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# DYNAMIC BUCKLING OF AN AXIALLY COMPRESSED CYLINDRICAL SHELL WITH DISCRETE RINGS AND STRINGERS

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DYNAMIC BUCKLING OF AN AXIALLY COMPRESSED CYLINDRICAL SHELL WITH DISCRETE RINGS AND STRINGERS

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| | |
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## SYMBOLS

| | |
|---|---|
| A | Stiffener cross-sectional area (in.$^2$) |
| d | Stringer center-to-center spacing (in.) |
| D | Bending stiffness (in.-lb.) |
| $\bar{e}$ | Distance from shell middle surface to line where $\hat{N}_x$ acts (in.) |
| E | Young's modulus (psi) |
| $\bar{F}$ | Nondimensional buckling load, $\bar{F} = \hat{N}_x R/Eh^2$ |
| g | Time step |
| G | Shear modulus (psi) |
| h | Unstiffened shell wall thickness (in.) |
| i | Integer |
| I | Moment of inertia of stiffener about its centroid (in.$^4$) |
| $I_o$ | Moment of inertia of stiffener about shell middle surface (in.$^4$) |
| $I_i$ | $\equiv im\pi/L$ (in.$^{-1}$) |
| $I_j$ | $\equiv im\pi j\ell/L$ (dimensionless) |
| $I_\zeta$ | $\equiv \zeta m\pi j\ell/L$ (dimensionless) |
| j | Integer |
| J | Polar moment of inertia (in.$^4$) |
| $J_j$ | $\equiv jn/R$ (in.$^{-1}$) |
| $J_k$ | $\equiv jnkd/R$ (dimensionless) |
| $J_\xi$ | $\equiv \xi nkd/R$ (dimensionless) |
| k | Integer |

| | |
|---|---|
| $\ell$ | Ring center-to-center spacing (in.) |
| L | Shell length (in.) |
| m | Number of half-waves in axial direction |
| M | Number of stringers on cylinder |
| $M_x, M_y, M_{xy}$ | Middle-surface moment resultants |
| n | Number of waves in circumferential direction |
| N | Number of rings on cylinder |
| $N_x, N_y, N_{xy}$ | Middle-surface stress resultants |
| $\hat{N}_x$ | Externally applied axial load resultant (positive in comp.) |
| Q | Galerkin "error functions" |
| r | Subscript referring to ring quantities |
| R | Radius of cylinder middle surface (in.) |
| s | Subscript referring to stringer quantities |
| t | Time (sec) |
| u,v,w | Displacements in axial, circumferential, and radial directions |
| U | Strain energy (in.-lb.) |
| W | Dependent variable used in Runge-Kutta algorithm |
| $\bar{W}$ | Dimensionless radial deflection of shell middle surface, $\bar{W} = w/w_{initial}$ |
| x,y,z | Circular cylindrical coordinates with origin lying in middle surface of shell and oriented in the axial, circumferential, and radial directions, respectively |
| $\bar{z}$ | Distance from centroid of stiffener to shell middle surface (in.) |
| Z | Batdorf parameter, $Z = L^2(1-\nu^2)/(Rh)$ |
| $\delta_{ij}$ | Kroneker delta |
| $\delta(x-j\ell)$ | Dirac delta function |

| | |
|---|---|
| $\Delta_1$ | $\delta_{i\zeta}\delta_{j\xi}$ (Double Kroneker delta) |
| $\Delta_2$ | $\delta_{i\zeta}\delta_{3j\xi}$ |
| $\Delta_3$ | $\delta_{3i\zeta}\delta_{j\xi}$ |
| $\Delta_4$ | $\delta_{3i\zeta}\delta_{3j\xi}$ |
| $\Delta_5$ | $\delta_{2i\zeta}\delta_{2j\xi}$ |
| $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ | Axial, circumferential and shear strains of shell middle surface |
| $\zeta$ | Integer |
| $\nu$ | Poisson's ratio |
| $\xi$ | Integer |
| $\rho$ | Density (lb-sec$^2$/in.$^4$) |
| $\sigma_x, \sigma_y, \tau_{xy}$ | Axial, circumferential and shear stresses |
| $\sum_j$ | Indicates $\sum_{j=1}^{N}$ |
| $\sum_k$ | Indicates $\sum_{k=1}^{M}$ |
| $\Sigma\Sigma$ | Indicates $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}$ |

A subscript preceded by a comma indicates partial differentiation with respect to the subscript.

Primes indicate total derivatives with respect to x.

# DYNAMIC BUCKLING OF AN AXIALLY COMPRESSED CYLINDRICAL SHELL WITH DISCRETE RINGS AND STRINGERS

## CHAPTER I

## INTRODUCTION

### 1.1 Survey of Cylindrical Shell Buckling

For the past decade, the aerospace and defense industry has been actively involved in studying the structural response of high-performance aircraft and missiles to blast-induced loadings. However, the structural response of a light aircraft (in the general aviation fleet) subjected to a crash loading has been largely ignored. Apparently, there are no current studies directed toward understanding the structural response of such aircraft to crash loadings. There are several reasons cited for this lack of analytical effort. These reasons are research cost, great variation in design of aircraft in the general aviation fleet, complexity of the crash-loading time-history delivered to the aircraft structure, and the lack of a procedure to treat the dynamic response of a complex sturcture.

The design of a more crashworthy aircraft will undoubtedly receive greater emphasis in the near future. In a recent article appearing in Time magazine [1], the high accident rate of the nation's 3,200 "third level" carriers was discussed. These include air taxis and commuter lines that usually fly smaller planes such as Cessna, Piper, Beechcraft and the like. According to Time, "last year 106

people died in third-level crashes. The accident death rate for every 100,000 hours flown is 1.31 for the third-levels, as compared with 0.09 for the nation's eleven first level trunk carriers and nine regionals." This high accident rate was further documented in a FAA report [2] which claimed that the death rate for general aviation is seven times that of automobiles. While many of these accidents could be avoided by improved procedures, many of the fatalities could probably have been avoided by an aircraft designed to withstand a certain level of crash loading.

At the present time, crash impact loading is not considered in the design of light aircraft. The problem is understandably complex, due to the great number of unknowns, and the lack of any controlled experimental data. However, the problem should be manageable if the structure is broken down into representative elements and if the crash loading is simplified. After the response of a representative element is correctly modeled and understood, a more rational design could perhaps be proposed to resist the effects of crash loading on the structure.

As a point of departure, the basic structural element of a light aircraft is assumed to be a thin cylinder, internally stiffened by both stringers and rings (bulkheads). One typical loading to be expected in a crash would be a suddenly applied axial compression loading of short to medium duration. If one could mathematically model the dynamic behavior of a stiffened cylinder to such a loading, one could then begin to rationally design a "crashworthy" aircraft.

Beginning first with the static axial buckling studies of unstiffened thin-walled cylinders, one is impressed by the incredible

number of papers written over the past several decades on this subject alone. This fact is due in large part to the serious discrepancies between theory and experiment. Two authors who have provided extensive bibliographies on this subject are Hoff [3] and Stein [4].

The so-called classical linear buckling theory was probably first formulated by Timoshenko [5]. In this theory, effects due to nonlinear prebuckling and initial imperfections are neglected. Unfortunately, experimental results range from twenty per cent to eighty per cent of the values predicted by this linear small deflection theory. Investigators have attempted to resolve this discrepancy by including in their analysis one or more of the following general effects: in-plane boundary conditions, nonlinear prebuckling, nonlinear postbuckling and initial imperfections.

Apparently, the first effects considered were initial imperfections and nonlinear postbuckling. In 1934 Donnell [6] proposed a nonlinear finite-deflection theory, together with a consideration of initial imperfections present in the thin-walled cylinder. This theory was modified and refined by numerous authors including von Kármán and Tsien [7], Kempner [8], and Almroth [9]. In order to answer questions of convergence of various solutions in the postbuckling range, Hoff [10] solved the large displacement equations using a radial displacement expression containing 1100 terms. It should be noted, however, that all authors have implicitly ignored exact satisfaction of some boundary conditions. This is partly due to the nonlinear terms in the boundary conditions arising from the nonlinear strain-displacement relations. Hoff [10] stated that the assumption of $L/R>1$ makes it possible

In-plane boundary conditions can play a significant role in obtaining theoretical buckling loads lower than the classical value. This has been pointed out by numerous authors [3,15-17]. Apparently, Hoff [3] was one of the first to document four different sets of boundary conditions which could be identified as "simple-support boundary conditions". Experiments performed in the ordinary tension-compression testing machine could be represented by several of these sets, depending upon the friction between testing machine and test specimen.

It is now generally agreed that, given a realistic set of boundary conditions, the main reason that theory and experiment did not agree in the past was due to initial imperfections in the test cylinder. The agreement obtained between theoretical (linear) results for a perfect shell with correct boundary conditions and experimental results from very nearly perfect specimens indicate the validity of the linear theory [19,20].

Theoretical buckling predictions for stiffened cylinders have not shown the wide disparity with experimental results that their unstiffened counterparts have shown. However, numerous authors have investigated the effects of in-plane boundary conditions, nonlinear prebuckling, and initial imperfections. Nonlinear prebuckling deformations have been shown to have only a small effect on the buckling load of stiffened cylinders [16,21-24]. The effect of in-plane boundary conditions on the buckling load of stiffened cylinders has also been investigated [16]. As before, the choice of in-plane simple support boundary conditions can make a difference on the magnitude of the buckling load.

However, as noted in reference [16], the influence of in-plane boundary conditions diminishes for internally stiffened shells with increasing values of stiffener eccentricity and area. Of course, a light-aircraft structure would have internally located stiffeners.

As in the unstiffened case, initial imperfections can make a considerable difference in the experimentally determined buckling load of stiffened cylinders. The bulk of this research has been conducted using the imperfection sensitivity concepts first introduced by Koiter [25] and expanded by numerous authors [26-30]. However, as pointed out by reference [24], the predicted regions of large imperfection sensitivity shift from one study to another, and have not been verified by experiment. Also, the predicted sensitivities appear to depend strongly on quantities such as torsional stiffness of stiffeners, which only slightly affect the classical buckling load [32]. Fortunately, the studies have shown that an internally stiffened shell is relatively imperfection insensitive, whereas an externally stiffened shell is imperfection sensitive [29,30].

The importance of two other effects, stiffener eccentricity and stiffener discreteness, have been explored by numerous investigators of stiffened shells. The stiffener eccentricity or one-sideness has a significant effect on the buckling strength, as demonstrated in various ways by many authors [32-40]. Stiffener discreteness was shown by Block, *et al.* [21] to have a significant effect on the buckling load, even for structures with many stiffeners. Block [21] used a finite difference approach and considered discrete rings only. His conclusion is in direct contrast with Singer's concluding remark in reference [41]

that "in ring stiffened shells under axial compression, the discreteness effect is always very small." Singer used a linear Donnell theory, and treated the stiffeners as linear discontinuities represented by the Dirac delta function. Despite the disagreement as to the importance of stiffeners in most light-aircraft, both stiffener eccentricity and discreteness should be included.

Linear buckling theory has been used by nearly all investigators in their study of stiffened cylinder buckling, primarily because stiffeners are closely spaced in most aerospace applications. However, as demonstrated by reference [43], linear theory is only applicable when the stiffeners are closely spaced. Again, this will not be the case for the light-aircraft structure considered in this study.

This problem of static buckling of stiffened and unstiffened cylinders has been extensively researched over the past 60 years. By contrast, dynamic buckling of axially loaded cylinders has been studied only for the past 15-20 years, and the amount of published work is only a small fraction of its static buckling counterpart.

The first studies were primarily experimental and phenomenological in nature. Some of the earliest work was published by Schmidt [44] and Coppa [45]. Both experimenters considered only unstiffened cylinders.

One of the first analytical treatments of dynamic buckling is credited to Volmir [48]. In 1957, he investigated the buckling of a shallow circular cylindrical panel subjected to a rapidly applied axial load. He used the large deflection shell equations (generally attributed to Donnell) and reduced them to equations in time only, using a

Galerkin procedure. By making certain restrictions on the time-dependent coefficients of the assumed radial deflection function, the system of equations was reduced to a single equation, which was then solved numerically. The final results were presented in the form of dimensionless axial load versus time. A closely related study was one by Agamirov and Volmir [47]. As in all subsequent work, both the longitudinal and circumferential inertia were neglected.

In 1962, using essentially the same procedure as Volmir used, Coppa and Nash [48] studied the dynamically loaded thin cylinder. They used a two-term Galerkin procedure, in which the terms approximated the familiar diamond buckle shape. Rotatory and axial inertia terms were neglected. The assumed two-term solution contained a guess as to the axial and circumferential buckle wavelength. The procedure was repeated over a range of axial and circumferential wavelengths to find the combination that gave the lowest buckling load. A constant rate of axial end shortening was used as the dynamic loading mechanism. Only qualitative correlations with related experiments were made.

In 1964 Roth and Klosner [49] studied the same problem previously investigated by Coppa and Nash. Roth and Klosner constructed appropriate kinetic and potential energy relations. They used the middle-surface nonlinear strain-displacement relations for thin circular shells which was based on the work of Donnell [6]. Applying Hamilton's principle, they derived the equations of motion and appropriate boundary conditions. Next, they rewrote their governing equations of motion using an Airy-type stress function, and obtained the same equations as Coppa and Nash. Using a four-term radial-deflection function,

together with the Galerkin procedure, they reduced their problem to a set of four ordinary nonlinear differential equations. This final set was solved using Runge-Kutta procedures. As is the case with all prior large deflection theories, the assumed deflection modes did not exactly satisfy all of the boundary conditions. Also, no correlations with any experiments were made.

Lindberg [50] used a linear, small-deflection theory instead of a large deflection theory, to calculate the growth of normal modes of cylinders under axial impact. An inspection of Lindberg's experimental setup quickly reveals that he was using an impulsive rather than a step function loading. His shells were free at the end opposite the impact end. As a result, the compressive impact stress had a duration (at the impact end), which at most was equal to the transit time of the longitudinal stress wave up and down the shell.

In a recent dissertation, Howell [51] studied the transient response of stiffened cylinders to an impact load. He used a linear theory and smeared the effect of the stiffeners over the cylinder surface. However, he applied the impact to a point on the cylinder surface between stringers rather than to the ends of the cylinder. Also, his time duration of loading and subsequent transient response measurements indicate a wave propagation-type of study rather than a buckling analysis.

The amount of experimental data on the dynamic buckling of unstiffened shells, other than very early work (44,45], has been sparse. The most recent was that of Tennyson [52]. To date, no experimental data on the dynamic buckling of stiffened cylindrical shells have been found.

The influence of damping on the dynamic stability of shells has been investigated recently by several authors. Mescall and Tsui [53] found that damping always increased the critical dynamic buckling load for the thin cylinders, cones, and spheres considered in their analysis. Consequently, it should be expected that a dynamic buckling analysis that neglected damping would yield conservative results.

## 1.2 Research Objectives

The major objective of this research is to develop a dynamic buckling analysis capable of predicting the dynamic response and buckling load of a stiffened, thin, circular cylindrical shell under the action of a suddenly applied step axial-loading pulse. This pulse will be similar to that experienced by a light aircraft during a crash on take-off or landing. The cylinder will be stiffened with widely spaced rings and stringers, which will be representative of modern light-aircraft fuselage structures. The stiffeners will be considered as discrete elements with eccentricity due to their internal location.

In order to accomplish the research objectives it has been necessary to devise a number of new approaches and methods. The following specific original contributions are noted:

1. The first discretely stiffened cylinder buckling analysis in which nonlinear rather than linear strain-displacement relations are used.
2. The first cylinder buckling analysis to solve the resultant nonlinear algebraic and differential equations using a modified Gauss-Jordan technique in conjunction with a Runge-Kutta algorithm.

3. The first dynamic buckling analysis with the capability of handling any number of assumed deflection terms, limited only by the available computer capacity.
4. The first dynamic buckling analysis of a discretely stiffened cylinder.
5. The first dynamic analysis of any kind to use the Dirac delta function to account for stiffener discreteness.
6. The first dynamic buckling analysis to contain the effects of stiffener eccentricity or one-sideness.

# CHAPTER II

# FORMULATION OF THEORY

## 2.1 Method of Analysis

An energy approach is used to facilitate the writing of compatible governing equations for the unstiffened cylinder, stringers and rings, which all buckle as an unit. This procedure allows for treatment of stringers and rings as discrete elements, rather than the usual orthotropic "smeared" analysis. Appropriate expressions for the potential and kinetic energies of the unstiffened cylinder, stringers, and rings are formulated and presented in Appendix A. To allow for finite deflections of the cylinder during the buckling process, the appropriate nonlinear strain-displacement relations, as suggested by Donnell [6], are employed. Also, the strain-displacement relations for the rings and stringers are related to the mid-surface unstiffened shell displacements. Then, in Appendix B, Hamilton's principle is used to obtain the governing nonlinear differential equations of motion and the appropriate boundary conditions which govern the prebuckling and buckling of a stiffened cylinder. To obtain the prebuckling and buckling equations, the axial, circumferential and radial displacements of the shell (u,v, and w) are assumed to be separable into two parts as follows:

$$u(x,y,t) = u_A(x) + u_B(x,y,t)$$

$$v(x,y,t) = v_A(x) + v_B(x,y,t)$$

$$w(x,y,t) = w_A(x) + w_B(x,y,t) \qquad (2\text{-}1)$$

The subscript A denotes the axisymmetric prebuckling displacement; the subscript B denotes the time-varying unsymmetric buckling displacement.

The prebuckling equations and boundary conditions are obtained by substitution of the axisymmetric displacements in Equations (2-1) into Equations (B-10). In a like manner, the buckling equations and boundary conditions are obtained by substituting Equations (2-1) into Equations (B-11), and subtracting out the previously obtained prebuckling identities.

To solve the buckling equations, the prebuckling quantities (subscript A) are first determined directly. Galerkin's technique is then applied to the three buckling equations of motion. The result of this operation is a set of simultaneous, nonlinear, ordinary differential equations. The coupled equations are then solved by means of a Runge-Kutta technique with the aid of a digital computer.

## 2.2 Hypotheses

All of the following assumptions are implicit in the analysis:

1. The circular cylindrical shell and all stiffeners remain in the linear elastic range during buckling.
2. The cylinder undergoes a classical axisymmetric prebuckling deformation during axial loading.
3. Initial imperfections in the stiffened cylinder are neglected.
4. The Kirchoff-Love hypothesis is used for the shell; thus,

it is assumed to have a wall thickness which is small compared to its radius.

5. The Donnell shallow-shell assumptions [54] apply. The buckled cylinder contains at least several buckled circumferential wavelengths. See reference [55] for details.
6. The stiffeners are discretely located along the length and circumference of the cylinder, and the width of the stiffeners is small compared to the distance between them.
7. In-surface and rotatory inertia effects in the stiffened cylinder are neglected.
8. Stiffeners behave as beam elements, and displacements vary linearly across stiffener depth. This implies that the stiffener depth is small compared to the radius of the shell middle surface.
9. The total circumferential arc length of the ring is approximately the same as that of the middle surface of the shell.
10. The stiffeners are rigidly attached to the shell.
11. The stiffeners are symmetrical with respect to a normal from the shell middle surface passing through the stiffener centroid.
12. The boundary conditions at the shell ends are not necessarily satisfied exactly, as assumed by references [6,7,8, 10,45,48,49].
13. The cylinder has the characteristic buckling behavior of

a "long" cylinder. Thus, the Batdorf parameter is greater than thirty.

14. Effects of axial-wave propagation do not influence the wavelengths at which the buckles form.
15. All material damping, thermal, and initial-stress effects are neglected.
16. The shell and stiffener materials are homogeneous and isotropic.

## 2.3 Prebuckling Equations

The general nonlinear equations of motion governing the stiffened cylinder buckling are derived in Appendices A and B. The appropriate equations of motion and boundary conditions governing the cylinder prebuckling are obtained by substituting the A-subscript portion of Equations (2-1) into Equations (B-10) and (B-11). Primes indicate total differentiation with respect to x. The prebuckling equations become

$$N'_{xA} = 0 \tag{2-2}$$

$$N'_{xyA} = 0 \tag{2-3}$$

$$-M''_{xA} + (N_{yA}/R) = 0 \tag{2-4}$$

The appropriate boundary conditions at the cylinder ends are:

$$N_{xA} + \hat{N}_x = 0 \quad \text{or} \quad u_A = 0 \tag{2-5}$$

$$N_{xyA} = 0 \quad \text{or} \quad v_A = 0 \tag{2-6}$$

$$M_{xA} + \hat{N}_x \bar{e} = 0 \quad \text{or} \quad w'_A = 0 \tag{2-7}$$

$$M'_{xA} = 0 \quad \text{or} \quad w_A = 0 \tag{2-8}$$

In the above equations the following definitions apply:

$$N_{xA} = \frac{E}{1-\nu^2}(u'_A+\nu R^{-1}w_A)+\sum_k \delta(y-kd)E_sA_su'_A \tag{2-9}$$

$$N_{yA} = \frac{E}{1-\nu^2}(R^{-1}w_A+\nu u'_A)+\sum_j \delta(x-j\ell)E_rA_rR^{-1}w_A \tag{2-10}$$

$$N_{xyA} = Gv'_A \tag{2-11}$$

$$M_{xA} = -\sum_k \delta(y-kd)E_sA_s\bar{z}_su'_A \tag{2-12}$$

Equations (2-2) and (2-5) require that $N_{xA}$ must equal $-\hat{N}_x$. This condition satisfies boundary condition (2-5). In a similar manner, Equations (2-3) and (2-6) require that there be no applied shear. Thus, if $N_{xy}$ is equated to zero, boundary condition (2-6) will be satisfied. To satisfy the third boundary condition (2-7), $w_A$ is set equal to a constant. By making $w_A$ equal to a constant, a classical prebuckling membrane state is assumed for the stiffened cylinder. Finally, the fourth boundary condition (2-8) is satisfied by setting $M_{xA}$ equal to zero.

For later use in the formulation of the buckling equations, the following prebuckling identities will be used:

$$N_{yA} = 0 \tag{2-13}$$

$$N_{xA} = 0 \tag{2-14}$$

$$N_{xyA} = 0 \tag{2-15}$$

## 2.4 Buckling Equations

The governing equations and boundary conditions of a stiffened cylinder with discrete rings and stringers can be obtained by substituting

Equations (2-1) into Equations (B-10) and (B-11), subtracting out identities (2-2) through (2-4) and making use of Equations (2-13) through (2-15). The buckling equations become:

$$N_{xB,x} + N_{xyB,y} = 0 \tag{2-16a}$$

$$N_{yB,y} + N_{xyB,x} = 0 \tag{2-16b}$$

$$\begin{aligned} &-M_{xB,xx} - 2M_{xyB,xy} - M_{yB,yy} + R^{-1}N_{yB} + \hat{N}_x w_{B,xx} - N_{xB} w_{B,xx} \\ &- N_{yB} w_{B,yy} - 2N_{xyB} w_{B,xy} + \rho h w_{B,tt} + \sum_j \rho_r A_r w_{B,tt} \delta(x - j\ell) \\ &+ \sum_k \rho_s A_s w_{B,tt} \delta(y - kd) = 0 \end{aligned} \tag{2-16c}$$

The boundary conditions at the stiffened cylinder ends become

$$N_{xB} = 0 \qquad \text{or} \qquad u_B = 0 \tag{2-17a}$$

$$N_{xyB} = 0 \qquad \text{or} \qquad v_B = 0 \tag{2-17b}$$

$$M_{xB} = 0 \qquad \text{or} \qquad w_{B,x} = 0 \tag{2-17c}$$

$$M_{xB,x} + 2M_{xyB,y} - N_x w_{B,x} + N_{xB} w_{B,x} + N_{xyB} w_{B,x} = 0 \quad \text{or} \quad w_B = 0 \tag{2-17d}$$

The definitions for the various buckling terms may be found in Appendix B, Equations (B-4) through (B-9), where a subscript B is added to each term and displacement.

## 2.5 Application of Galerkin's Method

The buckling Equations (2-16) are a set of coupled, nonlinear partial differential equations. To reduce them to a set of ordinary differential equations, the Galerkin weighted average method will be used.

The Galerkin method is an approximate assumed-mode method similar to the Rayleigh-Ritz method when applied correctly [56]. The method should be applied only to the equations of motion which arise directly from application of either Hamilton's principle or Newton's second law. If the equations are escalated by differentiation and combined, the method often yields incorrect results [57].

To apply the Galerkin method, solutions are assumed for the unknown variables in the equations of motion. In general, the solutions will not satisfy the equations of motion exactly. In order to minimize the error, the assumed solutions are inserted into the equations of motion and a non-zero "error function" is generated. Each error function is then orthogonalized with respect to the assumed solution functions. This orthogonalization process will give rise to a set of equations which can be solved for the unknown solution function coefficients. The resulting set of equations can be either linear or nonlinear, algebraic or differential, depending on the equations of motion and the form of the assumed solution.

To apply Galerkin's method to the buckling equations, the following series of assumed modes will be used

$$u = \sum\sum u_{ij}(t)\sin(im\pi x/L)\cos(jny/R) \tag{2-18a}$$

$$v = \sum\sum v_{ij}(t)\cos(im\pi x/L)\sin(jny/R) \tag{2-18b}$$

$$w = \sum\sum w_{ij}(t)\cos(im\pi x/L)\cos(jny/R) \tag{2-18c}$$

Most researchers [6-9,45,48,49] who used a nonlinear shell theory to study buckling resorted to one or more terms of the above

series. As noted in Chapter I, these researchers were unable to satisfy all of the boundary conditions. In line with hypothesis 14, Equations (2-18) satisfy boundary conditions (B-11a), (B-11b), and (B-11c), but not boundary condition (B-11d).

Substitution of Equations (2-18) into the buckling Equations (2-16) yields a set of "error functions", $Q_x$, $Q_y$, $Q_z$, respectively. The orthogonalization process may be formulated as

$$\int_0^{2\pi R}\int_0^{L} Q_x(u,v,w)\sin(\zeta m\pi x/L)\cos(\xi ny/R)\,dxdy = 0 \qquad \begin{matrix}\zeta=1,2,3\cdots\\ \xi=1,2,3\cdots\end{matrix} \tag{2-19}$$

$$\int_0^{2\pi R}\int_0^{L} Q_y(u,v,w)\cos(\zeta m\pi x/L)\sin(\xi ny/R)\,dxdy = 0 \qquad \begin{matrix}\zeta=1,2,3\cdots\\ \xi=1,2,3\cdots\end{matrix} \tag{2-20}$$

$$\int_0^{2\pi R}\int_0^{L} Q_z(u,v,w)\cos(\zeta m\pi x/L)\cos(\xi ny/R)\,dxdy = 0 \qquad \begin{matrix}\zeta=1,2,3\cdots\\ \xi=1,2,3\cdots\end{matrix} \tag{2-21}$$

Because $Q_x$ and $Q_y$ are not dependent on time, Equations (2-19) and (2-20) will each yield a set of nonlinear algebraic equations. Since $Q_z$ is dependent on time, Equation (2-21) will yield a set of nonlinear ordinary differential equations in time. The resultant equations after some simplification are listed below, and the details of the Galerkin method are found in Appendix C.

$$\begin{aligned}
&\delta_{i\zeta}\Sigma\Sigma u_{ij}\left[I_i^2\delta_{j\xi}+\tfrac{1}{2}(1-\nu)J_j^2\delta_{j\xi}+(1-\nu^2)(\pi REh)^{-1}I_i^2\sum_k E_sA_s\cos J_k\cos J_\xi\right]\\
&+\Delta_1\Sigma\Sigma v_{ij}\tfrac{1}{2}(1+\nu)I_iJ_j+\delta_{i\zeta}\Sigma\Sigma w_{ij}\left[R^{-1}\nu I_i\delta_{j\xi}\right.\\
&\left.+(1-\nu^2)(\pi REh)^{-1}I_i^3\sum_k E_sA_s\bar{z}_s\cos J_k\cos J_\xi\right]-\delta_{2i\zeta}\Sigma\Sigma w_{ij}^2\left[\tfrac{1}{4}I_iJ_j^2\delta_{2j\xi}+\tfrac{1}{4}I_i^3\delta_{2j\xi}\right.\\
&\left.+(1-\nu^2)(2Eh\pi R)^{-1}I_i^3\sum_k E_sA_s\cos^2 J_k\cos J_\xi\right] = 0
\end{aligned} \tag{C-1}$$

$$\Delta_1 \Sigma\Sigma u_{ij} \tfrac{1}{2}(1+\nu) I_i J_j + \delta_{j\xi} \Sigma\Sigma v_{ij} [J_j^2 \delta_{i\zeta} + \tfrac{1}{2}(1-\nu) I_i^2 \delta_{i\zeta}$$

$$+2(1-\nu^2)(EhL)^{-1} J_j^2 \sum_j E_r A_r \cos I_j \cos I_\zeta] + \delta_{j\xi} \Sigma\Sigma w_{ij} [R^{-1} J_j \delta_{i\zeta}$$

$$+2(1-\nu^2)(EhL)^{-1} \sum_j (R^{-1} J_j E_r A_r + J_j^3 E_r A_r \bar{z}_r) \cos I_j \cos I_\zeta] - \delta_{2j\xi} \Sigma\Sigma w_{ij}^2 [\tfrac{1}{4} J_j^3 \delta_{2i\zeta}$$

$$+\tfrac{1}{4} I_i^2 J_j \delta_{2i\zeta} + (1-\nu^2)(EhL)^{-1} J_j^3 \sum_j E_r A_r \cos^2 I_j \cos I_\zeta] = 0 \qquad \text{(C-2)}$$

$$Eh(1-\nu^2)^{-1} (1/32) \Sigma\Sigma w_{ij}^3 [I_i^4 (3\Delta_1 + \Delta_2 - 3\Delta_3 - \Delta_4) + 2 I_i^2 J_j^2 (-\Delta_1 + \Delta_2 + \Delta_3 - \Delta_4 + 4\nu\Delta_1$$

$$-2\nu\Delta_2 - 2\nu\Delta_3) + J_j^4 (3\Delta_1 - 3\Delta_2 + \Delta_3 - \Delta_4)] + (2/L) \Sigma\Sigma w_{ij}^3 (\delta_{j\xi} - \delta_{3j\xi}) \tfrac{1}{8} J_j^4 \sum_j E_r A_r \cos^3 I_j$$

$$\cdot \cos I_\zeta + (1/\pi R) \Sigma\Sigma w_{ij}^3 (\delta_{i\zeta} - \delta_{3i\zeta}) \tfrac{1}{8} I_i^4 \sum_k E_s A_s \cos^3 J_k \cos J_\xi + \Delta_5 Eh(1-\nu^2)^{-1} (\nu/8R)$$

$$\cdot \Sigma\Sigma w_{ij} (I_i^2 + J_j^2) + \delta_{2j\xi} (2/L) \Sigma\Sigma w_{ij}^2 (\tfrac{1}{4} R^{-1} J_j^2 - \tfrac{1}{2} J_j^4 \bar{z}_r) \sum_j E_r A_r \cos^2 I_j \cos I_\zeta$$

$$-\delta_{2i\zeta} (1/\pi R) \Sigma\Sigma w_{ij}^2 \tfrac{1}{2} I_i^4 \sum_k E_s A_s \bar{z}_s \cos^2 J_k \cos J_\xi + \Delta_1 Eh(1-\nu^2)^{-1} \Sigma\Sigma w_{ij} [(h^2/12) I_i^4$$

$$+(h^2/6) I_i^2 J_j^2 + (h^2/12) J_j^4 + (1/R^2) - \hat{N}_x (1-\nu^2) I_i^2 (Eh)^{-1}] + \delta_{j\xi} (2/L) \Sigma\Sigma w_{ij}$$

$$\cdot [J_j^4 E_r I_{or} + 2R^{-1} J_j^2 E_r A_r \bar{z}_r + (1/R^2) E_r A_r] \sum_j \cos I_j \cos I_\zeta + \delta_{j\xi} (2/L) \Sigma\Sigma w_{ij} J_j^2$$

$$\cdot I_i \sum_j G_r J_r \zeta_m \pi L^{-1} \sin I_j \sin I_\zeta + \delta_{i\zeta} (1/\pi R) \Sigma\Sigma w_{ij} I_i^4 \sum_k E_s I_{os} \cos J_k \cos J_\xi$$

$$+\delta_{i\zeta} (1/\pi R) \Sigma\Sigma w_{ij} I_i^2 J_j \sum_k G_s J_s \xi_n R^{-1} \sin J_K \sin J_\xi + \Delta_1 Eh(1-\nu^2)^{-1} \nu R^{-1} \Sigma\Sigma u_{ij} I_i$$

$$+\delta_{i\zeta} (1/\pi R) \Sigma\Sigma u_{ij} I_i^3 \sum_k E_s A_s \bar{z}_s \cos J_k \cos J_\xi + \Delta_1 Eh(1-\nu^2)^{-1} R^{-1} \Sigma\Sigma v_{ij} J_j$$

$$+\delta_{j\xi} (2/L) \Sigma\Sigma v_{ij} (J_j^3 \bar{z}_r + J_j R^{-1}) \sum_j E_r A_r \cos I_j \cos I_\zeta + \Delta_5 \tfrac{1}{4} Eh(1-\nu^2)^{-1} \Sigma\Sigma u_{ij} w_{ij} (I_i^3$$

$$+I_i J_j^2) + \delta_{2i\zeta} (1/\pi R) \Sigma\Sigma u_{ij} w_{ij} \tfrac{1}{2} I_i^3 \sum_k E_s A_s \cos^2 J_k \cos J_\xi + \Delta_5 \; Eh(1-\nu^2)^{-1}$$

$$\cdot\Sigma\Sigma v_{ij}w_{ij}(J_j^3+I_i^2J_j)+\delta_{2j\xi}L^{-1}\Sigma\Sigma v_{ij}w_{ij}J_j^3\sum_j E_rA_r\cos^2I_j\cos I_\zeta$$

$$+\Sigma\Sigma\ddot{w}_{ij}[\Delta_1\rho h+\delta_{j\xi}\rho_rA_r(2/L)\sum_j\cos I_j\cos I_\zeta+\delta_{i\zeta}\rho_sA_s(1/\pi R)\sum_k\cos J_k\cos J_\xi]=0 \tag{C-3}$$

## 2.6 Numerical Solution

The buckling equations consist of a set of 2k nonlinear algebraic equations and a set of k nonlinear differential equations, where k represents the number of terms used in the assumed modes. Because of the complex nature of these sets of equations, no closed form solution is known to exist. However, with given initial conditions, the above sets of equations should be numerically solvable if we use a Gauss-Jordan technique on the algebraic equations and a Runge-Kutta technique on the nonlinear differential equations. In addition to the initial conditions on the dependent variables, the solution of the equations depends also on the externally applied axial load resultant, $\hat{N}_x$, and the circumferential wave number n. Of course, it will be necessary to vary n over a range of values to find the minimum critical load, since this will be the true dynamic buckling load of the stiffened cylinder.

An inspection of the set of algebraic Equations (C-1) and (C-2) reveals that the nonlinearity is in the radial deflection terms, $w_{ij}(t)$. Thus, if $w_{ij}(t)$ is specified initially, Equations (C-1) and (C-2) can be solved by a suitable technique for solving simultaneous linear equations. The results, together with the value of $w_{ij}(t)$, can be substituted into the set of nonlinear differential equations

(C-3). Using a fourth order Runge-Kutta algorithm and advancing it $\frac{1}{4}$ step, a new set of $w_{ij}(t)$ can be obtained. The new $w_{ij}(t)$ set can be resubstituted into Equations (C-1) and (C-2) and the process repeated three times to yield one complete step.

Since most Runge-Kutta algorithms are written specifically for first-order differential equations, it is necessary to convert the k second-order differential equations into 2k first-order equations by a simple change of variable. The following fourth-order algorithm attributed to Kutta is used in the analysis [58]:

$$w_{i+1} = w_i + (g/6)(k_1+2k_2+2k_3+k_4)$$

where

$$k_1 = f(t_i, y_i)$$

$$k_2 = f(t_i+\tfrac{1}{2}g, y_i+\tfrac{1}{2}gk_1)$$

$$k_3 = f(t_i+\tfrac{1}{2}g, y_i+\tfrac{1}{2}gk_2)$$

$$k_4 = f(t_i+g, y_i+gk_3)$$

In order to begin the numerical procedure, it is necessary to choose representative initial values for the set of dependent variables, $w_{ij}(t)$. Physically, this corresponds to giving the shell a slight initial radial deflection or velocity. This procedure was first suggested by Roth and Klosner [49].

Finally, a criterion for identifying dynamic buckling must be established. An inspection of the representative buckling curves found in Chapters III and IV make it apparent that at some critical load the structure is diverging from its initial displacement rather than returning to its equilibrium (or zero displacement) position. Thus, as

suggested in references [49] and [59], the buckling load can be defined as the load at which a large increase in the amplitude of the deflection occurs. Of course, as mentioned earlier in this section, it will be necessary to plot the buckling load as a function of circumferential wave number, n, and then pick the lowest resultant load as the true buckling load.

Further details concerning the numerical techniques employed may be obtained by consulting Appendices IV and V. One particularly important technique, which allowed a general formulation of the governing equation coefficients independent of the number of assumed mode terms, will be illustrated here. It should be noted that the coefficients of the various terms in Equations (C-1) through (C-3) can be written as functions of the assumed mode subscripts i, j, $\zeta$ and $\xi$. However, the identification of parameters by four rather than by two subscripts becomes very cumbersome when attempting to construct a general computer program. This problem was solved by making use of an algorithm suggested in reference [60]. The algorithm permits the rewriting of each coefficient in terms of only two subscripts P and Q, and then "decoding" the subscripts as necessary during the numerical procedure. If a typical coefficient is represented by $A_{\zeta\xi ij}$, the algorithm will convert it to $A_{QP}$ as follows:

$$i = P - \left(\frac{P-1}{i^*}\right)_T i^*$$

$$j = 1 + \left(\frac{P+1}{i^*}\right)_T$$

$$\zeta = Q - \left(\frac{Q-1}{i^*}\right)_T i^*$$

$$\xi = 1 + \left(\frac{Q+1}{i^*}\right)_T$$

where i* represents the maximum value of the i coefficient and the symbol $(\quad)_T$ represents the operation of integer truncation. An example of this calculation for a four-term assumed deflection mode is shown in Table 2.1. The operation of this algorithm can be observed in Appendix V.

Table 2.1 Coefficient Algorithm

| $\zeta$ | $\xi$ | i :<br>j : | 1<br>1 | 2<br>1 | 1<br>2 | 2<br>2 |
|---|---|---|---|---|---|---|
| 1 | 1 | | (Q=1,P=1) | (Q=1,P=2) | (Q=1,P=3) | (Q=1,P=4) |
| 2 | 1 | | (Q=2,P=1) | (Q=2,P=2) | (Q=2,P=3) | (Q=2,P=4) |
| 1 | 2 | | (Q=3,P=1) | (Q=3,P=2) | (Q=3,P=3) | (Q=3,P=4) |
| 2 | 2 | | (Q=4,P=1) | (Q=4,P=2) | (Q=4,P=3) | (Q=4,P=4) |

# CHAPTER III

# EVALUATION OF THEORY

## 3.1 Comparison with Static Buckling

During the literature search no studies were found that dealt exclusively with sparsely stiffened cylinders. Consequently, most studies used a linear buckling theory in conjunction with a smeared stiffener analysis. In the smeared analysis, the stiffener properties are averaged over the appropriate dimension by dividing by the stiffener spacing. In the more exact analysis contained in this investigation, the stiffener properties are considered as discrete and a Dirac delta function is used to handle the discreteness in the analysis. In order to reduce the present discrete analysis to a smeared one for comparison, the typical results in Table 3.1 can be used.

Table 3.1 Comparison of Smeared and Discrete Terms

| | SMEARED | DISCRETE |
|---|---|---|
| Typical Term | $E_s A_s u_{,xx}/d$ | $\sum_k E_s A_s u_{,xx} \delta(y-kd)$ |
| Result After Galerkin Procedure | $-\frac{1}{2}u_{ij} I_i^2 \pi R L E_s A_s/d$ | $-\frac{1}{2}u_{ij} I_i^2 L E_s A_s \sum_k \cos J_k \cos J_\xi$ |
| Equivalent Terms | $\pi R/d$ | $\sum_k \cos J_k \cos J_\xi$ |

Block, *et al.* [38] used a linear smeared analysis and a one term assumed mode approach. This procedure reduced the static buckling problem to a single algebraic equation containing the applied axial load and the circumferential and axial wave numbers. In order to compare Block's final equation with the present results, Equations (C-1), (C-2) and (C-3) were linearized, the stiffeners were smeared using results analogous to those presented in Table 3.1, the inertia terms were removed, and only the first displacement terms were included. When the resulting equations were reduced, they compared exactly to final Equation (15) of reference [38].

In a later reference, Block [21] completed a linear classical buckling solution for a cylinder with smeared stringers and discrete rings. He used a Dirac delta approach to model the discrete rings. His assumed modes consisted of a single circumferential term together with a large number of axial terms, all of which satisfied the boundary conditions of the classical simple support. Again, a term-by-term comparison with the present analysis yielded analogous results.

Unfortunately, after an extensive literature search, no theoretical or experimental work was found which dealt with dynamic buckling of stiffened cylinders. The only dynamic studies were theoretical studies of unstiffened cylinders, and these are treated in the next section. However, to obtain an approximate comparison between the present dynamic buckling analysis and previously published static analyses, a dynamic buckling analysis was conducted using the shell parameters reported in references [21] and [38]. In these references the cylinder which was used had closely spaced stiffeners representative of the

large-diameter liquid rocket booster structures. Details of the structure and results of the comparison are contained in Fig. 3.1. The dynamic analysis was made using one circumferential and three axial terms. Since the dynamic loading consisted of a step pulse loading of infinite time duration, the predicted buckling load should roughly correspond to the predicted static buckling load. As observed from Fig. 3.1, except for the smeared static analysis, the dynamic load is generally above the corresponding static load. The difference can be attributed to the inclusion of the radial inertia and the nonlinear terms in the dynamic analysis.

### 3.2 Comparison with Dynamic Buckling

The only dynamic buckling analyses found in the literature were for unstiffened shells exclusively [46,48,49], and were not experimentally verified. Generally, a nonlinear buckling theory together with a stress function approach was used to reduce the problem to one equilibrium and one compatability equation. These equations were solved by the Ritz-Galerkin method and then integrated numerically. Because of the sensitivity of the unstiffened cylinder to initial imperfections, the analyses concentrated on the influence of these imperfections on the buckling load. Since initial imperfections are not included in the present stiffened cylinder analysis, it was necessary to extend the unstiffened cylinder analyses for comparison purposes. In order to accomplish this, the equations of reference [49] were solved numerically for the case of zero imperfections, and plotted in Fig. 3.2. As a comparison, the same unstiffened cylinder was treated using the present analysis, and the results are shown in Fig. 3.2. Because of the

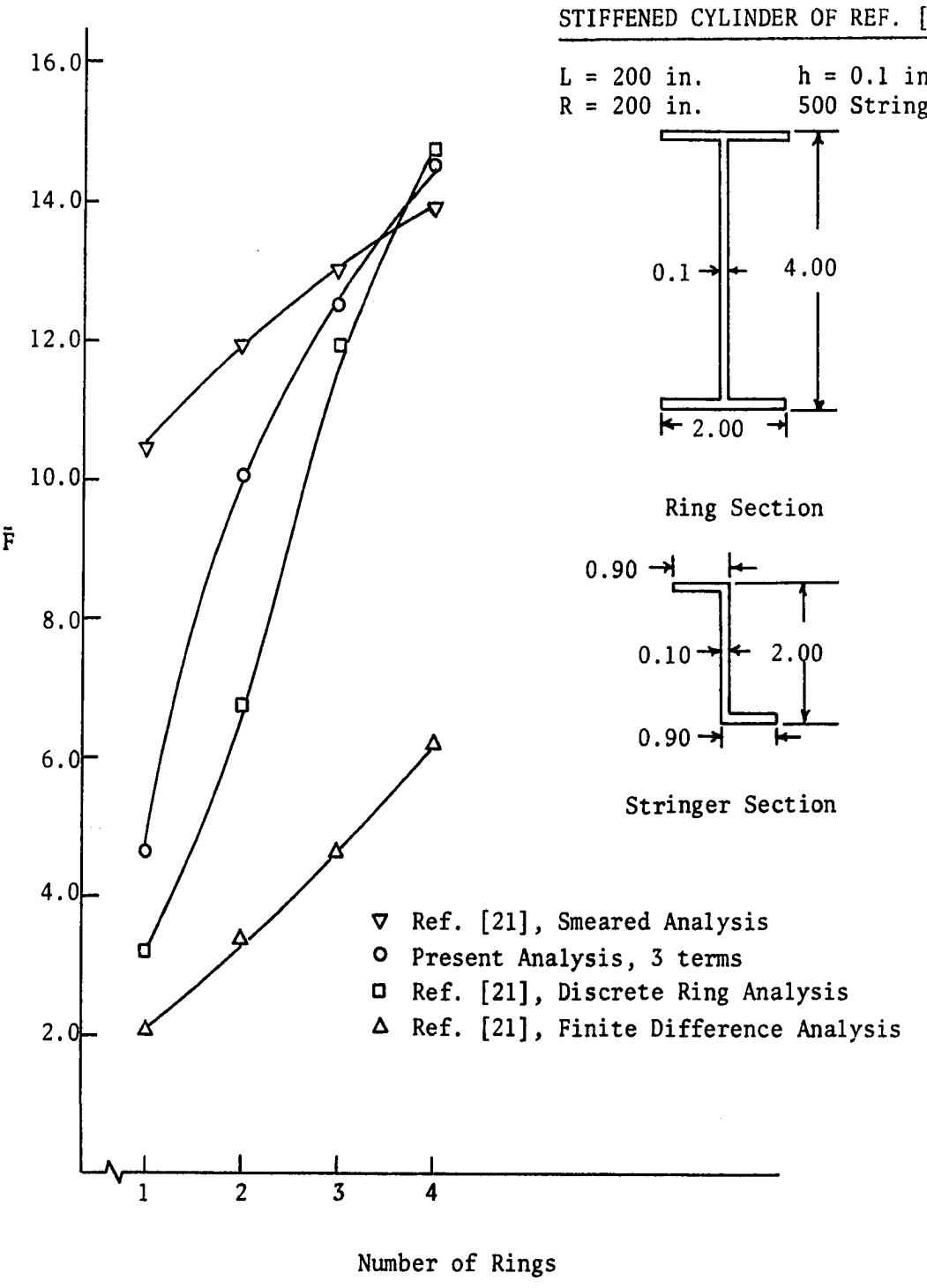


Figure 3.1 Static Buckling Comparison

different radial deflection functions used in reference [49], exact agreement would not be expected. However, the curves are in good agreement.

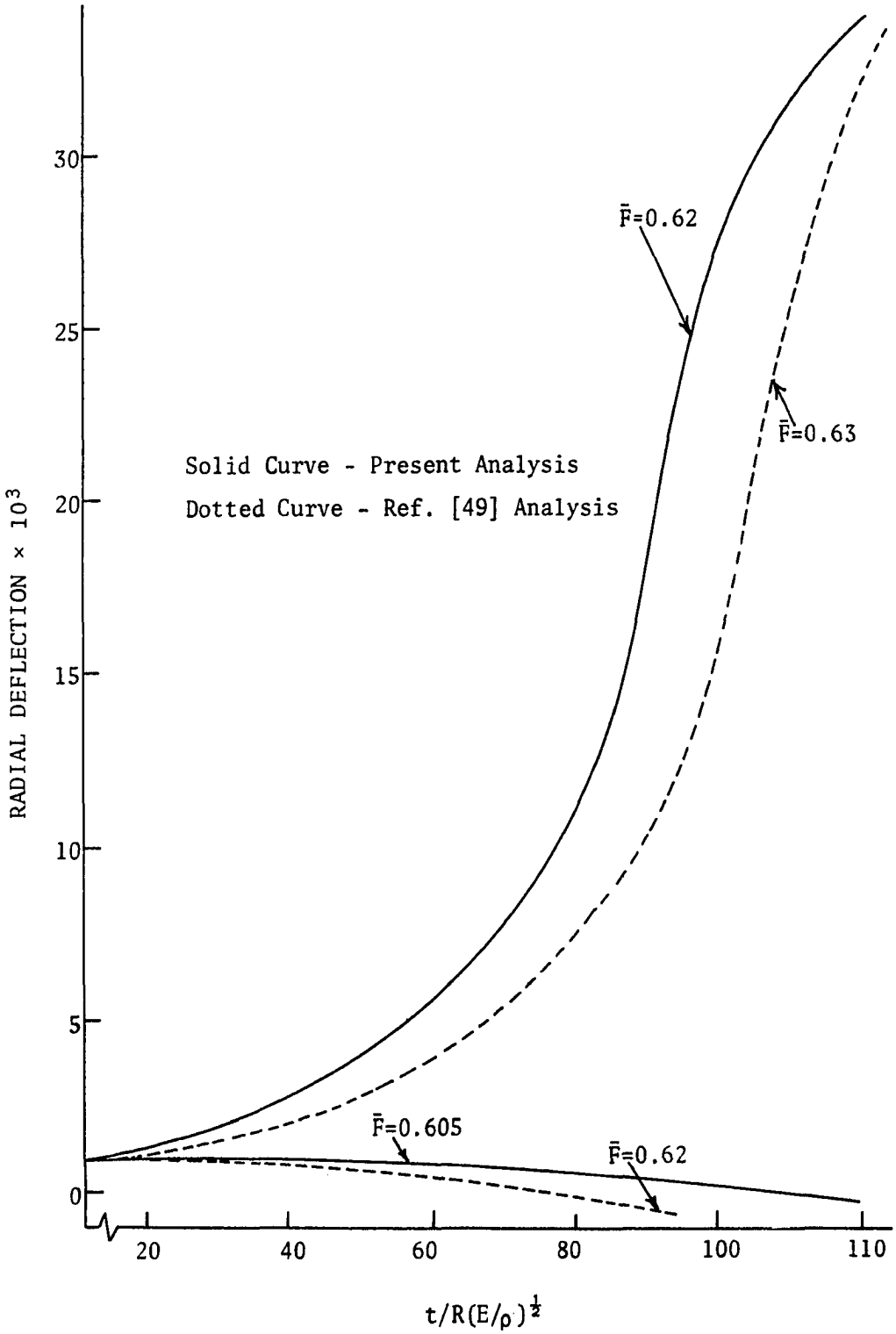


Figure 3.2 Dynamic Buckling Comparison, Unstiffened Shell

# CHAPTER IV

# RESULTS

## 4.1 Representative Aircraft Structure

Because of the large number of geometric and physical parameters involved in this investigation, it is impractical to present results of a general nature. However, it is of value to present some computed results for stiffened cylinders which are representative of light aircraft fuselage structures. In order to determine the geometric parameters of a typical light aircraft structure, four light aircraft companies were contacted. Based on the data they provided, Table 4.1 was constructed. It must be mentioned that the data shown in Table 4.1 is approximate only. It reflects engineering judgment made by the author, and not by personnel of the respective aircraft companies. Many of the actual geometric configurations have been simplified for this analysis. For example, actual rings or bulkheads are seldom uniform, of constant thickness, or without "lightening" holes to conserve weight. In this analysis, all rings and stringers are considered uniform, of constant thickness, and evenly spaced along the inside of the cylinder. No provision has been made for cutouts in the shell for doors or windows. However, despite these limitations, it is felt that a useful first-order parametric study can be made based on the approximate quantities contained in Table 4.1.

Table 4.1 Representative Light Aircraft Geometry

| Acft. | Type | Cabin Length (in.) | R (in.) | h (in.) | M | $A_s$ ($in^2$) | $I_{os}$ ($in^4$) | $J_s$ ($in^4$) | $\bar{z}_s$ (in.) | $A_r$ ($in^2$) | $I_{or}$ ($in^4$) | $J_r$ ($in^4$) | $\bar{z}_r$ (in.) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | Single Engine | 112 | 25 | 0.30 | 33 | 0.032 | 0.006 | 0.0049 | 0.308 | 0.094 | 0.088 | 0.039 | 0.763 |
| B | Single Engine | 110 | 25 | 0.25 | 22 | 0.065 | 0.015 | 0.0059 | 0.466 | 0.142 | 0.320 | 0.120 | 1.210 |
| C | Twin Engine | 168 | 34 | 0.30 | 27 | 0.046 | 0.013 | 0.0078 | 0.413 | 0.165 | 0.264 | 0.102 | 1.013 |
| D | Twin Engine | 217 | 31 | 0.34 | 30 | 0.035 | 0.003 | 0.0022 | 0.187 | 0.133 | 0.461 | 0.159 | 1.513 |

An inspection of Table 4.1 reveals that all four aircraft have approximately the same stiffener area, but that some of the other stiffener quantities vary by as much as 50 per cent. This is due to different geometrical shapes of the various stiffeners. For example, the stringers in aircraft C are channel-sections, while those in aircraft D are L-sections. However, the effect of variation of these parameters was shown to be small, as documented in Section 4.3.

Based on the data contained in Table 4.1, a representative set of stiffened cylinder parameters was chosen. To begin the study a 32-inch radius cylinder was used. The skin thickness was 0.04 inches and the largest spacing between rings was 32 inches. Thirty equally spaced stringers, together with the stiffener properties of aricraft C were used as a starting point. These various geometrical parameters were then varied, and the influence on the dynamic buckling behavior was noted.

In light of assumptions twelve and thirteen regarding the cylinder boundary conditions, the cylinder radius, thickness, and length were never such that the Batdorf parameter, Z, was less than 30. In fact, even if the stringers were smeared over the surface of the shortest cylinder, the smallest value of Z was 190. Thus, the expected buckling behavior of the stiffened cylinder would be that of a "long" cylinder.

Due to the relatively wide spacing between the rings on general-aviation light aircraft, it was felt that local buckling of the shell between the rings would be the mode of failure, rather than general instability. Local buckling between rings is defined as the buckling

mode in which the rings have little or no radial buckling deformation, and the cylinder buckles between the rings. General instability is defined as the buckling mode in which the rings deform radially and the cylinder wall and rings buckle as a composite wall. Since discreteness of the stiffeners has been accounted for, the present analysis can handle either type of instability.

Since an actual aircraft fuselage consists of a long cylinder with a number of repeating bays between the rings, it is reasonable to assume that buckling of the entire cylinder can be studied by considering a representative smaller portion or subshell of the cylinder. If this assumption is valid, fewer assumed mode terms in the deflection function would probably be needed to correctly model the dynamic behavior of the structure. Of course, fewer terms would mean a saving of computer time.

To validate the above assumption, the long cylinder shown in Fig. 4.1 was divided into three subshells. Subshell I consisted of one ring and an L/R of two. Subshell II consisted of two rings and an L/R of three. Subshell III consisted of the stringers between two rings and an L/R of one. A buckling analysis was performed on subshells I and II, and the results are shown on Fig. 4.2. Although the curves diverge somewhat for higher values of n, the lowest buckling load for both subshells was approximately the same and occurred for the same wave number, indicating the similarity of buckling shapes. As expected, subshell II required more terms than subshell I for convergence; subshell I required three terms while subshell II required five terms. It should be mentioned that the curves drawn through the calculated points on the various graphs are for ease in identifying the lowest buckling load.

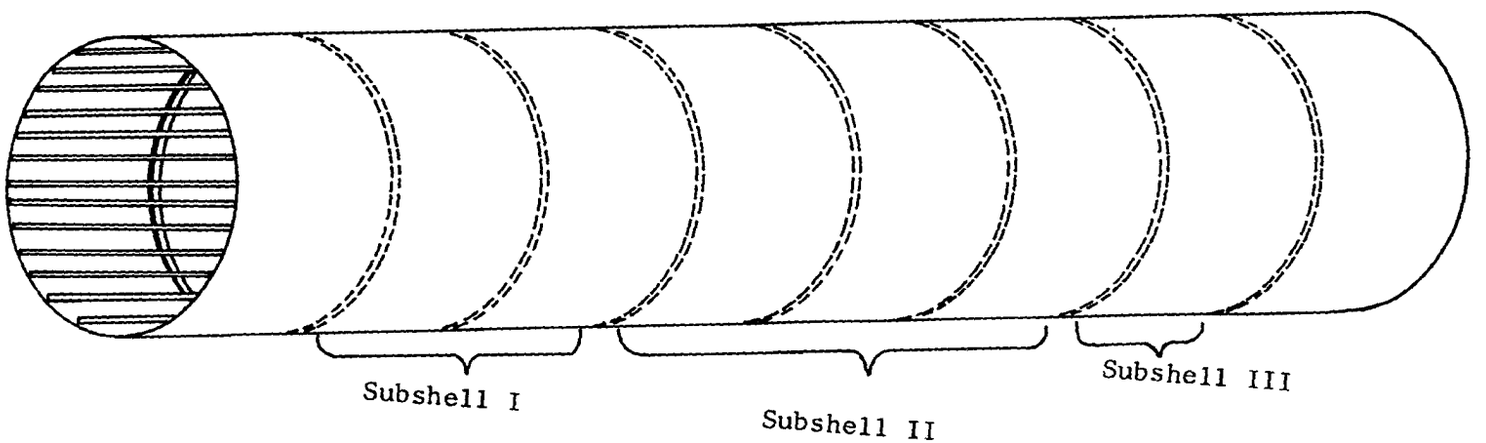


Figure 4.1 General Aircraft Structure

The curves are actually discrete points since n can take on integer values only.

Subshell III was used as a further check on the convergence of the curves presented in Fig. 4.2, and as a demonstration of local buckling of the subshell between the rings. When a stiffened cylinder undergoes local buckling between the rings, the rings can only exert a torsional restraint on the shell. Thus, a lower bound on the buckling load can be obtained by considering the stringer stiffened portion between the rings. This portion is shown as subshell III in Fig. 4.1, and the corresponding buckling curve is plotted on Fig. 4.2. The closeness of the subshell I and II curves to this lower bound curve demonstrates that local buckling between the rings is the mode of buckling for the representative cylinder considered in this analysis.

The effect of discreteness on the stringer stiffened shell can be seen in the region of n = 15. Since the shell consists of 30 stringers, n = 15 represents the case of local buckling between the stringers. However, when rings are used in addition to stringers, the buckling at n = 15 does not represent the lowest buckling load, as evidenced by the subshell I and II curves.

Based on this preliminary case, it is reasonable to assume that local buckling between the rings will be the dominant instability mode. Local buckling between the stringers will occur at some higher load and thus not be significant. Also, because of the great increase in computer running time and core space for each additional assumed mode term, the number of assumed mode terms in the circumferential direction will be limited to one. The number of terms in the axial direction will be

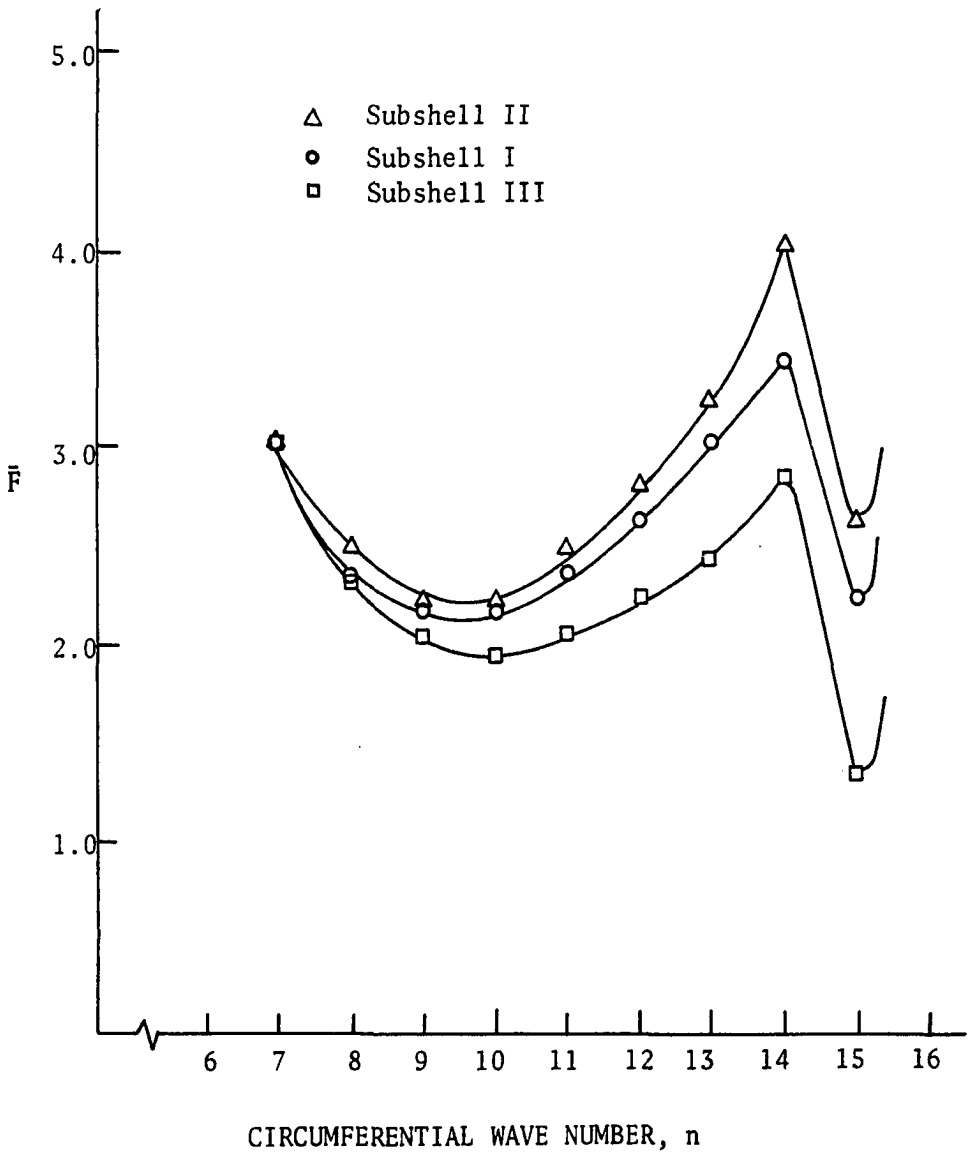


Figure 4.2 Comparison of Buckling Load for Different Cylinder Regions

increased until convergence is obtained.

## 4.2 Dynamic Considerations

As stated in Section 1.2, it is believed that this study represents the first dynamic buckling analysis of a discretely stiffened cylinder. Because of the relative newness of the dynamic buckling field, general procedures and criteria for identifying dynamic buckling have not been generally established as they have been for static buckling. Consequently, the results presented in this and subsequent sections should be considered exploratory in nature. Unlike static buckling analyses, where the lowest buckling load results from the solution of an eigenvalue problem, the shell deflection-time history curves must be inspected to determine buckling. Next, the buckling load must be cross-plotted as a function of the wave number n to determine the lowest buckling load. Because of the many parameters that must be varied, and because of the large amount of computer time required for the analysis, the following general calculational procedure was adopted:

Computer Run #1

a. Allow n to vary from one to twenty.
b. Establish a best guess as an upper and lower bound on the load, $\bar{F}$.
c. Establish an incrementing value for $\bar{F}$, usually beginning with 0.5.
d. Begin with a Runge-Kutta step increment of 0.1 msec and terminate computation after t reaches 5.0 msec.

Computer Run #2

a. Narrow region of n to those values yielding the lowest values of $\bar{F}$.

b. Narrow the upper and lower bounds and reduce the size of the incrementing step.

c. Examine the smoothness of the maximum deflection curves to determine whether the Runge-Kutta increment is adequate.

Computer Run #3

a. Identify and input the critical value of n causing the lowest buckling load.

b. Narrow the upper and lower bounds of $\bar{F}$ and reduce the size of the incrementing step.

c. Run the analysis for t>50 msec. to positively identify the lowest buckling load, and the time to the first maximum of the lowest buckling load.

d. Repeat the entire procedure with more terms in the assumed deflection function to determine convergence of the results.

Although the analysis and computer program allows one to consider an axial end load varying arbitrarily with time, the present study has been limited to loads having a step function variation in time. Since actual crash load durations are on the order of 100 milliseconds [61], the step function time duration was at least 100 milliseconds.

In order to start the numerical analysis it was necessary to establish the initial displacement and velocity conditions on the radial assumed mode terms. It was decided to set the initial deflection of the lowest mode equal to 0.001 which represented an initial displacement of less than five per cent of the shell thickness. All other initial conditions were set equal to zero. This procedure was suggested by reference [49] for studying dynamic buckling of unstiffened cylinders.

The initial deflection was varied with no change in buckling behavior, thus indicating the insensitivity of buckling to initial conditions.

As mentioned in Section 2.6, the buckling load for a particular wave number is defined as the load at which a large increase in the amplitude of the deflection occurs. The actual buckling load for a particular structure can then be determined from a plot of load versus wave number, such as shown in Fig. 4.2.

By considering the behavior of the lowest mode only as a function of time, it can be seen from Fig. 4.3 that there are two types of shell response. If the load is below the critical buckling load, the shell oscillates around its original equilibrium position. Since damping is neglected in the analysis, this oscillation would rapidly damp out in an actual shell. If the load is at or above the critical buckling load, there is a radical increase in the maximum shell deflection. If the load is increased further, the maximum radial deflection increases, and the time to maximum deflection decreases. This type of behavior was previously observed and documented for dynamic buckling of unstiffened shells by Roth and Klosner [49]. They used the same instability criterion by investigating the response of the lowest mode only.

In order to distinguish the lowest buckling load between two values of n, it was often necessary to observe the dynamic behavior for durations of up to 100 milliseconds. For example, from Fig. 4.2 the buckling load occurred either at a wave number of nine or ten. To

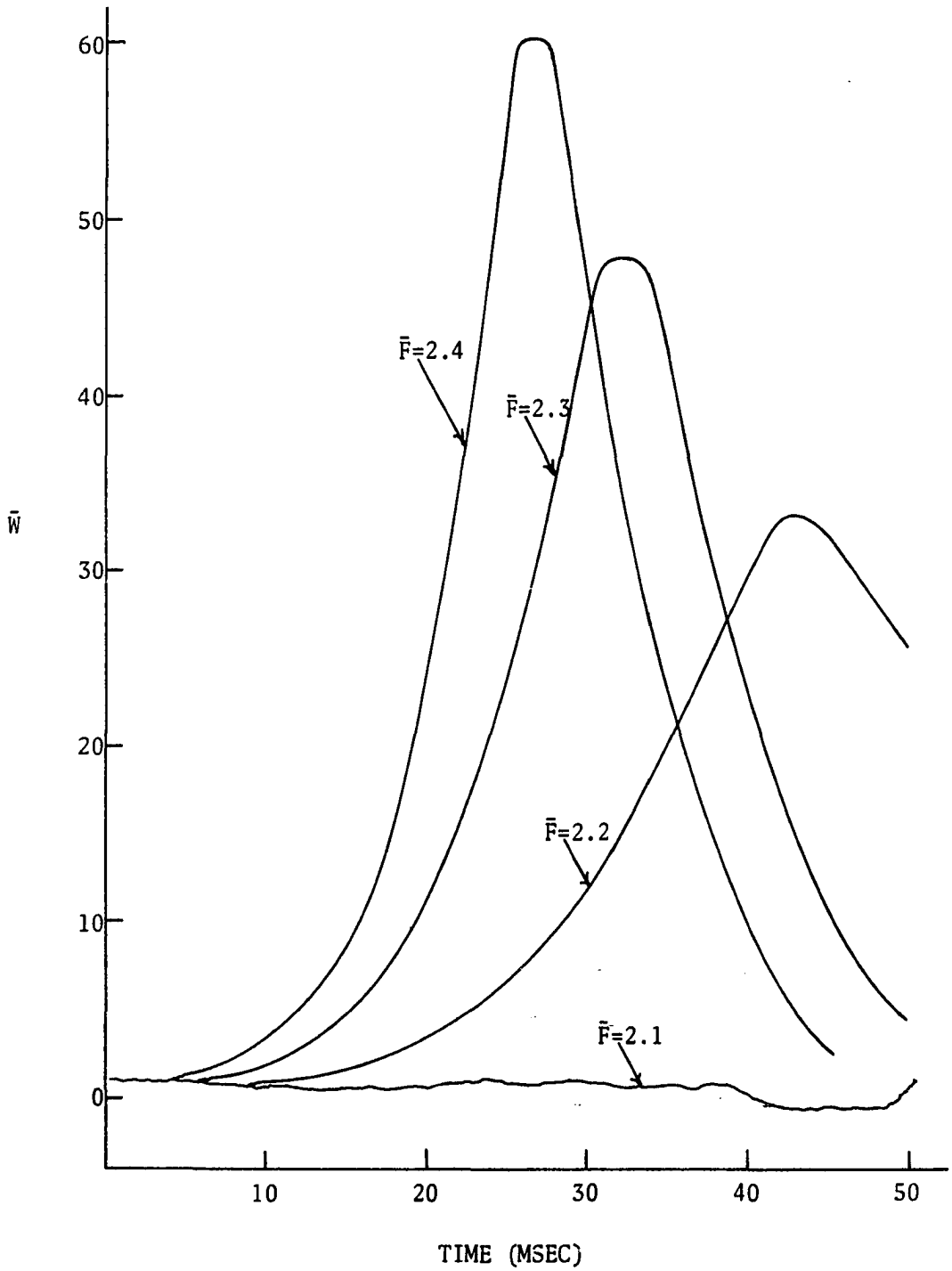


Figure 4.3 Maximum Radial Deflection Versus Time for Various End Loads, n = 9

identify the exact buckling load and the corresponding wave number, it was necessary to study the behavior of the shell in greater detail for longer time durations. This illustrative study is shown in Figs. 4.3 and 4.4. Based on these figures, the representative cylinder would buckle at a wave number of nine rather than ten. The shell reaches a greater deflection in Fig. 4.3 than in Fig. 4.4 for the same applied load.

As mentioned previously, the time required for maximum deflection to occur decreases as the applied load is increased. This time can be classed as the critical time required for buckling. If the time duration of the axial loading is short enough, the shell should be able to withstand loads higher than the minimum buckling load. Although this fact has not been experimentally verified for dynamically loaded cylinders, the same conclusion was reached in reference [49]. Figure 4.5 depicts the time duration curve for the representative stiffened cylinder.

## 4.3 Geometric Considerations

In an attempt to suggest design improvements on current light aircraft, various parameters (such as number of stiffeners, stiffener area and stiffener eccentricity) were varied and the effects on the buckling load were noted. The results of this limited parametric study are shown in Figs. 4.6 through 4.11.

The influence of the number of stringers on the buckling load of subshell II are shown in Figs. 4.6 and 4.7. Since local buckling between the stringers was not the dominant mode of buckling, only the region of importance is shown on the respective figures. However, the

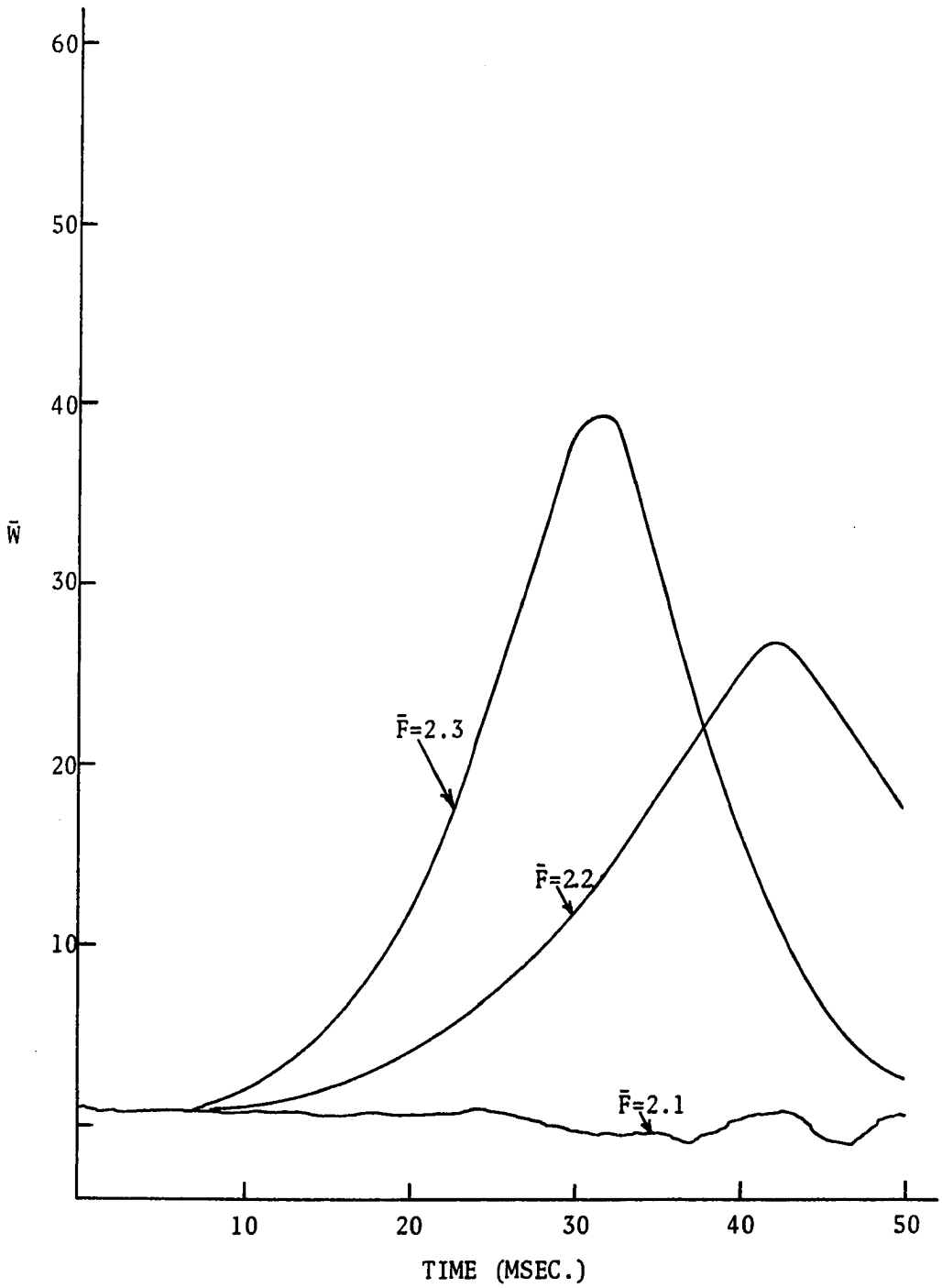


Figure 4.4 Maximum Radial Deflection Versus Time for Various End Loads, n = 10

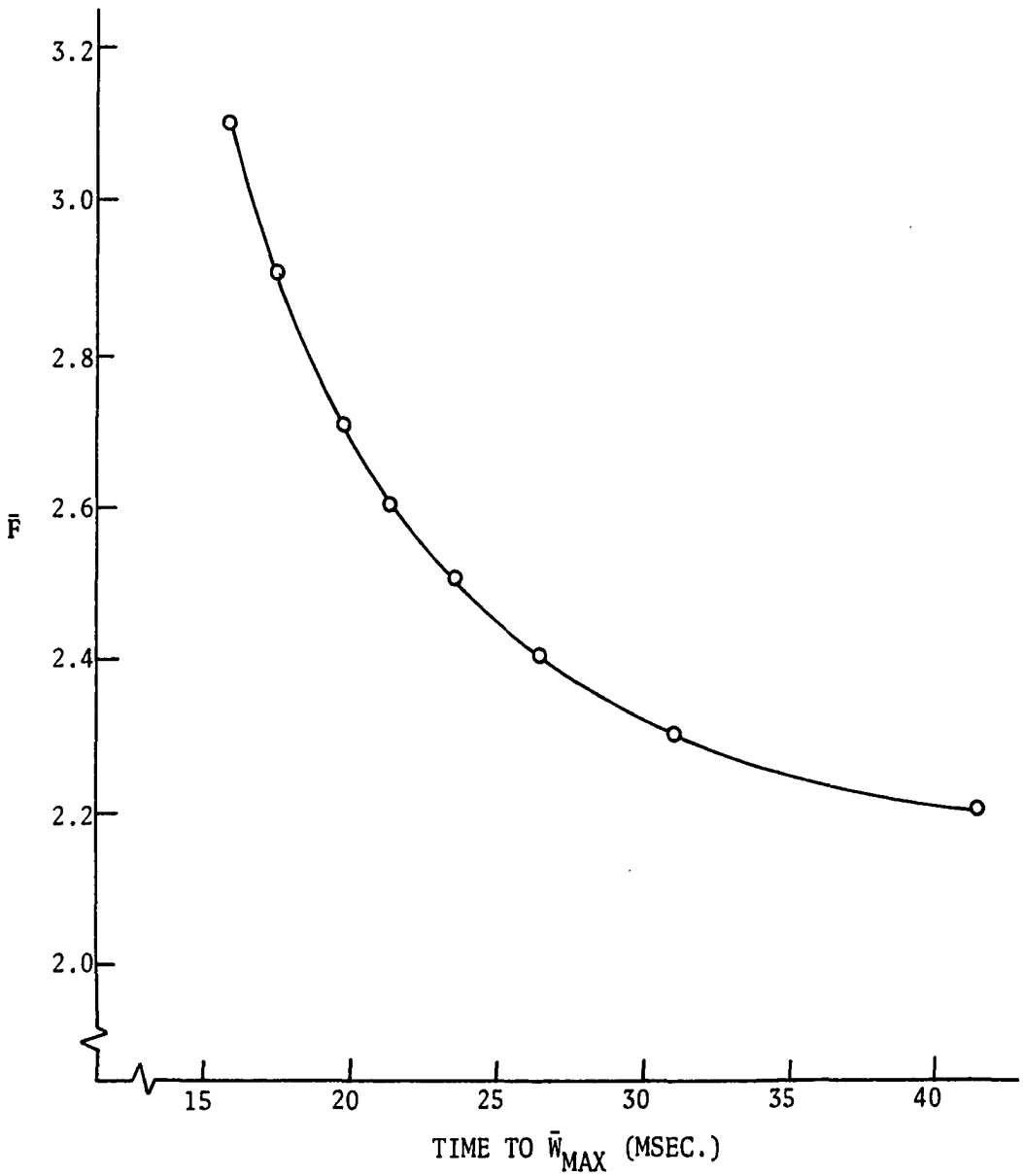


Figure 4.5 Buckling Load Versus Time Duration, n = 9

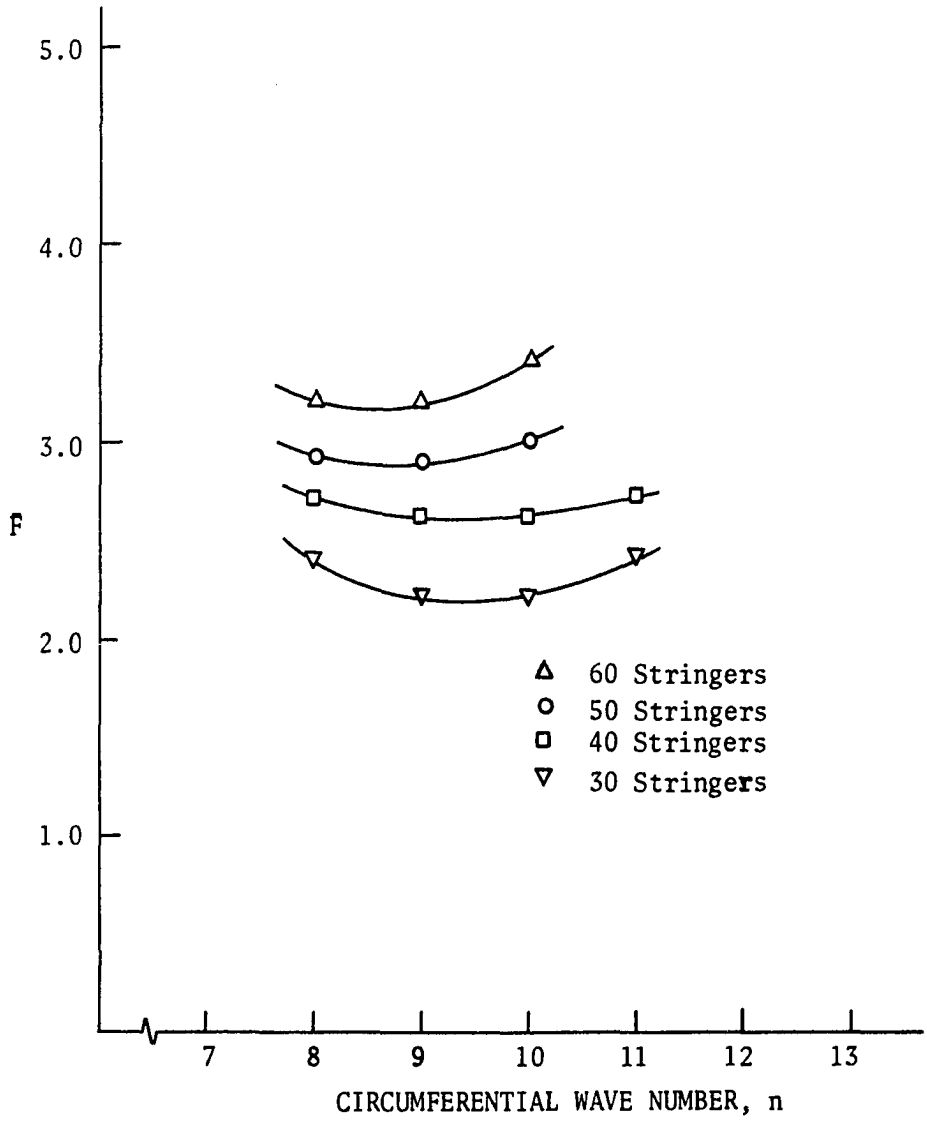


Figure 4.6 Axial Force Versus Wave Number with Increasing Number of Stringers

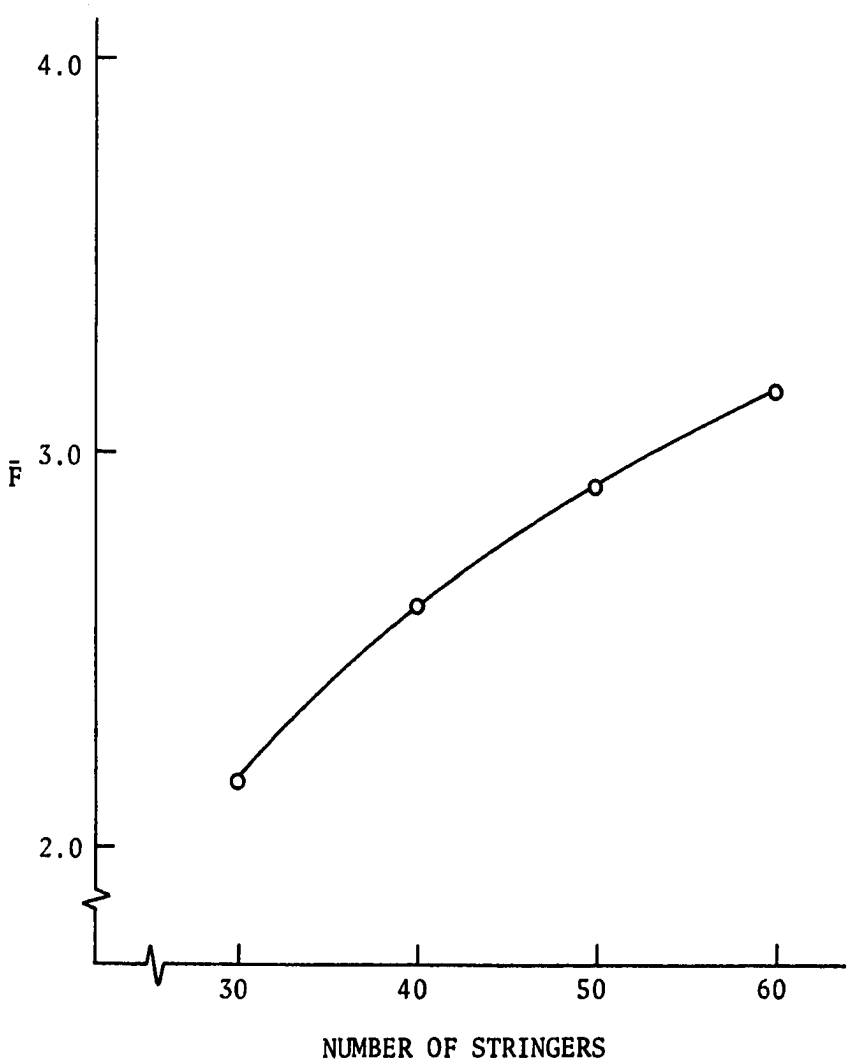


Figure 4.7 Influence of Number of Stringers on Buckling Load

calculations were also run for higher wave numbers to check this assumption. In performing the calculations all shell and stiffener parameters were held constant except for the number of stringers. It can be seen from Fig. 4.7, that if the number of stringers of the aircraft C type were doubled, the minimum dynamic buckling load would be increased by over 50 per cent. If the number of stringers was less than thirty, local buckling between the stringers would probably be the dominant mode of buckling, with a corresponding decrease in buckling load. This observation suggests a minimum weight design where the number of stringers was just enough to prevent local buckling between the stringers.

The influence of the number of rings on the buckling load is shown in Figs. 4.8 through 4.10. Figures 4.8 and 4.10 represent the lower bound on the calculations, since they were made using subshell III, which does not include the torsional rigidity of the rings. The calculations show the same trend previously observed when the number of stringers was increased. If the distance between rings is halved, the buckling load is increased by about 50 per cent. This same trend is also verified when the ring torsional rigidity is included, as shown in Fig. 4.9.

Finally, the ring and stringer cross-sectional areas and eccentricities were each varied independently and the analysis was repeated. No significant change in the buckling load was noted for increases of 25 to 50 per cent of each parameter. This somewhat surprising result was also observed by Singer, Baruch, and Harari [35] when they studied the static buckling of stiffened cylinders using a linear smeared

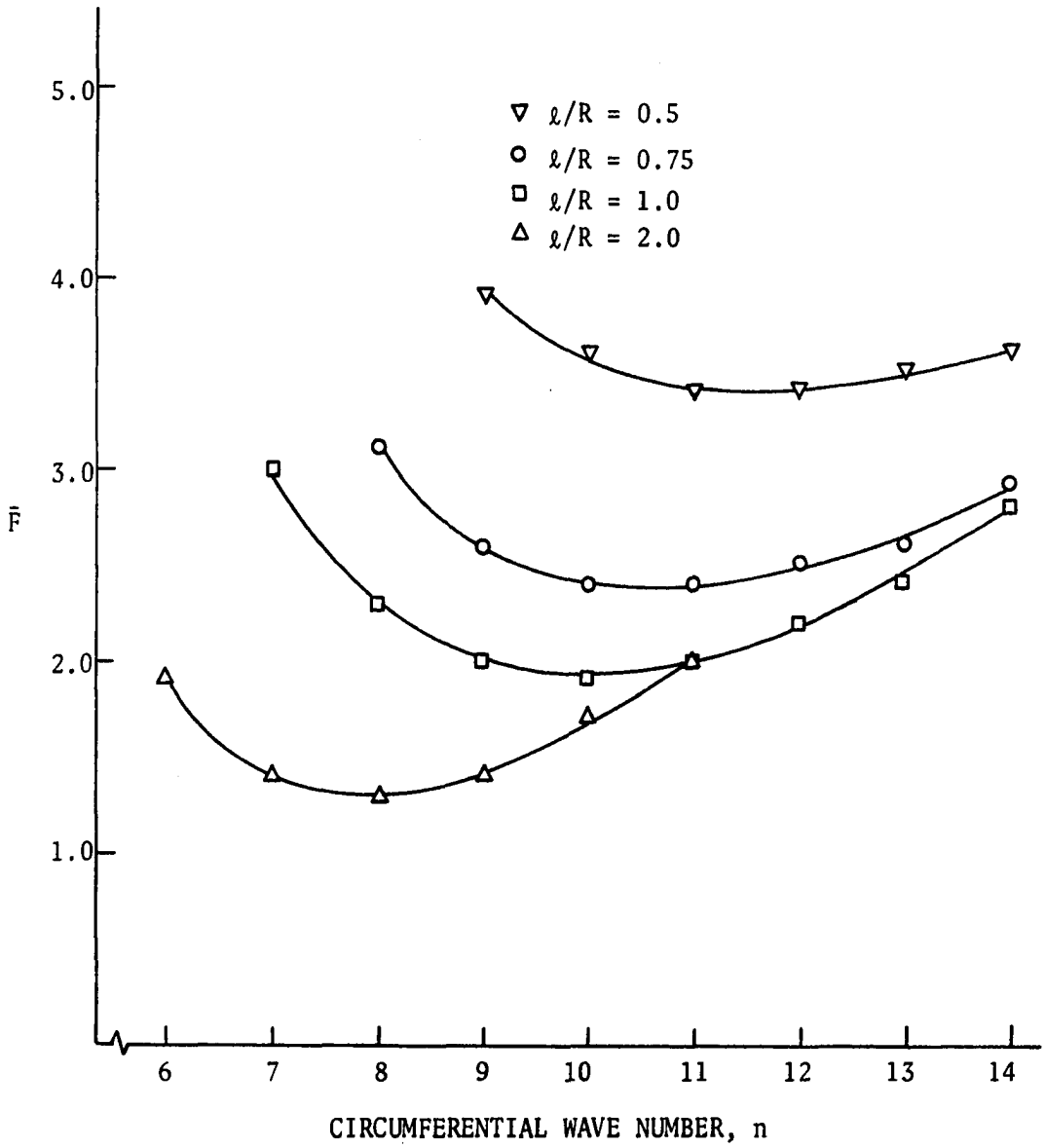


Figure 4.8 Axial Force Versus Wave Number for Different Ring Spacing, Subshell III

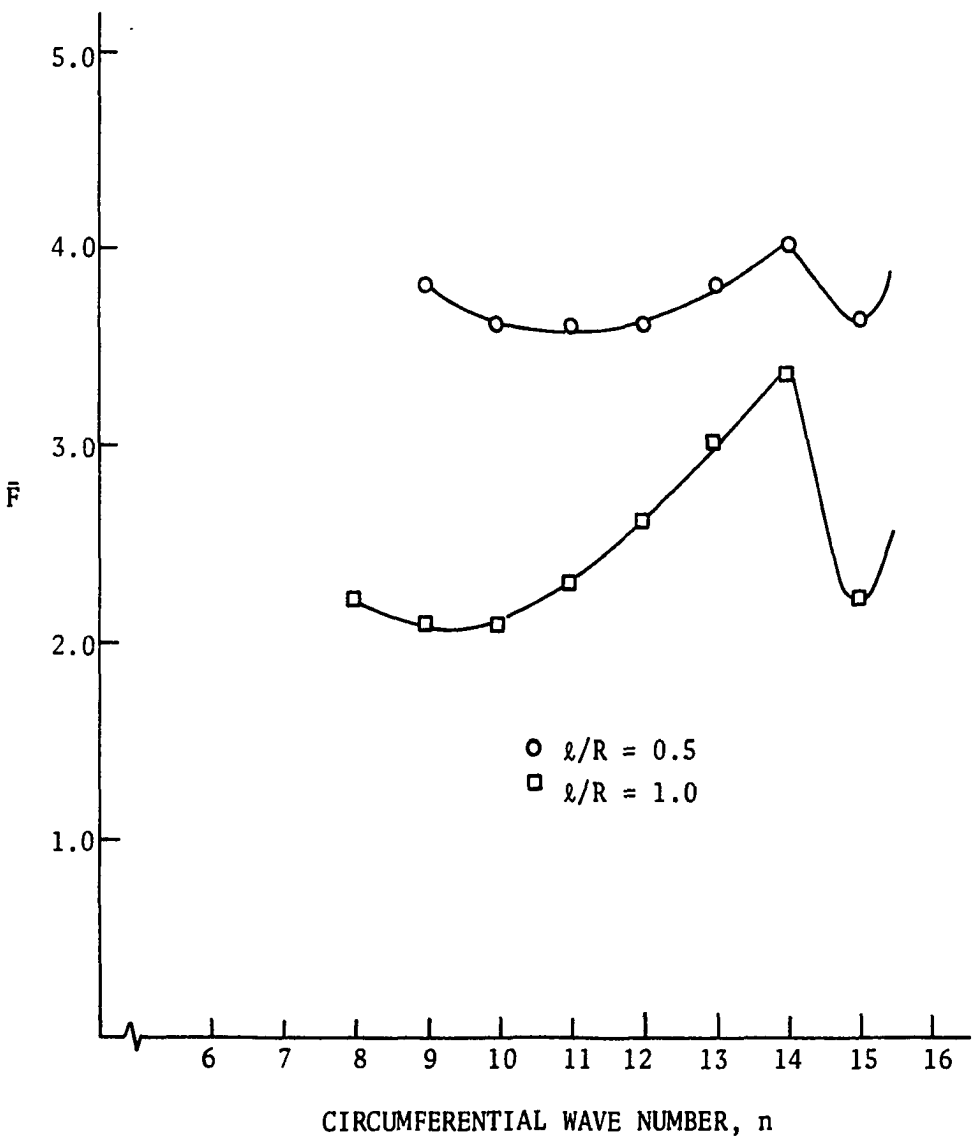


Figure 4.9 Axial Force Versus Wave Number for Different Ring Spacing, Subshell I

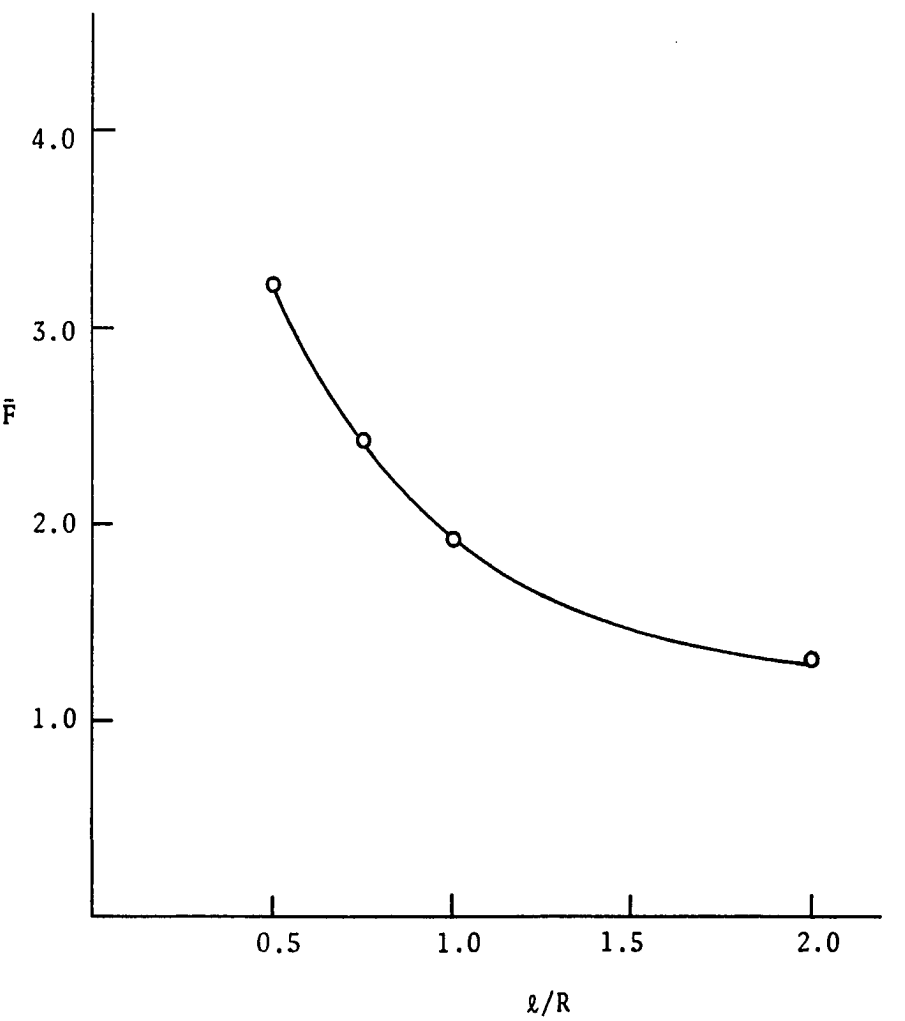


Figure 4.10 Influence of Number of Rings on Buckling Load

analysis. They stated that, except for very low values of the Batdorf parameters, Z, the buckling load was not increased appreciably by variation of stiffener area or eccentricity. Judging from the curves presented in reference [35], the value of Z had to be less than 25 for the variation to have any appreciable effect. As noted earlier, the smallest value of Z observed for any of the light aircraft structures was about 190.

The results of the parameter variation seem to suggest that stiffener eccentricity or one-sideness need not be included in the analysis. This, however, is definitely not the case as shown in Fig. 4.11. If the stiffeners were placed on the outside rather than on the inside of the cylinder, the dynamic buckling load is increased. The effect of stiffener eccentricity on the buckling load was first demonstrated by Koiter [25]. Of course, from a practical standpoint, light aircraft normally would not be designed with the stiffeners outside the fuselage!

The results of this section demonstrate that the number rather than the geometric configuration of the stiffeners is the important design consideration. This is not surprising, especially due to the local buckling behavior observed previously. The stiffeners can only exert a torsional restraint in the local buckling mode. Consequently, "beefing up" the stiffeners will only effect the buckling load slightly. The results of this section also suggest an aircraft design where the stiffener cross-section is reduced until general instability would be on the verge of predominating over local buckling. Then the reduction in stiffener weight could be used to increase the total number of stiffeners. The final design should be a more crashworthy aircraft with no increase in weight.

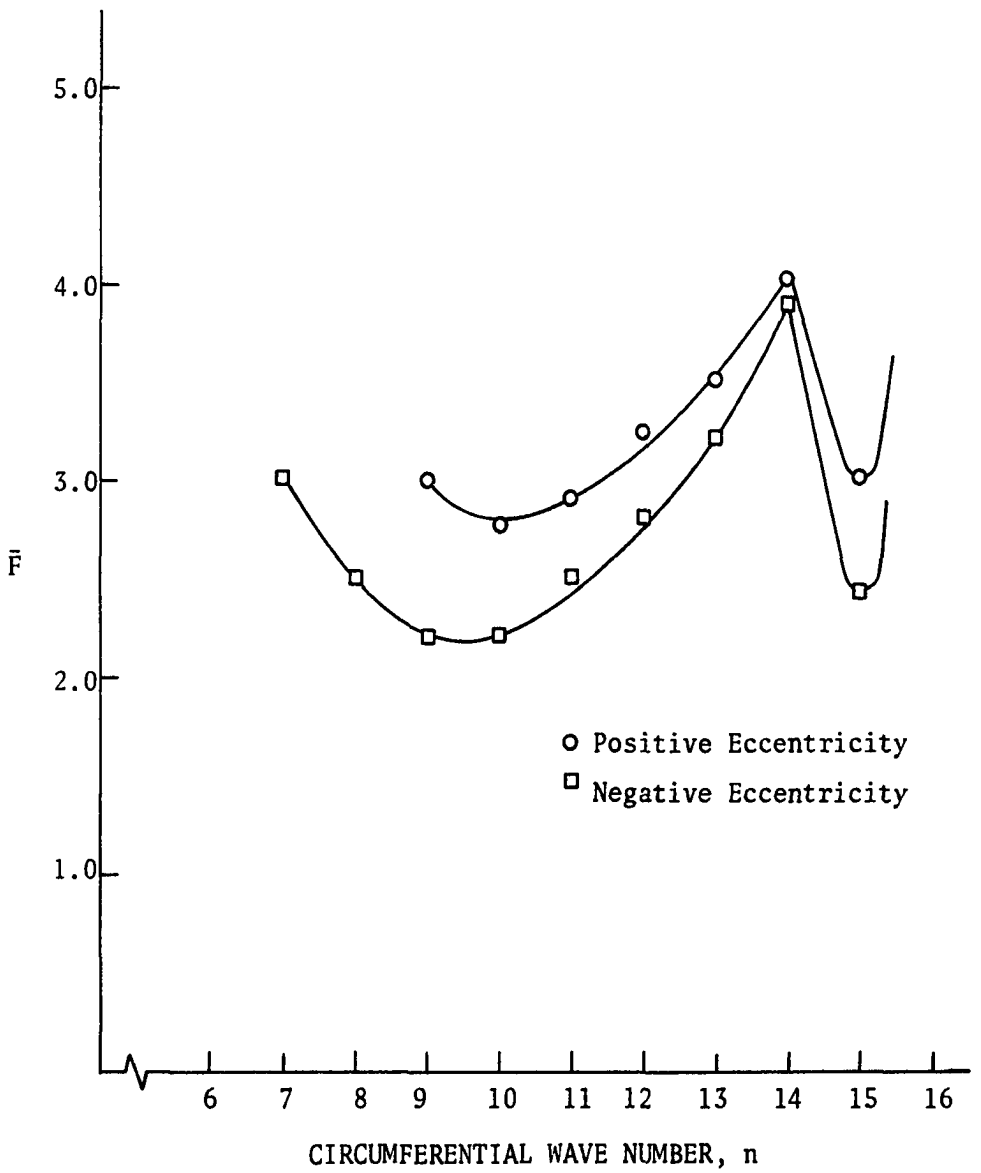


Figure 4.11 Influence of Stiffener Eccentricity on Buckling Load

## 4.4 Discreteness Considerations

Because of the wide spacing between some stiffeners in light aircraft structures, the stiffeners were treated as discrete elements, by use of the Dirac delta representation. However, it was found that most light aircraft have relatively closely spaced stringers. An inspection of Table 4.1 reveals that the number of stringers varies from 33 on aircraft A, which has a 25-inch radius, to 22 on aircraft C, which also has a 25-inch radius. As noted in Section 4.1, a stringer spacing of 30 caused some discreteness effects to be apparent, but not enough to dominate the buckling mode. This was due in part to the large number of stringers and in part to the restraining effect of the rings, as was observed in Fig. 4.2. However, if light aircraft are designed with fewer stringers it is recommended that stringer discreteness be checked using the present analysis.

The discreteness effects of the rings were observed using a suggestion from reference [41], where it was shown that the first term, the first approximation, of the discrete solution is equal to the "smeared" solution, for the case of equal rings. This fact also applies to the present analysis. If the stiffeners are smeared in the analysis, and the buckling Equations (2-16) are solved by the Galerkin method, one sees that the terms corresponding to the unstiffened shell are the same as those in the discrete analysis. The ring terms in the smeared analysis yield terms of the type

$$(E_r A_r/\ell)\int_0^L \cos(i\pi x/L)\cos(\zeta\pi x/L)dx = \tfrac{1}{2}(E_r A_r L/\ell)\delta_{i\zeta} \qquad (4\text{-}1)$$

The corresponding term for the discrete analysis would be

$$\int_{o}^{L} \sum_{j} E_{rj} A_{rj} \delta(x-j\ell) \cos(i\pi x/L) \cos(\zeta\pi x/L) dx = \sum_{j} E_{rj} A_{rj} \cos(i\pi x/L) \cos(\zeta\pi x/L) \tag{4-2}$$

In matrix form, Equation (4-1) would be an $i\zeta$ diagonal matrix, whereas Equation (4-2) would also be an $i\zeta$ matrix, but not necessarily diagonal. However, the diagonal terms of Equation (4-2) would be of the form

$$\sum_{j} E_{rj} A_{rj} \cos^2(i\pi x/L) = \tfrac{1}{2}(E_r A_r L/\ell) \tag{4-3}$$

Obviously Equation (4-3) is equal to Equation (4-1), demonstrating that the diagonal terms in the discrete case are equal to those of the smeared case.

Thus, instead of generating a smeared analysis to compare with the present analysis, it is only necessary to compare a one term solution with a multi-term solution to observe the discreteness effects.

The discreteness effects on subshell I are shown in Fig. 4.12. It is readily observed that a one-term solution was totally inadequate in predicting the buckling load, but that a three- or five-term solution rapidly converged to a lower bound answer. It is interesting to note that the three- and five-term solution predicted essentially the same critical buckling load at the same wave number, but differed somewhat at the higher wave numbers.

The discreteness effects on subshell II are shown in Fig. 4.13. They are of the same character as those for subshell I, but it took a five-term rather than a three-term solution to converge on the lowest buckling load. A seven-term solution is also shown in Fig. 4.13 which

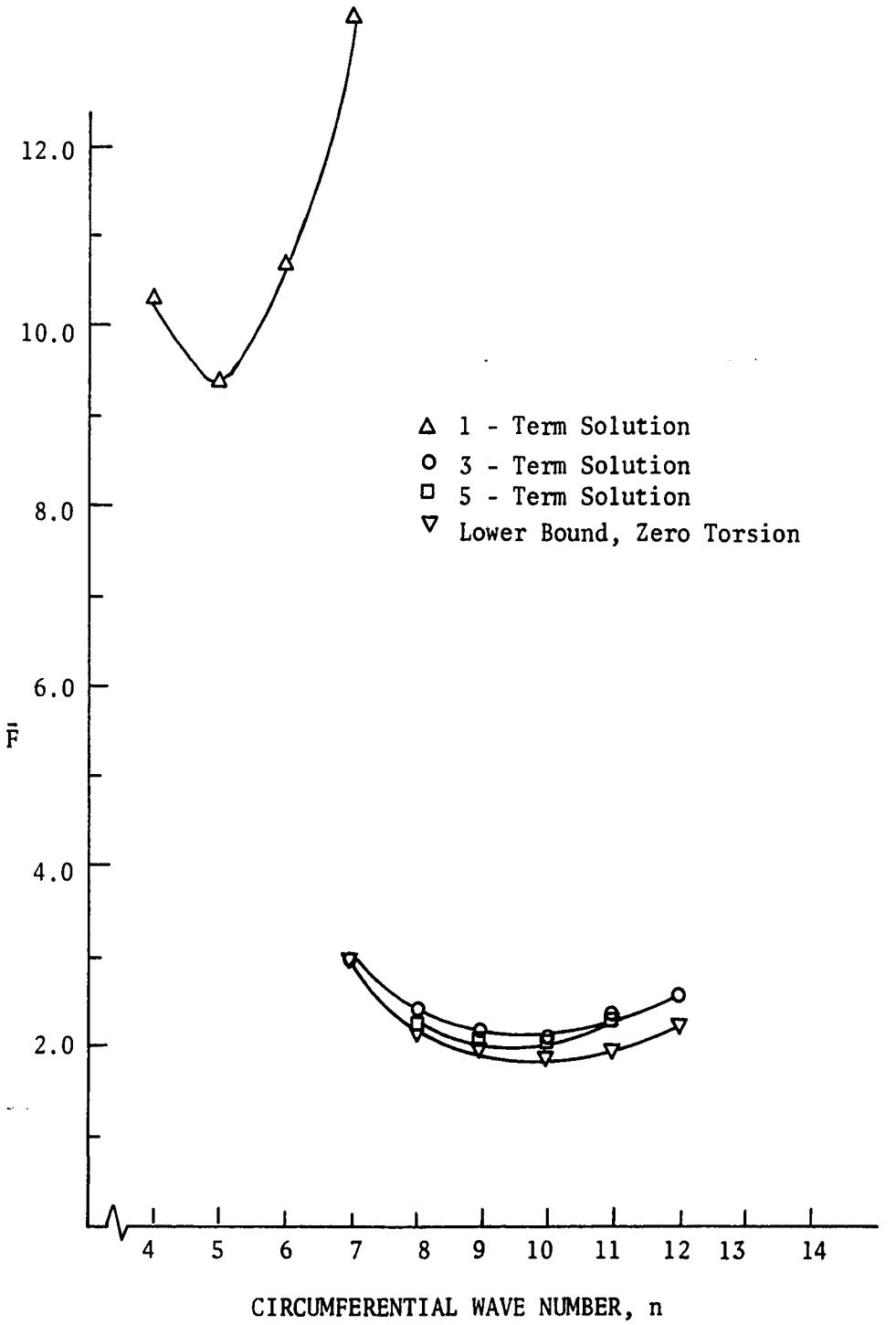


Figure 4.12 Convergence of Solution, Subshell I

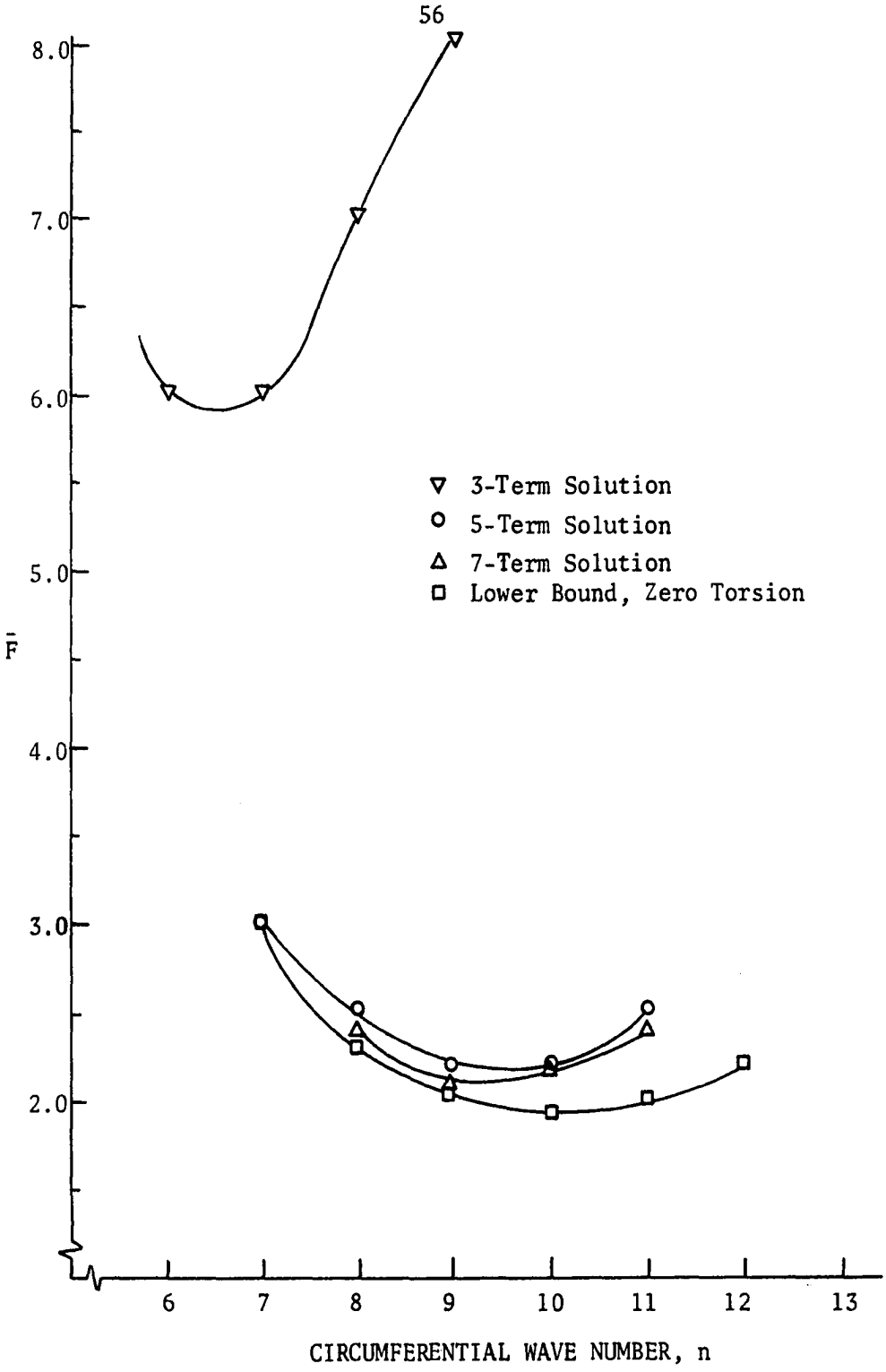


Figure 4.13 Convergence of Solution, Subshell II

further demonstrates the convergence. It is interesting to note that, for subshell I, only the third and fifth terms had a significant effect on the buckling load. Similarly, only the fifth and seventh terms had a significant effect on the subshell II buckling load. This behavior followed an empirical rule suggested by Dr. D. M. Egle for identifying the significant assumed mode terms. This rule can be deduced from Ref. [62] as:

$$1\text{st term},\ (2n_b \pm m)\ \text{term},\ (4n_b \pm m)\ \text{term, etc.}$$

where

$n_b$ = numbers of bays between rings

$m$ = axial wave number (normally 1)

This rule predicts the same important terms for subshell I and II as were actually observed.

Thus, as anticipated, the discreteness of the rings or bulkheads in light aircraft must be accounted for. The usual smeared analysis would calculate an incorrect wave number and a buckling load that could be an order of magnitude too high.

# CHAPTER V

## CLOSURE

The analysis developed in this dissertation can be applied to any light aircraft structure undergoing any dynamically varying axial compression loading. Since the stiffeners are treated as discrete elements, the analysis can be of great value in the design of new aircraft that can better withstand crash impact loads without any great increase in overall structural design weight. The computer program contained in the analysis can handle any dynamically varying load shape, such as ramp, exponential, or triangular. With minor modification, the program could handle stiffeners with varying geometric cross-sectional properties.

The present theory was compared with available dynamic unstiffened shell and static stiffened shell analyses, and good agreement was achieved. A limited parametric study was conducted on a stiffened cylinder which was representative of a present-day, light-aircraft cabin section. The results were presented primarily in graphical form, and the following general conclusions were obtained:

1. Since local buckling between bulkheads predominated over general shell instability in the test cases, the stiffeners in most aircraft are probably overdesigned. A more efficient design could be achieved

by designing the stiffeners such that general instability buckling would be on the verge of predominating over local buckling.

2. The number of rings and stringers should be increased in preference to increasing the size or cross-sectional shape of the stiffeners in current aircraft design.
3. Stiffener discreteness must be included in a dynamic analysis of the type presented here in order to adequately model a light aircraft structure in a crash environment.

Finally, it is recommended that this analysis be complimented by a series of experiments in which representative cylinders are subjected to a carefully controlled dynamic buckling environment. No experiments were found in the current literature in which dynamic axial compression buckling of stiffened shells was studied. Experiments of this type could study phenomena such as time duration of loading (Section 4.2), the effect of stiffener discreteness (Section 4.4), and the effect of geometric variations of the stiffened shell on the buckling load (Section 4.3). The present theory would provide the analytical basis for designing such experiments to extract maximum useful information.

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## APPENDIX A

## DERIVATION OF THE KINETIC AND POTENTIAL ENERGIES FOR A DISCRETELY STIFFENED CYLINDRICAL SHELL

### A.1 Nonlinear Strain-Displacement Formulation

The Donnell [4] nonlinear strain-displacement relations for the shell mid-surface are

$$\begin{aligned} \varepsilon_x &= u_{,x} = \tfrac{1}{2}(w_{,x})^2 \\ \varepsilon_y &= v_{,y} + wR^{-1} + \tfrac{1}{2}(w_{,y})^2 \\ \gamma_{xy} &= u_{,y} + v_{,x} + w_{,x}w_{,y} \end{aligned} \tag{A-1}$$

It is assumed that the stiffeners behave as beam elements and that displacements vary linearly across the stringer depth. Therefore, to satisfy compatibility of displacements where shell and stringer are joined, we may write

$$\begin{aligned} \varepsilon_{yr} &= \varepsilon_y - zw_{,yy} \\ \varepsilon_{xs} &= \varepsilon_x - zw_{,xx} \end{aligned} \tag{A-2}$$

### A.2 Unstiffened Cylinder Strain Energy

The strain energy of the unstiffened shell is found by considering a small element of a thin shell. Since plane stress is assumed to be a valid assumption for a thin shell and the shell material is homogeneous, isotropic, and linearly elastic, the following constitutive relations are appropriate

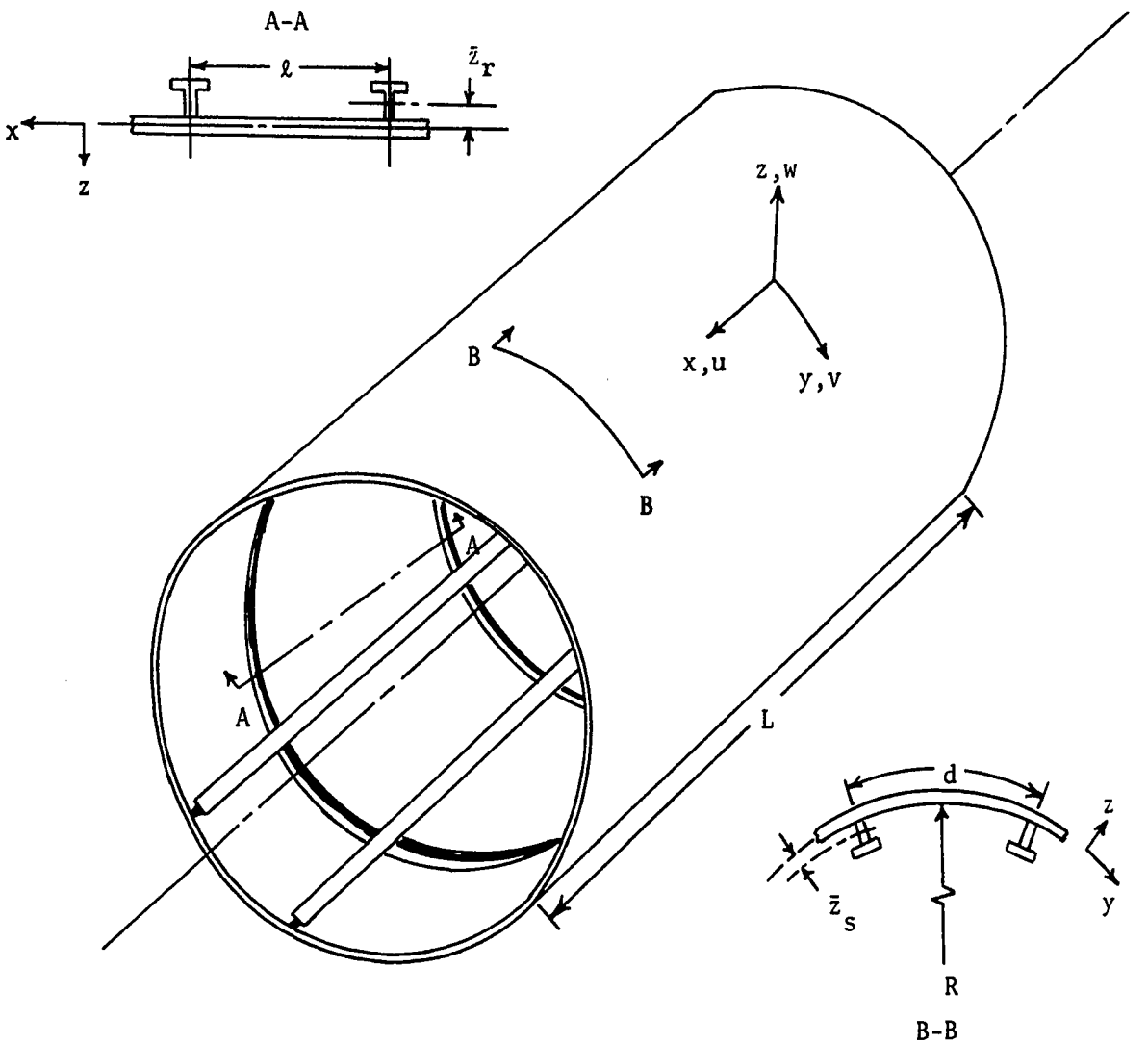


Figure A.1 Shell Geometry

$$\sigma_x = E(1-\nu^2)^{-1}(\varepsilon_x+\nu\varepsilon_y)$$

$$\sigma_y = E(1-\nu^2)^{-1}(\varepsilon_y+\nu\varepsilon_x) \qquad \text{(A-3)}$$

$$\tau_{xy} = G\gamma_{xy}$$

The incremental change in strain energy per unit volume for the small element is

$$dU_{vol.} = \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \tau_{xy} d\gamma_{xy} \qquad \text{(A-4)}$$

Substituting Equation (A-3) into (A-4) and integrating the result, one obtains the strain energy per unit volume

$$U_{vol.} = E(1-\nu^2)^{-1}[\tfrac{1}{2}\varepsilon_x^2 + \tfrac{1}{2}\varepsilon_y^2+\nu\varepsilon_x\varepsilon_y + \tfrac{1}{4}(1-\nu)\gamma_{xy}^2] \qquad \text{(A-5)}$$

If Equation (A-5) is integrated over the volume of the unstiffened cylinder, both the extensional and the bending strain energy will be found. This integration yields

$$U_{cylinder} = \tfrac{1}{2}\int_0^{2\pi R}\int_0^{L}[Eh(1-\nu^2)^{-1}(\varepsilon_x^2+2\varepsilon_x\varepsilon_y+\varepsilon_y^2 + \tfrac{1}{2}(1-\nu)\gamma_{xy}^2) + D(w_{,xx}+w_{,yy})^2 - 2D(1-\nu)(w_{,xx}w_{,yy}-w_{,xy}^2)]dxdy \qquad \text{(A-6)}$$

If the strain displacement relations (A-1) are substituted into Equation (A-6), one obtains

$$U_c = \int_0^{2\pi R}\int_0^{L}\{Eh(1-\nu^2)^{-1}[\tfrac{1}{2}u_{,x}^2+\tfrac{1}{2}u_{,x}w_{,x}^2+(\tfrac{1}{8})w_{,x}^4+\nu u_{,x}v_{,y}+\tfrac{1}{2}\nu R^{-1}ww_{,x}^2$$

$$+\tfrac{1}{2}\nu u_{,x}w_{,y}^2+\tfrac{1}{4}\nu w_{,x}^2w_{,y}^2+\nu R^{-1}wu_{,x}+\tfrac{1}{2}\nu w_{,x}^2v_{,y}+R^{-1}wv_{,y}+\tfrac{1}{2}v_{,y}^2+\tfrac{1}{2}v_{,y}w_{,y}^2$$

$$+\tfrac{1}{2}R^{-1}ww_{,y}^2+(\tfrac{1}{8})w_{,y}^4+\tfrac{1}{2}(w/R)^2+\tfrac{1}{2}(1-\nu)(\tfrac{1}{2}u_{,y}^2+\tfrac{1}{2}v_{,x}^2+u_{,y}w_{,x}w_{,y}+v_{,x}w_{,x}w_{,y}$$

$$+u_{,y}v_{,x}+\tfrac{1}{2}w_{,x}^{2}w_{,y}^{2})] + D(w_{,xx}+w_{,yy})^{2} - 2D(1-\nu)(w_{,xx}w_{,yy}-w_{,xy}^{2})\}dxdy \tag{A-7}$$

## A.3 Ring Strain Energy

The rings are considered as thin curved beam elements, with the strain energy of an individual ring consisting of strain energy due to flexure, extension, and torsion. Thus

$$U_r = \tfrac{1}{2}\sum_j \int_0^{2\pi R} (\int_{A_r} E_r \varepsilon_{yr}^2 dA_r + G_r J_r w_{,xy}^2)_{x=j\ell} dy \tag{A-8}$$

Next, the ring strain-displacement relations (A-2) are substituted into Equation (A-8) and integrations are performed over the ring area. The strain energy of N rings becomes

$$U_r = \tfrac{1}{2}\sum_j \int_0^{2\pi R} \int_0^{L} (E_r A_r \varepsilon_y^2 - 2E_r A_r \bar{z}_r w_{,yy}\varepsilon_y + E_r I_{or} w_{,yy}^2 + G_r J_r w_{,xy}^2)\delta(x-j\ell)\,dxdy \tag{A-9}$$

where $\delta(x-j\ell)$ is a Dirac delta function defined by

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta(x-j\ell)dx &= f(j\ell) \qquad (x=j\ell) \\ \delta(x-j\ell) &= 0 \qquad (x\neq j\ell) \end{aligned} \tag{A-10}$$

Now, Equation (A-9) is in terms of the strains of the shell middle surface. If the shell strain-displacement relations (A-1) are substituted into (A-9), one obtains

$$\begin{aligned} U_r = \sum_j \int_0^{2\pi R} \int_0^{L} \{ & E_r A_r [\tfrac{1}{2}v_{,y}^2 + v_{,y}R^{-1}w + \tfrac{1}{2}v_{,y}w_{,y}^2 + \tfrac{1}{2}R^{-1}ww_{,y}^2 + (\tfrac{1}{8})w_{,y}^4 \\ & + \tfrac{1}{2}(w/R)^2 - \bar{z}_r w_{,yy}(v_{,y} + R^{-1}w + \tfrac{1}{2}w_{,y}^2)] + \tfrac{1}{2}E_r I_{or} w_{,yy}^2 \\ & + \tfrac{1}{2}G_r J_r w_{,xy}^2\}\delta(x-j\ell)\,dxdy \end{aligned} \tag{A-11}$$

## A.4 Stringer Strain Energy

In an analogous fashion to the derivation of the ring strain energy, the stringer strain energy can be found as

$$U_s = \tfrac{1}{2}\sum_k \int_0^{2\pi R}\int_0^L (E_s A_s \varepsilon_x^2 - 2E_s A_s \bar{z}_s w_{,x}\varepsilon_x + E_s I_{os} w_{,xx}^2 + G_s J_s w_{,xy}^2)\,\delta(y-kd)\,dx\,dy \tag{A-12}$$

where

$$\int_{-\infty}^{\infty} f(y)\,\delta(y-kd)\,dy = f(kd) \qquad (y=kd)$$
$$\delta(y-kd) = 0 \qquad (y\neq kd) \tag{A-13}$$

As before, Equation (A-1) is substituted into Equation (A-12) to obtain

$$U_s = \sum_k \int_0^{2\pi R}\int_0^L \{E_s A_s (\tfrac{1}{2}u_{,x}^2 + \tfrac{1}{2}u_{,x}w_{,x}^2 + \frac{1}{8}w_{,x}^4 - \bar{z}_s w_{,xx}u_{,x} - z_s w_{,xx}w_{,x}^2)$$
$$+\tfrac{1}{2}E_s I_{os} w_{,xx}^2 + \tfrac{1}{2}G_s J_s w_{,xy}^2\}\,\delta(y-kd)\,dx\,dy \tag{A-14}$$

## A.5 Potential Energy of External Load

The potential energy of the externally applied load on the end of the stiffened shell is the same as the negative of the work done on the shell. Thus it can be calculated as the product of the applied force and the change of length of the cylinder. Finally, the potential energy due to the load resultant $\hat{N}_x$ applied at a distance $\bar{e}$ from the shell middle surface is

$$U_{\substack{\text{external}\\ \text{load}}} = \int_0^{2\pi R} \hat{N}_x (u - \bar{e}w_{,x})\Big|_0^L dy \tag{A-15}$$

## A.6 Total Potential Energy

The total potential energy is found by adding Equations (A-7), (A-11), (A-14) and (A-15) to obtain

$$
\begin{aligned}
U = \int_0^{2\pi R}\int_0^{L} \{ & Eh(1-\nu^2)^{-1}[\tfrac{1}{2}u_{,x}^2+\tfrac{1}{2}u_{,x}w_{,x}^2+\tfrac{1}{8}w_{,x}^4+\nu u_{,x}v_{,y}+\tfrac{1}{2}\nu R^{-1}ww_{,x}^2 \\
& +\tfrac{1}{2}\nu u_{,x}w_{,y}^2+\tfrac{1}{4}\nu w_{,x}^2w_{,y}^2+\nu R^{-1}wu_{,x}+\tfrac{1}{2}\nu w_{,x}^2v_{,y}+R^{-1}wv_{,y}+\tfrac{1}{2}v_{,y}^2+\tfrac{1}{2}v_{,y}w_{,y}^2 \\
& +\tfrac{1}{2}R^{-1}ww_{,y}^2+\tfrac{1}{8}w_{,y}^4+\tfrac{1}{2}(w/R)^2+\tfrac{1}{2}(1-\nu)(\tfrac{1}{2}u_{,y}^2+\tfrac{1}{2}v_{,x}^2+u_{,y}w_{,x}w_{,y}+v_{,x}w_{,x}w_{,y} \\
& +u_{,y}v_{,x}+\tfrac{1}{2}w_{,x}^2w_{,y}^2)]+D[(w_{,xx}+w_{,yy})^2-2(1-\nu)(w_{,xx}w_{,yy}-w_{,xy}^2)] \\
& +\sum_j \delta(x-j\ell)[\tfrac{1}{2}E_rI_{or}w_{,yy}^2+\tfrac{1}{2}G_rJ_rw_{,xy}^2+E_rA_r(\tfrac{1}{2}v_{,y}^2+R^{-1}v_{,y}w+\tfrac{1}{2}v_{,y}w_{,y}^2 \\
& +\tfrac{1}{2}R^{-1}ww_{,y}^2+\tfrac{1}{8}w_{,y}^4+\tfrac{1}{2}(w/R)^2-\bar{z}_rv_{,y}w_{,yy}-z_rR^{-1}ww_{,yy}-\tfrac{1}{2}\bar{z}_rw_{,yy}w_{,y}^2)] \\
& +\sum_k \delta(y-kd)[\tfrac{1}{2}E_sI_{os}w_{,xx}^2+\tfrac{1}{2}G_sJ_sw_{,xy}^2+E_sA_s(\tfrac{1}{2}u_{,x}^2+\tfrac{1}{2}u_{,x}w_{,x}^2+\tfrac{1}{8}w_{,x}^4 \\
& -\bar{z}_sw_{,xx}u_{,x}-\bar{z}_sw_{,xx}w_{,x}^2)]\}dxdy+\int_0^{2\pi R}\hat{N}_x(u-\bar{e}w_{,x})\Big|_0^L dy
\end{aligned}
\tag{A-16}
$$

## A.7 Total Kinetic Energy

Neglecting in-surface and rotatory inertia effects, one may write the kinetic energy of the unstiffened shell as

$$
T_{\substack{\text{unstiffened}\\ \text{shell}}} = \tfrac{1}{2}\rho h\int_0^{2\pi R}\int_0^{L} w_{,t}^2 dxdy \tag{A-17}
$$

In a like manner, the kinetic energy of rings and stringers (referenced to the shell middle surface) may be written as

$$
\begin{aligned}
T_r &= \tfrac{1}{2}\sum_j\int_0^{2\pi R}\int_0^{L}\delta(x-j\ell)\rho_rA_rw_{,t}^2dxdy \\
T_s &= \tfrac{1}{2}\sum_k\int_0^{2\pi R}\int_0^{L}\delta(y-kd)\rho_sA_sw_{,t}^2dxdy
\end{aligned}
\tag{A-18}
$$

## APPENDIX B

## APPLICATION OF HAMILTON'S PRINCIPLE TO DERIVE THE EQUATIONS OF MOTION

The governing equations of motion are obtained from Hamilton's principle, which requires that the first variation of the time-integrated difference between the potential and kinetic energies be zero.

$$\delta\int_{t_1}^{t_2}(U-T)dt = 0 \tag{B-1}$$

Substituting Equations (A-16), (A-17) and (A-18) into Equation (B-1) and performing the variational operation, one obtains

$$\begin{aligned}
\delta\int_{t_1}^{t_2}(U-T)dt = \int_o^t\int_o^{2\pi R}\int_o^L\Big(&Eh(1-\nu^2)^{-1}\{[u_{,x}+\tfrac{1}{2}w_{,x}^2+\nu v_{,y}+\tfrac{1}{2}\nu w_{,y}^2+R^{-1}\nu w]\delta u_{,x}\\
&+\tfrac{1}{2}[1-\nu][u_{,y}+w_{,x}w_{,y}+v_{,x}]\delta u_{,y}+\tfrac{1}{2}[1-\nu][u_{,y}+w_{,x}w_{,y}+v_{,x}]\delta v_{,x}\\
&+[\nu u_{,x}+\tfrac{1}{2}\nu w_{,x}^2+v_{,y}+R^{-1}w+\tfrac{1}{2}w_{,y}^2]\delta v_{,y}+[u_{,x}w_{,x}+\tfrac{1}{2}w_{,x}^3+R^{-1}\nu ww_{,x}\\
&+\tfrac{1}{2}w_{,x}w_{,y}^2+\nu w_{,x}v_{,y}+\tfrac{1}{2}(1-\nu)(u_{,y}w_{,y}+v_{,x}w_{,y})]\delta w_{,x}+[\nu u_{,x}w_{,y}\\
&+\tfrac{1}{2}\nu w_{,x}^2w_{,y}+v_{,y}w_{,y}+R^{-1}ww_{,y}+\tfrac{1}{2}w_{,y}^3+\tfrac{1}{2}(1-\nu)(u_{,y}w_{,x}+v_{,x}w_{,x}\\
&+w_{,y}w_{,x}^2)]\delta w_{,y}+[\tfrac{1}{2}\nu R^{-1}w_{,x}^2+R^{-1}\nu u_{,x}+R^{-1}v_{,y}+\tfrac{1}{2}R^{-1}w_{,y}^2\\
&+w(1/R)^2]\delta w\}+D\{[\tfrac{1}{2}w_{,xx}+\tfrac{1}{2}\nu w_{,yy}]\delta w_{,xx}+[\tfrac{1}{2}w_{,yy}+\tfrac{1}{2}\nu w_{,xx}]\\
&\cdot\delta w_{,yy}+[1-\nu]w_{,xy}\delta w_{,xy}\}+\sum_j\delta(x-j\ell)\{E_rA_r[v_{,y}+R^{-1}w+\tfrac{1}{2}w_{,y}^2
\end{aligned}$$

$$-\bar{z}_r w_{,yy}]\delta v_{,y}+E_r A_r[w_{,y}v_{,y}+R^{-1}ww_{,y}+\tfrac{1}{2}w_{,y}^3-\bar{z}_r w_{,yy}w_{,y}]\delta w_{,y}+E_r A_r[R^{-1}v_{,y}$$

$$+\tfrac{1}{2}R^{-1}w_{,y}^2+w(1/R)^2-\bar{z}_r R^{-1}w_{,yy}]\delta w+[E_r I_{or}w_{,yy}-E_r A_r\bar{z}_r(v_{,y}+R^{-1}w+\tfrac{1}{2}w_{,y}^2)]$$

$$\cdot\delta w_{,yy}+G_r J_r w_{,xy}\delta w_{,xy}\}+\sum_k\delta(y-kd)\{E_s A_s[u_{,x}+\tfrac{1}{2}w_{,x}^2-\bar{z}_s w_{,xx}]\delta u_{,x}+E_s A_s[w_{,x}u_{,x}$$

$$+\tfrac{1}{2}w_{,x}^3-\bar{z}_s w_{,xx}w_{,x}]\delta w_{,x}+[E_s I_{os}w_{,xx}-E_s A_s\bar{z}_s(u_{,x}+\tfrac{1}{2}w_{,x}^2)]\delta w_{,xx}+G_s J_s w_{,xy}\delta w_{,xy}\}$$

$$-\rho h w_{,t}\delta w_{,t}-\sum_j\delta(x-j\ell)\rho_r A_r w_{,t}\delta w_{,t}-\sum_k\delta(y-kd)\rho_s A_s w_{,t}\delta w_{,t}\Big]dxdydt = 0 \qquad \text{(B-2)}$$

The following illustrations of integrations by parts operations will be performed on Equation (B-2).

$$\int_0^t\int_0^L u_{,x}\delta u_{,x}dxdt = \int_0^t\{u_{,x}\delta u\Big|_0^L-\int_0^L u_{,xx}\delta udx\}dt$$

$$\int_0^t\int_0^L w_{,x}^2\delta u_{,x}dxdt = \int_0^t\{w_{,x}^2\delta u\Big|_0^L-\int_0^L (w_{,x})_{,x}^2\delta udx\}dt$$

$$\int_0^t\int_0^L v_{,y}\delta u_{,x}dxdt = \int_0^t\{v_{,y}\delta u\Big|_0^L-\int_0^L v_{,xy}\delta udx\}dt$$

$$\int_0^t\int_0^L w_{,y}^2\delta u_{,x}dxdt = \int_0^t\{w_{,y}^2\delta u\Big|_0^L-\int_0^L (w_{,y})_{,x}^2\delta udx\}dt$$

$$\int_0^t\int_0^{2\pi R} w_{,x}w_{,y}\delta u_{,y}dydt = \int_0^t\{w_{,x}w_{,y}\delta u\Big|_0^{2\pi R}-\int_0^{2\pi R}(w_{,x}w_{,yy}+w_{,xy}w_{,y})\delta udx\}dt$$

$$\int_0^t\int_0^L w_{,xx}\delta w_{,xx}dxdt = \int_0^t\{w_{,xx}\delta w_{,x}\Big|_0^L-w_{,xxx}\delta w\Big|_0^L+\int_0^L w_{,xxxx}\delta wdx\}dt$$

$$\int_0^t\int_0^{2\pi R}\int_0^L w_{,xy}\delta w_{,xy}dxdydt = \int_0^t\{\int_0^L w_{,xy}\delta w_{,x}\Big|_0^{2\pi R}dx-\int_0^{2\pi R} w_{,xyy}\delta w\Big|_0^L dy$$

$$+\int_0^{2\pi R}\int_0^L w_{,xxyy}\delta wdxdy\}dt$$

$$\int_o^t w_{,t}\delta w_{,t}dt = -\int_o^t w_{,tt}\delta w dt$$

Making use of the above integration by parts forms, and performing the indicated operations in Equation (B-2), one obtains the following equation

$$\delta\int_{t_1}^{t_2}(U-T)dt = \int_o^t\Bigg\{-\int_o^{2\pi R}\int_o^L\{Eh(1-\nu^2)^{-1}[u_{,xx}+w_{,x}w_{,xx}+\nu v_{,xy}+\nu w_{,y}w_{,xy}$$

$$+R^{-1}\nu w_{,x}+\tfrac{1}{2}(1-\nu)(u_{,yy}+w_{,xy}w_{,y}+w_{,x}w_{,yy}+v_{,xy})]$$

$$+\sum_k\delta(y-kd)E_sA_s[u_{,xx}+w_{,x}w_{,xx}-\bar{z}_sw_{,xxx}]\}\delta u dxdy$$

$$+\int_o^{2\pi R}\{Eh(1-\nu^2)^{-1}[u_{,x}+\tfrac{1}{2}w_{,x}^2+\nu v_{,y}+\tfrac{1}{2}\nu w_{,y}^2+R^{-1}\nu w]$$

$$+\sum_k\delta(y-kd)E_sA_s[u_{,x}+\tfrac{1}{2}w_{,x}^2-\bar{z}_sw_{,xx}]+\hat{N}_x\}\delta u\Big|_o^L dy$$

$$+\int_o^L\tfrac{1}{2}Eh(1+\nu)^{-1}\{u_{,x}+w_{,x}w_{,y}+v_{,x}\}\delta u\Big|_o^{2\pi R}dx-\int_o^{2\pi R}\int_o^L\{Eh(1-\nu^2)^{-1}$$

$$\cdot[\nu u_{,xy}+\nu w_{,x}w_{,xy}+v_{,yy}+R^{-1}w_{,y}+w_{,y}w_{,yy}+\tfrac{1}{2}(1-\nu)(w_{,xx}w_{,y}$$

$$+w_{,x}w_{,xy}+u_{,xy}+v_{,xx})]+\sum_j\delta(x-j\ell)E_rA_r[v_{,yy}+R^{-1}w_{,y}+w_{,y}w_{,yy}$$

$$-\bar{z}_rw_{,yyy}]\}\delta v dxdy+\int_o^{2\pi R}\tfrac{1}{2}Eh(1+\nu)^{-1}\{w_{,x}w_{,y}+u_{,y}+v_{,x}\}\delta v\Big|_o^L dy$$

$$+\int_o^L\{Eh(1-\nu^2)^{-1}[\nu u_{,x}+\tfrac{1}{2}\nu w_{,x}^2+v_{,y}+R^{-1}w+\tfrac{1}{2}w_{,y}^2]+\sum_j\delta(x-j\ell)E_rA_r$$

$$\cdot[v_{,y}+R^{-1}w+\tfrac{1}{2}w_{,y}^2-\bar{z}_rw_{,yy}]\}\delta v\Big|_o^{2\pi R}dx+\int_o^{2\pi R}\int_o^L\{D\nabla^4w+\rho hw_{,tt}$$

$$+\sum_j\delta(x-j\ell)\rho_rA_rw_{,tt}+\sum_k\delta(y-kd)\rho_sA_sw_{,tt}\}\delta w dxdy$$

$$-\int_o^{2\pi R}\int_o^L\{Eh(1-\nu^2)^{-1}[u_{,xx}w_{,x}+u_{,x}w_{,xx}+1.5w_{,x}^2w_{,xx}+R^{-1}\nu w_{,x}^2+R^{-1}\nu ww_{,xx}$$

$$+2w_{,x}w_{,y}w_{,xy}+\tfrac{1}{2}w_{,y}^{2}w_{,xx}+\nu w_{,xx}v_{,y}+\nu w_{,x}v_{,xy}+\tfrac{1}{2}\nu u_{,xy}w_{,y}+\tfrac{1}{2}u_{,xy}w_{,y}$$

$$+\nu u_{,x}w_{,yy}+\tfrac{1}{2}\nu w_{,x}^{2}w_{,yy}+v_{,yy}w_{,y}+v_{,y}w_{,yy}+R^{-1}w_{,y}^{2}+R^{-1}ww_{,yy}+1.5w_{,y}^{2}w_{,yy}$$

$$+(1-\nu)(u_{,y}w_{,xy}+\tfrac{1}{2}v_{,xx}w_{,y}+v_{,x}w_{,xy}+\tfrac{1}{2}u_{,yy}w_{,x}+\tfrac{1}{2}v_{,xy}w_{,x}$$

$$+\tfrac{1}{2}w_{,yy}w_{x}^{2})-R^{-1}(v_{,y}+R^{-1}w+\tfrac{1}{2}w_{,y}^{2}+\nu u_{,x}+\tfrac{1}{2}\nu w_{,x}^{2})]+\sum_{k}\delta(y-kd)E_{s}A_{s}[w_{,xx}u_{,x}$$

$$+w_{,x}u_{,xx}+1.5w_{,x}^{2}w_{,xx}-\bar{z}_{s}w_{,xx}^{2}-\bar{z}_{s}w_{,x}w_{,xx}+\bar{z}_{s}u_{,xxx}-\bar{z}_{s}w_{,xx}^{2}-\bar{z}_{s}w_{,x}w_{,xxx}$$

$$-(I_{os}/A_{s})w_{,xxxx}-(G_{s}J_{s}/E_{s}A_{s})w_{,xxyy}]+\sum_{j}\delta(x-j\ell)E_{r}A_{r}[w_{,yy}v_{,y}$$

$$+w_{,y}v_{,yy}+R^{-1}w_{,y}^{2}+R^{-1}ww_{,yy}+1.5w_{,y}^{2}w_{,yy}-\bar{z}_{r}w_{,yy}^{2}-\bar{z}_{r}w_{,y}w_{,yyy}-R^{-1}v_{,y}$$

$$-\tfrac{1}{2}R^{-1}w_{,y}^{2}-w(1/R)^{2}+\bar{z}_{r}R^{-1}w_{,yy}+\bar{z}_{r}v_{,yyy}+\bar{z}_{r}R^{-1}w_{,yy}-(I_{or}/A_{r})w_{,yyyy}$$

$$-(G_{r}J_{r}/E_{r}A_{r})w_{,xxyy}+\bar{z}_{r}w_{,yy}^{2}+\bar{z}_{r}w_{,y}w_{,yyy}]+\sum_{k}\delta_{,y}(y-kd)G_{s}J_{s}w_{,xxy}$$

$$+\sum_{j}\delta_{,x}(x-j\ell)G_{r}J_{r}w_{,xyy}\}\delta w\,dx\,dy+\int_{o}^{L}\{D[-w_{,yyy}-\nu w_{,xxy}]+Eh(1-\nu^{2})^{-1}$$

$$\cdot[\nu u_{,x}w_{,y}+\tfrac{1}{2}\nu w_{,x}^{2}w_{,y}+v_{,y}w_{,y}+R^{-1}ww_{,y}+\tfrac{1}{2}w_{,y}^{3}+\tfrac{1}{2}(1-\nu)(u_{,y}w_{,x}+v_{,x}w_{,x}$$

$$+w_{,y}w_{,x}^{2})]+\sum_{j}\delta(x-j\ell)E_{r}A_{r}[w_{,y}v_{,y}+R^{-1}ww_{,y}+\tfrac{1}{2}w_{,y}^{3}-\bar{z}_{r}w_{,yy}w_{,y}-(I_{or}/A_{r})$$

$$\cdot w_{,yyy}+\bar{z}_{r}v_{,yy}+\bar{z}_{r}R^{-1}w_{,y}+\bar{z}_{r}w_{,y}w_{,yy}]\}\delta w\Big|_{o}^{2\pi R}dx+\int_{o}^{2\pi R}\{D[-w_{,xxx}-2w_{,xyy}$$

$$+\nu w_{,xyy}]+Eh(1-\nu^{2})^{-1}[u_{,x}w_{,x}+\tfrac{1}{2}w_{,x}^{3}+\nu R^{-1}ww_{,x}+\tfrac{1}{2}w_{,y}^{2}w_{,x}+\nu w_{,x}v_{,y}$$

$$+\tfrac{1}{2}(1-\nu)(u_{,y}w_{,y}+v_{,x}w_{,y}+w_{,x}w_{,y}^{2})]-\sum_{k}\delta_{,y}(y-kd)G_{s}J_{s}w_{,xy}-\sum_{j}\delta(x-j\ell)$$

$$\cdot G_{r}J_{r}w_{,xyy}+\sum_{k}\delta(y-kd)E_{s}A_{s}[w_{,x}u_{,x}+\tfrac{1}{2}w_{,x}^{3}+\bar{z}_{s}w_{,xx}w_{,x}+\bar{z}_{s}u_{,xx}-(I_{os}/A_{s})$$

$$\cdot w,_{xxx}-(G_sJ_s/E_sA_s)w,_{xyy}]\}\delta w\Big|_o^L dy+\int_o^L\{2D(1-\nu)w,_{xy}+\sum_j\delta(x-j\ell)G_rJ_r$$

$$\cdot w,_{xy}+\sum_k\delta(y-kd)G_sJ_sw,_{xy}\}\delta w,_x\Big|_o^{2\pi R}dx+\int_o^{2\pi R}\{D[w,_{xx}+\nu w,_{yy}]$$

$$-\sum_k\delta(y-kd)E_sA_s[\bar{z}_su,_x+\bar{z}_sw,_x^2-(I_{os}/A_s)w,_{xx}]-\hat{N}_x\bar{e}\}\delta w,_x\Big|_o^L dy$$

$$+\int_o^L\{D[w,_{yy}+\nu w,_{xx}]-\sum_j\delta(x-j\ell)E_rA_r[\bar{z}_rv,_y+\bar{z}_rR^{-1}w+\tfrac{1}{2}\bar{z}_rw,_y^2-(I_{or}/A_r)$$

$$\cdot w,_{yy}]\}\delta w,_y\Big|_o^{2\pi R}dx\Big)dt = 0 \qquad \text{(B-3)}$$

Since the variation of the function with $\theta$ is periodic, all of the terms in Equation (B-3) that are evaluated between zero and $2\pi R$ reduce to zero. The three equations of motion (u,v,w) and all relevant boundary conditions can now be extracted from Equation (B-3).

To simplify the governing equations, we make use of the notation advanced by Kraus [50], and define the following terms

$$N_x = Eh(1-\nu^2)^{-1}[u,_x+\tfrac{1}{2}w,_x^2+\nu(v,_y+R^{-1}w+\tfrac{1}{2}w,_y^2)]+\sum_k\delta(y-kd)E_sA_s(u,_x+\tfrac{1}{2}w,_x^2$$
$$-\bar{z}_sw,_{xx}) \qquad \text{(B-4)}$$

$$N_y = Eh(1-\nu^2)^{-1}[v,_y+R^{-1}w+\tfrac{1}{2}w,_y^2+\nu(u,_x+\tfrac{1}{2}w,_x^2)]+\sum_j\delta(x-j\ell)E_rA_r$$
$$(v,_y+R^{-1}w+\tfrac{1}{2}w,_y^2-\bar{z}_rw,_{yy}) \qquad \text{(B-5)}$$

$$N_{xy} = \tfrac{1}{2}Eh(1+\nu)^{-1}(u,_y+v,_x+w,_xw,_y) \qquad \text{(B-6)}$$

$$M_x = -D(w,_{xx}+\nu w,_{yy})-\sum_k\delta(y-kd)E_sA_s[(I_{os}/A_s)w,_{xx}-\bar{z}_su,_x-\tfrac{1}{2}\bar{z}_sw,_x^2] \qquad \text{(B-7)}$$

$$M_y = -D(w,_{yy}+\nu w,_{xx})-\sum_j\delta(x-j\ell)E_rA_r[(I_{or}/A_r)w,_{yy}-\bar{z}_rv,_y-\bar{z}_rR^{-1}w$$
$$-\tfrac{1}{2}\bar{z}_rw,_y^2] \qquad \text{(B-8)}$$

$$M_{xy} = -D(1-\nu)w,_{xy}-\tfrac{1}{2}\sum_k\delta(y-kd)G_sJ_sw,_{xy}-\tfrac{1}{2}\sum_j\delta(x-j\ell)G_rJ_rw,_{xy} \qquad \text{(B-9)}$$

The three equations of motion (u,v,w) can now be extracted from Equation (B-3) and written as

$$N_{x,x} + N_{xy,y} = 0 \tag{B-10a}$$

$$N_{y,y} + N_{xy,x} = 0 \tag{B-10b}$$

$$-M_{x,xx} - 2M_{xy,xy} - M_{y,yy} + R^{-1}N_y - N_x w_{,xx} - N_y w_{,yy} - 2N_{xy} w_{,xy}$$

$$+\rho h w_{,tt} + \sum_j \delta(x-j\ell)\rho_r A_r w_{,tt} + \sum_k \delta(y-kd)\rho_s A_s w_{,tt} = 0 \tag{B-10c}$$

In a like manner, the boundary conditions which apply at the ends of the cylinder can be extracted from Equation (B-3) and written as

$$N_x + \hat{N}_x = 0 \qquad \text{or} \quad u = 0 \tag{B-11a}$$

$$N_{xy} = 0 \qquad \text{or} \quad v = 0 \tag{B-11b}$$

$$M_x + \hat{N}_x \bar{e} = 0 \qquad \text{or} \quad w_{,x} = 0 \tag{B-11c}$$

$$M_{x,x} + 2M_{xy,y} + N_x w_{,x} + N_{xy} w_{,y} = 0 \qquad \text{or} \quad w = 0 \tag{B-11d}$$

## APPENDIX C

## APPLICATION OF GALERKIN'S METHOD TO OBTAIN A SET OF ORDINARY NONLINEAR GOVERNING DIFFERENTIAL EQUATIONS

To correctly apply Galerkin's method, one must have the error functions $Q_x$, $Q_y$ and $Q_z$ orthogonal to the assumed mode solutions. The error functions are formed by substituting the assumed displacement functions (2-18) into the buckling equations of motion (2-16). The orthogonalization equations are Equations (2-19), (2-20) and (2-21).

Obviously, a number of terms in each orthogonalization equation will integrate to zero. For ease in identifying and keeping track of the various terms, a table is constructed for each of the orthogonalization equations. In the following development, the B subscript is dropped as understood.

Table C.1 contains the final integrated results of Equation (2-19), which is repeated below for easy reference

$$\int_0^{2\pi R}\int_0^{L} Q_x(u,v,w)\sin(\zeta m\pi x/L)\cos(\xi ny/R)\,dxdy = 0 \qquad \begin{matrix}\zeta = 1,2,3\cdots \\ \xi = 1,2,3\cdots\end{matrix} \qquad (2\text{-}19)$$

Thus, after integration, Equation (2-19) becomes:

$$\delta_{i\zeta}\Sigma\Sigma u_{ij}[I_i^2\delta_{j\xi}+\tfrac{1}{2}(1-\nu)J_j^2\delta_{j\xi}+(1-\nu^2)(\pi REh)^{-1}I_i^2\sum_k E_s A_s \cos J_k \cos J_\xi]$$

Table C.1. Integrated Terms from Axial Equation of Motion.

| TERM | FINAL INTEGRATED RESULT |
|---|---|
| $u_{,xx}$ | $-\frac{1}{2}\Delta_1 Eh\pi RL(1-\nu^2)^{-1}\Sigma\Sigma u_{ij}I_i^2$ |
| $w_{,x}w_{,xx}$ | $\Delta_5(Eh\pi RL/8)(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}^2 I_i^3$ |
| $\nu v_{,xy}$ | $-\frac{1}{2}\Delta_1 Eh\pi RL\nu(1-\nu^2)^{-1}\Sigma\Sigma v_{ij}I_iJ_j$ |
| $\nu R^{-1}w_{,x}$ | $-\frac{1}{2}\Delta_1 Eh\pi L\nu(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}I_i$ |
| $\nu w_{,y}w_{,xy}$ | $\Delta_5(Eh\pi RL\nu/8)(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}^2 I_iJ_j^2$ |
| $u_{,yy}$ | $-\frac{1}{4}\Delta_1 Eh\pi RL(1+\nu)^{-1}\Sigma\Sigma u_{ij}J_j^2$ |
| $v_{,xy}$ | $-\frac{1}{4}\Delta_1 Eh\pi RL(1+\nu)^{-1}\Sigma\Sigma v_{ij}I_iJ_j$ |
| $w_{,xy}w_{,y}$ | $\Delta_5(Eh\pi RL/16)(1+\nu)^{-1}\Sigma\Sigma w_{ij}^2 I_iJ_j^2$ |
| $w_{,x}w_{,yy}$ | $\Delta_5(Eh\pi RL/16)(1+\nu)^{-1}\Sigma\Sigma w_{ij}^2 I_iJ_j^2$ |
| $u_{,xx}$ | $-\frac{1}{2}\delta_{i\zeta}L\Sigma\Sigma u_{ij}I_i^2\sum_k E_sA_s\cos J_k\cos J_\xi$ |
| $w_{,x}w_{,xx}$ | $\frac{1}{4}\delta_{2i\zeta}L\Sigma\Sigma w_{ij}^2 I_i^3\sum_k E_sA_s\cos^2 J_k\cos J_\xi$ |
| $-\bar{z}_s w_{,xxx}$ | $-\frac{1}{2}\delta_{i\zeta}L\Sigma\Sigma w_{ij}I_i^3\sum_k E_sA_s\bar{z}_s\cos J_k\cos J_\xi$ |

$$+\Delta_1\Sigma\Sigma v_{ij}\tfrac{1}{2}(1+\nu)I_iJ_j+\delta_{i\zeta}\Sigma\Sigma w_{ij}[R^{-1}\nu I_i\delta_{j\xi}+(1-\nu^2)(\pi REh)^{-1}I_i^3\sum_k E_sA_s\bar{z}_s$$

$$\cdot\cos J_k\cos J_\xi]-\delta_{2i\zeta}\Sigma\Sigma w_{ij}^2[\tfrac{1}{4}I_iJ_j^2\delta_{2j\xi}+\tfrac{1}{4}I_i^3\delta_{2j\xi}$$

$$+(1-\nu^2)(2E\pi R)^{-1}I_i^3\sum_k E_sA_s\cos^2J_k\cos J_\xi] = 0 \qquad \text{(C-1)}$$

In a similar manner, Equation (2-20) may be evaluated. (The details of integration are contained in Table C.2.)

$$\int_0^{2\pi R}\int_0^L Q_y(u,v,w)\cos(\zeta m\pi x/L)\sin(\xi ny/R)\,dxdy = 0 \qquad \begin{matrix}\zeta = 1,2,3\cdots\\ \xi = 1,2,3\cdots\end{matrix} \qquad \text{(2-20)}$$

After integration, Equation (2-20) becomes

$$\Delta_1\Sigma\Sigma u_{ij}\tfrac{1}{2}(1+\nu)I_iJ_j+\delta_{j\xi}\Sigma\Sigma v_{ij}[J_j^2\delta_{i\zeta}+\tfrac{1}{2}(1-\nu)I_i^2\delta_{i\zeta}+2(1-\nu^2)(EhL)^{-1}$$

$$\cdot J_j^2\sum_j E_rA_r\cos I_j\cos I_\zeta]+\delta_{j\xi}\Sigma\Sigma w_{ij}[R^{-1}J_j\delta_{i\zeta}+2(1-\nu^2)(EhL)^{-1}\sum_j(R^{-1}J_jE_rA_r$$

$$+J_j^3E_rA_r\bar{z}_r)\cos I_j\cos I_\zeta]-\delta_{2j\xi}\Sigma\Sigma w_{ij}^2[\tfrac{1}{4}J_j^3\delta_{2i\zeta}+\tfrac{1}{4}I_i^2J_j\delta_{2i\zeta}+(1-\nu^2)(EhL)^{-1}$$

$$\cdot J_j^3\sum_j E_rA_r\cos^2I_j\cos I_\zeta] = 0 \qquad \text{(C-2)}$$

Finally, Equation (2-21) may be evaluated. Details of this integration are contained in Tables C.3, C.4 and C.5.

$$\int_0^{2\pi R}\int_0^L Q_z(u,v,w)\cos(\zeta m\pi x/L)\cos(\xi ny/R)\,dxdy = 0 \qquad \begin{matrix}\zeta = 1,2,3\cdots\\ \xi = 1,2,3\cdots\end{matrix} \qquad \text{(2-21)}$$

The last entries in Tables C.4 and C.5 are expanded here for clarity. They hinge on the definition of the derivative of a Dirac delta function found in reference [62].

$$\int_{-\infty}^{\infty} x(t)\delta^{(k)}(t-\psi)\,dt = (-1)^k x^{(k)}(\psi)$$

Table C.2. Integrated Terms from Circumferential Equation of Motion

| TERM | FINAL INTEGRATED RESULT |
|---|---|
| $v_{,yy}$ | $-\frac{1}{2}\Delta_1 Eh\pi RL(1-\nu^2)^{-1}\Sigma\Sigma v_{ij}J_j^2$ |
| $R^{-1}w_{,y}$ | $-\frac{1}{2}\Delta_1 Eh\pi L(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}J_j$ |
| $w_{,y}w_{,yy}$ | $\Delta_5(Eh\pi RL/8)(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}^2J_j^3$ |
| $\nu u_{,xy}$ | $-\frac{1}{2}\Delta_1 Eh\pi RL\nu(1-\nu^2)^{-1}\Sigma\Sigma u_{ij}I_iJ_j$ |
| $\nu w_{,x}w_{,xy}$ | $\Delta_5(Eh\pi RL\nu/8)(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}^2I_iJ_j$ |
| $u_{,xy}$ | $-\frac{1}{4}\Delta_1 Eh\pi RL(1+\nu)^{-1}\Sigma\Sigma u_{ij}I_iJ_j$ |
| $v_{,xx}$ | $-\frac{1}{4}\Delta_1 Eh\pi RL(1+\nu)^{-1}\Sigma\Sigma v_{ij}I_i^2$ |
| $w_{,xx}w_{,y}$ | $\Delta_5(Eh\pi RL/16)(1+\nu)^{-1}\Sigma\Sigma w_{ij}^2I_i^2J_j$ |
| $w_{,x}w_{,xy}$ | $\Delta_5(Eh\pi RL/16)(1+\nu)^{-1}\Sigma\Sigma w_{ij}^2I_i^2J_j$ |
| $v_{,yy}$ | $-\delta_{j\xi}\pi R\Sigma\Sigma v_{ij}J_j^2\sum_j E_rA_r\cos I_j\cos I_\zeta$ |
| $R^{-1}w_{,y}$ | $-\delta_{j\xi}\pi\Sigma\Sigma w_{ij}J_j\sum_j E_rA_r\cos I_j\cos I_\zeta$ |
| $w_{,y}w_{,yy}$ | $\frac{1}{2}\delta_{2j\xi}\pi R\Sigma\Sigma w_{ij}^2J_j^3\sum_j E_rA_r\cos^2 I_j\cos I_\zeta$ |
| $-\bar{z}_r w_{,yyy}$ | $-\delta_{j\xi}\pi R\Sigma\Sigma w_{ij}J_j^3\sum_j E_rA_r\bar{z}_r\cos I_j\cos I_\zeta$ |

Table C.3 Unstiffened Cylinder Integrated Terms from Radial Equation of Motion

| TERM | FINAL INTEGRATED RESULT |
|---|---|
| $-\frac{1}{2}w,_x^2 w,_{xx}$ | $Eh(1-\nu^2)^{-1}(\pi RL/64)(3\Delta_1+\Delta_2-3\Delta_3-\Delta_4)\Sigma\Sigma w_{ij}^3 I_i^4$ |
| $-\frac{1}{2}\nu w,_y^2 w,_{xx}$ | $Eh(1-\nu^2)^{-1}(\nu\pi RL/64)(3\Delta_1-3\Delta_2+\Delta_3-\Delta_4)\Sigma\Sigma w_{ij}^3 I_i^2 J_j^2$ |
| $-\frac{1}{2}w,_y^2 w,_{yy}$ | $Eh(1-\nu^2)^{-1}(\pi RL/64)(3\Delta_1-3\Delta_2+\Delta_3-\Delta_4)\Sigma\Sigma w_{ij}^3 J_j^4$ |
| $-\frac{1}{2}\nu w,_x^2 w,_{yy}$ | $Eh(1-\nu^2)^{-1}(\nu\pi RL/64)(3\Delta_1+\Delta_2-3\Delta_3-\Delta_4)\Sigma\Sigma w_{ij}^3 I_i^2 J_j^2$ |
| $-w,_x w,_y w,_{xy}$ | $Eh(1-\nu^2)^{-1}(\pi RL/32)(1-\nu)(-\Delta_1+\Delta_2+\Delta_3-\Delta_4)\Sigma\Sigma w_{ij}^3 I_i^2 J_j^2$ |
| $w,_{xxxx}$ | $\frac{1}{2}\Delta_1 D\pi RL\Sigma\Sigma w_{ij} I_i^4$ |
| $2w,_{xxyy}$ | $\Delta_1 D\pi RL\Sigma\Sigma w_{ij} I_i^2 J_j^2$ |
| $w,_{yyyy}$ | $\frac{1}{2}\Delta_1 D\pi RL\Sigma\Sigma w_{ij} J_j^2$ |
| $-u,_x w,_{xx}$ | $\Delta_5 Eh\pi RL(1-\nu^2)^{-1}\frac{1}{8}\Sigma\Sigma u_{ij} w_{ij} I_i^3$ |
| $-\nu v,_y w,_{xx}$ | $\Delta_5 Eh\nu\pi RL(1-\nu^2)^{-1}\frac{1}{8}\Sigma\Sigma v_{ij} w_{ij} I_i^2 J_j$ |
| $-\nu R^{-1} ww,_{xx}$ | $\Delta_5 Eh\nu\pi L(1-\nu^2)^{-1}\frac{1}{8}\Sigma\Sigma w_{ij}^2 I_i^2$ |
| $-v,_y w,_{yy}$ | $\Delta_5 Eh\pi RL(1-\nu^2)^{-1}\frac{1}{8}\Sigma\Sigma v_{ij} w_{ij} J_j^3$ |
| $-wR^{-1} w,_{yy}$ | $\Delta_5 Eh\pi L(1-\nu^2)^{-1}\frac{1}{8}\Sigma\Sigma w_{ij}^2 J_j^2$ |
| $-\nu u,_x w,_{yy}$ | $\Delta_5 Eh\nu\pi RL(1-\nu^2)^{-1}\frac{1}{8}\Sigma\Sigma u_{ij} w_{ij} I_i J_j^2$ |
| $-u,_y w,_{xy}$ | $\Delta_5 Eh\pi RL(1+\nu)^{-1}\frac{1}{8}\Sigma\Sigma u_{ij} w_{ij} I_i J_j^2$ |

Table C.3 Continued

Table C.3 (Cont'd.)

| TERM | FINAL INTEGRATED RESULT |
|---|---|
| $-v_{,x}w_{,xy}$ | $\Delta_5 Eh\pi RL(1+\nu)^{-1}\ \frac{1}{8}\Sigma\Sigma v_{ij}w_{ij}I_i^2J_j$ |
| $v_{,y}$ | $\frac{1}{2}\Delta_1 Eh\pi L(1-\nu^2)^{-1}\Sigma\Sigma v_{ij}J_j$ |
| $R^{-1}w$ | $\frac{1}{2}\Delta_1 Eh\pi LR^{-1}(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}$ |
| $\frac{1}{2}w_{,y}^2$ | $-\Delta_5 Eh\pi L(1-\nu^2)^{-1}(1/16)\Sigma\Sigma w_{ij}^2J_j^2$ |
| $\nu u_{,x}$ | $\frac{1}{2}\Delta_1 Eh\pi L\nu(1-\nu^2)^{-1}\Sigma\Sigma u_{ij}I_i$ |
| $\frac{1}{2}\nu w_{,x}^2$ | $-\Delta_5 Eh\nu\pi L(1-\nu^2)^{-1}(1/16)\Sigma\Sigma w_{ij}^2I_i^2$ |
| $\rho hw_{,tt}$ | $\frac{1}{2}\Delta_1\rho h\pi RL\Sigma\Sigma\ddot{w}_{ij}$ |
| $\hat{N}_x w_{,xx}$ | $-\frac{1}{2}\Delta_1\hat{N}_x\pi RL\Sigma\Sigma w_{ij}I_i^2$ |

Table C.4 Ring Integrated Terms from Radial Equation of Motion.

| TERM | FINAL INTEGRATED RESULT |
|---|---|
| $(I_{or}/A_r)w_{,yyyy}$ | $\delta_{j\xi}\pi R\Sigma\Sigma w_{ij}J_j^4\sum_j E_r I_{or}\cos I_j\cos I_\zeta$ |
| $-\bar{z}_r v_{,yyy}$ | $\delta_{j\xi}\pi R\Sigma\Sigma v_{ij}J_j^3\sum_j E_r A_r\bar{z}_r\cos I_j\cos I_\zeta$ |
| $-2\bar{z}_r R^{-1}w_{,yy}$ | $2\delta_{j\xi}\pi\Sigma\Sigma w_{ij}J_j^2\sum_j E_r A_r\bar{z}_r\cos I_j\cos I_\zeta$ |
| $-\bar{z}_r w_{,y}w_{,yyy}$ | $-\frac{1}{2}\delta_{2j\xi}\pi R\Sigma\Sigma w_{ij}^2J_j^4\sum_j E_r A_r\bar{z}_r\cos^2 I_j\cos I_\zeta$ |
| $(G_r J_r/E_r A_r)w_{,xxyy}$ | $\delta_{j\xi}\pi R\Sigma\Sigma w_{ij}J_j^2I_i^2\sum_j G_r J_r\cos I_j\cos I_\zeta$ |
| $-v_{,y}w_{,yy}$ | $\frac{1}{2}\delta_{2j\xi}\pi R\Sigma\Sigma v_{ij}w_{ij}J_j^3\sum_j E_r A_r\cos^2 I_j\cos I_\zeta$ |
| $-wR^{-1}w_{,yy}$ | $\frac{1}{2}\delta_{2j\xi}\pi\Sigma\Sigma w_{ij}^2J_j^2\sum_j E_r A_r\cos^2 I_j\cos I_\zeta$ |
| $-\frac{1}{2}w_{,y}^2w_{,yy}$ | $(\delta_{j\xi}-\delta_{3j\xi})(\pi/8)\Sigma\Sigma w_{ij}^3J_j^4\sum_j E_r A_r\cos^3 I_j\cos I_\zeta$ |
| $v_{,y}$ | $\delta_{j\xi}\pi\Sigma\Sigma v_{ij}J_j\sum_j E_r A_r\cos I_j\cos I_\zeta$ |
| $R^{-1}w$ | $\delta_{j\xi}\pi R^{-1}\Sigma\Sigma w_{ij}\sum_j E_r A_r\cos I_j\cos I_\zeta$ |
| $\frac{1}{2}w_{,y}^2$ | $-\frac{1}{4}\delta_{2j\xi}\pi\Sigma\Sigma w_{ij}^2J_j^2\sum_j E_r A_r\cos^2 I_j\cos I_\zeta$ |
| $\rho_r A_r w_{,tt}$ | $\delta_{j\xi}\rho_r A_r\pi R\Sigma\Sigma\ddot{w}_{ij}\sum_j\cos I_j\cos I_\zeta$ |
| $G_r J_r w_{,xyy}\delta_{,x}(x-j\ell)$ | $\delta_{j\xi}\pi R\Sigma\Sigma w_{ij}J_j^2I_i\sum_j G_r J_r(\zeta m\pi L^{-1}\sin I_j\sin I_\zeta - I_i\cos I_j\cos I_\zeta)$ |

Table C.5 Stringer Integrated Terms from Radial Equation of Motion.

| TERM | FINAL INTEGRATED RESULT |
|---|---|
| $(I_{os}/A_s)w_{,xxxx}$ | $\frac{1}{2}\delta_{i\zeta}L\Sigma\Sigma w_{ij}I_i^4\sum_k E_s I_{os}\cos J_k \cos J_\xi$ |
| $-\bar{z}_s u_{,xxx}$ | $\frac{1}{2}\delta_{i\zeta}L\Sigma\Sigma u_{ij}I_i^3\sum_k E_s A_s \bar{z}_s \cos J_k \cos J_\xi$ |
| $-\bar{z}_s w_{,x} w_{,xxx}$ | $-\frac{1}{4}\delta_{2i\zeta}L\Sigma\Sigma w_{ij}^2 I_i^4\sum_k E_s A_s \bar{z}_s \cos^2 J_k \cos J_\zeta$ |
| $(G_s J_s/E_s A_s)w_{,xxyy}$ | $\frac{1}{2}\delta_{i\zeta}L\Sigma\Sigma w_{ij}I_i^2 J_i^2\sum_k G_s J_s \cos J_k \cos J_\xi$ |
| $-u_{,x} w_{,xx}$ | $\frac{1}{4}\delta_{2i\zeta}L\Sigma\Sigma u_{ij} w_{ij} I_i^3\sum_k E_s A_s \cos^2 J_k \cos J_\xi$ |
| $-\frac{1}{2}w_{,x}^2 w_{,xx}$ | $(\delta_{i\zeta}-\delta_{3i\zeta})L/16\Sigma\Sigma w_{ij}^3 I_i^4\sum_k E_s A_s \cos^3 J_k \cos J_\xi$ |
| $\rho_s A_s w_{,tt}$ | $\frac{1}{2}\delta_{i\zeta}\rho_s A_s L\Sigma\Sigma \ddot{w}_{ij}\sum_k \cos J_k \cos J_\xi$ |
| $G_s J_s w_{,xxy}\delta_{,y}(y-kd)$ | $\frac{1}{2}\delta_{i\zeta}L\Sigma\Sigma w_{ij}I_i^2 J_j\sum_k G_s J_s(\xi n R^{-1}\sin J_k \sin J_\xi - J_j \cos J_k \cos J_\xi)$ |

Applying the above equation to the last term in Table C.5, one obtains the following results:

$$\int_0^{2\pi R}\int_0^{L}[\Sigma\Sigma w_{ij}G_sJ_sI_i^2J_j\sum_k\delta_{,y}(y-kd)\cos(I_ix)\cos(\zeta m\pi x/L)\sin(J_jy)\cos(\xi ny/R)]dxdy$$

$$= \Sigma\Sigma w_{ij}\tfrac{1}{2}G_sJ_sLI_i^2J_j\sum_k[-J_j\cos(J_jy)\cos(\xi ny/R)\Big|_{y=kd}+(\xi n/R)\sin(J_jy)\sin$$

$$\cdot(\xi ny/R)\Big|_{y=kd}]$$

Thus, after integration of Equation (2-21), one obtains the following expression:

$$Eh(1-\nu^2)^{-1}\frac{1}{32}\Sigma\Sigma w_{ij}^3[I_i^4(3\Delta_1+\Delta_2-3\Delta_3-\Delta_4)+2I_i^2J_j^2(-\Delta_1+\Delta_2+\Delta_3-\Delta_4+4\nu\Delta_1-2\nu\Delta_2$$

$$-2\nu\Delta_3)+J_j^4(3\Delta_1-3\Delta_2+\Delta_3-\Delta_4)]+(2/L)\Sigma\Sigma w_{ij}^3(\delta_{j\xi}-\delta_{3j\xi})\frac{1}{8}J_j^4\sum_j E_rA_r\cos^3I_j\cos I_\zeta$$

$$+(1/\pi R)\Sigma\Sigma w_{ij}^3(\delta_{i\zeta}-\delta_{3i\zeta})\frac{1}{8}I_i^4\sum_k E_sA_s\cos^3J_k\cos J_\xi+\Delta_5Eh(1-\nu^2)^{-1}(\nu/8R)$$

$$\cdot\Sigma\Sigma w_{ij}(I_i^2+J_j^2)+\delta_{2j\xi}(2/L)\Sigma\Sigma w_{ij}^2(\tfrac{1}{4}R^{-1}J_j^2-\tfrac{1}{2}J_j^4\bar{z}_r)\sum_j E_rA_r\cos^2I_j\cos I_\zeta-\delta_{2i\zeta}$$

$$\cdot(1/\pi R)\Sigma\Sigma w_{ij}^2\tfrac{1}{2}I_i^4\sum_k E_sA_s\bar{z}_s\cos^2J_k\cos J_\xi+\Delta_1Eh(1-\nu^2)^{-1}\Sigma\Sigma w_{ij}[(h^2/12)I_i^4$$

$$+(h^2/6)I_i^2J_j^2+(h^2/12)J_j^4+(1/R^2)-\hat{N}_x(1-\nu^2)I_i^2(Eh)^{-1}]+\delta_{j\xi}(2/L)\Sigma\Sigma w_{ij}$$

$$\cdot[J_j^4E_rI_{or}+2R^{-1}J_j^2E_rA_r\bar{z}_r+(1/R^2)E_rA_r]\sum_j\cos I_j\cos I_\zeta+\delta_{j\xi}(2/L)\Sigma\Sigma w_{ij}$$

$$\cdot J_j^2I_i\sum_j G_rJ_r\zeta m\pi L^{-1}\sin I_j\sin I_\zeta+\delta_{i\zeta}(1/\pi R)\Sigma\Sigma w_{ij}I_i^4\sum_k E_sI_{os}\cos J_k\cos J_\xi$$

$$+\delta_{i\zeta}(1/\pi R)\Sigma\Sigma w_{ij}I_i^2J_j\sum_k G_sJ_s\xi nR^{-1}\sin J_k\sin J_\xi+\Delta_1Eh(1-\nu^2)^{-1}\nu R^{-1}\Sigma\Sigma u_{ij}I_i$$

$$+\delta_{i\zeta}(1/\pi R)\Sigma\Sigma u_{ij}I_i^3\sum_k E_sA_s\bar{z}_s\cos J_k\cos J_\xi+\Delta_1Eh(1-\nu^2)^{-1}R^{-1}\Sigma\Sigma v_{ij}J_j$$

$$+\delta_{j\xi}(2/L)\Sigma\Sigma v_{ij}(J_j^3\bar{z}_R+J_jR^{-1})\sum_j E_rA_r\cos I_j\cos I_\zeta+\Delta_5\tfrac{1}{4}Eh(1-\nu^2)^{-1}\Sigma\Sigma u_{ij}w_{ij}$$

$$\cdot(I_i^3+I_iJ_j^2)+\delta_{2i\zeta}(1/\pi R)\Sigma\Sigma u_{ij}w_{ij}\tfrac{1}{2}I_i^3\sum_k E_sA_s\cos^2J_k\cos J_\xi+\Delta_5\tfrac{1}{4}Eh(1-\nu^2)^{-1}$$

$$\cdot\Sigma\Sigma v_{ij}w_{ij}(J_j^3+I_i^2J_j)+\delta_{2j\xi}L^{-1}\Sigma\Sigma v_{ij}w_{ij}J_j^3\sum_j E_rA_r\cos^2I_j\cos I_\zeta+\Sigma\Sigma\ddot{w}_{ij}$$

$$\cdot[\Delta_1\rho h+\delta_{j\xi}\rho_rA_r(2/L)\sum_j\cos I_j\cos I_\zeta+\delta_{i\zeta}\rho_sA_s(1/\pi R)\sum_k\cos J_k\cos J_\xi]=0 \qquad \text{(C-3)}$$

## APPENDIX D

## COMPUTER PROGRAM DOCUMENTATION

The program was written in FORTRAN IV, and was run using an IBM System 360, Model 50 computer. The core storage required for the program was dependent upon the number of terms in the assumed mode function. The maximum number of terms run was ten, which required E-level system operation.

The flow of the program can be observed in Fig. D.1 and is summarized as follows: The main program reads the input data which defines the various material and geometric parameters, the number of assumed modal functions, the range of the circumferential wave number, the range of the applied axial load, the Runge-Kutta step size, and the maximum time duration. The main program then calls Subroutines UVCOF and WCOEF in succession. These general subroutines calculate the coefficients for the required number of u, v, and w equations of motion. (The number of equations is dependent upon the number of assummed modal functions, k. The two subroutines will generate coefficients for any number of equations. This number is limited only by the available computer core space. Subroutine UVCOF generates the coefficients for k algebraic u equations and k algebraic v equations. Subroutine WCOEF generates the coefficients for k nonlinear differential w equations.) The main program then calculates the terms in the w

equation of motion containing the nondimensional load as a parameter. Subroutine RUNGE is called to solve the set of nonlinear differential equations. Subroutine RUNGE also uses subroutine FCT to solve the two sets of algebraic equations. For each step in the Runge-Kutta procedure, subroutine FCT must be called four times to solve the two sets of algebraic equations. The results are displayed in the form of shell displacement as a function of time for a particular axial end load and circumferential wave number. The output can be punched on cards for use in plotting the data.

The input data deck is set up as follows:

1. Title card identifying case being run.
2. Number of axial and circumferential assumed mode terms being used.
3. Maximum and minimum values of n being run.
4. Unstiffened shell parameters.
5. Ring parameters.
6. Stringer parameters.
7. Dynamic parameters.
8. Dynamic parameters (Cont.)
9. Maximum value of time and Runge-Kutta step size.
10. Maximum and minimum values of axial load, and load step size.

The program is set up so that cases may be stacked, and all parameters may be varied.

The formats for the above cards are:

1. (80A1)

   All eighty columns are available to assign a descriptive title to the problem being run.

2. (2I4) ISTAR, JSTAR

   ISTAR = maximum number of assumed mode terms in the axial direction, i.

   JSTAR = maximum number of assumed mode terms in the circumferential direction, j.

3. (2I4) NCRG, NCRS

   NCRG = 1 + minimum value of the circumferential wave number, n.

   NCRS = 1 + maximum value of n.

4. (5E15.8) ESHEL, HSHEL, PNU, RSHEL, SHLT

   ESHEL = Young's modulus of shell, E. (psi)

   HSHEL = Thickness of shell, h. (in.)

   PNU = Poisson's ratio, $\nu$. (dimensionless)

   RSHEL = Radius of Shell, R (in.)

   SHLT = Length of shell, L. (in.)

5. (I15,3E15.8) NRNG, ARING, ERING, ZRING

   NRNG = Number of Rings, N.

   ARING = Cross-sectional area of one ring, $A_r$. ($in.^2$)

   ERING = Young's modulus of rings, $E_r$. (psi)

   ZRING = Distance from ring centroid to shell mid-surface, $\bar{z}_r$ (in.)

6. (I15,3E15.8) MSTR, ASTR, ESTR, ZSTR

   MSTR = Number of Stringers, M.

   ASTR = Cross-sectional area of one stringer, $A_s$. ($in.^2$)

   ESTR = Young's modulus of stringers, $E_s$. (psi)

   ZSTR = Distance from stringer centroid to shell mid-surface, $\bar{z}_s$. (in.)

7. (5E15.8) SHDEN, RGDEN, STDEN, RGMI, STMI

   SHDEN = Unstiffened cylinder density, $\rho$. ($lb\text{-}sec^2/in.^4$)

   RGDEN = Ring density, $\rho_r$. ($lb\text{-}sec^2/in.^4$)

   STDEN = Stringer density, $\rho_s$. ($lb\text{-}sec^2/in.^4$)

   RGMI = Moment of inertia of one ring about shell mid-surface, $I_{or}$. ($in.^4$)

   STMI = Moment of inertia of the stringer about shell mid-surface, $I_{os}$. ($in.^4$)

8. (4E15.8) GRING, GSTR, RGJ, STJ

   GRING = Modulus of rigidity of ring, $G_r$. (psi)

   GSTR = Modulus of rigidity of stringer, $G_s$. (psi)

   RGJ = Polar moment of inertia of ring, $J_r$. ($in.^4$)

   STJ = Polar moment of inertia of stringer, $J_s$. ($in.^4$)

9. (3E15.8) X,XMAX, H

   X = Starting value of independent variable, t. (sec.)

   XMAX = Maximum value of independent variable, t. (sec.)

   H = Runge-Kutta increment size.

10. (3I10) ILOAD, ISTOP, INT

    ILOAD = Smallest value of load, $\bar{F}_{min}$ (dimensionless)

    ISTOP = Largest value of load, $\bar{F}_{max}$ (dimensionless)

    INT = Increment for increasing load value.

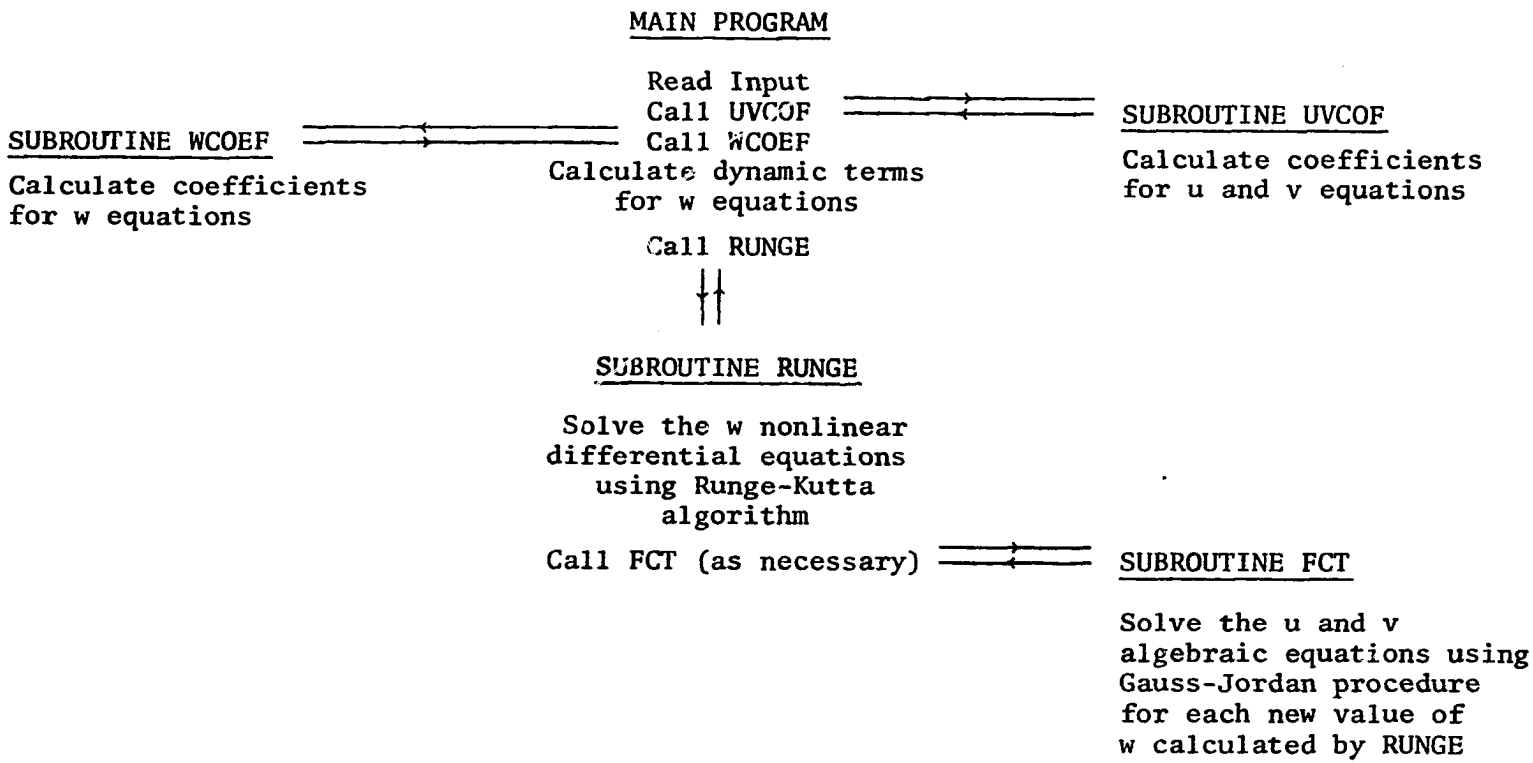


Figure D.1 Computer Program Flow Chart

# APPENDIX E

# COMPUTER PROGRAM LISTING

```
C       MAIN PROGRAM                                                    MAIN  1
        EXTERNAL FCT                                                    MAIN  2
        INTEGER P,Q                                                     MAIN  3
C                                                                       MAIN  4
        DIMENSION Y(10),DERY(10),B(5,40),D(5,40)                        MAIN  5
        DIMENSION PRMT(3),ICASE(80)                                     MAIN  6
C                                                                       MAIN  7
C       THE DIMENSION SIZE FOR WCOF(II,NN) MUST BE EQUAL TO             MAIN  8
C       OR LARGER THAN ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR    MAIN  9
C       LARGER THAN 8*ISIZE FOR THE SECOND SUBSCRIPT.                   MAIN 10
C       THE DIMENSION SIZE FOR C(II,NN) MUST BE EQUAL TO                MAIN 11
C       OR LARGER THAN 2*ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO     MAIN 12
C       OR LARGER THAN 4*ISIZE FOR THE SECOND SUBSCRIPT.                MAIN 13
        COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,1SHEL,PNU,MAIN 14
       1NRNG,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,MAIN 15
       2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ,MAIN 16
       3PLOAD,RSHEL,SHLT,IFLAG,NDOT                                    MAIN 17
C                                                                       MAIN 18
C       ISTAR=MAX. VALUE OF I . JSTAR=MAX. VALUE OF J                   MAIN 19
C       ARING=RING CROSS-SECT. AREA, ASTR=STRINGER CROSS-SECT. AREA     MAIN 20
C       ERING = YOUNGS MODULUS OF RING, ESTR=YOUNGS MODULUS OF STRINGER MAIN 21
C       DRING=DISTANCE BETWEEN RINGS, DSTR=DISTANCE BETWEEN STRINGERS   MAIN 22
C       ZRING=DISTANCE FROM RING CONTROID TO SHELL MID SURFACE          MAIN 23
C       ZSTR=DISTANCE FROM STRINGER CENTROID TO SHELL MID SURFACE       MAIN 24
C       NRNG=NUMBER OF RINGS, MSTR=NUMBER OF STRINGERS                  MAIN 25
C       ESHEL=YOUNGS MODULUS OF SHELL                                   MAIN 26
C       HSHEL=SHELL THICKNESS, RSHEL=SHELL RADIUS                       MAIN 27
C       PNU=POISSONS RATIO                                              MAIN 28
C       NCIR=NUMBER OF CIRCUMFERENTIAL WAVES (INTEGER)                  MAIN 29
C       MAX=NUMBER OF AXIAL HALF WAVES(INTEGER)                         MAIN 30
C       SHLT=SHELL LENGTH                                               MAIN 31
C       N1=NCIR/RSHEL (COMPUTED)...FN1                                  MAIN 32
C       N2=N1*N1 (COMPUTED)...FN2                                       MAIN 33
C       M1=MAX*3.14/SHLT (COMPUTED)...FM1                               MAIN 34
C       M2=M1*M1(COMPUTED)...FM2                                        MAIN 35
```

```
C      SHDEN=MASS DENSITY OF SHELL                                          MAIN 36
C      RGDEN=MASS DENSITY OF RING, STDEN=MASS DENSITY OF STRINGER           MAIN 37
C      RGMI=MOMENT OF INERTIA OF RING, STMI=MOMENT OF INERTIA OF STRINGERMAIN 38
C      GSTR=MODULUS OF RIGIDITY OF STRINGER                                 MAIN 39
C      GRING=MODULUS OF RIGIDITY OF RING                                    MAIN 40
C      RGJ=RING POLAR MOMENT OF INERTIA, STJ=STRINGER POLAR MOMENT          MAIN 41
C      PLOAD=AXIAL COMPRESSIVE FORCE                                        MAIN 42
C                                                                           MAIN 43
C      READ AND WRITE CASE IDENTIFICATION PARAMETERS                        MAIN 44
C                                                                           MAIN 45
 900   FORMAT(1H1,80A1)                                                     MAIN 46
 999   FORMAT(' ')                                                          MAIN 47
 1000  FORMAT(80A1)                                                         MAIN 48
 1001  FORMAT(2I4)                                                          MAIN 49
 1002  FORMAT(1H ,8HISTAR = ,I4,5X,8HJSTAR = ,I4/)                          MAIN 50
 1003  FORMAT(5E15.8)                                                       MAIN 51
 1005  FORMAT(I15,3E15.8)                                                   MAIN 52
 1007  FORMAT(5E15.8)                                                       MAIN 53
 1008  FORMAT(3I10)                                                         MAIN 54
 1009  FORMAT(5X,I15,3E15.8)                                                MAIN 55
 1012  FORMAT(5X,5E15.8)                                                    MAIN 56
 1013  FORMAT('  PRMT VALUES ARE...PRMT(1)=X,PRMT(2)=XMAX,PRMT(3)=H ')     MAIN 57
 1018  FORMAT(' SHELL INPUT-ESHEL,HSHEL,PNU,RSHEL,SHLT ')                   MAIN 58
 1019  FORMAT(' RING INPUT-NRNG,ARNG,ERING,ZRING ')                         MAIN 59
 1020  FORMAT(' STRINGER INPUT-MSTR,ASTR,ESTR,ZSTR ')                       MAIN 60
 1021  FORMAT(' DYNAMIC-SHDEN,RGDEN,STDEN,RGMI,STMI ')                      MAIN 61
 1022  FORMAT(' DYNAMIC-GRING,GSTR,RGJ,STJ ')                               MAIN 62
 1035  FORMAT(' RANGE OF BUCKLING LOAD...ILOAD,ISTOP,INT ')                 MAIN 63
 1040  FORMAT(1X,3I10)                                                      MAIN 64
 1045  FORMAT(' BUCKLING DID NOT OCCUR UNTIL AFTER MAX. LOAD WAS REACHED    MAIN 65
     1**********SORRY ')                                                    MAIN 66
 1055  FORMAT(//30H NUMBER OF AXIAL HALF-WAVES = ,I4,5X,39HNUMBER OF CIRCMAIN 67
     1UMFERENTIAL FULL WAVES = ,I4/)                                       MAIN 68
 5     READ(5,1000)ICASE                                                    MAIN 69
       WRITE(6,900)ICASE                                                    MAIN 70
       READ(5,1001)ISTAR,JSTAR                                              MAIN 71
```

```
      IF(ISTAR-100)6,15,15                                              MAIN 72
    6 WRITE(6,1002)ISTAR,JSTAR                                          MAIN 73
C                                                                       MAIN73A
C     READ IN NCRG,NCRS                                                 MAIN73B
C                                                                       MAIN73C
      READ(5,1001)NCRG,NCRS                                             MAIN73D
C                                                                       MAIN 74
C     READ AND WRITE SHELL INPUT PARAMETERS                             MAIN 75
C                                                                       MAIN 76
      READ(5,1003)ESHEL,HSHEL,PNU,RSHEL,SHLT                            MAIN 77
      WRITE(6,1018)                                                     MAIN 78
      WRITE(6,1007)ESHEL,HSHEL,PNU,RSHEL,SHLT                           MAIN 79
C                                                                       MAIN 80
C     READ AND WRITE RING INPUT PARAMETERS                              MAIN 81
C                                                                       MAIN 82
      READ(5,1005)NRNG,ARING,ERING,ZRING                                MAIN 83
      WRITE(6,1019)                                                     MAIN 84
      WRITE(6,1009)NRNG,ARING,ERING,ZRING                               MAIN 85
C                                                                       MAIN 86
C     READ AND WRITE STRINGER INPUT PARAMETERS                          MAIN 87
C                                                                       MAIN 88
      READ(5,1005)MSTR,ASTR,ESTR,ZSTR                                   MAIN 89
      WRITE(6,1020)                                                     MAIN 90
      WRITE(6,1009)MSTR,ASTR,ESTR,ZSTR                                  MAIN 91
C                                                                       MAIN 92
C     READ AND WRITE DYNAMIC INPUT PARAMETERS                           MAIN 93
C                                                                       MAIN 94
      READ(5,1007)SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ        MAIN 95
      WRITE(6,1021)                                                     MAIN 96
      WRITE(6,1012)SHDEN,RGDEN,STDEN,RGMI,STMI                          MAIN 97
      WRITE(6,1022)                                                     MAIN 98
      WRITE(6,1012)GRING,GSTR,RGJ,STJ                                   MAIN 99
      NSIZE=2*ISTAR*JSTAR                                               MAIN100
C                                                                       MAIN101
C     READ AND WRITE VALUES OF PRMT AND DERY                            MAIN102
C                                                                       MAIN103
```

```
      READ(5,1007)PRMT                                                  MAIN104
      DO 4 J2=1,NSIZE                                                   MAIN105
    4 DERY(J2)=1.00                                                     MAIN106
      WRITE(6,1013)                                                     MAIN107
      WRITE(6,1012)PRMT                                                 MAIN108
C                                                                       MAIN109
C     CALCULATE INTERMEDIATE VALUES                                     MAIN110
C                                                                       MAIN111
      MAX=1                                                             MAIN112
      ISIZE=ISTAR*JSTAR                                                 MAIN113
      PI=3.1415926535892                                                MAIN114
      DRING=SHLT/(NRNG+1.0)                                             MAIN115
      DSTR=2.0*PI*RSHEL/MSTR                                            MAIN116
C                                                                       MAIN117
C     READ INITIAL EXTERNAL LOAD, MAX LOAD, LOAD STEP SIZE             MAIN118
C     EXAMPLE LOAD VALUES...PLOAD=.610, ILOAD=610                       MAIN119
C                                                                       MAIN120
      READ(5,1008)ILOAD,ISTOP,INT                                       MAIN121
      WRITE(6,999)                                                      MAIN122
      WRITE(6,1035)                                                     MAIN123
      WRITE(6,1040)ILOAD,ISTOP,INT                                      MAIN124
      DO 10 NCR=NCRG,NCRS                                               MAIN125
      NCIR=NCR-1                                                        MAIN127
C                                                                       MAIN128
C     WRITE AXIAL AND CIRCUMFERENTIAL WAVE NUMBERS                      MAIN129
C                                                                       MAIN130
      WRITE(6,1055)MAX,NCIR                                             MAIN131
      FNCIR=NCIR                                                        MAIN132
      FN1=FNCIR/RSHEL                                                   MAIN133
      FN2=FN1*FN1                                                       MAIN134
      FMAX=MAX                                                          MAIN135
      FM1=FMAX*PI/SHLT                                                  MAIN136
      FM2=FM1*FM1                                                       MAIN137
      CALL UVCOF                                                        MAIN138
C                                                                       MAIN139
      CALL WCOEF                                                        MAIN140
```

```
 17    JX95=ILOAD                                                        MAIN145
 18    AJ=JX95                                                           MAIN146
       NDOT=0                                                            MAIN146A
       PLOAD=AJ/1000.0                                                   MAIN147
       PLOAD=PLOAD*HSHEL*HSHEL*ESHEL/RSHEL                               MAIN148
C                                                                        MAIN149
C      COMPUTE DYNAMIC TERMS OF WCOEF SUBROUTINE MATRIX                  MAIN150
C                                                                        MAIN151
C      SET WORKING MATRICES EQUAL TO ZERO                                MAIN152
C                                                                        MAIN153
       KOLX=3*ISIZE                                                      MAIN154
       KOL=2*ISIZE+1                                                     MAIN155
       DO 9 J4=KOL,KOLX                                                  MAIN156
       DO 8 I4=1,ISIZE                                                   MAIN157
       B(I4,J4)=0.0                                                      MAIN158
       D(I4,J4)=0.0                                                      MAIN159
 8     WCOF(I4,J4)=0.0                                                   MAIN160
 9     CONTINUE                                                          MAIN161
       II=0                                                              MAIN162
       DO 190 Q=1,ISIZE                                                  MAIN163
       II=II+1                                                           MAIN164
       INTGQ=(Q-1)/ISTAR                                                 MAIN165
       IZETA=Q-INTGQ*ISTAR                                               MAIN166
       IXI=INTGQ+1                                                       MAIN167
       FZETA=IZETA                                                       MAIN168
       FXI=IXI                                                           MAIN169
       NN=2*ISIZE                                                        MAIN170
       DO120P=1,ISIZE                                                    MAIN171
       NN=NN+1                                                           MAIN172
       INTGP=(P-1)/ISTAR                                                 MAIN173
       I=P-INTGP*ISTAR                                                   MAIN174
       J=INTGP+1                                                         MAIN175
       FI=I                                                              MAIN176
       FJ=J                                                              MAIN177
C                                                                        MAIN178
C      COMPUTE STRINGER TERMS                                            MAIN179
```

```
C                                                                       MAIN180
      IF(I-IZETA) 116,111,116                                           MAIN181
 111  SUM1=0.00                                                         MAIN182
      SUM2=0.00                                                         MAIN183
      DO112 K=1,MSTR                                                    MAIN184
      FK=K                                                              MAIN185
      SUM1=SUM1+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)                MAIN186
 112  SUM2=SUM2+SIN(FJ*FN1*FK*DSTR)*SIN(FXI*FN1*FK*DSTR)               MAIN187
      B(II,NN)=(FI*FM1)**4*ESTR*STMI*SUM1/(PI*RSHEL)+FI*FI*FM2*FJ*FN2*  MAIN188
     1GSTR*STJ*FXI*SUM2/(PI*RSHEL)                                      MAIN189
C                                                                       MAIN190
C     COMPUTE RING TERMS                                                MAIN191
C                                                                       MAIN192
 113  IF(J-IXI) 117,114,117                                             MAIN193
 114  SUM1=0.00                                                         MAIN194
      SUM2=0.00                                                         MAIN195
      DO 115 K=1,NRNG                                                   MAIN196
      FK=K                                                              MAIN197
      SUM1=SUM1+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)            MAIN198
 115  SUM2=SUM2+SIN(FI*FM1*FK*DRING)*SIN(FZETA*FM1*FK*DRING)           MAIN199
      D(II,NN)=2*((FJ*FN1)**4*ERING*RGMI*SUM1+2*FJ*FJ*FN2*ERING*ARING   MAIN200
     1*ZRING*SUM1/RSHEL+ERING*ARING*SUM1/RSHEL**2+FJ*FJ*FN2*FI*FZETA*   MAIN201
     2FM2*GRING*RGJ*SUM2)/SHLT                                          MAIN202
      IF(I-IZETA)118,119,118                                            MAIN203
 116  IF(J-IXI)120,113,120                                              MAIN204
 117  WCOF(II,NN)=B(II,NN)                                              MAIN205
      GO TO 120                                                         MAIN206
 118  WCOF(II,NN)=D(II,NN)                                              MAIN207
      GO TO 120                                                         MAIN208
C                                                                       MAIN209
C     COMPUTE SHELL TERMS                                               MAIN210
C                                                                       MAIN211
 119  WCOF(II,NN)=ESHEL*HSHEL*(HSHEL*HSHEL*(FI*FM1)**4/12.0            MAIN212
     1+HSHEL*HSHEL*FI*FI*FM2*FJ*FJ*FN2/6.0+HSHEL*HSHEL*(FJ*FN1)**4/12.0 MAIN213
     2+1.0/RSHEL**2-PLOAD*FI*FI*FM2*(1.0-PNU*PNU)/(ESHEL*HSHEL))/      MAIN214
     3(1.0-PNU*PNU)+B(II,NN)+D(II,NN)                                   MAIN215
```

```
120   CONTINUE                                                        MAIN216
190   CONTINUE                                                        MAIN217
      Y(1)=0.001                                                      MAIN218
      DO 3 L=2,NSIZE                                                  MAIN219
3     Y(L)=0.00                                                       MAIN220
      CALL RUNGE(PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)                  MAIN221
      IF(NDOT)22,22,10                                                MAIN222
22    JX95=JX95+INT                                                   MAIN227
      IF(JX95-ISTOP)18,18,10                                          MAIN228
10    CONTINUE                                                        MAIN230
      GO TO 5                                                         MAIN231
15    CALL EXIT                                                       MAIN232
      STOP                                                            MAIN233
      END                                                             MAIN234
```

```
      SUBROUTINE UVCOF                                                  UVCF   1
C                                                                       UVCF   2
C     THIS SUBROUTINE CALCULATES THE U,V,W,W**2 COEFFICIENTS FOR THE    UVCF   3
C     U AND V EQUATIONS.                                                UVCF   4
C                                                                       UVCF   5
      INTEGER P,Q                                                       UVCF   6
C     THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR     UVCF   7
C     LARGER THAN 2*ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR       UVCF   8
C     LARGER THAN 4*ISIZE FOR THE SECOND SUBSCRIPT.                     UVCF   9
C                                                                       UVCF  10
      DIMENSION B(10,20)                                                UVCF  11
      COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,HSHEL,PNU,  UVCF  12
     1NRNG,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,   UVCF  13
     2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ,   UVCF  14
     3PLOAD,RSHEL,SHLT,IFLAG,NDOT                                       UVCF  15
C                                                                       UVCF  16
C     SET WORKING MATRICES EQUAL TO ZERO                                UVCF  17
C                                                                       UVCF  18
      NSIZE=2*ISIZE                                                     UVCF  19
      MSIZE=4*ISIZE                                                     UVCF  20
      DO 2 J=1,MSIZE                                                    UVCF  21
      DO 1 I=1,NSIZE                                                    UVCF  22
      B(I,J)=0.00                                                       UVCF  23
    1 C(I,J)=0.00                                                       UVCF  24
    2 CONTINUE                                                          UVCF  25
C     COMPUTE COMPOSITE U-V EQUATION COEFFICIENT MATRIX                 UVCF  26
C                                                                       UVCF  27
      II=0                                                              UVCF  28
C                                                                       UVCF  29
C     COMPUTE U EQUATION MATRIX                                         UVCF  30
C                                                                       UVCF  31
      DO 130 Q=1,ISIZE                                                  UVCF  32
      II=II+1                                                           UVCF  33
      INTGQ=(Q-1)/ISTAR                                                 UVCF  34
      IZETA=Q-INTGQ*ISTAR                                               UVCF  35
      IXI=INTGQ+1                                                       UVCF  36
```

```
      FZETA=IZETA                                                        UVCF 37
      FXI=IXI                                                            UVCF 38
C                                                                        UVCF 39
C     SUBMATRIX U...U EQUATION                                           UVCF 40
C                                                                        UVCF 41
      NN=0                                                               UVCF 42
      DO 105 P=1,ISIZE                                                   UVCF 43
      NN=NN+1                                                            UVCF 44
      INTGP=(P-1)/ISTAR                                                  UVCF 45
      I=P-INTGP*ISTAR                                                    UVCF 46
      J=INTGP+1                                                          UVCF 47
      FI=I                                                               UVCF 48
      FJ=J                                                               UVCF 49
C                                                                        UVCF 50
C     COMPUTE STRINGER TERMS.                                            UVCF 51
C                                                                        UVCF 52
      IF(I-IZETA)105,101,105                                             UVCF 53
 101  SUM2=0.00                                                          UVCF 54
      DO 102 K=1,MSTR                                                    UVCF 55
      FK=K                                                               UVCF 56
 102  SUM2=SUM2+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)                 UVCF 57
      B(II,NN)=(1.0-PNU*PNU)*FI*FI*FM2*ESTR*ASTR*SUM2/(PI*RSHEL*ESHEL    UVCF 58
     1*HSHEL)                                                            UVCF 59
C                                                                        UVCF 60
C     COMPUTE SHELL TERMS                                                UVCF 61
C                                                                        UVCF 62
      IF(J-IXI)103,104,103                                               UVCF 63
 103  C(II,NN)=B(II,NN)                                                  UVCF 64
      GO TO 105                                                          UVCF 65
 104  C(II,NN)=B(II,NN)+FI*FI*FM2+(1.0-PNU)*FJ*FJ*FN2/2.0                UVCF 66
 105  CONTINUE                                                           UVCF 67
C                                                                        UVCF 68
C     SUBMATRIX V...U EQUATION                                           UVCF 69
C                                                                        UVCF 70
      DO 110 P=1,ISIZE                                                   UVCF 71
      NN=NN+1                                                            UVCF 72
```

```
      INTGP=(P-1)/ISTAR                                                 UVCF 73
      I=P-INTGP*ISTAR                                                   UVCF 74
      J=INTGP+1                                                         UVCF 75
      FI=I                                                              UVCF 76
      FJ=J                                                              UVCF 77
C                                                                       UVCF 78
C     COMPUTE SHELL TERMS                                               UVCF 79
C                                                                       UVCF 80
      IF(I-IZETA)110,106,110                                            UVCF 81
  106 IF(J-IXI)110,107,110                                              UVCF 82
  107 C(II,NN)=FI*FM1*FJ*FN1*(1.0+PNU)/2.0                              UVCF 83
  110 CONTINUE                                                          UVCF 84
C                                                                       UVCF 85
C     SUBMATRIX W...U EQUATION                                          UVCF 86
C                                                                       UVCF 87
      DO 115 P=1,ISIZE                                                  UVCF 88
      NN=NN+1                                                           UVCF 89
      INTGP=(P-1)/ISTAR                                                 UVCF 90
      I=P-INTGP*ISTAR                                                   UVCF 91
      J=INTGP+1                                                         UVCF 92
      FI=I                                                              UVCF 93
      FJ=J                                                              UVCF 94
C                                                                       UVCF 95
C     COMPUTE STRINGER TERMS                                            UVCF 96
C                                                                       UVCF 97
      IF(I-IZETA)115,111,115                                            UVCF 98
  111 SUM2=0.00                                                         UVCF 99
      DO 112 K=1,MSTR                                                   UVCF100
      FK=K                                                              UVCF101
  112 SUM2=SUM2+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)                UVCF102
      B(II,NN)=(1.0-PNU*PNU)*FI**3*FM2*FM1*ESTR*ASTR*ZSTR*SUM2/(        UVCF103
     1PI*RSHEL*ESHEL*HSHEL)                                             UVCF104
C                                                                       UVCF105
C     COMPUTE SHELL TERMS                                               UVCF106
C                                                                       UVCF107
      IF(J-IXI)113,114,113                                              UVCF108
```

```
  113   C(II,NN)=B(II,NN)                                                    UVCF109
        GO TO 115                                                            UVCF110
  114   C(II,NN)=B(II,NN)+PNU*FI*FM1/RSHEL                                   UVCF111
  115   CONTINUE                                                             UVCF112
C                                                                            UVCF113
C       SUBMATRIX W**2...U EQUATION                                          UVCF114
C                                                                            UVCF115
        DO 120 P=1,ISIZE                                                     UVCF116
        NN=NN+1                                                              UVCF117
        INTGP=(P-1)/ISTAR                                                    UVCF118
        I=P-INTGP*ISTAR                                                      UVCF119
        J=INTGP+1                                                            UVCF120
        FI=I                                                                 UVCF121
        FJ=J                                                                 UVCF122
C                                                                            UVCF123
C       COMPUTE STRINGER TERMS                                               UVCF124
C                                                                            UVCF125
        IF(2*I-IZETA)120,116,120                                             UVCF126
  116   SUM2=0.00                                                            UVCF127
        DO 117 K=1,MSTR                                                      UVCF128
        FK=K                                                                 UVCF129
  117   SUM2=SUM2+COS(FJ*FN1*FK*DSTR)**2*COS(FXI*FN1*FK*DSTR)               UVCF130
        B(II,NN)=(1.0-PNU*PNU)*(FI*FM1)**3*SUM2*ESTR*ASTR/(2.0*PI*           UVCF131
       1RSHEL*ESHEL*HSHEL)                                                   UVCF132
C                                                                            UVCF133
C       COMPUTE SHELL TERMS                                                  UVCF134
C                                                                            UVCF135
        IF(2*J-IXI)118,119,118                                               UVCF136
  118   C(II,NN)=-B(II,NN)                                                   UVCF137
        GO TO 120                                                            UVCF138
  119   C(II,NN)=-FI*FM1*FJ*FJ*FN2/4.0-FI**3*FM2*FM1/4.0-B(II,NN)            UVCF139
  120   CONTINUE                                                             UVCF140
  130   CONTINUE                                                             UVCF141
C                                                                            UVCF142
C       COMPUTE V EQUATION MATRIX                                            UVCF143
C                                                                            UVCF144
```

```
      DO 160 Q=1,ISIZE                                                  UVCF145
      II=II+1                                                           UVCF146
      INTGQ=(Q-1)/ISTAR                                                 UVCF147
      IZETA=Q-INTGQ*ISTAR                                               UVCF148
      IXI=INTGQ+1                                                       UVCF149
      FZETA=IZETA                                                       UVCF150
      FXI=IXI                                                           UVCF151
C                                                                       UVCF152
C     SUBMATRIX U...V EQUATION                                          UVCF153
C                                                                       UVCF154
      NN=0                                                              UVCF155
      DO 135 P=1,ISIZE                                                  UVCF156
      NN=NN+1                                                           UVCF157
      INTGP=(P-1)/ISTAR                                                 UVCF158
      I=P-INTGP*ISTAR                                                   UVCF159
      J=INTGP+1                                                         UVCF160
      FI=I                                                              UVCF161
      FJ=J                                                              UVCF162
C                                                                       UVCF163
C     COMPUTE SHELL TERMS                                               UVCF164
C                                                                       UVCF165
      IF(J-IXI)135,131,135                                              UVCF166
  131 IF(I-IZETA)135,132,135                                            UVCF167
  132 C(II,NN)=FI*FM1*FJ*FN1*(1.0+PNU)/2.0                              UVCF168
  135 CONTINUE                                                          UVCF169
C                                                                       UVCF170
C     SUBMATRIX V...V EQUATION                                          UVCF171
C                                                                       UVCF172
      DO 140 P=1,ISIZE                                                  UVCF173
      NN=NN+1                                                           UVCF174
      INTGP=(P-1)/ISTAR                                                 UVCF175
      I=P-INTGP*ISTAR                                                   UVCF176
      J=INTGP+1                                                         UVCF177
      FI=I                                                              UVCF178
      FJ=J                                                              UVCF179
C                                                                       UVCF180
```

```
C       COMPUTE RING TERMS                                                      UVCF181
C                                                                               UVCF182
        IF(J-IXI)140,136,140                                                    UVCF183
  136   SUM2=0.00                                                               UVCF184
        DO 137 K=1,NRNG                                                         UVCF185
        FK=K                                                                    UVCF186
  137   SUM2=SUM2+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)                  UVCF187
        B(II,NN)=2.0*(1.0-PNU*PNU)*FJ*FJ*FN2*ERING*ARING*SUM2/(ESHEL            UVCF188
       1*HSHEL*SHLT)                                                            UVCF189
C                                                                               UVCF190
C       COMPUTE SHELL TERMS                                                     UVCF191
C                                                                               UVCF192
        IF(I-IZETA)138,139,138                                                  UVCF193
  138   C(II,NN)=B(II,NN)                                                       UVCF194
        GO TO 140                                                               UVCF195
  139   C(II,NN)=B(II,NN)+FJ*FJ*FN2+FI*FI*FM2*(1.0-PNU)/2.0                     UVCF196
  140   CONTINUE                                                                UVCF197
C                                                                               UVCF198
C       SUBMATRIX W...V EQUATION                                                UVCF199
C                                                                               UVCF200
        DO 145 P=1,ISIZE                                                        UVCF201
        NN=NN+1                                                                 UVCF202
        INTGP=(P-1)/ISTAR                                                       UVCF203
        I=P-INTGP*ISTAR                                                         UVCF204
        J=INTGP+1                                                               UVCF205
        FI=I                                                                    UVCF206
        FJ=J                                                                    UVCF207
C                                                                               UVCF208
C       COMPUTE RING TERMS                                                      UVCF209
C                                                                               UVCF210
C                                                                               UVCF211
        IF(J-IXI)145,141,145                                                    UVCF212
  141   SUM2=0.00                                                               UVCF213
        DO 142 K=1,NRNG                                                         UVCF214
        FK=K                                                                    UVCF215
  142   SUM2=SUM2+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)                  UVCF216
```

```
      B(II,NN)=2.0*(1.0-PNU*PNU)*SUM2*(FJ*FN1*ERING*ARING/RSHEL+FJ**3*   UVCF217
     1FN1*FN2*ERING*ARING*ZRING)/(ESHEL*HSHEL*SHLT)                     UVCF218
C                                                                       UVCF219
C     COMPUTE SHELL TERMS                                               UVCF220
C                                                                       UVCF221
      IF(I-IZETA)143,144,143                                            UVCF222
  143 C(II,NN)=B(II,NN)                                                 UVCF223
      GO TO 145                                                         UVCF224
  144 C(II,NN)=B(II,NN)+FJ*FN1/RSHEL                                    UVCF225
  145 CONTINUE                                                          UVCF226
C                                                                       UVCF227
C     SUBMATRIX W**2...V EQUATION                                       UVCF228
C                                                                       UVCF229
C                                                                       UVCF230
      DO 150 P=1,ISIZE                                                  UVCF231
      NN=NN+1                                                           UVCF232
      INTGP=(P-1)/ISTAR                                                 UVCF233
      I=P-INTGP*ISTAR                                                   UVCF234
      J=INTGP+1                                                         UVCF235
      FI=I                                                              UVCF236
      FJ=J                                                              UVCF237
C                                                                       UVCF238
C     COMPUTE RING TERMS                                                UVCF239
C                                                                       UVCF240
      IF(2*J-IXI)150,146,150                                            UVCF241
  146 SUM2=0.00                                                         UVCF242
      DO 147 K=1,NRNG                                                   UVCF243
      FK=K                                                              UVCF244
  147 SUM2=SUM2+COS(FI*FM1*FK*DRING)**2*COS(FZETA*FM1*FK*DRING)         UVCF245
      B(II,NN)=(1.0-PNU*PNU)*FJ**3*FN2*FN1*ERING*ARING*SUM2/(ESHEL      UVCF246
     1*HSHEL*SHLT)                                                      UVCF247
C                                                                       UVCF248
C     COMPUTE SHELL TERMS                                               UVCF249
C                                                                       UVCF250
      IF(2*I-IZETA)148,149,148                                          UVCF251
  148 C(II,NN)=-B(II,NN)                                                UVCF252
```

```
      GO TO 150                                                          UVCF253
149   C(II,NN)=-B(II,NN)-(FJ*FN1)**3/4.0-FI*FI*FJ*FM2*FN1/4.0            UVCF254
150   CONTINUE                                                           UVCF255
160   CONTINUE                                                           UVCF256
      RETURN                                                             UVCF257
      END                                                                UVCF258
```

```
      SUBROUTINE WCOEF                                                  WCOF   1
C                                                                       WCOF   2
C     THIS SUBROUTINE CALCULATES THE W EQUATION COEFFICIENTS            WCOF   3
C     (U,V,W,W**2,W**3,U*W,V*W,D2W/DT2) AND PLACES THEM                 WCOF   4
C     IN A MATRIX CALLED WCOF                                           WCOF   5
C                                                                       WCOF   6
      INTEGER P,Q                                                       WCOF   7
C                                                                       WCOF   8
C     THE DIMENSION SIZE FOR THE SUBSCRIPTED VARIABLES MUST BE EQUAL TO WCOF   9
C     OR LARGER THAN ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR      WCOF  10
C     LARGER THAN 8*ISIZE FOR THE SECOND SUBSCRIPT.                     WCOF  11
C                                                                       WCOF  12
      DIMENSION B(5,40),D(5,40)                                         WCOF  13
      COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,HSHEL,PNU,  WCOF  14
     1NRNG,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,   WCOF  15
     2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ,   WCOF  16
     3PLOAD,RSHEL,SHLT,IFLAG,NDOT                                       WCOF  17
C                                                                       WCOF  18
C     SET WORKING MATRICES EQUAL TO ZERO                                WCOF  19
C                                                                       WCOF  20
      MSIZE=8*ISIZE                                                     WCOF  21
      DO 2 J=1,MSIZE                                                    WCOF  22
      DO 1 I=1,ISIZE                                                    WCOF  23
      B(I,J)=0.00                                                       WCOF  24
      D(I,J)=0.00                                                       WCOF  25
    1 WCOF(I,J)=0.00                                                    WCOF  26
    2 CONTINUE                                                          WCOF  27
C                                                                       WCOF  28
C     COMPUTE COMPLETE W EQUATION MATRIX                                WCOF  29
C                                                                       WCOF  30
      II=0                                                              WCOF  31
      DO 190 Q=1,ISIZE                                                  WCOF  32
      II=II+1                                                           WCOF  33
      INTGQ=(Q-1)/ISTAR                                                 WCOF  34
      IZETA=Q-INTGQ*ISTAR                                               WCOF  35
      IXI=INTGQ+1                                                       WCOF  36
```

```
      FZETA=IZETA                                                          WCOF 37
      FXI=IXI                                                              WCOF 38
C                                                                          WCOF 39
C     SUBMATRIX U...W EQUATION                                             WCOF 40
C                                                                          WCOF 41
      NN=0                                                                 WCOF 42
      DO 105 P=1,ISIZE                                                     WCOF 43
      NN=NN+1                                                              WCOF 44
      INTGP=(P-1)/ISTAR                                                    WCOF 45
      I=P-INTGP*ISTAR                                                      WCOF 46
      J=INTGP+1                                                            WCOF 47
      FI=I                                                                 WCOF 48
      FJ=J                                                                 WCOF 49
C                                                                          WCOF 50
C     COMPUTE STRINGER TERMS                                               WCOF 51
C                                                                          WCOF 52
      IF(I-IZETA)105,101,105                                               WCOF 53
  101 SUM2=0.00                                                            WCOF 54
      DO 102 K=1,MSTR                                                      WCOF 55
      FK=K                                                                 WCOF 56
  102 SUM2=SUM2+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)                   WCOF 57
      B(II,NN)=(FI*FM1)**3*ESTR*ASTR*ZSTR*SUM2/(PI*RSHEL)                  WCOF 58
C                                                                          WCOF 59
C     COMPUTE SHELL TERMS                                                  WCOF 60
C                                                                          WCOF 61
      IF(J-IXI)103,104,103                                                 WCOF 62
  103 WCOF(II,NN)=B(II,NN)                                                 WCOF 63
      GO TO 105                                                            WCOF 64
  104 WCOF(II,NN)=B(II,NN)+ESHEL*HSHEL*PNU*FI*FM1/(RSHEL*(1.0-PNU*PNU))   WCOF 65
  105 CONTINUE                                                             WCOF 66
C                                                                          WCOF 67
C     SUBMATRIX V...W EQUATION                                             WCOF 68
C                                                                          WCOF 69
      DO 110 P=1,ISIZE                                                     WCOF 70
      NN=NN+1                                                              WCOF 71
      INTGP=(P-1)/ISTAR                                                    WCOF 72
```

```
      I=P-INTGP*ISTAR                                                          WCOF 73
      J=INTGP+1                                                                WCOF 74
      FI=I                                                                     WCOF 75
      FJ=J                                                                     WCOF 76
C                                                                              WCOF 77
C     COMPUTE RING TERMS                                                       WCOF 78
C                                                                              WCOF 79
      IF(J-IXI)110,106,110                                                     WCOF 80
 106  SUM2=0.00                                                                WCOF 81
      DO 107 K=1,NRNG                                                          WCOF 82
      FK=K                                                                     WCOF 83
 107  SUM2=SUM2+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)                   WCOF 84
      B(II,NN)=2*ERING*ARING*SUM2*((FJ*FN1)**3*ZRING+FJ*FN1/RSHEL)/SHLT        WCOF 85
C                                                                              WCOF 86
C     COMPUTE SHELL TERMS                                                      WCOF 87
C                                                                              WCOF 88
      IF(I-IZETA)108,109,108                                                   WCOF 89
 108  WCOF(II,NN)=B(II,NN)                                                     WCOF 90
      GO TO 110                                                                WCOF 91
 109  WCOF(II,NN)=B(II,NN)+ESHEL*HSHEL*FJ*FN1/(RSHEL*(1.0-PNU*PNU))           WCOF 92
 110  CONTINUE                                                                 WCOF 93
C                                                                              WCOF 94
C     SUBMATRIX W...W EQUATION                                                 WCOF 95
C                                                                              WCOF 96
      DO120P=1,ISIZE                                                           WCOF 97
      NN=NN+1                                                                  WCOF 98
 120  CONTINUE                                                                 WCOF 99
C                                                                              WCOF100
C     SUBMATRIX W**2...W EQUATION                                              WCOF101
C                                                                              WCOF102
      DO 130 P=1,ISIZE                                                         WCOF103
      NN=NN+1                                                                  WCOF104
      INTGP=(P-1)/ISTAR                                                        WCOF105
      I=P-INTGP*ISTAR                                                          WCOF106
      J=INTGP+1                                                                WCOF107
      FI=I                                                                     WCOF108
```

```
      FJ=J                                                              WCOF109
C                                                                       WCOF110
C     COMPUTE STRINGER TERMS                                            WCOF111
C                                                                       WCOF112
      IF(2*I-IZETA)126,121,126                                          WCOF113
 121  SUM1=0.00                                                         WCOF114
      DO 122 K=1,MSTR                                                   WCOF115
      FK=K                                                              WCOF116
 122  SUM1=SUM1+COS(FJ*FN1*FK*DSTR)**2*COS(FXI*FN1*FK*DSTR)             WCOF117
      B(II,NN)=(FI*FM1)**4*ESTR*ASTR*ZSTR*SUM1/(PI*RSHEL*2.0)           WCOF118
C                                                                       WCOF119
C     COMPUTE RING TERMS                                                WCOF120
C                                                                       WCOF121
 123  IF(2*J-IXI)127,124,127                                            WCOF122
 124  SUM1=0.00                                                         WCOF123
      DO 125 K=1,NRNG                                                   WCOF124
      FK=K                                                              WCOF125
 125  SUM1=SUM1+COS(FI*FM1*FK*DRING)**2*COS(FZETA*FM1*FK*DRING)         WCOF126
      D(II,NN)=2.0*SUM1*(FJ*FJ*FN2/(4.0*RSHEL)-(FJ*FN1)**4*FZETA/2.0)*  WCOF127
     1ERING*ARING/SHLT                                                  WCOF128
      IF(2*I-IZETA)128,129,128                                          WCOF129
 126  IF(2*J-IXI)130,123,130                                            WCOF130
 127  WCOF(II,NN)=-B(II,NN)                                             WCOF131
      GO TO 130                                                         WCOF132
 128  WCOF(II,NN)=D(II,NN)                                              WCOF133
      GO TO 130                                                         WCOF134
C                                                                       WCOF135
C     COMPUTE SHELL TERMS                                               WCOF136
C                                                                       WCOF137
 129  WCOF(II,NN)=-B(II,NN)+D(II,NN)+ESHEL*HSHEL*PNU*(FI*FI*            WCOF138
     1FM2+FJ*FJ*FN2)/(8.0*RSHEL*(1.0-PNU*PNU))                          WCOF139
 130  CONTINUE                                                          WCOF140
C                                                                       WCOF141
C     SUBMATRIX W**3...W EQUATION                                       WCOF142
C                                                                       WCOF143
      DO 160 P=1,ISIZE                                                  WCOF144
```

```
      NN=NN+1                                                          WCOF145
      INTGP=(P-1)/ISTAR                                                WCOF146
      I=P-INTGP*ISTAR                                                  WCOF147
      J=INTGP+1                                                        WCOF148
      FI=I                                                             WCOF149
      FJ=J                                                             WCOF150
C                                                                      WCOF151
C     COMPUTE STRINGER TERMS                                           WCOF152
C                                                                      WCOF153
      DEL1=0.00                                                        WCOF154
      DEL2=0.00                                                        WCOF155
      IF(I-IZETA)134,131,134                                           WCOF156
  131 DEL1=+1.0                                                        WCOF157
  132 SUM1=0.00                                                        WCOF158
      DO 133 K=1,MSTR                                                  WCOF159
      FK=K                                                             WCOF160
  133 SUM1=SUM1+COS(FJ*FN1*FK*DSTR)**3*COS(FXI*FN1*FK*DSTR)            WCOF161
      B(II,NN)=DEL1*SUM1*ESTR*ASTR*(FI*FM1)**4/(PI*RSHEL*8.0)          WCOF162
      GO TO 136                                                        WCOF163
  134 IF(3*I-IZETA)136,135,136                                         WCOF164
  135 DEL1=-1.0                                                        WCOF165
      GO TO 132                                                        WCOF166
  136 IF(J-IXI)147,137,147                                             WCOF167
  137 DEL2=+1.0                                                        WCOF168
C                                                                      WCOF169
C     COMPUTE RING TERMS                                               WCOF170
C                                                                      WCOF171
  138 SUM1=0.00                                                        WCOF172
      DO 139 K=1,NRNG                                                  WCOF173
  139 SUM1=SUM1+COS(FI*FM1*FK*DRING)**3*COS(FZETA*FM1*FK*DRING)        WCOF174
      D(II,NN)=DEL2*2.0*SUM1*(FJ*FN1)**4*ERING*ARING/(8.0*SHLT)        WCOF175
      IF(DEL1)140,151,140                                              WCOF176
C                                                                      WCOF177
C     COMPUTE SHELL TERMS                                              WCOF178
C                                                                      WCOF179
  140 IF(DEL1*DEL2)141,141,144                                         WCOF180
```

```
  141   IF(DEL1)142,142,143                                                    WCOF181
  142   WCOF(II,NN)=B(II,NN)+D(II,NN)+(ESHEL*HSHEL)*(-3.0*(FI*FM1)**4+         WCOF182
       12.0*FI*FI*FM2*FJ*FJ*FN2*(1.0-2*PNU)+(FJ*FN1)**4)/                     WCOF183
       2(32.0*(1.0-PNU*PNU))                                                  WCOF184
        GO TO 160                                                             WCOF185
  143   WCOF(II,NN)=B(II,NN)+D(II,NN)+(ESHEL*HSHEL)*((FI*FM1)**4              WCOF186
       1+2.0*FI*FI*FM2*FJ*FJ*FN2*(1.0-2.0*PNU)-3.0*(FJ*FN1)**4)/              WCOF187
       2(32.0*(1-PNU*PNU))                                                    WCOF188
        GO TO 160                                                             WCOF189
  144   IF(DEL1)145,145,146                                                    WCOF190
  145   WCOF(II,NN)=B(II,NN)+D(II,NN)+(ESHEL*HSHEL)*(-(FI*FM1)**4              WCOF191
       1-2.0*FI*FI*FM2*FJ*FJ*FN2-(FJ*FN1)**4)/(32.0*(1.0-PNU*PNU))            WCOF192
        GO TO 160                                                             WCOF193
  146   WCOF(II,NN)=B(II,NN)+D(II,NN)+(ESHEL*HSHEL)*(3.0*(FI*FM1)**4           WCOF194
       1-2.0*FI*FI*FM2*FJ*FJ*FN2*(1.0-4.0*PNU)+3*(FJ*FN1)**4)/(32.0           WCOF195
       2*(1.0-PNU*PNU))                                                       WCOF196
        GO TO 160                                                             WCOF197
  147   IF(3*J-IXI)149,148,149                                                 WCOF198
  148   DEL2=-1.0                                                              WCOF199
        GO TO 138                                                             WCOF200
  149   IF(DEL1)150,160,150                                                    WCOF201
  150   WCOF(II,NN)=B(II,NN)                                                   WCOF202
        GO TO 160                                                             WCOF203
  151   WCOF(II,NN)=D(II,NN)                                                   WCOF204
  160   CONTINUE                                                               WCOF205
C                                                                            WCOF206
C       SUBMATRIX U*W...W EQUATION                                           WCOF207
C                                                                            WCOF208
        DO 165 P=1,ISIZE                                                     WCOF209
        NN=NN+1                                                              WCOF210
        INTGP=(P-1)/ISTAR                                                    WCOF211
        I=P-INTGP*ISTAR                                                      WCOF212
        J=INTGP+1                                                            WCOF213
        FI=I                                                                 WCOF214
        FJ=J                                                                 WCOF215
C                                                                            WCOF216
```

```
C       COMPUTE STRINGER TERMS                                                  WCOF217
C                                                                               WCOF218
        IF(2*I-IZETA)165,161,165                                                WCOF219
 161    SUM1=0.00                                                               WCOF220
        DO 162 K=1,MSTR                                                         WCOF221
        FK=K                                                                    WCOF222
 162    SUM1=SUM1+COS(FJ*FN1*FK*DSTR)**2*COS(FXI*FN1*FK*DSTR)                   WCOF223
C                                                                               WCOF224
C       COMPUTE SHELL TERMS                                                     WCOF225
C                                                                               WCOF226
        B(II,NN)=(FI*FM1)**3*SUM1*ESTR*ASTR/(2.0*PI*RSHEL)                      WCOF227
        IF(2*J-IXI)163,164,163                                                  WCOF228
 163    WCOF(II,NN)=B(II,NN)                                                    WCOF229
        GO TO 165                                                               WCOF230
 164    WCOF(II,NN)=B(II,NN)+ESHEL*HSHEL*((FI*FM1)**3+FI*FM1*FJ*FJ*FN2)/        WCOF231
     1(4.0*(1.0-PNU*PNU))                                                       WCOF232
 165    CONTINUE                                                                WCOF233
C                                                                               WCOF234
C       SUBMATRIX V*W...W EQUATION                                              WCOF235
C                                                                               WCOF236
        DO 170 P=1,ISIZE                                                        WCOF237
        NN=NN+1                                                                 WCOF238
        INTGP=(P-1)/ISTAR                                                       WCOF239
        I=P-INTGP*ISTAR                                                         WCOF240
        J=INTGP+1                                                               WCOF241
        FI=I                                                                    WCOF242
        FJ=J                                                                    WCOF243
C                                                                               WCOF244
C       COMPUTE RING TERMS                                                      WCOF245
C                                                                               WCOF246
        IF(2*J-IXI)170,166,170                                                  WCOF247
 166    SUM1=0.00                                                               WCOF248
        DO 167 K=1,NRNG                                                         WCOF249
        FK=K                                                                    WCOF250
 167    SUM1=SUM1+COS(FI*FM1*FK*DRING)**2*COS(FZETA*FM1*FK*DRING)               WCOF251
        B(II,NN)=SUM1*ERING*ARING*(FJ*FN1)**3/SHLT                              WCOF252
```

```
C                                                                       WCOF253
C       COMPUTE SHELL TERMS                                             WCOF254
C                                                                       WCOF255
        IF(2*I-IZETA)168,169,168                                        WCOF256
  168   WCOF(II,NN)=B(II,NN)                                            WCOF257
        GO TO 170                                                       WCOF258
  169   WCOF(II,NN)=B(II,NN)+ESHEL*HSHEL*((FJ*FN1)**3+FI*FI*FM2*        WCOF259
       1FJ*FN1)/(4.0*(1-PNU*PNU))                                       WCOF260
  170   CONTINUE                                                        WCOF261
C                                                                       WCOF262
C       SUBMATRIX D2W/DT2...W EQUATION                                  WCOF263
C                                                                       WCOF264
        DO 180 P=1,ISIZE                                                WCOF265
        NN=NN+1                                                         WCOF266
        INTGP=(P-1)/ISTAR                                               WCOF267
        I=P-INTGP*ISTAR                                                 WCOF268
        J=INTGP+1                                                       WCOF269
        FI=I                                                            WCOF270
        FJ=J                                                            WCOF271
C                                                                       WCOF272
C       COMPUTE STRINGER TERMS                                          WCOF273
C                                                                       WCOF274
        IF(I-IZETA)176,171,176                                          WCOF275
  171   SUM1=0.00                                                       WCOF276
        DO 172 K=1,MSTR                                                 WCOF277
        FK=K                                                            WCOF278
  172   SUM1=SUM1+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)              WCOF279
        B(II,NN)=SUM1*STDEN*ASTR/(PI*RSHEL)                             WCOF280
C                                                                       WCOF281
C       COMPUTE RING TERMS                                              WCOF282
C                                                                       WCOF283
  173   IF(J-IXI)177,174,177                                            WCOF284
  174   SUM1=0.00                                                       WCOF285
        DO 175 K=1,NRNG                                                 WCOF286
        FK=K                                                            WCOF287
  175   SUM1=SUM1+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)          WCOF288
```

```
      D(II,NN)=SUM1*RGDEN*ARING*2.0/SHLT                                WCOF289
      IF(I-IZETA)178,179,178                                            WCOF290
  176 IF(J-IXI)180,173,180                                              WCOF291
  177 WCOF(II,NN)=B(II,NN)                                              WCOF292
      GO TO 180                                                         WCOF293
  178 WCOF(II,NN)=D(II,NN)                                              WCOF294
      GO TO 180                                                         WCOF295
C                                                                       WCOF296
C     COMPUTE SHELL TERMS                                               WCOF297
C                                                                       WCOF298
  179 WCOF(II,NN)=B(II,NN)+D(II,NN)+SHDEN*HSHEL                         WCOF299
  180 CONTINUE                                                          WCOF300
  190 CONTINUE                                                          WCOF301
      RETURN                                                            WCOF302
      END                                                               WCOF303
```

```
      SUBROUTINE RUNGE(PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)                RUNG  1
C                                                                         RUNG  2
C     THIS SUBROUTINE TAKES THE PLACE OF SUBROUTINE RKGS CONTAINED IN     RUNG  3
C     THE IBM SCIENTIFIC SUBROUTINE PACKAGE.                              RUNG  4
C     FOR A DESCRIPTION OF THE VARIABLES CONTAINED IN THIS SUBROUTINE,    RUNG  5
C     SEE THE IBM SUBROUTINE RKGS WRITEUP.                                RUNG  6
C                                                                         RUNG  7
C     THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR       RUNG  8
C     GREATER THAN 2*ISIZE                                                RUNG  9
C                                                                         RUNG 10
      DIMENSION Y(10),DERY(10),PHI(10),SAVEY(10),SDATA(4,2),PRMT(3)       RUNG 11
      COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,HSHEL,PNU,    RUNG 12
     1NRNG,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,     RUNG 13
     2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ,     RUNG 14
     3PLOAD,RSHEL,SHLT,IFLAG,NDOT                                         RUNG 15
 1000 FORMAT(1X,E15.8)                                                    RUNG 16
 1005 FORMAT('**********BUCKLING HAS OCCURRED...MAX,NCIR,PLOAD******RUNG 17
     1********************************************** ')                  RUNG 18
 1010 FORMAT(1X,2I5,2X,E15.8)                                             RUNG 19
 1015 FORMAT(' PLOAD = ')                                                 RUNG 20
 1020 FORMAT(10(2X,E11.5))                                                RUNG 21
 1025 FORMAT(' EXPONENT OVERFLOW WILL PROBABLY OCCUR BECAUSE OF RUNGE-    RUNG21A
     1KUTTA STEP BEING TOO BIG.....CASE TERMINATED ')                     RUNG218
      X=PRMT(1)                                                           RUNG 22
      XMAX=PRMT(2)                                                        RUNG 23
      H=PRMT(3)                                                           RUNG 24
      N=2*ISIZE                                                           RUNG 25
      PNX=PLOAD*RSHEL/(HSHEL*HSHEL*ESHEL)                                 RUNG 26
      WRITE(6,1015)                                                       RUNG 27
      WRITE(6,1000)PNX                                                    RUNG 28
 10   CALL FCT(X,Y,DERY)                                                  RUNG 29
      DO 22 J=1,N                                                         RUNG 30
      SAVEY(J)=Y(J)                                                       RUNG 31
      PHI(J)=DERY(J)                                                      RUNG 32
 22   Y(J)=SAVEY(J)+0.5*H*DERY(J)                                         RUNG 33
      X=X+0.5*H                                                           RUNG 34
```

```
      CALL FCT(X,Y,DERY)                                                RUNG 35
      DO 33 J=1,N                                                       RUNG 36
      PHI(J)=PHI(J)+2.0*DERY(J)                                         RUNG 37
33    Y(J)=SAVEY(J)+0.5*H*DERY(J)                                       RUNG 38
      CALL FCT(X,Y,DERY)                                                RUNG 39
      DO 44 J=1,N                                                       RUNG 40
      PHI(J)=PHI(J)+2.0*DERY(J)                                         RUNG 41
44    Y(J)=SAVEY(J)+H*DERY(J)                                           RUNG 42
      X=X+0.5*H                                                         RUNG 43
      CALL FCT(X,Y,DERY)                                                RUNG 44
      DO 55 J=1,N                                                       RUNG 45
55    Y(J)=SAVEY(J)+(PHI(J)+DERY(J))*H/6.0                              RUNG 46
      N9=N-1                                                            RUNG 47
      WRITE(6,1020)X,(Y(J),J=1,N9,2)                                    RUNG 48
      IF(ABS(Y(1))-0.5)56,57,57                                         RUNG 49
56    IF (X-XMAX)10,60,60                                               RUNG49A
57    WRITE(6,1025)                                                     RUNG49B
      NDOT=1                                                            RUNG49C
      GO TO 80                                                          RUNG49D
60    IF(Y(1)-0.001)80,70,70                                            RUNG 50
70    WRITE(6,1005)                                                     RUNG 51
      WRITE(6,1010)MAX,NCIR,PNX                                         RUNG 52
80    RETURN                                                            RUNG 54
      END                                                               RUNG 55
```

```
      SUBROUTINE FCT(X,Y,DERY)                                          FNCT   1
C                                                                       FNCT   2
C     THIS SUBROUTINE CONTAINS NO READ AND WRITE STATEMENTS.            FNCT   3
C                                                                       FNCT   4
C     THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR     FNCT   5
C     GREATER THAN 2*ISIZE, WHERE ISIZE=ISTAR*JSTAR                     FNCT   6
C                                                                       FNCT   7
      DIMENSION Y(10),DERY(10),A(10,10),R(10),T(10),Z(10)               FNCT   8
C                                                                       FNCT   9
C     THE DIMENSION SIZE FOR C(II,NN) MUST BE EQUAL TO OR GREATER       FNCT  10
C     THAN 2*ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR GREATER      FNCT  11
C     THAN 4*ISIZE FOR THE SECOND SUBSCRIPT.                            FNCT  12
C     THE DIMENSION SIZE FOR WCOF(II,NN) MUST BE EQUAL TO OR GREATER    FNCT  13
C     THAN ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR GREATER THAN   FNCT  14
C     8*ISIZE FOR THE SECOND SUBSCRIPT.                                 FNCT  15
C                                                                       FNCT  16
      COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,HSHEL,PNU,  FNCT  17
     1NRNG,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,   FNCT  18
     2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ,   FNCT  19
     3PLOAD,RSHEL,SHLT,IFLAG,NDOT                                       FNCT  20
C                                                                       FNCT  21
C     PLACE U AND V COEFFICIENTS INTO THE GAUSS-JORDAN MATRIX A(N)      FNCT  22
C                                                                       FNCT  23
      MARK=0                                                            FNCT  24
      NSIZE=2*ISIZE                                                     FNCT  25
      DO 16 J=1,NSIZE                                                   FNCT  26
      DO 15 I=1,NSIZE                                                   FNCT  27
 15   A(I,J)=C(I,J)                                                     FNCT  28
 16   CONTINUE                                                          FNCT  29
C                                                                       FNCT  30
C     PLACE PROPER U AND V COEFFICIENTS INTO R(N) MATRIX                FNCT  31
C                                                                       FNCT  32
      DO 19 K=1,NSIZE                                                   FNCT  33
 19   R(K)=0.00                                                         FNCT  34
      DO 21 K=1,NSIZE                                                   FNCT  35
      DO 20 J=1,ISIZE                                                   FNCT  36
```

```
      J1=J+2*ISIZE                                                      FNCT 37
      J2=J+3*ISIZE                                                      FNCT 38
 20   R(K)=R(K)+C(K,J1)*Y(2*J-1)+C(K,J2)*Y(2*J-1)*Y(2*J-1)              FNCT 39
 21   CONTINUE                                                          FNCT 40
C                                                                       FNCT 41
C     CHANGE SIGN OF R(I) MATRIX                                        FNCT 42
C                                                                       FNCT 43
      DO 35 I=1,NSIZE                                                   FNCT 44
 35   R(I)=-R(I)                                                        FNCT 45
 22   N=NSIZE                                                           FNCT 46
C                                                                       FNCT 47
C  SOLVES N SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS BY A GAUSS-JORDAN   FNCT 48
C  REDUCTION.              A IS THE COEFFICIENT MATRIX, R IS THE CONSTANT FNCT 49
C  VECTOR.  A IS DESTROYED AND R IS REPLACED BY THE SOLUTION            FNCT 50
      DO 11 J=1,N                                                       FNCT 51
 11   T(J)=1.0                                                          FNCT 52
      M=N-1                                                             FNCT 60
      DO 1 J=1,M                                                        FNCT 61
      AD=A(J,J)                                                         FNCT 62
      DO 3 K=J,N                                                        FNCT 63
    3 A(J,K)=A(J,K)/AD                                                  FNCT 64
      R(J)=R(J)/AD                                                      FNCT 65
      L=N-J                                                             FNCT 66
      NN=0                                                              FNCT 67
      DO 1 I=1,L                                                        FNCT 68
      NN=J+I                                                            FNCT 69
      DA=A(NN,J)                                                        FNCT 70
      DO 2 K=J,N                                                        FNCT 71
 2    A(NN,K)=A(NN,K)-A(J,K)*DA                                         FNCT 72
 1    R(NN)=R(NN)-R(J)*DA                                               FNCT 73
      DO 4 J=1,N                                                        FNCT 74
      K=N+1-J                                                           FNCT 75
      S=0.0                                                             FNCT 76
      IF(K-N)9,4,9                                                      FNCT 77
 9    L=K+1                                                             FNCT 78
      DO 5 I=L,N                                                        FNCT 79
```

```
    5 S=A(K,I)*T(I)+S                                                    FNCT 80
    4 T(K)=(R(K)-S)/A(K,K)                                               FNCT 81
      DO 6 J=1,N                                                         FNCT 82
    6 R(J)=T(J)                                                          FNCT 83
C                                                                        FNCT 84
      IF(MARK-1)23,30,30                                                 FNCT 85
C                                                                        FNCT 86
C     SET A(N,N) EQUAL TO ZERO                                           FNCT 87
C                                                                        FNCT 88
 23   DO 32 I=1,NSIZE                                                    FNCT 89
      DO 33 J=1,NSIZE                                                    FNCT 90
 33   A(I,J)=0.00                                                        FNCT 91
 32   CONTINUE                                                           FNCT 92
C                                                                        FNCT 93
C     MAKE A(N,N) INTO AN IDENTITY MATRIX                                FNCT 94
C                                                                        FNCT 95
      DO 34 IJ=1,NSIZE                                                   FNCT 96
 34   A(IJ,IJ)=1.00                                                      FNCT 97
C                                                                        FNCT 98
C     PLACE W COEFFICIENTS INTO GAUSS-JORDAN MATRIX  A(N)                FNCT 99
C                                                                        FNCT100
      DO 25 J=1,ISIZE                                                    FNCT101
      DO 24 I=1,ISIZE                                                    FNCT102
      JN=J+7*ISIZE                                                       FNCT103
 24   A(2*I,2*J)=WCOF(I,JN)                                              FNCT104
 25   CONTINUE                                                           FNCT105
C                                                                        FNCT106
C     SET Z(N)=ZERO                                                      FNCT107
C                                                                        FNCT108
      DO 26 I=1,NSIZE                                                    FNCT109
 26   Z(I)=0.00                                                          FNCT110
C                                                                        FNCT111
C     PLACE PROPER W COEFFICIENTS INTO R(N)                              FNCT112
C                                                                        FNCT113
      DO 28 K=1,ISIZE                                                    FNCT114
      Z(2*K-1)=Y(2*K)                                                    FNCT115
```

```
      DO 27 J=1,ISIZE                                                    FNCT116
      J1=J+ISIZE                                                         FNCT117
      J2=J+2*ISIZE                                                       FNCT118
      J3=J+3*ISIZE                                                       FNCT119
      J4=J+4*ISIZE                                                       FNCT120
      J5=J+5*ISIZE                                                       FNCT121
      J6=J+6*ISIZE                                                       FNCT122
 27   Z(2*K)=Z(2*K)+WCOF(K,J)*R(J)+WCOF(K,J1)*R(J1)                      FNCT123
     1+WCOF(K,J2)*Y(2*J-1)+WCOF(K,J3)*Y(2*J-1)*Y(2*J-1)                  FNCT124
     2+WCOF(K,J4)*Y(2*J-1)**3+WCOF(K,J5)*R(J)*Y(2*J-1)                   FNCT125
     3+WCOF(K,J6)*R(J1)*Y(2*J-1)                                        FNCT126
 28   CONTINUE                                                           FNCT127
C                                                                       FNCT128
C     SET R(N)=Z(N)                                                     FNCT129
C                                                                       FNCT130
      DO 29 I=1,NSIZE                                                   FNCT131
 29   R(I)=Z(I)                                                         FNCT132
      MARK =1                                                           FNCT133
C                                                                       FNCT134
C     CHANGE SIGN OF R(2*K)                                             FNCT135
C                                                                       FNCT136
      DO 36 K=1,ISIZE                                                   FNCT137
 36   R(2*K)=-R(2*K)                                                    FNCT138
      GO TO 22                                                          FNCT139
C                                                                       FNCT140
C     PLACE R(N) RESULTS IN DERY                                        FNCT141
C                                                                       FNCT142
 30   DO 31 I=1,NSIZE                                                   FNCT143
 31   DERY(I)=R(I)                                                      FNCT144
      RETURN                                                            FNCT145
      END                                                               FNCT146
```