NEAREST AND FARTHEST POINTS
OF CONVEX SETS

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PREFACE

The basic problem of this thesis is the study of the structure of z-farthest point sets. In a normed linear space the z-farthest point set of a set S is the set of all points which are at least as far from the element z of S as from any other element of S. This type of set is analogous to one defined by Motzkin, [28] (numbers in square brackets refer to the bibliography at the end of the paper), which will be called a z-nearest point set of S in this paper. Phelps, [31], Motzkin, and numerous others have found z-nearest point sets to be a fruitful and interesting topic of research. In this paper, it is shown that z-farthest point sets have many properties analogous to those of z-nearest point sets and some properties which have no counterpart in the theory of z-nearest point sets. Also, further properties of z-nearest point sets are developed.

Chapter I is a brief survey of the research which has been done on nearest point sets and z-nearest point sets. A nearest point set of S relative to a point z is the set of all points of S which are at least as near z as are any other points of S. The main topics of interest and some open questions concerning nearest point sets and z-nearest point sets are pointed out and explained. In Chapter II, farthest point sets, sets analogous to nearest point sets, are defined, and research topics are discussed. It is the purpose of Chapter I and Chapter II to provide motivation for this study of z-farthest point sets. It is hoped that the inclusion of these two chapters will bring about some unification of the theory of these four different types of sets.

In Chapter III, the properties of z-farthest point sets are developed. It is shown that z-farthest point sets are closed and inverse starlike. It is also shown that the set S can be considered to be closed and convex when dealing with the z-farthest point set of S. Other results relate to translation and multiplication by a positive scalar of z-farthest point sets and z-nearest point sets. Properties of the element z of S which has a nonempty z-farthest point set are also discussed in Chapter III. The element z must be a boundary point of S. If the linear space is strictly convex then a z-farthest point set of S is nonempty if and only if z is a boundedly exposed point of S.

The main topic of interest in Chapter IV is a characterization of inner-product spaces in terms of z-farthest point sets. A normed linear space is an inner-product space if and only if for each set S and each element z of S, the z-farthest point set of S is convex. Other results in Chapter IV relate to the representation of z-farthest point sets and z-nearest point sets as intersections of closed half-spaces and unions of closed rays.

Chapter V deals with the approximation of a z-farthest point set of S by a z-farthest point set of a polytope contained in S. Similar results are shown for a z-nearest point set. Finally, Chapter VI is a summary of the paper and lists some unsolved and partially solved problems that have been raised in the course of the investigation.

All notation and terminology which is not defined in this paper can be found in Valentine, [36].

I wish to express my appreciation to all those who assisted me in the preparation of this thesis. In particular, I would like to thank

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CHAPTER I

THE DEVELOPMENT OF NEAREST POINTS

The initial step in the study of nearest points came in 1935 in the form of two articles by Motzkin, [27], which appeared in the same journal. In these articles two types of sets were discussed; the first set is defined as follows:

Definition 1.1. Let X be a normed linear space and let $S \subset X$. If $z \in X$, then

$$\mathfrak{N}(z, S) = \{x \in S : ||z - x|| = \inf \{ ||z - y|| : y \in S\}\}.$$

The elements of $\mathfrak{N}(z, S)$ are called projections of z onto S, and the set $\mathfrak{N}(z, S)$ is called the set of nearest points of S relative to z. Simply stated, the set $\mathfrak{N}(z, S)$ is the set of all points $x \in S$ which are at least as near z as are any other points of S.

Example 1.1. Let X be the space E_2 and let S be the set

$$\{(x, y) : x = -\sqrt{1 - y^2}, \quad -1 \leq y \leq 1\}.$$

If z is the point (1, 0), then $\mathfrak{N}(z, S) = \{(0, 1), \quad (0, -1)\}$. This is illustrated in Figure 1.1. It is apparent that for each point w = (a, 0), a > 0, that $\mathfrak{N}(w, S) = \mathfrak{N}(z, S)$; however, $\mathfrak{N}(0, S) = S$. If T denotes the open set bounded by the arc S and the line segment, $\{(0, y): -1 \leq y \leq 1\}$, then $\mathfrak{N}(z, T)$ is empty.

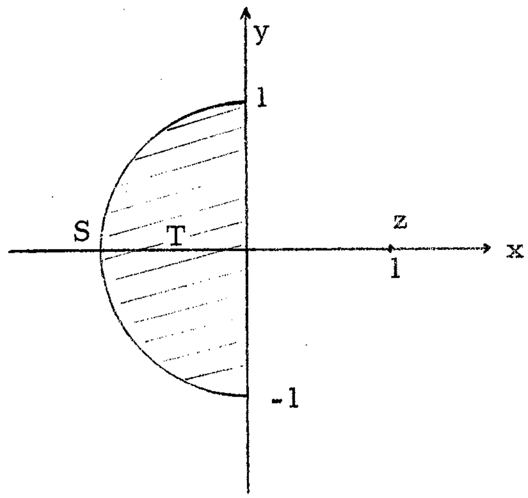


Figure 1. 1.

Since $\mathfrak{N}(z,S)$ may sometimes be empty Phelps, [31], devised a name for those sets S for which $\mathfrak{N}(z,S)$ is not empty for any z.

Definition 1. 2. Let S be a subset of the normed linear space X, then S is proximinal if and only if for each $z \in X$, $\mathfrak{N}(z,S)$ is not empty.

If a set S is proximinal and $\mathfrak{N}(z,S)$ is always a singleton then an acceptable terminology would be uniquely proximinal; however, the name "Chebyshev set" seems to be predominant. Since the person primarily responsible for the study of this type of set is Motzkin, the terminology used here will be Chebyshev-Motzkin set. For example a closed interval [a, b] on the real line is a Chebyshev-Motzkin set.

Definition 1. 3. A subset S of the normed linear space X is a Chebyshev-Motzkin set if and only if $\mathfrak{N}(z,S)$ is a singleton for each $z \in X$.

The following theorem appeared in 1935 in Motzkin's paper, [27]. Although not mentioned in the statement of the theorem, Motzkin also verified the converse in the same article.

Theorem 1.1. If each point of the plane outside a closed set E has a single projection on E, then E is convex.

This theorem was extended to sets in E_n by Jessen, [19], in 1940 and to straight line spaces by Busemann, [7], in 1947. Later authors considered more general spaces and tried to find the relationship of certain geometrical properties of the unit ball and Chebyshev-Motzkin sets. An interesting result of this type is the following theorem by Valentine, [36].

Theorem 1.2. Let X be a smooth and strictly convex finite dimensional normed linear space, and let S be a nonempty closed subset of X. Then S is convex if and only if S is a Chebyshev-Motzkin set.

A characterization of Chebyshev-Motzkin sets in terms of closed convex sets is not possible as shown by Valentine, [36]; however, Busemann, [7], showed that the implication in Theorem 1.2 can be improved in one direction as follows.

Theorem 1.3. Let S be a closed set in the smooth finite dimensional normed linear space. If S is a Chebyshev-Motzkin set then S must be convex.

Theorems 1.1, 1.2, and 1.3 place conditions on the norm of X and then show the relationship of Chebyshev-Motzkin sets to convex sets. Motzkin, [27], noted that the relationship of Chebyshev-Motzkin sets to convex sets determines a geometrical property of the unit ball in the case of two dimensional spaces. This theorem pointed to another avenue of research.

Theorem 1.4. A two dimensional Banach space X is smooth if and only if every Chebyshev-Motzkin set in X is convex.

By reasoning similar to that used by Motzkin in Theorem 1.4 it can be shown that in a finite-dimensional Banach space every Chebyshev-Motzkin set is convex. However, the possible validity of the converse, i.e., that if every Chebyshev-Motzkin set in a finite-dimensional Banach space is convex then the space must be smooth, was not resolved until some years later. Klee believed that the converse was true (see [21]), but it was later proved by Brøndsted, [5], in 1965 to be false. In fact, Brøndsted showed that counter-examples exist for any dimension at least as large as three. Thus the question of whether it is possible to characterize those finite-dimensional spaces which are smooth in terms of Chebyshev-Motzkin sets was raised. Brøndsted, [6], gave a partial answer in the following theorem which appeared in a later article.

Theorem 1.5. Let X be a three-dimensional Banach space with unit ball B. Then every Chebyshev-Motzkin set in X is convex if and only if every exposed point of B is a smooth point of B.

Thus smoothness does not seem to give an entirely satisfactory characterization, and a stronger theorem would be desirable. This is accomplished by substituting the condition of strict convexity for smoothness in Motzkin's theorem, Theorem 1.4. With this substitution the theorem is even true in arbitrary finite dimensional spaces. The theorem, as stated below, appears in Brønsted's paper, [5], but it is not proved there.

Theorem 1.6. A finite dimensional Banach space X is strictly convex if and only if every nonempty closed convex set in X is a Chebyshev-Motzkin set.

The reader will notice that Theorem 1.6 leaves open the possibility that not every Chebyshev-Motzkin set is convex even though X is strictly convex. The Russian mathematicians, Efimov and Stechkin, [11], showed that the conditions of smoothness and strict convexity together removes this possibility.

Theorem 1.7. A finite dimensional Banach space X is strictly convex and smooth if and only if the Chebyshev-Motzkin sets are the nonempty closed convex sets in X.

Theorems 1.1 through 1.6 indicate that one of the main topics of interest in the theory of Chebyshev-Motzkin sets has been the relationship of Chebyshev-Motzkin sets and convex sets in a finite dimensional Banach space. Clearly the topic has been thoroughly explored in the finite-dimensional case, but the infinite-dimensional situation is more delicate. Several men such as Klee, Efimov, Stechkin, and Vlasov have worked on this problem; however, even in Hilbert space it remains unknown whether a Chebyshev-Motzkin set must be convex. One of the first published results concerning spaces of arbitrary dimension is the following which is due to Efimov and Stechkin, [15]:

Theorem 1.8. Let X be a smooth, uniformly convex Banach space, then every boundedly compact Chebyshev-Motzkin set is convex.

In a later paper by Klee, [21], a result similar to Theorem 1.8 is given. The conditions on the Banach space were strengthened somewhat, and the conditions on the Chebyshev-Motzkin set were relaxed to produce a theorem which is the first infinite-dimensional characterization of closed convex sets in terms of the Chebyshev-Motzkin property.

Theorem 1.9. In a Banach space which is uniformly smooth and uniformly convex, a set is closed and convex if and only if it is a weakly closed Chebyshev-Motzkin set.

Another interesting theorem, due to Professor Ficken, but never published by him, was also in the Klee article, [21], in which Theorem 1.9 appeared. Ficken's method, which applies only in inner-product spaces, establishes a close connection between the problem of nearest points - "Must a Chebyshev-Motzkin set be convex?" - and a related problem involving farthest points. This relationship will be explained in more detail later, but the theorem due to Ficken is stated below.

Theorem 1.10. In a Hilbert space, every compact Chebyshev-Motzkin set is convex.

The theorems from Theorem 1.1 to Theorem 1.10 represent the more interesting and perhaps the most important conclusions drawn from the theory of Chebyshev-Motzkin sets. It is obvious that the theory of Chebyshev-Motzkin sets is not complete especially since large gaps are present in the theory for infinite-dimensional spaces.

One open question is - "Can the infinite-dimensional Banach spaces in which every Chebyshev-Motzkin set is convex be characterized?" - and another question mentioned before is - "Is every Chebyshev-Motzkin set convex in a Hilbert space?". Theorems 1.7, 1.8, and 1.9 are all efforts in the direction of one of the two questions stated above.

The class of non-Chebyshev-Motzkin sets is another facet in the study of sets of nearest points. These sets have not been as interesting as Chebyshev-Motzkin sets, and accordingly there is a paucity of results. However, the following two theorems, due to Erdös, [16], are results of this kind.

Theorem 1.11. Let S be a closed set in E_n. Denote by M the set of points $z \in E_n$ for which $\mathfrak{N}(z, S)$ consists of more than one point. Then the set M has Lebesgue measure zero.

The other interesting theorem in Erdös' paper states that the union of all sets of nearest points in a closed set S has Lebesgue measure zero. At first glance this does not seem too surprising since one expects the measure of the boundary of a closed set to be zero.

Theorem 1.12. Let S be a closed set in E_n, and let $x \in E_n \setminus S$. Then

$$\bigcup_{x \notin S} \mathfrak{N}(x, S)$$

has Lebesgue measure zero.

Valentine, [36], gives a more conventional type of theorem concerning sets of nearest points with the following theorem which again deals with the properties of the set S rather than with $\mathfrak{N}(z, S)$.

Theorem 1.13. Let S be a closed set in E_n. Let P denote the set of points z for which $\mathfrak{N}(z, S)$ contains two or more points. If P consists of only isolated points, then each bounded component of the complement of S is a solid open sphere whose center belongs to P. Moreover, bd conv S $\subset$ S.

Thus, according to Theorem 1.13, if a nonconvex set S has "holes" in it, then they must be "perfectly round," provided the set P consists of only isolated points. This theorem and the two immediately preceding it seem to be the major theorems relative to nearest points of closed sets that may be non-Chebyshev-Motzkin sets. Some other work has also been done by Pauc, [30]. Studies of this type are difficult since the structure of nonconvex sets is so general.

Existence of Sets of Nearest Points

Up to now, nothing has been said about the existence of the set $\mathfrak{N}(z, S)$ even though each preceding theorem has been concerned with the properties of S as related to $\mathfrak{N}(z, S)$. If S is an open set in a normed space then it is easily seen that $\mathfrak{N}(z, S)$ is empty whenever $z \notin S$, and if S is neither open nor closed $\mathfrak{N}(z, S)$ will be nonempty for some points z and empty for others. On the real line, if x is a number greater than b, then x has no nearest point in the half-open interval [a, b), but if x is less than a, then x has a nearest point, namely a. The following three theorems which are stated by Phelps, [31], give some instances when $\mathfrak{N}(z, S)$ is not empty.

Theorem 1.14. If S is a compact set in a normed linear space X, then $\mathfrak{N}(z, S)$ is not empty for any $z \in X$.

Theorem 1.15. If X is a finite dimensional normed linear space then each closed set is proximinal.

Theorem 1.16. If S is closed and convex and if a normed space X is reflexive, then S is proximinal.

The three theorems above are rather old, and their origins are difficult to trace. However, work is still being done on finding sufficient conditions for $\mathfrak{N}(z, S)$. Most recently Edelstein, [12], has shown under certain conditions that even though $\mathfrak{N}(z, S)$ may be empty for some z in a Banach space there are still sufficiently many points for which $\mathfrak{N}(z, S)$ is not empty to form a dense set in X.

Theorem 1.17. Let S be a nonempty closed set in a uniformly convex Banach space X. Then the set C of all points c in X for which there is a point $s \in S$ with

$$||s - c|| = \inf \{ ||x - c|| : x \in S \}$$

is dense in X.

This theorem followed an earlier theorem by Edelstein, [11], in which he showed that if S is a closed set in a uniformly convex Banach space X the set C of all points c such that

$$||s - c|| = \sup \{ ||x - c|| : x \in S \}$$

is dense in X. In response to this theorem, Asplund, [2], published a paper in which he proved that if S is closed and bounded in a reflexive and locally uniformly convex Banach space X, the complement of the above set C is of first Baire category. Thus, since Edelstein was able to prove a theorem about nearest points analogous to his first theorem concerning farthest points, a logical conjecture would be as follows: If S is a closed set in a reflexive and locally uniformly convex Banach space, then, except on a set of first Baire category, the points in X have nearest points in S. Of course, more generally, the open problem here is to characterize those spaces in which each closed set has sets of nearest points.

The Theory of z-Nearest Point Sets

The set $\mathfrak{N}(z, S)$ is always a subset of S. From Example 1.1 it is obvious that if x is in $\mathfrak{N}(z, S)$ then there are possibly more points w in X such that $x \in \mathfrak{N}(w, S)$. Hence, for a given $x \in S$ another set of interest related to x and S is the set of all $z \in X$ such that $x \in \mathfrak{N}(z, S)$. This is defined more formally below.

Definition 1.4. Let S be a subset of the normed linear space X and let $z \in S$. Then let N(z, S) denote the set of all points x in X such that

$$||x - z|| = \inf \{ ||x - y|| : y \in S \}.$$

This set was also introduced by Motzkin, [28], and was later studied by Pauc, [30], Phelps, [31], and Klee, [23]. In this article the set N(z, S) will be referred to as the z-nearest point set of S as opposed to sets of nearest points for the set $\mathfrak{N}(z, S)$. The elements of N(z, S) will be called z-nearest points. In order to make clear the meaning of the definition, consider the following example.

Example 1.2. Let S be the closed unit disk in E_2 and let z be the point (1, 0) which lies on the boundary of S. Then N(z, S) is the ray $\{(x, 0): x \geq 1\}$ (cf. Figure 1.2). Notice that S is convex and that $N(z, S)$ is a closed, convex cone.

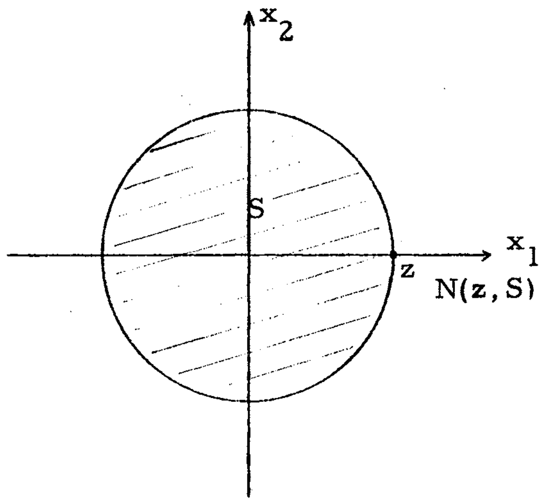


Figure 1.2.

Example 1.3. Let S be the set of points in E_2 whose first coordinates are not greater than -1 together with the point (1, 0). Let z be the point (1, 0), then the set of points equidistant from z and $S \setminus \{z\}$ is the parabola $\{(x_1, x_2) : x_2^2 = 4x_1\}$. Thus, it follows that $N(z, S)$ is the set $\{(x_1, x_2) : x_2^2 \leq 4x_1\}$. Note that again $N(z, S)$ is convex but that it is not a cone (cf. Figure 1.3).

Motzkin, [28], first studied this set and provided the first important theorem concerning them. His theorem, Theorem 1.18, shows that the z-nearest point sets in Example 1.2 and Example 1.3 have to be convex because E_2 is an inner product space.

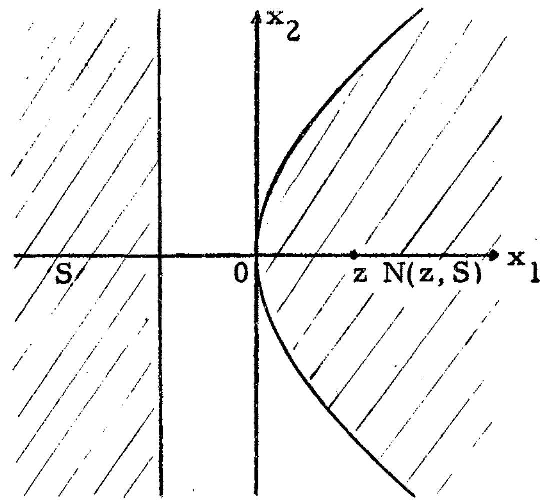


Figure 1.3.

Theorem 1.18. Suppose X is a two dimensional normed real linear space. Then X is an inner-product space if and only if for each set S and $z \in S$, $N(z,S)$ is convex.

Actually Motzkin required that X be a two dimensional space in which the unit ball is an ellipse, but this is known to be equivalent to an inner product space (cf. Day, [8]). Motzkin's result was extended by Phelps, [31], to include any finite dimensional inner product space. Examples which illustrate Theorem 1.18 are easily found. The following example shows a z-nearest point set which is not convex in a normed linear space which is not an inner-product space.

Example 1.4. Let X be the space R_2 with $||x|| = \max\{|x_1|, |x_2|\}$, $x = (x_1, x_2)$. Let $S = \{(0,0), (1,0)\}$, then if $z = (0,0)$ the set $N(z,S) = A \cup B$, where

$$A = \{t : t = (x_1, x_2),\ x_1 \leq 1/2\}$$

and

$$B = \{t : t = (x_1, x_2),\ x_1 > 1/2,\ |x_2| \geq x_1\}$$

(cf. Figure 1.4).

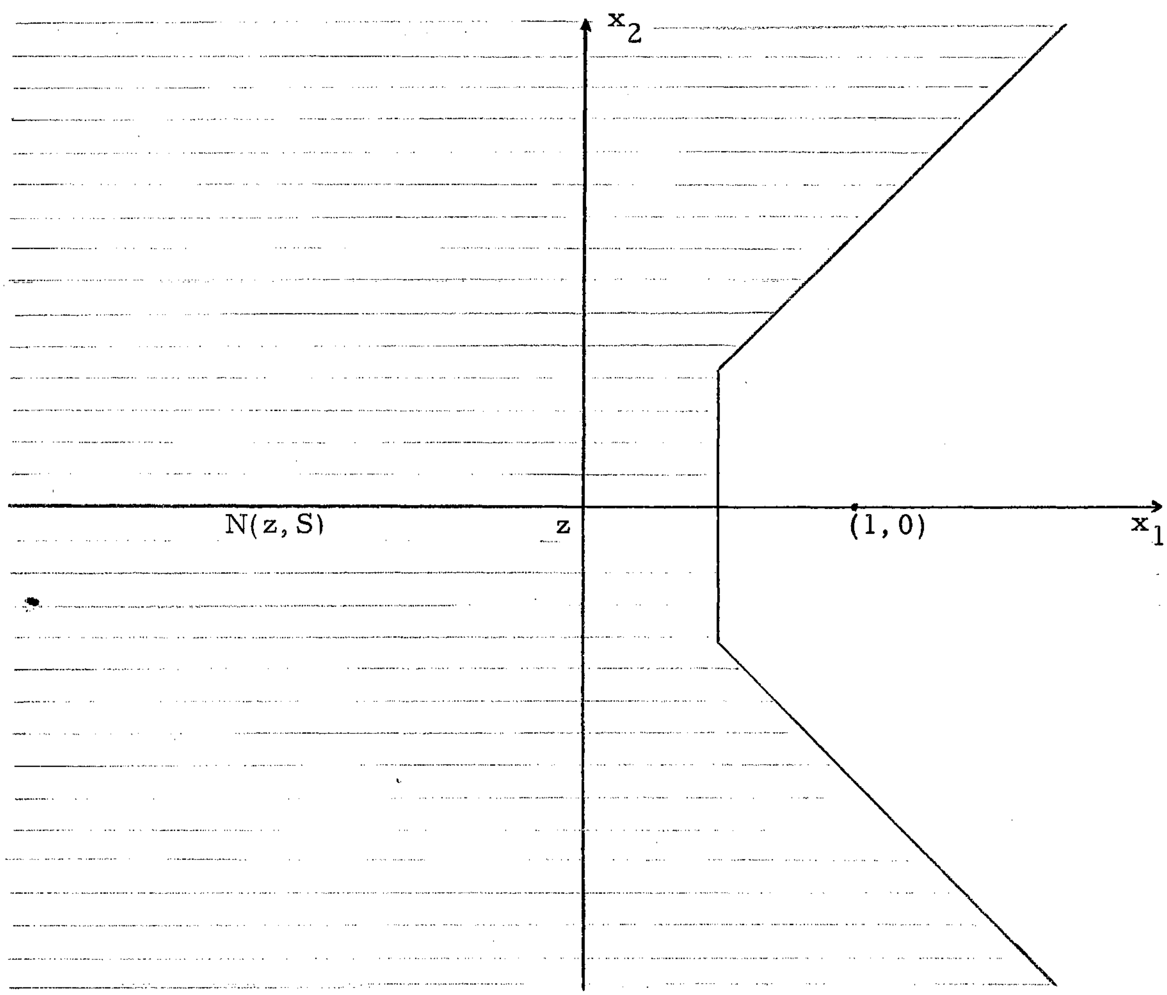


Figure 1.4.

In the following extension of Motzkin's theorem, Theorem 1.18, Phelps required convexity of S.

Theorem 1.19. Suppose that the dimension of the normed linear space X is at least three (equal to two). Then X is an inner-product (strictly convex) space if and only if for each convex set S and $z \in S$, N(z, S) is convex.

Other authors have sought after the geometrical properties of N(z, S). One of the earliest workers on this problem was Pauc, [30]. Pauc's theorems dealt with the boundedness and the interior points of N(z, S).

Theorem 1.20. Let S be a subset of the Euclidean space E_n, then each interior point of N(z, S) has only a single projection on S, namely z.

Pauc also showed that, although z is always an element of N(z, S), the only way for z to be an interior point of N(z, S) is to be an isolated point of S.

Theorem 1.21. In the space E_n, the element z of the set S is interior to N(z, S) if and only if z is an isolated point of S.

Pauc further developed the geometrical picture of N(z, S) with the following theorem:

Theorem 1.22. In the space E_n, the set N(z, S) is bounded for a set S if and only if z is interior to the convex hull of S.

This type of research was taken up much later by Klee. Klee preferred to consider more general spaces, and his theorem illuminates even better the geometrical shape of $N(z,S)$. The following theorem of Klee, [23], shows that the set $N(z,S)$ is a cone if S is convex.

Theorem 1.23. Let X be a normed linear space and $S \subset X$. If S is convex and $z \in S$ then $N(z,S)$ is a cone with vertex z.

Example 1.4 illustrates a nonconvex set S where $N(z,S)$ is not a cone. If the norm on R_n is changed to the usual Euclidean norm in Example 1.4, then $N(z,S)$ becomes the set of points (x_1, x_2) such that x_1 is not greater than one half. This set is a cone, but its vertex is not z. Klee, [23], went on to state a partial converse of Theorem 1.23; however, he did not prove it. A proof can be found in Phelps' first paper, [31], on nearest points. The theorem is as follows:

Theorem 1.24. Suppose S is closed and proximinal and that the normed linear space X is smooth. Then S is convex if for each $z \in S$, $N(z,S)$ is a cone with vertex z.

Since every closed subset of (smooth) E_n is proximinal, Theorems 1.23 and 1.24 combine, as shown by Phelps, to prove the following characterization of convexity of a closed set.

Theorem 1.25. A closed set S in E_n is convex if and only if for each $z \in S$, $N(z,S)$ is a cone with vertex z.

From the preceding theorems it is seen that the property that $N(z,S)$ is a cone has a similar relationship to S as the Chebyshev-Motzkin property has to S. That is, the two properties are both equivalent to convexity under suitable conditions, and one might suspect that if N(z, S) is a cone that each point of N(z, S) has one nearest point in S, namely z. This seems especially possible in light of Pauc's theorem, Theorem 1.20, which says that the interiors of two sets N(z, S) and N(w, S), $w \neq z$, do not intersect. Phelps showed in [31], that if S is a convex set in a strictly convex space X, then $N(z, S) \cap N(w, S)$ is empty for $z, w \in S$ and $z \neq w$.

The sets N(z, S) have been shown to be convex, unbounded, and cones, given favorable conditions. So a possible question at this point is "what characterizes the spaces X such that for a set M there exists a set S and a point z such that M = N(z, S)?" Must all these sets be convex, cones, or unbounded? The answer to these questions was provided by Phelps, [32], in his second paper on nearest points. These theorems are interesting inasmuch as their statements closely parallel those in the Chebyshev-Motzkin series.

Theorem 1.26. In a complete inner-product space X for each closed convex set T there is a set $S \subset X$ and a point $z \in S$ such that T = N(z, S).

For dimensions greater than three the converse of Theorem 1.26 is true; and hence, the first question asked above is answered for finite dimensional spaces.

Theorem 1.27. Suppose that the dimension of X is not less than three and that every closed convex subset T of X has the property that there is a set $S \subset X$ and a point $z \in S$ such that T = N(z, S). Then X is a complete inner-product space.

Then from Motzkin's theorem, Theorem 1.1, it is known that in a finite dimensional inner-product space each set N(z, S) must be convex, so this together with Theorems 1.26 answers the second question. Phelps also presented some other results of a different nature in his second paper.

Closest-Points

<u>Definition 1.5.</u> Let A be a subset of the normed linear space X, then $y \in X$ is said to be point-wise closer to A than is x provided $||y - a|| < ||x - a||$ for each $a \in A$. If x is such that no point of X is point-wise closer to A than x then x is called a closest-point to A.

<u>Example 1.5.</u> Let X be the space E_2, then if A is the open unit disk, each point of the boundary of A is a closest-point to A.

The concept of closest-points and sets of closest-points was originated by Fejér, [18], who proved the following theorem.

<u>Theorem 1.28.</u> If A is a subset of the complete inner-product space X then conv(A) is equal to the set of all closest-points to A.

Phelps, [32], obtained a partial converse of Fejér's theorem which showed that the complete inner-product spaces of finite dimension greater than two can be characterized in terms of closest points.

<u>Theorem 1.29.</u> Suppose that the dimension of the space X is at least three and that for each closed convex set $T \subset X$ there exists a set $S \subset X$ and a point $z \in S$ such that $T = N(z, S)$. Then X is a complete inner-product space.

A theorem similar to Theorem 1.28 and Theorem 1.29 was also obtained by Phelps, [32], for spaces of dimension two by merely requiring that X be strictly convex. The following theorem was also shown by Phelps in the same article.

Theorem 1.30. Let X be a normed linear space of dimension two, then for each subset A of X the set of closest-points to A is a subset of conv A if and only if X is strictly convex.

The subject of closest-points does not seem to be well explored as evidenced by the small number of articles written concerning them. An open question here is to characterize the spaces such that the set of closest points of any set A coincides with conv A.

Nearest Point Maps

A function f can be defined on a space X given a closed proximinal set S as follows: If $x \in X$ let $f(x)$ be a point $z \in S$ such that $x \in N(z, S)$. This nearest point map can exist if and only if S is proximinal and a Chebyshev-Motzkin set. The continuity of this function, when S is a Chebyshev-Motzkin set, has been found by Klee, Phelps, Fan, and Glicksberg to be closely related to the convexity of S. The following two theorems are stated by Klee, [21].

Theorem 1.31. In an arbitrary normed linear space, the nearest point map onto a boundedly compact Chebyshev-Motzkin set is continuous.

Theorem 1.32. In every uniformly convex Banach space X, the nearest point map onto a closed convex set is continuous.

When a set S is merely a Chebyshev-Motzkin set or when S is a convex Chebyshev-Motzkin set, it is not known what circumstances will cause the nearest-point map to be continuous. Even when S is a Chebyshev-Motzkin set in a Hilbert space, it is not known whether the associated nearest-point map must be continuous. However, continuity of the nearest-point map can be used to demonstrate the convexity of Chebyshev-Motzkin sets as shown by Klee, [21], in Theorem 1.33. This theorem is a generalization of an earlier theorem by Klee, [25], in which the Chebyshev-Motzkin set S was required to have a continuous and weakly continuous nearest-point map.

Theorem 1.33. Let S be a Chebyshev-Motzkin set in a smooth reflexive Banach space X, and each point of $X \setminus S$ admits a neighborhood on which the (restricted) nearest-point map is both continuous and weakly continuous. Then S is convex.

An interesting concept in the theory of nearest-point maps is that of a "sun". Some of the previous theorems could have been stated using this term.

Definition 1.6. Let S be a Chebyshev-Motzkin set in a space X, and let f(x) be the nearest-point map of X onto S. Then S is a sun if $f(z) = f(x)$ for every $x \in X \setminus S$ and every z on the ray emanating from f(x) and passing through x.

Thus, Definition 1.6 says that S is a sun if N(z, S) is a cone for each $z \in$ bdS. Hence Theorem 1.25 by Phelps could be changed to read--"A closed set S in E_n is convex if and only if S is a sun." Klee [21] also proved the following theorem concerning nearest point maps.

Theorem 1.34. If S is a Chebyshev-Motzkin set in a reflexive Banach space X, and if each point in $X \setminus S$ has a neighborhood on which the restriction of the nearest-point map is continuous and weakly continuous, then S is a sun.

Alternatively, L. P. Vlasov, [37], has shown that in any Banach space every boundedly compact Chebyshev-Motzkin set is a sun. No example is known of a Chebyshev-Motzkin set which is not a sun or does not have a continuous nearest-point map. However, Brøndsted, [5], was able to prove the following theorem.

Theorem 1.35. In any smooth normed linear space every sun is convex.

Since nearest points and the structure of the norm are closely related, as demonstrated by Theorem 1.18, it follows that the nearest-point map should be related to the norm. This has been shown by Phelps, [31], who makes the following definitions.

Definition 1.7. Let f be the nearest-point map defined by the Chebyshev-Motzkin set S, then f is said to shrink distances if $||f(x) - f(y)|| \leq ||x - y||$ whenever $x, y \in X$.

Definition 1.8. The normed linear space X is said to have property P if a nearest-point map shrinks distances whenever it exists for a closed convex set $S \subset X$.

Phelps, [31], proved a rather interesting theorem concerning property P. The proof of this theorem is also interesting in that it depends on a type of orthogonality defined by Birkhoff, [4], which is meaningful in a general normed space and coincides with the usual type in an inner-product space.

Theorem 1.36. Let the dimension of the normed linear space X be at least three (respectively, equal to two). Then X is an inner-product space (respectively, strictly convex and orthogonality is symmetric) if and only if X has the property P.

Phelps also showed that this "shrinking" property of nearest-point maps is restricted to those which exist for convex sets.

Theorem 1.37. Let the normed linear space X be strictly convex and assume that a nearest-point map f exists for the closed set $S \subset X$. Then S is convex if f shrinks distances.

The theorems presented in the preceding pages represent the main stream of research in the theory of nearest points. Not all theorems by all authors working in this area have been presented, but an effort has been made to present those which best illustrate the general trend of research. The bibliography presented in this paper is not complete, but it is extensive.

CHAPTER II

THE DEVELOPMENT OF FARTHEST POINTS

The obvious question at this point is whether or not analogous sets of points, a set of farthest points and a z-farthest point set, could be defined which would have some, or possibly all, of the analogous properties of sets of nearest points and z-nearest point sets, respectively. Sets of farthest points have been defined and considered by several authors; however, the properties of this set seem to be less developed than those of sets of nearest points. In Chapter III, sets analogous to z-nearest point sets will be defined and considered.

The definition of sets of farthest points is as follows:

Definition 2.1. Let X be a normed linear space and let $S \subset X$. If $z \in X$, then $\mathfrak{F}(z, S) = \{x \in S: \|z - x\| = \sup\{\|z - y\| : y \in S\}\}$.

It is obvious that the set S must be bounded in order for $\mathfrak{F}(z, S)$ to be nonempty. Although the elements of the set $\mathfrak{F}(z, S)$ have been named, there appears to be no terminology in general usage. Let us call the elements of $\mathfrak{F}(z, S)$ the farthest points of z in S. The set $\mathfrak{F}(z, S)$ will be called the set of farthest points of S relative to z.

Example 2.1. Let X be E_2 and let

$$S = \{(x, y) : x = -\sqrt{1 - y^2}, \quad -1 \le y \le 1\}.$$

If z is the point (1, 0), then $\mathfrak{F}(z, S) = \{(-1, 0)\}$, (cf. Figure 2.1).

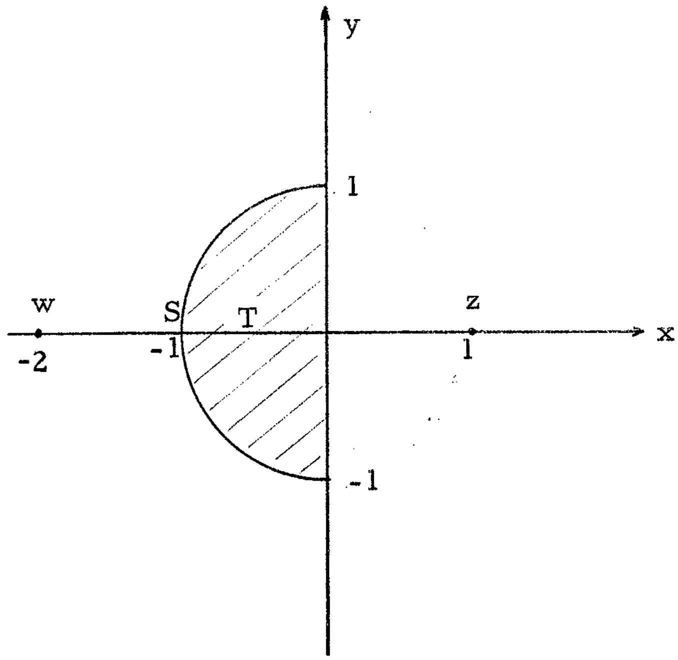


Figure 2. 1.

If $w = (-2, 0)$, then $\mathfrak{F}(w, S) = \{(0, 1), (0, -1)\}$. Moreover, if $p = (x, 0)$ is any point such that $x > 0$, then $\mathfrak{F}(p, S) = \{(-1, 0)\}$, and if $q = (x, 0)$ such that $x < 0$, then $\mathfrak{F}(q, S) = \{(0, 1), (0, -1)\}$. If T denotes the open set bounded by the arc S and the line segment, $\{(0, y) : -1 \leq y \leq 1\}$, then $\mathfrak{F}(z, T)$ is empty.

Example 2. 1 shows that $\mathfrak{F}(z, S)$ may sometimes be empty; however, no one has bothered to name those sets S for which $\mathfrak{F}(z, S)$ is not empty for any $z \in X$. Following Phelps' lead in defining proximinal, a good name would be remotal, a combination of the words remote and maximal.

Definition 2. 2. Let S be a subset of the normed linear space X, then S is remotal if and only if for each $z \in X$, $\mathfrak{F}(z, S)$ is not empty.

If a set S is remotal then there is the possibility that $\mathfrak{Z}(z, S)$ is a singleton set for each $z \in X$, in this case S is said to be uniquely remotal. If for each point $z \in M$, where $M \subset X$, $\mathfrak{Z}(z, S)$ is a singleton set, then S is said to be uniquely remotal with respect to the set M.

If the set $\mathfrak{Z}(z, S)$ is to be closely analogous to $\mathfrak{N}(z, S)$ then it is necessary that $\mathfrak{Z}(z, S)$ is related to the convexity of S and the structure of the norm of X. Very little has been done in this direction, perhaps because of the difficulty of the problems or possibly the problems have not been considered interesting. However, a few authors have pursued the solutions of analogous problems to those of Motzkin. One of the earliest such writers was Jessen, [19], who proved the following theorem.

Theorem 2. 1. In a Euclidean space, a bounded, closed, convex set S is uniquely remotal with respect to its complement if and only if it has interior points and contains the centers of all osculating spheres of its boundary.

An osculating circle in the plane is a circle that is tangent to a given curve K at a point p of K which has a higher degree of contact with K at p than has any other circle. This is similar to the case when considering surfaces in spaces of greater dimension. To find the osculating spheres at a point q of the boundary F of a closed, bounded, convex set K in E_n, let H be a support hyperplane to K at q and let h be the ray with end-point q which is perpendicular to H and lies on the same side of H as K. (cf. Figure 2. 2). Then for each point $p \in F$ let Q' be that sphere with center p' on h which passes through q and p. Every limit point q' of this set of centers is called an osculating center

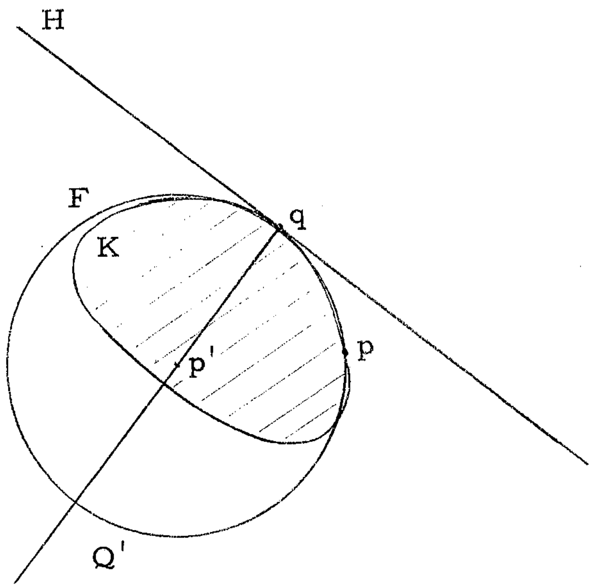


Figure 2.2.

for F on h. The corresponding spheres passing through q are called the osculating spheres for F for the element q of H.

Theorem 2.1 seems to be the only theorem which characterizes convex sets as sets which are uniquely remotal with respect to their complements. Most authors have been content to study sets which are uniquely remotal with respect to the entire space X. Such sets are really not as structurally interesting since the results indicate that if a set is uniquely remotal, then it is a singleton set, [36]; however, this has not been shown in very general spaces, in fact, it has not been shown for a Hilbert space.

An interesting result along this line is one by Ficken, which was never published by Ficken but appears in an article by Klee, [21]. As mentioned before, Ficken's method of proof relates a basic problem in the theory of nearest points to a basic problem in the theory of farthest points. By a process involving an inversion in spheres he establishes a close connection between the problem--"Must a Chebyshev-Motzkin set be convex?"--and the related problem--"Must a set which is uniquely remotal be a singleton?"

In order to present Ficken's result it is necessary to make the following definition.

Definition 2.3. In a normed space X, a set M is Chebyshevian at a point $z \in X$ provided $z \notin M$ and M is uniquely proximinal for each point $y \in X$ for which

$$\|y - z\| < \inf \{\|y - x\| : x \in M\}.$$

Ficken's theorem, with some sharpening and embellishment by Klee, is as follows:

Theorem 2.2. Let E be an inner product space, Λ and Δ classes of subsets of E such that Λ and Δ are related as follows:

Whenever $X \in \Lambda$, $x \in (\text{conv } X) \setminus X$, and ξ is the inversion of E in a sphere centered at x, then $\text{conv } \xi X \in \Delta$;

Whenever $Y \in \Delta$, y is an inner point of a line segment in Y, and η is the inversion of E in a sphere centered at y, then $\eta(Y \setminus \{y\}) \in \Delta$

Then the following two statements are equivalent:

1. If $X \in \Lambda$ and X is Chebyshevian at y, then $y \notin \text{conv } X$;
2. If $Y \in \Delta$ and Y is uniquely remotal, then Y is a single point.

A less complicated result by Motzkin, Straus, and Valentine, [29], is stated here as Theorem 2.3.

Theorem 2.3. If a subset Y of E_n is uniquely remotal, then Y must be a single point.

Klee, [21], points out that the convexity of Chebyshev-Motzkin sets in E_n may be deduced from Theorem 2.2 and Theorem 2.3; moreover, Theorem 2.2 indicates that if the unique nearest point problem can be solved in a Hilbert space then the corresponding unique farthest point problem will be solved. Thus, in the setting of an inner-product space Ficken has tied the theory of nearest and farthest points together.

Motzkin, Straus, and Valentine, [29], have contributed to the theory of farthest points by not only considering sets which are uniquely remotal, but also sets for which $\mathfrak{F}(z, S)$ has a constant, finite number of elements. Their results give some insight into the makeup of the boundary of a remotal set and the shape of some sets. The following theorem describes the boundary of a certain type of remotal set.

Theorem 2.4. Suppose S is a continuum in a two-dimensional normed linear space. If S is uniquely remotal with respect to S, then $\bigcup_{x \in S} \mathfrak{F}(x, S) = \text{bd conv } S$.

The following theorem shows the structure of S when $\mathfrak{F}(x, S)$ consists of exactly two points for each $x \in S$.

Theorem 2.5. Suppose S is a compact set in the plane E_2, and suppose that for each $x \in S$ the set of farthest points $\mathfrak{F}(x, S)$ has at least two points. Then S is contained in the union of a finite number of line segments. If $\mathfrak{F}(x, S)$ has exactly two elements for each $x \in S$, then S must be disconnected.

Farthest Point Maps

Motzkin, Straus, and Valentine in their paper, [29], considered a farthest point map which is analogous to the nearest point map defined in previous paragraphs. Although they did not demonstrate very many properties of this map, the map was useful in the proofs of some of their theorems.

Definition 2.3. Let S be a remotal subset of the normed linear space X, then the map Y, such that $Y(x) = \mathfrak{F}(x, S)$, $x \in X$, is called the farthest point map of X onto S.

Most of the properties and defintions given for the nearest point map have no analogies here. It is obvious that closest-points could have no analogy. But even so, the farthest point map does have some similar properties such as the following by Motzkin, Straus, and Valentine, [29].

Theorem 2.6. Suppose S is a subset of the normed linear space X and suppose S is uniquely remotal with respect to $T \subset X$. Then ϕ, where $\phi(x) = ||x - y(x)||$, for $x \in T$ and $y(x) \in Y(x)$, is continuous on T.

Finally, to close the discussion of the properties of $\mathfrak{F}(z, S)$, notice that Jessen's theorem, Theorem 2.1, shows that the convexity of a set S depends on $\mathfrak{F}(z, S)$, but the set $\mathfrak{F}(z, S)$ can also be shown to determine the convex set S in a manner similar to that of the extreme points and exposed points of S. The Krein-Milman theorem, see Valentine, [36], states that under suitable conditions, the closed convex hull of a set S is equal to the closed convex hull of its extreme points.

Straszewicz, [34], showed the exposed points of S could replace the extreme points of S in the Krein-Milman theorem. More recently Asplund, [1], has shown the following theorem, but the theorem can be deduced from Straszwicz's theorem or from a theorem by Klee, [22].

Theorem 2.7. Let S be a closed, bounded, and convex set in a Hilbert space X, then $S = \overline{\text{conv}} \bigcup_{z \in X} \mathfrak{F}(z, S)$.

It is evident from Theorem 2.7 and the previous theorems that farthest points are important building blocks of a convex set. Thus it is unfortunate that so little has been done with the theory of farthest points. The articles by Asplund, Jessen, Klee, and the article by Motzkin, Straus, and Valentine appear to be the only papers which relate farthest points to convex sets.

The Existence of Farthest Points

Finally, to close this discussion, the existence of farthest points will be considered. Again, as in the case of nearest points, there are certain theorems dealing with the existence of $\mathfrak{F}(z, S)$ which cannot be attributed to any one person. An example of this is the following theorem.

Theorem 2.8. A compact subset of a finite-dimensional normed linear space is remotal.

It is also true that if a set S is compact in a normed linear space then S is remotal; however, not all closed and bounded sets are compact. The reader will recall that every closed convex set S is proximinal if the normed space X is reflexive; however, apparently it is not known whether a closed, bounded, and convex set S is remotal in a reflexive space.

Edelstein, [11], has worked on the problem of the existence of $\mathfrak{F}(z, S)$, and although he has not shown that a closed, bounded set must be remotal, he has shown that the points x in a certain type of space such that $\mathfrak{F}(x, S)$ is not empty must be dense in the space. This theorem is very similar to Theorem 1.17.

Theorem 2.9. Let S be a nonempty closed and bounded set in a uniformly convex Banach space X. Then S is remotal with respect to a dense subset of X.

Asplund, [2], following Edelstein's lead, discovered a similar theorem. Instead of a dense set, Asplund's theorem deals with a set of Baire category one, a set that is the union of a countable number of nowhere dense sets.

Theorem 2.10. If S is a bounded, closed subset of a reflexive, locally uniformly convex Banach space X, then, except on a set of first Baire category, S is remotal.

Edelstein's theorem and Asplund's theorem are both interesting, but they fail to answer the basic question here--"What conditions on the space X will insure that each closed and bounded set is remotal?" Perhaps the only answer is that X must be finite dimensional.

The preceding paragraphs and theorems demonstrate the direction of the research in the theory of nearest and farthest points. This chapter is meant to be only a survey so many results had to be omitted.

CHAPTER III

PROPERTIES OF THE SET F(z, S)

In 1935, T. Motzkin, [27] defined, for a given set S and a point $z \in S$, the sets N(z, S) and $\mathfrak{N}(z, S)$. Later authors such as Asplund, Edelstein, and Klee investigated the set $\mathfrak{F}(z, S)$, which was defined in a natural manner analogous to that of $\mathfrak{N}(z, S)$. Hence an obvious extension in the theory of nearest and farthest points would be a definition analogous to that of N(z, S). This definition is as follows:

Definition 3.1. Let S be a subset of the normed linear space X and let $z \in S$, then

$$F(z,S) = \{x \in X : ||z - x|| = \sup \{||y - x|| : y \in S\}\}.$$

Simply speaking, the set F(z, S) is the set of all $x \in X$ which are at least as far from z as from any other point of S or, alternatively, the set F(z, S) is the set of all $x \in X$ such that z is an element of $\mathfrak{F}(x, S)$. The elements of the set F(z, S) will be called z-farthest points of S, and the set F(z, S) will be called the z-farthest point set of S. The following examples should illustrate the concept of F(z, S).

Example 3.1. Let X be the space E_2 and let S be the closed unit disk (cf. Figure 3.1). Then if $z = (-1, 0)$, $F(z, S) = \{(x, 0) : x \geq 0\}$.

If $w = (t, 0) \in \{(x, 0) : x \geq 0\}$, then the circle having equation $||p - w|| = ||w|| + 1$ passes through the point z and contains in its

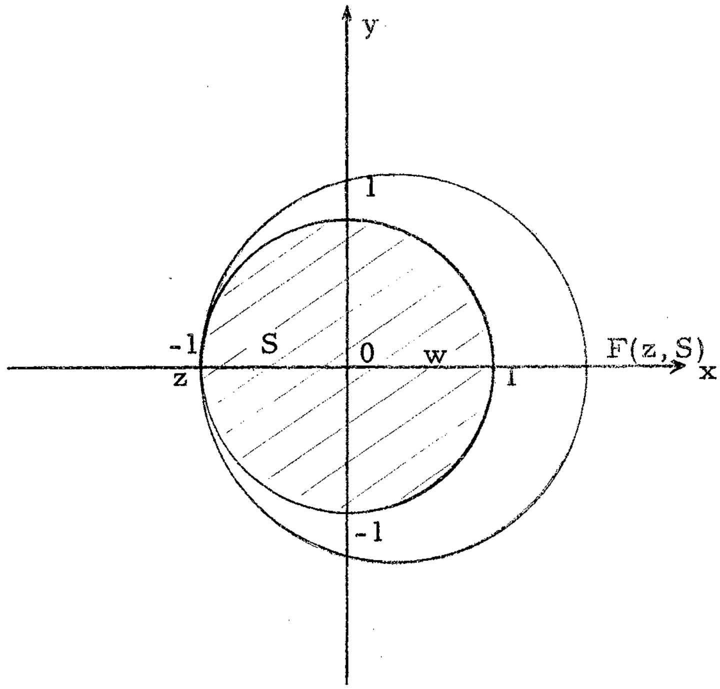


Figure 3. 1.

interior the set S; hence, $||z - w|| \geq ||p - w||$ for all $p \in S$. Since $z \in S$, $||z - w||$ is equal to $\sup \{ ||p - w|| : p \in S\}$ and, therefore, $w \in F(z, S)$. On the other hand, if $w = (u, v) \notin F(z, S)$ then the circle $||p - w|| = ||w|| + 1$ passes not through z, but through

$$w' = \left(\frac{-u}{|w|}, \frac{-v}{|v|} \right) \in S,$$

so that $||z - w|| < ||w' - w||$. Hence $w \notin F(z, S)$ and, therefore, $F(z, S) = \{(x, 0) : x \geq 0\}$.

<u>Example 3. 2.</u> Let X be the space E_2 and let

$$S = \{(0, 0), (1, 0), (0, 1), (2, 2)\}.$$

Then if $z = (0, 0)$,

$$F(z,S) = \{(x,0) : x \geq 1/2\} \cap \{(0,y) : y \geq 1/2\} \cap \{(x,y) : x + y \geq 2\},$$

(cf. Figure 3.2).

In this example, the set of points farther from z than from (1, 0) is $\{(x,0) : x \geq 1/2\}$; the set of points farther from z than (0, 1) is $\{(0,y) : y \geq 1/2\}$; and similarly the set of points farther from z than (2, 2) is the set $\{(x,y) : x + y \geq 2\}$. Hence, the intersection of these three sets is F(z, S). Note that F(z, S) is closed and convex, but it is not a cone.

Although a z-farthest point set need not be a cone, it must always be closed. This is shown by the following theorem. The theorem is proved for more general sets than z-farthest point sets by not requiring z to be an element of S.

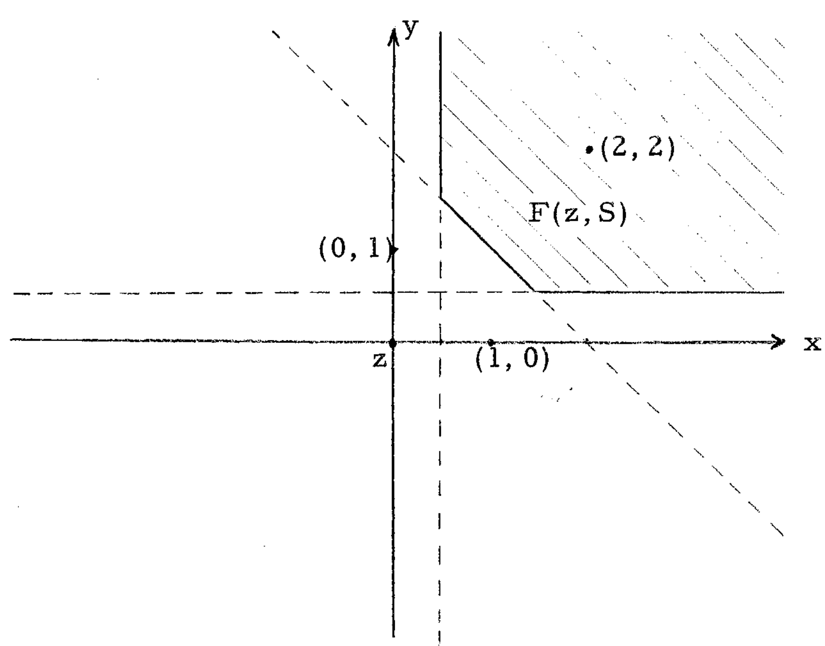


Figure 3.2.

Theorem 3.1. Let X be a normed linear space, $S \subset X$, and $z \in X$. Then

$$F = \{x \in X : \|x - z\| = \sup \{\|x - y\| : y \in S\}\}$$

is a closed set.

Proof: If F is empty then the theorem is true. Suppose F is not empty and assume that w is a limit point of F. There exists a sequence $\{x_n\}$ of points of F such that $\{x_n\}$ converges to w. So for each real number $\epsilon > 0$ there exists a positive integer N such that

$$\|x_n - w\| < \epsilon/2, \quad n \geq N. \tag{3.1}$$

Suppose that $y \in S$ and that $n \geq N$, then from (3.1)

$$\begin{aligned} \|z - x_n\| &= \|z - w + w - x_n\| \\ &\leq \|z - w\| + \|w - x_n\| \\ &< \|z - w\| + \epsilon/2. \end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned} \|y - w\| &= \|y - x_n + x_n - w\| \\ &\leq \|y - x_n\| + \|x_n - w\|. \end{aligned} \tag{3.3}$$

But since $x_n \in F$, $\|y - x_n\| \leq \|z - x_n\|$. Hence from (3.2), (3.3) becomes

$$\begin{aligned} \|y - w\| &\leq \|z - x_n\| + \|x_n - w\| \\ &< \|z - w\| + \epsilon/2 + \epsilon/2 \\ &= \|z - w\| + \epsilon. \end{aligned}$$

Therefore, since ϵ is arbitrary

$$||y - w|| \leq ||z - w||. \tag{3.4}$$

If $z \in S$, then the theorem is proved; however, if $z \notin S$ then consider the following:

Let $\epsilon > 0$ be given, then there exists a positive integer N such that

$$||w - x_n|| < \epsilon/3, \quad n \geq N. \tag{3.5}$$

This means that

$$\begin{aligned} ||z - x_N|| &= ||z - w + w - x_N|| \\ &\geq ||z - w|| - ||w - x_N|| \\ &> ||z - w|| - \epsilon/3. \end{aligned} \tag{3.6}$$

Since $x_N \in F$, there exists $y_0 \in S$ such that

$$||z - x_N|| - \epsilon/3 < ||y_0 - x_N||. \tag{3.7}$$

Then from (3.5), (3.6), and (3.7),

$$\begin{aligned} ||y_0 - w|| &= ||y_0 - x_N + x_N - w|| \\ &\geq ||y_0 - x_N|| - ||x_N - w|| \\ &> ||y_0 - x_N|| - \epsilon/3 \\ &> ||z - x_N|| - 2\epsilon/3 \\ &> ||z - w|| - \epsilon. \end{aligned}$$

Therefore, for each $\epsilon > 0$ there exists $y_0 \in S$ such that

$$||z - w|| - \epsilon < ||y_0 - w||. \tag{3.8}$$

Thus, from (3.4) and (3.8),

$$||z - w|| = \sup \{ ||y - w|| : y \in S \} .$$

Hence $w \in F$ and F is closed.

In addition to being closed, the set $F(z, S)$ must be inverse starlike relative to a point y, that is, there exists a point y such that if $x \in F(z, S)$ then

$$\infty x\, a = \{a + \alpha(x - a) : \alpha \geq 1\} \subset F(z, S).$$

Theorem 3.2. If X is a normed linear space, $z \in S \subset X$, such that $F(z, S) \neq 0$, then $F(z, S)$ is inverse starlike with respect to z.

Proof: The set $F(z, S)$ is assumed to be nonempty so let $x \in F(z, S)$, then

$$||z - x|| = \sup \{||y - x|| : y \in S\}. \tag{3.9}$$

Let $w = x + \alpha (x - z)$, where $\alpha > 0$, then

$$\begin{aligned} ||w - z|| &= ||(1 + \alpha)x - \alpha z - z|| \\ &= (1 + \alpha) ||x - z||. \end{aligned} \tag{3.10}$$

For each $y \in S$ it follows from (3.9) and (3.10) that

$$\begin{aligned} ||w - y|| &= ||x + \alpha(x - z) - y|| \\ &\leq ||x - y|| + \alpha ||x - z|| \\ &\leq ||x - z|| + \alpha ||x - z|| \\ &= (1 + \alpha) ||x - z|| \\ &= ||w - z||. \end{aligned} \tag{3.11}$$

Since $z \in S$ it follows that

$$||w - z|| = \sup \{ ||w - y|| : y \in S\}$$

and; therefore, $w \in F(z, S)$. Since $w \in \infty x z$ is arbitrary, $\infty x z \subset F(z, S)$.

A similar situation exists for z-nearest point sets, since it is well known (cf. Phelps, [31]) that if $x \in N(z,S)$ then the line segment $\{\alpha x + (1-\alpha)z : 0 \leq \alpha \leq 1\}$ is a subset of $N(z,S)$. Hence $N(z,S)$ is always starlike with respect to z; moreover, if S is convex then $N(z,S)$ is inverse-starlike also, which means that $N(z,S)$ is a cone.

Pauc, [30], has shown that in Euclidean space $N(z,S)$ and $N(w,S)$, $w \neq z$, z and w elements of S, do not intersect except possibly at boundary points. Phelps, [31], has shown that this is also true in case X is strictly convex. A similar situation holds true for z-farthest point sets as the following theorem shows.

Theorem 3.3. Let X be a strictly convex normed space and let $S \subset X$. If $z \in S$ and $w \in S$, $z \neq w$, such that $F(z,S) \neq 0$ and $F(w,S) \neq 0$, then $F(z,S)$ and $F(w,S)$ have only boundary points in common.

Proof: Let $x \in F(z,S) \cap F(w,S)$ and assume that x is an interior point of $F(z,S)$. Then there exists a number $\epsilon > 0$ such that

$$\{y : ||x - y|| < \epsilon\} \subset F(z,S).$$

Since $x \in F(z,S)$,

$$||x - z|| = \sup \{ ||x - y|| : y \in S\},$$

and since $x \in F(w,S)$,

$$||x - w|| = \sup \{ ||x - y|| : y \in S\};$$

hence,

$$||x - w|| = ||x - z||. \tag{3.12}$$

Let $d = ||w - x||$, then for each α such that $0 < \alpha < \epsilon$,

$$||w - [x + \alpha/d\ (x - w)]|| = ||(w - x) + \alpha/d(w - x)||$$

$$= (1 + \alpha/d)\ ||w - x|| \qquad (3.13)$$

$$= (1 + \alpha/d)\ ||z - x||$$

$$> ||z - x||. \qquad (3.14)$$

But $x + \alpha/d(x - w) \in F(z,S)$ since $0 < \alpha < \epsilon$. Hence,

$$||z - [x + \alpha/d(x - w)]|| = \sup \{ ||y - [x + \alpha/d(x - w)]|| : y \in S \}.$$

Now note that if there exists a scalar $\lambda > 0$ such that $(z - x) = \lambda(w - x)$, then $||z - x|| = \lambda||w - x||$, and (3.12) implies that $\lambda = 1$. But this implies that $x = w$, which is contrary to hypothesis. Furthermore, note that in a strictly convex space (cf. Wilansky, [38]), $||u + v|| = ||u|| + ||v||$ for vectors u and v if and only if there exists $\lambda > 0$ such that $u = \lambda v$. Then it follows from (3.12), (3.13), and (3.14)

$$||z - [x + \alpha/d(x - w)]|| = ||(z - x) + \alpha/d(w - x)||$$

$$< ||z - x|| + (\alpha/d)||w - x||$$

$$= ||w - x|| + (\alpha/d)\ ||w - x||$$

$$= ||\ w - [x + \alpha/d(x - w)]\ ||.$$

Hence, $x + \alpha/d(x - w)$ is farther from $w \in S$ than z so that $x + \alpha/d(x - w)$ cannot be an element of $F(z, S)$. But this contradicts (3.12); therefore, an interior point of either $F(z, S)$ or $F(w, S)$ cannot be an element of $F(z, S) \cap F(w, S)$.

Theorem 3.3 depends on the fact that in a strictly convex space, $||u+v|| = ||u|| + ||v||$ if and only if u and v are linearly dependent. It can be shown that this property, $||u+v|| = ||u|| + ||v||$ if and only if u and v are linearly dependent, implies that the space X is strictly convex (cf. Wilansky, [38]). Hence, the technique of proof indicates that X must be at least a strictly convex space in Theorem 3.3. The following example shows that $F(z,S) \cap F(w,S)$, $z \neq w$, can contain interior points if X is not a strictly convex space.

Example 3.3. Let X be the space $\ell^{\infty}(2)$ and let $S = \{z, w\}$, where $z = (1,0)$ and $w = (0,0)$. Then it can be shown that $F(z,S) = A \cup B$ and $F(w,S) = C \cup D$, where

$$A = \{(x,y) : x \leq 1/2\}$$

$$B = \{(x,y) : x > 1/2, \ |y| \geq x\}$$

$$C = \{(x,y) : x \geq 1/2\}$$

$$D = \{(x,y) : x < 1/2, \ |y| \geq |x - 1|\}.$$

To see that $F(z,S) = A \cup B$, let $t = (x,y) \in A$, then

$$||t - z|| = \max\{|x - 1|, \ |y|\}. \tag{3.15}$$

Since $||t - w|| = ||t||$ it follows that

$$||t - w|| = \max\{|x|, \ |y|\}. \tag{3.16}$$

Suppose $||t|| = |y|$, if $||t - z|| = |y|$ then $||t - z|| > ||t - w||$ and therefore $t \in F(z,S)$. If $||t - z|| = |x - 1|$ then, from (3.15), $|x - 1| \geq |y| = ||t||$. Hence $||t - z|| \geq ||t - w||$, which implies $t \in F(z,S)$.

If $||t|| = |x|$ and $||t - z|| = |y|$, then from (3.15) and (3.16), $|x| \geq |x - 1|$. Suppose $|x| > |x - 1|$ and $x > 0$, then since $t \in A$, $|x - 1| = -x + 1$. Hence, $x > -x + 1$, which implies that $x > 1/2$ and contradicts the fact that $t \in A$. If $|x| > |x - 1|$ and $x \leq 0$, then $|x| = -x$ and $|x - 1| = -x + 1$. Hence, $-x > -x + 1$ which implies $0 \geq 1$. Thus,

$|x| = |x - 1|$ and it follows that $|x| = |y|$. Therefore, $||t|| = |y|$, which is the preceding case.

Suppose $||t|| = |x|$ and $||t - z|| = |x - 1|$. If $0 < x \leq 1/2$ then $2x \leq 1$ which implies $x \leq -x + 1$ or that $|x| \leq |x - 1|$. If $x \leq 0$, then $-x + 1 \geq -x$ and so $|x - 1| \geq |x|$. Thus, in any case, $||t - z|| \geq ||t||$ so that $t \in F(z, S)$. Hence, $A \subset F(z, S)$.

Let $t = (x, y) \in B$, then again $||t - z||$ and $||t||$ are given by (3.15) and (3.16), respectively. Now $||t|| = |y|$ since, from the definition of B, $|y| \geq x > 1/2$. From (3.15), $||t - z|| \geq ||t||$ which implies $t \in F(z, S)$ which in turn implies that $B \subset F(z, S)$. Thus, $A \cup B \subset F(z, S)$.

Let $t = (x, y) \in F(z, S)$, then either $x \leq 1/2$ or $x > 1/2$. If $x \leq 1/2$ then t is an element of A. If $x > 1/2$ and $||t|| = |y|$ then, from (3.16), $x \leq |y|$ which implies that $t \in B$. If $x > 1/2$ and $||t|| = |x| = x$, then, since $||t - z|| \geq ||t||$ it follows that $||t - z|| = |x - 1|$. Hence $|x - 1| \geq x = |x|$. But, from this, if $1/2 < x \leq 1$, then $-x + 1 \geq x$, or $1/2 \geq x$ which is a contradiction. If $1 < x$, then $x - 1 \geq x$ or $-1 \geq 0$ which is again a contradiction. So if $x > 1/2$, $|y| \geq x$ which implies that $t \in B$. Therefore, $F(z, S) \subset A \cup B$ which implies that $F(z, S) = A \cup B$.

By a similar argument, $F(w, S) = C \cup D$. Thus

$$F(z, S) \cap F(w, S) = B \cup E \cup D,$$

where

$$E = \{(x, y) : x = 1/2\}.$$

The point $t_0 = (1/2, 1)$ is common to both $F(z, S)$ and $F(w, S)$ and a neighborhood N of radius 1/4 about t_0 is properly contained in $F(z, S) \cap F(w, S)$, (cf. Figure 3.3).

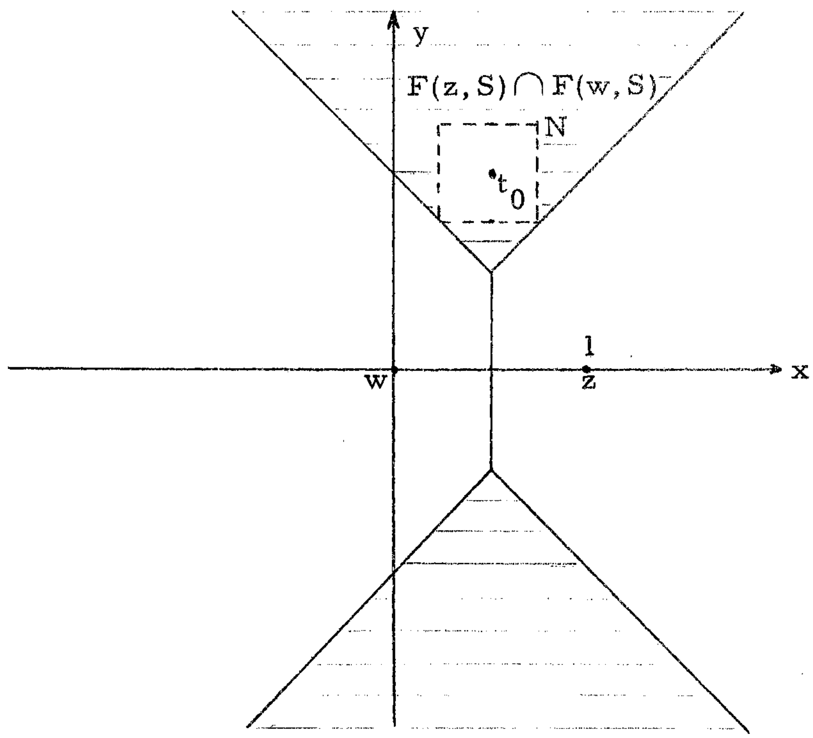


Figure 3. 3.

The preceding theorems dealt with the structure of a z-nearest point set and its relationship to the norm of the space X. It is also of interest to determine how F(z, S) might be related to $F(z, S_1)$, where $S \subset S_1$. For example, S_1 might be conv S or cl S. A special case of the following theorem shows that F(z, S) = F(z, conv S).

Theorem 3.4. Let S be a subset of the normed space X and let $z \in X$. Then E = F, where

$$E = \{x : ||x - z|| = \sup \{ ||x - y|| : y \in S\}\}$$

$$F = \{x : ||x - z|| = \sup \{ ||x - y|| : y \in \text{conv } S\}\}.$$

Proof: Let $x \in E$, then $||z - x|| \geq ||y - x||$ for each $y \in S$.

Suppose $w \in \text{conv } S$, then there exists a finite set $\{y_1, \ldots, y_n\} \subset S$ such that

$$w = \sum_{i=1}^{n} \alpha_i y_i, \quad \alpha_i \geq 0, \quad 1 \leq i \leq n, \text{ and } \sum_{i=1}^{n} \alpha_i = 1.$$

Then

$$\begin{aligned} \|w - x\| &= \left\| \sum_{i=1}^{n} \alpha_i y_i - \left(\sum_{i=1}^{n} \alpha_i\right) x \right\| \\ &\leq \left(\sum_{i=1}^{n} \alpha_i\right) \|y_i - x\| \\ &\leq \left(\sum_{i=1}^{n} \alpha_i\right) \|z - x\| \\ &= \|z - x\| \end{aligned} \tag{3.17}$$

Hence, $\|z - x\|$ is an upper bound for the set $\{\|y - x\| : y \in \text{conv } S\}$. If $z \in S$, then

$$\|z - x\| = \sup \{\|y - x\| : y \in \text{conv } S\}$$

and; therefore, $E \subset F$. But suppose $z \notin S$. Then if $\epsilon > 0$, there exists $y_0 \in S$ such that

$$\|z - x\| - \epsilon < \|y_0 - x\| \leq \|z - x\|.$$

But $y_0 \in S \subset \text{conv } S$; hence, $y_0 \in \text{conv } S$ and

$$\|z - x\| - \epsilon < \|y_0 - x\| \leq \|z - x\|. \tag{3.18}$$

Therefore, from (3.17) and (3.18),

$$\|z - x\| = \sup \{\|y - x\| : y \in \text{conv } S\}$$

which means that $x \in F$. This implies that $E \subset F$.

Now assume that $x \in F$, then $\|z - x\| \geq \|y - x\|$ for $y \in \text{conv } S$. So $\|z - x\| \geq \|y - x\|$ for $y \in S$ since $S \subset \text{conv } S$. If $z \in S$ then

$$||z - x|| = \sup \{ ||y - x|| : y \in S\}$$

and, therefore, $F \subset E$. However, if $z \notin S$, then since

$$||z - x|| = \sup \{ ||y - x|| : y \in \text{conv } S\},$$

for each $\epsilon > 0$ there exists $y_0 \in \text{conv } S$ such that

$$||z - x|| - \epsilon < ||y_0 - x|| \leq ||z - x||. \qquad (3.19)$$

Since $y_0 \in \text{conv } S$ there exists a finite set $\{y_1, y_2, \ldots, y_n\} \subset S$ such that

$$y_0 = \sum_{i=1}^{n} \alpha_i y_i, \quad \alpha_i \geq 0, \quad 1 \leq i \leq n, \quad \sum_{i=1}^{n} \alpha_i = 1. \qquad (3.20)$$

Let $y \in \{ y_1, \ldots, y_n\}$ such that

$$||y - x|| = \max \{ ||y_1 - x||, ||y_2 - x||, \ldots, ||y_n - x||\}. \qquad (3.21)$$

Then from (3.20) and (3.21)

$$\begin{aligned} ||z - x|| - \epsilon &< ||y_0 - x|| \\ &= || \sum_{i=1}^{n} \alpha_i y_i - \left(\sum_{i=1}^{n} \alpha_i \right) x|| \\ &\leq \left(\sum_{i=1}^{n} \alpha_i \right) ||y_i - x|| \\ &\leq \left(\sum_{i=1}^{n} \alpha_i \right) ||y - x|| \\ &= ||y - x||. \end{aligned} \qquad (3.22)$$

Note that $y \in S$. If $\epsilon > 0$, from (3.22) there exists $y \in S$ such that

$$||z - x|| - \epsilon < ||y - x|| \leq ||z - x||.$$

Therefore,

$$||z - x|| = \sup \{ ||y - x|| : y \in S\},$$

which means that $x \in E$ and $F \subset E$. Hence, $F = E$.

Note that Theorem 3.4 means that there is no loss in generality if S is assumed to be convex when properties of F(z, S) are being considered. This is not true; however, in the case of N(z, S). A theorem analogous to Theorem 3.4, where "sup" is changed to "inf" in the definition of the sets E and F is not possible. However, if z is required to be an element of S then a set inclusion is possible. Note that in the following theorem, N(z, conv S) and N(z, S) are analogous to F and E, respectively, of Theorem 3.4.

Theorem 3.5. If S is a subset of the normed linear space X and $z \in S$, then $N(z, \text{conv } S) \subset N(z, S)$.

Proof: Let $x \in N(z, \text{conv } S)$, then

$$||x - z|| = \inf \{||x - y|| : y \in \text{conv } S\}$$

which implies that $||x - z|| \leq ||y - x||$ for each y in conv S. Since $S \subset \text{conv } S$, it follows that $||x - z|| \leq ||y - x||$ for each $y \in S$. Since $z \in S$,

$$||x - z|| = \inf \{ ||x - y|| : y \in S\};$$

hence, $x \in N(z, S)$. Therefore, $N(z, \text{conv } S) \subset N(z, S)$.

The following examples show that Theorem 3.5 is the strongest result that can be obtained.

Example 3.4. Let X be the space E_2, $S = \{w, t\}$, and $z = (1, 0)$, where $w = (-1, 1)$ and $t = (-1, -1)$. Then $A = B \cup C$, where

$$A = \{p : p = (x, y), \ ||p - z|| = \inf \{ ||p - q|| : q \in S\}\}$$

$$B = \{p : p = (x, y),\ y = 2x + 1/2,\ x \geq -1/4\}$$

$$C = \{p : p = (x, y),\ y = -2x - 1/2,\ x \geq -1/4\}$$

(cf. Figure 3.4). A point p can be in A if and only if $||p - z|| = ||p - t||$ when $||p - t|| \leq ||p - w||$ or $||p - z|| = ||p - w||$ when $||p - w|| \leq ||p - t||$. Thus, if p is in the upper half-plane, then $||p - w|| \leq ||p - t||$ which means that p must lie on the perpendicular bisector of the line segment zw. Hence p must be an element of B. Likewise, if p is in the lower half-plane, $p \in C$. Therefore, $A = B \cup C$.

Now, conv S is the line segment tw (cf. Figure 3.5). It can be shown by reasoning similar to that above that $D = E \cup F \cup G$, where

$$D = \{p : p = (x, y),\ ||p - z|| = \inf\{||p - q|| : q \in \text{conv } S\}\}$$

$$E = \{p : p = (x, y),\ y^2 = 4x,\ 0 \leq x \leq 1/4\}$$

$$F = \{p : p = (x, y),\ y = 2x + 1/2,\ x \geq 1/4\}$$

$$G = \{p : p = (x, y),\ y = -2x - 1/2,\ x \geq 1/4\}.$$

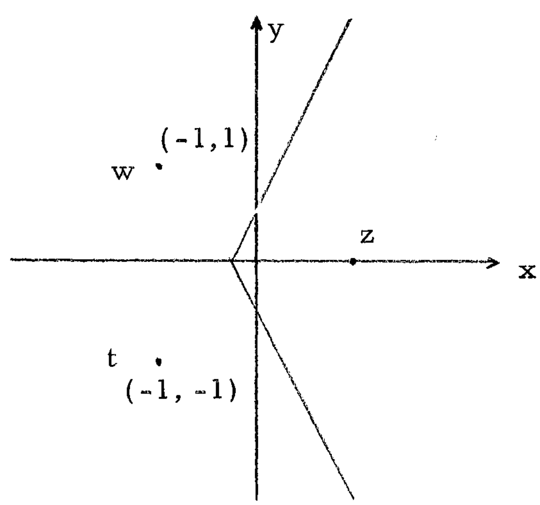


Figure 3.4.

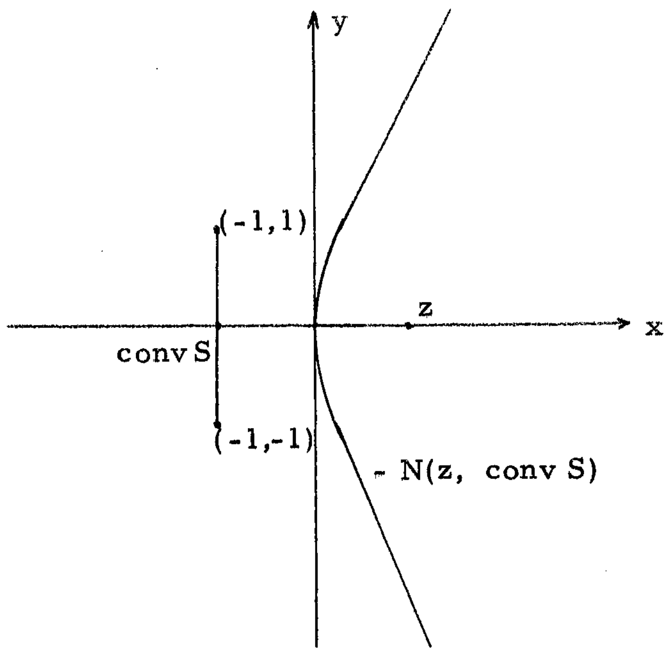


Figure 3.5.

The point (0, 0) is an element of D, but it is not an element of A.

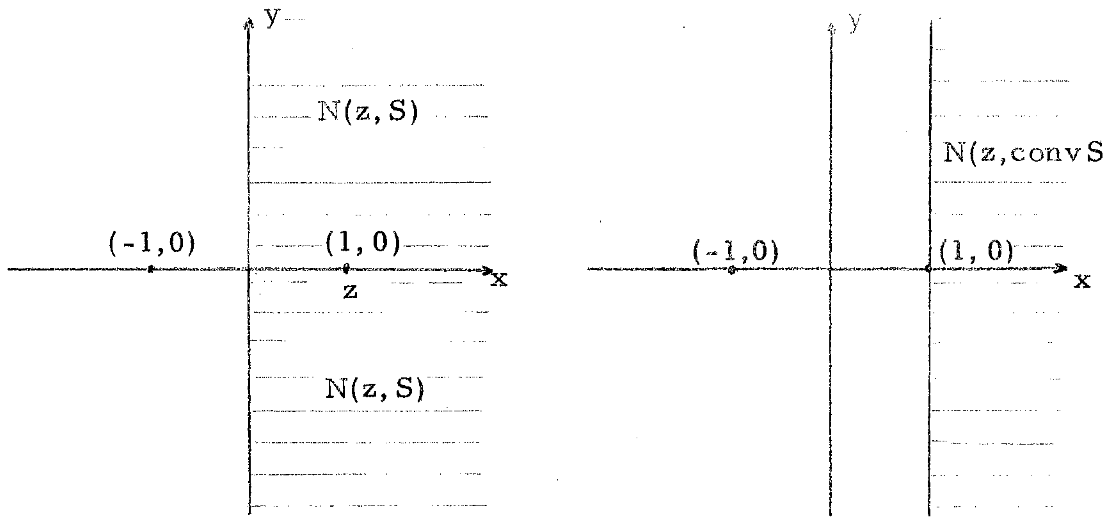
The next example will show that N(z, conv S) can be a proper subset of N(z, S).

Example 3.5. Let X be the Euclidean space E_2, let $S = \{(-1, 0), (1, 0)\}$, and let $z = (1, 0)$. Then $N(z, S) = \{(x, y) : x \geq 0\}$; but, since

$$\text{conv } S = \{\alpha(-1, 0) + (1-\alpha)(1, 0) : 0 \leq \alpha \leq 1\},$$

N(z, conv S) is the set $\{(x, y) : x \geq 1\}$. Obviously, N(z, S) properly contains N(z, conv S), (cf. Figure 3.6).

As was mentioned before, another set of interest which is closely related to S is cl S. It will be seen that cl S fits S so closely



(a) (b)

Figure 3.6.

that any point which is farther from z than any point of S must also be farther from z than any point of cl S. This is shown as a special case of the following theorem.

Theorem 3.6. Let X be a normed linear space, let $S \subset X$, and let $z \in X$. Then F = E, where

$$F = \{x : ||x - z|| = \sup \{ ||x - y|| : y \in \text{cl } S\}\}$$

$$E = \{x : ||x - z|| = \sup \{ ||x - y|| : y \in S\}\}.$$

Proof: Let $x \in F$, then $||x - z|| \geq ||x - y||$ for $y \in$ cl S; hence since $S \subset$ cl S

$$||x - z|| \geq ||x - y||, \quad y \in S. \tag{3.23}$$

If $z \in S$, then

$$||z - x|| = \sup \{ ||y - x|| : y \in S\},$$

which implies $x \in E$. If $z \notin S$, then for each $\epsilon > 0$, there exists $w \in \mathrm{cl}\,S$ such that

$$||z - x|| - \epsilon/2 < ||z - w|| \leq ||z - x||, \tag{3.24}$$

since $x \in F$. There exists $y_0 \in S$ such that $||w - y_0|| < \epsilon/2$, since $w \in \mathrm{cl}\,S$. Then

$$\begin{aligned} ||z - x|| - \epsilon/2 &< ||z - w|| \\ &= ||z - y_0 + y_0 - w|| \\ &\leq ||z - y_0|| + ||y_0 - w|| \\ &< ||z - y_0|| + \epsilon/2. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25),

$$||z - x|| - \epsilon < ||z - y_0|| \leq ||z - x||. \tag{3.26}$$

Therefore, from (3.23) and (3.26),

$$||z - x|| = \sup \{ ||y - x|| : y \in S\}$$

which implies that $x \in E$. Hence, $F \subseteq E$.

Suppose $x \in E$, then $||x - z|| \geq ||x - y||$ for $y \in S$. Let $w \in \mathrm{cl}\,S$, then for each $\epsilon > 0$ there is a $y_0 \in S$ such that $||w - y_0|| < \epsilon$. Then

$$\begin{aligned} ||x - w|| &= ||(x - y_0) + (y_0 - w)|| \\ &< ||x - y_0|| + ||y_0 - w|| \\ &< ||x - z|| + \epsilon. \end{aligned} \tag{3.27}$$

Since $\epsilon > 0$ was assumed to be arbitrary, it follows from (3.27) that $||x - w|| \leq ||x - z||$. Thus, for each $w \in \mathrm{cl}\,S$, $||x - w|| \leq ||x - z||$ and, therefore, $||z - x||$ is an upper bound for the set $\{ ||x - y|| : y \in S\}$.

For each $\epsilon > 0$ there exists $y \in S$ such that

$$||z - x|| - \epsilon < ||y - x|| \leq ||z - x||.$$

But $y \in S \subset \text{cl } S$, so that for each $\epsilon > 0$ there exists $y \in \text{cl } S$ such that

$$||z - x|| - \epsilon < ||y - x|| \leq ||z - x||.$$

Therefore,

$$||z - x|| = \sup \{ ||x - y|| : y \in \text{cl } S\},$$

which implies that $E \subset F$. Hence, $E = F$.

It was shown that in the case of z-nearest point sets that it was not true in general that the sets $N(z, S)$ and $N(z, \text{conv } S)$ are equal. This might cause some doubt then as to the existence of an analogous theorem to Theorem 3.6. However, the analogous theorem here for $N(z, S)$ and $N(z, \text{cl } S)$ is true, figuratively speaking, because cl S fits S much more closely than does conv S. This is shown as a special case of the following theorem.

Theorem 3.7. Let X be a normed linear space, let $S \subset X$, and let $z \in X$. Then $N = M$ where

$$N = \{x : ||x - z|| = \inf \{ ||x - y|| : y \in S\}\}$$

$$M = \{x : ||x - z|| = \inf \{||x - y|| : y \in \text{cl } S\}\}.$$

Proof: Let $x \in M$, then $||x - z|| \leq ||x - y||$ for each $y \in \text{cl } S$; hence, since $S \subset \text{cl } S$

$$||x - z|| \leq ||x - y||, y \in S. \tag{3.28}$$

Let $\epsilon > 0$, then there must exist $y \in \text{cl } S$ such that

$$||z - x|| \leq ||y - x|| < ||z - x|| + \epsilon/2. \tag{3.29}$$

But since $y \in \text{cl } S$ there must also be $w \in S$ such that

$$||w - y|| < \epsilon/2. \tag{3.30}$$

Then from (3. 30)

$$\begin{aligned} ||y - x|| &= ||(y - w) - (x - w)|| \\ &\geq ||x - w|| - ||y - w|| \\ &> ||x - w|| - \epsilon/2. \end{aligned}$$

Hence, from (3. 29)

$$||x - w|| - \epsilon/2 < ||y - x|| < ||z - x|| + \epsilon/2,$$

$$||z - x|| \leq ||x - w|| < ||z - x|| + \epsilon. \tag{3.31}$$

Therefore, from (3. 28) and (3. 31)

$$||z - x|| = \inf \{ ||y - x|| : y \in S\}.$$

This implies that $x \in N$. Hence $M \subset N$.

If $x \in N$, then $||x - z|| \leq ||x - y||$ for each $y \in S$. Let $w \in \text{cl } S$, then for each $\epsilon > 0$ there exists a $y_0 \in S$ such that $||w - y_0|| < \epsilon$. Then

$$\begin{aligned} ||x - w|| &= ||(x - y_0) - (w - y_0)|| \\ &\geq ||x - y_0|| - ||w - y_0|| \\ &> ||x - y_0|| - \epsilon \\ &\geq ||x - z|| - \epsilon. \end{aligned}$$

Hence,

$$||x - z|| < ||x - w|| + \epsilon,$$

and since ϵ is arbitrary,

$$||x - w|| \geq ||x - z||. \tag{3.32}$$

Furthermore, for each $\epsilon > 0$ there must exist $y \in S$ such that

$$||y - x|| < ||z - x|| + \epsilon. \tag{3.33}$$

But $y \in S \subset \text{cl } S$, so from (3.32) and (3.33)

$$||z - x|| = \inf \{ ||y - x|| : y \in \text{cl } S \}.$$

Hence, $x \in M$ so that $N \subset M$. Therefore, $N = M$.

Theorem 3.6 and Theorem 3.7 mean that there is no loss in generality in assuming that S is closed when considering the sets F(z, S) and N(z, S). When considering F(z, S), S must be bounded; otherwise, F(z, S) would always be empty. Thus Theorem 3.4 and Theorem 3.6 mean that in a finite-dimensional space, the set S may be assumed to be compact and convex when considering the set F(z, S).

Since for each set S and $z \in S$, F(z, S) and N(z, S) are sets, we can consider F(z, S) and N(z, S) to be the images of functions whose domains are subsets of the cross product of the space X and the power set of X. The question now is, "What properties do these functions have?" The following theorems partially answer this question.

Theorem 3.8. Let X be a normed linear space, $S \subset X$ and $z \in S$. If $\lambda > 0$, then $\lambda F(z, S) = F(\lambda z, \lambda S)$.

Proof: If F(z, S) is empty, then $\lambda F(z, S)$ is empty. But if $F(\lambda z, \lambda S)$ is not empty, then there is $x \in F(\lambda z, \lambda S)$ such that

$$||\lambda z - x|| = \sup \{ ||y - x|| : y \in \lambda S \}.$$

This implies

$$||z - 1/\lambda\ x|| = (1/\lambda) \sup \{ ||y - x|| : y \in \lambda S \}.$$

Since $1/\lambda > 0$, it follows that

$$||z - 1/\lambda\, x|| = \sup \{(1/\lambda)||y - x|| : y \in \lambda S\}$$

$$= \sup \{||1/\lambda\,(y - x)|| : y \in \lambda S\}$$

$$= \sup \{||y - 1/\lambda\, x|| : y \in S\}.$$

Thus $1/\lambda\, x \in F(z, S)$, which is a contradiction. Hence, if $F(z, S)$ is empty, then $F(\lambda z, \lambda S)$ is also empty.

If $F(z, S)$ is not empty, let $x \in F(z, S)$. Then

$$||\lambda z - \lambda x|| = \lambda ||z - x||$$

$$= \lambda \sup \{||y - x|| : y \in S\}.$$

Since $\lambda > 0$,

$$||\lambda z - \lambda x|| = \sup \{\lambda ||y - x|| : y \in S\}$$

$$= \sup \{||\lambda(y - x)|| : y \in S\}$$

$$= \sup \{||y - \lambda x|| : y \in \lambda S\}.$$

Hence, $\lambda x \in F(\lambda z, \lambda S)$ and so $\lambda F(z, S) \subset F(\lambda z, \lambda S)$. Since $F(\lambda z, \lambda S)$ is not empty, let $x \in F(\lambda z, \lambda S)$. Then

$$||\lambda z - x|| = \sup \{||y - x|| : y \in \lambda S\}$$

$$= \sup \{||\lambda y - x|| : y \in S\}$$

$$= \sup \{\lambda ||y - 1/\lambda\, x|| : y \in S\}$$

$$= \lambda \sup \{||y - 1/\lambda\, x|| : y \in S\}.$$

It follows then that

$$||z - 1/\lambda\, x|| = \sup \{||y - 1/\lambda\, x|| : y \in S\}.$$

Hence, $1/\lambda\, x \in F(z, S)$ which implies that $x \in \lambda F(z, S)$. Therefore,

$$\lambda F(z, S) = F(\lambda z, \lambda S).$$

In Theorem 3.8 if $F(z,S)$ is not empty then λ cannot be zero because $0F(z,S) = \{\phi\}$, but $F(\phi, \{\phi\}) = X$. Further complications arise if $F(z,S)$ is empty, for then $0F(z,S) = 0$, but still $F(\phi, \{\phi\}) = X$. Hence we must restrict λ to only positive numbers.

The next theorem shows that $N(z,S)$ has the same multiplicative property as $F(z,S)$.

Theorem 3.9. Let X be a normed linear space, $S \subset X$, and $z \in S$. If $\lambda > 0$, then $\lambda N(z,S) = N(\lambda z, \lambda S)$.

Proof: If $N(z,S)$ is empty, then $\lambda N(z,S)$ is also empty. However, if $N(\lambda z, \lambda S)$ is not empty, then there is $x \in N(\lambda z, \lambda S)$ such that

$$||\lambda z - x|| = \inf \{||y - x|| : y \in S\}.$$

Hence

$$||z - 1/\lambda\, x|| = (1/\lambda)\inf \{||y - x|| : y \in \lambda S\},$$

and since $\lambda > 0$,

$$\begin{aligned} ||z - 1/\lambda\, x|| &= \inf \{(1/\lambda)\, ||y - x|| : y \in \lambda S\} \\ &= \inf \{||1/\lambda\, y - 1/\lambda\, x|| : y \in \lambda S\} \\ &= \inf \{||y - 1/\lambda\, x|| : y \in S\}. \end{aligned}$$

Therefore, $1/\lambda\, x \in N(z,S)$ which is a contradiction, hence $N(\lambda z, \lambda S)$ must also be empty.

Suppose $x \in N(z,S)$, then

$$||z - x|| = \inf \{||y - x|| : y \in S\}.$$

Hence

$$\begin{aligned} ||\lambda z - \lambda x|| &= \lambda ||z - x|| \\ &= \lambda \inf \{||y - x|| : y \in S\}. \end{aligned}$$

Since $\lambda > 0$,

$$\lambda \inf \{||y - x|| : y \in S\} = \inf \{\lambda ||y - x|| : y \in S\}$$
$$= \inf \{||\lambda y - \lambda x|| : y \in S\}$$
$$= \inf \{||y - \lambda x|| : y \in \lambda S\}.$$

Hence $\lambda x \in N(\lambda z, \lambda S)$ and so $\lambda N(z,S) \subset N(\lambda z, \lambda S)$. If $x \in N(\lambda z, \lambda S)$, then

$$||\lambda z - x|| = \inf \{||y - x|| : y \in \lambda S\}.$$

However, by factoring out λ,

$$\lambda ||z - 1/\lambda\, x|| = \lambda \inf \{||1/\lambda\, y - 1/\lambda\, x|| : y \in \lambda S\}.$$

Hence,

$$||z - 1/\lambda\, x|| = \inf \{||y - 1/\lambda\, x|| : y \in S\},$$

which implies that $1/\lambda\, x \in N(z,S)$. Thus, $x \in \lambda N(z,S)$, and it follows that $\lambda N(z,S) = N(\lambda z, \lambda S)$.

If, in Theorem 3.9, $\lambda = 0$ and $N(z,S) \neq 0$, then $\lambda N(z,S) = \{\phi\}$. But $N(\lambda z, \lambda S) = N(\phi, \{\phi\}) = X$, hence λ cannot be zero.

It seems intuitively obvious that given a set S and a point $z \in S$ one should be able to translate S and z by the same element y and the $(z + y)$ - farthest point set of $S + y$ would be equal to the translate of the z-farthest point set of S. This is a special case of the following theorem.

<u>Theorem 3.10.</u> Let X be a normed linear space $S \subset X$, and $z \in X$. If A and B are nonempty sets such that

$$A = \{x : ||x - z|| = \sup \{||x - v|| : v \in S\}\}$$

and

$$B = \{x : ||x - (z+y)|| = \sup \{||x - v|| : v \in S + y\}\},$$

then A + y = B.

Proof: Let $x \in A + y$, then $x - y \in A$. Hence, for $v \in S$,

$$\begin{aligned} ||(v + y) - x|| &= ||v - (x - y)|| \\ &\leq ||z - (x - y)|| \\ &= ||(z + y) - x||. \end{aligned} \tag{3.34}$$

For each $\epsilon > 0$, there is a $v \in S$ such that

$$||z - (x - y)|| - \epsilon < ||v - (x - y)||. \tag{3.35}$$

But (3.35) can be written as

$$||(z + y) - x|| - \epsilon < ||(v + y) - x||. \tag{3.36}$$

Hence, from (3.34) and (3.36),

$$||(z + y) - x|| = \sup \{||(v + y) - x|| : v \in S\}.$$

Therefore, $x \in B$ which implies $A + y \subset B$.

If $x \in B$, then for $v \in S$, $v + y \in S + y$, and

$$\begin{aligned} ||v - (x - y)|| &= ||(v + y) - x|| \\ &\leq ||(z + y) - x|| \\ &= ||z - (x - y)||. \end{aligned} \tag{3.37}$$

For each $\epsilon > 0$ there exists $v \in S + y$ such that

$$||(z + y) - x|| - \epsilon < ||v - x||. \tag{3.38}$$

Then since $v = w + y$, $w \in S$, (3.38) can be written

$$||z - (x - y)|| - \epsilon < ||w - (x - y)||. \tag{3.39}$$

Hence, from (3.37) and (3.39),

$$||z - (x - y)|| = \sup \{ ||v - (x - y)|| : v \in S\}.$$

Thus, $x - y \in A$, which implies that $x \in A + y$ and $A + y = B$.

The set $N(z, S)$ also has the same additive property shown for $F(z, S)$ by Theorem 3.10.

Theorem 3.11. Let X be a normed linear space, $S \subset X$, and $z \in X$. If A and B are nonempty sets such that

$$A = \{x : ||x - z|| = \inf \{ ||x - v|| : v \in S\}\}$$

and

$$B = \{x : ||x - (z + y)|| = \inf \{ ||x - v|| : v \in S + y\}\},$$

then $A + y = B$.

Proof: Let $x \in A + y$, then $x - y \in A$. Hence for $v \in S$,

$$\begin{aligned} ||(z + y) - x|| &= ||z - (x - y)|| \\ &\leq ||v - (x - y)|| \\ &= ||(v + y) - x||. \end{aligned} \tag{3.40}$$

For each $\epsilon > 0$, there is a $v \in S$ such that

$$||v - (x - y)|| < ||z - (x - y)|| + \epsilon. \tag{3.41}$$

Then (3.41) can be written as

$$||(v + y) - x|| < ||(z + y) - x|| + \epsilon. \tag{3.42}$$

Hence, from (3.40) and (3.42),

$$||(z + y) - x|| = \inf \{ ||(v + y) - x|| : v \in S\}.$$

Thus, $x \in B$ which implies $A + y \subset B$.

If $x \in B$, then for $v \in S$, $v + y \in S + y$, and

$$\begin{aligned} ||z - (x - y)|| &= ||(z + y) - x|| \\ &\leq ||(v + y) - x|| \\ &= ||v - (x - y)||. \end{aligned} \tag{3.43}$$

For each $\epsilon > 0$ there exists $v \epsilon S + y$ such that

$$||v - x|| < ||(z + y) - x|| + \epsilon. \tag{3.44}$$

Then since $v = w + y$, $w \epsilon S$, (3.44) may be written

$$||w - (x - y)|| < ||z - (x - y)|| + \epsilon. \tag{3.45}$$

Hence, from (3.43) and (3.45),

$$||z - (x - y)|| = \inf \{ ||v - (x - y)|| : v \epsilon S\}.$$

Thus, $x - y \epsilon A$, which implies that $x \epsilon A + y$ and $A + y = B$.

Also of interest is the element z as related to the set in question, S. If $z \epsilon S$, then can z be an interior point? If z is a boundary point of S then what type of boundary point must it be? The next series of theorems will shed some light on the properties of the element z.

Theorem 3.12. Let S be a subset of the normed linear space X and let $z \epsilon S$. If $F(z,S)$ is nonempty, then z is a boundary point of S.

Proof: Suppose that z is an interior point of S. Then there exists a number $r > 0$ such that

$$\{w : ||w - z|| < r\} \subset S. \tag{3.46}$$

Suppose $x \epsilon F(z,S)$, then $||x - z|| \geq ||y - x||$ for each $y \epsilon S$. Consider the element $z + d\,(z - x)$, where $d = r/(2\,||x - z||)$. Its distance from x is given by

$$||z + d(z - x) - x|| = ||(1 + d)(z - x)||$$
$$= (1 + d)\, ||z - x||$$
$$> ||z - x||. \tag{3.47}$$

Its distance from z is given by

$$||z - (z + d(z - x))|| = d\, ||z - x||$$
$$= r/2 < r. \tag{3.48}$$

Hence, from (3.46) and (3.48), $z + d(z - x) \in S$, but from (3.47) its distance from x is greater than $||z - x||$ which is a contradiction. Therefore, z must be a boundary point.

The property of z shown in Theorem 3.12 still does not pinpoint the nature of z. However, in order that z might be limited to some special type of boundary point it is necessary to place a restriction on the norm of the space X. This restriction is simply that X be strictly convex.

Theorem 3.13. Let X be a normed linear space, $S \subset X$, and $z \in S$ such that $F(z, S)$ is nonempty. If X is strictly convex then z must be an extreme point of S.

Proof: Suppose z is not an extreme point of S, then there exists $x \in S$, $y \in S$, $x \neq y$, such that $z = 1/2\, x + 1/2\, y$. Let $w \in F(z, S)$, then

$$||w - z|| = ||w - 1/2\, x - 1/2\, y||$$
$$\leq (1/2)||w - x|| + (1/2)||w - y||. \tag{3.49}$$

Suppose $||w - x|| < ||w - z||$ and $||w - y|| < ||w - z||$, then $(1/2)||w - x|| < (1/2)\,||w - z||$ and $(1/2)\,||w - y|| < (1/2)\,||w - z||$, so that

$$(1/2)\|w - x\| + (1/2)\|w - y\| < \|w - z\|.$$

Hence, we may assume without loss of generality that $\|w - z\| \leq \|w - x\|$. But, since $w \in F(z, S)$ and $x \in S$, $\|w - z\| \geq \|w - x\|$ so that $\|w - z\| = \|w - x\|$. Moreover, since

$$\|w - z\| \leq (1/2)\|w - x\| + (1/2)\|w - y\|$$

$$= (1/2)\|w - z\| + (1/2)\|w - y\|,$$

it follows that $\|w - z\| \leq \|w - y\|$. But since $\|w - z\| \geq \|w - y\|$ it follows that $\|w - z\| = \|w - y\|$. Thus all three points, x, y, and z are on the boundary of the sphere $\{p : \|w - p\| \leq \|w - x\|\}$, but since X is strictly convex, intv xy must be a subset of the interior of this sphere. Hence a contradiction exists since $z = 1/2\, x + 1/2\, y$ is a boundary point of the sphere. Therefore, z is an extreme point of S when X is strictly convex.

The following example shows that if X is not strictly convex then z need not be an extreme point of S.

Example 3.6. Let X be the Hilbert space, $\ell^{\infty}(2)$, let

$$S = \{(x, y) : x = -1, -1 \leq y \leq 1\},$$

and let $z = (-1, 0)$. Then the origin, $(0, 0)$, is an element of $F(z, S)$ since its distance from each element of S is one. Hence, $F(z, S)$ is nonempty, but z is not an extreme point of S.

It should be noted that no requirements were placed on the set S in the preceding theorem other than $F(z, S)$ be nonempty. So, the only restriction placed on S was the implicit restriction that S be bounded, for otherwise $\sup\{\|y - x\| : y \in S\}$ never exists. By placing more restrictions on the set S it is possible to relax conditions on X and to determine more precisely the character of z. These revisions are made in the following theorem.

Theorem 3.14. Let S be a closed, strictly convex set in the normed linear space X. If $z \in S$ such that F(z, S) is nonempty, then z is an exposed point of S.

Proof: By Theorem 3.12, z is a boundary point of S. Since in a linear topological space each boundary point of a closed, strictly convex set S is an exposed point of S, then z must be an exposed point of S (cf. Valentine, [36], p. 94).

The following example shows that if S is a convex body which is not strictly convex then z need not be an exposed point of S.

Example 3.7. Let X be the Hilbert space, $\ell^{\infty}(2)$, let S be the closed unit ball of X, and let z = (1, 0). Then F(z, S) is not empty since the distance of the origin from z is at least as great as its distance from any other point of S. However, z is not an exposed point of S.

Another type of boundary point, which is not as well known, is the boundedly exposed point. An element z of the subset S of the normed linear space X is a boundedly exposed point of S if and only if there exists an open sphere B such that z is a boundary point of B and $S \setminus z \subset B$. This definition, as well as theorems which verify the existence of these points for a closed bounded set in a Hilbert space are found in a paper by Edelstein, [10].

Theorem 3.15. Let S be a subset of the strictly convex space

X and let $z \in S$. Then $F(z,S)$ is nonempty if and only if z is a boundedly exposed point of S.

Proof: Suppose $F(z,S)$ is not empty and let $w \in F(z,S)$. Then $||w - z|| \geq ||w - y||$ for each $y \in S$; however, there may be some element y_0 of S for which $||w - z|| = ||w - y_0||$. Therefore, let $w_0 = w + d(w - z)$, where $d = ||w - z||^{-1}$. Since $z \in S$, Theorem 3.2 implies that $w_0 \in F(z,S)$, hence $||w_0 - y|| \leq ||w_0 - z||$ for each $y \in S$. If $y \in S$ is such that there does not exist a positive number λ such that $w - z = \lambda(w - y)$, then since X is strictly convex,

$$\begin{aligned} ||w_0 - y|| &= ||w + d(w - z) - y|| \\ &< ||w - y|| + d||w - z|| \\ &= ||w - y|| + 1 \\ &\leq ||w - z|| + 1 \\ &= ||w_0 - z||. \end{aligned} \tag{3.50}$$

If, on the other hand, $w - z = \lambda(w - y)$ for some $\lambda > 0$, then

$$||w - z|| = \lambda||w - y||.$$

Hence

$$1/\lambda = \frac{||w - y||}{||w - z||} \leq 1.$$

If $1/\lambda < 1$, then $||w - y|| < ||w - z||$ and

$$\begin{aligned} ||w_0 - y|| &= ||w + ||w - z||^{-1}(w - z) - y|| \\ &\leq ||w - y|| + 1 \\ &< ||w - z|| + 1 \\ &= ||w_0 - z||. \end{aligned} \tag{3.51}$$

If $1/\lambda = 1$ then $z = y$. Hence, for

$$B = \{x : ||w_0 - x|| < ||w_0 - z||\},$$

we have from (3.50) and (3.51) that $S \setminus \{z\} \subset B$ and z is a boundary point of B so that z is a boundedly exposed point of S.

If z is a boundedly exposed point of a set S then there exists an element w of X and a number $r > 0$ such that

$$S \setminus \{z\} \subset \{x : ||w - x|| < r\}$$

and such that $||z - w|| = r$. Hence $||z - w|| \geq ||y - w||$ for each $y \in S$ so that $w \in F(z, S)$.

Note that in Theorem 3.15, if z is a boundedly exposed point $F(z, S)$ is nonempty even if X is not strictly convex. The following example shows that X must be strictly convex in order to guarantee that if $F(z, S)$ is nonempty then z is a boundedly exposed point of S.

Example 3.8. Let X be $\ell^\infty(2)$, let S be the unit ball, and let $z = (1, 0)$. Then $F(z, S)$ is not empty since the origin is an element of it. Each sphere in X is a square similar to S except for size and a translation. Hence, any sphere which contains S and has z as a boundary point must have a side which intersects the boundary of S in a line segment.

CHAPTER IV

THE RELATIONSHIP OF F(z,S) AND THE NORM OF X

Having thus far discussed the properties of the sets F(z,S) and N(z,S) and the properties of the element z as related to the set S, it is now appropriate to consider the set F(z,S) as related to the norm of X. The first theorems will be concerned with geometric methods of constructing F(z,S) and N(z,S). These methods will aid in the proof of the main theorem of this chapter. The first two theorems show that F(z,S) and N(z,S) can be found from the intersection of a certain collection of sets.

Theorem 4.1. Let S be a subset of the normed linear space X and let $z \in S$. Then

$$F(z,S) = \bigcap_{y \in S} F(z, \{z,y\}).$$

Proof: For simplicity let

$$F = \bigcap_{y \in S} F(z, \{z,y\}).$$

Furthermore, let $x \in F(z,S)$ and let $y \in S$, then $||x - z|| \geq ||x - y||$. Since $z \in \{z, y\}$,

$$||x - z|| = \sup \{ ||x - w|| : w \in \{z,y\}\},$$

and it follows that $x \in F(z, \{z,y\})$. The element $y \in S$ was arbitrary so that $x \in F$. Hence, $F(z,S) \subset F$.

If $x \in F$ then for each $y \in S$, $x \in F(z, \{z, y\})$. Hence $||x - z|| \geq ||x - y||$ for each $y \in S$. Since $z \in S$ it follows that

$$||x - z|| = \sup \{ ||x - y|| : y \in S\}.$$

This in turn implies that $x \in F(z,S)$. Hence, $F \subset F(z,S)$ and therefore, $F = F(z,S)$.

Corollary 4.1. Let S be a compact set in a normed linear space X. If $z \in S$, then $F(z,S) = F$ where

$$F = \bigcap_{x \in E} F(z, \{z, x\})$$

and E denotes the set of extreme points of cl conv S.

Proof: By the Krein-Milman theorem (cf. Valentine, [36], p. 138), cl conv E = cl conv S. By Theorem 3.4 and Theorem 3.6, $F(z,S) = F(z, \text{cl conv } S)$. Theorem 3.4 and Theorem 3.6 also imply that $F(z,E) = F(z, \text{cl conv } E)$. By Theorem 4.1, $F(z,E) = F$. Therefore, $F(z,S) = F$.

Theorem 4.2. Let S be a subset of the normed linear space X and let $z \in S$. Then $N(z,S) = \bigcap_{y \in S} N(z, \{z, y\})$.

Proof: For simplicity let

$$N = \bigcap_{y \in S} N(z, \{z, y\}).$$

Furthermore, let $x \in N(z,S)$ and let $y \in S$. Then $||x - z|| \leq ||x - y||$. Since $z \in \{z, y\}$, it follows that

$$||x - z|| = \inf \{||x - w|| : w \in \{z, y\}\};$$

and, therefore, that $x \in N(z, \{z, y\})$. Since y was arbitrary, $x \in N$ and therefore $N(z,S) \subset N$.

If $x \in N$, then $||x - z|| \leq ||x - y||$ for each $y \in S$. Since $z \in S$,

$$||x - z|| = \inf \{ ||x - y|| : y \in S\},$$

and it follows that $x \in N(z,S)$. Hence, $N \subset N(z,S)$ and therefore, $N(z,S) = N$.

As was shown by Example 3.5, $N(z,S)$ is not in general equal to $N(z, \text{conv } S)$ so that we can say only that

$$N(z,S) = \cap_{y \in S} N(z, \{z,y\}).$$

Of course it does little good to know that $F(z,S)$ and $N(z,S)$ can be expressed as the intersection of certain sets if one does not know more about these sets. In E_2, as has been shown in preceding examples, $F(z, \{z,y\})$ and $N(z, \{z,y\})$ are closed half-spaces. This is shown by the next two theorems to be true in any inner-product space.

Theorem 4.3. Let X be a real inner-product space, then $F(z, \{z,x\})$ is a closed half-space for any pair of distinct elements z and x of X.

Proof: Let the function $f : X \to R$ be defined by $f(y) = (x - z, y)$, then f is a linear functional. Let H be the closed half-space defined by

$$H = \{y : f(y) \geq (1/2)[(x,x) - (z,z)]\}.$$

If $y \in H$ then from the bilinearity of the inner-product

$$(x,x) - 2(x,y) + (y,y) \leq (z,z) - 2(z,y) + (y,y). \qquad (4.1)$$

Hence, by symmetry of the inner product, (4.1) becomes

$$(x,x) - (x,y) - (y,x) + (y,y) \leq (z,z) - (z,y) - (y,z) + (y,y), \qquad (4.2)$$

and consequently,

$$(x, x - y) - (y, x - y) \leq (z, z - y) - (y, z - y),$$

or

$$(x - y, x - y) \leq (z - y, z - y).$$

Hence, $||x - y|| \leq ||z - y||$. Since $z \in S$,

$$||z - y|| = \sup \{ ||x - y|| : x \in S\}.$$

Therefore, $H \subset F(z, \{z, x\})$.

If $y \in F(z, \{z, x\})$, then $||x - y|| \leq ||z - y||$ from which it follows that

$$(x - y, x - y) \leq (z - y, z - y).$$

Hence,

$$(x, x) - 2(x, y) + (y, y) \leq (z, z) - 2(z, y) + (y, y),$$

or

$$(x - z, y) \geq (1/2) [(x, x) - (z, z)].$$

Hence $y \in H$; and, therefore, $H = F(z, \{z, x\})$.

A similar result is true for $N(z, \{z, x\})$ as the following theorem shows.

Theorem 4.4. Let X be a real inner-product space, then $N(z, \{z, x\})$ is a closed half-space for any pair of distinct elements z and x of X.

Proof: Let $y \in N(z, \{z, x\})$, then $||z - y|| \leq ||x - y||$. Hence, $y \in F(x, \{z, x\})$. Likewise, if $y \in F(x, \{z, x\})$, then $||z - y|| \leq ||x - y||$. Hence, $y \in N(z, \{z, x\})$. Therefore, $N(z, \{z, x\}) = F(x, \{z, x\})$ which is a closed half-space by Theorem 4.3.

Since it is possible to represent $F(z, S)$ and $N(z, S)$ as intersections, it is interesting to see if they can also be represented as unions. This is also true in an inner-product space as will be shown by the next two theorems, Theorems 4.5 and 4.6. However, first it is necessary to make two definitions.

Definition 4.1. Let X be a real inner-product space and let H be a hyperplane of support to the set S at the point $z \in S$. If $w \in X$, $w \neq z$, is such that $(w - z, x - z) = 0$ for each $x \in H$ and $(w - z, y - z) \leq 0$ for each $y \in S$, then the set

$$R_w(z) = \{z + \lambda(w - z) : \lambda \geq 0\}$$

is called an outward normal ray to H at z relative to S.

Similarly an inward normal can be defined.

Definition 4.2. Let X be a real inner-product space and let H be a hyperplane of support to the set S at the point $z \in S$. If $w \in X$, $w \neq z$, is such that $(w - z, x - z) = 0$ for each $x \in H$ and $(w - z, y - z) \geq 0$ for each $y \in S$, then the set

$$R_w(z) = \{z + \lambda(w - z) : \lambda \geq 0\}$$

is called an inner normal ray to H at z relative to S.

These two definitions will help us to state the theorems which will show that N(z, S) and F(z, S) can be represented as a union of sets. The first theorem will deal with N(z, S) and it shows that N(z, S), for a boundary point z of the compact convex set S, is just the polar cone of the supporting cone of S at z (cf. Valentine, [36], p. 135).

Theorem 4.5. Let X be a real inner-product space. Let S be a compact, convex set and let z be a boundary point of S. Then N(z, S) is the union of the outward normal rays at z of each plane of support of S at z.

Proof: Without loss of generality we may assume that $z = \phi$. Suppose that $x \in N(\phi, S)$, $x \neq \phi$, then $||x|| \leq ||x - y||$ for each $y \in S$. Consider the hyperplane, $H = \{w : (w, x) = 0\}$. Then $H \cap S \neq 0$ since $(\phi, x) = 0$ and $x \in H^{+} = \{w : (w, x) \geq 0\}$ since $(x, x) > 0$. Now suppose that $y \in S$ such that $(y, x) = \alpha > 0$. Then $\lambda y \in S$ for $0 \leq \lambda \leq 1$ since S is convex. Define the real valued function

$$f(\lambda) = ||\lambda y - x||^2.$$

Then f is just a second degree polynomial in λ since

$$(\lambda y - x, \lambda y - x) = \lambda^2(y, y) - 2\lambda(y, x) + (x, x),$$

Then $f(\lambda)$ has a minimum value at $\lambda_0 = (y, x)/(y, y)$ since $(y, y) > 0$, and since (y, x) is positive, λ_0 is positive. Since $||x - y|| \geq ||x||$ we have

$$\begin{aligned}(x - y, x - y) &= (x - y, x) - (x - y, y)\\ &= (x - y, x) + (y - x, y)\\ &\geq (x, x). \qquad (4.3)\end{aligned}$$

Then from (4.3),

$$\begin{aligned}(y - x, y) &\geq (x, x) - (x - y, x)\\ &= (x, x) + (y - x, x)\\ &= (y, x)\\ &\geq 0.\end{aligned}$$

Hence, $(y, y) > (x, y)$ and therefore $1 > (x, y)/(y, y)$. Thus, $\lambda_0 y$ is an element of S, but $\lambda_0 y$ is nearer to x than is ϕ. This is a contradiction.

since $x \in N(\phi, S)$. Hence, $(y, x) \leq 0$ and x is an element of an outward normal ray to H at ϕ relative to S. Therefore, $N(\phi, S)$ is a subset of the union of all outward normal rays to S at ϕ.

Now let x be an element of an outward normal ray of the hyperplane H relative to S. By definition, $(x, w) \leq 0$ for each $w \in S$. Since $\phi \in H$, $H = \{y : (x, y) = 0\}$. Let $y \in H$, then

$$\begin{aligned} ||x - y||^2 &= (x - y, x - y) \\ &= (x, x) - 2(x, y) + (y, y) \\ &= ||x||^2 + ||y||^2 \\ &\geq ||x||^2. \end{aligned}$$

Hence, $||x - y|| \geq ||x||$ for each $y \in H$. If $w \in S$, the segment

$$\{\lambda w + (1 - \lambda)x : 0 \leq \lambda \leq 1\}$$

must intersect H for some λ_0 between zero and one since w and x are on opposite sides of H. Let $y = \lambda_0 w + (1 - \lambda_0)x$ be this element of H. Then

$$\begin{aligned} ||x|| &\leq ||x - y|| \\ &= ||x - (\lambda_0 z + (1 - \lambda_0)x)|| \\ &= \lambda_0 ||x - z|| \\ &< ||x - z||. \end{aligned} \tag{4.4}$$

Hence, x is an element of $N(\phi, S)$. Therefore, $N(\phi, S)$ is equal to the union of the outward normal rays relative to S at ϕ.

In an inner-product space it was possible to use all outward normal rays since $N(z, S)$ must be a cone when S is convex (cf. Phelps, [31]). However, in general $F(z, S)$ is not a cone in an inner-product space even though S is convex (cf. Example 3.2 and Theorem 3.4). Hence, in the case of $F(z,S)$, it is more difficult to state this type of theorem and it is somewhat more difficult to prove.

Theorem 4.6. Let X be a real inner-product space and let S be a compact convex subset of X. Let I be the collection of all inward normal rays to S at $z \in S$. Let

$$N = \{n \in X : R_n(z) \in I,\ ||z - n|| = 1\},$$

let

$$\lambda_n = \sup \{ ||x - z||^2 / [2(n - z, x - z)] : x \in S, x \neq z\}$$

for $n \in N$, and let

$$N' = \{n \in N : \lambda_n \text{ is finite}\}.$$

Then

$$F(z,S) = \bigcup_{n \in N'} \{\lambda(n - z) + z : \lambda \geq \lambda_n\}.$$

Proof: Without loss of generality we may assume that $z = \phi$. For simplicity, let

$$U = \bigcup_{n \in N'} \{\lambda(n - z) + z : \lambda \geq \lambda_n\},$$

let $y \in F(\phi, S)$, and let $H = \{x : (y, x) = 0\}$. Then H is a hyperplane and $\phi \in H$ since $(y, \phi) = 0$. Suppose $w \in S$, then $||y - w|| \leq ||y||$, and so

$$(y, y) - 2(y, w) + (w, w) \leq (y, y).$$

This implies that $(y, w) \geq 1/2(w, w) \geq 0$. If $H \cap S \neq \{\phi\}$, then there exists $w \neq \phi$ such that $w \in H \cap S$. But then

$$(y - w, y - w) = (y, y) - 2(y, w) + (w, w)$$

$$= (y, y) + (w, w). \qquad (4.5)$$

Hence, from (4.5),

$$||y - w||^2 = ||y||^2 + ||w||^2.$$

Since $||w|| > 0$, $||y-w|| > ||y||$ which is a contradiction. Thus, $H \cap S = \{\phi\}$ which implies that H is a support hyperplane to S. Therefore y belongs to an inward normal ray relative to S at ϕ.

Let $n = ||y||^{-1}y$, then $n \in N$ and $y = \lambda n$ where $\lambda = ||y||$. Furthermore, since $||y|| \geq ||y - w||$ for each $w \in S$ we have

$$(\lambda n, \lambda n) \geq (\lambda n, \lambda n) - 2(\lambda n, w) + (w, w). \tag{4.6}$$

From (4.6) it follows that $\lambda \geq (w, w) / 2(n, w)$ for each $w \in S$. Hence, λ_n is finite which implies that $n \in N'$. Therefore, $y \in U$ and $F(z, S) \subset U$.

Let $y \in U$, then $y = \lambda n$ where $n \in N'$ and $\lambda \geq \lambda_n$. Hence, $\lambda \geq (w, w)/2(n, w)$ for each $w \in S$, $w \neq \phi$. Hence

$$2\lambda(n, w) \geq (w, w)$$

or

$$-2\lambda(n, w) + (w, w) \leq 0. \tag{4.7}$$

By addition of $||y||^2$, (4.7) becomes

$$\lambda^2(n, n) - 2\lambda(n, w) + (w, w) \leq \lambda^2(n, n)$$

or

$$(y - w, y - w) \leq (y, y).$$

Hence, $||y - w|| < ||y||$ which means that $y \in F(\phi, S)$. Therefore, $F(\phi, S) = U$.

A boundedly exposed point is an exposed point in a real inner-product space, but Theorems 3.14 and 4.6 permit us to give an example of an exposed point which is not a boundedly exposed point.

<u>Example 4.1</u>. Let X be the space E_2, let

$$S = \{(x, y) ; 0 \leq x \leq 1,\ y^4 \leq x\},$$

and let $z = \phi$. Then $\phi \in S$ and ϕ is an exposed point of S. Since S is smooth at ϕ, the only hyperplane of support to S at ϕ is the y-axis. Hence the only inward normal relative to S at ϕ is $R_n(\phi)$, where $n = (1, 0)$. Let $w \in S$ be denoted by (x, y), then

$$\frac{w \cdot w}{2n \cdot w} = \frac{x^2 + y^2}{2x} .$$

On the curve $y^4 = x$ this becomes

$$\frac{w \cdot w}{2n \cdot w} = \frac{y^8 + y^2}{2y^4}$$

$$= \frac{1}{2}\left(y^4 + \frac{1}{y^2} \right) .$$

Hence $w \cdot w/2n \cdot w$ tends to infinity as y tends to zero for points (x, y) on this curve. Therefore,

$$\sup \{w \cdot w/2n \cdot w : w \in S,\ w \neq \phi\} = \infty.$$

Hence, $F(\phi, S) = 0$ and ϕ is not a boundedly exposed point.

Lastly, let us discuss the relationship of F(z, S) to the norm of the space X. It has been shown by Motzkin, [28], that a two-dimensional space is an inner-product space if and only if each set N(z, S) is convex for each set S when $z \in S$. Phelps, [31], was able to extend this to any finite dimensional space. The analogous theorem is also true for F(z, S), and this will be the object of the following discussion. To simplify the proof of the theorem two lemmas will be presented first.

<u>Lemma 4.1</u>. Let X be a two-dimensional normed linear space and let $S = \{b, -b\}$ where $b = (\beta, 0)$, $\beta > 0$. If $F(b,S) \cap F(-b,S)$ are convex, then $F(b,S) \cap F(-b,S)$ is a symmetric closed convex subset of a line passing through the origin.

Proof: The set $F(b,S) \cap F(-b,S)$ is closed and convex since both $F(b,S)$ and $F(-b,S)$ are closed and convex. Now

$$F(b,S) \cap F(-b,S) = \{x \in X : ||x - b|| = ||x + b||;$$

hence, the set is not empty since $\phi \in F(b,S) \cap F(-b,S)$. Let

$$\begin{aligned} z &= \lambda b + (1 - \lambda)(-b) \\ &= -b + 2\lambda b \end{aligned}$$

and assume that $||z - b|| = ||z + b||$. Then for $\lambda < 0$,

$$||-2b + 2\lambda b|| = ||2\lambda b||. \tag{4.8}$$

Then (4.8) becomes

$$(1 - \lambda)\ ||b|| = |\lambda|\ ||b||. \tag{4.9}$$

Thus, from (4.9), $1 - \lambda = -\lambda$ which implies $1 = 0$. Therefore, λ cannot be negative. If $0 < \lambda < 1$, then

$$(1 - \lambda)\ ||b|| = \lambda ||b|| \tag{4.10}$$

which implies that $\lambda = 1/2$, or that $z = \phi$. If $1 < \lambda$, then

$$(\lambda - 1)\ ||b|| = \lambda ||b|| \tag{4.11}$$

which means that $-1 = 0$. Hence, λ cannot be greater than one. Finally, note that neither b nor -b are elements of $F(b,S) \cap F(-b,S)$. Therefore, the only point $(x, 0)$ which is equidistant from b and -b is the origin.

Now let $z_1 \neq \phi$ and $z_2 \neq \phi$ be elements of $F(b, S) \cap F(-b, S)$ such that z_1 and z_2 are not collinear with the origin. If z_1 and z_2 have second coordinates with opposite signs, then the line segment z_1z_2 must contain a point $(x, 0)$, $x \neq 0$, but

$$z_1z_2 \subset F(b,S) \cap F(-b,S).$$

Hence, there is a contradiction. If z_1 and z_2 have second coordinates with the same signs then let $z_2' = -z_2$. Comparing the distances of z_2' from b and -b we find that

$$\begin{aligned} ||z_2' - b|| &= ||-z_2 - b|| \\ &= ||z_2 + b|| \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} ||z_2' + b|| &= ||-z_2 + b|| \\ &= ||z_2 - b||. \end{aligned} \tag{4.13}$$

Hence, from (4.12) and (4.13),

$$||z_2' - b|| = ||z_2' + b||$$

and

$$z_2' \in F(b,S) \cap F(-b,S).$$

This shows that $F(b,S) \cap F(-b,S)$ is symmetric. If there exists a real number λ such that $\phi = \lambda z_2' + (1 - \lambda)z_1$ then it can be shown that $z_2 = [(1 - \lambda)/\lambda]\, z_1$. But this means that z_1 and z_2 are collinear with the origin which is a contradiction. Therefore, the line segment z_1z_2' must contain a point $(x, 0)$ with $x \neq 0$, but this also is a contradiction. Therefore, all elements of $F(b, S) \cap F(-b, S)$ must be collinear with ϕ. Hence, $F(b, S) \cap F(-b, S)$ is a subset of a line passing through the origin.

Now the tools are available to prove the next lemma.

Lemma 4.2. Let X be a two-dimensional normed linear space such that F(z, S) is convex for each set S and each $z \in S$. Then X is an inner-product space.

Proof: Since, by hypothesis, F(z, S) is convex for each set S, let $S = \{b, -b\}$ where $b = (\beta, 0)$, $\beta > 0$. Then $F(b, S) \neq 0$ since $-b \in F(b, S)$, and F(b, S) and F(-b, S) are both closed and convex. Let

$$F(b, S)^0 = \{x \in X : ||x - b|| > ||x + b||\}.$$

Then $F(b, S)^0 \neq 0$ since $-b \in F(b, S)^0$ and $F(b, S)^0$ is open since

$$F(b, S)^0 = X \setminus F(-b, S).$$

Furthermore,

$$F(b, S) = F(b, S)^0 \cup (F(b, S) \cap F(-b, S)). \tag{4.14}$$

Let $x \in F(b, S) \cap F(-b, S)$ and let $\epsilon > 0$. Let $w = x + t(x + b)$, where $t = \epsilon/(2||x + b||)$. Then

$$\begin{aligned} ||w + b|| &= ||x + t(x + b) + b|| \\ &= (1 + t) ||x + b||, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} ||w - b|| &= ||x + t(x + b) - b|| \\ &\leq ||x - b|| + t||x + b|| \\ &= ||x + b|| + t||x + b|| \\ &= (1 + t) ||x + b|| \\ &= ||w + b||. \end{aligned} \tag{4.16}$$

Hence, from (4.16), $||w + b|| \geq ||w - b||$ which implies that $w \in F(-b,S)$.

If $||w + b|| = ||w - b||$, then w, x, and ϕ must be collinear from Lemma 4.1. But this is impossible since w, x, and -b are collinear. Therefore, $||w + b|| > ||w - b||$ and $w \notin F(b, S)$. Furthermore,

$$\begin{aligned} ||w - x|| &= ||x + t(x + b) - x|| \\ &= t||x + b|| \\ &= \epsilon/2 \\ &< \epsilon. \end{aligned} \tag{4.17}$$

Since ϵ was arbitrary, each neighborhood of x contains a point of $X \setminus F(b, S)$. Thus, x is a boundary point of F(b, S), and

$$F(b, S) \cap F(-b, S) \subset \text{bd } F(b, S).$$

Then, since $F(b, S)^0$ is open and can contain no boundary points of F(b, S), (4.14) implies that

$$\text{bd } F(b, S) = F(b, S) \cap F(-b, S). \tag{4.18}$$

Therefore, $F(b, S)^0$ is the interior of a convex body and must be convex. Hence $F(b, S)^0$ and F(-b, S) are complementary convex sets since

$$F(b, S)^0 \cup F(-b, S) = X$$

and

$$F(b, S)^0 \cap F(-b, S) = 0.$$

Then, $V = \text{lin } F(b, S)^0 \cap \text{lin } F(-b, S)$ is either a hyperplane or it is the entire space X (cf. Valentine, [36]). According to Valentine ([36], p. 11), $\text{lin } F(b, S)^0 \subset \text{cl } F(b,S)^0 = F(b,S)$ and $\text{lin } F(-b,S) \subset F(-b, S)$. Therefore, $V \subset F(b, S) \cap F(-b, S)$ which implies that V is a hyperplane. Furthermore, $F(b, S) \cap F(-b, S)$ is a subset of a line by Lemma 4.1. Hence $V = F(-b, S) \cap F(b, S)$ and $F(-b, S) \cap F(b, S)$, the set of points equidistant from b and -b, is a straight line.

Obviously, the preceding discussion can be applied to any pair of points x and y of X after a suitable rotation and translation of axes. Hence, the set of points equidistant from any pair of points of X is a straight line. A theorem of Day, [8], states that a normed linear space L is an inner-product space if and only if the set of points equidistant from any pair of points of L is a flat. Therefore, the space X is an inner-product space.

Now we are ready to prove the theorem.

Theorem 4.7. A normed linear space X is an inner-product space if and only if for each set S and $z \in S$, $F(z, S)$ is convex.

Proof: Suppose X is an inner-product space and that $S \subset X$ and $z \in S$. If $F(z, S)$ is empty then it is convex. If $F(z, S)$ is not empty, then by Theorem 4.1 and Theorem 4.3, $F(z, S)$ is the intersection of closed half-spaces. A closed half-space is always convex and the intersection of convex sets is always convex. Therefore, $F(z, S)$ must be convex.

Suppose that $F(z, S)$ is convex for each set S and $z \in S$. Let L be any two-dimensional subspace of X and let $S \subset L$. If $z \in S$, $F(z, S)$ must be convex which implies that $F(z, S) \cap L$ is convex. $F(z, S) \cap L$ is just $F(z, S)$ for the space L. Hence, in the space L, $F(z, S)$ is convex for each set S and $z \in S$. Therefore, by Lemma 4.2, L is an inner-product space. Day, [8], has shown that a normed linear space is an inner-product space if and only if every two-dimensional subspace of the space is an inner-product space. Hence, the space X is an inner-product space.

In Example 3.3, X is the normed linear space, $\ell^{\infty}(2)$, which is not an inner-product space. For the set $S = \{z, w\}$, where $z = (1, 0)$ and $w = (0, 0)$, it was shown that $F(z, S)$ is not convex.

CHAPTER V

APPROXIMATIONS BY POLYTOPES

The problem of determining the set F(z, S) for a given set S and a point z is usually difficult unless in an inner-product space the set S is a smooth convex set. Hence, it would be desirable to develop geometric methods for finding F(z, S) or for approximating F(z, S) in some sense. In order to approximate F(z, S) the procedure will be to approximate S by some set W, then consider the set of z-farthest points, F(z, W), which, hopefully, will be nearly equal to F(z, S). Obviously, W must be a set such that F(z, W) is readily found. Polytopes have been used to approximate sets. Thus, if F(z, P) for a polytope $P \subset S$ can be found easily, then this might lead to an approximation of F(z, S).

By a polytope, we mean a bounded convex set which is the intersection of a finite number of closed half-spaces. This definition has been shown by Klee, [24], to be equivalent to the definition that a polytope is the convex hull of a finite number of points. These two equivalent definitions will be used interchangeably throughout the remainder of the discussion.

For a polytope P and a point $z \in P$, we shall determine F(z, P). The following theorem by Fan, [17], will be useful: Let L be a real linear space of arbitrary dimension, finite or infinite. Then a system of inequalities

$$f_i(x) \geq \alpha_i, \quad 1 \leq i \leq p,$$

where $f_1, f_2, \ldots, f_p$ are linear functionals on L and $\alpha_1, \alpha_2, \ldots, \alpha_p$ are real numbers, is consistent if and only if for any p non-negative numbers λ_i the relation

$$\sum_{i=1}^{p} \lambda_i f_i = 0$$

implies

$$\sum_{i=1}^{p} \lambda_i \alpha_i \leq 0.$$

Of course, by consistent, we mean that there exists a point $x_0 \in L$ such that

$$f_i(x_0) \geq \alpha_i, \quad 1 \leq i \leq p.$$

Now the tools are available to determine F(z, P) for a polytope P and $z \in P$.

Theorem 5.1. Let $P = \text{conv}\{z, x_1, \ldots, x_m\}$ be a polytope in the real inner-product spaceX such that z is an extreme point of P and $z \neq x_i$, $1 \leq i \leq m$. Then F(z, P) is not empty and is the intersection of a finite number of half-spaces.

Proof: Whether F(z, P) is empty or not,

$$F(z, P) = F(z, \{z, x_1, \ldots, x_m\})$$

by Theorem 3.4. Then by Corollary 4.1,

$$F(z, \{z, x_1, \ldots, x_m\}) = F(z, \{z, z\}) \cap \bigcap_{i=1}^{m} F(z, \{z, x_i\}).$$

However, F(z, {z, z}) is just the space X so that

$$F(z, \{z, x_1, \ldots, x_m\}) = \bigcap_{i=1}^{m} F(z, \{z, x_i\}).$$

From Theorem 4.3 we see that each set $F(z, \{z, x_i\})$, $1 \leq i \leq m$, is the half space

$$H_i = \{x ; (x_i - z, x) \geq (1/2)(x_i - z, x_i + z)\}, \quad 1 \leq i \leq m.$$

Hence, $F(z, P)$ is the intersection of a finite number of half-spaces, and $F(z, P)$ will be nonempty if the system of inequalities

$$(x_i - z, x) \geq (1/2)(x_i - z, x_i + z), \quad 1 \leq i \leq m \tag{5.1}$$

is consistent. So assume there exist real numbers $\lambda_i \geq 0$, $1 \leq i \leq m$, such that

$$\sum_{i=1}^{m} \lambda_i (x_i - z, x) = 0$$

for each $x \in X$. Then by the bilinearity of the inner product

$$\left(\sum_{i=1}^{m} \lambda_i (x_i - z), x\right) = 0$$

for each $x \in X$. Hence

$$\sum_{i=1}^{m} \lambda_i (x_i - z) = \phi.$$

If there is some $\lambda_j > 0$, then we may write

$$z = \sum_{i=1}^{m} \frac{\lambda_i}{\sum_{k=1}^{m} \lambda_k} x_i .$$

If just $\lambda_j > 0$, then $z = x_j$, a contradiction. If more than one number λ_j is greater than zero then $z \in \operatorname{conv}\{x_1, \ldots, x_m\}$ since

$$\sum_{i=1}^{m} \frac{\lambda_i}{\sum_{k=1}^{m} \lambda_k} = 1,$$

However, in such a case z cannot be an extreme point of P. Hence, $\lambda_i = 0$ for each i such that $1 \leq i \leq m$. Thus,

$$\sum_{i=1}^{m} \lambda_i (x_i - z, x_i + z) = 0,$$

and the system (5.1) is consistent by Fan's theorem.

Corollary 5.1. Each vertex of a polytope P in an inner-product space X is a boundedly exposed point of P.

Proof: By Theorem 5.1, F(z, P) is nonempty for a vertex z of P. By Theorem 3.14, $F(z, P) \neq 0$ if and only if z is a boundedly exposed point.

From Theorem 5.1 we see that F(z, P) is easily found in an inner-product space by intersecting the half-spaces determined by the perpendicular bisectors of the line segments joining z to each of the other vertices or extreme points of P. Since F(z, P) is easily found it seems possible to determine when F(z, P) is a cone. The following theorem gives sufficient conditions for F(z, P) to be a cone.

Theorem 5.2. Let X be a real inner-product space and let $P = \text{conv}\{z, x_1, \ldots, x_m\}$ be a polytope such that z is an extreme point of P and $z \neq x_i$, $1 \leq i \leq m$. If, after a suitable rearrangement of the set $\{x_1, \ldots, x_m\}$, there exists a point $x_0 \in F(z, P)$ and an integer n, $0 < n < m$, such that

$$(x_i - z, x_0) = (1/2)(x_i - z, x_i + z),$$

$1 \leq i \leq n$, and the points x_r, $n + 1 \leq r \leq m$, are in the convex cone with vertex z and extremal rays $z + \mathbf{R}x_i$, $1 \leq i \leq n$, then F(z, P) is a convex cone with vertex x_0.

Proof: Without loss of generality, we may assume that $z = \phi$. Then the set of inequalities (5.1) which define $F(\phi, P)$ becomes

$$(x_i, x) \geq (1/2)\,||x_i||^2, \quad 1 \leq i \leq m.$$

Furthermore, this system is consistent since $x_0 \in F(\phi, P)$. Since each x_r, $n + 1 \leq r \leq m$, is in the cone with vertex ϕ and extremal rays Rx_i, $1 \leq i \leq n$, we have that

$$x_r = \sum_{i=1}^{n} \lambda_{ir} x_i, \quad \lambda_{ir} \geq 0, \quad 1 \leq i \leq n. \tag{5.3}$$

Then since $(x_i, x_0) = (1/2)\,||x_i||^2$, $1 \leq i \leq n$, it follows that

$$\begin{aligned} \sum_{i=1}^{n} \lambda_{ir} (1/2)||x_i||^2 &= \sum_{i=1}^{n} \lambda_{ir} (x_i, x_0) \\ &= \left(\sum_{i=1}^{n} \lambda_{ir} x_i, x_0 \right) \\ &= (x_r, x_0) \\ &\geq (1/2)||x_r||^2. \end{aligned} \tag{5.4}$$

To show that $F(\phi, P)$ is a cone it is necessary to show that for each $x \in F(\phi, P)$, $\{(1 - \lambda)x_0 + \lambda x : \lambda \geq 0\}$ is a subset of $F(\phi, P)$. So let $x \in F(\phi, P)$, and let $\lambda \geq 0$. If $0 \leq \lambda \leq 1$, then $(1 - \lambda)x_0 + \lambda x \in F(\phi, P)$ since, as shown by Theorem 4.7, $F(\phi, P)$ is convex. If $\lambda > 1$, then for each x_i, $1 \leq i \leq n$, we have

$$\begin{aligned} (x_i, (1 - \lambda)x_0 + \lambda x) &= (1 - \lambda)(x_i, x_0) + \lambda(x_i, x) \\ &= (1 - \lambda)(1/2)\,||x_i||^2 + \lambda(x_i, x) \\ &\geq (1 - \lambda)(1/2)||x_i||^2 + \lambda(1/2)||x_i||^2 \\ &= (1/2)||x_i||^2. \end{aligned} \tag{5.5}$$

If $n + 1 \leq r \leq m$, then from (5.3) and (5.4) it follows that

$$(x_r, (1-\lambda)x_0 + \lambda x) = (1-\lambda)(x_r, x_0) + \lambda(x_r, x).$$

$$= (1-\lambda)\left(\sum_{i=1}^{n} \lambda_{ir} x_i, x_0\right) + \lambda\left(\sum_{i=1}^{n} \lambda_{ir} x_i, x\right)$$

$$= (1-\lambda) \sum_{i=1}^{n} \lambda_{ir}(x_i, x_0) + \lambda \sum_{i=1}^{n} \lambda_{ir}(x_i, x)$$

$$= (1-\lambda) \sum_{i=1}^{n} \lambda_{ir}(1/2)||x_i||^2 + \lambda \sum_{i=1}^{n} \lambda_{ir}(x_i, x)$$

$$\geq (1-\lambda) \sum_{i=1}^{n} \lambda_{ir}(1/2)||x_i||^2 + \lambda \sum_{i=1}^{n} \lambda_{ir}(1/2)||x_i||^2$$

$$= (1/2) \sum_{i=1}^{n} \lambda_{ir} ||x_i||^2$$

$$\geq (1/2)||x_r||^2. \tag{5.6}$$

Thus by (5.5) and (5.6), $(1-\lambda)x_0 + \lambda x$ always satisfies the system of inequalities (5.2) when $\lambda \geq 0$ and $x \in F(\phi, P)$. Therefore, $F(\phi, P)$ is a convex cone with vertex x_0.

In a geometric sense, Theorem 5.2 says that the points x_i, $1 \leq i \leq n$ all lie on the surface of a sphere with center x_0 and radius $||x_0||$. The other vertex points of the polytope P lie within this sphere and also within the convex cone with vertex ϕ and extremal rays Rx_i, $1 \leq i \leq n$.

Since now the structure of $F(z, P)$ for a polytope P has been determined, we are ready to approximate $F(z, S)$, for a compact convex body S. The distance between two closed bounded sets A and B is denoted here by $d(A, B)$ where

$$d(A, B) = \inf \{\rho : A \subset B\rho,\ B \subset A\rho\},$$

and

$$A\rho = \bigcup_{a \in A} K(a, \rho), \quad 0 \leq \rho \in R,$$

and

$$K(a, \rho) = \{x : ||x - a|| \leq \rho\}.$$

The functional d satisfies all the properties of a metric (cf. Valentine, [36]). We shall say that a sequence $\{A_i\}$ of sets converges to the set A if and only if

$$\lim_{i \to \infty} d(A_i, A) = 0$$

and we shall write $A_i \to A$.

An important theorem related to this metric is the Blaschke selection theorem which is as follows: Let M be a uniformly bounded infinite collection of closed convex sets in a finite-dimensional normed linear space X_n. Then M contains a sequence which converges to a nonempty compact convex set. A uniformly bounded collection of sets is a collection which is contained within some solid sphere (cf. Valentine, [36]). This theorem provides a method of approximating a set S with polytopes with the additional property that a given boundary point z is a vertex of each polytope.

Theorem 5.3. Let S be a compact, convex body in the normed space X and let z be an extreme point of S. Then there exists a sequence of polytopes $\{P_n\}$ such that

1. $P_n \subset S$, $n = 1, 2, \ldots,$
2. $P_n \subset P_{n+1}$, $n = 1, 2, \ldots,$
3. z is a vertex of each P_n, $n = 1, 2, \ldots,$ and
4. $\lim_{n \to \infty} d(P_n, S) = 0$

Proof: Consider the sphere, $B_1 = \{x : ||x - z|| < 1\}$. This set is open so that $S \setminus B_1$ is compact if it is not empty. If $S \setminus B_1$ is empty, let $P_1 = \{z\}$. If $S \setminus B_1$ is not empty then cover $S \setminus B_1$ with open spheres of radius one and centers in $S \setminus B_1$. Since $S \setminus B_1$ is compact, there exists a finite subcovering of n open spheres with centers $\{x_1, x_2, \ldots, x_n\}$. Then consider $P_1 = \text{conv}\ \{z, x_1, \ldots, x_n\}$. We have $P_1 \subset S \subset (P_1)_1$; and, furthermore, z is an extreme point of P_1 since it is an extreme point of S.

Assume that $P_1, P_2, \ldots, P_{n-1}$ have been chosen such that P_{n-1} is a polytope with vertices $\{z, x_1, \ldots, x_N\}$. Assume that $P_i \subset P_{i+1}$, $1 \leq i \leq n - 2$; $P_i \subset S \subset (P_i)_{1/i}$, $1 \leq i \leq n - 1$; and that z is a vertex of each P_i, $1 \leq i \leq n - 1$. Let

$$B_n = \bigcup_{i=0}^{N} \{x : ||x - x_i|| < 1/n\}$$

where $x_0 = z$. Then $S \setminus B_n$ is compact or empty. If $S \setminus B_n$ is empty then let $P_n = P_{n-1}$; however, if $S \setminus B_n$ is not empty then cover $S \setminus B_n$ with open spheres of radius $1/n$ and centers in $S \setminus B_n$. There exists a finite subcovering which defines a finite set of points $\{x_{N+1}, \ldots, x_{N+t}\}$. Let $P_n = \text{conv}\ \{z, x_1, \ldots, x_{N+t}\}$. Then, whether $S \setminus B_n$ is empty or compact, $P_{n-1} \subseteq P_n$; z is a vertex of P_n; and $P_n \subset S \subset (P_n)_{1/n}$. Therefore, by induction, a sequence of polytopes having the properties (1), (2), and (3) has been defined. Note that if $S \setminus B_1$ is empty then there must exist some integer n such that $S \setminus \{x : ||x - z|| < 1/n\}$ is not empty. If not, then $S = \{z\}$ and would not be a convex body. Hence, there exists an integer, n, such that P_n is not a degenerate polytope.

We have that $P_n \subset S$ for each integer n; hence, $P_n \subset S\rho$ for any $\rho > 0$. Also $S \subset (P_n)_{1/n}$ for each n, hence

$$d(P_n, S) = \inf\{\rho : P_n \subset S\rho,\ S \subset (P_n)\rho\} \leq 1/n.$$

Therefore, $\lim_{n \to \infty} d(P_n, S) = 0$.

Theorem 5.3 really gives a non-constructive method, which is usually not practical, of finding a sequence satisfying the properties (1), (2), (3) and (4) of Theorem 5.3. If the space X is finite-dimensional, then it is possible to use a much more systematic method to achieve the same results.

Theorem 5.4. Let S be a bounded, convex body in a finite-dimensional normed space X and let z be an extreme point of S. Then there exists a sequence of convex polytopes $\{P_n\}$ such that

1. $P_n \subset S$, $n = 1, 2, \ldots,$
2. $P_n \subset P_{n+1}$, $n = 1, 2, \ldots,$
3. z is a vertex of each P_n, $n = 1, 2, \ldots,$ and
4. $\lim_{n \to \infty} d(P_n, S) = 0$.

Proof: Since a linear topological space of dimension r is always linearly isomorphic to E_r, we may assume $X = E_r$ (cf. Valentine, [36]). For each m, $m = 0, 1, 2, \ldots,$ let

$$L_m = \left\{ \left(\frac{p_1}{2^m}, \ldots, \frac{p_r}{2^m} \right) : p_i = 0, \pm 1, \pm 2, \ldots,\ i = 1, 2, \ldots, r \right\} \quad (5.7)$$

Then for each m there exists only a finite number of points from L_m which are contained in S. Let $L'_m \subset L_m$ be that set.

Let $P_m = \operatorname{conv}(L'_m \cup \{z\})$, then $P_m \subset S$ since $L'_m \cup \{z\} \subset S$ and S is convex. Again z is an extreme point of P_m since z is an extreme point of S.

Now $L'_m \subset L'_{m+1}$ since if

$$\left(\frac{p_1}{2^m}, \ldots \frac{p_r}{2^m}\right) \in L'_m$$

then

$$\left(\frac{2p_1}{2^{m+1}}, \ldots, \frac{2p_r}{2^{m+1}}\right) \in L'_{m+1} .$$

Hence $P_m \subset P_{m+1}$.

Each P_m is closed and convex and the sequence $\{P_m\}$ is uniformly bounded since S is bounded. Therefore, the Blashke convergence theorem implies that there exists a compact, convex set C and a subsequence $\{P_{m_k}\}$ such that

$$\lim_{k \to \infty} d(P_{m_k}, C) = 0. \tag{5.8}$$

We shall show that $P_{m_k} \subset C$ for each k. Suppose there exists $x \in P_{m_k} \setminus C$ for some $k > 0$. Let

$$\delta(x, C) = \inf \{ ||y - x|| : y \in C\}.$$

Then $C_{\delta/2}$ does not contain P_{m_k}. By (5.8), there is some integer $h > k$ such that

$$d(C, P_{m_h}) < \delta/2.$$

Since

$$d(C, P_{m_h}) = \inf \{\rho : C \subset (P_{m_h})\rho, \ P_{m_h} \subset C\rho\}$$

and since $C_{\delta/2}$ does not contain P_{m_h}, $d(C, P_{m_h}) \geq \delta/2$. But this is a contradiction; hence $P_{m_k} \subset C$ for each $k > 0$.

Let $\epsilon > 0$, then there exists an integer $k > 0$ such that

$d(P_{m_h}, C) < \epsilon$ for each $h \geq k$. Now suppose $m > m_h$, then $P_{m_h} \subset P_m$. Since $\{P_{m_k}\}$ is a sequence, there is an integer t such that $m < m_t$; hence, $P_m \subset P_{m_t} \subset C$. If the number ρ is such that $C \subset (P_{m_k})\rho$, then $C \subset (P_m)\rho$. Hence

$$\{\rho : C \subset (P_{m_k})\rho,\ P_{m_k} \subset C\rho\} \subset \{\rho : C \subset (P_m)\rho,\ P_m \subset C\rho\}$$

and it follows that

$$d(P_m, C) \leq d(P_{m_k}, C) < \epsilon.$$

Therefore,

$$\lim_{n \to \infty} (P_n, C) = 0.$$

Now $S \subset C$, for suppose there exists $x \in S \setminus C$. Then since C is compact there exists an open sphere of radius δ and center x, $N_\delta(x)$, such that $N_\delta(x) \cap C = 0$. Since $x \in S$ and S is a convex body, $N_\delta(x)$ must intersect the interior S^0 of S; hence, there must exist a point

$$\left(\frac{p_1}{2^{m_1}}, \ldots, \frac{p_r}{2^{m_r}}\right) \in N_\delta(x) \cap S^0$$

since this set of points is dense in E_r. But then

$$\left(\frac{2^{M_1} p_1}{2^M}, \ldots, \frac{2^{M_r} p_r}{2^M}\right) \in P_M \subset C,$$

where

$$M = \sum_{i=1}^{r} m_i$$

and

$$M_j = \sum_{\substack{i=1 \\ i \neq j}}^{r} m_i, \quad 1 \leq j \leq r.$$

But this is a contradiction; hence, $S \subset C$ and

$$\lim_{n \to \infty} d(P_n, S) = 0.$$

Now that a set S can be "approximated" by polytopes, it is possible to "approximate" $F(z,S)$ and $N(z,S)$ in an inner-product space. In fact, the intersection of the sets $F(z, P_n)$ is $F(z,S)$.

Theorem 5. 5. Let S be a convex body in an inner-product space X such that $z \in S$ is a boundedly exposed point of S. Let $\{P_n\}$ be a sequence of polytopes such that

1. $P_n \subset S$, $n = 1, 2, \ldots$,
2. $P_n \subset P_{n+1}$, $n = 1, 2, \ldots$,
3. z is a vertex of each P_n, $n = 1, 2, \ldots$, and
4. $\lim_{n \to \infty} d(P_n, S) = 0$.

Then

$$F(z,S) = \bigcap_{n=1}^{\infty} F(z, P_n).$$

If X is finite-dimensional and R is a closed sphere with center z such that $R \cap F(z,S) \neq 0$, then

$$\lim_{n \to \infty} d(R \cap F(z,S),\ R \cap F(z, P_n)) = 0.$$

Proof: First it will be shown that

$$F(z,S) = \bigcap_{n=1}^{\infty} F(z, P_n).$$

Suppose $x \in F(z,S)$, then $||x - z|| \geq ||y - x||$ for each $y \in S$. Since $P_n \subset S$ for each n, $||x - z|| \geq ||y - x||$ for each $y \in P_n$. Since $z \in P_n$, $x \in F(z, P_n)$. Hence $F(z,S) \subset F(z, P_n)$ for each integer n.

Now suppose

$$x \in \bigcap_{n=1}^{\infty} F(z, P_n),$$

then $||z - x|| \geq ||y - x||$ for $y \in P_n$, $n = 1, 2, \ldots$. But suppose $y \in S$ and there is not an n such that $y \in P_n$. Let $\epsilon > 0$, then there is an $N > 0$ such that for each $n \geq N$, $d(P_n, S) < \epsilon$. Hence, there exists $w \in P_N$ such that $||w - y|| < \epsilon$. Then

$$\begin{aligned} ||x - y|| &\leq ||x - w|| + ||w - y|| \\ &< ||x - z|| + \epsilon. \end{aligned} \tag{5.9}$$

Since ϵ was arbitrary, $||x - y|| \leq ||x - z||$ which means that $x \in F(z, S)$. Hence

$$F(z, S) = \bigcap_{n=1}^{\infty} F(z, P_n).$$

Suppose now that X is also finite-dimensional and that R is a closed sphere with center z which intersects $F(z, S)$. Let $R_0 = F(z, S) \cap R$. Since $F(z, S) \subset F(z, P_n)$ for each n, $F(z, P_n) \cap R$ is not empty. Let $R_n = F(z, P_n) \cap R$, then since $F(z, P_{n+1}) \subseteq F(z, P_n)$, $R_{n+1} \subseteq R_n$. Each set R_n is closed and convex and the collection of sets is uniformly bounded. By the Blashke Convergence Theorem, there exists a subsequence $\{R_{n_k}\}$ which converges to a nonempty, compact, convex set C.

Suppose that there exists a positive integer N such that $C \setminus R_N \neq 0$. Let $x \in C \setminus R_N$. Then

$$\delta(x, R_N) = \inf \{ ||x - y|| : y \in R_N \} = \epsilon > 0$$

since R_N is closed. There exists $n > N$ such that $d(C, R_m) < \epsilon/2$ for $m \geq n$. By definition

$$d(C, R_m) = \inf \{ \rho : C \subset (R_m)\rho, \ R_m \subset C\rho \},$$

but for $\rho = \epsilon/2$ we have

$$(R_n)_{\epsilon/2} \subset (R_N)_{\epsilon/2}$$

which implies that $(R_n)_{\epsilon/2} \setminus C \neq 0$. Hence $\epsilon/2$ is a lower bound for the set

$$\{\rho : C \subset (R_m)\rho,\ R_m \subset C\rho\}.$$

Hence $d(C, R_m) \geq \epsilon/2$, but this is a contradiction. Therefore $C \subset R_n$ for $n = 1, 2, \ldots,$ which means that $C \subset R_0$.

Now if ρ is such that $R_n \subset C\rho$, then $R_n \subset (R_0)\rho$ and so $d(R_0, R_n) \leq d(C, R_n)$. Hence

$$\lim_{n\to\infty} d(R_0, R_n) = 0.$$

Suppose $R_0 \setminus C \neq 0$. If $x \in R_0 \setminus C$ then $\delta(x, C) = \epsilon > 0$ and there exists $N > 0$ such that for $n \geq N$, $d(C, R_n) < \epsilon/3$. So $R_N \subset C_{\epsilon/2}$, but $R_0 \subset R_N$ which implies that $x \in C_{\epsilon/2}$. Hence, there is a contradiction. Therefore, $R_0 = C$.

The next theorem shows that the same type of result is true for nearest point sets.

Theorem 5.6. Let S be a convex body in the real-inner product space X and let $z \in S$. Let $\{P_n\}$ be a sequence of polytopes such that

1. $P_n \subset S$, $n = 1, 2, \ldots,$
2. $P_n \subset P_{n+1}$, $n = 1, 2, \ldots,$ and
3. $\lim_{n\to\infty} d(P_n, S) = 0$.

Then

$$N(z, S) = \bigcap_{n=1}^{\infty} N(z, P_n).$$

If X is also finite-dimensional and R is a closed sphere with center z, then

$$\lim_{n \to \infty} d(R \cap N(z,S),\ R \cap N(z,P_n)) = 0.$$

Proof: First it will be shown that

$$N(z,S) = \bigcap_{n=1}^{\infty} N(z,P_n).$$

Suppose $x \in N(z,S)$, then $||x - z|| \leq ||y - x||$ for each $y \in S$. Since $P_n \subset S$ for each n, $||x - z|| \leq ||y - x||$ for each $y \in P_n$. Since $z \in P_n$, $x \in N(z,P_n)$. Hence $N(z,S) \subset N(z,P_n)$ for each positive integer n. Now suppose

$$x \in \bigcap_{n=1}^{\infty} N(z,P_n),$$

then $||z - x|| \leq ||y - x||$ for $y \in P_n$, $n = 1, 2, \ldots$. But if $y \in S$ such that $y \notin P_n$ for any n, then let $\epsilon > 0$ be an arbitrary number. There exists an integer N such that for $n \geq N$, $d(P_n, S) < \epsilon$. Hence, there exists $w \in P_N$ such that $||w - y|| < \epsilon$. Then

$$\begin{aligned} ||x - y|| &= ||(x - w) - (y - w)|| \\ &\geq ||x - w|| - ||y - w|| \\ &> ||x - w|| - \epsilon. \end{aligned} \tag{5.10}$$

But $w \in P_N$ implies that $||x - w|| \geq ||x - z||$; hence, from (5.10), $||x - y|| > ||x - z|| - \epsilon$. Since ϵ is arbitrary, it follows that $||x - y|| \geq ||x - z||$. Thus, if $y \in S$, $||x - y|| \geq ||x - z||$. Therefore, $x \in N(z,S)$ from which it follows

$$N(z,S) = \bigcap_{n=1}^{\infty} N(z,P_n).$$

Now suppose X is finite-dimensional and that R is a closed sphere with center z. Then $R_0 = R \cap N(z,S)$ is a nonempty compact convex set. Let $R_n = N(z, P_n) \cap R$, then since $P_n \subseteq P_{n+1}$, $N(z, P_{n+1}) \subseteq N(z, P_n)$ and $N(z, P_{n+1}) \cap R \subseteq N(z, P_n) \cap R$. The sequence $\{R_n\}$ is a uniformly bounded collection of compact convex sets; therefore, the Blashke selection theorem implies that there exists a subsequence $\{R_{n_k}\}$ which converges to a nonempty, compact, and convex set C.

Suppose there exists an integer $k > 0$ such that $C \setminus R_{n_k} \neq 0$. Let $x \in C \setminus R_{n_k}$ and let

$$\delta(x, R_{n_k}) = \inf \{ ||x - y|| : y \in R_{n_k} \}$$

Then $\delta(x, R_{n_k}) > 0$ since R_{n_k} is closed. There exists $h > k$ such that $d(C, R_{n_h}) < \epsilon/2$. But $(R_{n_h})_{\epsilon/2} \subset (R_{n_k})_{\epsilon/2}$ which implies that $C \setminus R_{n_h} \neq 0$. Hence, $\epsilon/2$ is a lower bound for the set

$$\{\rho : C \subset (R_{n_h})\rho, \; R_{n_h} \subset C\rho\}.$$

Thus, $d(C, R_{n_h}) \geq \epsilon/2$ which is a contradiction. Therefore, $C \subset R_{n_k}$, $k = 1, 2, \ldots$. Since the sequence $\{R_n\}$ is monotone it follows that $\lim_{n \to \infty} R_n = C$ and $C \subset R_n$ for each positive integer n. Since $C \subset R_n$ for each positive integer n, it follows that $C \subset R_0$. Hence, by reasoning similar to that of Theorem 5.5, it follows that $\lim_{n \to \infty} d(R_n, R_0) = 0$.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The basic purpose of this study has been to examine the structure of z-farthest point sets and to determine properties of z-farthest point sets which are analogous to properties of z-nearest point sets. It was found that in a normed linear space, a z-farthest point set must be closed. Further investigation showed that a z-farthest point set is inverse starlike with respect to z. In a strictly convex normed linear space, F(z, S) and F(w, S), $z \neq w$, have only boundary points in common. The z-farthest point set of S is equal to the z-farthest point set of conv S and cl S. The z-nearest point set of S is equal to the z-nearest point set of cl S, but it is not, in general, equal to the z-nearest point set of conv S.

Another topic of interest was the element z which determines the z-farthest point set of S. If the z-farthest point set is nonempty, then z is a boundary point of S. If the normed linear space is strictly convex and the z-farthest point set of S is nonempty, then z is an extreme point of S. A z-farthest point set of S is nonempty in a strictly convex space if and only if z is a boundedly exposed point of S.

It was shown that in a real inner-product space a z-nearest point set and a z-farthest point set of S can be represented as the intersection of closed half-spaces and the union of closed rays. This led to a characterization of inner-product spaces in terms of z-farthest point sets. A normed linear space is an inner-product space if and only if for each set S and each element z of S, the z-farthest point set of S is convex.

Finally, the structure of the z-farthest point set of a polytope P was found. Sufficient conditions for the z-farthest point set of P to be a cone were developed. Then, methods of approximating z-farthest point sets and z-nearest point sets were found.

There are several problems which have been raised by this study which would be of interest for further consideration. One such problem is the characterization of sets which contain at least one point z whose z-farthest point set is a cone. The problem of completely characterizing the points z of a given set S whose z-farthest point set of S is nonempty has not been solved. It would be desirable to extend Theorem 2.1 to infinite dimensional inner-product spaces.

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