CHARACTERIZATIONS OF THE CONTINUOUS IMAGES OF ARCS, PSEUDO-ARCS, AND PSEUDO-CIRCLES

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LIST OF SYMBOLS

D Boundary of Set D

$\overline{xy}$ An Arc Between Points x and y

xy A Segment Between Points x and y

1_G The Identify Map of G onto Itself

CHAPTER I

PRELIMINARIES

Introduction

The main theme of this thesis is to expose the characterizations of continuous images of certain continua. The continua that will be considered in this paper are the following:

(1) arcs,

(2) pseudo-arcs,

(3) chainable continua,

(4) pseudo-circles,

(5) circularly chainable continua, and

(6) indecomposable continua.

Various aspects of the continuous images of these continua will be explored throughout the thesis. In Chapter II, a brief historical sketch of this study will be given. This sketch will start with Peano's (35) discovery that the square and its interior are the continuous image of the arc. Chapter II will contain detailed presentations of the characterizations of the continuous images of the pseudo-arc and the pseudo-circles. These presentations will take different mathematical approaches, and a comparison of these different methods will also be contained in Chapter III. In Chapter IV the inclusion relationships between these classes of continuous images in the plane and various other known classes of plane continua will be revealed. Also several examples will be presented to illustrate these relationships. Finally, in Chapter V, several unsolved problems related to the mathematics in Chapters III and IV will be mentioned. In addition, a sampling of recent research concerning continuous images of other continua, and the dual topic, continuous preimages of certain continua, will be briefly discussed.

A background in topology equivalent to the background gained by taking the usual first six-hour sequence in topology is assumed. For terms undefined in this paper, consult one of the following standard references: Kelley (20), R. L. Moore (33), or Dugundji (10).

Pseudo-Arcs and Pseudo-Circles

Before the definitions of pseudo-arc and pseudo-circle can be made, it will be useful to define the concepts of chainability and circular chainability for continua.

<u>Definition 1.1</u> A chain is a finite sequence of open sets of X, $d_1, d_2, \ldots, d_n$, such that $d_i \cap d_j \neq \emptyset$ if and only if $|i - j| = 1$. The set d_i of the chain, $d_1, \ldots, d_n$, is called a link. The notation $D = d_1, \ldots, d_n$ will be used to name the chain, $d_1, \ldots, d_n$, by D. A chain D is said to be an ε-chain if the diameter of each link is less than ε. Finally, a continuum H is said to be chainable or snake-like if, for each $\varepsilon > 0$, H can be covered by an ε-chain.

Definition 1.2 A circular chain is a chain in which the first and the last links intersect. A circular chain D is said to be a circular ε-chain if the diameter of each link is less than ε. Finally, a continuum H is said to be circularly chainable or circle-like if, for each $\varepsilon > 0$, H can be irreducibly covered by a circular ε-chain.

In Chapter IV, one may find a proof that the arc is a continuum that is chainable but not circularly chainable. The circle is an example of a continuum that is circularly chainable but not chainable. There are continua that are both circularly chainable and chainable. The pseudo-arc (3) is such a continuum.

In 1947, E. E. Moise (29) found an example of indecomposable plane continuum that is homeomorphic to each of its nondegenerate subcontinua. For this reason, Bing (1) called this continuum a pseudo-arc. This example will be presented to motivate the definition of a pseudo-arc in this thesis. First the notion of a chain D being crooked in another chain E must be defined.

Definition 1.3 Let $D = d_1, \ldots, d_n$ and $E = e_1, \ldots, e_m$ be chains in X. Then E is crooked in D provided that if $k - h > 2$, $e_i \subseteq d_h$, and $e_j \subseteq d_k$, then there exists links $e_r \subseteq d_{k-1}$ and $e_s \subseteq d_{h+1}$ such that either $i > r > s > j$ or $i < r < s < j$.

In Figure 1, the chain E is crooked in the chain D. It

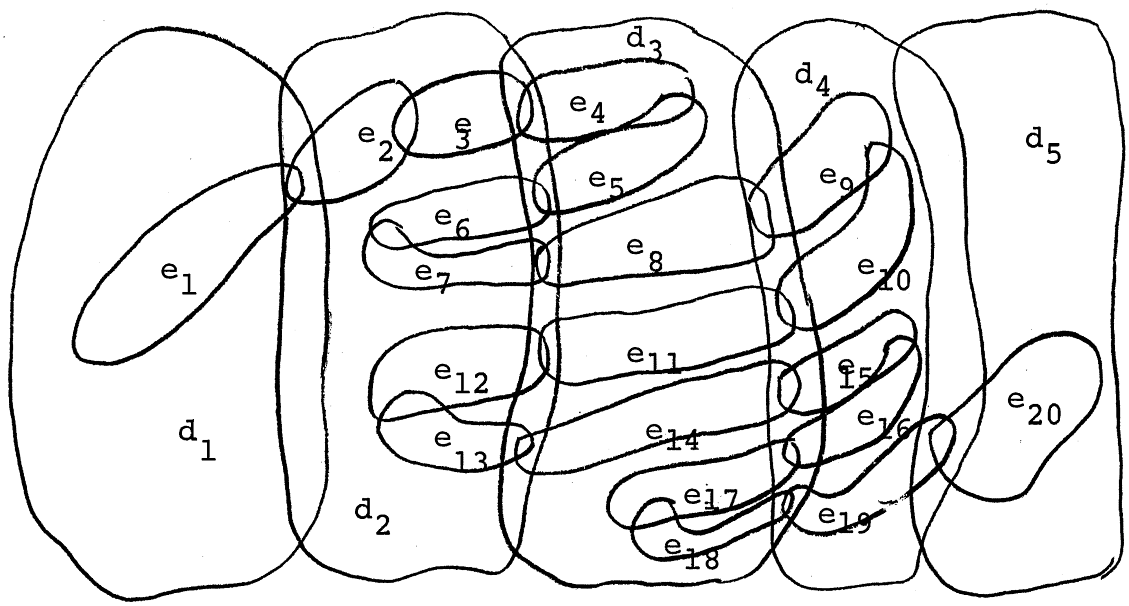


Figure 1. E Crooked in D

can be seen from this example that E must be pretty "nervous" to be crooked in D. Now the description of Moise's continuum can be made.

<u>Definition 1.4</u> Let p and q be two distinct points in the plane. Also let D_1, D_2, ... be a sequence of chains with the following properties for each positive integer j:

(i) D_j is a chain that contains p in its first link and q in its last link,

(ii) D_{j+1} is crooked in D_j,

(iii) D_j is an 1/j-chain, and

(iv) the closure of each link of D_{j+1} is a subset of a link of D_j.

Then the common part of this sequence D_1, D_2, ... is defined to be Moise's continuum M.

As previously mentioned, Moise (29) proved the following theorem:

<u>Theorem 1.1</u> The continuum M defined in Definition 1.4 is homeomorphic to each of its nondegenerate subcontinua.

Bing (1) was able to prove the following theorem:

<u>Theorem 1.2</u> The continuum M defined in Definition 1.4 is hereditarily indecomposable.

<u>Proof</u> Suppose the conclusion is not true. Then there exists a subcontinuum M' of M that is not indecomposable. Hence $M' = H \cup K$ where H and K are proper subcontinua of M'. Now by properties of metric spaces, there exists points p and q of M' and an integer j such that the distance from p to H is greater than 2/j and the distance from q to K is greater than 2/j. Let $D_j(j,k)$ and $D_{j+1}(u,v)$ be subchains of D_j and D_{j+1}, respectively, such that p and q are contained in end links of each subchain. Suppose that $p \in d^j_h$ where d^j_h is the first link of $D_j(h,k)$. Since M' is a connected set then each link of $D_j(h,k)$ and $D_{j+1}(u,v)$ contains a point of M'. Also it is known that d^j_{h+1} contains a point of K but none of H, and d^j_{k-1} contains a point of H but none of K.

Since D_{j+1} is crooked in D_j, $D_{j+1}(u,v)$ contains three links d^{j+1}_r, d^{j+1}_s, d^{j+1}_t such that $r < s < t$, d^{j+1}_s is a subset of d^j_{h+1}, and d^{j+1}_r, $d^{j+1}_t \subseteq d^j_{k-1}$. But H is not connected since both d_r^{j+1} and d_t^{j+1} contains points of H but d_s^{j+1} does not contain a point of H. This is a contradiction, and so M' must be indecomposable. //

Finally, Bing (2) was able to show the following theorem:

Theorem 1.3 Each hereditarily indecomposable chainable continuum is homeomorphic to Moise's continuum M.

The proof is omitted, but if you are interested, see (2) for a complete proof.

Now the precise definition of the pseudo-arc can be made. It is the culmination of several events. The first one was the description of Moise's continuum M having the property that it is homeomorphic to each of its nondegenerate subcontinua. The second event was Bing's proof that M was hereditarily indecomposable. And the final one was Bing's proof that all hereditarily indecomposable chainable continua are homeomorphic to Moise's continuum M.

Definition 1.5 A pseudo-arc is any hereditarily indecomposable and chainable continuum.

In 1950, R. H. Bing (2) described a hereditarily indecomposable circularly chainable continuum M' that was not the pseudo-arc. Because M' was circularly chainable, Bing called it a pseudo-circle.

Definition 1.6 Let D_1, D_2, ... be a sequence of circular chains with the following properties for each positive integer j:

(i) each link of D_j is the interior of a circle of diameter less than $1/j$,

(ii) the closure of each link of D_{j+1} is contained in a link of D_j,

(iii) each complimentary domain of the union of links of D_{j+1} contains a complimentary domain of the union of links of D_j, and

(iv) if E_j is a proper subchain of D_j, E_{j+1} a proper subchain of D_{j+1}, and E_{j+1} is contained in E_j, then E_{j+1} is crooked in E_j.

Then M' is defined to be the common part of the sequence D_1, D_2, ...

From Definition 1.6 it is not clear how to construct M'. It is hoped that this problem can now be resolved.

<u>Construction of M'</u> Suppose that the chain D_i has been constructed having all the desired properties of Definition 1.6. Let d_1, ..., d_n denote the links of D_i. Now a procedure for constructing D_{i+1} will be revealed. In this process it will be helpful to consider a circular chain $D_i' = d_1'$, ..., d_{3n}' which has the following properties: d_1', d_{n+1}', d_{2n+1}' are subsets of d_1; d_2', d_{n+2}', d_{3n}' are subsets of d_2; d_3', d_{n+3}', d_{3n-1}' are subsets of d_3; ... ; d_n', d_{2n}', d_{2n+1}' are subsets of d_n. Intuitively, D_i' "goes through" D_i twice in one direction and once in the opposite direction. This

process is pictured in Figure 2, and for the purposes of the illustration n = 5. Finally, D_{i+1} can be defined as the union of two chains, one which is crooked in the chain, d'_1, d'_2, ..., d'_{2n+1}, and the other which is crooked in the chain, d'_{2n+1}, d'_{2n+2}, ..., d'_{3n}, d'_1. Then D_{i+1} is a circular chain, and with a little effort on the part of the reader it can be seen that D_{i+1} has the desired properties.

The following two theorems by Bing (2) give two properties of the continuum M' that will eventually lead to the definition of the pseudo-circles:

Theorem 1.4 The continuum M' defined in Definition 1.6 is hereditarily indecomposable.

Proof Follows closely to the proof of Theorem 1.2. //

Theorem 1.5 The continuum M' defined in Definition 1.6 is not chainable.

Proof The conclusion follows from the fact the M' separates the plane.

As a consequence to Bing's continuum M' and its properties, J. T. Rogers, Jr. (37) defined a pseudo-circle as follows:

Definition 1.7 A continuum H is said to be a pseudo-circle if it is hereditarily indecomposable, circularly chainable, but not chainable.

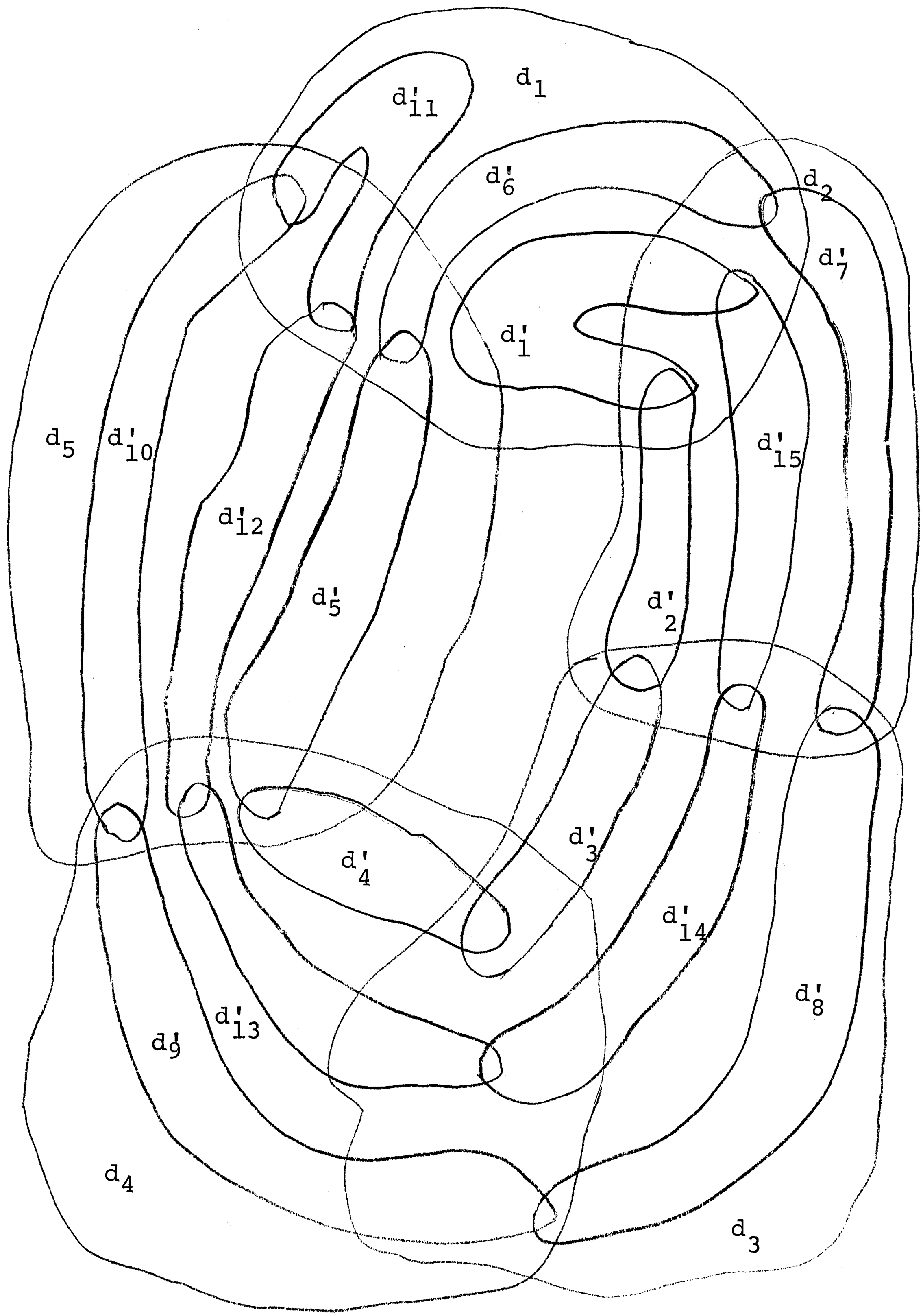


Figure 2. Illustration of D'_i in D_i

Inverse Limits

The full generality of inverse limits is not needed, and so only a special case will be presented. For a treatment of the most general case, consult Dugundji (10). Proofs to fundamental properties will be supplied only when they will help in understanding the concept of inverse limits. Finally, examples will be presented throughout this section to aid in understanding. All product spaces have the usual topology.

<u>Definition 1.8</u> A collection of topological spaces X_1, X_2, ... together with a collection of continuous maps β_n^{n+1}: $X_{n+1} \to X_n$, $n = 1, 2, \ldots$, is said to be an inverse spectrum. This inverse spectrum is denoted by $\{X_n, \beta_n^m\}$, where β_n^m will denote the map $\beta_{m-1}^m \beta_{m-2}^{m-1} \cdots \beta_n^{n+1}$. Let $\prod_{n=1}^{\infty} X_n$ be the product of the X_n's and P_n the projection map of $\prod_{n=1}^{\infty} X_n$ onto the n^{th} factor X_n. Then the inverse limit of the inverse spectrum is the following subspace of $\prod_{n=1}^{\infty} X_n$:

$$\{ (x_1, x_2, \ldots) \mid \text{for each pair of positive integers } j < k,\ x_j = \beta_j^k(x_k) \}.$$

Denote this by $\varprojlim\{X_n, \beta_n^m\}$, and simply call it the inverse limit. Equivalently, $\varprojlim\{X_n, \beta_n^m\}$ is the collection of points $(x_1, x_2, \ldots)$ that, for each pair of positive integers $j < k$, commute in the following diagram:

$$\begin{tikzcd} X_1 & X_2 \arrow[l] & X_3 \arrow[l] & \cdots \arrow[l] & X_j \arrow[l] & \cdots \arrow[l] & X_k \arrow[l] & \cdots \arrow[l] \\ & & & & \prod_{n=1}^{\infty} X_n \arrow[u, "P_j"] \arrow[urr, "P_k"'] & & & \end{tikzcd}$$

From Definition 1.8, it is only revealed that the inverse limit is a subspace of a product space, somewhat nonintuitive. This product space seems to be so big that it is hard to visualize it and any of its subspaces. Several examples will now be revealed to indicate that the inverse limit acts in some way like a limit, and this can make the inverse limit easier to visualize.

Let $X_n = X$, where X is a nonempty topological space, and $\beta_n^{n+1} = 1_X$, for each positive integer n. Then the $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$ is homeomorphic to X. This can be seen by looking at the map h of $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$ onto X defined by taking $(x, x, \ldots)$ to x. Clearly $(x, x, \ldots)$ is an element of $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$, and in fact, it is easy to see that $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\} = \{(x, x, \ldots) \mid x \in X\}$. With little effort, the map h can be seen to be one-to-one, continuous, and have a continuous inverse. Therefore, h is a homeomorphism. So in some sense the inverse limit of a "constant sequence" is the obvious "limit".

The next example shows that the inverse limit of a decreasing sequence of topological spaces, with bonding maps being inclusions, is just the intersection of the spaces. Let $X_n = \{ z \in R^2 \mid |z| < (2+1/n)^2\}$ and β_n^{n+1} the inclusion map of X_{n+1} onto X_n, for each positive integer n. Then $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ is homeomorphic to X_n which in turn is homeomorphic to $\{z \in R^2 \mid |z| \leq 4\}$. Clearly the only points of ΠX_n in $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ are the points $(z, z, \dots)$ where $|z| \leq 4$. Hence there is an obvious mapping of $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ onto $\{z \in R^2 \mid |z| \leq 4\}$. This mapping is a homeomorphism. You are urged to give this some thought.

The next example should give you the feeling that it would not be too uncommon to find the inverse limit contain no points.

Let $X_n = \{z \in R^2 \mid |z| < 1/n \text{ and } z \neq 0\}$ and β_n^{n+1} is the inclusion map of X_{n+1} onto X_n, for each positive integer n. Then $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ is homeomorphic to $\cap X_n$ as the last example indicates. But $\cap X_n = \cap \{z \in R^2 \mid |z| < 1/n \text{ and } z \neq 0\} = \emptyset$. This example leads one to ask what conditions on the spaces X_n would guarantee that the inverse limit be nonempty. The following theorem indicates one such condition.

<u>Theorem 1.6</u> Let X_n be nonempty, compact, and Hausdorff with β_n^{n+1} a continuous map of X_{n+1} into X_n, for each positive integer n. Then $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\} \neq \emptyset$.

<u>Proof</u> For each positive integer $m \geq 2$, let $S_m = \{(x_1, x_2, \dots) \mid \text{for each positive integer } n < m, x_n = \beta_n^m(x_m)\}$. It is not hard to see that $S_m \neq \emptyset$, and observe that for positive integers $k < j$, $S_j \subsetneq S_k$. Assert that S_m is closed for each positive integer m. Let $(x_1, x_2, \dots) \in \Pi X_n - S_m$. Then there exists a positive integer $j < m$ such that $x_j \neq \beta_j^m(x_m)$.

Since X_j is Hausdorff, open sets U and V of X_j can be found such that $x_j \varepsilon U$, $\beta_j^m(x_m) \varepsilon V$, and $U \cap V = \emptyset$. From the continuity of β_j^m there exists an open set W of X_m such that $x_m \varepsilon W$ and $\beta_j^m(W) \subseteq V$. Then the open set $0 = P_j^{-1}(U) \cap P_m^{-1}(W)$ contains $(x_1, x_2, \ldots)$ and $S_m \cap 0 = \emptyset$. Hence $\Pi X_n - S_m$ is open and S_m is closed. It is easily seen that the family $\{S_k\}$ has the finite intersection property. Since each X_n is compact, then ΠX_n is compact, and hence $\bigcap S_k \neq \emptyset$. But looking at the definition of S_k it can be seen that $\bigcap S_k \subseteq \underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$. Therefore, $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ is not empty. //

Another fundamental question of inverse limits is: When is the inverse limit a closed subspace of ΠX_n? The answer is found in the following theorem:

<u>Theorem 1.7</u> Let X_n be a nonempty Hausdorff space and β_n^{n+1} a continuous map of X_{n+1} into X_n, for each positive integer n. Then $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ is a closed subspace of ΠX_n.

<u>Proof</u> The proof is similar to showing that S_m was closed in the proof of Theorem 1.6. //

It is now time to consider the question: When is an inverse limit a continuum? The answer to this question was of great importance to J. T. Rogers, Jr. in his presentation revealed in Chapter III. First, an example will be given to show that the inverse limit of connected sets need not be connected.

Let $X_n = \{z \varepsilon R^2 | \ 1+1/n \leq |z| \leq 1$, and $z \neq -1$ or $+1\}$

and β_n^{n+1} the inclusion map of X_{n+1} onto X_n. Then as in previous examples $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$ is homeomorphic to X_n. But $X_n = \{z \in R^2 \mid |z| = 1, z \neq 1, \text{ and } z \neq -1\}$ which is not connected.

When is the inverse limit connected? The answer is provided in Theorem 1.8 whose proof has been omitted for the sake of continuity.

Theorem 1.8 Let X_n be nonempty, compact, connected, and Hausdorff with β_n^{n+1} a continuous map of X_{n+1} into X_n, for each positive integer n. Then $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$ is connected.

Theorem 1.8 has an important role in showing when an inverse limit is a continuum. It turns out that the inverse limit of nonempty metric continua is itself a nonempty metric continua.

Theorem 1.9 If X_n are nonempty metric continua, then $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$ is a nonempty metric continuum.

Proof Follows directly from Theorems 1.6, 1.7, and 1.8. //

The fundamental properties of inverse limits can now be applied to give some special properties specifically needed in Chapter III. The first property that will be revealed is the characterization of subcontinua of a continuum that is an inverse limit. J. T. Rogers, Jr. uses this property in his work presented in Chapter III.

Theorem 1.10 Let $X = \underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$ be a nonempty metric continuum, where the X_n's are nonempty metric continua. Then A is a subcontinuum of X if and only if $A = \underleftarrow{\text{limit}}\{A_n, \alpha_n^m\}$, where A_n is a subcontinuum of X_n and $\alpha_n^{n+1} = \beta_n^{n+1}|A_n$.

<u>Proof</u> Clearly if $A = \underleftarrow{\text{limit}}\{A_n, \alpha_n^m\}$, where the A_n are subcontinua of X_n, then A is a nonempty metric continuum. This follows from Theorem 1.9. Since $\alpha_n^{n+1} = \beta_n^{n+1}|A_n$, then it is easy to see that A $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\} = X$.

Suppose that A is a subcontinuum of X. Consider the image of A under the n^{th} projection map, P_n. The map P_n is continuous and so $P_n(A)$ is a nonempty metric subcontinuum of X_n. Let $A_n = P_n(A)$ for each positive integer n. Assert that $\underleftarrow{\text{Limit}}\{A_n, \beta_n|A_m\}$ is equal to A. Let $(x_1, x_2, \ldots) \in A$. Then by definition of the projection map P_n, $x_n = P_n((x_1, x_2, \ldots)) \in P_n(A) = A_n$, for each positive integer n. Clearly by the nature of the bonding maps $\beta_n^{n+1}|A_{n+1}$, $(x_1, x_2, \ldots) \in \underleftarrow{\text{limit}}\{A_n, \beta_n^m|A_m\}$. Let $(x_1, x_2, \ldots) \in \underleftarrow{\text{limit}}\{A_n, \beta_n^m|A_m\}$. But $A_n = P_n(A)$ and so it is easy to see that $(x_1, x_2, \ldots) \in A$. Therefore, $\underleftarrow{\text{limit}}\{A_n, \beta_n^m|A_m\} = A$ is a subcontinuum of X.

The next property that will be presented is due to Mardesic and Segal (25).

<u>Theorem 1.11</u> The class of circularly chainable continua coincides with the class of $\underleftarrow{\text{limit}}\{X_n, \beta_m^m\}$, where X_n is the unit circle and β_n^{n+1} is a continuous map of X_{n+1} onto X_n, for each positive integer n.

Rogers used this property in his work presented in Chapter III. The proof of Mardesic and Segal's result is out of the realm of this thesis, and so it will be omitted. For a proof see Theorem 1 , page 148 of (25).

J. T. Rogers, Jr. was able to take Mardesic and Segal's result, Theorem 1.11, and refine it to a form that is vital to his work presented in Chapter III. Rogers was able to deduce from Theorem 1.11 the following theorem:

<u>Theorem 1.12</u> A continuum H is circularly chainable with a defining sequence of circular chains $\{C_n\}$ such that C_{n+1} circles k_n times in C_n, for each positive integer n if and only if there exists an inverse limit representation, $\underset{\leftarrow}{\text{limit}}\{X_n, \beta_n^m\}$, for H such that X_n is the unit circle and the degree of β_n^{n+1} is k_n, for each positive n.

I need to define what is meant by C_{i+1} circles k_i times in C_i. This concept is due to Bing (3).

<u>Definition 1.9</u> Let $C_i = b_1, \ldots, b_m$ and $C_{i+1} = a_1, \ldots, a_n$ be circular chains in X such that C_{i+1} is contained in C_i. Let $f'(a_k)$ be the subscript of one of the elements of C_i containing a_k. There may be two possible choices for $f'(a_k)$, but a definite choice is made. Now define a function f having domain $\{0, 1, \ldots, n\}$ as follows:

$$f(0) = f'(a_n)$$

$$f(i+1) = \begin{cases} f(i) - 1, \text{ if } f'(a_{i+1}) \text{ precedes } f'(a_i) \\ f(i), \text{ if } f'(a_{i+1}) = f'(a_i) \\ f(i) + 1, \text{ if } f'(a_{i+1}) \text{ follows } f'(a_i) \end{cases}$$

Then the number of times C_{i+1} circles C_i is defined to be $|f(n)-f(0)|/m$. Intuitively, this is just the winding number.

The final three properties are concerned with the mappings of inverse limits into inverse limits. These properties say that under certain conditions the continuous maps between inverse spectra induce "nice" maps between the inverse limits. One of these properties is fundamental and the other two are not fundamental properties of inverse limits. It will be helpful to define what is meant by a continuous map from an inverse spectrum into another inverse spectrum.

<u>Definition 1.10</u> Let $\{X_n, \beta_n^m\}$ and $\{Y_n, \alpha_n^m\}$ be two inverse spectra. Then the collection of continuous maps $\{h_n\}$, where $h_n: X_n \to Y_n$ for each positive integer n, is said to be a continuous map of $\{X_n, \beta_n^m\}$ into $\{Y_n, \alpha_n^m\}$ if the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[d, "h_1"'] & X_2 \arrow[l, "\beta_1^2"'] \arrow[d, "h_2"'] & X_3 \arrow[l, "\beta_2^3"'] \arrow[d, "h_3"'] & X_4 \arrow[l] \arrow[d, "h_4"'] & \cdots \arrow[l, leftrightarrow] \\
Y_1 & Y_2 \arrow[l, "\alpha_1^2"] & Y_3 \arrow[l, "\alpha_2^3"] & Y_4 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The first property that I will state deals with when the continuous map between inverse spectra induces a continuous map of one inverse limit onto the other. This property is a fundamental property of inverse limits and can be found in Eilenberg and Steenrod (11) or Dugundji (10).

<u>Theorem 1.13</u> Let $\{h_n\} : \{X_n, \beta_n^m\} \to \{Y_n, \alpha_n^m\}$ be a continuous map of inverse spectra. Then there exists a continuous map $h\colon \underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\} \to \underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$ having the property that for each positive integer n, the diagram

$$\begin{array}{ccc} \underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\} & \xrightarrow{h} & \underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\} \\ P_n \downarrow & & \downarrow P_n \\ X_n & \xrightarrow{h_n} & Y_n \end{array}$$

is commutative. Furthermore, if each h_nP_n is onto, then $h(\underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\})$ is dense in $\underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$.

<u>Proof</u> Define h in the obvious way. That is, if $(x_1, x_2, \ldots) \in \underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\}$, then define $h(x_1, x_2, \ldots) = (h_1(x_1), h_2(x_2), \ldots)$. It is easy to show that h is a continuous map of $\underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\}$ into $\underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$.

Let $P_n^{-1}(U)$ be a nonempty basic open set in $\underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$. Since h_nP_n is onto, then $P_n^{-1}h_n^{-1}(U_n) \neq \emptyset$. Then from the commutativity of the diagram, $h^{-1}P_n^{-1}(U_n) = 0$. Therefore, each open set in $\underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$ contains a point of $h(\underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\})$, and so $h(\underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\})$ is dense in $\underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$.

Rogers used a corollary of this last result in Chapter III. When the X_n's and Y_n's are compact, the β_n^m, α_n^m, and h_n onto, then Theorem 1.13 says that the map h is a continuous map of $\underleftarrow{\mathrm{limit}}\{X_n, \beta_n^m\}$ onto $\underleftarrow{\mathrm{limit}}\{Y_n, \alpha_n^m\}$.

Finally, two properties of inverse limits will be presented that state conditions on continuous maps of inverse spectra that guarantee the inverse limits to be homeomorphic. These two properties are due to Mioduszewski (28). They are highly technical and are not in the mainstream of this thesis. Therefore, no attempt will be made to prove these properties.

<u>Theorem 1.14</u> Let $\{X_n, \beta_n^m\}$ and $\{Y_n, \alpha_n^m\}$ be inverse spectra where $X_n = X = Y_n$ are compact metric spaces, α_n^m is onto, and β_m^m is onto. Also let $\{\varepsilon_n\}$ be a positive sequence of numbers tending to zero. Then if the following diagram can be constructed:

$$\begin{tikzcd} X_1 \arrow[d, "e_1"] & X_2 \arrow[l] \arrow[d, "e_2"] & X_3 \arrow[l] \arrow[d, "e_3"] & \cdots \arrow[l] \\ Y_1 & Y_2 \arrow[l] & Y_3 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

where $e_1, e_2, \ldots$ are identities, (1) $\alpha_j^n e_n \beta_n^m =_{\varepsilon_n} \alpha_j^{\prime} e_m$, and (2) $\beta_j^n e_n^{-1} \alpha_n^m =_{\varepsilon_n} \beta_j^m e_m^{-1}$, then $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ is homeomorphic to $\underleftarrow{\text{limit}}\{Y_n, \alpha_n^m\}$.

For Mioduszewski's statement of Theorem 1.14 see Theorem 4, page 43 of (28). The version I stated in Theorem 1.14 is a special case of Mioduszewski's Theorem 4.

<u>Theorem 1.15</u> Let $\{X_n, \beta_n^m\}$ and $\{Y_n, \alpha_n^m\}$ be inverse spectra, where X_n and Y_n are compact metric spaces, β_n^{n+1} is onto, and ${}_n^{n+1}$ is onto, for each positive integer n. If for every pair of positive integers m and n, for every mapping f_{mn}: $X_m \to Y_n$ belonging to a class of maps F, for every $\varepsilon > 0$ and

$m' > m$, there exists $n' > n$ and a mapping $g_{n'm'}\colon Y_{n'} \to X_{m'}$ belonging to a class of maps G such that the diagram

$$\begin{array}{ccc} X_m & \leftarrow & X_{m'} \\ \downarrow & & \uparrow \\ Y_n & \leftarrow & Y_{n'} \end{array}$$

is ε-commutative, and the same true after change X for Y, F for G, etc., then $\underleftarrow{\text{limit}}\{X_n, \beta_n^m\}$ is homeomorphic to $\underleftarrow{\text{limit}}\{Y_n, \alpha_n^m\}$.

For Mioduszewski's statement and proof of Theorem 1.15 see Theorem 4, page 43 of (28).

CHAPTER II

HISTORY

Continuous Images of the Arc

Intuitively I think of the continuous images of the interval [0,1] as being rather "thin" in a geometric sense. I would hazard to guess that most mathematicians have this same feeling. In the last half of the nineteenth century this same feeling may have been conjectured as fact. So it must have been rather startling when G. Peano (35) proved the following theorem published in 1890:

Theorem 2.1 The square plus its interior is the continuous image of the interval [0,1].

D. Hilbert (18) and E. H. Moore (30) also proved this theorem in separate articles published in 1891 and 1900, respectively. Theorem 2.1 says that the continuous image of the interval can be rather "fat" in a geometric sense. Since this goes against intuition it would be interesting at this time to see a sketch of the proof of this theorem.

Let S be the unit square plus its interior and for n = 1, 2, ... let S_n be the subdivision of S into 4^n equal squares. Order the 4^{n+1} squares of S_{n+1}, $S_{n+1}^1 < S_{n+1}^2 < \ldots < S_{n+1}^{4n+1}$, such that (i) if two squares are adjacent in the order, then they are adjacent geometrically, and (ii) if two squares S^j_{n+1} and S^{j+1}_{n+1} are such that $S^j_{n+1} \subseteq S^k_n$ and $S^{j+1}_{n+1} \subseteq S^m_n$, then either $m = k$ or $m = k+1$. Now let f_n be the continuous map that uniformly stretches [0,1] through the centers of the squares of S_n "in order". For example f_1, f_2, and f_3 are illustrated in Figure 3. Finally, define a map f from [0,1] into S by letting the image of $0 \leq x \leq 1$ be the point-wise limit of $f_n(x)$ as n gets large. Then f is the desired continuous map of [0,1] onto S.

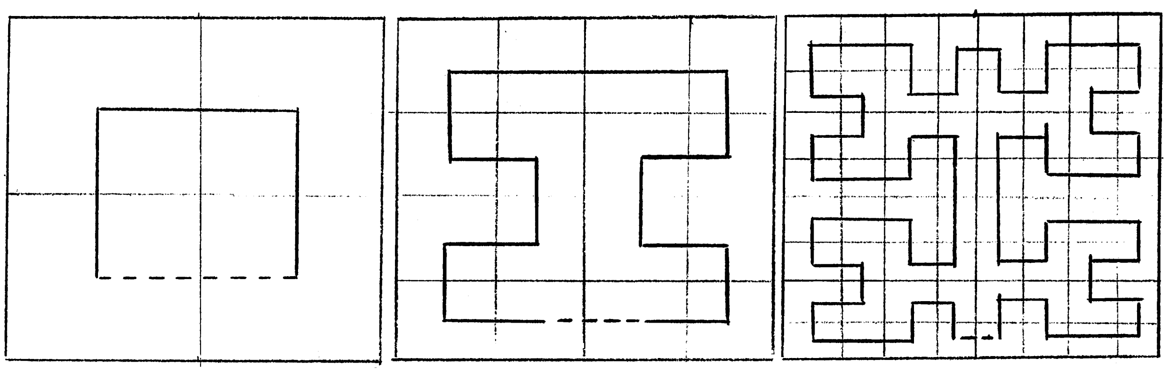


Figure 3. Illustration of f_1, f_2, and f_3

After Peano's discovery, the question, "What characterizes the continuous images of the arc?" became one of considerable interest. Peano's discovery had said that the notion of "thinness" was not the property to characterize the continuous images of the arc. Other conditions would have to be found. R. L. Moore (34) claims that some eighteen years later Schoenflies defined property P_1 as follows:

Definition 2.1 A continuum M R^2 is said to have property P_1 if and only if (i) for each $\varepsilon > 0$ there are not more than a finite number of components of R^2-M of diameter greater than ε, (ii) the boundary of each component of R^2-M is accessible at each of its points, and (iii) C is accessible at each point of $\partial C \cap \partial D$, where D is a component of R^2-M, C is a component of $D-xy$, x, $y \varepsilon \partial D$, and $xy \subseteq D$.

Using property P_1 Schoenflies was able to characterize the continuous images of the arc.

Theorem 2.2 A continuum $M \subseteq R^2$ is a continuous image of the arc if and only if M has property P_1.

It is easy to see that the square plus its interior is a plane continuum that satisfies property P_1. A plane continuum W that does not satisfy property P_1 is the Warsaw circle plus its interior pictured in Figure 4. The Warsaw circle W does satisfy conditions (i) and (ii) of the definition of property P_1, but fails to satisfy condition (iii). To see that W doesn't satisfy condition (iii) let $D = R^2-W$, $x = (0,1)$, $y = (\frac{1}{\pi},0)$, xy be the segment indicated in Figure 5 containing $(1/\pi,2)$, and C the bounded component of $D-xy$ shaded in Figure 5. Note that the origin is a point of $\partial C \cap \partial D$. But the origin is not accessible to any point of C.

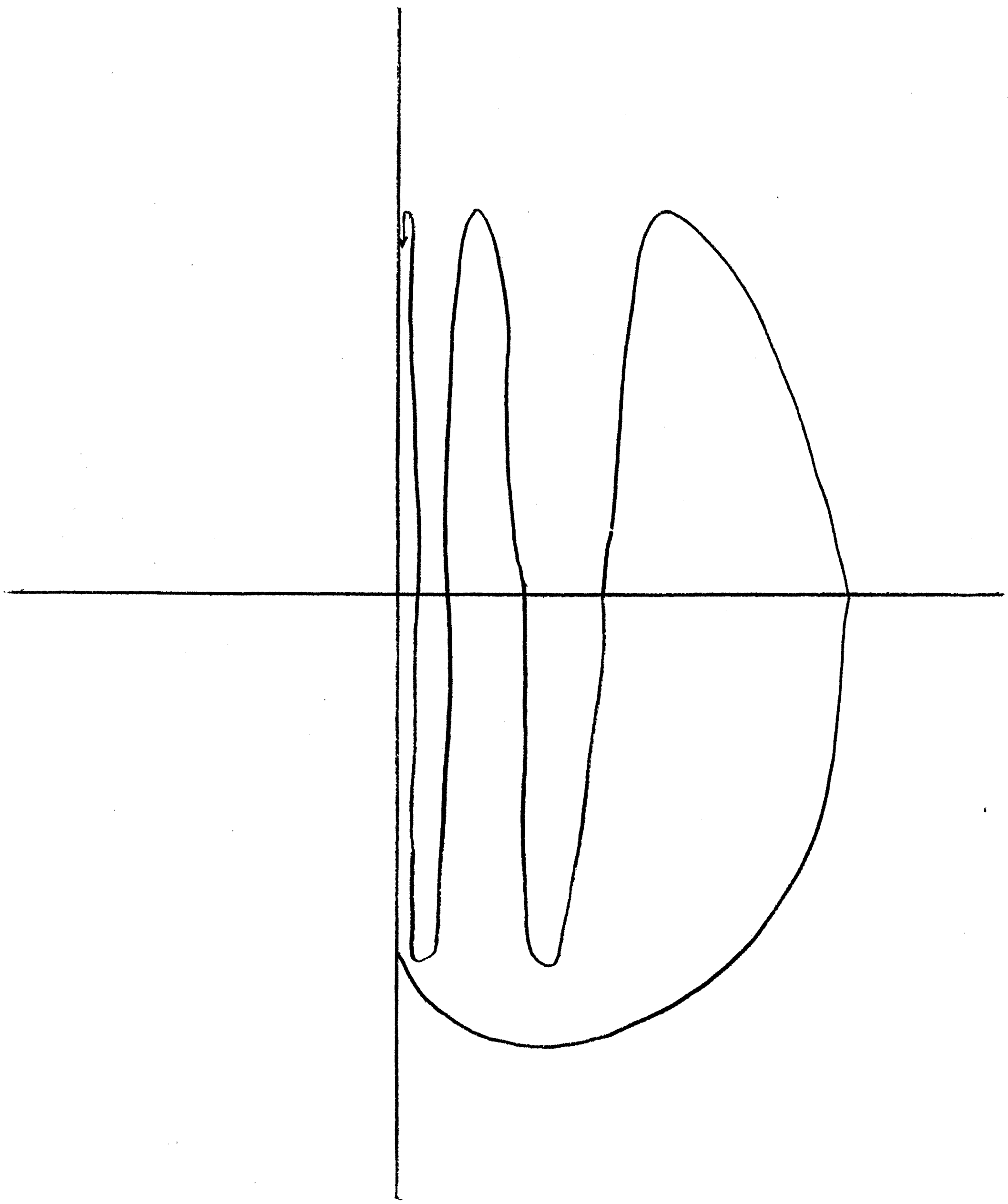


Figure 4. The Warsaw Circle

Hence Schoenflies' characterization says that W is not the continuous image of the arc.

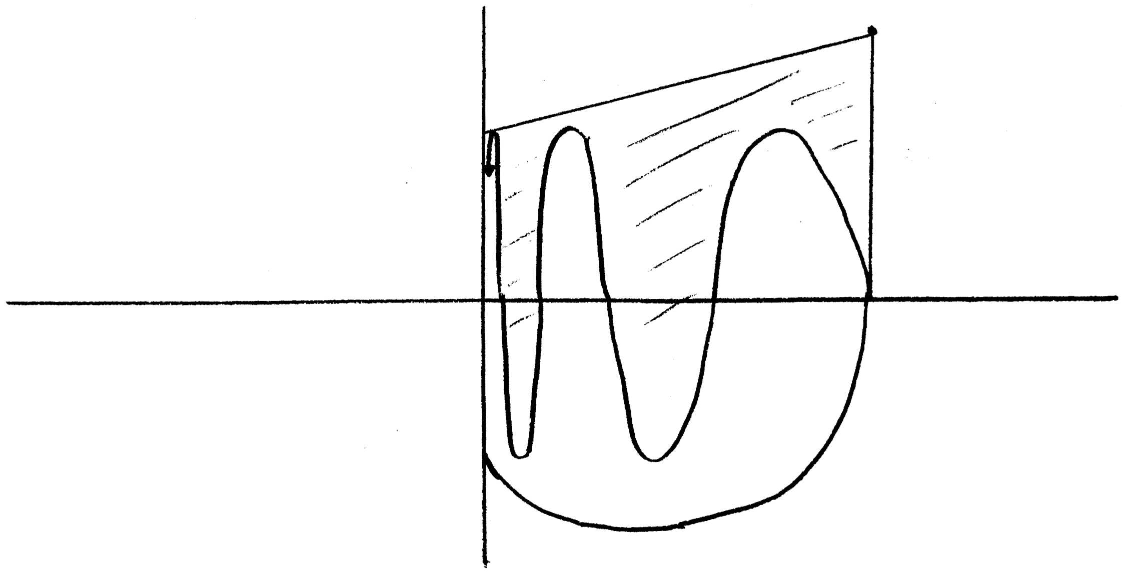


Figure 5. The Continuum C

Schoenflies' characterization is true only for images in R^2. However, in 1913 and 1914 S. Mazurkiewicz (26) and Hans Hahn (17) independently published articles, respectively, containing P_2.

<u>Definition 2.2</u> A continuum M has property P_2 if and only if for each point $x \varepsilon M$ and every open set D with respect to M such that $x \varepsilon D$ there is an open set D' with respect to M such that $x \varepsilon D'$ and D' is contained in a component of D.

The property P_2 allowed Hahn and Mazurkiewicz to characterize the continuous images of the arc in R^n, for n = 1, 2, ...

<u>Theorem 2.3</u> A continuum $M \subseteq R^n$, for n = 1, 2, ... , is the continuous image of the arc if and only if M has property P_2.

Again it is easy to see that the square plus its interior has property P_2. We also know that W must not have property P_2, since it is not the continuous image of the arc. To see this let $x = (0,0)$ and $D = \{y \varepsilon R^2 \mid d(x,y) < 1/2\}$ W. Then D is an open subset of W as illustrated in Figure 6. The component of D that contains x is the points on the vertical axis between 1/2 and -1/2. But this component contains no open subset of W.

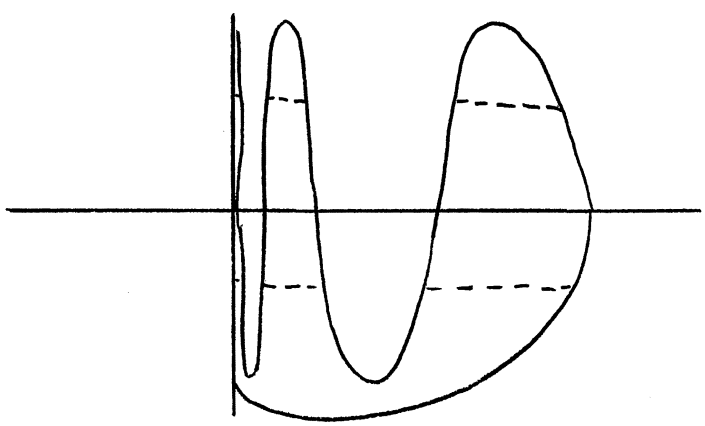


Figure 6. The Component D

In 1920, some six years after the Hahn-Mazurkiewicz characterization, W. Sierpinski (41) defined property P_3.

<u>Definition 2.3</u> A continuum M is said to have property P_3 if and only if for each $\varepsilon > 0$, M can be written as the finite union of subcontinua of diameter less than ε.

Property P_3 enabled Sierpinski to characterize the continuous images of the arc in R^n, for $n = 1, 2, \ldots$.

Theorem 2.4 A continuum $M \subseteq R^n$, for $n = 1, 2, \ldots$, is the continuous image of the arc if and only if M has property P_3.

Once more, let's go back to our previous two examples, the square plus its interior and W. Clearly the square plus its interior has property P_3, but it is slightly harder to see that W has property P_3. Let $\{ x_n \}$ be the sequence of "peak" points and $\varepsilon < 1$. Then, if W could be written as the union of subcontinua, W_j, $j = 1, \ldots, k$, with diameters all less than ε, then infinitely many of the x_n's must lie in one subcontinuum, W_j. Consider one $x_n \varepsilon W_j$. Note that since the diameter of W_j is less than ε and W_j is connected then W_j must be contained in the shaded area of Figure 7.

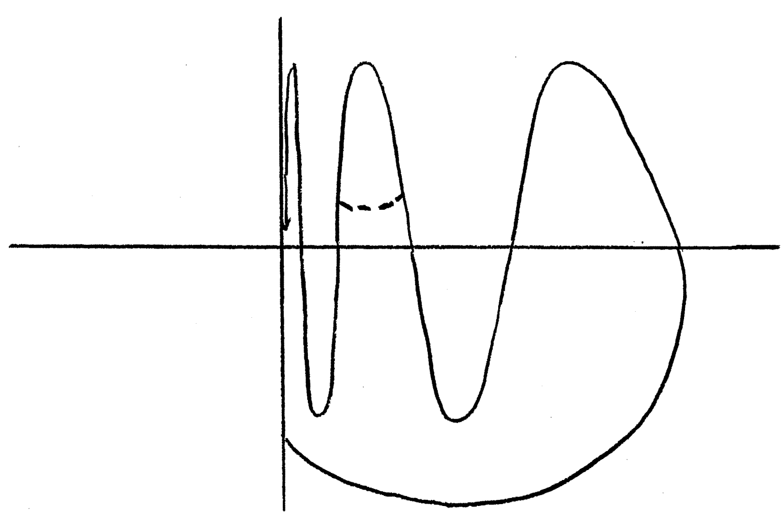


Figure 7. The Set That Contains W_j

But this is a contradiction since W_j contains infinitely many of the x_n's.

In 1922, R. L. Moore (31), S. Mazurkiewicz (26), and Tietze (42) all published articles, respectively, that contained a property P_4.

Definition 2.4 A continuum M is said to have property P_4 if and only if every component of an open subset of M is arcwise connected.

R. L. Moore (34) mentioned that his student, R. L. Wilder, in his thesis, used the property P_4 to characterize the continuous images of the arc in R^n, for n = 1, 2, ...

Theorem 2.5 A continuum $M \subsetneq R^n$, for n = 1, 2, ... , is the continuous image of the arc if and only if M has property P_4.

Again the square plus its interior can be easily seen to satisfy property P_4. W-{(0,0)} is an open subset of W and has only one component, but the points (0,1) and (0,1) can't be connected by an arc contained in W-{(0,0)}.

The Pseudo-Arc and the Pseudo-Circle

In the first years of publication, Fundamenta Mathematicae had a section consisting of unsolved problems that mathematicians of the time posed. These problems were numbered and usually placed at the end of the publication. One such section of problems is on page 285 of Fundamenta Mathematicae, Volume 2, 1921. Of interest to me in this section are Problems 14 and 15. Problem 14, "Does there exist a hereditarily indecomposable continuum other than the point?" was posed by Knaster and Kuratowski. Problem 15, "Is every plane continuum which is homeomorphic to each of its nondegenerate subcontinua an arc?" was posed by S. Mazurkiewicz.

Problem 14 was solved at about the same time that the question appeared in print. Bronislaw Knaster (21) in his Ph.D. thesis presented to the University of Varsovie in December of 1921, was able to prove the following theorem:

Theorem 2.6 There exists a nontrivial hereditarily indecomposable continuum.

Problem 15 remained unsolved for 26 years after the question appeared in print. It was not until E. E. Moise, in his Ph.D. thesis, written under the direction of R. L. Moore at the University of Texas in 1947, that Problem 15 was solved. Moise in his thesis described a class of continua, all homeomorphic, that he called pseudo-arcs. He was able to prove the following theorem:

Theorem 2.7 There exists a hereditarily indecomposable continuum that is homeomorphic to each of its nondegenerate subcontinua.

Also Moise conjectured that the continuum Knaster described was a pseudo-arc.

Shortly after Moise obtained his results concerning the pseudo-arc, R. H. Bing (1) used the pseudo-arc to answer a problem posed in 1920. This was Problem 2, "If a nondegenerate plane continuum is homogeneous, is it necessarily a simple closed curve?" of Fundamenta Mathematicae, Volume 1, posed by Knaster and Kuratowski. Bing's (1) answer to Problem 2 was negative, and the pseudo-arc was his example.

Theorem 2.8 There exists a nondegenerate homogeneous plane continuum that is not a simple closed curve.

An interesting sidelight to Problem 2 is the many wrong answers that were given by various mathematicians. Waraszkiewicz (43) and Choquet (9), in separate articles, both thought they had answered Problem 2 to the affirmative. In fact, Choquet even went further astray. He claimed that the following false theorem was true:

False-Theorem 2.1 Every homogeneous bounded closed plane set is of one of the following types:

(i) a finite number of points,

(ii) a totally disconnected perfect set,

(iii) and (iv) a set homeomorphic to the union of a collection of concentric circles such that the common part of this union and a line through the center of the circles is either a finite set or a totally disconnected perfect set.

In 1951, Bing (16) showed the following to be true:

<u>Theorem 2.9</u> Any two hereditarily indecomposable chainable continua are homeomorphic.

Therefore Moise's pseudo-arc and Knaster's hereditarily indecomposable chainable continuum were alike as Moise had conjectured. Also in the same article Bing described a circularly chainable hereditarily indecomposable plane continuum that is topologically different from the pseudo-arc. Bing was later to give this continuum the name pseudo-circle. He conjectured that two pseudo-circles are homeomorphic, and that the pseudo-circle is homogeneous.

In 1968 J. T. Rogers, Jr., in his Ph.D. thesis written under the direction of F. B. Jones, extended the definition of pseudo-circle. Rogers defined a pseudo-circle to be any hereditarily indecomposable circularly chainable continuum that is not chainable. Being had defined the pseudo-circle only in the plane. In 1970 Rogers (37) was able to show the following to be true:

<u>Theorem 2.10</u> Any two planar pseudo-circles are homeomorphic.

In 1970 Lawerence Fearnley (12) also proved Theorem 2.10. Although both men arrived at the same conclusion, both did so through different means. Rogers used inverse limits in his work, while Fearnley used circular chain properties in his work.

Continuous Images of the Pseudo-Arc

It has been shown in the previous section that in the late 40's and the early 50's the pseudo-arc was used to solve several long unsolved problems. So it is not hard to believe that one of the topics discussed at the Summer Institute on Set Theoretic Topology held in 1955 at Madison, Wisconsin, was the pseudo-arc. One of the mathematicians that gave special attention to the pseudo-arc was R. H. Bing. In one of his talks about the pseudo-arc Bing raised the question: "What characterizes the continuous images of the pseudo-arc?"

Bing's question was answered by two men, A. Lelek (24) and L. Fearnley (13), at about the same time. Fearnley may have arrived at the answer to Bing's question as early as 1959. It appears that Lelek came up with his answer in 1960 or 1961. Fearnley first presented his answer before the American Mathematical Society on January 29, 1960, but his answer wasn't published until 1964 when his article appeared in the _Transactions of the American Mathematical Society._ Lelek's answer appeared in an article published in 1962.

Both Fearnley's and Lelek's characterizations depend on a generalization of chainability. Fearnley used the notion of p-chainability, and Lelek the notion of weak chainability.

Definition 2.5 A p-chain is a finite sequence of sets in X, $p_1, \ldots , p_n$, each of which, except the last, intersects its successor in the sequence. A p-chain $p_1, \ldots , p_n$ is said to be a normal refinement of the p-chain $q_1, \ldots, q_m$ provided $p_i \quad q_{x_i}$, $i = 1, \ldots, n$, $x_i = 1$, $x_n = m$, and if $|i-j| \leq 1$, then $|x_i - x_j| \leq 1$. Finally, a continuum H is said to be p-chainable if there exists a sequence of p-chains, $P_1, P_2, \ldots$, such that for each positive integer i, (a) the union of elements of P_i is H, (b) P_{i+1} is a normal refinement of P_i, and (c) the diameter of each element of P_i is less than $1/i$.

<u>Definition 2.6</u> A finite sequence of sets in X, $p_1, \ldots, p_n$ is said to be a weak chain provided that if $|i-j| \leq 1$, then $p_i \cap p_j \neq \emptyset$. A weak chain $p_1, \ldots, p_n$ is said to be a refinement of a weak chain $q_1, \ldots, q_m$ provided that $p_i \subsetneq q_{xi}$, for each i, and if $|i-j| \leq 1$, then $|x_i - x_j| \leq 1$. Finally, a continuum H is said to be weakly chainable provided there exists a sequence of open covers $P_1, P_2, \ldots$, where each P_i is a weak chain, P_{i+1} is a refinement of P_i, and the diameter of each member of P_i is less than $1/i$.

The sets $q_1, \ldots, q_6$ in Figure 8 form both a p-chain and a weak chain, but not a chain. However, the sets $p_1, \ldots, p_5$ in Figure 9 form a p-chain, a weak chain, and a chain. It is clear from both Definition 2.5 and Definition 2.6 that the p-chain and the weak chain are generalized chains. Fearnley's notion of normal refinement of p-chains and Lelek's notion of refinement of weak chains need illustration to clarify. In Figure 10 the sets $p_1, \ldots, p_7$ are a weak chain and a p-chain, but they are not a normal

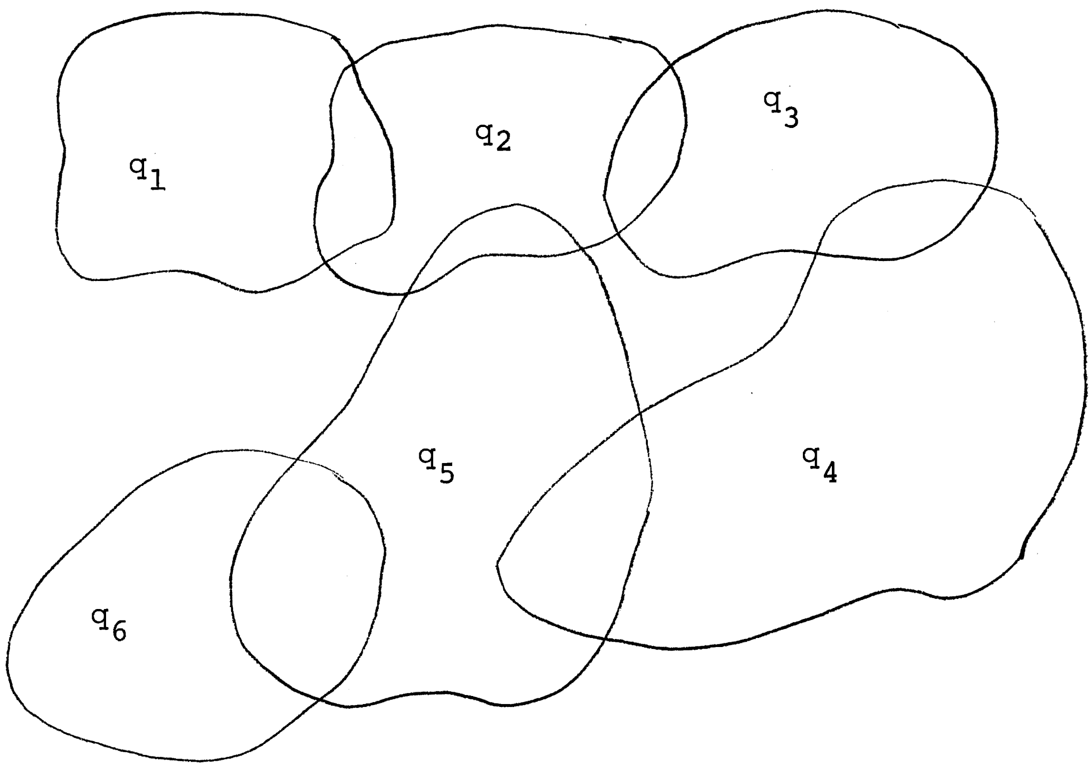


Figure 8. The P-Chain Q

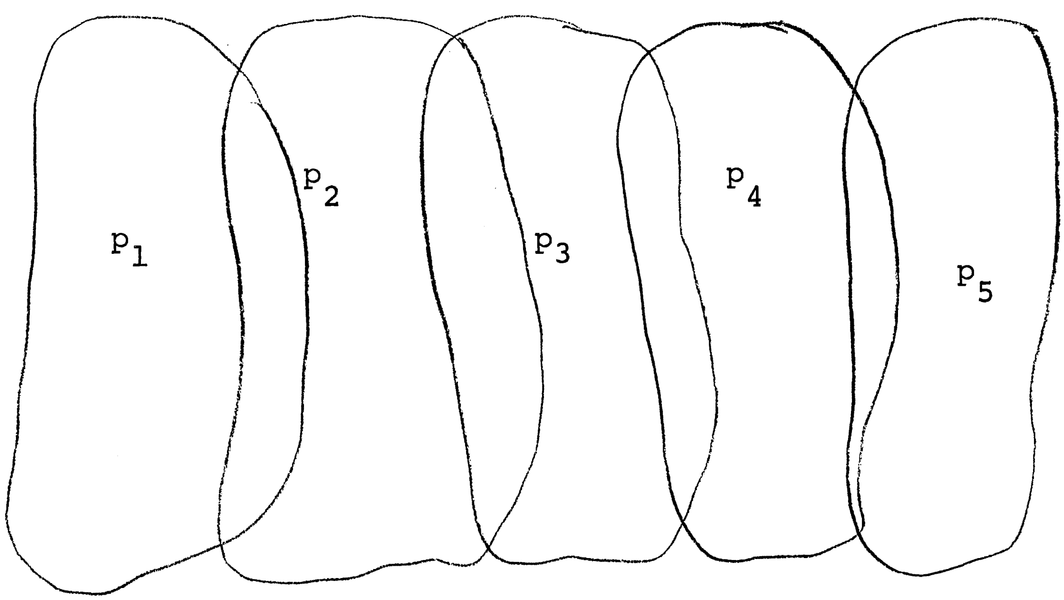


Figure 9. The Chain and P-Chain P

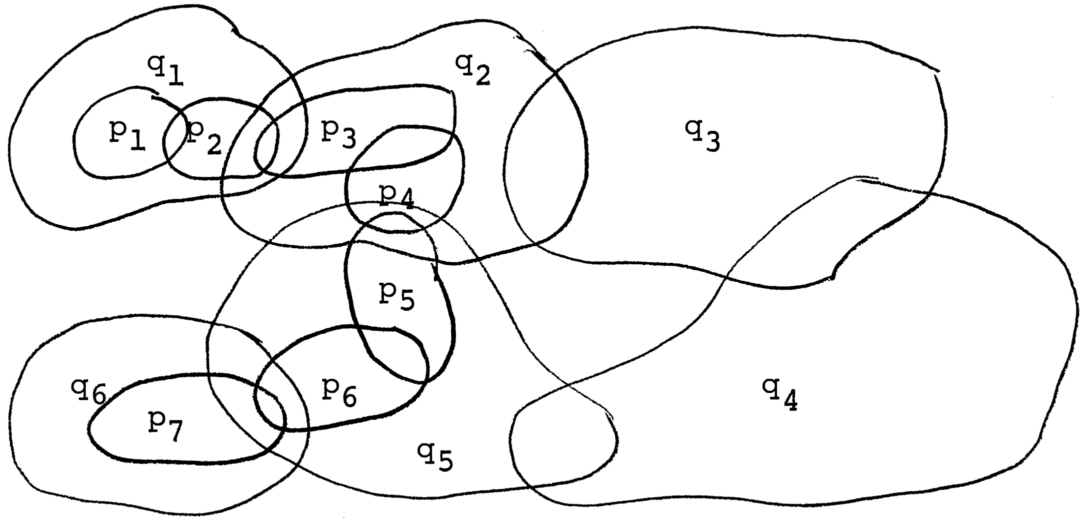


Figure 10. P Not a Normal Refinement of Q

refinement of the p-chain, $q_1, \ldots, q_6$, or a refinement of the weak chain, $q_1, \ldots, q_6$. To see that note that $p_4 \subseteq q_2$ and $p_5 \subseteq q_5$ and $|4-5| \leq 1$ but $|5-2| \leq 1$. However, in Figure 11 the p-chain and the weak chain, $p_1, \ldots, p_{11}$, is a normal refinement and a refinement of the p-chain and weak chain, $q_1, \ldots,$ q_6.

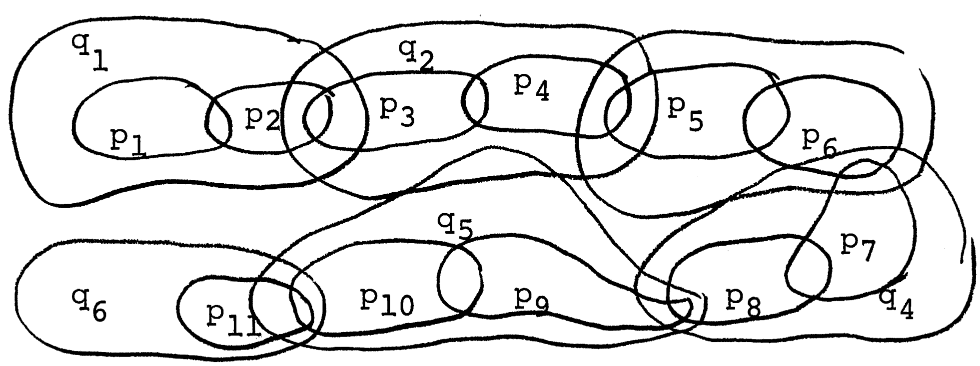


Figure 11. P a Normal Refinement of Q

Fearnley's characterization and Lelek's characterization of the continuous images of the pseudo-arc are as follows:

Theorem 2.11 A continuum H is the continuous image of the pseudo-arc if and only if it is p-chainable.

Theorem 2.12 A continuum H is the continuous image of the pseudo-arc if and only if it is weakly chainable.

From these two characterizations it is evident that the notions of p-chainability and weak chainability are equivalent. Also since p-chainability and weak chainability are generalizations of chainability, then each chainable continuum is the continuous image of the pseudo-arc. This is an important corollary to Fearnley's and Lelek's characterizations and is stated below.

Theorem 2.13 Every chainable continuum is the continuous image of the pseudo-arc.

From a historical point of view, it is interesting to note that Theorem 2.13 was also proved by J. Mioduszewski (27) used inverse limits in his proof of Theorem 2.13.

Continuous Images of Pseudo-Circles

Because of the similarities between the pseudo-arc and the planar pseudo-circle it was quite natural that, after the continuous images of the pseudo-arc had been characterized, mathematicians would find it interesting to investigate the continuous images of the pseudo-circles. Fearnley (12) mentions that the question, "What characterizes the continuous images of the pseudo-circles?" had been asked previously in the literature. However, I am unable to find any reference in the literature formally stating this question.

As mentioned in a previous section of this chapter, "the" pseudo-circle that Fearnley (12) worked with is just one in the class of pseudo-circles defined by Rogers (37), and can be described as the planar pseudo-circle. Fearnley was the first mathematician to solve a problem involving the continuous images of pseudo-circles when in (12) he was able to characterize the continuous images of the planar pseudo-circle. He characterized these images using the property of circular p-chainability.

Definition 2.7 A circular p-chain is a p-chain in which the first and last links intersect. A circular p-chain $p_0, \ldots, p_n$ is said to be a refinement of a circular p-chain $q_o, \ldots, q_m$ if

(a) $\overline{p}_i \subseteq q_{x_i}$ for some $0 \le x_i < m$ and each $0 \le i \le n$,

(b) if $|i-j| \pmod n \le 1$, $0 \le i, j \le n$, then $|x_i - x_j| \pmod m \le 1$, and

(c) $|$number of times $p_i \subseteq q_m$ and $p_{i+1} \subseteq q_0$ minus the number of times $p_i \subseteq q_0$ and $p_{i+1} \subseteq q_m$ for $i = 1, \ldots, n-1| = 1$.

Finally, a continuum H is said to be circularly p-chainable if there exists a sequence of circular p-chains $P_1, P_2, \ldots$ such that for each $i = 1, 2, \ldots$

(a) the union of links of P_i is H,

(b) P_{i+1} is a refinement of P_i, and

(c) the diameter of each link of P_i is less than $1/i$.

<u>Theorem 2.14</u> A continuum H is the continuous image of the planar pseudo-circle if and only if it is circulary p-chainable.

Some three years after Fearnley characterized the continuous images of the planar pseudo-circle, J. T. Rogers, Jr. (37) extended the definition of pseudo-circle and obtained a generalization of Fearnley's result, Theorem 2.14. Rogers' work was done under the direction of F. B. Jones, and relies heavily on the techniques used by J. Mioduszewski (27). He was able to characterize the continuous images of all pseudo-circles using the property of q-chainability.

<u>Definition 2.8</u> A q-chain is a finite sequence of sets in X, $q_0, \ldots, q_n$, such that $q_i \cap q_j \neq \emptyset$ whenever $|i-j| \pmod{n+1} \leq 1$. A q-chain, $p_0, \ldots, p_n$, is said to be a refinement of a q-chain, $q_0, \ldots, q_m$, if

(a) $\overline{p}_i \subseteq q_{x_i}$ for some $0 \leq x_i \leq m$ and each $0 \leq i \leq n$, and

(b) if $|i-j| \pmod{n+1} \leq 1$, $0 \leq i, j \leq n$, then $|x_i - x_j| \pmod{m+1} \leq 1$.

Finally, the continuum H is said to be q-chainable if there exists a sequence of q-chains Q_1, Q_2, ... such that for each $i = 1, 2, \ldots$

(a) Q_i covers H,

(b) Q_{i+1} is a refinement of Q, and

(c) each link of Q_i has diameter less than $1/i$.

Theorem 2.15 A continuum H is the continuous image of some pseudo-circle if and only if H is q-chainable.

Rogers (37) was able to characterize the continuous images of certain classes of pseudo-circles. The second section of Chapter III will give a more detailed account of these characterizations.

The last thing that will be done in this section is to make Definitions 2.7 and 2.8 clearer. In Figure 12, q_0, ..., q_5 and p_0, ..., p_9 are both circular p-chains and q-chains.

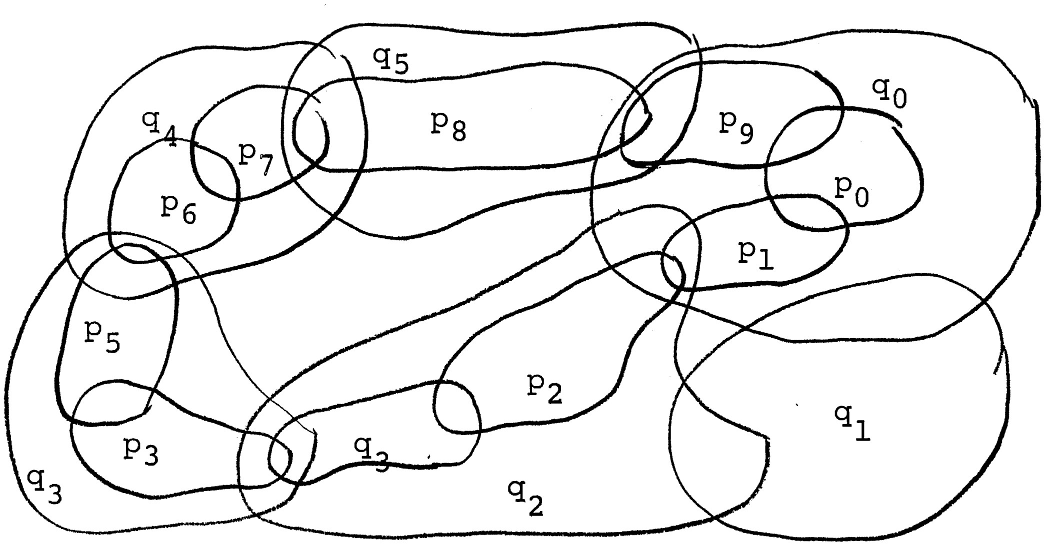


Figure 12. P Not a Refinement of Q

But $p_0, \ldots, p_9$ is not a refinement of $q_0, \ldots, q_5$ as a circular p-chain or as a q-chain. To see this, note that $|1-2| \leq 1$ but $|0-2| \leq 1$. In Figure 13, $q_0, \ldots, q_5$ and $p_0, \ldots, p_{12}$ are both q-chains and circular p-chains. Also, $p_0, \ldots, p_{12}$ is a refinement of $q_0, \ldots, q_5$ as a q-chain, but $p_0, \ldots, p_{12}$ is not a refinement of $q_0, \ldots, q_5$ as a circular p-chain. To see this, note that $p_0, \ldots, p_{12}$ winds around $q_0, \ldots, q_5$ twice, and hence violates part (c) of the definition of refinement of circular p-chains. In Figure 14, $q_0, \ldots, q_5$ and $p_0, \ldots, p_6$ are both q-chains and circular p-chains. Also, $p_0, \ldots, p_6$ is a refinement of $q_0, \ldots, q_5$, both as a q-chain and as a circular p-chain.

Continuous Images of Indeomposable Continua

J. W. Rogers, Jr., at the Auburn Topology Conference of 1969, raised the following questions: "Which continua are the continuous images of indecomposable continua?" and "Is there an indecomposable continuum of which every indecomposable continua is a continuous image?". The previous two sections have partially answered the first of Rogers' questions. In fact, the results of the last two sections are partially responsible for Rogers raising these two questions. The pseudo-arc and the pseudo-circles are all indecomposable continua and their continuous images are p-chainable continua and q-chainable continua, respectively. Hence the collection

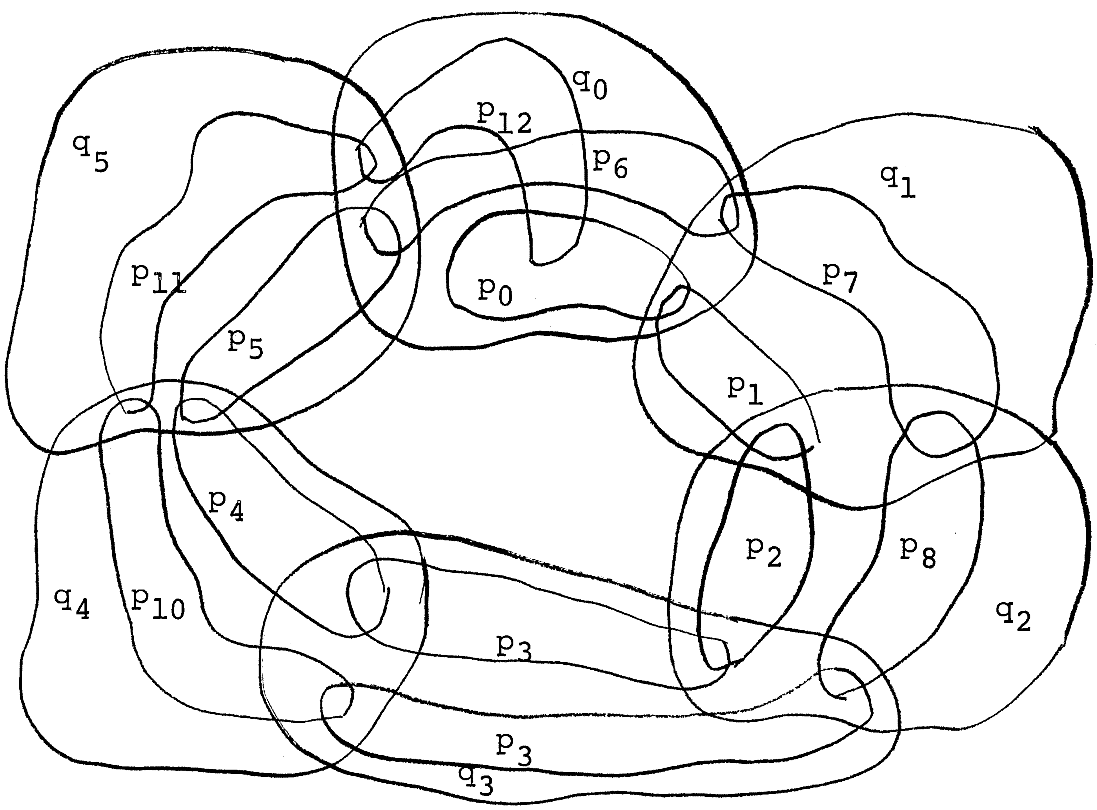


Figure 13. P Not a Refinement of Q

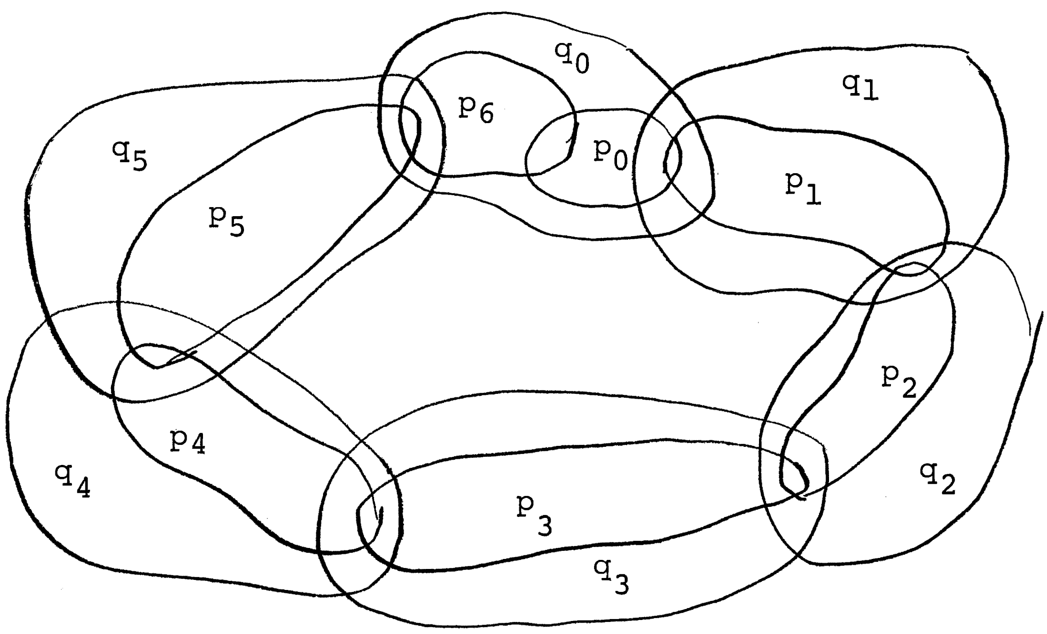


Figure 14. P a Refinement of Q

of continuous images of indecomposable continua must at least contain q-chainable continua and p-chainable continua.

In 1971 David Bellamy (8) was able to answer both of Rogers' questions. Bellamy was able to prove the following two theorems.

<u>Theorem 2.16</u> If H is a continuum, then there exists as indecomposable continuum M that maps continuously onto H.

<u>Theorem 2.17</u> There is no indecomposable continuum that maps continuously onto every indecomposable continua.

It would be interesting and informative to give a sketch of the proof of Theorem 2.16. First Bellamy proves that if H is a compact metric continuum, then there exists a metric continuum M such that M is irreducible between two points, and a continuous map of M onto H. I will not go into the proof of this first fact, but it can be found in Bellamy (8). Next, Bellamy proves that every continuum irreducible between two points is the continuous image of some indecomposable continuum. The proof of this fact is provided in the next paragraph.

Let M be a continuum irreducible between two points a and b, and D the well-known indecomposable continuum shown in Figure 15. Also let $D_a = \{(x,y) \varepsilon D \mid x \leq 2/5\}$, $D_b = \{(x,y) \varepsilon D \mid x \geq 3/5\}$, and $A = \{y \mid (3/5,y) \varepsilon D_b\}$. Note that $A = \{y \mid (2/5,y) \varepsilon D_a\}$. Now form the disjoint union of D_a, D_b, and $M \times A$ by identifying $(2/5,y) \varepsilon D_a$ with $(a,y) \varepsilon M \times A$ and identifying $(3/5,y) \varepsilon D_b$ with $(b,y) \varepsilon M \times A$, for each $y \varepsilon A$.

Intuitively what has been done is to remove from D a copy of a closed interval crossed with A and replace it with a copy of M crossed with A. This intuitive notion is illustrated in Figure 16. It turns out that this disjoint union, D_M, is an indecomposable continuum and that it can be mapped continuously onto M by the following map:

$g(p) = a$ for $p \varepsilon D_a$,

$g(p) = b$ for $p \varepsilon D_b$, and

$g(m,y) = m$ for $(m,y) \varepsilon M \times A$.

The proof of Theorem 2.17 follows from Theorem 2.16 and a result of Z. Waraszkiewicz (44) that says that no continuum maps continuously onto every continua.

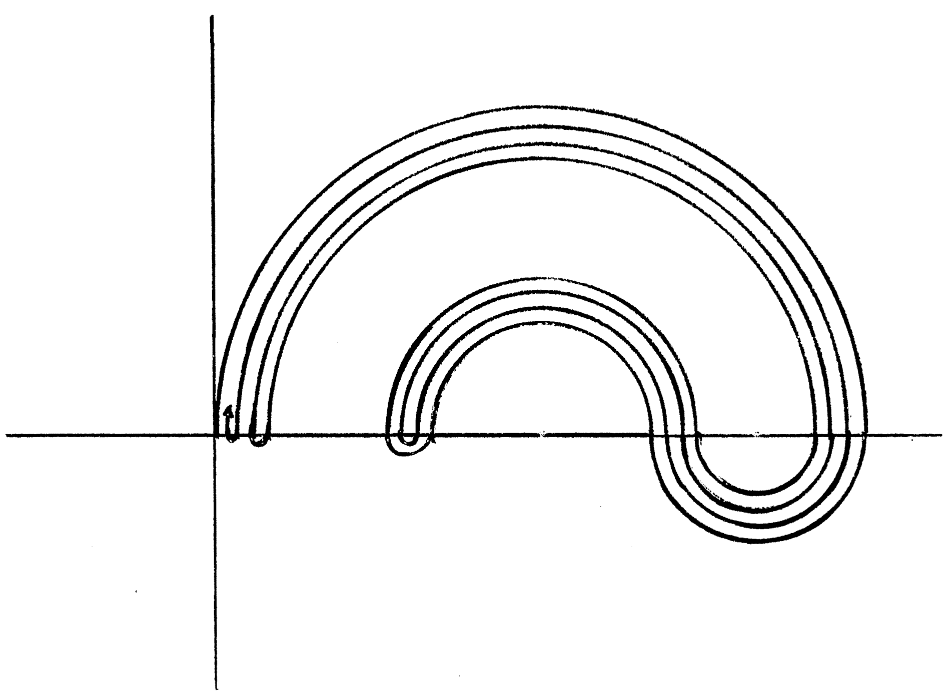


Figure 15. The Continuum D

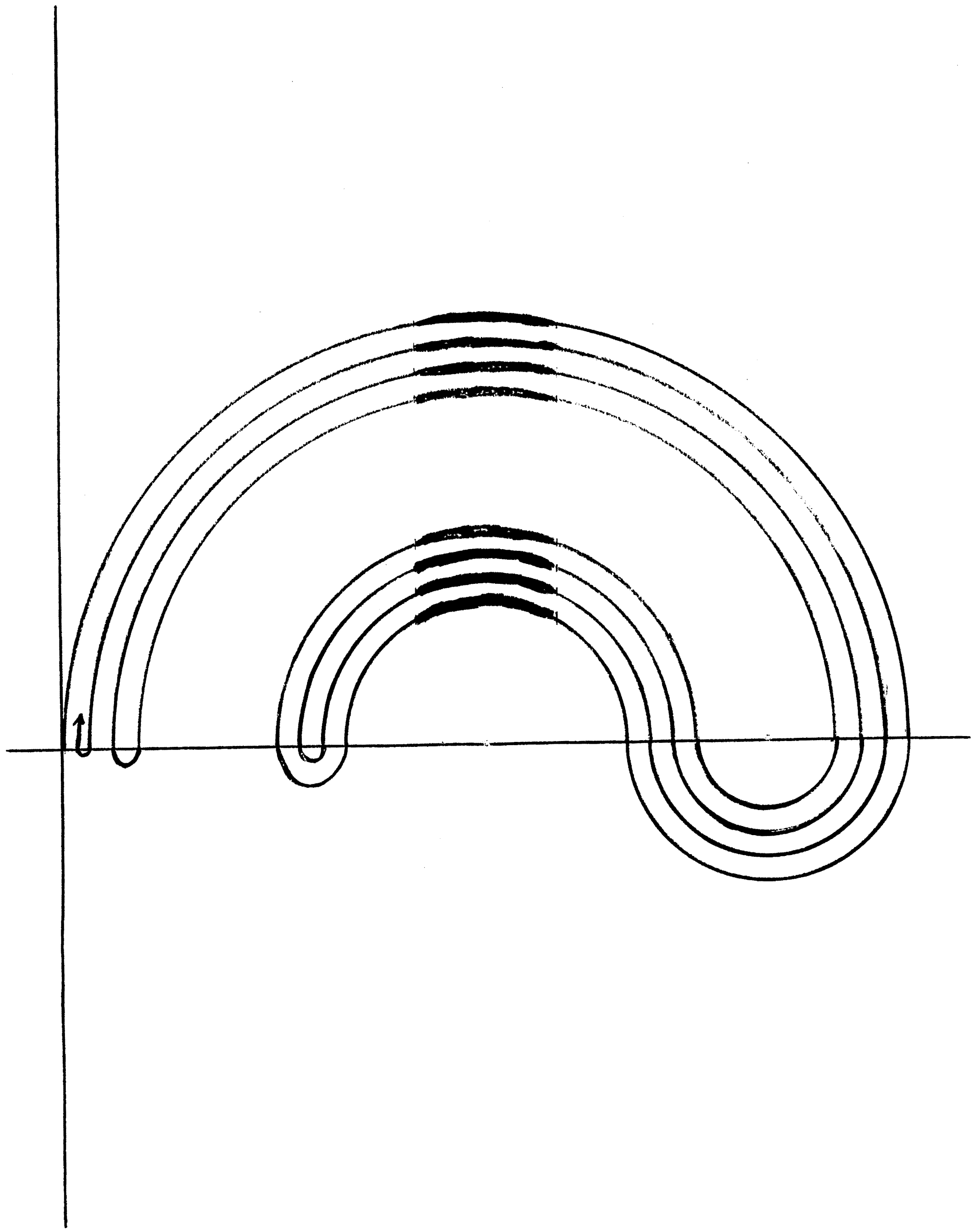


Figure 16. The Continuum D_M

CHAPTER III

CHARACTERIZATIONS

Continuous Images of the Pseudo-Arc

Remembering that in the late 50's and early 60's Fearnely proved Theorems 2.11 and 2.13 concerning the continuous images of the pseudo-arc is a good beginning for this section. His proofs will be exposed in detail for easy reading. Before this can be done an expansion of the idea of a p-chain is needed. This expansion is obtained by introducing quite a long list of related definitions. This collection of definitions appears to be a cold exposition of new facts. However, a compensation is that these definitions break the proofs of Theorems 2.11 and 2.13 into a collection of small digestible bites.

<u>Definition 3.1</u> If $p_1, \ldots, p_n$ is a p-chain, then the p-chain $p_n, \ldots, p_1$ is said to be the conjugate of the p-chain $p_1, \ldots, p_n$.

<u>Definition 3.2</u> If $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ are p-chains such that $p_i \subseteq q_{x_i}$ for each $1 \leq i \leq n$ and some $1 \leq x_i \leq m$, then the sequence of ordered pairs $(1,x_1), \ldots, (n,x_n)$ is said to be a pattern of $p_1, \ldots, p_n$, in $q_1, \ldots, q_m$. If, in this

pattern, $|x_i - x_j| \leq 1$ whenever $|i-j| \leq 1$, $1 \leq i, j \leq n$, then the pattern is said to be an r-pattern.

Definition 3.3 A p-chain, p-chain, $p_1, \ldots, p_n$, is said to be a refinement of a p-chain, $q_1, \ldots, q_m$, if there is an r-pattern of $p_1, \ldots, p_n$ in $q_1, \ldots, q_m$.

Definition 3.4 If a p-chain, $p_1, \ldots, p_n$, has an r-pattern of the form $(1, x_1=1)$, $(2, x_2)$, ..., $(n, x_n=m)$ in the p-chain, $q_1, \ldots, q_m$, then $p_1, \ldots, p_n$ is said to be a normal refinement of $q_1, \ldots, q_m$.

Definition 3.5 If a p-chain, $p_1, \ldots, p_n$, has an r-pattern $(1, x_1)$, ..., (n, x_n) in a p-chain, $q_1, \ldots, q_m$, such that for each $1 \leq i \leq n$, $p_i = q_{x_i}$ and for each $1 \leq j \leq m$, $j = x_i$ for some $1 \leq i \leq n$, then $p_1, \ldots, p_n$, is said to be a principal refinement of $q_1, \ldots, q_m$.

It will be convenient throughout the remainder of this chapter to adopt some notational shorthand.

Notation P will be the p-chain $p_1, \ldots, p_n$, Q will be the p-chain $q_1, \ldots, q_m$, and R will be the p-chain $r_1, \ldots,$ and r_k. Also $P(i,j)$ will be the sub-p-chain of P, $p_i, \ldots, p_j$. By $P(j,k)$ it is meant the sum of the p-chains $P(i,j)$ and $P(j,k)$ which yields the p-chain $P(i,k)$.

Looking at Figure 17, it can be seen that the p-chain $p_1, \ldots, p_7$ is not a refinement of the p-chain $q_1, \ldots, q_6$ because the pattern of P in Q, (1,1), (2,1), (3,2), (4,2), (5,5), (6,5), and (7,6) is not an r-pattern. However, in

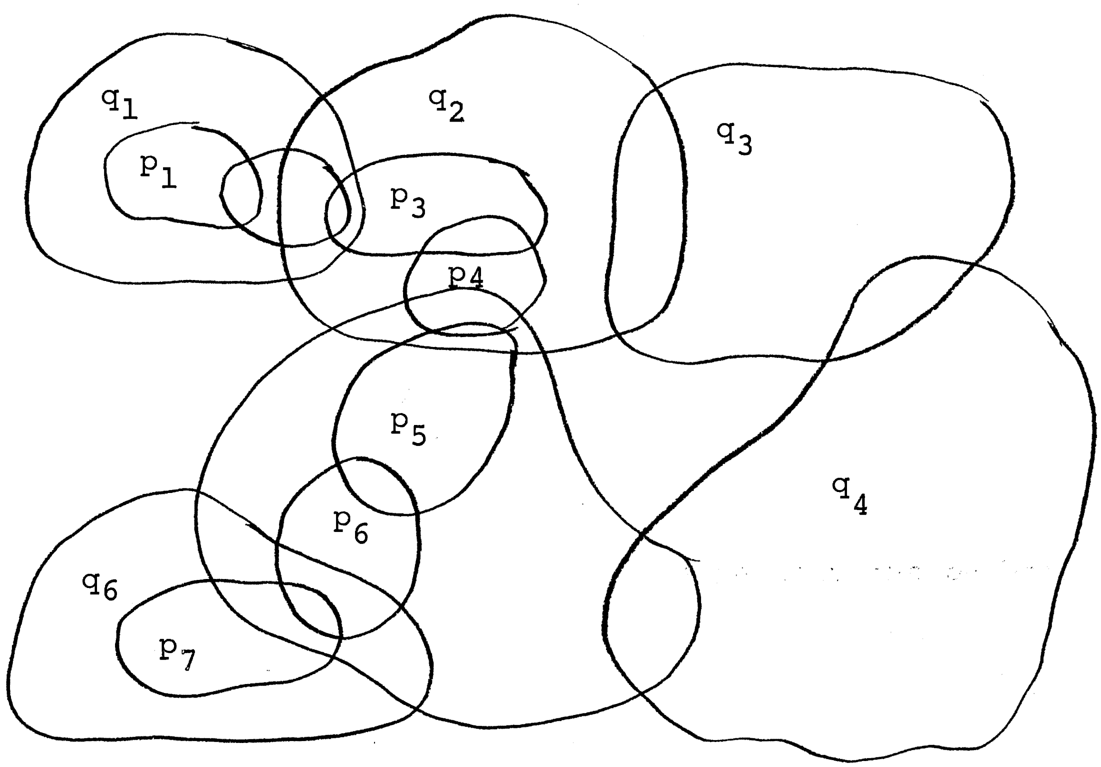


Figure 17. P Not a Refinement of Q

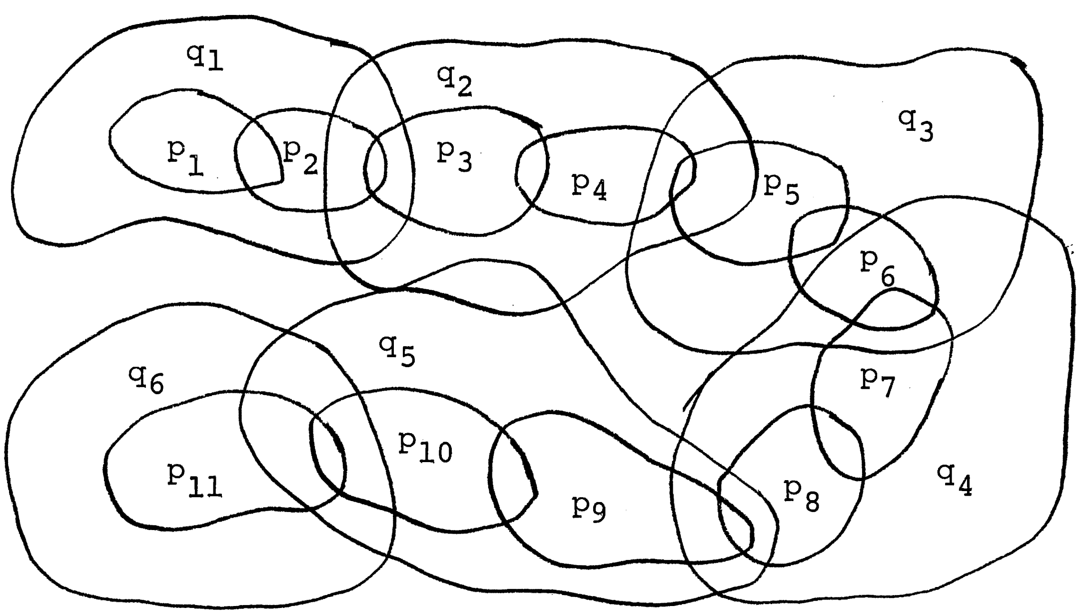


Figure 18. P is a Normal Refinement of Q

Figure 17 it can be seen that P is a normal refinement of Q, because the pattern of P in Q, (1,1), (2,1), (3,2), (4,2), (5,3), (6,3), (7,4), (8,4), (9,5), (10,5), and (11,6), is an r-pattern, x_1 is the first link of Q, and x_{11} is the last link of Q.

To illustrate the notion of a principal refinement, consider a p-chain, $q_1, \ldots, q_5$. Then, $p_1, \ldots, p_7$, where $p_1=q_1$, $p_2=q_1$, $p_3=q_2$, $p_4=q_2$, $p_5=q_3$, $p_6=q_4$, $p_7=q_5$ is also a p-chain, and its pattern in Q is (1,1), (2,1), (3,2), (4,2), (5,3), (6,4), and (7,5) which is an r-pattern. Also $p_i=q_{x_i}$, for i=1, ..., 7, and for each $1 \leq j \leq 5$, $j=x_i$ for some $1 \leq i \leq 7$. Hence P is a principal refinement of Q. Note also that P is a principal normal refinement of Q. If $p_1=q_1$, $p_2=q_2$, $p_3=q_3$, $p_4=q_3$, $p_5=q_2$, $p_6=q_3$, $p_7=q_4$, and $p_8=q_5$, then the p-chain $p_1, \ldots, p_8$ will be a principal normal refinement of Q under the pattern (1,1), (2,2), (3,3), (4,3), (5,2), (6,3), (7,4), and (8,5).

The ideas contained in the definitions above are very abstract and may seem without purpose. But these definitions aided Fearnley (13) in formulating a property of continua that characterized continua with respect to being the continuous images of the pseudo-arc. Fearnley called this property of continua, p-chainability.

<u>Definition 3.6</u> A continuum H is said to be p-chainable if there is a sequence $P_1, P_2, \ldots$ of p-chains such that for i=1, 2, ...

a) H is the union of links of P_i,

b) P_{i+1} is a normal refinement of P_i,

c) the diameter of each link of P_i is less than $1/i$, and

d) the closure of each link of P_{i+1} is a subset of the link of P_i which it corresponds under the r-pattern of P_{i+1} in P_i.

The sequence P_1, P_2, ... is said to be associated with H.

At this point it would be valuable to see examples of continua that are p-chainable and continua that are not p-chainable. Continua like arcs and simple closed curves are p-chainable. In fact, most continua that are usually visualized in the plane are p-chainable. It is much harder to find continua that are not p-chainable. Pseudo-circles are such continua. A more detailed look at what continua are and are not p-chainable will be given in Chapter IV.

Four theorems are needed to expose Fearnley's proof of Theorem 2.11. Theorem 3.1 is a very important technical tool to be used in many of the proofs that follows.

<u>Theorem 3.1</u> Each of the relations, "refinement", "normal refinement", and "principal normal refinement" between p-chains is transitive.

<u>Proof</u> Let P, Q, and R all be p-chains, and P a refinement of Q with r-pattern $(1,x_1)$, ..., (n,x_n), and Q a refinement of R with r-pattern $(1,y_1)$, ..., (m,y_m). Then for each

$1 \leq i \leq n$, p_i r_{y_x}, and so $(1,y_{x_1})$, ..., (n,y_{x_n}) is a pattern of P in R. Since $(1,y_1)$, ..., (m,y_m) is an r-pattern, then for each $1 \leq i, j \leq m$ such that $|i-j| \leq 1$, $|y_i-y_j| \leq 1$. In particular, if $|x_a-x_b| \leq 1$, then $|y_{x_a}-y_{x_b}| \leq 1$. But $(1,x_1)$, ..., (n,x_n) is an r-pattern and hence if $1 \leq a,b \leq n$ and $|a-b| \leq 1$, then $|x_a-x_b| \leq 1$. Therefore if $1 \leq a,b \leq n$ and $|a-b| \leq 1$, then $|y_{x_a}-y_{x_b}| < 1$. So $(1,y_{x_1})$, ..., (n,y_{x_n}) is an r-pattern, and by definition P a refinement of R. Thus the relation "refinement" is transitive.

Let P, Q, and R all be p-chains, and P a normal refinement of Q with r-pattern $(1,x_1=1)$, $(2,x_2)$, ..., $(n,x_n=m)$, and Q a normal refinement of R with r-pattern $(1,y_1-1)$, $(2,y_2)$, ..., $(m,y_m=k)$. Then as in the previous case $(1,y_{x_1})$, ..., (n,y_{x_n}) is an r-pattern of P in R. But x_1-1 and $y_.=1$ implies that $y_{x_1}=1$, and $x_n=m$ and $y_m=k$ implies that $y_{x_n}=k$. Therefore, by definition, P is a normal refinement of R, and the relation "normal refinement" is transitive.

It follows in a similar manner that the relation "principal normal refinement" is transitive. //

Fearnley (13) notes that the next theorem, Theorem 3.2, gives a critical refinemental relationship among p-chains. Given a normal refinement P of Q and a principal normal refinement R of Q, then a p-chain S can be found that is a principal normal refinement of P and a normal refinement of R. Being able to find such a p-chain, S plays a critical role in proving that p-chainability is a characterizing property for continua with respect to being the continuous images of the pseudo-arc, Theorem 2.11. But being able to find such a p-chain S also plays an important part in proving that every chainable continuum is the continuous images of the pseudo-arc, Theorem 2.13.

Theorem 3.2 If a p-chain P is a normal refinement of a p-chain Q, and R is a p-chain which is a principal normal refinement of Q, then there is a p-chain that is a principal normal refinement of P and a normal refinement of P and a normal refinement of R.

Proof Let $(1,x_1=1)$, $(2,x_2)$, ..., $(n,x_n=m)$ be an r-pattern of P in Q, and $(1,y_1=1)$, $(2,y_2)$, ..., $(k,y_k=m)$ be an r-pattern of R in Q. Note that for $i=1, 2, \ldots, k$, $r_i=q_{y_i}$. Since P is a normal refinement of Q, then a sequence of integers $a_1, \ldots, a_s$ may be defined in the following way: Let $a_1=1$. Then, if there exists integers j_2 and h_2 such that $j_2 < h_2 < n$, $x_{j_2}=m$ and $x_{h_2} < m$, then define a_2 to be the first integer such that $x_{a_2}=m$. Otherwise let $a_2=a_s=n$ and the sequence is defined. In the case $a_2 < n$ let j_3 be the first integer greater than a_2 such that $x_{j_3}=m$ and for which there is an integer h_3 such that $a_2 < h_3 < j_3$ and $x_{h_3} < m$. This is guaranteed by the integers j_2 and h_2. Now define a_3 to be the first integer greater than a_2 such that $x_{a_3} \leq x_w$ for $a_2 \leq w \leq j_3$. If there are integers j_4 and h_4 greater than a_3 such that $j_4 < h_4$, $x_{j_4}=m$ and $x_{h_4} < m$, then define a_4 to be the first integer greater than a_3 such that

$x_{a_4}=m$. Otherwise let $a_4=a_s=n$ and the sequence is defined. Continuing this process yields the desired increasing sequence of integers $a_1, \ldots, a_s$ with $a_s=n$. Also since R is a normal refinement of Q, then in a similar manner an increasing sequence of integers $b_1, \ldots, b_t$ with $b_t=k$ can be defined. Two other sequences of integers, $c_1, \ldots, c_t$, and $d_1, \ldots, d_s$ are to be defined. Let d_i be the greatest integer such that $y_{d_i}=x_{a_i}$ for $i=1, \ldots, s$, and c_i be the greatest integer such that $x_{c_i}=y_{b_i}$ for $i=1, \ldots, t$.

Note that the p-chain $P_1=P(a_1,a_2) + P(a_2,a_3) + \ldots + P(a_{s-1}, a_s)$ is a principal normal refinement of P. Also the p-chain $R_1=R(b_1,b_2) + R(b_2,b_3) + \ldots + R(b_{t-1},b_t)$ is a principal normal refinement of R.

Now the proof is completed by induction on the number, k, of links of the p-chain R. If k=1, then the p-chain R and the p-chain Q must be the same, by the definition of principal normal refinement, and k=m=1. But then the p-chain P is the desired p-chain that is a normal refinement of Q and a principal normal refinement of P.

Next suppose that the theorem is true for $k < f$ where $f > 1$, and consider the case when k=f. If $y_{k-1}=m$, then the induction hypothesis guarantees a p-chain with the desired properties. So it can be supposed that $y_{k-1}=m-1$, and two cases then must be considered.

In the first case suppose that t=2. If t=2, then $y_i < m$ for each $i=1, \ldots, k-1$, and hence the p-chain R(1,k-1) is a principal normal refinement of the p-chain Q(1,m-1). Let u be the integer such that the p-chain $P(1,u)$ is maximal with respect to being a sub-p-chain of $P(a_1, a_2)$, and with respect to being a normal refinement of $Q(1,m-1)$. Observe that the p-chain $R(k-1,k)$ is the same as the p-chain $Q(m-1,m)$. Also note that the p-chain $P(u,a_2)$ is a normal refinement of $Q(m-1,m)$, because of the properties of a_2. Since $P(1,u)$ is a normal refinement of $Q(1,m-1)$, and $R(1,k-1)$ is a principal normal refinement of $Q(1,m-1)$, the induction hypothesis guarantees a p-chain, S, that is a principal normal refinement of $P(1,u)$ and a normal refinement of $R(1,k-1)$. But then the p-chain $S_1 = S + P(u,a_2)$ is clearly a principal normal refinement of $P(1,a_2)$, and a normal refinement of R.

By definition of a_3, the p-chain $P(a_2,a_3)$ is a normal refinement of $Q(m,x_{a_3})$, and p_{a_3} is the first link of $P(a_2,a_3)$ that corresponds to the last link of $Q(m,x_{a_3})$. Likewise, by definition, the p-chain $R(k,d_3)$ is a principal normal refinement of $Q(m,x_{a_3})$, and r_{d_3} is the first link of $R(k,d_3)$ that is the same as the last link of $Q(m,x_{a_3})$. It can be noted that without loss of generality $d_3 > 1$. Now I have the same situation as above. Hence there exists a p-chain, S_2, that is a principal normal refinement of $P(a_2,a_3)$ and a normal refinement of $R(k,d_3)$. In a similar manner, it can be shown that there exists a p-chain, S_3, that is a principal normal refinement of $P(a_3,a_4)$, and a normal refinement of $R(d_3,k)$. Proceeding in this way, p-chains, S_1, S_2, ..., S_{s-1}, are obtained, which are principal normal refinements of $P(a_1,a_2)$, $P(a_2,a_3)$, ..., $P(a_{s-1},a_s)$, respectively, and normal refinements of $R(1,k)$, $R(k,d_3)$, $R(d_3,k)$, ..., $R(d_{s-1},k)$. Now consider the p-chain $S = S_1 + \ldots + S_{s-1}$. Then S is a principal normal refinement of P_1 and a normal refinement of R_1. By Theorem 3.1 it follows that S is a principal normal refinement of P and a normal refinement of R.

In the second case, assume that $t > 2$. Now the p-chains, $R(b_1,b_2)$, $R(b_2,b_3)$, ..., $R(b_{t-1},b_t)$, all have less than k links, and are respectively principal normal refinements of $Q(1,m)$, $Q(m,y_{b_3})$, ..., $Q(y_{b_{t-1}},y_{b_t}=m)$. At the same time, the p-chains, $P(1,n)$, $P(n,c_3)$, $P(c_3,n)$, ..., $P(c_{t-1},n)$, are normal refinements of $Q(1,m)$, $Q(y_{b_3},m)$, ..., $Q(y_{b_{t-1}},m)$, respectively. Hence by the induction hypothesis there exists p-chains, S_1, ..., S_{t-1} which are principal normal refinements of $P(1,n)$, $P(n,c_3)$, $P(c_3,n)$, ..., $P(c_{t-1},n)$, respectively, and are normal refinements of $R(b_1,b_2)$, ..., $R(b_{t-1},b_t)$, respectively. Then $S = S_1 + \ldots + S_{t-1}$ is a p-chain that is a principal normal refinement of P and a normal refinement of R. //

The next theorem will be used in the proof of Theorem 2.11 to show that p-chainability of a continuum H implies that H is the continuous image of the pseudo-arc.

<u>Theorem 3.3</u> If a p-chain P is a normal refinement of a p-chain Q, then there is a principal normal refinement R of P such that R is a normal refinement of Q and also crooked in Q.

<u>Proof</u> Let $(1,x_1 = 1)$, $(2,x_2)$, ..., $(n,x_n\text{-}m)$ be an r-pattern of P in Q. Now define R inductively, where the induction is done on the number of links, n, of P.

If $n = 1$, then let $R = P$. Hence R is a principal normal refinement of P, a normal refinement of Q, and crooked in Q. The crookedness follows vacuously from the definition, given previously in Chapter I. Next assume for $n < f$ and $f > 1$ that R has been chosen such that R is a principal normal refinement of P and crooked in P. Consider the case where $n = f$. In this case the p-chain $P(1,n-1)$, $P(n-1,2)$, and $P(2,n)$ each have less than f links, and so the induction hypothesis guarantees p-chains R_1, R_2, and R_3 which are principal normal refinements and crooked in $P(1,n-1)$, $P(n-1,2)$, and $P(2,n)$, respectively. Now let $R = R_1 + R_2 + R_3$.

Clearly R is a principal normal refinement of P. Let $(1, y_1 = 1)$, $(2,y_2)$, ..., $(k,y_k = n)$ be an r-pattern of R in P. It remains to show that R is crooked in P. It will suffice to show that if r_a and r_b are links of R such that r_a came from R_1, r_b came from R_3, $|y_a-y_b| > 2$, and $y_a < y_b$, then there exists integers s and t such that $a < s < t < b$, $y_t = y_a+1$ and $y_s = y_b-1$. Since r_a came from R_1 and R_1 is a normal refinement of $P(1,n-1)$, then $1 \leq y_a \leq n-3$. Also since r_b came from R_3 and R_3 is a normal refinement of $P(2,n)$, then $4 \leq y_b \leq n$. But the definition of normal refinement implies that there exists $s > a$ such r_s came from R_1 and $y_s = y_b-1$, and there exists $t < b$ such r_t came from R_3 and $y_t = y_a+1$. Finally, since $R = R_1 + R_2 + R_3$, r_s came from R_1, and r_t came from R_3, then $s < t$. Hence R is crooked in P, and by induction then R exists such that R is a principal normal refinement of P and crooked in P.

It now must be shown that R is crooked in Q. From Theorem 3.1 $(1,x_{y_1} = 1), (2,x_{y_2}), \ldots, (k,x_{y_k} = m)$ is an r-pattern of R in Q. Suppose r_a and r_b are links of R such that $a \ll b$ and $|x_{y_a} - x_{y_b}| > 2$. Then by definition of r-pattern $|x_{y_a} - x_{y_b}| > 2$ implies that $|y_a - y_b|$ 2. But R is crooked in P and so there exists integers s and t such that $a < s < t < b$ and if $y_a < y_b$, then $y_s = y_b - 1$ and $y_t = y_a + 1$, or if $y_a > y_b$, then $y_s = y_b + 1$ and $y_t = y_a - 1$. Hence $|y_s - y_b| = 1$ and $|y_a - y_t| = 1$, and by definition of normal refinement $|x_{y_s} - x_{y_b}| \leq 1$ and $|x_{y_a} - x_{y_b}| \leq 1$. By applying the definition of r-pattern to both R and P it is possible now to find $a < u < v < b$ such that if $x_{y_a} < x_{y_b}$, then $x_{y_u} = x_{y_b} - 1$ and $x_{y_v} - x_{y_a} + 1$, or if $x_{y_b} < x_{y_a}$, then $x_{y_u} = x_{y_b} + 1$ and $x_{y_v} = x_{y_a} - 1$. Therefore R is crooked in Q. //

Theorem 3.4 is a generalization of the necessity of Theorem 2.11. But the reason for proving Theorem 3.4 in its generality is that it will be used in Chapter IV to show that there is a continuum that is not p-chainable.

<u>Theorem 3.4</u> If a continuum H is the continuous image of the pseudo-arc and x is a point of H, then H is p-chainable and there is a sequence of p-chains associated with H such that x is contained in the first link of each p-chain.

<u>Proof</u> Let M be the pseudo-arc and f a continuous mapping of

M onto H. Also let p be a point of M that maps onto x. Recall that the pseudo-arc is a nondegenerate indecomposable continuum and that nondegenerate indecomposable continuum have uncountably many disjoint composants. Therefore there exists a point q of M such that p and q are contained in different composants. Now by a theorem of Bing's in his article (2) M is chainable between p and q. Hence there is a sequence of chains D_1, D_2, ... in M, considered as a space, such that for i = 1, 2, ..., (1) D_{i+1} is a refinement of D_i, (2) the diameter of each link of D_i is less than 1/i, (3) the closure of each link of D_{i+1} is contained in some link of D_i, and (4) the union of links of D_i is M.

For each positive integer i, let D_i be the chain, d_{i1}, d_{i2}, ..., d_{in_i}, and let $f(D_i)$ be the p-chain, $f(d_{i1})$, ..., $f(d_{in_i})$. The proof will be completed by showing that a subsequence of the sequence of p-chains $f(D_i)$, $f(D_2)$, ... is associated with H. Since M is compact and f continuous, then a classical theorem of topology says that f is uniformly continuous. Hency by property (2) above and the fact that f is uniformly continuous, a subsequence $f(D_{k1})$, $f(D_{k2})$, ... can be obtained such that for i = 1, 2, ..., the diameter of each link of $f(D_{k_i})$ is less than 1/i. Also for each positive integer i, $D_{k_{i+1}}$ is a refinement of D_{k_i} and the closure of each link of $D_{i_{i+1}}$ is a subset of some link of D_{k_i}. Therefore there is a pattern, $\pi(i,i+1)$, of $D_{k_{i+1}}$ in D_{k_i} such that the first and last links of $D_{k_{i+1}}$ are associated with the first and last links of D_{k_i}, respectively, and which associates each link of $D_{k_{i+1}}$ with the link of D_{k_i} that contains its closure. But the pattern is also a pattern of $f(D_{k_{i+1}})$, and with little trouble can be shown to be an r-pattern. Since the first and last links of $f(D_{k_{i+1}})$ and $f(D_{k_i})$ are associated under this r-pattern, then $f(D_{k_{i+1}})$ is a normal refinement of $f(D_{k_i})$. Furthermore, if $f(d_{k_{i+1}s})$ is a link of $f(D_{k_{i+1}})$ associated with the link $f(d_{k_i r})$ of $f(D_{k_i})$, then $d_{k_{i+1}s}$ $d_{k_i r}$, and $\overline{f(d_{k_{i+1}s})} = f(\bar{d}_{k_{i+1}s}) \subseteq f(d_{k_i r})$. Finally, it is clear from property (4) above and the definition of f that the union of links of $f(D_{k_i})$ is H for each positive integer i. Thus the sequence of p-chains, $f(D_{k_1})$, $f(D_{k_2})$, ... satisfies Definition 3.6 with respect to H and the first link of each p-chain contains x. //

Finally, Fearnley's proof of Theorem 2.11 characterizing the continuous images of the pseudo-arc can be given. The proof relies heavily on the previous four theorems and the definition of the pseudo-arc given in Chapter I.

Theorem 2.11 The continuum H is the continuous image of the pseudo-arc if and only if H is p-chainable.

Proof If H is the continuous image of the pseudo-arc, then Theorem 3.4 implies that H is p-chainable.

Suppose H is p-chainable and let P_1, P_2, ... be a sequence of p-chains associated with H. Now define two sequences of p-chains, T_1, T_2, ... and D_1, D_2, ... such that for each positive integer n, (1) T_n is a principal normal refinement of P_n, (2) D_n is a corresponding chain of open discs in the plane having the same number of links as T_n, and (3) for $n > 1$, T_n is crooked in T_{n-1}, D_n has the same pattern in D_{n-1} as the r-pattern of T_n in T_{n-1}, each link of D_n has diameter less than $1/n$, and the closure of each link of D_n is contained in the corresponding link of D_{n-1}. If $n = 1$, then it is easy to find T_1 and D_1 that satisfy properties (1), (2), and (3) above. Assume for $N < k$, $k > 1$, that the sequences $T_1, \ldots, T_n$ and $D_1, \ldots, D_n$ have been defined so that properties (1), (2), and (3) are satisfied. Now consider the case that $n = k$.

By induction hypothesis T_{k-1} is a principal normal refinement of P_{k-1}, and P_k is a normal refinement of P_{k-1}. Hence by Theorem 3.2 there exists a p-chain S_k that is a principal normal refinement of P_k and a normal refinement of T_{k-1}. Since S_k is a normal refinement of T_{k-1}, then Theorem 3.3 implies there exists a principal normal refinement T_k of S_k such that T_k is crooked in T_{k-1}. Also since S_k is a principal normal refinement of P_k, then Theorem 3.1 implies that T_k is a principal normal refinement of P_k. In addition, T_k could be tailored by the addition of duplicate links in any number and at any place in the sequence, and this would not effect T_k being a principal normal refinement of P_k or T_k being crooked in T_{k-1}. But this would allow a chain D_k of open discs in the plane to be found that has the same number of links as T_k, the same pattern in D_{k-1} as T_k in T_{k-1}, diameter of its links being less than $1/k$, and the closure of each of its links contained in the corresponding links of

D_{k-1}. Therefore by induction the sequences T_1, T_2, ... and D_1, D_2, ... have been defined with properties (1), (2), and (3) above. From these properties D_1, D_2, ... is a sequence of chains associated with a pseudo-arc M.

A continuous mapping f of M onto H is now needed. Let x be a point of M and let the links d_{1r_1} of D_1, d_{2r_2} of D_2, ... be a sequence of open sets closing down on x such that for each positive integer i, $d_{i+1r_{i+1}}$ corresponds to d_{ir_i} under the pattern of D_{i+1} in D_i. Then by conditions (c) and (d) of Definition 3.6, the intersection $\bigcap_{i=1}^{\infty} t_{ir_i}$ exists and is a single point. Define $f(x) = \bigcap_{i=1}^{\infty} t_{ir_i}$. First f must be shown to be a function. Suppose that d_{1r_1}, d_{2r_2}, ... and d_{1s_1}, d_{2s_2}, ... are two different sequences that close down on x in the manner described above. But definition of chain implies that $|r_i - s_i| \leq 1$ for each positive integer i. Hence t_{ir_i} t_{is_i} has diameter less than 2/i and $\bigcap_{i=1}^{\infty} (t_{ir_i} \cup t_{is_i})$ must be a single point. Therefore $\bigcap_{i=1}^{\infty} (t_{ir_i} \cup t_{is_i}) = \bigcap_{i=1}^{\infty} (t_{ir_i}) = \bigcap_{i=1}^{\infty} (t_{is_i})$ and f is a function. Next f must be shown to be continuous. Let g be an open set in H. Then for some integer k, g contains three consecutive links t_{ki-1}, t_{ki}, and t_{ki+1}. But if d_{1r_1}, d_{2r_2}, ... is a sequence of links that close down on a point of d_{ki}, then one of the links d_{ki-1}, d_{ki}, or d_{ki+1} is a member of this sequence. Thus d_{ki} M which is open in M is mapped into $t_{ki-1} \cup t_{ki} \cup t_{ki+1}$ which is contained in g. Therefore f is continuous, and from the above argument it can be deduced that f(M) is dense in H. But M is compact, so f(M) is a compact subset of H and hence must be closed in H. Since $f(M)$ is dense in H, then $f(M) = \overline{f(M)} = H$ and f is onto. Hence H is a continuous image of the pseudo-arc. //

Now that the continuous images of the pseudo-arc have been characterized in terms of p-chainability it can be easily shown that every chainable continuum is the continuous image of the pseudo-arc, which is exactly the statement of Theorem 2.13. To show that a chainable continuum H is the continuous image of the pseudo-arc Fearnley takes the sequence of chains $D_1, D_2, \ldots$ associated with H and finds principal refinements of these chains that form a sequence of p-chains associated with H. This proof will close out the section. Before presenting Fearnley's proof of Theorem 2.13, two technical lemmas needed in the proof of Theorem 2.13 are stated and proved.

Lemma 3.1 If P and Q are p-chains such that P is a refinement of Q and each of Q corresponds to at least one link of P, then there is a p-chain R such that R is a principal refinement of P and a normal refinement of Q.

Proof There exists integers s and t such that $1 \leq s,t \leq n$, p_s corresponds to q_1, and p_t corresponds to q_m, under the r-pattern of P in Q. Now define $R = P(s,1) + P(1,n) + P(n,t)$. Clearly R is a principal refinement of P, and R is a normal refinement of Q. //

Lemma 3.2 If P, Q, and R are p-chains with R a principal refinement of Q and P a refinement of Q in which each link of Q corresponds to at least one link of P, then there is a p-chain S such that S is a principal refinement of P and S is a refinement of R in which each link of R corresponds to at least one link of S.

<u>Proof</u> From Lemma 3.1 there exists p-chains P_1 and R_1 that are principal refinements of P and R, respectively, and are each normal refinements of Q. Also the transitivity of principal refinements, Theorem 3.1, implies that R_1 is a principal refinement of Q. Since P_1 is a normal refinement of Q and R_1 is a principal normal refinement of Q, then Theorem 3.2 guarantees a p-chain S that is a principal normal refinement of P_1 and a normal refinement of R_1. Again Theorem 3.2 implies that S is a principal refinement of P and S is a refinement of R. Fianlly, since S is a normal refinement of R_1 and R_1 is a principal refinement of R, then each link of R corresponds to at least one link of S. //

<u>Theorem 2.13</u> Every chainable continuum is a continuous image of the pseudo-arc.

<u>Proof</u> Let H be a chainable continuum and consider it a topological space. By definition of chainability there exists a sequence of chains D_1, D_2, ... such that for each positive i, (1) the union of links of D_i is H, (2) each link of D_i has diameter less than 1/i, (3) no link of D_i is a subset of any other link of D_i, and (4) there is an r-pattern of D_{1+1} in D_i such that the closure of each link of D_{i+1} is a subset

of the corresponding link of D_i. Now a sequence of p-chains P_1, P_2, ... will be constructed to be associated with H. Let P_1 be the chain D_1. The connectedness of H and property (3) above implies that each link of D_1 corresponds to at least one link of D_2, and therefore by Lemma 3.1 there exists a p-chain P_2 that is a principal refinement of D_2 and a normal refinement of P_1. Since each link of D_2 corresponds to at least one link of D_3, then Lemma 3.2 guarantees a p-chain S_3, such that S_3 is a principal refinement of D_3, and S_3 is a refinement of P_2 in which each link of P_2 corresponds to at least one link of S_3. Then from Lemma 3.1 there exists a p-chain P_3 that is a principal refinement of S_3 and hence D_3, and a normal refinement of P_2. Using induction and proceeding in the above manner a sequence of p-chains P_1, P_2, ... which are principal refinements of the chains D_1, D_2, ..., respectively, and which have the property that each is a normal refinement of the proceeding p-chain, is obtained. By definition of p-chainability and the fact that for each positive integer i, P_i is a principal refinement of D_i, the sequence of p-chains P_1, P_2, ... is associated with H. Therefore, Theorem 3.5 implies that H is the continuous image of the pseudo-arc. //

Continuous Images of Pseudo-Circles

In 1951, R. H. Bing (2) asked the question: "Is any pseudo-circle homogeneous?" At the University of California at Riverside, in the middle part of the 60's, F. B. Jones presented this problem to his seminar. One of the students in Jones' seminar, James T. Rogers, Jr., became interested in the problem and started working on its solution. In correspondence with Rogers, he has told me that he thought results about the continuous images of pseudo-circles would lead to progress on this homogeneity question. Rogers (37) obtained results concerning the continuous images of pseudo-circles and these results were incorporated into his Ph.D. thesis completed in 1968.

Later in 1968 Rogers was able to answer the original question concerning the homogeneity of the pseudo-circles. He was able to show that no pseudo-circle is homogeneous.

It is the purpose of this section to present Rogers' results concerning the continuous images of pseudo-circles. In particular, the proof of Theorem 2.15 which characterizes the continuous images of pseudo-circles will be shown. Another important result, all circularly chainable continua are the continuous image of some pseudo-circle, will be presented as a preliminary to Theorem 2.15.

The beginnings of Rogers' work are: (1) to define a special kind of category, (2) show the existence of such a category, and (3) prove that the inverse limit of certain objects and morphisms of this category is a pseudo-circle. An important tool used in this process is an Uniformation Theorem. This Uniformation Theorem requires considerable machinery to prove. But this machinery is not in the mainstream of Rogers' characterization of the continuous images of pseudo-circles, and so the development of the Uniformation Theorem will be omitted. See Rogers (37) for the complete development. However, the Uniformation Theorem will be stated. Before this Uniformation Theorem can be stated and understood, some preliminary definitions are needed.

Definition 3.7 A category $\mathcal{C}$ is a class of objects, $Ob(\mathcal{C})$, together with

(i) a family of disjoint sets, $Hom(A,B)$, one for each pair of objects $A, B \varepsilon Ob(\mathcal{C})$,

(ii) each triple of objects $A, B, C \varepsilon Ob(\mathcal{C})$, a function which assigns to $\alpha \varepsilon Hom(A,B)$ and $\beta \varepsilon Hom(B,C)$ one element $\beta\alpha \varepsilon Hom(A,C)$, and

(iii) a function which assigns to each object $A \varepsilon Ob(\mathcal{C})$ an element $1_A \varepsilon Hom(A,A)$;

all subject to the following two conditions:

(1) if $\alpha \varepsilon Hom(A,B)$, $\beta \varepsilon Hom(B,C)$, and $\delta \varepsilon Hom(C,D)$, then $\delta(\beta\alpha) = (\delta\beta)\alpha$ and

(2) if $\alpha \varepsilon Hom(A,B)$, then $\alpha 1_A = \alpha = 1_B \alpha$.

Elements of $Hom(A,B)$ are called morphisms.

An example of a category $\mathfrak{T}$ is to let $Ob(\mathfrak{T}) = \{X |\ X$ is a topological space$\}$, $Hom(X,Y) = \{f |\ f$ is a continuous map of X into $Y\}$, the function of (ii) to be the usual composition of maps, and the function of (iii) to assign the identity map of X as 1_X for each topological space X. Another example of a category $\mathfrak{A}$ is to let $Ob(\mathfrak{A}) = \{G |\ G$ is a group$\}$, $Hom(G,H) = \{f |\ f$ is a homomorphism of G into $H\}$, the function of (ii) to be the usual composition of maps, and the function of (iii) to assign the identity map of G as 1_G for each group G.

The next three definitions deal with maps of the unit circle onto itself, and a "winding" property of these maps. The first two definitions define a "simplicial" map of the circle onto itself. Most of the maps that will be encountered in this section will be of this type. The third definition defines the degree of a "simplicial" map of the circle onto itself. Intuitively, this degree is the winding number of the map. After the definitions have been presented, examples will be given to clarify the ideas involved in these definitions.

<u>Definition 3.8</u> Let C be the unit circle in the plane. A triangulation, T, of C is a decomposition of C into the union of arcs such that

(1) no arc of T sweeps 180° or more, and

(2) arcs of T intersect only at endpoints.

The endpoints of the members of T are called vertices of T.

<u>Definition 3.9</u> Let C be the unit circle in the plane. A continuous map f of C onto C is said to be a simplicial map of T onto T' if and only if T and T' are triangulations of C, f maps the vertices of T onto the vertices of T', and f maps each member of T onto a vertex of T' or homeomorphically (by uniform stretching or shrinking) onto a member of T'.

<u>Definition 3.10</u> Fix the positive direction of rotation on C, where C is the unit circle in the plane. Let f be a simplicial map of T onto T' and $\{x_0, x_1, \ldots, x_k\}$ be the vertices of T ordered by positive rotation. Next number the n vertices of T' ordered also by positive rotation in the following way: $\{f(x_0) = 0, 1, \ldots, n-1\}$. Now define for $0 \leq i \leq k-1$,

$$\rho(x_i) = \begin{cases} 1, \text{ if } f(x_{i+1})-f(x_i) = 1 \pmod{n} \\ 0, \text{ if } f(x_{i+1})-f(x_i) = 0 \pmod{n} \\ -1, \text{ if } f(x_{i+1})-f(x_i) = n-1 \pmod{n}. \end{cases}$$

Finally, define the degree of f, $\deg(f) = (\sum_{i=0}^{k-1}\rho(x_i))/n$.

Examples of triangulations of C are illustrated in Figure 19. The copy of C on the left has a triangulation T consisting of five vertices $x_0, \ldots x_4$ together with the corresponding five arcs, $\overline{x_0x_1}, \overline{x_1x_2}, \ldots, \overline{x_4x_0}$. The copy of C on the right has a triangulation T' consisting of three vertices y_0, y_1, and y_2 together with the corresponding arcs $\overline{y_0y_1}$, $\overline{y_1y_2}$, and $\overline{y_2y_0}$. Let f be a continuous map of C onto C that sends $\overline{x_0x_1}$ onto y_0, $\overline{x_1x_2}$ homeomorphically (by uniform stretching) onto $\overline{y_0y_1}$, $\overline{x_2x_3}$ onto y_1, $\overline{x_3x_4}$ homeomorphically (by uniform shrinking) onto $\overline{y_1y_2}$, and $\overline{x_4x_0}$ homeomorphically (by uniform stretching) onto $\overline{y_2y_0}$. Then f is a simplicial map of T onto T' and the degree of the map f is one. Let g be a continuous map of C onto C that sends x_0 to y_0, x_1 to y_1, x_2 to y_0, x_3 to y_2, x_4 to y_1, $\overline{x_0x_1}$ and $\overline{x_1x_2}$ homeomorphically (by uniform stretching) onto $\overline{y_0y_1}$, $\overline{x_2x_3}$ homeomorphically (by uniform stretching) onto $\overline{y_0y_2}$, $\overline{x_3x_4}$

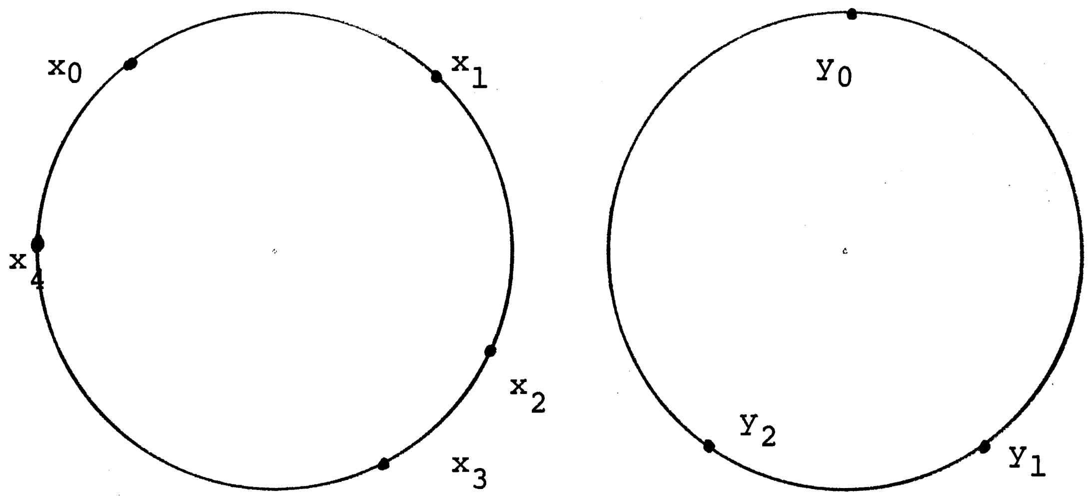


Figure 19. Triangulations of C

homeomorphically (by uniform shrinking) onto $\overline{y_1y_2}$, and $\overline{x_4x_0}$ homeomorphically (by uniform stretching) onto $\overline{y_0y_1}$. Then g is a simplicial map of T onto T' with deg(g) = -1.

It should be noted that the degree of a simplicial map of T onto T' does not depend on the way the vertices of T are sequenced. For example, if the copy of C in Figure 19 would have been labeled as in Figure 20 and f the same function of C onto C, then the degree of f is still going to be one. Another fact that should be mentioned here is that if h is a simplicial map of T_1 onto T_2 and also a simplicial map of T_3 onto T_4 where T_1, T_2, T_3, and T_4 are all triangulations of C, then the degree of h calculated by either pair of triangulations is the same. A proof of this can be found in a slightly different form in Dugundji (19).

Now a uniformation can be defined and Rogers' Uniformation Theorem stated.

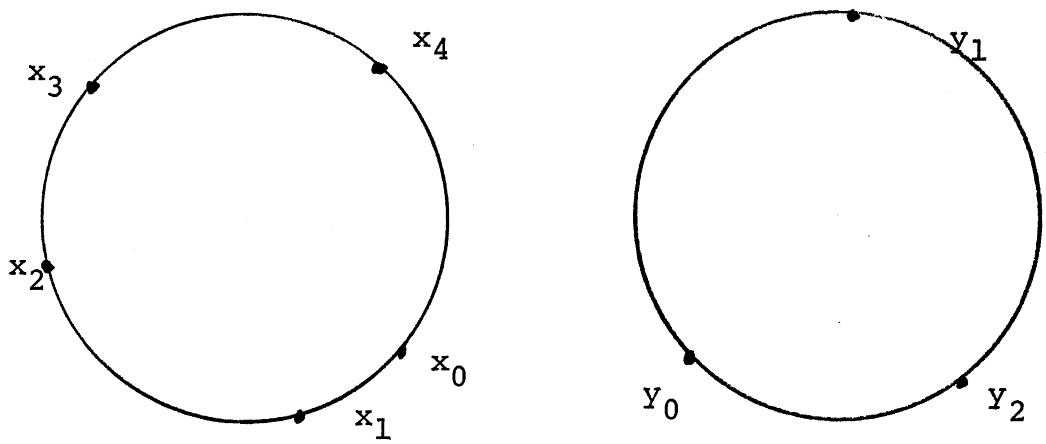


Figure 20. Different Labeling of Vertices

<u>Definition 3.11</u> Let $\mathcal{C}$ be a category and f and g morphisms of $\mathcal{C}$. A pair of morphisms, α and β, of $\mathcal{C}$ is said to be a uniformation of f and g if $f\alpha = g\beta$.

Two sets of "simplicial" maps of C onto C need to be labeled at this point. Let $U = \{f|$ there exists triangulations T and T' of C, f is a simplicial map of T onto T', and $\deg(f) \neq 0\}$ and $S = \{f \varepsilon U | \deg(f) > 0\}$.

<u>Theorem 3.5 (Uniformation Theorem)</u> Let $f_1, \ldots, f_n \varepsilon$ S and $\deg(f_i) = k_i$, for $1 \leq i \leq n$. Then there exists $\alpha_1, \ldots, \alpha_n \varepsilon$ S such that for $1 \leq j \leq n$, $f_1\alpha_1 = f_j\alpha_j$, the $\deg(\alpha_j) =$ the least common multiple of $\{k_i\}_{i=1}^{n}$ divided by k_j, and each α_j has the same triangulation in its domain as α_1.

A proof of Theorem 3.5 will not be given, because it would be lengthy and distract from the main purpose of this section. However, the proof can be found in Rogers (37).

Now the presentation of Rogers' work may begin. As mentioned above, the first item in Rogers' presentation is the defining of a special kind of category. Rogers calls this special category an A-category. He borrowed this name from J. Mioduszewski (27) who had used an A-category to show mapping properties of the pseudo-arc. The objects of Rogers' A-category are copies of the unit circle, and the morphisms are a subset of the class of "simplicial" maps, S, defined previously. But first the definition of a majorant is needed.

<u>Definition 3.12</u> A simplicial map $g \in S$ is said to be a majorant for Map(T'', T') if and only if for each pair of maps f, f' $\in$ Map(T'', T') there exists a map $\alpha \in S$ such that $f\alpha = f'g$.

<u>Definition 3.13</u> Let $\mathcal{W}$ denote the category about to be described. Then $Ob(\mathcal{W}) = \{ C_n \mid$ for $n = 1, 2, \ldots, C_n$ is a copy of the unit circle $\}$. Consider for every C_n a sequence of triangulations, $T_{n,n-1}, T_{n,n}, \ldots,$ of C_n into equal segments. Let $n(T_{p,q})$ denote the number of segments in $T_{p,q}$ and $Map(T_{p,q}, T_{r,s})$ the class of all simplicial maps of positive degree of $T_{p,q}$ onto $T_{r,s}$. Now assume the triangulations have the following properties:

(i) for $r \geq n-1$, $T_{n,r+1}$ is a triangulation of C_n with each member of $T_{n,r+1}$ at most half the length of a member of $T_{n,r}$, and each member of $T_{n,r+1}$ is a segment of a member of $T_{n,r}$,

(ii) $\mathrm{Map}(T_{n+1,r}, T_{n,r})$ is nonempty for $r,n = 1,$ $2, \ldots,$ and

(iii) $\mathrm{Map}(T_{m,p}, T_{n,p}) \quad \mathrm{Map}(T_{m,r}, T_{n,r})$ for $p < r$.

The morphisms of $\mathcal{W}$ may now be defined as follows:

$$\mathrm{Hom}(C_p, C_q) = \begin{cases} \emptyset, \text{ if } p < q \\ \mathrm{Map}(T_{p,p}, T_{p,p}), \text{ if } p = q \\ \mathrm{Map}(T_{p,p}, T_{q,p}), \text{ if } p > q. \end{cases}$$

Further assume that there exists a map $g^p_{p-1} \varepsilon \mathrm{Hom}(C_p, C_{p-1})$ that is a majorant for $\mathrm{Map}(T_{p,p}, T_{m,p})$, $m = 1, \ldots, p-1$. With usual function composition, $\mathcal{W}$ is a category, Finally, $\mathcal{W}$ is said to be an A-category if and only if $\lim_{r\to\infty}[n(T_{r,r})/ n(T_{m,r})] = \infty$ for $m = 1, 2, \ldots$.

The category $\mathcal{W}$, defined in Definition 3.13, is probably not clearly understood at the present time. This situation will be cleared up when an example is constructed in the proof of the existence of an A-category. However, some preliminary matters must be presented before this proof can be shown. So as to aid in understanding $\mathcal{W}$ now, the table in Figure 21 is offered. An arrow from the triangulation $T_{p,q}$ to the triangulation $T_{r,s}$ in the table indicates the existence of simplicial maps of positive degree from $T_{p,q}$ onto $T_{r,s}$. And in a column of triangulations, the triangulations become finer, by a factor of at least two, for each trinagulation down the column.

Now that an A-category has been defined, it becomes necessary to show that such categories exist. Rogers uses the same approach to show the existence of his A-categories

$$
\begin{array}{l|llllll}
 & \multicolumn{6}{c}{\text{Triangulations of}} \\
 & C_1 & C_2 & C_3 \; \ldots & & C_r & C_{r+1} \\
\hline
 & T_{1,0} & & & & & \\
 & T_{1,1} \leftarrow & T_{2,1} & & & & \\
\text{finer (down)} & T_{1,2} \leftarrow & T_{2,2} \leftarrow & T_{3,2} & & & \\
 & \cdot & & & & & \\
 & \cdot & & & & & \\
 & \cdot & & & & & \\
 & T_{1,r} \leftarrow & T_{2,r} \leftarrow & T_{3,r} & \cdots \leftarrow & T_{r,r} \leftarrow & T_{r+1,4} \\
 & \cdot & & & & & \\
 & \cdot & & & & & \\
 & \cdot & & & & &
\end{array}
$$

Figure 21. Triangulations Involved in an A-Category

that Mioduszewski (27) used to show the existence of a similar type of category. Before the existence can be shown, three preliminary lemmas must be presented. The first two lemmas will be used in the construction of an A-category. A crucial tool in their proofs is the Uniformation Theorem, Theorem 3.5.

<u>Lemma 3.3</u> Let T' and T'' be triangulations, into equal segments of the unit circle C such that Map(T'', T') is nonempty. Then there exists a trinagulation T''' of C into

equal segments and a simplicial map g of T''' onto T'' that is a majorant for Map(T'', T').

<u>Proof</u> By the definition of simplicial maps of T'' onto T' the class Map(T'', T') is finite. Let $f_1, f_2, \ldots, f_k$ be an enumeration of the maps of Map(T'', T'). Now according to the Uniformation Theorem, Theorem 3.5, there exists maps $\alpha_1, \alpha_2, \ldots, \alpha_k \in$ Map($T^{\#}$, T''), where $T^{\#}$ is a trinagulation of C, such that $f_1\alpha_1 = f_2\alpha_2 = \ldots = f_k\alpha_k$. Using the Uniformation Theorem again there exists a triangulation T''' of C and Maps $\beta_1, \beta_2, \ldots, \beta_k \in$ Map(T''', $T^{\#}$) such that $\alpha_1\beta_1 = \alpha_k\beta_k$. Let $g = \alpha_1\beta_1 = \ldots = \alpha_k\beta_k$. Then $g \in$ Map(T''', T'') and for any pair of maps f_i and $f_j \in$ Map(T'', T') there exists $\alpha = \alpha_j\beta_i \in$ Map(T''', T'') S such that $f_i g = f_i\alpha_i\beta_i = f_j\alpha_j\beta_i = f_j\alpha$. Hence g is a majorant for Map(T'', T'). //

One way to think of a majorant for Map(T'', T') is a map $g \in$ Map(T''', T'') so that for each pair of maps f, f' $\in$ Map(T'', T') there exists a map $\alpha \in$ Map(T''', T'') such that the following diagram is commutative:

$$\begin{array}{ccc} T''' & \xrightarrow{\alpha} & T'' \\ g \downarrow & & \downarrow f \\ T'' & \xrightarrow[f']{} & T' \end{array} .$$

<u>Lemma 3.4</u> Let $T_{n,r}$, for n = 1, ..., r, be triangulations of C with the properties outlined in Definition 3.13. Then there exists a triangulation $T^{\#}$ of C into equal segments and

a simplicial map g of $T^{\#}$ onto $T_{r,r}$ such that g is a majorant for $\mathrm{Map}(T_{r,r}, T_{n,r})$ for $n = 1, \ldots, r-1$.

Proof Let $f_{r,n}$ be the majorant of $\mathrm{Map}(T_{r,r}, T_{n,r})$ guaranteed by Lemma 3.3, for $n = 1, \ldots, r-1$, and T^n the triangulation of C such that $f_{r,n} \varepsilon \mathrm{Map}(T^n, T_{r,r})$. Now by the Uniformation Theorem there exists a triangulation $T^{\#}$ of C into equal segments and maps $\delta_n \varepsilon \mathrm{Map}(T^{\#}, T^n)$, $n = 1, \ldots, r-1$, such that $f_{r,1}\delta_1 = f_{r,2}{}_2 = \cdots = f_{r,r-1}\delta_{r-1}$. Let $g = f_{r,1}\delta_1 = \cdots = f_{r,r-1}\delta_{r-1}$, and note that $g \varepsilon \mathrm{Map}(T^{\#}, T_{r,r})$. Now let f', $f'' \varepsilon \mathrm{Map}(T_{r,r}, T_{n,r})$ for $n \leq r-1$. Because $f_{r,n}$ is a majorant for $\mathrm{Map}(T_{r,r}, T_{n,r})$, implies there exists a map $\beta \varepsilon \mathrm{Map}(T^n, T_{r,r})$ such that $f'f_{r,n} = f''\beta$. Hence $f'f_{r,n}\delta_n = f''\beta\delta_n$. But $f_{r,n}\delta_n = g$, and so $f'g = f''(\beta\delta_n)$ where $\beta\delta_n \varepsilon \mathrm{Map}(T^{\#}, T_{r,r})$. Therefore, g is a majorant for Map $\mathrm{Map}(T_{r,r}, T_{n,r})$ for $n = 1, \ldots, r-1$. //

The third and last lemma, before the existence of an A-category is presented, deals with the relative size of the triangulations, $T_{n-1,n}$, $T_{n,n}$, and $T_{n+1,n}$ of C, satisfying the properties outlined in Definition 3.12. This lemma will be used to show that the category, constructed in the proof of Theorem 3.6, is actually an A-category.

Lemma 3.5 Let $T_{m-1,m}$, $T_{m,m}$, and $T_{m+1,m}$ be triangulations of C satisfying the assumptions of Definition 3.12. Then $n(T_{m,m}) - n(T_{m-1,m}) \geq 2$ implies that $n(T_{m+1,m}) \geq 3n(T_{m,m})$.

<u>Proof</u> Let ρ_m be a member of $T_{m,m}$ and f', $f'' \varepsilon$ Map$(T_{m,m}, T_{m-1,m})$ be defined as follows: Define f' as the map that sends ρ_m onto a member ρ_{m-1} of $T_{m-1,m}$ and such that the inverse of the interior of ρ_{m-1} under f' is just the interior of ρ_m. This can be done since $n(T_{m,m}) \geq n(T_{m-1,m})$. Now using the fact that $n(T_{m,m}) - n(T_{m-1,m}) \geq 2$, there exists three adjacent members ρ_m^1, ρ_m^2, and ρ_m^3 of $T_{m,m}$ and a map'' ε Map$(T_{m,m}, T_{m-1,m})$ such that $f''(\rho_m^{\,i}) = \rho_{m-1}$, for i = 1, 2, and 3. By Lemma 3.4 there exists a majorant $g \varepsilon$ Map$(T_{m+1,m}, T_{m,m})$ for Map$(T_{m,m}, T_{m-1,m})$. By definition of a majorant there exists a map $\alpha \varepsilon$ Map$(T_{m+1,m}, T_{m,m})$ such that the following diagram is commutative:

$$\begin{tikzcd} T_{m,m} \arrow[d, "f'"'] & T_{m+1,m} \arrow[l, "g"'] \arrow[d, "\alpha"] \\ T_{m-1,m} & T_{m,m} \arrow[l, "f'"] \end{tikzcd}$$

Now let ρ_{m+1}^1, ρ_{m+1}^2, and ρ_{m+1}^3 be members of $T_{m+1,m}$ such that $\alpha(\rho_{m+1}^i) = \rho_m^i$ for I = 1, 2, and 3. But from the commutativity of the diagram above and the definition of f'', $f'g(\rho_{m+1}^i) = \rho_{m-1}$ for i = 1, 2, and 3. Using both the last fact and the fact that $f'^{-1}(\mathrm{int}(\rho_{m-1})) = \mathrm{int}(\rho_m)$ yields $g(\rho_{m+1}^i) = \rho_m$ for i = 1, 2, and 3. Hence each member ρ_m of $T_{m,m}$ is an image by g of at least three members of $T_{m+1,m}$. Therefore,

$$n(T_{m+1,m}) \geq 3n(T_{m,m}). \quad //$$

At long last the proof of the existence of an A-category can be shown. Rogers' proof follows closely Mioduszewski's proof of the existence of a similar category in (27).

<u>Theorem 3.6</u> There exists an A-category $\mathcal{W}$.

<u>Proof</u> The proof starts by constructing the triangulations $\{T_{n,r}\}$, of the unit circle C_n, for $r \geq n-1$ and $n = 1, 2, \ldots$, that satisfies the assumptions of Definition 3.13. Let $T_{0,1}$ and $T_{1,1}$ be triangulations of C_0 and C_1 consisting of three and six segments respectively. By Lemma 3.3 there exists a triangulation $T_{2,1}$ of C_2 and a majorant $g_1^2 \in \mathrm{Map}(T_{2,1}, T_{1,1})$ for the class $\mathrm{Map}(T_{1,1}, T_{0,1})$. Now look at the triangulation $T_{2,2}$ of C_2 created by dividing the segments of $T_{2,1}$ into three equal parts. Also triangulations $T_{1,2}$ and $T_{0,2}$ of C_1 and C_0, respectively, can be obtained in a similar manner. The following diagram indicates what has been constructed to this point:

$$\begin{array}{ccccc} T_{0,1} & \leftarrow & T_{1,1} & \leftarrow & T_{2,1} \\ \uparrow & & \uparrow & & \uparrow \\ T_{0,2} & \leftarrow & T_{1,2} & \leftarrow & T_{2,2} \end{array}$$

The arrows indicate the existence of simplicial maps between the triangulations. From the construction it is clear that $\mathrm{Map}(T_{2,1}, T_{1,1}) \subseteq \mathrm{Map}(T_{2,2}, T_{1,2})$, and hence $\mathrm{Map}(T_{m,p}, T_{n,p}) \subseteq \mathrm{Map}(T_{m,r}, T_{n,r})$, if $p < r$ for the integers in question.

Assume that triangulations $T_{p,r}$ of C_p and maps g_{p-1}^p are already defined for $p, r \leq n$ and $r \geq p-1$, and that the following properties are satisfied:

(i) $g_{p-1}^{p} \varepsilon \text{ Map}(T_{p,p}, T_{p-1,p})$ and is a majorant for $\text{Map}(T_{p-1,p-1}, T_{m,p-1})$ when $p \leq n$ and $M = 1, \ldots, p-2$, and

(ii) $\text{Map}(T_{p,r}, T_{m,r}) \subseteq \text{Map}(T_{p,q}, T_{m,q})$ for $r < q \leq n$ and $m < p \leq n$.

After the induction hypothesis has been assumed, the following diagram indicates the construction:

$$
\begin{array}{ccccccccccc}
T_{0,1} & \leftarrow & T_{1,1} & \leftarrow & T_{2,1} & & & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & & & \\
T_{0,2} & \leftarrow & T_{1,2} & \leftarrow & T_{2,2} & \leftarrow & T_{3,2} & & & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & & & \\
\cdot & & & & & & & & & & \\
\cdot & & & & & & & & & & \\
\cdot & & & & & & & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
T_{0,n} & \leftarrow & T_{1,n} & \leftarrow & T_{2,n} & \leftarrow & T_{3,n} & \leftarrow & \cdots & \leftarrow & T_{n,n}
\end{array}
$$

Now by Lemma 3.4 there exists a triangulation, $T_{n+1,n}$, of C_{n+1}, and a simplicial map $g_n^{n+1} \varepsilon \text{ Map}(T_{n+1,n}, T_{n,n})$ that is a majorant for the classes Map $(T_{n,n}, T_{m,n})$, $m = 1, \ldots, n-1$. Let $T_{n+1,n+1}$ be the triangulation of C_{n+1} created by dividing each segment of $T_{n+1,n}$ into three equal parts. The triangulations $T_{p,n+1}$ of C_p, $p = 0, 1, \ldots, n$, are obtained in a similar manner. Then it is clear that $\text{Map}(T_{p,n}, T_{m,n})$ $\text{Map}(T_{p,n+1}, T_{m,n+1})$ for $M < p \leq n+1$. Together with (ii) above yields $\text{Map}(T_{p,r}, T_{m,r}) \subseteq \text{Map}(T_{p,q}, T_{m,q})$ for $r < q \leq n+1$ and $m < p < n+1$. Hence, by induction, the sequence of triangulations $\{T_{p,q}\}$, $q \geq p-1$ and $p = 1, 2, \ldots,$ have been constructed with the properties outlined in Definition 3.13.

Also the sequence of majorants, g^p_{p-1}, for Map$(T_{p-1,p-1}, T_{m,p-1})$, $m = 1, 2, \ldots, p-2$, have been constructed so that $g^p_{p-1} \in$ Map$(T_{p,p}, T_{p-1,p})$ for $p = 2, 3, \ldots$.

Now all that remains is to show that the constructed category, $\mathcal{W}$, satisfies the condition: $\lim_{r\to\infty}[n(T_{r,r})/n(T_{m,r})] = \infty$ for $m = 1, 2, \ldots$. Assert that $n(T_{m,m}) - n(T_{m-1,m}) \geq 2$ for all $m = 1, 2, \ldots$. From the construction it is true for $m = 1$ because $n(T_{1,1}) = 6$ and $n(T_{0,1}) = 3$. So suppose it is true for $m \geq k$. Then by Lemma 3.5 it is true that $n(T_{k+1,k}) \geq 3n(T_{k,k})$, and by the method of constructing $T_{k+1,k+1}$ and $T_{k,k+1}$ it is also true that $n(T_{k+1,k+1}) \geq 3n(T_{k,k+1})$. Therefore $n(T_{k+1,k+1}) - n(T_{k,k+1}) \geq 2n(T_{k,k+1})$, and it is clear that $n(T_{k,k+1}) > 1$ for $k \geq 1$. Hence $n(T_{k+1,k+1}) - n\ (T_{k,k+1}) \geq 2$, and by induction $n(T_{m,m}) - n(T_{m-1,m}) \geq 2$ is true for all $m = 1, 2, \ldots$. Now by Lemma 3.5 $n(T_{m+1,m}) \geq 3n(T_{m,m})$ for $m = 1, 2, \ldots$. Also from the way the triangulations were constructed and the last fact it follows that for $m+j \leq r$, $n(T_{m+j,r})/n(T_{m+j,m+j})/n(T_{m+j-1,m+j}) \geq 3$. Finally, $n(T_{r,r})/n(T_{m,r}) = n(T_{m+1,r})n(T_{m+2,r}) \ldots n(T_{r,r})/n(T_{m,r})n(T_{m+1,r}) \ldots n(T_{r-1,r}) \geq 3^{r-m-1}$, and hence $\lim_{r\to\infty}[n(T_{r,r})/n(T_{m,r})/n(T_{m,r})/n(T_{m,r})] = \infty$ for $m = 1, 2, \ldots$. Therefore $\mathcal{W}$ is an A-category. //

Now that the existence of A-categories have been guaranteed, it is time to show what these categories have to do with pseudo-circles. Recalling Chapter I, it was stated that Mardesic and Segal (25) had shown that circularly chainable continua may be regarded as inverse limits of unit circles with onto bonding maps. Using this, Rogers (37) was able to deduce that a continuum H is circularly chainable with a defining sequence of circular chains $\{D_i\}$ such that D_{i+1} circles k_i times in D_i if and only if there exists an inverse limit representation, $\varprojlim\{C_i, g_i^m\}$, for H such that C_i is the unit circle and g_i^m is a continuous map of C_m onto C_i and $\deg(g_i^{i+1}) = k_i$. This was also presented in Chapter I. It should be noted that these facts are very important in what follows. If there are any questions about these facts, consult Chapter I or the appropriate reference. Now it can be seen that from $\mathcal{W}$ the inverse limit, $\varprojlim\{C_i, g_i^{i+1}\}$, where C_i is the unit circle and g_i^{i+1} the majorant for $\mathrm{Map}(T_{i,i}, T_{n,i})$, n = 1, 2, ..., i, is a circularly chainable but not chainable continuum. It will be shown later that this inverse limit is also hereditarily indecomposable. Hence it will be a pseudo-circle. This discussion leads to the following definition:

Definition 3.14 Let $\mathcal{W}$ be an A-category. Then the inverse limit, $\varprojlim\{C_i, g_i^{i+1}\}$, where C_i are unit circles and g_i^{i+1} the majorant for $\mathrm{Map}(T_{i,i}, T_{n,i})$, n = 1, 2, ..., i, is called the universal circularly chainable continuum in $\mathcal{W}$. Denote this continuum by $\mathrm{UCC}(\mathcal{W})$. Also if M is a set of natural numbers and if the morphisms of $\mathcal{W}$ are restricted to those of degree which is a product of nonnegative powers of elements of M, then $\mathcal{W}$ is called a A(M)-category and $\mathrm{UCC}(\mathcal{W})$ for this category is also denoted by $\mathrm{U(M)CC}(\mathcal{W})$.

Now when it is shown that $UCC(\mathcal{W})$ is hereditarily indecomposable, it will be true that $UCC(\mathcal{W})$ is a pseudo-circle. Also the beginnings of Rogers' presentation will be completed. Then all the machinery of Rogers that has been set up in the preceding pages can be put to use in showing that all circularly chainable continua are the continuous image of some pseudo-circle. This fact will in turn be used in the proof of the characterization of the continuous images of the pseudo-circles.

The proof of the hereditarily indecomposability of $UCC(\mathcal{W})$ depends on a notion of "oscillation" of simplicial maps of T' onto T where T' and T are triangulations of the unit circle C. This notion of "oscillation" parallels the idea of crookedness which was so crucial in Fearnley's characterization of the continuous images of the pseudo-arc. Also the characterization of subcontinua of $UCC(\mathcal{W})$ in terms of the inverse limit, $\underleftarrow{\mathrm{limit}}\{A_n, g_n'^{\,n+1}\}$, where A_n is a subcontinuum of C_n and $g_n'^{\,n+1} = g_n^{n+1}|A_n$, has an important part in the proof of the indecomposability of $UCC(\mathcal{W})$. Preliminary to this proof are two lemmas relating the above notions to $UCC(\mathcal{W})$. But since the exposition in this section has been long and dry, it will suit the purpose of this paper better if the proofs of these lemmas are omitted. However, the notion of "oscillation" will be defined formally, and an example will be presented to help clarify this definition. The example mentioned above will be constructed in a way that should show the parallel nature of "oscillation" and crookedness. Then the two lemmas mentioned previously will be stated without proof. Finally, the proof of the hereditarily indecomposability of UCC ($\mathcal{N}$) will be presented. In this manner one should get a taste of what is needed in the proof of the indecomposability of UCC($\mathcal{N}$) without getting too full to move onto more important matters.

<u>Definition 3.15</u> Let T' and T'' be triangulations of the unit circle C and f ε Map(T'', T'). Let J be a subcontinuum of C that is a union of members of T'. Then f is said to have full oscillation over J if, for every J' $\subsetneq$ J, where J' is a union of members of T', and every component K of f^{-} (J) such that K $\neq$ C and f maps the endpoints of K onto the endpoints of J, there exists two subcontinua K_1, $K_2 \subseteq$ K such that $f(K_1) = f(K_2) = J$.

Definition 3.15 is Rogers' generalization for the unit circle of Mioduszewski's (28) definition of "oscillation" for the unit interval. But the definition, by itself, is not easy to understand, nor is it easy to see how this notion of "oscillation" is parallel to the concept of crookedness. So now it will be shown how to constract a map of "full oscillation" from a chain crooked in another chain. In Figure 22 notice that the chain, Q = $\{q_1, \ldots, q_{19}\}$, is crooked in the chain, P = $\{p_1, \ldots, p_5\}$. Let T' be a triangulation of the unit circle C into five equal parts, t_1', ..., t_5', and T'' be a triangulation of C into 19 equal segments, t_1'', ..., t_{19}'', both pictured in Figure 23. Now let f ε Map(T'', T') be

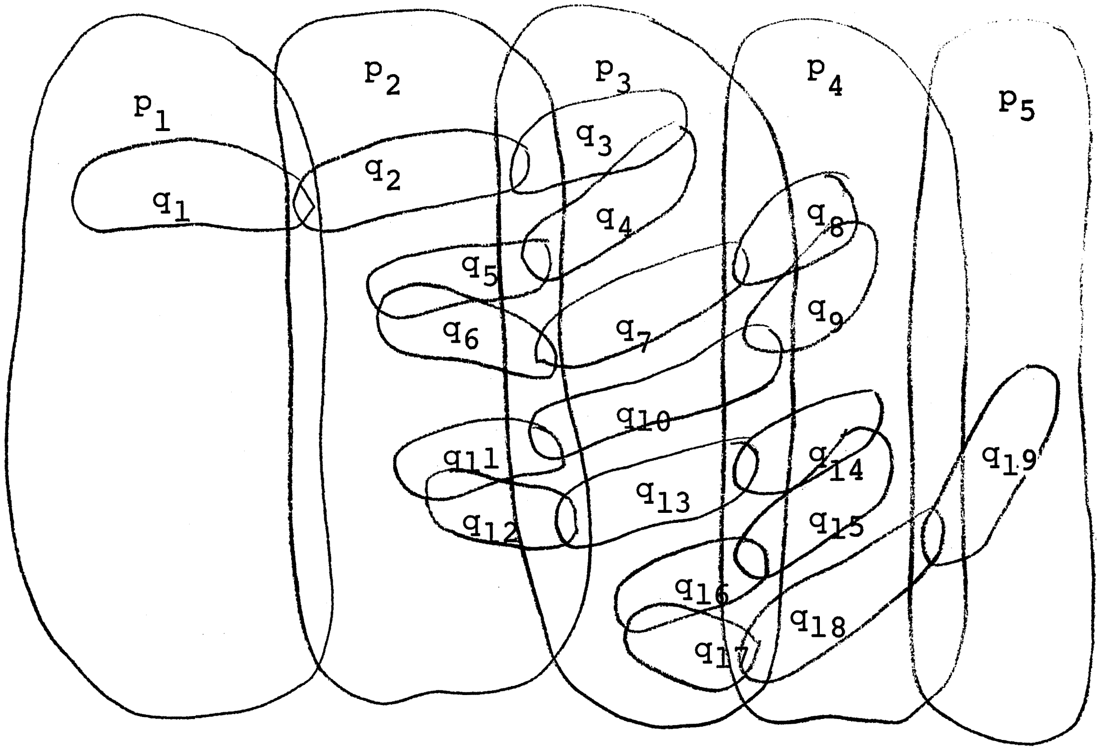


Figure 22. Q Crooked in P

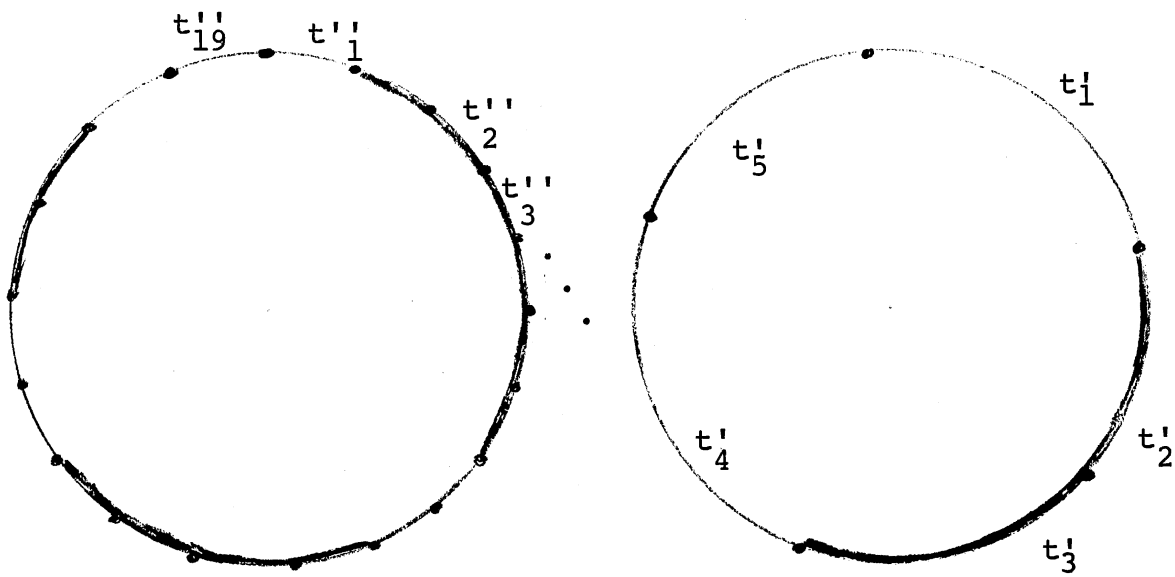


Figure 23. $f^{-1}(J)$ and J

defined such that f maps: t_1'' onto t_1'; t_2'', t_5'', t_6'', t_{11}'', and t_{12}'' onto t_2'; t_3'', t_4'', t_7'', t_{10}'', t_{13}'', t_{16}'', and t_{17}'' onto t_3'; t_8'', t_9'', t_{14}'', t_{15}'', t_{18}'' onto t_4'; and t_{19}'' onto t_5'. Note the correspondence this map has with the pattern of Q in P. Let $J = t_2' \quad t_3'$, and then $f^{-1}(J) = \bigcup_{i \varepsilon M}\{t_i''\}$, where $M = \{2, \ldots, 7, 10, \ldots, 13, 16, \text{ and } 17\}$. Both J and $f^{-1}(J)$ are illustrated in Figure 23. Notice that $K = t_2'' \cup \ldots \cup t_7''$ is the only component of $f^{-1}(J)$ such that f maps the endpoints of K onto the endpoints of J. The only possibilities for J' are t_2' or t_3'. If $J' = t_2'$, then there is two subcontinua, t_2'' and t_5'' of K, such that $f(t_2'') = f(t_5'') = J'$. If $J' = t_3'$, then there is two subcontinua, t_3'' and t_4'' of K, such that $f(t_3'') = f(t_4'') = J'$. Therefore f has a full oscillation over J.

For an example of a map that doesn't have a full oscillation over some interval it suffices to look at the map f just constructed. Let $J = t_1' \cup t_5'$ and thus $f^{-1}(J) = t_1'' \cup t_{19}''$. See Figure 24 for J and $f^{-1}(J)$. Clearly $f^{-1}(J)$, itself, is the only component of $f^{-1}(J)$, and so f maps the endpoints of this component onto the endpoints of $f^{-1}(J)$. But if $J' = t_1'$, then $J' \subsetneq J$ and there exists only one subcontinuum of $f^{-1}(J)$, t_1'', such that $f(t_1'') = J'$. Therefore, by Definition 3.15, f doesn't have a full oscillation over J.

The reason that f doesn't have a full oscillation over every interval is that it wasn't constructed from circular chains, one crooked in the other. The construction of f from Q crooked in P interrupted both P and Q as circular

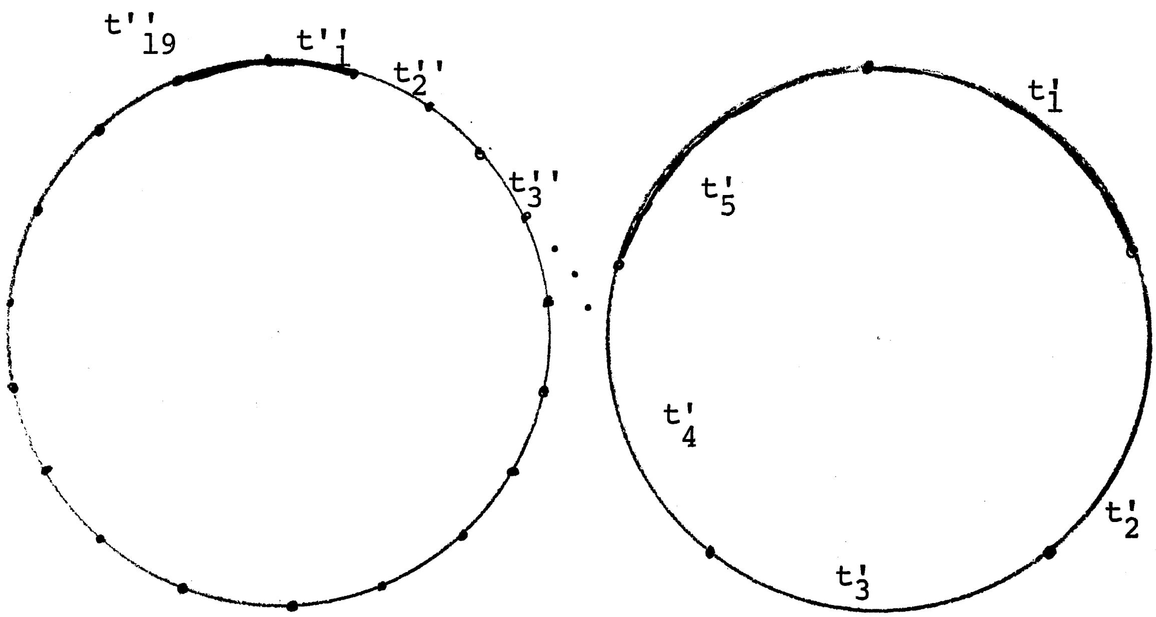


Figure 24. Another $f^{-1}(J)$ and J

chains. That is, the construction of f would have been the same if p_1 and p_5 had intersected and q_1 and q_{19} had intersected to form circular chains. But interrupted as circular chains Q is crooked only in some intervals of P.

Now the two lemmas relating the concept of "oscillation" to UCC($\mathcal{W}$) and subcontinua of UCC($\mathcal{W}$) will be stated.

<u>Lemma 3.6</u> Let $\mathcal{W}$ be an A-category and $\varprojlim\{C_n, g_n^{n+1}\} =$ UCC($\mathcal{W}$). Then g_n^{n+1} has full oscillation over any subcontinua of C_n consisting of members of $T_{n,n}$.

Recall from Chapter I that subcontinua of UCC($\mathcal{W}$) = $\varprojlim\{C_n, g_n^{n+1}\}$ are the inverse limits, $\varprojlim\{A_n, g_n'^{n+1}\}$, where A_n is a subcontinuum of C_n and $g_n^{'n+1}$ A_{n+1}. Then the next lemma relates the indecomposability of these subcontinua to the property of "oscillation".

<u>Lemma 3.7</u> Let $\mathcal{W}$ be an A-category, $\underleftarrow{\text{limit}}\{C_n, g_n^{n+1}\} = \text{UCC}(\mathcal{W})$, and $X = \underleftarrow{\text{limit}}\{A_n, g_n^{'n+1}\}$ be a subcontinuum of $\text{UCC}(\mathcal{W})$. Also let A_n' be the maximal subcontinuum consisting of members of $T_{n,n}$ contained in the interior of A_n. Then if $g_n^{'n+1}$ has full oscillation over A_n' for each n, then X is indecomposable.

Finally, the hereditary indecomposability of $\text{UCC}(\mathcal{W})$ can be shown. Rogers' (37) proof comes directly from a similar proof of Mioduszewski's (27) showing that the inverse limit of unit intervals with certain bonding maps is hereditarily indecomposable.

<u>Theorem 3.7</u> $\text{UCC}(\mathcal{W})$ is hereditarily indecomposable.

<u>Proof</u> Let $X = \underleftarrow{\text{limit}}\{A_n, g_n^{'n+1}\}$ be a subcontinuum of $\text{UCC}(\mathcal{W})$. It will be shown that X is indecomposable. Let A_n' be the maximal subcontinuum, consisting of members of $T_{n,n}$, that is contained in the interior of A_n. Without loss of generality there exists a component K of $(g_n^{n+1})^{-1}(A_n')$ such that K is contained in A_{n+1} and g_n^{n+1} maps the endpoints of K onto the endpoints of A_n'. Now let A_n'' be a proper subcontinuum of A_n' consisting of members of $T_{n,n}$. Then by Lemma 3.6, g_n^{n+1} has full oscillation over A_n'. Hence there exists subcontinua $K_1, K_2 \subseteq K$ such that $g_n^{n+1}(K_1) = g_n^{n+1}(K_2) = A_n''$. Because

$K \subseteq A_{n+1}$ then $K_1, K_2 \subseteq A_{n+1}$ and therefore $g_n^{'n+1}|A_n$ also has full oscillation over A_n'. Then by Lemma 3.7 the subcontinuum $X = \varprojlim\{A_n, g_n^{'n+1}\}$ is indecomposable. So every subcontinuum of UCC($\mathcal{W}$) is indecomposable and hence UCC ($\mathcal{W}$) is hereditarily indecomposable. //

In the previous pages it has been shown that a pseudo-circle can be obtained as an inverse limit of objects and morphisms of an A-category. This special inverse limit from an A-category $\mathcal{W}$ was given the name, universal circularly chainable continuum in $\mathcal{W}$, and denoted by UCC($\mathcal{W}$). The reason that Rogers went to all this trouble is that now using properties of inverse limits and the special bonding maps of UCC($\mathcal{W}$) he can show that there is a pseudo-circle that maps continuously onto every circularly chainable continuum. To present this work of Rogers is the next goal of this paper. It is hoped that this presentation will be nice enough to outweigh all the trouble needed to set up the necessary machinery. I will address myself more specifically to this point in the summary to Chapter III.

The first step in the presentation that there exists a pseudo-circle that maps continuously onto every circularly chainable continuum is to show that every circularly chainable continuum has an inverse limit representation where the objects and bonding maps are contained in some A-category. This is important because it will be shown later that there exists a pseudo-circle that maps continuously onto every circularly chainable continuum that is "contained" in some A-category. Three preliminary lemmas are needed to accomplish this. They all involve showing that continuous maps of the unit circle onto the unit circle can be "approximated" by "simplicial" maps. Now the idea of "approximate" will be made precise and the first lemma presented.

<u>Definition 3.16</u> Define the distance between two points on the unit circle C as the shorter of the arc length between the two points. Then for $\varepsilon > 0$ and f and g two continuous maps of C onto C, define $f =_\varepsilon g$ when $|f(x) - g(x)| \leq \varepsilon$ for each x in C.

<u>Lemma 3.8</u> Let f be a continuous map of C onto C and $\varepsilon > 0$ be given. Then there exists a map $g \in U$ such that $f =_\varepsilon g$.

<u>Proof</u> Let T' be a triangulation of C into equal segments such that the diameter of each segment is less than ε. Denote the vertices of T' by $\{x_i\}_{i=1}^{k'}$. Also denote the interior of the union of the two segments of T' that contain x_i by $St(x_i)$. Consider the collection $\{f^{-1}(St(x_i))\}_{i=1}^{k'}$. By definition of continuity this collection is an open cover of C. But since C is compact there exists a $\delta > 0$ such that for each $x \in C$, $\{y \in C| \ |x - y| < \delta\} = N_\delta(x) \subseteq f^{-1}(St(x_i))$ for some $1 \leq i \leq k'$. Now let T'' be a triangulation of C into equal segments such that the diameter of each segment is less than $\delta/2$. Denote the vertices of T'' by $\{y_i\}_{i=1}^{k''}$. Hence, for each $1 \leq i \leq k''$, $St(y_i) \subseteq N_\delta(y_i) \subseteq f^{-1}(St(x_j))$

for some $1 \leq j \leq k'$. Therefore $f(St(y_i)) \subseteq St(x_j)$ for each $1 \leq i \leq k''$ and some $1 \leq j \leq k'$. Now define a map g from the vertices of T'' into the vertices of T' in the following way: $g(y_i) = x_j$ for one x_j such that $f(St(y_i)) \subseteq St(x_j)$. Assert that g maps onto the vertices of T'. Suppose that x_l is a vertex of T' that is not the image of any vertex of T'' under g. Let $y \in C$ such that $y = f^{-1}(x_l)$. Then $y \in St(y_i)$ for some $1 \leq i \leq k''-1$. Also $f(St(y_i)) \subseteq St(x_m)$ for some $1 \leq m \leq k'$ such that $g(y_i) = x_m$. Hence $f(y) = x_l \in St(x_m)$, and so $x_l = x_m$. But this contradicts the fact that x_l has no image under g. Therefore the assertion is proved.

It will be true that g can be extended to a simplicial map of T'' onto T' if the image of two consecutive vertices of T'' under g is either one vertex of T' or two consecutive vertices of T'. Let y_i and y_{i+1} be two vertices of T''. Then $f(y_iy_{i+1}) \subseteq f(St(y_i)) \subseteq St(g(y_i))$ and $f(y_iy_{i+1}) \subseteq f(St(y_{i+1})) \subseteq St(g(y_{i+1}))$. Hence $St(g(y_i)) \cap St(g(y_{i+1})) \neq \emptyset$. Then by the properties of $St(\cdot)$ either $f(y_iy_{i+1}) = g(y_i) = g(y_{i+1})$ or $g(y_i)$ and $g(y_{i+1})$ are consecutive vertices of T' and $f(y_iy_{i+1}) = g(y_i)g(y_{i+1})$. Therefore g can be extended to a simplicial map of T'' onto T'. Denote this extension by g. Now let $x \in C$. If $x = y_i$ for some $1 \leq i \leq k''$, then $g(y_i) = x_j$ for some $1 \leq j \leq k'$ such that $f(St(y_i)) \subseteq St(x_j)$. Therefore, since $g(x), f(x) \in St(x_j)$ and the diameter of $St(x_j)$ is less than ε, $|g(x) - f(x)| \leq \varepsilon$. If x is not a vertex of T'', then $x \in y_iy_{i+1}$ for some $1 \leq i \leq k''-1$. But $x \in St(y_i)$ and $x \in St(y_{i+1})$ implies that $f(x) \in St(g(y_i)) \cap$

$St(g(y_{i+1})) = g(y_i)g(y_{i+1})$. Therefore $f(x), g(x) \varepsilon g(y_i)g(g(y_{i+1})$, and so $|f(x) - g(x)| \leq \varepsilon$. By Definition 3.16 then $f =_\varepsilon g$. //

The next lemma is a very technical lemma that relies heavily on Lemma 3.8 for its proof. This proof will add nothing to the presentation that is to follow, and so it will be omitted. The proof of this lemma can be found in Rogers (37).

Lemma 3.9 Let f be a continuous map of C onto C of positive degree and $\varepsilon > 0$. Then there exists an integer m such that whenever T' and T'' are triangulations of C such that $n(T'')/n(T') \geq m$ and the diameter of each segment of T' is less than $\varepsilon/4$, there exists a map $g \varepsilon \mathrm{Map}(T'',T')$ such that $f =_\varepsilon g$.

The last preliminary lemma, before it can be shown that every circularly chainable continuum has inverse limit representation where the objects and bonding maps are contained in some A-category, relates the facts of Lemma 3.8 and Lemma 3.9 to the familiar setting of an A-category.

Lemma 3.10 Let $0 < \varepsilon < 1$, an integer m, and a continuous map f of C onto C be given. Then there exists r_o such that for any $r \geq r_o$ there exists $g \varepsilon \mathrm{Map}(T_{r,r}, T_{m,r})$ such that $f =_\varepsilon g$, $\deg(f) = \deg(g)$.

Proof The proof follows immediately from Lemma 3.9 together with the fact that $\lim_{r\to\infty}[n(T_{r,r})/n(T_{p,r})] = \infty$ for $p = 1, 2, \ldots$. //

In this section of Chapter III it seems that every important theorem is preceded by enough preliminary lemmas to isolate these important theorems and make the presentation seem somewhat discontinuous. As a partial remedy to this I try to keep emphasizing, in the expository, the important theorems that have been presented as well as the ones to follow. At long last it can now be shown that every circularly chainable continuum has an inverse limit representation where the objects and bonding maps are contained in some A-category. The proof of this theorem uses a property of Mioduszewski's (28) of homeomorphisms between inverse limits that was seen in Chapter I.

<u>Theorem 3.8</u> Let $X = \varprojlim\{C_n, \alpha_n^m\}$ be a circularly chainable continuum such that each α_n^{n+1} has positive degree. Let P be a set of primes with the property that if p is a prime factor of the degree of some α_n^m, then $p \in P$. Then C_n and α_n^m are contained in an arbitrary A(P)-category $\mathcal{W}$. It is said that X is embeddable in $\mathcal{W}$.

<u>Proof</u> Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers tending to zero. The following diagram will be constructed:

$$\begin{tikzcd}
C_1 \arrow[d, "e_1"] & C_2 \arrow[l] \arrow[d, "e_2"] & C_3 \arrow[l, "\alpha_2^3"'] \arrow[d, "e_3"] & \cdots \arrow[l] \\
C_1 & C_2 \arrow[l, "\beta_1^2"] & C_3 \arrow[l, "\beta_2^3"] & \cdots \arrow[l]
\end{tikzcd}$$

where $e_1, e_2, \ldots$ are identities and (1) $\beta_j^n e_n \alpha_n^m =_{\varepsilon_n} \beta_j^m e_m$, (2) $\alpha_j^n e_n^{-1} \beta_n^m =_{\varepsilon_n} \alpha_j^m e_m^{-1}$, for $j < n < m$, and $\underleftarrow{\text{limit}}\{C_n, \beta_n^{n+1}\}$ is embeddable in $\mathcal{W}$. The existence of this diagram and properties guarantees that $\underleftarrow{\text{limit}}\{C_n, \alpha_n^{n+1}\}$ is homeomorphic to $\underleftarrow{\text{limit}}\{C_n, \beta_n^{n+1}\}$ is homeomorphic to limit$\{C_n, \beta_n^{n+1}\}$, because of a theorem of Mioduszewski's (28) concerning homeomorphisms between inverse limits. This theorem of Mioduszewski's was presented in Chapter I.

The construction will be done by induction. The map β_1^2 will be defined first. Let $\varepsilon_1' = \varepsilon_1/2$. Then by Lemma 3.10 with m = 1 there exists an integer r_2 and a map $\beta_1^2 \in \mathrm{Map}(T_{r_2,r_2}, T_{m,r_2})$ such that $\alpha_1^2 =_{\varepsilon_1'} \beta_1^2$. Let δ_2 be the delta of uniform continuity for α_1^2 and δ_2' be the delta of uniform continuity for β_1^2. The delta of uniform continuity for α_1^2 will be the positive number δ_2 such that if $x,y \in C_2$, and $|x-y| < \delta_2$, then $|\alpha_1^2(x)-\alpha_1^2(y)| < \varepsilon_1'$ and $|\alpha_1^2(x)-\alpha_1^2(y)| < \varepsilon_2$ for $1 \leq i < 2$. Now let $\varepsilon_2' = [\min(\delta_2, \delta_2', \varepsilon_2)]/2$. Again using Lemma 3.10 with $m = r_2$ there exists an integer r_3 and a map $\beta_2^3 \in \mathrm{Map}(T_{r_3,r_3}, T_{r_2,r_3})$ such that $\alpha_2^3 =_{\varepsilon_2'} \beta_2^3$. Now assume for $j \leq k$ that r_j, δ_{j-1}, δ_{j-1}', ε_{j-1}', and β_{j-1}^j have been defined in a manner similar to the cases described. Consider $j = k+1$. Let δ_k be the delta of uniform continuity for α_{k-1}^k and δ_k' the delta of uniform continuity for β_{k-1}^k. Define $\varepsilon_k' = [\min(\delta_k, \delta_k', \varepsilon_k)]/2$. Now by Lemma 3.10 with $m = r_k$ there exists an integer r_{k+1} and a map $\beta_k^{k+1} \in \mathrm{Map}(T_{r_{k+1},r_{k+1}}, T_{r_k,r_{k+1}})$ such that $\alpha_k^{k+1} =_{\varepsilon_k'} \beta_k^{k+1}$. Hence, by induction, r_k, δ_k, δ'_k, $\varepsilon'_,$, and β_k^{k+1} are defined for k = 1, 2, ...

Therefore, the construction of the diagram has been completed and all that remains is to show that properties (1) and (2) hold. Only property (1) will be proved in this paper. Property (2) can be proved in a similar manner.

Let $x \in C_m 1$ Then $|\alpha_{m-1}^m(x)-\beta_{m-1}^m(x)| < \varepsilon'_{m-1}$. But $\varepsilon'_{m-1} = [\min(\delta_{m-1}, \delta'_{m-1}, \varepsilon_{m-1})]/2$ and so $|\alpha_{m-2}^{m-1}(\alpha_{m-1}^m(x)) - \alpha_{m-2}^{m-1}(\alpha_{m-1}^m(x))| < \varepsilon'_{m-2}$. Also since $\alpha_{m-2}^{m-1} = _{\varepsilon'_{m-2}}\ \beta_{m-2}^{m-1}$, then $|\alpha_{m-2}^{m-1}(\beta_{m-1}^m(x)) - \beta_{m-2}^{m-1}(\beta_{m-1}^m(x))| < \varepsilon'_{m-2.}$ By the triangle inequality then $|\alpha_{m-2}^m(x) - \beta_{m-2}^m(x)| < 2\varepsilon'_{m-2.}$ But recall that $2\varepsilon'_{m-2} = \min(\delta_{m-2}, \delta'_{m-2}, \varepsilon_{m-2})$ and hence $|\alpha_{m-3}^{m-2}(\alpha_{m-2}^m(x)) - \alpha_{m-3}^{m-2}(\beta_{m-2}^m))| < \varepsilon'_{m-3}$. Again since $\alpha_{m-3}^{m-2} = _{\varepsilon'_{m-3}}\ \beta_{m-3}^{m-2}$, then $|\alpha_{m-3}^{m-2}(\beta_{m-2}^m(x)) - \beta_{m-3}^{m-2}(\beta_{m-2}^m(x))| < \varepsilon'_{m-3}$, and the triangel inequality implies $|\alpha_{m-3}^m(x)-\beta_{m-3}^m(x)| < \varepsilon'_{m-3}$. In a similar manner it can be shown that $|\alpha_n^m(x)-\beta_n^m(x)| < 2\varepsilon'_n = \min(\delta_n,\ '_n,\ _n)$. By the definition of δ'_n, finally, $|\beta_j^n(e_n\alpha_n^m(x)) - \beta_j^n(\beta_n^m e_m(x))| < \varepsilon_n$. But x was arbitrary and so $\beta_j^n e_n \alpha_n^m =_{\varepsilon_n} \beta_j^m e_m$.

Hence the diagram with the desired properties have been constricted. By the choose of β_n^{n+1}, note that $\deg(\beta_n^{n+1}) = \deg(\alpha_n^{n+1})$ by Lemma 3.10, it turns out that $\{C_n,\ _n^{n+1}\}$ is an inverse system and it is contained in . Therefore by Mioduszewski's theorem $X = \varprojlim\{C_n, \beta_n^{n+1}\}$ is homeomorphic to $\varprojlim\{C_n,\ _n^{n+1}\}$ which is embeddable in . //

The second step in Rogers' presentation that there is a pseudo-circle that maps continuously onto every circularly continuum is to show that if $\mathcal{U}$ and $\mathcal{W}$ are two A(N)-categories, then the universal circularly chainable continuum for $\mathcal{U}$ is homeomorphic to the universal circularly chainable continuum for $\mathcal{W}$. This will be used to prove that there is one pseudo-circle that maps continuously onto all circularly chainable continua that are embeddable in some A-category. The proof of this next theorem uses another property of inverse limits due to Mioduszewski (28). This property can be found in Chapter I of this paper along with other properties of inverse limits and a review, at this time, of the section of Chapter I that deals with the properties of inverse limits is suggested. Then the reading of the proofs in the remainder of this section will be easier. The property of the bonding maps of the universal circularly chainable continua is critical in the proof of the next theorem.

<u>Theorem 3.9</u> Let N be a set of natural numbers and $\mathcal{U}$ and $\mathcal{W}$ be A(N)-categories. Then U(N)CC($\mathcal{U}$) and U(N)CC($\mathcal{W}$) are homeomorphic.

<u>Proof</u> Let U(N)CC($\mathcal{U}$) $= \underset{\leftarrow}{\text{limit}}\{C_n, \alpha_n^{n+1}\}$ and U(N)CC($\mathcal{W}$) $= \underset{\leftarrow}{\text{limit}}\{C_n, \beta_n^{n+1}\}$. To show that U(N)CC(U) is homeomorphic to U(N)CC($\mathcal{W}$) it is sufficient to show that for every $\varepsilon > 0$, every m and n, and every continuous map f_{mn} of C_m onto C_n whose degree is the product of nonnegative powers of elements of N, there exists n', n'' > n and maps $g_{n,m}$ of $C_{n'}$ onto C_m and $g_{n''m}$ of $C_{n''}$ onto C_m, both of whose degree are the product of nonnegative powers of elements of N, such that (1) $f_{mn}g_{n'm} =_\varepsilon \alpha_n^{n'}$ and (2) $f_{mn}g_{n''m} = \beta_n^{n''}$. This condition

is due to Mioduszewski (28), and can be found in Chapter I along with other properties of inverse limits.

Because of the symmetry of (1) and (2) it is sufficient to show that there exists $n' > n$ and a map $g_{n'm}$ of $C_{n'}$ onto C_n such that $f_{mn}g_{n'm} =\varepsilon\ \alpha_n^{n'}$. By Lemma 3.10 there exists a map $f'_{mn} \varepsilon$ Map$(T_{r,r}, T_{n,r})$ such that $f_{mn} = f'_{mn}$ and $\deg(f_{mn}) = \deg(f'_{mn})$. But since the degree of f_{mn} is the product of non-negative powers of elements of N, then $f'_{mn} \varepsilon \mathcal{U}$. Let $n' = r+1$. Because α_r^{r+1} is a majorant for Map$(T_{r,r}, T_{n,r})$, there exists a map $g_{n'm} \varepsilon$ Map$(T_{r+1,r}, T_{r,r})$ such that $f'_{mn}g_{n'm} = \alpha_n^r\ _r^{r+1} = \alpha^{n'}_n$. However, $f_{mn} =\varepsilon\ f'_{mn}$ and thus $f_{mn}g_{n'm} = \varepsilon\ f'_{mn}g_{n'm} = \alpha_n^{n'}$. Therefore (1) has been proved and U(N)CC($\mathcal{U}$) is homeomorphic to U(N)CC($\mathcal{W}$). //

The proofs of Theorem 3.8 and Theorem 3.9 illustrate the power of the properties of inverse limits to Rogers' presentation. However, they have the effect of making the proofs unintuitive and very abstract. Unfortunately this pattern does not stop with Theorem 3.8 and Theorem 3.9.

Now it will be shown that if a circularly chainable continuum X is embeddable in an A-category $\mathcal{W}$, then X is a continuous image of UCC($\mathcal{W}$). This is a major step in showing that there is a pseudo-circle that maps continuously onto every chainable continuum. Again, the proof is Theorem 3.10 relies heavily on a basic property of inverse limits that can be found in Eilenberg and Steenrod (11) or Dugundji (10). Also this property is presented in Chapter I. The proof of Theorem 3.10 also shows the need of some of the more artificial properties of an A-category.

<u>Theorem 3.10</u> If the circularly chainable continuum X can be embedded in an A(N)-category $\mathcal{W}$, then X is the continuous image of U(N)CC($\mathcal{W}$).

<u>Proof</u> Let U(N)CC($\mathcal{W}$) = $\underleftarrow{\lim}\{C_n, \alpha_n^{n+1}\}$ is contained in the A(N)-category $\mathcal{W}$. Since X is embeddable in $\mathcal{W}$, then X = $\underleftarrow{\lim}\{C_{j_n}, \beta_j^j m\}$ where $\{C_j, \beta_j^j m\}$ is contained in $\mathcal{W}$. To show that X is the continuous image of U(N)CC($\mathcal{W}$) it will be sufficient to construct the following commutative diagram:

$$\begin{tikzcd}
C_{n_1} \arrow[d, "f_1"'] & C_{n_2} \arrow[l, "\alpha_m^{n_2}"'] \arrow[d, "f_2"'] & \cdots \arrow[l] & C_{n_k} \arrow[l] \arrow[d, "f_n"'] & C_{n_{k+1}} \arrow[l, "\alpha_{n_m}^{n_{k_m}}"'] \arrow[d, "f_{k+1}"'] & \cdots \arrow[l] \\
C_{j_1} & C_{j_2} \arrow[l, "\beta_{j_1}^{j_2}"] & \cdots \arrow[l] & C_{j_k} \arrow[l] & C_{j_{k+1}} \arrow[l, "\beta_{j_k}^{j_{k+1}}"] & \cdots \arrow[l]
\end{tikzcd},$$

where the vertical maps f_1, f_2, ... are continuous and onto. This is the property of inverse limits in Eilenberg and Steenrod (11) or Dugundji (10) that was mentioned before.

The construction is by induction. Let $n_1 = j_i+1$ and $f_1 \in \bigcup_{r=1}^{n_i-1} \mathrm{Map}(T_{n_1,n_1}, T_{r,n_1})$. Now assume that n_i and f_i have already been defined for $i \leq k$ such that $n_i = j_i+1$ and $f_i \in \bigcup_{r=1}^{n_i-1} \mathrm{Map}(T_{n_i,n_i}, T_{r,n_i})$. Let $n_{k+1} = j_{k+1} + 1$. By definition of U(N)CC($\mathcal{W}$), the mapping $\alpha_{n_{k+1}}^{n_{k+1}} - 1 = \alpha_{j_{k+1}}^{j_{k+1}} + 1$ is a majorant for $\mathrm{Map}(T_{j_{k+1},j_{k+1}}, T_{r,j_{k+1}})$, $r = 1, \ldots, j_{k+1} - 1$. But both

$f_k\alpha_{n_{k+1}}^{n_{k+1}-1}$ and $\beta_{j_k}^{j_{k+1}}$ belong to $\bigcup_{r=1}^{j_{k+1}-1}\mathrm{Map}(T_{j_{k+1},j_{k+1}}, T_{r,j_{k+1}})$ and so by Lemma 3.4 there exists a map $f_{k+1} \in \bigcup_{r=1}^{j_{k+1}}\mathrm{Map}(T_{j_{k+1}+1,j_{k+1}+1}, T_{r,j_{k+1}+1})$ such that $\beta_{j_k}^{j_{k+1}}f_{k+1} = (f_k\alpha_{n_{k+1}}^{n_{k+1}-1})\alpha_{n_{k+1}-1}^{n_{k+1}} = f_k\alpha_{n_k}^{n_{k+1}}$. Therefore by induction the diagram is constructed and is commutative. Hence X is the continuous image of $U(N)CC(\mathcal{W})$. //

With Theorem 3.10 out of the way, the journey to show that there exists a pseudo-circle that maps continuously onto every circularly chainable continuum, is now all downhill. If you have endured this presentation and my expository and are still with me, then the rest of the section will proceed easily. The next fact is an immediate consequen-e of Theorem 3.8, Theorem 3.9, and Theorem 3.10.

<u>Theorem 3.11</u> There is one universal circularly chainable continuum that maps continuously onto each circularly chainable continuum that is embeddable in an A-category.

<u>Proof</u> Let N be the set of all natural numbers and $\mathcal{W}$ an A(N)-category. Then $U(N)CC(\mathcal{W})$ is the desired universal circularly chainable continuum. Suppose X is a circularly chainable continuum that is embeddable in an A(M)-category $\mathcal{U}$, where M is a subset of N. Then by definition of an A(M)-category, $\mathcal{U}$ is contained in an A(N)-category $\mathcal{W}'$. So X is embeddable in $\mathcal{W}'$. By Theorem 3.10, X is the continuous

image of $U(N)CC(\mathcal{W}')$. But Theorem 3.9 implies that $U(N)CC(\mathcal{W})$ is homeomorphic to $U(N)CC(\mathcal{W}')$. Then X is the continuous image of $U(N)CC(\mathcal{W})$. Therefore each circularly chainable continuum that is embeddable in an A-category is the continuous image of one universal circularly chainable continuum. //

All that remains is to show that the circularly chainable continua not embeddable in some A-category are the continuous image of the pseudo-circle described in Theorem 3.11. Rogers (37) does this by first showing that the pseudo-arc is the continuous image of each universal circularly chainable continuum. Then using the result of Ingram's (19) that all circularly chainable continua not embeddable in some A-category are chainable, Rogers is able to show that these circularly chainable continua are the continuous image of the pseudo-arc and thus each universal circularly chainable continuum. Rogers' proof of this fact is very intuitive.

<u>Theorem 3.12</u> The pseudo-arc M is a continuous image of each universal circularly chainable continuum.

<u>Proof</u> Let M be assumed to be chainable between points p and q in the plane. Join M to another copy of M at p and q. That is the chain D_{i+1} circles inside of D_i one time, where $\{D_i\}$ is the sequence of chains associated with H. Hence H is the continuous image of $U(1)CC$ and thus each universal circularly chainable continuum. Now H can be mapped

continuously onto M by folding M into its copy. Therefore M is the continuous image of each universal circularly chainable continuum. //

It has been a long road that has brought you here. Most of the way was uphill and so it was never quite clear where you were going. But you've crossed the summit and the bottom of the hill is now in sight. All that is needed to show that a pseudo-circle maps continuously onto all circularly chainable continua is Ingram's result (19) concerning nonembeddable circularly chainable continuum.

<u>Theorem 3.13</u> Let N be the set of all natural numbers and $\mathcal{W}$ an A(N)-category. Then U(N)CC($\mathcal{W}$) maps continuously onto all circularly chainable continua.

<u>Proof</u> By Theorem 3.11, U(N)CC($\mathcal{W}$) maps continuously onto all circularly chainable continua embeddable in some A-category. But Ingram's result implies that all other circularly chainable continua are chainable. Hence the results of the previous section of this chapter implies that these circularly chainable continua, not embeddable in some A-category, are the continuous image of the pseudo-arc. And by Theorem 3.12 the pseudo-arc is the continuous image of U(N)CC($\mathcal{W}$). Hence each circularly chainable continuum, not embeddable in some A-category, is the continuous image of U(N)CC($\mathcal{W}$). Therefore all circularly chainable continua are the continuous image of U(N)CC($\mathcal{W}$). //

All that is needed now to characterize the continuous images of the pseudo-circles is to define Rogers' notion of q-chainability. This is the notion that he uses to characterize the continuous images of the pseudo-circles. The concept of q-chainability was briefly mentioned in Chapter II, and examples were given to clarify this notion. Here the definition of q-chainability from Chapter II will be expanded to a more workable state. This expansion is necessarily long, but examples will be given at the conclusion to help clarify the ideas given in the definitions. Finally, Rogers' proof that some pseudo-circle maps continuously onto a continuum M if and only if M is q-chainable will be presented to close out this section.

Definition 3.17 A finite sequence of sets in X, q_0, q_1, ..., q_n, is said to be a q-chain if $|i - j| \leq 1 \pmod{n+1}$ implies that $q_i \cap q_j \neq \emptyset$.

Definition 3.18 If P is the q-chain, p_0, ..., p_n, and Q is the q-chain, q_0, ..., q_m, and each link p_i is a subset of a link q_{x_i}, then the sequence of ordered pairs $(0,x_0)$, $(1,x_1)$, ..., (n,x_n) is said to be a pattern of P in Q if $|i - j| \leq 1$ $\pmod{n+1}$, $0 \leq i, j \leq n$, implies that $|x_i - x_j| \leq 1 \pmod{m+1}$.

Definition 3.19 Let P and Q be q-chains. Then P is said to refine Q if there is a pattern of P in Q.

Definition 3.20 A continuum M is said to be q-chainable if there exists a sequence of q-chains Q_1, Q_2, ... such that for each natural number i

(1) Q_i covers M,

(2) Q_{i+1} refines Q_i,

(3) each link of Q_i has diameter less than 1/i, and

(4) the closure of each link of Q_{i+1} is a subset of the link of Q_i to which it corresponds under the pattern of Q_{i+1} in Q_i.

The sequence $\{Q_i\}$ is said to be associated with M.

<u>Definition 3.21</u> Let P be the q-chain, $p_0, \ldots, p_n$, and Q the q-chain, $q_0, \ldots, q_m$, such that P refines Q. Then the degree of P in Q is the number of times in the pattern of P in Q that $x_j = m$ and $x_{j+1} = 0$, $j = 0, \ldots, m$, minus the number of times that $x_j = 0$ and $x_{j+1} = m$, $j = 0, \ldots, m$ and if $j = m$ then $j+1 = 0$.

<u>Definition 3.22</u> Let N be a set of natural numbers. Then the continuum M is said to be q-chainable of degree N if there is a sequence of q-chains $Q_1, Q_2, \ldots$ associated with M such that for each $j > 1$, the degree of Q_j in Q_{j-1} is zero or a product of nonnegative powers of elements of N.

In Figure 25 both P and Q are q-chains. Also there exists a pattern of P in Q. Namely the set of ordered pairs (0,0), (1,1), (2,2), (3,3), and (4,4). Hence, by Definition 3.19, P refines Q. From Definition 3.21, it is clear that the degree of P in Q is one. In Figure 26 both P and Q are q-chains. Also there exists a pattern of P in Q. This pattern of P in Q is (0,0), (1,0), (2,1), (3,1), (4,0), (5,4),

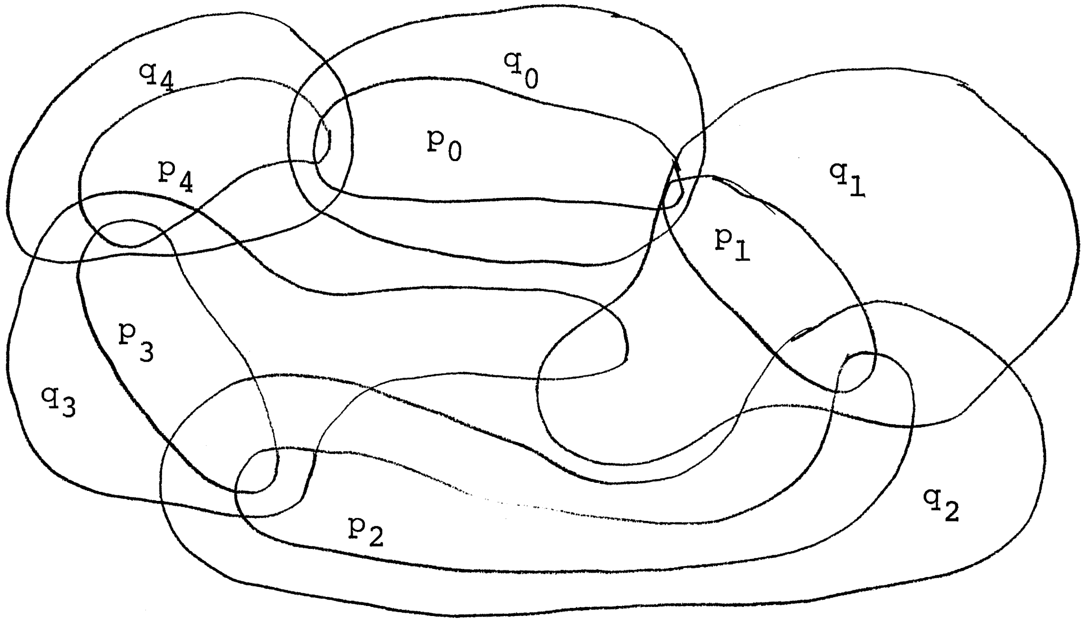


Figure 25. P refines Q

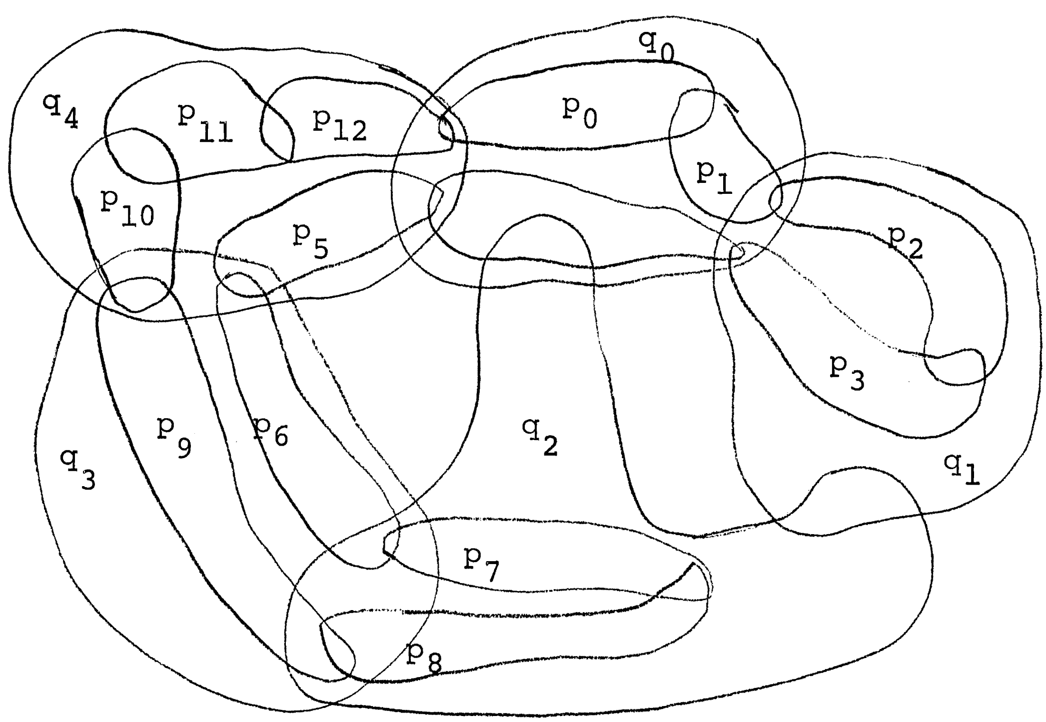


Figure 26. The Degree of P in Q is Zero

(6,3), (7,2), (8,2), (9,3), (10,4), (11,4), and (12,4). Therefore P refines Q and the degree of P in Q is zero.

Finally, the time has arrived to present Rogers' proof of Theorem 2.15, a continuum M is the continuous image of some pseudo-circle if and only if M is q-chainable. Rogers' proof follows closely to Fearnley's (13) proof that a continuum is the continuous image of the pseudo-arc if and only if it is p-chainable. However, in proving that the q-chainability of M implies that M is the continuous image of some pseudo-circle, Rogers uses Theorem 3.13, one pseudo-circle maps continuously onto all circularly chainable continua. So instead of constructing a pseudo-circle that maps continuously onto M, Rogers needs only to construct a circularly chainable continuum K that maps continuously onto M.

Theorem 2.15 A continuum M is the continuous image of some pseudo-circle if and only if M is q-chainable.

Proof Suppose that M is the continuous image of some pseudo-circle, M', under the map f. Let D_1, D_2, ... be a sequence of circular chains in M', considered as a space, such that for each i, (1) D_{i+1} refines D_i, (2) the diameter of each link of D_i is less than 1/i, (3) the closure of each link of D_{i+1} is a subset of some link of D_i, and (4) D_i covers M'. This is possible because M' is circularly chainable.

If D_i consists of the links d_0, d_{n_i}, then let $f(D_i)$ denote the sequence of sets $f(d_0)$, ..., $f(d_{n_j})$. By definition of a q-chain, $f(D_i)$ is a q-chain for each natural number i. Assert that $f(D_1)$, $f(D_2)$, ... is a sequence of q-chains associated with M. Since f is uniformly continuous, one may assume that the diameter of each link of $f(D_i)$ is less than $1/i$). Clearly $f(D_i)$ covers M since f was a map of M' onto M and D_i covered M'. Now Choose a pattern of D_i in D_{i-1} which assigns to each link of D_i a link of D_{i-1} that contains its closure. This can be done because of properties (1) and (3) of the sequence D_1, D_2, ... Now this pattern of D_i in D_{i-1} turns out to be a pattern of $f(D_i)$ in $f(D_{i-1})$. Hence $f(D_i)$ refines $f(D_{i-1})$ and the closure of each link of $f(D_i)$ is a subset of the link of $f(D_{i-1})$ to which it corresponds. Therefore by Definition 3.20, $f(D_1)$, $f(D_2)$, ... is a sequence of q-chains associated with M, and M is q-chainable.

Now suppose that M is a q-chainable continuum, and let Q_1, Q_2, ... be a sequence of q-chains associated with M. It will be shown now that M is the continuous image of U(N)CC where N is the set of all natural numbers. But by Theorem 3.13 each circularly chainable continuum is the continuous image of U(N)CC. Hence it will suffice to construct a circularly chainable continuum K which maps continuously onto M.

In order to construct K, define a new sequence P_1, P_2, ... of q-chains associated with M and a sequence of circular chains D_1, D_2, ... such that for each natural number n, (1) D_n is a circular chain of open balls in E^3, (2) D_{n+1} has the same pattern in D_n as P_{n+1} has in P_n, (3) the diameter of each link of D_n is less than $1/n$, and (4) the closure of each link of D_{n+1} is contained in the link of D_n to which it corresponds under the pattern of D_{n+1} in D_n. Then define K to be the common part of D_1, D_2, ...

The sequences $\{D_i\}$ and $\{P_i\}$ will be defined inductively. First let $P_1 = Q_1$ and let D_1 satisfy (1) and (3) above and have the same number of links as Q_1. Denote D_1 by d_0, ..., d_m. Let $(0, x_0)$, ..., (n, x_n) be the pattern of Q_2 in Q_1. Also let A_0 be an open ball of diameter less than 1/n such the closure of A_0 is a subset of d_{x_0}. Then A_0 will be the first link of D_2. Let A_0, ..., A_j be a chain of open balls of diameter less than 1/n that is a refinement of the chain d_{x_0}, d_{x_1} and the last link A_j is the first link of A_0, ..., A_j whose closure is contained in d_{x_1}. Let A_0, ..., A_j be the first j+1 links of D_2. Now define a new q-chain Q_2' as follows: $q_i' = q_0$ for $0 \leq i < j$, and $q_{i+j-1}' = q_i$ for $i \geq 1$. Then Q_2' will have the following pattern in Q_1: (i, x_0) for $0 \leq i < j$, and $(i+j-1, x_i)$ for $j \geq 1$. Continue this process to finish defining D_2. Also at the end let $P_2 = Q_2'$. Before the next step is started, all the chains Q_n, $n > 2$ must be modified to conform to the change from Q_2 to $Q_2' = P_2$. This is done to make Q_3 a refinement of Q_2', Q_4 a refinement of Q_3, and so on. The next step can now be performed in a similar manner to the last and thus the sequences $\{P_i\}$ and $\{D_i\}$ have been defined as desired.

Now a map f of K onto M will be defined using the circular chains D_1, D_2, ... for any $x \in K$, let $J_n(x)$ denote the union of the links of P_n such that x belongs to the links of D_n having the same indices. For each n, $J_n(x)$ is the union of at most links of P_n, and $J_n(x)$ has diameter less than 2/n. Also by the definition of $\{P_i\}$ and $\{D_i\}$ it is evident that $J_{n+1}(x) \subseteq J_n(x)$. So the sequence of closed sets $\overline{J_1(x)}$, $\overline{J_2(x)}$, ... forms a decreasing sequence of sets with diameters tending to zero. Hence define f(x) to be the unique intersection of this sequence.

It remains to show that f is continuous and onto. But these facts follow by a similar argument that Fearnley used in the proof of Theorem 2.11 presented in the first section of this chapter. Therefore f maps K continuously onto M. But U(N)CC maps continuously onto K and so U(N)CC maps continuously onto M. //

By using the basic proof of Theorem 2.15 above and considering the degree of q-chainability of the continuum M, Rogers was able to prove a more precise version of Theorem 2.15.

<u>Theorem 3.14</u> Let N be a set of natural numbers. Then the continuum M is the continuous image of U(N)CC if and only if M is q-chainable of degree N.

At last you are at the bottom of the hill. The journey has been long!

Comparison of Techniques

The presentation of Fearnley's characterization of the continuous images of the pseudo-arc and the presentation of Rogers' characterization of the continuous images of the pseudo-circles have made up this chapter so far. In this section I will talk to you about the advantages and disadvantages of both Fearnley's and Rogers' methods. It should be noted that the discussion to follow has been biased by my own struggle through their works. Also my taste in Mathematics has a part in the discussion. A disadvantage to me may seem like an advantage to you, or vice versa.

I see three advantages to the method Fearnley used in characterizing the continuous images of the pseudo-arc: (1) the geometrical nature of the method, (2) the little topological background needed in the method, and (3) the short presentation yielded by the method.

The proof of Theorem 2.11, the continuum H is the continuous image of the pseudo-arc if and only if H is p-chainable, relies heavily on the geometry of the p-chains. In specific, Theorem 3.3 shows that if the p-chain P is a normal refinement of the p-chain Q, then there exists a p-chain R that is a principal normal refinement of P and crooked in Q. This allowed Fearnley to geometrically construct a pseudo-arc to map continuously onto a p-chainable continuum in the proof of Theorem 2.11. These geometrical considerations make it easier to see that the notion of p-chainability characterizes the continuous images of the pseudo-arc. I feel that I wouldn't have gotten the handle on Fearnley's presentation without these geometrical considerations.

Fearnley, in his study, didn't use any more topological machiner than is normally found in a first course in topology. Thus the presentation of Fearnley's work is self-contained and easily accessible to most students of topology. This I feel is a definite advantage--an advantage that put me to ease after going through Fearnley's work in detail.

The presentation of Fearnley's work is relatively short. It is short enough that in going through the presentation you probably don't need to keep thumbing back for information already read. Thus, from any point in the presentation of Fearnley's work you are likely to have a handle on most of the previous information in the presentation. The advantage of this is that I felt I understood Fearnley's work after only one detailed reading.

I see only one slight disadvantage to the method Fearnley used in characterizing the continuous images of the pseudo-arc. The details in the proofs of some of Fearnley's preliminary facts are tedious. The details in these preliminary proofs are not messy enough to overshadow the concepts to be demonstrated. However, they are messy enough to almost lull you to sleep in places.

Overall, the advantages outweigh the disadvantages for the method Fearnley used in characterizing the continuous images of the pseudo-arc. Fearnley's presentation is short enough to read in detail with a minimum of rereading. It is also self-contained, which makes the presentation easily accessible to most students of topology. And last, but not the least, Fearnley's method uses enough geometrical notions to give a picture of what is happening.

I see three disadvantages to the method Rogers used in characterizing the continuous images of the pseudo-circles: (1) the method was very abstract, (2) the method required much topological machinery, and (3) the method yielded a long presentation.

The presentation of Rogers' work starts off with a very non-geometrical approach to obtaining pseudo-circles. The pseudo-circles that Rogers uses in his work are obtained as special inverse limits in an A-category. The notion of an inverse limit, alone, gives almost no geometrical feeling of any kind. So, when Rogers proves that the inverse limit of unit circles with the special bonding maps described in Definition 3.13 is a pseudo-circle, it should not be too hard to believe that I don't have any geometrical intuition about this pseudo-circle. However, abstractly I can believe that Rogers has obtained a pseudo-circle. After Rogers has obtained the pseudo-circles from the A-categories, he then proceeds to show that there is one pseudo-circle that maps continuously onto all circularly chainable continua. To demonstrate that there is one pseudo-circle that maps continuously onto all circularly chainable continua he relies heavily on some not too elementary properties of inverse limits. In particular, the proofs of Theorem 3.8 and Theorem 3.9 rely heavily on two very abstract and non-elementary properties of inverse limits due to Mioduszewski (28). Hence I also have no geometrical feelings for why there is a pseudo-circle that maps continuously onto all circularly chainable continua. However, abstractly I do believe it is true.

The presentation of Rogers' work was far from being self-contained. He relied heavily on results by Mioduszewski (28), Eilenberg and Steenrod (11), Mardesic and Segal (25), and Ingram (19). Each of these results were used in the presentation of Rogers' work in a crucial spot. As mentioned in the previous paragraph, Mioduszewski's (28) results about the mapping between inverse limits were a major factor in Rogers' showing that one pseudo-circle mapped continuously onto all circularly chainable continua. Also the result of Eilenberg and Steenrod (11) concerning the mappings between inverse limits was essential in showing that one pseudo-circle maps continuously onto all circularly chainable continua. Rogers used the result of Mardesic and Segal (25) to characterize all circularly chainable continua as the inverse limit of unit circles with onto bonding maps. Without this characterization Rogers wouldn't have been able to get to first base in showing that there is a pseudo-circle that maps continuously onto all circularly chainable continua. Lastly, Ingram's (19) result that all circularly chainable continua not embeddable in some A-category are chainable was the finishing touch on Rogers' proof that one pseudo-circle maps continuously onto all circularly chainable continua.

The presentation of Rogers' work was about three times longer than the presentation of Fearnley's work, and in the presentation of Rogers' work the development of the Uniformation Theorem was omitted. This development would have added many pages to the presentation of Rogers' work. Rogers' development was so long that you may have forgotten about the Uniformation Theorem. But it was a very important factor in showing the existence of A-categories. I was able to keep sight of all important facts in reading Fearnley's work, but this can't be said for Rogers' work. It was very difficult to keep in sight of all important facts in reading Rogers' paper for the first couple of times.

I can't see any advantages to Rogers' method other than the obvious advantages. That is, the method of Rogers did work in characterizing the continuous images of the pseudo-circles. It may not be clear to my why Rogers' method worked, but it did work. Also Rogers was able to take a great deal of existing mathematics and with a little foresight of his own make this existing mathematics work for characterizing the continuous images of the pseudo-circles.

Overall the disadvantages seem to overshadow the fact that Rogers' method did yield a characterization of the continuous images of the pseudo-circles. The fact that Rogers' method is very abstract, long, and not self-contaied left me without a firm grip on the ideas in the presentation of Rogers' work. I don't think that I could ever do anything with these ideas except use them at face value.

CHAPTER IV

INTERRELATIONSHIPS AND EXAMPLES

Relationships Between Characterizations

The inclusion relationships between chainable plane continua, circularly chainable plane continua, and the plane continuous images of arcs, pseudo-arcs, and pseudo-circles will be revealed in this section. These relationships are elementary consequences of Fearnley's and Rogers' results from Chapter III. It should be noted that the relationship will first be presented in a formal manner. This is done to show how easily these relationships follow from results of Chapter III. However, at the close of the section all the relationships will be tied together in a Venn diagram.

The first relationship that will be given is between the class of chainable plane continua and the continuous images, in the plane, of the pseudo-arc. It will be shown that the class of chainable plane continua is contained in the continuous images, in the plane, of the pseudo-arc. This is a special case of Theorem 2.13, every chainable continuum is the continuous image of the pseudo-arc, due to Fearnley.

Theorem 4.1 The class of chainable plane continua is contained in the continuous images, in the plane, of the pseudo-arc.

Proof This is just Theorem 2.13 restricted to plane continua. //

Because of the characterization of the continuous images of the pseudo-arc, Theorem 4.1 could be stated in the following way: the class of chainable plane continua is contained in the class of p-chainable plane continua. It will be shown, by example, in the next section that this containment is a proper containment.

The next relationship to be given is between the plane continuous images of the arc and the pseudo-arc. It will be shown that the continuous images, in the plane, of the arc is contained in the continuous images, in the plane, of the pseudo-arc. But the plane continuous images of the arc are just the class of locally connected plane continua, and so the class of locally connected plane continua is contained in the class of p-chainable plane continua. This result is an easy consequence of Theorem 4.1.

Theorem 4.2 The continuous images, in the plane, of the arc are contained in the continuous images, in the plane, of the pseudo-arc.

Proof Let X be a plane continuous image of the arc I under the map f. It is well known that the arc I is chainable.

Hence Theorem 4.1 implies that there exists a continuous map g of the pseudo-arc M onto the arc I. Consider the composition gf. Clearly this is a continuous map of the pseudo-arc onto X, and so X is a continuous image, in the plane, of the pseudo-arc. Therefore the plane continuous images of the arc are contained in the plane continuous images of the pseudo-arc. //

The containment mentioned in Theorem 4.2 is a proper containment. In the next section an example will be given to show this.

Now it is appropriate to state the relationship between the class of chainable plane continua and the class of locally connected plane continua. This relationship is not a consequence of anything in this paper, but it will be included in this paper in the event that you haven't come across it in your study of topology. It turns out that neither the class of locally connected plane continua is contained in the class of chainable plane continua, nor is the class of chainable plane continua contained in the class of locally connected plane continua. The proof of this fact will be put off until the next section. There, examples will be given to demonstrate this relationship.

The relationship between the plane continuous images of the pseudo-arc and the plane continuous images of some pseudo-circles will be given in the following theorem:

<u>Theorem 4.3</u> The continuous images, in the plane, of the

pseudo-arc are contained in the continuous images, in the plane, of U(N)CC, where N is a set of natural numbers.

Proof Let X be a plane continuous image of the pseudo-arc M under the map f. By Theorem 3.12 there exists a continuous map g such that g maps U(N)CC onto M. Then fg is a continuous map of U(N)CC onto X, and so X is a plane continuous image of U(N)CC. Therefore the plane continuous images of the pseudo-arc are contained in the plane continuous images of U(N)CC. //

An example will be given in the next section to show that this containment is proper.

An important special case of Theorem 4.3 is when N = {1}. It turns out that U(1)CC is the planar pseudo-circle, and so it is true that the plane continuous images of the pseudo-arc are contained in the plane continuous images of the planar pseudo-circle. To show that U(1)CC is the planar pseudo-circle requires an embedding result of Bing's (3) and the result of Fearnley's (14) that says all planar pseudo-circles are homeomorphic.

Theorem 4.4 The plane continuous images of the pseudo-arc are contained in the plane continuous images of the planar pseudo-circle.

Proof By Theorem 4.3 it will suffice to show that U(1)CC is the planar pseudo-circle. Bing's embedding result implies that U(1)CC is embeddable in the plane. Therefore Fearnley's

(14) result implies that all other planar pseudo-circles are homeomorphic to U(1)CC. //

As immediate corollaries to Theorem 4.1, Theorem 4.2, and Theorem 4.4 are the following:

Corollary 4.1 The class of locally connected plane continua is contained in the class of plane continuous images of the planar pseudo-circle.

Corollary 4.2 The class of chainable plane continua is contained in the class of plane continuous images of the planar pseudo-circle.

Next the relationship between circularly chainable plane continua and the plane continuous images of the planar pseudo-circle will be revealed. This relationship is a special case of Theorem 3.13 which says that all circularly chainable continua are the continuous image of U(N)CC, where N is the set of all natural numbers. In specific, the class of circularly chainable plane continua is contained in the class of plane continuous images of the planar pseudo-circle.

Theorem 4.5 The class of circularly chainable plane continua is contained in the class of plane continuous images of the planar pseudo-circle.

Proof Since all planar pseudo-circles are homeomorphic, then it suffices to consider the class of plane continuous images of U(1)CC. Now it follows that the circularly chainable plane continua are contained in the plane continuous images of U(1)CC by a special case to the proof of Theorem 3.12. //

Finally, the relationship the class of circularly chainable plane continua has with the class of p-chainable plane continua and the class of locally connected plane continua will be discussed. It is true that neither of these classes is contained in the class of circularly chainable plane continua or vice versa. Examples will be given, in the next section, to illustrate this point.

By now your head is probably spinning from all the relationships that have been discussed in the preceding pages. But I think it was necessary to get all these relationships down formally to see how they follow from the results of Chapter III. Now the best way to clarify all these relationships is with a picture, in the form of a Venn diagram, and with the examples that have been promised you. The Venn diagram, Figure 27, will close this section, and the examples will make up the next section. In Figure 27 the rectangle and its area represent the class of all plane continua, and each closed curve and its inside area represents a certain subclass of plane continua. Each subclass is labeled and it is hoped that no confusion occurs.

Examples

In the previous section the inclusion relationships between chainable plane continua, circularly chainable plane

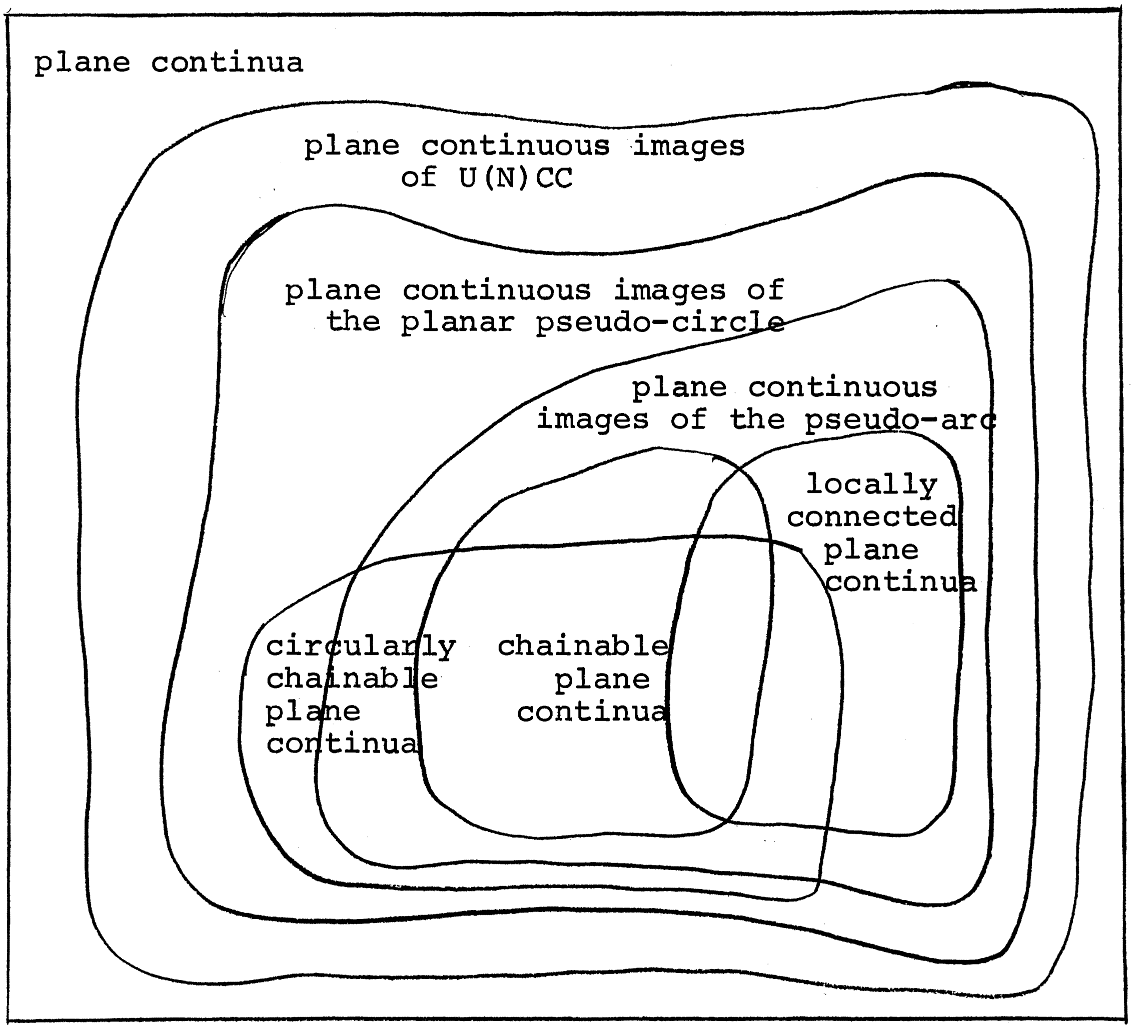
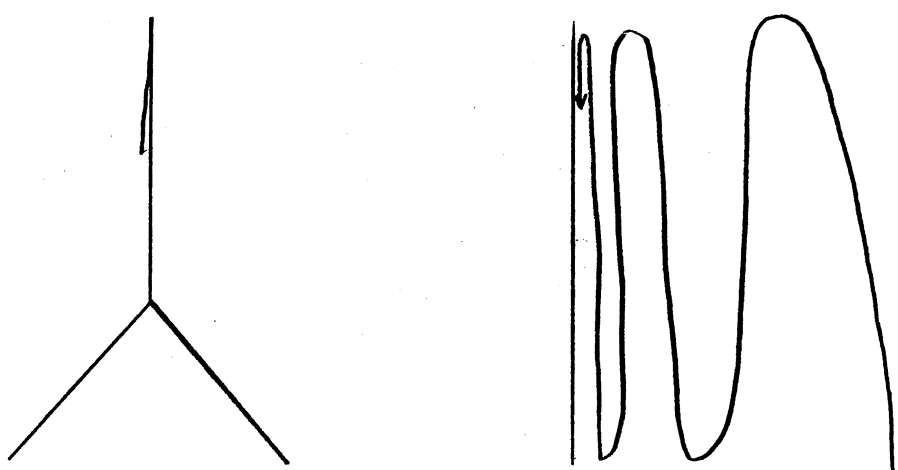


Figure 27. Venn Diagram of Inclusion Relationships

continua, and the plane continuous images of arcs, pseudo-arcs, and pseudo-circles were revealed. These relationships were summarized in Figure 27. The goal of this section, then, is to look at examples that will help clarify the relationships shown in Figure 27.

It was promised in the last section that examples would be given to show that the class of chainable plane continua and the class of locally connected plane continua were both properly contained in the plane continuous images of the pseudo-arc. Also it was promised that examples would be given to show that the class of chainable plane continua and the class of locally connected plane continua have no inclusion relationship. Now two examples will be presented to do all of this. Let A be the triod pictured in Figure 28 (a) and let B be the topologist sine curve pictured in Figure (b). Clearly A is a locally connected continua. And by



(a) (b)

Figure 28. The Triod and the Topologist Since Curve

classical results of R. L. Moore (32) and J. H. Roberts (36), A is not chainable because it contains a triod. Therefore, since A is contained in the class of locally connected plane continua but not in the class of chainable plane continua, the class of locally connected plane continua is not contained in the class of chainable plane continua, and the class of chainable plane continua is properly contained in the continuous images of the pseudo-arc. Now B is not locally connected, but it is chainable. To give you a feeling as to why B is chainable look at the chains in Figure 29. This chaining procedure in Figure 29 can be continued inductively and so B is chainable. Hence B shows that the class of chainable plane continua is not contained in the class of locally connected plane continua, and that the class of locally connected plane continua is properly contained in the plane continuous images of the pseudo-arc.

The next example to be presented will show that there exists plane continua that are neither chainable nor locally

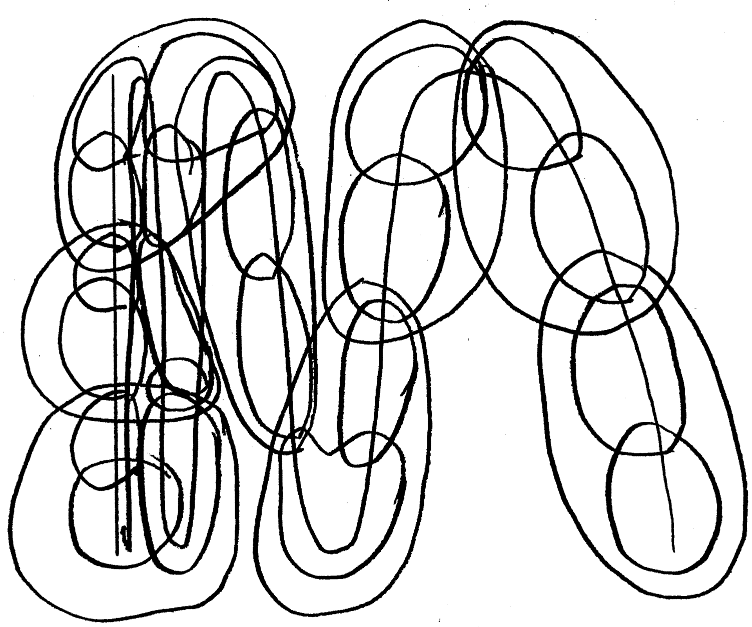


Figure 29. Chaining B

connected but are the continuous image of the pseudo-arc. Let C be the continuum pictured in Figure 30, a topologist sine curve with a tail. Clearly C is not locally connected.

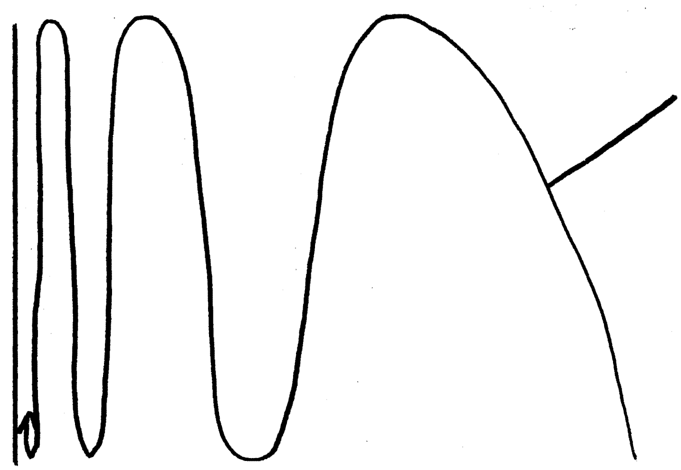


Figure 30. Continuum C

Also using the facts that a chainable continuum does not contain a triod and C does not contain a triod, then it can be deduced that C is not chainable. To see that C is the continuous image of the pseudo-arc I will show you how to define a sequence of p-chains associated with C. The first two p-chains of such a sequence are pictured in Figure 31, and it is clear that this process can be continued inductively. Therefore there exists plane continua that are not in the class of chainable continua and locally connected continua, but are contained in the class of p-chainable continua.

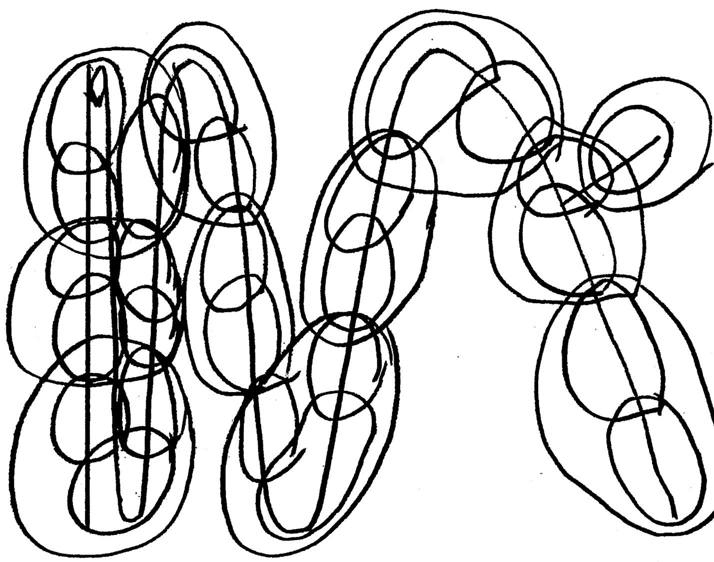


Figure 31. P-Chaining C

Finally, examples will be presented to illustrate the inclusion relationships the class of circularly chainable plane continua has with each of the following classes: (1) the p-chainable plane continua, (2) the chainable plane continua, and (3) the locally connected plane continua.

Let D be the planar pseudo-circle constructed in Chapter I. Then by the definition of D it is a circularly chainable plane continuum. But from Rogers' presentation in Chapter III, D is not the continuous image of the pseudo-arc. Hence D is not p-chainable, chainable, or locally connected. Therefore, the class of circularly chainable plane continua is not contained in the class of p-chainable plane continua chainable plane continua, or locally connected plane continua. Also D shows that the continuous images, in the plane, of the pseudo-arc are properly contained in the continuous images, in the plane, of the planar pseudo-circle.

The next example will show that none of the classes, p-chainable plane continua, locally connected plane continua, or chainable plane continua, are contained in the class of circularly chainable plane continua. Let E be the arc. Then the continuum E is the desired continuum for this example. Clearly E is a locally connected plane continuum, a chainable plane continuum, and a p-chainable continuum. All that is needed is to show that E is not circularly chainable. Intuitively this does seem true, but I feel some sort of argument is needed to prove this.

<u>Theorem 4.6</u> The arc is not circularly chainable.

<u>Proof</u> Let E denote the arc. Suppose that E is circularly chainable. Then by definition, for each $\varepsilon > 0$, E can be irreducibly covered by a circular chain whose links have diameter less than ε. Let $\varepsilon = 1/10$ be given and $d_0, d_1, \ldots, d_n$ the circular chain whose links have diameter less than and covers E. But by the circular nature of the chain there exists subchains, $d_i, \ldots, d_j$, and, $d_k, \ldots, d_m$, with the following properties: (1) either $j < k$ or $m < i$ and (2) $(1/10, 9/10)$ is contained in the union of the subchains, $d_i, \ldots, d_j$, and, $d_k, \ldots, d_m$. Let $X = \bigcup_{p=i}^{j} d_p$ and $Y = \bigcup_{p=k}^{m} d_p$. Then property (1) implies that $X \cap Y = \emptyset$. Definition of circular chain implies that X and Y are open. Hence $(1/10, 9/10) = [X \cap (1/10, 9/10)] \cup [Y \cap (1/10, 9/10)]$, is the union of two

nonempty nonintersecting open sets. Thus (1/10, 9/10) must not be connected. Obviously this is a contradiction and so E must not be circularly chainable. //

I think it is easy to see that this proof also implies that the continua A, B, and C are also not circularly chainable.

To summarize what the examples in this section show I feel it worthwhile to repeat the Venn diagram in Figure 27 with these examples placed in their appropriate place.

Relationship to Other Classes

The inclusion relationship between the continuous images of the pseudo-arc and other known classes of continua will be briefly touched upon in this section. These relationships will be illustrated with examples. The continua discussed in this section will not necessarily be plane continua.

The first example to be presented will be a tree-like plane continua that is not the continuous image of the pseudo-arc. Since tree-like continua may not be familiar, I will formally define them.

<u>Definition 4.1</u> A finite collection T of open sets is said to be a tree chain if: (1) no three of the open sets have a point in common, (2) no subcollection of T is a circular chain, and (3) each proper subcollection T' of T has an element that intersects an element of T - T'. A continuum M

is said to be tree-like if for each $\varepsilon > 0$ there exists a tree chain, whose links have diameter less than ε, that covers M.

It should be noted that the class of chainable continua is contained in the class of tree-like continua.

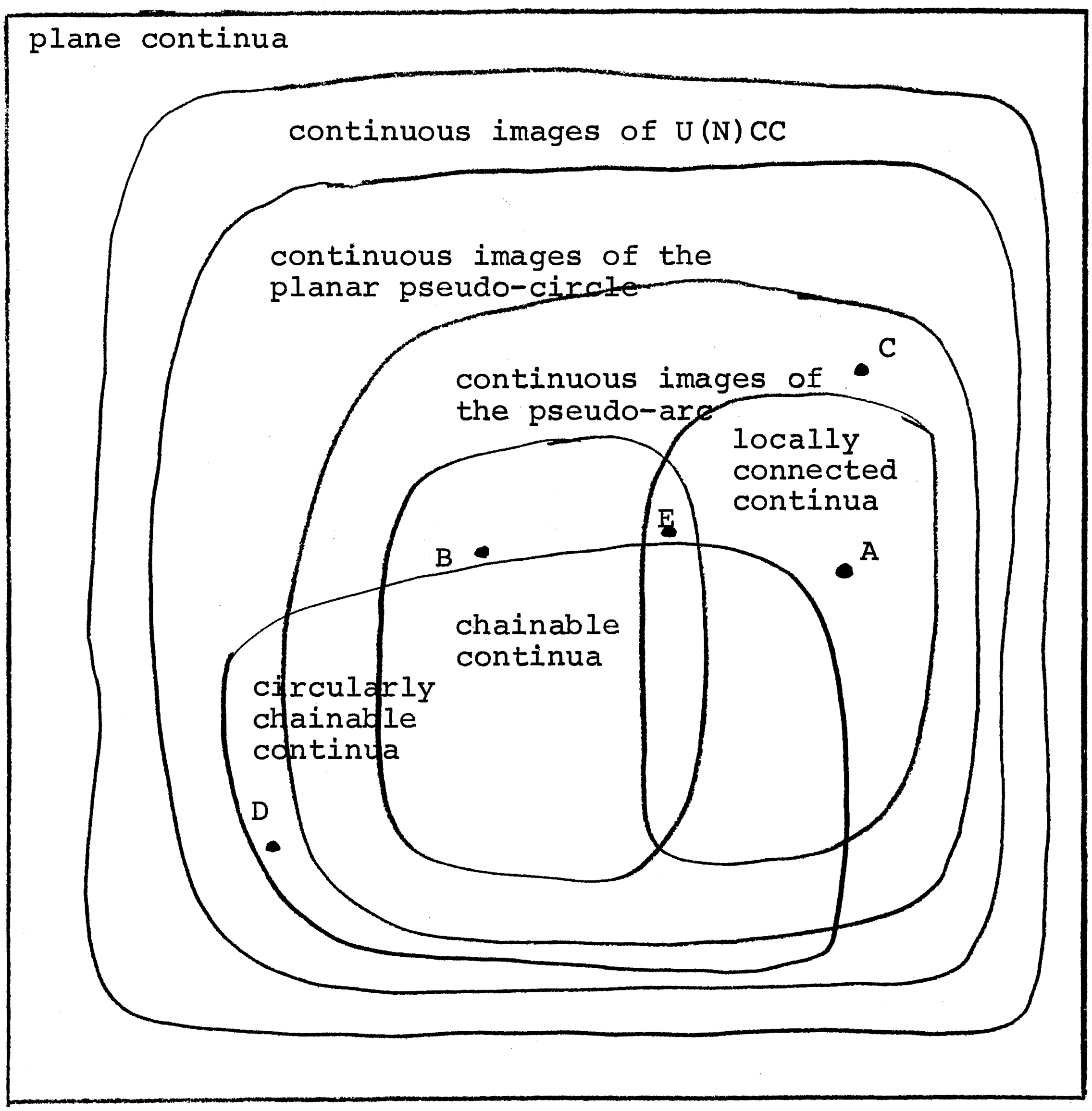


Figure 32. Venn Diagram Showing Continua A, B, C, D, and E.

Now let F be the continua pictured in Figure 33. I will try to convince you with another diagram that F is tree-like.

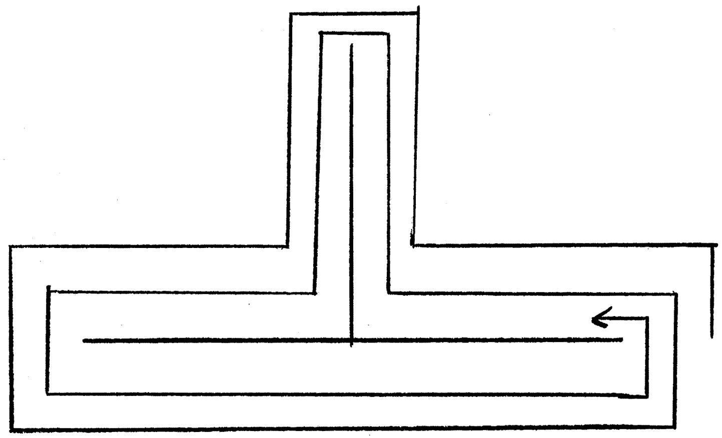


Figure 33. Continuum F

In Figure 34 it can be seen that the sequence of open sets covering F is a tree chain. Also from this diagram you should be able to see how to cover F with tree chains whose links have diameter less than $\varepsilon > 0$. To prove that F is not the continuous image of the pseudo-arc is not an easy job at all. I will not present the proof of this fact in this paper, because I don't think the proof adds any geometrical insight into why F is not the continuous image of the pseudo-arc. To see this proof see Fearnley (13), page 394. Therefore, the example F shows that the class of tree-like plane continua is not contained in the class of p-chainable plane continua. An interesting question, whose answer hasn't

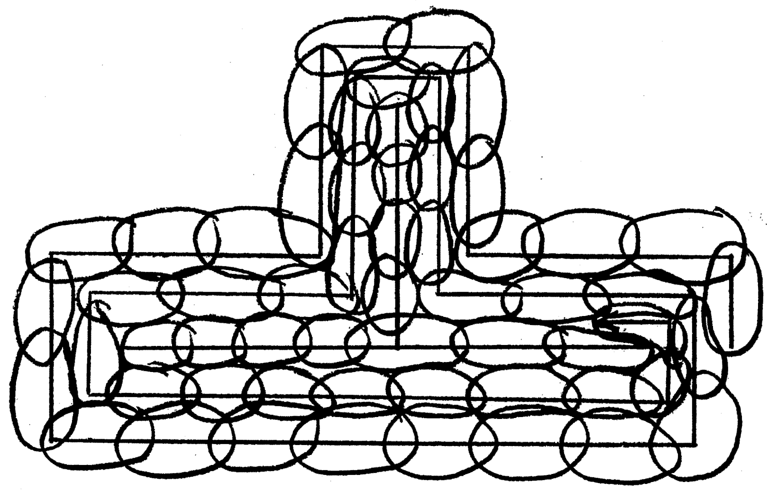


Figure 34. A Tree Chain Covering F

appeared in the journals, is: Does there exist three-like plane continua that are not the continuous image of the planar pseudo-circle?

The next example to be shown will be an arc-wise connected continuum that is not the continuous image of the pseudo-arc. Since arc-wise connected may be an idea not familiar, I will formally define it.

Definition 4.2 A continuum M is said to be arc-wise connected provided, for each pair of distinct points x, y ε M, there exists an arc from x to y contained in M.

In Figure 35 an arc-wise connected continuum is pictured. Denote this continuum by G. The continuum G is arc-wise connected, because for any two distinct points x, y ε G it is easy to see an arc in G from x to y. The

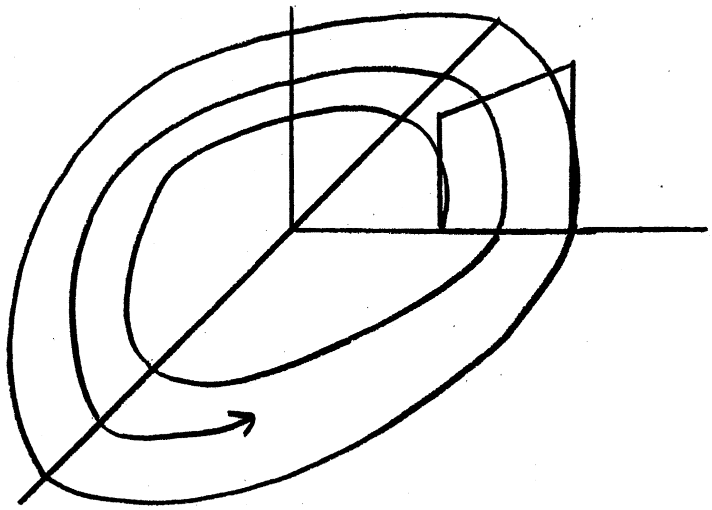


Figure 35. Continuum G

continuum G can be shown not to be the continuous image of the pseudo-arc by a similar argument to the one given by Fearnley (13), page 394 for the continuum F. Hence the continuum G illustrates that the class of arc-wise connected continua is not contained in the continuous images of the pseudo-arc. Several interesting questions, related to this example, come to mind: (1) Does there exist an arc-wise connected plane continuum that is not the continuous of the pseudo-arc?, and (2) Does there exist an arc-wise connected continuum that is not the continuous image of the planar pseudo-circel? I have been unable to find answers to these questions in the journals.

Finally, an example will be presented to show that the class of semi-locally connected continua is not contained in the continuous images of the pseudo-arc. The notion of semi-locally connected will formally be defined.

Definition 4.3 A connected set M is said to be semi-locally connected at a point $x \in M$ if for every $\varepsilon > 0$ there exists a neighborhood V of x in M of diameter less than ε such that $M - V$ has only a finite number of components. If M is semi-locally connected at each of its points, then M is said to be be semi-locally connected.

The example promised is pictured in Figure 36. Denote this continuum by H. For any x in H, it is easy to see that if V is the intersection of an ε-ball around x with H, then $H - V$ is connected. Hence by Definition 4.3, H is semi-locally connected. As was the case with the last two examples, the proof that H is not the continuous image of the pseudo-arc is quite lengthy and adds little geometric insight into why H is not the continuous image of the pseudo-arc. For these reasons the proof will not be presented in this paper. See Fearnley (13) for this proof. Hence, the example H shows that the class of semi-locally connected continua is not contained in the continuous images of the pseudo-arc. As before, several interesting questions have arisen: (1) Does there exist semi-locally connected plane continua that are not the continuous image of the pseudo-arc?, and (2) Does there exist a semi-locally connected continuum that is not the continuous image of the planar pseudo-circle? Again I have been unable to find answers to these questions in the journals.

It seems that the examples presented in this section have raised as many questions as they have answered. This

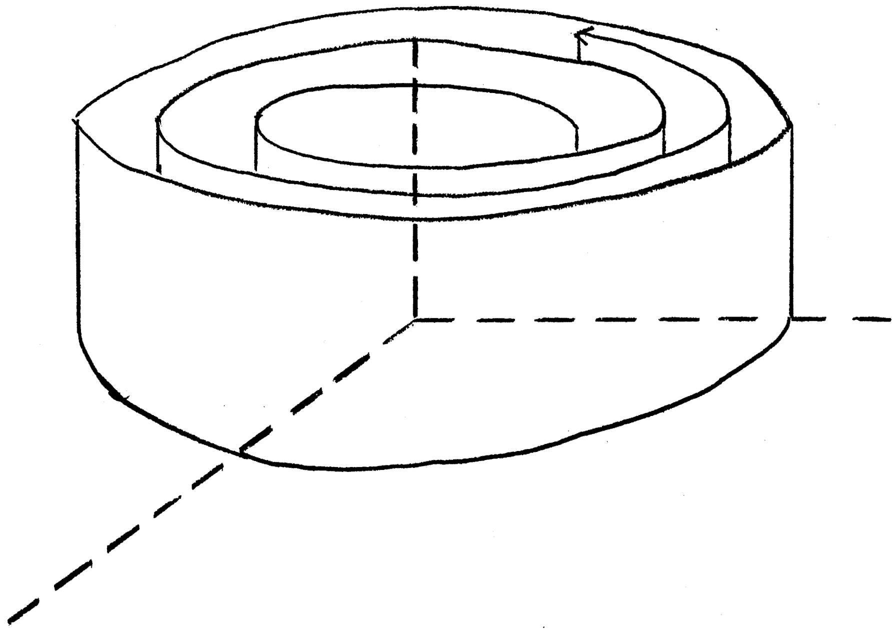


Figure 36. Continuum H

is just healthy mathematics. They story is never quite finished. There is always another chapter.

CHAPTER V

A POTPOURRI

Questions

While writing this thesis, several questions have come to mind, answers to which I have been unable to find. It's to these questions that this section is addressed.

In Chapter IV it was shown that the continuous images of the pseudo-arc are properly contained in the continuous images of U(N)CC, where N is the set of all natural numbers. Also it is known that the continuous images of U(N)CC is properly contained in the class of continua.

<u>Problem 5.1</u> Does there exist a continuum X such that the continuous images of X properly contain the continuous images of U(N)CC?

At this time, the answer to Problem 5.1 is not known. David P. Bellamy (8) gives a clue as to what kind of continuum X would be, if such a continuum exists. Bellamy (8) has shown that each continuum is the continuous image of some indecomposable continuum. Therefore, it seems likely that if there is a continuum X whose continuous images properly contained the continuous images of U(N)CC, then X is going to be indecomposable.

The presentation of Fearnley's work on characterizing the continuous images of the pseudo-arc was given in Chapter III. Omitted from this presentation in Chapter III was Fearnley's proof that there is no local property that characterizes the continuous images of the pseudo-arc. This presentation can be seen in Fearnley (13).

Problem 5.2 Can the continuous images of U(N)CC, where N is a set of natural numbers, be characterized in terms of a local property?

Problem 5.2 is unsolved at the present time. I would conjecture that the answer to Problem 5.2 is negative. A good place to start solving this problem is with the examples Fearnley used in showing that the continuous images of the pseudo-arc have no local property. These examples can be found on pages 293 and 393 of Fearnley (13). In fact, one of these examples was presented in Figure 33 of Chapter IV of this paper. If it can be shown that this continuum, in Figure 33 of Chapter IV, is not the continuous image of U(N)CC, then it will follow that the continuous images of U(N)CC cannot be characterized in terms of a local property. If, however, this example, in Figure 33 of Chapter IV, turns out to be the continuous image of U(N)CC, then it is not clear, to me, how to answer Problem 5.2.

In general, it is not easy to show that a continuum is not p-chainable. This fact was seen in Chapter IV. Since Fearnley has shown that the continuous images of the pseudo-arc cannot be characterized in terms of a local property, it is reasonable to ask the following question:

Problem 5.3 Are there other global properties that characterize the continuous images of the pseudo-arc?

As of the present time, Problem 5.3 has not been answered. It is hoped that Problem 5.3 can be answered affirmatively and that the new characterizing property will be easier to work with in showing that a continuum is not p-chainable. If such a property were found, it would be a benefit in understanding the class of continuous images of the pseudo-arc.

Problem 5.2 asks whether there are local properties that characterize the continuous images of U(N)CC. But it is not clear whether there are such properties. Hence, in light of Problem 5.3, it is reasonable to ask the following question:

Problem 5.4 Are there other global properties that characterize the continuous images of U(N)CC, where N is a set of natural numbers?

At present, Problem 5.4 has not been answered. Hopefully, there exist new characterizing properties for the continuous images of U(N)CC that lend themselves to easy determination of whether a continuum is or isn't the continuous image of U(N)CC. If such a property were found, it would be a benefit in understanding the class of continuous images of U(N)CC.

I have other questions related to the results in this paper, but I feel that the four problems listed are the most interesting. It is hoped that you have become interested enough, through reading this thesis, to attempt to solve one or more of these problems.

Other Research

Recently, research has been done concerning the continuous images of certain continua, where these continua have been different from the arc, pseudo-arc, and pseudo-circles. It is the purpose of this section to report on a sampling of this research.

Very recently, Wlodizimierz Kuperberg (22) characterized the continuous images of the cone over the Cantor set. The cone over the Cantor set is the continuum formed by the quotient space, (P x I)/R, where P is the Cantor set, I is the unit interval, and R is the equivalence relation that identifies all points with second coordinate 1. Kuperberg was able to characterize the images of the cone over the Cantor set with the property he calls "uniformly pathwise connected." This property will not be defined in this paper. See Kuperberg (22) for this definition.

Even more recently, Kuperberg and A. Lelek (23) have characterized the continuous images of pathwise connected continua X that have a special property relative to the group, A(X), defined in (23).

<u>Definition 5.1</u> A continuum M is said to be pathwise connected provided that each two points of M can be joined by a continuous image of the arc that is contained in M.

It can be noted that clearly an arc-wise connected continuum is also pathwise connected. The property that characterizes the continuous images of the images of the special pathwise connected continua also has to do with this group A. See Kuperberg and Lelek (23) for further details.

In the same article, Kuperberg and Lelek have also shown inclusion relationships between the following six classes of continua:

(I) locally connected continua,

(II) continuous images of the cone over the Cantor set,

(III) continuous images of dendroids,

(IV) continuous images of pathwise connected continua X with $\Pi^1(X) = 0$, where $\Pi^1(X)$ is a special group defined in (23),

(V) continuous images of pathwise connected continua X with $A(X) = 0$, where $A(X)$ is the group mentioned previously, and

(VI) pathwise connected continua.

These relationships are illustrated in Figure 37.

A question comes to mind immediately after looking at Figure 37: What are the inclusion relationships between Kuperberg and Lelek's clases II, III, IV, and V and the continuous images of the pseudo-arc and pseudo-circles?

The answer to this question is not known at the present time.

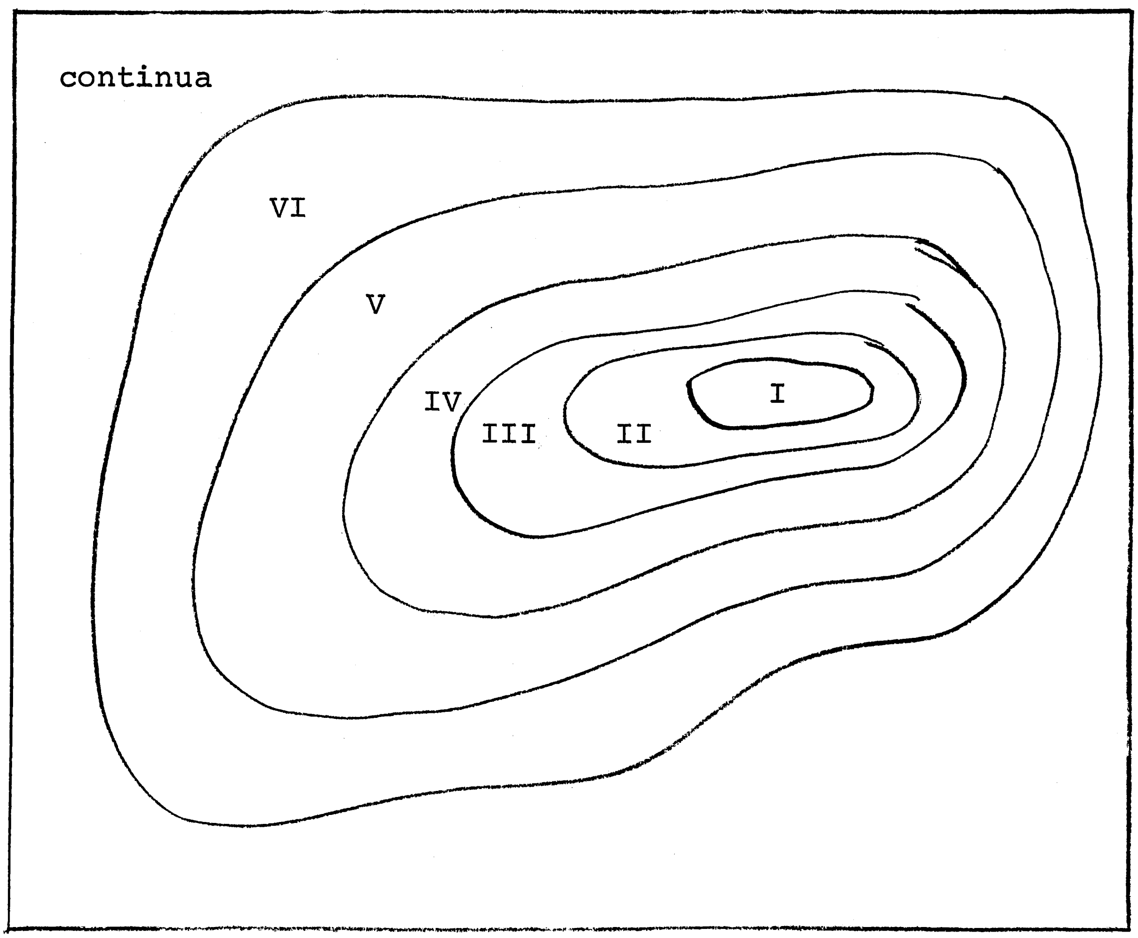


Figure 37. Classes of Continua

In August of 1975, Ray L. Russo (40) completed his Ph.D. thesis at Tulane University under the direction of J. T. Rogers, Jr. In this thesis Russo was able to prove several facts related to the topic, continuous images of certain continua, where the continua are not pseudo-arcs and pseudo-circles. One of the facts proved by Russo is that there is no continuum X that maps continuously onto all arc-wise connected continua. Another of the facts demonstrated by Russo in his thesis is that there is no continuum X that maps continuously onto all planar indecomposable tree-like continua. There are several more similar items to be found in Russo's thesis.

Immediately, it can be seen from Russo's results that the class of arc-wise connected continua is not contained in classes II, III, IV, and V of Figure 37. Also they are not contained in the continuous images of the pseudo-arc or the pseudo-circles. In fact, the class of arc-wise connected continua is not contained in the continuous images of any continuum. The same holds for the class of planar indecomposable tree-like continua.

The Dual Question

A. Lelek has mentioned to me, in a recent letter, that he feels the study of continuous images of certain continua is not really complete unless you also study the continuous preimages of certain continua. It has not been possible in this paper to study, in any depth, continuous preimages of certain continua, because of the length of the presentations in Chapter III. However, I feel that the least I can do is to mention a sampling of the recent research in this area.

Related to the results in Chapter III would be the following question:

<u>Problem 5.5</u> What characterizes the continuous preimages of the pseudo-arc?

At present, this problem has not been completely solved. However, there has been some partial results obtained. David P. Bellamy (7) has been able to show that the class of hereditarily indecomposable continua is contained in the class of continuous preimages of the pseudo-arc.

Another question that has received considerable recent interest is the following:

<u>Problem 5.6</u> What characterizes the continuous preimages of Knaster's indecomposable continuum, D, with one endpoint?

This continuum D of Knaster's is pictured in Figure 38.

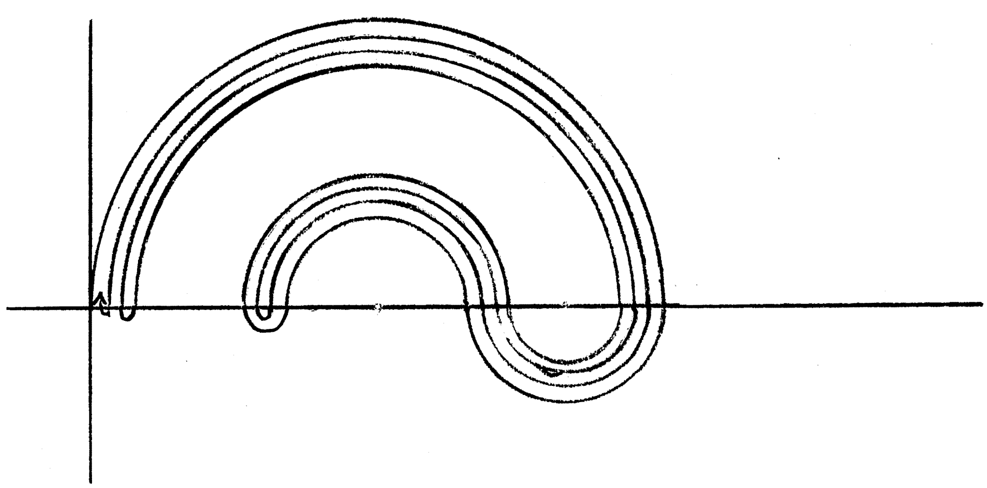


Figure 38. Knaster's Continuum D

Partial answers to this question have been given by David P. Bellamy (5), J. W. Rogers, Jr. (39), and Charles Hagopian (16). In 1969, J. W. Rogers, Jr. (39) was able to show that the class of indecomposable continua is contained in the continuous preimages of D. Next Bellamy (5) generalized Rogers's results to the case of Hausdorff spaces. That is, Bellamy was able to prove that every indecomposable Hausdorff continuum maps continuously onto D. Finally, in 1973, Charles Hagopian (16) was able to characterize the plane continuous preimages of D. The characterizing property of the plane continuous preimages of D is the property of being not "λ-connected".

Definition 5.2 A continuum M is said to be λ-connected if any two of its points can be joined by a hereditarily decomposable continuum in M.

The arc is a continuum that is λ-connected while the pseudo-arc is not λ-connected. Hagopian's result is that a plane continuum M is the continuous preimage of D if and only if M is not λ-connected. Therefore, in the plane the continuous preimages of D are precisely those continua that are not λ-connected.

No characterization of the continuous preimages of Knaster's continuum D have been found in general. Hargopian's results for the plane are the best results to date for Problem 5.6.

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