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CLASSIFICATION OF CODIMENSION ONE BIQUOTIENT FOLIATIONS IN LOW DIMENSIONS

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Contents

Abstract

In this thesis we study compact simply connected C1BFs (manifolds which arise as quotients of cohomogeneity one manifolds). In particular, we study various elementary properties of C1BFs including their topological and curvature properties. Moreover, we give a classification of their structures in low dimensions and also show that all simply connected manifolds which admit non-negative curvature in low dimensions admit a C1BF structure.

Chapter 0

Introduction

A very interesting class of Riemannian manifolds are those which admit metrics of non-negative sectional curvature. Elementary examples of such manifolds are the Euclidean spaces, spheres, and Lie groups equipped with a bi-invariant metric. Non-negatively curved manifolds have been of great interest in Riemannian geometry and a great deal of effort has been put into finding examples of manifolds of non-negative curvature. Aside from taking products, the easiest way of producing new examples is by taking quotients of manifolds with non-negative sectional curvature, which again have non-negative curvature by O'Neill's formula [O'N66]. In particular, a very large class of non-negatively curved manifolds are the *biquotient manifolds*. A biquotient manifold is any manifold which can be expressed as the quotient of a homogeneous space $M = G/H$ by a free isometric group action, where G is assumed to be compact. Note that by taking the free isometric group action to be the trivial action by the trivial group, we see that the class of biquotients contains all homogeneous spaces as a special case.

Biquotients first appeared in Riemannian geometry shortly after Milnor's discovery of the so called *exotic spheres* (manifolds which are homeomorphic but not diffeomorphic to the standard 7-dimensional sphere S^7) [Mil56]. Milnor's original construction of the exotic spheres realized them as S^3- bundles over S^4. Later, it was shown by Milnor, Smale, Kervaire–Milnor and Eells–Kuiper [KM63, Sma61, EK62] that there are in fact 28 possible oriented differentiable structures on S^7 and that 20 of these are diffeomorphic to S^3-bundles over S^4. With the discovery of the exotic spheres, a natural question was whether the exotic spheres admit metrics of non-negative sectional curvature. This question turned out to be quite difficult to answer. The first progress toward answering this question came when Gromoll and Meyer showed that one of the exotic spheres in dimension 7 is a biquotient, hence admits non-negative sectional curvature [GM74]. This is particularly interesting because a result due to Borel says that any homogeneous space that is homeomorphic to a sphere is necessarily diffeomorphic to a sphere. This in particular shows that the class of biquotients is strictly larger than the class of homogeneous spaces. It was later shown [Tot02, KZ04] that the Gromoll-Meyer sphere is the only exotic sphere which can be written as a biquotient. In particular, new techniques would need to be used to have any hope of putting metrics of non-negative curvature on the remaining exotic spheres.

Nearly three decades after the Gromoll-Meyer sphere was shown to admit non-negative curvature, it was shown by Grove-Ziller [GZ00] that all of the so called *Milnor spheres* (i.e. exotic spheres that are also S^3-bundles over S^4) in dimension 7 admit metrics of non-negative sectional curvature. In particular, they showed that any cohomogeneity one manifold with singular orbits of codimension at most two admits a metric of non-negative sectional curvature. They were then able to associate to each Milnor sphere a certain principal SO(3)-bundle over S^4 that is cohomogeneity one and satisfying this property and deduce that the Milnor spheres have non-negative sectional curvature. After this groundbreaking result, the question had been answered for all but 8 of the exotic spheres in dimension 7. The question of whether the remaining 8 exotic spheres admit metrics of non-negative sectional curvature remained unanswered for nearly two decades. Recently, Goette, Kerin, and Shankar were able to finish the problem using a new construction which yields all of the Grove-Ziller examples as a special case [GKS20].

As we have seen, cohomogeneity one manifolds and biquotients have played a very important role in finding new examples of manifolds with non-negative sectional curvature. One may then wonder about manifolds that are constructed from biquotients. It turns out that there is an interesting class of manifolds, which we will call *codimension one biquotient foliations*, or C1BFs, used in Goette, Kerin, and Shankar's work to show that the remaining exotic spheres admit non-negative curvature. In particular, a C1BF is a singular Riemannian foliation [Mol88] of a manifold where the principal leaf (i.e. the diffeomorphism class of leaves of maximal dimension forming an open dense set) are biquotients of codimension one. C1BFs can be easily constructed by taking quotients of any cohomogeneity one manifold (M,G) by any free isometric action contained within the group G. In particular, unless otherwise stated, a C1BF is such a quotient of a cohomogeneity one manifold. Note that by taking free isometric action to be the trivial action by the trivial subgroup of G, we see that every cohomogeneity one manifold is a C1BF. Thus cohomogeneity one manifolds are a special case of C1BFs.

C1BFs are quite ubiquitous and our first result provides a wealth of examples of

C1BFs.

Theorem 0.0.1. *Suppose $M \approx (S^{n_1} \times \cdots \times S^{n_r})/T^k$ is diffeomorphic to a quotient of a product of spheres by an effectively free linear torus action. Then M is a C1BF.*

Furthermore, we will see that all currently known examples of simply connected manifolds which admit metrics of non-negative sectional curvature up to dimension 6 give rise to a C1BF structure in a natural way. In fact, up to dimension 5 are biquotients, and we obtain as a byproduct of this result that every representation of such a manifold M^n, $n \leq 6$, as a *reduced biquotient* (defined in Chapter 1) gives rise to a C1BF structure on M.

Theorem 0.0.2. *All known examples of compact, simply connected manifolds of dimension at most 6 which admit a metric of non-negative sectional curvature are C1BFs. Furthermore, for all such examples which are diffeomorphic to biquotients, all representations of these manifolds as a reduced biquotient $G//H$ naturally give rise to a C1BF structure.*

C1BFs are a very natural generalization of cohomogeneity one manifolds and, as we will see later, C1BFs arising as quotients of cohomogeneity one manifolds have a structure very similar to that of cohomogeneity one manifolds. A great deal of effort has been put into classification results for the special case where the C1BF is cohomogeneity one. For instance, Hoelscher has given a classification of all cohomogeneity one actions in terms of their group diagram up to dimension 7 [Hoe10], Frank has given a classification of cohomogeneity one manifolds with positive Euler characteristic [Fra13], DeVito and Kennard have given a classification of cohomogeneity one manifolds with singly generated rational cohomology [DK20], and Straume and Wang [Str96, Wan60] have given classifications of cohomogeneity one actions on spheres as well as homology spheres. We note that Wang's classification contains a gap.

In this thesis we will do a certain classification of C1BFs in low dimensions. In particular, we will classify all possible *leaf structures* which occur in low dimensions for compact simply connected C1BFs, where by leaf structure we mean triples (P, X, Y) of diffeomorphism types of the principal leaf P and singular leaves X and Y. A leaf structure is said to be an *admissible leaf structure* provided that there exists a compact simply connected C1BF which realizes the leaf structure. Such a classification in higher dimensions is currently intractable because there is no complete classification of biquotients. However, there are several partial classifications such as DeVito's classification of compact simply connected biquotients in dimensions up to dimension 7 [DeV14, DeV17] and Kapovitch-Ziller's classification of biquotients with singly generated rational cohomology [KZ04]. Such classification of C1BFs is important because such a classification in dimension 6 may yield new examples of manifolds with nonnegative sectional curvature.

We present now our classification results. In dimension 4 we have the following theorem.

Theorem 0.0.3. *Let M be a compact, simply connected, 4-dimensional C1BF and let $\mathscr{L} = S^3/Q_8 \approx SO(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ denote the nonclassical lens space with fundamental group Q_8. The following list is the complete list of admissible leaf structures:*

1. (S^3, pt, pt)
2. (S^3, S^2, pt)
3. (S^3, S^2, S^2)
4. $(S^2 \times S^1, S^2, S^2)$
5. $(S^2 \times S^1, S^2, S^1)$
6. $(L_m(1), S^2, S^2) \quad (m \geq 2)$
7. $(\mathscr{L}, \mathbb{R}P^2, \mathbb{R}P^2)$
8. $(L_4(1), S^2, \mathbb{R}P^2)$

Furthermore, the topology of such C1BFs can be described as follows.

(i) *C1BFs of type 1,5, or 7 are diffeomorphic to* S^4

(ii) *C1BFs of type 2 or 8 are diffeomorphic to* $\mathbb{C}P^2$.

(iii) *C1BFs of type 6 are diffeomorphic to* $\mathbb{C}P^2\# -\mathbb{C}P^2$ *if* m *is odd. If* $m \geq 4$ *is even, they are diffeomorphic to* $S^2 \times S^2$*. In the special case* $m = 2$*, such a C1BF is diffeomorphic to either* $S^2 \times S^2$ *or* $\mathbb{C}P^2\#\mathbb{C}P^2$.

(iv) *A C1BF of type 4 is diffeomorphic to either* $S^2 \times S^2$ *or* $\mathbb{C}P^2\#\mathbb{C}P^2$

It is worth noting that Parker [Par86] has shown that $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is not cohomogeneity one. Thus we have the following corollary.

Corollary 0.0.4. $\mathbb{C}P^2\#\mathbb{C}P^2$ *the lowest dimensional example of a manifold which admits a C1BF structure but does not admit a cohomogeneity one structure.*

We note that the 5-dimensional classification has one exceptional case in the sense that it is, at the time of this thesis submission, incomplete. In particular, the case where the C1BF has principal leaf $L_m(r) \times S^1$ turns out to be rather complex compared to the other cases, as we will see below in Chapter 3. In particular, there is one infinite family of leaf structures which, at the time of submission of this thesis, we were unable to determine whether all leaf structures which get past the sphere bundle and van Kampen theorem obstructions are admissible. For a more detailed explanation of what is known about this infinite family, see Case C.6.1 of Chapter 3. We note that, due to the classification of simply connected disk bundles in dimension 5, that C1BFs of this type cannot produce new examples of non-negative curvature in the simply connected case.

Theorem 0.0.5. *Let* M *be a compact, simply connected, 5-dimensional C1BF and let* $S^2 \hat{\times} S^1$ *denote the unique nonorientable* S^2*-bundle over* S^1 *and "*$\equiv_m$*" denote congruence* *modulo m. With the exception of some leaf structures of type (11) potentially not being admissible, the following list is the complete list of admissible leaf structures:*

1. (S^4, pt, pt)
2. $(S^2 \times S^2, S^2, S^2)$
3. $(\mathbb{C}P^2 \# -\mathbb{C}P^2, S^2, S^2)$
4. $(S^3 \times S^1, S^3, S^3)$
5. $(S^3 \times S^1, L_m(r), L_n(s))$ $\gcd(m,n) = 1$
6. $(S^3 \times S^1, S^1, S^3)$
7. $(S^3 \times S^1, S^3, S^2 \times S^1)$
8. $(S^3 \times S^1, S^3, L_m(r))$
9. $(S^3 \times S^1, S^2 \times S^1, L_m(r))$
10. $(S^2 \times T^2, S^2 \times S^1, S^2 \times S^1)$
11. $(L_m(r) \times S^1, L_n(r); L_k(s))$ $m|n, k,$ *and* $s \equiv \pm_m r^{\pm 1}$
12. $(L_m(1) \times S^1, S^2 \times S^1, L_n(1))$ *where* $m|n$
13. $(L_2(1) \times S^1, L_2(1); S^2 \hat{\times} S^1)$

In dimension 6 the difficulty of such a classification increases substantially because the singular leaves can be 5-dimensional biquotients and they need not be simply connected. We have restricted our attention to the case where the principal leaf is simply connected which ensures that the singular leaves are also simply connected. With this restriction, we have the following classification theorem

Theorem 0.0.6. *Let M be a compact, simply connected, 6-dimensional C1BF. Let $S^3 \hat{\times} S^2$ denote the nontrivial S^3-bundle over S^2. With the possible exception of leaf structure (19), the following list is the complete list of admissible leaf structures for M:*

1. (S^5, pt, pt)
2. $(S^5, pt, \mathbb{C}P^2)$
3. $(S^5, \mathbb{C}P^2, \mathbb{C}P^2)$
4. $(S^3 \times S^2, S^2, S^2)$
5. $(S^3 \times S^2, S^3, S^2)$
6. $(S^3 \times S^2, S^2 \times S^2, S^2)$
7. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2)$
8. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^2)$
9. $(S^3 \times S^2, S^3, S^3)$
10. $(S^3 \times S^2, S^3, S^2 \times S^2)$
11. $(S^3 \times S^2, S^3, \mathbb{C}P^2 \# \mathbb{C}P^2)$
12. $(S^3 \times S^2, S^3, S^2 \times S^2)$
13. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^3)$
14. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^3)$
15. $(S^3 \times S^2, S^2 \times S^2, S^2 \times S^2)$
16. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2 \times S^2)$
17. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^2 \times S^2)$
18. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# \mathbb{C}P^2)$
19. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$
20. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$
21. $(S^3 \hat{\times} S^2, S^2, S^2)$
22. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^2)$
23. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2)$
24. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$
25. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$
26. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# \mathbb{C}P^2)$

In the case of leaf structure (19), there does not exist a representation of a C1BF as a group diagram with $G = Sp(1) \times Sp(1)$ *hence any C1BF which possibly admits such a leaf cannot do so with the principal or singular leaves given as reduced biquotients.*

Note that it is likely that leaf structure (19) does not occur as a C1BF arising as a quotient of a cohomogeneity one manifold. In particular, DeVito has outlined an approach to showing that such a leaf structure does not arise for any G (see chapter 3). However, there certainly exist double disk-bundles with this leaf structure. In particular, there exits C1BFs with principal leaf $S^3 \times S^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $\mathbb{C}P^2 \# - \mathbb{C}P^2$ as *one* of the singular leaves. One can simply take two such C1BFs, separate the two halves, and glue the two halves together to form a double disk-bundle with leaf structure (19). This would be a good candidate for a new example of non-negative curvature. Although it is worth noting that, since it is likely not a C1BF, it is not immediate that the codimension 2 singular leaves guarantee non-negative curvature.

Chapter 1

Preliminaries

1.1 Transformation Groups

In this section we will establish some notation that will be used throughout as well as introduce some basic results about transformation groups. We start by recalling some standard terminology for group actions. The theory of compact transformation groups is extensive, and this is not meant to be complete, but rather to introduce some notation and standard terminology that we will use. For a complete treatment of these topics, see [Bre72, Lee13]. Recall that if G is a group and M is a set, *a left action* of G on M is a map $G \times M \to M$, often written as $(g, p) \mapsto g \cdot p$, that satisfies

$$g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p \quad \text{for all } g_1, g_2 \in G \text{ and } p \in M$$
$$e \cdot p = p \quad \text{for all } p \in M$$

and a *right action* is defined similarly. The group G will always be assumed to be a compact Lie group and M a smooth manifold, unless otherwise stated, and we will usually want the action to be *smooth*; that is, defining map $G \times M \to M$ is smooth. A manifold M endowed with an action of a group G is often referred to as a G-*space*. Group actions will be used quite often throughout this thesis and sometimes there will be more than one group action present. Often times other notation will be used for a Lie group G acting on a manifold M. We may, for example, denote the action by $(g, p) \mapsto g \star p$. For each $p \in M$, the *orbit of* p will be denoted by $G \cdot p$ and is the set of all images of p under the action by elements of G; that is,

$$G \cdot p = \{g \cdot p : g \in G\}$$

Similarly, for each $p \in M$, the *isotropy group of* p, denoted by G_p, is the set of elements of G that fix p:

$$G_p = \{g \in G : g \cdot p = p\}.$$

Let $N = \{g \in G : g \cdot x = x \text{ for all } x \in X\}$. Then N is a closed normal subgroup of G, called the *ineffective kernel*. A group action is said to be *transitive* if for every pair of points $p, q \in M$, there exists a $g \in G$ such that $g \cdot p = q$, or equivalently, if the only orbit is all of M. A group action of G on M is said to be *effective* if the ineffective kernel N is trivial. A group action is said to be *free* if the only element of G that fixes any element of M is the identity; that is, $g \cdot p = p$ for some $p \in M$ implies $g = e$, or equivalently, if every isotropy group is trivial. We say that two group actions on a space M are *orbit equivalent* if the two actions have the same orbits. We observe that if G acts on a space M, then the induced action of the quotient group G/N on M is effective and is orbit equivalent to the action of G on M. We say that the action of a group G on a set M is *effectively free* if for all $g \in G$, if there is an $p \in M$ such that $g \cdot p = p$, then $g \in N$.

It is useful to have a notion of when two group actions are equivalent. We say that the action of G_1 on M_1 is *equivalent* to the action of G_2 on M_2 if there is a diffeomorphism $f : M_1 \to M_2$ and an isomorphism $\varphi : G_1 \to G_2$ such that $f(g \cdot p) = \varphi(g) \star f(p)$ for all $p \in M_1$ and $g \in G_1$. In the special case $G_1 = G_2$ there is a stronger type of equivalence that is usually preferred. A map $f : M_1 \to M_2$ between G-manifolds is *G-equivariant* if $f(g \cdot p) = g \star f(p)$ for all $p \in M_1$ and $g \in G$. Two group actions are equivalent if there exists a G-equivariant diffeomorphism between the two G-spaces. A G-equivariant map is often said to *intertwine* the two actions.

Recall that for an action of a group G on a manifold M, one can define an equivalence relation on M whose equivalence classes are precisely the orbits of G in M. The set of orbits is denoted by M/G and with the quotient topology, it is called the *orbit space* of the action. We recall that the quotient manifold theorem implies that if a compact group G acts smoothly and freely, on a smooth manifold M, then the orbit space is a smooth manifold of dimension equal to $\dim M - \dim G$. Since any effectively free action is orbit equivalent to a free action by taking the quotient of the acting group by the ineffective kernel, it follows that the quotient of a manifold by a compact, smooth, effectively free group action is again a smooth manifold.

The following elementary proposition is taken from [Bre72]. This important proposition tells us that the orbits of a group action are diffeomorphic to the quotient of the group by the isotropy at each point.

Proposition 1.1.1. *Suppose G acts on a smooth manifold M and let $p \in M$. The map $\alpha_p : G/G_p \to G \cdot p$ defined by $\alpha_p(gG_p) = g \cdot p$ is a diffeomorphism, where $G \cdot p$ denotes the orbit of p and G_p denotes the isotopy subgroup at p.*

The following proposition is often useful for computing quotients of manifolds by actions where the group G is disconnected. The proof is left to the reader.

Proposition 1.1.2. *Suppose a compact Lie group G acts on M and suppose N is a closed normal subgroup of G. Then M/G is canonically diffeomorphic to $(M/N)/(G/N)$, where G/N acts on M/N by $gN\cdot[p]=[g\cdot p]$.*

The above proposition is most often used to compute the diffeomorphism type of the quotient of a manifold M by the action of a disconnected group G. In this case, one takes $N = G_0$, where G_0 denotes the identity component of G. Hence one first computes the quotient of M by the identity component, followed by the quotient of the resulting space by the group of components of G.

Proposition 1.1.3. *Let G be a compact Lie group acting continuously and transitively on a connected manifold M. Then the identity component G_0 also acts transitively on M.*

Proof. Assume G acts on M transitively. Let $p\in M$. We show that the orbit of p by G_0 acting on p is all of M. Let $\theta_p : G\to M$ be the orbit map defined by $\theta_p(g) = g\cdot p$. In general the quotient map by a group action is an open map, therefore, the quotient map $G\to G/G_p$ is an open map (since G_p acts on G by $h\cdot g = gh^{-1}$). Moreover, the map $G/G_p \to G\cdot p = M$ given by $gG_p \mapsto g\cdot p$ is a homeomorphism hence their composition

$$G\to G/G_p \to G\cdot p$$

is open. Moreover, this composition is θ_p, hence θ_p is open. Since G is compact, θ_p is also closed. Now, the identity component G_0 is both open and closed. The restriction of an open map to an open set is an open map hence the restriction of θ_p to G_0 is open.

The same statement holds with “open" replaced with “closed". It follows that the orbit of p by G_0 is clopen and hence is all of M since M is connected. $\square$

Another useful tool that is often useful for computing quotients is the *associated bundle construction*:

Associated Bundle Construction:

Let $\pi : P \to M$ be a principal G-bundle arising from a right action of G on P and let $\star$ denote a smooth left action of G on a manifold F. Define a right action of G on $P \times F$ by $(p, q) \cdot g = (p \cdot g, g^{-1} \star q)$. We then take the quotient by this action to obtain the space $E = P \times_G F$. Note that $[p \cdot g, q] = [p, g \star q]$ for all $g \in G$. Then the projection map $\overline{\pi} : E \to M$ defined by $\overline{\pi}[p, q] = \pi(p)$ is a fiber bundle with fiber F and structure group G.

A fundamental theorem of compact transformation groups is the so-called *Slice Theorem*. Let M be a Riemannian manifold and let $S \subset M$ be a closed submanifold of M. Let $\nu(S)$ denote the normal bundle over S; that is, the subbundle of TM consisting of vectors based at points of S which are perpendicular to S. Furthermore let $\nu^{<\epsilon}(S) = \{v \in \nu(S) : |v| < \epsilon\}$ denote vectors of $\nu(S)$ of length less than some $\epsilon > 0$. Let $\nu_x(S) \subset \nu(S)$ denote vectors in $\nu(S)$ with basepoint $x \in S$. Finally, let $\mathcal{N}_\epsilon(S) = \exp(\nu^{<\epsilon}(S))$. With this notation, we now state a version of the slice theorem:

Theorem 1.1.4. *(The Classical Slice Theorem) Let G be a compact Lie group acting isometrically on a Riemannian manifold M. For all $p \in M$, the orbit $G \cdot p$ is an embedded submanifold of M. Moreover, for all $p \in M$, there exists an $\epsilon > 0$ so that the slices $S_x = \exp_x\left(\nu_x^{<\epsilon}(G \cdot p)\right)$ at $x \in G \cdot p$ and the tubular neighborhood $G \cdot S_p = \mathcal{N}_\epsilon(G \cdot p) = \bigcup_{x \in G \cdot p} S_x$ about $G \cdot p$ satisfy the following properties:*

(i) The slices S_x are pairwise disjoint; that is, $S_x \cap S_y = \emptyset$ for all $x, y \in G \cdot p$ and $x \neq y$.

(ii) $g \cdot S_x = S_{g\cdot x}$ *for all* $g \in G$ *and* $x \in G \cdot p$.

(iii) G_x *acts on* S_x*, and the action is* G*-equivariant via* $\exp_x$ *to the isotropy representation* $G_x \to Isom\big(\nu_x(G\cdot p)\big)$.

(iv) The map $[g,q] \mapsto g\cdot q$ *is a well defined diffeomorphism* $G \times_{G_p} S_p \to \mathcal{N}_\epsilon(G\cdot p)$ *where* $G\times_{G_p} S_p$ *is the quotient of the action of* G_p *on* $(G\times S_p)$ *by* $h \star (g,q) = (gh^{-1}, h\cdot q)$*, where the action on the second factor is the action given by (iii).*

(v) The map $[g,q] \mapsto gG_p$ *defines a* G*-equivariant fiber bundle projection* $G\times_{G_p} S_p \to G/G_p \approx G\cdot p$ *with fiber* S_p.

1.2 Homogeneous Spaces

Homogeneous spaces are a special case of a biquotients which will be essential to our study. Therefore, it is important that we review some basics about homogeneous spaces.

Definition 1.2.1. A smooth manifold endowed with a transitive smooth action by a Lie group G is called a **homogeneous space**.

Observe that if H and K are conjugate Lie subgroups of G, i.e. there exists $g \in G$ such that $K = gHg^{-1}$, then the map $h \mapsto ghg^{-1}$ is a Lie group isomorphism $H \to K$.

Proposition 1.2.2. *Let* M *be a topological space with an action by a Lie group* G*. If* $p, q \in M$ *are in the same orbit, then the isotropy groups* G_p *and* G_q *are conjugate. In particular, if* M *is a homogeneous space with Lie group* G*, then all of the isotropy groups under the action by* G *are conjugate.*

Proof. Suppose $g\cdot p = q$. Then it is easy to show that $gG_pg^{-1} = G_q$. The latter statement follows immediately because all points in a homogenous space lie in the same orbit. □

By the previous proposition, for a homogeneous space M, it makes sense to refer to *the* isotropy group of the action. If H is the isotropy group of a homogeneous space, observe that H has a smooth left action on G given by $h \cdot g = gh^{-1}$ and that this action is free. Therefore, the quotient space G/H is a smooth manifold and by Proposition 1.1.1 we have M is diffeomorphic to G/H.

Theorem 1.2.3. *(Classification of Homogeneous Spaces) There is a one-to-one correspondence between homogeneous spaces and manifolds of the form G/H where G is a Lie group and H is a closed Lie subgroup.*

Proof. If M is a homogeneous space with Lie group G, we have already shown that M is diffeomorphic to G/H where H is the isotropy group of G. Conversely, if G is a Lie group and H is a Lie subgroup, then G/H is a homogeneous space via the action of G on G/H given by $g_1 \cdot (g_2 H) = g_1 g_2 H$. □

This theorem, along with the closed subgroup theorem shows that the study of homogeneous spaces reduces to the study of Lie groups and their closed subgroups. Thus a homogeneous space is usually written as $M = G/H$.

1.3 Biquotients

In this section, we will recall some basic properties about biquotients. The basics of biquotients are discussed in detail in Eschenburg's habilitations [Esc84]. For more details about the topology of biquotients see, for instance, [Esc92] where Eschenburg has shown how to compute their cohomology rings and [Sin93]where Singhoff has described how to compute their characteristic classes.

Definition 1.3.1. A **biquotient** is any manifold which can be expressed as a quotient of a homogeneous space $M = G/H$ by an effectively free isometric action.

We are interested in non-negative curvature and when the group G for the homogeneous space is compact, the resulting biquotient has non-negative curvature. This follows from O'Neill's formula [O'N66], which applies more generally to Riemannian submersions. Recall that a *Riemannian submersion* is a smooth map $\pi : M \to N$ such that the pushforward π_* is an isometry on horizontal vectors (that is, vectors orthogonal to the fibers of π). For Riemannian submersions, O'Neill's formula tells us that

$$\sec_N(X,Y) = \sec_M(\tilde{X},\tilde{Y}) + \frac{3}{4}\|[\tilde{X},\tilde{Y}]^v\|^2$$

where X and Y are orthonormal vector fields on N, $\tilde{X}$ and $\tilde{Y}$ are their horizontal lifts to M,$[X,Y]^v$ is the projection of $[X,Y]$ to its vertical part (i.e. the part of [X,Y] tangent to the fibers of π).

When a compact Lie group G acts effectively freely and isometrically on a Riemannian manifold M, the quotient projection $\pi : M \to M/G$ is a Riemannian submersion when M/G is equipped with the quotient metric. Therefore, O'Neill's formula tells us that taking quotients by an effectively free isometric action causes curvature to increase. It follows that any biquotient that is the quotient of a homogeneous space $M = G/H$, with G compact, by a free isometric action admits a metric of non-negative curvature.

As noted in the introduction, every homogeneous space is itself a biquotient by taking the quotient of $M = G/H$ by the free action of the trivial group. In practice, a convenient way to construct biquotients is the following:

Construction 1.3.2. *Let $f = (f_1, f_2) : H \to G \times G$ be a homomorphism of groups. There is* *an induced action of H on G given by $h \cdot g = f_1(h) g f_2(h)^{-1}$. When this action is effectively free, the quotient space, denoted $G/\!/H$, is a biquotient.*

An action as in the construction above will be called a *biquotient action*. In the case where H is a subgroup of G, f can be taken to be the inclusion, therefore, any subgroup H of G gives a biquotient action on G by left and right translation, that is, $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. Furthermore, in the special case $H = K \times L \subset G \times G$ and f the inclusion, we denote the biquotient by $K \backslash G / L$.

Totaro [Tot02] has shown that if $M \simeq G/\!/H$ is a compact, simply connected biquotient, then M is also diffeomorphic to $G'/\!/H'$ where G' is simply connected, H' is connected, and no factor of H' acts transitively on any simple factor of G'. Such biquotients will be called *reduced biquotients*. Reduced biquotients have been classified in low dimensions by Kapovitch-Ziller and DeVito [KZ04, DeV14, DeV17]

The following proposition is sometimes useful for checking whether a biquotient action is effectively free.

Proposition 1.3.3. *Suppose that $f = (f_1, f_2) : U \to G \times G$ is a homomorphism of Lie groups and U acts on G via the corresponding biquotient action. Then the action is effectively free if and only if for all $u \in U$, if $f_1(u)$ is conjugate to $f_2(u)$ in G, then $f_1(u) = f_2(u) \in Z(G)$.*

Proof. Suppose that U acts on G effectively freely and suppose $u \in U$ with $f_1(u)$ conjugate to $f_2(u)$ in G. Then there exists $g_0 \in G$ such that $g_0 f_1(u) g_0^{-1} = f_2(u)$. But

$$g_0 f_1(u) g_0^{-1} = f_2(u) \iff f_1(u) g_0^{-1} f_2(u)^{-1} = g_0^{-1} \iff u \cdot g_0^{-1} = g_0^{-1}.$$

Therefore, u fixes an element and hence, since U acts effectively freely, we must in fact have $u \cdot g = g$ for all $g \in G$. That is, $f_1(u) g f_2(u)^{-1} = g$ and hence $f_1(u) g = g f_2(u)$, for all $g \in G$. In particular, this must hold for $g = e$, which says $f_1(u) = f_2(u)$ and hence it follows that $f_1(u) = f_2(u) \in Z(G)$. Suppose now that for $u \in U$ that $f_1(u)$ is conjugate to $f_2(u)$ if and only if $f_1(u) = f_2(u) \in Z(G)$. We wish to show that U acts on G effectively freely. Pick $g_0 \in G$ and assume $u \cdot g_0 = g_0$. Then $u \cdot g = g \iff f_1(u) g_0 f_2(u)^{-1} = g_0 \iff g_0^{-1} f_1(u) g_0 = f_2(u)$. In particular, $f_1(u)$ and $f_2(u)$ are conjugate. Therefore, $f_1(u) = f_2(u)$ lie in $Z(G)$. It follows immediately that $u \cdot g = g$ for all $g \in G$. □

Corollary 1.3.4. *The biquotient action as in the previous proposition is free if and only if for all $u \in U$, if $f_1(u)$ is conjugate to $f_2(u)$ in G, then $f_1(u) = f_2(u) = e$.*

Proof. Going through the proof of the previous proposition, one sees that when the action is free one must have $f_1(u) = f_2(u) = e$ rather than just lying in $Z(G)$. □

The following proposition shows that to check whether a biquotient action is effectively free, it suffices to check whether it is true on a maximal torus.

Proposition 1.3.5. *Suppose that U is a connected Lie group and suppose that $f = (f_1, f_2) : U \to G \times G$ is a homomorphism of Lie groups and U acts on G via the corresponding biquotient action. The action is effectively free if and only if the action is effectively free when restricted to a maximal torus $T_U \subset U$.*

Proof. If the action is effectively free then clearly it is free when restricted to a maximal torus. Now assume a maximal torus $T_U \subset U$ acts effectively freely on G. Pick $u \in U$ and assume $u \cdot g_0 = g_0$. By the Maximal Torus Theorem, there exists $x \in U$ such that

$xux^{-1} \in T_U$. Then $u \cdot g_0 = g_0 \implies f_1(u) g_0 f_2(u)^{-1} = g_0$ and hence

$$\begin{aligned}
xux^{-1} \cdot f_1(x) g_0 f_2(x)^{-1} &= f_1(xux^{-1}) f_1(x) g_0 f_2(x)^{-1} f_2(xux^{-1})^{-1} \\
&= f_1(x) f_1(u) f_1(x)^{-1} f_1(x) g_0 f_2(x)^{-1} f_2(x) f_2(u)^{-1} f_2(x)^{-1} \\
&= f_1(x) \left[f_1(u) g_0 f_2(u)^{-1} \right] f_2(x)^{-1} \\
&= f_1(x) g_0 f_2(x)^{-1}.
\end{aligned}$$

Therefore, xux^{-1} fixes $f_1(x) g_0 f_2(x)^{-1}$ and, therefore, since $xux^{-1} \in T_U$ and the action of T_U is effectively free, we must have $xux^{-1} \cdot g = g$ for all $g \in G$, that is

$$f_1(x) f_1(u) f_1(x)^{-1} g f_2(x) f_2(u)^{-1} f_2(x)^{-1} = g \tag{1.3.1}$$

for all $g \in G$. In particular, taking $g = f_1(x) f_2(x)^{-1}$ in (1.3.1) we see that $f_1(u) = f_2(u)$. Furthermore, since xux^{-1} has a fixed point, it follows that $f_1(xux^{-1})$ is conjugate to $f_2(xux^{-1})$. But $xux^{-1} \in T_U$ and, therefore, since the T_U action is effectively free, by Proposition 1.3.3 we must have $f_1(xux^{-1}) = f_2(xux^{-1}) \in Z(G)$. Thus, if $f_1(xux^{-1}) = z \in Z(G)$, then $f_1(x) f_1(u) f_1(x)^{-1} = z$ which implies that $f_1(u) = f_1(x)^{-1} z f_1(x) \in Z(G)$. Therefore, Proposition 1.3.3 tells us that the action by U is effectively free. □

Corollary 1.3.6. *A biquotient action is free if and only if the action is free when restricted to a maximal torus.*

Proof. In the proof of the previous proposition, if the maximal torus T_U acts freely, then since we had $xux^{-1} \cdot g = g$ and $xux^{-1} \in T_U$, we must have $xux^{-1} = e$. But then $u = e$ which implies that the action of U was free. □

We will often be computing quotients of manifolds by a compact Lie group acting freely or effectively freely by a biquotient action. The following proposition is extremely useful for such computations.

Proposition 1.3.7. *Let G be a compact Lie group acting smoothly on manifolds X and Y. Suppose that the action of G on X is transitive and the diagonal action of G on $X \times Y$ is free. Then for any $x \in X$ the action of the isotropy group G_x on Y is free and the quotient spaces $(X \times Y)/G$ and Y/G_x are canonically diffeomorphic. Moreover, if the action of G on $X \times Y$ is a biquotient action then the action of G_x on Y is again a biquotient action.*

Proof. We first show that G_x acts on Y freely. Suppose $g \in G_x$ such that $g \cdot y = y$ for some $y \in Y$. Then since $g \in G_x$ we have $g \cdot (x, y) = (g \cdot x, g \cdot y) = (x, y)$ so g fixes (x, y). But by freeness of the diagonal action of G on $X \times Y$ we must have $g = e$, so G_x acts freely on Y. Now we show the diffeomorphism statement. Fix $x \in X$ and consider the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"'] & X \times Y \arrow[d, "q"] \\ Y/G_x \arrow[r, dashed, "F"'] & (X \times Y)/G \end{tikzcd}$$

where $f(y) = (x, y)$ and π and q are the quotient maps. We wish to show that $q \circ f$ is constant on the fibers of π. Note that $\pi^{-1}[y] = \{g \cdot y : g \in G_x\}$. Then $(q \circ f)(g \cdot y) = q(x, g \cdot y) = q(g \cdot x, g \cdot y) = [g \cdot x, g \cdot y] = [x, y]$. Then by the universal property of quotient maps we get an induced map

$$F : Y/G_x \to (X \times Y)/G; \quad F[y] = [x, y]$$

which is well defined and smooth. To show F is surjective, suppose $[x', y] \in (X \times Y)/G$. Since G acts on X transitively, there exists $g \in G$ such that $g \cdot x = x'$. Then $F[g^{-1} \cdot y] =$

$[x, g^{-1} \cdot y] = [g \cdot x, g \cdot (g^{-1} \cdot y)] = [x' y]$ so F is surjective. To show F is injective, suppose that $[x, y] = [x, y']$. Then there exists $g \in G$ such that $(x, y') = (g \cdot x, g \cdot y)$ and hence $g \cdot x = x$ and $g \cdot y = y'$. Thus $g \in G_x$ so $[y'] = [g \cdot y] = [y]$ so F is injective. The inverse being smooth is obvious.

To prove the statement about biquotient actions, assume that the action of G on $X \times Y$ is a biquotient action induced by the homomorphism $f = \big((f_1, f_2), (f_3, f_4)\big) : G \to (X \times Y)^2$. Then

$$\begin{aligned} g \cdot (x, y) &= \big(f_1(g), f_2(g)\big)(x, y)\big(f_3(g)^{-1}, f_4(g)^{-1}\big) \\ &= \big(f_1(g) x f_3(g)^{-1}, f_2(g) y f_4(g)^{-1}\big) \end{aligned}$$

But by definition this is the diagonal action, so the second component is the action of G on Y which says $g \cdot y = f_2(g) y f_4(g)^{-1}$. This is the biquotient action induced by the homomorphism $\hat{f} = (f_2, f_4) : G \to Y \times Y$ and restricting this action to $G_x \subset G$ remains a biquotient action, completing the proof. $\square$

Corollary 1.3.8. *With the same hypothesis as Proposition 1.3.7, but with the diagonal action of G on $X \times Y$ being effectively free, then the action of G_x on Y is effectively free and we have the diffeomorphisms $(X \times Y)/G \approx Y/G \approx Y/\hat{G}_x$ where $\hat{G}_x = G_x/K$ and K is the ineffective kernel of diagonal the action, and G_x is the isotropy of any point $x \in X$.*

Proof. We first show that G_x acts on Y effectively freely. Suppose $g \in G_x$ such that $g \cdot y = y$ for some $y \in Y$. Then since $g \in G_x$ we have $g \cdot (x, y) = (g \cdot x, g \cdot y) = (x, y)$ so g fixes (x, y). But since the diagonal action is effectively free, we must have g fix every point of $X \times Y$, so G_x acts effectively freely on Y. Now, the same argument used in the proof of Proposition 1.3.7 gives the first diffeomorphism. For the second diffeomorphism, let K be the ineffective kernel of the diagonal action of G on $X \times Y$. Then $G' = G/K$ acts freely on $X \times Y$ so by Proposition 1.3.7 we have

$$(X \times Y)/G \approx (X \times Y)/G' \approx Y/(G')_x$$

where $(G')_x$ is the isotropy of some point $x \in X$ by the action of G' on X. Note that $(G')_x = \{[g] = gK \in G' : g \cdot x = x\}$. Now, we claim that $\hat{G}_x \approx (G')_x$, where $\hat{G}_x$ is as defined above, from which the result follows immediately. Define the natural homomorphism $\pi : G_x \to (G')_x$ by $g \mapsto [g]$, which is well defined because if $g \in G_x$ then $[g] \in (G')_x$ by definition. This homomorphism is clearly surjective and smooth and it is obvious that $K = \ker \pi$. □

1.4 Cohomogeneity One Manifolds

Here we will review some basic facts about cohomogeneity one manifolds. Let G be a compact Lie group and M a closed smooth manifold.

Definition 1.4.1. An action of G on M is said to be **cohomogeneity one** if the orbit space M/G is one dimensional or, equivalently, if there are orbits of codimension one.

Since M is compact, the quotient $M/G \simeq [-1,1]$ or $M/G \simeq S^1$. We will only consider the case where $M \simeq [-1,1]$. From now on, whenever we say *cohomogeneity one manifold*, we mean a cohomogeneity one manifold in which the quotient $M/G \simeq [-1,1]$.

Example: A rather simple example is obtained by taking $M = S^2$, the unit sphere, and considering the action of the unit circle S^1 on S^2 by rotation about the axis passing through the north and south poles. The orbit of any point on S^2 which does not lie on the north or south poles is a circle (corresponding to the lines of latitude) while the orbit of the two poles are both points. Since the circular orbits are codimension one, this is a cohomogeneity one action and, moreover, the quotient $S^2/S^1 \simeq [-1,1]$. The fact that the codimension one orbits of this action all have the same diffeomorphism type and forming an open dense subset of S^2 as well as the existence of two orbits differing from these is typical for simply connected cohomogeneity one actions.

The orbits of codimension one forming an open dense subset are called the *principal orbits* and any non-principal orbits are called *singular orbits*. In a general cohomogeneity one action the principal orbits are all diffeomorphic, so it makes sense to refer to *the* principal orbit of a cohomogeneity one action. Note that in a simply connected cohomogeneity one manifold, there will always be two singular orbits of codimension strictly greater than one. Furthermore, the diffeomorphism types of the singular orbits can differ from each other. In the non-simply connected case, it is possible for every orbit to be principal or for there to be singular orbits of codimension one. In the latter case, the singular orbits are called *exceptional orbits*.We will only be concerned with simply connected cohomogeneity one manifolds, so exceptional orbits will not occur this thesis. The isotropy groups corresponding to the principal and singular orbits are called the *principal isotropy group* and *singular isotropy group*, respectfully, and the principal orbits have isomorphic isotropy groups.

Mostert showed in [Mos57] that there are precisely two singular orbits corresponding to the endpoints of $I = [-1,1]$, and M can be decomposed as the union of two tubular neighborhoods of the singular orbits, with common boundary a principal orbit, and these tubular neighborhoods are disk bundles over their corresponding singular orbit. This gives cohomogeneity one manifolds what is called a *double disk-bundle* (DDB) structure:

Definition 1.4.2. A closed manifold is said to admit a *double disk-bundle* (DDB) decomposition if it can be written as the union of two disk bundles glued together along their common boundary by a diffeomorphism.

DDB structures have been studied extensively, for instance in [GH87, DGGK20, EU11]. In the former, they actually consider a more general object called the *double mapping cylinder* of which a DDB structure is a special case.

The following argument, taken from [GZ00], makes this description of a cohomogeneity one manifold as a DDB more precise in terms of an arbitrary but fixed G-invariant Riemannian metric on M, normalized so that $M/G \simeq [-1,1]$. Let $x_0 \in \pi^{-1}(0)$ and let $\gamma : [-1,1] \to M$ be the unique minimal geodesic with $\gamma(0) = x_0$ and $\pi \circ \gamma = Id$. Note that γ intersects all orbits orthogonally and, since $\pi \circ \gamma = Id$, it follows that γ is a minimal geodesic between the two singular orbits $B_\pm = \pi^{-1}(\pm 1) = G \cdot x_\pm$, where $x_\pm = \gamma(\pm 1)$. Let $K^\pm = G_{x_\pm}$ be the singular isotropy groups and $H = G_{x_0} = G_{\gamma(t)}$, -1 < t <1, be the principal isotropy group. Let $D^{\ell_\pm+1}$ be the unit normal disk at $x_\pm$. In the notation of the slice theorem, $D^{\ell_\pm+1} = S_{x_\pm} = \exp_{x_\pm}(\nu^1_x)$. Note that, by construction, $\partial D^{\ell_\pm+1}$ intersects the principal orbit $G \cdot x_0$. Note also that since G acts by isometries, G acts on each unit normal disk and g takes the unit normal disk $D^{\ell_\pm+1}$ at $x_\pm$ to the unit normal disk at $g \cdot x_\pm$. It follows that $G \cdot D^{\ell_-+1} = \pi^{-1}[-1,0] = \mathcal{N}_1(G \cdot x_-)$ and $G \cdot D^{\ell_++1} = \pi^{-1}[0,1] = \mathcal{N}_1(G \cdot x_+)$, so these are tubular neighborhoods with unit disks as the slices. By (iv) of the slice theorem, these tubular neighborhoods of the singular orbits have the form

$$D(B_\pm) = G \times_{K^\pm} D^{\ell_\pm+1}$$

Note that $D(B_\pm)$ are both disk bundles over the corresponding singular orbit with projection taking each disk to the point of the orbit over which it is centered. Thus we have obtained a DDB decomposition

$$M = D(B_-) \cup_E D(B_+)$$

where $E = \pi^{-1}(0) = G \cdot x_0 = G/H$ is canonically identified with the boundaries $\partial D(B_\pm) = G \times_{K^\pm} S^{\ell_\pm}$.

Let (M, G) be a cohomogeneity one manifold and let $K^\pm$ be the singular isotropy groups and H the principal isotropy group of the action. It is well known that any cohomogeneity one manifold (M, G) with $M/G \simeq [-1, 1]$ determines a group diagram

$$\begin{tikzcd}[column sep=small] & G & \\ K^- \arrow[ur, dash] & & K^+ \arrow[ul, dash] \\ & H \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

with $H \subset K^\pm \subset G$ and $K^\pm/H \approx S^{l_\pm}$. The spheres $S^{\ell_\pm}$ are called the *fiber spheres* of the cohomogeneity one manifold. For simplicity, a cohomogeneity one group diagram will often be denoted by $G \supset K^-, K^+, \supset H$. Conversely, any such diagram determines a cohomogeneity one manifold. The quotients $G/K^\pm$ and G/H are diffeomorphic to the singular and principal orbits, respectfully. Thus such group diagrams completely specify the manifold as well as the action. For example, the group diagram for the action of S^1 on S^2 by rotation mentioned above is

$$\begin{tikzcd}[column sep=small] & S^1 & \\ S^1 \arrow[ur, dash] & & S^1 \arrow[ul, dash] \\ & \{1\} \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Given an action of a group G on a manifold M, we will often want to compute whether the action is cohomogeneity one. Intuitively, if the dimension of G is much larger than that of M and the action of G is not transitive, the action has a good chance of being cohomogeneity one. The easiest way to show that an action is cohomogeneity one is if one can find a codimension one orbit. Another useful tool is the following well known proposition which allows us to construct new cohomogeneity one actions from old ones

Proposition 1.4.3. *Suppose G_1 acts by cohomogeneity one on M_1 and G_2 acts transitively on M_2. Then the product action of $G_1 \times G_2$ acts by cohomogeneity one on $M_1 \times M_2$.*

Proof. The dimension of the orbit of a product action is the sum of the dimensions of the orbits of each action. Since the orbit of G_1 on M_1 is codimension one and the action of M_2 is transitive, hence each orbit is codimension zero, it follows that the product action has a codimension-one orbit. □

Before giving another very useful way for showing that an action is cohomogeneity one, we need a definition.

Definition 1.4.4. Suppose a Lie group G acts on a Riemannian manifold M and let G_p be the isotropy group at $p \in M$. The differential of the action of G_p on M defines a linear representation of G_p on T_pM called the **isotropy representation**. The tangent space of the orbit $T_p(G \cdot p) \subset T_pM$ and its normal space $\nu_P(G \cdot p) \subset T_pM$ are invariant subspaces of the isotropy representation. The restriction of the isotropy representation to $\nu_p(G \cdot p)$ is called the **slice representation**.

One can restrict the action of G_p induced by slice representation to the unit sphere in the normal space. The following well known proposition tells gives us a sufficient condition for an action to be cohomogeneity one.

Proposition 1.4.5. *Suppose a Lie group G acts on a Riemannian manifold M and let $p \in M$. If the action of G_p on $\nu_p(G \cdot p)$ induced by the slice representation is transitive, then the action of G on M has cohomogeneity one.*

As a final note, we are interested in manifolds of non-negative sectional curvature, and the following theorem of Grove and Ziller [GZ00] gives a useful criterion which ensures that certain cohomogeneity one manifolds admit non-negative sectional curvature.

Proposition 1.4.6. *(Grove-Ziller) Any cohomogeneity one manifold with singular orbits of codimension at most 2 admits an invariant metric of non-negative sectional curvature*

1.5 Codimension One Biquotient Foliations

Here we introduce the main object of interest, namely *codimension one biquotient foliations.* A *singular Riemannian foliation* of a manifold M is a certain partition $\mathcal{F}$ of M into smooth, connected, locally equidistant submanifolds of M, called the *leaves* of the foliation (precise definitions can be found in either of[MC88, MR19]). When all of the leaves have the same dimension, the foliation is called a *regular foliation*, or simply a *foliation* of M. Foliation theory tells us that a singular Riemannian foliation has a diffeomorphism class of leaves, called the *principal leaf*, which form an open dense subset of M and have maximal codimension. Any leaf which is not a principal leaf is called a *singular leaf.*

Definition 1.5.1. If all of the leaves of a singular Riemannian foliation are biquotients then we call the foliation a *biquotient foliation.*

Definition 1.5.2. A biquotient foliation in which the principal leaf has codimension one is called a *codimension one biquotient foliation* or simply *C1BF*.

General C1BFs defined, as above, in terms of singular Riemannian foliations are rather complicated. We will restrict our attention to a (potentially) smaller class of C1BFs which will be easier to study. We will now motivate this restriction. We will use the following lemma:

Lemma 1.5.3. *Suppose G is a compact Lie group. Let $\Delta G \leq G \times G$ be the diagonal subgroup. Suppose H is another subgroup of $G \times G$ such that the induced biquotient action is free, giving a biquotient $G//H$. Then the biquotients $\Delta G \backslash\ G \times G\ /H$ and $G//H$ are canonically diffeomorphic.*

Proof. Define a map $F : \Delta G \backslash G \times G / H \to G//H$ by $F[g_1, g_2] = [g_1^{-1} g_2]$. Observe that this is well defined, since if $[g_1, g_2] = [\tilde{g}_1, \tilde{g}_2]$, then $(\tilde{g}_1, \tilde{g}_2) = (g g_1 h_1^{-1}, g g_2 h_2^{-1})$ for some $(g, g) \in \Delta G$ and $(h_1, h_2) \in H$. Thus we have

$$
\begin{aligned}
F[\tilde{g}_1, \tilde{g}_2] &= F[g g_1 h_1^{-1}, g g_2 h_2^{-1}] \\
&= [h_1 g_1^{-1} g^{-1} g g_2 h_2^{-1}] \\
&= [h_1 g^{-1} g_2 h_2^{-1}] \\
&= [g_1^{-1} g_2] \\
&= F[g_1, g_2]
\end{aligned}
$$

Clearly F is surjective because for $[g] \in G//H$ we have $[e, g] \mapsto [g]$. On the other hand, F is injective. Indeed, if $F[g_1, g_2] = F[\tilde{g}_1, \tilde{g}_2]$, then $[\tilde{g}_1^{-1} \tilde{g}_2] = [g_1^{-1} g_2]$ and hence $g_1^{-1} g_2 = h_1 \tilde{g}_1^{-1} \tilde{g}_2 h_2^{-1}$ for some $(h_1, h_2) \in H$. Thus we have

$$
\begin{aligned}
[\tilde{g}_1, \tilde{g}_2] &= [\tilde{g}_1 h_1^{-1}, \tilde{g}_2 h_2^{-1}] \\
&= [(g_1 h_1 \tilde{g}_1^{-1})\tilde{g}_1 h_1^{-1}, (g_1 h_1 \tilde{g}_1^{-1})\tilde{g}_2 h_2^{-1}] \\
&= [g_1, g_1 g_1^{-1} g_2] \\
&= [g_1, g_2]
\end{aligned}
$$

A standard smoothness argument shows that F and its inverse are smooth, making F a diffeomorphism. $\square$

Suppose now that we have a cohomogeneity one manifold M with group diagram

$$
\begin{tikzcd}[column sep=small]
& G \arrow[dl, dash] \arrow[dr, dash] & \\
K^{-} \arrow[dr, dash] & & K^{+} \arrow[dl, dash] \\
& H &
\end{tikzcd}
$$

where $G/K^{\pm}$ and G/H are the orbits and $K^{\pm}/H \simeq S^{\ell_{\pm}}$ are the fiber spheres.

Suppose we restrict the action of G on M to a subgroup $L \leq G$ which acts freely on M. Note that the action of L on M preserves each orbit of the G-action on M. This can be expressed in terms of L acting on the homogeneous spaces $G/K^{\pm}$ and G/H via the obvious action. Upon taking the quotient of M by the L action, the homogeneous quotients corresponding to the orbits of the cohomogeneity one action become biquotients foliating M/L and we get a corresponding group diagram

$$
\begin{tikzcd}[column sep=small]
& G \arrow[dl, equal] \arrow[dr, equal] & \\
L \times K^{-} \arrow[dr, dash] & & L \times K^{+} \arrow[dl, dash] \\
& L \times H &
\end{tikzcd}
$$

which completely specifies the leaf structure of the foliation. In particular, the principal leaf is the biquotient $L\backslash G/H$ and the singular leaves are $L\backslash G/K^{\pm}$. Furthermore, we observe that $(L\times K^{\pm})/(L\times H)\simeq K^{\pm}/H\simeq S^{\ell_{\pm}}$. This shows that every quotient of a cohomogeneity one manifold by a free isometric action contained within the group G is a C1BF and determines a group diagram analogous to the cohomogeneity one group diagram. Note that by taking the subgroup of G to the the trivial subgroup, we see that every cohomogeneity one manifold is itself a C1BF.

Conversely, suppose we have biquotients $G//K^{\pm}$ and $G//H$ where $K^{\pm}, H\leq G\times G$ act freely on G and satisfying the condition that $K^{\pm}/H\approx S^{\ell_{\pm}}$. Then one can form the corresponding group diagram and Wilking noticed this diagram corresponds to a quotient of a cohomogeneity one manifold by a free isometric action contained within G, namely

$$\begin{tikzcd} & G\times G \arrow[dl, dash] \arrow[dr, dash] & \\ K^- \arrow[dr, dash] & & K^+ \arrow[dl, dash] \\ & H & \end{tikzcd} \quad \xrightarrow{\Delta G} \quad \begin{tikzcd} & G \arrow[dl, equal] \arrow[dr, equal] & \\ K^- \arrow[dr, dash] & & K^+ \arrow[dl, dash] \\ & H & \end{tikzcd}$$

where we mean the diagonal subgroup $\Delta G\leq G$ acts on the cohomogeneity one manifold specified by the group diagram on the left. This follows from Lemma 1.5.3.

Definition 1.5.4. We call the cohomogeneity one manifold given by the group diagram on the left the **standard lift to cohomogeneity one** of the C1BF given by the diagram to the right.

This shows that every group diagram of the form

$$\begin{tikzcd} & G \arrow[dl, equal] \arrow[dr, equal] & \\ K^- \arrow[dr, dash] & & K^+ \arrow[dl, dash] \\ & H & \end{tikzcd}$$

with $H \subset K^{\pm} \subset G \times G$ and $K^{\pm}/H \simeq S^{\ell_{\pm}}$ determines a C1BF that is a quotient of a cohomogeneity one manifold (M, G) by a free isometric action contained within the group G. The singular leaves are the biquotients $G//K^{\pm}$ and the principal leaf is $G//H$.

Given a C1BF with group diagram as above, note that the embeddings of H and $K^{\pm}$ into $G \times G$ specify biquotient actions of these groups on G. We will, in general, allow the embeddings of the groups to have kernel, provided that the induced action remains effectively free. For example, if H is embedded in $G \times G$ via some homomorphism $f = (f_1, f_2) : H \to G \times G$, then the action of H on G is the biquotient action induced by f and the quotient by this action is the principal leaf $G//H$. Similarly, we get embeddings of $K^{\pm}$ into $G \times G$ giving the biquotients $G//K^{\pm}$.

Convention: We restrict our attention to codimension one biquotient foliation which arise as quotients of cohomogeneity one manifolds with group G by an effectively free action where the effectively free action comes from the restriction of the cohomogeneity one action to a subgroup of G. From now on, when we refer to C1BFs, we mean C1BFs that arise as these quotients, unless explicitly stated.

The following proposition, whose proof is taken from [DGGK20], shows that C1BFs which arise as quotients of cohomogeneity one manifolds have a DDB structure completely analogous to that of cohomogeneity one manifolds.

Proposition 1.5.5. *Any C1BF which is the quotient of a cohomogeneity one manifold by a free subaction admits a DDB structure.*

Proof. Suppose G acts on M' by cohomogeneity one. Then, as we have seen above, there are closed subgroups $H \subset K^{\pm} \subset G$ with $K^{\pm}/H \simeq S^{\ell_{\pm}}$ such that M' is equivariantly diffeomorphic to the union of the disk-bundles $G \times_{K^\pm} D^{\ell_\pm+1}$ glued equivariantly along their common boundary $G \times_{K^\pm} S^{\ell_\pm} \simeq G/H$. Suppose now that there is a subgroup $U \subset G$ which acts freely on M' with quotient the C1BF M. Observe that the U action on M' preserves the orbits of the G action. Now, via the equivariant diffeomorphism mentioned above, U acts freely on each of the disk bundles $G \times_{K^\pm} D^{\ell_\pm+1}$ by the action induced from the action of U by left multiplication on the first factor of the product $G \times D^{\ell_\pm+1}$. As the U action commutes with the action of $K^\pm$ on the right of the first factor, it follows that $U \backslash (G \times_{K^\pm} D^{\ell_\pm+1})$ is diffeomorphic to $(U\backslash G) \times_{K^\pm} D^{\ell_\pm+1}$. These disk bundles both have boundary diffeomorphic to the biquotient $U\backslash G/H$ and the equivariant gluing map in the DDB decomposition of M' now induces a gluing of the quotient disk bundles, yielding the desired DDB decomposition of M. □

Corollary 1.5.6. *A C1BF M with group diagram $\{G, K^-, K^+, H\}$ and $K^\pm/H \simeq S^{\ell_\pm}$ has a DDB decomposition $M = G \times_{K^-} D^{\ell_-+1} \cup_{G//H} G \times_{K^-} D^{\ell_++1}$, where the actions of $K^\pm$ on $G \times D^{\ell_\pm+1}$ are given by the biquotient action on the G factor and the on the disk factor, these are the same actions as the actions of $K^\pm$ on the disks in the DDB decomposition of the standard lift of M to cohomogeneity one.*

Proof. This follows from Proposition 1.5.3 as well as the proof of the previous theorem by taking M' to be the standard lift of M to cohomogeneity one and taking the quotient by the diagonal ΔG subaction of $G \times$G. □

In particular, a C1BF decomposes as two disk bundles, each of which have base space a singular leaf $G//K^\pm$, and the two disk bundles are glued together along a principal leaf $G//H$. Note that $\partial D^{\ell_\pm+1} = S^{\ell_\pm} = K^\pm/H$. The spheres $S^{\ell_\pm}$ are called the *fiber spheres* of the DDB.

Definition 1.5.7. Given a C1BF, we define the **leaf structure** of the C1BF to be the diffeomorphism types of the principal leaf $G//H$ and the singular leaves $G//K^{\pm}$, and will be denoted as a triple $(G//H, G//K^{-}, G//K^{+})$. A leaf structure is said to be an **admissible leaf structure** provided that there exists a compact simply connected C1BF which realizes the leaf structure.

The central goal of this thesis will be to answer the following question:

Question: Given a triple of biquotients $(G//H, G//K^{-}, G//K^{+})$, where $G//H$ has dimension $n \leq 5$, which triples occur as admissible leaf structures for a C1BF manifold M^{n+1}?

Another important fact coming from the DDB structure of the C1BF is that the principal leaf must be a sphere bundle over each singular leaf. In particular, for a DDB with fiber spheres $S^{\ell_{\pm}}$, principal leaf P, and singular leaves $G//K^{\pm}$, we get sphere bundles $S^{\ell_{\pm}} \to G//H \to G//K^{\pm}$. Moreover, the long exact sequence of homotopy groups

$$... \to \pi_n(S^{\ell}) \to \pi_n(G//H) \to \pi_n(G//K^{\pm}) \to \pi_{n-1}(S^{\ell}) \to ...$$

Sphere bundles have been studied extensively, for example, in [Ste44, Gib68, Tho74, DL05, Mel84]. Such classifications will frequently allow us to rule out many possibilities for B. Furthermore, Corollary 1.6.2 (see below) allows us to rule out the case $\ell = 0$. This fact will allow us to greatly reduce the number of possible C1BF structures on manifolds.

The following proposition is an easy generalization of Proposition 1.4.6 which gives a simple criterion to guarantee a C1BF has non-negative curvature.

Proposition 1.5.8. *Any C1BF with singular leaves of codimension at most 2 admits a metric of non-negative sectional curvature*

Proof. By Proposition 1.4.6, the analogous result holds for cohomogeneity one manifolds with singular orbits of codimension at most two. We observed above that any C1BF with group diagrams containing groups G, $K^{\pm}$, and H is the quotient of a cohomogeneity one manifold, namely

$$\begin{tikzcd}[column sep=small] & G\times G \arrow[dl, dash] \arrow[dr, dash] & & & & G \arrow[dl, equal] \arrow[dr, equal] & \\ K^- \arrow[dr, dash] & & K^+ \arrow[dl, dash] \arrow[rr, "\Delta G"] & & K^- \arrow[dr, dash] & & K^+ \arrow[dl, dash] \\ & H & & & & H & \end{tikzcd}$$

But the corresponding cohomogeneity one manifold given by the left diagram has singular orbits of the same codimension as the singular leaves of the C1BF. Thus by O'Neill's formula, the result follows for C1BFs. $\square$

We will now prove a nice theorem which gives vast amounts of examples of C1BFs. In particular, we will show that any quotient of a product of spheres by an effectively free torus action is a C1BF. We will first prove a lemma which shows that this statement is true for the quotient of a single sphere by such an action.

Lemma 1.5.9. *Suppose a torus T^k acts on a sphere by any linear torus action. Then the action of T^k is contained within a cohomogeneity one action on the sphere. In particular, the quotient of any sphere by an effectively free linear torus action is a C1BF.*

Proof. Suppose the torus T^k acts on the sphere S^{n-1} as above. The action being linear means that it is induced by a representation $\rho : T^k \to O(n)$. It follows that the action of T^k is orbit equivalent to a torus subgroup of $O(n)$ acting on S^{n-1}. We may assume that $k \leq \mathrm{rank} O(n)$ because the dimension of such a torus cannot exceed the rank of $O(n)$.

Now, write $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ and consider the natural action of $\mathrm{SO}(2)\mathrm{SO}(n-2)$ on $\mathrm{SO}(n)/\mathrm{SO}(n-1)$. This action is well known to be cohomogeneity one [Wan60, Str96]. Therefore, we have exhibited a cohomogeneity one action of $\mathrm{SO}(2)\mathrm{SO}(n-2)$ on $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$. Since $\mathrm{rank}\big(\mathrm{SO}(2)\mathrm{SO}(2n-2)\big) = \mathrm{rank} O(n)$, it follows that, up to conjugacy $T^k \subset \mathrm{SO}(2)\mathrm{SO}(n-2)$, so this action contains an action equivalent to the original T^k action on S^{n-1}, so the quotient S^{n-1}/T^k is a C1BF.

□

Theorem 1.5.10. *Suppose $M \approx (S^{n_1} \times \cdots \times S^{n_r})/T^k$ is diffeomorphic to a quotient of a product of spheres by an effectively free linear torus action. Then M is a C1BF.*

Proof. The theorem follows from the previous lemma. Indeed, suppose T^k acts on $S^{n_1} \times \cdots S^{n_r}$ as in the statement of the theorem. The projection of this action onto each factor yields a linear torus action T^k on S^{n_i}, $i \in \{1,...,r\}$, which may or may not be effective. Note that T^k cannot be effective for $k > \mathrm{rank} O(n_i+1)$, so on each factor, T^k acts as a subgroup of $O(n_i+1)$. Consider the action of T^k on S^{n_1} given by the projection of the action onto the first factor. By the previous lemma, we can extend this to a cohomogeneity one action G on S^{n_1}. Now, consider the action of $G \times O(n_2+1) \times \cdots O(n_r+1)$ on $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}$ via the product action, where each $O(n_i)$ acts via the standard transitive action. Note that $O(n_i+1)$ contains the effective portion of the T^k action, due to the fact that it is the isometry group of S^{n_i}. Therefore, since G acts by cohomogeneity one and $O(n_2+1) \times \cdots \times O(n_r+1)$ acts transitively on $S^{n_2} \times \cdots \times S^{n_r}$, so by Proposition 1.4.3, the product action is cohomogeneity one. Thus we have exhibited a group which acts by cohomogeneity one and contains an orbit equivalent action to the T^k action on $S^{n_1} \times \cdots \times S^{n_r}$, showing the quotient is a C1BF. □

This theorem has a plethora of applications. It can be, for example, applied immediately to Totaro's work [Tot03], as well as DeVito's work [DeV17], to obtain infinite families of C1BFs in dimension 6.

Given group diagrams for two cohomogeneity one manifolds, [GWZ08] gives a theorem which completely classifies when the diagrams are equivalent. One would like to have a theorem which does the same for general C1BFs. There is no complete answer for this, but we present now the first such theorem in this direction. In particular, the theorem gives sufficient conditions for two diagrams to determine the same C1BF up to diffeomorphism.

Theorem 1.5.11. *Suppose M_1 and M_2 are C1BFs with group diagrams $\{G_1, K^-, K^+, H\}$ and $\{G_2, L^-, L^+, J\}$, respectively. Suppose there exists a diffeomorphism $\varphi : G_1 \to G_2$ and isomorphisms (i)-(iii) which agree on their restrictions to H and satisfying conditions (a)-(c)*

(i) $\psi : K^- \to L^-$	*(a)* $\varphi(k_1 \cdot g) = \psi(k_1) \star \varphi(g)$
(ii) $\mu : K^+ \to L^+$	*(b)* $\varphi(k_2 \cdot g) = \mu(k_2) \star \varphi(g)$
(iii) $\rho : H \to J$	*(c)* $\varphi(h \cdot g) = \rho(h) \star \varphi(g)$

for all $k_1 \in K^-$, $k_2 \in K^+$, $h \in H$, and $g \in G$. Here $\cdot$ and $\star$ denotes the biquotient action of the subgroups of $G_1 \times G_1$ on G_1 and the subgroups of $G_2 \times G_2$ on G_2, respectively. Then there exists a diffeomorphism $F : M_1 \to M_2$.

Before proving the theorem, we note that it is perhaps somewhat surprising is that the map $\varphi : G_1 \to G_2$ is only required to be a diffeomorphism rather than an isomorphism of Lie groups. Moreover, we present an example of C1BFs satisfying the hypothesis of this theorem, where φ is a diffeomorphism but *not* an isomorphism after the proof of the theorem. The conditions that the (i)-(iii) along with (a)-(c) are precisely the conditions needed to ensure that principal leaves are taken to principal leaves and singular leaves are taken to singular leaves.

Proof. The idea of the proof is that we start by showing that φ induces diffeomorphisms taking the principal and singular leaves of M_1 to the corresponding leaf in M_2 and that these diffeomorphisms extend to a global diffeomorphism $M_1 \to M_2$. Observe that $\varphi : G_1 \to G_2$ induces a diffeomorphism $\overline{\varphi} : G_1 // H \to G_2 // J$ on the principal leaves given by $\overline{\varphi}[g] = [\varphi(g)]$. This is well defined since in $[g] = [x] \in G // H$ if and only if $x = h \cdot g$ for some $h \in H$. Thus $\overline{\varphi}[h \cdot g] = [\varphi(h \cdot g)] = [\rho(h) \star \varphi(g)] = [\varphi(g)] = \overline{\varphi}[g]$. Moreover, $\overline{\varphi}$ is bijective. To see this, observe that $\varphi^{-1} : G_2 \to G_1$ satisfies $\varphi^{-1}(j \star g) = \rho^{-1}(j) \cdot \varphi^{-1}(g)$ for all $j \in J$ and $g \in G_2$. Indeed, since $\varphi\big(\rho^{-1}(j) \cdot \varphi^{-1}(g)\big) = \rho\big(\rho^{-1}(j)\big) \star \varphi\big(\varphi^{-1}(g)\big) = j \star g$, this implies $\varphi^{-1}(j \star g) = \rho^{-1}(j) \cdot \varphi^{-1}(g)$. Thus φ^{-1} induces a map $\overline{\varphi}^{-1} : G_2 // J \to G_1 // H$ given by $\overline{\varphi}^{-1}[g] = [\varphi^{-1}(g)]$. Moreover, $\overline{\varphi}^{-1}$ is actually the inverse of $\overline{\varphi}$, so $\overline{\varphi}$ is bijective. Clearly $\overline{\varphi}$ and $\overline{\varphi}^{-1}$ are smooth because φ and φ^{-1} are smooth, so $\overline{\varphi}$ is a diffeomorphism. Similarly, φ induces diffeomorphism on the singular leaves.

We have shown that φ induces diffeomorphisms taking each leaf of M_1 to the corresponding leaf of M_2. We will now show that φ induces a global diffeomorphism $F : M_1 \to M_2$. Recall that we can decompose M_1 as a DDB

$$M_1 = G_1 \times_{K^-} D^{\ell_- + 1} \cup_{G//H} G_1 \times_{K^+} D^{\ell_+ + 1}$$

where $K^\pm / H \approx S^{\ell_\pm}$ and $\partial D^{\ell_\pm + 1} = S^{\ell_\pm}$. Let $I = [0, 1)$ and note that a disk D is diffeomorphic to $\big(\partial D \times I\big) / \sim$, where $\sim$ identifies $\partial D \times \{0\}$ to a point. By scaling the metric on M_1, it follows from Proposition 1.5.5 and its corollary that we can decompose M_1 as

$$
\begin{aligned}
M_1 &= \Big(G_1 \times_{K^-} \big(K^-/H \times [0,1)\big)/\sim\Big) \cup \Big(G_1//H \times (-1,1)\Big) \cup \Big(G_1 \times_{K^+} \big(K^+/H \times [0,1)\big)/\sim\Big) \\
&:= M_1^- \cup M_1^0 \cup M_1^+
\end{aligned}
$$

where the action of K^- is the biquotient action on the G_1 factor and on the second factor the action is given by $k_1 \cdot [k_2 H, t] = [k_1 k_2 H, t]$ and, similarly, we can decompose M_2 as

$$
\begin{aligned}
M_2 &= \Big(G_2 \times_{L^-} \big(L^-/J \times [0,1)\big)/\sim\Big) \cup \Big(G_2//J \times (-1,1)\Big) \cup \Big(G_2 \times_{L^+} \big(L^+/J \times [0,1)\big)/\sim\Big) \\
&:= M_2^- \cup M_2^0 \cup M_2^+
\end{aligned}
$$

Define $F_\pm : M_1^\pm \to M_2^\pm$ by $F_\pm\big[[g,[kH,t]\big] = \big[\varphi(g),[\psi(k)J,t]\big]$ and $F_0 = M_1^0 \to M_2^0$ by $F_0\big([g],t\big) = \big([\varphi(g)],t\big)$.

Clearly F_0 is well defined. We now show that $F_\pm$ are well defined. To check that F_- is well defined, note that $\big[g_1,[k_1 H,t_1]\big] = \big[g_2,[k_2 H,t_2]\big] \in M_1^-$ if and only if $g_2 = k \cdot g_1$ for some $k \in K^-$ and one of conditions (i) or (ii) below hold and, similarly, $\big[g_1,[l_1 J,t_1]\big] = \big[g_2,[l_2 J,t_2]\big] \in M_2^-$ if and only if $g_2 = l \star g_1$ for some $l \in L^-$ and one of conditions (a) or (b) below hold:

(i) $t_1 = t_2 = -1$,

(ii) $t_1 = t_2 \neq -1$ and $k_2^{-1} k_1 \in H$

(a) $t_1 = t_2 = -1$

(b) $t_1 = t_2 \neq -1$ and $k_2^{-1} k_1 \in J$

Assume $\big[g_1,[k_1 H,-1]\big] = \big[g_2,[k_2 H,-1]\big] \in M_1^-$. Then $g_2 = k \cdot g_1$ for some $k \in K^-$ and also $[k_1H,-1]=[k_2H,-1]$. Hence we have

$$\begin{aligned}
F_-\big[g_2,[k_2H,-1]\big] &= F_-\big[k\cdot g_1,[k_1H,-1]\big]\\
&= \big[\varphi(k\cdot g_1),[\psi(k_1)J,-1]\big]\\
&= \big[\psi(k)\cdot g_1,[\psi(k_1)J,-1]\big]\\
&= \big[g_1,[\psi(k_1)J,-1\big]
\end{aligned}$$

Similarly, if $t\neq -1$ and $\big[g_1,[k_1H,t]\big]=\big[g_2,[k_2H,t]\big]\in M_1^-$, then $g_2=k\cdot g_1$ and $k_2^{-1}k_1\in H$. Since $k_2^{-1}k_1\in H$ and ψ takes H to J, we have $\psi(k_2)^{-1}\psi(k_1)\in J$ and hence

$$\begin{aligned}
F_-\big[g_2,[k_2H,t]\big] &= F_-\big[k\cdot g_1,[k_2H,t]\big]\\
&= \big[\varphi(k\cdot g_1),[\psi(k_2)J,t]\big]\\
&= \big[\psi(k)\star\varphi(g_1),[\psi(k_2)J,t]\big]\\
&= \big[\varphi(g_1),[\psi(k_1)J,t]\big]
\end{aligned}$$

so F_- is well defined. The same argument shows that F_+ is well defined. Note that $M_i^{\pm}$ and M_i^0, $i\in\{1,2\}$, are open submanifolds of M_1 and M_2, respectively, and it is easy to see that $F_\pm$ and F_0 are smooth. Moreover, they have the obvious inverse maps, obtained by replacing the maps φ and ψ appearing in the definition of these maps with their inverses, so their inverse maps are smooth as well, hence $F_\pm$ and F_0 are diffeomorphisms. Therefore, they extend to a diffeomorphism $F:M_1\to M_2$ provided that they agree on the overlaps of $M_1^{\pm}$ and M_0.

To check that they agree on the overlap, note that the only overlaps of these submanifolds are the following two cases

(I) A point $\left[g,[kH,t]\right] \in M_1^-$ also lies in M_1^0 when $t \in (-1,0)$

(II) A point $\left[g,[kH,t]\right] \in M_1^+$ also lies in M_1^0 when $t \in (0,1)$

Since there is no identification in $\left(K^-/H \times [0,1)\right)/\sim$ for $t \neq -1$, it follows that

$$\begin{aligned} A &= M_1^- \cap M_1^0 \\ &= G_1 \times_{K^-} \left(K^-/H \times (-1,0)\right) \\ &= \left(G_1 \times_{K^-} K^-/H\right) \times (-1,0) \end{aligned}$$

where the last equality follows because K^- acts trivially on $(-1,0)$. Similarly, we have

$$\begin{aligned} B &= M_1^0 \cap M_1^+ \\ &= G_1 \times_{K^+} \left(K^+/H \times (0,1)\right) \\ &= \left(G_1 \times_{K^+} K^+/H\right) \times (0,1) \end{aligned}$$

We wish to identify the points in A and B with a point in $G_1//H \times (-1,1)$ in such a way that $F_\pm$ and F_0 agree on these points. Observe that there is a canonical way to identify $G_1 \times_{K^-} K^-/H$ with $G_1//H$. In particular, the K^- action on $G_1 \times K^-/H$ is transitive on the K^-/H factor with isotropy group H, so by Proposition 1.3.7 we have that $G_1 \times_{K^-} K^-/H \simeq G_1//H$ via the diffeomorphism

$$G_1//H \to G_1 \times_{K^-} K^-/H; \quad [g] \mapsto [g,eH]$$

Thus we identify $([g],t) \in G_1//H \times (-1,1)$ with $\left([g,eH],t\right) \in A$ and, similarly, $([g],t) \in$

$G_2//J \times (-1,1)$ with $\big([g, eJ], t\big) \in M_2^- \cap M_2^0$. Note that $F_0\big([g], t\big) = \big([\varphi(g)], t\big)$ while

$$\begin{aligned}
F_-\big([g, eh], t\big) &= \big([\varphi(g), \psi(e)J], t\big) \in G_2 \times_{L^-} L^-/J \\
&= \big([\varphi(g), eJ], t\big) \\
&= \big([\varphi(g)], t\big) \in G_2//J \times (-1,1)
\end{aligned}$$

where the last equality follows from the identifications above. The same argument shows that F_0 and F_+ agree on B. Thus F_0 and $F_\pm$ extend to a global diffeomorphism $F : M_1 \to M_2$ as desired.

□

As an application of this theorem, we can explicitly exhibit an example C1BF diagrams determining diffeomorphic manifolds, where $\varphi : G_1 \to G_2$ is a diffeomorphism but not an isomorphism. Consider the C1BF diagram for a manifold M_1 given by

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
S^1 \times \mathrm{Sp}(1) & & S^1 \times \mathrm{Sp}(1) \\
 & \mathrm{Sp}(1) \arrow[ul, no head] \arrow[ur, no head] &
\end{tikzcd}$$

where, in the notation of the above theorem, the embeddings are given by

$$\begin{aligned}
K^- \to G_1 \times G_1;\ (z,p) &\mapsto (z,z,1,p) \\
K^+ \to G_1 \times G_1;\ (z,p) &\mapsto (z,z,1,p) \\
H \to G_1 \times G_1;\ p &\mapsto (1,1,1,p)
\end{aligned}$$

Similarly, let M_2 be the C1BF determined by the C1BF diagram

$$\begin{tikzcd}
 & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[dl, no head, double] \arrow[dr, no head, double] & \\
\mathrm{Sp}(1) \times S^1 \arrow[dr, no head] & & \mathrm{Sp}(1) \times S^1 \arrow[dl, no head] \\
 & \mathrm{Sp}(1) &
\end{tikzcd}$$

where, in the notation of the above theorem, the embeddings are given by

$$L^- \to G_2 \times G_2;\ (p, z) \mapsto (z, p, 1, z)$$

$$L^+ \to G_2 \times G_2;\ (p, z) \mapsto (z, p, 1, z)$$

$$J \to G_2 \times G_2;\ p \mapsto (1, p, 1, 1)$$

In the notation of the above theorem, define $\varphi : G_1 \to G_2$ by $(q_1, q_2) \mapsto (q_1, \overline{q}_2)$. Then φ is clearly a diffeomorphism but not an isomorphism. If we define $\psi : K^- \to L^-$ and $\mu : K^+ \to L^+$ by $(z, p) \mapsto (p, z)$ and $\rho : H \to J$ to be the identity, we see that these homomorphisms agree on their restrictions to H. Moreover, we have

$$\begin{aligned}
\psi(z, p) \star \varphi(q_1, q_2) &= (p, z) \star (q_1, \overline{q}_2) \\
&= (zq_1, p\overline{q}_2\overline{z}) \\
&= \varphi(zq_1, zq_2\overline{p}) \\
&= \varphi\big((z, p) \cdot (q_1, q_2)\big)
\end{aligned} \tag{1.5.1}$$

showing that ψ satisfies condition (a) of the theorem. Similarly, it is easy to show that μ and ρ satisfy conditions (b) and (c) of the theorem, respectively. Thus M_1 and M_2 are diffeomorphic.

1.6 Additional Results About Codimension-One Biquotient Foliations

We are interested in classifying compact simply connected C1BFs. The following proposition, whose proof can be found in [DGGK20] , tells us that in compact simply connected C1BFs the fiber spheres always have strictly positive dimension, provided that the singular leaves are connected.

Proposition 1.6.1. *Let M be a compact simply connected manifold which admits a DDB decomposition with connected singular leaves $B_\pm$. Then $B_\pm$ are both of codimension at least two.*

Corollary 1.6.2. *Let M be a compact C1BF with group diagram $\{G, K^-, K^+, H\}$ with $K^\pm / H \approx S^{\ell_\pm}$. If M is simply connected and the singular leaves $G//K^\pm$ are connected, then $\ell_\pm \geq 1$. In particular, if M is simply connected and G connected, $\ell_\pm \geq 1$.*

We have the following well known fact, stated as a proposition, which puts a strong restriction on the principal leaf of a C1BF.

Proposition 1.6.3. *Let M be a simply connected closed manifold. Any codimension one submanifold of M is orientable. In particular, the principal leaf of a C1BF must be orientable.*

The following propositions give strong restrictions on the topology of the leaves of a C1BF.

Proposition 1.6.4. *Let M be a simply connected C1BF with principal leaf $G//H$. Then $\pi_1(G//H)$ is either abelian or the quaternion group Q_8.*

Proof. The work of Grove-Halperin in [GH87] implies the result as follows. In their terminology, let $F \to G//H \to M$ be the associated fibration with homotopy fiber F. Then by the long exact sequence (LES) of homotopy we get

$$\pi_2(M) \xrightarrow{\beta} \pi_1(F) \to \pi_1(G//H) \to 0$$

Thus $\pi_1(G//H) = \pi_1(F)/\ker\beta$ and, therefore, $\pi_1(F)$ is a quotient of one of the groups in Table 1.4 of the referenced paper and hence is either a quotient of an abelian group or a quotient of Q_8. Any quotient of Q_8 is either abelian or Q_8, so the result follows. □

Proposition 1.6.5. *Let M be any simply connected DDB. Then the singular leaves must have cyclic fundamental group. In particular, the singular leaves of C1BF must have cyclic fundamental group.*

Proof. Let P denote the principal leaf and $B_\pm$ denote the singular leaves. By the long exact sequence of homotopy, $\pi_1(P) \to \pi_1(B_\pm)$ are surjective. Therefore, if $\pi_1(B_\pm)$ are not cyclic, then neither is $\pi_1(P)$. Let F be the homotopy fiber of the inclusion $P \to M$. This means we have a homotopy fibration $F \to P \to M$ and hence by the long exact sequence of homotopy we have $\pi_1(F) \to \pi_1(P)$ is surjective. Thus, since $\pi_1(P)$ is not cyclic, we also have that $\pi_1(F)$ is not cyclic. It follows from Table 1.4 in [GH87] that $\pi_1(F) \in \{Q_8, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}\}$ and that we have circle bundles $S^1 \to P \to B_\pm$. If $\pi_1(F) \approx Q_8$ and $\pi_1(P)$ is a noncyclic quotient of Q_8, then $\pi_1(P) = Q_8$ or $\pi_1(P) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. In either case, since M is simply connected, by (3.7) of [GH87], the images of the two maps $\pi_1(S^1) \to \pi_1(P)$ generate $\pi_1(P)$. In the case of $\pi_1(P) \simeq Q_8$, since $\{\pm 1\} \subset Q_8$ is a subgroup of every nontrivial subgroup of Q_8, it follows that both maps have images which strictly contain $\{\pm 1\}$. In particular, they have images of order 4 or 8. Thus, $\pi_1(B_\pm) \approx \mathbb{Z}_2$ or is trivial, either of which is cyclic, a contradiction. In the case $\pi_1(P) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, as noted above, the images of the two maps $\pi_1(S^1) \to \pi_1(P)$ generate $\pi_1(P)$, so at least one of the maps has to be nontrivial, which implies $\pi_1(B_\pm)$ is again cyclic, a contradiction. For $\pi_1(P) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,as well asFor the remaining two cases, $\pi_1(F)$ is abelian, so $\pi_1(P)$ is abelian in this case. The images of $\pi_1(S^1) \to \pi_1(P)$ are two cyclic subgroups C_- and C_+ of $\pi_1(P)$, since they are quotients of $\pi_1(S^1)$. By assumption, $\pi_1(P)/C_+ \approx \pi_1(B_+)$ is noncyclic, so the image of C_- in $\pi_1(P)/C_+$ cannot be onto. But this means that the group generated by C_- and C_+ in $\pi_1(P)$ is a proper subgroup of $\pi_1(P)$. Indeed, if any $g \in \pi_1(P)$ can be written as $c_- + c_+$ for $c_\pm \in C_\pm$, then the image of g in $\pi_1(P)/C_-$ is the same as the image of c_+. But this contradicts that M is simply connected. □

Given a C1BF decomposition of a manifold M, it will be important for us to decide whether the manifold is simply connected either directly from its C1BF diagram or its leaf structure. Here we will prove a version of the van Kampen theorem for C1BFs which will allow us to do this. We first introduce a small amount of rational homotopy theory which will allow us to prove a corollary of the van Kampen theorem which will put further restrictions on the topology of the principal leaf of a C1BF.

Definition 1.6.6. Let X be a topological space and let $\pi_k(X)$ denote the k-th homotopy group of X. The corresponding **rational homotopy group** is obtained by tensoring with $\mathbb{Q}$. We will denote this by $\pi_k(X)_{\mathbb{Q}} := \pi_k(X) \otimes \mathbb{Q}$.

Remark: The tensor product $\pi_1(X) \otimes \mathbb{Q}$ only makes sense with the usual tensor product for $\pi_1(X)$ abelian. For $\pi_1(X)$ nonabelian the tensor product is interpreted using Example 2.52 in [FOT08]. In all instances in this thesis, $\pi_1(X)$ is abelian or the quaternion group Q_8 and, in the latter case, $\pi_1(X)_{\mathbb{Q}} = 0$.

For X a topological space and R a coefficient ring, define $H_*(X;R) = \bigoplus_{i=1}^{\infty} H_i(X;R)$ and

$\pi_*(X) = \bigoplus_{i=1}^{\infty} \pi_i(X)$. The following lemma allows us to compute the rational homotopy of spaces with finitely generated homotopy groups.

Lemma 1.6.7. *Let A_i be abelian groups. Then the following properties hold:*
(i) $(\bigoplus_{i=1}^{m} A_i) \otimes \mathbb{Q} \approx \bigoplus_{i=1}^{m} (A_i \otimes \mathbb{Q})$,
(ii) $\mathbb{Z}_n \otimes \mathbb{Q} = 0$,
(iii) $\mathbb{Z}^n \otimes \mathbb{Q} \approx \mathbb{Q}^n$.

We now state the classical version of the van Kampen theorem to establish the notation for the C1BF version of the theorem.

Theorem 1.6.8. *(Classical van Kampen Theorem) Suppose $X = A \cup B$, where A, B are open subsets and $x_0 \in A \cap B$ is the basepoint, and A, B, $A \cap B$ are path connected. Then we have four inclusion maps*

$$\begin{tikzcd} & A \arrow[dr, "i"] & \\ A \cap B \arrow[ur, "a"] \arrow[dr, "b"'] & & X \\ & B \arrow[ur, "j"'] & \end{tikzcd}$$

and the homomorphism $\Phi : \pi_1(A) * \pi_1(B) \to \pi_1(X)$ extending the induced homomorphisms a_* and b_* is surjective with kernel $N = \langle\langle a_*(w) b_*(w)^{-1} \rangle\rangle$ for $w \in \pi_1(A \cap B)$.

To prove our version of the van Kampen theorem, we will use the following lemma.

Lemma 1.6.9. *Suppose $X = A \cup B$ satisfies the hypothesis of the classical van Kampen theorem and let N be as in the classical van Kampen theorem. If the induced maps $a_* : \pi_1(A \cap B) \to \pi_1(A)$ and $b_* : \pi_1(A \cap B) \to \pi_1(B)$ are surjective, then*

$$(\pi_1(A) * \pi_1(B))/N \approx \pi_1(A \cap B)/N_A N_B$$

where $N_A = \ker a_*$ and $N_B = \ker b_*$.

Proof. Define $\varphi : \pi_1(A \cap B)/N_A N_B \to (\pi_1(A) * \pi_1(B))/N$ by $\varphi(gN_AN_B) = a_*(g)N$. Observe that $a_*(g)N = b_*(g)N$. This implies that φ is well defined. Also, φ is clearly a homomorphism. We wish to construct an inverse homomorphism for φ. Note that since $a_* : \pi_1(A \cap B) \to \pi_1(A)$ is surjective, a_* induces an isomorphism $\overline{a}_* : \pi_1(A \cap B)/N_A \to \pi_1(A)$ and, similarly, b_* induces an isomorphism $\overline{b}_* : \pi_1(A \cap B)/N_B \to \pi_1(B)$. Therefore, we get homomorphisms

$$\pi_1(A) \xrightarrow{\overline{a}_*^{-1}} \pi_1(A \cap B)/N_A \xrightarrow{j_a} \pi_1(A \cap B)/N_A N_B$$

where j_a is the quotient map $gN_A \mapsto gN_AN_B$. Similarly, we get homomorphisms

$$\pi_1(B) \xrightarrow{\overline{b}_*^{-1}} \pi_1(A \cap B)/N_A \xrightarrow{j_b} \pi_1(A \cap B)/N_A N_B$$

where j_b is the analogous quotient map. Let $\tilde{a} = j_a \circ \overline{a}_*^{-1}$ and $\tilde{b} = j_b \circ \overline{b}_*^{-1}$. By the universal property of free products $\tilde{a}$ and $\tilde{b}$ extend to a homomorphism $\psi : \pi_1(A) * \pi_1(B) \to \pi_1(A \cap B)/N_A N_B$ defined by $\psi(a_1 b_1 a_2 b_2 \ldots) = \tilde{a}(a_1)\tilde{b}(b_1)\tilde{a}(a_2)\tilde{b}(b_2)\ldots$. It is easy to check that $N \subset \ker\psi$ and, therefore, ψ induces a homomorphism $\overline{\psi} : (\pi_1(A) * \pi_1(B))/N \to \pi_1(A \cap B)/N_A N_B$ defined by $\overline{\psi}(gN) = \psi(g)$. It is straightforward to check that $\overline{\psi}$ is the inverse of φ hence φ is an isomorphism. □

We now give the C1BF analogue of the van Kampen theorem for cohomogeneity one manifolds, which can be found in [Hoe10]. Before stating the theorem, note that for any C1BF given by the group diagram $\{G, K^-, K^+, H\}$, we have sphere bundles $K^\pm/H \xrightarrow{i} G//H \xrightarrow{\pi} G//K$ where the projection map π is given by $\pi[g]_H = [g]_{K^\pm}$ and the inclusion map i is given by $(k_1, k_2)H \mapsto [k^{-1}gk_2]$ for any $g \in G$.

Theorem 1.6.10 (van Kampen Theorem for C1BFs)**.** *Let M be a C1BF with group diagram*

$$\begin{tikzcd}
 & G \arrow[dl, equal] \arrow[dr, equal] & \\
K^- \arrow[dr, dash] & & K^+ \arrow[dl, dash] \\
 & H &
\end{tikzcd}$$

with $K^\pm/H \approx S^{\ell_\pm}$ *with* $\ell_\pm \geq 1$. *Then* $\pi_1(M) \approx \pi_1(G//H)/N_- N_+$ *where*

$$N_\pm = \ker\{\pi_1(G//H) \to \pi_1(G//K^\pm)\} = Im\{\pi_1(K^\pm/H) \to \pi_1(G//H)\}.$$

In particular, M *is simply connected if and only if the images of* $K^\pm/H = S^{\ell_\pm}$ *generate* $\pi_1(G//H)$ *under the natural inclusions.*

Proof. We will compute the fundamental group of M using van Kampen's theorem. We know that M decomposes as a double disk bundle $M = B_+ \cup B_-$ glued along a principal leaf $G//H$; that is, $B_+ \cap B_- = G//H$. Let $a^\pm$ be the inclusions of $B_+ \cap B_-$ into $B\pm$. Assume the basepoint x_0 is contained in $B_+ \cap B_-$. Observe that the inclusions $a^\pm$ induce the same maps on homotopy as the projection maps in the sphere bundles $K^\pm H \to G//H \to G//K^\pm$. Indeed, we know that $B_\pm$ deformation retracts onto the singular leaves $G//K_\pm$ and it is easy to see that, under the deformation retractions, the inclusions $a^\pm : B_+ \cap B_- \to B_\pm$ become the projection map in the sphere bundle above. Thus, with a slight abuse of notation, we also call the projection maps $a^\pm$, giving us the sphere bundles

$$K^\pm/H \xrightarrow{i^\pm} G//H \xrightarrow{a^\pm} G//K^\pm.$$

where $K^\pm/H \approx S^{\ell_\pm}$ and $i^\pm$ are the inclusions. This gives the long exact sequence of homotopy groups

$$\ldots \to \pi_1(S^{\ell_\pm}) \xrightarrow{i^\pm_*} \pi_1(G//H) \xrightarrow{a^\pm_*} \pi_1(G//K^\pm) \to \pi_0(S^{\ell_\pm}).$$

Since $\ell_\pm > 0$ we have that $S^{\ell_\pm}$ is connected so $\pi_0(S^{\ell_\pm}) = 0$. Therefore $a_*^\pm : \pi_1(G//H) \to \pi_1(G//K^\pm)$ is surjective. The result now follows from the the previous lemma.

$\square$

Corollary 1.6.11. *If M is a C1BF as in the van Kampen theorem and is simply connected, the principal leaf $G//H$ has $\pi_1(G//H)_{\mathbb{Q}} \in \{0, \mathbb{Q}, \mathbb{Q}^2\}$. In particular, if M is simply connected and the principal leaf has a torus factor T^n, then $n \leq 2$.*

Proof. Since $\pi_1(M) = 0$ we have $N_- N_+ = \pi_1(G//H)$. Also note that

$$N_\pm = \mathrm{Im}(\pi_1(K^\pm/H)) \approx \mathrm{Im}(\pi_1(S^{\ell_\pm})).$$

Since $\pi_1(S^{\ell_\pm}) = 0$ or $\pi_1(S^{\ell_\pm}) \approx \mathbb{Z}$ we have $N_\pm \in \{0, \mathbb{Z}_n, \mathbb{Z}\}$ for some positive integer n. By Proposition 1.6.4, it follows that $\pi_1(G//H)$ is either abelian or Q_8. If $\pi_1(G//H)$ is abelian then $N_- N_+ \approx N_- \times N_+$ and hence $\pi_1(G//H)_{\mathbb{Q}} = \pi_1(N_- N_+)_{\mathbb{Q}} \in \{0, \mathbb{Q}, \mathbb{Q}^2\}$ depending on the values of $N_\pm$. On the other hand, if $\pi_1(G//H) = Q_8$ then $\pi_1(G//H)_{\mathbb{Q}} = 0$. $\square$

Chapter 2

Low-Dimensional Examples of Non-negative Curvature

Non-negatively curved (and positively curved) manifolds have been of great interest in Riemannian geometry and are the subject of several classical theorems such as the theorems of Bonnet-Myers and Synge, as well as the sphere theorem. Examples of non-negatively curved manifolds are limited, but new examples of such manifolds are frequently constructed as homogeneous spaces or biquotients. For example, the Gromoll-Meyer sphere [GM74] was the first exotic sphere shown to admit non-negative sectional curvature by showing it could be written as a biquotient. Grove-Ziller [GZ00] in the process of proving that the Milnor spheres (i.e. exotic spheres that are also S^3-bundles over S^4) in dimension 7 admit non-negative curvature have shown that cohomogeneity one manifolds with singular orbits of codimension at most two admit a metric of non-negative sectional curvature. The proof exploits the presence of a double disk-bundle (DDB) structure by putting metric of non-negative curvature on each half of the DDB that agrees on the common boundary. DDB decompositions are present in a plethora of other examples of manifolds which admit metrics of non-negative curvature; for example see, [GKS20, Dea11, GVZ11].

Given the prevalence of known examples of manifolds which admit metrics of non-negative curvature which are known to admit DDB decompositions, the Double-Soul Conjecture [Gro02] asks whether every non-negatively curved, closed, simply connected Riemannian manifold admits a DDB decomposition. It is important to note that there exist compact, simply connected manifolds which do not admit DDB decompositions, but have no currently known obstruction to admitting non-negative curvature. For example, in dimension 5, the complete list of manifolds which admit a DDB decomposition are S^5, $S^3 \times S^2$, the Wu manifold SU(3)/SO(3), and $S^3 \hat{\times} S^2$ (the unique nontrivial S^3-bundle over S^2 [DGGK20]. Therefore the connected sum of the Wu manifold with itself is not a DDB, yet there is no known obstruction to this manifold admitting non-negative curvature.

We have seen that, just like cohomogeneity one manifolds, C1BFs also admit a DDB structure and, if the singular leaves have codimension at most two, such a C1BF also admits a metric of non-negative curvature. Given the prevalence of DDB decompositions in non-negative curvature, it is natural to ask whether all currently known examples of manifolds which admit metrics of non-negative curvature admit a C1BF structure. This will be the topic of this chapter and we will see that this is true up to dimension 6. In particular, we will show that all known examples of compact simply connected manifolds M^n which admit metrics of non-negative sectional curvature admit a C1BF structure for $n \leq 6$. Moreover, as mentioned above, most such examples of non-negative curvature are biquotients and we will, in particular, show that every representation of M^n as a reduced biquotient gives rise to a C1BF structure in a natural way. For $n \leq 5$, all such examples are biquotients. DeVito [DeV14, DeV17] has given a classification of compact simply connected reduced biquotients up to dimension 7, so we refer to this (along with DeVito's PhD thesis) for the descriptions of low dimensional manifolds as biquotients. We note that Kapovitch and Ziller [KZ04] have previously classified biquotients with singly generated rational cohomology rings. We summarize the results of this chapter in the following theorem.

Theorem 2.0.1. *All known examples of compact, simply connected manifolds of dimension at most 6 which admit a metric of non-negative sectional curvature are C1BFs. Furthermore, for all such examples which are diffeomorphic to biquotients, all representations of these manifolds as a reduced biquotient $G//H$ naturally give rise to a C1BF structure.*

It is worth noting that if one drops the assumption that the manifold M is simply connected, it is not clear whether all examples of non-negative curvature in dimensions less than 6 are biquotients. In particular, Torres [Tor19] has constructed an example of a 4-dimensional manifold with fundamental group $\mathbb{Z}_2$ which is certainly not diffeomorphic to a biquotient $G//H$ for G connected. It is currently undecided whether this manifold is diffeomorphic to a biquotient for G disconnected. In higher dimensions (with the exception of dimension 7), it is unlikely that all known examples simply connected examples of manifolds with non-negative curvature are C1BFs. However, many higher dimensional examples are constructed using techniques that guarantee they are C1BFs, such as the infinite family of 8-dimensional cohomogeneity one examples constructed by Dessai [Des16].

In dimensions 2 and 3, the above theorem is trivial because only simply connected manifolds are S^2 and S^3 and their only descriptions as reduced biquotients are the homogeneous quotients $S^2 = \mathrm{SU}(2)/S^1$ and $S^3 = \mathrm{SU}(2)/\{e\}$. Theorem 1.5.10 trivially implies that both of these give rise to C1BF structures. We will break the work for dimensions 4-6 into separate sections.

2.1 Examples in Dimensions 4 and 5

Dimension 4:

From DeVito's classification, it follows that $M = G/\!/H$ is one of the spaces on the following list:

1. S^4
2. $\mathbb{C}\mathrm{P}^2$
3. $S^2 \times S^2$
4. $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$
5. $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$

For spaces (3)-(5), the only way to write these spaces as reduced biquotient is $M = \mathrm{Sp}(1)^2/\!/T^2$ for various effectively free linear torus actions, so each of their representations as reduced biquotients give rise to a C1BF structure. Thus it remains to handle spaces (1) and (2). In the case $M = S^4$, if these spaces are written as a reduced biquotient $G/\!/H$, then the groups G and H are one the following four pairs:

(i) $G = \mathrm{Sp}(2)$; $H = \mathrm{Sp}(1)^2$

(ii) $G = \mathrm{SU}(4)$; $H = \mathrm{SU}(3) \times \mathrm{SU}(2)$

(iii) $G = \mathrm{Spin}(8)$; $H = \mathrm{Spin}(7) \times \mathrm{SU}(2)$

(iv) $G = \mathrm{Spin}(7)$; $H = G_2 \times \mathrm{SU}(2)$

Similarly, if $M = \mathbb{C}\mathrm{P}^2$ then the groups G and H are one of the following two pairs:

(v) $G = \mathrm{SU}(3)$; $H = \mathrm{SU}(2) \times S^1$

(vi) $G = \mathrm{SU}(4)$; $H = \mathrm{Sp}(2) \times S^1$

According to DeVito's work, groups (i) give rise to a homogeneous space and a non-homogeneous biquotient, described in Cases 1 and 2 below, respectively.

Case 1:

The homogeneous space $\mathrm{Sp}(2)/\mathrm{Sp}(1)^2$ determined by groups (i) is given by the standard embedding $(p,q) \mapsto \mathrm{diag}(p,q)$ of $\mathrm{Sp}(1)^2$ into $\mathrm{Sp}(2)$. Equivalently, this is the quotient of the action of $\mathrm{Sp}(1)^2$ on $\mathrm{Sp}(2)$ by right translation. The action by right translation is contained within the action of $\Gamma = \mathrm{Sp}(1)^4$ on $\mathrm{Sp}(2)$ given by $(p,q,r,s) \cdot X = \mathrm{diag}(r,s) X \mathrm{diag}(\overline{p}, \overline{q})$. It is easy to compute that the isotropy Γ_Y of the matrix

$$Y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is $\Gamma_Y = \{(q,q,q,q) : q \in \mathrm{Sp}(1)\} \simeq \mathrm{Sp}(1)$, so the orbit Γ/Γ_Y is 9 dimensional, which is codimension 1 inside of $\mathrm{Sp}(2)$. Thus the action of Γ is cohomogeneity one and it follows that this homogeneous space is a C1BF.

Case 2:

The biquotient determined by groups (i) is given by the action of $\mathrm{Sp}(1)^2$ on $\mathrm{Sp}(2)$ by

$$(p,q) \cdot X = \mathrm{diag}(p,p) X \mathrm{diag}(\overline{q}, 1)$$

This biquotient action is contained within the same action as the homogenous action in Case 1, so is also a C1BF.

For groups (ii), according to DeVito's work there is no homogeneous spaces and only one biquotient, given by Case 3.

Case 3:

The biquotient determined by groups (ii) is given by the action of $SU(2) \times SU(3)$ on $SU(4)$ by

$$(A,B)\cdot X = \mathrm{diag}(A,A)X\mathrm{diag}(B^*,1)$$

Extend this to the action of $\Gamma = SU(2) \times SU(2) \times U(3)$ on $SU(4)$ given by

$$(A,B,C)\cdot X = \mathrm{diag}(A,B)X\mathrm{diag}(C^*,\overline{\det C^*})$$

To see that this action is cohomogeneity one, it is not difficult to compute that the isotropy of the identity $I \in SU(4)$ is

$$\Gamma_I = \{(A,I,\mathrm{diag}(A,I)) : A \in SU(2)\} \simeq SU(2)$$

which has codimension 3 inside of $SU(4)$. We wish to show that the action of G_I induced by the slice representation on the unit sphere in the normal space of the identity is transitive. The orbit has codimension 3, so the unit sphere in the normal space is S^2. Thus we are looking at a linear action of $\Gamma_I = SU(2)$ on S^2. But $\dim(SU(2)) = \dim(SO(3))$ so it follows from Myers-Steenrod that this action must be equivalent to the transitive action of $O(3)$ on S^2. Thus by Proposition 1.4.5 it follows that the action of Γ is cohomogeneity one, so the biquotient in this case is a C1BF.

For groups (iii), DeVito's work tells us that there are no homogeneous spaces and only one biquotient, described in Case 4.

Case 4:

To describe the biquotient given by groups (iii), recall that the standard embedding of SU(2) into SO(8) is given by taking a matrix $P \in \mathrm{SU}(2)$ and splitting P into its real and imaginary parts; that is, writing $P = B + Ci$ for some real matrices B and C and mapping P to the block matrix

$$\hat{P} = \begin{pmatrix} B & -C & \\ C & B & \\ & & I \end{pmatrix}$$

The biquotient determined by groups (iii) is then given by the action of $\mathrm{Spin}(7) \times \mathrm{SU}(2)$ on Spin(8) by

$$(A, P) \cdot X = \mathrm{diag}(A, 1) X \hat{P}^T$$

where we take the lift of $\mathrm{diag}(A, 1)$ and $\hat{P}^T$ in SO(8) to a corresponding element of Spin(8). To see that this action is contained within a cohomogeneity one action, observe that $\hat{P} \in \mathrm{SO}(7) \subset \mathrm{SO}(8)$, so its lift is contained in $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$. Thus we can extend the above action to the action of $\mathrm{Spin}(7) \times \mathrm{Spin}(7)$ on Spin(8) by left and right translation, which is the lift of the analogous action of $\mathrm{SO}(7) \times \mathrm{SO}(7)$ on SO(8), which is known to be cohomogeneity one.

For groups (iv), there is one biquotient, which is discussed in Case 5.

Case 5:

According to DeVito's work, up to finite cover, the biquotient determined by groups (iv) is given as $G_2 \backslash \mathrm{Spin}(7) / \mathrm{SU}(2)$ where $G_2 \to \mathrm{Spin}(7)$ via the lift of the standard embedding $G_2 \to \mathrm{SO}(7)$ and SU(2) is embedded into Spin(7) via the lift of the map $\mathrm{SU}(2) \to \mathrm{SO}(3) \to \mathrm{SO}(7)$, where the first map is the double covering map and the second map sends an element $B \in \mathrm{SO}(3)$ to $\mathrm{diag}(B, I)$. Thus the biquotient can be examined by the action of

$G_2 \times \mathrm{SO}(3)$ on $\mathrm{SO}(7)$ given by

$$(A, B) \cdot X = AX\mathrm{diag}(B^T, I)$$

To extend this action to a cohomogeneity one action is somewhat tricky. The natural thing to do is to extend the SO(3) factor to SO(6) or SO(5), but Onishchik [Oni94] has shown that these actions are both transitive. However, Kollross [Kol02] has shown that the action of $G_2 \times \mathrm{SO}(3) \times \mathrm{SO}(4)$ on SO(7) is cohomogeneity one, which is an extension of the above biquotient action. So the above biquotient yields a C1BF structure.

Groups (v) give rise to one homogeneous space and one biquotient, described in Cases 6 and 7, respectively.

Case 6:

The homogeneous space determined by groups (v) is $\mathrm{SU}(3)/\mathrm{U}(2)$ where U(2) embeds in SU(3) via the standard embedding $A \mapsto \mathrm{diag}(A, \overline{\det A})$. This homogeneous action is clearly contained within the action of $\mathrm{U}(2) \times \mathrm{U}(2)$ on SU(3) by left and right translation, which is well known to be cohomogeneity one. Thus we get a C1BF structure in this case.

Case 7:

The nonhomogeneous biquotient determined by groups (v) is given by the action of $\mathrm{SU}(2) \times S^1$ on SU(3) by

$$(A, z) \cdot X = \mathrm{diag}(zA, \overline{z}^2)X\mathrm{diag}(\overline{z}^4, \overline{z}^4, z^8)$$

To see that this biquotient action is contained within a cohomogeneity one action, observe that $\mathrm{diag}(\overline{z}^4, \overline{z}^4, z^8)$ us contained in $\mathrm{U}(2) \subset \mathrm{SU}(3)$ and, similarly, $\mathrm{diag}(zA, \overline{z}^2)$ is also contained within $\mathrm{U}(2) \subset \mathrm{SU}(3)$. Thus we can extend this biquotient action to the same cohomogeneity one action as in the previous case.

Finally, groups (vi) give rise to only a nonhomogeneous biquotient, which we will describe in Case 8.

Case 8: Recall that the standard embedding of Sp(2) in SU(4) is given by writing a matrix $P \in \mathrm{Sp}(2)$ as $P = A + Bj$ for complex matrices A and B and mapping P to the block matrix

$$\hat{P} = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

For groups (vi), the nonhomogeneous biquotient is is given by the action of $\mathrm{Sp}(2) \times S^1$ on SU(4) by

$$(P, z) \cdot X = \hat{P} X \mathrm{diag}(\overline{z}, \overline{z}, \overline{z}, z^3)$$

To see that this action is contained within a cohomogeneity one action, observe that $\mathrm{diag}(\overline{z}, \overline{z}, \overline{z}, z^3)$ is contained in $\mathrm{S}(\mathrm{U}(2)\mathrm{U}(2)) \subset \mathrm{SU}(4)$, so we can extend the action to the action of $\Gamma = \mathrm{Sp}(2) \times \mathrm{S}(\mathrm{U}(2)\mathrm{U}(2))$ on SU(4) given by

$$\left(\hat{P}, \mathrm{diag}(C, D)\right) \cdot X = \hat{P} X \mathrm{diag}(C^*, D^*)$$

Let us compute the orbit of the identity. It is not difficult to compute that the isotropy of the identity is

$$\Gamma_I = \left\{\mathrm{diag}(A, \overline{A}), \mathrm{diag}(A, \overline{A}) : A \in \mathrm{U}(2)\right\} \simeq \mathrm{U}(2)$$

Therefore the orbit of the identity has codimension 2 inside of SU(4), so the normal sphere to the orbit at the identity is a circle S^1. Thus $\Gamma_I \simeq \mathrm{SU}(2)$ acts on S^1 via the action induced by the slice representation. Moreover, because this action is nontrivial, it follows that it is effectively a circle action, so is transitive. Thus the action by Γ is cohomogeneity one and contains the original biquotient action, which shows that the biquotient is a C1BF.

Dimension 5:

Dimension 5 actually turns out to be much easier than dimension 4. According to Pavlov [Pav04], the only compact simply connected five dimensional biquotients are S^5, $S^3 \times S^2$, $S^3 \hat{\times} S^2$, and the Wu manifold $W = \mathrm{SU}(3)/\mathrm{SO}(3)$. According to DeVito's classification [DeV14], for $S^3 \times S^2$ and $S^3 \hat{\times} S^2$, all reduced biquotients are of the form $M = \mathrm{Sp}(1)^2/S^1$ for some linear action of S^1, so by Theorem 1.5.10 all give rise to C1BF structures. Moreover, in the cases where $M = S^5$ or the case where M is the Wu manifold, all reduced biquotients are homogeneous. In particular, S^5 can be written as $S^5 = \mathrm{SU}(4)/\mathrm{Sp}(2)$ or $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$, and the Wu manifold is, of course, the homogeneous space $W = \mathrm{SU}(3)/\mathrm{SO}(3)$. Each of these representations of S^5 and W as reduced biquotients admit cohomogeneity one actions. In particular, the action of $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ on W is well known to be cohomogeneity one [Hoe10]. It is also easy to see that the actions of $\mathrm{Sp}(2)$ on $S^5 = \mathrm{SU}(4)/\mathrm{Sp}(2)$ is cohomogeneity one and that the action of $\mathrm{SU}(2)$ on $\mathrm{SU}(3)$ by right translation is contained in the cohomogeneity one action of $\mathrm{U}(2) \times \mathrm{U}(2)$ on $\mathrm{SU}(3)$, therefore $\mathrm{SU}(3)/\mathrm{SU}(2)$ is also a C1BF.

2.2 Examples in Dimension 6

In dimension 6 we have the usual biquotient examples, which can be found in DeVito and Kapovitch-Ziller's classifications [DeV14, DeV17, KZ04]. We note that a certain family of biquotients of the form $\mathrm{Sp}(1)^3//T^3$ appearing in DeVito's classification were previously examined in detail by Totaro [Tot03]. Furthermore, in dimension 6 there are additional infinite families of examples which admit non-negative curvature discovered by Grove-Ziller [GZ00, GZ11] which are constructed as quotients of certain principal SO(3)-bundles over various 4-manifolds, namely S^4 or $\mathbb{C}\mathrm{P}^2$, $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$. For instance, in the case of $\mathbb{C}\mathrm{P}^2$, Grove-Ziller take the cohomogeneity one group diagram for $\mathbb{C}\mathrm{P}^2$

$$\begin{tikzcd}[column sep=small]
 & S^3 \arrow[dl, dash] \arrow[dr, dash] & \\
C_j \cup jC_i & & C_j \\
 & \mathbb{Z}_4 = \langle j \rangle \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

where $C_i = \{e^{i\theta} : \theta \in \mathbb{R}\}$ and $C_j = \{e^{j\theta} : \theta \in \mathbb{R}\}$, and lift this cohomogeneity one group diagram to a cohomogeneity one diagram which defines a principal SO(3)-bundle over $\mathbb{C}\mathrm{P}^2$ which is given by

$$\begin{tikzcd}[column sep=small]
 & \mathrm{SO}(3) \times S^3 \arrow[dl, dash] \arrow[dr, dash] & \\
R_{j,k}(p_-\theta), e^{i\theta}) \cup (R_{i,k}(\pi), j) \cdot S^1 & & (R_{i,k}(p_+\theta), e^{j\theta}) \\
 & \mathbb{Z}_4 = \langle R_{i,k}(\pi), j \rangle \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

where p_- is even and $p_+ \equiv 2 \pmod 4$. Here S^1 denotes the identity component of the group K^- in the diagram above and $R_{i,k}(\theta)$ and $R_{j,k}(\theta)$ denote the group of rotations by an angle θ in the 2-plane spanned by $\{i, k\}$ and $\{j, k\}$, respectively in the imaginary quaternions $\mathrm{Im}(\mathbb{H})$. Grove and Ziller use a result of Dold and Whitney [DW59] which says that the total space of any principal bundle over a simply connected 4-manifold is determined by its second Stiefel-Whitney class and its first Pontryagin class. Moreover, they show that the total space of every vector bundle over $\mathbb{C}\mathrm{P}^2$ with nontrivial second Stiefel-Whitney class admits a complete metric of non-negative curvature. That is, every vector bundle over $\mathbb{C}\mathrm{P}^2$ which is not spin admits a complete metric of non-negative curvature. In particular, Grove and Ziller show that the principal SO(3)-bundles defined by the diagram above has first Pontryagin class $p_1 = \frac{1}{4}(p_+^2 - p_-^2)$ and is spin if and only if $p_\pm \equiv 2 \pmod 4$. Therefore, "half" of these bundles are known to admit a metric non-negative curvature. It is currently unknown whether the other half admit a metric of non-negative curvature. We obtain additional 6-dimensional examples of non-negative curvature by restricting the action of SO(3) on the above 7-dimensional examples to the free SO(2)-action contained within it. These examples are obviously C1BFs by construction. Similarly, the other Grove-Ziller principal SO(3)-bundles are easily seen to be C1BFs as well by the same argument.

We now turn to the 6-dimensional compact simply connected reduced biquotients $M^6 = G//H$. According to DeVito's classification, for compact simply connected reduced biquotients whose cohomology ring is not singly generated, the following list of groups are all of the possible pairs of groups (G, H)

1. $G = \mathrm{SU}(2)^2$; $H = 1$
2. $G = \mathrm{SU}(4) \times \mathrm{SU}(2)$; $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times S^1$
3. $G = \mathrm{Sp}(2) \times \mathrm{SU}(2)$; $H = \mathrm{Sp}(1)^2 \times S^1$
4. $G = \mathrm{Spin}(7) \times \mathrm{SU}(2)$; $H = G_2 \times \mathrm{SU}(2) \times S^1$
5. $G = \mathrm{Spin}(8) \times \mathrm{SU}(2)$; $H = \mathrm{Spin}(7) \times \mathrm{SU}(2) \times S^1$
6. $G = \mathrm{SU}(3)$; $H = T^2$
7. $G = \mathrm{SU}(3) \times \mathrm{SU}(2)$; $H = \mathrm{SU}(2) \times T^2$
8. $G = \mathrm{SU}(4) \times \mathrm{SU}(2)$; $H = \mathrm{Sp}(2) \times T^2$

9. $G = \mathrm{SU}(2)^3$; $H = T^3$

Note that pair (1) is trivially a C1BF. Notice for pairs (2)-(5), all of the groups G have a SU(2) factor and all of the H groups have a circle factor. This motivates us to make the following proposition.

Proposition 2.2.1. *Suppose we have an action of $L \times S^1$ on $\Gamma \times SU(2)$ for some groups L and Γ such that the action is a product action with L acting on Γ transitively and S^1 acting on $SU(2)$ by any nontrivial linear action. Then the product action is contained within a cohomogeneity one action. Thus the quotient of $\Gamma \times SU(2)$ by any effectively free subaction of the action of $L \times S^1$ on $\Gamma \times SU(2)$ is a C1BF.*

Proof. G acts on Γ transitively, so by Proposition 1.4.3 the action can be extended to cohomogeneity one if and only if the action of S^1 on SU(2) can be extended to cohomogeneity one. But $\mathrm{SU}(2) \simeq S^3$, so this is a linear torus action on a sphere. The result then follows from Theorem 1.5.10. □

Therefore, to show biquotients arising from pairs (2)-(5) are C1BFs, it suffices to extend the actions determined by these biquotients to actions of the form of Proposition 2.2.1. We now describe the actions determined by pairs (2)-(5), as specified in DeVito's classification [DeV17], in the cases below. Note that in each case below, the actions require restrictions on the parameters to ensure that they are effectively free, but we will be able to extend them to cohomogeneity one actions regardless of the parameters, so we do not specifically mention the parameter restrictions.

Case 1: Group Pair (2)

The pair (2) determines one family of actions of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times S^1$ on $\mathrm{SU}(4) \times \mathrm{SU}(2)$

given by

$$(A, B, z) \cdot (X, Y) = \left(\mathrm{diag}(z^m A, \overline{z}^m) X \mathrm{diag}(\overline{z}^n B^*, z^n B^*), \mathrm{diag}(z^l, \overline{z}^l) Y\right)$$

for appropriate restrictions on the parameters to ensure the action is effectively free.

Case 2: Group Pair (3)

The pair (3) determines two families of actions of $\mathrm{Sp}(1)^2 \times S^1$ on $\mathrm{Sp}(2) \times \mathrm{SU}(2)$ where the first family is given by

$$(p, q, z) \cdot (X, Y) = \left(\mathrm{diag}(p, q) X \mathrm{diag}(\overline{z}^m, \overline{z}^n), \mathrm{diag}(z^l, \overline{z}^l) Y\right)$$

and the second family is given by

$$(p, q, z) \cdot (X, Y) = \left(R(m\theta) \mathrm{diag}(p, p) X \mathrm{diag}(\overline{q}, \overline{z}^n), \mathrm{diag}(z^l, \overline{z}^l) Y\right)$$

where in the second action, $z = e^{i\theta}$ and $R(\theta)$ is the standard rotation matrix. Note that we require appropriate restrictions on the parameters to ensure the action is effectively free.

Case 3: Group Pair (4)

The pair (4) determines one family of actions of $\mathrm{Spin}(7) \times \mathrm{SU}(2) \times S^1$ on $\mathrm{Spin}(8) \times \mathrm{SU}(2)$ given by

$$(A, B, z) \cdot (X, Y) = \left(\mathrm{diag}(A, 1) X \mathrm{diag}(\overline{z}^m B^*, \overline{z}^n B^*), \mathrm{diag}(z^l, \overline{z}^l) Y\right)$$

where the above notation means the lift of $\mathrm{diag}(A, 1) \in \mathrm{SO}(7) \subset \mathrm{SO}(8)$ to $\mathrm{Spin}(7) \subset$

Spin(8) and the notation $\mathrm{diag}(z^m B, z^m B) \in \mathrm{Spin}(8)$ means the lift of the block diagonal embedding $\mathrm{U}(2) \subset \Delta\mathrm{SO}(4) \subset \mathrm{SO}(4)^2 \subset \mathrm{SO}(8)$. Note that we require appropriate restrictions on the parameters to ensure the action is effectively free.

Case 4: Group Pair (5)

Recall that, up to conjugacy, there is a unique nontrivial embedding $G_2 \to \mathrm{SO}(7)$. We let $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ denote the double covering map. The family of actions determined by pair (5) is induced by the lift of the homomorphism $f : G_2 \times \mathrm{SU}(2) \times S^1 \times S^1 \to (\mathrm{SO}(7) \times \mathrm{SU}(2))^2$ where, with

$$z = e^{i\theta}; \quad f(A, B, z) = \big(A, \mathrm{diag}(z^l, \overline{z}^l), \mathrm{diag}(\pi(B), R(m\theta), R(n\theta)), I\big).$$

Thus the pair (5) determines one family of actions of $G_2 \times \mathrm{SU}(2) \times S^1$ on $\mathrm{SO}(7) \times \mathrm{SU}(2)$ given by

$$(A, B, z) \cdot (X, Y) = \big(AX \mathrm{diag}(\pi(B), R(m\theta), R(n\theta))^{-1}, \mathrm{diag}(z^l, \overline{z}^l) Y\big)$$

for appropriate restrictions on the parameters to ensure the action is effectively free.

In each of the above cases, we have some group $L \times S^1$ acting on some group $\Gamma \times \mathrm{SU}(2)$ where $L \times S^1$ acts on Γ by left and right translation by elements which are contained within Γ, and the action on $\mathrm{SU}(2)$ is given by multiplication by $\mathrm{diag}(z^l, \overline{z}^l)$, so we can extend each action to one of the form $\Gamma^2 \times S^1$ to an action of $\Gamma^2 \times S^1$ on $\Gamma \times \mathrm{SU}(2)$ given by $(\gamma_1, \gamma_2) \cdot (X, Y) = \big(\gamma_1 X \gamma_2^{-1}, \mathrm{diag}(z^l, \overline{z}^l) Y\big)$, which is an action of the form of Proposition 2.2.1, so all of their quotients are C1BFs.

The group pair (6) above determines only one biquotient, namely *Eschenburg's in-*

homogeneous flag manifold [Esc82]. A standard way of constructing this manifold is as the quotient by the action of T^2 on SU(3) given by

$$(z,w)\cdot X = \mathrm{diag}(z,w,zw)X\mathrm{diag}(1,1,\overline{z}^2\overline{w}^2) \tag{2.2.1}$$

To see that Eschenburg's inhomogeneous flag manifold is a C1BF, we recall that the action $\mathrm{U}(2)\times\mathrm{U}(2)$ on SU(3) by left and right translation is cohomogeneity one. Introduce ineffective kernel into the action (2.2.1) by cubing every instance of z and w so that we get an action of T^2 on SU(3) by

$$(z,w)\cdot X = \mathrm{diag}(z^3,w^3,z^3w^3)X\mathrm{diag}(1,1,\overline{z}^6\overline{w}^6) \tag{2.2.2}$$

Now, observe that $\mathrm{diag}(\overline{z}^2\overline{w}^2,\overline{z}^2\overline{w}^2,\overline{z}^2\overline{w}^2) = \overline{z}^2\overline{w}^2 I \in Z(\mathrm{U}(2))$, where $Z(G)$ denotes the center of a group G. Hence the action of T^2 on SU(3) by

$$(z,w)\cdot X = \mathrm{diag}(z\overline{w}^2,\overline{z}^2w,zw)X\mathrm{diag}(z^2w^2,z^2w^2,\overline{z}^4\overline{w}^4) \tag{2.2.3}$$

has the same quotient as action (2.2.2). But action (2.2.3) is now an action by left and right multiplication by elements which are contained in $\mathrm{U}(2)\subset\mathrm{SU}(3)$, so we can extend this to the cohomogeneity one action $\mathrm{U}(2)\times\mathrm{U}(2)$ on SU(3) mentioned above, showing that Eschenburg's inhomogeneous flag manifold is a C1BF.

Finally, group pairs (7) and (8), the work has already been done. In particular, DeVito shows that every such biquotient coming from pair (8) is diffeomorphic to a manifold of the form $S^5\times_{T^2}S^3$ for an effectively torus actions, while all biquotients coming from pair (9) come from linear actions of T^3 on $S^3\times S^3\times S^3$. Thus Theorem 1.5.10 shows that all such biquotients are C1BFs.

It remains to show that the 6-dimensional biquotients whose cohomology ring is singly generated are C1BFs. These were not explicitly handled in [DeV17] because such biquotients had previously been handled by Kapovitch and Ziller, as mentioned at the beginning of this chapter. However, the explicit results can be found in DeVito's dissertation. In particular, any compact simply connected 6-dimensional biquotient is diffeomorphic to either S^6 or $\mathbb{C}P^3$.

10. $G = \mathrm{Spin}(7)$; $H = \mathrm{Spin}(6)$
11. $G = G_2$; $H = \mathrm{SU}(3)$
12. $G = \mathrm{SU}(4)$; $H = \mathrm{SU}(3) \times S^1$
13. $G = \mathrm{Sp}(2)$; $H = \mathrm{SU}(2) \times S^1$
14. $G = \mathrm{Spin}(7)$; $H = G_2 \times S^1$
15. $G = \mathrm{Spin}(8)$; $H = \mathrm{Spin}(7) \times S^1$

Note that biquotients determined by groups (10) and (11) are always diffeomorphic to S^6 and the remaining groups determine biquotients diffeomorphic to $\mathbb{C}P^3$. According to DeVito's classification, the only biquotient determined by group pair (10) is the homogeneous space $\mathrm{Spin}(7)/\mathrm{Spin}(6)$. This is well known to admit a cohomogeneity one action by Spin(6) so is a C1BF. Similarly, the only biquotient determined by group pair (11) is the homogeneous space $G_2/\mathrm{SU}(3)$. According to Kollross [Kol02], the action of $\mathrm{SO}(4) \times \mathrm{SU}(3)$ on G_2 by left and right translation is cohomogeneity one, so it follows that $G_2/\mathrm{SU}(3)$ yields a C1BF structure.

Group pairs (12)-(15) require a bit more work, so we will handle these each in separate cases below.

Case 5: Group pair (12)

Group pair (12) determines two nonhomogeneous biquotients as well as a homogeneous biquotient. The first nonhomogeneous biquotient is given by the action of $\mathrm{SU}(3) \times$

S^1 on SU(4) by

$$(A, z) \cdot B = \mathrm{diag}(zA, \overline{z}^3) B \mathrm{diag}(\overline{z}^3, \overline{z}^3, \overline{z}^3, z^9) \tag{2.2.4}$$

and the other one is given by the action

$$(A, z) \cdot B = \mathrm{diag}(A, 1) B \mathrm{diag}(z, z, \overline{z}, \overline{z}) \tag{2.2.5}$$

For the homogeneous quotient, this can be thought of as SU(4)/U(3), where this arises from the group pair (12) by the map $\mathrm{SU}(3) \times S^1 \to U(3) \to \mathrm{SU}(4)$, where the first map is the triple covering map and the second map is the usual embedding. It is well known [Hoe10] that U(3)/SU(4) admits a cohomogeneity one action by SU(2) × SU(2) so gives rise to a C1BF structure. To see that biquotient (2.2.4) is a C1BF, observe that the matrices which multiply on the left and right are contained in $\mathrm{U}(3) \subset \mathrm{SU}(4)$. Thus we can extend this action to the usual cohomogeneity one U(3) × U(3) on SU(4), so this biquotient gives rise to a C1BF structure. The same argument shows the biquotient (2.2.5), so also gives rise to a C1BF structure.

Case 6: Group pair (13)

Group pair (13) determines one homogeneous biquotient and one nonhomogeneous biquotient. The homogeneous biquotient is $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times S^1$ where $\mathrm{Sp}(1) \times S^1$ is embedded via $(q, z) \mapsto \mathrm{diag}(q, z)$ and the nonhomogeneous biquotient is determined by the action of $\mathrm{Sp}(1) \times S^1$ on Sp(2) given by

$$(q, z) \cdot X = \mathrm{diag}(q, 1) X \mathrm{diag}(\overline{z}, \overline{z})$$

Both the homogeneous and the nonhomogeneous biquotient can be extended to the action of $\mathrm{Sp}(1)^4$ on Sp(2) given by $(p, q, r, s) \cdot X = \mathrm{diag}(p, q) X \mathrm{diag}(\overline{r}, \overline{s})$ which we have already seen to be cohomogeneity one in Case 8 of Section 2.1. Thus both of these biquotients give rise to a C1BF structure.

Case 6: Group pair (14)

The embedding of G_2 into Spin(7) is given by the lift of the unique (up to automorphism) embedding $G_2 \to \mathrm{SO}(7)$ and the embedding of $S^1 \simeq \mathrm{SO}(2)$ is given by the lift of the embedding $B \mapsto \mathrm{diag}(B, I)$. Thus the biquotient is given by the action of $G_2 \times \mathrm{SO}(2)$ on Spin(7) via

$$(A, B) \cdot X = AX\mathrm{diag}(B^T, I)$$

This action can be examined via the analogous action of $G_2 \times \mathrm{SO}(2)$ on SO(7), of which the above action is a lift of. This action can be extended to the cohomogeneity one action of $G_2 \times \mathrm{SO}(3)\mathrm{SO}(4)$ on SO(7) which Kollross [Kol02] has shown is cohomogeneity one. Thus this biquotient gives rise to a C1BF structure.

Case 7: Group pair (15)

For group pair (15), the only embedding is the lift of the usual block embedding $\mathrm{SO}(7) \to \mathrm{SO}(8)$. Furthermore, $S^1 \simeq \mathrm{SO}(2)$ embeds via the lift of $B \mapsto \mathrm{diag}(B, B, B, B)$. Thus the biquotient is given by the action of $\mathrm{Spin}(7) \times \mathrm{SO}(2)$ on Spin(8) by

$$(A, B) \cdot X = \mathrm{diag}(A, 1)X\mathrm{diag}(B^T, B^T, B^T, B^T)$$

We can examine this action by passing to the analogous action of $\mathrm{SO}(7) \times \mathrm{SO}(2)$ on SO(8), of which the above biquotient action is a lift. Observe that $\mathrm{diag}(B^T, B^T, B^T, B^T)$ is contained in $\mathrm{SO}(6) \times \mathrm{SO}(2) \subset \mathrm{SO}(8)$. Thus we can extend the above action to the action of $\mathrm{SO}(7) \times \mathrm{SO}(6) \times \mathrm{SO}(2)$ on SO(8) by

$$(A, B, C) \cdot X = \mathrm{diag}(A, 1)X\mathrm{diag}(B^T, C^T)$$

Observe that if we first take the quotient by $\mathrm{SO}(7)$, we are left with the action of $\mathrm{SO}(6) \times \mathrm{SO}(2)$ on $\mathrm{SO}(7)\backslash\mathrm{SO}(8) \simeq S^7$. The proof of Theorem 1.5.10 shows that this action is cohomogeneity one. Thus this biquotient gives rise to a C1BF structure.

Chapter 3

Classification of C1BFs

Let M^n, $n \leq 5$ be a compact simply connected C1BF. In this chapter we will classify all triples of biquotients $(G//H, G//K^-, G//K^+)$ which are admissible leaf structures for a C1BF in these dimensions. Furthermore, we will do a partial classification of such leaf structures in dimension $n = 6$.

For $n \leq 3$, there are not very many possibilities. In dimension 2, all such C1BFs must be diffeomorphic to S^2 and it is clear that the only possible leaf structure is the case where the principal leaf is a circle and both singular leaves are a point. This leaf structure is well known to be realized by the standard two fixed point cohomogeneity one action on S^2. In dimension 3, clearly all such C1BFs are diffeomorphic to S^3. The singular leaf is a compact biquotient of dimension 1 or 0 so must be a s circle or a point. Furthermore, the principal leaf must be a sphere bundle over each singular leaf and must also be orientable by Proposition 1.6.3, so it follows from Steenrod's classification of sphere bundles [Ste44] that if one of the singular leaves is a circle, then the principal leaf must be the torus T^2. In the case where a singular leave is a point, the principal leaf has to be a circle. Therefore, the only possible leaf structures are

(T^2, S^1, S^1) and (S^1, pt, pt). The former is realized by the cohomogeneity one action on S^2 with group diagram $T^2 \supset S^1 \times 1, 1 \times S^1 \supset 1$ and the latter is realized by the standard two fixed point cohomogeneity one action on S^3. We summarize these results in the following theorem.

Proposition 3.0.1. *Let M be a compact simply connected 2 or 3 dimensional C1BF. Then M is diffeomorphic to S^2 or S^3. Furthermore, the only admissible leaf structure in the case of S^2 is (S^1, pt, pt) and in the case of S^3, there are two admissible leaf structures, namely (S^2, pt, pt) and (T^2, S^1, S^1).*

3.1 Classification in Dimension 4

Let M^4 be a C1BF with three dimensional principal leaf P^3. By Ge and Radeschi's classification [GR15], M is diffeomorphic to one of S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$. Notably, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is not cohomogeneity one but, as we will see in Chapter 4, it is indeed a C1BF. Ge and Radeschi's classification greatly simplifies the classification of C1BFs in dimension 4 since, as a byproduct of their classification, they have obtained a classification of singular Riemannian foliations of codimension one on all simply connected closed 4-manifolds. Thus we need only consider the leaf structures obtained in their classification and determine which of these structures is realized as a C1BF. These are written in Table 1 of the aforementioned paper. The results of this section can be summarized by the following theorem:

Theorem 3.1.1. *Let M be a compact, simply connected, 4-dimensional C1BF and let $\mathscr{L} = S^3/Q_8 \approx SO(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ denote the nonclassical lens space with fundamental group Q_8. The following list is the complete list of admissible leaf structures:*

1. (S^3, pt, pt)
2. (S^3, S^2, pt)
3. (S^3, S^2, S^2)
4. $(S^2 \times S^1, S^2, S^2)$
5. $(S^2 \times S^1, S^2, S^1)$
6. $(L_m(1), S^2, S^2) \quad (m \geq 2)$
7. $(\mathscr{L}, \mathbb{R}P^2, \mathbb{R}P^2)$
8. $(L_4(1), S^2, \mathbb{R}P^2)$

Furthermore, the topology of such C1BFs can be described as follows.

(i) *C1BFs of type 1,5, or 7 are diffeomorphic to* S^4

(ii) *C1BFs of type 2 or 8 are diffeomorphic to* $\mathbb{C}P^2$.

(iii) *C1BFs of type 6 are diffeomorphic to* $\mathbb{C}P^2\# - \mathbb{C}P^2$ *if* m *is odd. If* $m \geq 4$ *is even, they are diffeomorphic to* $S^2 \times S^2$. *In the special case* $m = 2$, *such a C1BF is diffeomorphic to either* $S^2 \times S^2$ *or* $\mathbb{C}P^2\#\mathbb{C}P^2$.

(iv) *A C1BF of type 4 is diffeomorphic to either* $S^2 \times S^2$ *or* $\mathbb{C}P^2\#\mathbb{C}P^2$

It is important to note that while structures of type 3 can, in principle, be diffeomorphic to either $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ or $\mathbb{C}\mathrm{P}^2\# - \mathbb{C}\mathrm{P}^2$, We do not currently have an example where this occurs on $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$. Similarly, for case 4, this can, in principle be diffeomorphic to $\mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2$, but I don't have an explicit example of this, but I fully expect that it happens ($CP^2\# - \mathbb{C}\mathrm{P}^2$ generally has much more "freedom" for this sort of thing than $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$). Moreover, dimension 4 is the lowest dimension where there exists a manifold that is a C1BF which is not cohomogeneity one, namely $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is not cohomogeneity one [Par86].

Corollary 3.1.2. $\mathbb{C}P^2\#\mathbb{C}P^2$ *the lowest dimensional example of a manifold which admits a C1BF structure but does not admit a cohomogeneity one structure.*

To prove the above theorem, we need only go through each leaf structure appearing in Ge and Radeschi's classification and verify that there is a C1BF realizing each leaf structure. In fact, it turns out that each leaf structure can be realized as a cohomogeneity one structure.

Case B.1: ($P = S^3$)

From Ge and Radeschi's classification, there are three possible leaf structures with principal leaf S^3, which are examined in the cases below. Note that, from their classification, M is diffeomorphic to one of S^4, $\mathbb{C}\mathrm{P}^2$, or $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$.

Case B.1.1 $(P, B_-, B_+) = (S^3, pt, pt)$

Such a C1BF is easily seen to be diffeomorphic to S^4. And is well known to be realized by the standard two fixed point cohomogeneity one action on S^4.

Case B.1.2 $(P, B_-, B_+) = (S^3, S^2, pt)$

This leaf structure is realized as cohomogeneity one via the group cohomogeneity one group diagram $\mathrm{SU}(2) \supset \mathrm{SU}(2), \mathrm{S}(\mathrm{U}(1)\mathrm{U}(1)) \supset \mathrm{SU}(1)$. It is also realized as cohomogeneity one via the group diagram $\mathrm{U}(n) \supset U(n), U(n-1)U(1) \supset U(n-1)$. Hoelscher showed in [Hoe10] that both of these diagrams arise as an action on $\mathbb{C}\mathrm{P}^2$. In fact, from Ge and Radeschi's work, a C1BF with this leaf structure must always be diffeomorphic to $\mathbb{C}\mathrm{P}^2$. This leaf structure is also realized via a non-cohomogeneity one C1BF in Structure 4.1.4 in Chapter 4.

Case B.1.3 $(P, B_-, B_+) = (S^3, S^2, S^2)$

Hoelsher shows in [Hoe10] that the group diagram $S^3 \supset S^1, S^1 \supset \mathbb{Z}_m$ is an action on $\mathbb{C}\mathrm{P}^2\# -\mathbb{C}\mathrm{P}^2$, n is odd. In the case $m = 1$, such a cohomogeneity one manifold has this leaf structure and the principal orbit is S^3, which is an example of a cohomogeneity one manifold $\mathbb{C}\mathrm{P}^2\# -\mathbb{C}\mathrm{P}^2$ realizing this leaf structure. This is also realized as a non-cohomogeneity one structure on $\mathbb{C}P^2\# -\mathbb{C}P^2$ by Structure 4.1.2 in Chapter 4.

Case B.2 ($P = S^2 \times S^1$)

There are two possibilities for the leaf structures for M in this case. M is diffeomorphic to one of S^4, $S^2 \times S^2$, or $\mathbb{C}P^2\# -\mathbb{C}P^2$.

Case B.2.1 $(P, B_, B_+) = (S^2 \times S^1, S^2, S^2)$

By Ge and Radeschi's classification, a C1BF with this leaf structure is either $S^2 \times S^2$ or $\mathbb{C}P^2\# -\mathbb{C}P^2$. Hoelscher shows in [Hoe10] that the cohomogeneity one group diagram $S^3 \times S^1 \supset S^1 \times S^1, S^1 \times S^1 \supset S^1 \times \{1\}$ arises as a product action on $S^2 \times S^2$. Such an action has the desired leaf structure, so this case is realized as cohomogeneity one.

Case B.2.2 $(P, B_, B_+) = (S^2 \times S^1, S^2, S^1)$

Hoelscher shows in [Hoe10] that the group diagram $S^3 \times S^1 \supset S^1 \times S^1, S^3 \times 1 \supset S^1 \times 1$ arises as a sum action on S^4. Such an action has the desired leaf structure so this case arises as cohomogeneity one. In fact, by Ge and Radeschi's classification, a C1BF with this leaf structure is always diffeomorphic to S^4.

Case B.3: (P = Lens Space)

We now consider the case where the principal leaf is any three dimensional lens space. Recall that the three dimensional lens spaces $L_m(q)$, $\gcd(m, q) = 1$, are quotients of S^3 by free $\mathbb{Z}_m$-actions. In particular, for $S^3 \subset \mathbb{C}^2$, $L_m(q) = S^3/\mathbb{Z}_m$ where the $\mathbb{Z}_m$-action is generated by $(z_1, z_2) \mapsto (e^{\frac{2\pi i}{m}} z_1, e^{\frac{2\pi i q}{m}} z_2)$. Note that in the case $m = 1$ that $L_m(q) = S^3$ which we have already treated separately, so we assume $m \geq 2$. Note also that this case contains the special case $L_2(1) = \mathbb{R}P^2$. We also consider within this case the nonclassical lens space $\mathscr{L} = S^3/Q_8 \approx SO(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. We do *not* consider $S^2 \times S^1$ as a lens space. From Ge and Radeschi's classification, we get only the following cases.

Case B.3.1 $(P, B_-, B_+) = (L_m(1), S^2, S^2)$

The only possibilities for a C1BF with this leaf structure are $S^2 \times S^2$, and $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$. In fact, all three of these manifolds arise as C1BFs with this leaf structure. The same group diagram $S^3 \supset S^1, S^1 \supset \mathbb{Z}_m$ referenced in Case B.1.3 for $m \geq 2$ realizes this leaf structure as cohomogeneity one. This arises as an action on $S^2 \times S^2$ or $\mathbb{C}P^2 \# - \mathbb{C}P^2$ depending on whether m is even or odd, respectively. We can similarly realize this leaf structure as a non-cohomogeneity one C1BF on $S^2 \times S^2$ and $\mathbb{C}P^2 \# - \mathbb{C}P^2$ when m is even or odd, respectively, as shown in Structures 4.1.3 and 4.1.2 of Chapter 4, respectively. This is also realized by Structure 4.1.1 in the case $m = 2$ where $L_m(1) = \mathbb{R}P^3$. It is also notable is that by Ge and Radesci's classification, $m = 2$ is the only case where M is diffeomorphic $\mathbb{C}P^2 \# \mathbb{C}P^2$ for $m \geq 2$.

Case B.3.2 $(P, B_-, B_+) = (\mathscr{L}, \mathbb{R}P^2, \mathbb{R}P^2)$

This is realized as cohomogeneity one by the cohomogeneity one group diagram $S^3 \supset \{e^{i\theta}\} \cup \{je^{i\theta}\}, \{e^{j\theta}\} \cup \{ie^{j\theta}\} \supset Q_8$. This comes from an action of SO(3) on S^4 as described in [GZ00]. In fact, by Ge and Radeschi's classification [GR15], a C1BF with this leaf structure is always diffeomorphic to S^4.

Case B.3.3 $(P, B_-, B_+) = (L_4(1), S^2, \mathbb{R}P^2)$

This is realized as cohomogeneity one by the cohomogeneity one group diagram $S^3 \supset \{e^{i\theta}\}, \{e^{j\theta}\} \cup \{ie^{j\theta}\} \supset \langle i \rangle$. This diagram arises from an action of SO(3) on $\mathbb{C}P^2$, as described by Hoelscher in [Hoe10]. In fact, by Ge and Radesci's classification [GR15], any C1BF with this leaf structure is diffeomorphic to $\mathbb{C}P^2$.

3.2 Classification in Dimension 5

Let M^5 be a compact simply connected C1BF with principal leaf P^4. By the classification of 5-dimensional DDBs [DGGK20], M must be diffeomorphic to either S^5, $S^3 \times S^2$, $S^3 \hat{\times} S^2$ (the nontrivial S^3-bundle over S^2), or the the Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. We note that the 5-dimensional classification has one exceptional case in the sense that it is, at the time of this thesis submission, incomplete. In particular, the case where the C1BF has principal leaf $L_m(r) \times S^1$ turns out to be rather complex compared to the other cases, as we will see below. In particular, there is one infinite family of leaf structures which, at the time of submission of this thesis, we were unable to determine whether all leaf structures which get past the sphere bundle and van Kampen theorem obstructions are admissible. For a more detailed explanation of what is known about this infinite family, see Case C.6.1 below. The rest of the section will be devoted to proving the remaining statements in the following theorem.

Theorem 3.2.1. *Let M be a compact, simply connected, 5-dimensional C1BF and let $S^2 \hat{\times} S^1$ denote the unique nonorientable S^2-bundle over S^1 and "$\equiv_m$" denote congruence modulo m. With the exception of some leaf structures of type (11) potentially not being admissible, the following list is the complete list of admissible leaf structures:*

1. (S^4, pt, pt)
2. $(S^2 \times S^2, S^2, S^2)$
3. $(\mathbb{C}P^2 \# -\mathbb{C}P^2, S^2, S^2)$
4. $(S^3 \times S^1, S^3, S^3)$
5. $(S^3 \times S^1, L_m(r), L_n(s))$ $\gcd(m,n) = 1$
6. $(S^3 \times S^1, S^1, S^3)$
7. $(S^3 \times S^1, S^3, S^2 \times S^1)$
8. $(S^3 \times S^1, S^3, L_m(r))$
9. $(S^3 \times S^1, S^2 \times S^1, L_m(r))$
10. $(S^2 \times T^2, S^2 \times S^1, S^2 \times S^1)$
11. $(L_m(r) \times S^1, L_n(r); L_k(s))$ *$m|n, k$, and $s \equiv \pm_m r^{\pm 1}$*
12. $(L_m(1) \times S^1, S^2 \times S^1, L_n(1))$ *where $m|n$*
13. $(L_2(1) \times S^1, L_2(1); S^2 \hat{\times} S^1)$

To prove the above theorem, we first wish to determine what all of the possibilities for the singular leaves are in a 5-dimensional C1BF. To do this, first note that by Proposition 1.6.5, the singular leaf must have cyclic fundamental group. Using classifications of such manifolds in dimensions 1-3, it follows that, other than a point, the only manifolds which could possibly appear as the singular leaf of a 5-dimensional C1BF are given by the following list:

List A: (Potential Singular Leaves)

1. S^1
2. S^2
3. S^3
4. $L_n(r)$, $n \geq 2$
5. $S^2 \times S^1$
6. $S^2 \hat{\times} S^1$
7. $\mathbb{R}P^2$ (shown to not occur below)

Note: When "List A" is referred to below, we mean spaces (1)-(6), as $\mathbb{R}P^2$ is shown below to not occur. Additionally, note that in the case where $P = S^4$, we can also have a point as the singular leaf.

Now, the fact that the principal leaf P is a sphere bundle over each singular leaf with orientable total space puts very strong restrictions on the possibilities for the principal leaf. For S^2, we are interested in S^2-bundles over S^2. It follows from [Ste44] that the only two such possibilities are $S^2 \times S^2$ and $\mathbb{C}P^2 \# - \mathbb{C}P^2$. Similarly, it follows from the same reference that the only S^3-bundle over S^1 with orientable total space is $S^3 \times S^1$. To determine the possibilities for the principal leaf when one of the singular leaves is a lens space $L_n(r)$, consider the following theorem, taken from [Tho74]

Theorem 3.2.2 (Thornton)**.** *Let* $\{f_i : 0 < i \leq p-1\}$ *be representatives of the elements in*

$[L_p(q), \mathbb{C}P^\infty]$. *Then the total space of a principal* S^1*-bundle determined by* f_i *is homeomorphic to* $L_d(q) \times S^1$, *where* $d = \gcd(i, p)$.

The “homeomorphism" conclusion in the above theorem can be upgraded to “diffeomorphism" without too much trouble. Furthermore, we have the following important result, also from [Tho74]

Proposition 3.2.3 (Thornton)**.** *Any fiber bundle* $S^1 \to E \to B$ *with* E *and* B *orientable manifolds is a principal* S^1*-bundle.*

It now follows, any circle bundle of the principal leaf P over S^3 or $L_n(r)$ has to be diffeomorphic to $L_m(r) \times S^1$ for some m and r. Furthermore, Thornton also shows that the total space of a principal circle bundle over or $S^2 \times S^1$ is diffeomorphic to $L_n(1) \times S^1$ for some n.

Lemma 3.2.4. *There cannot exist a 5-dimensional C1BF manifold* M *with* $S^2 \hat{\times} S^1$ *as both singular leaves or* $\mathbb{R}P^2$ *as both singular leaves. Furthermore, there cannot exist such a C1BF manifold with* $S^2 \hat{\times} S^1$ *and* $\mathbb{R}P^2$ *as singular leaves together. In particular, at least one singular leaf must be orientable.*

Proof. Let P be the principal leaf and suppose $\mathbb{R}\mathrm{P}^2$ and $S^2 \hat{\times} S^1$ are the singular leaves. Then we get a bundle $S^2 \to P \to \mathbb{R}\mathrm{P}^2$ and the long exact sequence of homotopy tells us that $\pi_1(P) \approx \pi_1(\mathbb{R}\mathrm{P}^2)$. On the other hand, we also have a bundle $S^1 \to P \to S^2 \hat{\times} S^1$ and the long exact sequence of homotopy gives us $\pi_1(P) \to \pi_1(S^2 \hat{\times} S^1) \to 0$, so we get a surjection $\mathbb{Z}_2 \to \mathbb{Z}$, which is impossible. On the other hand, to rule out the possibility of both singular leaves being $S^2 \hat{\times} S^1$, note that in this case both fiber spheres are S^1. Since $S^2 \hat{\times} S^1$ is nonorientable, it follows that, in the notation of Table 1.4 of [GH87], both of the bundle projection maps are “twisted", so $\pi_1(F) \approx Q_8$, where F is the homotopy fiber of the inclusion. It follows from the long exact sequence of homotopy associated to the fibration $F \to P \to M$ that, since M is simply connected, $\pi_1(P)$ is a quotient of Q_8, so must be finite. On the other hand, we also have the circle bundle $S^1 \to P \to S^2 \hat{\times} S^1$, and the long exact sequence of homotopy implies $\pi_1(P) \to \pi_1(S^2 \hat{\times} S^1)$ is surjective, which is impossible since $\pi_1(P)$ is finite. In the case where both singular leaves are $\mathbb{R}\mathrm{P}^2$, Table 1.4 of [GH87] implies that $\pi_1(F) = 0$. On the other hand, the LES of homotopy associated to the sphere bundle $S^2 \to P \to \mathbb{R}\mathrm{P}^2$ implies $\pi_1(P) \simeq \pi_1(\mathbb{R}\mathrm{P}^2)$, a contradiction. □

Thus, by the lemma, it follows that if either of these spaces appear as a singular leaf, the other singular leaf is one of the cases we have already dealt with above. Thus we have proved the following proposition:

Proposition 3.2.5. *Let P be the principal leaf of a 5-dimensional C1BF. Then P is diffeomorphic to one of the following spaces:*

1. S^4
2. $S^2 \times S^2$
3. $\mathbb{C}P^2 \# -\mathbb{C}P^2$
4. $S^3 \times S^1$
5. $S^2 \times T^2$
6. $L_m(r) \times S^1$, $m \geq 2$

Corollary 3.2.6. *The biquotient $\mathbb{R}P^2$ does not appear as a singular leaf of any C1BF in dimension 5.*

Proof. From the proof of Lemma 3.2.4 above, the principal leaf P must have $\pi_1(P) \approx \mathbb{Z}_2$. No principal leaf on the above list has such a fundamental group. □

Now that we have significantly narrowed down the possibilities for the leaves of the C1BF, we wish to determine the admissible leaf structures (P, B_-, B_+). Given a fixed principal leaf P, recall that we must have a sphere bundle $S^\ell \to P \to B^b$ over the singular leaf B for $\ell > 0$. In the special case where $P = S^4$, we can get the bundle $S^4 \to$

$S^4 \to \{pt\}$. Otherwise, using List A, we will have the following three *sphere bundle cases*, which will be repeatedly referenced below.

(i) $S^3 \to P \to S^1$

(ii) $S^2 \to P \to S^2$

(iii) $S^1 \to P \to B^3$

We will also repeatedly use the fact that if $\pi_1(B) = 0$, then B is a compact simply connected two or three dimensional biquotient, hence is S^2 or S^3, respectively. We will frequently use the long exact sequence of homotopy associated to bundles, which we will refer to simply as the *LES of homotopy*.

Case C.1: $P = S^4$

Note that the sphere bundle cases (i) and (ii) are immediately ruled out by the LES of homotopy. Similarly, for case (iii), the LES of homotopy gives $\pi_1(B) = 0$, hence $B = S^3$. But the LES of homotopy also implies $\pi_3(B) = 0$, which is impossible. Therefore, the only possibility is sphere bundle $S^4 \to S^4 \to \{pt\}$. Thus the only possible leaf structure in this case is $(P, B_-, B_+) = (S^4, pt, pt)$. Additionally, it is easily seen that M must be diffeomorphic to S^5. This is well known to be realized by the standard two fixed point cohomogeneity one action on S^5.

Case C.2: $P = S^2 \times S^2$

Sphere bundle case (i) is clearly impossible by the LES of homotopy. For (iii), the LES of homotopy tells us that B is simply connected, hence $B = S^3$. On the other hand, the LES of homotopy gives an isomorphism $\pi_3(S^2 \times S^2) \to \pi_3(B)$ which is impossible. This only leaves case (ii) which is clearly possible via the trivial bundle. Thus the only possible leaf structure is the case where $(P, B_-, B_+) = (S^2 \times S^2, S^2, S^2)$. This is realized as a non-cohomogeneity one C1BF structure on $S^3 \times S^2$ in Structure 4.2.4. We also show at the end of Structure 4.2.6 in Chapter 4 that a C1BF with this leaf structure is necessarily diffeomorphic to $S^3 \times S^2$.

Case C.3: $P = \mathbb{C}P^2 \# - \mathbb{C}P^2$

Sphere bundle case (i) is immediately ruled out by the LES of homotopy. For case (iii), the LES of homotopy implies B is simply connected so $B = S^3$. However, a portion of the LES of homotopy also gives $\pi_2(S^1) \to \pi_2(\mathbb{C}P^2 \# - \mathbb{C}P^2) \to \pi_2(B)$ which implies an injection $\pi_2(\mathbb{C}P^2 \# - \mathbb{C}P^2) \to \pi_2(B)$ which is impossible since, by the Hureweicz theorem, $\pi_2(\mathbb{C}P^2 \# - \mathbb{C}P^2) \approx H_2(\mathbb{C}P^2 \# - \mathbb{C}P^2) \approx H_2(\mathbb{C}P^2) \oplus H_2(\mathbb{C}P^2) \approx \mathbb{Z} \oplus \mathbb{Z}$. Thus only case (ii) remains. It is well known that $\mathbb{C}P^2 \# - \mathbb{C}P^2$ is the unique nontrivial S^2-bundle over S^2. Thus the only possible leaf structure here is $(P, B_-, B_+) = (\mathbb{C}P^2 \# - \mathbb{C}P^2, S^2, S^2)$. This example is realized as Structure 4.2.6 in Chapter 4 and we also show that that a C1BF with this leaf structure is necessarily diffeomorphic to $S^3 \hat{\times} S^2$.

Case C.4: $P = S^3 \times S^1$

In contrast with the previous cases, in this case we get a plethora of different possible leaf structures. In fact, the only sphere bundle case that we can rule out is via the LES of homotopy is (ii). Indeed, the LES of homotopy gives $0 \to \pi_1(S^3 \times S^1) \to \pi_1(S^2) \to 0$ which implies $\pi_1(S^2) \approx \mathbb{Z}$, but this is clearly impossible. We now need to determine which spaces on List A can occur in a sphere bundle $S^\ell \to S^3 \times S^1 \to B$. Clearly $B = S^1$ and $B = S^3$ are possible via the trivial bundles. Moreover, the action of S^1 on $S^3 \times S^1$ via the Hopf action on the S^3 factor and trivial on the S^1 factor is free with quotient $S^2 \times S^1$, so we get a principal bundle $S^1 \to S^3 \times S^1 \to S^2 \times S^1$.

Lemma 3.2.7. *$S^3 \times S^1$ is a circle bundle over any lens space $L_m(r)$.*

Proof. Let m, r be integers which are relatively prime. Recall that the lens space $L_m(r) =$

$S^3/\mathbb{Z}_m$ is the quotient of $\mathbb{Z}_m = \langle \zeta_0 = e^{\frac{2\pi i}{m}} \rangle$ generated by the diffeomorphism $(z_1, z_2) \mapsto (\zeta_0 z_1, \zeta_0^r z_2)$. Consider the action of S^1 on $S^3 \times S^1$ by $w \cdot \big((z_1, z_2), \theta\big) = \big((w z_1, w^r z_2), w^m \theta\big)$. Since $\gcd(m, r) = 1$, this action is free. Furthermore, it is transitive on the S^1 factor, so by Proposition 1.3.7

$$(S^3 \times S^1)/S^1 \approx S^3/\Gamma_e$$

where Γ_e is the isotropy of the identity 1 of the transitive factor. Clearly, $\Gamma_e = \mathbb{Z}_m = \{\zeta \in S^1 \,:\, \zeta^m = 1\}$. If we choose $\zeta_0 = e^{\frac{2\pi i}{m}}$ to be the generator, then the action of $\mathbb{Z}_m$ on S^3 is generated by $\zeta_0 \cdot (z_1, z_2) = (\zeta_0 z_1, \zeta_0^r z_2)$, which is precisely the action giving $L_m(r)$ as the quotient. Thus we get a principal bundle $S^1 \to S^3 \times S^1 \to L_m(r)$. □

The only remaining possibility for the singular leaf is $S^2 \hat{\times} S^1$, which is ruled out via the following lemma.

Lemma 3.2.8. *$S^3 \times S^1$ is not a circle bundle over $S^2 \hat{\times} S^1$.*

Proof. We know that $S^2 \hat{\times} S^1 \approx (S^2 \times S^1)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ action is generated by $(q, z) \mapsto (-q, -z)$. It then follows immediately from the associated bundle construction that $S^2 \hat{\times} S^1$ is a circle bundle over $\mathbb{R}\mathrm{P}^2$. We claim that if $S^3 \times S^1$ is a circle bundle over $S^2 \hat{\times} S^1$, then $S^3 \times S^1$ is a bundle over $\mathbb{R}\mathrm{P}^2$, with fiber either T^2 or the Klein bottle K. To see this, we have circle bundles $S^1 \to S^3 \times S^1 \xrightarrow{\pi_1} S^2 \hat{\times} S^1$ and $S^1 \to S^2 \hat{\times} S^1 \xrightarrow{\pi_2} \mathbb{R}\mathrm{P}^2$. Let $\pi : S^3 \times S^1 \to \mathbb{R}\mathrm{P}^2$ denote the composition

$$S^3 \times S^1 \xrightarrow{\pi_1} S^2 \hat{\times} S^1 \xrightarrow{\pi_2} \mathbb{R}\mathrm{P}^2$$

This is a composition of surjective submersions, so is a surjective submersion. By Ehresmann's lemma, $S^3 \times S^1 \to \mathbb{R}\mathrm{P}^2$ is a fiber bundle. To see the fiber type, note that $\pi_2^{-1}(p) \approx S^1$. Now, $\pi_1^{-1}(S^1) \in S^3 \times S^1$ is precisely the pullback of the bundle $S^3 \times S^1 \to$

$S^2 \hat{\times} S^1$ via the inclusion $S^1 \hookrightarrow S^2 \hat{\times} S^1$. Thus, $\pi^{-1}(S^1)$ is some S^1-bundle over S^1. By Steenrod's classification of sphere bundles [Ste44], it follows that

$$\pi^{-1}(S^1) = \pi_1^{-1}(\pi_2^{-1}(p)) = \pi^{-1}(p)$$

is either T^2 or K. Finally to complete the proof of the lemma, we prove the following claim:

Claim: There is no bundle $F \to S^3 \times S^1 \to \mathbb{RP}^2$ with fiber T^2 or K.

This follows from the spectral sequence using $\mathbb{Z}_2$ coefficients. Indeed, we have that $H^1(T^2;\mathbb{Z}_2) \approx H^1(K;\mathbb{Z}_2) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Together with the fact that $H^1(\mathbb{RP}^2;\mathbb{Z}_2) \approx H^2(\mathbb{RP}^2;\mathbb{Z}_2) \approx \mathbb{Z}_2$, it follows that the $E_\infty^{(0,1)}$ is nontrivial as is $E_\infty^{(1,0)}$. This implies that $H^1(S^3 \times S^1;\mathbb{Z}_2)$ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, which is a contradiction. $\square$

Thus, by the work above, we have reduced the list of possibly admissible leaf structures to the following cases.

Case C.4.1: $(P, B_-, B_+) = (S^3 \times S^1, S^1, S^1)$

In this case, the fiber spheres $K^\pm / H \approx S^3$, thus the van Kampen theorem says that such a C1BF is not simply connected because their images do not generate $\pi_1(G/\!/H) \approx \mathbb{Z}$.

Case C.4.2: $(P, B_-, B_+) = (S^3 \times S^1, S^1, S^3)$

This is realized in Structure 4.2.1 as a non-cohomogeneity C1BF structure on S^5. This is also realized as cohomogeneity one by the group diagram $S^3 \times S^1 \supset \{e^{ip\theta}, e^{i\theta}\}, S^3 \times \mathbb{Z}_n \supset \mathbb{Z}_n$ for $\gcd(p, n) = 1$ which, according to Hoelscher's classification [Hoe10] arises as an action on S^5.

Case C.4.3: $(P, B_-, B_+) = (S^3 \times S^1, S^1, L_m(r))$

We assume for this case that $m \geq 2$. We claim that this case is not simply connected. To see this, note that one fiber sphere is S^3 and the other is S^1. WLOG say $K^+/H \approx S^3$ and $K^-/H \approx S^1$. Thus by van Kampen theorem, the map $\pi_1(K^-/H) \to \pi_1(S^3 \times S^1)$ induced by inclusion must be surjective if M is to be simply connected. But this is impossible. Indeed, consider the bundle $S^1 \xrightarrow{i} S^3 \times S^1 \xrightarrow{\pi} L_m(r)$ and the corresponding LES of homotopy

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(S^3 \times S^1) \xrightarrow{\pi_*} \pi_1(L_m(r)) \to 0$$

Since π_* is surjective, if i_* were surjective, then $\pi_* = 0$ by exactness. This would require $\pi_1(L_m(r)) = 0$ which is clearly not the case. Thus M is not simply connected.

Case C.4.4: $(P, B_-, B_+) = (S^3 \times S^1, S^1, S^2 \times S^1)$

We claim that this case is not simply connected. To see this, we use the same argument as the previous case. Note that one fiber sphere is S^3 and the other is S^1. WLOG say $K^+/H \approx S^3$ and $K^-/H \approx S^1$. Thus by van Kampen theorem, the map $\pi_1(K^-/H) \to \pi_1(S^3 \times S^1)$ induced by inclusion must be surjective if M is to be simply connected. But this is impossible. Indeed, consider the bundle $S^1 \xrightarrow{i} S^3 \times S^1 \xrightarrow{\pi} S^2 \times S^1$ and the corresponding LES of homotopy

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(S^3 \times S^1) \xrightarrow{\pi_*} \pi_1(S^2 \times S^1) \to 0$$

Since π_* is surjective, if i_* were surjective, then $\pi_* = 0$ by exactness. This would require $\pi_1(S^2 \times S^1) = 0$ which is clearly not the case. Thus M is not simply connected.

Case C.4.5: $(P, B_-, B_+) = (S^3 \times S^1, S^3, S^3)$

This structure is realized as cohomogeneity one.

From Hoelscher's classification [Hoe10], the cohomogeneity one group diagram $S^3 \times S^1 \supset \{e^{ip\theta}, e^{i\theta}\}, \{e^{ip\theta}, e^{i\theta}\} \supset \mathbb{Z}_n$, $\gcd(p,n) = 1$, arises as an action on $S^3 \times S^2$, and it is not difficult to compute that this cohomogeneity one action has the correct leaf structure using Proposition 1.3.7. This structure is also realized as a non-cohomogeneity one C1BF structure in exceptional case 1 of Structure 4.2.2.

Case C.4.6: $(P, B_-, B_+) = (S^3 \times S^1, S^3, L_m(r))$

This is realized as a C1BF as a special case of Structure 4.2.7.

Case C.4.7: $(P, B_-, B_+) = (S^3 \times S^1, S^3, S^2 \times S^1)$

This is realized as a cohomogeneity one C1BF in Structure 4.2.5 of Chapter 4.

Case C.4.8: $(P, B_-, B_+) = (S^3 \times S^1, L_m(r), L_n(s))$

Here we assume that $m, n \geq 2$. We prove that M is simply connected if and only if $\gcd(m,n) = 1$. By van Kampen theorem for C1BFs, M is simply connected if and only if the images of $\pi_1(K^\pm/H)$ generate $\pi_1(G//H)$ under the natural inclusions. Note that in this case $K^\pm/H \approx S^1$ and we have fiber bundles $S^1 \xrightarrow{i} S^3 \times S^1 \xrightarrow{\pi} L_m(r)$ and $S^1 \to S^3 \times S^1 \to L_n(s)$. By the LES of homotopy we have

$$0 \to \pi_1(S^1) \xrightarrow{i_*} \pi_1(S^3 \times S^1) \xrightarrow{\pi_*} \pi_1(L_m(r)) \to 0$$

or, equivalently

$$0 \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z}_m \to 0$$

By exactness π_* is surjective and hence $\mathbb{Z}/\ker \pi_* = \mathbb{Z}_m$ and thus $\mathrm{im}(i_*) = \ker \pi_* = m\mathbb{Z}$. Similarly, the image of the other fiber sphere is $n\mathbb{Z}$. But $m\mathbb{Z}$ and $n\mathbb{Z}$ generate $\mathbb{Z}$ if and only if $\gcd(m,n) = 1$. This C1BF structure is realized in Structure 4.2.7 of Chapter 4.

Case C.4.9: $(P, B_-, B_+) = (S^3 \times S^1, L_m(r), S^2 \times S^1)$

Here we assume that $m \geq 2$. This this leaf structure is realized as a C1BF in Structure 4.2.8 of Chapter 4. We note also that in the case that $r = 1$ and m and n are both odd, at the end of Structure 4.2.2 of Chapter 4 we show that all such leaf structures can be realized as a C1BF on $S^3 \times S^2$. The case where m and n have opposite parity and $r = 1$ we will see that all such leaf structures can be realized as a C1BF on $S^3 \hat{\times} S^2$, which we describe at the end of Structure 4.2.3 of Chapter 4.

Case C.4.10: $(P, B_-, B_+) = (S^3 \times S^1, S^2 \times S^1, S^2 \times S^1)$

We claim that this is not simply connected. Suppose we have such a leaf structure. Then we have fiber bundles $S^1 \xrightarrow{i_*} S^3 \times S^1 \xrightarrow{\pi_*} S^2 \times S^1$ so by the LES of homotopy we have

$$\mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \to 0$$

Therefore, $\mathbb{Z} \simeq \mathbb{Z}/\mathrm{Im}\, i_*$. But $\mathrm{Im}\, i_* = n\mathbb{Z}$ for some integer n. It follows that $n = 0$ so i_* is trivial. Therefore the inclusions $K^{\pm}/H$ cannot generate the fundamental group of the principal leaf.

Case C.5:$P = S^2 \times T^2$

For sphere bundle case (i), note that the LES of homotopy gives $\pi_1(S^2 \times T^2) \approx \pi_1(S^1)$ which is impossible. For sphere bundle case (ii), the LES of homotopy gives an isomorphism $\pi_1(S^2 \times T^2) \approx \pi_1(S^2)$ which is impossible. This proves the following lemma:

Lemma 3.2.9. *For a C1BF with principal leaf $S^2 \times T^2$, the singular leaves must be codimension two. In particular, the fiber spheres are both circles.*

Corollary 3.2.10. *The manifold $S^2 \hat{\times} S^1$ cannot be the singular leaf of a C1BF manifold M with principal leaf $P = S^2 \times T^2$.*

Proof. Let F be the homotopy fiber of the inclusion $P \to M$. Then we get the fibration $F \to P \to M$ and, since M is simply connected, the LES of homotopy associated with this fibration tells us that $\pi_1(P)$ is a quotient of $\pi_1(F)$. Because the fiber spheres are both circles, Table 1.4 of [GH87] tells us that if one of the singular leaves is nonorientable, then $\pi_1(F)$ is either $\mathbb{Z} \oplus \mathbb{Z}_2$ or Q_8. But $\pi_1(P) \approx \mathbb{Z} \oplus \mathbb{Z}$ is not a quotient of either of these groups. $\square$

The remaining possibilities that we have not yet considered on List A are S^3, $S^2 \times S^1$, and $L_m(r)$ for $m \geq 2$. We observe that S^3 is impossible because if we have a bundle $S^1 \to S^2 \times T^2 \to S^3$, then the LES of homotopy gives $\pi_1(S^1) \to \pi_1(S^2 \times T^2)$ is surjective, that is, a surjective homomorphism $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, which is impossible. More generally, we cannot have a sphere bundle $S^1 \to S^2 \times T^2 \to L_m(r)$. Indeed, note that $\pi_1(S^2 \times T^2) \approx \mathbb{Z} \oplus \mathbb{Z}$, so $H_1(S^2 \times T^2) \approx \mathbb{Z} \oplus \mathbb{Z}$. By universal coefficients we then have $H^1(S^2 \times T^2) \approx \mathbb{Z} \oplus \mathbb{Z}$. We also know that $H^1(L_m(r)) = 0$. Now, a portion of the Gysin sequence [Hat02] gives $0 \to H^1(S^2 \times T^2) \to H^1(L_m(r))$, so we have an injection $\mathbb{Z} \oplus \mathbb{Z} \to 0$, which is impossible. Finally, we note that $S^2 \times T^2$ admits a free circle action via $z \cdot (q, w_1, w_2) = (q, zw_1, zw_2)$ whose quotient is clearly $S^2 \times S^1$, so we get a principal bundle $S^1 \to S^2 \times T^2 \to S^2 \times S^1$.

We have thus shown that the only possible leaf structure is $(P, B_-, B_+) = (S^2 \times T^2, S^2 \times S^1, S^2 \times S^1)$. This is realized as a non-cohomogeneity one C1BF structure on $S^3 \times S^2$ in the exceptional case 2 of structure 4.2.2.

Case C.6: $P = L_m(r) \times S^1$

Here we handle the case where the principal leaf $P = L_m(r) \times S^1$ for $m \geq 2$. This is the most complicated case. Note first that for sphere bundle case (i), the LES of homotopy gives $0 \to \pi_1(L_m(r) \times S^1) \to \pi_1(S^1) \to 0$ which is impossible. For sphere bundle case (ii), by the LES we get $0 \to \pi_1(L_m(r) \times S^1) \to \pi_1(S^2) \to 0$ so $\pi_1(S^2) \approx \mathbb{Z}_m \times \mathbb{Z}$, which is impossible.

We have thus shown that any singular leaf of a C1BF with $P = L_m(r) \times S^1$ must be 3-dimensional, and thus the fiber spheres are both circles. This allows us to prove the following important lemma.

Lemma 3.2.11. *Let M be a C1BF manifold with principal leaf $P = L_m(r) \times S^1$. Then $S^2 \hat{\times} S^1$ cannot be the singular leaf for $m \neq 2$.*

Proof. Let F be the homotopy fiber of the inclusion $P \to M$. Since $\pi_1(M) = 0$, it follows from the LES of homotopy associated to the fibration $F \to P \to M$ that $\pi_1(P)$ is a quotient of $\pi_1(F)$. By the work above, both of the fiber spheres are circles. Furthermore, we know from Lemma 3.2.4 that at least one of the singular leaves must be orientable, so Table 1.4 of [GH87] tells us that $\pi_1(F) \approx \mathbb{Z} \oplus \mathbb{Z}_2$. Therefore, we need $\pi_1(P) \approx \mathbb{Z}_m \oplus \mathbb{Z}$ to be a quotient of $\pi_1(F) \approx \mathbb{Z}_2 \oplus \mathbb{Z}$, which implies $m \leq 2$. We have already ruled out the case $m = 1$, where the lens space is a sphere, in Lemma 3.2.8. □

Note that this lemma rules out the possibility of $S^2 \hat{\times} S^1$ in all cases but $m = 2$, but does not tell us whether this actually happens in the case $m = 2$. Recall that when $m = 2$ we have $L_2(r) \approx \mathbb{R}\mathrm{P}^2$ for any value of r. We also know from Lemma 3.2.4 that $S^2 \hat{\times} S^1$ can be at most one of the singular leaves in a C1BF.

We can certainly have $P = L_m(r) \times S^1$ be a circle bundle over $L_m(r)$ itself. But we can, in fact, have P circle fiber over many more lens spaces, as we will now show. Suppose we have a circle bundle $S^1 \to L_m(r) \times S^1 \to L_n(s)$. By Proposition 3.2.3 this bundle must be principal and, moreover, by Theorem 3.2.2 we may assume that $r = s$ and m divides

n since $m = \gcd(i, n)$ for some i. Conversely, suppose x is a divisor of n. We claim that $L_x(r) \times S^1$ circle fibers over $L_n(r)$. Indeed, x divides n, so $1 \le x \le n$, thus $x = \gcd(x, n)$, so $L_x(r) \times S^1$ circle fibers over $L_x(r)$ by the same theorem. In summary, we have shown that we can restate Thornton's theorem as follows:

Proposition 3.2.12. *There exists a circle bundle $S^1 \to L_m(r) \times S^1 \to L_n(r)$ if m divides n. Furthermore, the total space P of any circle bundle $S^1 \to P \to L_n(r)$ is diffeomorphic to $L_m(r) \times S^1$.*

A subtle question is whether it is possible to have a C1BF with principal leaf $P = L_m(r) \times S^1$ and singular leaves $B_- = L_n(r)$ and $B_+ = L_k(s)$ where $r \neq s$ and m divides n and k. Thornton's theorem implies that if such a C1BF exists, then $L_m(r) \times S^1$ is diffeomorphic to $L_m(s) \times S^1$. We would like to thank Igor Belegradek for providing the reference for the following fact, stated as a lemma.

Lemma 3.2.13. *Suppose L and L' are 3-dimensional lens spaces and $L \times S^1$ is diffeomorphic to $L' \times S^1$, then L is diffeomorphic to L'.*

Proof. Suppose L, L' are 3-dimensional lens spaces and $S^1 \times L$ is diffeomorphic to $S^1 \times L'$, then the covering space of $S^1 \times L$ corresponding to the torsion subgroup defines an h-cobordism between L and L' (we have embeddings of L and L' in the covering space with disjoint images, and the images bound an h-cobordsim). It follows from Atiyah-Singer fixed point theorem that h-cobordant lens spaces are diffeomorphic [AB68]. $\square$

Proposition 3.2.14. *Suppose there exists a C1BF with principal leaf $P = L_m(r) \times S^1$ and singular leaves $L_n(t)$ and $L_k(s)$. Then we may assume that $t = r$ and $s \equiv \pm r^{\pm 1} \pmod{m}$.*

Proof. By Thornton's theorem it follows that we may assume that $t = r$. Furthermore, we then have a circle bundle $S^1 \to L_m(r) \times S^1 \to L_k(s)$ which, by Thornton's theorem and the above lemma implies that $L_m(r)$ is diffeomorphic to $L_m(s)$, so by the classification of lens spaces [Bro60] implies the statement about congruence of r and s modulo m. $\square$

We also note that Thornton's work [Tho74] also implies that the total space of any principal circle bundle over $S^2 \times S^1$ is diffeomorphic to $L_m(1) \times S^1$.

It follows from the work above that we can deduce the following summary.

Summary: Let M be a C1BF manifold with principal leaf $P = L_m(r) \times S^1$. Then the only leaf structures which are possibly admissible are the following:

1. $(L_m(r) \times S^1, L_n(r), L_k(s))$
2. $(L_m(1) \times S^1, L_n(r), S^2 \times S^1)$; $r \equiv \pm 1^{\pm 1} \pmod{m}$
3. $(L_2(r) \times S^1, L_{2n}(s), S^2 \hat{\times} S^1)$
4. $(L_m(1) \times S^1, S^2 \times S^1, S^2 \times S^1)$

where m divides n and k and $s \equiv \pm r^{\pm 1} \pmod{m}$.

We now break these possibilities down by cases as we have for the previous principal leaves.

Case C.6.1: $(P, B_-, B_+) = \left(L_m(r) \times S^1, L_n(r), L_k(s)\right)$

By the work above we must have that m divides n and k and that $r \equiv \pm s^{\pm 1} \pmod{m}$. At the time of submission of this thesis, this case is incomplete. In particular, we do not know whether all leaf structures of this type can be realized as a simply connected C1BF. However, many of them (and maybe even all of them) can be realized as the examples given by Standard Case 1.2 of Structure 4.2.2 in conjunction with Case 2 of Structure 4.2.3.

Case C.6.2: $(P, B_-, B_+) = \left(L_m(1) \times S^1, L_n(r), S^2 \times S^1\right)$

Here we must have that m divides n. Moreover, we will show that if such a C1BF is to be simply connected, we must have $m = n$. In this case, because the principal leaf must be diffeomorphic to $L_m(r) \times S^1$ to fiber over $L_m(r)$ and also be diffeomorphic to $L_m(1) \times S^1$ in order to fiber over $S^2 \times S^1$. From the work above it follows that $L_m(r)$ is diffeomorphic to $L_m(1)$. Thus for simply connected C1BFs, we are looking for leaf structures of the form $(P, B_-, B_+) = \left(L_m(1) \times S^1, L_m(1), S^2 \times S^1\right)$. We exhibit C1BF structures on $S^3 \times S^2$ and $S^3 \hat{\times} S^2$ realizing this leaf structure in 4.2.2 and 4.2.3, respectively.

To prove the statement about simply connectedness, let $\pi_1(L_m(1) \times S^1) = \mathbb{Z}_m + \mathbb{Z}$ with generators a and b. The LES of homotopy groups corresponding to the fiber bundle $S^1 \xrightarrow{i} L_m(1) \times S^1 \xrightarrow{\pi_*} S^2 \times S^1$ gives

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}_m + \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \to 0$$

Note that because $\mathbb{Z}$ has no torsion, a must map to $0 \in \mathbb{Z}$ under π_*. Since π_* is surjective, b must map to a generator of $\mathbb{Z}$, so $\mathrm{Im}\, i_* = \mathrm{Ker}\pi_* = \mathbb{Z}_m + 0$. Now consider the LES of homotopy groups corresponding to the fiber bundle $S^1 \xrightarrow{j} L_m(1) \times S^1 \xrightarrow{p_*} L_n(1)$:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}_m + \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z}_n \to 0$$

If the images of both fiber circles are to generate $\pi_1(L_m(1) \times S^1)$, we must have $b \in \mathrm{Ker} p* = \mathrm{Im}\, j_*$. Using the fact that p is surjective now means that $m \geq n$, but since m divides n it follows that $m = n$.

Case C.6.3: $(P, B_-, B_+) = \left(L_2(r) \times S^1, L_{2n}(s), S^2 \hat{\times} S^1\right)$

We show first that such a C1BF is not simply connected for $n \neq 1$. In the case $n = 1$, the principal leaf is $\mathbb{RP}^3 \times S^1$ and the lens space singular leaf $\mathbb{RP}^3$, independent of the choice of the value r and s. We exhibit a simply connected cohomogeneity one C1BF with this leaf structure in Structure 4.2.10 in Chapter 4. If we have a C1BF with this leaf structure, then we have circle bundles

$$S^1 \xrightarrow{i} L_2(r) \times S^1 \to S^2 \hat{\times} S^1 \tag{3.2.1}$$

$$S^1 \xrightarrow{j} L_2(r) \times S^1 \to L_{2n}(s) \tag{3.2.2}$$

The first has LES of homotopy $\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}_2 \times \mathbb{Z} \to \mathbb{Z} \to 0$ and, because $\mathrm{Im}(i_*)$ has to be cyclic, we must have $\mathrm{Im}(i_*) = \langle (0,k) \rangle$ or $\mathrm{Im}(i_*) = \langle (1,k) \rangle$ for some $k \in Z$. But we must also have $(\mathbb{Z}_2 \times \mathbb{Z})/\mathrm{Im}\, i_* \simeq \mathbb{Z}$, so it follows that $\mathrm{Im}(i_*) = \langle (1,k) \rangle$. On the other hand, the same argument applied to the other LES implies that $\mathrm{Im}(j_*) = \langle 0, p \rangle$ for some integer p. But then $(\mathbb{Z}_2 \times \mathbb{Z})/\mathrm{Im}(j_*) \simeq \mathbb{Z}_2 \times \mathbb{Z}_p \simeq \mathbb{Z}_{2n}$ so in fact $p = n$ where n is forced to be odd. Thus the images of i_* and j_* are generated by $(1,k)$ and $(0,n)$ which is easily seen not to contain the element $(0,1) \in \mathbb{Z}_2 \times \mathbb{Z}$ unless $n = 1$, which by the van Kampen theorem for C1BFs proves that such a C1BF is not simply connected unless $n = 1$.

Case C.6.4: $(P, B_-, B_+) = \left(L_m(1) \times S^1, S^2 \times S^1, S^2 \times S^1\right)$

We show that such a C1BF is not simply connected if $m \geq 2$. In the case $m = 1$, this reduces to Case C.4.10 for which we have already shown is not simply connected. To see that this is not simply connected for $m \geq 2$, suppose we have such a C1BF. Then we have circle bundles

$$S^1 \xrightarrow{i} L_m(1) \times S^1 \xrightarrow{\pi} S^2 \times S^1$$

So by the LES of homotopy we get

$$\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}_m \times \mathbb{Z} \xrightarrow{\pi_*} \mathbb{Z} \to 0$$

so $\mathbb{Z} \simeq (\mathbb{Z}_m \times \mathbb{Z})/\mathrm{Ker}(\pi_*) \simeq (\mathbb{Z}_m \times \mathbb{Z})/\mathrm{Im}(i_*)$, so it follows that $\mathrm{Im}(i_*) \subset \mathbb{Z}_m$. Thus the images of the fiber spheres under the inclusions cannot generate $\pi_1(G//H) = \pi_1(L_m(1) \times S^1)$, so it is not simply connected if $m \geq 2$.

3.3 Classification in Dimension 6

Let M^6 be a C1BF with five dimensional principal leaf P^5. In this section, we will classify all possible leaf structures for such C1BFs under the additional assumption that P is simply connected. According to Pavlov [Pav04], the only compact simply connected five dimensional biquotients are S^5, $S^3 \times S^2$, $S^3 \hat{\times} S^2$, and the Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. Thus these are the only possible manifolds which can appear as the principal leaf of a six dimensional C1BF with simply connected singular leaf. The results of this section can be summarized by the following theorem.

Theorem 3.3.1. *Let M be a compact, simply connected, 6-dimensional C1BF. Let $S^3 \hat{\times} S^2$ denote the nontrivial S^3-bundle over S^2. With the possible exception of leaf structure (19), the following list is the complete list of admissible leaf structures for M:*

1. (S^5, pt, pt)
2. $(S^5, pt, \mathbb{C}P^2)$
3. $(S^5, \mathbb{C}P^2, \mathbb{C}P^2)$
4. $(S^3 \times S^2, S^2, S^2)$
5. $(S^3 \times S^2, S^3, S^2)$
6. $(S^3 \times S^2, S^2 \times S^2, S^2)$
7. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2)$
8. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^2)$
9. $(S^3 \times S^2, S^3, S^3)$
10. $(S^3 \times S^2, S^3, S^2 \times S^2)$
11. $(S^3 \times S^2, S^3, \mathbb{C}P^2 \# \mathbb{C}P^2)$
12. $(S^3 \times S^2, S^3, S^2 \times S^2)$
13. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^3)$
14. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^3)$
15. $(S^3 \times S^2, S^2 \times S^2, S^2 \times S^2)$

16. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2 \times S^2)$

17. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^2 \times S^2)$

18. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# \mathbb{C}P^2)$

19. $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$

20. $(S^3 \times S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$

21. $(S^3 \hat{\times} S^2, S^2, S^2)$

22. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, S^2)$

23. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2)$

24. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# -\mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$

25. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# -\mathbb{C}P^2)$

26. $(S^3 \hat{\times} S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# \mathbb{C}P^2)$

In the case of leaf structure (19), there does not exist a representation of a C1BF as a group diagram with $G = Sp(1) \times Sp(1)$ hence any C1BF which possibly admits such a leaf structure cannot do so with the principal or singular leaves given as reduced biquotients.

Note that it is likely that leaf structure (19) does not occur as a C1BF arising as a quotient of a cohomogeneity one manifold. In particular, DeVito has outlined an approach to showing that such a leaf structure does not arise for any G. In particular, if $M = G//H$ is simply connected, then we can write $M = G'//H'$ for G' and H' connected. According to DeVito, given a biqutoient $G//H$, one can pull back G to any cover of G to get an equivalent biquotient. Thus we may assume that $G = G_1 \times \cdots \times G_n \times T^k$ for G_i, $i \in \{1, ..., n\}$ simple. We will call a biquotient *semi-reduced* if, up to cover, $M = G//H$ where G, H are connected and H does not act transitively on any factor of G. Note that a biquotient $G//H$ for G, H connected is not semi-reduced if H acts transitively on some factor of G, hence we can cancel factors using Proposition 1.3.7. Given a C1BF diagram $\{G, K_-, K_+, H\}$, one can replace the groups in the diagram with connected groups to obtain another C1BF diagram $\{G', K'_-, K'_+, H'\}$ with the same leaves. One can then cancel any factors of the G' group in the diagram by getting rid of factors of G' for which H' acts transitively. Since $H' \leq K'_{\pm}$, it follows that $K'_{\pm}$ also act transitively on G'. Thus we can replace the C1BF diagram $\{G', K'_-, K'_+, H'\}$ with another C1BF diagram $\{G'', K''_-, K''_+, H''\}$ for which $G''//H''$ is a semi-reduced biquotient. However, $K''_{\pm}$ may still act transitively on some factor of G'', so these leaves may or may not be semi-reduced. We call such a C1BF a *reduced C1BF*. One needs to show that this reduction process actually yields C1BF diagrams which give rise to diffeomorphic manifolds. Once this has been shown, it should follow that one cannot get leaf structure (19) as a C1BF.

We wish to determine the admissible leaf structures (P, B_-, B_+). Given a fixed principal leaf P, recall that we must have a sphere bundle $S^\ell \to P \to B^b$ over the singular leaf B for $\ell > 0$. In the special case where $P = S^5$, we can get the bundle $S^5 \to S^4 \to \{pt\}$. Otherwise, it follows from the LES of homotopy associated to the above sphere bundle that $\pi_1(B) = 0$. Therefore, we will have the following three *sphere bundle cases*, which will be repeatedly referenced below.

(i) $S^3 \to P \to S^2$

(ii) $S^2 \to P \to S^3$

(iii) $S^1 \to P \to B^4$

For sphere bundle case (iii), we must have that B is simply connected, hence it follows from DeVito's classification of 4-dimensional biquotients that the only possibilities for B are the biquotients on the following list.

List B: (Potential Singular Leaves)

1. S^4
2. $\mathbb{C}P^2$
3. $S^2 \times S^2$
4. $\mathbb{C}P^2 \# \mathbb{C}P^2$
5. $\mathbb{C}P^2 \# -\mathbb{C}P^2$

The following lemma will often allow us to rule out spaces (3)-(5) as a possibility for a singular leaf with a single argument.

Lemma 3.3.2. *The manifolds $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# -\mathbb{C}P^2$ have the same homotopy groups. Likewise, the manifolds $S^3 \times S^2$ and $S^3 \hat{\times} S^2$ have the same homotopy groups.*

Proof. Note that by DeVito's classification of 4-dimensional biquotients, all three of these spaces can be written as as a biquotient $B = (S^3 \times S^3)//T^2$ for some free torus action. Therefore, we get a LES of homotopy associated to the bundle

$$T^2 \to S^3 \times S^3 \to B.$$

from which it follows that $\pi_n(B) \simeq \pi_n(S^3 \times S^3)$ for $n \geq 3$. In the case $n = 1$ we know that all of these spaces are simply connected, and in the case $n = 2$ it is well known that $\pi_2(B) \simeq \mathbb{Z} \times \mathbb{Z}$ for each of these spaces.

Similarly, according to DeVito's classification of 5-dimensional biquotients, we can write $S^3 \times S^2$ and $S^3 \hat{\times} S^2$ as a biquotient $(S^3 \times S^3)//S^1$. Applying a similar argument to the one above shows these spaces have the same homotopy groups as well.

□

Case D.1: $P = \mathrm{SU}(3)/\mathrm{SO}(3)$

We will show that the $P = \mathrm{SU}(3)/\mathrm{SO}(3)$ does not occur as the principal leaf of a C1BF. This follows immediately from the following lemma.

Lemma 3.3.3. *The Wu manifold $W = SU(3)/SO(3)$ is not a sphere bundle over any space.*

Proof. By the LES of homotopy associated to the bundle $\mathrm{SO}(3) \to \mathrm{SU}(3) \to \mathrm{SU}(3)/\mathrm{SO}(3)$ it is easy to compute $\pi_1(M) = 0$ and $\pi_2(M) \simeq \mathbb{Z}_2$. Suppose now that we have a sphere bundle $S^d \to W \to B^{5-d}$. Clearly $d = 4$ is impossible by the LES of homotopy associated to this sphere bundle. For $d = 3$, the LES implies implies $\pi_1(B^2) = 0$ so $B^2 = S^2$, which is impossible since there are only two S^3-bundles over S^2, namely $S^3 \times S^2$ and $S^3 \hat{\times} S^2$. Similarly, for $d = 3$, the LES implies $\pi_1(B^3) = 0$, so by the Poincaré conjecture $B^3 = S^3$, but by Steenrod's classification of sphere bundles, the only S^2 bundle over S^3 is $S^2 \times S^3$. The only remaining case is whether there is a circle bundle $S^1 \to W \to B^4$. By the LES of homotopy we have

$$0 \to \mathbb{Z}_2 \to \pi_2(B^4) \to \mathbb{Z} \to 0 \to \pi_1(B^4) \to 0$$

which implies $\pi_1(B^4) = 0$ and hence, by the Hurewicz theorem, $H_2(B^4) \simeq \pi_2(B^4)$. But by exactness, $\pi_2(B^4)/\mathbb{Z}_2 \simeq \mathbb{Z}$ hence, because $\pi_2(B^4) \simeq H_2(B^4)$ is free abelian, it follows that $H_2(\mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$. Now, since B^4 is simply connected we have that B^4 is orientable, so $H^2(B^4) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ by Poincaré duality. On the other hand, since $H_1(B) = 0$, the Universal Coefficient Theorem implies $H^2(B^4) \simeq \mathbb{Z}$, a contradiction. Thus the Wu manifold is not a sphere bundle. $\square$

Case D.2: $P = S^5$

As mentioned above, we can have a point as the singular leaf in this case. It is easy to rule out sphere bundle cases (i) and (ii) by the LES of homotopy. This only leaves sphere bundle case (iii). The LES immediately rules out the case where $B = S^4$. The well known Hopf fibration gives a circle bundle $S^1 \to S^5 \to \mathbb{C}\mathrm{P}^2$ showing that $B = \mathbb{C}\mathrm{P}^2$ is a possibility for the singular leaf. It follows from the LES of homotopy associated to sphere bundle case (iii) that $B = S^2 \times S^2$ cannot occur and hence, by Lemma 3.3.2 the remaining spaces on List B cannot occur as the singular leaf. Thus the only leaf structures which can possibly occur are the following three cases.

Case D.2.1 $(P, B_-, B_+) = (S^5, pt, pt)$ Such a C1BF is easily seen to always be diffeomorphic to S^6 and is realized by the standard two fixed point cohomogeneity one action on S^6.

Case D.2.2 $(P, B_-, B_+) = (S^5, \mathbb{C}\mathrm{P}^2, pt)$

This is realized as a C1BF in Structure 4.3.8 in Chapter 4.

Case D.2.2 $(P, B_-, B_+) = (S^5, \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2)$

This is realized as a C1BF in Structure 4.3.5 in Chapter 4.

Case D.3: $P = S^3 \times S^2$

Clearly sphere bundle cases (i) and (ii) are possible via the trivial bundles. For sphere bundle case (iii), it is easy to rule out the case $B = S^4$ via the LES of homotopy. Using the Hopf fibration $S^1 \to S^5 \to \mathbb{C}\mathrm{P}^2$, it follows that $\pi_3(\mathbb{C}\mathrm{P}^2) = 0$. It this then easy to to rule out the case $B = \mathbb{C}\mathrm{P}^2$ using the LES of homotopy. The case $B = S^2 \times S^2$ clearly occurs because $S^3 \times S^2$ admits a free circle action with quotient $S^2 \times S^2$. We handle the cases $B = \mathbb{C}\mathrm{P}^2 \# \pm \mathbb{C}\mathrm{P}^2$ in the following lemmas.

Lemma 3.3.4. *There exists principal circle bundles $S^1 \to S^3 \times S^2 \to \mathbb{C}P^2 \# \pm \mathbb{C}P^2$.*

Proof. According to DeVito's classification of 4-dimensional biquotients, we can write $\mathbb{C}\mathrm{P}^2\#{-}\mathbb{C}\mathrm{P}^2 \simeq (S^3 \times S^3)//T^2$ where T^2 acts on $S^3 \times S^3$ via $(z,w)\cdot(p,q) = (z^2 p, wz^n q \overline{z}^n)$ for n an odd integer. Fixing $z = 1$ yields a free circle action by the w-factor circle S^1_w on $S^3 \times S^3$ which obviously has quotient $(S^3 \times S^3)/S^1_w = S^3 \times S^2$. Now, the circle action obtained by fixing $w = 1$ in the above action yields an effectively free circle action by the z-factor circle S^1_z which commutes with the action of S^1_w, so the z-circle acts on the quotient $(S^3 \times S^3)/S^1_w \simeq S^3 \times S^2$ and must be effectively free because the original product action is effectively free. Thus we have a nontrivial effectively free circle action on $S^3 \times S^2$ with quotient $\mathbb{C}\mathrm{P}^2\# - \mathbb{C}\mathrm{P}^2$ which implies that we get a principal bundle as above.
We can apply a similar argument to get a principal bundle $S^1 \to S^3 \times S^2 \to \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ using (again from DeVito's classification) that $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2 \simeq (S^3 \times S^3)//T^2$ where T^2 acts on $S^3 \times S^3$ by $(z,w)\cdot(p,q) = (zwp\overline{w}, zw^2 q\overline{z})$. $\square$

In summary, we can have $B \in \{S^2, S^3, S^2 \times S^2, \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2\# - \mathbb{C}\mathrm{P}^2\}$, so each of the following leaf structures can possibly occur.

Case D.3.1: $(P, B_-, B_+) = (S^3 \times S^2, S^2, S^2)$

This is realized as a C1BF in Structure 4.3.3 in Chapter 4.

Case D.3.2: $(P, B_-, B_+) = (S^3 \times S^2, S^3, S^2)$

This is realized as a C1BF in Structure 4.3.2 in Chapter 4.

Case D.3.3: $(P, B_-, B_+) = (S^3 \times S^2, S^2 \times S^2, S^2)$

This is realized as a C1BF in Structure 4.3.6 in Chapter 4.

Case D.3.4: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, S^2)$

This is realized as a C1BF in Structure 4.3.9 in Chapter 4.

Case D.3.5: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2, S^2)$

This is realized as a C1BF in Structure 4.3.11 in Chapter 4.

Case D.3.6: $(P, B_-, B_+) = (S^3 \times S^2, S^3, S^3)$

This is realized as a C1BF in Structure 4.3.1 in Chapter 4.

Case D.3.7: $(P, B_-, B_+) = (S^3 \times S^2, S^2 \times S^2, S^3)$

This is realized as a C1BF in Structure 4.3.4 in Chapter 4.

Case D.3.8: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, S^3)$

This is realized as a C1BF in Structure 4.3.13 in Chapter 4.

Case D.3.9: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2, S^3)$

This is realized as a C1BF in Structure 4.3.14 in Chapter 4.

Case D.3.10: $(P, B_-, B_+) = (S^3 \times S^2, S^2 \times S^2, S^2 \times S^2)$

This is realized as a C1BF in Structure 4.3.7 in Chapter 4.

Case D.3.11: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, S^2 \times S^2)$

This is realized as a C1BF in Structure 4.3.20 in Chapter 4.

Case D.3.12: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2, S^2 \times S^2)$

This is realized as a C1BF in Structure 4.3.15 in Chapter 4.

Case D.3.13: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2)$

This is realized as a C1BF in Structure 4.3.16 in Chapter 4.

Case D.3.14: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2)$

We do not currently have a model of a C1BF realizing this leaf structure. We see in Chapter 4 that all other leaf structures in dimension 6 with simply connected principal leaf having $S^2 \times S^2$ or $\mathbb{C}\mathrm{P}^2 \# \pm \mathbb{C}\mathrm{P}^2$ as a singular leaf can be constructed from a group diagram using $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. We will prove the following proposition

Proposition 3.3.5. *A C1BF with leaf structure $(S^3 \times S^2, \mathbb{C}P^2 \# - \mathbb{C}P^2, \mathbb{C}P^2 \# \mathbb{C}P^2)$ does not have a representation as a group diagram with $G = Sp(1) \times Sp(1)$.*

The proof of this is rather long due to the fact that there are a lot of things to check. First, we note that DeVito classifies reduced biquotients up to the following equivalence.

Proposition 3.3.6 (DeVito)**.** *Suppose $f : H \to G \times G$ induces an effectively free biquotient action. Then, after any of the following modifications of f, the new induced action is effectively free and the quotients are naturally diffeomorphic.*
(i) For any automorphism f' of H, change f to $f \circ f'$,
(ii) For any element $g = (g_1, g_2) \in G \times G$, change f to $C_g \circ f$, where C_g denotes conjugation by g,
(iii) For any automorphism f' of G, change f to $(f', f') \circ f$.

Proof of Proposition 3.3.5. Consider the following homomorphisms

(1) $T^2 \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$; $\quad (z, w) \mapsto (zw, zw^2, w, z)$

(2) $T^2 \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$; $\quad (u, v) \mapsto (u^2, vu^n, 1, u^n)$

(3) $S^1 \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$; $\quad \theta \mapsto (\theta^a, \theta^b, \theta^c, \theta^d)$

According to DeVito's classification of reduced biquotients, (1) induces a biquotient diffeomorphic to $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and the homomorphism (2) induces a biquotient diffeomorphic to $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$ if n is odd and diffeomorphic to $S^2 \times S^2$ if n is even. Finally, (3) induces a biquotient diffeomorphic to $S^3 \times S^2$ if $\gcd(a, b, c, d) = \gcd(a^2 - c^2, b^2 - d^2) = 1$ and diffeomorphic to $S^3 \hat{\times} S^2$ if $\gcd(a, b, c, d) = 1$ and $\gcd(a^2 - c^2, b^2 - d^2) = 4$. Furthermore, up to the equivalence theorem above, these are the only ways to write these quotients as reduced biquotients, hence any C1BF diagram for the desired leaf structure using $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ would have $K^{\pm} = T^2$ and $H = S^1$, where the tori are embedded using the embeddings above, up to equivalence. We will show that a C1BF diagram with these groups cannot give the desired leaf structure.

Observe that the images of (1) and (2) do not intersect in a circle. In particular, any intersecting circle must have $w = 1$ and with $w = 1$ we have that the image of (2) becomes $(z, z, 1, z)$. By replacing z with an integer power of z, we may assume that the images of (1) and (2) intersect only when $(z, z, 1, z) = (u^2, vu^n, 1, u^n)$. In particular, we must have

$$z = u^2 \tag{3.3.1}$$

$$z = vu^n \tag{3.3.2}$$

$$z = u^n \tag{3.3.3}$$

Note that setting (3.3.2) equal to (3.3.3) implies $v = 1$ and then setting (3.3.1) equal to (3.3.3) implies $u^2 = u^n$. Hence these circles only intersect on the subgroup

$$\{(u,1) \,:\, u^{n-2} = 1\} \subset T^2.$$

Note that this is a circle only in the case $n = 2$, which would give a quotient diffeomorphic to $S^2 \times S^2$. It remains to show that there is no equivalent embeddings (in the sense of DeVito's theorem above) that can produce the desired leaf structure.

We start by showing that equivalence (ii) cannot produce a C1BF diagram with the desired leaf structure; that is, there is no way to conjugate the image of the tori by an element of $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))^2$ to get the desired leaf structure. It suffices to keep the torus determined by (2) fixed and conjugate the torus given by (1). First, note that for $q = a + bi + cj + dk \in \mathrm{Sp}(1)$ and $x + yi \in S^1 \subset \mathbb{C}$, it is not difficult to check that

$$q(x+yi)\overline{q} = x + (a^2y + b^2y - c^2y - d^2y)i + (2ady + 2bcy)j + (2bdy - 2acy)k \tag{3.3.4}$$

Since we assume that the tori for the embeddings (1) and (2) land in $\mathrm{Sp}(1) \cap \mathbb{C}$, it follows that we must have

$$2ady + 2bcy = 0 \tag{3.3.5}$$

$$2bdy - 2acy = 0 \tag{3.3.6}$$

in order to get an intersection of (2) with the image of (1) after conjugation by an element of $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))^2$. Note that we may assume $y \neq 0$ because this must hold for every $z \in S^1$. Therefore, equations (3.3.4) and (3.3.6) imply

$$ad + bc = 0 \tag{3.3.7}$$

$$bd - ac = 0 \tag{3.3.8}$$

Multiplying (3.3.7) by b and (3.3.8) by a gives

$$abd + b^2 c = 0 \tag{3.3.9}$$

$$abd - a^2 c = 0 \tag{3.3.10}$$

which implies $(b^2 - a^2)c = 0$ hence $b^2 - a^2 = 0$ or $c = 0$. We consider these as two separate cases.

Case 1: $(c = 0)$

In this case, equations (3.3.7) and (3.3.8) imply

$$ad = 0 \tag{3.3.11}$$

$$bd = 0 \tag{3.3.12}$$

These are satisfied if $d = 0$ or both $a = 0$ and $b = 0$. If $d = a = b = 0$ then $q = 0$ which is a contradiction. If $d = 0$ then $p \in \mathbb{C} \cap \mathrm{Sp}(1)$ so conjugation does nothing. If $a = 0$ and $b = 0$ then $d \neq 0$ and hence $p = \pm k$ and by formula (3.3.4), we have $kz\overline{k} = \overline{z}$ so q acts on S^1 by taking complex conjugates, so will not help us get an appropriate circle intersection in the images of (1) and (2).

Case 2: $(b^2 - a^2 = 0)$

In this case we will assume $c \neq 0$ otherwise we are back in Case 1. The condition $b^2 - a^2 = 0$ implies $b = \pm a$. In the case $b = a$, we have by equations (3.3.7) and (3.3.8) that

$$ad + ac = 0 \tag{3.3.13}$$

$$ad - ac = 0 \tag{3.3.14}$$

Note that it is easy to see from equations (3.3.7) and (3.3.8) that the cases $b = \pm a$ are equivalent. Adding equations (3.3.13) and (3.3.14) we see that $2ad = 0$ and hence $a = 0$ or $d = 0$. If $a = d = 0$ then $a = b$ and hence $b = 0$ and, therefore, $q = \pm j$ for which conjugation acts by taking complex conjugates which is not helpful. If $a = 0$ and $d \neq 0$ then $a = b$ implies $b = 0$ so $q = \pm k$ so conjugation by q again is not helpful. If $d = 0$ and $a \neq 0$ then $b \neq 0$ so $q = a + bi \in \mathbb{C} \cap \mathrm{Sp}(1)$ so conjugation again does nothing.

It follows that we cannot conjugate the image of the torus embedding (1) to get an S^1 intersection with (2) when n is odd. In particular, no embeddings of the tori up to the equivalence given by (ii) of DeVito's equivalence theorem will give us a C1BF diagram with the desired leaf structure.

For equivalences given by (ii), note that the automorphism group of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is

$$\mathrm{Aut}(\mathrm{Sp}(1) \times \mathrm{Sp}(1)) = \langle \sigma, K_{(p,q)} \,:\, (p,q) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rangle$$

where σ denotes the automorphism which swaps the first and second factors, as well as the third and fourth factors and $K_{(p,q)}$ is conjugation by $(p,q) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Note that in (iii), if we choose $f' = K_{(p,q)}$, then $(f', f') \circ f = C_g \circ f$ which we have already shown is not helpful because permissible conjugations either do nothing or take complex conjugates. Thus we need only focus on swapping factors. If we keep the torus (2) fixed again and swap the factors of torus (1) by $(\sigma,\sigma)\circ f$, the image of torus (1) becomes $(vu^n, u^2, u^n, 1) \subset (\mathrm{Sp}(1)\times\mathrm{Sp}(1))^2$. Therefore, we want to look for intersections of this torus with $(zw, zw^2, w, z) \subset (\mathrm{Sp}(1)\times\mathrm{Sp}(1))^2$. Clearly we must have $z = 1$ and in this case the second torus becomes $(w, w^2, w, 1)$. We then want

$$vu^n = w \tag{3.3.15}$$

$$u^2 = w^2 \tag{3.3.16}$$

$$u^n = w \tag{3.3.17}$$

It is not hard to see that these equations imply that the intersection is $\{(u,1) : u^{2n-2} = 1\} \subset T^2$ which is only a circle in the case $n = 1$. In the case $n = 1$, it is easy to see that the circle intersection gives a biquotient diffeomorphic to $S^3 \hat{\times} S^2$.

For equivalences given by (i) of DeVito's theorem, note that the automorphism group of T^2 is $\mathrm{GL}_2(\mathbb{Z})$, i.e. the group of invertible 2×2 integer matrices. The automorphism corresponds to the matrix multiplication

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z^a & w^b \\ z^c & w^d \end{pmatrix}$$

That is, the automorphism $(z,w)\mapsto (z^a w^b, z^c w^d)$ where the matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible over $GL_2(\mathbb{Z})$. Precomposing the embedding $(z,w)\mapsto(zw,zw^2,w,z)$ for the torus (1) with $(z,w)\mapsto(z^a w^b, z^c w^d)$ gives

$$(z,w)\mapsto(z^{a+c}w^{b+d}, z^{a+2c}w^{b+2d}, z^c w^d, z^a w^b) \tag{3.3.18}$$

In particular, this is a change of coordinates on the torus and we wish to determine whether there is a change of coordinates that gives a circle intersection with the torus $(u,v)\mapsto(vu^n,u^2,u^n,1)$ for n odd. In particular, it is clear that we need $z^a w^b = 1$. The question remains which subgroups of T^2 satisfy for this for all z,w within that subgroup. This subgroup also needs to be a circle and thus w has to be a power of z (essentially to restrict the torus to a circle subset). Thus we have $w=z^m$ for some integer m. Therefore $z^a w^b = z^a w^{mb} = 1$ which implies $z^{a+mb}=1$. This happens for all $z\in S^1$ if and only if $a=-mb$.

Thus we have shown that the only way we can get a circle intersection with the desired property is if $w=z^m$ for some $m\in\mathbb{Z}$ and $a=-mb$. Thus we set

$$X=\begin{pmatrix}-mb & b\\ c & d\end{pmatrix} \tag{3.3.19}$$

and consider the corresponding embedding

$$(z,z^m)\mapsto(z^{md+c}, z^{2md+2c}, z^{md+c}, 1) \tag{3.3.20}$$

We want $(u,v)\mapsto(vu^n,u^2,u^n,1)$, n odd, to have a circle intersection with (3.3.20). It is clear that we need $v=1$ so we restrict to $(u,1)\mapsto(u^n,u^2,u^n,1)$. By replacing z with an integer power of z, we may assume WLOG that

$$u^n = z^{2md+c} \tag{3.3.21}$$

$$u^2 = z^{2md+c} \tag{3.3.22}$$

Note that (3.3.22) implies that $u = \pm z^{md+c}$ and then these equations together imply that $(\pm z)^{n(md+c)} = z^{md+c}$. This needs to hold for all $z \in S^1$, so we have to use "+" and have either $n = 1$ or n an arbitrary odd integer and $md + c = 0$. If $c = -md$ then

$$X = \begin{pmatrix} -mb & b \\ -md & d \end{pmatrix}$$

has determinant zero so is not invertible. Thus we must have $n = 1$.

Thus we have shown that the tori

$$T^2 = (vu^n, u^2, u^n, 1)$$

$$T^2 = (z^{a+c}w^{b+d}, z^{a+2c}w^{b+2d}, z^c w^d, z^a w^b)$$

intersect in a circle only when the following conditions are satisfied

1. $a = -mb, m \in Z$
2. $w = z^m$
3. $n = 1$
4. $v = 1$
5. $u = z^{md+c}$
6. The matrix (3.3.19) is in $\mathrm{GL}_2(\mathbb{Z})$

Under these conditions, the corresponding circle intersection is

$$z \mapsto (z^{md+c}, z^{2md+2c}, z^{md+c}, 1) \tag{3.3.23}$$

It is easy to see using the gcd conditions for the circle embedding (3) that such a circle will always have quotient $S^3 \hat{\times} S^2$.

Finally, we note that in the argument for equivalences given by (iii), we used a torus embedding equivalent to the embedding (2) by swapping factors and showed that we could not get the desired leaf structure. If we had instead used embedding (2) without swapping factors, completely analogous work shows that you can only get a circle intersection with (1) in the case that $n = 2$ which corresponds to a leaf structure $(S^3 \times S^2, \mathbb{C}P^2 \# \mathbb{C}P^2, S^2 \times S^2)$.

$\square$

Case D.3.15: $(P, B_-, B_+) = (S^3 \times S^2, \mathbb{C}P^2 \# - \mathbb{C}P^2, \mathbb{C}P^2 \# - \mathbb{C}P^2)$

This is realized as a C1BF in Structure 4.3.18 in Chapter 4.

Case D.4: $P = S^3 \hat{\times} S^2$

Note that sphere bundle case (i) happens by definition of this space. By Steenrod's classification of sphere bundles, sphere bundle case (ii) must be a product bundle so cannot happen in this case. For sphere bundle case (iii), note that $S^3 \times S^2$ and $S^3 \hat{\times} S^2$ have the same homotopy by Lemma 3.3.2, all of the sphere bundle cases ruled out in Case D.3 via homotopy are also ruled out here. In particular, in sphere bundle case (iii), $B = S^4$ is ruled out as well as $B = \mathbb{C}P^2$. It follows from a theorem of Giblin [Gib68] that any simply connected total space of of a circle bundle over $S^2 \times S^2$ is homeomorphic to $S^3 \times S^2$, so $B = S^2 \times S^2$ does not occur. We handle the remaining two cases for B as lemmas.

Lemma 3.3.7. *There exist a principal circle bundle $S^1 \to S^3 \hat{\times} S^2 \to -\mathbb{C}P^2 \# -\mathbb{C}P^2$.*

Proof. The argument is similar to that of Lemma 3.3.4. In particular, by DeVito's classification of biquotients we have $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2 \simeq (S^3 \times S^3)//T^2$ where T^2 acts by $(z,w)\cdot(p,q) = (z^2 p, wz^n q\overline{z}^n)$ where n is odd, and $S^3 \hat{\times} S^2 \simeq (S^3 \times S^3)//S^1$, where S^1 acts on $S^3 \times S^3$ by $z\cdot(p,q) = (z^a p\overline{z}^c, z^b q\overline{z}^d)$ where $\gcd(a,b,c,d) = 1$ and $\gcd(a^2-c^2, b^2-d^2) = 4$. Choose $a = 2$, $c = 0$, and $b = d = n$ for n odd. Then the above gcd conditions are satisfied and the circle action is given by

$$z\cdot(p,q) = (z^2 p, z^n q\overline{z}^n) \tag{3.3.24}$$

For the above T^2 action, fixing $w = 1$ leaves precisely the circle action (3.3.24) with quotient $S^3 \hat{\times} S^2$. Using the same argument as in Lemma 3.3.4, it follows that we get a principal bundle $S^1 \to S^3 \hat{\times} S^2 \to \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$.

□

Lemma 3.3.8. *There exist a principal circle bundle $S^1 \to S^3 \hat{\times} S^2 \to \mathbb{C}P^2 \# \mathbb{C}P^2$.*

Proof. By DeVito's classification we can write $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 \simeq (S^3 \times S^3)//T^2$ where T^2 acts by $(z,w)\cdot(p,q) = (zwp\overline{w}, zw^2 q\overline{z})$ and we can write $S^3 \hat{\times} S^2 \simeq (S^3 \times S^3)//S^1$ where S^1 acts on $S^3 \times S^3$ by $w\cdot(p,q) = (w^a p\overline{w}^c, w^b q\overline{w}^d)$ where $\gcd(a,b,c,d) = 1$ and $\gcd(a^2-c^2, b^2-d^2) = 4$. Observe that if $a = c = 1$, $b = 2$, and $d = 0$ then the gcd conditions are satisfied and the circle action is given by

$$w\cdot(p,q) = (wp\overline{w}, w^2 q). \tag{3.3.25}$$

For the above T^2 action, fixing $z = 1$ leaves precisely the circle action (3.3.25) with quotient $S^3 \hat{\times} S^2$. Using the same argument as in Lemma 3.3.4, it follows that we get a principal bundle $S^1 \to S^3 \hat{\times} S^2 \to \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$.

□

In summary, we can have $B \in \left\{S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2\right\}$, so each of the following leaf structures can possibly occur.

Case D.4.1: $(P, B_-, B_+) = (S^3 \hat{\times} S^2, S^2, S^2)$

This is realized as a C1BF in Structure 4.3.21 in Chapter 4.

Case D.4.2: $(P, B_-, B_+) = (S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2, S^2)$

This is realized as a C1BF in Structure 4.3.12 in Chapter 4.

Case D.4.3: $(P, B_-, B_+) = (S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, S^2)$

This is realized as a C1BF in Structure 4.3.10 in Chapter 4.

Case D.4.4: $(P, B_-, B_+) = (S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2)$

This is realized as a C1BF in Structure 4.3.19 in Chapter 4.

Case D.4.5: $(P, B_-, B_+) = (S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2)$

This is realized as a C1BF in Structure 4.3.22 in Chapter 4.

Case D.4.6: $(P, B_-, B_+) = (S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2)$

This is realized as a C1BF in Structure 4.3.17 in Chapter 4.

Chapter 4

Models of C1BFs in Detail

In this chapter we will provide many explicit models of C1BFs which realize the leaf structures appearing in Chapter 3.

4.1 C1BFs in Dimension 4:

Dimension 4 is the lowest dimension where there exists a manifold that is a C1BF which is not cohomogeneity one, namely $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is not cohomogeneity one. As our first explicit C1BF structure in this chapter, we will show that $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ is a C1BF.

Structure 4.1.1. From DeVito's classification of compactly simply connected biquotients [DeV14], we can write $\mathbb{C}P^2\#\mathbb{C}P^2 = \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)//T^2$ where the action of T^2 on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ is given by

$$(z,w)\cdot(r_1,r_2) = (zwr_1\overline{w}, zw^2r_2\overline{z}) \tag{4.1.1}$$

Let $S^3 \subset \mathbb{C}^2$ and consider the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ on $\mathrm{Sp}(1) \times S^3$ given by

$$(p, q, x, y) \cdot (r, (a, b)) = (pr\overline{q}, (xa, yb)) \tag{4.1.2}$$

This is a product action of the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on the $\mathrm{Sp}(1)$ factor and the action of T^2 on the S^3. The former action is transitive, while the latter is by cohomogeneity one, hence the product action is cohomogeneity one. The goal here is to find a torus $\mathbb{T}$ contained within the group in the action given by (4.1.2) such that the action restricted to $\mathbb{T}$ is the action given by (4.1.1). This will exhibit $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ as a C1BF. To simplify things, we will rewrite the action (4.1.2) as an action on $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ so that in both cases the groups are acting on this space. We identify $S^3 \subset \mathbb{C}^2$ with $\mathrm{Sp}(1)$ via the diffeomorphism $(a, b) \mapsto a + bj$. It is easy to check that under this diffeomorphism of the S^3 factor, the action (4.1.2) is equivariant with the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ on $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ given by

$$(p, q, x, y) \cdot (r, a + bj) = (pr\overline{q}, xa + ybj) \tag{4.1.3}$$

Finally, writing $r_2 = a + bj$ in action 4.1.1 and using the fact that $j\overline{z} = zj$ for $j \in \mathrm{Sp}(1)$, action 4.1.1 becomes

$$(z, w) \cdot (r, a + bj) = (zwr\overline{w}, w^2 a + z^2 w^2 bj) \tag{4.1.4}$$

Comparing actions (4.1.3) and (4.1.4), it is easy to see that if we define a homomorphism $\Phi : T^2 \to \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ by $\Phi(z, w) = (zw, w, w^2, z^2 w^2)$, then the restriction of action 4.1.3 to the torus $\mathbb{T} = \mathrm{Im}(\Phi)$ is precisely the action 4.1.4. We have thus exhibited $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ as a C1BF.

We now wish to compute the group diagram and leaf structure for $\mathbb{C}P^2 \# \mathbb{C}P^2$ arising from this construction. To do this, we first compute the principal and singular orbits of the cohomogeneity one action (4.1.3). Once we know the principal and singular orbits of this action, we can compute the leaves of the C1BF by taking the quotient of each leaf by the action of $\mathbb{T}$. Let $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$. It is easy to check that the isotropy group of $y = (1,1)$ is $G_y = \Delta\mathrm{Sp}(1) \times S^1 \times \{1\}$ and gives a singular orbit G/G_y. The other singular orbit G/G_u corresponds to the point $u = (1, j)$ and it is easy to check that $G_u = \Delta\mathrm{Sp}(1) \times \{1\} \times S^1$. To get the principal orbit G/G_v, we can look at the point $v = (1, 1+j)$ and it is easy to check that $G_v = \Delta\mathrm{Sp}(1) \times \{1\} \times \{1\}$. Now, to compute the principal leaf, we use the following general method.

Method: Suppose G acts on M by cohomogeneity one and a subgroup H of G acts effectively freely on M by restricting the action. Let G_p be the isotropy group corresponding to a point p. Then the orbit containing p is G/G_p and the corresponding leaf in the C1BF M/H is obtained by taking the biquotient $H\backslash G/G_p$. To compute this quotient, we think of G/G_p as the quotient by the action of G_p on G by $k \cdot g = gk^{-1}$ and the quotient of G/G_p by H as the quotient by the action of H on G/G_p by $h\cdot[g] = [hg]$. Since these action commute, the biquotient $H\backslash G/G_p$ is the quotient by the action $H \times G_p$ on G given by left and right translation, that is $(h,k) \cdot g = hgk^{-1}$.

Using this and Construction 1.3.2, it is easy to see that the principal leaf is the biquotient induced by the embedding

$$T^2 \times \mathrm{Sp}(1) \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2)^2$$

given by $(z, w, p) \mapsto (zw, w, w^2, z^2w^2, p, p, 1, 1)$ or, more explicitly, the quotient by the action of $T^2 \times \mathrm{Sp}(1)$ on $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ given by

$$(z, w, p) \cdot (q_1, q_2, x, y) = (zwq_1\overline{p}, wq_2\overline{p}, z^2w^2y)$$

To compute this quotient, let $\Gamma = T^2 \times \mathrm{Sp}(1)$ and observe that Γ acts transitively on the second $\mathrm{Sp}(1)$ factor and the T^2 factor. Thus by Proposition 1.3.7 we have

$$(\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2)//\Gamma \approx \mathrm{Sp}(1)/\Gamma_e$$

where $e = (1,1,1) \in \mathrm{Sp}(1) \times T^2$. It is easy to compute that

$$\Gamma_e = \{(1,1,1), (1,-1,-1), (-1,1,1), (-1,-1,-1)\}.$$

Observe that under the action of Γ_e on $\mathrm{Sp}(1)$ that $(1,-1,-1)$ acts ineffectively and that $(-1,1,1)$ and $(-1,-1,-1)$ act by $q \mapsto -q$. It follows that the action of Γ_e on $\mathrm{Sp}(1)$ is equivalent to the $\mathbb{Z}_2$ action on $\mathrm{Sp}(1)$ generated by $q \mapsto -q$. Thus $\mathrm{Sp}(1)/\Gamma_e \approx \mathbb{RP}^3$. Similarly, we can easily compute that the singular leaf corresponding to the quotient of the singular orbits G/G_y and G/G_u are given by the biquotients induced by the embeddings $T^2 \times \mathrm{Sp}(1) \times S^1 \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2)^2$ given by

$$(z, w, p, t) \mapsto (zw, w, w^2, z^2w^2, p, p, t, 1)$$

$$(z, w, p, t) \mapsto (zw, w, w^2, z^2w^2, p, p, 1, t),$$

respectively. From Ge and Radeschi's classification [GR15], both singular leaves in this case are necessarily S^2.

To summarize, from the computations above, it follows that the C1BF group diagram in this case is

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2 \arrow[dl, dash, shift left] \arrow[dl, dash, shift right] \arrow[dr, dash, shift left] \arrow[dr, dash, shift right] & \\
T^2\times\mathrm{Sp}(1)\times S^1 \arrow[dr, dash] & & T^2\times\mathrm{Sp}(1)\times S^1 \arrow[dl, dash] \\
 & T^2\times\mathrm{Sp}(1) &
\end{tikzcd}
\qquad
\begin{array}{c}
M \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2\\
G//H \simeq \mathbb{R}\mathrm{P}^3\\
G//K^- \simeq S^2\\
G//K^+ \simeq S^2
\end{array}$$

where the embeddings are

$$K^- \to G\times G;\ (z,w,p,t)\mapsto (zw,w,w^2,z^2w^2,p,p,t,1)$$

$$K^+ \to G\times G;\ (z,w,p,t)\mapsto (zw,w,w^2,z^2w^2,p,p,1,t)$$

$$H \to G\times G;\ (z,w,p)\mapsto (zw,w,w^2,z^2w^2,p,p,1,1)$$

Structure 4.1.2. This construction will exhibit a non-cohomogeneity one structure on $\mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2$. The construction follows the same general procedure as Structure 4.1.1.

From DeVito's classification of compactly simply connected biquotients [DeV14], we can write $\mathbb{C}P^2\#-\mathbb{C}P^2=\big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)//T^2$ where the action of T^2 on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ is given by

$$(z,w)\cdot(r_1,r_2)=(z^2r_1,wz^nr_2\overline{z}^n) \tag{4.1.5}$$

where n is odd. As we did in Structure 4.1.1, we rewrite this action with $r_2=a+bj$ to get (4.1.5) in the form

$$(z,w)\cdot(r,a+bj)=(z^2r,wa+wz^{2n}bj) \tag{4.1.6}$$

We again consider the action (4.1.3) from structure 4.1.1 and wish to find a torus $\mathbb{T}$ contained within the acting group so that the restriction of the action to $\mathbb{T}$ is the same as action (4.1.5). Comparing actions (4.1.3) and (4.1.6), it is easy to see that if we define a homomorphism $\Phi : T^2 \to \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2$ by $\Phi(z,w) = (z^2,1,w,wz^{2n})$, then the restriction of action (4.1.3) to the torus $\mathbb{T} = \mathrm{Im}(\Phi)$ is precisely the action (4.1.6). We have thus exhibited $\mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2$ as a non-cohomogeneity one C1BF.

We will now compute the group diagram and leaf structure for $\mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2$ arising from this construction in the same general manner as in structure 4.1.1. We already computed the singular orbits of action cohomogeneity one action (4.1.3) in the previous structure. Using the same method as in the previous structure, it is easy to see that the principal leaf is the biquotient induced by the embedding $T^2\times\mathrm{Sp}(1)\to(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2)^2$ given by $(z,w,p)\mapsto(z^2,1,w,wz^{2n},p,p,1,1)$ or, more explicitly, the quotient by the action of $T^2\times\mathrm{Sp}(1)$ on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2$ by

$$(z,w,p)\cdot(q_1,q_2,x,y) = (z^2q_1\overline{p},q_2\overline{p},wx,wz^{2n}y) \qquad (n\ \text{odd})$$

To compute this quotient, let $\Gamma = T^2\times\mathrm{Sp}(1)$ and observe that Γ acts transitively on the last $\mathrm{Sp}(1)$ and T^2 factors, hence by Proposition 1.3.7 we have

$$(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2)/\Gamma \approx \mathrm{Sp}(1)/\Gamma_e$$

where $e = (1,1,1)\in\mathrm{Sp}(1)\times T^2$. It is easy to compute that $\Gamma_e = \{(\zeta,1,1) : \zeta^{2n} = 1\}$. To compute the quotient, note that Γ_e acts on $\mathrm{Sp}(1)$ by $(\zeta,1,1)\cdot q = \zeta^2 q$. This is equivalent to the action of the group $\mathbb{Z}_{2n}$ of $2n$th roots of unity acting on $\mathrm{Sp}(1)$ by $\zeta\cdot q = \zeta^2 q$ which is again equivalent to the action of the group of nth roots of unity $\mathbb{Z}_n$ acting by $\zeta\cdot q = \zeta q$. It follows that the quotient $\mathrm{Sp}(1)/\Gamma_e \approx \mathrm{Sp}(1)/\mathbb{Z}_n \approx L_n$, where L_n is the lens space $L_n(1)$ with n odd. Similarly, it is easy to check that the singular leaves are given by the two biquotients induced by the embeddings $T^2\times\mathrm{Sp}(1)\times S^1\to(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2)^2$ given by $(z,w,p,t)\mapsto(z^2,1,w,wz^{2n},p,p,t,1)$ and $(z,w,p,t)\mapsto(z^2,1,w,wz^{2n},p,p,1,t)$. The work in case B.3 in Chapter 3 implies that the singular leaves both have to be S^2.

To summarize, from the computations above, it follows that the C1BF group diagram in this case is

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2 & \\ T^2\times\mathrm{Sp}(1)\times S^1 \arrow[ur, equal] & & T^2\times\mathrm{Sp}(1)\times S^1 \arrow[ul, equal] \\ & T^2\times\mathrm{Sp}(1) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \qquad \begin{array}{c} M \simeq \mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2 \\ G/\!/H \simeq L_n(1);\ n \text{ odd} \\ G/\!/K^- \simeq S^2 \\ G/\!/K^+ \simeq S^2 \end{array}$$

where the embeddings are

$$K^- \to G\times G;\ (z,w,p,t)\mapsto (z^2,1,w,wz^{2n},p,p,t,1)$$

$$K^+ \to G\times G;\ (z,w,p,t)\mapsto (z^2,1,w,wz^{2n},p,p,1,t)$$

$$H \to G\times G;\ (z,w,p)\mapsto (z^2,1,w,wz^{2n},p,p,1,1)$$

In particular, in the case $n=1$ we have $G/\!/H \approx S^3$.

Structure 4.1.3. This construction will exhibit a non-cohomogeneity one structure on $S^2\times S^2$. This construction is essentially the same as structure 4.1.2. In particular, from DeVito's classification of compactly simply connected biquotients [DeV14], we can write $S^2\times S^2 = \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)/\!/T^2$ where the action of T^2 on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ is given by

$$(z,w)\cdot(r_1,r_2) = (z^2r_1, wz^n r_2\overline{z}^n)\ \ (n \text{ even})$$

where n is even. This is the same action as the one from structure 4.1.2, the only difference being that n is now even instead of odd. Thus the same argument as the previous case applies, and we get the same leaf structure for $S^2\times S^2$. In particular, the C1BF group diagram for $S^2\times S^2$ arising from this construction is

$$
\begin{tikzcd}[column sep=small]
& \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2 \arrow[dl, equal] \arrow[dr, equal] & \\
T^2\times\mathrm{Sp}(1)\times S^1 \arrow[dr, no head] & & T^2\times\mathrm{Sp}(1)\times S^1 \arrow[dl, no head] \\
& T^2\times\mathrm{Sp}(1) &
\end{tikzcd}
\qquad
\begin{array}{c}
M\simeq S^2\times S^2\\
G//H\simeq L_n(1);\ n \text{ even}\\
G//K^-\simeq S^2\\
G//K^+\simeq S^2
\end{array}
$$

where the embeddings are

$$K^- \to G\times G;\ (z,w,p,t)\mapsto (z^2,1,w,wz^{2n},p,p,t,1)$$

$$K^+ \to G\times G;\ (z,w,p,t)\mapsto (z^2,1,w,wz^{2n},p,p,1,t)$$

$$H \to G\times G;\ (z,w,p)\mapsto (z^2,1,w,wz^{2n},p,p,1,1)$$

Structure 4.1.4. This construction will exhibit a non-cohomogeneity one structure on $\mathbb{C}\mathrm{P}^2$. The construction requires some knowledge of Eschenburg spaces which we review now. For more details, see [GSZ06]. Let $\overline{a}=(a_1,a_2,a_3)$ and $\overline{b}=(b_1,b_2,b_3)$ be triples of integers such that $\sum a_i=\sum b_i=c$ for some integer c. Let

$$S^1_{\overline{a},\overline{b}}=\left\{\left(\mathrm{diag}(z^{a_1},z^{a_2},z^{a_3}),\mathrm{diag}(z^{b_1},z^{b_2},z^{b_3})\right)\,:\,z\in U(1)\right\}\subset \mathrm{SU}(3)\times\mathrm{SU}(3)$$

The action of $S^1_{\overline{a},\overline{b}}$ on SU(3) by left and right translation is free if and only if for every permutation $\sigma\in S_3$, $\gcd(a_1-b_{\sigma(1)},a_2-b_{\sigma(2)})=1$. In this case, the resulting 7-manifold $E_{\overline{a},\overline{b}}=\mathrm{SU}(3)//S^1_{\overline{a},\overline{b}}$ is called an *Eschenburg space.* The Eschenburg spaces contain the homogeneous *Aloff-Wallach spaces* $\mathscr{A}$, which correspond to the case that $b_i=0$, $i\in\{1,2,3\}$. Let $\mathscr{E}$ denote the set of Eschenburg spaces. Grove and Ziller noticed that $\mathscr{E}$ contains an infinite family $\mathscr{E}_1$ of cohomogeneity one manifolds

$$\mathscr{E}_1=\left\{E_p=E_{\overline{a},\overline{b}}\in\mathscr{E}\,:\,\overline{a}=(1,1,p),\ \overline{b}=(0,0,p+2),\ p>0\right\}$$

Observe that the actions by S^1_p and S^1_{-p-1} on SU(3) are equivalent. It follows that E_1 is the unique space in $\mathscr{E}_1 \cap \mathscr{A}$. Note that each E_p has cohomogeneity one since the action of $\mathrm{U}(2) \times \mathrm{SU}(2) \subset \mathrm{SU}(3) \times \mathrm{SU}(3)$ on SU(3) by left and right translation commutes with the S^1_p action, and that $\mathrm{U}(2) \backslash \mathrm{SU}(3)/\mathrm{SU}(2) = \mathbb{C}\mathrm{P}^2/\mathrm{SU}(2)$, which is an interval. The following proposition gives the group diagrams for these cohomogeneity one manifolds

Proposition 4.1.1. *(Grove-Shankar-Ziller)*
The cohomogeneity one action of $G = SU(2) \times SU(2)$ on E_p has principal isotropy group $H = \left\{(\pm I)^{p+1}, (\pm I)^p\right\} \approx \mathbb{Z}_2$ and singular isotropy groups $K^- = \Delta SU(2) \cdot H$ and $K^+ = S^1_{(p+1,p)}$ embedded with slope $(p+1, p)$ in a maximal torus of $SU(2) \times SU(2)$.

We are interested in the Aloff-Wallach space E_1 and the Eschenburg space E_2. Observe that E_1 and E_2 admit an SO(3) action in the following way. Write $E_1 = \mathrm{SU}(3)//S^1_1$ and $E_2 = \mathrm{SU}(3)//S^1_2$ where

$$S^1_1 = \left\{\left(\mathrm{diag}(z,z,z), \mathrm{diag}(1,1,z^3)\right) \,:\, z \in U(1)\right\} \subset \mathrm{SU}(3) \times \mathrm{SU}(3)$$

$$S^1_2 = \left\{\left(\mathrm{diag}(z,z,z^2), \mathrm{diag}(1,1,z^4)\right) \,:\, z \in U(1)\right\} \subset \mathrm{SU}(3) \times \mathrm{SU}(3)$$

Then the $\mathrm{SU}(2) \subset \mathrm{SU}(3)$ action on $E_1 = \mathrm{SU}(3)//S^1_1$ given by $A \cdot [X] = [AX]$ is well defined with kernel $\mathbb{Z}_2 = \{\pm I\}$, so the effective action is by $\mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(3)$. The authors show that in the case of E_1 and E_2, the SO(3) action is free. Moreover, this action preserves the orbits of the cohomogeneity one actions on these spaces. Thus taking the quotient of E_1 and E_2 by the SO(3) action we obtain a C1BF structure on each of these spaces. We will compute the leaf structure of these C1BFs.

The above proposition tells us that the group diagram for E_1 is

$$\begin{array}{ccc} & \mathrm{SU}(2)\times\mathrm{SU}(2) & \\ \diagup & & \diagdown \\ \Delta\mathrm{SU}(2)\cdot H & & S^1_{(2,1)} \\ \diagdown & & \diagup \\ & H & \end{array}$$

where $H = \{(I,I),(I,-I)\}$. Observe that since H commutes with everything in $\Delta\mathrm{SU}(2)$, it follows that $\Delta\mathrm{SU}(2)\cdot H$ is just the group of all products of things in $\Delta\mathrm{SU}(2)$ and H. From this observation, it follows that

$$\Delta\mathrm{SU}(2)\cdot H = \{(A,A)\ :\ A\in\mathrm{SU}(2)\}\cup\{(A,-A)\ :\ A\in\mathrm{SU}(2)\}$$

Therefore, to compute the singular orbit corresponding to $\Delta\mathrm{SU}(2)\cdot H$, we use proposition 1.1.2. In particular,

$$\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/\big(\Delta\mathrm{SU}(2)\cdot H\big)\approx\Big(\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/\Delta\mathrm{SU}(2)\Big)/\mathbb{Z}_2$$

Clearly $\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/\Delta\mathrm{SU}(2)\approx S^3$. It follows then that the quotient by $\mathbb{Z}_2$ is $\mathrm{SO}(3)\approx\mathbb{RP}^3$. Thus the singular orbit $\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/\big(\Delta\mathrm{SU}(2)\cdot H\big)\approx\mathbb{RP}^3$. For the other singular orbit, it follows from Barden's classification of simply connected 5 manifolds [Bar65] that $\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/S^1_{(2,1)}\approx S^3\times S^2$. For the principal orbit, $H=\{(I,I),(I,-I)\}$ so $\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/H$ is the quotient by the action of H on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ by $(I,\pm I)\cdot(X,Y)=(X,\pm Y)$ which is really just the antipodal $\mathbb{Z}_2$ action on the second factor, hence $\big(\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/H\approx\mathrm{SU}(2)\times\mathrm{SO}(3)\approx S^3\times\mathbb{RP}^3$. Similarly, the above proposition tells us that the group diagram for E_2 is

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{SU}(2)\times\mathrm{SU}(2) \arrow[dl, dash] \arrow[dr, dash] & \\
\Delta\mathrm{SU}(2)\cdot H \arrow[dr, dash] & & S^1_{(3,2)} \arrow[dl, dash] \\
 & H &
\end{tikzcd}
$$

and the same calculations as above tell us that the orbits are the same as in the case of E_1.

Now, to get the leaves, it is not too difficult to see that principal leaf is

$$\big(\mathrm{SU}(2)\times\mathrm{SO}(3)\big)/\mathrm{SO}(3)\approx\mathrm{SU}(2)\approx S^3$$

and the singular leaves are $\mathrm{SO}(3)/\mathrm{SO}(3)=pt$ and $S^3\times S^2/\mathrm{SO}(3)\approx S^2$. A C1BF with this leaf structure is necessarily $\mathbb{C}\mathrm{P}^2$.

4.2 C1BFs in Dimension 5

He we will give explicit models of C1BFs in dimension 5 which realize the leaf structures determined in Chapter 3.

Structure 4.2.1. In this structure we exhibit an non-cohomogeneity one C1BF structure on the sphere S^5. We start by embedding $\mathrm{U}(2)\to\mathrm{SU}(3)$ via the standard block diagonal embedding $\mathrm{S}(\mathrm{U}(1)\mathrm{U}(2))$. Then $G=\mathrm{U}(2)\times\mathrm{U}(2)$ acts on $\mathrm{SU}(3)$ by left and right translation $(A,B)\cdot X=AXB^*$. Consider the matrix

$$X=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & 0 & -1\end{pmatrix}\in\mathrm{SU}(3)$$

We show that the orbit of X under the action of G has codimension one, hence the action above is cohomogeneity one. Indeed, consider arbitrary elements of SU(2) $\subset$ SU(3)

$$A = \begin{pmatrix} z & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \qquad\qquad B = \begin{pmatrix} w & 0 & 0 \\ 0 & r & s \\ 0 & t & v \end{pmatrix}$$

where $z = \frac{1}{ad-bc}$ and $w = \frac{1}{rv-st}$ and $z, w \in \mathrm{U}(1)$. It is easy to compute

$$AXB^* = \begin{pmatrix} \frac{z\overline{w}}{\sqrt{2}} & \frac{z\overline{r}}{\sqrt{2}} & z\overline{t} \\ \frac{a\overline{w}}{\sqrt{2}} & -\frac{a\overline{r}}{\sqrt{2}} - b\overline{s} & -\frac{a\overline{t}}{\sqrt{2}} - b\overline{v} \\ \frac{c\overline{w}}{\sqrt{2}} & -\frac{c\overline{r}}{\sqrt{2}} - d\overline{s} & -\frac{c\overline{t}}{\sqrt{2}} - d\overline{v} \end{pmatrix} = X$$

if and only if

$$A = B = \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \overline{z}^2 \end{pmatrix}$$

It follows that the isotropy G_X is a circle and the orbit G/G_X is codimension one as desired. Similarly, it is easy to see that the identity matrix I and the matrix

$$Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

do not lie in the same orbit and $G_I = \Delta\mathrm{U}(2) \subset \mathrm{SU}(3) \times \mathrm{SU}(3)$. Moreover, a similar computation to the principal isotropy group above shows that

$$G_Y = \left\{ \left(\mathrm{diag}(\overline{zw}, z, z), \mathrm{diag}(w, z, \overline{zw}) \right) \; : \; z, w \in \mathrm{U}(1) \right\} \approx T^2$$

so the orbits of I and Y are singular orbits. As a simplification, we can write down the group diagram of the above cohomogeneity one action as

$$\begin{tikzcd}[column sep=small] & \mathrm{U}(2)\times\mathrm{U}(2) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{U}(2) & & T^2 \\ & S^1 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

where $\mathrm{U}(2)$ is embedded diagonally and $S^1 \times S^1$ is embedded via

$$(z, w) \mapsto \left(\mathrm{diag}(z, w), \mathrm{diag}(w, \overline{zw}\,) \right)$$

and S^1 is embedded via $z \mapsto (z, \overline{z}^2)$.

Now, consider the restriction of the cohomogeneity one action above to $\mathrm{SU}(2) \times \{I\} \subset \mathrm{U}(2) \times \mathrm{U}(2)$, embedded as above. It is easy to see that the restriction of the above action to this subgroup is free, hence the quotient of $\mathrm{SU}(3)$ by this subaction is a C1BF and, in fact, the quotient is diffeomorphic to $\mathrm{SU}(3)/\mathrm{SU}(2)$ which is necessarily diffeomorphic S^5. We will now compute the leaves of the action using the same method as in structure 4.1.1. The principal leaf will be the quotient of $\mathrm{U}(2) \times \mathrm{U}(2)$ by the biquotient action induced by the embedding $\mathrm{SU}(2) \times S^1 \to (\mathrm{U}(2) \times \mathrm{U}(2))^2$ via

$$(A, z) \mapsto \left(A, I, \mathrm{diag}(z, \overline{z}^2), \mathrm{diag}(z, \overline{z}^2) \right)$$

or, more explicitly, the biquotient action of $\Gamma = \mathrm{SU}(2) \times S^1$ on $\mathrm{U}(2) \times \mathrm{U}(2)$ by $(A, z) \cdot (X, Y) = \left(AX\mathrm{diag}(z, \overline{z}^2), Y\mathrm{diag}(z, \overline{z}^2) \right)$. Clearly this action is transitive on the first $\mathrm{U}(2)$

factor. Thus, by proposition 1.3.7

$$(\mathrm{U}(2)\times\mathrm{U}(2))/(\mathrm{SU}(2)\times S^1)\approx\mathrm{U}(2)/\Gamma_I$$

where Γ_I is the isotropy of the identity $I\in\mathrm{U}(2)$ with respect to the action of Γ on the first U(2) factor. But $(A,z)\cdot I = A\mathrm{diag}(\overline{z},z^2) = I$ if and only if $A=\mathrm{diag}(z,\overline{z}^2)$. But $A\in\mathrm{SU}(2)$ so $\det A=\overline{z}=1$, hence $z=1$ and $A=I$. Thus Γ_I is trivial, so the principal leaf is U(2) which, as a manifold, is diffeomorphic to $S^3\times S^1$. Now let us compute the singular leaves.

The singular leaves are the biquotients induced by the embedding $\mathrm{SU}(2)\times\mathrm{U}(2)\to(\mathrm{U}(2)\times\mathrm{U}(2))^2$ given by $(A,B)\mapsto(A,I,B,B)$ and the embedding $\mathrm{SU}(2)\times T^2\to(\mathrm{U}(2)\times\mathrm{U}(2))^2$ given by $(A,z,w)\mapsto\big(A,I,\mathrm{diag}(z,w),(\overline{zw}\,)\big)$. The first induces the action of $L=\mathrm{SU}(2)\times\mathrm{U}(2)$ on $\mathrm{U}(2)\times\mathrm{U}(2)$ given by

$$(A,B)\cdot(X,Y)=(AXB^*,YB^*)$$

and the second induces the action of $F=\mathrm{SU}(2)\times T^2$ on $\mathrm{U}(2)\times\mathrm{U}(2)$ by

$$(A,z,w)\cdot X=\big(AX\mathrm{diag}(\overline{z},\overline{w}\,),Y\mathrm{diag}(\overline{z},zw)\big)$$

For the first of these actions, clearly the action is transitive on the first factor, so again by 1.3.7

$$(\mathrm{U}(2)\times\mathrm{U}(2))/(\mathrm{SU}(2)\times\mathrm{U}(2))\approx\mathrm{U}(2)/\Delta\mathrm{SU}(2)$$

where $\Delta\mathrm{SU}(2)$ is the isotropy of the identity $I\in\mathrm{U}(2)$ with respect to the action of L on the first U(2) factor. Clearly this quotient is diffeomorphic to $\mathrm{U}(2)/\mathrm{SU}(2)\approx S^1$. For the other singular leaf, again the action of F on the first factor is transitive, so by the same lemma

$$(\mathrm{U}(2)\times\mathrm{U}(2))/(\mathrm{SU}(2)\times T^2)\approx \mathrm{U}(2)/F_I$$

where F_I is the isotropy of the identity $I\in\mathrm{U}(2)$ with respect to the action of F on the first factor. It is easy to compute

$$F_I=\left\{\left(\mathrm{diag}(z,\overline{z}),z,\overline{z}\right)\;:\;z\in\mathrm{U}(1)\right\}\approx\mathrm{U}(1)$$

Now, the quotient $\mathrm{U}(2)/F_I$ is clearly diffeomorphic to $\mathrm{U}(2)/\mathrm{U}(1)\approx S^3$.

To summarize, from the computations above, it follows that the C1BF diagram in this case is

$$\begin{tikzcd}[column sep=small] & \mathrm{U}(2)\times\mathrm{U}(2) \arrow[dl, equal] \arrow[dr, equal] & \\ \mathrm{SU}(2)\times\mathrm{U}(2) \arrow[dr, no head] & & \mathrm{SU}(2)\times T^2 \arrow[dl, no head] \\ & \mathrm{SU}(2)\times S^1 & \end{tikzcd} \qquad \begin{array}{c} M\simeq S^5 \\ G//H\simeq S^3\times S^1 \\ G//K^-\simeq S^1 \\ G//K^+\simeq S^3 \end{array}$$

where the embeddings are

$$K^-\to G\times G;\ (A,B)\mapsto(A,I,B,B)$$

$$K^+\to G\times G;\ (A,z,w)\mapsto\left(A,I,\mathrm{diag}(z,w),(\overline{z}\,\overline{w}\,)\right)$$

$$H\to G\times G;\ (A,z)\mapsto\left(A,I,\mathrm{diag}(z,\overline{z}^2),\mathrm{diag}(z,\overline{z}^2)\right)$$

Structure 4.2.2. In this structure we exhibit several non-cohomogeneity one C1BF structure on $S^3\times S^2$ which arise by varying the parameters in the equation below. From DeVito's classification of simply connected biquotients [DeV14], we can write $S^3\times S^2=(\mathrm{Sp}(1)\times\mathrm{Sp}(1))//S^1$ where the action of S^1 on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ is given by

$$z\cdot(r_1,r_2)=(z^a r_1\overline{z}^c,z^b r_2\overline{z}^d) \tag{4.2.1}$$

where a, b, c, d are integers with $\gcd(a, b, c, d) = 1$ and $\gcd(a^2 - c^2, b^2 - d^2) = 1$. Writing $r_2 = a + bj$, action 4.2.1, becomes

$$z \cdot (r_1, a + bj) = (z^a r_1, \overline{z}^c, z^{b-d} x + z^{b+d} yj) \tag{4.2.2}$$

We again consider the same cohomogeneity one action (4.1.3) that we used in structure 4.1.1. Comparing action (4.1.3) to action (4.2.2), we see that if we define a homomorphism $\Phi : S^1 \to \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ by $\Phi(z, w) = (z^a, z^c, z^{b-d}, z^{b+d})$, then the restriction of action (4.1.3) to the circle $\mathrm{Im}(\Phi)$ is precisely the action (4.2.2). We have thus exhibited $S^3 \times S^2$ as a C1BF.

We now wish to compute the group diagram and leaf structure for $S^3 \times S^2$ arising from this construction. Using the principal and singular orbits of action (4.1.3), we see that the principal leaf is the biquotient induced by the homomorphism $S^1 \times \mathrm{Sp}(1) \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1))^2$ by $(z, p) \mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, 1, 1)$. More precisely, the principal leaf is the quotient by the action of $S^1 \times \mathrm{Sp}(1)$ on $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ by

$$(z, p) \cdot (q_1, q_2, w_1, w_2) = (z^a q_1 \overline{p}, z^c q_2 \overline{p}, z^{b-d} w_1, z^{b+d} w_2) \tag{4.2.3}$$

To compute the quotient, there are several cases to consider. In particular, we would like to be able to assume that at least one of a or c is nonzero and that one of $b - d$ and $b + d$ is nonzero. The cases where a and c are both zero, as well as the cases where $b - d$ and $b + d$ are both zero will be referred to as *exceptional cases* and the cases where at least one of a or c is nonzero and at least one of $b - d$ or $b + d$ is nonzero will be referred to as *standard cases*. We will first deal with the exceptional cases since these are easier.

When $a = c = 0$ or $b - d = b + d = 0$,the gcd conditions imply that the following list gives the only permissible values of the parameters:

1. $(a,b,c,d) = (0,\pm 1,0,0)$
2. $(a,b,c,d) = (0,0,0,\pm 1)$
3. $(a,b,c,d) = (\pm 1,0,0,0)$
4. $(a,b,c,d) = (0,0,\pm 1,0)$

It is easy to see that the family of actions given by (1) and (2) are equivalent and that the family of actions given by (3) and (4) are equivalent. Thus we need only consider the cases where $(a,b,c,d) = (0,1,0,0)$ and $(a,b,c,d) = (1,0,0,0)$ separately.

We will also like to compute the singular leaves in the exceptional cases. We note that the singular leaves are the biquotients induced by the embeddings $S^1 \times \mathrm{Sp}(1) \times S^1 \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2)^2$ by $(z_1, p, z_2) \mapsto (z_1^a, z_1^c, z_1^{b-d}, z_1^{b+d}, p, p, z_2, 1)$ and $S^1 \times \mathrm{Sp}(1) \times S^1 \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2)^2$ by $(z_1, p, z_2) \mapsto (z_1^a, z_1^c, z_1^{b-d}, z_1^{b+d}, p, p, 1, z_2)$ or, more precisely, the actions of $S^1 \times \mathrm{Sp}(1) \times S^1$ on $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ given, respectively, by the following

$$(z_1, p, z_2) \cdot (q_1, q_2, w_1, w_2) = (z_1^a q_1 \overline{p}, z_1^c q_2 \overline{p}, z_1^{b-d} w_1 \overline{z}_2, z_1^{b+d} w_2) \tag{4.2.4}$$

$$(z_1, p, z_2) \cdot (q_1, q_2, w_1, w_2) = (z_1^a q_1 \overline{p}, z_1^c q_2 \overline{p}, z_1^{b-d} w_1, z_1^{b+d} w_2 \overline{z}_2) \tag{4.2.5}$$

Exceptional Case 1: $(a,b,c,d) = (0,1,0,0)$

In this case, the principal leaf is the quotient by the action of $S^1 \times \mathrm{Sp}(1)$ on $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ given by

$$(z, p) \cdot (q_1, q_2, w_1, w_2) = (q_1 \overline{p}, q_2 \overline{p}, z w_1, z w_2)$$

This action is clearly transitive on the first and third factors so by proposition 1.3.7 we have that the quotient is diffeomorphic to $(\mathrm{Sp}(1) \times S^1)/\Gamma_e$ where Γ_e is the isotropy of the identity $e = (1, 1)$ of the transitive factors. But it is easy to see that Γ_e is trivial, so we have

$$(\mathrm{Sp}(1) \times S^1)/\Gamma_e \approx S^3 \times S^1$$

For the first singular leaf, note that in this case action 4.2.4 becomes

$$(z_1, p, z_2) \cdot (q_1, q_2, w_1, w_2) = (q_1\overline{p}, q_2\overline{p}, z_1 w_1 \overline{z}_2, z_1 w_2)$$

This action is transitive on the last three factors, so again by 1.3.7 we have that the quotient is diffeomorphic to $\mathrm{Sp}(1)/\Gamma_e$ where Γ_e is the isotropy on the identity $e = (1, 1, 1)$ on the transitive factors. It is again easy to see that Γ_e is trivial so

$$\mathrm{Sp}(1)/\Gamma_e \approx S^3$$

Essentially the same computation shows that the other singular leaf is S^3 as well. To summarize, from the computations above, it follows that the C1BF group diagram for $S^3 \times S^2$ in this exceptional case is

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2 \arrow[dl, dash, double] \arrow[dr, dash, double] & \\ S^1\times\mathrm{Sp}(1)\times S^1 \arrow[dr, dash] & & S^1\times\mathrm{Sp}(1)\times S^1 \arrow[dl, dash] \\ & S^1\times\mathrm{Sp}(1) & \end{tikzcd} \qquad \begin{gathered} M \simeq S^3 \times S^2 \\ G//H \simeq S^3 \times S^1 \\ G//K^- \simeq S^3 \\ G//K^+ \simeq S^3 \end{gathered}$$

where the embeddings are

$$K^- \to G \times G;\ (z, p, w) \mapsto (1, 1, z, z, p, p, w, 1)$$

$$K^+ \to G \times G;\ (z, p, w) \mapsto (1, 1, z, z, p, p, 1, w)$$

$$H \to G \times G;\ (z, p) \mapsto (1, 1, z, z, p, p, 1, 1)$$

Exceptional Case 2: $(a, b, c, d) = (1, 0, 0, 0)$

In this case, the principal leaf is the quotient by the action of $S^1 \times \mathrm{Sp}(1)$ on $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2$ given by

$$(z, p) \cdot (q_1, q_2, w_1, w_2) = (zq_1\overline{p}, q_2\overline{p}, w_1, w_2)$$

This is transitive on the third factor so by proposition 1.3.7 the quotient is diffeomorphic to $(\mathrm{Sp}(1) \times T^2)/\Gamma_e$ where Γ_e is the isotropy on $e = 1$ of the transitive factor. It is easy to see that $\Gamma_e = \{(z, 1) : z \in S^1\} \approx S^1$ and that the quotient is

$$(\mathrm{Sp}(1) \times T^2)/\Gamma_e \approx S^2 \times T^2$$

For the first singular leaf, action 4.2.4 reduces to

$$(z_1, p, z_2) \cdot (q_1, q_2, w_1, w_2) = (z_1q_1\overline{p}, q_2\overline{p}, w_1\overline{z}_2, w_2)$$

which, using essentially the same argument we see that the singular leaf corresponding to this action is $S^2 \times S^1$. Similarly, the action for the other singular leaf reduces to

$$(z_1, p, z_2) \cdot (q_1, q_2, w_1, w_2) = (z_1q_1\overline{p}, q_2\overline{p}, w_1, w_2\overline{z}_2)$$

and it is easy to compute the singular leaf in this case is also $S^2 \times S^1$. To summarize, from the computations above, it follows that the C1BF group diagram for $S^3 \times S^2$ in this exceptional case is

$$\begin{tikzcd}[column sep=tiny]
& \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2 \arrow[dl, dash, shift left] \arrow[dl, dash, shift right] \arrow[dr, dash, shift left] \arrow[dr, dash, shift right] & \\
S^1\times\mathrm{Sp}(1)\times S^1 & & S^1\times\mathrm{Sp}(1)\times S^1 \\
& S^1\times\mathrm{Sp}(1) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{array}{c}
M \simeq S^3\times S^2\\
G//H \simeq S^2\times T^2\\
G//K^- \simeq S^2\times S^1\\
G//K^+ \simeq S^2\times S^1
\end{array}$$

where the embeddings are

$$K^- \to G\times G;\ (z,p,w)\mapsto(z,1,1,1,p,p,w,1)$$

$$K^+ \to G\times G;\ (z,p,w)\mapsto(z,1,1,1,p,p,1,w)$$

$$H \to G\times G;\ (z,p)\mapsto(z,1,1,1,p,p,1,1)$$

We now examine several standard cases which are much more complicated than the exceptional cases.

Standard Case 1: In this case we will assume $b-d\neq 0$, at least one of a and c is nonzero, and additionally that $a\neq \pm c$. The principal leaf is given by action (4.2.3) above. In this case, since $b-d\neq 0$, the action is clearly transitive on the first and third factors, so by Proposition 1.3.7 we have

$$(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2)/S^1\times\mathrm{Sp}(1)\simeq(\mathrm{Sp}(1)\times S^1)/\Gamma_e$$

where Γ_e is the isotropy of the action on the identity $e=(1,1)$ of the transitive factors. It is easy to compute that

$$\Gamma_e=\left\{(\zeta,\zeta^a):\zeta^{b-d}=1\right\}\simeq\mathbb{Z}_{b-d}=\{\zeta:\zeta^{b-d}=1\}\subset S^1$$

Therefore it follows that the principal leaf is given by the quotient of the action of $\mathbb{Z}_{b-d}$

on $\mathrm{Sp}(1)\times S^1$ by

$$\zeta\cdot(q,w)=(\zeta^c q\overline{\zeta}^a,\zeta^{b+d}w) \tag{4.2.6}$$

The quotient of the above action depends on the parameter values, so we break the analysis into a couple of cases.

Standard Case 1.1: In this case we will assume $b-d\neq 0$, $b+d=0$, and at least one of a and c is nonzero, and additionally that $a\neq\pm c$. In this case, the quotient which determines the principal leaf reduces to the action of $\mathbb{Z}_{b-d}$ on $\mathrm{Sp}(1)\times S^1$ by

$$\zeta\cdot(q,w)=(\zeta^c q\overline{z}^a,w)$$

In particular, the action is trivial on the second factor so

$$(\mathrm{Sp}(1)\times S^1)/\mathbb{Z}_{b-d}\simeq\left(\mathrm{Sp}(1)/\mathbb{Z}_{b-d}\right)\times S^1$$

But $b+d=0$ implies that $b-d=2b$ hence $b^2-d^2=0$. Hence $1=\gcd(a^2-c^2,0)$ implies that $a^2-c^2=\pm 1$ so $(a,c)=(\pm 1,0)$. Thus we assume $(a,c)=(1,0)$ since the parameter values will clearly have the same quotients. It follows that the quotient for the principal leaf is the quotient by the action of $\mathbb{Z}_{2b}$ on $\mathrm{Sp}(1)\times S^1$ by $\zeta\cdot(q,w)=(q\overline{\zeta},w)$. It follows immediately that the principal leaf is

$$(\mathrm{Sp}(1)\times S^1)/\mathbb{Z}_{2b}\simeq L_{2b}(1)\times S^1$$

where b is any positive integer. Now let us compute the singular leaves under the assumptions of Standard Case 1.1. The quotients for the singular leaves are given by actions (4.2.4) and (4.2.5) above. Under the assumptions of this case however, the action

(4.2.4) reduces to

$$(z_1, p, z_2)\cdot(q_1, q_2, w_1, w_2) = (z_1 q_1 \overline{p}, q_2 \overline{p}, z_1^{2b} w_1 \overline{z}_2, w_2)$$

The action is transitive on the first and third factors so by Proposition 1.3.7 we have that this singular leaf is given by the quotient

$$(\mathrm{Sp}(1)\times \mathrm{Sp}(1)\times T^2)/(S^1\times \mathrm{Sp}(1)\times S^1) \simeq (\mathrm{Sp}(1)\times S^1)/\Gamma_e$$

where Γ_e is the isotropy of the identity $(1,1)$ of the transitive factors. It is easy to compute that

$$\Gamma_e = \left\{(z, z, z^{2b}) : z\in S^1\right\} \simeq S^1$$

so it follows that the action of Γ_e on $\mathrm{Sp}(1)\times S^1$ is given by

$$z\cdot(q, w) = (q\overline{z}, w).$$

Therefore, this singular leaf is clearly diffeomorphic to

$$(\mathrm{Sp}(1)\times S^1)/S^1 \simeq S^2\times S^1.$$

For the other singular leaf, this is the quotient by (4.2.5) which under our assumptions reduces to

$$(z_1, p, z_2)\cdot(q_1, q_2, w_1, w_2) = (z_1 q_1 \overline{p}, q_2 \overline{p}, z_1^{2b} w_1, w_2 \overline{z}_2)$$

This action is transitive on the first, third, and fourth factors, so by Proposition 1.3.7 we have

$$(\mathrm{Sp}(1)\times \mathrm{Sp}(1)\times T^2)/(S^1\times \mathrm{Sp}(1)\times S^1) \simeq \mathrm{Sp}(1)/\Gamma_e$$

where Γ_e is the isotropy of the identity $(1,1,1)$ of the transitive factors. It is easy to compute here that

$$\Gamma_e = \left\{(\zeta, \zeta, 1) : \zeta^{2b} = 1\right\} \simeq \mathbb{Z}_{2b} \subset S^1$$

It follows that this singular leaf is equivalent to the quotient of $\mathbb{Z}_{2b}$ on $\mathrm{Sp}(1)$ by $\zeta \cdot q = q\bar{\zeta}$, so clearly this singular leaf is diffeomorphic to

$$\mathrm{Sp}(1)/\mathbb{Z}_{2b} \simeq L_{2b}(1)$$

To summarize, from the computations above, it follows that the C1BF group diagram for $S^3 \times S^2$ in Standard case 1.1 is

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2 \arrow[dl, dash, double] \arrow[dr, dash, double] & \\ S^1 \times \mathrm{Sp}(1) \times S^1 & & S^1 \times \mathrm{Sp}(1) \times S^1 \\ & S^1 \times \mathrm{Sp}(1) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \qquad \begin{gathered} M \simeq S^3 \times S^2 \\ G//H \simeq L_{2b}(1) \times S^1 \\ G//K^- \simeq S^2 \times S^1 \\ G//K^+ \simeq L_{2b}(1) \end{gathered}$$

where the embeddings are

$$K^- \to G \times G;\ (z, p, w) \mapsto (z, 1, z^{2b}, 1, p, p, w, 1)$$

$$K^+ \to G \times G;\ (z, p, w) \mapsto (z, 1, z^{2b}, 1, p, p, 1, w)$$

$$H \to G \times G;\ (z, p) \mapsto (z, 1, z^{2b}, 1, p, p, 1, 1)$$

Standard Case 1.2:

Now we assume that in addition to the assumptions made for Standard Case 1 above, that we also have $b + d \neq 0$. It follows that the action (4.2.6) for the principal leaf is now nontrivial on the second factor. To determine the quotient, we write $q = x + yj$ for $x, y \in \mathbb{C}$. Then $\zeta^c q \bar{\zeta}^a = \zeta^c (x + yj) \bar{\zeta}^a = \zeta^{c-a} x + \zeta^{c+a} yj = x + yj$ if and only if at least one of ζ^{c-a} or ζ^{c+a} is 1. In particular, ζ is a $(b-d)^{th}$ root of unity and either a $(c+a)^{th}$ or $(c-a)^{th}$ root of unity. But $\gcd(c\pm a, b-d) \leq \gcd(a^2-c^2, b^2-d^2) = 1$ so $\gcd(c\pm a, b-d) = 1$ and hence $\zeta = 1$ using the fact that if $\zeta^n = \zeta^m = 1$, then $\zeta^{\gcd(m,n)} = 1$. Thus the action giving the principal leaf is free when restricted to the action on the first coordinate. Thus the associated bundle construction implies that we get a bundle

$$S^1 \to \mathrm{Sp}(1) \times_{\mathbb{Z}_{b-d}} S^1 \to S^3/\mathbb{Z}_{b-d}$$

That is, the quotient is a circle bundle over a lens space. It follows from Theorem 3.2.2 that the quotient is diffeomorphic to $L \times S^1$ for some lens space L.

To determine the the lens space L, we consider the action of $\mathbb{Z} \times \mathbb{Z}$ on $S^3 \times \mathbb{R}$ (the universal cover of $L \times S^1$) given by

$$(n,m)\cdot(p,t) = \left(z_0^n \star p, t+m+n(\frac{b+d}{b-d})\right) \tag{4.2.7}$$

where $z_0 = e^{\frac{2\pi i}{b-d}}$ is the generator of $\mathbb{Z}_{b-d}$ and $\star$ is any free action of $\mathbb{Z}_{b-d}$ on S^3.

Lemma 4.2.1. *The action (4.2.7) is effectively free with kernel K generated by* $(b-d, -(b+d))$. *In particular, the action is effectively a free action by* $\mathbb{Z}_e \times \mathbb{Z}$, *where* $e = \gcd(b-d, b+d)$.

Proof. Suppose $(n,m)\cdot(p,t) = (p,t)$. This implies $\left(z_0^n \star p, t+m+n(\frac{b+d}{b-d})\right) = (p,t)$. Thus we have the following two conditions hold:

1. $z_0^n \star p = p$
2. $t+m+n(\frac{b+d}{b-d}) = t$

Condition (1) implies $z_0^n = 1$ hence $e^{\frac{2\pi i n}{b-d}} = 1$ so $b-d = n$ and, therefore $n = k(b-d)$ but then condition (2) implies $m + k(b+d) = 0$ so $m = k(-(b+d))$. Thus $(n,m) =$

$\big(k(b-d), k(-(b+d))\big)$ for some $k \in \mathbb{Z}$. In other words, the kernel K is generated by $(b-d, -(b+d))$. Finally, it follows from Smith Normal Form that $(\mathbb{Z}\times\mathbb{Z})/K \simeq \mathbb{Z}_e \times \mathbb{Z}$ where $e = \gcd(b-d, b+d)$. □

We now claim that $(S^3 \times \mathbb{R})/(\mathbb{Z}\times\mathbb{Z})$ is diffeomorphic to the principal leaf $(\mathrm{Sp}(1) \times S^1)/\mathbb{Z}_{b-d}$ provided we choose $\star$ to be the action given by the action on the first coordinate of action (4.2.6). Now, since the $\mathbb{Z}$ actions on $S^3 \times \mathbb{R}$ given by n and m commute, we can compute the quotient by first taking the quotient by the action of $\mathbb{Z}$ by m, followed by the quotient of the action of $\mathbb{Z}$ given by n. The former is the quotient of the action of $\mathbb{Z}$ on $S^3 \times \mathbb{R}$ given by $m \cdot (p, t) = (p, t+m)$. Then we have that the quotient by this action is

$$(S^3 \times \mathbb{R})/\mathbb{Z} \simeq S^3 \times (\mathbb{R}/\mathbb{Z}) \simeq S^3 \times S^1.$$

Now, the $\mathbb{Z}$ factor corresponding to n acts on $S^3 \times \mathbb{R}$ by $n \cdot (p, t) = \big(z_0^n \star p, t + n + (\frac{b+d}{b-d})\big)$ so acts on $S^3 \times (\mathbb{R}/\mathbb{Z})$ by

$$n \cdot (p, [t]) = \Big(z_0^n \star p, \big[t + n(\frac{b+d}{b-d})\big]\Big) \tag{4.2.8}$$

Let $F : S^3 \times (\mathbb{R}/\mathbb{Z}) \to S^3 \times S^1$ be the diffeomorphism $(p, [t]) \mapsto (p, e^{2\pi i t})$. We observe that this diffeomorphism is equivariant with respect to the action (4.2.8) and the action of $\mathbb{Z}$ on $S^3 \times S^1$ defined by

$$n \diamond (p, w) = (z_0^n \star p, (z_0^n)^{b+d} w)$$

But z_0 is a generator of $\mathbb{Z}_{b-d} \subset S^1$ and the action $\star$ was defined to be the action given by the action on the first coordinate of of action (4.2.6), so this is in fact the same as the action (4.2.6). This shows that $(S^3 \times \mathbb{R})/\mathbb{Z}\times\mathbb{Z}$ is diffeomorphic to the principal leaf $(\mathrm{Sp}(1) \times S^1)/\mathbb{Z}_{b-d}$. In particular, putting this all together, we have that the principal leaf is

$$(\mathrm{Sp}(1)\times S^1)/\mathbb{Z}_{b-d}\simeq (S^3\times\mathbb{R})/(\mathbb{Z}\times\mathbb{Z})\simeq (S^3\times\mathbb{R})/(\mathbb{Z}_e\times\mathbb{Z})\simeq L_e(q)\times S^1$$

for some q, where $e=\gcd(b-d,b+d)$. To determine this value of q, we recall that the action of $\mathbb{Z}_{b-d}$ on S^3 which determines the lens space was given by $\zeta\cdot p=\zeta^c q\bar{\zeta}^a$. Writing $p=z_1+z_2 j$ we see that $\zeta\cdot(z_1+z_2 j)=\zeta^{c-a}z_1+\zeta^{c+a}z_2 j$. Therefore, this action is equivalent to the action of $\mathbb{Z}_{b-d}$ on $S^3\subset\mathbb{C}^2$ given by $\zeta(z_1,z_2)=(\zeta^{c-a}z_1,\zeta^{c+a}z_2)$. It follows that the quotient is

$$S^3/\mathbb{Z}_{b-d}\simeq L_{b-d}(\mu_-(c+a)).$$

where μ_- is the multiplicative inverse of $c-a$ in $\mathbb{Z}_{b-d}$.

We now compute the singular leaves. The first singular leaf is given by the quotient of the action (4.2.4) above. In particular, this action is transitive on all but the second factor, so

$$(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2)/(S^1\times\mathrm{Sp}(1)\times S^1)\simeq\mathrm{Sp}(1)/\Gamma_e$$

where Γ_e is the isotropy of the identity $(1,1,1)$ of the transitive factors. It is easy to compute that

$$\Gamma_e=\{(\zeta,\zeta^a,\zeta^{b-d}):\zeta^{b+d}=1\}\simeq\mathbb{Z}_{b+d}\subset S^1$$

Note that $\mathbb{Z}_{b+d}=\Gamma_e$ acts on S^3 by $\zeta\cdot q=\zeta^c q\bar{\zeta}^a$ or, equivalently, $\mathbb{Z}_{b+d}$ acts on $S^3\subset\mathbb{C}^2$ by $\zeta(z_1,z_2)=(\zeta^{c-a}z_1,\zeta^{c+a}z_2)$. It follows that

$$S^3/\mathbb{Z}_{b+d}\simeq L_{b+d}(\mu_+(c+a))$$

where μ_+ is the multiplicative inverse of $c-a$ in $\mathbb{Z}_{b+d}$.

Finally, the other singular leaf is the quotient by action (4.2.5) and similar work shows that this singular leaf is diffeomorphic to

$$S^3/\mathbb{Z}_{b-d} \simeq L_{b-d}(\mu_-(c+a))$$

where μ_- is the same as above.

To summarize, from the computations above, it follows that the C1BF group diagram for Standard Case 1.2 is

$$\begin{tikzcd}[column sep=small]
& \mathrm{Sp}(1)\times\mathrm{Sp}(1)\times T^2 \arrow[dl, dash, shift left=0.5ex] \arrow[dl, dash, shift right=0.5ex] \arrow[dr, dash, shift left=0.5ex] \arrow[dr, dash, shift right=0.5ex] & \\
S^1\times\mathrm{Sp}(1)\times S^1 \arrow[dr, dash] & & S^1\times\mathrm{Sp}(1)\times S^1 \arrow[dl, dash] \\
& S^1\times\mathrm{Sp}(1) &
\end{tikzcd}
\qquad
\begin{aligned}
M &\simeq S^3\times S^2\\
G/\!/H &\simeq L_e(\mu_-(c+a))\times S^1\\
G/\!/K^- &\simeq L_{b+d}(\mu_+(c+a))\\
G/\!/K^+ &\simeq L_{b-d}(\mu_-(c+a))
\end{aligned}$$

where the embeddings are

$$\begin{aligned}
K^- \to G\times G;\ (z,p,w) &\mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, w, 1)\\
K^+ \to G\times G;\ (z,p,w) &\mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, 1, w)\\
H \to G\times G;\ (z,p) &\mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, 1, 1)
\end{aligned}$$

where $\gcd(a,b,c,d)=1$, $\gcd(a^2-c^2, b^2-d^2)=1$ and $b\pm d\neq 0$ and $a\pm c\neq 0$. The leaves are $G/\!/H \approx L_e(\mu_-(c+a))\times S^1$ and $G/\!/K^- \approx L_{b+d}(\mu_+(c+a))$ and $G/\!/K^+ \simeq L_{b-d}(\mu_-(c+a))$ where $e=\gcd(b-d, b+d)$, μ_- is the multiplicative inverse of $c-a$ in $\mathbb{Z}_{b-d}$ and μ_+ is the multiplicative inverse of $c-a$ in $\mathbb{Z}_{b+d}$.

As a special case of Standard Case 1.2, we observe that if we choose $c=1$ and $a=0$, then $\gcd(a,b,c,d)=\gcd(a^2-c^2, b^2-d^2)=1$ for any choice of b and d and $\mu_\pm=1$. Thus

$G//K^- = L_{b+d}(1)$ and $K^+ = L_{b-d}(1)$. If we choose $b+d$ and $b-d$ to be relatively prime, then $e = 1$ and hence $G//H = S^3 \times S^1$, which gives us models of C1BFs in Case C.4.8 of Chapter 3 with leaf structure $(S^3 \times S^1, L_m(r), L_n(r))$ in the special case that $r = 1$. Observe that if $b - d = m$ and $b + d = n$, then $2b = m + n$ and hence $m + n$ is even, so m and n are either both even or both odd. However, they can't both be even since $\gcd(m, n) = 1$. Moreover, if m and n are both odd, and $b - d = m$ and $b + d = n$, then b and d have opposite parity. Since m and n are both odd, the midpoint $\frac{m+n}{2}$ is an integer. We set $b = \frac{m+n}{2}$ and $d = \frac{n-m}{2}$ and observe then that $b + d = n$ and $b - d = m$ and note that if the midpoint b is even then d is odd and if d is even then the midpoint b is odd. It follows that we can get all C1BFs in case C.4.8 in the case where $r = 1$ and m and n are both odd from the Standard Case 1.2.

Structure 4.2.3. This is an extension of the work from Structure 4.2.2. In particular, if we now assume that $\gcd(a, b, c, d) = 1$ and $\gcd(a^2 - c^2, b^2 - d^2) = 4$ we will get C1BFs which are diffeomorphic to $S^3 \hat{\times} S^2$. We will consider two cases below.

Case 1: Here we make the same assumptions on the parameters (aside from the gcd conditions) as in Standard Case 1.1 of Structure 4.2.2. Note that $b + d = 0$ implies that $b^2 - d^2 = 0$ so $4 = \gcd(a^2 - c^2, 0)$ which implies $(a, c) = (\pm 2, 0)$. We assume that $b = 2$. But we then also have $4 = \gcd(a, b, c, d) = \gcd(2, b)$ which implies that b is odd. Now, the action for the principal leaf, which is given by action (4.2.6) is an action of $\mathbb{Z}_{2b}$ on $\mathrm{Sp}(1) \times S^1$ given by

$$\zeta \cdot (q, w) = (q\bar{\zeta}^2, w)$$

This action is effectively free with kernel $\mathbb{Z}_2 = \pm 1$. Thus the action is effectively a $\mathbb{Z}_{2b}/\mathbb{Z}_2 \simeq \mathbb{Z}_b$ action given by

$$\zeta \cdot (q, w) = (q\bar{\zeta}, w).$$

Thus the principal leaf is

$$(\mathrm{Sp}(1) \times S^1)/\mathbb{Z}_b \simeq (\mathrm{Sp}(1)/\mathbb{Z}_b) \times S^1 \simeq L_b(1) \times S^1.$$

for b an odd integer. For the singular leaf given by action (4.2.4), the action reduces to

$$(z_1, p, z_2) \cdot (q_1, q_2, w_1, w_2) = (z_1^2 q_1, \overline{p}, q_2 \overline{p}, z_1^{2b} w_1 \overline{z}_2, w_2)$$

It is not difficult to compute that the quotient for this singular leaf is

$$(\mathrm{Sp}(1) \times S^1)/S^1 \simeq S^2 \times S^1.$$

The other singular leaf is given by action (4.2.5) and the same sort of argument as in Standard Case 1.1 of Structure 4.2.2 allows us to compute the corresponding singular leaf is

$$\mathrm{Sp}(1)/\mathbb{Z}_b \simeq L_b(1)$$

for b odd.

To summarize, from the computations above, it follows that the C1BF group diagram for $S^3 \hat{\times} S^2$ in this case is

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2 & \\
S^1 \times \mathrm{Sp}(1) \times S^1 \arrow[ur, equal] & & S^1 \times \mathrm{Sp}(1) \times S^1 \arrow[ul, equal] \\
 & S^1 \times \mathrm{Sp}(1) \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{array}{c}
M \simeq S^3 \hat{\times} S^2 \\
G//H \simeq L_b(1) \times S^1;\ b \text{ odd} \\
G//K^- \simeq S^2 \times S^1 \\
G//K^+ \simeq L_b(1);\ b \text{ odd}
\end{array}$$

where the embeddings are

$$K^- \to G\times G;\ (z,p,w)\mapsto (z,1,z^{2b},1,p,p,w,1)$$

$$K^+ \to G\times G;\ (z,p,w)\mapsto (z,1,z^{2b},1,p,p,1,w)$$

$$H \to G\times G;\ (z,p)\mapsto (z,1,z^{2b},1,p,p,1,1)$$

Case 2: Here we make the same assumptions on the parameters as in Standard Case 1.2 of Structure 4.2.2. The principal leaf is given by action (4.2.6). Write $q = x + yj$ for $x, y \in \mathbb{C}$. Then $\zeta \cdot q = \zeta^{c-a}x + \zeta^{c+a}yj = x + yj$ if and only if at least one of ζ^{c-a} or ζ^{c+a} is 1. Thus ζ is a $(b-d)^{th}$ root of unity and either a $(c+a)^{th}$ or $(c-a)^{th}$ root of unity. Since $\gcd(c\pm a, b-d)$ divides a^2-c^2 and b^2-d^2 and any divisor divisor of two numbers divides their gcd, it follows that $\gcd(a\pm c, b-d)$ both divide $\gcd(a^2-c^2, b^2-d^2) = 4$. Thus $\gcd(a\pm c, b-d) \in \{1,2,4\}$. It is easy to se that the case where this gcd is equal to 1 or 4 is impossible, so it has to be 2. Since $\zeta^{b-d} = \zeta^{a\pm c} = 1$, we must have $\zeta^{\gcd(a\pm c, b-d)} = 1$. It follows that $\zeta = \pm 1$. But then 2 divides $(a+c)(a-c)$ and 2 divides $(b+d)(b-d)$. But 2 divides $a+c$ if and only if 2 divides $a-c$ and similarly 2 divides $b+d$ if and only if 2 divides $b-d$. Thus $\gcd(a\pm c, b-d) = 2$. Thus the action (4.2.6) is effectively free with kernel $\mathbb{Z}_2 = \{\pm 1\}$. Note also that the above work implies that $b\pm d$ and $c\pm a$ are all even. Now, rewrite action (4.2.6) in terms of an action of $S^3 \subset \mathbb{C}^2$, which we see is an action of $\mathbb{Z}_{b-d}$ on $S^3\times S^1$ given by

$$\zeta\cdot(z_1,z_2,w) = (\zeta^{c-a}z_1, \zeta^{c+a}y, \zeta^{b+d}w).$$

Note that $\mathbb{Z}_2 = \{\pm 1\}$ is a normal subgroup of the kernel of this action. Thus $\mathbb{Z}_{b-d}/\mathbb{Z}_2 \simeq \mathbb{Z}_{\frac{b-d}{2}}$ acts on $S^3\times S^1$ with the added benefit that the action by $\mathbb{Z}_{\frac{b-d}{2}}$ is free on the S^3 factor. Note that the isomorphism $\mathbb{Z}_{b-d}/\mathbb{Z}_2 \simeq \mathbb{Z}_{\frac{b-d}{2}}$ is given by $[\zeta]\mapsto \zeta^2$ and hence $\mathbb{Z}_{\frac{b-d}{2}}$

acts on $S^3 \times S^1$ by

$$\zeta^2 \cdot (z_1, z_2, w) = [\zeta] \cdot (z_1, z_2, w) = (\zeta^{c-a} z_1, \zeta^{c+a} z_2, \zeta^{b-d} w) = \left((\zeta^2)^{\frac{c-a}{2}}, (\zeta^2)^{\frac{c+a}{2}}, (\zeta^2)^{\frac{b-d}{2}}\right).$$

It follows that the principal leaf is the quotient by the action of $\mathbb{Z}_{\frac{b-d}{2}}$ on $S^3 \times S^1$ by

$$\zeta \cdot (z_1, z_2, w) = (\zeta^{\frac{c-a}{2}} z_1, \zeta^{\frac{c+a}{2}} z_2, \zeta^{\frac{b-a}{2}} w).$$

A similar argument to that in Standard Case 1.2 of Structure 4.2.2 implies that the principal leaf is diffeomorphic to $L_{e/2}(\mu_- \frac{c+a}{2})$ where μ_- is the multiplicative inverse of $\frac{c-a}{2}$ in $\mathbb{Z}_{\frac{b-d}{2}}$ and $e = \gcd(b-d, b+d)$. Note also that $\frac{e}{2} = \gcd(\frac{b-d}{2}, \frac{b+d}{2})$. Similarly, one can show that the other singular leaf corresponding to action (4.2.4) is diffeomorphic to $L_{\frac{b+d}{2}}(\mu_+ \frac{c+a}{2})$ where μ_+ is the multiplicative inverse of $\frac{c-a}{2}$ in $\mathbb{Z}_{\frac{b+d}{2}}$ and the singular leaf corresponding to (4.2.5) is diffeomorphic to $L_{\frac{b-d}{2}}(\mu_- \frac{c+a}{2})$ where μ_+ is the multiplicative inverse of $\frac{c+a}{2}$ in $\mathbb{Z}_{\frac{b+d}{2}}$.

To summarize, from the computations above, it follows that the C1BF group diagram for $S^3 \hat{\times} S^2$ in this case is

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times T^2 \arrow[dl, equal] \arrow[dr, equal] & \\ S^1 \times \mathrm{Sp}(1) \times S^1 \arrow[dr, dash] & & S^1 \times \mathrm{Sp}(1) \times S^1 \arrow[dl, dash] \\ & S^1 \times \mathrm{Sp}(1) & \end{tikzcd}$$

$$\begin{gathered} M \simeq S^3 \hat{\times} S^2 \\ G//H \simeq L_{\frac{e}{2}}(\mu_- \tfrac{c+a}{2}) \times S^1 \\ G//K^- \simeq L_{\frac{b+d}{2}}(\mu_+ \tfrac{c+a}{2}) \\ G//K^+ \simeq L_{\frac{b-d}{2}}(\mu_- \tfrac{c+a}{2}) \end{gathered}$$

where the embeddings are

$$K^- \to G \times G;\ (z,p,w) \mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, w, 1)$$

$$K^+ \to G \times G;\ (z,p,w) \mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, 1, w)$$

$$H \to G \times G;\ (z,p) \mapsto (z^a, z^c, z^{b-d}, z^{b+d}, p, p, 1, 1)$$

where $\gcd(a,b,c,d) = 1$, $\gcd(a^2-c^2, b^2-d^2) = 4$ and $b \pm d \neq 0$ and $c \pm a \neq 0$. Note that these conditions imply that $b \pm d$ and $c \pm a$ are even. The leaves are $G//H \approx L_{\frac{e}{2}}(\mu_- \frac{c+a}{2}) \times S^1$ and $G//K^- \approx L_{\frac{b+d}{2}}(\mu_+ \frac{c+a}{2})$ and $G//K^+ \simeq L_{\frac{b-d}{2}}(\mu_- \frac{c+a}{2})$ where $e = \gcd(b-d, b+d)$, μ_- is the multiplicative inverse of $\frac{c-a}{2}$ in $\mathbb{Z}_{\frac{b-d}{2}}$ and μ_+ is the multiplicative inverse of $\frac{c-a}{2}$ in $\mathbb{Z}_{\frac{b+d}{2}}$.

As a special case of Case 2, if we choose $c = 2$ and $a = 0$, then $\mu_\pm = 1$ so $G//K^- = L_{\frac{b+d}{2}}(1)$ and $G//K^+ = L_{\frac{b-d}{2}}(1)$. Choosing b and d so that $\gcd(b-d, b+d) = 2$ gives $\frac{e}{2} = 1$ so $G//H = S^3 \times S^1$ which will give us models C1BFs of Case C.4.8 of Chapter 3 with leaf structure $(S^3 \times S^1, L_m(r), L_n(r))$. Note that we must also have $\gcd(a,b,c,d) = 1$ and $\gcd(a^2-c^2, b^2-d^2) = 4$, and if we want $\gcd(a,b,c,d) = 1$ then at least one of b or d must be odd. Moreover, $\gcd(a^2-c^2, b^2-d^2) = \gcd(4, b^2-d^2) = 4$ implies 4 divides b^2-d^2. Thus 2 divides $b+d$ or 4 divides $b-d$, so b and d must both be odd. Now, we want $\frac{b+d}{2} = m$ and $\frac{b-d}{2} = n$ so $b+d = 2m$ and $b-d = 2n$. Hence $b = m+n$ and, moreover, since b is odd we must have that m and n have opposite parity. We assume that WLOG that $m < n$. Set $b = \frac{2m+2n}{2} = m+n$ and $d = \frac{2n-2m}{2} = n-m$. Then $\frac{b+2}{2} = n$ and $\frac{b-d}{2} = m$ so we get every C1BF of Case C.4.8 of Chapter 3 for m and n with opposite parity.

Structure 4.2.4. Here we will exhibit a C1BF structure on $S^3 \times S^2$. Consider the cohomogeneity one action of $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ by

$$(r,s)\cdot(p,q) = (rp\bar{r}, rq\bar{s}) \tag{4.2.9}$$

Recall that the quotient by the action of S^1 on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ by

$$z\cdot(p,q) = (z^a p\bar{z}^c, z^b q\bar{z}^d) \tag{4.2.10}$$

gives $S^3\times S^2$ if $\gcd(a,b,c,d)=1$ and $\gcd(a^2-c^2, b^2-d^2)=1$, and gives $S^3\hat{\times}S^2$ when $\gcd(a,b,c,d)=1$ and $\gcd(a^2-c^2,b^2-d^2)=4$. We see that if $a=b=c$, $r=z^a$ and $s=z^d$, then these two actions are the same. In this case, it is easy to check that the gcd conditions imply that the only permissible values for the parameters are

1. $(a,b,c,d)=(1,1,1,0)$
2. $(a,b,c,d)=(-1,-1,-1,0)$
3. $(a,b,c,d)=(0,0,0,1)$
4. $(a,b,c,d)=(0,0,0,-1)$

Thus restricting action 4.2.9 to $S^1=\mathrm{Im}\big(z\mapsto(z^a,z^d)\big)\subset\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ for $(a,d)=(\pm1,0)$ or $(a,d)=(0,\pm1)$ and taking the quotient gives us C1BF structures on $S^3\times S^2$. It is clear that parameter values will yield the same results. Thus we need only consider parameter values (1). It is easy to compute that the singular and principal isotropy groups corresponding to $y=(1,1)$, $u=(-1,1)$ and $v=(i,1)$, respectively, are $G_e=G_u=\Delta\mathrm{Sp}(1)$ and $G_v=\Delta S^1$. Now, let us compute the leaves of the corresponding C1BF in the case $(a,b,c,d)=(1,0,0,0)$. In this case, the principal leaf is the biquotient induced by the embedding $T^2\to(\mathrm{Sp}(1)\times\mathrm{Sp}(1))^2$ by $(z,w)\mapsto(z,1,w,w)$ or, more explicitly, the the action of T^2 on $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ by

$$(z,w)\cdot(p,q)=(zp\bar{w}, q\bar{w})$$

To compute the quotient, it is easy to see that the diffeomorphism $f : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ given by $f(p,q) = (p\overline{q}, q)$ intertwines the above action with the action $(z,w) \star (p,q) = (zp, q\overline{w})$. It follows that the principal leaf is $S^2 \times S^2$. On the other hand, the singular leaves are, in both cases, are the biquotients induced by the embedding $S^1 \times \mathrm{Sp}(1) \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1))^2$ by $(z,p) \mapsto (z,1,p,p)$. It is easy to compute that both singular leaves are S^2 directly, or deduce this using case C.3 of Chapter 3. To summarize, from the computations above, it follows that the C1BF group diagram is

$$
\begin{tikzcd}[column sep=tiny]
 & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
S^1 \times \mathrm{Sp}(1) \arrow[dr, dash] & & S^1 \times \mathrm{Sp}(1) \arrow[dl, dash] \\
 & T^2 &
\end{tikzcd}
\qquad
\begin{array}{c}
M \simeq S^3 \times S^2 \\
G//H \simeq S^2 \times S^2 \\
G//K^- \simeq S^2 \\
G//K^+ \simeq S^2
\end{array}
$$

where the embeddings are

$$K^- \to G \times G;\ (z,p) \mapsto (z,1,p,p)$$

$$K^+ \to G \times G;\ (z,w) \mapsto (z,1,w,w)$$

$$H \to G \times G;\ (z,w) \mapsto (z,1,w,w)$$

Structure 4.2.5. Here we will exhibit a C1BF with leaf structure $(S^3 \times S^1, S^3, S^2 \times S^1)$. Consider the cohomogeneity one diagram

$$
\begin{tikzcd}[column sep=tiny]
 & S^3 \times S^1 \arrow[dl, dash] \arrow[dr, dash] & \\
\{1\} \times S^1 \arrow[dr, dash] & & S^1 \times \{1\} \arrow[dl, dash] \\
 & \{1\} &
\end{tikzcd}
\qquad
\begin{array}{c}
G/H \simeq S^3 \times S^1 \\
G/K^- \simeq S^3 \\
G/K^+ \simeq S^2 \times S^1
\end{array}
$$

This is manifold is easily seen to be simply connected. To see this, recall that for the cohomogeneity one case, we have the sphere bundles $K^\pm/H \xrightarrow{i} G/H \xrightarrow{\pi} G/K^\pm$ where the inclusion map is given by $kH \mapsto kH$. Consider the loop $\alpha_- : [0,1] \to K^-/H$ given by $\alpha_-(t) = (1, e^{2\pi i t})$. This is clearly a generating loop for K^-/H and the image of this loop under the natural inclusion $K^-/H \to G/H$ generates $\pi_1(G/H)$ on its own, so the manifold determined by the above diagram is simply connected.

Structure 4.2.6. Here we will exhibit $S^3 \hat{\times} S^2$ as a C1BF with leaf structure $(\mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2, S^2, S^2)$. Recall from DeVito's classification [DeV14] that we can write $\mathbb{C}\mathrm{P}^2\#-\mathbb{C}\mathrm{P}^2$ as the biquotient induced by the embedding

$$T^2 \to \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)^2; \qquad (z,w) \mapsto (z^2, wz^n, 1, z^n)$$

for n an odd integer. We will construct a C1BF diagram using these groups which yields the correct leaf structure. In order to make this work, we will need reparametrize the torus action above. Let $\theta = z$ and $\varphi = wz^n$. This coordinate transformation is easily seen to be invertible, so gives a change of coordinates of the torus. Then the above, torus embedding is equivalent to $(\theta,\varphi) \mapsto (\theta^2, \varphi, 1, \theta^n)$. This shows that the biquotient induced by the torus embedding

$$T^2 \to \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)^2; \qquad (z,w) \mapsto (z^2, w, 1, z^n)$$

is also diffeomorphic to $\mathbb{C}\mathrm{P}^2\# - \mathbb{C}\mathrm{P}^2$. Therefore, we consider the group diagram

$$\begin{tikzcd}[column sep=small]
& \mathrm{Sp}(1)\times\mathrm{Sp}(1) & \\
S^1\times\mathrm{Sp}(1) \arrow[ur, equal] & & S^1\times\mathrm{Sp}(1) \arrow[ul, equal] \\
& T^2 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,p) \mapsto (z^2, p, 1, z^n)$$

$$K^+ \to G \times G;\ (z,w) \mapsto (z^2, p, 1, z^n)$$

$$H \to G \times G;\ (z,w) \mapsto (z^2, w, 1, z^n)$$

Because the biquotient action induced from H is effectively free, it follows from Proposition 1.3.5 that the biquotient actions induced from the embeddings of $K^\pm$ are effectively free as well. It follows immediately from Proposition 1.3.7 that $G//K^\pm \approx S^2$ and, by above, $G//H \approx \mathbb{C}P^2 \# -\mathbb{C}P^2$.

We now wish to determine the diffeomorphism type of the manifold M given by the group diagram. We know that M must be one of S^5, $S^3 \times S^2$, $S^3 \hat{\times} S^2$, or the Wu manifold $W = SU(3)/SO(3)$. Note that $H^2(S^5) = 0$ and $H^3(W) = \mathbb{Z}_2$. We fist show that $H^3(M) \approx \mathbb{Z}$, hence M must be either $S^3 \times S^2$ or $S^3 \hat{\times} S^2$. Decompose $M = B_- \cup B_+$ as a DDB and note that each disk bundle $B_\pm$ deformation retracts onto S^2 and $P := B_- \cap B_+ = \mathbb{C}P^2 \# -\mathbb{C}P^2$. Consider the following portion of the Mayer-Vietoris sequence

$$0 \to H^2(M) \xrightarrow{\Psi} H^2(B_-) \oplus H^2(B_+) \xrightarrow{\Phi} H^2(P) \to H^3(M)$$

Note that $H^2(B_-) \oplus H^2(B_+) \approx H^2(P) \approx \mathbb{Z} \oplus \mathbb{Z}$. By exactness, Ψ is injective, thus $H^2(M) \in \{0, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}\}$. Furthermore, $\mathrm{Ker}(\Phi) = \mathrm{Im}(\Psi) \approx H^2(M)$. Thus to compute $H^2(M)$, it suffices to compute $\mathrm{Ker}(\Phi)$. Note that $\Phi = i^* - j^*$ where i^* and j^* are induced from the inclusion maps $i : P \to B_-$ and $j : P \to B_+$. Note also that because $B_- = B_+$ (and, more specifically, are the exact same quotient), it follows that $i = j$. Observe that since $i^* = j^*$, it follows that $\Delta\mathbb{Z} \subset \mathbb{Z}^2 \subset \mathrm{Ker}(\Phi)$; that is, $\mathrm{Ker}(\Phi)$ contains a copy of $\mathbb{Z}$. It follows that $H^2(M) \approx \mathbb{Z}$.

We note that $k^* : H^2(M) \to H^2(P)$ induced by the inclusion $k : P \to M$ is injective (on cohomology with $\mathbb{Z}_2$ coefficients). To see this, we claim that the map i^* above is injective. First note that the maps $i : P \to B_-$ and and the bundle projection of the principal leaf onto the singular leaf, $S^2 \to P \xrightarrow{\pi} S^2$, induce the same homomorphism on cohomology because composing i with the deformation retraction f_t of B_- onto the singular leaf S^2 gives $f_t \circ i = \pi$.

Consider now the Gysin sequence associated to the sphere bundle $S^2 \to P \xrightarrow{\pi} S^2$:

$$0 \to H^2(B_-) \xrightarrow{\pi^*} H^2(P) \to H^0(B_-) \xrightarrow{\cup e} H^3(P)$$

By exactness, $\pi^* = i^*$ is injective, as desired. It now follows that $k^* : H^2(M) \to H^2(P)$ is injective. Indeed, recall that $\Psi = (\alpha^*, \beta^*)$ where α^* and β^* are induced by inclusions $\alpha : B_- \to M$ and $\beta : B_+ \to M$. Thus Ψ is injective if and only if α^* and β^* are injective. Thus $k^* = i^* \circ \alpha^*$ is injective.

To determine whether M is $S^3 \times S^2$ or $S^3 \hat{\times} S^2$, recall that these two spaces are distinguished by their second Stiefel-Whitney classes w_2 associated to their tangent bundles. In particular, w_2 is zero for $S^3 \times S^2$ and nonzero for $S^3 \hat{\times} S^2$ (for a detailed treatment of Stiefel-Whitney classes, see [MS74]). We will show that the second Stiefel-Whitney class is nonzero. To do this, consider the pullback of the tangent bundle k^*TM. We recall the following facts about Stiefel-Whitney classes:

Fact 1: For a submanifold $L \subset M$, and $k : L \to M$ the inclusion, $k^*TM = TL \oplus \nu L$, where νL is the normal bundle to L.

Fact 2: For a codimension one submanifold of M with M simply connected, then $\nu L = \mathbb{1}$, where $\mathbb{1}$ denotes the trivial bundle.

Using these facts, using standard properties of Stiefel-Whitney classes, we have

$$\begin{aligned}
k^* w_2(TM) &= w_2(k^* TM)\\
&= w_2(TP \oplus \nu P)\\
&= w_2(TP \oplus \mathbb{1})\\
&= w_2(TP)\, w_0(\mathbb{1}) + w_1(TP)\, w_1(\mathbb{1}) + w_0(TP)\, w_2(\mathbb{1})\\
&= w_2(TP) + w_1(TP)\, w_1(\mathbb{1}) + w_2(\mathbb{1})\\
&= w_2(TP)
\end{aligned}$$

But $w_2(TP) \neq 0$ for $P = \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2$, so $w_2(TM) \neq 0$, so we must have $M \approx S^3 \hat{\times} S^2$.

We note that this same argument can be used to show that if $P = S^2 \times S^2$ instead of $\mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2$, then $M = S^3 \times S^2$. Indeed, since k^* is injective, and $w_2(TP) = 0$ for $P = S^2 \times S^2$, it follows that $w_2(TM) = 0$, so $M = S^3 \times S^2$.

In summary, in this case we get the group diagram

$$\begin{tikzcd}[column sep=small]
& \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \\
S^1 \times \mathrm{Sp}(1) \arrow[ur, equal] & & S^1 \times \mathrm{Sp}(1) \arrow[ul, equal] \\
& T^2 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{aligned}
M &\simeq S^3 \hat{\times} S^2\\
G/\!/H &\simeq \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2\\
G/\!/K^- &\simeq S^2\\
G/\!/K^+ &\simeq S^2
\end{aligned}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, p) \mapsto (z^2, p, 1, z^n)$$

$$K^+ \to G \times G;\ (z, w) \mapsto (z^2, p, 1, z^n)$$

$$H \to G \times G;\ (z, w) \mapsto (z^2, w, 1, z^n)$$

Structure 4.2.7. Here we will exhibit a C1BF with leaf structure $\big(S^3 \times S^1, L_m(s), L_n(r)\big)$, where $\gcd(m, n) = 1$. To begin, note that the action of S^1 on $S^3 \times S^1$ by

$$w \cdot \big((z_1, z_2), \theta\big) = \big((wz_1, w^r z_2), w^m\theta\big)$$

is free when $\gcd(m, r) = 1$ it follows from Proposition 1.3.7 that the quotient $(S^3 \times S^1)/S^1 \simeq L_m(r)$. Applying the proof of Proposition 2.13 in [DeV17] to the above action, it follows that the biquotient action induced by

$$T^2 \to \big(\mathrm{U}(2) \times S^1\big)^2; \quad (z, w) \mapsto \big(\mathrm{diag}(z, z^r), z^m, \mathrm{diag}(w, 1), 1\big)$$

is free with quotient $(\mathrm{U}(2) \times S^1)/T^2 \simeq L_m(r)$. Using this as motivation, we consider the C1BF diagram

$$\begin{tikzcd}
& \mathrm{U}(2) \times S^1 \arrow[dl, dash, double] \arrow[dr, dash, double] & \\
T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\
& S^1 &
\end{tikzcd}
\qquad
\begin{array}{l}
G/\!/H \simeq S^3 \times S^1 \\
G/\!/K^- \simeq L_m(r) \\
G/\!/K^+ \simeq L_n(s)
\end{array}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,w) \mapsto \big(\mathrm{diag}(z,z^r), z^m, \mathrm{diag}(w,1), 1\big)$$
$$K^+ \to G \times G;\ (z,w) \mapsto \big(\mathrm{diag}(z,z^s), z^n, \mathrm{diag}(w,1), 1\big)$$
$$H \to G \times G;\ w \mapsto \big(I, 1, \mathrm{diag}(w,1), 1\big)$$

where $\gcd(m,r) = \gcd(n,s) = 1$. Note that H is the restriction of $K^{\pm}$ to $z = 1$, so the diagram is consistent. It follows easily from the work above along with Proposition 1.3.7 that the leaves are $G//H \simeq S^3 \times S^1$, $G//K^- \simeq L_m(r)$, and $G//K^+ \simeq L_n(s)$. By the work in case C.4.8 of Chapter 3, this is simply connected provided that $\gcd(m,n) = 1$. Note that if $\gcd(m,n) \neq 1$, the above diagram still defines a C1BF but it is not simply connected. Finally, observe that in the special case $n = 1$ we get a C1BF with leaf structure $\big(S^3 \times S^1, L_m(r), S^3\big)$.

Structure 4.2.8. Here we will exhibit a C1BF with leaf structure $\big(S^3 \times S^1, L_m(r), S^2 \times S^1\big)$, where $\gcd(m,r) = 1$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small]
 & \mathrm{U}(2) \times S^1 \arrow[dl, equal] \arrow[dr, equal] & \\
T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\
 & S^1 &
\end{tikzcd}
\qquad
\begin{aligned}
G//H &\simeq S^3 \times S^1 \\
G//K^- &\simeq L_m(r) \\
G//K^+ &\simeq S^2 \times S^1
\end{aligned}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,w) \mapsto \big(\mathrm{diag}(z,z^r), z^m, \mathrm{diag}(w,1), 1\big)$$
$$K^+ \to G \times G;\ (z,w) \mapsto \big(I, 1, \mathrm{diag}(w,z), 1\big)$$
$$H \to G \times G;\ w \mapsto \big(I, 1, \mathrm{diag}(w,1), 1\big)$$

It is clear that the actions induced by $K^{\pm}$ and H are free and that the embedding of H is obtained simply by setting $z = 1$, so the diagram is consistent. From the work in Structure 4.2.7 we have $G//K^{-} \simeq L_m(r)$ and it is easy to see that $G//K^{+} \simeq S^2 \times S^1$ and $G//H \simeq S^3 \times S^1$.

It remains to check whether the C1BF coming from the above group diagram is simply connected. By the van Kampen theorem for C1BFs, it is simply connected if and only if $\pi_1(K^{\pm}/H)$ generates $\pi_1(G//H)$ under the natural inclusions. To check this, consider the curve $\alpha_- : [0,1] \to K^-$ defined by $\alpha_-(t) = \left(\mathrm{diag}(e^{\frac{2\pi ti}{m}}, e^{\frac{2\pi rti}{m}}), e^{2\pi ti}, I, 1\right)$. We claim that the curve α_- in K^- pushes down via the quotient map to to a generating loop in $K^-/H \simeq S^1$. To see this, observe that this curve pushes down to the following curve in K^-/H, which we also call α_-

$$\alpha_-(t) = \left(\mathrm{diag}(e^{\frac{2\pi ti}{m}}, e^{\frac{2\pi rti}{m}}), e^{2\pi ti}, I, 1\right)H$$

which is now a loop in K^-/H because

$$\alpha_-(1) = \left(\mathrm{diag}(e^{\frac{2\pi ti}{m}}, e^{\frac{2\pi rti}{m}}), 1, I, 1\right)H = (I, 1, I, 1)H = \alpha_-(0).$$

Furthermore, we know that $K^-/H \simeq S^1$ and this loop goes around exactly once, so it follows that it is a generating loop for $G//H$.

Now, by the remarks above the van Kampen theorem for C1BFs, the inclusion map for the sphere bundle $K^{\pm}/H \to G//H \to G//K^{\pm}$ is, in general, given by

$$(k_1, k_2)H \mapsto [k_1^{-1} g k_2]$$

for any fixed $g \in G$. It suffices to check whether the inclusion of the fiber spheres at the identity (that is, $g = e$) generate $\pi_1(G//H)$. Note that $G//H \simeq S^3 \times S^1$ via the diffeomorphism $[B, z] \mapsto \big((b_1, b_2), z\big)$ where $(b_1, b_2)^T$ is the first column of B and consider the following diagram

$$\begin{tikzcd}
K^-/H \arrow[r] & G//H \arrow[r] \arrow[d, "\simeq"'] & G//K^- \arrow[d, "\simeq"] \\
& S^3 \times S^1 \arrow[r, dashed] & L_m(r)
\end{tikzcd}$$

By above, a generating loop for $K^-/H \simeq T^3/T^2$ is $\alpha_-(t)$, which maps via the inclusion map at the identity to the loop

$$t \mapsto \left[\operatorname{diag}(e^{\frac{-2\pi t i}{m}}, e^{\frac{-2\pi r t i}{m}}), 1, e^{-2\pi t i}\right]$$

in $G//H$. Composing this loop with the left vertical diffeomorphism in the diagram this loop becomes

$$t \mapsto \left([e^{\frac{-2\pi t i}{m}}, 0], e^{-2\pi t i}\right)$$

Note that in $\pi_1(L_m(r) \times S^1) \simeq \mathbb{Z}$ that this loop corresponds to 1, so is a generator for $\pi_1(G//H)$.

Structure 4.2.9. Here we will exhibit a C1BF with leaf structure $\big(L_m(r) \times S^1, L_m(r), L_m(r)\big)$, where $\gcd(m, r) = 1$. We know that the biquotient induced by

$$T^2 \to \big(\mathrm{U}(2) \times S^1\big)^2; \quad (z, w) \mapsto \big(\operatorname{diag}(z, z^r), z^m, 1, \operatorname{diag}(w, 1), 1, 1\big)$$

is diffeomorphic to $L_m(r)$. Let us extend this to the biquotient action induced from the embedding

$$T^2 \to \big(\mathrm{U}(2)\times T^2\big)^2; \qquad (z,w)\mapsto \big(\mathrm{diag}(z,z^r), z^m, 1, \mathrm{diag}(w,1), 1, 1\big)$$

It is clear that $(\mathrm{U}(2)\times T^2)//T^2 \simeq L_m(r)\times S^1$. Consider the group diagram

$$\begin{tikzcd} & \mathrm{U}(2)\times T^2 \arrow[dl, equal] \arrow[dr, equal] & \\ T^3 \arrow[dr, dash] & & T^3 \arrow[dl, dash] \\ & T^2 & \end{tikzcd} \qquad \begin{aligned} G//H &\simeq L_m(r)\times S^1\\ G//K^- &\simeq L_m(r)\\ G//K^+ &\simeq L_m(r)\end{aligned}$$

where the embeddings are given by

$$\begin{aligned} K^- \to G\times G;\ (z,w,\theta) &\mapsto \big(\mathrm{diag}(\theta z, \theta^r z^r), z^m, \theta^m, \mathrm{diag}(w,1), 1, 1\big)\\ K^+ \to G\times G;\ (z,w,\theta) &\mapsto \big(\mathrm{diag}(z, z^r), z^m, \theta, \mathrm{diag}(w,1), 1, 1\big)\\ H \to G\times G;\ (z,w) &\mapsto \big(\mathrm{diag}(z, z^r), z^m, 1, \mathrm{diag}(w,1), 1, 1\big)\end{aligned}$$

Note that we have seen previously that the $G//K^+ \simeq L_m(r)$ and $G//H \simeq L_m(r)\times S^1$. But we have not seen the biquotient induced by the embedding of K^- previously. We will show that $G//K^- \simeq L_m(r)$. To see this, consider the action of S^1 on $L_m(r)\times S^1$ given by

$$\theta\cdot\big([w_1,w_2],x\big) = \big([\theta w_1, \theta^r w_2], \theta^m x\big). \tag{4.2.11}$$

Note that this action is well defined since $[w_1,w_2] = [w_1', w_2']$ if and only if $[w_1', w_2'] = [\zeta w_1, \zeta^r w_2]$ for some $\zeta\in\mathbb{Z}_m\subset S^1$ and, furthermore, θ commutes with ζ. Observe also that $\theta\cdot\big([w_1,w_2],x\big) = \big([w_1,w_2],x\big)$ if and only if $\theta\in\mathbb{Z}_m$, which acts trivially, so the action is effectively free. Thus by Proposition 1.3.7, since the action is transitive on the S^1

factor, we have

$$(L_m(r)\times S^1)/S^1 \simeq L_m(r)/\mathbb{Z}_m \simeq L_m(r).$$

Note that the diffeomorphism $G/\!/H \simeq (\mathrm{U}(2)\times T^2)/\!/T^2 \simeq L_m(r)\times S^1$ is given by $[B,x,y]\mapsto \big([b_1,b_2],y\big)$ where $(b_1,b_2)^T$ is the first column of B. It is not difficult to see that, under this diffeomorphism, action (4.2.11) becomes S^1 acting on $(\mathrm{U}(2)\times T^2)/\!/T^2$ via $\theta\cdot[B,x,y] = \big[\mathrm{diag}(\theta,\theta^r)B,x,\theta^m y\big]$. Since this action commutes with the action of T^2 on $\mathrm{U}(2)\times T^2$, it follows that the quotient by this action is equivalent to the biquotient induced by $K^-\to G\times G$.

It remains to check that C1BF defined by the above diagram is simply connected. By the van Kampen theorem for C1BFs, it is simply connected if and only if $\pi_1(K^\pm/H)$ generates $\pi_1(G/\!/H)$ under the natural inclusions. Consider $\alpha_-:[0,1]\to K^-$ given by $\alpha_-(t)=\big(\mathrm{diag}(e^{\frac{2\pi ti}{m}},e^{\frac{2\pi rti}{m}}),1,e^{2\pi ti},I,1,1\big)$.

We claim that the curve α_- in K^- pushes down via the quotient map to to a generating loop in $K^-/H\simeq S^1$. To see this, observe that this curve pushes down to the following curve in K^-/H, which we also call α_-

$$\alpha_-(t)=\big(\mathrm{diag}(e^{\frac{2\pi ti}{m}},e^{\frac{2\pi rti}{m}}),1,e^{2\pi ti},I,1,1\big)H$$

which is now a loop in K^-/H because

$$\alpha_-(1)=\big(\mathrm{diag}(e^{\frac{2\pi ti}{m}},e^{\frac{2\pi rti}{m}}),1,1,I,1,1\big)H=(I,1,1,I,1,1)H=\alpha_-(0).$$

Furthermore, we know that $K^-/H\simeq S^1$ and this loop goes around exactly once, so it follows that it is a generating loop for K^-/H. Similarly, the loop $\alpha_+(t)=(I,1,e^{2\pi ti},I,1,1)$ pushes down to a generating loop for $K^+/H\simeq S^1$.

Now, by the remarks above the van Kampen theorem for C1BFs, the inclusion map for the sphere bundle $K^{\pm}/H \to G//H \to G//K^{\pm}$ is, in general, given by

$$(k_1, k_2)H \mapsto [k_1^{-1} g k_2]$$

for any fixed $g \in G$. It suffices to check whether the inclusion of the fiber spheres at the identity (that is, $g = e$) generate $\pi_1(G//H)$. Consider the following diagram

$$\begin{tikzcd}
K^{\pm}/H \arrow[r] & G//H \arrow[r] \arrow[d, "\simeq"'] & G//K^{\pm} \arrow[d, "\simeq"] \\
 & L_m(r) \times S^1 \arrow[r, dashed] & L_m(r)
\end{tikzcd}$$

By above, a generating loop for $K^-/H \simeq T^3/T^2$ is $\alpha_-(t)$, which maps via the inclusion map at the identity to the loop

$$t \mapsto \left[\operatorname{diag}(e^{\frac{-2\pi ti}{m}}, e^{\frac{-2\pi rti}{m}}), 1, e^{-2\pi ti}\right]$$

in $G//H$. Composing this loop with the left vertical diffeomorphism in the diagram this loop becomes

$$t \mapsto \left([e^{\frac{-2\pi ti}{m}}, 0], e^{-2\pi ti}\right)$$

Note that in $\pi_1(L_m(r) \times S^1) \simeq \mathbb{Z}_m \times \mathbb{Z}$ that this loop corresponds to $(1,1)$. Similarly, doing the same thing with $\alpha_+(t)$, we see that this loop corresponds to the loop

$$t \mapsto \left([1,0], e^{-2\pi ti}\right)$$

in $L_m(r) \times S^1$, which corresponds to $(0,1)$ in the fundamental group. Therefore, $\alpha_{\pm}(t)$

together generate $\pi_1(G//H)$, so M is simply connected.

Structure 4.2.10. Here we will exhibit a C1BF with leaf structure $\left(L_2(1)\times S^1, L_2(1), S^2\hat{\times} S^1\right)$. Consider the following cohomogeneity one diagram, taken from [Hoe10], which is known to be simply connected.

$$\begin{tikzcd} & S^3\times S^1 \arrow[dl, dash] \arrow[dr, dash] & \\ \left\{(e^{i\theta},1)\right\}\cdot H & & \left\{(e^{jn\theta},e^{2i\theta})\right\} \\ & \left\langle (j,-1)\right\rangle \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \qquad \begin{array}{c} G/H \simeq L_2(1)\times S^1 \\ G/K^- \simeq S^2\hat{\times} S^1 \\ G/K^+ \simeq L_2(1) \end{array}$$

where n is an odd integer. We wish to compute G/K^+. This can be computed as the quotient by the action of S^1 on $S^3\times S^1$ by $(e^{jn\theta}, e^{2i\theta})\cdot(q,w) = (qe^{jn\theta}, we^{2i\theta})$. Let us compute the kernel of this action. We want

$$qe^{jn\theta} = q \tag{4.2.12}$$

$$we^{2i\theta} = w \tag{4.2.13}$$

The first of these equations implies $e^{j\theta}$ is an n^{th} root of 1; that is, $\theta = e^{\frac{2\pi k}{n}}$ and the second equation implies that $\theta = 0$ or $\theta = \pi$. But n is odd so this implies that the action is in fact free. Furthermore, the above action is transitive on the circle factor so by Proposition 1.3.7 we have

$$(S^3\times S^1)/K^+ \simeq S^3/\Gamma_e$$

where Γ_e is the isotropy of the identity of the transitive factor. It is easy to compute that $\Gamma_e = \left\{(1,1),(-1,1)\right\}$. Thus it is easy to see that $S^3/\Gamma_e \simeq \mathbb{RP}^3 \simeq L_2(1)$.

Now let us compute G/K^-. Note that H is a normal subgroup of G so $\left\{(e^{i\theta},1)\right\}\cdot H$ is just products of elements of $\left\{(e^{i\theta},1)\right\}$ with elements of H. The identity component of K^- is the circle $K_0^- = \{(e^{i\theta}, 1)\}$. Clearly $(S^3 \times S^1)/K_0^- \simeq S^2 \times S^1$ via the diffeomorphism $[q, w] \mapsto ([q], w)$. Observe that in K^-/K_0^- we have $[e^{i\theta}, 1] = [-e^{i\theta}, 1]$ and $[je^{i\theta}, -1] = [-je^{i\theta}, 1]$ and $K^-/K_0^- \simeq \mathbb{Z}_2$ acts by

$$[je^{i\theta}, -1] \cdot ([q], w) = ([je^{i\theta}q], -w) = ([jq], -w).$$

This is equivalent to the antipodal map on both factors, so $S^2 \times S^1/\mathbb{Z}_2 \simeq S^2 \hat{\times} S^1$. Finally, it follows from the work in Chapter 3 that there is no other choice but for the principal leaf to be $G/H \simeq L_2(1) \times S^1$.

4.3 C1BFs in Dimension 6 With Simply Connected Principal Leaf

He we will give explicit models of C1BFs in dimension 6 which realize the leaf structures determined in Chapter 3. Note that when the principal leaf is simply connected, any consistent C1BF diagram is necessarily simply connected by the van Kampen theorem for C1BFs.

Structure 4.3.1. Here we will exhibit a C1BF with leaf structure $(S^3 \times S^2, S^3, S^3)$. According to Hoelscher's classification of cohomogeneity one manifolds, the diagram

$$\begin{tikzcd}[arrows=-] & S^3 \times S^3 \arrow[dl] \arrow[dr] & \\ \Delta S^3 \arrow[dr] & & \Delta S^3 \arrow[dl] \\ & \Delta S^1 & \end{tikzcd} \qquad \begin{array}{c} M \simeq S^3 \times S^3 \\ G/H \simeq S^3 \times S^2 \\ G/K^- \simeq S^3 \\ G/K^+ \simeq S^3 \end{array}$$

arises as a cohomogeneity one action of $S^3 \times S^3$ on itself. It is clear that $G/H \simeq S^3 \times S^2$ and $G/K^\pm \simeq S^3$

Structure 4.3.2. Here we will exhibit a C1BF with leaf structure $(S^3 \times S^2, S^3, S^2)$. According to Hoelscher's classification of cohomogeneity one manifolds, the diagram

$$\begin{tikzcd}[column sep=small] & S^3 \times S^3 \arrow[dl, dash] \arrow[dr, dash] & \\ \Delta S^3 \arrow[dr, dash] & & S^3 \times S^1 \arrow[dl, dash] \\ & \Delta S^1 & \end{tikzcd} \qquad \begin{array}{c} M \simeq S^6 \\ G/H \simeq S^3 \times S^2 \\ G/K^- \simeq S^3 \\ G/K^+ \simeq S^2 \end{array}$$

arises as a cohomogeneity one action of $S^3 \times S^3$ on S^6. It is clear that $G/H \simeq S^3 \times S^2$, $G/K^- \simeq S^3$, and $G/K^+ \simeq S^2$.

Structure 4.3.3. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, S^2, S^2)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[dl, dash, shift left] \arrow[dl, dash, shift right] \arrow[dr, dash, shift left] \arrow[dr, dash, shift right] & \\ S^1 \times \mathrm{Sp}(1) \arrow[dr, dash] & & S^1 \times \mathrm{Sp}(1) \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{array}{c} G/\!/H \simeq S^3 \times S^2 \\ G/\!/K^- \simeq S^2 \\ G/\!/K^+ \simeq S^2 \end{array}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, p) \mapsto (z^2, z^n, 1, p)$$

$$K^+ \to G \times G;\ (z, p) \mapsto (z^2, z^n, 1, p)$$

$$H \to G \times G;\ z \mapsto (z^2, z^n, 1, z^n)$$

for any even integer n. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G/\!/H \simeq S^3 \times S^2$ (because n is even) and it follows from Proposition 1.3.7 that $G/\!/K^\pm \simeq S^2$.

Structure 4.3.4. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, S^2 \times S^2, S^3)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\ T^2 \arrow[dr, dash] & & \mathrm{Sp}(1) \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{array}{c} G/\!/H \simeq S^3 \times S^2 \\ G/\!/K^- \simeq S^2 \times S^2 \\ G/\!/K^+ \simeq S^3 \end{array}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, w) \mapsto (z^2, wz^n, 1, z^n)$$

$$K^+ \to G \times G;\ p \mapsto (1, p, 1, 1)$$

$$H \to G \times G;\ w \mapsto (1, w, 1, 1)$$

for n even. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G/\!/K^- \simeq S^2 \times S^2$. It is clear that $G/\!/H \simeq S^3 \times S^2$ and $G/\!/K^+ \simeq S^3$.

Structure 4.3.5. Here we will exhibit a C1BF with leaf structure $(S^5, \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2)$. According to Hoelscher's classification of cohomogeneity one manifolds, the diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{SU}(3) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{S(U(2)U(1))} \arrow[dr, dash] & & \mathrm{S(U(2)U(1))} \arrow[dl, dash] \\ & \mathrm{SU(2)SU(1)} & \end{tikzcd} \qquad \begin{array}{c} M \simeq \mathbb{C}\mathrm{P}^3 \# -\mathbb{C}\mathrm{P}^3 \\ G/H \simeq S^5 \\ G/K^- \simeq \mathbb{C}\mathrm{P}^2 \\ G/K^+ \simeq \mathbb{C}\mathrm{P}^2 \end{array}$$

arises as a cohomogeneity one action on $\mathbb{C}\mathrm{P}^3 \# -\mathbb{C}\mathrm{P}^3$. It is easy to see that $G/H \simeq S^5$. It then follows dimension considerations and from the work in Case D.2 of Chapter 3 that $G/K^{\pm} \simeq \mathbb{C}\mathrm{P}^2$.

Structure 4.3.6. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, S^2 \times S^2, S^2)$.

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
T^2 \arrow[dr, dash] & & \mathrm{Sp}(1)\times S^1 \arrow[dl, dash] \\
 & S^1 &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \times S^2 \\
G//K^- \simeq S^2 \times S^2 \\
G//K^+ \simeq S^2
\end{array}
$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, w) \mapsto (z^2, wz^n, 1, z^n)$$

$$K^+ \to G \times G;\ (p, w) \mapsto (1, w, p, 1)$$

$$H \to G \times G;\ w \mapsto (1, w, 1, 1)$$

for n even. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq S^2 \times S^2$ and $G//H \simeq S^3 \times S^2$ and it follows from Proposition 1.3.7 that $G//K^+ \simeq S^2$.

Structure 4.3.7. Now we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, S^2 \times S^2, S^2 \times S^2)$. Consider the C1BF diagram

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\
 & S^1 &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \times S^2 \\
G//K^- \simeq S^2 \times S^2 \\
G//K^+ \simeq S^2 \times S^2
\end{array}
$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, w) \mapsto (z^2, wz^n, 1, z^n)$$

$$K^+ \to G \times G;\ (z, w) \mapsto (z^2, wz^n, 1, z^n)$$

$$H \to G \times G;\ w \mapsto (1, w, 1, 1)$$

for n even. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^{\pm} \simeq S^2 \times S^2$ and clearly $G//H \simeq S^3 \times S^2$.

Structure 4.3.8. Here we will exhibit a C1BF with leaf structure $(S^5, \mathbb{C}\mathrm{P}^2, pt)$. We will simply make a small modification to Structure 4.3.5. Consider the cohomogeneity one diagram

$$
\begin{tikzcd}
 & \mathrm{SU}(3) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{S(U(2)U(1))} & & \mathrm{SU}(3) \\
 & \mathrm{SU(2)SU(1)} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{array}{c}
M \simeq \mathbb{C}\mathrm{P}^3 \\
G/H \simeq S^5 \\
G/K^- \simeq \mathbb{C}\mathrm{P}^2 \\
G/K^+ \simeq pt
\end{array}
$$

We already know from Structure 4.3.5 that $G/K^- \simeq \mathbb{C}\mathrm{P}^2$ and $G/H \simeq S^5$ and it is clear that $G/K^+ = pt$. It is easy to see that such a C1BF is diffeomorphic to $\mathbb{C}\mathrm{P}^3$.

Structure 4.3.9. Here we will exhibit a C1BF with leaf structure $(S^3 \times S^2, \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2, S^2)$. Consider the C1BF diagram

$$
\begin{tikzcd}
 & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[dl, Rightarrow, no head] \arrow[dr, Rightarrow, no head] & \\
T^2 & & S^1 \times \mathrm{Sp}(1) \\
 & S^1 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \times S^2 \\
G//K^- \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2 \\
G//K^+ \simeq S^2
\end{array}
$$

where the embeddings are given by

$$
\begin{aligned}
K^- \to G \times G;\ (z, w) &\mapsto (zw, zw^2, w, z) \\
K^+ \to G \times G;\ (z, p) &\mapsto (z, p, 1, z) \\
H \to G \times G;\ z &\mapsto (z, z, 1, z)
\end{aligned}
$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}P^2\#\mathbb{C}P^2$ and $G//H \simeq S^3 \times S^2$ and it follows from Proposition 1.3.7 that $G//K^+ \simeq S^2$.

Structure 4.3.10. Here we will exhibit a C1BF with leaf structure $(S^3 \hat{\times} S^2, \mathbb{C}P^2\#\mathbb{C}P^2, S^2)$. This will simply be a slight modification of Structure 4.3.9. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\ T^2 \arrow[dr, dash] & & S^1\times\mathrm{Sp}(1) \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{array}{c} G//H \simeq S^3 \hat{\times} S^2 \\ G//K^- \simeq \mathbb{C}P^2\#\mathbb{C}P^2 \\ G//K^+ \simeq S^2 \end{array}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, w) \mapsto (zw, zw^2, w, z)$$
$$K^+ \to G \times G;\ (w, p) \mapsto (w, w^2, p, 1)$$
$$H \to G \times G;\ w \mapsto (w, w^2, w, 1)$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}P^2\#\mathbb{C}P^2$ and $G//H \simeq S^3 \hat{\times} S^2$ and it follows from Proposition 1.3.7 that $G//K^+ \simeq S^2$.

Structure 4.3.11. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, \mathbb{C}P^2\# - \mathbb{C}P^2, S^2)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\ T^2 \arrow[dr, dash] & & \mathrm{Sp}(1)\times S^1 \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{array}{c} G//H \simeq S^3 \times S^2 \\ G//K^- \simeq \mathbb{C}P^2\# - \mathbb{C}P^2 \\ G//K^+ \simeq S^2 \end{array}$$

where the embeddings are given by

$$
\begin{aligned}
K^- \to G \times G;\ (z,w) &\mapsto (z^2, wz^n, 1, z^n)\\
K^+ \to G \times G;\ (p,w) &\mapsto (1, w, p, 1)\\
H \to G \times G;\ w &\mapsto (1, w, 1, 1)
\end{aligned}
$$

for n odd. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2$ and $G//H \simeq S^3 \times S^2$ and it follows from Proposition 1.3.7 that $G//K^+ \simeq S^2$.

Structure 4.3.12. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2, S^2)$. This will be a slight modification of Structure 4.3.11. Consider the C1BF diagram

$$
\begin{tikzcd}
 & \mathrm{Sp}(1)\times\mathrm{Sp}(1) & \\
T^2 \arrow[ur, equal] & & S^1\times\mathrm{Sp}(1) \arrow[ul, equal] \\
 & S^1 \arrow[ul, no head] \arrow[ur, no head] &
\end{tikzcd}
\qquad
\begin{aligned}
G//H &\simeq S^3 \hat{\times} S^2\\
G//K^- &\simeq \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2\\
G//K^+ &\simeq S^2
\end{aligned}
$$

where the embeddings are given by

$$
\begin{aligned}
K^- \to G \times G;\ (z,w) &\mapsto (z^2, wz^n, 1, z^n)\\
K^+ \to G \times G;\ (z,p) &\mapsto (z^2, z^n, 1, p)\\
H \to G \times G;\ z &\mapsto (z^2, z^n, 1, z^n)
\end{aligned}
$$

for n odd. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# - \mathbb{C}\mathrm{P}^2$ and $G//H \simeq S^3 \hat{\times} S^2$ and it follows from Proposition 1.3.7 that $G//K^+ \simeq S^2$.

Structure 4.3.13. Here we will exhibit a C1BF with leaf structure $(S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, S^3)$. Consider the C1BF diagram

$$\begin{tikzcd}
& \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \\
T^2 \arrow[ur, equal] & & \mathrm{Sp}(1) \arrow[ul, equal] \\
& S^1 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \times S^2 \\
G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 \\
G//K^+ \simeq S^3
\end{array}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, w) \mapsto (zw, zw^2, w, z)$$

$$K^+ \to G \times G;\ p \mapsto (p, p, 1, p)$$

$$H \to G \times G;\ z \mapsto (z, z, 1, z)$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $G//H \simeq S^3 \times S^2$ and it follows from Proposition 1.3.7 that $G//K^+ \simeq S^3$.

Structure 4.3.14. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2, S^3)$. Consider the C1BF diagram

$$\begin{tikzcd}
& \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \\
T^2 \arrow[ur, equal] & & \mathrm{Sp}(1) \arrow[ul, equal] \\
& S^1 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \times S^2 \\
G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2 \\
G//K^+ \simeq S^3
\end{array}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,w) \mapsto (z^2, wz^n, 1, z^n)$$
$$K^+ \to G \times G;\ p \mapsto (1, p, 1, 1)$$
$$H \to G \times G;\ w \mapsto (1, w, 1, 1)$$

for n odd. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}P^2\# - \mathbb{C}P^2$. It is clear that $G//H \simeq S^3 \times S^2$ and $G//K^+ \simeq S^3$.

Structure 4.3.15. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, \mathbb{C}P^2\# - \mathbb{C}P^2, S^2 \times S^2)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\ T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\ & S^1 & \end{tikzcd}$$

$$G//H \simeq S^3 \times S^2$$
$$G//K^- \simeq \mathbb{C}P^2\# - \mathbb{C}P^2$$
$$G//K^+ \simeq S^2 \times S^2$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,w) \mapsto (z^2, wz^n, 1, z^n)$$
$$K^+ \to G \times G;\ (z,w) \mapsto (z^2, wz^m, 1, z^m)$$
$$H \to G \times G;\ w \mapsto (1, w, 1, 1)$$

with n odd and m even. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}P^2\# - \mathbb{C}P^2$ and $G//K^+ \simeq S^2 \times S^2$. It is also clear that $G//H \simeq S^3 \times S^2$.

Structure 4.3.16. Here we will exhibit a C1BF with leaf structure
$(S^3 \times S^2, \mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{C}P^2\#\mathbb{C}P^2)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\
 & S^1 &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \times S^2 \\
G//K^- \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2 \\
G//K^+ \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2
\end{array}$$

where the embeddings are given by

$$K^- \to G\times G;\ (z,w)\mapsto (zw, zw^2, w, z)$$

$$K^+ \to G\times G;\ (z,w)\mapsto (zw, zw^2, w, z)$$

$$H \to G\times G;\ w\mapsto (z, z, 1, z)$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^{\pm} \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $G//H \simeq S^3\times S^2$.

Structure 4.3.17. Here we will exhibit a C1BF with leaf structure $(S^3\hat{\times}S^2, \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2)$. This is a slight modification of Structure 4.3.16. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\
 & S^1 &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \hat{\times} S^2 \\
G//K^- \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2 \\
G//K^+ \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2
\end{array}$$

where the embeddings are given by

$$K^- \to G\times G;\ (z,w)\mapsto (zw, zw^2, w, z)$$

$$K^+ \to G\times G;\ (z,w)\mapsto (zw, zw^2, w, z)$$

$$H \to G\times G;\ w\mapsto (w, w^2, w, 1)$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^{\pm} \simeq \mathbb{C}P^2 \# \mathbb{C}P^2$ and $G//H \simeq S^3 \hat{\times} S^2$.

Structure 4.3.18. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \times S^2, \mathbb{C}P^2\# - \mathbb{C}P^2, \mathbb{C}P^2\# - \mathbb{C}P^2)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, dash, double] \arrow[dr, dash, double] & \\ T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{gathered} G//H \simeq S^3 \times S^2 \\ G//K^- \simeq \mathbb{C}P^2\# - \mathbb{C}P^2 \\ G//K^+ \simeq \mathbb{C}P^2\# - \mathbb{C}P^2 \end{gathered}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, w) \mapsto (z^2, wz^n, 1, z^n)$$

$$K^+ \to G \times G;\ (z, w) \mapsto (z^2, wz^n, 1, z^n)$$

$$H \to G \times G;\ w \mapsto (1, w, 1, 1)$$

for n odd. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^{\pm} \simeq \mathbb{C}P^2\# - \mathbb{C}P^2$ and clearly $G//H \simeq S^3 \times S^2$.

Structure 4.3.19. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \hat{\times} S^2, \mathbb{C}P^2\# - \mathbb{C}P^2, \mathbb{C}P^2\# - \mathbb{C}P^2)$. Consider the C1BF diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, dash, double] \arrow[dr, dash, double] & \\ T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{gathered} G//H \simeq S^3 \hat{\times} S^2 \\ G//K^- \simeq \mathbb{C}P^2\# - \mathbb{C}P^2 \\ G//K^+ \simeq \mathbb{C}P^2\# - \mathbb{C}P^2 \end{gathered}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,w) \mapsto (z^2, wz^n, 1, z^n)$$

$$K^+ \to G \times G;\ (z,w) \mapsto (z^2, wz^n, 1, z^n)$$

$$H \to G \times G;\ z \mapsto (z^2, z^n, 1, z^n)$$

for n odd. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^{\pm} \simeq \mathbb{C}\mathrm{P}^2\# - \mathbb{C}\mathrm{P}^2$ and $G//H \simeq S^3 \hat{\times} S^2$.

Structure 4.3.20. Here we will exhibit a C1BF with leaf structure $(S^3 \times S^2, \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2, S^2 \times S^2)$. Consider the C1BF diagram

$$\begin{tikzcd} & \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, dash, shift left] \arrow[dl, dash, shift right] \arrow[dr, dash, shift left] \arrow[dr, dash, shift right] & \\ T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\ & S^1 & \end{tikzcd} \qquad \begin{aligned} G//H &\simeq S^3 \times S^2 \\ G//K^- &\simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2 \\ G//K^+ &\simeq S^2\#S^2 \end{aligned}$$

where the embeddings are given by

$$K^- \to G \times G;\ (z,w) \mapsto (z^2 w, z^2 w^2, w, z^2)$$

$$K^+ \to G \times G;\ (z,w) \mapsto (z^2, z^2 w, 1, z^2)$$

$$H \to G \times G;\ z \mapsto (z^2, z^2, 1, z^2)$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//K^- \simeq \mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $G//K^+ \simeq S^2 \times S^2$. Note also that according to DeVito's classification the biquotient induced by the embedding

$$S^1 \to \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)^2; \quad z \mapsto (z, z, 1, z)$$

is diffeomorphic to $S^3 \times S^2$. The biquotient induced by the embedding $H \to G \times G$ in our diagram is orbit equivalent to this action, so we have $G/\!/H \simeq S^3 \times S^2$.

Structure 4.3.21. Here we will exhibit an infinite family of C1BFs with leaf structure $(S^3 \hat{\times} S^2, S^2, S^2)$. Consider the C1BF diagram

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[dl, equal] \arrow[dr, equal] & \\
S^1 \times \mathrm{Sp}(1) & & S^1 \times \mathrm{Sp}(1) \\
 & S^1 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}
\qquad
\begin{gathered}
G/\!/H \simeq S^3 \hat{\times} S^2 \\
G/\!/K^- \simeq S^2 \\
G/\!/K^+ \simeq S^2
\end{gathered}
$$

where the embeddings are given by

$$K^- \to G \times G;\ (z, p) \mapsto (z^2, z^n, 1, p)$$

$$K^+ \to G \times G;\ (z, p) \mapsto (z^2, z^n, 1, p)$$

$$H \to G \times G;\ z \mapsto (z^2, z^n, 1, z^n)$$

for any odd integer n. It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G/\!/H \simeq S^3 \hat{\times} S^2$ It follows from Proposition 1.3.7 that $G/\!/K^{\pm} \simeq S^2$.

Structure 4.3.22. Here we will exhibit a C1BF with leaf structure
of the form $(S^3 \hat{\times} S^2, \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2)$.
From DeVito's classification [DeV14] the homomorphisms $T^2 \to (\mathrm{Sp}(1) \times \mathrm{Sp}(1))^2$ given by $(z, w) \mapsto (zw, zw^2, w, z)$ and $(z, w) \mapsto (z^2, wz, 1, z)$ induce biquotients diffeomorphic to $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$, respectively. According to the equivalences taken advantage of in DeVito's classification, the biquotient induced by $(z, w) \mapsto (wz, z^2, z, 1)$ also has quotient $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$.

Consider the C1BF diagram

$$
\begin{tikzcd}[column sep=small]
& \mathrm{Sp}(1)\times\mathrm{Sp}(1) \arrow[dl, dash, double] \arrow[dr, dash, double] & \\
T^2 \arrow[dr, dash] & & T^2 \arrow[dl, dash] \\
& S^1 &
\end{tikzcd}
\qquad
\begin{array}{c}
G//H \simeq S^3 \hat{\times} S^2 \\
G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 \\
G//K^+ \simeq \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2
\end{array}
$$

where the embeddings are given by

$$
\begin{aligned}
K^- \to G \times G;\ (z,w) &\mapsto (zw, zw^2, w, z) \\
K^+ \to G \times G;\ (z,w) &\mapsto (wz, z^2, z, 1) \\
H \to G \times G;\ z &\mapsto (z, z^2, z, 1)
\end{aligned}
$$

It is clear that the diagram is consistent. By DeVito's classification of biquotients we have $G//H \simeq S^3 \hat{\times} S^2$ and $G//K^- \simeq \mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ and $G//K^+ \simeq \mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$.

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