteen months to about six or seven years of age, the child learns language, which serves a symbolic function and therefore requires thought or representation. However at this stage he lacks the quality of reversibility in this thought. For example, if a liquid is poured from a tall narrow container into a short wide one, the child will say there is more in the tall narrow vessel even if he has watched the pouring. He accepts his perception of the situation: since the tall container looks as if it contains more, he believes it does. He is not yet capable of realizing the relevance of the fact that the pouring can be reversed. As a result of this absence of reversibility, the child also does not understand conservation of quantity. It does not bother him that according to his perception some of the liquid must have been destroyed in the pouring. Since reversibility is essential to an operation, this stage is called pre-operational.

In the third stage, the individual is able to perform intellectual operations but these operations are on objects rather than verbally expressed hypotheses. Thus this stage is called concrete operational. In the final stage the individual can operate intellectually with hypotheses, and not only on objects. He can construct new operations of propositional logic, and can develop new more complicated structures.

This stage is called the formal operational level.

Although Piaget originally felt that most people attain the formal operational level between the ages of eleven and fifteen, a recent article indicates that he now considers the following as the most probable of several possible alternatives: "all normal subjects attain the stage of formal operations or structuring if not between 11-12 to 14-15 years, in any case between 15 and 20 years. However, they reach this stage in different areas according to their aptitudes and their professional specilizations" (24, p.10). This implies the possibility of some people never attaining the formal operational level in areas where they have neither aptitude nor professional training. In particular, there can be adults of normal intelligence who are not at the formal level in, say, mathematics, although they may be quite capable of advanced thought in other fields.

Piaget's theory goes further and claims that until an individual has attained a given level of development, it is impossible for him to operate at that level (23). This has important implications for education. If it is true, instruction on the formal operational level for an individual whose level of intellectual development is at the concrete operational stage can only result in frustration for both pupil and teacher.

The Mathematics Laboratory

Long before the formulation of this theory, educators had attempted to help students by letting them work with concrete objects as well as with ideas. In mathematics education pedagogy, this philosophy is often put into practice by means of a mathematics laboratory. The current popularity of this method can be traced back to the early 1960's when the so-called "new Math" emerged in response to the needs of a technological society for better trained scientists and professional mathematicians. This "new Math" was, for the most part, quite abstract and, although logically correct and aesthetically pleasing, "constituted" according to Robert Davis, "an entirely inadequate response to these needs" (9, p. 4).

This inadequacy seemed to be not in the curriculum itself, which was well thought out by many of the best mathematicians and educators in the country, but in the method in which it was taught. For the most part, the material was presented at the formal operational level, a level with which its creators were quite comfortable but which was incomprehensible not only to the majority of elementary and secondary students but also to many of their teachers. The mathematics laboratory approach has come into the educational spotlight as an attempt to meet students of all ages where they are intellectually, which for many of them is the concrete operational level. It is hoped that this approach will form a more adequate response to the still pressing need for better mathematics education.

The concept of a mathematics laboratory goes back to the eighteenth and nineteenth centuries when progressives like Frederick Foebel, Heinrich Pestalozzi, and Jean Rosseau made a belief in the principle of learning by doing the basis for demanding drastic changes in the educational structures of the schools. Although changes were slow in coming, the seed was sown, and between 1890 and 1900 John Perry succeeded in changing England's technical schools so that mathematics instruction was based on experimentation and laboratory work. His efforts, together with those of John Dewey on a more general scale, provided an impetus to United State educators to begin thinking along similar lines.

Mathematics education literature prior to 1950 contains several articles discussing the use of manipulative materials in learning mathematics. In England, the Nuffield Project provided a model for large scale implementation of such an approach to the learning of mathematics while the Madison Project, headed by Robert Davis, was one of the first formal programs in this country which advocated active learning of mathematics in an informal atmosphere (30).

A mathematics laboratory is both an approach to learning and a physical environment for learning. The approach to learning has been described as a "spirit of inquiry" which prevails in both students and teachers.

> This spirit of inquiry so fundamental to the laboratory approach, involves students in a "learning by doing" situation. It provides an informal climate for learning that creates and maintains exploration and discovery. It also provides an atmosphere conducive to problem solving and a place where real problems at varying levels of sophistication can be solved by pupils (30, p. 10).

The physical environment is some space set aside in which are available various aids for "learning by doing". These aids are frequently manipulative materials, objects which can be handled by learners and which embody mathematical concepts. These manipulatives can be made or purchased at a relatively low cost.

As a result of these theoretical and practical factors, mathematics laboratories have grown in popularity on the elementary level over the past five years. Although research on their effectiveness has been limited, there are indications that students can learn mathematical ideas from a laboraory setting as well as they do in a traditional one.

Also, it appears that student attitudes toward mathematics are better as a result, or, at least, that they prefer the laboratory approach to the more traditional method (37).

The Role of Attitude in Mathematics

The relationship of attitudes toward and achievement in mathematics has been studied in some depth (2). In general, there seems to be a reciprocal influence, that is, not only does achievement help determine attitude, but attitude affects achievement. A student's innate mathematical talent and his achievement level will both affect his attitude toward the subject. So also will the attitudes of those around him: his parents, his peers and his teachers. Of these the most important factor appears to be the teacher attitudes toward the subject.

This prompted Lewis Aiken (2, p. 232) to recommend improvement of teacher attitude as a means of building more positive attitudes on the part of students. Thus the attitudes of elementary teachers toward mathematics are relevant not only to their own mathematical progress but also to that of their students, present and future.

Institutional Background

Xavier University is a small (approximately 1800 students), private, predominantly Black institution situated in New Orleans, Louisiana. The majority of students attending Xavier University are from the state of Louisiana and for the most part have graduated from predominantly Black high schools. Their mathematical backgrounds range from poor to excellent; however entrance examination scores on Xavier Mathematics Placement tests as well as ACT mathematics scores are low for a large percentage of them.

In 1972-73, Xavier University received a grant from the National Science Foundation to establish a mathematics laboratory. Materials were purchased to aid in remediation for the poorer student and to provide stimulation to original thinking for the better student. At the present time, the laboratory is equipped with terminals attached to an IBM 1130 computer, a large programmable calculator, the Wang 2200, several smaller Wang programmable calculators, hand-held calculators, games and models. There are also study carrels equipped with slide projector-cassette recorder combinations where students can make use of the many commercial remedial lessons availble. The facility has been fairly well used in the two years of its existence, serving mainly as a tutorial center. Few of the mathematics faculty, however, have capitalized on its full potential for effective teaching of existing courses. For the most part, this seems to be caused by a lack of knowledge of how to integrate the course content meaningfully with available physical apparatus.

Mathematics for Elementary School Teachers, Math 105, is the first course of the two-semester mathematics content sequence required of all elementary education majors and is intended to fulfill the Level I Recomendations of the Committee on the Undergraduate Program in Mathematics (8). There are two main reasons why Math 105 seems to be an appropriate mathematics course in which to initiate a mathematics laboratory approach.

First the mathematics laboratory approach is currently popular in the elementary school and probably will continue to be in the future. It has been frequently voiced by various educators (3, 21) that teachers tend to teach not as their methods courses recommend but as they themselves have been taught. Thus, a laboratory approach to the mathematics content course should furnish the future teacher with a pattern for mathematics teaching. The UNESCO publication New Trends in Mathematics Teaching urges "Trainers of teachers should themselves put into practice what they recommend or describe as possibly valuable methods or techniques" (36, p. 110).

The second reason is that both pre-service and in-service elementary teachers tend to perform poorly in mathematics, although they usually are people of normal intelligence (10). This fact viewed in the light of Piaget's theory makes it seem likely that a good number of them have not yet reached the formal operational level with respect to mathematics. "A concrete operational thinker does not become formal operational by constantly being confronted with formal operational tasks" (20, p. 558), but by meeting situations at the concrete level which challenge his ability to move to higher levels. Thus a course taught using a variety of concrete operational activities is more likely to be conducive to the mathematical growth of these students than a course taught in the traditional manner.

Problem Definition

Statement of the Problem

As a pilot project, Math 105 was taught using a mathematics laboratory approach at Xavier University of Louisiana during the fall semester of the 1975-76 school year by the researcher.

The developmental aspect of this study consisted of preparing activities which involved the use of manipulative materials and which were integrated into Math 105. The content of the course was chosen according to the most recent guidelines of the Committee on the Undergraduate Program in Mathematics. As an adjunct to these activities and as an aid to the student, although not a part of the study itself, lecture notes, definition sets and exercise sets were also prepared.

The experimental aspect of the study consisted of assessing achievement and attitudinal changes of the students enrolled in Math 105. Achievement was measured by means of an achievement test developed by the researcher for Math 105. Attitude was measured by an attitude scale developed by Aichele (1). This scale is designed to assess student attitudes in four areas: the learning of mathematics, mathematics as a process, the place of mathematics in society, and school and learning generally.

Hypotheses

The hypotheses are arranged according to the areas described in the preceding section. For purposes of statistical analysis, the hypotheses are reported in the null form.

H_1: There is no significant difference between initial performance of students in Math 105 on the Achievement Test and final performance of students in Math 105 on the Achievement Test.

H_2: There is no significant difference between initial attitudes of students in Math 105 concerning the learning of mathematics and the terminal attitudes of students in Math 105 concerning the learning of mathematics.

H_3: There is no significant difference between initial attitudes of students in Math 105 concerning mathematics as a process and terminal attitudes of students in Math 105 concerning mathematics as a process.

H_4: There is no significant difference between initial attitudes of students in Math 105 concerning the place of mathematics in society and terminal attitudes of students in Math 105 concerning the place of mathematics in society.

H_5: There is no significant difference between initial attitudes of students in Math 105 concerning school and learning generally and terminal attitudes of students in Math 105 concerning school and learning generally.

Importance of the Study

Low mathematics achievement of elementary teachers (10) and the obvious influence which they have on the future of mathematics education make it imperative to investigate avenues of possible improvement of their preparation. One such avenue is the mathematics laboratory approach which does not assume that the students are able, in Piagetian terms, to operate on a formal logical level. This study is important since it adds new data to previous research on the subject. It is particularly valuable since it is conducted with minority students in a predominantly Black institution.

The staff of the Mathematics Department at Xavier University has been striving to make the best use of its existing facilities, especially the Mathematics Laboratory. This study is important to them since it investigates the value of the laboratory in course work as opposed to that in a remedial or tutorial program. It is also important in this aspect to other colleges of similar size and population which are considering the initiation of mathematics laboratories.

Assumptions

It is assumed in this study that errors in grading students are normally distributed throughout the sample. An assumption is also made that students having highly favorable attitudes toward mathematics are more likely to agree with statements having highly favorable scale values than they are with statements which do not. Similarly, students with less favorable attitudes toward mathematics are more likely to endorse statements scaled near their own positions than they are statements having highly favorable scaled values.

The results of the Achievement Test are assumed to be a reliable index of a student's mathematical knowledge. The results of the Attitude Scale are assumed to be a reliable index of a student's attitude toward mathematics.

Limitations

This study is limited to those students enrolled in the course, Mathematics for Elementary School Teachers, Math 105, during the fall semester of the 1975-76 academic year at Xavier University of Louisiana who complete the initial Attitude Scale, terminal Attitude Scale, initial Achievement Test and final Achievement Test. This study is also restricted to the extent that assessed attitude reflects true attitude and to the extent that scores on the Achievement Test reflect true achievement.

Definitions

Mathematics Laboratory

A mathematics laboratory is both an approach to learning and a physical environment. The approach to learning is an informal one where a spirit of inquiry must prevail in both pupils and teachers and which involves students in a "learning by doing" situation. The physical environment is some space set aside in which are available various aids for this "learning by doing."

Mathematics for Elementary School Teachers

Mathematics for Elementary School Teachers is a three hour mathematics content course required of all elementary education majors at Xavier University. Throughout the study Mathematics for Elementary School Teachers will be referred to as Math 105. In the fall semester of the 1975-76 school year the material of the course was divided into four sections: Geometry; Natural Numbers, Whole Numbers and Integers; Number Theory; and the Rational Number System.

Activities

Activities are materials developed by the researcher and integrated into Math 105. They are designed to be performed in small groups and make use of the type of manipulative materials frequently found in mathmatics laboratories.

Achievement Test

The Achievement Test is an examination constructed by the researcher covering the material taught in Math 105. It was administered to each student enrolled in Math 105 during the first week of the semester as a diagnostic test and again during the last week of the semester as the final examination of the course.

Intermediate Achievement Tests

The intermediate achievement tests are four examinations constructed by the researcher which were given at the end of each section of Math 105 immediately after completion of that section.

Attitude Scale

The Attitude Scale (1) is an instrument which contains statements reflecting attitudes toward mathematics. Each item has a scale value which is used in determining the intensity of feeling regarding the situation described in the statement. The total score is the median value of those statements with which the student agrees. The attitudes of each student enrolled in Math 105 were assessed during the first week of the semester and again during the final week of the semester.

Intermediate Attitude Tests

The intermediate attitude tests are open ended instruments to which students were asked to respond four times during the semester. Each asks for the affective reactions of the student to the activities connected with the section of the course just completed.

Overview

This study is divided into five chapters. Chapter I serves as an introduction, giving the background of the study as well as the problem definition. A survey of selected related research is presented in Chapter II. The research procedures are discussed in Chapter III. This discussion is composed of two parts: the developmental aspect of the study and the experimental aspect of the study. Chapter IV gives the results of the study, again in two parts. The results of the developmental aspect, the activities used in Math 105, are followed by the results of the experimental aspect, evaluation of achievement and attitudinal changes on the part of students in Math 105. Chapter V contains the summary, conclusions and implications related to this investigation and some suggestions for further study.

CHAPTER II

RELATED LITERATURE

The purpose of this chapter is to review selected research findings related to the present study. This research is divided into three main areas: the presence of the formal operational level in normal adults; the relation of Piagetian concepts to achievement in minority situations; and effects of a mathematics laboratory approach on attitude and achievement of pre-service elementary teachers.

The Presence of the Formal Operational Level in Normal Adults

The notion of using a laboratory approach with college age students is based on the belief that a good number of these students are not yet at the formal operational level at least with respect to mathematics. Research which indicates that not all people of normal intelligence move to the formal level during early adolescence has caused Piaget himself to modify his theory. His newer ideas are discussed below, as well as research by C. Tomlinson-Keasey and K. Lovell regarding the age at which transfer from concrete operational to formal operational thought takes place. A related matter of interest also studied by Tomlinson-Keasey and Lovell is whether or not training is effective in aiding in this transfer.

Modification of Piaget's Theory

As indicated in Chapter I, Piaget originally theorized that people of normal intelligence reach the formal operational level between the ages of eleven and fifteen years. The experiments which led to this conclusion were all conducted on upper middle class children chosen from prestigious secondary schools in Geneva, Switzerland. When these experiments were repeated on other less privileged populations, however, the results were not the same. Many subjects older than fifteen years did not exhibit formal operational thought, although they were able to function quite adequately in their society.

Faced with this data, Piaget hypothesized three possible explanations of which he considers the last the most probable (24). The first is that although all individuals go through the same sequence of stages, they proceed at different rates of speed. This explanation implies the possibility that in extremely disadvantageous situations formal thought may never develop.

The second explanation is that the ability to construct formal thought structures is a talent or an ability which some people possess and others do not. Those who do possess it begin much the same as those who do not, i.e., they begin with concrete operational thought structures, but develop their talent as they mature. This is analogous to, say, artistic talent. The work of a great artist and an ordinary person might be indistinguishable in early childhood but, as time progresses, the talent of the artist matures and the difference in quality becomes more apparent.

The third hypothesis, which Piaget himself accepts as the most plausible, is that all people attain the formal operational level in some but not necessarily in every area or discipline. Areas in which they attain the formal level are determined by their particular aptitudes and professional specializations. Although they may not be expressly trained to think on a formal level, their experience nurtures the transition from concrete to formal thought in those areas. In other disciplines, however, lack of experience may require them to operate on a concrete level.

A Study by C. Tomlinson-Keasey

C. Tomlinson-Keasey (35) of the University of Nebraska designed a study to determine the level of cognitive development of three age groups of women. She also investigated whether training acted as a facilitator in the acquisition of formal operations.

The experiments were performed on a total of eighty-nine sixth grade girls, college coeds, and women with mean ages of 11.9, 19.7 and 54 respectively. To determine their level of cognitive development, they were asked to solve the pendulum, balance and flexibility problems as described by Inhelder and Piaget (16). Then an experimental group consisting of twenty-four subjects from each age level were given a short-term training experience on the same three problems. This was followed immediately by a repetition of the test. A week later all eighty-nine were given another test which contained a different version of the flexibility problem, as well as the chemical and inclined plane problems of Piaget and Inhelder.

The results of the first test showed that 32% of the sixth grade girls' responses were at the formal level with 4% at the most advanced stage. Of the coeds' responses, 67% were at the formal level with 23% on the highest level, while of the women's responses 54% were at the formal level with 17% at the most advanced stage.

The training procedures led to significant increases from the first test to the immediate post-test for all three groups. On the two new tasks of the delayed post-test, however, there was no significant difference between the experimental subjects who had received training and the control group who did not. On the repeated task of the delayed post-test, the experimental groups of girls and coeds differed significantly from their control counterparts.

The results of the experiment indicate that the attainment of formal thought is not as wide-spread as one might believe and that the attainment of the highest level might even be called rare. The presence of the formal level in an individual seems to be dependent on available mental structures, experiences, use, and, according to Tomlinson-Keasey, even preference. Training, however, does not seem to be a factor. Although it is useful in paeticular problems, it does not generalize so that a complete transition from concrete to formal thought takes place.

A Study by K. Lovell

K. Lovell (22) of the Institute of Education at the University of Leeds in England undertook a follow-up study of the work described by Inhelder and Piaget in The Growth of Logical Thinking (16). Ten of the fifteen experiments described by them were repeated under more controlled conditions. The population consisted of thirty-four average and bright primary school pupils; fourteen average and bright preparatory school pupils (eight to eleven years of age); thirty-nine grammar school pupils; fifty secondary school pupils; fifty comprehensive school pupils; ten training college students; and three able adults whose ages ranged from twenty-five to thirty-two years of age. This made two hundred subjects in all. Approximately equal numbers were drawn from the higher and lower ability sections in the comprehensive and secondary school groups. Each subject was examined individually by means of four experiments using a semi-structured procedure, i.e., the experimenter sometimes asked supplementary questions, or prompted, or performed the experiment slightly differently if he thought it would be helpful. As a result of the experiments, Lovell made a number of conclusions and observations.

The main stages in the development of logical thinking proposed by Inhelder and Piaget were confirmed. However, the least able of the secondary and comprehensive school pupils were at a low level of logical thought even at fifteen years of age, and many of them did not seem to progress beyond the concrete operational level of thinking. By getting each subject to participate in four experiments and analyzing the results by means of a non-parametric statistical technique, Lovell was able to conclude that there was considerable agreement between the levels of thinking that subjects displayed in the four experiments. The majority of the subjects showed much the same kind of reasoning as those of Inhelder and Piaget. Thus the study supported many of the statements made by them.

Teaching, in the sense of instruction, did not seem to affect the results as much as had been expected. The value of teaching seemed greatest when the required thinking skills were already, or nearly, available to the subject. If the power to think at the requisite level was not present, knowledge gained by instruction was either forgotten or remained rote knowledge. Although instruction itself did not seem to influence greatly the development of thinking skills, it appears from this study that the atmosphere of the classroom and the way that the teacher poses and discusses problems, can be of great consequence in the development of thought processes.

The Relation of Piagetian Concepts to Achievement in Minority Situations

It is hoped by many educators that the insights provided by Piagetian research can be helpful in promoting growth in academic achievement. This is particularly true at institutions like Xavier University whose students come from a population whose academic achievement has been dramatically below the norm for the general population (7). The sociological and historical reasons for this are well known and will not be discussed here. But two studies investigating the relation of Piagetian concepts to achievement in minority situations are discussed below.

A Study with College Students

Ronald J. Raven, Arthur J. Hannah and Rodney Doran (28) conducted a study, the major purpose of which was to investigate the relationships between Piaget's logical operations and achievement in the physical and biological sciences. Subjects in this study were 123 black freshmen (average age, 19.4 years) enrolled in a required introductory physical science course at Clark College, Atlanta, Georgia. All of the students were high school graduates, primarily from Georgia and other southeastern states.

Science achievement was measured by two tests: the Earth Science Curriculum Project "Test of Science Knowledge" and the Biological Science Curriculum "Comprehensive Final Examination." The level of cognitive development was determined by "Raven's Test of Logical Operations" (RTLO). This test measures cognitive attainment at the pre-operational, concrete operational, and formal operational stages. It is divided into seven subtests, each of which measures a different logical operation.

The results of the RTLO were correlated with achievement measures. The correlation indicated that "the operative structures defined by Jean Piaget and used on the RTLO can provide the basis for determining if students will have problems with certain types of science content" (28, p. 567). This result encourages the belief that achievement can be improved by giving consideration to the Piagetian level of a student's thought structure.

A Study with Ninth-Grade Mathematics Students

Emma Jane Dixon Johanson (18) of the University of Toledo conducted a study on a mathematics curriculum for ninth-graders. This curriculum was based on Piagetian concepts in the sense that it did not assume that the students were thinking on a formal operational level. Use of manipulative materials designed to help the learners acquire mathematical concepts and skills was built into the course. The curriculum was designed so that the students worked in pairs or small groups, as an encouragement to discussion of relevant ideas.

The curriculum was tested in an inner-city class which was ninety-five percent black and where the vast majority of the students had very low achievement records in mathematics. A control group was taught using a traditional approach. At the end of the nine-week period covered by the curriculum both groups were tested using standardized achievement tests. In addition, attitudes toward mathematics in both groups were measured by means of the Prouse Preference Survey. In both achievement and attitude, the experimental group scored higher than the control group.

Effects of a Mathematics Laboratory Approach on Attitudes and Achievement of Pre-Service Elementary Teachers

Although mathematics laboratories have been used fairly extensively in the last several years on the elementary and secondary levels, relatively little has been done on the college level. Most college work in mathematics laboratories, as they are defined in this study, has been with pre-service elementary teachers. Two studies investigating the effects of such an approach on these students are discussed below.

A Study at the University of Chicago

Karen Fuson (13) conducted a study with a threefold aim: 1) development of mathematics curriculum material designed for pre-service elementary teachers and emphasizing the use of manipulative materials; 2) creation or adaptation of instruments and techniques for evaluating teacher learning of such materials; 3) examination of the effects of such teacher learning in several areas. Of these three aims, discussion in this chapter will be limited to the third.

The course under consideration was a combination mathematics content and methods course consisting of twenty sessions of two and a half hours each. It was offered at the University of Chicago in the fall quarter, 1971. The sixteen students who took the course were studying for a Master of Science in Teaching degree and were college graduates. For the most part they were higher in IQ and verbal ability than most pre-service elementary teachers, but there was a considerable range in their preparation, ability and attitude with respect to mathematics.

Among the effects of this course as listed by Fuson (13) are the following:

1. The students expressed an increased desire to use manipulative materials in teaching mathematics.

2. They actually increased their ability to demonstrate mathematical concepts with such materials.

3. They used manipulative materials to a considerable extent in practice teaching.

4. The students expressed an increased desire to teach mathematics in learner-focused ways, and videotaped teaching sessions indicated that they actually did increase the extent to which they taught in learner-focused ways.

5. Although the measure is imprecise, students both believed they improved and actually seemed to improve in their understanding of elementary mathematical concepts.

6. The students showed significant mean changes toward more positive attitudes to teaching mathematics, both with respect to enjoying the teaching and with respect to a feeling of competence.

Fuson felt that one of the most important effects of the study was one that could not be measured. This type of learning seemed to bring moments of intense involvement and great excitement. The verbal structure of mathematical knowledge was given concrete references which made it more meaningful for the students. This seemed to change student beliefs and behavior in the direction of increased use of manipulatives in mathematics teaching.

A Study at Arizona State University

Gary Warkentin (41) in a study at Arizona State University in 1972 investigated the viability of a laboratory/models approach to the mathematics content course for pre-service elementary teachers as an alternative to the traditional lecture/textbook method. Attitudes and achievement of the students in an experimental group were compared with those of the students in a control group. The attitudes of all students were measured before and after the course to determine if any change took place in either group. Attitude was determined by a twenty item instrument which measured the students' liking or disliking of mathematics, their tendency to engage in or avoid mathematical activity, their belief that they had or did not have mathematical ability, and their belief that mathematics is useful or useless. Achievement was determined by the comprehensive final examination over the content of the textbook used by the control group. In addition, all subjects took the arithmetic section of the advanced level Stanford Achievement Test prior to the course to determine whether there was any initial significant difference between the experimental and control group.

The subjects were students enrolled in the theory of arithmetic course at Arizona State University during the fall semester of 1972. Six sections with a total of 149 students were taught with the laboratory/models approach, and nine sections with 197 students served as the control group. Both groups met for fifty minutes three times a week. The same subject matter, which included the usual topics of the course in the theory of arithmetic, was considered by both groups. However the control sections were able to cover more content than the laboratory sections in the same amount of time.

There was no initial significant difference between the control and the experimental groups for either the Standord Achievement or the attitude tests. A comparison of the mean scores of the two groups on the attitude posttest, however, indicated that there was a difference between them at the .01 level of significance with the experimental group having the more favorable attitude. A comparison of attitude change during the semester indicated that there was a significant positive change in the experimental group but none in the control group. However, with respect to achievement, the results were reversed. There was a significant difference at the .01 level here favoring the control group. This may be explained in part by the fact that there were some items on the examination that were covered by the control sections that were not covered in the laboratory.

Summary

The research discussed in this chapter covered three areas chosen for their relevance to the present study. The areas are: the presence of the formal operational level in normal adults; the relation of Piagetian concepts to achievement in minority situations; and effects of a mathematics laboratory approach on attitude and achievement of pre-service elementary teachers.

C. Tomlinson-Keasey (35) found that attainment of the highest stage of formal operations was rare in a group of adolescent girls and adult women. K. Lovell (22) confirmed the existence of the stages of development described by Inhelder and Piaget (16), although he found evidence that the transition from concrete operational to formal operational does not always take place at the age determined by Inhelder and Piaget.

Both Tomlinson-Keasey and Lovell found that formal thought operations cannot be directly taught. Lovell indicated, however, that the atmosphere of the classroom and conduct of the teacher is an aid to its attainment.

Raven, Hannah and Doran (28) found that the presence or lack of formal operational thought in black students was a predictor of the level of success in a college science course. Johanson (18) found that a Piagetian mathematics curriculum was instrumental in improving attitude and achievement of a low-achieving ninth grade class that was ninety-five percent black.

Research by Fuson (13) indicated that a combined mathematics method and content course making use of manipulative materials contributed to the improvement of attitude and achievement of pre-service elementary teachers. Warkentin (41) found that a laboratory/models approach to the mathematics content course for pre-service elementary teachers resulted in better attitudes but lower achievement than the traditional lecture/textbook method.

CHAPTER III

RESEARCH PROCEDURES

The study was divided into two main parts, the developmental and the experimental. Included in the developmental aspect were the selections of course content, of the format of the written materials and of the manipulative materials to be used as well as the writing of the materials. The experimental section consisted of the actual teaching of the course using a mathematics laboratory approach, the evaluation of growth in mathematics achievement, and the evaluation of changes in attitude on the part of the students with respect to mathematics.

Developmental Aspect of the Study

Selection of Course Content

The Committee on the Undergraduate Program in Mathematics (CUPM) is a committee of the Mathematical Association of America whose purpose is to work for the improvement of college and university mathematics curricula. Created in the early days of the so-called "new math revolution", CUPM issued its first set of recommendations for the mathematics training of teachers in 1961. Five levels of teachers were considered including teachers of elementary school mathematics (grades K to 6), referred to in the report as Level I teachers.

The recommendations of CUPM with respect to training of teachers were well received and put into practice in many undergraduate institutions around the country. In fact, by 1966, only 8.1% of colleges having programs for training elementary teachers required no mathematics courses as opposed to 22.7% in 1962, while those which required five or more semester hours of mathematics rose from 31.8% to 51.5% in the same period (8, p. 2). In spite of this success, the increasing pace of change not only in elementary school curricula but also in pedagogical methods caused CUPM to make a review of their recommendations; an updated report was issued in 1971.

These new recommendations deal mainly with course content although other aspects of teacher training are considered briefly. In particular, it is urged that future teachers "be led to regard mathematics as a creative activity — something which one <u>does</u> rather than merely something which one learns" and that courses "be taught in ways that foster active student involvement" (8, p. 20). In the same vein, the committee suggest

> It must be kept in mind that a prospective teacher is profoundly influenced by what he observes and experiences as a student. Later his own methods and philosophy of teaching will reflect that experience. Hence it is of paramount importance that these courses be conducted in a manner which encourages active participation in mathematical discovery. Frequent and substantial assignments which expose and drive home the attendant manipulative and computational skills should also be the rule (8, p. 24).

The crux of the report with respect to Level I pre-service teachers lies in the recommendations for course content. Two sequences of four 3 semester hour courses are outlined; the same mathematical content is included in both sequences although the approaches are different. The first sequence does not emphasize any single area of mathematics in a particular course but integrates them throughout; the second sequence stresses a different mathematical field in each course. The topics developed in each sequence include number systems, algebra, geometry, probability, statistics, functions, mathematical systems, and the role of deduction and induction in mathematics.

Sequence 1 consists of the four courses: Number and Geometry with Applications I and II, and Mathematical Systems with Applications I and II. Throughout this sequence, arithmetic and geometry are presented in such a way that the interaction of the two areas illustrates the unity of mathematics. To some extent, this sequence utilizes the spiral approach, i.e., certain ideas appear several times, each time on a deeper level than that of the preceding presentation. The members of CUPM see this as an advantage for students who will not be able to take all four of the courses, since at the end of the first two courses, they will have been exposed to most of the important concepts although not in the desired depth.

Sequence 2 consists of the following four courses: Number Systems and their Origins; Geometry, Measurement and Probability; Mathematical Systems; Functions. In spite of the fact that each course emphasizes a particular area, topics from most of the other areas are covered as well. Thus, in this sequence as well as in the first, a student who completes only two of the courses will have covered the entire range of topics, although not in the depth the committee recommended.

The committee proposed two distinct sequences to make clear that there are many ways of organizing the recommended material. Whether one chooses Sequence 1 or 2, or decides on yet another arrangement, is largely a matter of preference. For the most part, the course developed for Math 105 follows the outline of Course I, Sequence 1, Number and

Geometry with Applications I.

This course, as envisioned by CUPM, is divided into three main sections: Elementary Ideas of Space, Measurement and the Number Line; The Rational Number System and Subsystems; Probability, Statistics and Other Applications. Because of the amount of material to be covered, it was decided by the researcher to regroup it into five parts: Elementary Ideas of Space, Measurement and the Number Line; Natural Numbers, Whole Numbers and Integers; Number Theory; The Rational Number System; Probability, Statistics and Other Applications. Activities as well as supplementary lecture notes, sets of relevant terms, exercise sets and assignments were developed for all five parts. The progress of the course as it was actually taught, however, made it evident that it would be impossible to cover that much material adequately. Therefore, the last section on Probability, Statistics and Other Applications was postponed until the spring semester and covered in the next course of the sequence, Math 106. This course is also required of all elementary education majors at Xavier University. Further discussion in this report of the material and activities will be limited to the first four sections.

Format of Written Materials

The activities used in Math 105 were developed in the summer of 1975. Each one is designed to make use of manipulative materials in a laboratory setting and thus to implement the "learning by doing" philosophy. The activities are unevenly distributed throughout the four parts of the course, since some topics, e.g., geometry, are more conducive to such an approach than others, e.g., number theory. Thus, Part I contains sixteen activities and Part II contains twelve activities while Parts III and IV have only four and eight activities, respectively.

Each of the forty activities is introduced by a list of materials needed for its performance. Following the materials list is a series of directions for experiments, modelings, or discussions, as well as questions relating to the activities. Since an essential ingredient of the activities is the interplay of ideas between students, the activities are all designed to be performed in small group situations. An opportunity for discussion of pertinent concepts with the instructor is also built into the activities to guard against the danger that the performance of the activities will become mechanical and that the students will fail to see their significance.

As these activities were being developed, the need for supplemental written materials became apparent. Since only part of the content of the course was covered by the activities, it seemed unfair, as well as unreasonable, to expect students to depend on notetaking for the other portion. Therefore, lecture notes and sets of relevant terms were prepared. These are not considered a part of the developmental aspect of the study itself, although necessary for an efficient conducting of the course. Thus they are not contained in the body of this report.

Although many of the presentations of the course are made on the concrete operational level, the object of this approach is to help students move to the formal operational level. Thus, large portions of the achievement tests are abstract. As an aid in the necessary transition, exercise sets and assignments were developed to follow each topic. These sections contain problems similar to those found in traditional texts designed for a course of this nature. As with the lecture notes and sets of relevant terms, they are not included in this report.

Manipulative Materials

There are many manipulative materials available commercially for mathematics laboratory experiences. In addition, ordinary objects can frequently be used to embody mathematical ideas. Such ordinary objects that were chosen for use in the activities for Math 105 included pipe cleaners, pins, cardboard, string, pencils, straws, popsicle sticks, tissue paper, tracing paper, rulers, graph paper, erasers, computer cards, paperclips, colored pens, and dice. The content of the course determined to a large extent which materials could and should be used.

Some commercially available objects, which are not manipulatives in the strict sense of the word, were also chosen. These included a set of models of geometric solids and a set of models of plane geometric figures. The latter are referred to in the activities as "Geometric Figures". A globe made of blackboard slate was also used and was helpful in the sections involving spherical geometry.

In addition, the following commercial manipulative materials were used: geostrips, geoboard, Mira, Multi-base blocks, colored chips, Cuisenaire rods, mosaic tiles, hundreds board, up to twenty blocks, fraction circles, and peg board. Each will be described below.

Geostrips

Geostrips (Figure 1) were designed in England. They are strips of heavy weight plastic with from two to nine evenly spaced holes which can be joined together by brass paper fasteners to form polygons. A special protractor is also provided so that it is possible to make accurate figures.

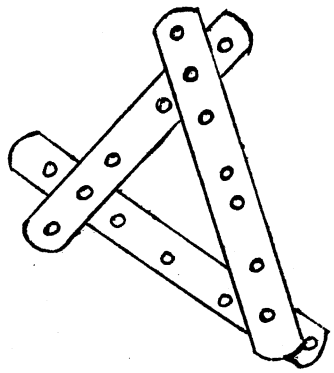


Figure 1. Geostrips

Geoboard

One of the most frequently used manipulative materials is the geoboard, available in either plastic or wood and in many sizes. Rubber bands can be stretched on pegs lined in evenly spaced rows to form various geometric figures. Figure 2 shows a 5 x 5 geoboard on which are displayed a square, rectangle and right triangle.

Mira

Mira (Figure 3) is a transparent plastic device which has the reflective characteristics of a mirror. This combination of transparency and reflection make it useful in studying such topics as transformations and symmetry. Small and light-weight, it is intended for individual use.

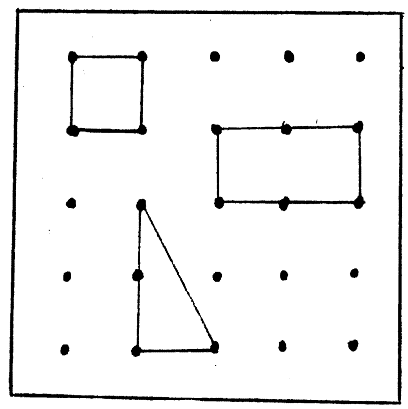


Figure 2. Square Geoboard

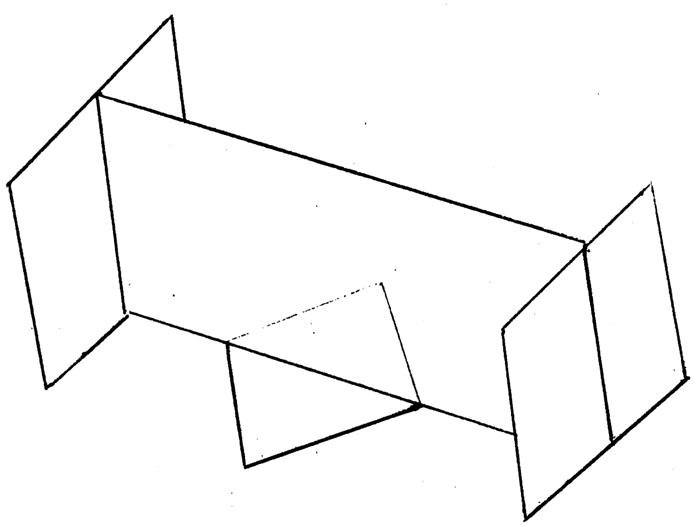


Figure 3. Mira

<u>Multi-base Blocks</u>

Developed by Zoltan Dienes, multi-base blocks are designed as a concrete representation of the place value number system. Any number may be chosen as the base; however, the sets used in Math 105 were restricted to base two, three, four, five, six and ten. For any given base, n, there are four shapes: a small cube called a unit; a rectangular block of width one unit and length n units called a long; a square n x n block called a flat; and a large cube of dimensions n x n x n called a block. Figure 4 shows a set of base three multi-base blocks.

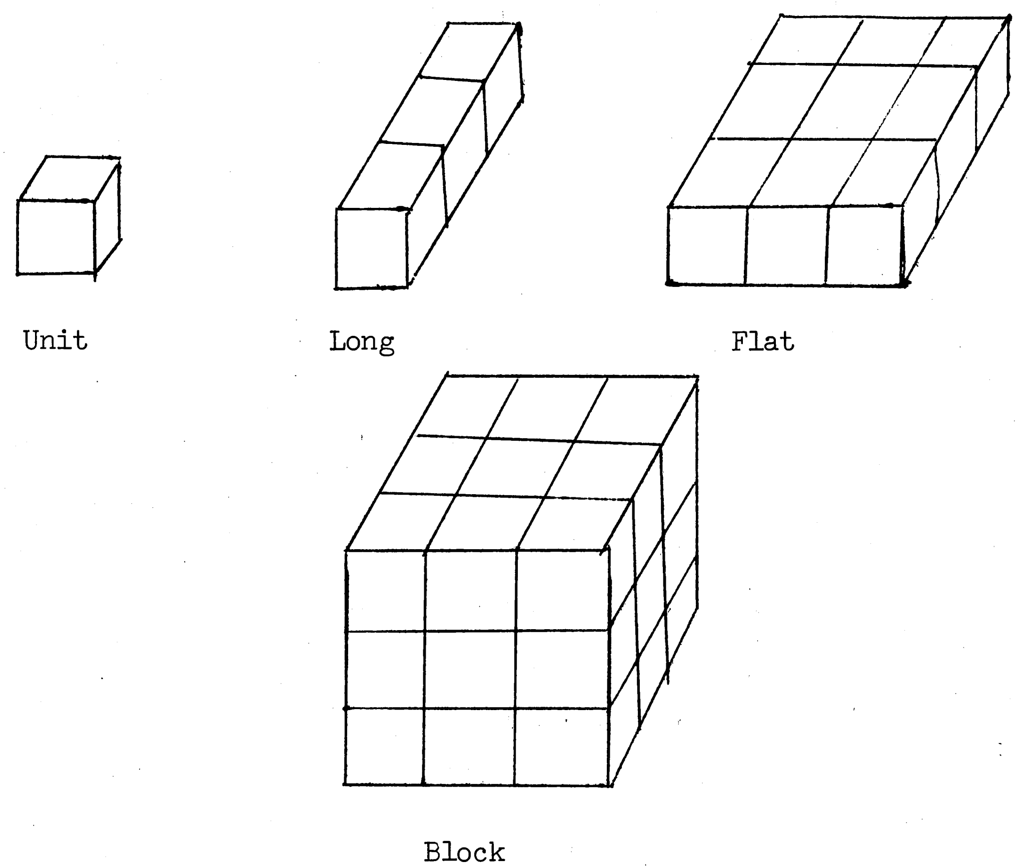


Figure 4. Base three multi-base blocks

Colored Chips

Colored chips are round plastic chips of various colors useful in counting problems and for demonstrating properties of whole numbers.

Cuisenaire Rods

Cuisenaire rods use length as a model for number and can be very effective in illustrating various number relationships. They were developed in Belguim by M. Georges Cuisenaire, an educator and musician, and have been widely accepted by elementary mathematics teachers. The rods range in length from 1 cm to 10 cm, with each rod a different color (Figure 5).

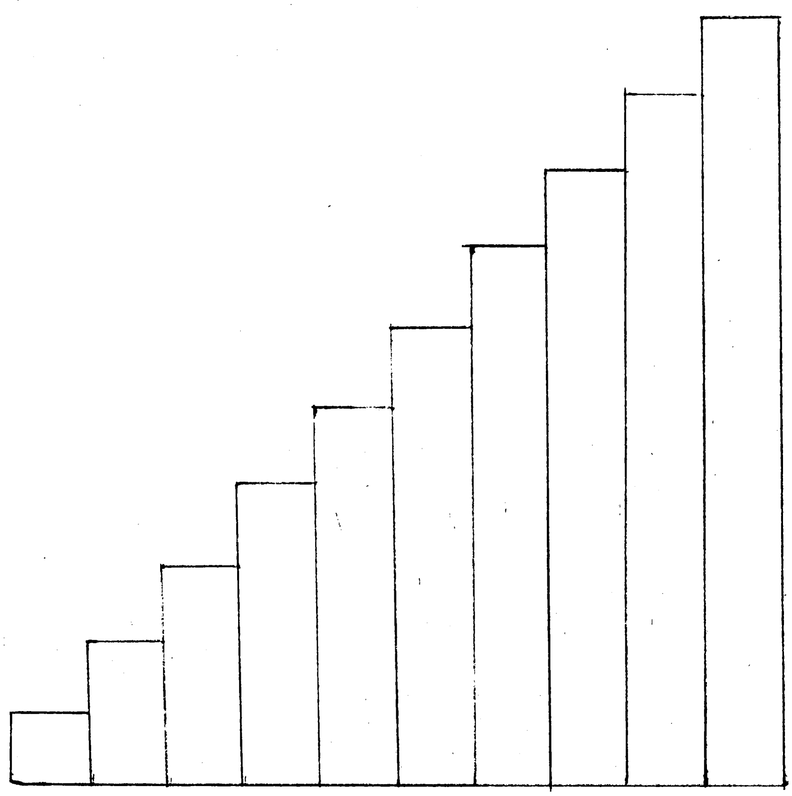


Figure 5. Cuisenaire rods

Mosaic Tiles

Mosaic tiles are small plastic squares of various colors. Approximately 2 cm x 2 cm, they can be useful in arithmetic and geometry.

Hundreds Board

A strong plastic square, approximately 1 ft. x 1 ft., the hundreds board is divided into 100 squares, ten rows of ten each. Beginning at the upper left hand corner and moving to the right, each square is numbered consecutively from one to one hundred. Small square chips which fit exactly in the numbered spaces are provided. Of these, one hundred are also numbered consecutively and twenty-five are unmarked. A hundreds board is shown in Figure 6.

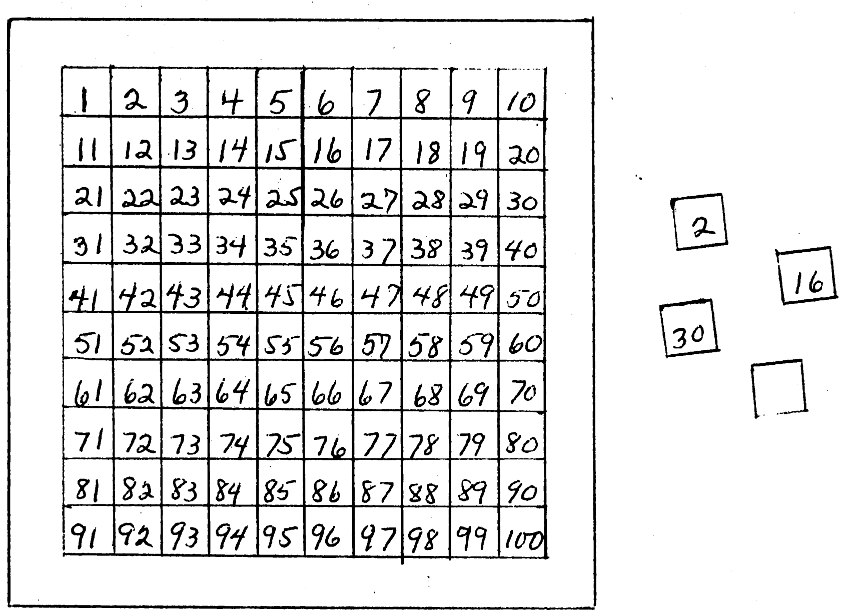


Figure 6. Hundreds Board

Up to Twenty Blocks

Up to twenty blocks are similar to Cuisenaire rods in that they use length to represent number. They differ in the fact that they contain twenty lengths rather than ten and in the way color is used. All powers of a number are the same color, e.g., the rods representing 2,4,8 and 16 are all yellow.

Fraction Circles

Fraction circles are a set of six congruent plastic circles. One is whole and the others are divided into halves, thirds, fourths, sixths, or eighths.

Pegboard

A pegboard can be of any shape and of any durable material. The one chosen for Math 105 was plastic with 100 holes arranged in a 10 x 10 array. Pegs of various colors fit into the holes.

Experimental Aspect of the Study

Teaching of the Course

Math 105 was taught in two sections by the researcher in the fall semester of the 1975-76 school year. The first section was taught at 10 o'clock in the morning and the second at 3 o'clock in the afternoon. Of the fifty-one students who completed the course, thirty-two were in Section 1 and nineteen were in Section 2.

All of the fifty-one students were high school graduates and fifty were enrolled as elementary education majors at Xavier University. The other was an elementary education major at Dominican College in New Orleans. Dominican and Xavier University are members of the New Orleans Consortium, an association which permits students at any member institution to take courses at any other participating school. There were forty-one women and ten men involved. Eight of the men and one of the women attended Xavier under the Career Opportunities Program (COP). COP is designed to train residents of urban neighborhoods as elementary teachers for those same neighborhoods. Forty of the students were residents of New Orleans; the remaining eleven came from cities of Louisiana outside of New Orleans or from other states including Illinois, Maryland, Alabama, Mississippi, and Missouri. Forty-eight of the students were black; three white.

Many of the classes were held in the Xavier University Mathematics Laboratory. The room used was equipped with six tables, each seating a maximum of six students. Although there were two blackboards in the area, they were so placed that it was difficult for all of the students to see them at the same time. Therefore, on days when no activities were scheduled, class was held in a traditional classroom.

The students were asked on the first day of class to place themselves in small groups for activity work and to remain in those same groups throughout the course. Since the groups were self-chosen, for the most part the students in them worked well together. A disadvantage of the self-selection procedure was that in at least one of the groups all of the members were quite slow. Consequently, that group needed more external guidance than the others.

Each laboratory period was introduced with an explanation of the day's agenda by the instructor. In the first weeks of the semester these introductions were quite short, consisting only of an announcement of the work to be covered and a demonstration of the manipulative materials involved. However, as time progressed, a more detailed explanation of the activities required and of the type of results expected was seen to make the students feel more secure and able to progress further on their own. Since a laboratory approach to mathematics was a new experience for all of the students, this security seemed an important factor.

Following the introduction, each group began work on the activities. During this time the instructor moved from group to group asking questions, clarifying instructions, and discussing results. In the morning section, which was relatively large, it was difficult for the instructor to get to each group as much as some of the students wished. This was the biggest source of dissatisfaction and frustration on the part of the students, especially during the first half of the semester when the whole approach was unfamiliar to them. Later, they became more independent, but the instructor-student dialogue remained an important aspect of the course.

Each student kept a Lab Notebook in which he recorded the results of his work. These notebook s were collected and corrected by the instructor several times during the semester. In addition, exercise sets were assigned, collected and corrected regularly. Although there was no penalty for handing the assignments in late, the students were encouraged to keep abreast of them and to use them as practice immediately following the presentation of the material.

Evaluation of Growth in Mathematics Achievement

The achievement test which was administered at the beginning and the end of the course was constructed by the researcher. This test, based on 160 points, covered the four sections of the course equally; that is, 40 points each on Geometry, The Set of Integers and Its Subsets, Number Theory, and the Rational Number System. The test is given in Appendix A. As far as possible, physical conditions such as room size, lighting etc. were identical. A basic difference between the two testings, however, resided in the fact that the terminal test was the final examination of the course, for which the students had prepared, while the initial test was given without previous notice. Forty-seven of the fifty-one subjects took both tests.

The raw scores from the two administrations were compared by a t-test for paired data (5). The test was conducted as follows. The score of each student on the first administration of the test, X_{iA}, was paired with his score on the second administration, X_{iB}, where i ranged from 1 to 47. For each student, $D_i = X_{iB} - X_{iA}$, was computed and the mean difference, $\bar{D} = \frac{\sum D_i}{n}$ calculated. A standard deviation of the differences was computed according to the formula

$$S_D = \sqrt{\frac{\sum D_i^2 - \frac{\left(\sum D_i\right)^2}{n}}{n-1}}$$

The standard error, $S_{\bar{D}} = \frac{S_D}{\sqrt{n}}$, was also calculated and $t = \frac{\bar{D}}{S_{\bar{D}}}$. Because of the large number of cases, the normal distribution was used to approximate the t distribution.

In addition to the initial and terminal achievement tests, four intermediate tests were constructed by the researcher and administered at the end of each section of the course. These are found in Appendix B.

The final grade assigned to each student in the course was based on the scores of the four in-term tests, the final examination, and the homework assignments. The final examination had twice the weight of a single in-term test and the homework the same weight. The lowest score of the tests was dropped in computing the average. Letter grades were assigned according to the following scale:

87% - 100%	A
79% - 86%	B
67% - 78%	C
56% - 66%	D
Below 56%	F

Evaluation of Changes in Attitude toward Mathematics

The Attitude Scale used was developed, tested and validated at the University of Missouri, Columbia, in 1968 by D.B. Aichele (1). It contains twenty-seven statements which were derived from similar statements developed in the International Study of Achievement in Mathematics and which are subdivided into four categories. These are: views concerning the learning of mathematics; views concerning mathematics as a process; views concerning the place of mathematics in society; and views concerning school and learning generally. Each statement has a numerical value scaled from least favorable to most favorable, where high positive scale values are associated with favorable statements. The Attitude Scale is given in Appendix C.

The Attitude Scale was administered to the students in Math 105 during the first meeting of the class and again during the last meeting prior to the final examination. Of the fifty-one students who completed the course, forty-one responded on both the initial and terminal attitude scale. For each student, an attitude score for each of the four areas was calculated by finding the median of the scale values with which the student agreed. This served as an index of the intensity of the student's feelings toward each of the four categories.

Hypotheses H_2, H_3, H_4 and H_5 were tested by means of a Sign Test for the comparison of medians and a McNemar Test for significance of item change.

For the Sign Test, the initial attitude score of each student was matched with his terminal attitude score on each of the four sections. If the terminal score was higher than the initial one, the student was assigned a plus, if lower, a minus and if equal, a zero. The distribution of differences was then tested to see if any change which occured could be regarded as significant. N was taken as the total number of signs that changed. If $N \leq 10$, the null hypothesis was tested by use of the binomial distribution; otherwise, by the normal approximation to the binomial distribution. In that case the z value for the number of pluses (X) is given by

$$z = \frac{(X + .5) - .5N}{\sqrt{N(.25)}}$$

where $X + .5$ is used if $X < .5N$ and $X - .5$ if $X > .5N$ (5, p. 187).

The McNemar Test for significance of change was used to analyze each item on the Attitude Scale for significant changes between the initial and terminal testings. The fourfold table below was used to categorize the responses to each item.

TABLE I

FORM OF FOURFOLD TABLE OF FREQUENCIES USED IN TESTING SIGNIFICANCE OF CHANGES IN PERFORMANCE ON ATTITUDE SCALE

		Terminal Presentation	
		Agree	Disagree
Initial Presentation	Agree	B	A
	Disagree	D	C

Cells A and D represent the total number of students who changed responses to an item on the initial and terminal testings. The McNemar test is given according to the formula

$$\chi^2 = \frac{(A - D)^2}{A + D}$$

with one degree of freedom. This becomes more reliable if a correction for continuity is performed. This correction is necessary since the chi-square distribution is continuous while the sampling distribution is discrete. The Yates correction for continuity was thus used in the analysis and chi-square computed by

$$\chi^2 = \frac{(|A - D| - 1)^2}{A + D}$$

(32, p. 64). If the expected frequency, .5(A + D) was less than 5, the binomial test was used rather than the corrected McNemar Test.

In addition to the initial and terminal Attitude Scale, each student was asked to respond to an open-ended informal attitude evaluation instrument four times during the semester. These were administered following each of the four sections of the course and were similar to each other in construction. The first is given below.

> Please comment on how you _feel_ about the activities you performed for this unit. Which ones, if any, did you find helpful? Which ones were confusing? Have you enjoyed any of them? Which ones?

All of the intermediate attitude evaluation instruments are found in Appendix D.

CHAPTER IV

RESULTS OF THE STUDY

The results of the study are discussed in two parts. The first part reports the results of the developmental aspect of the study: the activities and games involving manipulative materials which formed an essential ingredient of Math 105. These activities and games are given below in the order in which they were used in the course and are grouped according to the four sections of the course. Some of them contain references to buildings and areas on the Xavier campus which were familiar to the students. The associated lecture notes, sets of relevant terms, assignments and exercise sets are indicated in the introductory sections.

The second part of this chapter considers the findings of the experimental aspect of the study. These are given in three areas: teaching of the course, growth in achievement as a result of the course, and changes of attitudes toward mathematics on the part of students enrolled in the course.

Activities Developed for Math 105

Part I: Elementary Ideas of Space, Measurement and the Number Line

History and Geometric Figures as Sets of Points

This section is covered by Lecture Notes 1, Relevant Terms - Set 1, and Exercise Set 1.

Curves

This section consists of Activities 1,2,3,4 and 5, Game 1, Relevant Terms - Set 2, and Exercise Set 2.

Activity 1. Materials: Pipe cleaners, paper.

A. By means of pipecleaners, make a representation of the path from the Math Lab Building to the middle of the Quadrangle. Place two dots, one on one side of the path and one on the other. Can you connect the dots without crossing the path? If it is possible to get from one side of a path to the other without crossing it, the path is called an open path or an open curve. Model two additional open curves. Is a line segment an open curve? Justify your answer.

B. Is it possible (not "convenient") to get from downtown New Orleans to Gretna without crossing the Mississippi River? Demonstrate by means of the pipe cleaners. What kind of curve does the Mississippi represent?

C. You have been asked to take a visitor on a tour of campus. You meet him at the Math Lab and are to make sure he returns to the Lab a half-hour later. With the pipe cleaners represent the path on which you will take him. Once again, place a dot on one side of the path and one on the other. Can you connect the dots without crossing the path? If it is impossible to get from one side of a path to the other without crossing it, the path is called a closed path or a closed curve. Model at least three other closed curves with the pipe cleaners.

D. It is possible for a path to cross itself. With the pipe cleaners, demonstrate both an open and a closed curve which cross themselves. Show them to the instructor. A curve which does not cross itself, except possibly at the endpoints, is called a simple curve. Most of the curves we will deal with in this course will be simple curves.

Activity 2. Materials: Pin, cardboard, string, pencil, globe.

A. Probably the most familiar simple closed curve is the circle. It is familiar because it is commonly used in many aspects of our lives. Name five non-mathematical objects which appear to be circular.

B. Tie one end of a string around a pencil and with a pin, secure the other end on a piece of cardboard. Pull the string taut, and keeping it that way, move the pencil through the only path it can follow. What figure have you drawn on the cardboard?

C. This brings us to the formal definition of a circle. A circle is a set of points all of which are equidistant from a fixed point within called the center. In B above, what represents the center of the circle? The distance from the center to any point of the circle is called the radius. In B above, what represents the radius?

D. Notice that the definition of a circle says that the center must be "within" the circle. We also know that every circle has exactly one center. In plane geometry, this is the only way it can be, but in spherical geometry, it is possible to find a set of points equidistant from two points, neither of which are "within" the circle in the usual sense of the word. Take the globe and see if you can find such a "circle". What are the two points that are the "centers"?

E. Only the simple closed curve itself is the circle, and so the center is not part of the circle. If one is speaking of the circle as well as all the space inside (called the interior), one is referring to the circular region. How many of the objects which you named in A above are circles and how many are circular regions? Make sure you can name at least three non-mathematical objects which are models of circles and three which are models of circular regions.

Activity 3. Materials: Geostrips.

A. A polygon is a simple, closed curve made up of line segments. Take two geostrips and try to form a polygon with them. (The connecting points of the geostrips are considered to be the endpoints of the line segments.) Having trouble? Take another and try with the three. What is the smallest number of geostrips you can use to form a polygon?

B. Make a triangle and also a four sided polygon (called a quadrilateral) with the geostrips. Can you change the shape of the triangle by pushing on its sides? How about the quadrilateral? This property of triangles, called rigidity or stability, has important applications in all kinds of construction. Why? Take the quadrilateral you just formed and try to make it stable by adding one more geostrip. However, do not change the number of sides of the polygon.

Can you do it? What did you really do to the quadrilateral? Show your work to the instructor before proceeding.

C. Take geostrips of length 1,2 and 4, and form a triangle with them. Everything OK? Try with geostrips of length 2,2 and 4. Did that work? How about 2,3 and 4? See if you can figure out what relationship must hold between the lengths of the segments so that it is possible to form a triangle. Discuss this in your group and share your thoughts with the instructor.

<u>Activity 4.</u> Materials: Geoboard, globe, chalk, cards for this activity, paper. (The cards contain the definitions of scalene, isosceles, equilateral, acute, right and obtuse triangles.)

A. Take the cards for this activity and form a model of each triangle on the geoboard. Try to get them all on the board at one time. Call the instructor over and tell her which model goes with each card. Be able to explain why you think you are correct.

B. On the geoboard, form as many of the following as you can. It is not possible to form all of them.

1. isosceles right triangle 2. scalene right triangle
3. equilateral right triangle 4. right obtuse triangle

Which ones were you able to form? Do you think you know why you were unable to form the others? The reason for your failure is the same in all cases. As a hint, fold a piece of paper in the shape of a triangle. Tear off the three angles and put them together, vertex to vertex. What do they form? Now with a piece of paper, form an obtuse angle and a right angle. (To get the right angle, form a line on the paper, and then fold the line onto itself. Figure out for yourself how to get the obtuse angle.) After you have formed the angles, tear them off and put them together vertex to vertex. What do they form? What fact about triangles does this illustrate?

C. Notice that the geoboard is a model of a plane. If, instead, we would deal with the surface of a sphere, it might be possible for us to form some of the figures which were impossible in the plane. Take the globe and chalk and see if you can make an equilateral right triangle and a right obtuse triangle. What can you say about the sum of the angles of a "triangle" on a sphere?

Activity 5. Materials: Geoboard, two sets of cards for this activity. (Set I contains definitions of a quadrilateral, parallelogram, trapezoid, isosceles trapezoid, rectangle, square, and rhombus. Set II contains the definitions of a pentagon, hexagon, octagon, decagon and n-gon.)

A. Take Set I of the cards for this activity and form a model for each on the geoboard. Try to get them all on the board at the same time. Call the instructor and tell her which model goes with each card. Be able to explain why you think you are correct.

B. Find one figure which is a quadrilateral, parallelogram, rectangle and rhombus all at the same time. Does it have a special name? What is it?

C. Take Set II of the cards for this activity and form a model for each on the geoboard. For the n-gon, let n = 7. Keep them on the board for use in D and E below.

D. A polygon is called regular if all of its sides and all of its angles are congruent to each other. Look at the models you made in C above. Which ones are regular and which irregular? Call the instructor and tell her what you think.

E. For each model you formed in C above, form another based on the following rule: if the polygon you formed originally was regular, make the new one irregular, and vice versa.

Game 1. Materials: Two decks of cards, one with pictures of various polygons and one with names of the various polygons. Number of players: two or more. Rules: Both decks of cards are shuffled, and the deck with names is put face down in the center. The deck with pictures is dealt evenly to the players. Any extra are put face up in the discard pile. The dealer turns up the top card in the name deck and the first player on his left must show and then place in the discard pile any cards in his hand which are pictures of the polygon named on the card. Any pictures which fulfill all conditions of the definition may be discarded even if they also fulfill other conditions as well. E.g., if a player is holding the picture of a square and the word "rectangle" is turned up, he may discard the square because every square is a rectangle. However, he may not discard a picture of a non-square rectangle if the word "square" is turned up. If he attempts to discard any card which is not a picture of the named polygon, he may be called by any other player and must then take the entire discard pile into his hand. After his play, each player turns up the top card on the name pile for the next player on his left. The first player to discard all of his cards wins.

Regions and Solids

This section consists of Activities 6 and 7, Relevant Terms - Set

3, and Exercise Set 3.

Activity 6. Materials: Pipe cleaners, geoboard.

A. With the pipe cleaners, map the outline of Xavier's Administration Building. (This map does not have to be exact, but should show the general design of the building.) Place a dot in the Religious Center and one at the point nearest the Math Lab. Join these points by a straight line. Does that line lie entirely inside the region? A region such as the one referred to above, i.e., where there exists two points such that the line segment joining them does not lie entirely inside the region is called a concave region.

B. With the pipe cleaners, model at least three other concave regions and prove they are concave. Show them to the instructor.

C. A region which is not concave is convex. Most of the regions we have considered thus far are convex regions. With the geoboard and/or the pipe cleaners, model at least three convex regions and demonstrate that they are convex.

Activity 7. Materials: Geometric figures, models for various solids.

A. Take two equilateral triangles from the geometric figures. Can you make a closed surface with these? Try three. Any better? How about four? What is the least number of regions that can be used to form a closed surface?

B. Just as there are regular polygons, there are also regular polyhedra. A polyhedron is regular if and only if all of its faces are congruent, regular polygons. There are only five regular polyhedra. Examine the models available in the Lab and pick out those that fit the definition of regular polyhedra. Show the instructor the ones you chose and explain the reasons for your choice.

Congruence and Similarity

This section consists of Activities 8,9,10,11,12,13 and 14, Relevant Terms - Set 4, and Exercise Set 4.

Activity 8. Materials: Tracing paper, regular paper, ruler.

A. On your regular paper, draw a simple open curve. Then copy the curve on the tracing paper. Now move the tracing paper approximately three inches to the left or to the right and recopy the figure. Remove the tracing paper and copy over the trace. Surely the two curves are the same size and shape, i.e., congruent, since the second is an exact replica of the first. Now take pencil and straightedge and join the corresponding points of the two figures. What kind of lines do you have?

B. On your regular paper, draw some simple closed curve - a polygon might make life easier for you, but any kind will do. Copy on the tracing paper as in A above, but this time move the paper in a northeast or southeast direction before you retrace the figure. Once again join the corresponding points of the two figures and notice what kind of lines you get.

C. On your paper, draw a triangle ABC. Using the tracing paper, draw a congruent triangle and label it A'B'C' so that A' corresponds to A, B' to B, and C' to C. The new triangle A'B'C' is called the image of the triangle ABC. Form $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$. As in A and B above, they appear to be parallel. Does $\overline{AA'}$ appear to be congruent to $\overline{BB'}$?

$\overline{CC'}$?

D. The transformation of one figure into another in such a way that the lines joining corresponding points are parallel and all points are moved the same distance in the same direction is called a translation or slide. Of course, one does not have to use tracing paper to construct a slide image of a figure. Use the conditions of the definition and no tracing paper to translate a rectangle ABCD to a congruent rectangle A'B'C'D' approximately two inches away from ABCD.

Activity 9. Materials: Paper, ruler.

A. Draw a figure, as elaborate or simple as you please, near the upper left hand corner of your paper. Translate it approximately three inches to the right. Now translate the image about six inches down. A series of translations, one following the other, is called a product of translations. Could you have obtained that same image by means of a single translation? Demonstrate.

B. Repeat the above experiment, but this time take your original figure through three translations. Does it seem that we can always say that the product of translations is another translation? Discuss this with the instructor.

Activity 10. Materials: Pipe cleaners, cardboard, pins.

A. On the cardboard, draw a non-simple open curve. Then place a congruent pipe cleaner version of the curve on top of it, fastening one end point with a pin. Now rotate the pipe cleaner curve on the cardboard through a quarter turn, i.e., about $\frac{1}{4}$ of a circle. Clearly, the curves are still congruent. This type of transformation is called a rotation.

B. On the cardboard draw a triangle and label its vertices A,B, and C. Make a congruent pipe cleaner triangle and label its vertices A', B', and C'. Pin A' to A so that B' corresponds to B and rotate triangle A'B'C' through a quarter turn. Trace A'B'C' on the paper. Replace the pipe cleaner triangle to its original position and try to slide it (without turning) so that it is in its new position. Can you do it? Does there seem to be any product of translations that would give you this image? Join A to A', B to B', and C to C'. Do the lines seem to be parallel? congruent? Now you can be more secure in your belief that a rotation is fundamentally different from a translation.

C. Take a triangle ABC through a product of rotations where the first rotation is a quarter turn and the second is a half turn. Could you have accomplished this by means of a single rotation? Take products of several rotations several times. Does there seem to be a pattern here? I.e., see if you can complete the sentence: The product of rotations is always a ______________________________.

Activity 11. Materials: Paper, Mira.

A. On your paper draw a triangle and label its vertices A,B and C. Place the Mira so that it is parallel to line segment $\overline{AB}$. The reflection is surely congruent to the original triangle. Quite naturally, this type of transformation is called a reflection.

B. Using the Mira, draw the image of triangle ABC under the reflection. Label the image of A, A'; of B, B'; and of C, C'. It will probably not surprise you to learn that a reflection is sometimes called a flip. Now remove the Mira and in its place draw a line, l, which we will call the mirror line. Draw $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. What relation does it seem that the segments $\overline{AA'}$, $\overline{BB'}$ and $\overline{CC'}$ have to the mirror line? Let M be the point where $\overline{AA'}$ crosses l. What is the relation between $\overline{AM}$ and $\overline{A'M}$? Consider the lines connecting other points of ABC with their images. In all cases, where do the midpoints lie? Check with the instructor before proceeding.

C. Draw an irregular quadrilateral and label its vertices A,B,C, and D. Now construct the image of ABCD in a line l parallel to $\overline{AB}$ applying the information you acquired in B above. Use the Mira to check your work, and label the points corresponding to A,B,C, and D as A', B', C' and D' respectively. Now repeat the process and find the image of A'B'C'D' in a line, m, parallel to $\overline{A'B'}$, but not on the same side of A'B'C'D' as line l. Name this figure EFGH. Could you have obtained this image from ABCD by some other transformation? What kind? Is it true to say that the product of reflections is always a reflection? Discuss this with the instructor before proceeding.

Activity 12. Materials: Mira, paper, ruler.

A. On your paper, print the capital letter A. Place the Mira on the letter in such a way that you can see the whole letter A, half in the Mira and half on the paper. We say that such a figure is symmetric with respect to a line, the mirror line.

B. Symmetry with respect to a line is very common in nature. Name five non-mathematical objects which are symmetric with respect to a line. Symmetrical designs are pleasing to the eye and thus symmetry is frequently used in art, architecture, landscaping and other types of decorating. Can you give some examples? Talk about this in your group and invite the instructor over and share your thoughts with her.

C. Go through the letters of the alphabet (printed capital letters) and find all those which are symmetrical with respect to a line. Show the line of symmetry in each case. It is possible that there is more than one line of symmetry, so if there is, show as many as you can find. Use the Mira if you find it helpful.

D. There is also another type of symmetry called symmetry with respect to a point. To see what this means, draw on your paper a polygon and place a point, P, somewhere off of the polygon, perhaps at a distance of about one inch. P will be the point of symmetry. Connect one vertex of the polygon with P. Note the distance between the two points and extend the line segment that same distance again. The end-point of the new segment is the image of the original point. Repeat this process with all the vertices of the original polygon. Connect the new points. The entire figure, i.e., the original and its image, is symmetric with respect to a point, the point P.

E. Go through the letters of the alphabet again. Can you find any which are symmetric with respect to a point? The point may be part of the letter, but it does not have to be.

Activity 13. Materials: Paper, geoboard, Mira.

A. On the geoboard, model the following figures:

1. scalene triangle 2. isosceles triangle 3. equilateral triangle

Look for lines of symmetry and use the Mira to check yourself.

B. Form a square on the geoboard and find all lines of symmetry. How many can you find? Make a table with a place for the name of the figure and one for the number of lines of symmetry. Fill in the table for a square, a non-square rectangle, a non-rectangular rhombus, a parallelogram, an isosceles trapezoid and a concave quadrilateral. Make sure the data on your table is true for every figure of the type under consideration. When you are finished, discuss the results with the instructor.

C. How many of the figures that you made in A and B above are symmetric with respect to a point? Add a column to your table for this property and, for each figure, fill the column with either "yes" or "no" depending on whether it is symmetric with respect to a point or not.

D. Perhaps the "most symmetrical" geometric figure is the circle. Why do you think the circle could be called the "most symmetrical"?

Activity 14. Materials: Geoboard, graph paper.

A. Compare your right hand with the right hands of others in the group. Think of a baby's right hand and of Bill Russell's right hand. What is the same and what is different about the right hands of people?

B. On the geoboard, form a right triangle whose legs have length 3 and 4, one whose legs have length 1 and 5, and one whose legs have length 6 and 8. Are any of the triangles congruent? Why or why not? Is there any way that two of the triangles are alike? When you have an answer, call the instructor and discuss it with her.

C. Figures which have the same shape but not necessarily the same size are called similar to each other. Are all rectangles similar? Give a reason for your answer and discuss your thinking with the instructor. Fill in the table below with "yes" if all such figures are similar and with "no" if it is possible that two such figures are not similar. Be able to demonstrate the truth of your answer if you say "no".

Figures	Similar
Rectangles	
Squares	
Triangles	
Equilateral Triangles	
Trapezoids	
Isosceles Trapezoids	

D. An important property of similar figures is that corresponding angles of the two figures are congruent, and all sets of corresponding segments are in the same proportion to each other. To see this, once again form on the geoboard a right triangle whose legs are of length 3 and 4. Measurement will probably convince you that the third side has length 5. Now construct a right triangle whose legs have length 6 and 8. Its third side has length 10. Notice that these triangles have the property of similar triangles given above, and they also fulfill the condition of the definition, i.e., they have the same shape but not necessarily the same size.

E. This property can be very useful in enlarging and reducing figures. Take a piece of graph paper and draw a polygon on it. Now redraw the figure so that it is twice as large as the original. The grid marks on the graph paper should be an aid in this. Make it three times as large, and then draw it so that it is half as large. All four of the polygons you have drawn should be similar to each other.

Are they?

Measurement and the Number Line

This section consists of Activity 15, Assignment 1, Activity 16, Lecture Notes 2, Relevant Terms - Set 5, and Exercise Set 5.

Activity 15. Materials: Pencil, eraser, computer card, paper clip.

A. With each of the materials above measure the length of the desk at which you are working. Make a table of the lengths according to the various measuring devices. Each device is called a unit of measure.

B. Have each one in your group measure the length of the desk using their hand as the unit of measure. Make a table with the lengths according to the various people.

C. Suppose you had to purchase a tablecloth for the desk you measured. Would any of the measuring work you've done so far help you? Discuss this in the group, and invite the instructor to listen to your discussion.

Activity 16. Materials: Cardboard unmarked straightedge, paper clips, colored pens, string or paper.

A. Place the paper clip at the end of the straightedge and using it as a unit, make the straightedge into a ruler, so that it only has units marked. Start at either end and number the marks consecutively. This numbering of the marks is called calibrating the straightedge.

B. With your new ruler, measure the length and width of this paper to the "nearest paper clip", i.e., if it does not come out evenly, take the number to which it is closest. How long is it? Does that mean it is exactly that long or only approximately? Can you give a range of values so that you feel sure the exact length is somewhere in that range? What would be the smallest range that you feel would contain the exact measurement? Discuss this and the reasons for your choice in your group and with the instructor.

C. Take a piece of string or a piece of folded paper the same length as the paper clip. Fold it so that the two end points are on top of each other. With this as a unit, and a different color from the one you originally used to make your ruler, calibrate the straightedge in terms of paper clips. What numbers should you put on your new marks?

D. Once again measure the length and width of this paper. What is its measurement to the "nearest half paper clip"? Is it exact? Is it closer to being exact than the first measurement?

E. Suppose you wished your approximation to be even closer to the exact value than the one you obtained in D above. What would be your next logical step? Could you continue this indefinitely? Discuss this with the instructor.

Part II: Natural Numbers (N), Whole Numbers (W), and Integers (Z)

Natural Numbers and Whole Numbers

This section is composed of Relevant Terms - Set 6, Exercise Set 6, Lecture Notes 3, Relevant Terms - Set 7, and Exercise Set 7.

Place-Value Systems

This section consists of Games 2 and 3, Activity 17, Game 4, Activities 18 and 19, Assignment 2, Activity 20, Assignment 3, Relevant Terms

- Set 8, and Exercise Set 8.

Game 2. Materials: Multi-base blocks, dice. Players: two or more. Object: To obtain a block or a flat according to the table below, where n is the base of the blocks being used.

n	Number of Dice	Object of Game
2,3	1	Block
4,5	2	Block
6,7	1	Flat
8,9,10	2	Flat

Rules: "Bank" has supply of units, longs, flats and blocks. Each player, in turn, throws the dice and takes from the bank the number of units shown on the top of the dice. Whenever it is possible to trade for a larger piece, it must be done during the current turn. To signal that one has finished a turn, the dice are passed to the next player. If all trades are not made, any opponent can "call" the player and may take any pieces which have not been changed. For example, suppose the game is being played in base five and Player One has acquired 3 flats, 4 longs, and 3 units. Throwing the dice he obtains a 3 and takes 3 units, giving him 6 units. He exchanges 5 of these for a long and passes the dice to the next player. During the subsequent play and only then, any other player may call him for failing to change the 5 longs for a flat, make the required change and take the flat. If the original player has failed to make any trade, any other player can call him, exchange the 5 units for a long, then the 5 longs for a flat and take the flat. The first player to obtain the goal piece wins.

Game 3. Materials: Multi-base blocks, dice, special cards for the game. Players: Two or more. Object: To form models for each number on the card. Rules: Each player takes a card at the beginning of play which he does not show to his fellow players. The card contains representation of several numbers given in terms of units, longs, and flats. Each player in turn throws the dice and takes the number of units shown on the top of the dice, in an attempt to form models of the representations indicated on his card. These models may be built in any order and it is permissable to build several simultaneously. As in Game 2, trading up must be done, except when a player needs more than the base of a particular block. For example, if he is required to represent

1. 2 longs, 3 units
2. 4 longs, 3 units
3. 3 flats, 2 longs, 4 units

he may keep up to 10 units and 8 longs, no matter what base is being used. Each player's blocks must be kept in the view of the other players, and if he fails to trade up, he may be challenged by any other player. He then must show his card. If he needs the disputed blocks for his models, he may take the equivalent number of blocks from the challenger, but if he does not, he must give them up to him. The first player to form exact models, with no extra blocks, wins. As the game progresses, a roll of the dice might give a player more blocks than he needs. In that case, he must pass and try for the exact number required on subsequent throws.

It is conceivable that some players will feel that this game is unfair, since one player may receive a card requiring three representations and another four. Therefore, when the first game is ended, each player should take from the bank what he needs to form all of his numbers. Trading up should then be done.

Activity 17. Materials: Multi-base blocks (bases four, five and six).

A. Take the base five blocks and set out 3 flats, 2 longs and 4 units. An easier way to write out the name of this representation is to omit the words (which always follow the same order) and simply write 324. Set out 3 flats and 1 unit. This can be written 301 where 0 indicates that there are no longs.

B. Represent 322 with the base four, the base five and the base six blocks. Does 322 stand for the same number of units in each of these cases? A way to eliminate possible confusion about the meaning of 322 is to put the base as a subscript to the number, e.g., 322_{four}, 322_{five}, or 322_{six}. Custom dictates that if no subscript is written, ten is assumed.

C. In working with base five, how many different numerals will be used in writing number names? What are they? Why are no more needed? Discuss this in your group and with the instructor. Answer the same questions for bases four, six, two, nine and ten.

Game 4 (optional). Materials: Multi-base blocks, dice, cards containing numbers written in a given base, e.g., 133_{four}. Players: Two or more. Rules: At the beginning of play, a card is drawn. Each player attempts to form the representation of the number on that card. In turn, each player throws the dice and either takes or gives back to the bank the number of units shown on the dice, whichever is most appropriate to get the chosen number. Rules for trading up are the same as in

Games 2 and 3. No player may ever have more blocks than are needed for the number. E.g., if the number is 133_{four}, no player may keep more than 3 units.

<u>Activity 18.</u> Materials: Multi-base blocks (any base), paper.

A. You have been working with units, longs, flats and blocks in various bases. In this activity each group will concentrate on one base, but what we will do applies to every other base as well. Take all the blocks (i.e., the representations of 1000) of a given base. If you had a great many of these, it would be convenient to put them in groups. What would be a good way to do this in keeping with the patterns we have developed so far? What kind of new piece could be formed with the group? What might be a good name for the piece? What would be the number name for it? Discuss this with the instructor before proceeding.

B. Now imagine that you have a large collection of the objects you described in A above. What would be a good way to group them? Sketch a picture of one such group. What relation does it have to the object you formed in A above? What might be a good name for it? What would be its number name?

C. Could we continue a process like this? How long?

<u>Activity 19.</u> Materials: Multi-base blocks (all bases).

A. Take a long from each base and write its number name. What is the same and what is different about the names? What does this reflect about the longs themselves?

B. Take 2 blocks, 4 flats, 3 longs and 1 unit from the base six blocks. We know that we can write this name as 2431_{six}. The 3 stands for 3 longs, and in A above we gave the name 10_{six} to the long. Thus 3 longs can be written as 30_{six} or as $3 \times 10_{six}$. What are six longs? As well as a flat, or 100_{six}, we could simply translate six longs as $10_{six} \times 10_{six}$ or 10^2_{six}. We can proceed like this for all the places in the number. It is not hard to see now that we could write

$$2431_{six} = 2 \times 10^3_{six} + 4 \times 10^2_{six} + 3 \times 10_{six} + 1$$

This is called the <u>expanded form</u> of the number.

C. Write each of the following numbers in expanded form. Be sure to indicate the base. If it helps you to make a model for the number with the blocks, be sure to do so.

1041_{five}	100101_{two}	213_{four}
2120_{four}	4932	5041_{six}

<u>Activity 20.</u> Materials: Popsicle sticks, cards, paper. In Assignment 2 we formalized a method of changing a number written in a base other than ten to a base ten number. In this activity, we will work in the other direction, i.e., we will find a method of writing a base ten number in any other base. This is a little sticky, so get ready to think.

A. Suppose you wished to write 43 in base six notation. As a first step make a representation of 43 (remember 43 is a base ten number) using the popsicle sticks. Since you wish to use base six in the new representation, group those same sticks in groups of six. Now you have seven groups of 10_{six} and one stick left over. Is this as simple as it can be? No, since six of the seven groups of 10_{six} can be grouped to form a group of 100_{six} with one group of 10_{six} left over. Thus 43 is the same as $1 \times 10^2_{six} + 1 \times 10_{six} + 1$ or 111_{six}.

B. What you did with the popsicle sticks can be done mentally by means of repeated division. That is,

$$\begin{array}{r}
7 \\
6\overline{)43} \\
\underline{42} \\
1
\end{array}$$

Then the quotient 7 represents the number of groups of 10_{six} and the remainder 1 represents the number of units. Since there are enough groups of 10_{six} to regroup, we repeat.

$$\begin{array}{r}
1 \\
6\overline{)7} \\
\underline{6} \\
1
\end{array}$$

showing that we get 1 group of 10^2_{six} (represented by the quotient) with one group of 10_{six} (represented by the remainder) left over.

C. Take the cards and use the directions of Steps A and B as guides to change the numbers to the indicated base. Try, by the time you get to the third problem, to get by with step B only, i.e., do not use the sticks unless you have to.

<u>Properties of Operations in W</u>

This section is made up of Activities 21,22, 23, and 24, Relevant Terms - Set 9, Assignment 4 and Exercise Set 9.

<u>Activity 21.</u> Materials: Colored chips, Cuisenaire rods.

A. Form a set containing five black chips and one containing three green chips. Are the sets disjoint? Form the union of the sets by joining the three green to the five black. What addition problem have you represented?

B. Now form two sets, one containing three red chips and the other five blue chips. Are the sets disjoint? Form the union of the sets by joining the five blue chips to the three red. What addition problem have you represented?

C. Compare the sets formed in A and B above. Do they have the same cardinality? What can you say about the two sums?

D. If it is always true that the order in which an operation is performed does not affect the outcome, we say that the operation is <u>commutative</u>. The above activity gives a clue that addition is commutative. Now we will use the Cuisenaire rods to look for further clues. Take a red rod and place a dark green one next to it. What addition problem does it represent? Take a dark green rod and place a red one next to it. What addition problem does it represent? Are the two equivalent? Is there a single rod which could replace either set? Does $2 + 6 = 6 + 2$?

E. Although we have not <u>proven</u> that addition is commutative (this is beyond the scope of this course) are you willing to accept it as true? If not, repeat the above activities using different numbers until you are satisfied that for all whole numbers a and b, $a + b = b + a$.

F. Take a set consisting of ten red chips. Remove three of the chips. What subtraction problem is represented? Now take a set of three yellow chips. Remove ten of the chips. (Remember it is not always possible to do as one wishes!) What subtraction problem does it represent? Does it have an answer in the set of whole numbers? Is subtraction commutative in the set of whole numbers?

<u>Activity 22.</u> Materials: Mosaic tiles, paper.

A. With the tiles form a rectangular array to represent the problem 3×5 and one to represent 5×3. Are the two arrays congruent? What geometrical transformation will take the first into the second?

Does $3 \times 5 = 5 \times 3$?

B. Use the Cartesian product model to represent 4×2 and 2×4. Are the resulting sets equivalent? Does $2 \times 4 = 4 \times 2$?

C. State the Commutative Law for Multiplication of Whole Numbers.

D. Take six pennies and separate them into groups of two each. How many such groups are there? What arithmetic problem does this represent? Now take two pennies and try to group them in groups of six each. Can this be done? What arithmetic problem would it represent? Can it be done in the set of whole numbers? Do you think division is commutative in the set of whole numbers? Does this example prove that you are right or only give you a clue? Discuss this in your group and then with the instructor.

Activity 23. Materials: Colored chips, Cuisenaire rods.

A. Mary, Jim and Sue attend the movies together. During the first half of the film, Jim puts his arm around Mary but during the second half around Sue. Half-way through a change has occured, but not in the order in which they are sitting — only in the way they are associated. A similar situation can occur in mathematics. To see this, form six disjoint sets, two with five red chips, two with four blue chips and two with seven green chips. Now unite one set of blue chips with one of red. Write down the addition problem it represents. Consider this as a new set. Put parentheses around the additon problem you wrote to show that it represents that whole set. Now unite one set of green chips with your newly formed set. Write the addition problem this union represents. Keep this set and the problem you just wrote while you proceed with the activity. Unite the remaining green chips with the set of blue chips and write the problem it represents. Unite this new set with the last set of red chips and write the problem it represents, being careful to remember to place parentheses where needed. Compare the two resulting sets. Are they equivalent? Does $(5 + 4) + 7 = 5 + (4 + 7)$?

B. Take the Cuisenaire rods and represent $(2 + 1) + 3$ as well as $2 + (1 + 3)$. Does the end result look different? When was the only time there was a difference in what you had? Discuss this in your group and with the instructor.

C. You are probably now ready to accept as true the Associative Law of Addition of Whole Numbers, i.e., for all whole numbers a,b, and c, $(a + b) + c = a + (b + c)$. Write five examples of this law and show that it is true for those cases. E.g.,

$$\begin{aligned}(a + b) + c &= (2 + 3) + 6 \\ &= 5 + 6 \\ &= 11 \\ a + (b + c) &= 2 + (3 + 6) \\ &= 2 + 9 \\ &= 11\end{aligned}$$

D. The question arises as to whether all operations are associative. Let's investigate subtraction. Form six sets of chips as follows: two sets of seven blue chips, two sets of four red chips and two sets of two green chips. First remove from one set of red chips a set equivalent to the set of green chips, i.e., represent the problem $4 - 2$. Now remove from one of the sets of blue chips, a set equivalent to the set with $(4 - 2)$ chips. What subtraction problem have you represented and what is your answer? Record it for future reference. Now from the other set of blue chips remove a set equivalent to the remaining set of red chips, i.e., represent 7 - 4. From this set remove a set equivalent to the remaining set of green chips. What subtraction problem have you represented and what is your answer? Is subtraction associative, i.e., is it true that for <u>all</u> whole numbers a,b, and c, $(a - b) - c = a - (b - c)$? Can you prove your opinion?

E. On your own, using either sets of chips or Cartesian products, decide whether multiplication and division are associative. When you have made a decision and have a reason for it, call the instructor and discuss it with her.

<u>Activity 24.</u> Materials: Mosaic Tiles.

A. With the tiles, form a rectangular array to represent 6 x 3 and one to represent 6 x 4. Unite the two sets, retaining the rectangular shape. You now have a representation of (6 x 3) + (6 x 4). Can you express this sum as a single multiplication problem? What is it? Repeat with rectangular arrays for 4 x 2 and 4 x 7, 2 x 6 and 2 x 2, 3 x 5 and 3 x 8.

B. You have just demonstrated one of the most basic laws of the mathematics of whole numbers. It combines the two operations of addition and multiplication and is called the Distributive Law of Multiplication over Addition. Formally stated, it says: for all whole numbers a,b, and c, $a(b + c) = ab + ac$. Give five examples with numbers to show the truth of the Law, e.g.,

$$\begin{aligned} a(b + c) &= 3(6 + 8) \\ &= 3(14) \\ &= 42 \end{aligned}$$

$$ab + ac = 3(6) + 3(8)$$
$$= 18 + 24$$
$$= 42$$

Algorithms for Operations in W

This section is composed of Activity 25, Assignment 5, Activities 26 and 27, Assignment 6, Activity 28 and Exercise Set 10.

Activity 25. Materials: Multi-base blocks (all bases including ten), paper.

A. The purpose of this part of the activity is to help you think about the algorithm, or rule, for adding numbers, and in particular, to further your understanding of regrouping, which is sometimes called "carrying". You are to form the representations of each number with the Multi-base blocks, then by combining and trading for larger blocks where possible, you are to form the model for the sum. Be sure you have made all possible trades. On your paper, write the number that corresponds to your final representation.

21_{six}	23_{five}	93	232_{five}	113_{four}
$\underline{32_{six}}$	$\underline{42_{five}}$	$\underline{28}$	$\underline{423_{five}}$	$\underline{303_{four}}$

B. Put the blocks aside and with your answer paper covered, add the numbers again. Check these answers with the ones you obtained with the blocks.

C. Regrouping is an important concept in subtraction also, where it is sometimes called "borrowing". To model subtraction with the Multi-base blocks, form representations of both numbers. Then from the representation of the minuend, remove the blocks that represent the subtrahend. It might be necessary to "trade down" the minuend to do this. On your paper, write the number that corresponds to your final representation.

401_{six}	10011_{two}	3658	1141_{five}	321_{four}
-43_{six}	-1001_{two}	-793	-324_{five}	-33_{four}

D. Put the blocks aside and with your answer paper covered, subtract the numbers again. Check your answers with the ones you obtained with the blocks.

Activity 26. Materials: Multi-base blocks, paper.

A. Recall that multiplication is repeated addition. Take the base five set of blocks, and by means of addition, represent the following problems. You may wish to use a tally to keep track of how many times you have added. Be sure to make all possible trades. Write the numbers corresponding to your final representations on your paper.

$23_{five} \times 1_{five}$ $23_{five} \times 2_{five}$ $23_{five} \times 3_{five}$ $23_{five} \times 4_{five}$

B. Now consider $23_{five} \times 10_{five}$. To find the answer, you may wish to think of 10_{five} as five units. Notice that the representation of 23_{five} contains 2 longs and 3 units. What does the representation of $23_{five} \times 10_{five}$ contain? Does there seem to be a relationship between them?

C. Perform the following operations and keep a record of the representations of the first number and the answer.

$123_{five} \times 10_{five}$ $324_{five} \times 10_{five}$ $132_{four} \times 10_{four}$

$513_{six} \times 10_{six}$

Can you see a pattern?

D. With the blocks, now represent $23_{five} \times 100_{five}$. To find the answer you may wish to think of 100_{five} as five 10_{five}'s. As in B above, compare the representation of the answer with the representation of 23_{five}. Does there seem to be a relationship between them?

E. Work the following problems and keep a record of the representations of the first number and the answer. If you think you see what will happen, you do not need to use the blocks.

$123_{five} \times 100_{five}$ $\quad$ $34_{six} \times 100_{six}$ $\quad$ $20_{four} \times 100_{four}$

$121_{three} \times 100_{three}$

F. Do you think you could write a rule for multiplying by powers of the base in any place value system? Go ahead and try. Check with the instructor when you think you have one.

<u>Activity 27.</u> Materials: Multi-base blocks, paper.

A. In Activity 26, you formed $23_{five} \times 10_{five}$. What would be a good way to form $23_{five} \times 20_{five}$? The Associative Law suggests $23_{five} \times 2_{five} \times 10_{five}$. Form the representation for this with the blocks and write the number indicated on your paper.

B. Now let us consider the problem $23_{five} \times 12_{five}$. There are two ways to think about this problem. The first is to consider 12_{five} as units and, by using a tally to keep track of how many times you add, actually add 23_{five} that many times. Do this with the blocks and write the number name of your final representation on your paper. The second and more sophisticated way is to think of 12_{five} as $(10_{five} + 2_{five})$ and to make use of the Distributive Law. Then

$$23_{five} \times 12_{five} = 23_{five}(10_{five} + 2_{five})$$
$$= (23_{five} \times 10_{five}) + (23_{five} \times 2_{five})$$

Perform these operations using the blocks if you find them helpful.

Write the answer you obtained on your paper. Are the answers from the two ways the same? If not, check your work thoroughly.

C. You are now in a position to understand the rule for multiplication. For any product, take one factor (usually the smallest) and think of it in additive form. For example, if the problem is $231_{four} \times 32_{four}$ think of 32_{four} as $30_{four} + 2_{four}$. Then multiply the larger factor by each part and add the answers together. What Law is being used here? Since

$$231_{four} \times 2_{four} = 1122_{four}$$

and

$$231_{four} \times 30_{four} = 20130_{four}$$

we have $231_{four} \times 32_{four} = 1122_{four} + 20130_{four} = 21312_{four}$.

To make our addition easier, we line up the problem as follows:

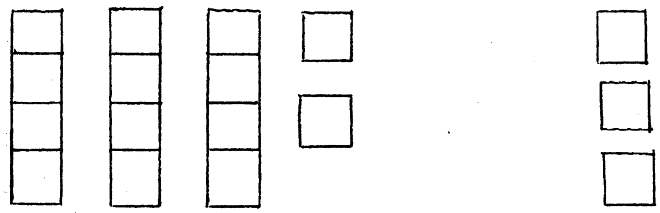
$$\begin{array}{r} 231_{four} \\ \times\ 32_{four} \\ \hline 1122_{four} \\ 20130_{four} \\ \hline 21312_{four} \end{array}$$

D. Explain the rule for multiplication using

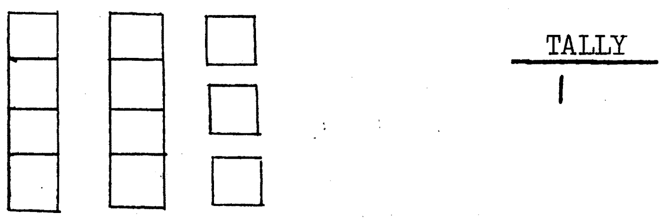
172×84 $\qquad$ 36×22 $\qquad$ $101_{three} \times 21_{three}$

Activity 28. Materials: Multi-base blocks (base four, five and ten). This activity is designed to help you understand *why* the long division rule you learned in elementary school works. We know that when we divide, say 7 into 58, we wish to determine how many groups of 7 we can obtain out of 58 things. We can, of course, actually find 58 things and place them in groups of 7, but that is unnecessarily time consuming. So instead, we subtract the divisor, 7, as long as we can and keep track of how many groups we <u>would</u> get if we actually formed them.

A. Let's take the problem $32_{four} \div 3_{four}$. Form the representation of the dividend, 32_{four}, and also of the divisor, 3_{four}. You should have something like this.



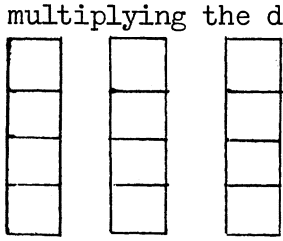
Now subtract 3_{four} from the dividend and make a tally, i.e., you now have

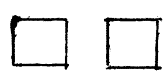


Continue like this until you can no longer subtract 3_{four}. Your final answer should look like this:

The tally is the quotient. Writing it in base four, we see we have that $32_{four} \div 3_{four} = 10_{four}$ with a remainder of 2_{four}.

B. Notice that multiplying the divisor by 10_{four} gives



Subtracting these from the dividend gives  .

But really we subtracted the divisor 10_{four} times. That is, your tally should now say

TALLY
10_{four}

C. Use the base ten blocks, if necessary, to help you understand the repeated subtraction approach to dividing 5232 by 38. Keep a tally on your paper.

```
   5232                TALLY
 - 3800                 100
   1432
 -  380                  10
   1052
 -  380                  10
```

Continue the problem.

D. How many times did you subtract 3800? When you subtracted 3800, you really subtracted 38 how many times? Think about the way you learned long division in grade school. The first step looked like this.

```
       1
38 ) 5232
     38
     143
```

What does that <u>really</u> mean? Really, it means

```
      100
38 ) 5232
     3800
     1432
```

Also, instead of subtracting 10 x 38 three times, in the next step we subtract 30 x 38 or 1140.

```
       30
      100
38 ) 5232
     3800
     1432
     1140
      292
```

Now rather than subtract 38 seven times, we subtract 7 x 38 or 266.

```
             7 ┐
            30 ├   137
           100 ┘
38 ) 5232
     3800
     1432
     1140
      292
      266
       26
```

Now you should be able to see more clearly where our rule for long division comes from.

Open and Closed Sentences

This section consists of Lecture Notes 4, Assignment 7, and Exercise Set 11.

Integers

This section is composed of Lecture Notes 5, Assignment 8 and Exercise Set 12.

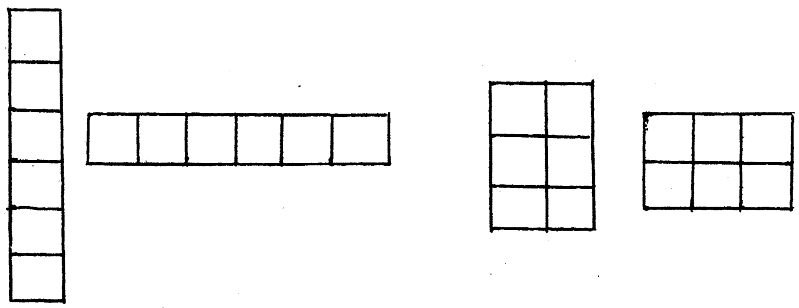
Part III: Number Theory

Factorization

This section consists of Relevant Terms - Set 10, Activities 29 and 30, Lecture Notes 6, Activities 31 and 32, Lecture Notes 7, Assignment 9, Lecture Notes 8, and Exercise Set 13.

Activity 29. Materials: Mosaic tiles, paper.

A. In this activity you will be considering the set of Natural Numbers, i.e., $N = \{1,2,3,\ldots\}$. Take a box of mosaic tiles and for each number of the set $\{1,2,\ldots, 15\}$ form all possible rectangles using that number of tiles and record the multiplication problem it represents. For example, for 6 you can form the following rectangles.



In this example, we notice that expressing 6 as 6 x 1 and 1 x 6 gives rise to congruent rectangles, where one is simply a rotation of the other, and, similarly, for the rectangles representing 2 x 3 and 3 x 2. However, the figures for 6 x 1 and 2 x 3 are different from each other geometrically and also in the factors they contain. Together they give us all the factors of 6, which are 1,2,3, and 6. List the factors of the first fifteen counting numbers. Your table should look like this.

n	Products	Factors
⋮	⋮	⋮
6	1x6, 2x3, 3x2, 6x1	1,2,3,6
⋮	⋮	⋮

B. Of the numbers you dealt with above, which ones are prime? What are the composite numbers less than or equal to 15? (Note: 1 is neither prime nor composite.)

Activity 30. Materials: Hundreds board.

A. It frequently is useful (you will see several uses later in this section) to know which numbers are prime and which are composite. One way of making this decision dates back to an early Greek mathematician, Eratosthenes, and is known as the Sieve of Eratosthenes, since the prime numbers are "caught", as it were, in a sieve while the composites "slip through". Although this method is very simple, it has proven so efficient that modern computer searches for prime numbers use essentially the same method today. To see how it works, take the Hundreds board and put in all the numbers from 2 to 100. (Since 1 is neither prime nor composite, we don't consider it.) Leave the 2 in place, but remove every number which has 2 as a factor, that is, every second number. Since each number removed has two as a factor, none of them can be prime.

B. Now the next number after 2 which is left on the board is 3. Since 3 does not have 2 (the only number under consideration less than it) as a factor, it is prime. Now remove all numbers which have 3 as a factor, that is, every third number. Some of the numbers may be gone already, such as 6, because they also have 2 as a factor.

C. What is the next prime after 3? How can you be sure it is prime? Continue the sieving process by removing all numbers which have this prime as a factor.

D. Continue the sieving process as long as you can. What is the last number you removed? Which prime did it have as a factor? Do you have any idea why this factor is so small? Discuss this in your group and with the instructor.

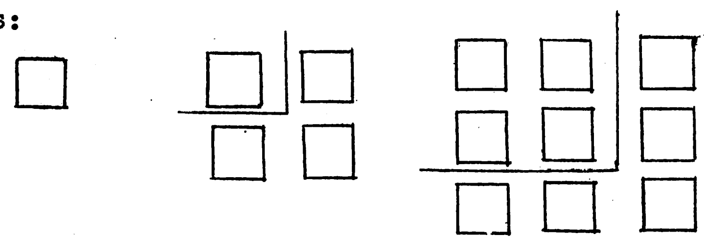
E. How many prime numbers less than 100 are there? Write them down.

Could you find all the prime numbers less than 500? 1000? How?

Activity 31. Materials: Mosaic tiles, chips.

A. In Activity 29 you used the mosaic tiles to find all factors of a given number. For most numbers, there turned out to be an even number of factors, or equivalently, an even number of rectangles which could be formed. But for some there were an odd number. Review that activity and find those numbers for which there were an odd number of rectangles. Form the rectangular representations for them.

B. Do you see now why we call 1,4, and 9 perfect squares? Notice that if you have one square, you can form the next higher one from it, as follows:



Using this method, form the next three squares, and complete the following table.

n	n^2	Difference between n^2 and $(n + 1)^2$
1	1	
2	4	3
3	9	5
4		
5		
6		
7		
8		
9		

Can you see a pattern in the number of blocks added? Express it in terms of n. Do you think this pattern will continue? Why or why not?

C. We have seen square and rectangular numbers. Is there such a thing as a triangular number? Play around with the chips for awhile and see if you can find any triangular numbers. Look also for patterns which connect each triangular number to the one following in somewhat the same way you did with the square numbers in B above. When you think you have something, call the instructor and discuss it with her.

Activity 32. Materials: Mosaic tiles. The whole numbers are separated into two subsets according to whether they are divisible by 2 or not. Of course, these subsets are the even numbers (those divisible by 2) and the odd numbers (those not divisible by 2). Although they are very familiar, they have such interesting properties that we will investigate some of them in this activity.

A. With the tiles show that the numbers 4,12, 20 and 32 fulfill the definition of even numbers and that 7,13,21 and 27 fulfill the definition of odd numbers.

B. An integer, n, is even if there exists an integer k such that $n = 2k$. From your work with the tiles, write an equation to describe an odd number. Show your equation to the instructor before proceeding.

C. Take two sets of tiles which represent even numbers. Form the representation of the sum of those two numbers by uniting the sets. What kind of number is it? Make a guess about whether addition is closed in the set of even numbers. What about in the set of odd numbers? Try to prove these guesses using the equation definitions of odd and even numbers as well as the hints derived from the activity. Show your work to the instructor before proceeding.

E. Form the representation for the sum of an odd and an even number. Is it odd or even? Will this always be true? Justify your answer.

Divisibility

This section is composed of Lecture Notes 9 and Exercise Set 14.

Greatest Common Divisor and Lowest Common Multiple

This section consists of Lecture Notes 10, Assignment 10, Lecture Notes 11 and Exercise Set 15.

Part IV: The Rational Number System

Basic Ideas

This section consists of Activities 33 and 34, Assignments 11, 12, and 13, and Relevant Terms - Set 11.

Activity 33. Materials: Cuisenaire rods. Up until now we have only considered division of a smaller integer into a larger one. We are now ready to proceed to any division.

A. Take an orange Cuisenaire rod and consider it to represent one. Divide it into five equal parts, i.e., represent $1 \div 5$. Of course, you could cut the given rod (and this is a very good way to do it, theoretically) but perhaps you can find some smaller rods, so that exactly 5 of them will equal the one whole orange rod. What color are the rods you have found? The number each of these rods represents is the <u>rational number</u>, or <u>fraction</u>, 1/5.

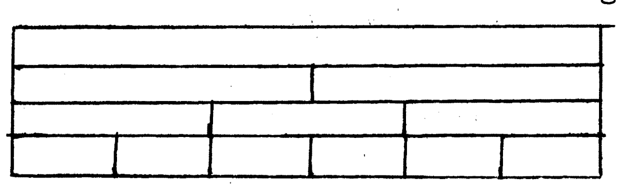
B. To find 3/5, place three orange rods in a row and then see if you can find a rod, so that five of them will equal the three orange rods. That is, can you find a rod which will represent $3 \div 5$? Compare this rod with the rod of A which represents 1/5. What relation holds between them? Can you think of a way to describe 3/5 other than $3 \div 5$?

C. Take a yellow rod as a unit and find a rod which represents $1 \div 5$ in this case. Also find 3/5 in this case by the two methods used above. Compare the rod which represents 1/5 here with the one that represented 1/5 in A. Are they the same or different? Why? What about the 3/5 rod here and the one in A? Think back to our work with measurement. Was there anything analogous to this in whole numbers? Discuss these ideas in your group and with the instructor.

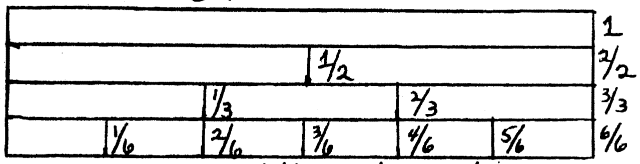
D. Consider the orange Cuisenaire rod as 1 and find the rods which represent 1/10, 1/5 and 1/2. How many 1/10 rods does it take to form a train of rods congruent to the orange rod? What equation can you write? How many 1/5 rods? How many 1/2 rods? Write equations for these two relationships also. Try to generalize this relationship for any integer n. I.e., how many 1/n rods does it take to form a train of rods congruent to a unit rod?

<u>Activity 34.</u> Materials: Up to twenty blocks.

A. Take the six stick as a unit and find all rods which when put in a train with similar rods can be made congruent to the unit. Form a rectangle with the rods. You should have something like this.



Now name each rod. E.g.,



Notice that not only does 6/6 = 3/3 = 2/2 = 1 but also 2/6 = 1/3, 1/2 = 3/6, and 4/6 = 2/3.

B. Now take the eight stick as a unit and repeat A above, i.e., find all fractions that can be represented with the rods and find all equal fractions. Do the same with the 9,10,12,15,16,18 and the 20 rod. Use the information to fill in the table below.

Fraction	Equal Fractions
1/2	
1/3	
2/3	
1/4	
3/4	
1/5	
2/5	
1/6	
1/8	
3/8	
5/8	
7/8	
1/9	
2/9	
4/9	
1/10	

C. Examine the table closely. There is a computational way of obtaining not only the fractions above but an infinite set of equal fractions for each fraction given. Find this method and describe the set formed by means of it. Discuss this in your group and with the instructor.

Addition in Q

This section is composed of Activities 35 and 36, Assignment 14, and Exercise Set 16.

Activity 35. Materials: Number line constructed in Assignment 12, fraction circles. This activity and the one that follows deal with the operation of addition in Q. Since $Z \subset Q$, we expect that the ways we have described addition in Z, namely by means of slides of the number line and union of sets, will still be valid.

A. Take the number line you constructed in D of Assignment 12. By means of slides, find the following and keep a record of your answers. For each problem draw a sketch of the number line and reduce all answers to lowest terms.

$1/4 + 5/4$ $-3/4 + 7/4$ $5/12 + 11/12$ $11/12 + -17/12$

B. Notice that in each problem of A, the fractions involved contained the same denominator. What seems to be the rule for adding such fractions? Write out what you think it is either in words or in other symbols. Test your rule by using it on the following problems and then reworking them on the number line.

$-1/6 + 7/6$ $7/3 + -4/3$ $3/4 + -5/4$ $11/12 + 5/12$

C. Take the fraction circles and find pieces which represent 1/2 and

1/6. To add the numbers, you need to form the union of the two sets. Can you give a number name to the result as it appears here? What exchange must you perform first? How much is 1/2 + 1/6? Use the same technique to find:

1/3 + 1/6   1/4 + 3/8   1/2 + 3/8

Keep a record of the answers you get.

D. Study the problems and the answers of C above. What did you have to do in every case with one of the fractions? Can you write a rule for obtaining an answer when the fractions do not have the same denominator?

<u>Activity 36.</u> Materials: Number line constructed in Assignment 12, Up to twenty blocks, fraction circles.

A. Use the number line to find 1/3 + 1/4. What is the answer? How about 3/4 + 1/6? These problems are different from the ones you dealt with in Activity 35. How? What is the relation between the denominators of the addends and the denominator of the sum? Write your answer and keep it for future reference.

B. Suppose you wish to add 1/3 and 1/5. What is the smallest rod you can use as a unit so that it can be divided evenly into both thirds and fifths? What relation does this number have to 3 and 5?

C. Consider 1/6 + 2/9. What is the smallest rod you can use as a unit so that it can be divided evenly into both sixths and ninths? What relation does this number have to 6 and 9?

D. Look at part C of Activity 35 in the light of the relationship you feel exists between the denominators of the addends and the denominator of the sum. Does the same relationship hold? How about the problems where all the denominators were the same? Express in general terms this relationship. After you agree in your group, call the instructor and discuss it with her.

Multiplication and Division in Q

This section consists of Activity 37, Assignment 15, Lecture Notes 12, and Exercise Set 17.

Activity 37. Materials: Number line, Mira. We already considered computation in Q in terms of geometric transformations when we thought of addition as a translation, or slide, along the number line. We will now extend this interrelationship of geometry and numerical computation to multiplication.

A. Consider 3/4 x 7. Take the number line, and, since multiplication is repeated addition, show this as 3/4 + 3/4 + 3/4 + 3/4 + 3/4 + 3/4 + 3/4 on it. I.e., make 7 slides to the right of length 3/4 beginning at 0. Where did you stop? This answer can be expressed in two ways: as a whole number and a fraction (this is sometimes called a mixed number) or a single fraction. Express it both ways.

B. 3/4 x 7 can be written as

$$\frac{3}{4} \times \frac{7}{1}$$

Study this and the answer you obtained in A when it was written as a single fraction. What pattern do you see? Write a tentative rule for multiplying fractions, either in words or in other symbols.

C. Commuting the problem above we get 7 x 3/4. To think of this as 3/4 of a slide to the right of length 7 beginning at 0 is valid, but a bit messy. Let's think of it instead in terms of "stretching" or multiplying, and then "shrinking" or dividing. Take a number line and by slides "stretch" 7 three times. Now divide the result by 4. Where did you stop? Is the answer the same as 3/4 x 7? Does the tentative rule you wrote in B give that answer too? What would be your guess about the commutativity of multiplication in rational numbers?

D. Take the number line and place Mira at zero. What is the reflection of 7? of -7? of 1/2? of -1/2? We already know that the additive inverse, symbolized by a negative sign, of 7 is -7 and of -7 is 7. I.e., -(7) = -7 and -(-7) = 7. So we can think of the negative sign geometrically as a flip or reflection about zero. Using Mira, find -(-6), -[-(-6)], -3/4, -(-1/2). Record your answers. On which side of zero does a number lie if it has an even number of negative signs? If it has an odd number?

E. Using the idea of a negative sign as a flipper, find -7 x 2. (First find 7 x 2 and then apply the sign.) Find (-7) x (-2). What is the sign of the product of two numbers with opposite signs? What is the sign of the product of two negative numbers? Discuss this with the instructor before proceeding.

F. Combine the methods of A and D above to demonstrate how to obtain -3/4 x -7. Show your work to the instructor before proceeding.

<u>Properties of Addition and Multiplication in Q</u>

This section is composed of Lecture Notes 13, Relevant Terms - Set 12 and Exercise Set 18.

Decimal Numeration and Percentage

This section consists of Activity 38, Lecture Notes 14, Activity 39, Assignment 16 and Exercise Set 19.

Activity 38. Materials: Multi-base blocks (base ten). In this activity, the base ten place value number system will be extended to cover the rational numbers.

A. Take one of each type of the base ten multi-base blocks and arrange them so that their order from left to right is block, flat, long and unit. In previous work we have thought of these materials in terms of "multiplying", as it were. That is, a unit multiplied by 10 gave a long, a long multiplied by 10 gave a flat etc. But suppose we moved in the other direction. I.e., if we started at the block, what numerical operation would need to be performed to get a flat? What would we do to a flat to get a long? To a long to get a unit? Would we have to stop there? Why or why not? Discuss this in the group and with the instructor.

B. Now think of the large block as a unit, i.e., as representing 1. In this case, what does a flat represent? A long? A small block? Set out 1 large block, 3 flats, 4 longs and 7 small blocks. Write in expanded form the number this represents if the large block stands for 1. Do the same for the following:

   a) 7 flats, 3 small blocks   b) 3 longs

   c) 1 large block, 2 longs, 1 small block

   Check your answers with the instructor before proceeding to part C below.

C. As with whole numbers, the expanded forms of these numbers can be abbreviated. A point, usually referred to as a decimal point, is placed between the whole number and the fraction. E.g., 1 + 3/10 + 4/100 + 7/1000 is 1.347. As before, zero acts as a place holder. So 4/100 must be thought of as 0/10 + 4/100 or .04. Why must the tenths place be held? Suppose we just put .4 instead? What would that mean?

Activity 39. Materials: Peg board and pegs, graph paper. Percent comes from two Latin words, per centum, meaning "for each hundred". Thus 25% means 25 for each hundred or 25/100. The symbol " /100" is shortened to " % " and 25/100 becomes 25%.

A. Take the peg board and count the holes. How many are there? Put 12 red pegs, 7 blue pegs, 4 yellow pegs and 10 green pegs on the board. What percentage of the board is red? blue? yellow? green? covered? uncovered?

B. With the pegs, make a design on the peg board so that 25% of the board is covered with red pegs and 30% with green. Leave 10% uncovered and fill the remainder of the holes with colors of your choice but make sure that 7% are of color 1, 8% of color 2 and 20% of color 3. Show the instructor your work before you proceed.

C. Clear the peg board. Then place a peg in every other hole, i.e., fill ½ of the holes. What percentage have you filled? After clearing the board, fill every third hole. Can you do that evenly? 1/3 is about what percent? exactly what percent? (Hint: what part of the next three are left? By means of the peg board, complete the following table.

Fraction	Approximate percent	Exact percent
1/2		
1/3		
1/4		
1/5		
1/6		
1/7		
1/8		
1/9		
1/10		

Can you think of a way to change from a fraction to a percent without a pegboard, i.e., by an arithmetical operation? Discuss this in your group and with the instructor. What relation must hold between the denominator of a fraction and 100 if the fraction is to be expressed as a percentage evenly?

D. Place pegs on .5 of the peg board. What percentage is that? On the whole board. What percentage? Find a rule for changing decimals to percents and discuss it with the instructor.

Mathematical Sentences in Q

This section consists of Activity 40, Lecture Notes 16, and Assignment 17.

Activity 40. Materials: Number line, Mira.

A. Mark -1/2 and -3/4 on the number line. By the Law of Trichotomy, one of three possible relations must hold between these two numbers.

Which one does? Now multiply both numbers by -1. We know that multiplying by a negative number results in a flip, so place Mira at zero to help you visualize the reflection. What relation holds between the new numbers?

B. Repeat the above procedure but this time mark 6 and 8 on the number line and divide by -2.

C. Place -4 and 3 on the number line and see what relationship holds between them. Then multiply both by -3 and note the relationship between the answers. What seems to happen to inequalities that are multiplied or divided by negative numbers? Check your answer with the instructor before proceeding.

D. Suppose you had the following mathematical sentence: $-3x < 9$. What would be the first thing you would think of doing to both sides? Remembering your work in A,B, and C above, what do you think is the relation between the quotients if you divide both sides by -3?

Experimental Aspect of the Study

Teaching of the Course

For the most part, the teaching of the course went smoothly. The greatest tension for the students seemed to result from the insecurity they felt as they tried to cope with a method of learning different from any they had ever met before. Their feelings with respect to this and other aspects of the course will be further discussed in the section on attitude changes.

For the instructor, the biggest problem was that of time. The activities, if thoroughly thought out and conscientiously done by the students, often took longer than the fifty minutes allotted for class. Frequently, in this case, the instructor opted to temporarily abandon the discovery approach and simply demonstrated the principle in question to the students using the manipulatives. Although this was not regarded as an optimum solution to the problem, the amount of content to be covered seemed to make it necessary. Since these students were pre-service elementary teachers, it was the instructor's judgment that they should be exposed to as much of the content recommended by CUPM as possible.

In fact, however, there were several sections where the demonstrations seemed to be more effective, as well as faster, than the activities. For example, even those students who had done Activity 20 on the algorithm for changing base ten numbers to other bases and Activities 27 and 28 on the multiplication and division algorithms found a demonstration necessary to really understand the process. This appeared to be related to the students' inability to read the instructions with understanding rather than to any inherent superiority of demonstration over discovery. Because these particular activities required an abstract understanding of words, many of the students simply did not realize what was expected of them.

Many of the activities relied heavily on students' past knowledge as well as on their ability to draw implications from previous work and to transfer everyday knowledge and vocabulary to the mathematical ideas under consideration. This proved to be an obstacle for a good many of them, because of their apparently under-developed reading skills. Another problem arose from the fact that the students tended to get caught up in details. For example, if the directions of an activity said to move a figure a given distance, say 3 cm, some students would redo these directions several times to make sure the distance was 3 cm as opposed to, say, 2 cm or 4 cm. As the semester progressed, however, they became more attuned to the spirit of the course and began to look instead for the main ideas of each activity.

Every teacher is familiar with the experience of having a "good" lesson or a "bad" one. Although one is not able to measure scientifically the differences in quality of a lesson, one can describe them. A "good" lesson "clicks"; the students respond enthusiastically and correctly, ask probing questions, and are visibly thrilled with their new knowledge. A "bad" class, on the other hand, "drags"; the students may be bored, confused, and sometimes, even angry. The material is incomprehensible to them, or at best, is judged irrelevant. Both "good" and "bad" classes occured in the conducting of Math 105. The instructor's evaluation together with student assessment formed a basis for the selection of the activities cited below.

Activities 2D and 4C were alike in that they were designed to show the possibility in spherical geometry of things impossible in plane geometry. In both cases, although almost all of the students did not understand the material while they were reading the directions, about half of them suddenly realized the distinctions when they began to work with the globe. These students became quite enthusiastic and began spontaneously to explain profusely to those of their peers who had not yet comprehended the idea.

Game 1 was quite effective in those groups which consisted of students with varied abilities, since those who really understood the distinctions between the figures and the inclusion relation among them were able to challenge the moves of those less able. These, in turn, were stimulated to pay more attention to details and to be more careful in their plays. However, in groups where all the students were knowledgeable, there was no opportunity to challenge, and in groups where all were slow, the chances to challenge were not recognized. This constituted a definite drawback to the success of the game.

The materials dealing with congruence, Activities 8 - 12, were also stimulating to the students and were the source of considerable excitement and interest. Especially effective were the activities dealing with reflections and making use of the Mira. As a result, the distinctions among a translation, rotation and reflection were grasped almost immediately by the majority of the students and were not forgotten throughout the course.

Activity 3C, making use of geostrips, was one of the most confusing sections for the students. The main problem seemed to lie in the lack of the concept of a unit of measurement. Geostrips contain holes at varying intervals and to do the activity correctly it was necessary to use strips which had the same distance between the holes. Even after this was pointed out to the students, many of them continued to use different units, and hence got wrong answers. One thing about this activity that did prove valuable, however, was that it preceeded the section on measurement and hence could be referred to as an example of the need of a basic unit of measurement for accuracy.

All of the material on the non-decimal place value systems, in the instructor's opinion, tended to confuse the students more than it enlightened them. Although they appeared to understand the relations between the various Multi-base blocks, they failed to abstract from this to the place value number systems. Activity 38, using the multi-base blocks as an introduction to decimal numbers less than one, was so difficult as to be discouraging to the majority of the students.

Growth in Achievement

Hypothesis H_1 was tested by means of a comparison between the initial and terminal grades on the Achievement Test. In addition, intermediate tests and final grades in the course were used as a determination of growth in mathematics achievement.

The Achievement Test

Forty-seven students took the Achievement Test at both the beginning and end of the course. Table II presents the initial and terminal scores and the difference between these scores for each student. According to a t-test for paired data, hypothesis H_1 was rejected at the .05 level of significance, i.e., there was a significant difference between the initial performance of students enrolled in Math 105 on the Achievement Test and the final performance of students enrolled in Math 105 on the Achievement Test. The Achievement Test is found in Appendix A.

Intermediate Achievement Tests

At the end of each of the four sections of the course, an achievement test covering the material of that section was administered. The second and fourth tests had two different forms, Form I taken by the morning class and Form II by the afternoon class. The first and third had a single form taken by both classes. The tests are found in Appendix B.

TABLE II

INITIAL AND TERMINAL ACHIEVEMENT SCORES

INITIAL SCORE	TERMINAL SCORE	DIFFERENCE
2	109	107
29	75	46
43	97	54
31	111	80
35	117	82
30	139	109
22	110	88
33	115	82
28	143	115
4	84	80
13	85	72
14	86	72
29	118	89
45	140	95
17	137	120
46	139	93
7	77	70
12	121	109
25	140	115
22	107	85
31	109	78
22	69	47
23	118	95
37	91	54
24	110	86
27	129	102
26	97	71
40	140	100
26	72	46
15	132	117
27	106	79
19	76	57
22	116	94
19	73	54
24	126	102
45	135	90
27	84	57
26	92	66
8	92	84
26	89	63
30	75	45
25	105	80
15	100	85
34	116	82
20	144	124
31	108	77
13	104	91

Each of the tests contained 100 points and grades were assigned based on the following scale: 90% - 100%, A; 80% - 90%, B; 70% - 80%, C; 60% - 70%, D; below 60%, F. Table III gives the results of the testings. Although there were 32 students in the morning section and 19 in the afternoon, the students were given the option of taking the tests at which ever period was most convenient for them. Thus in Table III, $N_2 > 19$ for Tests 2,3, and 4. However in all cases $N_1 + N_2 \leq 51$.

Final Grades in the Course

Each student's final grade in the course was based on the average of the four intermediate tests, the homework assignments and the final examination. The homework grade had the weight of a single intermediate test and the final examination had the weight of two intermediate tests. The lowest test grade of each student was dropped before the average was computed. Table IV gives the statistics concerning the final grades of the course where letter grades were based on the following scale: 87% - 100%, A; 79% - 86%, B; 67% - 78%, C; 56% - 66%, D; and below 56%, F.

TABLE III

INTERMEDIATE ACHIEVEMENT TEST DATA

TEST	SECTION	DISTRIBUTION	RANGE	N	MEAN	MEDIAN
1	1	A - 0 B - 3 C - 6 D - 9 F -14	31-89	32	61.00	62
1	2	A - 2 B - 2 C - 6 D - 6 F - 3	39-97	19	70.68	72
2	1	A - 2 B - 3 C - 9 D - 6 F -11	22-90	31	61.03	67
2	2	A - 0 B - 5 C - 5 D - 5 F - 5	38-88	20	67.40	71
3	1	A - 6 B - 8 C - 6 D - 3 F - 6	28-96	29	72.86	79
3	2	A - 2 B - 5 C - 7 D - 2 F - 4	26-93	20	72.50	77
4	1	A - 2 B - 4 C - 3 D - 8 F - 9	18-92	26	61.88	64
4	2	A - 4 B - 2 C - 6 D - 4 F - 8	27-100	24	65.88	68

TABLE IV

DISTRIBUTION OF FINAL GRADES ON THE COURSE

SECTION 1		SECTION 2	
A	2	A	4
B	6	B	3
C	9	C	5
D	8	D	6
F	7	F	1
Range 37 - 90		Range 35 - 92	
Mean 67.31		Mean 71.68	
Median 68		Median 73	

Discussion of Achievement Results

The growth in achievement as indicated by the differences in the initial and terminal test scores is significant at the .05 level of significance. However, approximately 15% of the students involved still did not attain a level of achievement sufficient to enable them to pass the course. This is comparable with the over-all failure rate in this course as taught at Xavier University over the past five years, but lower than that of the immediate past year. Table V givens the pertinant information.

The quality of the preparation of incoming college students has decreased generally in the past few years (31). This is as true at Xavier as at other institutions. Thus, the increased failure rate from 1972 to

1973 to 1974 is not surprising. The 9% decrease, however, from 1974 to 1975 is unexpected and is an encouraging sign that the laboratory approach may indeed be a valuable technique for increasing achievement.

TABLE V

FAILURE HISTORY OF MATH 105 AT XAVIER UNIVERSITY

YEAR	NUMBER ENROLLED	NUMBER OF FAILURES	PERCENTAGE OF FAILURES
1971	42	4	9%
1972	38	1	3%
1973	25	3	12%
1974	53	13	24%
1975	51	8	15%
Total	259	29	12%

Changes in Attitude Concerning Mathematics

The forty-one students enrolled in Math 105 who took both the initial and terminal presentation of the Attitude Scale were assigned values which were the medians of the scale values of those statements with which the students agreed. The initial and terminal median values for the respective parts of the Attitude Scale were compared by a sign test. Also each item on the Attitude Scale was analyzed for changes between initial presentation and terminal presentation. In addition to the Attitude Scale, informal attitude questionaires were given at the end of each of the four sections of the course.

Comparison of Medians

Hypotheses H_2, H_3, H_4 and H_5 were not rejected at the .05 level of significance. There was no significant difference between the initial and terminal attitudes of students in Math 105 concerning the learning of mathematics, mathematics as a process, the place of mathematics in society, and school and learning generally.

Item Change Analysis

Table VI gives a percentage representation of the responses to statements on the initial and terminal presentations of the Attitude Scale. Category A represents the percentage of students in Math 105 who agreed with the item only on initial presentation; B, the percentage who agreed with the item on both initial and terminal presentation; C, the percentage who did not agree with the item on either the initial or terminal presentation; and D, the percentage who agreed with the item only on terminal presentation. A + B, the total percentage who agreed initially and B + D, the total percentage who agreed terminally are also given. In cases where the McNemar test was not appropriate and the Binomial test was used, a blank appears in the χ^2 column.

One of the two items which showed a significant change at the .05 level of significance was statement 13 (Almost all present day mathematics was known at least a century ago). Table VI shows that 51.3% of the students in Math 105 agreed with statement 13 initially, while only 29.2% agreed with it at the end of the course. This change is particularly noteworthy in the light of the content of Math 105. Practically all of the mathematics covered has, in fact, been known for well over a century, and the historical background of the material was emphasized in the course.

TABLE VI

CATEGORICAL PERCENTAGE REPRESENTATION OF RESPONSES TO STATEMENTS ON INITIAL AND TERMINAL PRESENTATION OF ATTITUDE SCALE

	Item	A	B	C	D	A+B	B+D	χ^2
	1.	17.1	46.3	17.1	19.5	63.4	65.8	0.000
	5.	14.6	48.8	19.5	17.1	63.4	65.9	0.000
	9.	12.2	7.3	78.1	2.4	19.5	9.7	———
Part I	16.	9.8	73.2	4.8	12.2	83.0	85.4	———
	20.	14.6	63.5	7.3	14.6	78.1	78.1	.083
	24.	7.3	78.1	4.8	9.8	85.4	87.9	———
	27.	2.4	0	92.8	4.8	2.4	4.8	———
	2.	12.2	80.5	0	7.3	92.7	87.8	———
	6.	24.4	22.0	41.4	12.2	26.4	34.2	1.066
	10.	14.6	65.9	12.2	7.3	80.5	73.2	———
Part II	13.	26.9	24.4	43.9	4.8	51.3	29.2	4.923 *
	17.	17.1	24.4	46.3	12.2	41.5	36.6	.083
	21.	12.2	9.8	65.8	12.2	22.0	22.0	.100
	25.	7.3	36.6	36.6	19.5	43.9	56.1	1.454
	3.	12.2	7.3	73.2	7.3	19.5	14.6	———
	7.	12.2	65.8	12.2	9.8	78.0	75.6	———
	11.	12.2	4.8	75.7	7.3	17.0	12.2	———
Part III	14.	19.5	34.1	24.4	22.0	53.6	56.1	0.000
	18.	22.0	41.4	22.0	14.6	63.4	56.0	.266
	22.	24.4	12.2	58.6	4.8	36.6	17.0	4.083 *
	26.	24.4	31.7	24.4	19.5	56.1	51.2	.055
	4.	9.8	68.3	7.3	14.6	78.1	82.9	.100
	8.	2.4	0	97.6	0	2.4	0	———
	12.	7.3	26.9	58.5	7.3	34.2	34.2	———
Part IV	15.	4.8	73.2	9.8	12.2	78.0	85.4	———
	19.	17.0	63.4	9.8	9.8	80.4	73.2	.363
	23.	12.2	36.6	29.2	22.0	48.8	48.6	.642

*Significant at the .05 level

The other item which showed a significant change at the .05 level of significance was statement 22 (Unless one is planning to become a mathematician or scientist, the study of advanced mathematics is not very important.) Table VI shows that 36.6% agreed with this statement initially while only 17% agreed with it at the end of the semester, and 24.4% changed from agreement to disagreement. Thus, at the end of the course, 83% of the students felt that mathematics was important to all and not just to a select few. This certainly was a desirable change.

Although not significant at the .05 level, there was also an increase in the percentage of students who agreed with statement 6 (Mathematics helps one think according to strict rules) from 26.4% initially to 34.2% terminally. The probability that this change occured by chance is approximately .3. The change is especially interesting since the course was taught using a discovery approach where the strict rules of mathematics might not be as apparent as in a traditional approach.

Informal Attitude Questionaires

Responses to the informal attitude questionaires were positive for the most part. In general, there about three times as many positive responses as negative, with most of the negative reactions following the section on Whole Numbers, Natural Numbers and Integers. This was the section covering non-decimal place value number systems, and it seemed to be this topic which the students found most threatening and confusing.

Students did not identify themselves on the questionaires, but the distinctive handwriting of one student made it possible to follow the evolution of her feelings about the course and the laboratory approach. On the first questionaire, she expressed the opinion that the students should be given an option between the laboratory method and the traditional lecture approach. Then she wrote, "When I signed up for the course I did not know that it would be taught under this method. If I would have I would not have taken the course." At the end of the second section, she still did not find the method satisfactory answering, "They (the activities) didn't really help me." However, by the end of third section she wrote, "I thought that this section was the best. It really helped me. These were the best activities. Right on!" In answer to the fourth questionaire she responded, "Yes, they (the activities) did help me. I never enjoy the activities, but that doesn't mean that they aren't good. They were (sic)." It is likely that the insecurity of dealing with an unfamiliar classroom situation was responsible, at least in part, for her initial dissatisfaction. As she grew in knowledge of what was expected of her, however, she also began to see the value of the activities even though she "never enjoy(ed)" them.

Some other quotes indicative of the general flavor of the responses are given below.

> I disliked Math before I took this course.
>
> In all, I think seeing and performing, rather than just listening is more helpful.
>
> I feel that the activities are somewhat helpful, <u>but</u> there are too many to do in one day.
>
> I feel the Activities were very confusing in respect to how they were written.
>
> I find them a great help, although I began to use the concrete object as a crutch.
>
> I really enjoyed this and it has given me ideas about what to do in my own classroom.
>
> Seems to me that I'm beginning to understand it - it's beginning to come to me. I am now enjoying Math, better than when I first came in and hated it.

For once I saw what I had been taught to do from memorization.

Discussion of Attitude Changes

Although hypotheses H_2, H_3, H_4 and H_5 were not rejected on the basis of the Attitude Scale, the informal attitude questionaires showed a growth in positive feelings toward mathematics, or at least toward the course. Some possible reasons for this are discussed below.

The Attitude Scale was developed and validated for a very different population from that used in the present study. The University of Missouri - Columbia is a large, public institution with a predominantly white, middle-class student body. Xavier University, on the other hand, is small and private with a predominantly black student body, many of whom come from low-income families.

Also, the climate of the American college campus in the 1960's when the Attitude Scale was developed and validated, was considerably different from that of the present time. A disillusionment with society in general, and the university as one of its most influential components, was rampant. Students especially doubted the relevance of the courses they were required to take and whether they were, in fact, of any use in preparing them for the "real world". At the present time, partly because of the universities' attempt to revamp their programs, few question, at least publicly, the content of the curriculum.

In light of these considerations, it is interesting to note that on each of the four parts of the Attitude Scale, the majority of the students in Math 105 were assigned a scale value higher than the median value of all statements in that section. This was especially noticable in Part IV, Views Concerning School and Learning Generally, where the scale values assigned to the students were consistently highly positive. This is probably because most of the subjects regard education as an essential ingredient of their progress to economic prosperity and a middle-class life style. This intellectual assent to the value of education and, in particular, mathematics education, could be responsible for the high, and virtually unchanging, scores on the Attitude Scale, even though the informal questionaires showed that the feelings of the students did not always correspond to what they accepted as true.

CHAPTER V

SUMMARY, CONCLUSIONS AND IMPLICATIONS, AND SUGGESTIONS FOR FURTHER STUDY

This chapter contains a summary of the study, conclusions and implications of this investigation, as well as some suggestions for further study.

Summary

Chapter I serves as an introduction giving the background of the study as well as the problem definition. A survey of selected related research is presented in Chapter II. The research procedures are discussed in Chapter III. The discussion is presented in two parts: the developmental aspect of the study and the experimental aspect of the study. Chapter IV gives the results of the study, again in two parts. The results of the developmental aspect, the activities used in Math 105, are followed by the results of the experimental aspect, evaluation of achievement and attitudinal changes on the part of students enrolled in Math 105.

Background of the Study

The work of Jean Piaget indicates that people pass through four stages in their intellectual development. These phases always occur in the same order but continue for varying lengths of time in different individuals. They are the sensory-motor level, preoperational level, concrete operational level and formal operational level. Of these, the last two are the most relevant to this study.

In the concrete operational stage, which usually begins about age seven, the individual is able to perform intellectual operations but these operations are on physical objects rather than verbally expressed hypotheses. In the formal operational stage, the individual can operate intellectually with hypotheses and not only on objects. There is evidence to indicate that some people reach the formal operational level by age eleven while others still have not reached it by adulthood. This is important since Piagetian research further points out that until an individual has attained a given level of development, it is impossible for him to operate at that level. Thus, a person who is on the concrete operational level is not able to respond adequately to instruction given on a formal operational level.

A development in mathematics education which fits in well with this theory, although it did not evolve from it, is the mathematics laboratory. Both an approach to learning and a physical environment, a mathematics laboratory frequently contains manipulative materials which can be handled by learners and which embody mathematical concepts. Until recently, mathematics laboratories were used mainly on the elementary school level. However, in the last several years they have been found on higher levels as well. One such higher level is in the training of pre-service elementary teachers.

There is evidence (10) that both pre-service and in-service elementary teachers tend to perform poorly in mathematics, although they are usually people of normal intelligence. In light of Piaget's theory, this fact makes it seem likely that a good number of them have not yet reached the formal operational level with respect to mathematics. Thus a mathematics content course taught using a variety of concrete operational activities is more likely to be conducive to the mathematical growth of these students than one taught on the formal operational level, as most traditional college mathematics classes are.

Achievement in mathematics is related to attitude toward mathematics in a reciprocal manner, i.e., not only does achievement help determine attitude but attitude affects achievement. A teacher's attitude toward mathematics seems to be transmitted to her students (2). Thus, the attitudes of elementary teachers toward mathematics are relevant not only to their own mathematical progress but also to that of their students, present and future.

Problem Definition

As a pilot project, Math 105 was taught using a mathematics laboratory approach at Xavier University of Louisiana during the fall semester of the 1975-76 school year by the researcher. The developmental aspect of this study consisted of preparing activities which involved the use of manipulative materials and which were integrated into Math 105. The experimental aspect of the study consisted of assessing achievement and attitudinal changes of the students enrolled in Math 105.

Achievement was measured by means of an achievement test developed by the researcher for Math 105. Attitude was measured by an attitude scale designed by Aichele (1) to assess student attitudes in four areas: the learning of mathematics, mathematics as a process, the place of mathematics in society, and school and learning generally.

The hypotheses of the study are arranged according to the preceding areas. For purposes of statistical analysis, the hypotheses are reported in the null form.

H_1: There is no significant difference between initial performance of students in Math 105 on the Achievement Test and final performance of students in Math 105 on the Achievement Test.

H_2: There is no significant difference between initial attitudes of students in Math 105 concerning the learning of mathematics and terminal attitudes of students in Math 105 concerning the learning of mathematics.

H_3: There is no significant difference between initial attitudes of students in Math 105 concerning mathematics as a process and terminal attitudes of students in Math 105 concerning mathematics as a process.

H_4: There is no significant difference between initial attitudes of students in Math 105 concerning the place of mathematics in society and terminal attitudes of students in Math 105 concerning the place of mathematics in society.

H_5: There is no significant difference between initial attitudes of students in Math 105 concerning school and learning generally and terminal attitudes of students in Math 105 concerning school and learning generally.

It is assumed in this study that errors in evaluating students are normally distributed throughout the sample. The results of the Achievement Test are assumed to be a reliable index of a student's mathematical knowledge. Likewise, the results of the Attitude Scale are assumed to be a reliable index of a student's attitude toward mathematics.

This study is limited to those students enrolled in the course, Mathematics for Elementary School Teachers, Math 105, during the fall semester of the 1975-76 academic year at Xavier University of Louisiana who completed the initial Attitude Scale, terminal Attitude Scale, initial Achievement Test and final Achievement Test. The conclusions of the study are also restricted to the extent that assessed attitude reflects true attitude and scores on the Achievement Test reflect true achievement.

Related Research

The related research chosen for discussion in this study covered three areas: the presence of the formal operational level in normal adults; the relation of Piagetian concepts to achievement in minority situations; and the effects of a mathematics laboratory approach on attitude and achievement of pre-service elementary teachers.

C. Tomlinson-Keasey (35) found that attainment of the highest stage of formal operations was rare in a group of adolescent girls and adult women. K. Lovell (22) confirmed the existence of the stages of development described by Inhelder and Piaget (16), although he found evidence that the transition from concrete operational to formal operational does not always take place at the age determined by Inhelder and Piaget. Both Tomlinson-Keasey and Lovell found that formal thought operations cannot be directly taught. Lovell indicated, however, that the atmosphere of the classroom and conduct of the teacher is an aid to their attainment.

Raven, Hannah and Doran (28) found that the presence or lack of formal operational thought in black students was a predictor of success in a college science course. Johanson (18) found that a Piagetian mathematics curriculum was instrumental in improving attitude and achievement of a low-achieving ninth grade class that was ninety-five percent black.

Research by Fuson (13) indicated that a combined mathematics methods and content course making use of manipulative materials contributed to the improvement of attitude and achievement of pre-service elementary teachers. Warkentin (41) found that pre-service elementary teachers attained better attitudes but lower achievement as a result of a laboratory/models approach to the mathematics content course than the traditional lecture/textbook method.

Research Procedures

The study was divided into two main parts, developmental and experimental. The developmental aspect consisted of selecting the course content, planning the format of the written materials, choosing the manipulative materials to be used, and writing of the activities. Included in the experimental part were the actual teaching of the course using a mathematics laboratory approach, the evaluation of growth in mathematics achievement, and the evaluation of changes in attitude on the part of the students.

The content of Math 105 was chosen to fulfill the 1971 Level I CUPM recommendations (8). As developed, the course contained four sections: Elementary Ideas of Space, Measurement, and the Number Line; Natural Numbers, Whole Numbers and Integers; Number Theory; and, the Rational Number System. Activities designed to make use of manipulative materials in a laboratory setting were written in the summer of 1975. Both commercially available manipulative materials and ordinary objects which embody mathmatical ideas were chosen for use in the course.

Math 105 was taught in two sections during the fall semester of 1975 by the researcher at Xavier University of Louisiana. Fifty-one students, of whom forty-eight were black, were enrolled in the course. Many of the classes were held in the Xavier University Mathematics Laboratory, but on days when no activities were scheduled, a traditional classroom was used. In the laboratory situation, the students worked in small groups so that discussion and student interaction were encouraged.

The Achievement Test covering the four sections of the course equally was administered at the beginning and the end of the course. Hypothesis H_1 was tested by a t-test for paired data on the raw scores of the two administrations. In addition, four intermediate tests were given, one at the end of each section of the course. The final grade assigned to each student in the course was based on the scores of the four in-term tests, the final examination and homework assignments.

The Attitude Scale was administered to the students in Math 105 during the first meeting of the class and again during the last meeting prior to the final examination. For each student, an attitude score for each of the four areas was calculated by finding the median of the scale values with which the student agreed. Hypotheses H_2, H_3, H_4, and H_5 were tested by means of a Sign Test for the comparison of medians and a McNemar Test for significance of item change. In addition, each student was asked to respond to an open-ended informal attitude evaluation instrument four times during the semester.

Conclusions and Implications

The conclusions and implications of the study are considered in three areas: those related to the teaching of the course, those related to the measurements of achievement, and those related to the evaluations of attitude.

Teaching of the Course

The mathematics laboratory approach as tested by this study seems to be a sound one. This was indicated by the fact that growth in achievement was definitely apparent and that both the instructor and the students had basically positive feelings about the experience. There were difficulties, however, which will need to be resolved in future attempts at this method.

The biggest problem was that of time. Performance of the activities frequently took longer than the fifty minutes allotted for class. A possible solution might be to have Math 105 run for two seventy-five minute periods a week rather than for three fifty minute periods. The more pressing problem with respect to time, however, eminates from the fact that the discovery approach necessarily takes longer than a lecture approach. Thus, it is difficult to cover the required amount of content. This is particularly true when the instructor needs to be available to answer questions and offer suggestions to thirty students at once. In the researcher's opinion, trained teacher assistants working with each group on a permanent basis and leaving the instructor free to move from group to group would make more progress possible. These assistants would not be students presently enrolled in the course, but students who had previously taken the course and had become familiar with the laboratory spirit.

Growth in Achievement

Hypothesis H_1 was tested by means of a comparison between the initial and terminal grades on the Achievement Test. According to a t-test for paired data it was rejected at the .05 level of significance. The differences between the initial and terminal scores of the students were impressive, the smallest being an increase of 45 points out of a possible 160 and the largest 124. The average growth was 82.7 points. In spite of this, 15% of the students received an F as their final grade in the course and 28% received a D. Possibly this was due to the fact that their incoming preparation was so poor that a single semester was insufficient to bring them to the required level. It should be noted, however, that this failure rate was a decrease from the 24% who received an F in Math 105 in 1974.

Changes in Attitude

The analysis of the initial and terminal scores of students in Math 105 on the Attitude Scale showed little change. None of the four hypotheses concerning attitude changes were rejected at the .05 level and only a few statements showed a significant change in the item analysis. However, on each of the four parts of the Attitude Scale, the majority of the students were assigned an initial scale value higher than the median value of all statements in that section. Thus it might have been unreasonable to have expected a significant rise.

On the informal attitude evaluation instruments, positive responses outnumbered negative ones three to one. These instruments measured attitude toward the course and the laboratory method more than toward mathematics itself, however. Thus, it would seem from this study that this method has very little effect upon attitudes toward mathematics.

Suggestions for Further Study

The comparisons in this study were of the students' achievement and attitude before taking Math 105 with their achievement and attitude after taking Math 105. The results of the laboratory approach as opposed to those of the traditional method were compared only by looking at the failure rate of the 1975 sections of Math 105 and those of previous years. Though other studies (41) have been done using a control group, there is need for several repetitions of that kind of study before any definite conclusions can be drawn.

The present study was based on Piagetian theory. However, the students were not tested prior to the course to determine their intellectual operational level. It is suggested that a study be conducted comparing the Piagetian levels of students in Math 105 with their success in the course. Another possible study is the comparison of the Piagetian levels of students entering Math 105 with their levels at the end of the course.

Responses on the Attitude Scale were highly positive and virtually unchanging. Yet discussion with the students and the informal attitude questionaires indicated that many did indeed have negative feelings toward mathematics. What is the cause of the apparent contradiction? Some possible reasons come to mind.

First the Attitude Scale was developed and validated for a very different population from that of the present study. In addition, it was developed at a time when the climate of thought among college students was quite different from that of the present time. Therefore, it would be interesting and helpful to repeat the present study using a different instrument to evaluate the initial and terminal attitudes of students in the course toward mathematics.

There is another possible explanation for the apparent contradiction. The students enrolled in Math 105 come from a population which believes strongly in the value of education as a vehicle to economic prosperity and a better life. Perhaps on formal written instruments like the Attitude Scale, such students repsond in accordance with their convictions rather than with their feelings. Informally or orally, however, they permit their true attitude to show. An interesting study would be to investigate the truth of this possibility and to determine what type of instrument would be effective in measuring attitudes in this kind of situation.

It is also possible that the reading skills of the students enrolled in the course were so poor that they did not adequately understand the statements on the Attitude Scale. Thus they might have indicated that they agreed with something while in reality they did not. A valuable study might be to run a multiple correlation on reading skills, assessed attitude on a written instrument and oral attitudes.

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APPENDIX A

INITIAL AND TERMINAL ACHIEVEMENT TEST

I. For each number below, state whether it is prime or composite. If it is composite, list all of its factors.

a) 28 b) 97 c) 169 d) 102 e) 144

II. Two geometric figures are congruent to each other if one can be transformed into the other by one of three transformations. Name these transformations and draw figures to demonstrate each one.

III. The four basic operations of arithmetic are addition, subtraction, multiplication, and division. Tell which of the sets, N, W, Z, Q, the even integers, and the odd integers, are closed under each operation.

IV. Work the following problem using scientific notation.

$$\frac{62{,}000{,}000 \times .000012}{.0008 \times 3{,}000}$$

V. Find the following by the prime factorization method.

a) GCD(18,30) b) LCM(18,30) c) GCD(108,81)

d) LCM(60,42) e) GCD(5,9)

VI. Decide whether the following statements are true or false. If a statement if false, rewrite it so that it is true.

a) Every square is a rhombus.

b) An isosceles triangle is symmetric with respect to a line.

c) A line segment is a closed curve.

d) Every prism has at least five faces.

e) A ray is a half-line without its endpoint.

f) Two lines in the same plane which never intersect are called parallel.

g) In spherical geometry it is not possible to have an equilateral right triangle.

h) Every trapezoid is isosceles.

i) Every square is a rectangle.

j) All circles are similar to each other.

VII. Decide whether the following are open or closed sentences. Then decide whether they are true or false, or if you can't decide, find all values of Z that will make them true.

a) $3x < 9$ b) $3 + 6 > 4 - 1$ c) $2 \neq 6$

d) $1 = 4x + 9$ e) $1 + 2(6) - 4 > 86 - 103$

VIII. Write the following numbers in expanded form.

a) 376.14 b) 101_{two} c) 341_{six}

d) 1001_{three} e) 100

IX. Solve the following in Q. Check your answers.

a) $\frac{1}{2}x - 6 < \frac{3}{4}x + 4$

b) $\frac{3}{8}x + \frac{7}{12} = \frac{1}{4}x - \frac{1}{18}$

X. For each pair below, decide if the first is less than, equal to, or greater than the second. If they are not equal, find one number which lies between the two.

a) 1/2, -2/3 b) -2/3, 2/-3 c) 1/4, 8/9

d) -(1/-4), 6/-91 e) -101/200, 202/400

XI. State the following laws and give examples of each.

a) Commutative Law of Addition of Whole Numbers

b) Associative Law of Addition of Whole Numbers

c) Commutative Law of Multiplication of Whole Numbers

d) Associative Law of Multiplication of Whole Numbers

e) Distributive Law of Multiplication over Addition of Whole Numbers.

XII. Change the following to meters.

a) 17 dekameters   b) 14 centimeters   c) 206 millimeters

d) 45 kilometers   e) 16 hectometers

XIII. $A = \{1,3,2,4\}$ $B = \{a,b,c\}$

a) What is n(A)? n(B)?

b) Use A and B to demonstrate that $4 + 3 = 7$.

c) Use A and B to demonstrate that $4 \times 3 = 12$.

d) Use A to demonstrate the difference between an ordinal and a cardinal number.

e) Construct a set which is equivalent to B and demonstrate the one-to-one correspondence.

XIV. Without actually dividing, tell which of the following numbers are divisible by 2, by 3, by 5, by 6, by 9, and by 10.

a) 4620   b) 10131   c) 26   d) 411216

e) 84

XV. A line is measured to be $7\frac{1}{2}$ inches long. Is that exact or approximate? If a ruler is used whose precision is $\frac{1}{2}$ inch, express the measurement in terms of the greatest possible error.

XVI. Express each number below in lowest terms and with at most one minus sign. Also give its reciprocal.

a) $-4/12 - 5/8$ b) $-(-3/17)$ c) $4\frac{3}{7} + 1$

d) $-1\frac{1}{17}$ e) $18/36$ f) $1/2 + 3/4$

g) $(1/4 \times 3/4) \div 6/10$

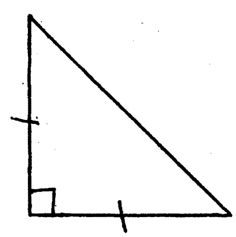
APPENDIX B

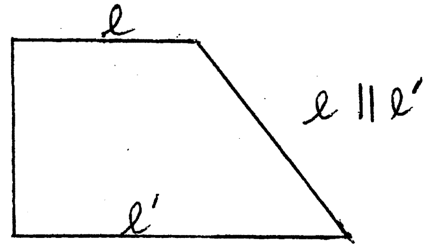
INTERMEDIATE ACHIEVEMENT TESTS

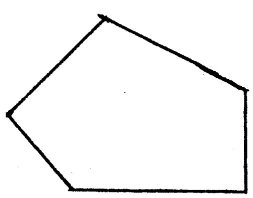
Test One

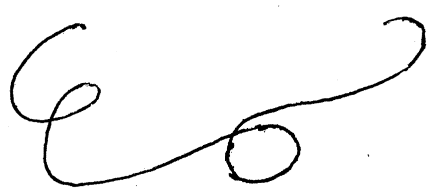
I. Name the three transformations of a figure that preserve size and shape, i.e., transformations where the image is congruent to the original. <u>Illustrate each one.</u>

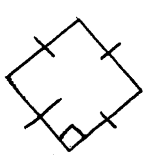
II. Name each of the following by the most descriptive name possible.

a) 

b) 

c) 

d) 

e) 

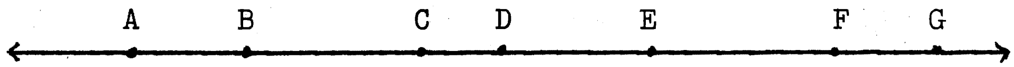
III. How many a) faces; b) edges; c) corners

does an octagonal prism have?

How many d) faces; e) edges

does a square pyramid have?

IV.



a) $\overline{AC} \cap \overleftrightarrow{BD} =$ b) $\overline{AC} \cup \overleftrightarrow{BF} =$ c) $\overleftrightarrow{EG} \cap \overleftrightarrow{FD} =$

d) $\overleftrightarrow{EG} \cup \overleftrightarrow{FD} =$ e) $\overline{AC} \cap \overline{CE} =$

V. Draw an example of the following and show any lines of symmetry. Be sure to indicate any lines or angles you mean to be congruent, any right angles, and any lines you mean to be parallel.

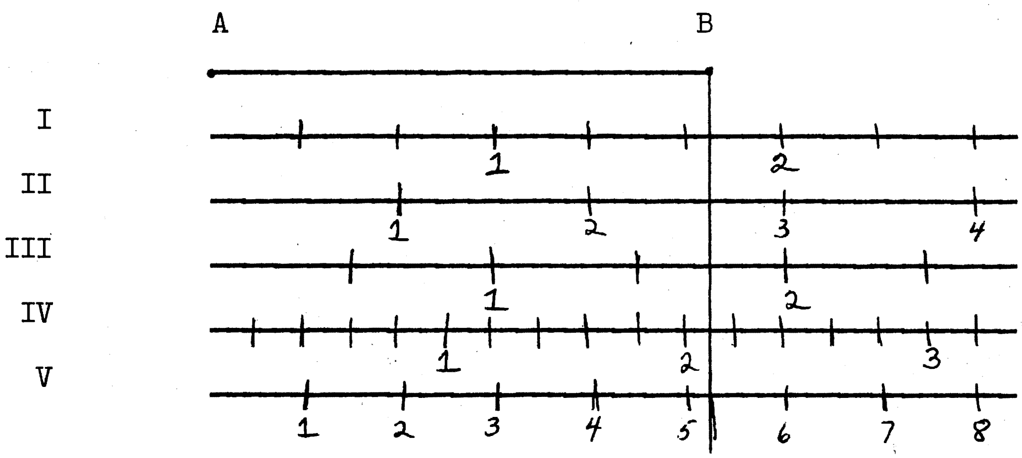
a) square b) rectangle c) trapezoid

d) parallelogram which is not a rectangle e) circle

f) scalene triangle g) a non-simple closed curve

h) isosceles triangle i) pentagon j) rhombus

VI. Measure segment AB according to each of the following rulers. Give the precision of each ruler and express each measure exactly in terms of the greatest possible error.



VII. Give a non-mathematical example of each of the following:

a) a figure symmetrical with respect to a line

b) a cylinder

c) a circular region

d) a pair of skew lines

e) intersecting planes

VIII. Determine whether the following statements are true or false.

a) The product of slides is not always a slide.

b) All squares are similar to each other.

c) It is possible for two figures to be similar without being congruent.

d) The letter A is symmetric with respect to a point.

e) A circle is concave.

f) A simple closed curve composed of a union of line segments is called a polygon.

g) A square is a regular quadrilateral.

h) An obtuse right triangle is impossible in every geometry.

i) All measurement is approximate.

j) A plane divides a line into two half-lines.

Test Two - Form I

I. a) Give two sets which can be used to demonstrate $4 + 7 = 11$, and show how they demonstrate it.

b) Use the Cartesian product model of multiplication and the sets $A = \{red, black\}$ and $B = \{a, b, c\}$ to demonstrate $2 \times 3 = 6$.

c) For each of the problems in a) and b) above, give the related inverse problem.

II. a) Change 432 to base five.

b) Change 124_{seven} to base ten.

III. Find the following:

a) $20 - (-11) + 7$ b) $-14 + 2 - 6 + 9$

c) $-7 + (-7) - -14$

IV. Give the additive inverse of each of the answers you found in III.

V. Put the answers you found in III in order beginning with the smallest and ending with the largest.

VI. A = $\{1,2,3,4,5\}$ B = $\{2,4,6,8\}$ C = $\{1,3,5,9\}$

D = $\{6,7,8,9,10\}$ E = $\{5,4,3,2,1\}$

a) Which of the sets above are equivalent?

b) Which are equal?

c) Set up a 1-1 correspondence between A and some other set.

d) Find n(A), n(B), $n(A \cap B)$, $n(A \cap C)$, $n(D \cap E)$ and $n(B \cup C)$.

VII. Find solution sets in Z of the following open sentences and check your solutions.

a) $13x + 12 = 9x + 72$ b) $4x + 5(x + 1) - 8 < 87$

c) $3(x + 9) - 6 > 25$

VIII. Explain each step in the following problems.

a)
```
            7 ⎫
           40 ⎬  147
          100 ⎭
    14 ) 2046
         1400
          646
          560
           86
           84
            2
```

b) $422 \times 36 = 422(30 + 6)$

$= (422 \times 30) + (422 \times 6)$

```
   422
  x 36
  2532
 1266
 15196
```

IX. Give an example which shows the following laws using Whole Numbers.

a) Distributive Law of Multiplication over Addition

b) Commutative Law of Addition

c) Commutative Law of Multiplication

d) Associative Law of Multiplication

e) Associative Law of Addition.

X. Determine whether the following statements are true or false.

a) Subtraction is not closed on the set of Whole Numbers.

b) $123_{five} + 412_{five} = 535_{five}$

c) Division is the inverse of multiplication.

d) $2 < 5$ because $2 + 3 = 5$.

e) $3 - 7 = 7 - 3$.

f) Subtraction is closed on the set of Integers.

g) $-(-8) = -8$.

h) Zero is the multiplicative identity in the set of Whole Numbers.

i) $7 = 6$ is a closed sentence.

j) Division is not closed on the set of Integers.

Test Two - Form II

I. Find the following.

a) $18 - (-9) + 5$ b) $-16 + 2 - 6 + 7$

c) $-9 + (-9) - -18$

II. Give the additive inverse of each of the answers you found in I.

III. Put the answers you found in I in order beginning with the smallest and ending with the largest.

IV. a) Give two sets which can used to demonstrate $3 + 8 = 11$, and show how they demonstrate it.

b) Use the Cartesian product model of multiplication and the sets $A = \{red, black, green\}$ and $B = \{1,2\}$ to demonstrate $3 \times 2 = 6$.

c) For each of the problems in a) and b) above, give the related inverse problem.

V. a) Change 432 to base seven.

b) Change 124_{six} to base ten.

VI. Explain each step of the following problems.

a)

$$\begin{array}{r@{\,}l}
 & \left.\begin{array}{r} 7 \\ 40 \\ 100 \end{array}\right\} \; 147 \\
14 \overline{)} & \overline{2046} \\
 & \underline{1400} \\
 & 646 \\
 & \underline{560} \\
 & 86 \\
 & \underline{84} \\
 & 2
\end{array}$$

b) $422 \times 36 = 422(30 + 6)$

$= (422 \times 30) + (422 \times 6)$

$$\begin{array}{r}
422 \\
\underline{\times\ 36} \\
2532 \\
\underline{1266} \\
15196
\end{array}$$

VII. $A = \{a,b,c,d,e\}$ $\quad B = \{b,d,f,h\}$ $\quad C = \{a,c,e,i\}$

$D = \{f,g,h,i,j\}$ $\quad E = \{e,d,c,b,a\}$

a) Which of the sets above are equivalent?

b) Which are equal?

c) Set up a 1-1 correspondence between set A and some other set.

d) Find $n(A)$, $n(B)$, $n(A \cap B)$, $n(A \cap C)$, $n(D \cap E)$ and $n(B \cup C)$.

VIII. Find solution sets in Z of the following open sentences and check your solutions.

a) $13x + 12 = 7x + 72$ $\qquad$ b) $4x + 4(x + 2) - 8 < 120$

c) $3(x + 9) - 6 > 25$

IX. Determine whether the following statements are true or false.

a) Subtraction is closed on the set of Whole Numbers.

b) $123_{five} + 412_{five} = 535_{five}$.

c) Division is the inverse of multiplication.

d) $2 < 5$ because $2 + 3 = 5$.

e) $3 - 7 = 7 - 3$.

f) Subtraction is closed on the set of Integers.

g) $-(-8) = -8$.

h) Zero is the multiplicative identity in the set of Whole Numbers.

i) $7 = 6$ is a closed sentence.

j) Division is closed on the set of Integers.

X. Give an example which shows the following laws using Whole Numbers.

a) Distributive Law of Multiplication over Addition

b) Commutative Law of Addition

c) Commutative Law of Multiplication

d) Associative Law of Multiplication

e) Associative Law of Addition.

Test Three

I. Define the following and give an example of each.

a) divisor
b) multiple
c) prime number
d) composite number
e) even number
f) odd number
g) perfect number
h) square number
i) triangular number
j) greatest common divisor

II. State the Fundamental Theorem of Arithmetic and give an example of it.

III. Give the prime factorizations of the following numbers.

a) 30 b) 24 c) 123 d) 69 e) 18

IV. Find the following by means of the algorithms.

a) GCD (18,30)
b) LCM (18,30)

V. Find the following by means of prime factorization.

a) GCD (84, 96)
b) LCM (27,63)
c) GCD (16, 44)
d) LCM (33, 121)

VI. Determine whether the following statements are true or false.

a) Every triangular number is odd.

b) 16 and 49 are relatively prime.

c) LCM (16, 32) = 32.

d) 123 is divisible by nine.

e) There are 4,637,826 prime numbers.

f) To find all factors of 86, one need only test numbers up to and including 9.

g) GCD(3,7) = 0.

h) The even numbers are closed under addition.

i) The sum of the digits of a number gives the remainder when the number is divided by six.

j) The odd numbers are closed under multiplication.

VII. Find <u>all</u> the factors of the following numbers.

a) 30 b) 24 c) 123 d) 69 e) 18

VIII. Without dividing decide whether the following numbers are divisible by 2,3,5,6,9 or 10.

a) 30 b) 41100

Test Four - Form I

I. Perform the following operations and reduce answers to lowest terms.

a) 4/21 + 3/7 b) 3/4 + 5/6 c) (1/9 + 4/12) - 3/4

II. For each fraction below, give

a) the additive inverse and b) the multiplicative inverse.

Identify each one by marking the additive inverse with A and the multiplicative inverse with M.

a) 4/9 b) -17 c) -(-1) d) -1 e) 0

III. Determine whether the following statements are true or false.

a) 17% = .17 = 17/100

b) The multiplicative inverse of 0 is 0.

c) The rational number which immediately follows 1/5 is 2/5.

d) The rational numbers are closed under division.

e) 1.6 + 6.1 = .77

f) 1.6 x .1 = 1.6

g) 7/8 = 49/56

h) The lowest common denominator of 1/4 and 3/12 is 24.

i) Cancellation is based on the fact that zero is the identity element in addition.

j) 3/200 = 1.5%

IV. In each case below decide whether the first fraction is less than, equal to, or greater than the second. If they are not equal, find one fraction between them.

a) 1/6, -3/4 b) 4/10, 2/5 c) 3/8, 4/9

d) 7/8, 8/7 e) 6/9, 27/18 f) 14/5, 42/15

V. Perform the following operations. Make sure all answers are in lowest terms.

a) 5/7 x -12/6 b) -1/12 ÷ -4/3 c) -9/-2 x -7/15

VI. Solve the following for x.

a) $3/2\ x - 6 < 15/4\ x + 9$

b) $x + \frac{1}{2} = 7/9\ x - 4$

VII. Write the following as percents.

a) .016 b) 17/34 c) 3.02

d) 1/9 e) 8

VIII. Solve the following using scientific notation. Write the answer without scientific notation.

$$\frac{3000 \times .000009 \times 480000}{16{,}000 \times 300}$$

IX. Express the following as a single decimal.

a) 16% b) 1.4 x 1.06 c) 1.4 ÷ 1.06

d) 31.6% e) 3/4

X. In these problems simply <u>set up</u> the equations. You do not have to solve them.

a) A sofa is marked up 15% of its cost to a retailer to cover operating expenses. If it is marked up $30, how much did it cost the retailer?

b) Another piece cost the retailer $120. It is also marked up 15%. How much is it marked up?

c) A third piece cost the retailer $500. He marked it $50 higher. What was the percent of mark-up?

Test Four - Form II

I. Perform the following operations. Make sure all answers are in lowest terms.

a) 10/9 x -12/8 b) -1/2 ÷ -4/3 c) -9/2 x -7/12

II. Express the following as a single decimal.

a) 15% b) 1.4 x 1.08 c) 1.60 ÷ 1.04

d) 71.5% e) 3/8

III. For each fraction below, give

a) the additive inverse, and b) the multiplicative inverse.

Identify the additive inverse with A and the multiplicative with M.

a) -4/9 b) 1/17 c) -(-2) d) - 1 e) 0

IV. In each case below decide whether the first fraction is less than equal to, or greater than the second. If they are not equal, find one number between them.

a) 14/5, 42/15 b) 6/9, 27/18 c) 7/8, 8/7

d) 3/8, 4/9 e) 4/10, 2/5 f) 1/6, -3/4

V. Determine whether the following statements are true or false.

a) .16 = 16%

b) The multiplicative inverse of 0 is 0.

c) The rational number which immediately follows 1/7 is 2/7.

d) The rational numbers are closed under division.

e) 6.1 + 1.6 = .77

f) 1.6 x .1 = 1.6

g) 3/4 = 21/28

h) The lowest common denominator of 3/4 and 1/12 is 24.

i) Cancellation is based on the fact that zero is the identity element for addition.

j) 1.5% = 3/200.

VI. Perform the following operations and reduce answers to lowest terms.

a) 6/25 + 3/5 b) 3/7 + 5/6 c) (1/8 + 4/12) - 3/2

VII. Set up the equations to solve each of the following. You do <u>not</u> have to solve them.

a) A sofa is marked up 15% of its cost to a retailer in order to cover operating expenses. If it is marked up $30, how much did it cost the retailer?

b) Another piece cost the retailer $120. It is also marked up 15%. How much is it marked up?

c) A third piece cost the retailer $500. He marked it $50 higher. What was the percent of mark-up?

VIII. Solve the following using scientific notation. Write the answer without scientific notation.

$$\frac{4000 \times .000012 \times 480000}{16{,}000 \times 300}$$

IX. Solve the following for x.

a) $2/3\ x - 6 < 15/4\ x + 9$

b) $2x + \frac{1}{2} = 7/9\ x - 4$

X. Write the following as percents.

a) .012 b) 18/36 c) 4.06 d) 1/11

e) 7

APPENDIX C

ATTITUDE SCALE

The statements of the Attitude Scale are given below. The numbers of the statements indicate their placement on the scale when the students responded to it. The scale value of each statement is given in parenthesis at the end of the statement.

The directions for the students responding to the scale were: Read each of the statements given below. Place a check immediately to the left in the space provided of those statements which express your present feeling toward mathematics.

Part I: Views Concerning the Learning of Mathematics

9. Very few people can learn mathematics. (.000)

27. Only people with a special talent can learn mathematics. (.463)

1. Most work in mathematics is the memorizing of information. (1.107)

5. Anyone can learn mathematics. (1.231)

20. Mathematics can be made understandable and useful to every college student. (2.064)

16. Any person of average intelligence can learn to understand a good deal of mathematics. (2.306)

24. Almost any student can learn mathematics if it is properly taught. (2.314)

Part II: Views Concerning Mathematics
as a Process

21. There is little place for originality in mathematics. (.000)
13. Almost all present-day mathematics was known at least a century ago. (.179)
17. Mathematics is a good field for creative people to enter. (.392)
25. Mathematics will change rapidly in the future. (.502)
6. Mathematics helps one think according to strict rules. (1.058)
2. In mathematics there is always a rule to follow in solving problems. (1.106)
10. Mathematics helps one develop a good sense of logic. (2.643)

Part III: Views Concerning the Place of
Mathematics in Society

11. Mathematics (algebra, geometry, etc.) is <u>not</u> useful for problems of everyday life. (.000)
3. Outside of sciences and engineering, there is little place for mathematics (algebra, geometry, etc.) in most jobs. (.112)
22. Unless one is planning to become a mathematician or scientist, the study of mathematics is not very important. (.185)
14. A thorough knowledge of advanced mathematics is a key to an understanding of our world in the twentieth century. (.887)
26. In the near future most jobs will require knowledge of advanced mathematics. (.905)
18. It is important to know mathematics (algebra, geometry etc.) in order to get a good job. (.934)

7. Mathematics is of great importance to a country's development. (1.226)

Part IV: Views Concerning School and Learning Generally

8. I dislike school and will leave it as soon as possible. (.000)
23. Most school work is the memorizing of information. (1.104)
4. I generally like my school work. (1.354)
19. I find school interesting and challenging. (1.532)
12. School is not very enjoyable, but I can see value in getting a good education. (1.598)
15. Although school is difficult, I want as much education as I can get. (1.902)

APPENDIX D

INFORMAL ATTITUDE EVALUATION INSTRUMENTS

Questionaire One

Please comment on how you feel about the Activities you performed for this unit. Which ones, if any, did you find helpful? Which ones were confusing? Have you enjoyed any of them? Which ones?

Questionaire Two

Please comment on how you feel about the Activities you performed for the unit on Natural Numbers, Whole Numbers and Integers. Which ones, if any, did you find helpful? Which ones were confusing? Have you enjoyed any of them? Which ones?

Questionaire Three

Although the chapter on Number Theory has very few Activities, it would still be helpful to have your opinion of them. Did they aid you in understanding the material or just make it more confusing? Please review pages 78 - 83 before answering.

Questionaire Four

Did the activities with the Cuisenaire rods and the Up to twenty blocks help you in your understanding of equal fractions? Did you enjoy the activities?

Did the activities with the number line help you in your understanding of addition of fractions? of lowest terms? Did you enjoy the activities?

Did the activities with Mira help you in your understanding of multiplication of negative numbers? Did you enjoy the activities?

VITA

Sister Rosemarie Kleinhaus

Candidate for the Degree of

Doctor of Education

Thesis: A DEVELOPMENT OF MATERIALS TO BE USED IN A LABORATORY APPROACH TO A MATHEMATICS CONTENT COURSE FOR PRE-SERVICE ELEMENTARY TEACHERS AND THE EFFECTS OF THIS APPROACH ON ACHIEVEMENT AND ATTITUDE

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