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A DISSERTATION
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in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

BY
CARY A. FISHER
Norman, Oklahoma
1972
DYNAMIC BUCKLING OF AN AXIALLY COMPRESSED CYLINDRICAL SHELL WITH DISCRETE RINGS AND STRINGERS

APPROVED BY

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DISSERTATION COMMITTEE
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TABLE OF CONTENTS

Page

ACKNOWLEDGMENT .............................................................. iii

LIST OF TABLES ................................................................. vii

LIST OF ILLUSTRATIONS .................................................... viii

LIST OF SYMBOLS ............................................................. ix

CHAPTER

I. INTRODUCTION ............................................................... 1

1.1 Survey of Shell Buckling ............................................. 1
1.2 Research Objectives ............................................... 10

II. FORMULATION OF THEORY ........................................... 12

2.1 Method of Analysis ................................................... 12
2.2 Hypotheses ............................................................. 13
2.3 Prebuckling Equations ............................................. 15
2.4 Buckling Equations ................................................. 16
2.5 Application of Galerkin's Method ............................. 17
2.6 Numerical Solution ............................................... 21

III. EVALUATION OF THEORY ............................................. 25

3.1 Comparison with Static Buckling .............................. 25
3.2 Comparison with Dynamic Buckling ......................... 27

IV. RESULTS ................................................................. 31

4.1 Representative Aircraft Structure ........................... 31
4.2 Dynamic Considerations ......................................... 38
4.3 Geometric Considerations ...................................... 42
4.4 Discreteness Considerations ................................. 53

V. CLOSURE ................................................................. 58

REFERENCES ............................................................... 60
TABLE OF CONTENTS (Cont'd.)

APPENDICES

A. DERIVATION OF KINETIC AND POTENTIAL ENERGIES OF A DISCRETELY STIFFENED CYLINDRICAL SHELL .......................... 66
   A.1 Nonlinear Strain Displacement Formulation ................. 66
   A.2 Unstiffened Cylinder Strain Energy ....................... 66
   A.3 Ring Strain Energy .......................................... 69
   A.4 Stringer Strain Energy ...................................... 70
   A.5 External Load Potential Energy ............................. 70
   A.6 Total Potential Energy ...................................... 71
   A.7 Total Kinetic Energy ........................................ 71

B. APPLICATION OF HAMILTON'S PRINCIPLE TO DERIVE THE EQUATIONS OF MOTION ........................................... 72

C. APPLICATION OF GALERKIN'S METHOD TO OBTAIN A SET OF ORDINARY NONLINEAR GOVERNING DIFFERENTIAL EQUATIONS ......................... 78

D. COMPUTER PROGRAM DOCUMENTATION ...................................... 88

E. COMPUTER PROGRAM LISTING ........................................ 93
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Coefficient Algorithm</td>
<td>24</td>
</tr>
<tr>
<td>3.1 Comparison of Smeared and Discrete Terms</td>
<td>25</td>
</tr>
<tr>
<td>4.1 Representative Light Aircraft Geometry</td>
<td>32</td>
</tr>
<tr>
<td>C.1 Integrated Terms from Axial Equation of Motion</td>
<td>79</td>
</tr>
<tr>
<td>C.2 Integrated Terms from Circumferential Equation of Motion</td>
<td>81</td>
</tr>
<tr>
<td>C.3 Unstiffened Cylinder Integrated Terms from Radial Equation of Motion</td>
<td>82</td>
</tr>
<tr>
<td>C.4 Ring Integrated Terms from Radial Equation of Motion</td>
<td>84</td>
</tr>
<tr>
<td>C.5 Stringer Integrated Terms from Radial Equation of Motion</td>
<td>85</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Static Buckling Comparison</td>
<td>28</td>
</tr>
<tr>
<td>3.2</td>
<td>Dynamic Buckling Comparison, Unstiffened Shell</td>
<td>30</td>
</tr>
<tr>
<td>4.1</td>
<td>General Aircraft Fuselage</td>
<td>35</td>
</tr>
<tr>
<td>4.2</td>
<td>Comparison of Buckling Load for Different Cylinder Regions</td>
<td>37</td>
</tr>
<tr>
<td>4.3</td>
<td>Maximum Radial Deflection Versus Time for Various End Loads, n = 9</td>
<td>41</td>
</tr>
<tr>
<td>4.4</td>
<td>Maximum Radial Deflection Versus Time for Various End Loads, n = 10</td>
<td>43</td>
</tr>
<tr>
<td>4.5</td>
<td>Buckling Load Versus Time Duration, n = 9</td>
<td>44</td>
</tr>
<tr>
<td>4.6</td>
<td>Axial Force Versus Wave Number with Increasing Number of Stringers</td>
<td>45</td>
</tr>
<tr>
<td>4.7</td>
<td>Influence of Number of Stringers on Buckling Load</td>
<td>46</td>
</tr>
<tr>
<td>4.8</td>
<td>Axial Force Versus Wave Number for Different Ring Spacing, Subshell III</td>
<td>48</td>
</tr>
<tr>
<td>4.9</td>
<td>Axial Force Versus Wave Number for Different Ring Spacing, Subshell I</td>
<td>49</td>
</tr>
<tr>
<td>4.10</td>
<td>Influence of Number of Rings on Buckling Load</td>
<td>50</td>
</tr>
<tr>
<td>4.11</td>
<td>Influence of Stiffener Eccentricity on Buckling Load</td>
<td>52</td>
</tr>
<tr>
<td>4.12</td>
<td>Convergence of Solution, Subshell I</td>
<td>55</td>
</tr>
<tr>
<td>4.13</td>
<td>Convergence of Solution, Subshell II</td>
<td>56</td>
</tr>
<tr>
<td>A.1</td>
<td>Shell Geometry</td>
<td>67</td>
</tr>
<tr>
<td>D.1</td>
<td>Computer Program Flow Chart</td>
<td>92</td>
</tr>
<tr>
<td>SYMBOLS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>Stiffener cross-sectional area (in.²)</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>Stringer center-to-center spacing (in.)</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>Bending stiffness (in.-lb.)</td>
<td></td>
</tr>
<tr>
<td>ẻ</td>
<td>Distance from shell middle surface to line where ( \hat{N}_x ) acts (in.)</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>Young's modulus (psi)</td>
<td></td>
</tr>
<tr>
<td>argout</td>
<td>Nondimensional buckling load, ( \tilde{F} = \hat{N}_x R / Eh^2 )</td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>Time step</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>Shear modulus (psi)</td>
<td></td>
</tr>
<tr>
<td>h</td>
<td>Unstiffened shell wall thickness (in.)</td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>Integer</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>Moment of inertia of stiffener about its centroid (in.⁴)</td>
<td></td>
</tr>
<tr>
<td>²</td>
<td>Moment of inertia of stiffener about shell middle surface (in.⁴)</td>
<td></td>
</tr>
<tr>
<td>ływ</td>
<td>( \equiv imn/L ) (in.⁻¹)</td>
<td></td>
</tr>
<tr>
<td>ływ</td>
<td>( \equiv imnjk/L ) (dimensionless)</td>
<td></td>
</tr>
<tr>
<td>ływ</td>
<td>( \equiv \xi mnj/L ) (dimensionless)</td>
<td></td>
</tr>
<tr>
<td>j</td>
<td>Integer</td>
<td></td>
</tr>
<tr>
<td>J</td>
<td>Polar moment of inertia (in.⁴)</td>
<td></td>
</tr>
<tr>
<td>ływ</td>
<td>( \equiv jn/R ) (in.⁻¹)</td>
<td></td>
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<td>( \equiv jnk/R ) (dimensionless)</td>
<td></td>
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<tr>
<td>ły</td>
<td>( \equiv \xi nk/R ) (dimensionless)</td>
<td></td>
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<td>Integer</td>
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ix
\[ \begin{align*}
\xi & \quad \text{Ring center-to-center spacing (in.)} \\
L & \quad \text{Shell length (in.)} \\
m & \quad \text{Number of half-waves in axial direction} \\
M & \quad \text{Number of stringers on cylinder} \\
M_x, M_y, M_{xy} & \quad \text{Middle-surface moment resultants} \\
n & \quad \text{Number of waves in circumferential direction} \\
N & \quad \text{Number of rings on cylinder} \\
N_x, N_y, N_{xy} & \quad \text{Middle-surface stress resultants} \\
N_x & \quad \text{Externally applied axial load resultant (positive in comp.)} \\
Q & \quad \text{Galerkin "error functions"} \\
r & \quad \text{Subscript referring to ring quantities} \\
R & \quad \text{Radius of cylinder middle surface (in.)} \\
s & \quad \text{Subscript referring to stringer quantities} \\
t & \quad \text{Time (sec)} \\
u, v, w & \quad \text{Displacements in axial, circumferential, and radial directions} \\
U & \quad \text{Strain energy (in.-lb.)} \\
W & \quad \text{Dependent variable used in Runge-Kutta algorithm} \\
\tilde{W} & \quad \text{Dimensionless radial deflection of shell middle surface, } \tilde{W} = \frac{w}{w_{\text{initial}}} \\
x, y, z & \quad \text{Circular cylindrical coordinates with origin lying in middle surface of shell and oriented in the axial, circumferential, and radial directions, respectively} \\
\bar{z} & \quad \text{Distance from centroid of stiffener to shell middle surface (in.)} \\
Z & \quad \text{Batdorf parameter, } Z = L^2(1-v^2)/(Rh) \\
\delta_{ij} & \quad \text{Kroneker delta} \\
\delta(x-j\xi) & \quad \text{Dirac delta function} 
\end{align*} \]
(Double Kronecker delta)

\[ \delta_{ij} \delta_{jk} \]

Axial, circumferential and shear strains of shell middle surface

\[ \varepsilon_x, \varepsilon_y, \gamma_{xy} \]

Integer

\[ \nu \]

Poisson's ratio

\[ \zeta \]

Integer

\[ \rho \]

Density (lb-sec^2/in.^4)

Axial, circumferential and shear stresses

\[ \sigma_x, \sigma_y, \tau_{xy} \]

Indicates

\[ \sum_j \quad j=1 \]

\[ \sum_k \quad k=1 \]

\[ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \]

A subscript preceded by a comma indicates partial differentiation with respect to the subscript.

Primes indicate total derivatives with respect to x.
1.1 Survey of Cylindrical Shell Buckling

For the past decade, the aerospace and defense industry has been actively involved in studying the structural response of high-performance aircraft and missiles to blast-induced loadings. However, the structural response of a light aircraft (in the general aviation fleet) subjected to a crash loading has been largely ignored. Apparently, there are no current studies directed toward understanding the structural response of such aircraft to crash loadings. There are several reasons cited for this lack of analytical effort. These reasons are research cost, great variation in design of aircraft in the general aviation fleet, complexity of the crash-loading time-history delivered to the aircraft structure, and the lack of a procedure to treat the dynamic response of a complex structure.

The design of a more crashworthy aircraft will undoubtedly receive greater emphasis in the near future. In a recent article appearing in Time magazine [1], the high accident rate of the nation's 3,200 "third level" carriers was discussed. These include air taxis and commuter lines that usually fly smaller planes such as Cessna, Piper, Beechcraft and the like. According to Time, "last year 106
people died in third-level crashes. The accident death rate for every 100,000 hours flown is 1.31 for the third-levels, as compared with 0.09 for the nation's eleven first level trunk carriers and nine regionals."

This high accident rate was further documented in a FAA report [2] which claimed that the death rate for general aviation is seven times that of automobiles. While many of these accidents could be avoided by improved procedures, many of the fatalities could probably have been avoided by an aircraft designed to withstand a certain level of crash loading.

At the present time, crash impact loading is not considered in the design of light aircraft. The problem is understandably complex, due to the great number of unknowns, and the lack of any controlled experimental data. However, the problem should be manageable if the structure is broken down into representative elements and if the crash loading is simplified. After the response of a representative element is correctly modeled and understood, a more rational design could perhaps be proposed to resist the effects of crash loading on the structure.

As a point of departure, the basic structural element of a light aircraft is assumed to be a thin cylinder, internally stiffened by both stringers and rings (bulkheads). One typical loading to be expected in a crash would be a suddenly applied axial compression loading of short to medium duration. If one could mathematically model the dynamic behavior of a stiffened cylinder to such a loading, one could then begin to rationally design a "crashworthy" aircraft.

Beginning first with the static axial buckling studies of unstiffened thin-walled cylinders, one is impressed by the incredible
number of papers written over the past several decades on this subject alone. This fact is due in large part to the serious discrepancies between theory and experiment. Two authors who have provided extensive bibliographies on this subject are Hoff [3] and Stein [4].

The so-called classical linear buckling theory was probably first formulated by Timoshenko [5]. In this theory, effects due to nonlinear prebuckling and initial imperfections are neglected. Unfortunately, experimental results range from twenty per cent to eighty per cent of the values predicted by this linear small deflection theory. Investigators have attempted to resolve this discrepancy by including in their analysis one or more of the following general effects: in-plane boundary conditions, nonlinear prebuckling, nonlinear postbuckling and initial imperfections.

Apparently, the first effects considered were initial imperfections and nonlinear postbuckling. In 1934 Donnell [6] proposed a nonlinear finite-deflection theory, together with a consideration of initial imperfections present in the thin-walled cylinder. This theory was modified and refined by numerous authors including von Kármán and Tsien [7], Kempner [8], and Almroth [9]. In order to answer questions of convergence of various solutions in the postbuckling range, Hoff [10] solved the large displacement equations using a radial displacement expression containing 1100 terms. It should be noted, however, that all authors have implicitly ignored exact satisfaction of some boundary conditions. This is partly due to the nonlinear terms in the boundary conditions arising from the nonlinear strain-displacement relations. Hoff [10] stated that the assumption of $L/R > 1$ makes it possible
In-plane boundary conditions can play a significant role in obtaining theoretical buckling loads lower than the classical value. This has been pointed out by numerous authors [3,15-17]. Apparently, Hoff [3] was one of the first to document four different sets of boundary conditions which could be identified as "simple-support boundary conditions". Experiments performed in the ordinary tension-compression testing machine could be represented by several of these sets, depending upon the friction between testing machine and test specimen.

It is now generally agreed that, given a realistic set of boundary conditions, the main reason that theory and experiment did not agree in the past was due to initial imperfections in the test cylinder. The agreement obtained between theoretical (linear) results for a perfect shell with correct boundary conditions and experimental results from very nearly perfect specimens indicate the validity of the linear theory [19,20].

Theoretical buckling predictions for stiffened cylinders have not shown the wide disparity with experimental results that their unstiffened counterparts have shown. However, numerous authors have investigated the effects of in-plane boundary conditions, nonlinear prebuckling, and initial imperfections. Nonlinear prebuckling deformations have been shown to have only a small effect on the buckling load of stiffened cylinders [16,21-24]. The effect of in-plane boundary conditions on the buckling load of stiffened cylinders has also been investigated [16]. As before, the choice of in-plane simple support boundary conditions can make a difference on the magnitude of the buckling load.
However, as noted in reference [16], the influence of in-plane boundary conditions diminishes for internally stiffened shells with increasing values of stiffener eccentricity and area. Of course, a light-aircraft structure would have internally located stiffeners.

As in the unstiffened case, initial imperfections can make a considerable difference in the experimentally determined buckling load of stiffened cylinders. The bulk of this research has been conducted using the imperfection sensitivity concepts first introduced by Koiter [25] and expanded by numerous authors [26-30]. However, as pointed out by reference [24], the predicted regions of large imperfection sensitivity shift from one study to another, and have not been verified by experiment. Also, the predicted sensitivities appear to depend strongly on quantities such as torsional stiffness of stiffeners, which only slightly affect the classical buckling load [32]. Fortunately, the studies have shown that an internally stiffened shell is relatively imperfection insensitive, whereas an externally stiffened shell is imperfection sensitive [29,30].

The importance of two other effects, stiffener eccentricity and stiffener discreteness, have been explored by numerous investigators of stiffened shells. The stiffener eccentricity or one-sideness has a significant effect on the buckling strength, as demonstrated in various ways by many authors [32-40]. Stiffener discreteness was shown by Block, et al. [21] to have a significant effect on the buckling load, even for structures with many stiffeners. Block [21] used a finite difference approach and considered discrete rings only. His conclusion is in direct contrast with Singer's concluding remark in reference [41]
that "in ring stiffened shells under axial compression, the discreteness effect is always very small." Singer used a linear Donnell theory, and treated the stiffeners as linear discontinuities represented by the Dirac delta function. Despite the disagreement as to the importance of stiffeners in most light-aircraft, both stiffener eccentricity and discreteness should be included.

Linear buckling theory has been used by nearly all investigators in their study of stiffened cylinder buckling, primarily because stiffeners are closely spaced in most aerospace applications. However, as demonstrated by reference [43], linear theory is only applicable when the stiffeners are closely spaced. Again, this will not be the case for the light-aircraft structure considered in this study.

This problem of static buckling of stiffened and unstiffened cylinders has been extensively researched over the past 60 years. By contrast, dynamic buckling of axially loaded cylinders has been studied only for the past 15-20 years, and the amount of published work is only a small fraction of its static buckling counterpart.

The first studies were primarily experimental and phenomenological in nature. Some of the earliest work was published by Schmidt [44] and Coppa [45]. Both experimenters considered only unstiffened cylinders.

One of the first analytical treatments of dynamic buckling is credited to Volmir [48]. In 1957, he investigated the buckling of a shallow circular cylindrical panel subjected to a rapidly applied axial load. He used the large deflection shell equations (generally attributed to Donnell) and reduced them to equations in time only, using a
Galerkin procedure. By making certain restrictions on the time-dependent coefficients of the assumed radial deflection function, the system of equations was reduced to a single equation, which was then solved numerically. The final results were presented in the form of dimensionless axial load versus time. A closely related study was one by Agamirov and Volmir [47]. As in all subsequent work, both the longitudinal and circumferential inertia were neglected.

In 1962, using essentially the same procedure as Volmir used, Coppa and Nash [48] studied the dynamically loaded thin cylinder. They used a two-term Galerkin procedure, in which the terms approximated the familiar diamond buckle shape. Rotatory and axial inertia terms were neglected. The assumed two-term solution contained a guess as to the axial and circumferential buckle wavelength. The procedure was repeated over a range of axial and circumferential wavelengths to find the combination that gave the lowest buckling load. A constant rate of axial end shortening was used as the dynamic loading mechanism. Only qualitative correlations with related experiments were made.

In 1964 Roth and Klosner [49] studied the same problem previously investigated by Coppa and Nash. Roth and Klosner constructed appropriate kinetic and potential energy relations. They used the middle-surface nonlinear strain-displacement relations for thin circular shells which was based on the work of Donnell [6]. Applying Hamilton's principle, they derived the equations of motion and appropriate boundary conditions. Next, they rewrote their governing equations of motion using an Airy-type stress function, and obtained the same equations as Coppa and Nash. Using a four-term radial-deflection function,
together with the Galerkin procedure, they reduced their problem to a set of four ordinary nonlinear differential equations. This final set was solved using Runge-Kutta procedures. As is the case with all prior large deflection theories, the assumed deflection modes did not exactly satisfy all of the boundary conditions. Also, no correlations with any experiments were made.

Lindberg [50] used a linear, small-deflection theory instead of a large deflection theory, to calculate the growth of normal modes of cylinders under axial impact. An inspection of Lindberg's experimental setup quickly reveals that he was using an impulsive rather than a step function loading. His shells were free at the end opposite the impact end. As a result, the compressive impact stress had a duration (at the impact end), which at most was equal to the transit time of the longitudinal stress wave up and down the shell.

In a recent dissertation, Howell [51] studied the transient response of stiffened cylinders to an impact load. He used a linear theory and smeared the effect of the stiffeners over the cylinder surface. However, he applied the impact to a point on the cylinder surface between stringers rather than to the ends of the cylinder. Also, his time duration of loading and subsequent transient response measurements indicate a wave propagation-type of study rather than a buckling analysis.

The amount of experimental data on the dynamic buckling of unstiffened shells, other than very early work [44,45], has been sparse. The most recent was that of Tennyson [52]. To date, no experimental data on the dynamic buckling of stiffened cylindrical shells have been found.
The influence of damping on the dynamic stability of shells has been investigated recently by several authors. Mescall and Tsui [53] found that damping always increased the critical dynamic buckling load for the thin cylinders, cones, and spheres considered in their analysis. Consequently, it should be expected that a dynamic buckling analysis that neglected damping would yield conservative results.

1.2 Research Objectives

The major objective of this research is to develop a dynamic buckling analysis capable of predicting the dynamic response and buckling load of a stiffened, thin, circular cylindrical shell under the action of a suddenly applied step axial-loading pulse. This pulse will be similar to that experienced by a light aircraft during a crash on take-off or landing. The cylinder will be stiffened with widely spaced rings and stringers, which will be representative of modern light-aircraft fuselage structures. The stiffeners will be considered as discrete elements with eccentricity due to their internal location.

In order to accomplish the research objectives it has been necessary to devise a number of new approaches and methods. The following specific original contributions are noted:

1. The first discretely stiffened cylinder buckling analysis in which nonlinear rather than linear strain-displacement relations are used.

2. The first cylinder buckling analysis to solve the resultant nonlinear algebraic and differential equations using a modified Gauss-Jordan technique in conjunction with a Runge-Kutta algorithm.
3. The first dynamic buckling analysis with the capability of handling any number of assumed deflection terms, limited only by the available computer capacity.

4. The first dynamic buckling analysis of a discretely stiffened cylinder.

5. The first dynamic analysis of any kind to use the Dirac delta function to account for stiffener discreteness.

6. The first dynamic buckling analysis to contain the effects of stiffener eccentricity or one-sideness.
CHAPTER II

FORMULATION OF THEORY

2.1 Method of Analysis

An energy approach is used to facilitate the writing of compatible governing equations for the unstiffened cylinder, stringers and rings, which all buckle as an unit. This procedure allows for treatment of stringers and rings as discrete elements, rather than the usual orthotropic "smeared" analysis. Appropriate expressions for the potential and kinetic energies of the unstiffened cylinder, stringers, and rings are formulated and presented in Appendix A. To allow for finite deflections of the cylinder during the buckling process, the appropriate nonlinear strain-displacement relations, as suggested by Donnell [6], are employed. Also, the strain-displacement relations for the rings and stringers are related to the mid-surface unstiffened shell displacements. Then, in Appendix B, Hamilton's principle is used to obtain the governing nonlinear differential equations of motion and the appropriate boundary conditions which govern the prebuckling and buckling of a stiffened cylinder. To obtain the prebuckling and buckling equations, the axial, circumferential and radial displacements of the shell (u,v, and w) are assumed to be separable into two parts as follows:

\[ u(x,y,t) = u_A(x) + u_B(x,y,t) \]
\[ v(x,y,t) = v_A(x) + v_B(x,y,t) \]
\[ w(x, y, t) = w_A(x) + w_B(x, y, t) \]  

(2-1)

The subscript \( A \) denotes the axisymmetric prebuckling displacement; the subscript \( B \) denotes the time-varying unsymmetric buckling displacement.

The prebuckling equations and boundary conditions are obtained by substitution of the axisymmetric displacements in Equations (2-1) into Equations (B-10). In a like manner, the buckling equations and boundary conditions are obtained by substituting Equations (2-1) into Equations (B-11), and subtracting out the previously obtained prebuckling identities.

To solve the buckling equations, the prebuckling quantities (subscript \( A \)) are first determined directly. Galerkin's technique is then applied to the three buckling equations of motion. The result of this operation is a set of simultaneous, nonlinear, ordinary differential equations. The coupled equations are then solved by means of a Runge-Kutta technique with the aid of a digital computer.

2.2 Hypotheses

All of the following assumptions are implicit in the analysis:

1. The circular cylindrical shell and all stiffeners remain in the linear elastic range during buckling.

2. The cylinder undergoes a classical axisymmetric prebuckling deformation during axial loading.

3. Initial imperfections in the stiffened cylinder are neglected.

4. The Kirchoff-Love hypothesis is used for the shell; thus,
it is assumed to have a wall thickness which is small compared to its radius.


6. The stiffeners are discretely located along the length and circumference of the cylinder, and the width of the stiffeners is small compared to the distance between them.

7. In-surface and rotatory inertia effects in the stiffened cylinder are neglected.

8. Stiffeners behave as beam elements, and displacements vary linearly across stiffener depth. This implies that the stiffener depth is small compared to the radius of the shell middle surface.

9. The total circumferential arc length of the ring is approximately the same as that of the middle surface of the shell.

10. The stiffeners are rigidly attached to the shell.

11. The stiffeners are symmetrical with respect to a normal from the shell middle surface passing through the stiffener centroid.

12. The boundary conditions at the shell ends are not necessarily satisfied exactly, as assumed by references [6,7,8, 10,45,48,49].

13. The cylinder has the characteristic buckling behavior of
a "long" cylinder. Thus, the Batdorf parameter is greater than thirty.

14. Effects of axial-wave propagation do not influence the wavelengths at which the buckles form.

15. All material damping, thermal, and initial-stress effects are neglected.

16. The shell and stiffener materials are homogeneous and isotropic.

2.3 Prebuckling Equations

The general nonlinear equations of motion governing the stiffened cylinder buckling are derived in Appendices A and B. The appropriate equations of motion and boundary conditions governing the cylinder prebuckling are obtained by substituting the A-subscript portion of Equations (2-1) into Equations (B-10) and (B-11). Primes indicate total differentiation with respect to \( x \). The prebuckling equations become

\[
N'_{xA} = 0 \tag{2-2}
\]

\[
N'_{xyA} = 0 \tag{2-3}
\]

\[
-M''_{xA} + \left( N_{yA}/R \right) = 0 \tag{2-4}
\]

The appropriate boundary conditions at the cylinder ends are:

\[
N_{xA} + \hat{N}_x = 0 \quad \text{or} \quad u_A = 0 \tag{2-5}
\]

\[
N_{xyA} = 0 \quad \text{or} \quad v_A = 0 \tag{2-6}
\]

\[
M_{xA} + \hat{N}_x \dot{\theta} = 0 \quad \text{or} \quad w'_A = 0 \tag{2-7}
\]

\[
M'_{xA} = 0 \quad \text{or} \quad w_A = 0 \tag{2-8}
\]
In the above equations the following definitions apply:

\[ N_{xA} = \frac{E}{1-\nu^2}(u'_A + \nu u'_A) + \sum_k \delta(y-kd)E_A s_s u'_A \]  
\[ N_{yA} = \frac{E}{1-\nu^2}(R^{-1} w_A + \nu u'_A) + \sum_j \delta(x-j\lambda)E_A R^{-1}w_A \]  
\[ N_{xyA} = Gv'_A \]  
\[ M_{xA} = -\sum_k \delta(y-kd)E_A s_s s'_A \]  

Equations (2-2) and (2-5) require that \( N_{xA} \) must equal \( -N_x \).
This condition satisfies boundary condition (2-5). In a similar manner, Equations (2-3) and (2-6) require that there be no applied shear. Thus, if \( N_{xy} \) is equated to zero, boundary condition (2-6) will be satisfied. To satisfy the third boundary condition (2-7), \( w_A \) is set equal to a constant. By making \( w_A \) equal to a constant, a classical prebuckling membrane state is assumed for the stiffened cylinder. Finally, the fourth boundary condition (2-8) is satisfied by setting \( M_{xA} \) equal to zero.

For later use in the formulation of the buckling equations, the following prebuckling identities will be used:

\[ N_{yA} = 0 \]  
\[ N_{xA} = 0 \]  
\[ N_{xyA} = 0 \]  

2.4 Buckling Equations

The governing equations and boundary conditions of a stiffened cylinder with discrete rings and stringers can be obtained by substituting
Equations (2-1) into Equations (B-10) and (B-11), subtracting out identities (2-2) through (2-4) and making use of Equations (2-13) through (2-15). The buckling equations become:

\[ N_{xB,x} + N_{xyB,y} = 0 \]  
\[ N_{yB,y} + N_{xyB,x} = 0 \]  
\[ -M_{xB,xx} - 2M_{xyB,xy} - M_{yB,yy} + R^{-1}N_{yB} + N_{xB,xx} - N_{xB,xx}^B \]
\[ -N_{yB,yy} - 2N_{xyB,B,xy} + \rho w_{B,tt} + \sum_j^r A_j x_{B,B,tt} \delta(x-j\lambda) \]
\[ + \sum_k^s k_{B,B,tt} \delta(y-k\delta) = 0 \]  

The boundary conditions at the stiffened cylinder ends become

\[ N_{xB} = 0 \quad \text{or} \quad u_B = 0 \]  
\[ N_{xyB} = 0 \quad \text{or} \quad v_B = 0 \]  
\[ M_{xB} = 0 \quad \text{or} \quad w_B,x = 0 \]  
\[ M_{xB,x} + 2M_{xyB,y} - N_{w,B,x} + N_{xB,w}_B,x + N_{xyB,w}_B,x = 0 \quad \text{or} \quad w_B = 0 \]

The definitions for the various buckling terms may be found in Appendix B, Equations (B-4) through (B-9), where a subscript B is added to each term and displacement.

### 2.5 Application of Galerkin's Method

The buckling Equations (2-16) are a set of coupled, nonlinear partial differential equations. To reduce them to a set of ordinary differential equations, the Galerkin weighted average method will be used.
The Galerkin method is an approximate assumed-mode method similar to the Rayleigh-Ritz method when applied correctly [56]. The method should be applied only to the equations of motion which arise directly from application of either Hamilton's principle or Newton's second law. If the equations are escalated by differentiation and combined, the method often yields incorrect results [57].

To apply the Galerkin method, solutions are assumed for the unknown variables in the equations of motion. In general, the solutions will not satisfy the equations of motion exactly. In order to minimize the error, the assumed solutions are inserted into the equations of motion and a non-zero "error function" is generated. Each error function is then orthogonalized with respect to the assumed solution functions. This orthogonalization process will give rise to a set of equations which can be solved for the unknown solution function coefficients. The resulting set of equations can be either linear or nonlinear, algebraic or differential, depending on the equations of motion and the form of the assumed solution.

To apply Galerkin's method to the buckling equations, the following series of assumed modes will be used

\[ u = \sum_{i} \sum_{j} u_{ij}(t) \sin\left(\frac{i\pi x}{L}\right) \cos\left(\frac{j\pi y}{R}\right) \]  
(2-18a)

\[ v = \sum_{i} \sum_{j} v_{ij}(t) \cos\left(\frac{i\pi x}{L}\right) \sin\left(\frac{j\pi y}{R}\right) \]  
(2-18b)

\[ w = \sum_{i} \sum_{j} w_{ij}(t) \cos\left(\frac{i\pi x}{L}\right) \cos\left(\frac{j\pi y}{R}\right) \]  
(2-18c)

Most researchers [6-9,45,48,49] who used a nonlinear shell theory to study buckling resorted to one or more terms of the above
series. As noted in Chapter I, these researchers were unable to satisfy all of the boundary conditions. In line with hypothesis 14, Equations (2-18) satisfy boundary conditions (B-11a), (B-11b), and (B-11c), but not boundary condition (B-11d).

Substitution of Equations (2-18) into the buckling Equations (2-16) yields a set of "error functions", \( Q_x, Q_y, Q_z \), respectively. The orthogonalization process may be formulated as

\[
\int_0^{2\pi R} \int_0^L Q_x(u,v,w) \sin(\xi \pi x/L) \cos(\xi ny/R) \, dx \, dy = 0 \\
\xi = 1,2,3 \ldots
\]  
(2-19)

\[
\int_0^{2\pi R} \int_0^L Q_y(u,v,w) \cos(\xi \pi x/L) \sin(\xi ny/R) \, dx \, dy = 0 \\
\xi = 1,2,3 \ldots
\]  
(2-20)

\[
\int_0^{2\pi R} \int_0^L Q_z(u,v,w) \cos(\xi \pi x/L) \cos(\xi ny/R) \, dx \, dy = 0 \\
\xi = 1,2,3 \ldots
\]  
(2-21)

Because \( Q_x \) and \( Q_y \) are not dependent on time, Equations (2-19) and (2-20) will each yield a set of nonlinear algebraic equations. Since \( Q_z \) is dependent on time, Equation (2-21) will yield a set of nonlinear ordinary differential equations in time. The resultant equations after some simplification are listed below, and the details of the Galerkin method are found in Appendix C.

\[
\delta_{ij} \sum_{k} \left[ I_{1}^{2} \delta_{ij} + \frac{1}{2} (1-v) I_{2} J_{j}^{2} \delta_{ij} + (1-v^2) (\pi REh)^{-1} I_{1}^{2} S_{1} A_{s} \cos J_{k} \cos J_{\xi} \right] \\
+ \Delta_{ij} \sum_{k} \left[ I_{1}^{2} \delta_{ij} + \frac{1}{2} (1+v) I_{2} J_{j}^{2} \delta_{ij} \right] \\
+ (1-v^2) (\pi REh)^{-1} I_{1}^{2} S_{1} A_{s} \sum_{k} \left[ I_{2} S_{1} \cos J_{k} \cos J_{\xi} \right] \\
-2 I_{2} \sum_{k} \left[ I_{1} J_{j}^{2} \delta_{ij} \right] S_{1} \cos 2J_{\xi} \\
+ (1-v^2) (2Eh\pi R)^{-1} I_{1}^{2} S_{1} A_{s} \cos 2J_{k} \cos J_{\xi} = 0
\]  
(C-1)
\[ \Delta_1 \Sigma \varepsilon_{ij} \left\{ 1 + (1 + \nu) \frac{I_{ij}}{I_{jj}} + \frac{1}{2} \delta_{ij} \right\} \sum_i \left[ J_{ij}^2 \delta_{ij} + \frac{1}{2} (1 - \nu) I_{ij}^2 \delta_{ij} \right] + 2 (1 - \nu^2) (EhL)^{-1} \sum_j \sum_i J_{ij}^2 \left( R_1 - 2 \delta_{ij} \right) + 2 (1 - \nu^2) \frac{EhL}{I_{ij}} \left[ (R_1 - 2 \delta_{ij}) \sum_i J_{ij}^2 \right] + 4 I_{ij}^2 \delta_{ij} + (1 - \nu^2) (EhL)^{-1} J_{ij}^2 \left( R_1 - 2 \delta_{ij} \right) \sum_i J_{ij}^2 \left( R_1 - 2 \delta_{ij} \right) = 0 \] 

(C-2)
21

\[ \sum_{i,j} \psi_{ij} w_{ij} (J_j^2 + T_{ij}) + \delta_{ij} L^{-1} \sum_{i,j} \psi_{ij} J_j^2 E \rho A \cos^2 I_j \cos I_z + \sum_{i,j} \rho A (2/L)^2 \cos I_j \cos I_z = 0 \]

(C-3)

2.6 Numerical Solution

The buckling equations consist of a set of 2k nonlinear algebraic equations and a set of k nonlinear differential equations, where k represents the number of terms used in the assumed modes. Because of the complex nature of these sets of equations, no closed form solution is known to exist. However, with given initial conditions, the above sets of equations should be numerically solvable if we use a Gauss-Jordan technique on the algebraic equations and a Runge-Kutta technique on the nonlinear differential equations. In addition to the initial conditions on the dependent variables, the solution of the equations depends also on the externally applied axial load resultant, \( \hat{N}_x \), and the circumferential wave number n. Of course, it will be necessary to vary n over a range of values to find the minimum critical load, since this will be the true dynamic buckling load of the stiffened cylinder.

An inspection of the set of algebraic Equations (C-1) and (C-2) reveals that the nonlinearity is in the radial deflection terms, \( w_{ij}(t) \). Thus, if \( w_{ij}(t) \) is specified initially, Equations (C-1) and (C-2) can be solved by a suitable technique for solving simultaneous linear equations. The results, together with the value of \( w_{ij}(t) \), can be substituted into the set of nonlinear differential equations.
Using a fourth order Runge-Kutta algorithm and advancing it \( \frac{1}{4} \) step, a new set of \( w_{ij}(t) \) can be obtained. The new \( w_{ij}(t) \) set can be resubstituted into Equations (C-1) and (C-2) and the process repeated three times to yield one complete step.

Since most Runge-Kutta algorithms are written specifically for first-order differential equations, it is necessary to convert the \( k \) second-order differential equations into \( 2k \) first-order equations by a simple change of variable. The following fourth-order algorithm attributed to Kutta is used in the analysis [58]:

\[
\begin{align*}
    w_{i+1} &= w_i + \frac{(g/6)(k_1 + 2k_2 + 2k_3 + k_4)} \quad \text{where} \\
    k_1 &= f(t_i, y_i) \\
    k_2 &= f(t_i + \frac{1}{2}g, y_i + \frac{1}{2}gk_1) \\
    k_3 &= f(t_i + \frac{1}{2}g, y_i + \frac{1}{2}gk_2) \\
    k_4 &= f(t_i + g, y_i + gk_3)
\end{align*}
\]

In order to begin the numerical procedure, it is necessary to choose representative initial values for the set of dependent variables, \( w_{ij}(t) \). Physically, this corresponds to giving the shell a slight initial radial deflection or velocity. This procedure was first suggested by Roth and Klosner [49].

Finally, a criterion for identifying dynamic buckling must be established. An inspection of the representative buckling curves found in Chapters III and IV make it apparent that at some critical load the structure is diverging from its initial displacement rather than returning to its equilibrium (or zero displacement) position. Thus, as
suggested in references [49] and [59], the buckling load can be defined as the load at which a large increase in the amplitude of the deflection occurs. Of course, as mentioned earlier in this section, it will be necessary to plot the buckling load as a function of circumferential wave number, \( n \), and then pick the lowest resultant load as the true buckling load.

Further details concerning the numerical techniques employed may be obtained by consulting Appendices IV and V. One particularly important technique, which allowed a general formulation of the governing equation coefficients independent of the number of assumed mode terms, will be illustrated here. It should be noted that the coefficients of the various terms in Equations (C-1) through (C-3) can be written as functions of the assumed mode subscripts \( i, j, \zeta \) and \( \xi \). However, the identification of parameters by four rather than by two subscripts becomes very cumbersome when attempting to construct a general computer program. This problem was solved by making use of an algorithm suggested in reference [60]. The algorithm permits the rewriting of each coefficient in terms of only two subscripts \( P \) and \( Q \), and then "decoding" the subscripts as necessary during the numerical procedure. If a typical coefficient is represented by \( A_{\zeta \xi ij} \), the algorithm will convert it to \( A_{QP} \) as follows:

\[
i = P - \left( \frac{P-1}{i^*} \right)_{i^*} \\
j = 1 + \left( \frac{P+1}{i^*} \right)_{i^*} \\
\zeta = Q - \left( \frac{Q-1}{i^*} \right)_{i^*}
\]
$\xi = 1 + \left( \frac{Q+1}{i^*} \right)_T$

where $i^*$ represents the maximum value of the $i$ coefficient and the symbol $(\_)_T$ represents the operation of integer truncation. An example of this calculation for a four-term assumed deflection mode is shown in Table 2.1. The operation of this algorithm can be observed in Appendix V.

**Table 2.1 Coefficient Algorithm**

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\xi$</th>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>$j$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1 1</td>
<td>1 1</td>
<td>(Q=1,P=1)</td>
<td>(Q=1,P=2)</td>
<td>(Q=1,P=3)</td>
<td>(Q=1,P=4)</td>
<td></td>
</tr>
<tr>
<td>2 1</td>
<td>2 1</td>
<td>(Q=2,P=1)</td>
<td>(Q=2,P=2)</td>
<td>(Q=2,P=3)</td>
<td>(Q=2,P=4)</td>
<td></td>
</tr>
<tr>
<td>1 2</td>
<td>3 1</td>
<td>(Q=3,P=1)</td>
<td>(Q=3,P=2)</td>
<td>(Q=3,P=3)</td>
<td>(Q=3,P=4)</td>
<td></td>
</tr>
<tr>
<td>2 2</td>
<td>4 1</td>
<td>(Q=4,P=1)</td>
<td>(Q=4,P=2)</td>
<td>(Q=4,P=3)</td>
<td>(Q=4,P=4)</td>
<td></td>
</tr>
</tbody>
</table>
CHAPTER III
EVALUATION OF THEORY

3.1 Comparison with Static Buckling

During the literature search no studies were found that dealt exclusively with sparsely stiffened cylinders. Consequently, most studies used a linear buckling theory in conjunction with a smeared stiffener analysis. In the smeared analysis, the stiffener properties are averaged over the appropriate dimension by dividing by the stiffener spacing. In the more exact analysis contained in this investigation, the stiffener properties are considered as discrete and a Dirac delta function is used to handle the discreteness in the analysis. In order to reduce the present discrete analysis to a smeared one for comparison, the typical results in Table 3.1 can be used.

Table 3.1 Comparison of Smeared and Discrete Terms

<table>
<thead>
<tr>
<th>Typical Term</th>
<th>SMEARED</th>
<th>DISCRETE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Result After Galerkin Procedure</td>
<td>$-\pi u_{ij} \int I_s^2 \pi R \frac{E A_s}{d}$</td>
<td>$-\pi u_{ij} \int I_s^2 \pi R \frac{E A_s}{d}$</td>
</tr>
<tr>
<td>Equivalent Terms</td>
<td>$\pi R/d$</td>
<td>$\sum_k \cos J_k \cos J_\xi$</td>
</tr>
</tbody>
</table>
Block, et al. [38] used a linear smeared analysis and a one term assumed mode approach. This procedure reduced the static buckling problem to a single algebraic equation containing the applied axial load and the circumferential and axial wave numbers. In order to compare Block's final equation with the present results, Equations (C-1), (C-2) and (C-3) were linearized, the stiffeners were smeared using results analogous to those presented in Table 3.1, the inertia terms were removed, and only the first displacement terms were included. When the resulting equations were reduced, they compared exactly to final Equation (15) of reference [38].

In a later reference, Block [21] completed a linear classical buckling solution for a cylinder with smeared stringers and discrete rings. He used a Dirac delta approach to model the discrete rings. His assumed modes consisted of a single circumferential term together with a large number of axial terms, all of which satisfied the boundary conditions of the classical simple support. Again, a term-by-term comparison with the present analysis yielded analogous results.

Unfortunately, after an extensive literature search, no theoretical or experimental work was found which dealt with dynamic buckling of stiffened cylinders. The only dynamic studies were theoretical studies of unstiffened cylinders, and these are treated in the next section. However, to obtain an approximate comparison between the present dynamic buckling analysis and previously published static analyses, a dynamic buckling analysis was conducted using the shell parameters reported in references [21] and [38]. In these references the cylinder which was used had closely spaced stiffeners representative of the
large-diameter liquid rocket booster structures. Details of the structure and results of the comparison are contained in Fig. 3.1. The dynamic analysis was made using one circumferential and three axial terms. Since the dynamic loading consisted of a step pulse loading of infinite time duration, the predicted buckling load should roughly correspond to the predicted static buckling load. As observed from Fig. 3.1, except for the smeared static analysis, the dynamic load is generally above the corresponding static load. The difference can be attributed to the inclusion of the radial inertia and the nonlinear terms in the dynamic analysis.

3.2 Comparison with Dynamic Buckling

The only dynamic buckling analyses found in the literature were for unstiffened shells exclusively [46,48,49], and were not experimentally verified. Generally, a nonlinear buckling theory together with a stress function approach was used to reduce the problem to one equilibrium and one compatibility equation. These equations were solved by the Ritz-Galerkin method and then integrated numerically. Because of the sensitivity of the unstiffened cylinder to initial imperfections, the analyses concentrated on the influence of these imperfections on the buckling load. Since initial imperfections are not included in the present stiffened cylinder analysis, it was necessary to extend the unstiffened cylinder analyses for comparison purposes. In order to accomplish this, the equations of reference [49] were solved numerically for the case of zero imperfections, and plotted in Fig. 3.2. As a comparison, the same unstiffened cylinder was treated using the present analysis, and the results are shown in Fig. 3.2. Because of the
**STIFFENED CYLINDER OF REF. [21]**

- \( L = 200 \text{ in.} \)
- \( h = 0.1 \text{ in.} \)
- \( R = 200 \text{ in.} \)
- 500 Stringers

**Figure 3.1 Static Buckling Comparison**

- ▼ Ref. [21], Smeared Analysis
- ○ Present Analysis, 3 terms
- □ Ref. [21], Discrete Ring Analysis
- △ Ref. [21], Finite Difference Analysis

**Number of Rings**
different radial deflection functions used in reference [49], exact agreement would not be expected. However, the curves are in good agreement.
Solid Curve - Present Analysis
Dotted Curve - Ref. [49] Analysis

Figure 3.2 Dynamic Buckling Comparison, Unstiffened Shell
CHAPTER IV
RESULTS

4.1 Representative Aircraft Structure

Because of the large number of geometric and physical parameters involved in this investigation, it is impractical to present results of a general nature. However, it is of value to present some computed results for stiffened cylinders which are representative of light aircraft fuselage structures. In order to determine the geometric parameters of a typical light aircraft structure, four light aircraft companies were contacted. Based on the data they provided, Table 4.1 was constructed. It must be mentioned that the data shown in Table 4.1 is approximate only. It reflects engineering judgment made by the author, and not by personnel of the respective aircraft companies. Many of the actual geometric configurations have been simplified for this analysis. For example, actual rings or bulkheads are seldom uniform, of constant thickness, or without "lightening" holes to conserve weight. In this analysis, all rings and stringers are considered uniform, of constant thickness, and evenly spaced along the inside of the cylinder. No provision has been made for cutouts in the shell for doors or windows. However, despite these limitations, it is felt that a useful first-order parametric study can be made based on the approximate quantities contained in Table 4.1.
<table>
<thead>
<tr>
<th>Acft. Type</th>
<th>Cabin Length (in.)</th>
<th>R (in.)</th>
<th>h (in.)</th>
<th>M</th>
<th>A_s (in^2)</th>
<th>I_os (in^4)</th>
<th>J_s (in^4)</th>
<th>( \bar{s} ) (in.)</th>
<th>A_r (in^2)</th>
<th>I_or (in^4)</th>
<th>J_r (in^4)</th>
<th>( \bar{z} ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Single Engine</td>
<td>112</td>
<td>25</td>
<td>0.30</td>
<td>33</td>
<td>0.032</td>
<td>0.006</td>
<td>0.0049</td>
<td>0.308</td>
<td>0.094</td>
<td>0.088</td>
<td>0.039</td>
<td>0.763</td>
</tr>
<tr>
<td>B Single Engine</td>
<td>110</td>
<td>25</td>
<td>0.25</td>
<td>22</td>
<td>0.065</td>
<td>0.015</td>
<td>0.0059</td>
<td>0.466</td>
<td>0.142</td>
<td>0.320</td>
<td>0.120</td>
<td>1.210</td>
</tr>
<tr>
<td>C Twin Engine</td>
<td>168</td>
<td>34</td>
<td>0.30</td>
<td>27</td>
<td>0.046</td>
<td>0.013</td>
<td>0.0078</td>
<td>0.413</td>
<td>0.165</td>
<td>0.264</td>
<td>0.102</td>
<td>1.013</td>
</tr>
<tr>
<td>D Twin Engine</td>
<td>217</td>
<td>31</td>
<td>0.34</td>
<td>30</td>
<td>0.035</td>
<td>0.003</td>
<td>0.0022</td>
<td>0.187</td>
<td>0.133</td>
<td>0.461</td>
<td>0.159</td>
<td>1.513</td>
</tr>
</tbody>
</table>
An inspection of Table 4.1 reveals that all four aircraft have approximately the same stiffener area, but that some of the other stiffener quantities vary by as much as 50 per cent. This is due to different geometrical shapes of the various stiffeners. For example, the stringers in aircraft C are channel-sections, while those in aircraft D are L-sections. However, the effect of variation of these parameters was shown to be small, as documented in Section 4.3.

Based on the data contained in Table 4.1, a representative set of stiffened cylinder parameters was chosen. To begin the study a 32-inch radius cylinder was used. The skin thickness was 0.04 inches and the largest spacing between rings was 32 inches. Thirty equally spaced stringers, together with the stiffener properties of aircraft C were used as a starting point. These various geometrical parameters were then varied, and the influence on the dynamic buckling behavior was noted.

In light of assumptions twelve and thirteen regarding the cylinder boundary conditions, the cylinder radius, thickness, and length were never such that the Batdorf parameter, Z, was less than 30. In fact, even if the stringers were smeared over the surface of the shortest cylinder, the smallest value of Z was 190. Thus, the expected buckling behavior of the stiffened cylinder would be that of a "long" cylinder.

Due to the relatively wide spacing between the rings on general-aviation light aircraft, it was felt that local buckling of the shell between the rings would be the mode of failure, rather than general instability. Local buckling between rings is defined as the buckling
mode in which the rings have little or no radial buckling deformation, and the cylinder buckles between the rings. General instability is defined as the buckling mode in which the rings deform radially and the cylinder wall and rings buckle as a composite wall. Since discreteness of the stiffeners has been accounted for, the present analysis can handle either type of instability.

Since an actual aircraft fuselage consists of a long cylinder with a number of repeating bays between the rings, it is reasonable to assume that buckling of the entire cylinder can be studied by considering a representative smaller portion or subshell of the cylinder. If this assumption is valid, fewer assumed mode terms in the deflection function would probably be needed to correctly model the dynamic behavior of the structure. Of course, fewer terms would mean a saving of computer time.

To validate the above assumption, the long cylinder shown in Fig. 4.1 was divided into three subshells. Subshell I consisted of one ring and an L/R of two. Subshell II consisted of two rings and an L/R of three. Subshell III consisted of the stringers between two rings and an L/R of one. A buckling analysis was performed on subshells I and II, and the results are shown on Fig. 4.2. Although the curves diverge somewhat for higher values of n, the lowest buckling load for both subshells was approximately the same and occurred for the same wave number, indicating the similarity of buckling shapes. As expected, subshell II required more terms than subshell I for convergence; subshell I required three terms while subshell II required five terms. It should be mentioned that the curves drawn through the calculated points on the various graphs are for ease in identifying the lowest buckling load.
Figure 4.1 General Aircraft Structure
The curves are actually discrete points since \( n \) can take on integer values only.

Subshell III was used as a further check on the convergence of the curves presented in Fig. 4.2, and as a demonstration of local buckling of the subshell between the rings. When a stiffened cylinder undergoes local buckling between the rings, the rings can only exert a torsional restraint on the shell. Thus, a lower bound on the buckling load can be obtained by considering the stringer stiffened portion between the rings. This portion is shown as subshell III in Fig. 4.1, and the corresponding buckling curve is plotted on Fig. 4.2. The closeness of the subshell I and II curves to this lower bound curve demonstrates that local buckling between the rings is the mode of buckling for the representative cylinder considered in this analysis.

The effect of discreteness on the stringer stiffened shell can be seen in the region of \( n = 15 \). Since the shell consists of 30 stringers, \( n = 15 \) represents the case of local buckling between the stringers. However, when rings are used in addition to stringers, the buckling at \( n = 15 \) does not represent the lowest buckling load, as evidenced by the subshell I and II curves.

Based on this preliminary case, it is reasonable to assume that local buckling between the rings will be the dominant instability mode. Local buckling between the stringers will occur at some higher load and thus not be significant. Also, because of the great increase in computer running time and core space for each additional assumed mode term, the number of assumed mode terms in the circumferential direction will be limited to one. The number of terms in the axial direction will be
Figure 4.2 Comparison of Buckling Load for Different Cylinder Regions
increased until convergence is obtained.

4.2 Dynamic Considerations

As stated in Section 1.2, it is believed that this study represents the first dynamic buckling analysis of a discretely stiffened cylinder. Because of the relative newness of the dynamic buckling field, general procedures and criteria for identifying dynamic buckling have not been generally established as they have been for static buckling. Consequently, the results presented in this and subsequent sections should be considered exploratory in nature. Unlike static buckling analyses, where the lowest buckling load results from the solution of an eigenvalue problem, the shell deflection-time history curves must be inspected to determine buckling. Next, the buckling load must be cross-plotted as a function of the wave number \( n \) to determine the lowest buckling load. Because of the many parameters that must be varied, and because of the large amount of computer time required for the analysis, the following general calculational procedure was adopted:

**Computer Run #1**

a. Allow \( n \) to vary from one to twenty.

b. Establish a best guess as an upper and lower bound on the load, \( \bar{F} \).

c. Establish an incrementing value for \( \bar{F} \), usually beginning with 0.5.

d. Begin with a Runge-Kutta step increment of 0.1 msec and terminate computation after \( t \) reaches 5.0 msec.

**Computer Run #2**

a. Narrow region of \( n \) to those values yielding the lowest values of \( \bar{F} \).
b. Narrow the upper and lower bounds and reduce the size of the incrementing step.

Examining the smoothness of the maximum deflection curves to determine whether the Runge-Kutta increment is adequate.

**Computer Run #3**

a. Identify and input the critical value of n causing the lowest buckling load.

b. Narrow the upper and lower bounds of $\bar{F}$ and reduce the size of the incrementing step.

c. Run the analysis for $t>50$ msec to positively identify the lowest buckling load, and the time to the first maximum of the lowest buckling load.

d. Repeat the entire procedure with more terms in the assumed deflection function to determine convergence of the results.

Although the analysis and computer program allows one to consider an axial end load varying arbitrarily with time, the present study has been limited to loads having a step function variation in time. Since actual crash load durations are on the order of 100 milliseconds [61], the step function time duration was at least 100 milliseconds.

In order to start the numerical analysis it was necessary to establish the initial displacement and velocity conditions on the radial assumed mode terms. It was decided to set the initial deflection of the lowest mode equal to 0.001 which represented an initial displacement of less than five per cent of the shell thickness. All other initial conditions were set equal to zero. This procedure was suggested by reference [49] for studying dynamic buckling of unstiffened cylinders.
The initial deflection was varied with no change in buckling behavior, thus indicating the insensitivity of buckling to initial conditions.

As mentioned in Section 2.6, the buckling load for a particular wave number is defined as the load at which a large increase in the amplitude of the deflection occurs. The actual buckling load for a particular structure can then be determined from a plot of load versus wave number, such as shown in Fig. 4.2.

By considering the behavior of the lowest mode only as a function of time, it can be seen from Fig. 4.3 that there are two types of shell response. If the load is below the critical buckling load, the shell oscillates around its original equilibrium position. Since damping is neglected in the analysis, this oscillation would rapidly damp out in an actual shell. If the load is at or above the critical buckling load, there is a radical increase in the maximum shell deflection. If the load is increased further, the maximum radial deflection increases, and the time to maximum deflection decreases. This type of behavior was previously observed and documented for dynamic buckling of unstiffened shells by Roth and Klosner [49]. They used the same instability criterion by investigating the response of the lowest mode only.

In order to distinguish the lowest buckling load between two values of \( n \), it was often necessary to observe the dynamic behavior for durations of up to 100 milliseconds. For example, from Fig. 4.2 the buckling load occurred either at a wave number of nine or ten. To
Figure 4.3 Maximum Radial Deflection Versus Time for Various End Loads, n = 9
identify the exact buckling load and the corresponding wave number, it was necessary to study the behavior of the shell in greater detail for longer time durations. This illustrative study is shown in Figs. 4.3 and 4.4. Based on these figures, the representative cylinder would buckle at a wave number of nine rather than ten. The shell reaches a greater deflection in Fig. 4.3 than in Fig. 4.4 for the same applied load.

As mentioned previously, the time required for maximum deflection to occur decreases as the applied load is increased. This time can be classed as the critical time required for buckling. If the time duration of the axial loading is short enough, the shell should be able to withstand loads higher than the minimum buckling load. Although this fact has not been experimentally verified for dynamically loaded cylinders, the same conclusion was reached in reference [49]. Figure 4.5 depicts the time duration curve for the representative stiffened cylinder.

4.3 Geometric Considerations

In an attempt to suggest design improvements on current light aircraft, various parameters (such as number of stiffeners, stiffener area and stiffener eccentricity) were varied and the effects on the buckling load were noted. The results of this limited parametric study are shown in Figs. 4.6 through 4.11.

The influence of the number of stringers on the buckling load of subshell II are shown in Figs. 4.6 and 4.7. Since local buckling between the stringers was not the dominant mode of buckling, only the region of importance is shown on the respective figures. However, the
Figure 4.4 Maximum Radial Deflection Versus Time for Various End Loads, n = 10
Figure 4.5  Buckling Load Versus Time Duration, n = 9
Figure 4.6 Axial Force Versus Wave Number with Increasing Number of Stringers
Figure 4.7 Influence of Number of Stringers on Buckling Load
calculations were also run for higher wave numbers to check this assumption. In performing the calculations all shell and stiffener parameters were held constant except for the number of stringers. It can be seen from Fig. 4.7, that if the number of stringers of the aircraft C type were doubled, the minimum dynamic buckling load would be increased by over 50 per cent. If the number of stringers was less than thirty, local buckling between the stringers would probably be the dominant mode of buckling, with a corresponding decrease in buckling load. This observation suggests a minimum weight design where the number of stringers was just enough to prevent local buckling between the stringers.

The influence of the number of rings on the buckling load is shown in Figs. 4.8 through 4.10. Figures 4.8 and 4.10 represent the lower bound on the calculations, since they were made using subshell III, which does not include the torsional rigidity of the rings. The calculations show the same trend previously observed when the number of stringers was increased. If the distance between rings is halved, the buckling load is increased by about 50 per cent. This same trend is also verified when the ring torsional rigidity is included, as shown in Fig. 4.9.

Finally, the ring and stringer cross-sectional areas and eccentricities were each varied independently and the analysis was repeated. No significant change in the buckling load was noted for increases of 25 to 50 per cent of each parameter. This somewhat surprising result was also observed by Singer, Baruch, and Harari [35] when they studied the static buckling of stiffened cylinders using a linear smeared
Figure 4.8 Axial Force Versus Wave Number for Different Ring Spacing, Subshell III
Figure 4.9 Axial Force Versus Wave Number for Different Ring Spacing, Subshell I
Figure 4.10 Influence of Number of Rings on Buckling Load
analysis. They stated that, except for very low values of the Batdorf parameters, $Z$, the buckling load was not increased appreciably by variation of stiffener area or eccentricity. Judging from the curves presented in reference [35], the value of $Z$ had to be less than 25 for the variation to have any appreciable effect. As noted earlier, the smallest value of $Z$ observed for any of the light aircraft structures was about 190.

The results of the parameter variation seem to suggest that stiffener eccentricity or one-sideness need not be included in the analysis. This, however, is definitely not the case as shown in Fig. 4.11. If the stiffeners were placed on the outside rather than on the inside of the cylinder, the dynamic buckling load is increased. The effect of stiffener eccentricity on the buckling load was first demonstrated by Koiter [25]. Of course, from a practical standpoint, light aircraft normally would not be designed with the stiffeners outside the fuselage!

The results of this section demonstrate that the number rather than the geometric configuration of the stiffeners is the important design consideration. This is not surprising, especially due to the local buckling behavior observed previously. The stiffeners can only exert a torsional restraint in the local buckling mode. Consequently, "beefing up" the stiffeners will only effect the buckling load slightly. The results of this section also suggest an aircraft design where the stiffener cross-section is reduced until general instability would be on the verge of predominating over local buckling. Then the reduction in stiffener weight could be used to increase the total number of stiffeners. The final design should be a more crashworthy aircraft with no increase in weight.
Figure 4.11 Influence of Stiffener Eccentricity on Buckling Load
4.4 Discreteness Considerations

Because of the wide spacing between some stiffeners in light aircraft structures, the stiffeners were treated as discrete elements, by use of the Dirac delta representation. However, it was found that most light aircraft have relatively closely spaced stringers. An inspection of Table 4.1 reveals that the number of stringers varies from 33 on aircraft A, which has a 25-inch radius, to 22 on aircraft C, which also has a 25-inch radius. As noted in Section 4.1, a stringer spacing of 30 caused some discreteness effects to be apparent, but not enough to dominate the buckling mode. This was due in part to the large number of stringers and in part to the restraining effect of the rings, as was observed in Fig. 4.2. However, if light aircraft are designed with fewer stringers it is recommended that stringer discreteness be checked using the present analysis.

The discreteness effects of the rings were observed using a suggestion from reference [41], where it was shown that the first term, the first approximation, of the discrete solution is equal to the "smeared" solution, for the case of equal rings. This fact also applies to the present analysis. If the stiffeners are smeared in the analysis, and the buckling Equations (2-16) are solved by the Galerkin method, one sees that the terms corresponding to the unstiffened shell are the same as those in the discrete analysis. The ring terms in the smeared analysis yield terms of the type

\[
\left(\frac{E A}{L}\right) \int_0^L \cos(i\pi x/L) \cos(\xi \pi x/L) dx = \frac{1}{2} (E A L/\lambda) \delta_{i, \xi} \tag{4-1}
\]
The corresponding term for the discrete analysis would be

\[
\int_0^L E_{rj} A_{rj} \delta(x-j\lambda) \cos(i\pi x/L) \cos(\xi\pi x/L) \, dx = \sum_{j} E_{rj} A_{rj} \cos(i\pi x/L) \cos(\xi\pi x/L)
\]

\( (4-2) \)

In matrix form, Equation (4-1) would be an \( i\xi \) diagonal matrix, whereas Equation (4-2) would also be an \( i\xi \) matrix, but not necessarily diagonal. However, the diagonal terms of Equation (4-2) would be of the form

\[
\sum_{j} E_{rj} A_{rj} \cos^2(i\pi x/L) = \frac{1}{2}(E_{r} A_{r} L/\lambda)
\]

\( (4-3) \)

Obviously Equation (4-3) is equal to Equation (4-1), demonstrating that the diagonal terms in the discrete case are equal to those of the smeared case.

Thus, instead of generating a smeared analysis to compare with the present analysis, it is only necessary to compare a one term solution with a multi-term solution to observe the discreteness effects.

The discreteness effects on subshell I are shown in Fig. 4.12. It is readily observed that a one-term solution was totally inadequate in predicting the buckling load, but that a three- or five-term solution rapidly converged to a lower bound answer. It is interesting to note that the three- and five-term solution predicted essentially the same critical buckling load at the same wave number, but differed somewhat at the higher wave numbers.

The discreteness effects on subshell II are shown in Fig. 4.13. They are of the same character as those for subshell I, but it took a five-term rather than a three-term solution to converge on the lowest buckling load. A seven-term solution is also shown in Fig. 4.13 which
Figure 4.12 Convergence of Solution, Subshell I
Figure 4.13 Convergence of Solution, Subshell II
further demonstrates the convergence. It is interesting to note that, for subshell I, only the third and fifth terms had a significant effect on the buckling load. Similarly, only the fifth and seventh terms had a significant effect on the subshell II buckling load. This behavior followed an empirical rule suggested by Dr. D. M. Egle for identifying the significant assumed mode terms. This rule can be deduced from Ref. [62] as:

\[ \text{1st term, (2n_b \pm m) term, (4n_b \pm m) term, etc.} \]

where

\[ n_b = \text{numbers of bays between rings} \]
\[ m = \text{axial wave number (normally 1)} \]

This rule predicts the same important terms for subshell I and II as were actually observed.

Thus, as anticipated, the discreteness of the rings or bulkheads in light aircraft must be accounted for. The usual smeared analysis would calculate an incorrect wave number and a buckling load that could be an order of magnitude too high.
CHAPTER V

CLOSURE

The analysis developed in this dissertation can be applied to any light aircraft structure undergoing any dynamically varying axial compression loading. Since the stiffeners are treated as discrete elements, the analysis can be of great value in the design of new aircraft that can better withstand crash impact loads without any great increase in overall structural design weight. The computer program contained in the analysis can handle any dynamically varying load shape, such as ramp, exponential, or triangular. With minor modification, the program could handle stiffeners with varying geometric cross-sectional properties.

The present theory was compared with available dynamic unstiffened shell and static stiffened shell analyses, and good agreement was achieved. A limited parametric study was conducted on a stiffened cylinder which was representative of a present-day, light-aircraft cabin section. The results were presented primarily in graphical form, and the following general conclusions were obtained:

1. Since local buckling between bulkheads predominated over general shell instability in the test cases, the stiffeners in most aircraft are probably over-designed. A more efficient design could be achieved

58
by designing the stiffeners such that general instability buckling would be on the verge of predominating over local buckling.

2. The number of rings and stringers should be increased in preference to increasing the size or cross-sectional shape of the stiffeners in current aircraft design.

3. Stiffener discreteness must be included in a dynamic analysis of the type presented here in order to adequately model a light aircraft structure in a crash environment.

Finally, it is recommended that this analysis be complimented by a series of experiments in which representative cylinders are subjected to a carefully controlled dynamic buckling environment. No experiments were found in the current literature in which dynamic axial compression buckling of stiffened shells was studied. Experiments of this type could study phenomena such as time duration of loading (Section 4.2), the effect of stiffener discreteness (Section 4.4), and the effect of geometric variations of the stiffened shell on the buckling load (Section 4.3). The present theory would provide the analytical basis for designing such experiments to extract maximum useful information.
REFERENCES


APPENDIX A

DERIVATION OF THE KINETIC AND POTENTIAL ENERGIES FOR A DISCRETELY STIFFENED CYLINDRICAL SHELL

A.1 Nonlinear Strain-Displacement Formulation

The Donnell [4] nonlinear strain-displacement relations for the shell mid-surface are

\[ \varepsilon_x = u,_{xx} = \frac{1}{2}(w,_{xx})^2 \]
\[ \varepsilon_y = v,_{yy} + wR^{-1} + \frac{1}{2}(w,_{yy})^2 \]  \hspace{1cm} (A-1)
\[ \gamma_{xy} = u,_{yx} + v,_{xy} + w,_{xx}w,_{yy} \]

It is assumed that the stiffeners behave as beam elements and that displacements vary linearly across the stringer depth. Therefore, to satisfy compatibility of displacements where shell and stringer are joined, we may write

\[ \varepsilon_{yr} = \varepsilon_y - zW,_{yy} \]  \hspace{1cm} (A-2)
\[ \varepsilon_{xs} = \varepsilon_x - zW,_{xx} \]

A.2 Unstiffened Cylinder Strain Energy

The strain energy of the unstiffened shell is found by considering a small element of a thin shell. Since plane stress is assumed to be a valid assumption for a thin shell and the shell material is homogeneous, isotropic, and linearly elastic, the following constitutive relations are appropriate
Figure A.1 Shell Geometry
\[ \sigma_x = E(1-v^2)^{-1}(\varepsilon_x + \nu \varepsilon_y) \]
\[ \sigma_y = E(1-v^2)^{-1}(\varepsilon_y + \nu \varepsilon_x) \]
\[ \tau_{xy} = G \gamma_{xy} \]

The incremental change in strain energy per unit volume for the small element is

\[ dU_{\text{vol.}} = \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \tau_{xy} d\gamma_{xy} \]  
(A-4)

Substituting Equation (A-3) into (A-4) and integrating the result, one obtains the strain energy per unit volume

\[ U_{\text{vol.}} = E(1-v^2)^{-1}\left[\frac{1}{2} \varepsilon_x^2 + \frac{1}{2} \varepsilon_y^2 - \varepsilon_x \varepsilon_y + \frac{1}{2} (1-v) \gamma_{xy}^2\right] \]  
(A-5)

If Equation (A-5) is integrated over the volume of the unstiffened cylinder, both the extensional and the bending strain energy will be found. This integration yields

\[ U_{\text{cylinder}} = \frac{1}{4} \int_0^{2\pi R} \int_0^L \left[ E h (1-v^2)^{-1}\left( \varepsilon_x^2 + 2 \varepsilon_x \varepsilon_y + \varepsilon_y^2 + \frac{1}{2} (1-v) \gamma_{xy}^2 \right) + D (w_{,xx} + w_{,yy}) - 2D (1-v) \left( \frac{w}{R} \right)^2 \right] dx dy \]  
(A-6)

If the strain displacement relations (A-1) are substituted into Equation (A-6), one obtains

\[ U_c = \int_0^{2\pi R} \int_0^L \left[ E h (1-v^2)^{-1}\left( \frac{1}{2} u_{,x}^2 + u_{,y}^2 + \frac{1}{8} w_{,x}^2, w_{,y}^2 + \frac{1}{2} \nu u_{,x} \varepsilon_{,y} + \frac{1}{2} \nu u_{,y} \varepsilon_{,x} + \frac{1}{2} R^{-1} w_{,x} \varepsilon_{,x} + \frac{1}{2} R^{-1} w_{,y} \varepsilon_{,y} \right) \right] dx dy \]
A.3 Ring Strain Energy

The rings are considered as thin curved beam elements, with the strain energy of an individual ring consisting of strain energy due to flexure, extension, and torsion. Thus

\[ U_r = \frac{1}{2} \int_0^{2\pi R} \left( \int_0^L \left( E_A \varepsilon_y^2 - 2EA \varepsilon_y \varepsilon_y' + GJ \varepsilon_y^2 \right) dy + \frac{1}{2} \left( \frac{w}{R} \right)^2 \varepsilon_y^2 + \varepsilon_y' \varepsilon_y'' \right) dx dy \]  

(A-8)

Next, the ring strain-displacement relations (A-2) are substituted into Equation (A-8) and integrations are performed over the ring area. The strain energy of N rings becomes

\[ U_r = \frac{1}{2} \int_0^{2\pi R} \left( \int_0^L \left( E_A \varepsilon_y^2 - 2EA \varepsilon_y \varepsilon_y' + GJ \varepsilon_y^2 \right) dy + \frac{1}{2} \left( \frac{w}{R} \right)^2 \delta(x-j\ell) \right) dx dy \]  

(A-9)

where \( \delta(x-j\ell) \) is a Dirac delta function defined by

\[ \int_{-\infty}^{\infty} f(x) \delta(x-j\ell) dx = f(j\ell) \quad (x=j\ell) \]

\[ \delta(x-j\ell) = 0 \quad (x \neq j\ell) \]  

(A-10)

Now, Equation (A-9) is in terms of the strains of the shell middle surface. If the shell strain-displacement relations (A-1) are substituted into (A-9), one obtains

\[ U_r = \frac{1}{2} \int_0^{2\pi R} \left( \int_0^L \left( E_A \varepsilon_y^2 - 2EA \varepsilon_y \varepsilon_y' + GJ \varepsilon_y^2 \right) dy + \frac{1}{2} \left( \frac{w}{R} \right)^2 \delta(x-j\ell) \right) dx dy \]  

(A-11)
A.4 Stringer Strain Energy

In an analogous fashion to the derivation of the ring strain energy, the stringer strain energy can be found as

\[ U_s = \frac{1}{2} \int_0^{2\pi} \int_0^L \left( \frac{E_A}{R_x} \right) \left( \frac{\partial u}{\partial x} \right)^2 + \frac{2E_A}{R_x} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{8} \left( \frac{\partial w}{\partial x} \right)^4 + \frac{G_{J_w}}{\rho_x} \left( \frac{\partial w}{\partial y} \right)^2 \delta (y-kd) \, dx \, dy \]

(A-12)

where

\[ \int_{-\infty}^{\infty} f(y) \delta (y-kd) \, dy = f(kd) \quad (y=kd) \]
\[ \delta (y-kd) = 0 \quad (y\neq kd) \]

As before, Equation (A-1) is substituted into Equation (A-12) to obtain

\[ U_s = \int_0^{2\pi} \int_0^L \left( \frac{E_A}{R_x} \right) \left( \frac{\partial u}{\partial x} \right)^2 + \frac{2E_A}{R_x} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{1}{8} \left( \frac{\partial w}{\partial x} \right)^4 + \frac{G_{J_w}}{\rho_x} \left( \frac{\partial w}{\partial y} \right)^2 \delta (y-kd) \, dx \, dy \]

(A-14)

A.5 Potential Energy of External Load

The potential energy of the externally applied load on the end of the stiffened shell is the same as the negative of the work done on the shell. Thus it can be calculated as the product of the applied force and the change of length of the cylinder. Finally, the potential energy due to the load resultant \( \hat{N}_x \) applied at a distance \( \hat{e} \) from the shell middle surface is

\[ U_{\text{external}} = \int_0^{2\pi} \int_0^L \hat{N}_x \left( u - \hat{e} w \right) \, dx \, dy \]

(A-15)
The total potential energy is found by adding Equations (A-7), (A-11), (A-14) and (A-15) to obtain

\[
U = \int_0^{2\pi R} \int_0^L \left\{ Eh(1-v^2)^{-1} \left[ \frac{1}{8} w_x^2 + \frac{1}{8} w_y^2 + \frac{1}{8} w_z^2 + \frac{1}{8} w_{xx}^2 + \frac{1}{8} w_{yy}^2 + \frac{1}{8} w_{zz}^2 \right] \right. \\
+ \left. \frac{1}{8} v_u w_x w_y + \frac{1}{8} v_w w_x w_z + \frac{1}{8} v_w w_y w_z \right\} dx dy (A-16)
\]

A.7 Total Kinetic Energy

Neglecting in-surface and rotatory inertia effects, one may write the kinetic energy of the unstiffened shell as

\[
T_{\text{unstiffened}} = \frac{1}{2} \rho_h \int_0^{2\pi R} \int_0^L \dot{u}^2 dx dy (A-17)
\]

In a like manner, the kinetic energy of rings and stringers (referenced to the shell middle surface) may be written as

\[
T_r = \frac{1}{2} \int_0^{2\pi R} \int_0^L \left( \frac{\rho A w}{\rho} \right)^2 dx dy (A-18)
\]

\[
T_s = \frac{1}{2} \int_0^{2\pi R} \int_0^L \left( \frac{\rho A w}{\rho} \right)^2 dx dy (A-18)
\]
APPENDIX B

APPLICATION OF HAMILTON'S PRINCIPLE TO DERIVE
THE EQUATIONS OF MOTION

The governing equations of motion are obtained from Hamilton's principle, which requires that the first variation of the time-integrated difference between the potential and kinetic energies be zero.

\[ \delta \int_{t_1}^{t_2} (U-T) \, dt = 0 \quad (B-1) \]

Substituting Equations (A-16), (A-17) and (A-18) into Equation (B-1) and performing the variational operation, one obtains

\[ \delta \int_{t_1}^{t_2} (U-T) \, dt = \int_{t_1}^{t_2} \int_{\omega_0}^{2\pi R L} \left[ Eh(1-v^2)^{-1} \{ [u, x, x] + \frac{1}{2} w, x, y + v, x, y + v, x, y + \frac{1}{2} v, x, y + v, x, y \} \delta u, x + v, x, y + v, x, y \right] \, dt \]

\[ + \left[ u, y, + w, x, x, y + v, x, y \right] \delta u, y + \left[ u, y, + w, x, x, y + v, x, y \right] \delta v, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]

\[ + \left[ v, x, + \frac{1}{2} w, x, y, y + R^{-1} w, y + v, x, y \right] \delta v, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, x \]

\[ + \left[ u, w, x, x, y + v, x, y \right] \delta w, y + \left[ u, w, x, x, y + v, x, y \right] \delta w, y \]
The following illustrations of integrations by parts operations will be performed on Equation (B-2).

\[
\int_0^L \int_0^L u_x \delta u_x \, dx \, dt = \int_0^L \int_0^L \{u_x \delta u_x \} \, dx \, dt
\]

\[
\int_0^L \int_0^L w_x \delta u_x \, dx \, dt = \int_0^L \int_0^L \{w_x \delta u_x \} \, dx \, dt
\]

\[
\int_0^L \int_0^L v_y \delta u_y \, dx \, dt = \int_0^L \int_0^L \{v_y \delta u_y \} \, dx \, dt
\]

\[
\int_0^L \int_0^L w_y \delta u_y \, dx \, dt = \int_0^L \int_0^L \{w_y \delta u_y \} \, dx \, dt
\]

\[
\int_0^{2\pi R} \int_0^{2\pi R} w \delta u \, dy \, dt = \int_0^{2\pi R} \int_0^{2\pi R} \{w \delta u \} \, dy \, dt
\]

\[
\int_0^L \int_0^{2\pi R} w \delta w \, dx \, dt = \int_0^L \int_0^{2\pi R} \{w \delta w \} \, dx \, dt
\]

\[
\int_0^L \int_0^{2\pi R} w \delta w \, dx \, dt = \int_0^L \int_0^{2\pi R} \{w \delta w \} \, dx \, dt
\]

\[
+ \int_0^L \int_0^{2\pi R} w \delta w \, dx \, dt
\]
Making use of the above integration by parts forms, and performing the indicated operations in Equation (8-2), one obtains the following equation:

\[
\int_{t_1}^{t} (U-T) dt = \int_{0}^{t} \left[ - \int_{0}^{2\pi R/L} \left\{ \frac{2\pi R}{L} \right\} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ w_{xx} + w_{xy} + w_{yx} + w_{yy} \right] \, dt
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ u_{xx} + u_{xy} + u_{yx} + u_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ v_{xx} + v_{xy} + v_{yx} + v_{yy} \right] \, dy
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ w_{xx} + w_{xy} + w_{yx} + w_{yy} \right] \, dz
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \rho_{xx} + \rho_{xy} + \rho_{yx} + \rho_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \rho_{zz} + \rho_{z\phi} + \rho_{\phi z} + \rho_{\phi\phi} \right] \, dz
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \sigma_{xx} + \sigma_{xy} + \sigma_{yx} + \sigma_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \sigma_{zz} + \sigma_{z\phi} + \sigma_{\phi z} + \sigma_{\phi\phi} \right] \, dz
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \tau_{xx} + \tau_{xy} + \tau_{yx} + \tau_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \tau_{zz} + \tau_{z\phi} + \tau_{\phi z} + \tau_{\phi\phi} \right] \, dz
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \epsilon_{xx} + \epsilon_{xy} + \epsilon_{yx} + \epsilon_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \epsilon_{zz} + \epsilon_{z\phi} + \epsilon_{\phi z} + \epsilon_{\phi\phi} \right] \, dz
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \gamma_{xx} + \gamma_{xy} + \gamma_{yx} + \gamma_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \gamma_{zz} + \gamma_{z\phi} + \gamma_{\phi z} + \gamma_{\phi\phi} \right] \, dz
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \delta_{xx} + \delta_{xy} + \delta_{yx} + \delta_{yy} \right] \, dx
\]

\[
+ \int_{0}^{2\pi R} \left[ \frac{2\pi R}{L} \right] \frac{1}{\sqrt{1 - \nu^2}} \left[ \delta_{zz} + \delta_{z\phi} + \delta_{\phi z} + \delta_{\phi\phi} \right] \, dz
\]
Since the variation of the function with $\theta$ is periodic, all of the terms in Equation (B-3) that are evaluated between zero and $2\pi R$ reduce to zero. The three equations of motion $(u,v,w)$ and all relevant boundary conditions can now be extracted from Equation (B-3).

To simplify the governing equations, we make use of the notation advanced by Kraus [50], and define the following terms

$$N_x = \frac{Eh}{s} (1-v^2)^{-1} [u_x + \frac{z}{R} w,^2 + v(v_x + \frac{R}{2} v,^2 + w_x + \frac{R}{2} w,^2)] + \sum_k \delta(y-kd) E A_s (u_x + \frac{z}{R} w,^2 + v(v_x + \frac{R}{2} v,^2 + w_x + \frac{R}{2} w,^2)] + \frac{Eh}{s} (1-v^2) [u_y + \frac{z}{R} w,^2 + v(v_y + \frac{R}{2} v,^2 + w_y + \frac{R}{2} w,^2)] + \sum_k \delta(x-kj) E A_r (v_y + \frac{R}{2} v,^2 - \frac{z}{R} w,^2)]$$

$$N_y = \frac{Eh}{s} (1-v^2)^{-1} [v_y + \frac{R}{2} v,^2 - \frac{z}{R} w,^2 + v(u_x + \frac{z}{R} w,^2 + w_x + \frac{R}{2} w,^2)] + \sum_k \delta(x-kj) E A_r (v_y + \frac{R}{2} v,^2 - \frac{z}{R} w,^2)]$$

$$N_{xy} = \frac{Eh}{s} (1+v) [u_y + v_x + w_x,^2 + v_x,^2]$$

$$M_x = -D(w_{xx} + u_{yy}) - \sum_k \delta(y-kd) E A_s [(I_{os}/A_s)w_{xx} - \frac{z}{R} u_x + \frac{z}{R} w,^2]$$

$$M_y = -D(w_{yy} + u_{xx}) - \sum_k \delta(x-kj) E A_r [(I_{or}/A_r)w_{yy} - \frac{z}{R} v_y + \frac{z}{R} w,^2]$$

$$M_{xy} = -D(1-v)w_{xy} - \sum_k \delta(y-kd) G J_s w_{xy} - \sum_k \delta(x-kj) G J_r w_{xy}$$
The three equations of motion \((u,v,w)\) can now be extracted from Equation (B-3) and written as

\[
\begin{align*}
N_{x,x} + N_{xy,y} &= 0 \\
N_{y,y} + N_{xy,x} &= 0
\end{align*}
\]  

\[(B-10a)\]

\[-M_{x,xx} - 2M_{xy,xy} - M_{y,yy} + R^{-1}N_{w,xx} - N_{w,yy} - 2N_{w',xy}w,xy \]

\[+ \rho hw,tt + \int \delta(x-j) \rho A w,tt + \int \delta(y-kd) \rho A w,tt = 0 \]

\[(B-10c)\]

In a like manner, the boundary conditions which apply at the ends of the cylinder can be extracted from Equation (B-3) and written as

\[
\begin{align*}
N_x + \hat{N}_x &= 0 \quad \text{or} \quad u = 0 \\
N_{xy} &= 0 \quad \text{or} \quad v = 0 \\
M_x + \hat{N}_x &= 0 \quad \text{or} \quad w_x = 0 \\
M_{x,x} + 2M_{xy,y} + N_{x,x} + N_{xy,y} &= 0 \quad \text{or} \quad w = 0
\end{align*}
\]  

\[(B-11a)-d)\]
APPENDIX C

APPLICATION OF GALERKIN'S METHOD TO OBTAIN A SET OF ORDINARY NONLINEAR GOVERNING DIFFERENTIAL EQUATIONS

To correctly apply Galerkin's method, one must have the error functions $Q_x$, $Q_y$, and $Q_z$ orthogonal to the assumed mode solutions. The error functions are formed by substituting the assumed displacement functions (2-18) into the buckling equations of motion (2-16). The orthogonalization equations are Equations (2-19), (2-20) and (2-21).

Obviously, a number of terms in each orthogonalization equation will integrate to zero. For ease in identifying and keeping track of the various terms, a table is constructed for each of the orthogonalization equations. In the following development, the $B$ subscript is dropped as understood.

Table C.1 contains the final integrated results of Equation (2-19), which is repeated below for easy reference

$$\int_0^{2\pi R/L} \int_0^L Q_x(u,v,w) \sin(\xi m x/L) \cos(\xi n y/R) dx dy = 0 \quad \xi = 1,2,3\ldots$$

Thus, after integration, Equation (2-19) becomes:

$$\delta_{i\xi} \sum_{ij} \left[ I_i^2 \delta_j + \frac{1}{2} (1-v) J_j^2 \delta_i + (1-v^2) (\pi R E h)^{-1} I_i^2 \right] E_{ik} \cos \xi_k \cos \xi_k$$

78
Table C.1. Integrated Terms from Axial Equation of Motion.

<table>
<thead>
<tr>
<th>TERM</th>
<th>FINAL INTEGRATED RESULT</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_{,xx})</td>
<td>(-\frac{1}{2}\Delta_1 \Delta_2 \Delta_1 E \pi RL (1-\nu^2)^{-1} \sum u_{ij} I_i^2)</td>
</tr>
<tr>
<td>(w_{,x} w_{,xx})</td>
<td>(\Delta_3 (E \pi RL/8) (1-\nu^2)^{-1} \sum w_{ij}^2 I_i^3)</td>
</tr>
<tr>
<td>(\nu v_{,xy})</td>
<td>(-\frac{1}{2}\Delta_1 \Delta_2 \Delta_1 E \pi LV (1-\nu^2)^{-1} \sum v_{ij} I_i J_j)</td>
</tr>
<tr>
<td>(\nu R^{-1} w_{,x})</td>
<td>(-\frac{1}{2}\Delta_1 \Delta_2 \Delta_1 E \pi LV (1-\nu^2)^{-1} \sum w_{ij}^2 I_i^2)</td>
</tr>
<tr>
<td>(\nu w_{,y} w_{,xy})</td>
<td>(\Delta_3 (E \pi RL/8) (1-\nu^2)^{-1} \sum w_{ij}^2 I_i J_j^2)</td>
</tr>
<tr>
<td>(u_{,yy})</td>
<td>(-\frac{1}{2}\Delta_1 \Delta_2 \Delta_1 E \pi RL (1+\nu) \sum u_{ij} J_j^2)</td>
</tr>
<tr>
<td>(v_{,xy})</td>
<td>(-\frac{1}{2}\Delta_1 \Delta_2 \Delta_1 E \pi RL (1+\nu) \sum v_{ij} I_i J_j)</td>
</tr>
<tr>
<td>(w_{,xy} w_{,y})</td>
<td>(\Delta_3 (E \pi RL/16) (1+\nu)^{-1} \sum w_{ij}^2 I_i^2 J_j^2)</td>
</tr>
<tr>
<td>(w_{,x} w_{,yy})</td>
<td>(\Delta_3 (E \pi RL/16) (1+\nu)^{-1} \sum w_{ij}^2 I_i J_j^2)</td>
</tr>
<tr>
<td>(u_{,xx})</td>
<td>(-\frac{1}{2}\delta_{ij} \zeta \sum I_i^2 E_A s_k \cos J_k \cos J_\zeta)</td>
</tr>
<tr>
<td>(w_{,x} w_{,xx})</td>
<td>(\frac{1}{2}\delta_{ij} \zeta \sum I_i^2 E_A s_k \cos^2 J_k \cos J_\zeta)</td>
</tr>
<tr>
<td>(-\zeta w_{,xxx})</td>
<td>(-\frac{1}{2}\delta_{ij} \zeta \sum I_i^2 E_A \zeta s_k s_s \cos J_k \cos J_\zeta)</td>
</tr>
</tbody>
</table>
In a similar manner, Equation (2-20) may be evaluated. (The details of integration are contained in Table C.2.)

After integration, Equation (2-20) becomes

Finally, Equation (2-21) may be evaluated. Details of this integration are contained in Tables C.3, C.4 and C.5.

The last entries in Tables C.4 and C.5 are expanded here for clarity. They hinge on the definition of the derivative of a Dirac delta function found in reference [62].
<table>
<thead>
<tr>
<th>TERM</th>
<th>FINAL INTEGRATED RESULT</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_{yy} )</td>
<td>(-\frac{1}{2} \Delta_1 \mathcal{E}n \pi RL (1-v^2)^{-1} \sum v_{ij} J^2 )</td>
</tr>
<tr>
<td>( R^{-1} w_{,y} )</td>
<td>(-\frac{1}{2} \Delta_1 \mathcal{E}n \pi L (1-v^2)^{-1} \sum w_{ij} J )</td>
</tr>
<tr>
<td>( w_{,y} w_{,yy} )</td>
<td>( \Delta_5 (\mathcal{E}n \pi RL/8) (1-v^2)^{-1} \sum w_{ij} J^3 )</td>
</tr>
<tr>
<td>( w_{,y} u_{,xy} )</td>
<td>(-\frac{1}{2} \Delta_1 \mathcal{E}n RL v (1-v^2)^{-1} \sum u_{ij} I )</td>
</tr>
<tr>
<td>( w_{,xy} w_{,xy} )</td>
<td>( \Delta_5 (\mathcal{E}n \pi RL v/8) (1-v^2)^{-1} \sum w_{ij} I J )</td>
</tr>
<tr>
<td>( u_{,xy} )</td>
<td>(-\frac{1}{2} \Delta_1 \mathcal{E}n RL (1+v) (1-v^2)^{-1} \sum u_{ij} I )</td>
</tr>
<tr>
<td>( v_{,xx} )</td>
<td>(-\frac{1}{2} \Delta_1 \mathcal{E}n RL (1+v) (1-v^2)^{-1} \sum v_{ij} I^2 )</td>
</tr>
<tr>
<td>( w_{,xx} w_{,y} )</td>
<td>( \Delta_5 (\mathcal{E}n \pi RL/16) (1+v) (1-v^2)^{-1} \sum w_{ij} I^2 J )</td>
</tr>
<tr>
<td>( w_{,x} w_{,xy} )</td>
<td>( \Delta_5 (\mathcal{E}n \pi RL/16) (1+v) (1-v^2)^{-1} \sum w_{ij} I^2 J )</td>
</tr>
<tr>
<td>( v_{,yy} )</td>
<td>(-\delta J \cos \zeta \sum v_{ij} A \cos I \cos \zeta )</td>
</tr>
<tr>
<td>( R^{-1} w_{,y} )</td>
<td>(-\frac{1}{2} \Delta_1 \mathcal{E}n RL v (1+v) (1-v^2)^{-1} \sum w_{ij} I )</td>
</tr>
<tr>
<td>( w_{,y} w_{,yy} )</td>
<td>( \frac{1}{2} \delta J \cos \zeta \sum w_{ij} A \cos^2 I \cos \zeta )</td>
</tr>
<tr>
<td>(-\frac{1}{2} w_{,y} w_{,yy} )</td>
<td>(-\delta J \sin \zeta \sum w_{ij} A \cos \zeta \cos \zeta )</td>
</tr>
</tbody>
</table>
Table C.3 Unstiffened Cylinder Integrated Terms from Radial Equation of Motion

<table>
<thead>
<tr>
<th>TERM</th>
<th>FINAL INTEGRATED RESULT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{1}{2}w, w, x, xx$</td>
<td>$E_h(1-\nu^2)^{-1}(\pi RL/64)(3\Delta_1+\Delta_2-3\Delta_3-\Delta_4)\Sigma Ew_{ij}I_4^4$</td>
</tr>
<tr>
<td>$-\frac{1}{2}w, y, w, xx$</td>
<td>$E_h(1-\nu^2)^{-1}(\pi RL/64)(3\Delta_1-3\Delta_2+\Delta_3-\Delta_4)\Sigma Ew_{ij}I_2^2J_2^2$</td>
</tr>
<tr>
<td>$-\frac{1}{2}w, y, y, y$</td>
<td>$E_h(1-\nu^2)^{-1}(\pi RL/64)(3\Delta_1-3\Delta_2+\Delta_3-\Delta_4)\Sigma Ew_{ij}I_2^2J_2^2$</td>
</tr>
<tr>
<td>$-\frac{1}{2}w, x, w, y$</td>
<td>$E_h(1-\nu^2)^{-1}(\pi RL/64)(3\Delta_1+\Delta_2-3\Delta_3-\Delta_4)\Sigma Ew_{ij}I_2^2J_2^2$</td>
</tr>
<tr>
<td>$-w, x, w, y, w, xy$</td>
<td>$E_h(1-\nu^2)^{-1}(\pi RL/32)(1-\nu)(-\Delta_1+\Delta_2-\Delta_3-\Delta_4)\Sigma Ew_{ij}I_2^2J_2^2$</td>
</tr>
<tr>
<td>$w, xxxx$</td>
<td>$\frac{1}{4}\Delta_1 DnRL\Sigma Ew_{ij}I_4^4$</td>
</tr>
<tr>
<td>$2w, xxyy$</td>
<td>$\Delta_1 DnRL\Sigma Ew_{ij}I_2^2J_2^2$</td>
</tr>
<tr>
<td>$w, yyyy$</td>
<td>$\frac{1}{4}\Delta_1 DnRL\Sigma Ew_{ij}J_2^2$</td>
</tr>
<tr>
<td>$-u, x, w, xx$</td>
<td>$\Delta_5 E_h\pi RL (1-\nu^2)^{-1} \frac{1}{8} \Sigma u_{ij} w_{ij} l_3^3$</td>
</tr>
<tr>
<td>$-uv, y, w, xx$</td>
<td>$\Delta_5 E_h\pi RL (1-\nu^2)^{-1} \frac{1}{8} \Sigma u_{ij} w_{ij} l_2^2 J_2^2$</td>
</tr>
<tr>
<td>$-uR^{-1}ww, xx$</td>
<td>$\Delta_5 E_h\pi RL (1-\nu^2)^{-1} \frac{1}{8} \Sigma w_{ij}^2 J_2^2$</td>
</tr>
<tr>
<td>$-v, y, w, yy$</td>
<td>$\Delta_5 E_h\pi RL (1-\nu^2)^{-1} \frac{1}{8} \Sigma w_{ij}^2 J_2^2$</td>
</tr>
<tr>
<td>$-wR^{-1}w, yy$</td>
<td>$\Delta_5 E_h\pi RL (1-\nu^2)^{-1} \frac{1}{8} \Sigma w_{ij}^2 J_2^2$</td>
</tr>
<tr>
<td>$-vu, x, w, yy$</td>
<td>$\Delta_5 E_h\pi RL (1-\nu^2)^{-1} \frac{1}{8} \Sigma u_{ij} w_{ij} l_2^2 J_2^2$</td>
</tr>
<tr>
<td>$-u, y, w, xy$</td>
<td>$\Delta_5 E_h\pi RL (1+\nu)^{-1} \frac{1}{8} \Sigma u_{ij} w_{ij} l_2^2 J_2^2$</td>
</tr>
</tbody>
</table>

Table C.3 Continued
### Table C.3 (Cont'd.)

<table>
<thead>
<tr>
<th>TERM</th>
<th>FINAL INTEGRATED RESULT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-v_x w_{xy}$</td>
<td>$\Delta_5 \pi \rho RL (1+\nu) (1-\nu^2)^{-1} \frac{1}{8} \epsilon \Sigma v_{ij} w_{ij} I_{12} j$</td>
</tr>
<tr>
<td>$v_y$</td>
<td>$\frac{1}{2} \Delta_1 \pi \rho L (1-\nu^2)^{-1} \epsilon \Sigma v_{ij} J_j$</td>
</tr>
<tr>
<td>$R^{-1} w$</td>
<td>$\frac{1}{2} \Delta_1 \pi \rho L R^{-1} (1-\nu^2)^{-1} \epsilon \Sigma w_{ij}$</td>
</tr>
<tr>
<td>$\hat{s} w_{y}^2$</td>
<td>$-\Delta_5 \pi \rho v L (1-\nu^2)^{-1} (1/16) \epsilon \Sigma w_{ij}^2 j^2 j$</td>
</tr>
<tr>
<td>$u u_x$</td>
<td>$\Delta_1 \pi \rho \nu L v (1-\nu^2)^{-1} \epsilon \Sigma u_{ij} I_i$</td>
</tr>
<tr>
<td>$\hat{s} v w_x^2$</td>
<td>$-\Delta_5 \pi \rho v L (1-\nu^2)^{-1} (1/16) \epsilon \Sigma w_{ij}^2 I_{12} I_i$</td>
</tr>
<tr>
<td>$\rho h w_{tt}$</td>
<td>$\frac{1}{2} \Delta_1 \pi \rho R L \epsilon \Sigma \hat{w}_{ij}$</td>
</tr>
<tr>
<td>$N_x w_{xx}$</td>
<td>$-\frac{1}{2} \Delta_1 \pi \rho R L \epsilon \Sigma \hat{w}<em>{ij} I</em>{12} I_1$</td>
</tr>
</tbody>
</table>
Table C.4 Ring Integrated Terms from Radial Equation of Motion.

<table>
<thead>
<tr>
<th>TERM</th>
<th>FINAL INTEGRATED RESULT</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1_{\text{r}}/A_{\text{r}})w,yyyy)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(-z_{\text{r}}v,yyy)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(-2z_{\text{r}}R^{-1}w,yy)</td>
<td>(2\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(-\tilde{z}_{\text{r}}w,w,yyyy)</td>
<td>(-\tilde{\delta} \delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>((G_{\text{r}}J/L_{\text{r}})w,xyy)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(-v_{\text{r}}w',yy)</td>
<td>(\tilde{\delta} \delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(-wR^{-1}w',yy)</td>
<td>(\tilde{\delta} \delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(-\tilde{w}_{\text{r}}w,yyyy)</td>
<td>((\delta \eta -\delta \eta -\delta \eta \pi (\pi/8) \times E \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(v_{\text{r}}y,yyyy)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(R^{-1}w,yyyy)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(Iw',yyyy)</td>
<td>(\tilde{\delta} \delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(G_{\text{r}}J_{\text{r}}w,xtt)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>(G_{\text{r}}J_{\text{r}}xyy,yyyy)</td>
<td>(\delta \eta \pi r \cos I \cos I \zeta)</td>
</tr>
<tr>
<td>TERM</td>
<td>FINAL INTEGRATED RESULT</td>
</tr>
<tr>
<td>------</td>
<td>--------------------------</td>
</tr>
<tr>
<td>$(I_{os}/A_s) w',xxx$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^4 E_{ij}^4 E_{ki}^4 \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$-e_{s}^{u',xxx}$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^3 E_{ij}^4 A_{i} \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$-e_{s}^{w',xxx}$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^3 E_{ij}^4 A_{i} \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$(G_{s}/E_{A}) w',xyy$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^2 J_{ij}^2 G_{J} \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$-u_{x}^{w',xx}$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^3 E_{ij}^4 A_{i} \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$-e_{x}^{w',xx}$</td>
<td>$(\delta_{i} - \delta_{i}) L / 16 E_{LEEW} I_{ij}^3 E_{ij}^4 A_{i} \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$\rho_{s}^{A} w',tt$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^4 \cos J_{i} \cos J_{k} \cos \zeta_{i}$</td>
</tr>
<tr>
<td>$G_{s} w',xyy' (y-kd)$</td>
<td>$\frac{1}{4}E_{LEEW} I_{ij}^4 G_{J} (\xi_{n} R_{i}^2 \sin J_{i} \sin J_{k} - J_{i} \cos J_{i} \cos J_{k})$</td>
</tr>
</tbody>
</table>
Applying the above equation to the last term in Table C.5, one obtains the following results:

\[
\int_0^{2\pi R/L} \int_0^L \left[ \Sigma \omega \sum_{ij} \sum_{s=1}^{J \in k} \delta_{ij} (y-kd) \cos(\zeta_{m+n}/L) \sin(Jy) \cos(\xi_{ny}/R) \right] \, dx \, dy \\
= \Sigma \omega \sum_{ij} \sum_{s=1}^{J \in k} \left[ -J_y \cos(Jy) \cos(\xi_{ny}/R) \right] \left. \right|_{y=kd} + (\xi_{ny}/R) \sin(Jy) \sin(Jy) \left. \right|_{y=kd} \\
\cdot (\xi_{ny}/R) |_{y=kd}
\]

Thus, after integration of Equation (2-21), one obtains the following expression:

\[
\begin{align*}
\text{Eh}(1-\nu^2)^{-1} & \frac{1}{32} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \left[ I^h_1 (3\Delta_1+\Delta_2-3\Delta_3-\Delta_4) + 2I^2_j \sum_{S=1}^{J \in k} (-\Delta_1+\Delta_2+\Delta_3+4\nu\Delta_4-2\nu\Delta_2) \\
& -2\nu\Delta_2 + J^h_1 (3\Delta_1+3\Delta_2+3\Delta_3-\Delta_4) \right] + (2/L) \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \left[ E^A \cos^3 I_j \cos I_{\xi} - \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \text{Eh}(1-\nu^2)^{-1} (\nu/8R) \\
& + (1/\pi R) \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \left[ \delta_{ij} \cos \zeta \cos \Delta_5 \text{Eh}(1-\nu^2)^{-1} \sum_{j=1}^{J \in k} \left( \text{h}^2/12 \right) I^4_i \\
& + (h^2/6) I^2_j + (h^2/12) J^h_1 + (1/R^2) - \hat{N}_x (1-\nu^2) \left( \text{h}^2/12 \right) I^4_i \right] + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \left[ J^h_1 E^I \cos \zeta + \frac{2}{3} R^{-1} J^h_1 \cos \zeta \cos \Delta_5 \text{Eh}(1-\nu^2)^{-1} + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \right] \sum_{j=1}^{J \in k} \left[ \delta_{ij} \cos \zeta + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \right] \sum_{j=1}^{J \in k} \left[ \delta_{ij} \cos \zeta + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \right]
\end{align*}
\]

\[
\begin{align*}
& + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& + \delta_{ij} (1/\pi R) \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \sum_{j=1}^{J \in k} \left[ \delta_{ij} \cos \zeta + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \right] \sum_{j=1}^{J \in k} \left[ \delta_{ij} \cos \zeta + \frac{1}{6} \Sigma \omega \sum_{i,j} \sum_{s=1}^{J \in k} \cos \zeta \cos \Delta_5 \left( \text{h}^2/12 \right) I^4_i \\
& \right]
\end{align*}
\]
\[ +\delta_{ij} (2/L) \sum_{\nu, \eta} \left( \frac{J_{3,\nu}^2 + J_{3,\eta}^2}{2} \right) \sum_j E_{A,\nu} \cos I_{\nu,\eta} + \Delta_4 \frac{4}{n} E_h (1 - n^2)^{-1} \sum_{\nu, \eta} I_{\nu,\eta} \]

\[ + \left( J_{1,1}^2 + J_{1,2}^2 \right) + \delta_{2,1} (1/\pi R) \sum_{\nu, \eta} \sum_{i, j} E_{A,\nu} \cos I_{\nu,\eta} + \Delta_5 \frac{4}{n} E_h (1 - n^2)^{-1} \]

\[ \cdot \sum_{\nu, \eta} \sum_{i, j} E_{A,\nu} \cos I_{\nu,\eta} + \sum_{\nu, \eta} I_{\nu,\eta} \]

\[ \cdot \left[ J_{1,1}^2 + J_{1,2}^2 \right] + \delta_{2,1} \frac{L}{2} \sum_{\nu, \eta} \sum_{i, j} E_{A,\nu} \cos I_{\nu,\eta} + \sum_{\nu, \eta} I_{\nu,\eta} \]

\[ \cdot \left[ \Delta_{1,1} \rho + \delta_{1,1} \right] \sum_{i, j} E_{A,\nu} \cos I_{\nu,\eta} + \delta_{1,1} \rho A \left( \frac{1}{2} \right) \sum_{i, j} E_{A,\nu} \cos I_{\nu,\eta} \]

\[ = 0 \]

(C-3)
APPENDIX D

COMPUTER PROGRAM DOCUMENTATION

The program was written in FORTRAN IV, and was run using an IBM System 360, Model 50 computer. The core storage required for the program was dependent upon the number of terms in the assumed mode function. The maximum number of terms run was ten, which required E-level system operation.

The flow of the program can be observed in Fig. D.1 and is summarized as follows: The main program reads the input data which defines the various material and geometric parameters, the number of assumed modal functions, the range of the circumferential wave number, the range of the applied axial load, the Runge-Kutta step size, and the maximum time duration. The main program then calls Subroutines UVCOF and WCOEF in succession. These general subroutines calculate the coefficients for the required number of u, v, and w equations of motion. (The number of equations is dependent upon the number of assumed modal functions, k. The two subroutines will generate coefficients for any number of equations. This number is limited only by the available computer core space. Subroutine UVCOF generates the coefficients for k algebraic u equations and k algebraic v equations. Subroutine WCOEF generates the coefficients for k nonlinear differential w equations.) The main program then calculates the terms in the w
equation of motion containing the nondimensional load as a parameter. Subroutine RUNGE is called to solve the set of nonlinear differential equations. Subroutine RUNGE also uses subroutine FCT to solve the two sets of algebraic equations. For each step in the Runge-Kutta procedure, subroutine FCT must be called four times to solve the two sets of algebraic equations. The results are displayed in the form of shell displacement as a function of time for a particular axial end load and circumferential wave number. The output can be punched on cards for use in plotting the data.

The input data deck is set up as follows:

1. Title card identifying case being run.
2. Number of axial and circumferential assumed mode terms being used.
3. Maximum and minimum values of $n$ being run.
4. Unstiffened shell parameters.
5. Ring parameters.
7. Dynamic parameters.
8. Dynamic parameters (Cont.)
10. Maximum and minimum values of axial load, and load step size.

The program is set up so that cases may be stacked, and all parameters may be varied.
The formats for the above cards are:

1. (80A1)
   All eighty columns are available to assign a descriptive title to the problem being run.

2. (2I4) ISTAR, JSTAR
   ISTAR = maximum number of assumed mode terms in the axial direction, i.
   JSTAR = maximum number of assumed mode terms in the circumferential direction, j.

3. (2I4) NCRG, NCRS
   NCRG = 1 + minimum value of the circumferential wave number, n.
   NCRS = 1 + maximum value of n.

4. (5E15.8) ESHEL, HSHEL, PNU, RSHEL, SHLT
   ESHEL = Young's modulus of shell, E. (psi)
   HSHEL = Thickness of shell, h. (in.)
   PNU = Poisson's ratio, v. (dimensionless)
   RSHEL = Radius of Shell, R (in.)
   SHLT = Length of shell, L. (in.)

5. (I15,3E15.8) NRNG, ARING, ERING, ZRING
   NRNG = Number of Rings, N.
   ARING = Cross-sectional area of one ring, A_r. (in.^2)
   ERING = Young's modulus of rings, E_r. (psi)
   ZRING = Distance from ring centroid to shell mid-surface, \( \bar{z}_r \) (in.)
6. (I15,3E15.8) MSTR, ASTR, ESTR, ZSTR

MSTR = Number of Stringers, M.
ASTR = Cross-sectional area of one stringer, $A_S$. (in.$^2$)
ESTR = Young's modulus of stringers, $E_S$. (psi)
ZSTR = Distance from stringer centroid to shell mid-surface, $z_S$. (in.)

7. (5E15.8) SHDEN, RGDEN, STDEN, RGMI, STMI

SHDEN = Unstiffened cylinder density, $\rho$. (lb-sec$^2$/in$^4$)
RGDEN = Ring density, $\rho_T$. (lb-sec$^2$/in.$^4$)
STDEN = Stringer density, $\rho_S$. (lb-sec$^2$/in.$^4$)
RGMI = Moment of inertia of one ring about shell mid-surface, $I_{or}$. (in.$^4$)
STMI = Moment of inertia of the stringer about shell mid-surface, $I_{os}$. (in.$^4$)

8. (4E15.8) GRING, GSTR, RGJ, STJ

GRING = Modulus of rigidity of ring, $G_T$. (psi)
GSTR = Modulus of rigidity of stringer, $G_S$. (psi)
RGJ = Polar moment of inertia of ring, $J_T$. (in.$^4$)
STJ = Polar moment of inertia of stringer, $J_S$. (in.$^4$)

9. (3E15.8) X, XMAX, H

X = Starting value of independent variable, t. (sec.)
XMAX = Maximum value of independent variable, t. (sec.)
H = Runge-Kutta increment size.

10. (3I10) ILOAD, ISTOP, INT

ILOAD = Smallest value of load, $\bar{F}_{\text{min}}$ (dimensionless)
ISTOP = Largest value of load, $\bar{F}_{\text{max}}$ (dimensionless)
INT = Increment for increasing load value.
SUBROUTINE WCOEF
Calculate coefficients for w equations

SUBROUTINE UVCOF
Calculate coefficients for u and v equations

SUBROUTINE RUNGE
Solve the w nonlinear differential equations using Runge-Kutta algorithm

SUBROUTINE FCT
Solve the u and v algebraic equations using Gauss-Jordan procedure for each new value of w calculated by RUNGE

Figure D.1 Computer Program Flow Chart
APPENDIX E

COMPUTER PROGRAM LISTING
C MAIN PROGRAM
EXTERNAL FCT
INTEGER P,Q

C DIMENSION Y(10),DERY(10),B(5,40),D(5,40)
DIMENSION PRMT(3),ICASE(80)

C THE DIMENSION SIZE FOR WCOF(i,n) MUST BE EQUAL TO OR LARGER THAN ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR LARGER THAN 8*ISIZE FOR THE SECOND SUBSCRIPT.
C THE DIMENSION SIZE FOR C(i,n) MUST BE EQUAL TO OR LARGER THAN 4*ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR LARGER THAN 4*ISIZE FOR THE SECOND SUBSCRIPT.
COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,ISHE3,PNU,
1INRNG,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,
2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,GM1,SM1,GRING,GSTR,RGJ,STJ,
3PLOAD,RSHEL,SHLT,IFLAG,NDOT
C ISTAR=MAX. VALUE OF I  .  JSTAR=MAX. VALUE OF J
C ARING=RING CROSS-SECT. AREA, ASTR=STRINGER CROSS-SECT. AREA
C ERING = YOUNGS MODULUS OF RING, ESTR=YOUNGS MODULUS OF STRINGER
C DRING=DISTANCE BETWEEN RINGS, DSTR=DISTANCE BETWEEN STRINGERS
C ZRING=DISTANCE FROM RING CENTROID TO SHELL MID SURFACE
C ZSTR=DISTANCE FROM STRINGER CENTROID TO SHELL MID SURFACE
C NRNG=NUMBER OF RINGS, MSTR=NUMBER OF STRINGERS
C ESHEL=YOUNGS MODULUS OF SHELL
C HSHEL=SHELL THICKNESS, RSHEL=SHELL RADIUS
C PNU=POISSONS RATIO
C NCIR=NUMBER OF CIRCUMFERENTIAL WAVES (INTEGER)
C MAX=NUMBER OF AXIAL HALF WAVES (INTEGER)
C SHLT=SHELL LENGTH
C N1=NCIR/RSHEL (COMPUTED)...FN1
C N2=N1*N1 (COMPUTED)...FN2
C M1=MAX*3.14/SHLT (COMPUTED)...FM1
C M2=M1*M1 (COMPUTED)...FM2
SHDEN=MASS DENSITY OF SHELL
RGDEN=MASS DENSITY OF RING, STDEN=MASS DENSITY OF STRINGER
RGMI=MOMENT OF INERTIA OF RING, STMI=MOMENT OF INERTIA OF STRINGER
GSTR=MODULUS OF RIGIDITY OF STRINGER
GRING=MODULUS OF RIGIDITY OF RING
RGJ=RING POLAR MOMENT OF INERTIA, STJ=STRINGER POLAR MOMENT
PLOAD=AXIAL COMpressive FORCE

READ AND WRITE CASE IDENTIFICATION PARAMETERS

900 FORMAT(1H1,80A1) MAIN 36
999 FORMAT(' ') MAIN 37
1000 FORMAT(80A1) MAIN 38
1001 FORMAT(214) MAIN 39
1002 FORMAT(1H,8HISTAR = ,14.5X,8HJSTAR = ,14/9) MAIN 40
1003 FORMAT(5E15.8) MAIN 41
1005 FORMAT(15,3E15.8) MAIN 42
1007 FORMAT(5E15.8) MAIN 43
1008 FORMAT(3110) MAIN 44
1009 FORMAT(5X,15,3E15.8) MAIN 45
1012 FORMAT(5X,5E15.8) MAIN 46
1013 FORMAT(' PRMT VALUES ARE...,PRMT(1)=X,PRMT(2)=X,MAX,PRMT(3)=H ') MAIN 47
1018 FORMAT(' SHELL INPUT-ESHEL,HSHEL,PNU,RSHEL,SHLT ') MAIN 48
1019 FORMAT(' RING INPUT-ARNG,ERNG,ERING,GRING ') MAIN 49
1020 FORMAT(' STRINGER INPUT-MSTR,ASTR,ESTR,ZSTR ') MAIN 50
1021 FORMAT(' DYNAMIC-SHDEN,RGGEN,STDEN,RGMI,STMI ') MAIN 51
1022 FORMAT(' DYNAMIC-GRING,GSTR,RGJ,STJ ') MAIN 52
1035 FORMAT(' RANGE OF BUCKLING LOAD...,LOAD,ISTOP,INT ') MAIN 53
1040 FORMAT(1X,3110) MAIN 54
1045 FORMAT(' BUCKLING DID NOT OCCUR UNTIL AFTER MAX. LOAD WAS REACHED MAIN 55
1**********SORRY ') MAIN 56
1055 FORMAT(//30H NUMBER OF AXIAL HALF-WAVES = ,14.5X,39HNUMBER OF CIRC ) MAIN 57
1Umerential FULL WAVES = ,14/9) MAIN 58
5 READ(5,1000)ICASE MAIN 59
WRITE(6,900)ICASE MAIN 60
READ(5,1001)JSTAR,JSTAR MAIN 61
IF (ISTAR - 100) 6, 15, 15
.write (6, 1002) ISTAR, JSTAR

C READ IN NCRG, NCRS
C.read (5, 1001) NCRG, NCRS
C READ AND WRITE SHELL INPUT PARAMETERS
C.read (5, 1003) ESHEL, HSHEL, PNU, RSHEL, SHLT
.write (6, 1018)
.write (6, 1007) ESHEL, HSHEL, PNU, RSHEL, SHLT
C READ AND WRITE RING INPUT PARAMETERS
C.read (5, 1005) NRNG, ARING, ERING, ZRING
.write (6, 1019)
.write (6, 1009) NRNG, ARING, ERING, ZRING
C READ AND WRITE STRINGER INPUT PARAMETERS
C.read (5, 1005) MSTR, ASTR, ESTR, ZSTR
.write (6, 1020)
.write (6, 1009) MSTR, ASTR, ESTR, ZSTR
C READ AND WRITE DYNAMIC INPUT PARAMETERS
C.read (5, 1007) SHDEN, RGDEN, STDEN, RGMI, STMI, GRING, GSTR, RGJ, STJ
.write (6, 1021)
.write (6, 1012) SHDEN, RGDEN, STDEN, RGMI, STMI
.write (6, 1022)
.write (6, 1012) GRING, GSTR, RGJ, STJ
NSIZE = 2 * ISTAR * JSTAR
C C READ AND WRITE VALUES OF PRMT AND DERY
C

Main 72
Main 73
Main 73A
Main 73B
Main 73C
Main 73D
Main 74
Main 75
Main 76
Main 77
Main 78
Main 79
Main 80
Main 81
Main 82
Main 83
Main 84
Main 85
Main 86
Main 87
Main 88
Main 89
Main 90
Main 91
Main 92
Main 93
Main 94
Main 95
Main 96
Main 97
Main 98
Main 99
Main100
Main101
Main102
Main103
READ(5,1007)PRMT
DO 4 J2=1,NSIZE
  DERY(J2)=1.00
WRITE(6,1013)
WRITE(6,1012)PRMT
C
C CALCULATE INTERMEDIATE VALUES
C
MAX=1
ISIZE=ISTAR*JSTAR
PI=3.1415926535892
DRING=SHL/((NRNG+1.0)
DSTR=2.0*PI*RSHEL/MSTR
C
C READ INITIAL EXTERNAL LOAD, MAX LOAD, LOAD STEP SIZE
C EXAMPLE LOAD VALUES...PLOAD=.610, ILOAD=610
C
READ(5,1008)ILOAD,ISTOP,INT
WRITE(6,999)
WRITE(6,1035)
WRITE(6,1040)ILOAD,ISTOP,INT
DO 10 NCR=NCRG,NCRS
  NCIR=NCR-1
WRITE AXIAL AND CIRCUMFERENTIAL WAVE NUMBERS
C
WRITE(6,1055)MAX,NCIR
FNCR=NCIR
FN1=FNCR/RSHEL
FN2=FN1*FN1
FMAX=MAX
FM1=FMAX*PI/SHLT
FM2=FM1*FM1
CALL UVCOF
C
CALL WCOEF
COMPUTE STRINGER TERMS

C

Fj=1
Fj=1
j=IGP+1
I=IGP*IGSTAR
IGP=P-P1/IGSTAR
NN=NN+1
DO120 P=1, I SIZE
NN=2*I SIZE
FX1=1*I
IZETA=IGP+1
IZETA=0-IGP+1
IZETA=0-IGP+1
I=I+1
DO 190 0=1*I SIZE
I=0
CONTINUE

C

C

C

SET WORKING MATRICES EQUAL TO ZERO

C

C

C

COMPUTE DYNAMIC TERMS OF WEDGE SUBROUTINE MATRIX

PLOAD=PLAD+HSHED*HSHED/RSHED
PLAD=PLAD+F / 1000.0
NOIL=10
A1=2X95
X95=LOAD
120 CONTINUE
190 CONTINUE
   Y(1)=0.001
   DO 3 L=2,NSIZE
   3  Y(L)=0.00
      CALL RUNGE(PRMT,Y,DERY,NDIM,HLF,FCT,OUTP,AUX)
      IF(NDOT)22,22,10
      JX9S=JX9S+INT
      IF(JX9S-ISTOP)18,18,10
10 CONTINUE
   GO TO 5
15 CALL EXIT
   STOP
   END

MAIN216
MAIN217
MAIN218
MAIN219
MAIN220
MAIN221
MAIN222
MAIN227
MAIN228
MAIN230
MAIN231
MAIN232
MAIN233
MAIN234
SUBROUTINE UVCOF

C THIS SUBROUTINE CALCULATES THE U.*V.*W**2 COEFFICIENTS FOR THE C U AND V EQUATIONS.
C
INTEGER P,Q

C THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR C LARGER THAN 2*ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR C LARGER THAN 4*ISIZE FOR THE SECOND SUBSCRIPT.
C
DIMENSION B(10,20)
COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESHE3,HSHEL,PNU,
INRING,ARING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,
2FN1,FM2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STM1,GRING,GSTR,STJ,
3PLOAD,RSHEL,SHLT,IFLAG,NDOT

C SET WORKING MATRICES EQUAL TO ZERO

NSIZE=2*ISIZE
MSIZE=4*ISIZE
DO 2 J=1,MSIZE
DO 1 I=1,NSIZE
B(I,J)=0.00
1 CONTINUE

C COMPUTE COMPOSITE U-V EQUATION COEFFICIENT MATRIX

II=0

C COMPUTE U EQUATION MATRIX

DO 130 Q=1,ISIZE
II=II+1
INTGQ=(Q-1)/ISTAR
IZETA=Q-INTGQ*ISTAR
IXI=INTGQ+1

UVCF 1
UVCF 2
UVCF 3
UVCF 4
UVCF 5
UVCF 6
UVCF 7
UVCF 8
UVCF 9
UVCF 10
UVCF 11
UVCF 12
UVCF 13
UVCF 14
UVCF 15
UVCF 16
UVCF 17
UVCF 18
UVCF 19
UVCF 20
UVCF 21
UVCF 22
UVCF 23
UVCF 24
UVCF 25
UVCF 26
UVCF 27
UVCF 28
UVCF 29
UVCF 30
UVCF 31
UVCF 32
UVCF 33
UVCF 34
UVCF 35
UVCF 36
DO 110 P=1,SIZE
C SUBMATRX V.EQ.0 EQUATION
C CONTINUE
105 CONTINUE
104 (II',NN)=(II'NN)=B(II',NN) +F1+F2+F3+F4+F5*/M2/2*N
60 TO 105
C IF(J-I)1,03,14,103
C COMPUTE SHELL TERMS
C IH=HELL
B(II',NN)=(II',NN)=B(II',NN)+F1+F2+F3+F4+F5*/M2/2*N2
C SUM2=SUM2+CO5(F1+F1+F2)2+DSTR*CO5(S(F1+F1+F2))2
C FK=K
00 102 K=1,MAXR
101 SUM2=0.00
C IF(P-1)ZETA(105,103,103)
C COMPUTE STRINGER TERMS.
C F=J
FJ=1
F=1
I=INGP+1
106 I=1-P*INGP*ISTAR
105 (P-1)*INGP*ISTAR=1
104 (N+1)*NN=NN
00 105 P=1,SIZE
00 105 P=1,SIZE
C SUMMATRX V.EQ.0 EQUATION
C FZETA=ZETA
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
FJ=J

C COMPUTE SHELL TERMS
C
IF(I-IZETA)110,106,110
106 IF(J-IXI)110,107,110
107 C(IJ,NN)=FI*FM1*FJ*FN1*(1.0+PNU)/2.0
110 CONTINUE
C
C SUBMATRIX W...U EQUATION
C
DO 115 P=1,ISIZE
NN=NN+1
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
FJ=J

C COMPUTE STRINGER TERMS
C
IF(I-IZETA)115,111,115
111 SUM2=0.00
DO 112 K=1,MSTR
FK=K
112 SUM2=SUM2+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)
B(IJ,NN)=(1.0-PNU*PNU)*FI**3*FM2*FM1*ESTR*ASTR*ZSTR*SUM2/
{1PI*RSHEL*ESHEL*HSHEL}
C
C COMPUTE SHELL TERMS
C
IF(J-IXI)113,114,113
COMPUTATE A EQUATION MATRIX

CONTINUE

120 CONTINUE

119 ([II*NN] = [II*NN] + [II*NN] - [II*NN])

GO TO 120

118 (II*NN) = -8(II*NN)

IF (2*PI-1) 118, 118, 119

COMPUTE SHELL TERMS

117 SUM2 = SUM2 + COS(FXI*PI/2)*COS(FXI*PI/2)*FXI*PI/2)

DO 117 K=1, NSTR

116 SUM2 = 0.00

IF (2*PI-IETA) 120, 116, 120

COMPUTE STRINGS TERMS

115 CONTINUE

114 ([II*NN] = [II*NN] + [II*NN] + FNU*PI/2)*FNU*PI/2)

GO TO 115

113 ([II*NN] = [II*NN])

SUBMATRX **2.0 EQUATION

CONTINUE

120 CONTINUE

119 ([II*NN] = [II*NN] + [II*NN] + [II*NN] - [II*NN])

GO TO 120

118 (II*NN) = -8(II*NN)

IF (2*PI-1) 118, 118, 119

COMPUTE SHELL TERMS

117 SUM2 = SUM2 + COS(FXI*PI/2)*COS(FXI*PI/2)*FXI*PI/2)

DO 117 K=1, NSTR

116 SUM2 = 0.00

IF (2*PI-IETA) 120, 116, 120

COMPUTE STRINGS TERMS

115 CONTINUE

114 ([II*NN] = [II*NN] + [II*NN] + FNU*PI/2)*FNU*PI/2)

GO TO 115

113 ([II*NN] = [II*NN])
B(I1,NN) = 2.0*(1.0-PNU*PNU)*SUM2*(FJ*FN1*ERING*ARING/RS*EL+FJ**3*
1FN1*FN2*ERING*ARING*ZRING)/(ESHEL*HSHEL*SHTL)
C
C COMPUTE SHELL TERMS
C
IF(I=IZETA)143,144,143
143 C(I1,NN)=B(I1,NN)
GO TO 145
144 C(I1,NN)=B(I1,NN)+FJ*FN1/RSHEL
145 CONTINUE
C
SUBMATRIX W**2...V EQUATION
C
DO 150 P=1,ISIZE
NN=NN+1
INTGP=(P-1)/STAR
I=P-INTGP*STAR
J=INTGP+1
FI=I
FJ=J
C
C COMPUTE RING TERMS
C
IF(2*J-IXI)150,146,150
146 SUM2=0.00
DO 147 K=1,NRNG
FK=K
147 SUM2=SUM2+COS(FI*FN1*FK*DRING)**2*COS(FZETA*FM1*FK*DRING)
B(I1,NN)=(1.0-PNU*PNU)*FJ**3*FN2*ERING*ARING*SUM2/(ESHEL*
1*HSHEL*SHTL)
C
C COMPUTE SHELL TERMS
C
IF(2*I=IZETA)148,149,148
148 C(I1,NN)=-B(I1,NN)
GO TO 150
149 \( C_{II,NN} = -B_{II,NN} - (F_{J*FN1})**3/4.0 - F_{I*F_{I*F_{J*FM2*FN1}/4.0}} \)
150 CONTINUE
160 CONTINUE
RETURN
END
UVCF253
UVCF254
UVCF255
UVCF256
UVCF257
UVCF258
SUBROUTINE WCOEF

C THIS SUBROUTINE CALCULATES THE W EQUATION COEFFICIENTS
C IN A MATRIX CALLED WCOF

INTEGER P,Q

THE DIMENSION SIZE FOR THE SUBSCRIPTED VARIABLES MUST BE EQUAL TO
C OR LARGER THAN ISIZE FOR THE FIRST SUBSCRIPT AND LARGER THAN 8*ISIZE
C FOR THE SECOND SUBSCRIPT.

DIMENSION 8(5,40),0(5,40)
COMMON WCOF(5,40),C(10,20)•ISTAR,JSTAR,MX,NCIR,ESHE3,HSHEL,PNUM,
C INRING,ERING,ERING,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,
C 2FN1,FM2,FM1,FM2,SMDEN,RGDEN,STDEN,RGMI,STNI,GRING,GSTR,RGJ,STJ,
C 3PLOAD,RSHFL,SHLT,FLAG,NDOT

C SET WORKING MATRICES EQUAL TO ZERO

MSIZE=8*ISIZE
DO 2 J=1,MSIZE
DO 1 I=1,ISIZE
B(I,J)=0.00
D(I,J)=0.00
WCOF(I,J)=0.00
CONTINUE

1 CONTINUE

2 COMPUTE COMPLETE W EQUATION MATRIX

II=0
DO 190 Q=1,ISIZE
II=II+1
INTGQ=(Q-1)/ISTAR
IZETA=Q-INTGQ*ISTAR
IXI=INTGQ+1

WCOF 1
WCOF 2
WCOF 3
WCOF 4
WCOF 5
WCOF 6
WCOF 7
WCOF 8
WCOF 9
WCOF 10
WCOF 11
WCOF 12
WCOF 13
WCOF 14
WCOF 15
WCOF 16
WCOF 17
WCOF 18
WCOF 19
WCOF 20
WCOF 21
WCOF 22
WCOF 23
WCOF 24
WCOF 25
WCOF 26
WCOF 27
WCOF 28
WCOF 29
WCOF 30
WCOF 31
WCOF 32
WCOF 33
WCOF 34
WCOF 35
WCOF 36
FZETA=IZETA
FXI=IXI

C SUBMATRIX U...W EQUATION

NN=0
DO 105 P=1,ISIZE
NN=NN+1
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
FJ=J

C COMPUTE STRINGER TERMS

IF(I-IZETA)105,101,105
101 SUM2=0.00
DO 102 K=1,MSTR
FK=K
102 SUM2=SUM2+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)
     B(I+NN)=(FI*FM1)**3*ESTR*ASTR*ZSTR*SUM2/(PI*RSHEL)

C COMPUTE SHELL TERMS

IF(J-IXI)103,104,103
103 WCOF(I+NN)=B(I+NN)
GO TO 105
104 WCOF(I+NN)=B(I+NN)+ESHEL*HSHEL*PNU*FI*FM1/(RSHEL*(1.0-PNU*PNU))
105 CONTINUE

C SUBMATRIX V...W EQUATION

DO 110 P=1,ISIZE
NN=NN+1
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
FJ=J

C COMPUTE RING TERMS

C IF(J-IXI)110,106,110
106 SUM2=0.00
DO 107 K=1,NRNG
FK=K
107 SUM2=SUM2+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)
B(II,NN)=2*ERING*ARING*SUM2*((FJ*FN1)**3*RING+FJ*FN1/RSHEL)/SHLT

C COMPUTE SHELL TERMS

C IF(I-IZETA)108,109,108
108 WCOF(II,NN)=B(II,NN)
GO TO 110
109 WCOF(II,NN)=B(II,NN)+ESH*HSHEL*FJ*FN1/(RSHEL*(1.0-PNU*PNU))
110 CONTINUE

C SUBMATRIX W...W EQUATION

C DO120P=1,ISIZE
NN=NN+1
120 CONTINUE

C SUBMATRIX W**2...W EQUATION

C DO 130 P=1,ISIZE
NN=NN+1
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
DO 160 P=1,100
130 CONTINUE

IF(RHESK*(6.0-RHESK)*(1.0-PN/PN+PN))
139 WCPF(J,NN)=(1.0*NN+D(J,NN)+EHEL+HEL*PN+U*PN*(F*PN)

WRITE(10,128)
128 WCPF(J,NN)=G(D(J,NN))

WRITE(10,129)
129 WCPF(J,NN)=B(J,NN)

WRITE(10,130)
130 IF(J-0.1*II+1.0*II,126,128)

131 IF(J-0.1*II+1.0*II,126,128)

WRITE(10,132)
132 SMW=SMW+0.1*CRING*

WRITE(10,133)
133 IF(J-0.1*II+1.0*II,127,134)

 testify your terms

WRITE(10,134)
134 S=K=1.1*SMW

WRITE(10,135)
135 IF(J-0.1*II+1.0*II,127,134)

F=J
NN=NN+1
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
FJ=J

C COMPUTE STRINGER TERMS
C
DEL1=0.00
DEL2=0.00
IF(I-IZETAl134,134
132 SUM1=0.00
131 DO 133 K=1,MSTR
FK=K
133 SUM1=SUM1+COS(FJ*FN1*FK*DSTR)**3*COS(FXI*FN1*FK*DSTR)
B(II,NN)=DEL1*SUM1*ESTR*ASTR*(F*FM1)**4/(PI*RSHEL*8.0)
GO TO 136
134 IF(3*I-IZETA)136,135,136
135 DEL1=-1.0
GO TO 132
136 IF(3-I-IXI)147,137,147
137 DEL2=+1.0
C COMPUTE RING TERMS
C
138 SUM1=0.00
137 DO 139 K=1,NRNG
139 SUM1=SUM1+COS(FI*FM1*FK*DRING)**3*COS(FZETA*FM1*FK*DRING)
D(II,NN)=DEL2*2.0*SUM1*(FJ*FN1)**4*ERING*ARING/(8.0*SHLT)
IF(DEL1)140,151,140
C COMPUTE SHELL TERMS
C
140 IF(DEL1*DEL2)141,141,144
C     COMPUTE STRINGER TERMS
C
161  IF(2*I-IZETA)165,161,165
     SUM1=0.00
     DO 162 K=1,MSTR
     FK=K
     162  SUM1=SUM1+COS(FJ*FN1*FK*DRING)**2*COS(FXI*FN1*FK*DSRT)
C     COMPUTE SHELL TERMS
C
163  WCOF(I1,NN)=B(I1,NN)
     GO TO 165
164  WCOF(I1,NN)=B(I1,NN)+ESHEL*HSHEL*(((FI*FM1)**3+FI*FM1*FJ*FJ*FN2)/
     1(4.0*(1.0-PNU*PNU)))
     CONTINUE
C     SUBMATRIX V*W...W EQUATION
C
165  DO 170 P=1,ISIZE
     NN=NN+1
     INTGP=(P-1)/ISTAR
     I=P-INTGP*ISTAR
     J=INTGP+1
     FI=I
     FJ=J
C     COMPUTE RING TERMS
C
166  IF(2*J-IXI)170,166,170
     SUM1=0.00
     DO 167 K=1,NRNG
     FK=K
     167  SUM1=SUM1+COS(FI*FM1*FK*DRING)**2*COS(FZETA*FM1*FK*DRING)
     B(I1,NN)=SUM1*ERING*ARING*((FJ*FN1)**3)/SHLT
     CONTINUE
COMPUTE SHELL TERMS

IF(2*I-1-ZETA)168,169,168
168 WCOF(II,NN)=B(II,NN)
GO TO 170
169 WCOF(II,NN)=B(II,NN)+ESHEL*HSHEL*((FJ*FN1)**3+FI*FI*FM2*
1FJ*FN1)/(4.0*(1-PNU*PNU))
170 CONTINUE

SUBMATRIX D2W/DT2...W EQUATION

DO 180 P=1,ISIZE
NN=NN+1
INTGP=(P-1)/ISTAR
I=P-INTGP*ISTAR
J=INTGP+1
FI=I
FJ=J

COMPUTE STRINGER TERMS

IF(I-ZETA)176,171,176
171 SUM1=0.00
DO 172 K=1,MSTR
FK=K
172 SUM1=SUM1+COS(FJ*FN1*FK*DSTR)*COS(FXI*FN1*FK*DSTR)
B(II,NN)=SUM1*STDEN*ASTR/(PI*RSHEL)

COMPUTE RING TERMS

IF(J-IXI)177,174,177
173 SUM1=0.00
DO 175 K=1,NRNG
FK=K
175 SUM1=SUM1+COS(FI*FM1*FK*DRING)*COS(FZETA*FM1*FK*DRING)
D(I1,NN)=SUM1*RGDEN*ARING*2.0/SHLT
IF(I-IZETA)178.179.178
176 IF(J-IXI)180.173.180
177 WCOF(I1,NN)=B(I1,NN)
GO TO 180
178 WCOF(I1,NN)=D(I1,NN)
GO TO 180
179 MCOF(I1,NN)=B(I1,NN)+D(I1,NN)+SHDEN*HSHEL
180 CONTINUE
190 CONTINUE
RETURN
END
SUBROUTINE RUNGE(PRMT,Y,DERY,NDIM,IHLF,FCT,OUTP,AUX)
C
C THIS SUBROUTINE TAKES THE PLACE OF SUBROUTINE RKGS CONTAINED IN
C THE IBM SCIENTIFIC SUBROUTINE PACKAGE.
C FOR A DESCRIPTION OF THE VARIABLES CONTAINED IN THIS SUBROUTINE,
C SEE THE IBM SUBROUTINE RKGS WRITEUP.
C
C THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR
C GREATER THAN 2*ISIZE
C
DIMENSION Y(10),DERY(10),PHI(10),SAVEY(10),SDATA(4,2),PRMT(3)
COMMON WCOFS,MOFS,ISTAR,JSTAR,MNMAX,NCIR,ESHEL,HSHEL,PNU,
INRING,ARING,ERING,ZRING,MSURE,STR,ESHEL,ESHEL,ISIZE,PN,DRING,GINS,
2FN1,FM1,FM2,SHDEN,RGDN,STDEN,REGM,STM,GRING,GSTR,GRGJ,SGJ,
3PLOAD,RSHEL,SHLT,IFLAG,NDOT

1000 FORMAT(1X,E15.8) RUNG 1
1005 FORMAT(11*11) RUNG 2
1005 FORMAT(11*11) RUNG 3
1005 FORMAT(11*11) RUNG 4
1005 FORMAT(11*11) RUNG 5
1005 FORMAT(11*11) RUNG 6
1005 FORMAT(11*11) RUNG 7
1005 FORMAT(11*11) RUNG 8
1005 FORMAT(11*11) RUNG 9
1005 FORMAT(11*11) RUNG 10
1005 FORMAT(11*11) RUNG 11
1005 FORMAT(11*11) RUNG 12
1005 FORMAT(11*11) RUNG 13
1005 FORMAT(11*11) RUNG 14
1005 FORMAT(11*11) RUNG 15
1005 FORMAT(11*11) RUNG 16
1005 FORMAT(11*11) RUNG 17
1005 FORMAT(11*11) RUNG 18
1005 FORMAT(11*11) RUNG 19
1005 FORMAT(11*11) RUNG 20
1005 FORMAT(11*11) RUNG 21
1005 FORMAT(11*11) RUNG 22
1005 FORMAT(11*11) RUNG 23
1005 FORMAT(11*11) RUNG 24
1005 FORMAT(11*11) RUNG 25
1005 FORMAT(11*11) RUNG 26
1005 FORMAT(11*11) RUNG 27
1005 FORMAT(11*11) RUNG 28
1005 FORMAT(11*11) RUNG 29
1005 FORMAT(11*11) RUNG 30
1005 FORMAT(11*11) RUNG 31
1005 FORMAT(11*11) RUNG 32
1005 FORMAT(11*11) RUNG 33
1005 FORMAT(11*11) RUNG 34

1000 FORMAT(1X,E15.8) RUNG 1
1005 FORMAT(11*11) RUNG 2
1005 FORMAT(11*11) RUNG 3
1005 FORMAT(11*11) RUNG 4
1005 FORMAT(11*11) RUNG 5
1005 FORMAT(11*11) RUNG 6
1005 FORMAT(11*11) RUNG 7
1005 FORMAT(11*11) RUNG 8
1005 FORMAT(11*11) RUNG 9
1005 FORMAT(11*11) RUNG 10
1005 FORMAT(11*11) RUNG 11
1005 FORMAT(11*11) RUNG 12
1005 FORMAT(11*11) RUNG 13
1005 FORMAT(11*11) RUNG 14
1005 FORMAT(11*11) RUNG 15
1005 FORMAT(11*11) RUNG 16
1005 FORMAT(11*11) RUNG 17
1005 FORMAT(11*11) RUNG 18
1005 FORMAT(11*11) RUNG 19
1005 FORMAT(11*11) RUNG 20
1005 FORMAT(11*11) RUNG 21
1005 FORMAT(11*11) RUNG 22
1005 FORMAT(11*11) RUNG 23
1005 FORMAT(11*11) RUNG 24
1005 FORMAT(11*11) RUNG 25
1005 FORMAT(11*11) RUNG 26
1005 FORMAT(11*11) RUNG 27
1005 FORMAT(11*11) RUNG 28
1005 FORMAT(11*11) RUNG 29
1005 FORMAT(11*11) RUNG 30
1005 FORMAT(11*11) RUNG 31
1005 FORMAT(11*11) RUNG 32
1005 FORMAT(11*11) RUNG 33
1005 FORMAT(11*11) RUNG 34
CALL FCT(X,Y,DERY)
DO 33 J=1,N
PHI(J)=PHI(J)+2.0*DERY(J)
33 Y(J)=SAVEY(J)+0.5*H*DERY(J)
CALL FCT(X,Y,DERY)
DO 44 J=1,N
PHI(J)=PHI(J)+2.0*DERY(J)
44 Y(J)=SAVEY(J)+H*DERY(J)
X=X+0.5*H
CALL FCT(X,Y,DERY)
DO 55 J=1,N
55 Y(J)=SAVEY(J)+(PHI(J)+DERY(J))*H/6.0
N9=N-1
WRITE(6,1020)X,(Y(J),J=1,N9*2)
IF(ABS(Y(1))<0.5)56,37,57
IF(X-XMAX)10,60,60
56 WRITE(6,1025)
NDT=1
GO TO 80
57 WRITE(6,1025)
NDT=1
GO TO 80
60 IF(Y(1)<0.001)80,70,70
70 WRITE(6,1005)
WRITE(6,1010)MAX,NCIR,PNX
80 RETURN
END
SUBROUTINE FCT(X,Y,DERY)
C
THIS SUBROUTINE CONTAINS NO READ AND WRITE STATEMENTS.
C
THE DIMENSION SIZE FOR THE NEXT STATEMENT MUST BE EQUAL TO OR
GREATER THAN 2*ISIZE, WHERE ISIZE=ISTAR*JSTAR
C
DIMENSION Y(10),DERY(10),A(10,10),R(10),T(10),Z(10)
C
THE DIMENSION SIZE FOR C(nn) MUST BE EQUAL TO OR GREATER
THAN 2*ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR GREATER
THAN SIZE FOR THE SECOND SUBSCRIPT.
C
THE DIMENSION SIZE FOR WCOF(nn) MUST BE EQUAL TO OR GREATER
THAN ISIZE FOR THE FIRST SUBSCRIPT AND EQUAL TO OR GREATER THAN
8*ISIZE FOR THE SECOND SUBSCRIPT.
C
COMMON WCOF(5,40),C(10,20),ISTAR,JSTAR,MAX,NCIR,ESME3,HSHEL,PNU,
INRNG,ARING,ERNG,ZRING,MSTR,ASTR,ESTR,ZSTR,ISIZE,PI,DRING,DSTR,
2FN1,FN2,FM1,FM2,SHDEN,RGDEN,STDEN,RGMI,STMI,GRING,GSTR,RGJ,STJ,
3PLDAD,RSHEL,SHLT,IFLAG,NDOT
C
PLACE U AND V COEFFICIENTS INTO THE GAUSS-JORDAN MATRIX A(N)
C
MARK=0
NSIZE=2*ISIZE
DO 16 J=1,NSIZE
DO 15 I=1,NSIZE
15 A(I,J)=C(I,J)
16 CONTINUE
C
PLACE PROPER U AND V COEFFICIENTS INTO R(N) MATRIX
C
DO 19 K=1,NSIZE
R(K)=0.00
DO 21 K=1,NSIZE
DO 20 J=1,ISIZE
J1 = J + 2 * ISIZE
J2 = J + 3 * ISIZE
20 R(K) = R(K) + C(K, J1) * Y(2 * J - 1) + C(K, J2) * Y(2 * J - 1) * Y(2 * J - 1)
21 CONTINUE
C
C CHANGE SIGN OF R(I) MATRIX
C
DO 35 I = 1, NSIZE
35 R(I) = -R(I)
22 N = NSIZE
C
C SOLVES N SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS BY A GAUSS-JORDAN
C REDUCTION. A IS THE COEFFICIENT MATRIX, R IS THE CONSTANT
C VECTOR. A IS DESTROYED AND R IS REPLACED BY THE SOLUTION
DO 11 J = 1, N
11 T(J) = 1.0
M = N - 1
DO 1 J = 1, M
AD = A(J, J)
DO 3 K = J, N
3 A(J, K) = A(J, K) / AD
R(J) = R(J) / AD
L = N - J
NN = 0
DO 1 I = 1, L
NN = J + I
DA = A(NN, J)
DO 2 K = J, N
2 A(NN, K) = A(NN, K) - A(J, K) * DA
1 R(NN) = R(NN) - R(J) * DA
DO 4 J = 1, N
K = N + 1 - J
S = 0.0
IF(K-N)9, 4, 9
9 L = K + 1
DO 5 I = L, N
5 \( S = A(K,I) * T(I) + S \)
4 \( T(K) = (R(K) - S) / A(K,K) \)
6 \( R(J) = T(J) \)
C
5 IF(MARK-1)23,30,30
C
C
C
23 DO 32 I=1,NSIZE
DO 33 J=1,NSIZE
33 A(I,J) = 0.00
32 CONTINUE
C
C
C
MAKEM(A(N,N))INTOANIDENTITYMATRIX
C
23 DO 34 IJ=I,NSIZE
34 A(IJ,IJ) = 1.00
C
C
PLACEWCOEFFICIENTSINTOGAUSS-JORDANMATRIXA(N)
C
DO 25 J=1,ISIZE
DO 24 I=1,ISIZE
JN=J+7*ISIZE
24 A(2*I,2*J)=WCOF(I,JN)
25 CONTINUE
C
C
C
SETZ(N)=ZERO
C
DO 26 I=1,NSIZE
26 Z(I) = 0.00
C
C
PLACEPROPERWCOEFFICIENTSINTOR(N)
C
DO 28 K=1,ISIZE
Z(2*K-1) = Y(2*K)
DO 27 J=1,N SIZE
30 D31=[1]*N SIZE
DEBY1=RI1
END

CHANGE SIGN OF R(2*K)
DO 36 K=1,N SIZE
R(2*K)=R(2*K)
GO TO 22

CONTINUE
2*W(COF(K-J,J)+(2*J-1))
3*W(COF(K-J,J)+(2*J-1))