OKLAHOMA STATE UNIVERSITY

HONORS THESIS

Color Symmetry in the Platonic Solids

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Abstract

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Rotationally-invariant colorings of the Platonic solids are considered. Permutation representations of the symmetry groups of the Platonic solids are constructed using group actions on rotationally-invariant colorings of faces, edges, or vertices.

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Chapter 1

Introduction

1.1 The Problem

Is there a way to define color symmetry and can we use this definition to create color symmetric Platonic solids?

1.2 Introduction

One of the most important and beautiful themes unifying many areas of modern mathematics is the study of symmetry. The history of the mathematical theory of color symmetry began in the late 1920s when the concept of two-color symmetry was introduced to describe the symmetries of repeating patterns in a two-sided plane in three-dimensional space [9]. However, this new concept didn't attract much attention at this time. The true roots of color symmetry can be traced back to the 1950's under the work of Soviet crystallographer A. V. Shubnikov [9]. Shubnikov investigated polyhedra and other figures whose faces could be colored black and white by appending, when possible, an "anti-symmetry operation" to their symmetries. Because this use of color opened the door refinements of the usual classification of spatial patterns by their symmetries, and thus to the solution of certain problems concerning atomic patterns in crystals, the idea was quickly extended by Shubnikov's colleagues to other groups of crystallographic interest, and to patterns with more than two colors.

In the discussion that follows in this thesis, we review the concept of symmetry, and then discuss its generalization to color symmetry and show how the Platonic solids can be colored symmetrically.

Chapter 2

Groups and Symmetry

2.1 Symmetry

Let's begin by providing some background information that will provide context to many terms used throughout this paper when defining "color symmetry."

Definition 2.1 The function $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is called an **isometry** in Euclidean Space if it preserves distances. That is,

$$||\vec{x} - \vec{y}|| = ||T(\vec{x}) - T(\vec{y})|| \tag{2.1}$$

for all $\vec{x}$, $\vec{y} \in \mathbb{R}^n$. Isometries include rotations, reflections, translations, glide reflections, and the identity map.

A **symmetry**, in Euclidean Space, of a figure F is an isometry of the space that preserves F. That is, if $F \subset \mathbb{R}^n$ and T is an isometry of $\mathbb{R}^n$, then T is a symmetry of F if $T(F) = F$. Thus, in this sense, after applying a symmetry, the figure will look the same as the original and remains in the same location in space, so a symmetry is an "undetectable motion."

All bounded figures in any dimension may possess rotational symmetries, and the collection of all rotational symmetries is a group since the composition of two rotations in intersecting lines is another rotation. This is the group of rotational symmetries of the figure. All figures may also possess symmetries which reverse orientation such as reflections or rotary reflections. The full symmetry group of any figure may also be considered. For figures in space, reflections are abstract symmetry operations that are often not able to be implemented in 3-space since they would require turning a figure inside-out. So, it is common to restrict to discussing rotational symmetries, due to the difficulty in realizing orientation-reversing symmetries physically. In addition, the phrase "finite figure" often refers to a bounded figure determined by finitely many of a certain type of attribute such as a polyhedron with finitely many faces

2.2 Symmetry in $\mathbb{R}^2$

Consider a square. It has four rotations, through angles 90°, 180°, 270°, 360° = 0°, which preserve the square. We call the 0° rotation the **identity symmetry** and write 1 for the identity map. We write r for the the counterclockwise rotation through 90°. Likewise, the counterclockwise rotation through 180° is obtained by applying the rotation 90° twice, so the 180° can be denoted as $r \cdot r = r^2$. Thus, 270° can be denoted as $r \cdot r \cdot r = r^3$. Notice that $r^4 = 1$ as well as $r^{-1} = r^3$.

The square also has four lines of bilateral symmetry, so that reflection through each of these lines preserves the square. We denote the reflection across the vertical lines of bilateral symmetry as m and notice that $m^2 = 1$, since reflecting across the same line twice is the same thing as doing nothing.

We shall compose symmetries from right to left since we consider this as a composition of functions, thus mr is a rotation r, then followed by a rotation m. Thus, we can determine that $rm = mr^3$ since our convention is to label m on the left in labeling reflections.

The symmetry group of the square is the dihedral group D_4 of order 8, where the word "dihedral" refers to the presence of both rotational and reflection symmetry. Thus, the eight symmetries of a square are written as $1, r, r^2, r^3, m, mr, mr^2$, and mr^3. Then,

$$D_4 = \{1, r, r^2, r^3, m, mr, mr^2, mr^3\}.$$

2.3 Groups and Subgroups

A **group** is a nonempty set G possessing a binary operation $*$ that satisfies the following four axioms:

1. **Closure**: $a * b \in G$ for all $a, b \in G$.
2. **Associativity**: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.
3. **Existence of Identity Element**: There exists an element $1 \in G$ such that $a * 1 = 1 * a = a$ for all $a \in G$.
4. **Existence of Inverses**: For every $a \in G$ there exists an element $d \in G$ such that $a * d = d * a = 1$.

   A group is said to be **abelian** if it satisfies the following additional axiom:
5. **Commutativity**: $a * b = b * a \in G$ for all $a, b \in G$.

A group G is said to be **finite** if the group has a finite number of elements. The **order of G** is said to be n if there are n elements in G, which is denoted as $|G| = n$. Otherwise, a group with infinitely many elements is said to have infinite order [5].

A subset H of a group G is a **subgroup** of G if H is itself a group under the same operation as G [5].

2.4 The Symmetric Group

Definition 2.2 Let X be any finite, nonempty set. A **permutation** of X is a reposition of the elements of X, in other words, a bijective function from X to X. Let $S(X)$ set of all bijective functions from X to itself. Consider the operation of composition of functions:

1. If $f, g \in S(X)$ are both permutations of X, then the composition $f \circ g : X \to X$ is again a bijection on X and so $f \circ g \in S(X)$.
2. The identity function $e : X \to X$ defined by $e(x) = x$ for all $x \in X$ satisfies $f \circ e = e \circ f = f$ for any $f \in S(X)$, and so $e \in S(X)$ is the identity permutation.
3. Also, if $f \in S(X)$ is any permutation, the inverse function $f^{-1} : X \to X$ is well-defined and a bijection, and $f \circ f^{-1} = f^{-1} \circ f = e$, so the inverse function f^{-1} is the inverse permutation of f.
4. The composition of functions is also known to be associative.

This shows that $S(X)$ is a group, for any set X. This group $S(X)$ is called the **symmetric group** on X. In particular, we denote by S_n the set of permutations of the set $\{1, 2, \ldots, \mathrm{n}\}$. A standard notation for the permutation that sends $i \to \ell_i$ is

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \ell_1 & \ell_2 & \ell_3 & \ldots & \ell_n \end{pmatrix}$$

and a k-cycle is a permutation of the form

$$f(\ell_1) = \ell_2, f(\ell_2) = \ell_3, \ldots, f(\ell_{k-1}) = \ell_k, \text{ and } f(\ell_k) = \ell_1$$

for distinct $l_1, \ldots, \ell_k$ among $\{1, 2, \ldots, \text{n}\}$, and $f(i) = i$ for i not among the ℓ_j. There is standard notation for this cycle:

$$f = (\ell_1 \ \ \ell_2 \ \ \ell_3 \ \ . \ . \ . \ \ \ell_k).$$

Using cycle notation, we may still multiply, or compose, permutations, and we keep the convention that we apply the permutations from right to left, as in composition of functions. For example, if $\tau = (1\,4\,5)(2\,3)$ and $\rho = (1\,2\,4)$, then

$$\rho\tau = (1\,2\,4)(1\,4\,5)(2\,3) = (2\,3\,4\,5)$$

since the rightmost cycle sends 2 to 3, which remains unchanged by other cycles to the left; 3 is sent on the right to 2, which is sent to 4; 4 is sent to 5; and finally 5 is sent to 1, then to 2. In the product of these permutations, 1 is sent to 4, then back to 1, so is unchanged and hence is not listed.

Theorem 2.1 S_n is a finite group with order $n!$ [6].

Proof. Using standard notation, any permutation $\sigma \in S_n$ is described by the matrix

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

It is clear that there are n choice for $\sigma(1)$, $n-1$ choices remaining for $\sigma(2)$, and so on, until there is only one choice for $\sigma(n)$, thus there are $n!$ choices in total for all permutation of n objects.

We provide the full multiplication table in cycle decomposition notation for the group S_3 acting on the set $\{1\ , 2, 3\}$. The convention followed here is that the column element is on the right and the row element is on the left, and functions act on the left. Hence, to determine the effect of the composite permutation on any element of $\{1, 2, 3\}$, we must first apply the permutation given by the column element and then apply the permutation given by the row element.

Lemma 2.1 Every permutation in S_n is a product of transpositions [5].

Proof. See page 231 of [5].

A permutation of S_n is said to be **even** if it can be written as the product of an even number of transposition, and **odd** if it can be written as the product of an odd number of transposition.

Lemma 2.2 No permutation in S_n is both even and odd [5].

Proof. See page 231 - 232 of [5].

The set of all even permutations in S_n is denoted A_n and is called the alternating group of degree n, which is shown by the Lemma 2.3.

S_3	(1)	(1 2 3)	(1 3 2)	(1 2)	(1 3)	(2 3)
(1)	(1)	(1 2 3)	(1 3 2)	(1 2)	(1 3)	(2 3)
(1 2 3)	(1 2 3)	(1 3 2)	(1)	(1 3)	(2 3)	(1 2)
(1 3 2)	(1 3 2)	(1)	(1 2 3)	(2 3)	(1 2)	(1 3)
(1 2)	(1 2)	(2 3)	(1 3)	(1)	(1 3 2)	(1 2 3)
(1 3)	(1 3)	(1 2)	(2 3)	(1 2 3)	(1)	(1 3 2)
(2 3)	(2 3)	(1 3)	(1 2)	(1 3 2)	(1 2 3)	(1)

FIGURE 2.1: The multiplication table for S_3.

Lemma 2.3 A_n is a subgroup of S_n [5].

Proof. If a and b are in A_n, then $a = \sigma_1\sigma_2 \ldots \sigma_p$ and $b = \tau_1\tau_2 \ldots \tau_q$ with each σ_i, τ_j a transposition and p, q even. Thus, $ab = \sigma_1\sigma_2 \ldots \sigma_p\tau_1\tau_2 \ldots \tau_q$. Since $p + q$ is even, $ab \in A_n$. So, A_n is closed under multiplication. Also, $a^{-1} = \sigma_p\sigma_{p-1} \ldots \sigma_1$. Since p is even, $a^{-1} \in A_n$. Therefore, A_n is a subgroup.

Lemma 2.4 $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$.

We need to show that half the elements of S_n are even. Let α be any two-cycle of S_n e.g. $\alpha = (12)$. Consider the mapping $f : S_n \to S_n$ by $f(\beta) = \alpha\beta$. Notice, that f is both surjective and injective.

First, to prove surjectivity, let $\gamma \in S_n$. We must find $\beta \in S_n$ so that $f(\beta) = \gamma$. Notice, $\alpha\beta = f(\beta) = \gamma$, which implies $\beta = \alpha^{-1}\gamma$.

Now, to prove injectivity, let $g(\beta_1) = g(\beta_2)$. Thus, $\alpha\beta_1 = \alpha\beta_2$, or $\beta_1 = \beta_2$.

Notice that f maps the even permutations to the odd ones and vise versa in a one-to-one, onto fashion. Hence, there must be the same number of even and odd permutations, that is half of S_n is even.

Lastly, we provide a full listing of all elements of S_4 and which 12 form the subgroup A_4.

The Symmetric Group S_4		
One-line Notation	Cycle Decomposition Notation	Even or Odd Permutation
1234	(1)	Even
1243	(3 4)	Odd
1324	(2 3)	Odd
1342	(2 3 4)	Even
1423	(2 4 3)	Even
1432	(2 4)	Odd
2134	(1 2)	Odd
2143	(1 2)(3 4)	Even
2314	(1 2 3)	Even
2341	(1 2 3 4)	Odd
2413	(1 2 4 3)	Odd
2431	(1 2 4)	Even
3124	(1 3 2)	Even
3142	(1 3 4 2)	Odd
3214	(1 3)	Odd
3241	(1 3 4)	Even
3412	(1 3)(2 4)	Even
3421	(1 3 2 4)	Odd
4123	(1 4 3 2)	Odd
4132	(1 4 2)	Even
4213	(1 4 3)	Even
4231	(1 4)	Odd
4312	(1 4 2 3)	Odd
4321	(1 4)(2 3)	Even

2.5 Homomorphisms and Isomorphisms

Definition 2.3 Let G and H be groups and $f : G \to H$ a function from G to H. We say that f is a **homomorphism** if

$$f(g_1) \cdot f(g_2) = f(g_1 * g_2) \tag{2.2}$$

for all g_1, $g_2 \in G$. The asterisk operation represents the operation in G, while the dot operation represents the operation in H. If $f : G \to H$ is a homomorphism and also a bijection, then f is an **isomorphism**.

Lemma 2.5 Suppose two finite groups $G = \{1 = g_1, g_2, \ldots, g_n\}$ and $H = \{1 = h_1, h_2, \ldots, h_n\}$ have the same number of elements. Let $A = (a_{ij})$ be the operation table for G and $B = (b_{ij})$ the operation table for H, so that $a_{ij} = g_i \cdot g_j$ and $b_{ij} = h_i \cdot h_j$ for each i and j. Let $f : G \to H$ be a bijection defined so that $f(a_{ij}) = (b_{ij})$ for each i and j. Then, f is an isomorphism [8].

Proof. Given that $b_{ij} = f(a_{ij}) = f(g_i \cdot g_j)$ for each i and j, then, since B is the operation table for H, it shows that $b_{ij} = h_i \cdot h_j = f(g_i) \cdot f(g_j)$. Hence, $f(g_i) \cdot f(g_j) = f(g_i \cdot g_j)$ for each i and j. Thus, f is a homomorphism but also since f is a bijection by construction, then it follows that f is an isomorphism.

2.6 Group Action

A **group action** is a representation of the elements of a group as symmetries of a set. Many groups have a natural group action coming from their construction; e.g. the dihedral group D_4 acts on the vertices of a square because the group is given as a set of symmetries of the square. A group action of a group on a set is an abstract generalization of this idea, which can be used to derive useful facts about both the group and the set it acts on.

Formally, a group action of a group G on a set X is a function $f : G \times X \to X$ satisfying the following properties:

1. $f(1, x) = x$ for all $x \in X$ and
2. $f(gh)(x) = f(g)(f(h)(x))$ for all $g, h \in G$ and $x \in X$.

When the action is clear, the function $f(g)(x)$ is often written as $g.x$ and with this notation, the axioms become

1. $1.x = x$ and
2. $g.(h.x) = (gh).x$.

The standard example of a group action is when G equals the symmetry group S_n (or a subgroup of S_n) and $X = \{1, 2, \ldots, n\}$. Then G acts on X by the formula $g \cdot x = g(x)$. The properties are clear: $e \cdot x = e(x) = x$ when e is the identity of S_n, and $g \cdot (h \cdot x) = g \cdot h(x) = g\big(h(x)\big) = (g \circ h)(x)$.

Definition 2.4 Let G be a group acting on a set X. A **fixed point** of an element $g \in G$ is an element $x \in X$ such that $g \cdot x = x$.

Definition 2.5 Let G be a group acting on a set X. The **stabilizer** G_x of a point $x \in X$ is the set of elements $g \in G$ such that x is a fixed point of g.

Definition 2.6 Let G be a group acting on a set X. The **orbit** of an element $x \in X$ is the set of elements $y \in X$ such that $g \cdot x = y$ for some $g \in G$.

Definition 2.7 Let G be a group acting on a set X. The action is **transitive** if there is only one orbit: for any $x, y \in X$, there is an element $g \in G$ such that $g \cdot x = y$.

Definition 2.8 Let G be a group acting on a set X. The action is **faithful** if the intersection of the stabilizers G_x for $x \in X$ consists only of the trivial element e_G.

Induced Homomorphism Theorem Let a group G act on a set X, with mapping $f_g(x) = g.x$ for each $g \in G$ and $x \in X$. Then,

1. f_g is a bijection, that is, f_g is a permutation of the set X, and
2. the map $f : G \to S(x)$ given by $g \mapsto f_g \in S(X)$ for $g \in G$ is a homomorphism from G into $S(X)$, the group of permutations of X [8].

Proof. We begin by proving (1), that is, we must show that $f(g)$ is a permutation of X or in other words, f_g is a bijection. First, let $y \in X$ and define x such that $x = g^{-1}.y \in X$. Then $f_g(x) = f_g(g^{-1}.y) = g.(g^{-1}.y) = y$. This proves surjectivity. Now, to show injectivity, suppose $x_1, x_2 \in X$ and let $f_g(x_1) = f_g(x_2)$. Then, $g.x_1 = g.x_2$, so $g^{-1}.(g.x_1) = g^{-1}.(g.x_2)$, thus $x_1 = x_2$, which shows that f_g is injective. Hence, this shows that f_g is a bijection, that is, f_g is a permutation of X.

Now, we must prove (2), that is, $f : G \to S(x)$ is a homomorphism from G into $S(X)$. By property (2) of a group action $f_g(f_h(x)) = g.(h.x) = (gh).x = f_{gh}(x)$ for all $x \in X$. So, $f_g f_h = f_{gh}$ for all $g, h \in G$. Thus, $f : G \to S(X)$ is a homomorphism.

Orbit-Stabilizer Theorem Let G be a group acting on a set X. Let G_x be the stabilizer of an element $x \in X$. Suppose that the orbit O_x of x is finite. Then, the index $[G : G_x]$ is finite and equal to $|O_x|$. If G is finite, then

$$|G_x| \cdot |O_x| = |G|.$$

Definition 2.9 An element b in a group G is **conjugate** to an element a if there is a $g \in G$ such that $a = gbg^{-1}$. (Alternatively, one says that a is a conjugate of b.)

To frame this in the language of group actions, consider the function $f : G \times G \to G$ defined by

$$f(g, b) = gbg^{-1}.$$

Then, $f(e, b) = ebe^{-1} = b$ for all b, and

$$f\big(g, f(h, b)\big) = f\big(g, hbh^{-1}\big) = g\big(hbh^{-1}\big)g^{-1} = (gh)b\big(h^{-1}g^{-1}\big) = (gh)b(gh)^{-1} = f(gh, b).$$

This shows that f defines a group action of G on itself. The conjugates of b are precisely the members of the orbit of b under the action. The stabilizer of b is the subgroup of elements g such that $gbg^{-1} = b$, or $gb = bg$. This is called the **centralizer** of b, the subgroup of elements of G which commute with b.

The **conjugacy classes** of G are the equivalence classes produced by the relation of conjugation. So, a conjugacy class in G is a subset of G consisting of elements which are all conjugate to one another.

Chapter 3

The Platonic Solids

3.1 Preliminaries

Definition 3.1 A **regular polyhedron** is a three-dimensional solid whose faces consist of congruent regular polygons, and whose vertex configurations (the number and types of polygons meeting at each vertex) is the same for every vertex [6]. The regular polyhedra are also called the **Platonic solids**.

There are exactly five such solids: the cube (hexahedron), dodecahedron, icosahedron, octahedron, and tetrahedron as we will show below. These solids were studied by the ancient Greeks extensively. Some sources, such as Proclus, credit Pythagoras with their discovery [2]. Other evidence suggests that he may have only been familiar with the tetrahedron, cube, and dodecahedron and that the discovery of the octahedron and icosahedron belong to Theaetetus, a contemporary of Plato. In any case, Theaetetus gave a mathematical description of all five and may have been responsible for the first known proof that no other convex regular polyhedra exist. However, the Platonic solids are prominent in the philosophy of Plato, their namesake, in one of his dialogues *Timaeus*.

Timaeus makes conjectures on the composition of the four elements which some ancient Greeks thought constituted the physical universe: earth, water, air, and fire [2]. Timaeus links each of these elements to a certain Platonic solid: the element of earth would be a cube, of air an octahedron, of water an icosahedron, and of fire a tetrahedron. Each of these perfect polyhedra would be in turn composed of triangular faces the 30-60-90 and the 45-45-90 triangles. The faces of each element could be broken down into its component right-angled triangles, either isosceles or scalene, which could then be put together to form all of physical matter. Particular characteristics of matter, such as water's capacity to extinguish fire, was then related to shape and size of the constituent triangles.

The Platonic Solids			
Name	Faces	Edges	Vertices
Tetrahedron	4	6	4
Cube	6	12	8
Octahedron	8	12	6
Dodecahedron	12	30	20
Icosahedron	20	30	12

The fifth element was the dodecahedron, whose faces are not triangular, and which was taken to represent the shape of the Universe as a whole, possibly because of all the elements it most approximates a sphere, which Timaeus has already noted was the shape into which God had formed the Universe.

Remark: Recall that the sum S of all internal angles of a n-gon is:

$$S = (n-2)180^\circ. \tag{3.1}$$

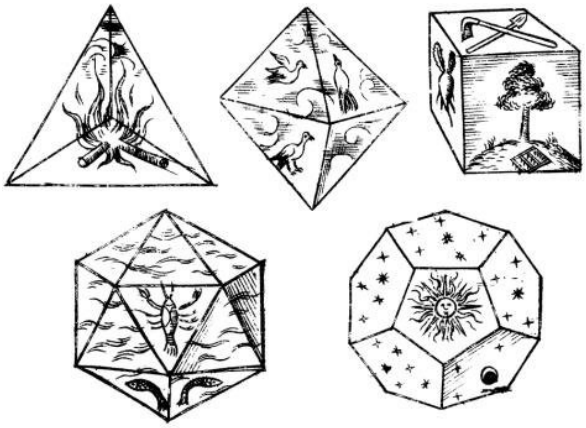


FIGURE 3.1: Johannes Kepler's drawing of the five Platonic solids representing the elements.

Thus, the size A of each internal angle of a regular polygon with n sides is:

$$A = \frac{(n-2)180^\circ}{n} = 180^\circ - \frac{360^\circ}{n}. \tag{3.2}$$

Lemma 3.1 There are exactly five Platonic solids. These are tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

Proof. Consider a convex regular polyhedron P. Let x be the number of sides of each of the regular polygons that form the faces of P. Let y be the number of those polygons which meet at each vertex of P, so $y \geq 3$ in order to form a closed three-dimensional solid.

Thus, by (3.2), the internal angles of each face of P measure 180° - $\frac{360^\circ}{x}$. The sum of interior angles incident with each vertex must be less than 360° to avoid flatness. Since each face is the regular polygon of the same type, this condition puts an upper bound on how many faces can be incident at a single vertex. Therefore,

$$y(180^\circ - \tfrac{360^\circ}{x}) < 360^\circ$$

$$y(1 - \tfrac{2}{x}) < 2$$

$$y(x-2) < 2x$$

$$y(x-2) - 2x < 0$$

$$yx - 2x - 2y + 4 < 4$$

$$(x-2)(y-2) < 4$$

Since, x and y must be greater than 2. Then, we look at all possible cases.

1. If $x = 3$, y can only be 3, 4, or 5.
2. If $x = 4$, y can only be 3.
3. If $x = 5$, y can only be 3.

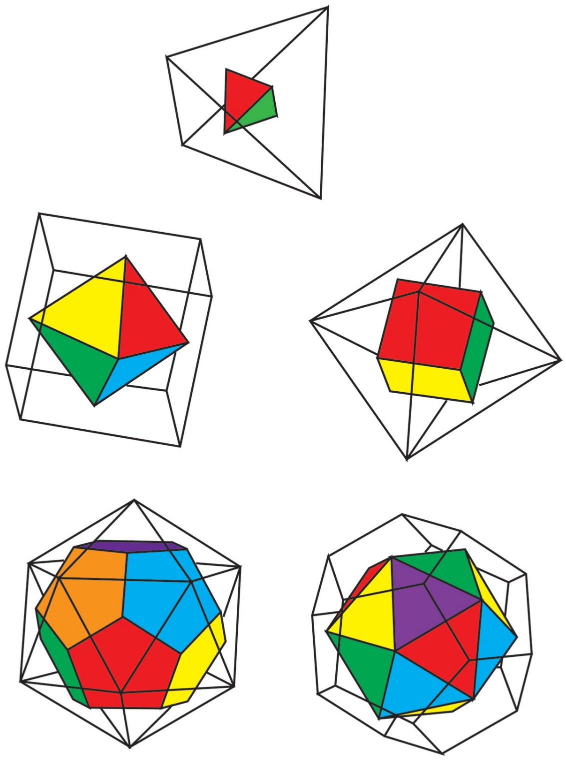


FIGURE 3.2: Dual compounds.

Thus, we've exhausted all cases so there are five possibilities in all and all Platonic solids have been accounted for.

Note the similarities in the numbers in the table *The Platonic Solids*. For example, the numbers of faces, edges, and vertices for the cube are the same as the numbers for the vertices, edges, and faces for the octahedron. Thus, the octahedron has a vertex for every face on the cube, the number of edges, and a face for every vertex of the cube. The cube and the octahedron are *duals*. If one take a cube and places a vertex at the center of each face, one gets an octahedron neatly embedded inside the cube. Similarly, if one places a new vertex in the center of each triangular face of an octahedron and connects these new vertices, one gets a cube embedded inside the octahedron.

Definition 3.2 For each regular polyhedron, the **dual polyhedron** is defined to be the polyhedron constructed by placing a point in the center of each face of the original polyhedron, connecting each new point with the new points of its neighboring faces, and erasing the original polyhedron [6].

The tetrahedron is self-dual (i.e. its dual is another tetrahedron). The cube and the octahedron form a dual pair and the dodecahedron and the icosahedron form a dual pair.

3.2 Symmetries

The **symmetry group** of a geometric object is the group of all isometries under which the object is invariant, endowed with the group operation of composition as referenced back in section 2.1. So, how many rotational symmetries does a cube have and what is the group of symmetries of the cube?

Let O be the group of symmetries of the cube. Then, O acts on the set F of faces of the cube. There are six faces, and the action is transitive, so the size of an orbit O_f of a given face is 6. The order of the stabilizer of a face is 4, because there are four rotations of a cube that fix a face (the rotations around the axis perpendicular to the face). So, $|S(C)| = 4 \cdot 6 = 24$, by the previously stated Orbit-Stabilizer Theorem.

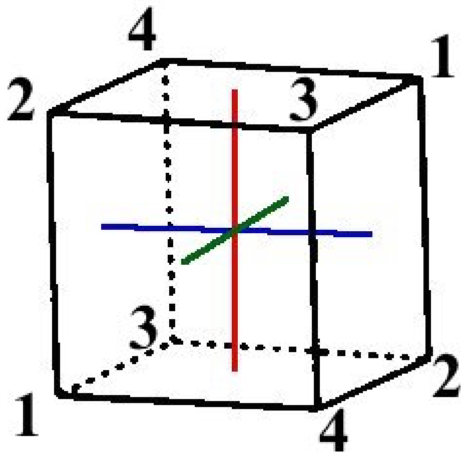
Also, O also acts on the edges of a cube. There are twelve edges, and the action is transitive, so the size of an orbit O_e of a given edge is 12. The stabilizer of an edge has order two; there is the identity of O and the unique element of O, which switches the vertices of the edge. So, again $|O| = 2 \cdot 12 = 24$.

Lastly, O also acts on pairs of opposite vertices of a cube in which the pairs of opposite vertices sit at either end of the body diagonals inside a cube. There are eight vertices, and again the action is transitive, so the size of an orbit O_v of a given vertex is 8. What is the order of the stabilizer of a vertex? The only rotations of a cube that leave one vertex fixed are the three rotations of 120°, 240° and 360° around the diagonal of the cube. So, the rotational symmetries preserve the configuration that two vertices are opposite, so we get an action on pairs of opposite vertices and $|O| = 3 \cdot 8 = 24$.

In fact, O is isomorphic to the symmetry group S_4, and the isomorphism uses yet another action of O, namely the action of O on the pairs of opposite vertices.

Theorem 3.1 The rotational symmetry group of a cube is S_4 [7].

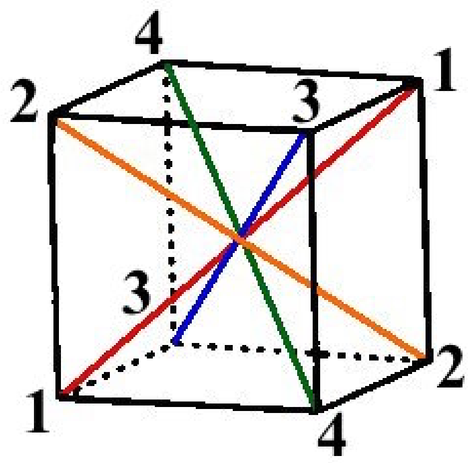
Proof. We must show that there is an isomorphism of the two groups, the group O of rotational symmetries of the cube, and the group S_4 of permutations of 4 objects. Now, the group O is being shown to act on the set $\{1, 2, 3, 4\}$ of pairs of opposite vertices when it rotates the cube. Therefore, by the Induced Homomorphism Theorem, there must therefore be a homomorphism from O into S_4. We will label the opposite vertices of the cube in the first figure as 1, 2, 3, and 4 because if one of the vertices in the pair moves, the other pair would move correspondingly to remain an opposite vertex. We know that S_4 contains 24 elements from section 2.4 and the maximum rotational symmetries is 24 (from answering the question up above), provided that all the permutation of these four pairs of vertices can be found. Inspection shows that this can be done by considering the cube below with numbered vertices, specifically the closest square in which the number 1 is labeled as the lower left-hand corner, 2 as the upper left-hand corner, 3 as the upper right-hand corner, and 4 as the lower right-hand corner.



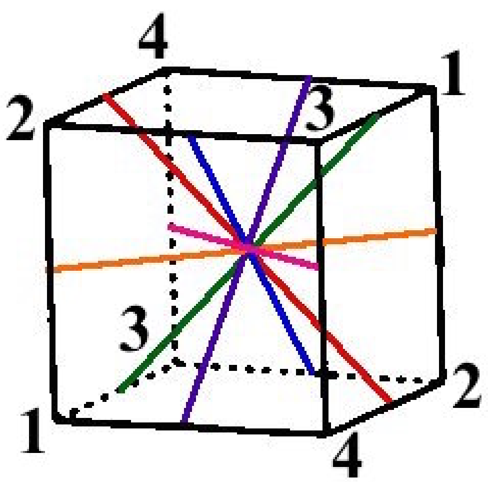
First of all, the identity rotation: that is, no rotation at all. There is only one such rotation or viewed on our square as (1), the identity element synonymous with (1) in the table with the elements of S_4 in cycle decomposition in section 2.4.

Next, consider the 9 rotations of the cube about the 3 axes. We can either rotate by 90°, 180° or 270° around either the red, blue or green axes. First, consider the green axis. If we rotate about the axis 90° clockwise, we obtain the element (1 4 3 2) in S_4 in our starting square. If we rotate the cube 90° more in the same clockwise direction, we obtain the element (1 3)(2 4), and 90° more, we obtain the element (1 2 3 4). Notice that the 6 rotations by 90° or 270° will have the same cycle structure and the 3 rotations of 180° about each of the axes will have the same cycle structure i.e. any rotation about one of those pictured axes above is either a 4 cycle, if it is by 90° or 270° , or a product of disjoint transpositions if it is by 180°. For example, a rotation about the blue axis 180° will produce the element (1 4)(2 3). In fact, using the face axis we can see that all 6 4-cycle are obtained, and all 3 pairs of disjoint transpositions.

Next, we have rotations by 120° and 240° around the 4 axes by joining a vertex and the opposite vertex. This gives 8 permutations. All 8 of these permutations will have the same cycle structure, that is, all 8 3-cycles are obtained in this way since since the vertex number that determines the axis of rotation is fixed and you cycle through the 3 others. For example, when we rotate about the red axis, we obtain (2 4 3) and (2 3 4).



Finally, consider the 6 rotations of the cube by 180° around the 6 axes formed by joining the center of an edge to the center of an opposite edge. Each of these 6 permutations will have the same cycle structure, that is, all 6 transpositions are obtained in this way in S_4.



Since the induced homomorphism is surjective, and both groups have 24 elements, it is also injective and therefore an isomorphism because once two groups have the same number of elements and a map is surjective, then it is also injective.

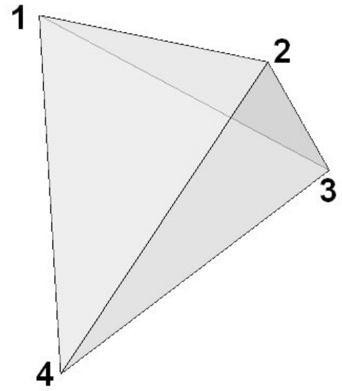
Theorem 3.2 The rotational symmetry group of a tetrahedron is A_4 [5].

Proof. Let T denote the rotational symmetry group of a tetrahedron. We must show that there is an isomorphism of the two groups, the group T of rotational symmetries of the tetrahedron, and the group A_4 of even permutations of 3 objects. Now, the group T is being shown to act on the vertices when it rotates the tetrahedron. Therefore, by the Induced Homomorphism Theorem, there must be a homomorphism from T into S_4. We will show this homomorphism is injective, and its image is A_4. We will label the vertices of the tetrahedron in the figure as 1, 2, 3, and 4 as shown below.

Let's use the convention of a cyclic notation: the identity is then (1).

Now, (1 2 3), and the other seven of this kind: namely, (1 3 2), (1 2 4), (1 4 2), (1 3 4), (1 4 3), (2 3 4), and (2 4 3) is a rotation where the tetrahedron is rotated through the axis formed by a vertex and the centroid of the face not containing the vertex at an angle of 120° in either clockwise or counterclockwise directions.

Finally, (1 2)(3 4) and the other two of this kind: namely, (1 3)(2 4) and (1 4)(2 3) is also a rotation, where the axis of rotation goes through a center of an edge to a center of an another edge that is not adjacent to the aforementioned edge.



Hence we have obtained 1 + 8 + 3 = 12 different permutations in the image of T under the induced homomorphism, and they were all distinct, so the induced homomorphism is injective. Also note that the permutations we obtained are the identity, all 8 3-cycles, which are even, and all 3 products of disjoint transpositions, which are even (see the Table on page 6). So the induced homomorphism has as its range A_4, the subgroup of all even permutations in S_4. We have shown that the induced homomorphism maps T into A_4, and that the mapping is both injective and surjective. Therefore it is an isomorphism.

Theorem 3.3 The rotational symmetry group of a icosahedron is A_5 [7][8].

Proof. For this last examination of symmetry groups of Platonic solids, we will focus on the case of the icosahedron. Let I denote the rotational symmetry group of an icosahedron. We must show that there is an isomorphism of the two groups, the group I of rotational symmetries of the icosahedron, and the group A_5 of even permutations of 5 objects. We begin by numbering the 20 faces of the icosahedron 1 through 5, in other words, we are assembling an origami Triangle Edge module of an icosahedron. It takes 30 sheets of paper to construct this module with each slip folding such that it takes three slips to build one face with each slip over an edge so that one slip helps construct an additional, consecutive face. (See figure 4.3 as an example of a 6-coloring Triangle Edge module [1].) Since each slip folds over, there are 5-choose-2 ways to number the end of the modules from 1 to 5 or 10 ways. However, we need 30 slips so we need three of each, 3 with 1 and 2, 3 with 1 and 3, 3 with 1 and 4, 3 with 1 and 5, 3 with 2 and 3, 3 with 2 and 4, 3 with 2 and 5, 3 with 3 and 4, 3 with 3 and 5, and, finally, 3 with 4 and 5. Thus, we can assemble the icosahedron in such a way that there exists four faces numbered 1. The four faces numbered 1 should be equidistant from each other, with their centers being located at the vertices of a regular tetrahedron embedded inside the icosahedron. Then, the four faces numbered 2 are located similarly, and likewise for the faces numbered 3, 4, and 5. This shows how the faces can be distributed in a symmetric arrangement, with the numbers 1 through 5 indicating the vertices of five distinct regular tetrahedra embedded inside the icosahedron. See [4] for a picture of the five intersecting tetrahedra in an origami model created by Thomas Hull. Now, the rotation group I maintains the arrangement of faces numbered 1 at the vertices of a tetrahedron, and the same holds true for faces numbered 2 through 5. The rotations of the icosahedron permute embedded tetrahedra, so they permute the five groups of four faces with the same number. In other words, this numbering of faces of the icosahedron creates an action of the group I on the set $\{1, 2, 3, 4, 5\}$. Therefore, by the Induced Homomorphism Theorem, there is a homomorphism from I into S_5. We must prove that this homomorphism is injective, and that its image is exactly A_5, the group of even permutations in S_5.

Now, we observe the rotations of the icosahedron. First, note that there are 20 faces, so there are 10 pairs of opposite faces. We observe that the opposite faces have different numbers, that is, the opposite face for a face labeled with a 1 will not be a 1. In fact, all 5-choose-2 ways to pair up two numbers from $\{1, 2, 3, 4, 5\}$ occur. When rotating about an axis connecting centers of opposite faces, the rotations are through 120° or 240°. They fix the two numbers on the faces through which the axis is constructed, and they cause a permutation of the three remaining numbers. We obtain all 20 3-cycles in S_5 in this way.

Next, recall that there are 12 vertices on an icosahedron, so we have 6 axes that connect opposite vertices. Around these 6 axes, we can have 4 nontrivial rotations by n(72°), where n is 1, 2, 3, or 4. These all produce 5 cycles, because all five numbers of faces touch a given vertex. We obtain 6 x 4 = 24 5-cycles this way, all of the 5-cycles in S_5.

Finally, there are 30 edges so 15 pairs of opposite edges. When an axis connects opposite edges, there is only a rotation by 180°, the same as for the other solids. Now, one of the edges is between two numbers i and j on the adjacent faces, and the other is between numbers k and l. The rotation by 180° swaps these numbers in pairs, and fixing the 5th number, so we obtain a product of disjoint transpositions, $(i\ k)(j\ l)$. There are 15 of these in S_5 and we obtain them all.

Thus, the induced homomorphism is injective since all permutations obtained were distinct. And we obtained all even permutations, and only even permutations. This shows the map is surjective onto A_5. We conclude it is an isomorphism between I and A_5.

Chapter 4

The Platonic Solids and Color Symmetry

4.1 Color Symmetry

The analysis about each of the rotational symmetries can provide us more information about the theory of group actions in regards to color symmetry. For example, recall that the cube has 6 faces, 8 vertices, and 12 edges. Thus, it is possible to color the cube in 3 colors by coloring each pair of opposite faces in a different color as shown in Figure 4.1, or by coloring groups of 3 edges in different colors, so long as we choose the coloring's in a pattern that define color symmetry. That is, we can define color symmetry as the division into colored regions being invariant (or preserved) under rotations. Note, that the number of regions in each individual color must divide the total number of regions being colored.

Let's first look at all the possible coloring of the cube with respect to the faces, vertices, and edges. We begin by observing all the possible coloring of the cube with respect to the faces:

1. If we color the faces of the cube all a single color, then by inspection we can see that this is homomorphic to S_1, however the action is not faithful. So, we will not consider this a color symmetric object.
2. If we color the faces of the cube with two colors, then there are two possibilities for the coloring. However, neither one of these produces a symmetric coloring because it fails to hold that the division into colored regions being invariant under rotations.

FIGURE 4.1: Color symmetric cube in 3 colors.

3. If we color the faces of the cube with three coloring, then there are three possibilities for the coloring. However, only one of these coloring is symmteric, i.e. the coloring of the cube in which each pair of opposite faces is similar. We can see a homomorphism from O into S_3 induced by the action of O on this type of symmetrically 3-colored cube that is surjective but not injective, also known as an epimorphism.

4. Clearly, we cannot the faces of a cube with only four or five colors since there will always be more colors that have more coloring's than the other colors which will always fail the definition of color symmetry.

5. If we color the faces of the cube with different colors, that is six, then we can consider this a color symmetric object because there is a homomorphism from O into S_6 induced by the action of O on this type of symmetrically 6-colored cube that is injective but not surjective, also known as a monomorphism.

Permutations from a 6-Colored Face Cube induced by the action of O					
123456	215634	312645	413652	514623	625431
134526	231564	326415	436512	521463	632541
145236	256314	341265	451362	546213	643251
152346	263154	364125	465132	562143	654321

Now, we begin by observing all the possible colorings of the cube with respect to the vertices:

1. If we color the vertices of the cube all a single color, then by inspection we can see that this is homomorphic to S_1, however the action is not faithful. So, we will not consider this a color symmetric object.

2. If we color the vertices of the cube with two colors, then there are multiple possibilities for the coloring. However, only one of these produces a symmetric coloring because there is a epimorphism from O into S_2 induced by the action of O.

3. There does not exist a symmetric coloring of the vertices with three, five, six, or seven colors.

4. We have already shown that O is isomorphic to the symmetry group S_4 by the action of O on the body diagonals, so if we color these vertices, each being its own color, then it is 4-colored symmetric cube.

5. If we color the vertices of the cube with different colors, that is eight, then this a color symmetric object because there is a monomorphism from O into S_6 induced by the action of O on this type of symmetrically 8-colored cube.

Now, we begin by observing all the possible colorings of the cube with respect to the edges.

1. If we color the edges of the cube with one, two, five, seven, eight, nine, ten, or eleven color(s), then it fails to be color symmetric.

2. If we color the cube with three colors, then there exists away such that the coloring is isomorphic to the 3-colored symmetric face cube, thus there is epimorphism from O into S_3 induced by the action of O on this edge colored cube.

3. With 4 colors, there exists an isomorphism from O into S_4 as shown in Figure 4.2 [3].

4. If we color the edges of the cube with different colors, that is twelve, then there is a monomorphism from O into S_{12} induced by the action of O on this type of symmetrically 12-colored cube.

5. If we color the edges of the cube with six different colors, then this is a color symmetric object because of the monomorphism from O into S_6 induced by the action of O.

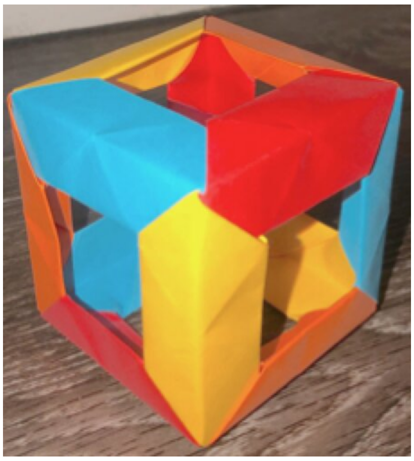


FIGURE 4.2: Edge coloring of a color symmetric cube with four colors [3].

Permutations from a 6-Colored Edge Cube induced by the action of $S(C)$					
123456	214356	321465	412365	516324	615342
143265	243165	341256	432156	526413	625431
153624	254613	351642	452631	536142	635124
163542	264531	361524	462513	546231	645213

Next, we look at all the possible coloring of the tetrahedron with respect to the faces, vertices, and edges. We begin by observing all the possible coloring of the tetrahedron with respect to the faces or vertices:

1. If we color the faces or vertices of the tetrahedron all a single, two, or three color(s), the it fails to hold that the division into colored regions being invariant under rotations and not faithful.

2. If we color the faces of the tetrahedron with four colors, we've already shown that T is isomorphic to the symmetry group A_4 by the action of T on the faces, so each face being its own color creates a 4-colored symmetric tetrahedron.

Now, we observe all the possible coloring of the tetrahedron with respect to the edges.

1. If we color the faces of the tetrahedron all a single, two, four or five color(s), the it fails to hold that the division into colored regions being invariant under rotations and not faithful.

2. If we color the faces of the tetrahedron with 3 colors, then there are three possibilities for the coloring. However, only one of these coloring is symmetric, i.e the coloring of the tetrahedron in which each pair of opposite edges is similar. We can see a epimorphism from T into A_4 induced by the action of A_4 on this type of symmetrically 3-colored tetrahedron.

Lastly, We begin the generalizations of color symmetry of the icosahedron by observing all the possible coloring's of the icosahedron with respect to the faces, vertices, and edges:

1. We know that there can only exists a 5, 10, and 20 coloring of the faces of the icosahedron to make it color symmetric because the other coloring would have too little or too many colors.

2. This holds true for the 3, 4, 6, and 12 color symmetric icosahedron with respect to the vertices.

3. This also holds true for the 5, 6, 10, 15, and 30 color symmetric icosahedron with respect to the edges.

FIGURE 4.3: Vertex coloring of an icosahedron from six colors with each vertex treated as the absent of the sixth color [1].

4.2 Conclusions and Discussions

In conclusion, the permutation representation of the symmetry group of the Platonic solids can be constructed using group actions on rotationally-invariant colorings as defined by color symmetry of the faces, edges, or vertices. Not only does this allow us to understand the symmetry group of the Platonic solids, but it allows us to understand group actions to further illustrate crucial mathematical definition using hands on objects.

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