Oklahoma State University

Honors Thesis

Graphs and Rhythmic Canons in Musical Composition

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Contents

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1 Idea

The goal of this paper is to describe the different methods of producing rhythmic canons. A **rhythmic canon** is a music compositional technique in which a specific melody or string of notes is repeated throughout the piece of music. The melody, also called the motif, can either be repeated as an exact replica of the original duration, or it can be repeated after being submitted to some form of transformation.

In this paper, rhythmic canons will be broken down into two different sets: $M \subseteq \mathbb{Z}$ will be the motif, or the inner voice, so that each $m \in M$ corresponds to a specific note. Then $T \subseteq \mathbb{Z}$ will be the set of translations, or the outer voice, such that each $t \in T$ will translate each m by t beats. Together M and T will create the map:

$$M \times T \to \mathbb{Z}_n$$

$$(m, t) \mapsto m + t$$

For all $m \in M$ and for all $t \in T$.
There are two types of canons that we will be working with:

A **packing** is a canon that which has at most one note on every beat, thus some beats may have nothing. Therefore, the map of $M \times T \hookrightarrow \mathbb{Z}_n$ for some $n \in \mathbb{N}$ is injective.

A **tiling** is a form of packing that has exactly one note on every beat. Then the map for a tiling will be $M \times T \leftrightarrow \mathbb{Z}_n$ for some $n \in \mathbb{N}$ is bijective

This paper will analyze different techniques of creating packings, and then it will explore ways to expand packings into tilings with the analysis of undirected graphs and their cliques.

2 Construction of Packings

2.0.1 Basic Translations

Let $M = \{0, 2\}$ and $T = \{0, 3, 6\}$. Now assign each element of M to a note. Say the element 0 corresponds to A and say 2 corresponds to C. Then if we apply the translations given in T to the elements in M, we will produce a new set $Z = M \oplus T$ where $\oplus$ is defined as: $x \oplus y = \{x + y : x \in X, y \in Y\}$. Below is what the canon will look like: (The motif is the first measure and Z is the following series of notes)

Note that the pitch of each beat of the composition is determined by the pitch corresponding to the element of the motif that is added to the translation that outputs the specific beat. Furthermore, if there is any beat that does not have a corresponding element in Z, then the pitch of that beat is carried over from the most previous pitch before it.

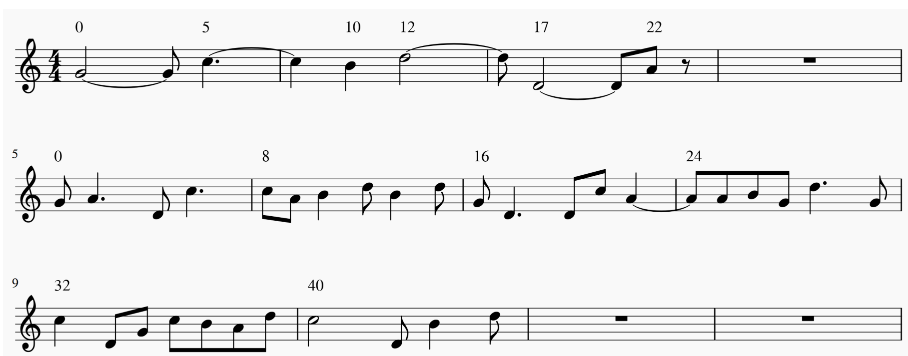
2.0.2 Wrapping Around

Sometimes when choosing a specific modulus, say $\mathbb{Z}_n$, we may result in a case where the sum of one element of M and one element of T is greater than n. Then, this sum would take on its congruent value within the specified modulus. As a result, a motif may **wrap around** the canon as it carries out its translations.

Example 1. Let $M \oplus T \subseteq \mathbb{Z}_{48}$ where

$$M = \{0, 5, 10, 12, 17, 22\} \text{ and } T = \{0, 3, 16, 27, 35\}$$

Then we will have that $22 + 35 = 57 > 48$, but $57 \equiv 9 \pmod{48}$, so the note corresponding with the element $\{22\}$ from M will be translated to the 9th spot, instead of the 57th. Thus, $M \oplus T \subseteq \mathbb{Z}_{48}$ will look like this:



Example 2.

$$M = \{0, 2, 5, 11, 13, 34, 43, 54\}, T = \{6, 28, 28, 45, 64\}, \mathbb{Z}_{72}$$

2.1 Translation Techniques

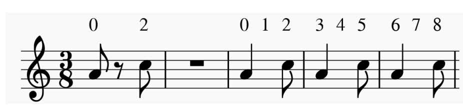
2.1.1 Stuttering

We can carry out the act of **stuttering** by replacing each element in the motif with k repetitions of itself.
Given the inner voice M and the outer voice T, this idea can be algebraically written as:

$$Stut(M, k) = kM \oplus \{0, 1, ..., k-1\}$$

where $kM = \{km : m \in M\}$. T is also augmented to kT and $\mathbb{Z}_n$ is augmented to $\mathbb{Z}_{kn}$.

Example 3. Let $M \oplus T \subseteq \mathbb{Z}_9$, where $M = \{0, 2\}$, and $T = \{0, 3, 6\}$, with the $\{1\}$ corresponding with an A and $\{2\}$ corresponding with C. Then we will have the same canon as previously:



Now let $k = 2$, then we will have:

$$Stut(M, 2) = 2M \oplus \{0, 1\} = \{0, 4\} \oplus \{0, 1\} = \{0, 1, 4, 5\}$$

$$2T = \{0, 6, 12\} \text{ and } \mathbb{Z}_9 \text{ becomes } \mathbb{Z}_{18}$$

Now the new canon is: $\{0, 1, 4, 5\} \oplus \{0, 6, 12\} \subseteq \mathbb{Z}_{18}$. Then, if we assign a note to each element in M, say $\{0\}$ corresponds to G, $\{1\}$ corresponds to A, $\{4\}$ corresponds to B, and $\{5\}$ corresponds to C. Then by carrying out the stutter on M, our packing changes from one in Z_9 to one in Z_{18}:

2.1.2 Even-Odd Overlapping

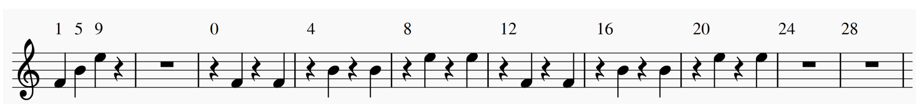
Let M_1 contain only odd integers, let M_2 contain only even integers, and let T_1 and T_2 contain only even integers so that $M_1 \oplus T_1 = Z_1$ and $M_2 \oplus T_2 = Z_2$ will be disjoint subsets, i.e. $Z_1 \cap Z_2 = \{\emptyset\}$. Then if we assign a motif to M_1 and a motif to M_2, we can overlap their transpositions to have a more densely tiled canon.

Example 4. Consider:

$$M_1 = \{1, 5, 9\} \quad \textit{and} \quad T_1 = \{0, 2, 12, 14\}$$

$$M_2 = \{0, 2, 8, 10\} \quad \textit{and} \quad T_2 = \{0, 4, 16, 20\}$$

Then Z_1 would look like this:



Z_2 would look like this:

But together, they look like this:

<u>Note:</u> Even though $Z_1 \cup Z_2$ creates a packing, $(M_1 \cup M_2) \oplus (T_1 \cup T_2)$ does not necessarily create a packing.

2.1.3 Switching

The act of **switching** involves switching the elements of M and T. So in other words, the motif becomes the translations and the translations become the motif.

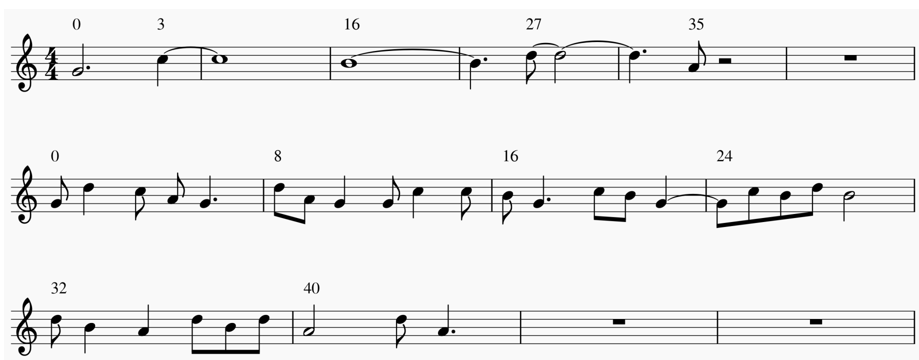
Example 5. Consider the sets defined above from Example 1. Originally we have in $\mathbb{Z}_{48}$:

$$M = \{0, 5, 10, 12, 17, 22\} \quad and \quad T = \{0, 3, 16, 27, 35\}$$

But let's switch them to obtain:

$$M' = \{0, 3, 16, 27, 35\} \quad and \quad T' = \{0, 5, 10, 12, 17, 22\}$$

Then we can carry out the normal process of translating the motif to create a new canon:



Notice that there are the same number of notes in the composition, but since there are less elements in the motif and more elements in the translation set after switching, each single element is repeated one extra time.

2.1.4 Packing of a Set of Motifs

Instead of assigning each element of M to a specific note, we can assign each element of M to a different motif.

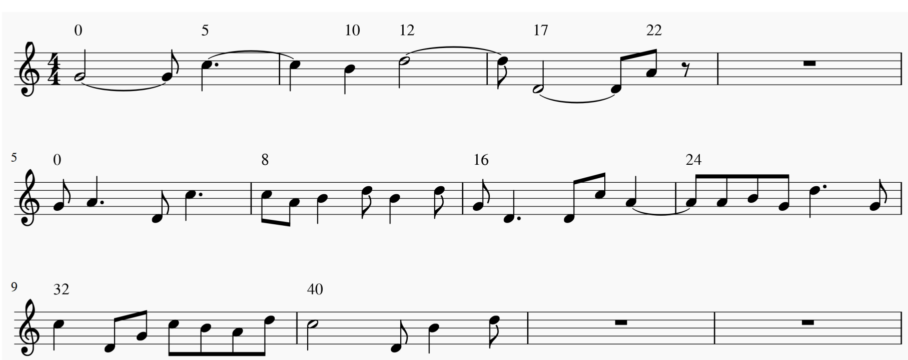
Example 6. Let $M = \{0, 3, 6, 12, 17\}$ and $T = \{0, 8, 16, 24\}$ in $\mathbb{Z}_{32}$

Then we can assign each element of M to a unique series of notes, but each of the same length. Say each element corresponds with four eighth notes. Then a resulting canon may look something like this:

2.1.5 Changing the Elements of a Translation Set

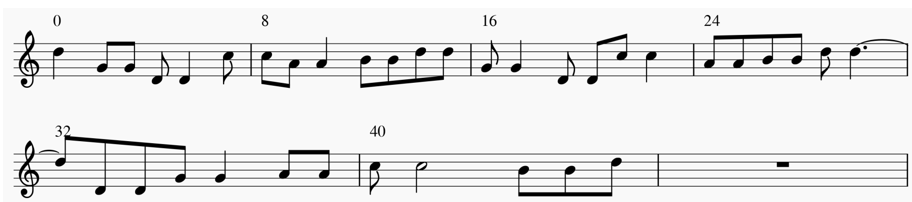
After choosing a motif and translation set, if one were to wish to make the packing "more dense", then it may be possible to remove some elements of the translation set and replace them with new elements.

Example 7. Consider $M = \{0, 5, 10, 12, 17, 22\}$ and $T = \{0, 3, 16, 27, 35\}$ in $\mathbb{Z}_{48}$. Let the motif and translation look like:



Note that we cannot add any more elements to the translation set because 10 and 12 only have a difference of 2, and the only empty beats with a distance of 2 apart have a corresponding translation that overlaps with previously made elements of $\mathbb{Z}_{48}$. So, what if instead we removed $\{0\}$ from T and added the elements $\{2, 36\}$ to get $T_1 = (T \setminus \{0\}) \cup \{2, 36\} = \{2, 3, 16, 27, 35, 36\}$. This will create the new composition:

Then the process can be repeated to get some $T_2 = (T_1 \setminus \{27\}) \cup \{17\} = \{2, 3, 16, 17, 35, 36\}$ which will look like:



Of course, these were not the only possible paths to change the original translation of M. However, note that as a "denser" packing of M was created, the elements of T grew to be more adjacent to one another.

2.2 Beyond Packings

So far all of the musical compositions have had at most one note per beat, which satisfies the definiton of a packing. Now we will look into some translation techniques that result with more than one note per beat.

2.2.1 One Motif, Two Translations

An interesting technique to transform a canon is to take some motif set M and give it two translation sets, say T_1 and T_2. This will result in three potential compositions: $M \oplus T_1$, $M \oplus T_2$ and $M \oplus (T_1 \cup T_2)$.

Example 8. Let $M = \{0, 3, 8, 11\}$, $T_1 = \{1, 5, 7, 14\}$, and $T_2 = \{2, 6, 16, 20\}$, all in $\mathbb{Z}_{26}$.
Let the motif look like:

Then this creates the following compositions:

$$M \oplus T_1 :$$

$$M \oplus T_2 :$$

$$M \oplus (T_1 \cup T_2) :$$

Note that both $M \oplus T_1$ and $M \oplus T_2$ creates a packing, while $M \oplus (T_1 \cup T_2)$ does not since there are at most two notes per beat. Lets call this type of composition a **2-packing**, a canon with at most two notes per beat. Then furthermore, a **k-packing** is a canon with at most k notes per beat.

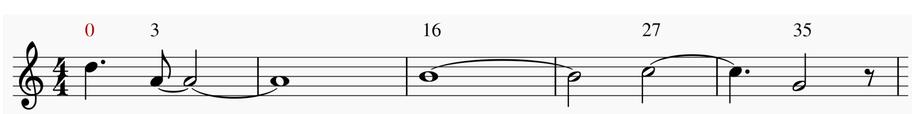
This leads to the conclusion that the union of k unique packing-producing translation sets for a motif will result in at most a k-packing. Even if there exists one element per translation set that overlaps with at least one element in every other translation set, there can still only be k notes per beat at the maximum.

On the other hand, an interesting question to ask is whether or not the translation set for every packing with at most k notes per beat be split into k disjoint translation sets.

2.2.2 Doubling

A spin-off of giving one motif two translations is to take some motif set M and its respective translation set T, then double each element in T to create T_2. This will still result in three potential compositions: $M \oplus T$, $M \oplus T_2$ and $M \oplus (T \cup T_2)$.

Example 9. Consider $M = \{0, 3, 16, 27, 35\}$, and $T = \{5, 10, 12, 17, 22\}$ in $\mathbb{Z}_{48}$. This results with $T_2 = \{10, 20, 24, 34, 44\}$. Let the motif look like:

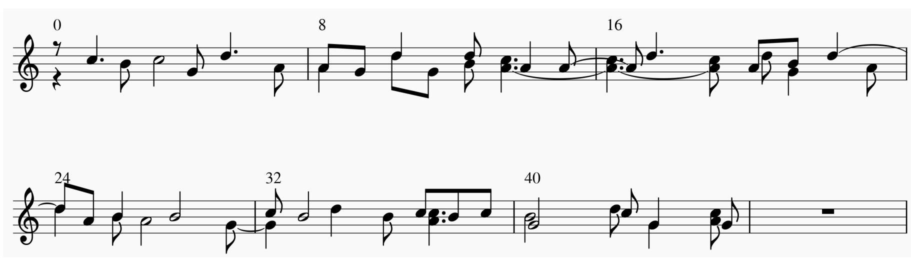


Then the translations will look like:

$$M \oplus T:$$

$$M \oplus T_2:$$

$$M \oplus (T \cup T_2):$$



Notice that $M \oplus T$ creates a packing while $M \oplus T_2$ creates a 2-packing. If we were to remove the last two elements from T, to get $T' = T \setminus \{17, 22\} = \{5, 10, 12\}$, then $T_2' = \{10, 20, 24\}$, which will then become a packing. However, this will result in $M \oplus T'$ and $M \oplus T_2'$ to be less musically dense and therefore subjectively "less interesting".

2.2.3 Splitting Up the Translation Set

Idea: Each translation set can be broken apart into disjoint subsets whose union make up the whole translation set.

Example 10. Consider the packing that has at most two notes per beat with $M = \{0, 5, 7, 13, 18\}$ and $T = \{0, 3, 7, 11, 12, 15, 23, 24, 27, 32, 36, 38, 39, 44\}$ in $\mathbb{Z}_{48}$. Then we can say that actually $T = T_1 \cup T_2 \cup T_3$ where $T_1 = \{0, 12, 15, 24, 36\}$, $T_2 = \{3, 7, 23, 27, 39\}$, and $T_3 = \{11, 32, 44\}$. Let the motif look like:

Then, we can have two different compositions:

$$M \oplus T:$$

$$M \oplus (T_1 \cup T_2 \cup T_3):$$

Note that these are technically the same compositions. The only difference is how long each pitch is held. Furthermore, T could also be split up into $T = T_1' \cup T_2'$ where $T_1' = \{0, 3, 12, 15, 24, 27, 36\}$ and $T_2' = \{7, 11, 23, 32, 39, 44\}$. This new union will create yet another variation of the original $M \oplus T$ piece.

2.3 Summary

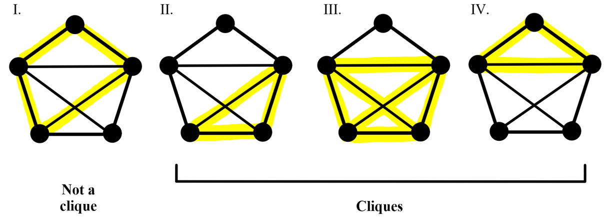
As different variations of these musical compositions are created using the various techniques, many questions arise:

- As more elements are added to the motif or translation set, more notes appear in the music. How can we pick elements for M and T such that we turn a packing into a tiling?
- Is there a pattern with how we choose the elements of M and T that would allow us to predict how the composition will look or sound?
- What are some other ways to mathematically notate what is happening to create these compositions?

These questions transition us into the next section, which will be the more mathematical part of these musical compositions.

3 Graphs and Cliques

An **undirected graph** is a set of vertices that are connected with bidirectional edges. Then a **clique** is a subset of vertices of an undirected graph with every two distinct pairs of vertices being adjacent. Furthermore, a **maximal clique** is a clique that is not a proper subset of another clique.

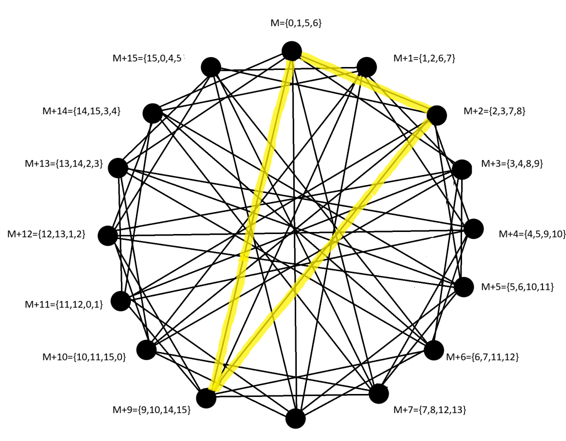


From Figure 1. above, we can see that (III) and (IV) are maximal cliques of G. Also note that (III) is a maximal clique of size 4 while (IV) is a maximal clique of size 3. This shows that not all max cliques for a single graph have the same size.

3.1 Construction of Graphs and Cliques

Consider some $M \subseteq \mathbb{Z}_n$. Then for all $0 \leq i \leq n-1$, let each set $M+i$ be a vertex of the graph. Then connect each pair of vertices $M+i$ and $M+j$ that do not share any common elements. This graph will be notated as $\Gamma(M,n)$. Any subgraph of $\Gamma(M,n)$ that has each pair of vertices connected by an edge is a clique of $\Gamma(M,n)$.

Example 11. Consider $M = \{0,1,5,6\}$ in $\mathbb{Z}_{16}$. Then the graph $\Gamma(M,16)$ looks like:



By observing $\Gamma(M, 16)$, it can be seen that both M, $M+8$ form a clique and M, $M+2$, and $M+9$ form a clique. Note that these are also both maximal cliques.

3.2 The Interval Set

Consider $M \in \mathbb{Z}_n$. Let $I(M) \subseteq \mathbb{Z}$ where:

$$I(M) = \{a - b \in \mathbb{Z}_n : a, b \in M\}$$

We will call this the interval set. Now take the complement of $I(M)$ to get:

$$[I(M)]^c = \mathbb{Z}_n \setminus I(M)$$

It is from this complement set where we will find the potential vertices for cliques and maximal cliques.

Idea: If $C \subseteq [I(M)]^c$ is a clique of $\Gamma(M, n)$, then each element of C is contained in $[I(M)]^c$, and each interval between every two elements of C is contained in $[I(M)]^c$.

Proof. Let $\Gamma(M, n)$ be the graph for some motif $M \subseteq \mathbb{Z}_n$. Recall that $\Gamma(M, n)$ is created by connecting all $M + i_m$ that do not share a common element. Therefore, for all $a \in M$ and for all $b + i \in M + i$, it must be that $a \neq b + i$ in order for M and $M + i$ to be connected. This implies that $a - b \neq i$. Then since $a - b \in I(M)$, $i \notin I(M)$, so $i \in [I(M)]^c$. Thus, the elements of $[I(M)]^c$ compose $\Gamma(M, n)$. Furthermore, since a clique is a subset of $\Gamma(M, n)$, the vertices of the clique must be found in $[I(M)]^c$ as well.

Furthermore, cliques also have the property that each pair of their elements are connected. This means that the difference between each i and j in C must also be in $[I(M)]^c$. So for all $a + j \in M + j$ and $b + i \in M + i$, we must have that $a + j \neq b + i$ in order for the vertices to be connected. Thus, $a - b \neq j - i$. Then since $a - b \in I(M)$, $j - i \in [I(M)]^c$. Therefore the intervals between each vertex of a clique are contained in the interval set as well. □

From here we can define a clique to be:

$$C = \{i, i - j : i \in [I(M)]^c, i - j \in [I(M)]^c\}$$

Example 12. Let $M = \{0, 5, 10, 20\} \subseteq \mathbb{Z}_{40}$. This motif produces:

$$I(M) = \{5, 10, 15, 20, 25, 30, 35\}$$

$$[I(M)]^c = \{0 \ldots 4, 6 \ldots 9, 11 \ldots 14, 16 \ldots 19, 21 \ldots 24, 26 \ldots 29, 31 \ldots 34\}$$

Now consider $C = \{0, 9, 18, 27\}$. Note that each of these elements are contained in $[I(M)]^c$, along with their differences. Thus C is a clique. Since 10 is not contained in $[I(M)]^c$, we can already conclude that any element with a 0, 9, 8, or 7 in the one's place cannot be added to the clique. And also since 5 is not contained in $[I(M)]^c$, any element with a 4, 3, 2, 1 in the one's place cannot be added to the clique. Thus no more elements can be added to C, so C is a maximal clique.

3.3 Relationships Between Graphs, Cliques, and Sets

3.3.1 Cliques and Translation Sets

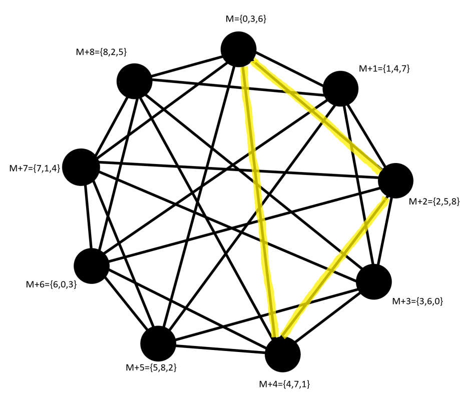
In order for $M \oplus T$ to produce a packing, we must have that no $M + i_m$ share any common elements. This means that each i_m is contained in $[I(m)]^c$ and each $i_{m_1} - i_{m_2}$ is contained in $I[(M)]^c$ as well (for similar reasons as stated previously). Then if we take each i_m that satisfies this condition to make the translation set $T = \{i_1, i_2, \ldots, i_m\}$, T is now equivalent to a clique.

Idea: $M \oplus T$ is a packing if and oly if T is a clique of $\Gamma(M, n)$

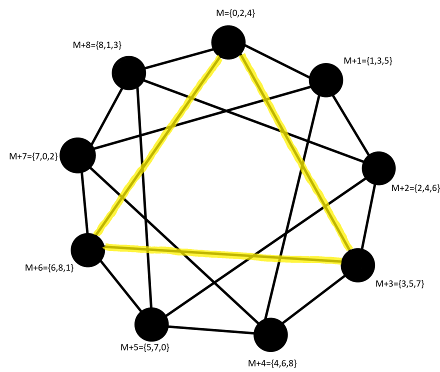
3.3.2 Motif and Translation Sets

Recall that the act of switching involves swapping the elements of M and T. Now consider the graph $\Gamma(M, n)$ with a clique whose elements compose the set T. If the elements of M and T were swapped, say to get M' and T', then the graph $\Gamma(M', n)$ will contain a clique whose elements compose T'.

Example 13. Let $M = \{0, 3, 6\}$ where $M \subseteq \mathbb{Z}_9$. Then the graph $\Gamma(M, 9)$ will look like:



Notice that the vertices $0, 2$, and 4 create a complete subgraph, thus they make a clique. Let $T = \{0, 2, 4\}$ be this clique and therefore a translation set. Now if we switch $M = \{0, 3, 6\}$ and $T = \{0, 2, 4\}$ to get $M' = \{0, 2, 4\}$ and $T' = \{0, 3, 6\}$, the graph $\Gamma(M', 9)$ will look like:



Note that the vertices $0, 3$, and 6 create a clique in this new graph.

So, in conclusion T is a clique of $\Gamma(M, n)$ and M is a clique of $\Gamma(T, n)$, i.e. M and T are each cliques of the other's graph. This is particularly useful when trying to add an element to either the motif or translation set. After graphing Γ, we can visually spot the vertices that create a larger clique. These vertices can be added to either M or T, depending on which graph Γ we are looking at.

3.3.3 The Motif Set and $\Gamma(M, n)$

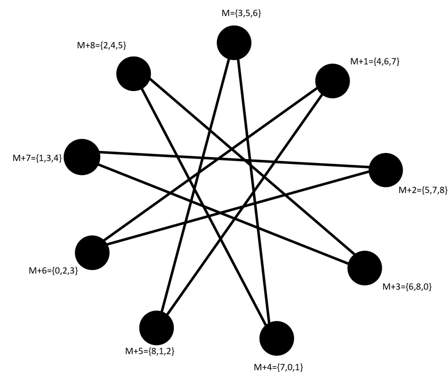
Idea: If $I(M_1) = I(M_2)$ for $M_1, M_2 \subseteq \mathbb{Z}_n$, then $\Gamma(M_1, n) = \Gamma(M_2, n)$. In other words, different motif sets can create the same graph.

When we add an element to M, we add a pair of elements to $I(M)$. There are n different elements that can be added to M, but there are only $\frac{n}{2}$ or $\frac{n+1}{2}$ element pairs that can be added to $I(M)$, depending on if n is even or odd. Therefore, there are more combinations of elements that can be added to M than the combinations of elements that can be added to $I(M)$. This gives that there will be more than one motif set that will result in the same $I(M)$. Then furthermore, the complement to the created interval set, $[I(M)]^c$ will be the same for different motif sets. Since it is from $[I(M)]^c$ that the $\Gamma(M, n)$'s are made, it must be that multiple motif sets will create the same graph.

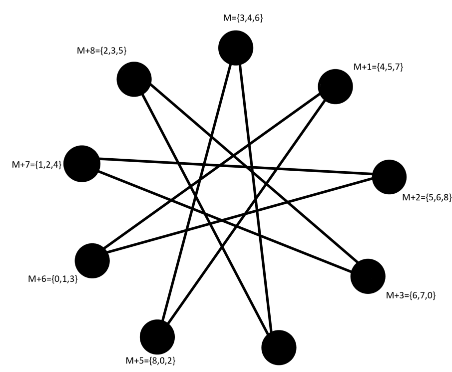
This leads to the creation of **classes** for the motif set. The class of some set $M \subseteq \mathbb{Z}_n$ can be defined as:

$$Cl(M) = \{M_1 \subseteq \mathbb{Z}_n : I(M) = I(M_1)\}$$

Example 14. Consider $M_1 = \{3, 5, 6\} \subseteq \mathbb{Z}_9$. This makes $\Gamma(M_1, 9)$:



Note that $I(M_1) = \{1, 2, 3, 6, 7, 8\}$. Now consider $M_2 = \{3, 4, 6\} \subseteq \mathbb{Z}_9$. This makes $\Gamma(M_2, 9)$:



Note that $I(M_2) = \{1, 2, 3, 6, 7, 8\}$ as well. So, M_1 and M_2 are two different motifs, but they both produce the same graph. Thus, they are in the same class.

3.4 Clique Sizes

3.4.1 The Maximum Size of a Clique

Let M be a motif for $\mathbb{Z}_n$, giving $\Gamma(M, n)$. Then T is a clique for $\Gamma(M, n)$ iff $M \oplus T$ does not have any repeating elements,which implies $M \oplus T$ is a packing. Assume that $M \oplus T$ has one note on every beat in $\mathbb{Z}_n$. Then $M \oplus T$ has n elements. Since each of these elements are created by adding each $m \in M$ with each $t \in T$, $|M \oplus T| = |M||T| < n$. So $|T| < \frac{n}{|M|}$. Note that not all motifs and translation sets will create a packing with exactly one note on every beat, and the cardinality of $M \oplus T$ must be represented by an integer. Thus, the largest size of the maximal clique of $\Gamma(M, n)$ is less than or equal to $\lfloor \frac{n}{|M|} \rfloor$.

3.4.2 There exist arbitrarily large cliques

Let M and T be motif and translation sets that together create a packing in $\mathbb{Z}_n$. Note that T is then a clique of $\Gamma(M, n)$. Then if the sum of the largest element of M and the largest element of T is less than n (i.e. no "wrapping around" occurs) then $T^a = (T+(a-1)n) \cup (T+(a-2)n) \cup ... \cup T$ is a clique for $\Gamma(M, an)$.

<u>Note:</u> It follows that $|T^a| = a|T|$

Example 15. Let $M = \{0, 1, 5, 6\}$ and $T = \{0, 2, 9\}$ in $\mathbb{Z}_{16}$, giving $\Gamma(M, 16)$. Note that $6+9 < 16$, so no "wrapping around" occurs. This creates the packing:

0	0	**5**	5	**10**	1
1	1	**6**	6	**11**	
2	0	**7**	5	**12**	
3	1	**8**	6	**13**	
4		**9**	0	**14**	5
				15	6

Now let $a = 3$. This gives $\Gamma(M, a(16)) = \Gamma(M, 48)$ so

$$T^3 = (T + (3-1)n) \cup (T + (3-2)n) \cup (T + (3-3)n)$$

$$= (T + 2n) \cup (T + n) \cup T$$

$$= \{32, 34, 41\} \cup \{16, 18, 25\} \cup \{0, 2, 9\}$$

Then by the translation of M by T^3 creates the packing:

0	0	**10**	1	**20**		**30**	5	**40**	6
1	1	**11**		**21**	5	**31**	6	**41**	0
2	0	**12**		**22**	6	**32**	0	**42**	1
3	1	**13**		**23**	5	**33**	1	**43**	
4		**14**	5	**24**	6	**34**	0	**44**	
5	5	**15**	6	**25**	0	**35**	1	**45**	
6	6	**16**	0	**26**	1	**36**		**46**	5
7	5	**17**	1	**27**		**37**	5	**47**	6
8	6	**18**	0	**28**		**38**	6		
9	0	**19**	1	**29**		**39**	5		

3.4.3 Graphs $\Gamma(M, n)$ can have arbitrarily far apart cliques

Note that even though T^a can be a clique for $\Gamma(M, an)$, the original T is still a clique for $\Gamma(M, an)$ (though not a maximal clique). Thus since $|T^a| = a|T|$, $|T^a|$ is arbitrarily larger than $|T|$ as a becomes arbitrarily large.

3.4.4 Graphs $\Gamma(M, n)$ can have arbitrarily far apart maximal cliques.

Recall that $\Gamma(M, n)$ can have more than one maximal clique. So let T and S both be two different max cliques for $\Gamma(M, n)$. Then $|T^a| = a|T|$ and $|S^a| = a|S|$. Let $|T| = t$ and $|S| = s$ with $s > t$. Then

$$\lim_{a\to\infty}(|T^a| - |S^a|) = \lim_{a\to\infty}(a|T| - a|S|) = \lim_{a\to\infty} a(t - s)$$

Since this goes to infinity, $|T^a|$ and $|S^a|$ can be arbitrarily far apart. Thus maximal cliques can be arbitrarily far apart.

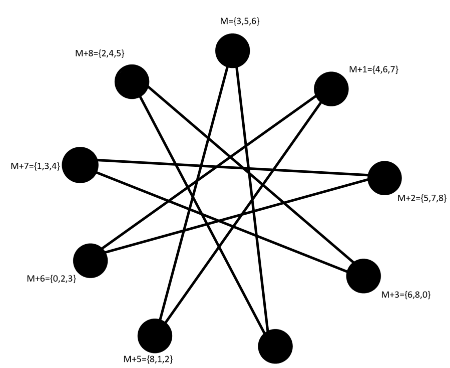
Example 16. Recall the set $M = \{0, 1, 5, 6\} \subseteq \mathbb{Z}_{16}$ from Example 11. Both the sets $T_1 = \{0, 8\}$ and $T_2 = \{0, 2, 9\}$ form maximal cliques in $\Gamma(M, 16)$. Thus, as $a \to \infty$, T_1^a and T_2^a will become arbitraily large in $\Gamma(M, an)$, but they will be arbitrairly far apart, as shown above.

4 Questions and Future Ideas

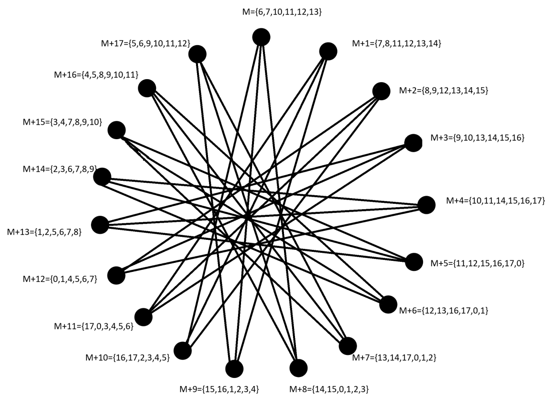
4.1 Stuttering Cyclic Graphs

When graphing many different motifs, it was often found that if $\Gamma(M, n)$ is cyclic, then $\Gamma(Stut(M, k), kn)$ is also cyclic. (**Cyclic:** $\Gamma(M, n)$ contains at least one path between vertices where the first vertex corresponds with the last). Though, some stuttered graphs created a single cycle between all the vertices while other ones created two disjoint cycles through the vertices whose union makes up the whole graph. A counter example has not yet been found to disprove this idea, so the question arises as to whether or not this is a property of stuttering the graphs.

Example 17. Let $M_1 = \{3, 5, 6\} \subseteq \mathbb{Z}_9$ to create $\Gamma(M_1, 9)$:

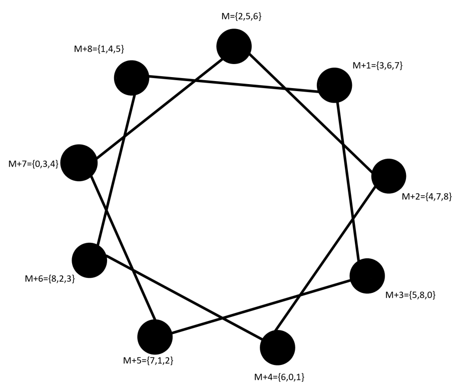


Notice that $\Gamma(M_1, 9)$ is composed of just a single cycle. Then if we stutter $\Gamma(M_1, 9)$ by $k = 2$, we will obtain $2M_1 = \{6, 7, 10, 11, 12, 13\} \subseteq \mathbb{Z}_{18}$. This creates $\Gamma(2M_1, 18)$ which looks like this:



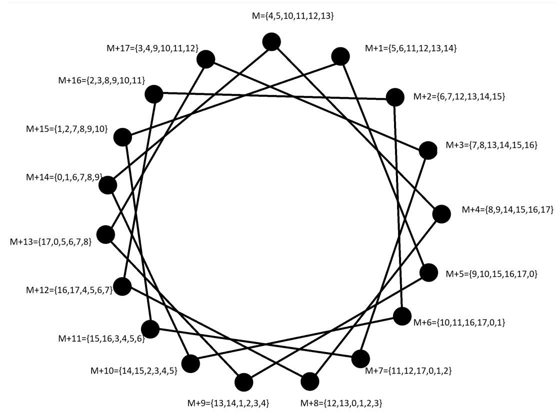
Notice that $\Gamma(2M_1, 18)$ is also composed of one single cycle.

Now let $M_2 = \{2, 5, 6\} \subseteq \mathbb{Z}_9$ to create $\Gamma(M, 9)$:



Then we will also stutter this graph by $k = 2$ to get $2M_2 = \{4, 5, 10, 11, 12, 13\} \subseteq$

$\mathbb{Z}_{18}$. This creates $\Gamma(2M_2, 18)$ which looks like this:



$\Gamma(2M_2, 18)$, however is composed of two disjoint cycles that together make up the entirety of $\Gamma(2M_2, 18)$. How do we know when a single cycle or two disjoint cycles will appear after stuttering some $\Gamma(M, n)$?

Some other questions that were asked about stuttered graphs were:

- Is there a way to predict that if $\Gamma(M, n)$ is composed of a single cycle, then $\Gamma(Stut(M, k), n)$ has either one cycle or two disjoint cycles?
- Would looking at $I[M]$ give us any information?
- Is there a relationship between the sizes of cliques in $\Gamma(M, n)$ and $\Gamma(Stut(M, k), n)$?
- What about the relationship between the degrees of the vertices?

4.2 Affine Maps

Originally, we used the map $(m, t) \mapsto m + t$ to translate the motif and make a musical composition. Instead, if we chose some $k \in \mathbb{Z}$ such that $gcd(k, n) = 1$, we could change the map to:

$$m \mapsto km + t$$

It is important to make k and n relatively prime since as you multiply all $m \in M$ by k, you do not want the product to be an equivalent value of m in the modulus. This new map would result in a different $\Gamma(M, n)$ for each different k. We could then find a clique among the vertices of the different $\Gamma(M, n)$'s, rather than just vertices within the same $\Gamma(M, n)$, to create a translation set and furthermore a musical composition.

Example 18. Consider the motif $M = \{3, 5, 7\} \subseteq \mathbb{Z}_{11}$. Then as

$$k = 1, M = \{3, 5, 7\}$$

$$k = 2, 2M = \{6, 10, 3\}$$

$$k = 6, 6M = \{7, 8, 9\}$$

More specifically,

$$2M + 9 = \{4, 8, 1\}$$

and

$$6M + 2 = \{9, 10, 0\}$$

Since M, $2M + 9$, and $6M + 2$ do not share any common elements, we can translate each element of the motif by the functions m, $2m+9$, and $6m+2$. Let 3 correspond with the note C, let 5 correspond with a B, and let 7 correspond with an E. Then our composition composed with this new map will look like:

4.3 Two Dimensional Compositions

The musical compositions produced by $M \oplus T$ have so far been one-dimensional, meaning that the motif is only translated across a number line of beats. We could add a second dimension to this by using the map:

$$(m, t, s) \mapsto (m + t, m + s)$$

For all $m \in M$, $t \in T$, and $s \in S$. In this case, we have two translation sets S and T. T would move the motif along the x-axis while S would move the motif along the y-axis. So, for example, S could produce the pitch, depending on how far up or down the motif is moved, and T will produce the placement of the pitch on a specific beat.

References

[1] Emmanuel Amiot,

Structures, Algorithms and Algebraic Tools for Rhythmic Canons, Perspectives of New Music, R. Graham, St. Nazaire, France, 2011.