CLOSED SEQUENTIAL PROCEDURES FOR TESTING HYPOTHESES ABOUT THE PARAMETER OF A RAYLEIGH DISTRIBUTION

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TABLE OF CONTENTS

LIST OF TABLES

Table		Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION AND BACKGROUND

General Discussion of Sequential Analysis

The characteristic feature of sequential analysis is that the number of observations required by the procedure is not determined in advance of the experiment. At stage m (m=1,2,...) of the experiment, an observation is taken and one of three decisions is made:

1. Accept the null hypothesis
2. Reject the null hypothesis
3. Continue the experiment by taking another observation.

The decision to terminate the experiment at any given stage depends on the results of observations previously made; consequently, the number of observations required to make a terminal decision (called the decisive sample number or DSN) is a random variable.

The principal advantage of sequential methods is that test procedures can be constructed which require, on the average, a substantially smaller number of observations than the most efficient test procedures based upon a predetermined sample size. The dollar savings associated with using a smaller number of samples can often be significant, depending on the cost of the test items and the related testing costs. Since many test programs have multiple objectives, sequential procedures may result in test items being available for accomplishing secondary objectives when these objectives could not ordinarily be achieved through fixed sample testing because of constraints on test item availability. Finally, in medical trials to determine which of two treatments is superior, there are strong ethical considerations for stopping the test as soon as possible (administering the inferior treatment to as few patients as possible).

The principal difficulty with fixed sample tests is that, inherent in their structure, they fail to take advantage of information accumulated <u>during</u> the course of the experiment. As an example, suppose one were interested in determining whether the reliability of a lot of electronic components was of an acceptable level. Suppose further that once the null and alternative hypotheses were specified, together with the required probabilities of Type I and Type II error, the test procedure turned out to be: test the reliability of 100 components and if five or more fail, reject the lot; otherwise, accept it. What if all of the first five components tested fail? Clearly, it would be imprudent to test the remaining 95 components. What if four of the first five items tested fail? One would certainly have compelling evidence to terminate the experiment at that point since it would be regarded as highly unlikely that, should the remaining 95 components be put on test, none would fail.

Sequential procedures are not always appropriate, as the following two examples should illustrate:

1. A situation which occurs (unfortunately) far too often is that the researcher comes to the statistician, data in hand, and requests assistance in analyzing the results. At this juncture, any discussion of fixed versus sequential testing is academic and every piece of data should be examined. Some value may be salvaged, however, if the data

were taken sequentially in that the statistician may analyze the data as though it had been collected in accordance with a sequential design and indicate the potential savings (if any) to the researcher so that he might use this information in the design of future trials.

2. Consider the case of an agronomist who hypothesizes that the infestation rate of weevils in pecan trees strongly depends on the soil chemistry of the ground in which the trees are grown and that he believes a particular soil chemistry would drastically reduce the weevil infestation rate of pecan trees grown therein. It would obviously not make sense for him to plant a pecan tree seed in the proposed type of soil, wait the several years required for the tree to reach maturity and bear pecans and observe the infestation rate; then based on this infestation rate, decide to accept his null hypothesis, reject it, or plant a new tree and repeat the process.

The point to be made by the above discussion is that sequential procedures are not always appropriate, but when they are they generally offer a viable alternative to fixed-sample procedures.

Although sequential statistical methods were known for some time before, until World War II these were mainly very simple or ad hoc rules. The formal theory known today as "sequential analysis" began in 1943 with Abraham Wald (1945) in America and G. A. Barnard (1946) in England, both men working in war-time industrial advisory groups. The most important discovery was Wald's sequential probability ratio test (SPRT) and an elegant body of theory surrounding this was soon developed. The basic reference to sequential analysis is the book by Wald (1947); comprehensive surveys of the field are given by Jackson (1960) and by Johnson (1961). Books by Ghosh (1970), Wetherill (1975) and Govindarajulu (1975) may be consulted for more recent developments.

The SPRT is usually constructed as a sequential test of one simple hypothesis against another. In most cases, a parametric form is assumed for the underlying probability density or probability mass function and the two simple hypotheses are completely specified by two values of the single unknown parameter of interest. The SPRT has an optimum property for these two hypotheses: there is no other test with at least as low probabilities of Type I and Type II error and with smaller expected sample size under either or both of the two hypotheses. Oftentimes, however, one is interested in the performance of this sequential procedure for values of the unknown parameter other than the two which have been specified. Unfortunately, one generally finds that the expected sample size of the SPRT is relatively large for values of the parameter between the two specified ones; i.e., a larger number of observations is expected for those cases in which one does not particularly care which decision is taken.

Another, sometimes more serious, difficulty with the SPRT is that it is an "open" procedure. This means that the number of observations is a random variable which is unbounded and which has positive probability of being greater than any given constant. Since it is usually difficult to provide for taking an arbitrarily large number of observations, the SPRT is frequently "closed" (also termed "truncated" or "restricted"). This effectively bounds the sample size regardless of the underlying parameter value. In addition to Wald (1947), other authors who have considered closed sequential procedures include Stockman and Armitage (1946), Bross (1952), Armitage (1957), Schneiderman and Armitage (1962a, 1962b), Spicer (1962), Choi (1968), Aroian (1968), Aroian and Robison

(1969), Elfring and Schultz (1973), Öksoy (1973), Goss (1974) and Schmee (1974).

Since decisions (accept the null hypothesis, reject the null hypothesis, continue testing) made at any particular stage of the SPRT are conditioned upon what has happened during previous stages, some rather complex conditional probabilities are involved. Historically, there have been three distinctly different approaches to the solution of the attendant problems:

1. <u>Approximation by a Continuous Process</u>. This is the original and still most widely used approach. The sequential observations are assumed to occur at constant intervals of time and the discrete ($\Delta t=1$) process is approximated by the continuous parameter process obtained in the limit as $\Delta t \to 0$. This approach leads to a process whose probabilistic properties are well known (normal diffusion process, Wiener process, Wiener-Levỳ process) by regarding the SPRT as a (continuous parameter) random walk between two absorbing barriers and a correspondence is established between decisive sample number and first passage time and between Type I and Type II error rates and absorption probabilities. This approach is known to lead to a conservative procedure (actual error rates less than those specified in determining the boundaries) but has persisted largely due to theoretical convenience. Although the theory surrounding this approach gives (rather wide) bounds on the actual error rates and the expected or average sample number (ASN), it fails to give any information on the distribution of the DSN and this may be regarded as a major shortcoming. In addition to Wald (1945, 1957), other authors who have elected to use this approach include Page (1954), Anderson (1960), Weiss (1962) and Suich (1968).

2. Monte Carlo Simulation. This term refers to large scale sampling experiments whereby many repetitions of the discrete process are conducted via computer simulation and the relative frequency of various terminal decisions and trial number of occurrence of these decisions is recorded. The extent to which the Monte Carlo trials approximate the true process is a function of the number of times the process is simulated; unlike the first approach, this technique leads to an (empirical) distribution of the DSN. The Monte Carlo technique is the one which was employed by the author during the study documented by this report. The first, and perhaps most significant, instance where this technique was employed was in the paper by Baker (1950). In an empirical investigation of the SPRT for testing the mean of a normal distribution with known variance, Baker demonstrated that:

a. The actual error rates are approximately thirty percent less than the error rates specified in determining the acceptance and rejection boundaries.

b. The average sample number (ASN) is underestimated when the SPRT is approximated by a continuous parameter Markov process.

c. The distribution of the DSN is positively skewed (long right tail), providing additional motivation for the use of truncated procedures.

The Monte Carlo approach has been also used by Genzi (1965), Read (1971), Monahan (1973), Alexander and Suich (1973) and Madsen (1974).

3. Direct Method. This method is due to Aroian (1968) and consists of numerical evaluation of truncated convolution integrals involving sums of random variables. The technique is quite general and permits calculation of error rates and the DSN distribution to any desired degree of

accuracy. It has been used very successfully in addressing many common problems by Aroian (1968), Aroian and Robison (1969), Öksoy (1973), Goss (1974) and Schmee (1974) among others. The author's attempts to use Aroian's direct method were unsuccessful. The exact nature of the problem and the algebraic complications encountered are discussed in detail in Appendix B.

The Sequential Probability Ratio Test (SPRT)

Let the random variable X have probability density (or mass) function $f(x,\theta)$ which is completely specified except for the value of the single unknown parameter θ and consider a test of the simple null hypothesis $H_0:\theta=\theta_0$ against the simple alternative hypothesis $H_1:\theta=\theta_1$. For any positive integer n, the probability that the random sample $x_1,\cdots,x_n$ is obtained is given by

$$p_{in}=f(x_1,\theta_i)\cdot f(x_2,\theta_i)\cdots f(x_n,\theta_i) \text{ where } \theta=\theta_i \quad (i=0,1).$$

The SPRT for testing H_0 against H_1 is conducted as follows:

1. Two constants a and r $(0<a<1<r<\infty)$ are chosen to give the test strength (α,β). By strength (α,β) it is meant that

$$P(\text{reject } H_0|\theta=\theta_0)=\alpha \text{ and}$$

$$P(\text{accept } H_0|\theta=\theta_1)=\beta.$$

In other words, a test has strength (α,β) if the probabilities of committing a Type I and Type II error are α and β, respectively.

2. At stage n $(n=1,2,\cdots)$ of the experiment, the probability ratio p_{1n}/p_{0n} is computed and:

   a. if $p_{1n}/p_{0n}\leq a$, H_0 is accepted

   b. if $p_{1n}/p_{0n}\geq r$, H_0 is rejected

   c. if $a<p_{1n}/p_{0n}<r$, another observation is taken.

In many instances, it is more convenient to work with the natural logarithm of the probability ratio rather than the probability ratio itself. In these cases, let $A=\log a$, $R=\log r$; then at stage n of the experiment, compute $\lambda_n=\log(p_{1n}/p_{0n})$ and:

a. if $\lambda_n \leqslant A$, H_0 is accepted

b. if $\lambda_n \geqslant R$, H_0 is rejected

c. if $A<\lambda_n<R$, another observation is taken.

As an example, consider testing the mean μ of a normal distribution with unit variance and suppose that the hypotheses of interest are $H_0:\mu=0$ and $H_1:\mu=1$. The logarithm of the probability ratio is easily shown to be

$$\lambda_n = \sum_1^n x_i - \frac{n}{2}\ .$$

At stage n $(n=1,2,\cdots)$ one computes λ_n and:

a. if $\sum_1^n x_i \leqslant A + \frac{n}{2}$, H_0 is accepted

b. if $\sum_1^n x_i \geqslant R + \frac{n}{2}$, H_0 is rejected

c. if $A + \frac{n}{2} < \sum_1^n x_i < R + \frac{n}{2}$, another observation is taken.

Notice that the acceptance and rejection boundaries are functions of the sample number. This is shown graphically in Figure 1. The dependence of the boundaries on the sample number may be removed if we make the transformation $Z_i=X_i - \frac{1}{2}$. This is shown in Figure 2.

At this point, it seems natural to ask "What assurance does one have that this (open) procedure will not continue indefinitely?" Wald (1947) proved that if the underlying observations are independent, then the probability is 1 that the sequential probability ratio test procedure will eventually terminate. More generally, Ghosh (1970) shows that if

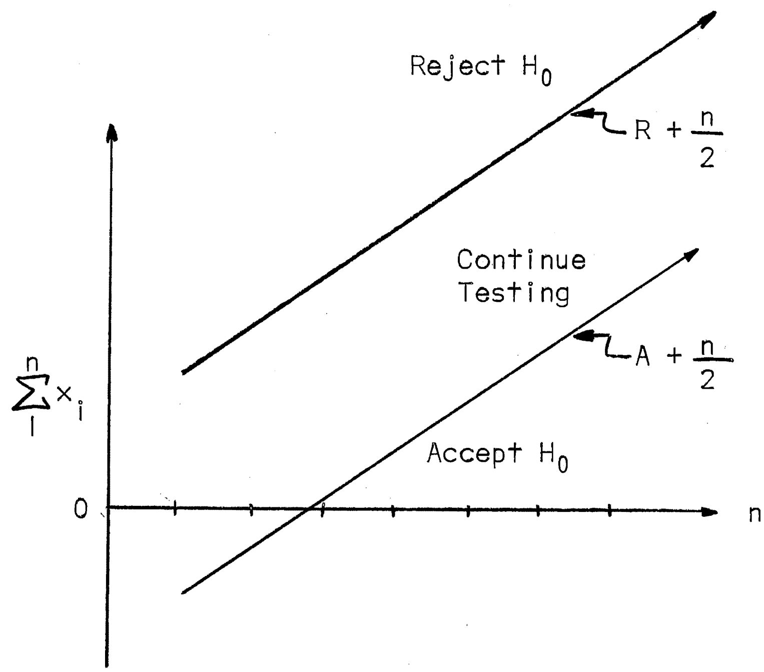

Figure 1. SPRT Boundaries (Raw Data)

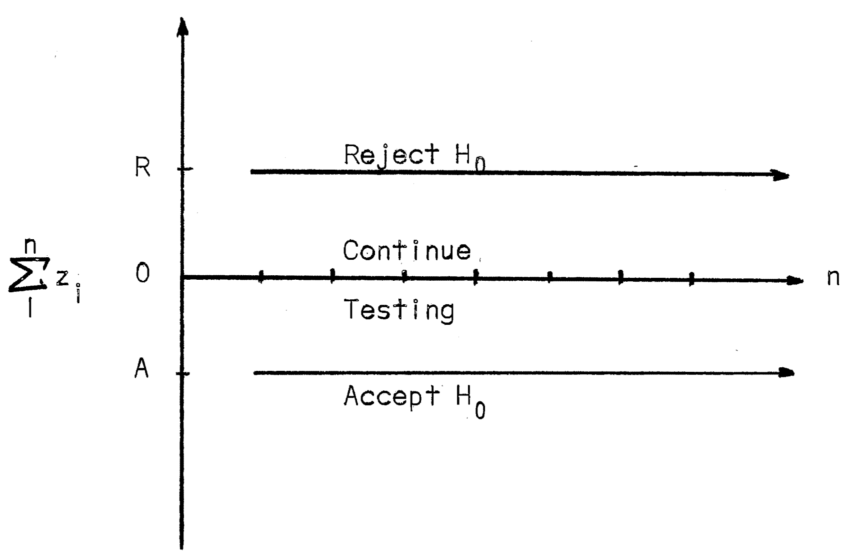

Figure 2. SPRT Boundaries (Transformed Data)

the underlying distribution is of the Koopman-Darmois or exponential form, then the process will eventually terminate with probability 1. In many fields of experimentation, however, this assurance carries little weight because of the uncertainty in knowing how long a particular sequential experiment may continue. There is, therefore, a natural tendency to truncate the SPRT at some particular point by specifying an upper limit, say m, for the number of observations to be taken. Wald (1947) gives a simple and reasonable rule for truncation at n=m: if the SPRT has not resulted in a terminal decision by the m-th trial, accept H_0 if $\sum_1^n z_i \leq 0$, otherwise reject H_0. Any SPRT truncated in the manner suggested by Wald will be denoted by TSPRT to distinguish it from closed boundaries obtained by other methods. The TSPRT is illustrated in Figure 3.

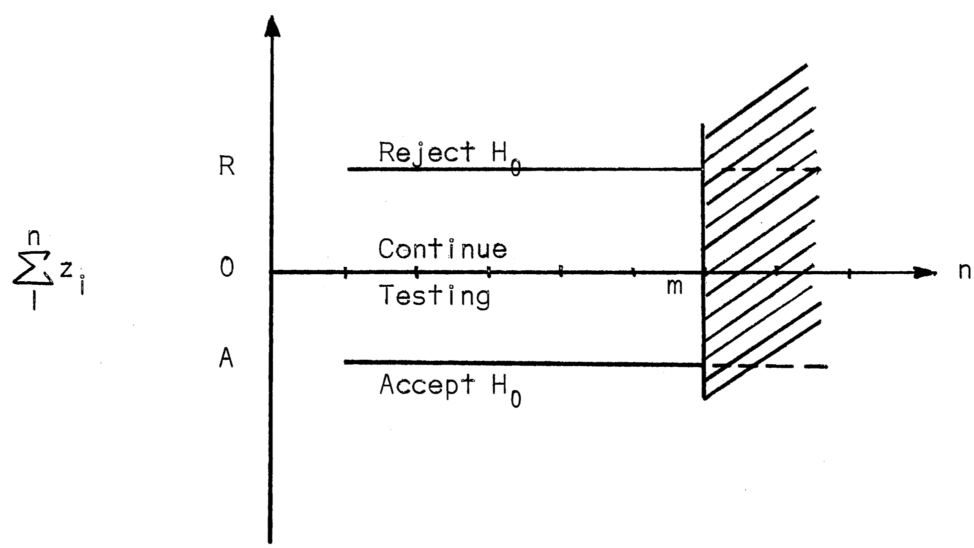

Figure 3. TSPRT Boundaries

Truncating the SPRT has two effects: it reduces the ASN for all values of the unknown parameter and it increases the probability of committing Type I and Type II errors. Wald gives upper bounds on the error rates when the SPRT is truncated in the manner described above.

Up to this point, discussion of how the two constants a and r are determined to give the test strength (α,β) has been avoided. Exact determination of the values a and r is often very laborious if not analytically intractable. Wald (1947) establishes the bounds

$$a \geqslant \beta/(1-\alpha) \quad \text{and} \quad r \leqslant (1-\beta)/\alpha.$$

He argues that these bounds are quite close to the required values and suggests treating the above inequalities as though they were equalities, i.e.,

$$a = \beta/(1-\alpha) \quad \text{and} \quad r = (1-\beta)/\alpha.$$

These approximations lead to a conservative test in the sense that the actual Type I and Type II error rates are less than the error rates specified in determining the boundaries. If one uses the continuous parameter Markov process approximation (as Wald did), then, at the termination of the test, the probability ratio p_{1n}/p_{0n} equals $(1-\beta)/\alpha$ or $\beta/(1-\alpha)$ exactly. The approximation arises from the fact that, at the termination of the discrete process, the ratio p_{1n}/p_{0n} exceeds or overshoots the boundaries. Wald states that the effect of "ignoring the excess over the boundaries" is, for all practical purposes, negligible. The author contends that the effect is <u>not</u> negligible and there is some evidence which suggests that the extent to which the SPRT is conservative is problem dependent (Baker, 1950; Goss, 1974; Schmee, 1974).

Problem Definition

A problem of interest in weapons effectiveness testing is the determination of whether a contractor has met the accuracy specifications in the production of a new munition. Accuracy is normally stated in terms of Circular Error Probable (CEP) which is defined as the median of the distribution of radial miss distances. It is the radius of the smallest circle about the target which is expected to contain fifty percent of the weapons delivered against that target. Although there are more meaningful ways of describing munition accuracy, CEP is the widely accepted and commonly used standard. Figure 4 gives a simple illustration of the relation between the components of miss distance and the radial miss distance.

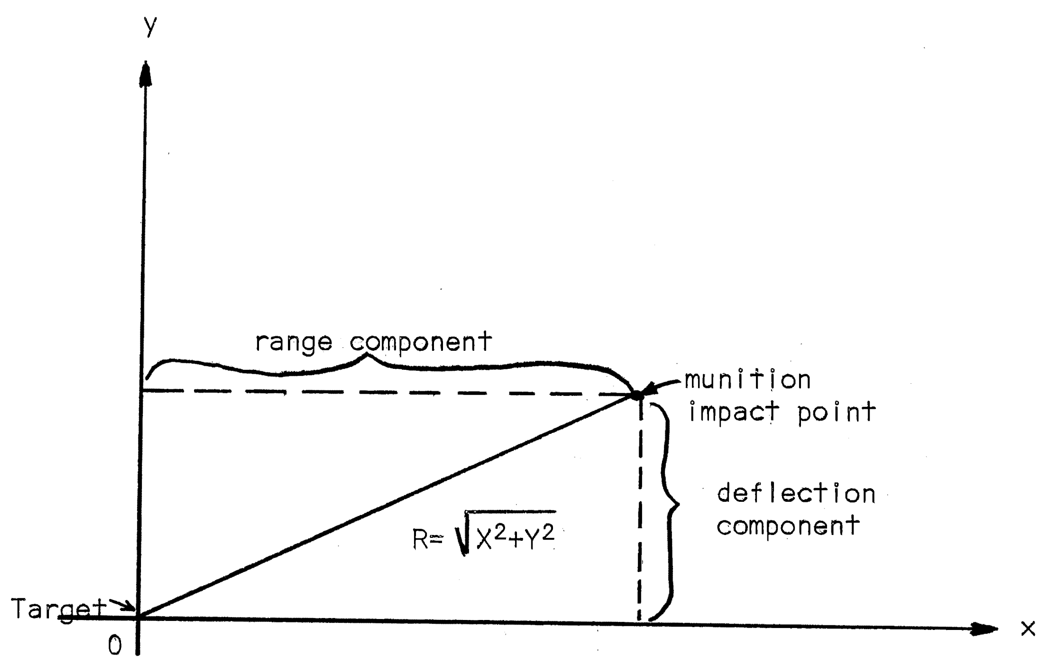

Figure 4. Relation Between Miss Distance Components

For unguided weapons, the range (X) and deflection (Y) components of miss distance are usually regarded as being independent random variables having a bivariate normal distribution with zero mean and common variance $\sigma_X^2=\sigma_Y^2=\sigma^2$; more succinctly,

$$(X,Y)\sim N_2(0,\sigma^2 I).$$

If one transforms from rectangular to polar coordinates, it may be shown (Lindgren, 1968) that the radial miss distance $R=\sqrt{X^2+Y^2}$ has a Rayleigh distribution with density function

$$f(R)=(R/\sigma^2)\exp(-R^2/2\sigma^2),\ R>0$$

and cumulative distribution function

$$F(R)=1-\exp(-R^2/2\sigma^2),\ R>0.$$

The linear relationship between the CEP (median) and the parameter σ of the Rayleigh distribution is easily found using the cumulative distribution function F(R):

$$CEP=\sqrt{2\cdot\log 2}\sigma \doteq 1.17741\sigma.$$

Thus, instead of testing CEP, one may equivalently test σ or σ^2. Consider testing $H_0:\sigma=\sigma_0$ versus $H_1:\sigma=\sigma_1$ $(\sigma_0<\sigma_1)$. For the Raleigh distribution

$$\frac{P_{1n}}{P_{0n}} = \frac{\left(\frac{1}{\sigma_1^2}\right)^n\left(\prod_1^n R_i\right)\exp\left(-\frac{1}{2\sigma_1^2}\sum_1^n R_i^2\right)}{\left(\frac{1}{\sigma_0^2}\right)^n\left(\prod_1^n R_i\right)\exp\left(-\frac{1}{2\sigma_0^2}\sum_1^n R_i^2\right)}$$

$$= \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^n\exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_1^n R_i^2\right].$$

Since it was assumed that X and Y are independent, $X \sim N(0,\sigma^2)$ and $Y \sim N(0,\sigma^2)$, a sample of n Rayleigh observations could alternatively be regarded as being a sample of 2n observations from $N(0,\sigma^2)$ (n observations on X and n observations on Y). Doing so gives

$$\frac{p_{1n}}{p_{0n}} = \frac{\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma_1^2}\sum_1^n x_i^2\right)\cdot\left(\frac{1}{\sigma_1\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma_1^2}\sum_1^n y_i^2\right)}{\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma_0^2}\sum_1^n x_i^2\right)\cdot\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma_0^2}\sum_1^n y_i^2\right)}$$

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^{2n} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_1^n (x_i^2+y_i^2)\right]$$

which is exactly the same as the previous result. This shows that the SPRT for testing the parameter of a Rayleigh distribution is equivalent to the SPRT for testing the variance of a normal distribution with zero mean <u>if</u> in the latter case one regards the observations as being taken in groups of size two.

At this point, it is appropriate to consider some complications which arise when performing a SPRT in groups of size g rather than on individual samples. Clearly, grouping can only increase the number of observations required by the test. Recall that the acceptance and rejection boundaries were approximated by neglecting the "excess over the boundaries" and that this resulted in a conservative test. The effect of grouping is to increase the "excess over the boundaries" with the result that the procedure becomes even more conservative. Wald offers the flip comment that "this feature of grouping compensates, to some extent, for the increase in the number of observations." To quote Ghosh (1970), "As g increases, the Wald approximations progressively overestimate the true risks but underestimate the true ASN of the grouped SPRT."

This thesis documents a detailed empirical investigation of the (open) SPRT and several closed sequential tests of hypotheses for the parameter of a Rayleigh distribution, using Monte Carlo simulation. The problem of adjusting the acceptance and rejection boundaries to give the specified error rates is studied in detail. This amounts to choosing an α' and a β' and then constructing boundaries

$$a'=\beta'/(1-\alpha') \quad \text{and} \quad r'=(1-\beta')/\alpha'$$

which result in the test actually having strength (α,β). For the SPRT, boundary adjustment is felt to be highly significant in its own right because it reveals the extent to which the unadjusted SPRT is conservative and because it indicates the (average) savings which may be realized by adjusting the boundaries to give the specified error rates. For other sequential procedures (closed and open), boundary adjustment permits a direct DSN distribution comparison among procedures having (approximately) the same power. This boundary adjustment problem has not been previously addressed in the literature.

Additionally, closed procedures are examined to determine the effect of delaying accept or reject decisions until several observations have been taken and the effect of continuing the sequential procedure beyond the fixed-sample size n. Letting (d,m) denote a sequential probability ratio test for which no decisions are made until after d observations have been taken and for which m items are available for testing, (n,n) then refers to a fixed-sample size test and a $(1,\infty)$ plan is an open SPRT. The TSPRT closed at the fixed-sample size is then a (1,n) plan. In addition to $(1,\infty)$ and $(1,n)$ plans, the author also examined $(n/3,n)$, $(1,2n)$ and $(n/3,2n)$ plans. The last three types of plans will be collectively referred to as extended sequential probability ratio tests (EXSPRT's).

Chapter II of this thesis describes the ten cases studied and the Monte Carlo procedures employed. Chapter III presents the results of Monte Carlo simulation of the SPRT for these cases and treats the SPRT boundary adjustment problem. Chapter IV gives parallel results for the TSPRT. The EXSPRT plans are examined and contrasted with the SPRT and TSPRT plans in Chapter V. Chapter VI provides a summary of this study, together with conclusions and recommendations for future work.

Appendix A contains tables of unadjusted and adjusted $(1,\infty)$, $(1,n)$, $(n/3,n)$, $(1,2n)$ and $(n/3,2n)$ plans. Appendix B provides a discussion of the author's unsuccessful attempts to use Aroian's direct method. The computer program used for this study is documented in Appendix C.

CHAPTER II

DESCRIPTION OF CASES STUDIED

Ten cases were selected for comparing the various sequential procedures, representing two different discrimination ratios (σ_1/σ_0) and five different strengths (α,β):

Case	σ_1/σ_0	α	β	Case	σ_1/σ_0	α	β
1	2.0	0.100	0.100	6	1.5	0.100	0.100
2	2.0	0.050	0.100	7	1.5	0.050	0.100
3	2.0	0.050	0.050	8	1.5	0.050	0.050
4	2.0	0.025	0.050	9	1.5	0.025	0.050
5	2.0	0.025	0.025	10	1.5	0.025	0.025

For convenience $\sigma_0=1.0$, $\sigma_1=2.0$ and $\sigma_0=1.0$, $\sigma_1=1.5$ were used for discrimination ratios of 2.0 and 1.5, respectively. To a large extent, choice of cases was arbitrary and was tempered by the amount of computer time available to perform the Monte Carlo simulations.

Determination of Fixed-Sample Size

Since some of the sequential procedures addressed in this study are closed at the fixed-sample size, determination of the number of observations required for a fixed-sample size test of strength (α,β) is a necessary prerequisite. For the Rayleigh density function and hypotheses $H_0:\sigma=\sigma_0$, $H_1:\sigma=\sigma_1$ ($\sigma_0<\sigma_1$), the likelihood ratio L_n for a sample of size n is given by

$$L_n = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^n \exp\left[\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_1^n R_i^2\right]$$

and the critical region is determined by some positive constant K and smallest sample size n which make the following two inequalities simultaneously true:

$$P(L_n > K | H_0) \leq \alpha \quad \text{and} \quad P(L_n < K | H_1) \leq \beta.$$

Inequalities, rather than equalities, are used since n is required to be an integer. Now

$$\begin{aligned} L_n > K &\Rightarrow \log L_n > \log K \\ &\Rightarrow 2n \log\left(\frac{\sigma_0}{\sigma_1}\right) + \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_1^n R_i^2 > \log K \\ &\Rightarrow \sum_1^n R_i^2 > \left[\log K + 2n \log\left(\frac{\sigma_1}{\sigma_0}\right)\right] \Big/ \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \\ &\Rightarrow \sum_1^n R_i^2 > K' \quad \text{where } K' = \left[\log K + 2n \log\left(\frac{\sigma_1}{\sigma_0}\right)\right] \Big/ \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right). \end{aligned}$$

Since X,Y are assumed to be independently and identically distributed as $N(0,\sigma^2)$, the quantity

$$\frac{1}{\sigma^2}\sum_1^n R_i^2 = \frac{1}{\sigma^2}\sum_1^n (X_i^2 + Y_i^2)$$

has a chi-square distribution with 2n degrees of freedom (χ^2_{2n}). For

$$Z \sim \chi^2_{2n},$$

$$g_Z(z) = \frac{1}{2^n \Gamma(n)} z^{n-1} e^{-z/2}, \quad z > 0;$$

$$G_Z(z) = \frac{1}{2^n \Gamma(n)} \int_0^z t^{n-1} e^{-t/2}\, dt;$$

$$\text{and} \quad G_Z(z) = \gamma \Rightarrow z = \chi^2_{2n,\gamma}.$$

Now $P(L_n > K | H_0) \leq \alpha$

$$\Rightarrow P(L_n \leq K | H_0) \geq 1-\alpha$$

$$\Rightarrow P\left(\sum_1^n R_i^2 \leq K' | H_0\right) \geq 1-\alpha$$

$$\Rightarrow P\left(\frac{1}{\sigma^2}\sum_1^n R_i^2 \leq \frac{K'}{\sigma^2}\right) \geq 1-\alpha$$

$$\Rightarrow G\left(\frac{K'}{\sigma_0^2}\right) \geq 1-\alpha$$

$$\Rightarrow \frac{K'}{\sigma_0^2} \geq \chi^2_{2n,1-\alpha} .$$

Similarly, $P(L_n \leq K | H_1) \leq \beta$

$$\Rightarrow \frac{K'}{\sigma_1^2} \leq \chi^2_{2n,\beta}.$$

From the above inequalities it follows that

$$\sigma_0^2 \cdot \chi^2_{2n,1-\alpha} \leq \sigma_1^2 \cdot \chi^2_{2n,\beta}$$

or $$\frac{\chi^2_{2n,1-\alpha}}{\chi^2_{2n,\beta}} \leq \left(\frac{\sigma_1}{\sigma_0}\right)^2 .$$

Thus for α, β, σ_0, σ_1 given, the required sample size, say m, is the smallest integral value of n satisfying the above inequality.

Example. For Case I, $\alpha=\beta=0.100$, $\sigma_0=1.0$, $\sigma_1=2.0$. The above inequality becomes

$$\frac{\chi^2_{2n,0.90}}{\chi^2_{2n,0.10}} \leq 4.0 .$$

n	$\chi^2_{2n,0.90}$	$\chi^2_{2n,0.10}$	$\chi^2_{2n,0.90}/\chi^2_{2n,0.10}$
1	4.605	0.211	21.825
2	7.779	1.064	7.311
3	10.645	2.204	4.830
4	13.362	3.490	3.829

From the above data, it is seen that m=4 Rayleigh observations are required for a fixed-sample test of $H_0:\sigma=1$ versus $H_1:\sigma=2$ at strength (α,β).

A computer program which employed the International Mathematics and Statistics Library (IMSL) subroutine for the cumulative χ^2 distribution (MDCHI) was written to obtain the required sample sizes for the ten cases under consideration. This could also have been accomplished using standard chi-square tables for 2n<30 and one of several approximations for 2n>30; however, for the range of values of n used for the cases under consideration, these approximations were not particularly satisfactory. The sample-size determination results are presented in Table I.

TABLE I

FIXED-SAMPLE SIZES FOR THE TEN CASES

Case	σ_1/σ_0	α	β	n
1	2.0	0.100	0.100	4
2	2.0	0.050	0.100	5
3	2.0	0.050	0.050	7
4	2.0	0.025	0.050	7
5	2.0	0.025	0.025	9
6	1.5	0.100	0.100	11
7	1.5	0.050	0.100	13
8	1.5	0.050	0.050	17
9	1.5	0.025	0.050	20
10	1.5	0.025	0.025	24

For each of the ten cases, all of the sequential procedures were evaluated at seven distinct values of the unknown parameter σ. The seven values used depended upon whether the discrimination ratio was 2.0 or 1.5:

Cases 1–5 $(\sigma_1/\sigma_0=2.0)$

	(H_0)				(H_1)	
0.500	1.000	1.250	1.500	1.750	2.000	2.500

Cases 6–10 $(\sigma_1/\sigma_0=1.5)$

	(H_0)				(H_1)	
0.750	1.000	1.125	1.250	1.375	1.500	1.750

By evaluating the various procedures at these seven parameter values, it was possible to construct reasonably accurate OC and ASN curves. The number of values of σ was to some extent dictated by available computer time; the choice of values was rather arbitrary and did not include the value of σ for which the ASN is a maximum. Maximum ASN occurs for that value of σ for which the stochastic process has zero drift; i.e.,

$$E(\log L_n) = 0 = 2n\log\left(\frac{\sigma_0}{\sigma_1}\right) + \frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{1}^{n} E(R_i^2)$$

$$= 2n\log\left(\frac{\sigma_0}{\sigma_1}\right) + n\sigma^2\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)$$

$$\Rightarrow \sigma = \sqrt{\frac{\log(\sigma_1^2/\sigma_0^2)}{\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)}}\ .$$

For discrimination ratios of 2.0 and 1.5, maximum ASN occurs for $\sigma=1.35956$ and $\sigma=1.20817$, respectively.

Number of Monte Carlo Iterations

The extent to which the Monte Carlo trials approximate the true process is proportional to the number of times the process is simulated. A preliminary trade-off study between computer time requirements and desired accuracy of empirical error rate estimates led to the selection of 10,000 Monte Carlo iterations of the sequential process for each of the ten cases. For the range of σ and β values considered, 10,000 iterations results in a standard error (in estimating α and β) on the order of 0.0025. Loosely speaking, this means that the error rates are estimated to within ± 0.005 with probability 0.95.

Generation of Uniform Random Variables

A standard simulation technique for generating random numbers distributed according to any absolutely continuous probability law is to first generate random numbers which are uniform on the interval (0,1) and then to map these uniform random numbers into random numbers obeying the desired probability law via the appropriate probability integral transform.

Many uniform random number generators exist. Some are reportedly better than others; unfortunately, reports typically fail to provide substantive evidence of the comparisons which have allegedly been conducted. The author attempted to compare three uniform random number generators in hopes of making an intelligent choice among them. The three routines were:

1. RANDU - the routine used in IBM's Scientific Subroutine Package (SSP)

2. GGUB - The routine used in the International Mathematics and Statistics Library (IMSL); "reportedly" better than RANDU

3. RANF - the routine developed by J. P. Chandler of Oklahoma State University's Computer Science Department; "reportedly" better than GGUB.

Ten sets of 10,000 random numbers were generated using each of these three routines. Histograms with class intervals of 0.01, 0.05 and 0.10 were constructed and the appropriate chi-square statistics computed. The means and variances, maximum and minimum values for each set of 10,000 random numbers as well as for the combined group of 100,000 numbers were calculated. Unhappily, the results of the comparison were inconclusive. The author finally elected to use RANF which, although approximately 30 percent slower than the other generators, does not exhibit the serial correlation between random numbers which is "reportedly" a shortcoming of the other two routines.

Generation of Rayleigh Random Variables

It is well known that if X is an absolutely continuous random variable with density function f(x) and distribution function F(x), then Y=F(x) is uniformly distributed on the unit interval. This result, known as the probability integral transformation, is especially useful in computer simulation. Recall that the Rayleigh distribution function is $F(R)=1-\exp(-R^2/2\sigma^2)$, $R>0$. Equating Y=F(R) and solving for R gives

$$R = \sigma\sqrt{-2\cdot\log(1-Y)}.$$

Thus, to generate a Rayleigh random variable with parameter σ_j, we first generate a uniform (0,1) random number Y and then compute

$$R^{(j)} = \sigma_j\sqrt{-2\cdot\log(1-Y)}.$$

To save computer time, the author generated seven Rayleigh random variables $R^{(1)},\cdots,R^{(7)}$ for each Y (one for each of the seven parameter values under consideration). Although $R^{(1)},\cdots,R^{(7)}$ are clearly functionally dependent, this dependence does not vitiate comparisons between procedures and these are the comparisons of interest.

Description of Monte Carlo Procedure

For each of the ten cases and for each sequential procedure, 10,000 Monte Carlo trials were performed. A trial may be described as follows. At stage i $(i=1,2,\cdots)$ generate a uniform random variable, then transform to obtain seven Rayleigh random variables. For $j=1,\cdots,7$ determine whether the likelihood ratio was in the continuation region prior to this stage. If so, compute the new likelihood ratio and determine if it results in an accept, reject or continue decision. If an accept or reject decision is made, record the trial number and the type decision made. If one or more of the seven processes are still in the continuation region, generate another uniform random number and repeat the above procedure, otherwise go on to the next trial.

At the end of the 10,000 Monte Carlo trials, the program generates, for each parameter value, the empirical OC curve value, standard error, mean (ASN) and standard deviation of the DSN distribution. A summary of the stopping history for each parameter value is also printed.

A listing of the FORTRAN source program developed for this study is given in Appendix C.

CHAPTER III

THE SEQUENTIAL PROBABILITY RATIO TEST (SPRT)

This chapter presents the result of performing 10,000 Monte Carlo trials of Wald's SPRT [$(1,\infty)$ plans] for each of the ten cases described in Chapter II. The extremely conservative nature of the SPRT is revealed and the problem of adjusting the Wald boundaries to obtain the desired error rates is discussed. Finally, unadjusted and adjusted SPRT procedures are contrasted to illustrate the payoff realized from removing the "conservatism" of the unadjusted SPRT.

The conservative nature of the SPRT is shown in Table II where α_S, β_S and $\hat{\alpha}$, $\hat{\beta}$ denote specified and observed error rates, respectively.

TABLE II

SPECIFIED VERSUS OBSERVED ERROR RATES FOR $(1,\infty)$ PLANS

Case	α_S	$\hat{\alpha}$	$\hat{\alpha}/\alpha_S$	β_S	$\hat{\beta}$	$\hat{\beta}/\beta_S$
1	.1000	.0280	.2800	.1000	.0666	.6660
2	.0500	.0121	.2420	.1000	.0629	.6290
3	.0500	.0123	.2460	.0500	.0331	.6620
4	.0250	.0054	.2160	.0500	.0312	.6240
5	.0250	.0075	.3000	.0250	.0135	.5400
6	.1000	.0440	.4400	.1000	.0815	.8150
7	.0500	.0238	.4760	.1000	.0762	.7620
8	.0500	.0236	.4720	.0500	.0374	.7480
9	.0250	.0111	.4440	.0500	.0362	.7240
10	.0250	.0102	.4080	.0250	.0186	.7440

It should be noted that the observed Type I error rates range from .22 to .30 of the specified values for Cases 1-5 and from .41 to .48 of the specified values for Cases 6-10. The Type II error rates range from .54 to .67 and from .77 to .82 of the specified values for Cases 1-5 and Cases 6-10, respectively. Two observations may be made:

1. The SPRT is substantially more conservative under H_0 than under H_1.

2. The SPRT is more conservative for a higher discrimination ratio (Cases 1-5) than for a lower discrimination ratio (Cases 6-10).

Figure 5 illustrates an interesting conjecture that the sum of the observed error rates $(\hat{\alpha}+\hat{\beta})$ appears to be bounded by the sum of the specified error rates $(\alpha_s+\beta_s)$ divided by the discrimination ratio.

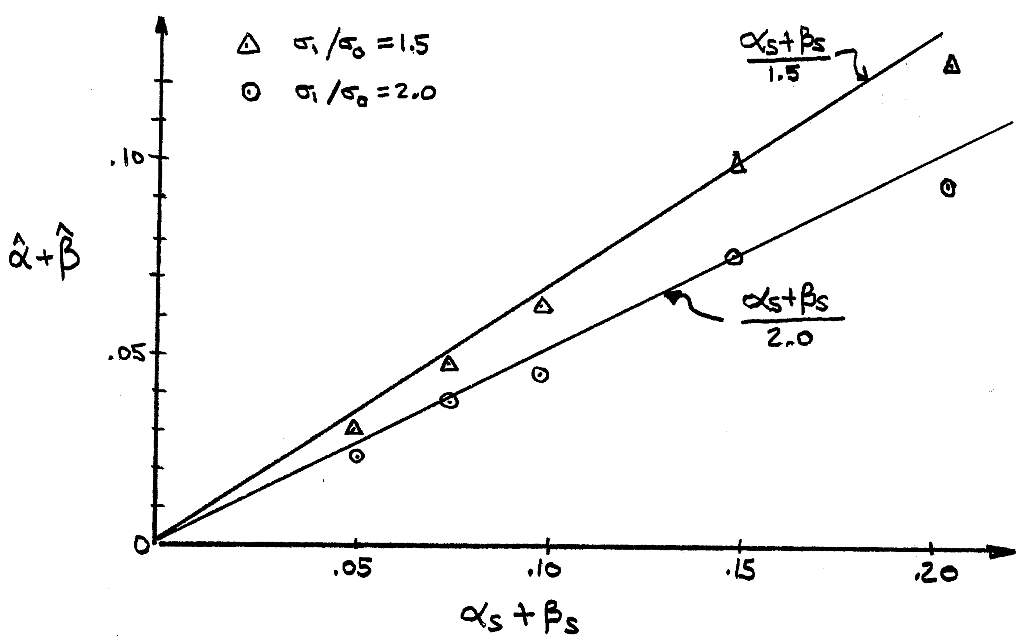

Figure 5. Relation Between Specified and Observed Error Rates

Further discussion and display of data for the (unadjusted) SPRT will be deferred until after the boundary adjustment issue is addressed to permit a more convenient comparison of unadjusted and adjusted SPRT procedures.

SPRT Boundary Adjustment

Recall that when working with the logarithm of the likelihood ratio, the Wald acceptance and rejection boundaries are given by $A=\log[\beta/(1-\alpha)]$ and $R=\log[(1-\beta)/\alpha]$, respectively. For each of the ten selected cases, $\hat{\alpha}<\alpha_s$ and $\hat{\beta}<\beta_s$. Suppose it is desired to "adjust" the SPRT procedure by stipulating an α' and a β' and then computing an acceptance boundary A' and a rejection boundary R' which result in $\hat{\alpha}=\alpha_s$ and $\hat{\beta}=\beta_s$. How should one proceed? For definiteness, consider Case I in which $\alpha_s=\beta_s=0.10$ gives $A=2.197$, $R=2.197$; the corresponding error rates are $\hat{\alpha}=0.0280$ and $\hat{\beta}=0.0666$. What values of α', β' should be chosen to give $\hat{\alpha}=0.100$, $\hat{\beta}=0.100$? The answer is not at all clear. The problem is complicated by the fact that if we fix α and change β, both A and R are affected; similarly, for fixed β, a change in α results in changes in both A and R.

It seems a bit more intuitive to adjust the boundaries, rather than α and β. This can be done by rewriting the equations for A and R in terms of α and β as equations for α and β in terms of A and R. Doing so yields $\alpha=(1-e^A)/(e^R-e^A)$ and $\beta=[e^A(e^R-1)]/(e^R-e^A)$.

Largely through trial and error and guided by trends as they arose, the author succeeded in selecting boundaries which gave "approximately" the specified error rates. By "approximately" it is meant that the difference between specified and observed error rates is on the order of one standard error (of the observed error rates).

Basically, for Cases 1-5, A was reduced by 0.50 and R by 1.35; for Cases 6-10, A was reduced by 0.25 and R by 0.80. If this adjustment did not result in $\hat{\alpha},\hat{\beta}$ within about one standard error of α_S,β_S, a second adjustment was made.

Comparison of Unadjusted and Adjusted SPRT's

Tables VII through XVI of Appendix A provide a comparison of the unadjusted and adjusted $(1,\infty)$ plans for each of the ten selected cases. For every case, the operating characteristic $L(\sigma)$, its standard error, the mean (ASN) and standard deviation of the DSN and the probability of requiring more observations than the corresponding fixed-sample test are presented for each of the seven values of the unknown parameter σ specified in Chapter II.

For illustrative purposes, a comparison between the unadjusted and adjusted SPRT for Case 1 will now be given in some detail. Similar comparisons for the other nine cases may be made using the data provided in Tables VIII - XVI of Appendix A.

Figure 6 is a graph of the ASN for the unadjusted and adjusted SPRT's plotted against the seven values of the unknown parameter. Two observations may be made:

1. For values of the unknown parameter between $\sigma=1.0$ and $\sigma=1.7$, the unadjusted SPRT requires more observations (on the average) to reach a decision than the corresponding fixed-sample test. It is believed that the reason the unadjusted SPRT appears to require as many observations as the fixed-sample test under H_0 is due strictly to sampling error. This situation did not occur for any of the other

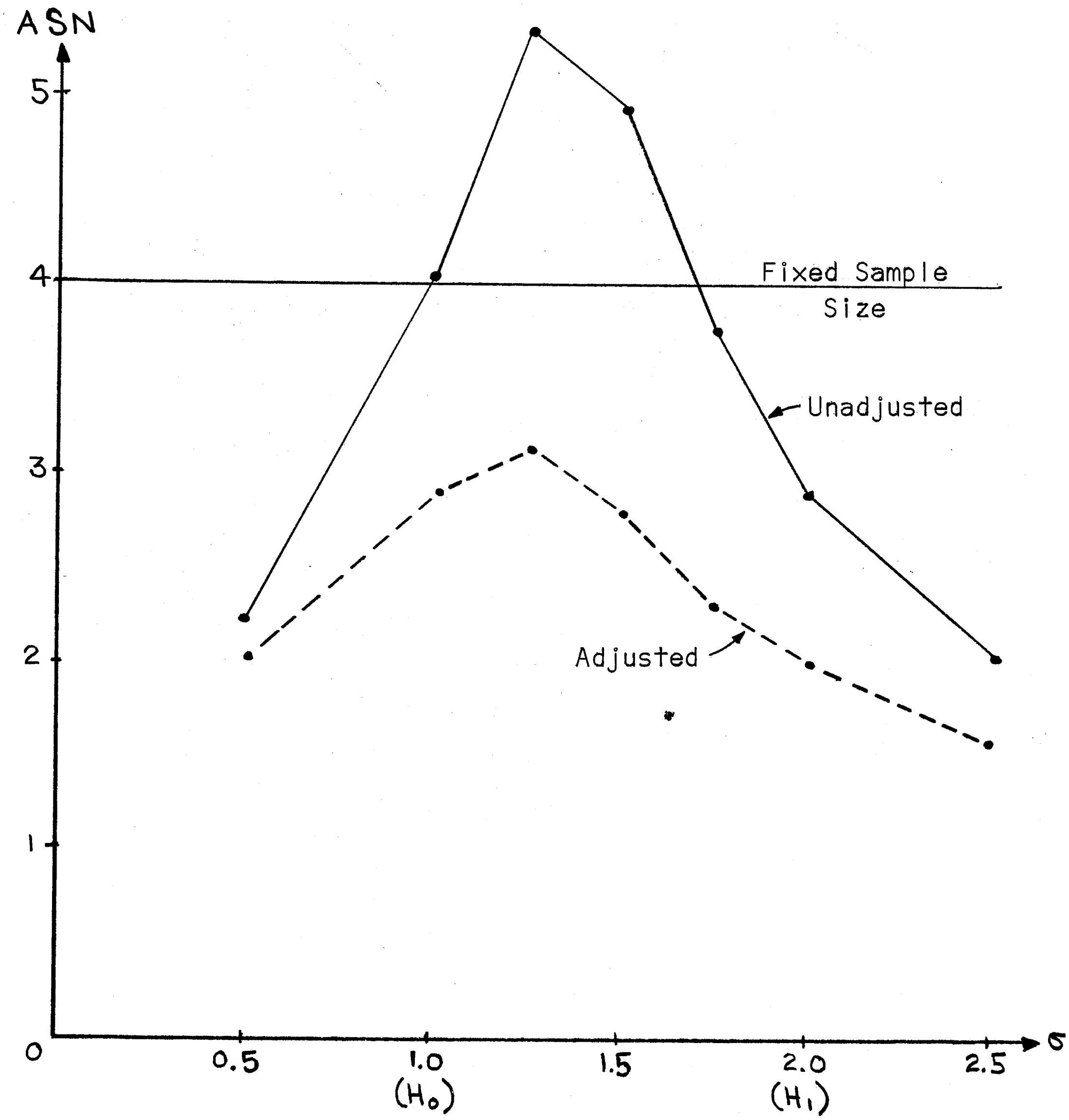

Figure 6. Average Sample Number (ASN) for Selected Values of Sigma

nine cases.

2. The adjusted SPRT requires substantially fewer observations to reach a decision than the unadjusted SPRT for all parameter values considered and never exceeds (on the average) the number of samples required for the corresponding fixed-sample test.

Figure 7 represents an alternative method of judging the significance of adjusting the SPRT to give the specified error rates. It graphically illustrates the probability of requiring more than the fixed-sample number of observations if one elects to use an (open) SPRT.

Importance of SPRT Boundary Adjustment

Adjusting the SPRT boundaries to give (approximately) the specified error rates may be regarded as important for two reasons:

1. In situations where an occasional large sample is acceptable (quality control, lot inspection, etc.) and an open sequential procedure is appropriate, substantial savings may be realized using the adjusted SPRT boundaries rather than the unadjusted boundaries. Depending on the cost of the test items, time and other resources required to conduct the test, and whether the testing is of a destructive nature, the savings afforded through using the adjusted boundaries may have a high dollar value. If one is reasonably certain that the assumptions underlying the test are met and if the specified error rates are meaningful, it would appear to be difficult to justify <u>not</u> using the adjusted boundaries. The time and money required to determine the proper adjustment via Monte Carlo simulation should generally be negligible compared with the dollar savings resulting from their use. It is hoped that further research in this area will produce algorithms which

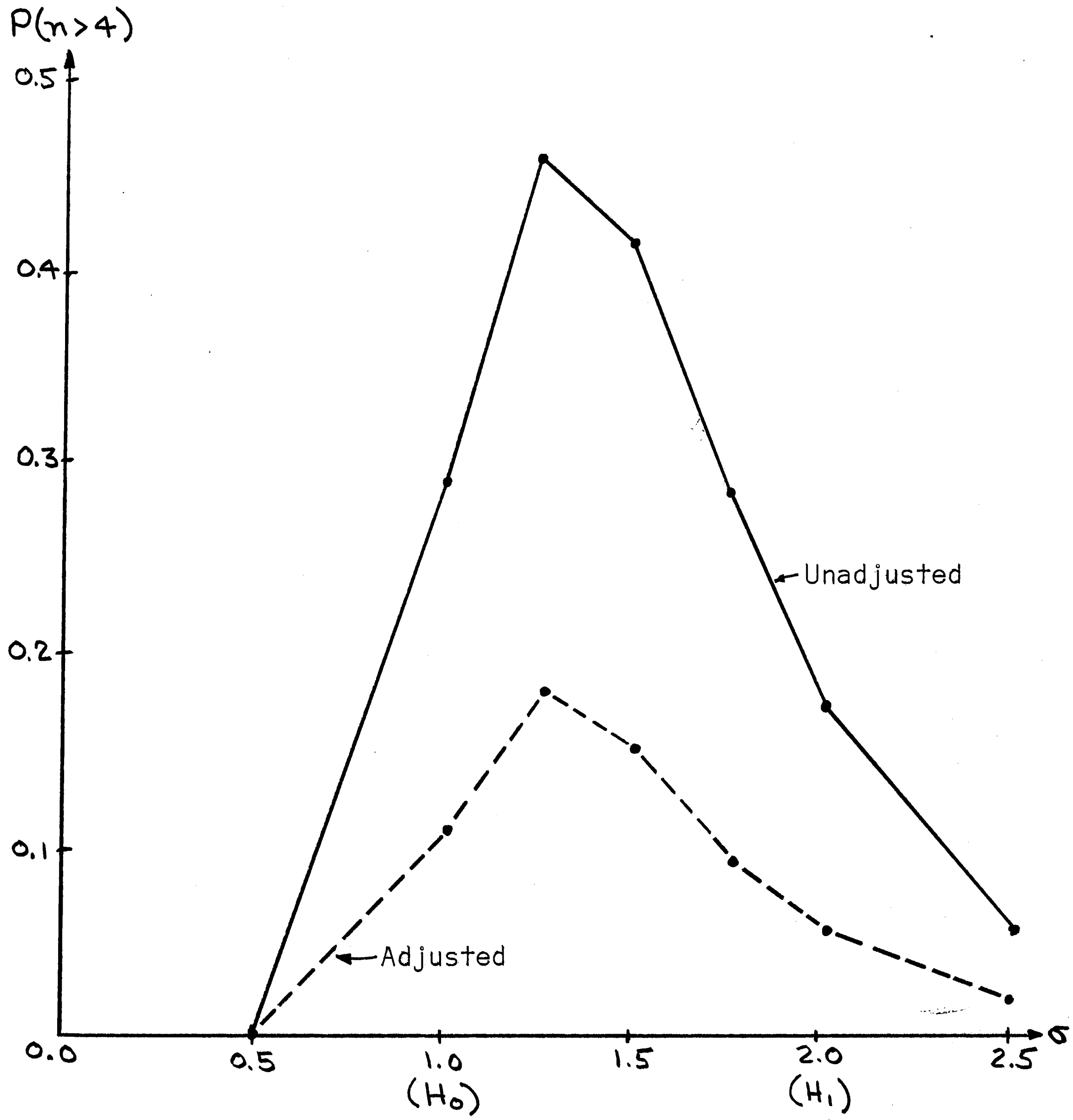

Figure 7. Probability That SPRT Requires More Observations Than Fixed-Sample Test

permit boundary adjustment without recourse to Monte Carlo simulation.

2. There does not appear to be any generally agreed upon method for comparing alternative sequential (or fixed-sample size) procedures when those procedures have different OC curves (or, equivalently, different power curves). By adjusting the SPRT boundaries to give the specified error rates and by performing the same adjustment on the TSPRT and other closed procedures described later in this report, one insures that at least two points on the OC curve (at $\sigma=\sigma_0$ and at $\sigma=\sigma_1$) will be the same. As it turns out, with this adjustment there does not appear to be any practical difference in the OC curves over the entire range of parameter values considered. Consequently, one may compare the various (adjusted) procedures directly and select the one whose DSN distribution has the most desirable properties. This point is regarded as quite significant and will be treated more fully in the ensuing chapters.

CHAPTER IV

THE TRUNCATED SEQUENTIAL PROBABILITY RATIO TEST (TSPRT)

This chapter presents the result of performing 10,000 Monte Carlo trials of Wald's TSPRT [(1,n) plans] for the ten cases defined in Chapter II. In every case, the TSPRT was obtained by truncating the SPRT at the fixed-sample size as shown in Figure 3. The problem of adjusting the boundaries to obtain the desired error rates is addressed and the unadjusted and adjusted TSPRT's compared. Finally, adjusted TSPRT's are contrasted with the adjusted SPRT's and the relative merits discussed.

The extent to which the specified error rates are achieved for the unadjusted TSPRT is shown in Table III where α_s, β_s and $\hat{\alpha}$,$\hat{\beta}$ denote the specified and observed error rates, respectively.

Note that the observed Type I error rates range from 0.58 to 1.14 of the specified values for Cases 1-5 and from 0.82 to 1.20 of the specified values for Cases 6-10. The Type II error rates range from 1.05 to 1.37 and from 1.17 to 1.35 of the specified values for Cases 1.5 and 6-10 respectively. The ratio of observed to specified error rates tends to be slightly higher for Cases 6-10 than for Cases 1-5 and this ratio tends to be higher for Type II error rates than for Type I error rates.

TABLE III

SPECIFIED VERSUS OBSERVED ERROR RATES FOR (1,n) PLANS

Case	α_s	$\hat{\alpha}$	$\hat{\alpha}/\alpha_s$	β_s	$\hat{\beta}$	$\hat{\beta}/\beta_s$
1	0.1000	0.0634	0.6340	0.1000	0.1304	1.3040
2	0.0500	0.0473	0.9460	0.1000	0.1050	1.0500
3	0.0500	0.0291	0.5820	0.0500	0.0562	1.1240
4	0.0250	0.0285	1.1400	0.0500	0.0594	1.1880
5	0.0250	0.0199	0.7960	0.0250	0.0343	1.3720
6	0.1000	0.0823	0.8230	0.1000	0.1260	1.2600
7	0.0500	0.0587	1.1740	0.1000	0.1167	1.1670
8	0.0500	0.0444	0.8880	0.0500	0.0669	1.3380
9	0.0250	0.0299	1.1960	0.0500	0.0586	1.1720
10	0.0250	0.0239	0.9560	0.0250	0.0338	1.3520

TSPRT Boundary Adjustment

One might argue that when the SPRT is truncated at the fixed-sample size, the specified error rates are met or only slightly exceeded. Consequently, one may make the practical assumption that this procedure gives approximately the specified error rates and use the procedure without adjustment. This argument notwithstanding, the author elected to adjust the TSPRT boundaries for two reasons:

1. If the specified error rates have quantitative meaning, the researcher may be quite concerned with the possibility of exceeding these error rates by as much as 37 percent.

2. If the TSPRT is adjusted to give the specified error rates, then the adjusted procedure may be directly compared with the adjusted SPRT since both procedures have essentially the same OC curve.

Largely through trial and error and guided by trends as they arose, the author succeeded in selecting TSPRT boundaries which gave "approximately" the specified error rates. By "approximately" it is meant that the difference between specified and observed error rates is on the order of one standard error (of the observed error rates).

Comparison of Unadjusted and Adjusted TSPRT's

Tables XVII - XXVI of Appendix A provide a comparison of the unadjusted and adjusted TSPRT for each of the ten cases. In each case, the operating characteristic L(σ), its standard error and the mean (ASN) and standard deviation of the DSN are presented for each of the seven values of the unknown parameter specified in Chapter II.

For illustrative purposes, a comparison between the unadjusted and adjusted (l,n) plans for Case I will now be given in some detail. Similar comparisons for the other nine cases may be made using the data provided in Tables XVIII - XXVI of Appendix A.

Figure 8 is a graph of the OC for the adjusted and unadjusted TSPRT plotted against the seven selected values of the unknown parameter. Note that the adjusted TSPRT is uniformly more powerful than the unadjusted TSPRT for the range of parameter values considered. This same trend holds for the other nine cases.

Figure 9 depicts the ASN for the adjusted and unadjusted TSPRT's. The fixed-sample size is also indicated on the graph for reference. The unadjusted TSPRT has a lower ASN than the adjusted TSPRT for parameter values less than 1.25. For parameter values greater than 1.25, the reverse is true. This occurs because the boundary adjustment makes the null hypothesis easier to reject for low parameter values and harder to accept for high parameter values. These trends are not true for all

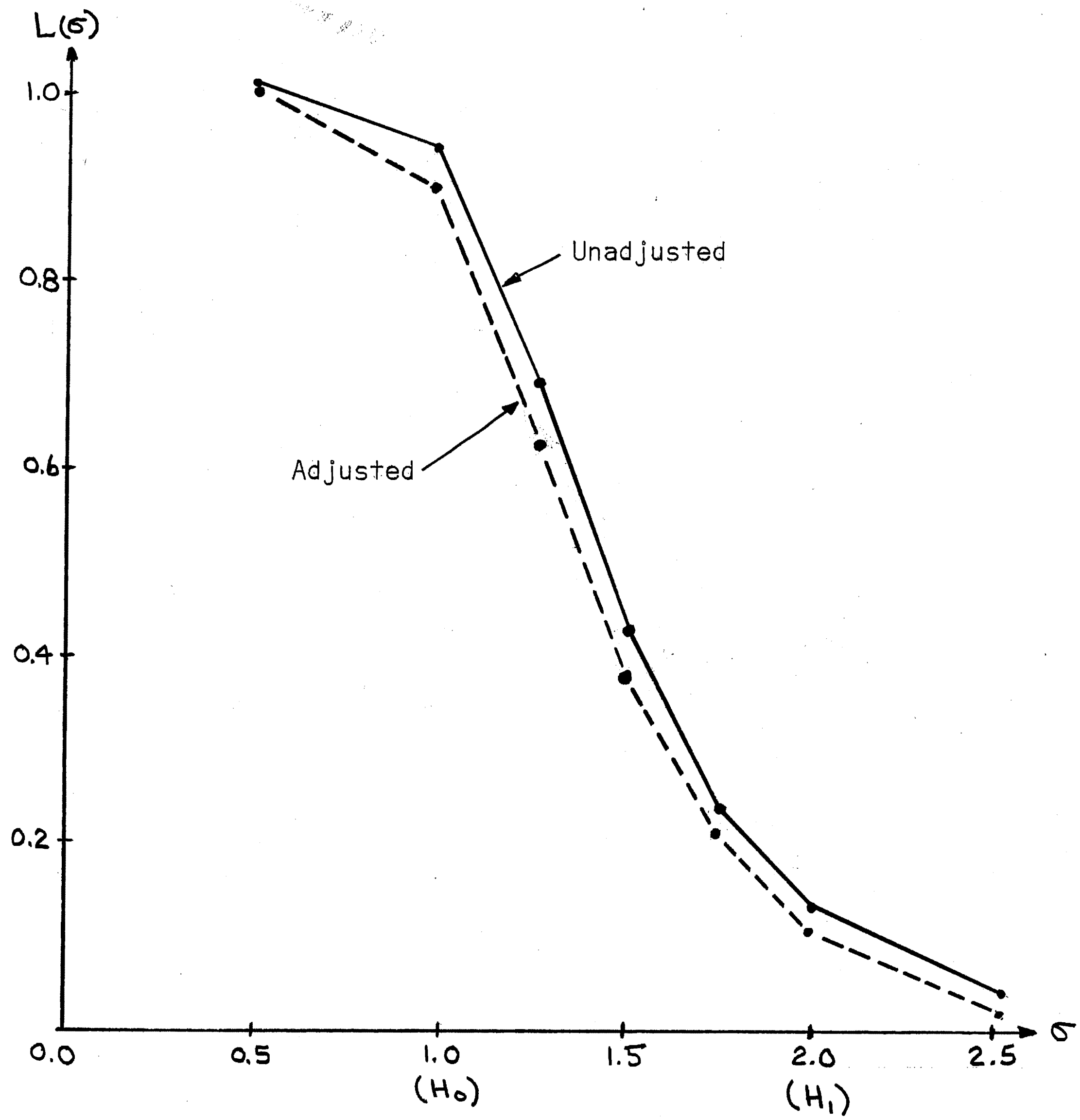

Figure 8. Comparison of OC Curve for Adjusted and Unadjusted TSPRT

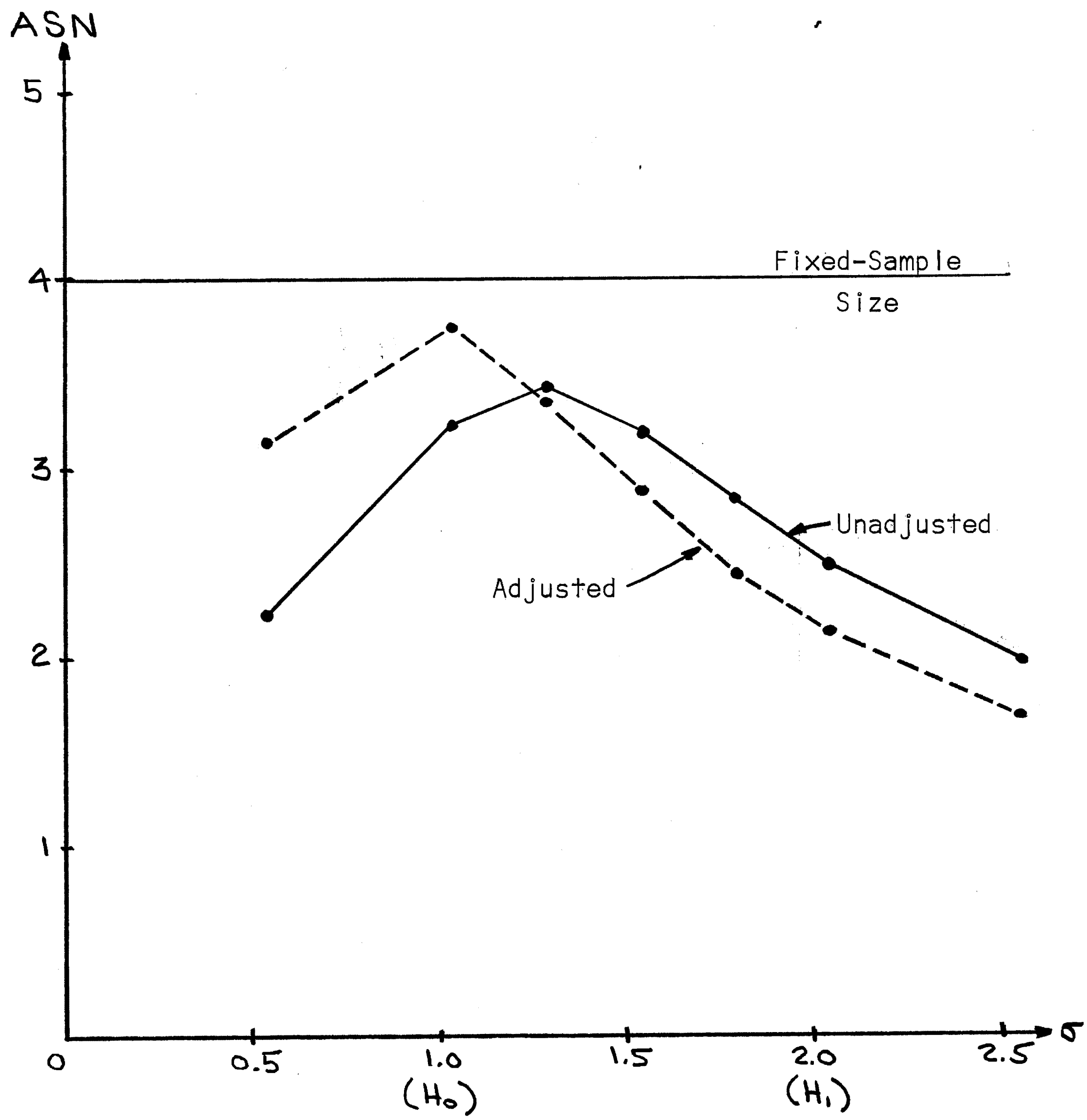

Figure 9. Comparison of ASN for Adjusted and Unadjusted TSPRT

cases; rather, they depend upon the direction and extent of the boundary adjustment for each case. For example, in Cases 4, 7 and 9 the unadjusted TSPRT resulted in both specified error rates being exceeded. The appropriate adjustment gave new boundaries that were outside the unadjusted boundaries; consequently, the ASN for the adjusted procedure is higher than for the unadjusted procedure for all parameter values.

Comparison of Adjusted TSPRT's With Adjusted SPRT's

A comparison of the OC curve for the adjusted TSPRT and the adjusted SPRT for Case I is shown in Figure 10. Recall that the procedures were adjusted to have the same OC value at two points on the OC curve (at H_0 and at H_1). The effect of this adjustment is to give the two procedures (approximately) the same power for all parameter values. This adjustment consequently permits a direct comparison of ASN for the SPRT and TSPRT as shown in Figure 11.

Notice that the ASN is consistently lower for the adjusted SPRT than for the adjusted TSPRT and that the difference is rather large for lower parameter values. For example, under H_0 ($\sigma=1$) the ASN is 2.89 and 3.68 for the SPRT and TSPRT, respectively. In this instance, one might expect a 22 percent savings by using the SPRT. From Table VII however, one observes that the SPRT will require more observations than the fixed-sample size test about 11 percent of the time.

Further discussion of the tradeoff between expected savings and probability of exceeding the fixed-sample size test will be given in the following chapter.

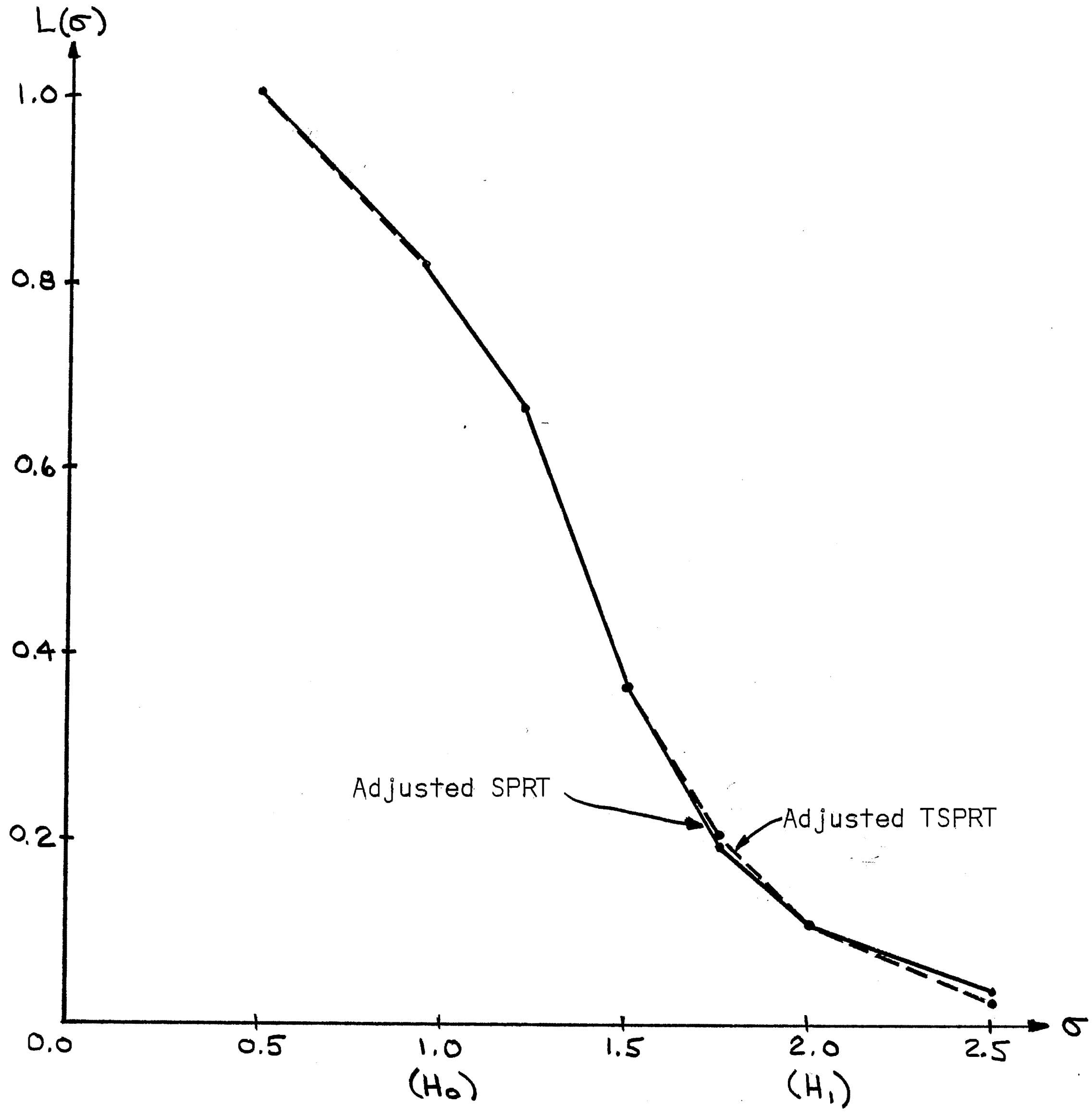

Figure 10. Comparison of OC Curve for Adjusted SPRT and Adjusted TSPRT

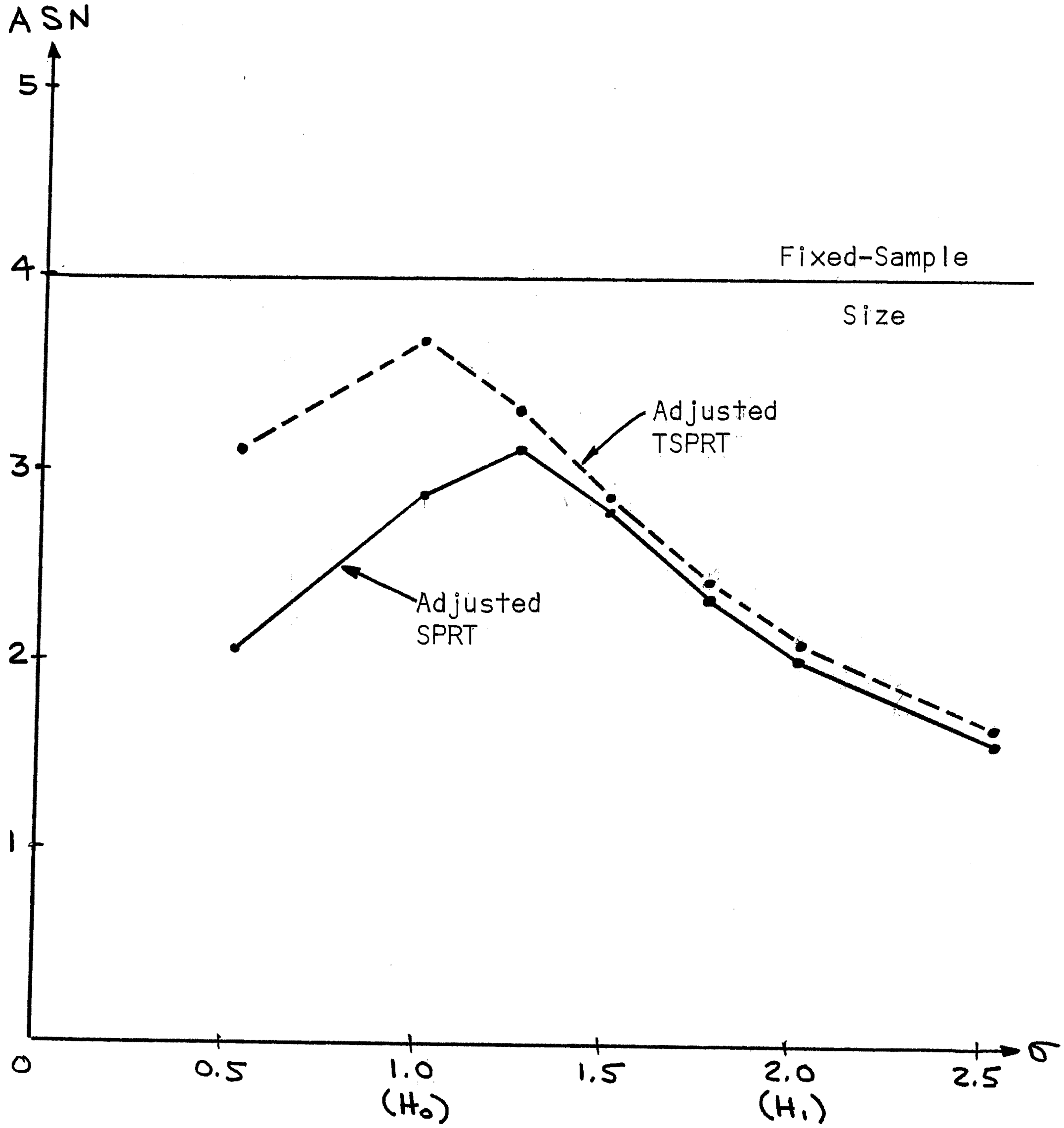

Figure 11. Comparison of ASN for Adjusted SPRT and Adjusted TSPRT

CHAPTER V

THE EXTENDED SEQUENTIAL PROBABILITY RATIO TEST (EXSPRT)

Clearly, the tradeoff between expected savings and probability of exceeding the fixed-sample size test can only be made in the context of a particular testing situation. In many quality control applications, sequential procedures are used repeatedly in judging product quality. In these instances, where an occasional large sample is acceptable, one should use the (adjusted) SPRT and take advantage of the attendant savings which it affords.

Now consider those situations in which the test procedure is going to be employed only once. How much emphasis should be placed on expected savings in the number of observations required to reach an accept or reject decision? If there is a limit to the number of items available for testing (and this is frequently the case), how should one proceed? This problem will be examined more critically in the following paragraphs.

Assume that, in the context of the problem at hand, meaningful probabilities of Type I and Type II error have been determined. Further suppose that, together with these error rates, null and alternative hypotheses have been specified which permit calculation of the number of observations, say n, required to conduct a fixed-sample size test. Let the number of items available for test be m and assume that m>n. (If

$m<n$, it will generally not be possible to achieve specified error rates.)

If, for economic or other reasons, it is desired to restrict the test to at most n test items then one must decide between a fixed-sample size test and the adjusted TSPRT (or some other procedure which is closed at the fixed-sample size and adjusted to give the specified error rates). Neither the adjusted nor the unadjusted SPRT is an admissible candidate since both give positive probability that more than n observations will be required. Moreover, the unadjusted TSPRT should not be employed since the probability of committing a Type I and/or Type II error is greater than the stipulated value.

If, on the other hand, it is permissible to use up to m test items, one may still eliminate the adjusted and unadjusted SPRT's for precisely the same reason given in the above paragraph. Now, however, several new candidates must be considered. If m is larger than n by a factor of two or more, then the unadjusted TSPRT (truncated after m samples) will probably not cause the specified error rates to be exceeded (Baker, 1950). Also, adjusted TSPRT's closed at $n+1$, $n+2, \cdots$, m will have progressively better ASN properties while maintaining the specified error rates.

In contrasting the fixed-sample size test with adjusted TSPRT's, the choice would appear to be an easy one since the TSPRT will usually require fewer observations than the fixed-sample size alternative. There remains, however, one largely psychological factor which opposes use of the TSPRT or, for that matter, any sequential procedure. By way of illustration, consider the following example. Suppose that $n=10$ and that one has elected to use the adjusted TSPRT. Suppose further that the commission of a Type I or Type II error has severe financial (or other) implications. (In fairness, one must assume that these severe implications were thoroughly considered in the process of determining the specified error rates.) If the actual test procedure required six or eight observations to reach a decision in this instance, the responsible researchers would likely be mildly euphoric over the wise choice of a test procedure. Now, suppose instead that an accept or reject decision was indicated after only one or two observations had been taken. Pity the poor statistician who tries to convince the researcher and other decision makers that such an important decision could properly be based on so little data! The tendency would appear to be that of wanting (or insisting on) more data just to insure that the early indication was no fluke.

By way of compromise, one could choose some integer, say d, such that $d<n$ and conduct a sequential test as follows. Make no decisions until d observations have been taken, then proceed with the sequential test. According to Wald (1947), this would be an "ineffective" procedure; that is, it would not be as efficient (in terms of minimizing ASN) as the standard ($d=1$) sequential plan. That fact notwithstanding, such a procedure would represent a compromise between the fixed-sample size test and an adjusted TSPRT which might be more palatable to many decision makers.

In an attempt to quantify the effects on a sequential procedure of continuing beyond the fixed-sample size and of delaying accept or reject decisions, the author elected to examine $(1,2n)$, $(n/3,n)$ and $(n/3,2n)$ plans for the ten cases defined in Chapter II. Clearly, this selection of plans was arbitrary and was largely tempered by the amount of computer time available. One could just have easily chosen $d=2,3,\cdots$

or d=n/4,n/2,⋯ rather than d=n/3. Similarly, one could have alternatively selected m=n+1,n+2,⋯ or m=n+d or m=k·n rather than m=2n. The remainder of this chapter will be devoted to a discussion of Monte Carlo simulation results for the (1,2n), (n/3,n) and (n/3,2n) plans.

(1,2n) Plans

The extent to which the specified error rates are achieved for the (unadjusted) (1,2n) plans is shown in Table IV.

TABLE IV

SPECIFIED VERSUS OBSERVED ERROR RATES FOR (1,2n) PLANS

Case	α_s	$\hat{\alpha}$	$\hat{\alpha}/\alpha_s$	β_s	$\hat{\beta}$	$\hat{\beta}/\beta_s$
1	0.1000	0.0316	0.3160	0.1000	0.0792	0.7920
2	0.0500	0.0178	0.3560	0.1000	0.0733	0.7330
3	0.0500	0.0158	0.3160	0.0500	0.0354	0.7080
4	0.0250	0.0081	0.3240	0.0500	0.0325	0.6500
5	0.0250	0.0065	0.2600	0.0250	0.0155	0.6200
6	0.1000	0.0483	0.4830	0.1000	0.0863	0.8630
7	0.0500	0.0263	0.5260	0.1000	0.0825	0.8250
8	0.0500	0.0258	0.5160	0.0500	0.0414	0.8280
9	0.0250	0.0123	0.4920	0.0500	0.0375	0.7500
10	0.0250	0.0102	0.4080	0.0250	0.0194	0.7760

Note that the unadjusted (1,2n) plans are conservative for all cases; more conservative under H_0 than under H_1; more conservative for a higher discrimination ratio (Cases 1-5) than for a lower discrimination ratio (Cases 6-10). Tables XXVII - XXXVI of Appendix A provide

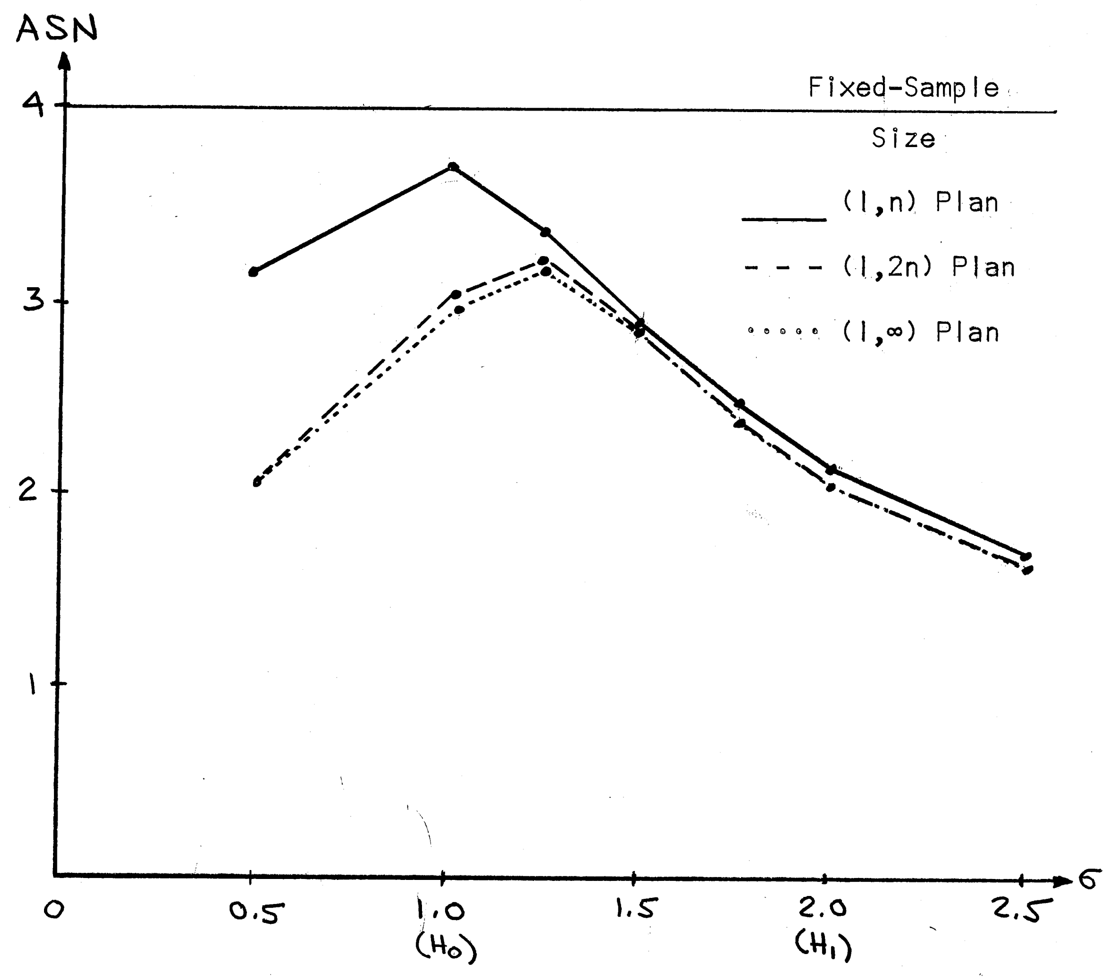

Figure 12. ASN Comparison for Adjusted (1,n), (1,2n), (1,∞) Plans

a comparison of the unadjusted and adjusted (1,2n) plans for each of the ten selected cases.

An ASN comparison between adjusted (1,n), (1,2n) and (1,∞) plans for Case I is shown in Figure 12. Two comments are suggested by it:

1. The ASN curve for the (1,2n) and (1,∞) plans are practically the same. This is because most of the Monte Carlo trials did not require more than 2n observations to reach a decision; for those that did, truncation at 2n observations resulted in the same decision that the untruncated trial would have reached in many cases.

2. Truncation at n, rather than 2n, observations tends to result in more errors for smaller values of the unknown parameter; for larger parameter values, the sequential procedure tends to terminate rapidly and very little is to be gained from extending the truncation point from n to 2n.

(n/3,n) Plans

The extent to which the specified error rates are attained for the unadjusted (n/3,n) plans is given in Table V.

From Table V it may be seen that use of unadjusted (n/3,n) plans tends to result in overestimation of the actual Type I error rate and in underestimation of the true Type II error rate. Tables XXXVII - XLVI of Appendix A give a comparison of the unadjusted and adjusted (n/3,n) plans for the ten cases defined in Chapter II. Figure 13 gives an ASN comparison between adjusted (1,n) and (n/3,n) plans for Case I. The only difference between the two plans for parameter values less than specified under the null hypothesis is due to sampling error. As σ increases beyond this value, the difference in ASN between the two plans

TABLE V

SPECIFIED VERSUS OBSERVED ERROR RATES FOR (n/3,n) PLANS

Case	α_s	$\hat{\alpha}$	$\hat{\alpha}/\alpha_s$	β_s	$\hat{\beta}$	$\hat{\beta}/\beta_s$
1	0.1000	0.0634	0.6340	0.1000	0.1297	1.2970
2	0.0500	0.0458	0.9160	0.1000	0.1045	1.0450
3	0.0500	0.0279	0.5580	0.0500	0.0585	1.1700
4	0.0250	0.0281	1.1240	0.0500	0.0602	1.2040
5	0.0250	0.0164	0.6560	0.0250	0.0343	1.3720
6	0.1000	0.0769	0.7690	0.1000	0.1268	1.2680
7	0.0500	0.0567	1.1340	0.1000	0.1158	1.1580
8	0.0500	0.0413	0.8260	0.0500	0.0672	1.3440
9	0.0250	0.0293	1.1720	0.0500	0.0599	1.1980
10	0.0250	0.0224	0.8960	0.0250	0.0341	1.3640

progressively increases due to delaying the decision to reject the null hypothesis.

(n/3,2n) Plans

A comparison between specified and observed error rates for unadjusted (n/3,2n) plans is given in Table VI.

Notice that the unadjusted (n/3,2n) plans are conservative for both Type II and especially Type I error rates. The effect of extending the test to 2n observations has more than compensated for the effect of delaying accept and reject decisions until after n/3 observations have been taken.

Tables XLVII - LVI of Appendix A give comparisons of unadjusted and adjusted (n/3,2n) plans for the ten selected cases. Figure 14 shows the ASN comparison between adjusted (1,2n) and (n/3,2n) plans for Case 1.

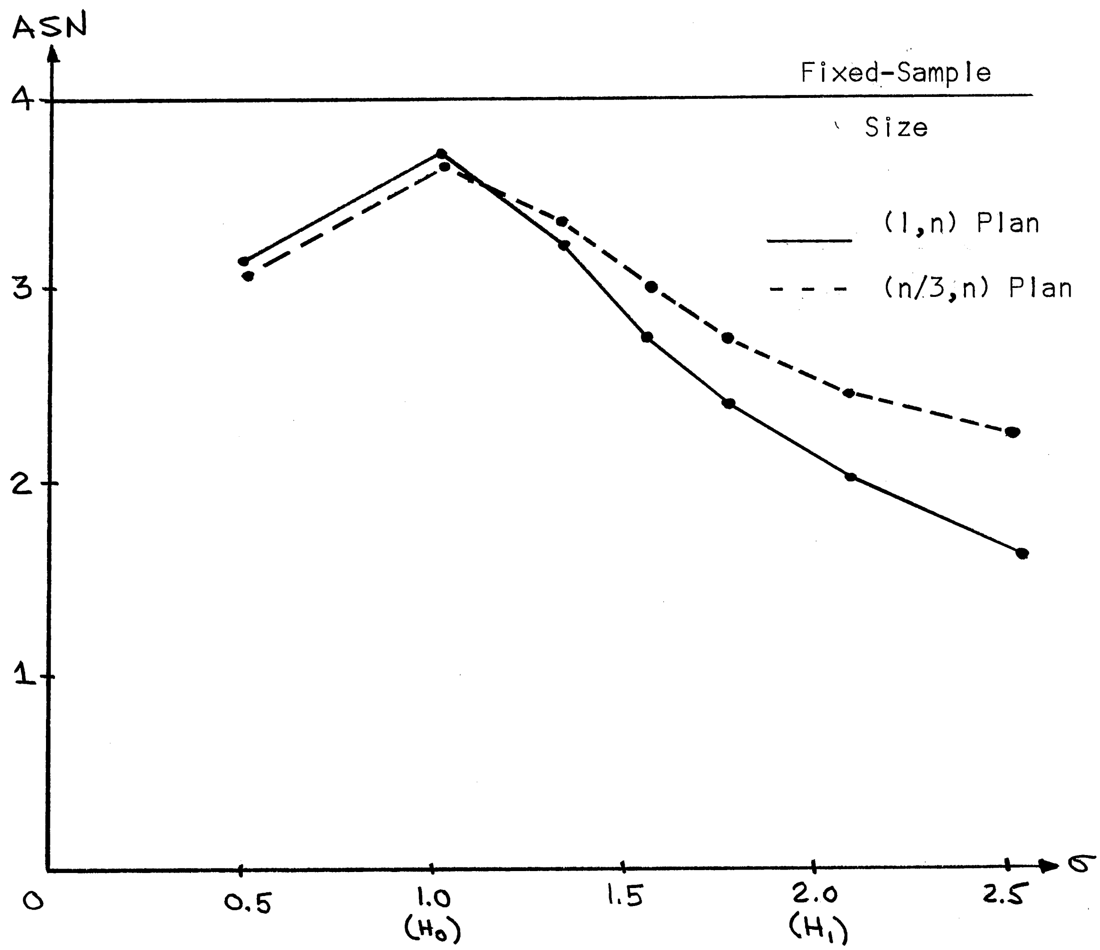

Figure 13. ASN Comparison for Adjusted (1,n) and (n/3,n) Plans

TABLE VI

SPECIFIED VERSUS OBSERVED ERROR RATES FOR (n/3,2n) PLANS

Case	α_s	$\hat{\alpha}$	$\hat{\alpha}/\alpha_s$	β_s	$\hat{\beta}$	$\hat{\beta}/\beta_s$
1	0.1000	0.0247	0.2470	0.1000	0.0805	0.8050
2	0.0500	0.0157	0.3140	0.1000	0.0741	0.7410
3	0.0500	0.0122	0.2440	0.0500	0.0360	0.7200
4	0.0250	0.0071	0.2840	0.0500	0.0343	0.6860
5	0.0250	0.0044	0.1760	0.0250	0.0149	0.5960
6	0.1000	0.0459	0.4590	0.1000	0.0820	0.8200
7	0.0500	0.0232	0.4640	0.1000	0.0760	0.7600
8	0.0500	0.0216	0.4320	0.0500	0.0416	0.8320
9	0.0250	0.0113	0.4520	0.0500	0.0364	0.7280
10	0.0250	0.0086	0.3440	0.0250	0.0189	0.7560

Note the similarity between Figure 13 and Figure 14.

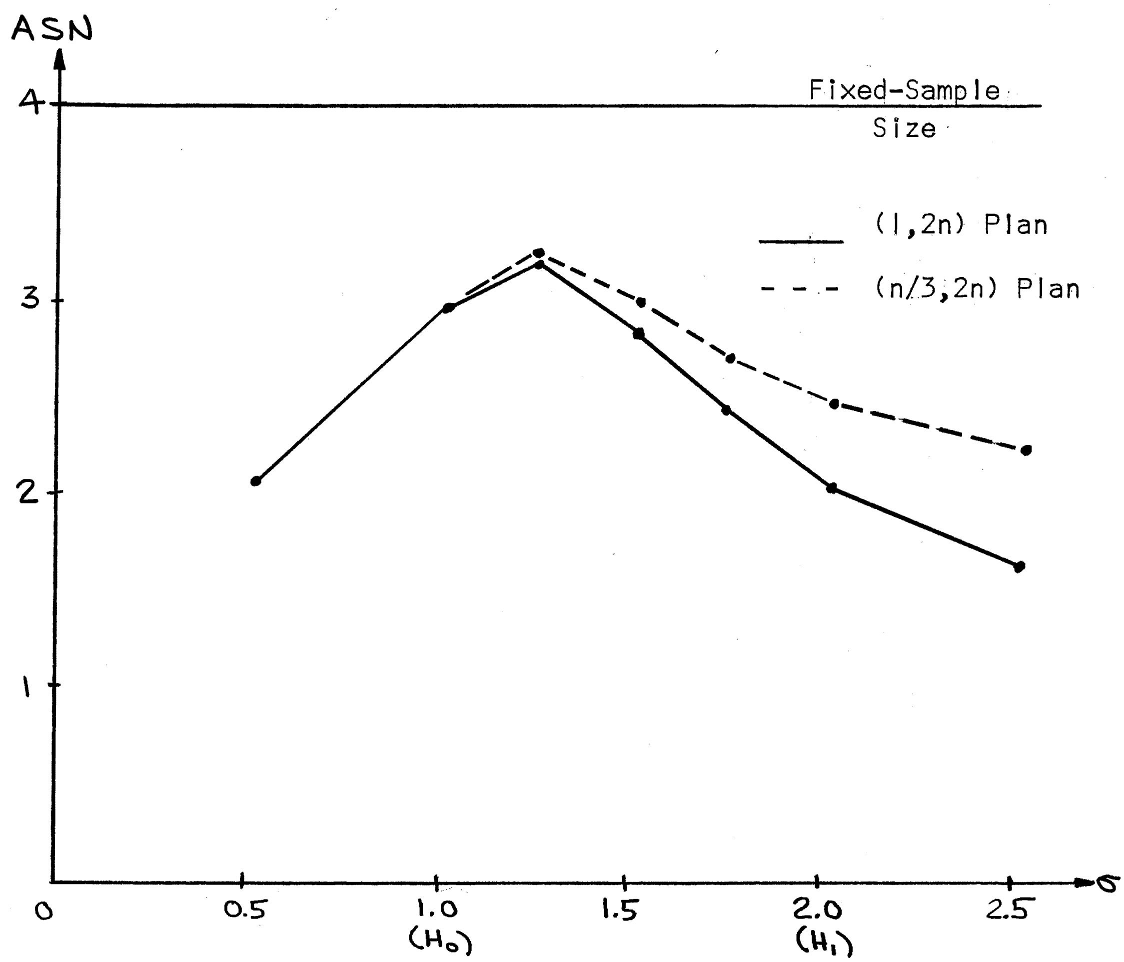

Figure 14. ASN Comparison for Adjusted (1,2n) and (n/3,2n) Plans

CHAPTER VI

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

This thesis documents an empirical investigation of the properties of the SPRT and several closed sequential procedures for testing the parameter of a Rayleigh distribution. The most significant problem addressed was that of adjusting the boundaries of sequential procedures to attain the specified error rates. It was argued that, for the SPRT, boundary adjustment was important in its own right. Of even greater importance, however, is the fact that when considering several alternative sequential plans in a particular testing context, one may adjust all these plans to give (approximately) the same power curve, thereby providing a direct DSN comparison between plans and a more relevant basis for selecting from among the competing plans.

Chapter I presented a general background for sequential probability ratio tests, together with a literature review and definition of the problem to be investigated. Chapter II gave a detailed description of the ten cases studied and of the Monte Carlo simulation procedures employed. Chapter III introduced the boundary adjustment problem and gave comparisons of unadjusted and adjusted SPRT's. Chapter IV gave parallel results for the TSPRT. Chapter V introduced the extended plans [(1,2n), (n/3,n), (n/3,2n)] and provided pre- and post-adjustment comparisons for them.

Based upon the results of this investigation, the author recommends the following areas for future research:

1. Although the SPRT has been known to be conservative for several years (Baker, 1950), the extent of this conservatism is not generally known. Through the technique of boundary adjustment, removal of the conservatism can be directly translated into expected savings in test items. Boundary adjustment should be applied to the "standard" class of sequential tests; e.g., testing the mean of a normal distribution with known variance, t-test, etc.

2. Effort should be expended to develop "rules-of-thumb" or algorithms for boundary adjustment without recourse to Monte Carlo simulation. A natural candidate for developing such algorithms would appear to be Aroian's direct method (Aroian, 1968; Aroian and Robison, 1969). Although the author was unsuccessful in his attempts to use this technique (see Appendix B), it seems clear that, had he succeeded, the boundary adjustment problem would have been significantly simpler since the proper adjustments would not have been masked by sampling error as was the case with Monte Carlo simulation.

3. The robustness of sequential procedures should be addressed. This is very simple to do (mechanically) with Monte Carlo simulation; it would be much more difficult to do with Aroian's direct method. The hard issue is the determination of distributional alternatives which make sense in the context of the particular problem under study. It may well be the case that adjusted sequential procedures are much less robust than their unadjusted counterparts. If this is the case, one might be understandably unwilling to give up robustness for an expected savings in test items.

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APPENDIX A

CASE SUMMARY TABLES FOR ALL PLANS

TABLE VII

CASE I SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 4

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-1.700
Reject Boundary:	2.197	0.800
Specified Alpha:	0.1000	0.4001
Specified Beta:	0.1000	0.1096
Observed Alpha:	0.0280	0.1001
Observed Beta:	0.0666	0.1027

σ	L(σ)	S.E.	ASN	S.D.	P(n>4)
		Unadjusted:			
0.500	1.0000	0.0000	2.19	0.40	0.000
1.000	0.9720	0.0016	4.01	2.07	0.288
1.250	0.7445	0.0044	5.33	3.70	0.463
1.500	0.3660	0.0048	4.93	3.85	0.415
1.750	0.1521	0.0036	3.74	2.89	0.284
2.000	0.0666	0.0025	2.90	2.17	0.172
2.500	0.0185	0.0013	2.02	1.35	0.056
		Adjusted:			
0.500	1.0000	0.0000	2.02	0.15	0.000
1.000	0.8999	0.0030	2.89	1.36	0.108
1.250	0.6389	0.0048	3.12	1.91	0.181
1.500	0.3573	0.0048	2.78	1.90	0.151
1.750	0.1885	0.0039	2.31	1.56	0.091
2.000	0.1027	0.0030	1.99	1.35	0.054
2.500	0.0372	0.0019	1.58	0.94	0.016

TABLE VIII

CASE 2 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0   Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 5

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-1.800
Reject Boundary:	2.890	1.600
Specified Alpha:	0.0500	0.1743
Specified Beta:	0.1000	0.1365
Observed Alpha:	0.0121	0.0470
Observed Beta:	0.0629	0.1005

σ	L(σ)	S.E.	ASN	S.D.	P(n>5)
		Unadjusted:			
0.500	1.0000	0.0000	2.24	0.43	0.000
1.000	0.9879	0.0011	4.20	2.17	0.191
1.250	0.7939	0.0040	6.28	4.60	0.427
1.500	0.3782	0.0048	5.99	4.69	0.411
1.750	0.1469	0.0035	4.38	3.37	0.266
2.000	0.0629	0.0024	3.28	2.44	0.146
2.500	0.0166	0.0013	2.22	1.48	0.038
		Adjusted:			
0.500	1.0000	0.0000	2.04	0.19	0.000
1.000	0.9530	0.0021	3.28	1.70	0.098
1.250	0.7144	0.0045	4.01	2.72	0.214
1.500	0.3940	0.0049	3.69	2.73	0.184
1.750	0.1960	0.0040	2.99	2.19	0.116
2.000	0.1005	0.0030	2.45	1.73	0.063
2.500	0.0321	0.0018	1.82	1.14	0.014

TABLE IX

CASE 3 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.450
Reject Boundary:	2.944	1.600
Specified Alpha:	0.0500	0.1877
Specified Beta:	0.0500	0.0701
Observed Alpha:	0.0123	0.0490
Observed Beta:	0.0331	0.0525

σ	L(σ)	S.E.	ASN	S.D.	P(n>7)
		Unadjusted:			
0.500	1.0000	0.0000	3.04	0.20	0.000
1.000	0.9877	0.0011	5.30	2.52	0.148
1.250	0.7743	0.0042	7.98	5.65	0.402
1.500	0.3074	0.0046	7.22	5.81	0.353
1.750	0.0944	0.0029	4.86	3.83	0.185
2.000	0.0331	0.0018	3.49	2.69	0.079
2.500	0.0051	0.0007	2.29	1.56	0.012
		Adjusted:			
0.500	1.0000	0.0000	2.49	0.51	0.000
1.000	0.9510	0.0022	4.26	2.01	0.069
1.250	0.6803	0.0047	5.15	3.46	0.190
1.500	0.3118	0.0046	4.51	3.49	0.158
1.750	0.1282	0.0033	3.37	2.62	0.076
2.000	0.0525	0.0022	2.65	2.01	0.034
2.500	0.0134	0.0011	1.87	1.24	0.004

TABLE X

CASE 4 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-2.500
Reject Boundary:	3.638	2.300
Specified Alpha:	0.0250	0.0928
Specified Beta:	0.0500	0.0745
Observed Alpha:	0.0054	0.0251
Observed Beta:	0.0312	0.0490

σ	L(σ)	S.E.	ASN	S.D.	P(n>7)
		Unadjusted:			
0.500	1.0000	0.0000	3.05	0.21	0.000
1.000	0.9946	0.0007	5.40	2.61	0.155
1.250	0.8191	0.0038	8.87	6.48	0.450
1.500	0.3265	0.0047	8.37	6.73	0.423
1.750	0.0982	0.0030	5.51	4.22	0.236
2.000	0.0312	0.0017	3.90	2.93	0.109
2.500	0.0057	0.0008	2.48	1.68	0.016
		Adjusted:			
0.500	1.0000	0.0000	2.58	0.51	0.000
1.000	0.9749	0.0016	4.53	2.19	0.091
1.250	0.7380	0.0044	6.15	4.28	0.269
1.500	0.3346	0.0047	5.50	4.36	0.228
1.750	0.1259	0.0033	4.03	3.17	0.124
2.000	0.0490	0.0022	3.04	2.32	0.054
2.500	0.0130	0.0011	2.04	1.36	0.006

TABLE XI

CASE 5 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 9

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-3.150
Reject Boundary:	3.664	2.300
Specified Alpha:	0.0250	0.0964
Specified Beta:	0.0250	0.0387
Observed Alpha:	0.0075	0.0258
Observed Beta:	0.0135	0.0240

σ	L(σ)	S.E.	ASN	S.D.	P(n>9)
		Unadjusted:			
0.500	1.0000	0.0000	3.52	0.52	0.000
1.000	0.9925	0.0009	6.43	2.83	0.118
1.250	0.8034	0.0040	10.74	7.71	0.433
1.500	0.2642	0.0044	9.68	8.01	0.380
1.750	0.0589	0.0024	5.86	4.58	0.171
2.000	0.0135	0.0012	4.01	3.04	0.059
2.500	0.0021	0.0005	2.51	1.74	0.005
		Adjusted:			
0.500	1.0000	0.0000	3.10	0.30	0.000
1.000	0.9742	0.0016	5.52	2.49	0.072
1.250	0.7132	0.0045	7.54	5.20	0.258
1.500	0.2716	0.0044	6.38	5.28	0.203
1.750	0.0794	0.0027	4.31	3.53	0.086
2.000	0.0240	0.0015	3.11	2.47	0.028
2.500	0.0054	0.0007	2.07	1.40	0.002

TABLE XII

CASE 6 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 11

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-1.900
Reject Boundary:	2.197	1.400
Specified Alpha:	0.1000	0.2177
Specified Beta:	0.1000	0.1170
Observed Alpha:	0.0440	0.0987
Observed Beta:	0.0815	0.1020

σ	L(σ)	S.E.	ASN	S.D.	P(n>11)
		Unadjusted:			
0.750	0.9999	0.0001	5.04	1.43	0.007
1.000	0.9560	0.0021	8.76	5.19	0.218
1.125	0.7736	0.0042	11.39	8.19	0.365
1.250	0.4405	0.0050	11.44	9.19	0.368
1.375	0.1957	0.0040	9.11	7.42	0.273
1.500	0.0815	0.0027	6.83	5.47	0.159
1.750	0.0182	0.0013	4.25	3.19	0.037
		Adjusted:			
0.750	0.9981	0.0004	4.41	1.36	0.001
1.000	0.9013	0.0030	6.93	4.02	0.116
1.125	0.6976	0.0046	7.96	5.63	0.196
1.250	0.4170	0.0049	7.56	5.89	0.193
1.375	0.2109	0.0041	6.27	5.16	0.133
1.500	0.1020	0.0030	4.90	4.00	0.073
1.750	0.0263	0.0016	3.32	2.65	0.015

TABLE XIII

CASE 7 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 13

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-2.000
Reject Boundary:	2.890	2.000
Specified Alpha:	0.0500	0.1192
Specified Beta:	0.1000	0.1192
Observed Alpha:	0.0238	0.0522
Observed Beta:	0.0762	0.1006

σ	L(σ)	S.E.	ASN	S.D.	P(n>13)
		Unadjusted:			
0.750	1.0000	0.0000	5.18	1.46	0.001
1.000	0.9762	0.0015	9.39	5.71	0.176
1.125	0.8210	0.0038	13.50	10.48	0.358
1.250	0.4734	0.0050	14.28	11.61	0.394
1.375	0.1964	0.0040	11.15	8.97	0.289
1.500	0.0762	0.0027	8.22	6.33	0.161
1.750	0.0153	0.0012	5.03	3.62	0.032
		Adjusted:			
0.750	0.9998	0.0001	4.64	1.39	0.000
1.000	0.9478	0.0022	7.89	4.69	0.111
1.125	0.7600	0.0043	9.85	7.09	0.222
1.250	0.4519	0.0050	9.82	7.81	0.230
1.375	0.2188	0.0041	8.10	6.66	0.160
1.500	0.1006	0.0030	6.23	4.96	0.087
1.750	0.0236	0.0015	4.03	3.08	0.015

TABLE XIV

CASE 8 SUMMARY FOR $(1,\infty)$ PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=1.5$
Number of Observations Required for Fixed-Sample Size Test: 17

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.600
Reject Boundary:	2.944	2.150
Specified Alpha:	0.0500	0.1088
Specified Beta:	0.0500	0.0662
Observed Alpha:	0.0236	0.0472
Observed Beta:	0.0374	0.0480

σ	$L(\sigma)$	S.E.	ASN	S.D.	$P(n>17)$
		Unadjusted:			
0.750	1.0000	0.0000	6.56	1.63	0.001
1.000	0.9764	0.0015	12.05	6.57	0.164
1.125	0.8040	0.0040	17.67	13.03	0.371
1.250	0.4005	0.0049	18.04	14.84	0.386
1.375	0.1236	0.0033	12.83	10.62	0.236
1.500	0.0374	0.0019	8.81	7.00	0.106
1.750	0.0053	0.0007	5.15	3.81	0.013
		Adjusted:			
0.750	0.9998	0.0001	5.83	1.54	0.000
1.000	0.9528	0.0021	10.20	5.60	0.098
1.125	0.7485	0.0043	13.29	9.55	0.236
1.250	0.3945	0.0049	12.86	10.25	0.240
1.375	0.1466	0.0035	9.73	8.19	0.129
1.500	0.0480	0.0021	7.07	5.89	0.062
1.750	0.0073	0.0009	4.25	3.26	0.006

TABLE XV

CASE 9 SUMMARY FOR $(1,\infty)$ PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=1.5$
Number of Observations Required for Fixed-Sample Size Test:20

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-2.700
Reject Boundary:	3.638	2.850
Specified Alpha:	0.0250	0.0542
Specified Beta:	0.0500	0.0636
Observed Alpha:	0.0111	0.0255
Observed Beta:	0.0362	0.0478

σ	$L(\sigma)$	S.E.	ASN	S.D.	P(n>20)
		Unadjusted:			
0.750	1.0000	0.0000	6.61	1.63	0.000
1.000	0.9889	0.0010	12.46	7.06	0.118
1.125	0.8506	0.0036	19.95	15.54	0.345
1.250	0.4235	0.0049	21.42	17.36	0.387
1.375	0.1264	0.0033	15.33	12.28	0.230
1.500	0.0362	0.0019	10.40	7.88	0.101
1.750	0.0047	0.0007	5.93	4.17	0.008
		Adjusted:			
0.750	1.0000	0.0000	6.06	1.61	0.000
1.000	0.9745	0.0016	11.03	6.22	0.079
1.125	0.8096	0.0039	15.94	11.77	0.247
1.250	0.4222	0.0049	16.38	13.25	0.272
1.375	0.1476	0.0035	12.18	9.96	0.160
1.500	0.0478	0.0021	8.52	6.66	0.059
1.750	0.0084	0.0009	5.02	3.69	0.000

TABLE XVI

CASE 10 SUMMARY FOR (1,∞) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:24

	Unadjusted:	Adjusted:
Accept Boundary:	-3.663	-3.400
Reject Boundary:	3.664	2.900
Specified Alpha:	0.0250	0.0533
Specified Beta:	0.0250	0.0316
Observed Alpha:	0.0102	0.0263
Observed Beta:	0.0186	0.0244

σ	L(σ)	S.E.	ASN	S.D.	P(n>24)
		Unadjusted:			
0.750	1.0000	0.0000	7.98	1.76	0.000
1.000	0.9898	0.0010	15.14	7.85	0.113
1.125	0.8406	0.0037	24.63	17.88	0.373
1.250	0.3636	0.0048	25.54	20.59	0.392
1.375	0.0824	0.0027	16.73	13.40	0.207
1.500	0.0186	0.0014	10.82	8.23	0.068
1.750	0.0016	0.0004	5.98	4.25	0.003
		Adjusted:			
0.750	1.0000	0.0000	7.47	1.75	0.000
1.000	0.9737	0.0016	13.80	7.33	0.084
1.125	0.7893	0.0041	20.00	14.60	0.255
1.250	0.3562	0.0048	20.00	16.66	0.287
1.375	0.0941	0.0029	13.57	11.39	0.143
1.500	0.0244	0.0015	8.98	7.21	0.041
1.750	0.0021	0.0005	5.12	3.84	0.002

TABLE XVII

CASE I SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 4

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-3.200
Reject Boundary:	2.197	1.000
Specified Alpha:	0.1000	0.3583
Specified Beta:	0.1000	0.0262
Observed Alpha:	0.0634	0.1000
Observed Beta:	0.1304	0.1014

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.19	0.40
1.000	0.9362	0.0024	3.18	0.78
1.250	0.7025	0.0046	3.39	0.88
1.500	0.4283	0.0049	3.15	1.08
1.750	0.2373	0.0043	2.78	1.19
2.000	0.1304	0.0034	2.43	1.19
2.500	0.0390	0.0019	1.91	1.05
		Adjusted:		
0.500	1.0000	0.0000	3.11	0.32
1.000	0.9000	0.0030	3.68	0.71
1.250	0.6410	0.0048	3.34	1.08
1.500	0.3768	0.0048	2.84	1.26
1.750	0.2021	0.0040	2.41	1.26
2.000	0.1014	0.0030	2.07	1.17
2.500	0.0266	0.0016	1.64	0.93

TABLE XVIII

CASE 2 SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 5

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-2.400
Reject Boundary:	2.890	2.600
Specified Alpha:	0.0500	0.0680
Specified Beta:	0.1000	0.0845
Observed Alpha:	0.0473	0.0492
Observed Beta:	0.1050	0.1001

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.24	0.43
1.000	0.9527	0.0021	3.67	1.08
1.250	0.7192	0.0045	4.05	1.17
1.500	0.4142	0.0049	3.85	1.38
1.750	0.2136	0.0041	3.37	1.51
2.000	0.1050	0.0031	2.87	1.59
2.500	0.0278	0.0016	2.14	1.25
		Adjusted:		
0.500	1.0000	0.0000	2.42	0.50
1.000	0.9508	0.0022	3.82	1.03
1.250	0.7093	0.0045	4.11	1.15
1.500	0.4102	0.0049	3.81	1.39
1.750	0.2024	0.0040	3.28	1.52
2.000	0.1001	0.0030	2.77	1.49
2.500	0.0241	0.0015	2.08	1.24

TABLE XIX

CASE 3 SUMMARY FOR (l,n) PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=2.0$
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-3.000
Reject Boundary:	2.944	1.800
Specified Alpha:	0.0500	0.1584
Specified Beta:	0.0500	0.0419
Observed Alpha:	0.0291	0.0508
Observed Beta:	0.0562	0.0483

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.05	0.21
1.000	0.9709	0.0017	4.83	1.48
1.250	0.7236	0.0045	5.51	1.72
1.500	0.3730	0.0048	5.02	2.06
1.750	0.1492	0.0036	4.09	2.16
2.000	0.0562	0.0023	3.26	1.98
2.500	0.0089	0.0009	2.28	1.48
		Adjusted:		
0.500	1.0000	0.0000	3.06	0.23
1.000	0.9492	0.0022	4.75	1.51
1.250	0.6863	0.0046	5.06	1.95
1.500	0.3351	0.0047	4.33	2.24
1.750	0.1353	0.0034	3.38	2.12
2.000	0.0483	0.0021	2.68	1.82
2.500	0.0073	0.0009	1.93	1.27

TABLE XX

CASE 4 SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-3.400
Reject Boundary:	3.638	5.000
Specified Alpha:	0.0250	0.0065
Specified Beta:	0.0500	0.0332
Observed Alpha:	0.0285	0.0271
Observed Beta:	0.0594	0.0523

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.05	0.22
1.000	0.9715	0.0017	4.88	1.57
1.250	0.7318	0.0044	5.71	1.61
1.500	0.3754	0.0058	5.36	1.96
1.750	0.1499	0.0036	4.45	2.14
2.000	0.0594	0.0024	3.55	2.03
2.500	0.0095	0.0010	2.46	1.56
		Adjusted:		
0.500	1.0000	0.0000	3.23	0.43
1.000	0.9729	0.0016	5.36	1.35
1.250	0.7268	0.0045	6.20	1.30
1.500	0.3684	0.0048	5.99	1.65
1.750	0.1400	0.0035	5.13	2.04
2.000	0.0523	0.0022	4.15	2.08
2.500	0.0065	0.0008	2.83	1.71

TABLE XXI

CASE 5 SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 9

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-4.600
Reject Boundary:	3.664	2.900
Specified Alpha:	0.0250	0.0545
Specified Beta:	0.0250	0.0095
Observed Alpha:	0.0199	0.0250
Observed Beta:	0.0343	0.0273

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.52	0.52
1.000	0.9801	0.0014	6.03	1.86
1.250	0.7381	0.0044	7.20	2.19
1.500	0.3409	0.0047	6.49	2.71
1.750	0.1092	0.0031	5.06	2.82
2.000	0.0343	0.0018	3.84	2.47
2.500	0.0031	0.0006	2.52	1.68
		Adjusted:		
0.500	1.0000	0.0000	4.27	0.46
1.000	0.9750	0.0016	7.03	1.68
1.250	0.7156	0.0045	7.52	2.23
1.500	0.3202	0.0047	6.26	2.92
1.750	0.0940	0.0029	4.64	2.85
2.000	0.0273	0.0016	3.44	2.37
2.500	0.0013	0.0004	2.28	1.54

TABLE XXII

CASE 6 SUMMARY FOR (1,n) PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=1.5$
Number of Observations Required for Fixed-Sample Size Test: 11

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-3.450
Reject Boundary:	2.197	1.800
Specified Alpha:	0.1000	0.1609
Specified Beta:	0.1000	0.0266
Observed Alpha:	0.0823	0.1013
Observed Beta:	0.1260	0.1025

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.750	0.9999	0.0001	5.06	1.40
1.000	0.9177	0.0027	7.52	2.66
1.125	0.7269	0.0045	8.19	2.92
1.250	0.4675	0.0050	7.90	3.24
1.375	0.2557	0.0044	6.98	3.49
1.500	0.1260	0.0033	5.88	3.43
1.750	0.0265	0.0016	4.13	2.84
		Adjusted:		
0.750	0.9994	0.0002	7.48	1.55
1.000	0.8987	0.0030	9.52	2.17
1.125	0.6938	0.0046	9.28	2.83
1.250	0.4302	0.0050	8.23	3.47
1.375	0.2248	0.0042	6.90	3.71
1.500	0.1025	0.0030	5.61	3.53
1.750	0.0173	0.0013	3.79	2.74

TABLE XXIII

CASE 7 SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 13

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-2.650
Reject Boundary:	2.890	5.000
Specified Alpha:	0.0500	0.0063
Specified Beta:	0.1000	0.0702
Observed Alpha:	0.0587	0.0579
Observed Beta:	0.1167	0.1011

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	0.9998	0.0001	5.15	1.41
1.000	0.9413	0.0024	8.28	3.21
1.125	0.7519	0.0043	9.56	3.44
1.250	0.4686	0.0050	9.55	3.70
1.375	0.2489	0.0043	8.52	3.99
1.500	0.1167	0.0032	7.18	3.98
1.750	0.0209	0.0014	4.94	3.29
		Adjusted:		
0.750	1.0000	0.0000	5.93	1.53
1.000	0.9421	0.0023	9.38	3.01
1.125	0.7449	0.0044	10.91	2.83
1.250	0.4650	0.0050	11.44	2.68
1.375	0.2386	0.0043	10.93	3.11
1.500	0.1011	0.0030	9.71	3.61
1.750	0.0132	0.0011	7.01	3.63

TABLE XXIV

CASE 8 SUMMARY FOR (1,n) PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=1.5$
Number of Observations Required for Fixed-Sample Size Test: 17

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-6.000
Reject Boundary:	2.944	2.850
Specified Alpha:	0.0500	0.0577
Specified Beta:	0.0500	0.0023
Observed Alpha:	0.0444	0.0488
Observed Beta:	0.0669	0.0518

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	6.56	1.62
1.000	0.9556	0.0021	10.83	4.04
1.125	0.7579	0.0043	12.68	4.47
1.250	0.4383	0.0050	12.30	5.05
1.375	0.1938	0.0040	10.34	5.42
1.500	0.0669	0.0025	8.11	5.08
1.750	0.0084	0.0009	5.09	3.60
		Adjusted:		
0.750	1.0000	0.0000	12.56	2.03
1.000	0.9512	0.0022	16.10	2.14
1.125	0.7407	0.0044	15.62	3.50
1.250	0.4218	0.0049	13.53	5.10
1.375	0.1700	0.0038	10.77	5.66
1.500	0.0518	0.0022	8.25	5.24
1.750	0.0038	0.0006	5.05	3.62

TABLE XXV

CASE 9 SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:20

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-3.400
Reject Boundary:	3.638	4.400
Specified Alpha:	0.0250	0.0119
Specified Beta:	0.0500	0.0330
Observed Alpha:	0.0299	0.0288
Observed Beta:	0.0586	0.0512

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	6.58	1.60
1.000	0.9701	0.0017	11.50	4.76
1.125	0.7836	0.0041	14.34	5.38
1.250	0.4418	0.0050	14.60	5.76
1.375	0.1818	0.0039	12.41	6.21
1.500	0.0586	0.0023	9.67	5.79
1.750	0.0055	0.0007	5.89	4.01
		Adjusted:		
0.750	1.0000	0.0000	7.43	1.71
1.000	0.9712	0.0017	12.82	4.69
1.125	0.7740	0.0042	15.80	4.87
1.250	0.4362	0.0050	16.20	5.14
1.375	0.1679	0.0037	14.02	5.97
1.500	0.0512	0.0022	11.09	5.95
1.750	0.0033	0.0006	6.81	4.31

TABLE XXVI

CASE 10 SUMMARY FOR (1,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:24

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-5.500
Reject Boundary:	3.664	3.600
Specified Alpha:	0.0250	0.0272
Specified Beta:	0.0250	0.0040
Observed Alpha:	0.0239	0.0265
Observed Beta:	0.0338	0.0261

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	7.96	1.77
1.000	0.9761	0.0015	14.14	5.57
1.125	0.7848	0.0041	17.74	6.31
1.250	0.4167	0.0049	17.34	7.11
1.375	0.1361	0.0034	13.98	7.55
1.500	0.0338	0.0018	10.32	6.60
1.750	0.0017	0.0004	6.01	4.15
		Adjusted:		
0.750	1.0000	0.0000	11.64	2.15
1.000	0.9735	0.0016	19.11	4.70
1.125	0.7666	0.0042	21.08	5.06
1.250	0.3932	0.0049	18.73	7.09
1.375	0.1221	0.0033	14.37	7.80
1.500	0.0261	0.0016	10.28	6.77
1.750	0.0007	0.0003	5.95	4.14

TABLE XXVII

CASE I SUMMARY FOR (1,2n) PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=2.0$
Number of Observations Required for Fixed-Sample Size Test: 4

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-1.750
Reject Boundary:	2.197	0.830
Specified Alpha:	0.1000	0.3898
Specified Beta:	0.1000	0.1060
Observed Alpha:	0.0316	0.0975
Observed Beta:	0.0792	0.0986

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.19	0.40
1.000	0.9684	0.0017	3.87	1.69
1.250	0.7315	0.0044	4.67	2.24
1.500	0.3918	0.0049	4.33	2.38
1.750	0.1756	0.0038	3.52	2.21
2.000	0.0792	0.0027	2.85	1.93
2.500	0.0208	0.0014	2.01	1.30
		Adjusted:		
0.500	1.0000	0.0000	2.03	0.16
1.000	0.9025	0.0030	2.92	1.33
1.250	0.6463	0.0048	3.16	1.81
1.500	0.3624	0.0048	2.79	1.77
1.750	0.1857	0.0039	2.36	1.59
2.000	0.0986	0.0030	2.00	1.31
2.500	0.0361	0.0019	1.60	0.95

TABLE XXVIII

CASE 2 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 5

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-1.800
Reject Boundary:	2.890	1.550
Specified Alpha:	0.0500	0.1836
Specified Beta:	0.1000	0.1349
Observed Alpha:	0.0178	0.0498
Observed Beta:	0.0733	0.1020

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.23	0.43
1.000	0.9822	0.0013	4.11	1.95
1.250	0.7747	0.0042	5.49	2.81
1.500	0.3944	0.0049	5.27	3.01
1.750	0.1670	0.0037	4.14	2.68
2.000	0.0733	0.0026	3.23	2.23
2.500	0.0195	0.0014	2.24	1.50
		Adjusted:		
0.500	1.0000	0.0000	2.04	0.19
1.000	0.9502	0.0022	3.22	1.58
1.250	0.7170	0.0045	3.83	2.31
1.500	0.4022	0.0049	3.53	2.33
1.750	0.2020	0.0040	2.92	2.03
2.000	0.1020	0.0030	2.41	1.68
2.500	0.0325	0.0018	1.80	1.15

TABLE XXIX

CASE 3 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.500
Reject Boundary:	2.944	1.600
Specified Alpha:	0.0500	0.1884
Specified Beta:	0.0500	0.0666
Observed Alpha:	0.0158	0.0488
Observed Beta:	0.0354	0.0505

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.04	0.20
1.000	0.9842	0.0012	5.25	2.35
1.250	0.7621	0.0043	7.26	3.87
1.500	0.3224	0.0047	6.52	4.11
1.750	0.1056	0.0031	4.72	3.40
2.000	0.0354	0.0018	3.47	2.58
2.500	0.0073	0.0009	2.29	1.57
		Adjusted:		
0.500	1.0000	0.0000	2.57	0.51
1.000	0.9512	0.0022	4.34	2.00
1.250	0.6792	0.0047	5.16	3.17
1.500	0.3134	0.0046	4.49	3.22
1.750	0.1235	0.0033	3.42	2.63
2.000	0.0505	0.0022	2.65	2.00
2.500	0.0131	0.0011	1.87	1.24

TABLE XXX

CASE 4 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=2.0$
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-2.500
Reject Boundary:	3.638	2.400
Specified Alpha:	0.0250	0.0839
Specified Beta:	0.0500	0.0752
Observed Alpha:	0.0081	0.0231
Observed Beta:	0.0325	0.0511

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.04	0.21
1.000	0.9919	0.0009	5.30	2.39
1.250	0.7957	0.0040	7.74	3.95
1.500	0.3482	0.0048	7.38	4.21
1.750	0.1071	0.0031	5.41	3.63
2.000	0.0325	0.0018	3.95	2.81
2.500	0.0055	0.0007	2.51	1.69
		Adjusted:		
0.500	1.0000	0.0000	2.58	0.51
1.000	0.9769	0.0015	4.48	2.11
1.250	0.7538	0.0043	5.97	3.54
1.500	0.3530	0.0048	5.45	3.69
1.750	0.1355	0.0034	4.09	3.03
2.000	0.0511	0.0022	3.13	2.34
2.500	0.0133	0.0011	2.13	1.42

TABLE XXXI

CASE 5 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 9

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-3.150
Reject Boundary:	3.664	2.300
Specified Alpha:	0.0250	0.0964
Specified Beta:	0.0250	0.0387
Observed Alpha:	0.0065	0.0246
Observed Beta:	0.0155	0.0256

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.51	0.52
1.000	0.9935	0.0008	6.43	2.77
1.250	0.7863	0.0041	9.72	5.02
1.500	0.2875	0.0045	8.61	5.34
1.750	0.0632	0.0024	5.79	4.22
2.000	0.0155	0.0012	4.03	2.99
2.500	0.0011	0.0003	2.52	1.72
		Adjusted:		
0.500	1.0000	0.0000	3.09	0.29
1.000	0.9754	0.0015	5.44	2.45
1.250	0.7286	0.0044	7.27	4.40
1.500	0.2877	0.0045	6.24	4.56
1.750	0.0844	0.0028	4.33	3.41
2.000	0.0256	0.0016	3.19	2.51
2.500	0.0038	0.0006	2.11	1.45

TABLE XXXII

CASE 6 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 11

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-1.950
Reject Boundary:	2.197	1.400
Specified Alpha:	0.1000	0.2192
Specified Beta:	0.1000	0.1111
Observed Alpha:	0.0483	0.0994
Observed Beta:	0.0863	0.1030

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	0.9998	0.0001	5.05	1.39
1.000	0.9517	0.0021	8.62	4.62
1.125	0.7582	0.0043	10.58	6.04
1.250	0.4490	0.0050	10.31	6.42
1.375	0.2081	0.0041	8.56	5.95
1.500	0.0863	0.0028	6.64	4.96
1.750	0.0165	0.0013	4.27	3.22
		Adjusted:		
0.750	0.9981	0.0004	4.51	1.37
1.000	0.9006	0.0030	7.08	3.93
1.125	0.6910	0.0046	8.00	5.14
1.250	0.4162	0.0049	7.46	5.30
1.375	0.2140	0.0041	6.29	4.91
1.500	0.1030	0.0030	4.93	3.95
1.750	0.0257	0.0016	3.36	2.66

TABLE XXXIII

CASE 7 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: $\sigma=1.0$ Alternative Hypothesis: $\sigma=1.5$
Number of Observations Required for Fixed-Sample Size Test: 13

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-1.950
Reject Boundary:	2.890	2.100
Specified Alpha:	0.0500	0.1069
Specified Beta:	0.1000	0.1271
Observed Alpha:	0.0263	0.0500
Observed Beta:	0.0825	0.1033

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	5.17	1.43
1.000	0.9737	0.0016	9.33	5.25
1.125	0.7997	0.0040	12.28	7.19
1.250	0.4660	0.0050	12.68	7.72
1.375	0.2072	0.0041	10.52	7.09
1.500	0.0825	0.0028	8.12	5.89
1.750	0.0160	0.0013	4.99	3.58
		Adjusted:		
0.750	0.9997	0.0002	4.54	1.36
1.000	0.9500	0.0022	7.78	4.58
1.125	0.7663	0.0042	9.66	6.31
1.250	0.4661	0.0050	9.67	6.74
1.375	0.2243	0.0042	7.96	5.97
1.500	0.1033	0.0030	6.25	4.82
1.750	0.0255	0.0016	4.11	3.10

TABLE XXXIV

CASE 8 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 17

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.600
Reject Boundary:	2.944	2.150
Specified Alpha:	0.0500	0.1088
Specified Beta:	0.0500	0.0662
Observed Alpha:	0.0258	0.0510
Observed Beta:	0.0414	0.0513

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	6.54	1.63
1.000	0.9742	0.0016	11.86	6.19
1.125	0.7956	0.0040	16.26	9.35
1.250	0.4133	0.0049	16.22	10.10
1.375	0.1412	0.0035	12.42	8.91
1.500	0.0414	0.0020	8.89	6.77
1.750	0.0056	0.0007	5.19	3.76
		Adjusted:		
0.750	0.9996	0.0002	5.85	1.55
1.000	0.9490	0.0022	10.14	5.43
1.125	0.7455	0.0044	12.92	8.31
1.250	0.3953	0.0049	12.42	8.93
1.375	0.1516	0.0036	9.39	7.45
1.500	0.0513	0.0022	6.98	5.84
1.750	0.0086	0.0009	4.20	3.29

TABLE XXXV

CASE 9 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:20

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-2.650
Reject Boundary:	3.638	2.900
Specified Alpha:	0.0250	0.0513
Specified Beta:	0.0500	0.0670
Observed Alpha:	0.0123	0.0237
Observed Beta:	0.0375	0.0479

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	6.63	1.61
1.000	0.9877	0.0011	12.46	6.80
1.125	0.8308	0.0037	18.44	11.11
1.250	0.4154	0.0049	19.28	11.91
1.375	0.1346	0.0034	14.63	10.27
1.500	0.0375	0.0019	10.28	7.53
1.750	0.0040	0.0006	5.88	4.10
		Adjusted:		
0.750	1.0000	0.0000	5.94	1.56
1.000	0.9763	0.0015	10.90	6.16
1.125	0.8083	0.0039	15.31	10.00
1.250	0.4302	0.0050	15.55	10.68
1.375	0.1542	0.0036	11.97	9.07
1.500	0.0479	0.0021	8.59	6.65
1.750	0.0072	0.0008	5.10	3.68

TABLE XXXVI

CASE 10 SUMMARY FOR (1,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:24

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-3.300
Reject Boundary:	3.664	2.800
Specified Alpha:	0.0250	0.0587
Specified Beta:	0.0250	0.0347
Observed Alpha:	0.0102	0.0270
Observed Beta:	0.0194	0.0265

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	7.98	1.79
1.000	0.9898	0.0010	15.09	7.72
1.125	0.8242	0.0038	22.69	13.06
1.250	0.3728	0.0048	22.84	14.27
1.375	0.0903	0.0029	16.10	11.72
1.500	0.0194	0.0014	10.71	7.98
1.750	0.0014	0.0004	6.05	4.19
		Adjusted:		
0.750	1.0000	0.0000	7.24	1.72
1.000	0.9730	0.0016	13.25	6.94
1.125	0.7833	0.0041	18.64	11.80
1.250	0.3704	0.0048	18.07	12.80
1.375	0.1054	0.0031	12.94	10.45
1.500	0.0265	0.0016	8.79	7.14
1.750	0.0027	0.0005	5.05	3.76

TABLE XXXVII

CASE I SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 4

	Unadjusted	Adjusted
Accept Boundary:	-2.197	-3.000
Reject Boundary:	2.197	0.600
Specified Alpha:	0.1000	0.5361
Specified Beta:	0.1000	0.0231
Observed Alpha:	0.0634	0.1001
Observed Beta:	0.1297	0.0997

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.19	0.40
1.000	0.9366	0.0024	3.29	0.77
1.250	0.7060	0.0046	3.46	0.76
1.500	0.4321	0.0050	3.31	0.85
1.750	0.2397	0.0043	3.04	0.90
2.000	0.1297	0.0034	2.77	0.87
2.500	0.0405	0.0020	2.41	0.71
		Adjusted		
0.500	1.0000	0.0000	3.06	0.23
1.000	0.8999	0.0030	3.65	0.59
1.250	0.6317	0.0048	3.41	0.83
1.500	0.3659	0.0048	3.04	0.92
1.750	0.1976	0.0040	2.72	0.88
2.000	0.0997	0.0030	2.49	0.77
2.500	0.0285	0.0017	2.23	0.56

TABLE XXXVIII

CASE 2 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 5

	Unadjusted	Adjusted
Accept Boundary:	-2.251	-2.350
Reject Boundary:	2.890	2.400
Specified Alpha:	0.0500	0.0828
Specified Beta:	0.1000	0.0875
Observed Alpha:	0.0450	0.0488
Observed Beta:	0.1045	0.0999

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.24	0.43
1.000	0.9542	0.0021	3.68	1.07
1.250	0.7206	0.0045	4.09	1.10
1.500	0.4163	0.0049	3.96	1.21
1.750	0.2132	0.0041	3.56	1.28
2.000	0.1045	0.0031	3.15	1.22
2.500	0.0285	0.0017	2.57	0.95
		Adjusted:		
0.500	1.0000	0.0000	2.35	0.49
1.000	0.9512	0.0022	3.76	1.04
1.250	0.7101	0.0045	4.09	1.09
1.500	0.4085	0.0049	3.87	1.23
1.750	0.2075	0.0041	3.42	1.27
2.000	0.0999	0.0030	3.03	1.19
2.500	0.0253	0.0016	2.51	0.89

TABLE XXXIX

CASE 3 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.950
Reject Boundary:	2.944	1.250
Specified Alpha:	0.0500	0.2756
Specified Beta:	0.0500	0.0379
Observed Alpha:	0.0279	0.0512
Observed Beta:	0.0585	0.0503

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.05	0.22
1.000	0.9721	0.0016	4.84	1.46
1.250	0.7281	0.0044	5.63	1.55
1.500	0.3810	0.0049	5.36	1.67
1.750	0.1514	0.0036	4.62	1.66
2.000	0.0585	0.0023	3.99	1.42
2.500	0.0090	0.0009	3.37	0.89
		Adjusted:		
0.500	1.0000	0.0000	3.05	0.22
1.000	0.9488	0.0022	4.73	1.43
1.250	0.6801	0.0047	5.16	1.62
1.500	0.3335	0.0047	4.64	1.67
1.750	0.1331	0.0034	3.97	1.43
2.000	0.0503	0.0022	3.55	1.12
2.500	0.0083	0.0009	3.17	0.59

TABLE XL

CASE 4 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-3.450
Reject Boundary:	3.638	5.100
Specified Alpha:	0.0250	0.0059
Specified Beta:	0.0500	0.0316
Observed Alpha:	0.0281	0.0259
Observed Beta:	0.0602	0.0484

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.05	0.22
1.000	0.9719	0.0017	4.90	1.46
1.250	0.7336	0.0044	5.79	1.50
1.500	0.3780	0.0048	5.60	1.63
1.750	0.1537	0.0036	4.85	1.70
2.000	0.0602	0.0024	4.17	1.51
2.500	0.0097	0.0010	3.44	0.97
		Adjusted:		
0.500	1.0000	0.0000	3.28	0.46
1.000	0.9741	0.0016	5.43	1.35
1.250	0.7204	0.0045	6.26	1.20
1.500	0.3607	0.0048	6.12	1.41
1.750	0.1391	0.0035	5.39	1.68
2.000	0.0484	0.0021	4.61	1.64
2.500	0.0077	0.0009	3.65	1.15

TABLE XLI

CASE 5 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 9

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-4.700
Reject Boundary:	3.664	2.400
Specified Alpha:	0.0250	0.0900
Specified Beta:	0.0250	0.0083
Observed Alpha:	0.0164	0.0234
Observed Beta:	0.0343	0.0260

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.52	0.52
1.000	0.9836	0.0013	6.07	1.86
1.250	0.7411	0.0044	7.31	2.04
1.500	0.3376	0.0047	6.71	2.38
1.750	0.1103	0.0031	5.46	2.39
2.000	0.0343	0.0018	4.45	1.97
2.500	0.0031	0.0006	3.49	1.11
		Adjusted:		
0.500	1.0000	0.0000	4.35	0.50
1.000	0.9766	0.0015	7.14	1.62
1.250	0.7121	0.0045	7.56	2.03
1.500	0.3076	0.0046	6.33	2.53
1.750	0.0907	0.0029	4.94	2.29
2.000	0.0260	0.0016	4.06	1.75
2.500	0.0015	0.0004	3.33	0.91

TABLE XLII

CASE 6 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:11

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-3.600
Reject Boundary:	2.197	1.550
Specified Alpha:	0.1000	0.2077
Specified Beta:	0.1000	0.0216
Observed Alpha:	0.0769	0.0984
Observed Beta:	0.1268	0.1008

σ	$L(\sigma)$	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	5.10	1.36
1.000	0.9231	0.0027	7.64	2.58
1.125	0.7320	0.0044	8.41	2.67
1.250	0.4733	0.0050	8.26	2.79
1.375	0.2611	0.0044	7.54	2.90
1.500	0.1268	0.0033	6.63	2.75
1.750	0.0267	0.0016	5.28	2.07
		Adjusted:		
0.750	0.9999	0.0001	7.76	1.52
1.000	0.9016	0.0030	9.75	1.89
1.125	0.6969	0.0046	9.51	2.39
1.250	0.4270	0.0049	8.52	2.91
1.375	0.2205	0.0041	7.31	3.01
1.500	0.1008	0.0030	6.25	2.72
1.750	0.0151	0.0012	4.96	1.85

TABLE XLIII

CASE 7 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 13

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-2.600
Reject Boundary:	2.890	4.000
Specified Alpha:	0.0500	0.0170
Specified Beta:	0.1000	0.0730
Observed Alpha:	0.0567	0.0567
Observed Beta:	0.1158	0.1006

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	0.9999	0.0001	5.59	1.14
1.000	0.9433	0.0023	8.44	3.04
1.125	0.7515	0.0043	9.77	3.18
1.250	0.4721	0.0050	9.95	3.22
1.375	0.2457	0.0043	9.16	3.32
1.500	0.1158	0.0032	8.10	3.16
1.750	0.0205	0.0014	6.37	2.28
		Adjusted:		
0.750	0.9999	0.0001	5.99	1.37
1.000	0.9433	0.0023	9.24	2.98
1.125	0.7506	0.0043	10.69	2.89
1.250	0.4619	0.0050	11.08	2.80
1.375	0.2329	0.0042	10.38	3.12
1.500	0.1006	0.0030	9.19	3.27
1.750	0.0157	0.0012	7.04	2.68

TABLE XLIV

CASE 8 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 17

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-6.000
Reject Boundary:	2.944	2.600
Specified Alpha:	0.0500	0.0741
Specified Beta:	0.0500	0.0023
Observed Alpha:	0.0413	0.0482
Observed Beta:	0.0672	0.0562

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	6.85	1.35
1.000	0.9587	0.0020	11.00	3.93
1.125	0.7616	0.0043	13.00	4.13
1.250	0.4420	0.0050	12.87	4.35
1.375	0.1937	0.0040	11.22	4.49
1.500	0.0672	0.0025	9.39	4.00
1.750	0.0071	0.0008	7.17	2.39
		Adjusted:		
0.750	1.0000	0.0000	12.57	2.03
1.000	0.9518	0.0021	16.14	1.90
1.125	0.7406	0.0044	15.70	3.05
1.250	0.4164	0.0049	13.77	4.39
1.375	0.1709	0.0038	11.35	4.68
1.500	0.0562	0.0023	9.24	4.05
1.750	0.0044	0.0007	7.05	2.29

TABLE XLV

CASE 9 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:20

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-3.400
Reject Boundary:	3.638	4.000
Specified Alpha:	0.0250	0.0177
Specified Beta:	0.0500	0.0328
Observed Alpha:	0.0293	0.0296
Observed Beta:	0.0599	0.0504

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	7.47	1.09
1.000	0.9707	0.0017	11.80	4.55
1.125	0.7869	0.0041	14.67	5.05
1.250	0.4424	0.0050	15.14	5.14
1.375	0.1823	0.0039	13.31	5.27
1.500	0.0599	0.0024	11.03	4.65
1.750	0.0050	0.0007	8.27	2.62
		Adjusted:		
0.750	1.0000	0.0000	7.86	1.41
1.000	0.9704	0.0017	12.90	4.59
1.125	0.7797	0.0041	15.77	4.73
1.250	0.4343	0.0050	16.09	4.85
1.375	0.1706	0.0038	14.07	5.28
1.500	0.0504	0.0022	11.57	4.87
1.750	0.0036	0.0006	8.48	2.82

TABLE XLVI

CASE 10 SUMMARY FOR (n/3,n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:24

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-6.000
Reject Boundary:	3.664	3.500
Specified Alpha:	0.0250	0.0301
Specified Beta:	0.0250	0.0024
Observed Alpha:	0.0224	0.0241
Observed Beta:	0.0341	0.0255

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	8.66	1.28
1.000	0.9776	0.0015	14.35	5.37
1.125	0.7881	0.0041	18.12	5.95
1.250	0.4151	0.0049	18.11	6.27
1.375	0.1384	0.0035	15.06	6.34
1.500	0.0341	0.0018	11.95	5.20
1.750	0.0019	0.0004	9.02	2.50
		Adjusted:		
0.750	1.0000	0.0000	12.66	2.26
1.000	0.9759	0.0015	20.15	4.19
1.125	0.7726	0.0042	21.70	4.32
1.250	0.3971	0.0049	19.43	6.15
1.375	0.1223	0.0033	15.45	6.59
1.500	0.0255	0.0016	12.06	5.34
1.750	0.0011	0.0003	8.97	2.46

TABLE XLVII

CASE I SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 4

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-1.700
Reject Boundary:	2.197	0.500
Specified Alpha:	0.1000	0.5575
Specified Beta:	0.1000	0.0808
Observed Alpha:	0.0247	0.0976
Observed Beta:	0.0805	0.1014

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.18	0.39
1.000	0.9753	0.0016	3.92	1.71
1.250	0.7463	0.0044	4.78	2.15
1.500	0.4044	0.0049	4.55	2.24
1.750	0.1790	0.0038	3.78	2.00
2.000	0.0805	0.0027	3.18	1.66
2.500	0.0235	0.0015	2.52	1.03
		Adjusted:		
0.500	1.0000	0.0000	2.02	0.14
1.000	0.9024	0.0030	2.93	1.20
1.250	0.6354	0.0048	3.19	1.51
1.500	0.3541	0.0048	2.96	1.41
1.750	0.1846	0.0039	2.65	1.15
2.000	0.1014	0.0030	2.43	0.93
2.500	0.0353	0.0018	2.19	0.58

TABLE XLVIII

CASE 2 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 5

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-1.800
Reject Boundary:	2.890	1.250
Specified Alpha:	0.0500	0.2510
Specified Beta:	0.1000	0.1238
Observed Alpha:	0.0157	0.0520
Observed Beta:	0.0741	0.0987

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	2.23	0.43
1.000	0.9843	0.0012	4.14	1.99
1.250	0.7742	0.0042	5.50	2.77
1.500	0.3985	0.0049	5.33	2.85
1.750	0.1699	0.0038	4.39	2.53
2.000	0.0741	0.0026	3.55	2.04
2.500	0.0190	0.0014	2.68	1.24
		Adjusted:		
0.500	1.0000	0.0000	2.03	0.18
1.000	0.9480	0.0022	3.26	1.55
1.250	0.7001	0.0046	3.88	2.11
1.500	0.3864	0.0049	3.59	2.01
1.750	0.1932	0.0039	3.10	1.66
2.000	0.0987	0.0030	2.73	1.34
2.500	0.0335	0.0018	2.30	0.76

TABLE XLIX

CASE 3 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.450
Reject Boundary:	2.944	1.100
Specified Alpha:	0.0500	0.3131
Specified Beta:	0.0500	0.0593
Observed Alpha:	0.0122	0.0481
Observed Beta:	0.0360	0.0500

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.04	0.20
1.000	0.9878	0.0011	5.30	2.36
1.250	0.7701	0.0042	7.46	3.76
1.500	0.3272	0.0047	6.93	3.82
1.750	0.1082	0.0031	5.30	3.02
2.000	0.0360	0.0019	4.23	2.13
2.500	0.0073	0.0009	3.39	1.03
		Adjusted:		
0.500	1.0000	0.0000	3.01	0.08
1.000	0.9519	0.0021	4.41	1.81
1.250	0.6649	0.0047	5.20	2.66
1.500	0.3067	0.0046	4.71	2.48
1.750	0.1204	0.0033	3.97	1.83
2.000	0.0500	0.0022	3.53	1.27
2.500	0.0119	0.0011	3.16	0.60

TABLE L

CASE 4 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 7

	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-2.450
Reject Boundary:	3.638	1.900
Specified Alpha:	0.0250	0.1384
Specified Beta:	0.0500	0.0743
Observed Alpha:	0.0071	0.0227
Observed Beta:	0.0343	0.0516

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.05	0.22
1.000	0.9929	0.0008	5.38	2.47
1.250	0.7792	0.0041	7.85	3.89
1.500	0.3471	0.0048	7.58	3.99
1.750	0.1077	0.0031	5.77	3.27
2.000	0.0343	0.0018	4.52	2.33
2.500	0.0058	0.0008	3.50	1.16
		Adjusted:		
0.500	1.0000	0.0000	3.00	0.07
1.000	0.9773	0.0015	4.56	2.02
1.250	0.7243	0.0045	5.88	3.14
1.500	0.3314	0.0047	5.42	3.04
1.750	0.1245	0.0033	4.43	2.31
2.000	0.0516	0.0022	3.77	1.58
2.500	0.0102	0.0010	3.25	0.78

TABLE LI

CASE 5 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=2.0
Number of Observations Required for Fixed-Sample Size Test: 9

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-3.200
Reject Boundary:	3.664	1.850
Specified Alpha:	0.0250	0.1518
Specified Beta:	0.0250	0.0346
Observed Alpha:	0.0044	0.0258
Observed Beta:	0.0149	0.0246

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.500	1.0000	0.0000	3.52	0.52
1.000	0.9956	0.0007	6.46	2.75
1.250	0.7787	0.0042	9.84	4.96
1.500	0.2807	0.0045	8.91	5.13
1.750	0.0639	0.0024	6.22	3.89
2.000	0.0149	0.0012	4.64	2.55
2.500	0.0020	0.0004	3.50	1.16
		Adjusted:		
0.500	1.0000	0.0000	3.12	0.33
1.000	0.9742	0.0016	5.60	2.43
1.250	0.6921	0.0046	7.22	4.05
1.500	0.2660	0.0044	6.24	3.91
1.750	0.0776	0.0027	4.75	2.81
2.000	0.0246	0.0015	3.87	1.79
2.500	0.0035	0.0006	3.26	0.84

TABLE LII

CASE 6 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:11

	Unadjusted:	Adjusted:
Accept Boundary:	-2.197	-1.900
Reject Boundary:	2.197	1.100
Specified Alpha:	0.1000	0.2979
Specified Beta:	0.1000	0.1050
Observed Alpha:	0.0459	0.1023
Observed Beta:	0.0820	0.1003

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	5.11	1.36
1.000	0.9541	0.0021	8.83	4.61
1.125	0.7616	0.0043	10.91	5.89
1.250	0.4399	0.0050	10.76	6.12
1.375	0.2041	0.0040	9.05	5.48
1.500	0.0820	0.0027	7.35	4.48
1.750	0.0157	0.0012	5.36	2.50
		Adjusted:		
0.750	0.9991	0.0003	4.68	1.14
1.000	0.8977	0.0030	7.07	3.54
1.125	0.6790	0.0047	7.83	4.32
1.250	0.4070	0.0049	7.57	4.42
1.375	0.2046	0.0040	6.63	3.77
1.500	0.1003	0.0030	5.73	2.97
1.750	0.0253	0.0016	4.69	1.68

TABLE LIII

CASE 7 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 13

	Unadjusted:	Adjusted:
Accept Boundary:	-2.251	-1.900
Reject Boundary:	2.890	1.900
Specified Alpha:	0.0500	0.1301
Specified Beta:	0.1000	0.1301
Observed Alpha:	0.0232	0.0481
Observed Beta:	0.0760	0.1012

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	0.9999	0.0001	5.59	1.13
1.000	0.9768	0.0015	9.56	5.18
1.125	0.8020	0.0040	12.66	7.06
1.250	0.4695	0.0050	13.15	7.31
1.375	0.2038	0.0040	11.32	6.63
1.500	0.0760	0.0026	9.04	5.21
1.750	0.0132	0.0011	6.46	2.77
		Adjusted:		
0.750	1.0000	0.0000	5.34	0.89
1.000	0.9519	0.0021	8.06	4.09
1.125	0.7717	0.0042	9.86	5.56
1.250	0.4627	0.0050	10.12	5.98
1.375	0.2251	0.0042	8.73	5.06
1.500	0.1012	0.0030	7.36	3.84
1.750	0.0220	0.0015	5.86	2.02

TABLE LIV

CASE 8 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test: 17

	Unadjusted:	Adjusted:
Accept Boundary:	-2.944	-2.600
Reject Boundary:	2.944	1.800
Specified Alpha:	0.0500	0.1549
Specified Beta:	0.0500	0.0628
Observed Alpha:	0.0216	0.0496
Observed Beta:	0.0416	0.0493

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	6.85	1.37
1.000	0.9784	0.0015	12.07	6.16
1.125	0.8028	0.0040	16.69	9.17
1.250	0.4148	0.0049	16.92	9.66
1.375	0.1412	0.0035	13.29	8.19
1.500	0.0416	0.0020	10.17	5.97
1.750	0.0054	0.0007	7.22	2.61
		Adjusted:		
0.750	1.0000	0.0000	6.51	1.09
1.000	0.9504	0.0022	10.41	5.14
1.125	0.7382	0.0044	12.95	7.33
1.250	0.3886	0.0049	12.70	7.69
1.375	0.1479	0.0036	10.29	6.10
1.500	0.0493	0.0022	8.38	4.32
1.750	0.0076	0.0009	6.66	1.84

TABLE LV

CASE 9 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0	Alternative Hypothesis: σ=1.5	
Number of Observations Required for Fixed-Sample Size Test:20		
	Unadjusted:	Adjusted:
Accept Boundary:	-2.970	-2.550
Reject Boundary:	3.638	2.550
Specified Alpha:	0.0250	0.0724
Specified Beta:	0.0500	0.0724
Observed Alpha:	0.0113	0.0257
Observed Beta:	0.0364	0.0483

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	7.48	1.10
1.000	0.9887	0.0011	12.73	6.69
1.125	0.8339	0.0037	18.75	10.66
1.250	0.4181	0.0049	20.08	11.42
1.375	0.1306	0.0034	15.65	9.51
1.500	0.0364	0.0019	11.68	6.50
1.750	0.0034	0.0006	8.27	2.82
		Adjusted:		
0.750	1.0000	0.0000	7.25	0.79
1.000	0.9743	0.0016	11.20	5.67
1.125	0.7916	0.0041	15.14	9.09
1.250	0.4198	0.0049	15.26	9.23
1.375	0.1502	0.0036	12.46	7.51
1.500	0.0483	0.0021	9.96	5.09
1.750	0.0047	0.0007	7.75	2.12

TABLE LVI

CASE 10 SUMMARY FOR (n/3,2n) PLAN

Null Hypothesis: σ=1.0 Alternative Hypothesis: σ=1.5
Number of Observations Required for Fixed-Sample Size Test:24

	Unadjusted:	Adjusted:
Accept Boundary:	-3.664	-3.400
Reject Boundary:	3.664	2.400
Specified Alpha:	0.0250	0.0880
Specified Beta:	0.0250	0.0304
Observed Alpha:	0.0086	0.0259
Observed Beta:	0.0189	0.0239

σ	L(σ)	S.E.	ASN	S.D.
		Unadjusted:		
0.750	1.0000	0.0000	8.66	1.28
1.000	0.9914	0.0009	15.32	7.57
1.125	0.8305	0.0038	23.26	12.83
1.250	0.3779	0.0048	23.93	13.76
1.375	0.0894	0.0029	17.49	11.07
1.500	0.0189	0.0014	12.51	6.89
1.750	0.0010	0.0003	9.03	2.55
		Adjusted:		
0.750	1.0000	0.0000	8.48	1.12
1.000	0.9741	0.0016	14.02	6.74
1.125	0.7636	0.0042	18.96	10.98
1.250	0.3424	0.0047	18.35	11.35
1.375	0.0968	0.0030	13.90	8.59
1.500	0.0239	0.0015	10.77	5.24
1.750	0.0017	0.0004	8.60	1.90

APPENDIX B

ATTEMPTS TO USE AROIAN'S DIRECT METHOD

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ATTEMPTS TO USE AROIAN'S DIRECT METHOD

This appendix documents the author's attempts to use Aroian's direct method as an alternative to Monte Carlo simulation. Aroian (1968) and especially Aroian and Robison (1969) were used as a guide. It was believed that the direct method would be particularly appropriate for use with the Rayleigh distribution since the cumulative distribution function for the test statistic could be written in closed form, eliminating the requirement for complicated numerical integration of truncated convolution integrals involving sums of random variables.

If the author had succeeded in implementing the direct method, the problem of adjusting the boundaries to achieve the specified error rates would have been greatly simplified since the effect of changes in stopping boundaries on observed error rates would not have been obscured by sampling error. More interesting from a theoretical viewpoint is that since the stepwise error rates are explicit functions of the stopping boundaries, it might be possible to "invert" the problem and solve for stopping boundaries as functions of specified stepwise error rates. That is, for any specified error rate schedule, it might be possible to solve for the boundaries which would produce those error rates.

The fact that the author failed in trying to implement the direct method should <u>not</u> be used to infer that the method is inappropriate for this problem. It was abandoned because the approach <u>appeared</u> to become more complicated at each step and the author was not sufficiently clever to see an emerging pattern and simplify the computations.

The remainder of this appendix will present direct method derivations for the first three stages of a sequential procedure and a partial comparison of direct method and Monte Carlo results for Case I of the SPRT.

Consider the events of acceptance (A), rejection (R) and continuation (C) and denote their probabilities as follows:

$P_m(A)=Pr(\text{accepting } H_0 \text{ on trial } m)$

$P_m(R)=Pr(\text{rejecting } H_0 \text{ on trial } m)$

$P_m(C)=Pr(\text{continuing at trial } m)$

The above three probabilities are dependent on σ which has been suppressed in the notation.

Denote by A_m and R_m the acceptance and rejection boundaries at stage m, respectively and let the boundary functions be arbitrary to within the following restrictions:

$A_1 \leq A_2 \leq \cdots \leq A_m \leq \cdots$

$R_1 \leq R_2 \leq \cdots \leq R_m \leq \cdots$

$A_i \leq R_i, \quad i=1,2,\cdots,m,\cdots.$

Let $V \sim$ Rayleigh (σ). Then

$h(v)=(v/\sigma^2)\exp(-v^2/2\sigma^2), \ v>0$ and

$H(v)=1-\exp(-v^2/2\sigma^2), \ v>0.$

Now if one makes the substitution $X=V^2$ and $\theta=1/2\sigma^2$, an elementary transformation yields

$f(x)=\exp(-\theta x), \ x>0$

$F(x)=1-\exp(-\theta x), \ x>0.$

Although $Y_m=\sum_1^m X_i$ conveys the idea for the test statistic, the method of computation requires modified definitions and notation to account for the fact that Y_{m-1} must come from the continuation region at stage (m-1). The random variables of interest may be denoted by W_m given by

$$W_0 = 0$$

$$W_m = W_{m-1}^T + X_m, \quad m = 1,2,3,\cdots, \text{where}$$

$$W_0^T = 0$$

$W_{m-1}^T = W_{m-1}$ truncated to coming from the continuation interval (A_{m-1}, R_{m-1})

X_m = the m-th independent squared Rayleigh observation.

<u>Computations for Stage 1:</u>

$$W_1 = X_1$$

$$P_1(A) = Pr(W_1 \leq A_1) = F(A_1) = 1 - e^{-\theta A_1}$$

$$P_1(R) = Pr(W_1 \geq R_1) = 1 - F(R_1) = e^{-\theta R_1}$$

$$P_1(C) = 1 - P_1(A) - P_1(R) = e^{-\theta A_1} - e^{-\theta R_1}$$

<u>Computations for Stage 2:</u>

$$W_1^T \varepsilon (A_1, R_1)$$

$$f(W_1^T) = \theta e^{-\theta W_1^T} / P_1(C), \quad A_1 < W_1^T < R_1$$

$$\text{Let } W_2 = W_1^T + X_2$$

$$Z = W_1^T$$

$$f(W_2, Z) = \theta^2 e^{-\theta W_2} / P_1(C)$$

From Figure 15, it may be seen that

$$f(W_2) = \begin{cases} 0, & W_2 \leq A_1 \\ \dfrac{1}{P_1(C)} \displaystyle\int_{A_1}^{W_2} \theta^2 e^{-\theta W_2} dz = (W_2 - A_1)\theta^2 e^{-\theta W_2}/P_1(C), & A_1 < W_2 \leq R_1 \\ \dfrac{1}{P_1(C)} \displaystyle\int_{A_1}^{R_1} \theta^2 e^{-\theta W_2} dz = (R_1 - A_1)\theta^2 e^{-\theta W_2}/P_1(C), & W_2 > R_1 \end{cases}$$

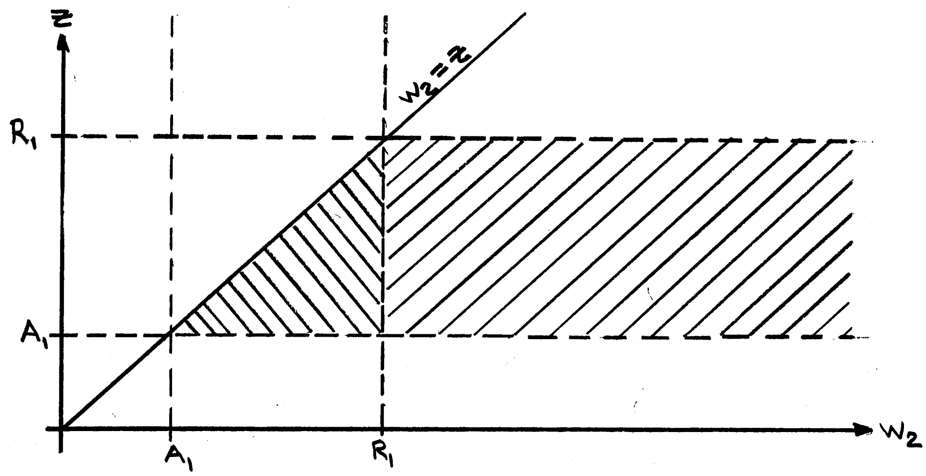

Figure 15. Region of Definition for W_2.

If $A_2 \le R_1$,

$$P_2(A) = \int_{A_1}^{A_2} (w_2 - A_1)\,\theta^2 e^{-\theta w_2}\,dw_2$$

$$= e^{-\theta A_1} - [\theta(A_2 - A_1) + 1]\,e^{-\theta A_2} \;;$$

if $A_2 > R_1$,

$$P_2(A) = \int_{A_1}^{R_1} (w_2 - A_1)\,\theta^2 e^{-\theta w_2}\,dw_2 + \int_{R_1}^{A_2} (R_1 - A_1)\,\theta^2 e^{-\theta w_2}\,dw_2$$

$$= e^{-\theta A_1} - e^{-\theta R_1} - \theta(R_1 - A_1)\,e^{-\theta A_2}.$$

$$P_2(R) = P_1(c)\,P_r(w_2 \ge R_2 \mid A_1 < w_1 < R_1)$$

$$= P_1(c)\left[1 - P_r(w_2 < R_2 \mid A_1 < w_1 < R_1)\right]$$

$$= P_1(c) - \int_{A_1}^{R_1} (w_2 - A_1)\,\theta^2 e^{-\theta w_2}\,dw_2 - \int_{R_1}^{R_2} (R_1 - A_1)\,\theta^2 e^{-\theta w_2}\,dw_2$$

$$= \theta(R_1 - A_1)\,e^{-\theta R_2},$$

$$P_2(C) = P_1(C) - P_2(A) - P_2(R).$$

If $A_2 \le R_1$, $P_2(C) = [\theta(A_2 - A_1)+1]e^{-\theta A_2} - e^{-\theta R_1} - \theta(R_1 - A_1)e^{-\theta R_2}$;

if $A_2 > R_1$, $P_2(C) = \theta(R_1 - A_1)(e^{-\theta A_2} - e^{-\theta R_2})$.

Computations for Stage 3:

$$w_2^T \in (A_2, R_2)$$

If $A_2 \le R_1$,

$$f(w_2^T) = \begin{cases} (w_2^T - A_1)\theta^2 e^{-\theta w_2^T}/P_2(C), & A_2 < w_2^T \le R_1 \\ (R_1 - A_1)\theta^2 e^{-\theta w_2^T}/P_2(C), & R_1 < w_2^T < R_2; \end{cases}$$

if $A_2 > R_1$,

$$f(w_2^T) = (R_1 - A_1)\theta^2 e^{-\theta w_2^T}/P_2(C), \quad A_2 < w_2^T < R_2.$$

Let $w_3 = w_2^T + X_3$,

$z = w_2^T$.

If $A_2 \le R_1$,

$$f(w_3, z) = \begin{cases} (z - A_1)\theta^3 e^{-\theta w_3}/P_2(C), & A_2 < z \le R_1 \\ (R_1 - A_1)\theta^3 e^{-\theta w_3}/P_2(C), & R_1 < z < R_2. \end{cases}$$

With the help of Figure 16, one may write

$$f(w_3) = \frac{1}{P_2(C)} \int_{A_2}^{w_3} (z - A_1)\theta^3 e^{-\theta w_3}\, dz$$

$$= \frac{1}{P_2(C)}\Big[\frac{\theta^3}{2} w_3^2 e^{-\theta w_3} - \theta^3 A_1 w_3 e^{-\theta w_3}$$

$$+ \theta^3 A_2\left(A_1 - \frac{A_2}{2}\right)e^{-\theta w_3}\Big], \quad A_2 < w_3 \le R_1;$$

$$f(w_3) = \frac{1}{P_2(C)}\left[\int_{A_2}^{R_1} (z-A_1)\theta^3 e^{-\theta w_3} dz + \int_{R_1}^{w_3} (R_1-A_1)\theta^3 e^{-\theta w_3} dz \right]$$

$$= \frac{1}{P_2(C)}\left[\theta^3 (R_1-A_1) w_3 e^{-\theta w_3} + \theta^3 (A_1A_2 - \frac{A_2^2-R_1^2}{2} e^{-\theta w_3} \right], \quad R_1 < w_3 \leq R_2;$$

$$f(w_3) = \frac{1}{P_2(C)}\left[\int_{A_2}^{R_1} (z-A_1)\theta^3 e^{-\theta w_3} dz + \int_{R_1}^{R_2} (R_1-A_1)\theta^3 e^{-\theta w_3} dz \right]$$

$$= \frac{1}{P_2(C)}\left[\theta^3 \left(A_1A_2 + R_1R_2 - A_1R_2 - \frac{R_1^2+A_2^2}{2}\right) e^{-\theta w_3} \right], \quad w_3 > R_2.$$

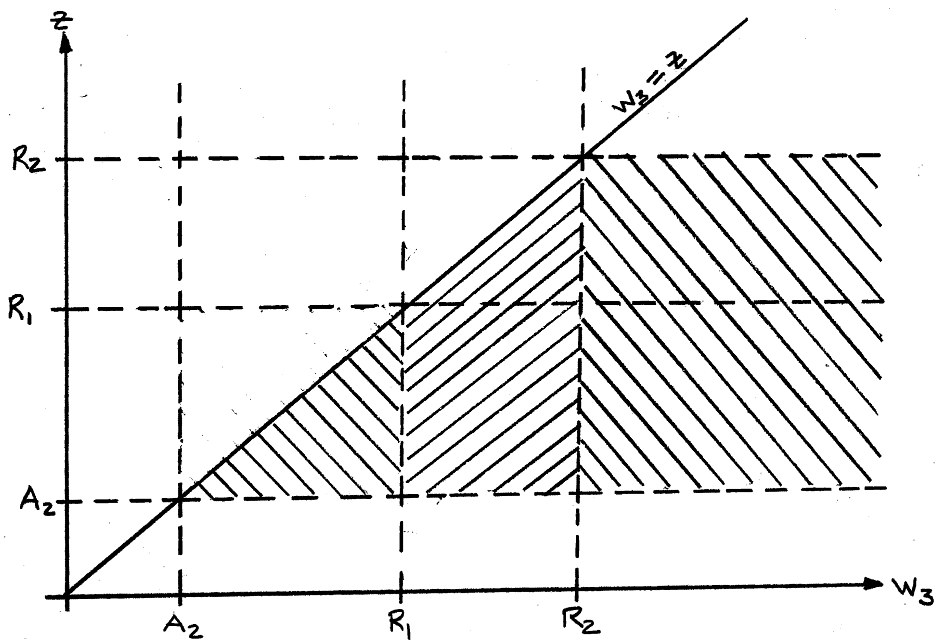

Figure 16. Region of Definition for W_3 $(A_2 \leq R_1)$

If $A_2 > R_1$, $f(w_3, z) = (R_1-A_1)\theta^3 e^{-\theta w_3} / P_2(c)$, $A_2 < z < R_2$,

and from Figure 17,

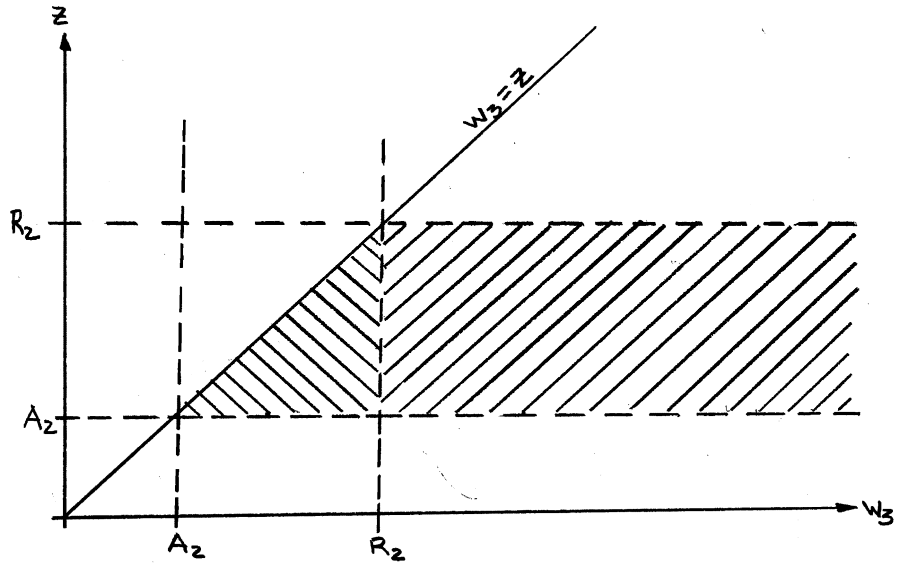

Figure 17. Region of Definition for W_3 $(A_2 > R_1)$

$$f(w_3) = \frac{1}{P_2(c)} \int_{A_2}^{w_3} (R_1 - A_1)\theta^3 e^{-\theta w_3}\, dz$$

$$= \frac{1}{P_2(c)} \left[\theta^3 (R_1 - A_1) w_3 e^{-\theta w_3} - \theta^3 A_2 (R_1 - A_1) e^{-\theta w_3} \right], \quad A_2 < w_3 \le R_2;$$

$$f(w_3) = \frac{1}{P_2(c)} \int_{A_2}^{R_2} (R_1 - A_1)\theta^3 e^{-\theta w_3}\, dz$$

$$= \frac{1}{P_2(c)} \theta^3 (R_2 - A_2)(R_1 - A_1) e^{-\theta w_3}, \quad w_3 > R_2 .$$

If $A_2 \le A_3 \le R_1$,

$$P_3(A) = \int_{A_2}^{A_3} \left[\frac{\theta^3}{2} w_3^2 e^{-\theta w_3} - \theta^3 A_1 w_3 e^{-\theta w_3} + \theta^3 A_2 \left(A_1 - \frac{A_2}{2}\right) e^{-\theta w_3} \right] dw_3$$

$$= \left[\theta (A_2 - A_1) + 1\right] e^{-\theta A_2} + \left\{ \theta^2 \left[A_2 \left(\frac{A_2}{2} - A_1\right) - A_3 \left(\frac{A_3}{2} - A_1\right) \right] - \theta (A_3 - A_1) - 1 \right\} e^{-\theta A_3};$$

If $A_2 \le R_1 < A_3$,

$$P_3(A) = \int_{A_2}^{R_1} \left[\frac{\theta^3}{2} w_3^2 e^{-\theta w_3} - \theta^3 A_1 w_3 e^{-\theta w_3} + \theta^3 A_2 \left(A_1 - \frac{A_2}{2}\right) e^{-\theta w_3} \right] dw_3$$

$$+ \int_{R_1}^{A_3} \left\{ \theta^3 (R_1 - A_1) w_3 e^{-\theta w_3} + \theta^3 \left[A_1 A_2 - \frac{A_2^2 + R_1^2}{2} \right] e^{-\theta w_3} \right\} dw_3 .$$

If $A_2 \le R_1$,

$$P_3(R) = \int_{R_3}^{\infty} \theta^3 \left[A_1 A_2 + R_1 R_2 - A_1 R_2 - \frac{R_1^2 + A_2^2}{2} \right] e^{-\theta w_3} dw_3$$

$$= \theta^2 \left[A_1 A_2 + R_1 R_2 - A_1 R_2 - \frac{R_1^2 + A_2^2}{2} \right] e^{-\theta R_3}$$

If $A_2 > R_1$,

$$P_3(R) = \int_{R_3}^{\infty} \theta^3 (R_2 - A_2)(R_1 - A_1) e^{-\theta w_3} dw_3$$

$$= \theta^2 (R_1 - A_1)(R_2 - A_2) e^{-\theta R_3} .$$

At this point, derivation of $P_m(A)$, $P_m(R)$ and $P_m(C)$ for $m>3$ was abandoned because it appeared that the algebra involved was becoming increasingly overwhelming with each additional stage. For comparative purposes, Table LVII shows results for the direct method (DM) and 10,000 Monte Carlo (MC) iterations for the first three stages of the SPRT for Case I. Note that both methods give $P_1(A)=0$ since $A_1<0$.

TABLE LVII

COMPARISON OF DIRECT METHOD WITH MONTE CARLO

Item	Method	Unknown Parameter Value 0.500	1.000	1.250	1.500	1.750	2.000	2.500
$P_1(A)$	DM	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$P_1(A)$	MC	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$P_1(R)$	DM	0.000	0.008	0.047	0.120	~~0.210~~	0.303	0.466
$P_1(R)$	MC	0.000	0.009	0.049	0.120	0.212	0.305	0.469
$P_2(A)$	DM	0.811	0.179	0.087	0.046	0.027	0.016	0.007
$P_2(A)$	MC	0.808	0.179	0.087	0.044	0.025	0.016	0.007
$P_2(R)$	DM	0.000	0.006	0.044	0.117	0.179	0.228	0.265
$P_2(R)$	MC	0.000	0.006	0.046	0.118	0.183	0.224	0.263
$P_3(A)$	DM	0.187	0.328	0.171	0.084	0.042	0.022	0.007
$P_3(A)$	MC	0.191	0.327	0.171	0.079	0.039	0.019	0.007
$P_3(R)$	DM	0.000	0.004	0.036	0.091	0.134	0.150	0.132
$P_3(R)$	MC	0.000	0.005	0.035	0.089	0.131	0.151	0.133

APPENDIX C

COMPUTER PROGRAM

APPENDIX C

COMPUTER PROGRAM

Program Input:

ICASE - End of case flag. Set ICASE=0 if only one run is desired or if the current run is the last run in a job. Set ICASE=1 if another run follows the present one.

IBOUND - Boundary option. If IBOUND=0, α_s and β_s should be input; acceptance and rejection boundaries will be computed. If IBOUND=1, acceptance and rejection boundaries should be input; α_s and β_s will be computed.

ITER - Number of Monte Carlo iterations desired.

ALPHA - α or acceptance boundary (See IBOUND).

BETA - β or rejection boundary (See IBOUND).

NSTART - Number of observations to be taken prior to making acceptance or rejection decisions.

NFIX - Fixed-sample size.

NTEST - Number of items available for testing.

SIG (1),···,SIG(7) - Seven values of the unknown parameter for which computations are desired. SIG(2) and SIG(6) must be the parameter values under H_0 and H_1, respectively.

Arrays:

AN(I) - ASN for SIG(I).

KK(I) - Flag to indicate if procedure has made an accept or reject decision for SIG(I).

OC(I) - OC curve value for SIG(I).

RIS(I) - Cumulative sum of squared Rayleigh observations for SIG(I).

SE(I) - Standard error associated with SIG(I).

SD(I) - Standard deviation of DSN for SIG(I).

SIG(I) - Population parameter

NA(I) - Number of truncated cases resulting in acceptance of H_0 for SIG(I).

NT(I) - Number of truncated cases for SIG(I).

LOC(I) - OC counter (Number of times H_0 is accepted for SIG(I)).

NN(I) - DSN counter for AN(I).

NNS(I) - DSN counter for SD(I).

SIGS(I) - Squared value of SIG(I).

NACC(I,J) - Counter for number of accept decisions at stage J for SIG(I).

NSTP(I,J) - Counter for number of stops at stage J for SIG(I). The appropriate value of J in the arrays NACC(I,J) and NSTP(I,J) is problem dependent. For closed procedures, J=NTEST; for the open SPRT, use J$\geq$(10)(NFIX) to preclude truncation of all but the most extreme cases.

Program Output:

1. Case information (input data which identifies case).

2. Case summary: SIG(I), OC(I), SE(I), AN(I), SD(I), I=1,$\cdots$,7.

3. DSN history: NSTP(I,J), NACC(I,J), NN(I), NA(I), I=1,$\cdots$,7, J=1,$\cdots$,M where M is the dimensioned array size.

Sample Output: Adjusted (1,∞) Plan for Case 1

```
****************** CASE INFORMATION *****************
*                                                    *
*   ACC. BOUNDARY=-1.700  REJ. BOUNDARY= 0.800       *
*   NULL HYPOTHESIS: SIGMA =   1.00                  *
*   ALT. HYPOTHESIS: SIGMA =   2.00                  *
*   SPECIFIED ALPHA: 0.4001        OBSERVED: 0.1001  *
*   SPECIFIED BETA:  0.1096        OBSERVED: 0.1027  *
*   OBSNS. REQUIRED FOR FIXED SAMPLE SIZE TEST:  4   *
*   NUMBER OF TEST ITEMS AVAILABLE FOR TESTING: 30   *
*   NUMBER OF OBSERVATIONS PRIOR TO DECISIONS:   1   *
*                                                    *
*   NUMBER OF MONTE CARLO ITERATIONS:    10000       *
*   NUMBER OF RANDOM NUMBERS GENERATED:  49322       *
*                                                    *
******************************************************
```

```
******************** CASE SUMMARY ********************
*                                                    *
*    SIGMA   L(SIGMA)   STD ERR   A.S.N.   STD DEV   *
*                                                    *
*     0.50    1.0000     0.0       2.02     0.15     *
*     1.00    0.8999     0.0030    2.89     1.36     *
*     1.25    0.6389     0.0048    3.12     1.91     *
*     1.50    0.3573     0.0048    2.78     1.90     *
*     1.75    0.1885     0.0039    2.31     1.56     *
*     2.00    0.1027     0.0030    1.99     1.35     *
*     2.50    0.0372     0.0019    1.58     0.94     *
*                                                    *
******************************************************
```

	SIGMA = 0.50		SIGMA = 1.00		SIGMA = 1.25		SIGMA = 1.50		SIGMA = 1.75		SIGMA = 2.00		SIGMA = 2.50	
N	STP	ACC	STP	ACC	STP	ACC	STP	ACC	STP	ACC	STP	ACC	STP	ACC
1	0	0	506	0	1503	0	2726	0	3857	0	4818	0	6277	0
2	9786	9786	4494	4238	3219	2334	2861	1365	2785	830	2718	515	2332	225
3	213	213	2664	2539	2184	1687	1875	935	1578	476	1285	268	919	104
4	1	1	1259	1196	1280	968	1033	501	875	287	638	122	308	32
5	0	0	548	519	752	574	617	312	444	137	274	59	108	9
6	0	0	271	260	449	362	364	179	229	72	135	30	34	1
7	0	0	134	130	277	210	227	131	119	37	62	18	14	0
8	0	0	77	72	142	109	135	64	61	26	41	7	6	1
9	0	0	32	30	82	67	82	49	27	10	13	3	1	0
10	0	0	9	9	43	27	31	14	13	4	8	3	1	0
11	0	0	3	3	33	24	17	8	6	3	1	0	0	0
12	0	0	2	2	17	12	15	8	0	0	7	2	0	0
13	0	0	0	0	8	6	5	2	3	2	0	0	0	0
14	0	0	1	1	4	3	4	0	2	0	0	0	0	0
15	0	0	0	0	2	1	4	3	1	1	0	0	0	0
16	0	0	0	0	1	1	1	1	0	0	0	0	0	0
17	0	0	0	0	1	1	1	0	0	0	0	0	0	0
18	0	0	0	0	1	1	1	0	0	0	0	0	0	0
19	0	0	0	0	1	1	1	1	0	0	0	0	0	0
20	0	0	0	0	1	1	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0

```
      DIMENSION AN(7),KK(7),OC(7),RIS(7),SE(7),SD(7),SIG(7),NA(7),NT(7),
     *NACC(7,50),NSTP(7,50),LOC(7),NN(7),NNS(7),SIGS(7)
    1 READ(5,100) ICASE,IBOUND,ITER,ALPHA,BETA,NSTART,NFIX,NTEST,SIG(1),
     *SIG(2),SIG(3),SIG(4),SIG(5),SIG(6),SIG(7)
      WRITE(6,101)
C
C                                 INITIALIZATION
C
      NRN=0
      T1=SIG(2)
      T2=SIG(6)
      T1=T1*T1
      T2=T2*T2
      T=ALOG(T1/T2)
      GAMMA=(T2-T1)/(2.*T1*T2)
      IF(IBOUND) 2,2,3
    2 ACC=ALOG(BETA/(1.-ALPHA))
      REJ=ALOG((1.-BETA)/ALPHA)
      GO TO 4
    3 ACC=ALPHA
      REJ=BETA
      EA=EXP(ACC)
      ER=EXP(REJ)
      ED=EA-ER
      ALPHA=(EA-1.)/ED
      BETA=EA*(1.-ER)/ED
    4 DO 6 I=1,7
      SIGS(I)=SIG(I)*SIG(I)
      LOC(I)=0
      NA(I)=0
      NT(I)=0
      NN(I)=0
      NNS(I)=0
      DO 5 J=1,NTEST
      NSTP(I,J)=0
    5 NACC(I,J)=0
    6 CONTINUE
      SAVRI=0.
      SAVRIS=0.
C
C                                 BEGIN OUTSIDE LOOP
C
      DO 21 L=1,ITER
      DO 8 I=1,7
      RIS(I)=0.
    8 KK(I)=-1
      KTHRU=0
      TERM=0.
C
C                                 BEGIN INSIDE LOOP
C
      DO 15 K=1,NTEST
      IF(KTHRU-7) 9,21,21
    9 CALL RAYLEE(RAY)
```

```
      RAYS=RAY*RAY
      NRN=NRN+1
      TERM=TERM+T
      DO 14 I=1,7
      IF(KK(I)) 10,14,14
   10 CON=RIS(I)+RAYS*SIGS(I)
      RIS(I)=CON
      IF(K.LT.NSTART) GO TO 14
      STAT=TERM+GAMMA*CON
      IF(ACC-STAT) 12,11,11
   11 NACC(I,K)=NACC(I,K)+1
      LOC(I)=LOC(I)+1
      GO TO 13
   12 IF(STAT-REJ) 14,13,13
   13 NSTP(I,K)=NSTP(I,K)+1
      KK(I)=1
      KTHRU=KTHRU+1
      NN(I)=NN(I)+K
      NNS(I)=NNS(I)+K*K
   14 CONTINUE
   15 CONTINUE
      IF(KTHRU-7) 16,21,21
   16 DO 20 I=1,7
      IF(KK(I)) 17,20,20
   17 STAT=TERM+GAMMA*RIS(I)
      IF(STAT) 18,18,19
   18 NACC(I,NTEST)=NACC(I,NTEST)+1
      LOC(I)=LOC(I)+1
      NA(I)=NA(I)+1
   19 NSTP(I,NTEST)=NSTP(I,NTEST)+1
      NN(I)=NN(I)+NTEST
      NNS(I)=NNS(I)+NTEST*NTEST
      NT(I)=NT(I)+1
   20 CONTINUE
   21 CONTINUE
      TRIALS=FLOAT(ITER)
      DO 22 I=1,7
      OCC=FLOAT(LOC(I))/TRIALS
      OC(I)=OCC
      SE(I)=SQRT((OCC*(1.-OCC))/TRIALS)
      ANN=FLOAT(NN(I))/TRIALS
      ANNS=FLOAT(NNS(I))/TRIALS
      AN(I)=ANN
   22 SD(I)=SQRT(ANNS-ANN*ANN)
      AHAT=1.-OC(2)
      BHAT=OC(6)
C
C                                  OUTPUT SECTION
C
      WRITE(6,116)
      WRITE(6,116)
      WRITE(6,103)
      WRITE(6,104)
      WRITE(6,113) ACC,REJ
```

```
      WRITE(6,106) SIG(2)
      WRITE(6,107) SIG(6)
      WRITE(6,108) ALPHA,AHAT
      WRITE(6,109) BETA,BHAT
      WRITE(6,110) NFIX
      WRITE(6,102) NTEST
      WRITE(6,123) NSTART
      WRITE(6,104)
      WRITE(6,111) ITER
      WRITE(6,112) NRN
      WRITE(6,104)
      WRITE(6,115)
      WRITE(6,116)
      WRITE(6,116)
      WRITE(6,116)
      WRITE(6,116)
      WRITE(6,117)
      WRITE(6,104)
      WRITE(6,118)
      WRITE(6,104)
      DO 23 I=1,7
   23 WRITE(6,119) SIG(I),OC(I),SE(I),AN(I),SD(I)
      WRITE(6,104)
      WRITE(6,115)
      WRITE(6,101)
      WRITE(6,120) SIG(1),SIG(2),SIG(3),SIG(4),SIG(5),SIG(6),SIG(7)
      WRITE(6,105)
      WRITE(6,121)
      WRITE(6,105)
      DO 24 I=1,NTEST
      WRITE(6,122) I,NSTP(1,I),NACC(1,I),NSTP(2,I),NACC(2,I),NSTP(3,I),
     *NACC(3,I),NSTP(4,I),NACC(4,I),NSTP(5,I),NACC(5,I),NSTP(6,I),
     *NACC(6,I),NSTP(7,I),NACC(7,I)
   24 CONTINUE
      WRITE(6,116)
      WRITE(6,114) NT(1),NA(1),NT(2),NA(2),NT(3),NA(3),NT(4),NA(4),
     *NT(5),NA(5),NT(6),NA(6),NT(7),NA(7)
C
C                                  END-OF-RUN TEST
C
      IF(ICASE.EQ.1) GO TO 1
      WRITE(6,101)
C
C                                  FORMAT STATEMENTS
C
  100 FORMAT(2I1,I5,2F5.4,3I2,7F7.4)
  101 FORMAT(1H1)
  102 FORMAT(' ',37X,'*    NUMBER OF TEST ITEMS AVAILABLE FOR TESTING:',
     *I3,3X,'*')
  103 FORMAT(' ',37X,'****************** CASE INFORMATION **************
     *****')
  104 FORMAT(' ',37X,'*
     *   *')
```

```
  105 FORMAT(1H )
  106 FORMAT(' ',37X,'*   NULL HYPOTHESIS: SIGMA = ',F6.2,18X,'*')
  107 FORMAT(' ',37X,'*   ALT. HYPOTHESIS: SIGMA = ',F6.2,18X,'*')
  108 FORMAT(' ',37X,'*   SPECIFIED ALPHA:',F7.4,7X,'OBSERVED:',F7.4,3X,
     *'*')
  109 FORMAT(' ',37X,'*   SPECIFIED BETA:'F8.4,7X,'OBSERVED:',F7.4,3X,
     *'*')
  110 FORMAT(' ',37X,'*   OBSNS. REQUIRED FOR FIXED SAMPLE SIZE TEST:',
     *I3,3X,'*')
  111 FORMAT(' ',37X,'*   NUMBER OF MONTE CARLO ITERATIONS:',I9,7X,'*')
  112 FORMAT(' ',37X,'*   NUMBER OF RANDOM NUMBERS GENERATED:',I7,7X,'*'
     *)
  113 FORMAT(' ',37X,'*   ACC. BOUNDARY=',F6.3,'  REJ. BOUNDARY=',F6.3,7
     *X,'*')
  114 FORMAT(' ',5X,14I8)
  115 FORMAT(' ',37X,'*************************************************
     *****')
  116 FORMAT(1H0)
  117 FORMAT(' ',37X,'******************** CASE SUMMARY ****************
     *****')
  118 FORMAT(' ',37X,'*    SIGMA   L(SIGMA)   STD ERR   A.S.N.   STD DEV
     *   *')
  119 FORMAT(' ',37X,'*',4X,F5.2,4X,F6.4,5X,F6.4,3X,F5.2,5X,F5.2,4X,'*')
  120 FORMAT(' ',8X,7('SIGMA =',F6.2,3X))
  121 FORMAT(' ',3X,'N',5X,7('STP     ACC',5X))
  122 FORMAT(' ',2X,I3,14I8)
  123 FORMAT(' ',37X,'*   NUMBER OF OBSERVATIONS PRIOR TO DECISIONS:',
     *I4,3X,'*')
      STOP
      END

      SUBROUTINE RAYLEE(R)
C
C        THIS SUBROUTINE CALLS FOR A UNIFORM (0,1) RANDOM NUMBER FROM
C        FUNCTION RANF.  IT RETURNS A STANDARD (SIGMA=1) RAYLEIGH
C        RANDOM NUMBER VIA THE PROBABILITY INTEGRAL TRANSFORMATION.
C
      Y=RANF(0)
      R=SQRT(-2.*ALOG(1.-Y))
      RETURN
      END

      FUNCTION RANF(NARG)
C
C  GENERATES PSEUDO-RANDOM NUMBERS, UNIFORMLY DISTRIBUTED ON (0,1).
C  THIS VERSION IS FOR THE IBM 360.
C
C  J. P. CHANDLER, COMPUTER SCIENCE DEPT., OKLA. STATE U.
C
C  METHOD...  COMPOSITE OF THREE MULTIPLICATIVE CONGRUENTIAL GENERATORS
C  G. MARSAGLIA AND T. A. BRAY, COMM. A.C.M. 11 (1968) 757.
C
C  IF RANF IS CALLED WITH NARG=0, THE NEXT RANDOM NUMBER IS RETURNED.
C  IF RANF IS CALLED WITH NARG.NE.0, THE GENERATOR IS RE- INITIALIZED
```

```
C  USING IABS(2*NARG+1), AND THE FIRST RANDOM NUMBER FROM THE NEW
C  SEQUENCE IS RETURNED.
      EQUIVALENCE (RAN,JRAN)
      DIMENSION N(128)
C
C     DATA NFIRST/7/,K/7654321/,L/7654321/,M/7654321/
      DATA NFIRST/7/,K/7654321/,L/3141593/,M/271828183/
C
C  MULTIPLIERS USED BY VAN GELDER....
C
C     DATA MK/105005B/,ML/10405B/,MM/20005B/
C     DATA MK/282629/,ML/34821/,MM/65541/
C
C                                   CHANDLER-S MULTIPLIERS....
      DATA MK/231525/,ML/282629/,MM/253125/
C
      IF(NARG) 20,10,20
   10 IF(NFIRST) 30,60,30
C                                   RE-INITIALIZE USING NARG.
   20 KLM=IABS(2*NARG+1)
      K=KLM
      L=KLM
      M=KLM
C                                   INITIALIZE.
   30 NFIRST=0
C                                   2**24 ....
      NDIV=16777216
C                                   EXACT REAL REPRESENTATION OF 2**31 ....
      RDIV=32768.*65536.
C                                   FILL THE TABLE.
      DO 50 J=1,128
      K=K*MK
   50 N(J)=K
C                                   COMPUTE THE NEXT RANDOM NUMBER.
   60 L=L*ML
      J=1+IABS(L)/NDIV
      M=M*MM
      NR=IABS(N(J)+L+M)
      RAN=FLOAT(NR)/RDIV
C                                   FIX UP THE LEAST SIGNIFICANT BIT.
      IF(J.GT.64  .AND.  RAN.LT.1.)  JRAN=JRAN+1
      RANF=RAN
C                                   REFILL THE J-TH PLACE IN THE TABLE.
      K=K*MK
      N(J)=K
      RETURN
      END
```

VITA

James Edward Gibson

Candidate for the Degree of

Doctor of Philosophy

Thesis: CLOSED SEQUENTIAL PROCEDURES FOR TESTING HYPOTHESES ABOUT THE PARAMETER OF A RAYLEIGH DISTRIBUTION

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