ONE STAGE PROCUREMENT MODELS WITH UNCERTAIN DELIVERY DATE

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PREFACE

In the procurement of large or expensive components for a large job shop manufacturing facility many factors can arise to prevent on-time delivery by subcontractors or vendors. Many situations also arise in the construction industry in which an expensive, critical item with an uncertain delivery time must be procured to meet a fixed production schedule. Recognizing the high costs of late delivery and the chance that vendors may not be able to meet the specified delivery date, materials management personnel typically specify a delivery date to vendors that is several days or even weeks prior to the actual requirement date. This "safety time allowance," or buffer, is used to insure on-time delivery; but the use of long buffer periods can result in very high holding costs for parts that do arrive on the contracted delivery date and then must be kept in inventory for long periods before being used. The problem approached in this dissertation is that of determining the optimal safety time allowance to be used in procurement situations where the delivery date may be considered as a random variable. Although developed for a single-stage procurement situation, the models can be applied in any situation involving uncertain delivery time where the number of items needed is fixed and a specific requirement date is known. The models developed in this dissertation provide a useful decision aid for determining the proper safety time allowance to use in specifying the delivery date that will minimize the expected total variable cost of procurement.

The models developed in this dissertation employ a new method for dealing with the costs of lateness in delivery which may be useful in future research in the field of procurement and inventory theory. In particular, the models could be utilized in a vendor rating system that would quantify direct costs of uncertainty in delivery time as well as bid prices and cost of quality. They could also be used to evaluate alternative expediting strategies. The methodology used could also form the basis for the development of models in which both the requirement date and the delivery date are random variables. It is the hope of the author that the models developed in this dissertation may be utilized in future research in this area.

My graduate work and research was made possible by the financial support of the National Science Foundation through its Graduate Traineeship Program. Additional support was received from the University Development Foundation and the Department of Industrial Engineering and Management of Oklahoma State University. I am grateful to each of these organizations for their assistance in my graduate study and research.

Several individuals have contributed greatly to the successful completion of my graduate program. Professor James E. Shamblin has provided advice and encouragement as Chairman of my Doctoral Committee and deserves special thanks for suggesting the topic of this dissertation. His guidance during the research was particularly helpful and is reflected in many ways in this treatise. I would like to express my sincere gratitude to Dr. Shamblin for his inspiration and assistance. I am also deeply indebted to Professor Wilson J. Bentley for his inspiration and advice during my graduate work. I am especially grateful for the opportunity to teach which was provided by Professor Bentley during my tenure as a graduate assistant and for the many other ways in which he enriched my years of graduate study. Special thanks are also due the other members of my doctoral committee, Professors G. T. Stevens, Jr. and David E. Bee for their advice and encouragement. I am very grateful to each of these men for their guidance and counsel during my graduate program. The contribution of these men in the classroom and the academic profession is substantial, but an even greater contribution lies in the inspiration they provide to their many students and associates. It has been a great pleasure and privilege to have worked under them.

I would also like to thank the staff of the Engineering Computing Laboratory for their cooperation in the use of the laboratory and the IBM 1620 computer during the testing of the computer programs, the sensitivity analysis of parameters, and the calculation of decision tables. Special credit is due Mrs. Frank Roberts for her excellent typing and the preparation of this work in its final form.

Lastly, I would like to express my deep gratitude to my wife, Judy, for her encouragement and assistance throughout the years of graduate study. Many sacrifices were required of my wife and my daughters, Stacy and Courtney, and I appreciate their willingness to make them. I would also like to thank my parents, Mr. and Mrs. Leon Stout, for instilling in me the value of education and for their sacrifice in making that education possible, and my wife's parents, Mr. and Mrs. Harold Stump, for their assistance during my graduate study.

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CHAPTER I

INTRODUCTION

The objective of this research is to develop a new tool to aid in arriving at an optimal decision in specifying the delivery date for large critical components or subsystems in a large job shop procurement situation. It involves a new approach for dealing with lateness costs in a single-stage inventory model with a fixed order size and probabilistic delivery date.

The Problem

The problem is to determine the optimal safety time allowance, or buffer, between the requirement date for the item being ordered and the delivery date when the delivery date is uncertain. The typical procurement situation in which this problem arises is when a one-time order is being placed for an important subsystem or group of components in a large manufacturing job shop or for a construction project. These subsystems are usually very expensive and/or large, and very high holding costs are incurred while they are in inventory. But these subsystems are also very critical to the project; substantial expediting, rescheduling and other lateness costs are incurred if delivery is made behind schedule. Because many factors may arise which may prevent on-time delivery, a buffer period is generally allowed between the date when the part is required and the delivery date specified to the subcontractor.

A short buffer period would tend to minimize holding costs, but would result in high expediting and lateness costs. A long buffer would minimize lateness costs but would often result in high holding costs. What should be the proper buffer length to minimize the total variable cost of procurement, composed of inventory carrying costs and lateness costs?

Examples of the items under consideration might be a special purpose computer, a generator with unique specifications, the air conditioning equipment for a large building under construction, the leading edge of a wing for an experimental aircraft or any made-to-order item. In addition to specially built subsystems, the model to be developed can be used for any situation in which a single order is being placed to procure an expensive critical item or fixed number of items with a probabilistic lead time.

Background of the Research

It should be noted that this research was stimulated by a real world problem discussed with Dr. James E. Shamblin, Associate Professor of Industrial Engineering and Management at Oklahoma State University on a consulting visit with a large aircraft manufacturer. One of the company's problems was that large amounts of capital were being tied up in inventory. For any given project, payment for procured items had to be made upon delivery, while income was delayed until work was completed. If f(t) represents expenditures for inventory items and g(t) represents income, the investment in materials and inventory for any given project might be represented by the shaded area in Figure 1. When the size of the shaded area is large, this means large amounts of capital are invested over extended periods of time. Although there must always be some "work in process" inventory, at first glance it would appear desirable to attempt to minimize this investment because capital is a scarce resource that generally must be rationed among the multitudinous needs for capital within a large firm.

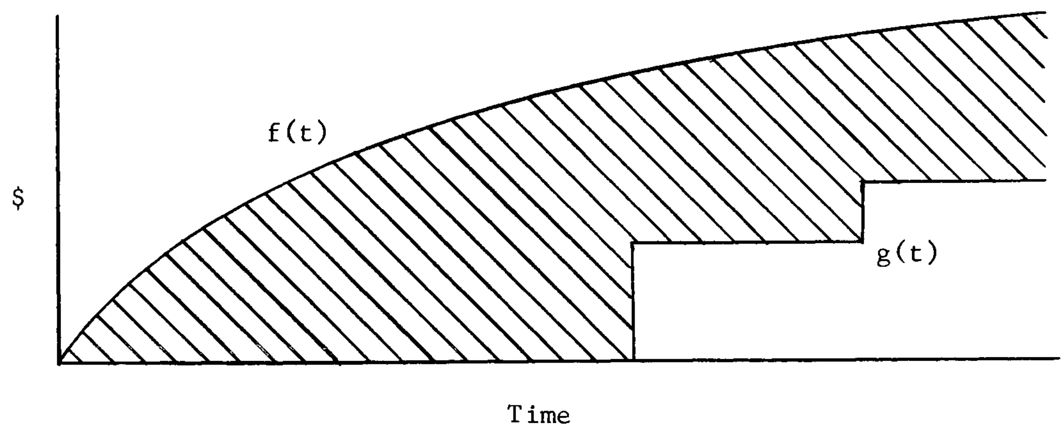


Figure 1. Typical Relationship Between Expenditures f(t) and Income g(t) for a Large Job Shop Project

One practice resulting in an extended lag between the rises in f(t) and g(t) was the practice of ordering large items or subsystems with a large "safety time" allowance for delivery. Because of this practice, large items were usually arriving two months or more before being needed. By cutting this safety time allowance, the shaded area could be reduced considerably. But complete elimination of the safety time allowance on these items might result in suboptimization for the overall project because some items would arrive later than expected. The resulting delays and rescheduling of work could incur far greater expense than the costs of the static capital investment in inventory.

This problem of finding an optimal safety allowance was then presented by Dr. Shamblin in a graduate course in inventory theory at Oklahoma State University. No papers have previously been written on this problem in the literature, as will be discussed shortly. In fact, most of the literature to date has dealt with models of retail distributors; and Iglehart (1967) states in a recent survey of inventory theory that one "large area that could stand more work is that of inventory installations run in conjunction with production facilities."[1]

The problem defined earlier occurs frequently in any large job shop, and similar situations occur often in large construction projects where the lead time on certain items is not constant. Common practice under these conditions is to add a safety time allowance, determined by intuition, to the lead time in placing the order. Because of the furor and problems caused when a critical part arrives late, it is probably the case that most safety time allowances are much larger than optimal in order to assure an "on-time" delivery and, thus, preclude procurement personnel from unnecessary embarrassment. An alternative procedure might be to use the longest possible lead time as the expected lead time. This procurement procedure should certainly result in on-time delivery in most instances, but in so doing the unnecessarily large safety time allowances used would tend to inflate costs by tying up large amounts of capital and storage space. In the light of increasing competition in industry it is increasingly important for firms to utilize improved decision aids and cost models of the procurement process in order to improve their procurement policies and lower costs. Even in situations involving cost-plus and other large federal contracts, the federal government is reducing contract prices when costly inefficiencies are found.[2] The model to be developed should be of considerable value in improving procurement decisions related to the problem of determining the optimal buffer period when the delivery date is probabilistic.

Review of the Literature

At this point it would be helpful to review the literature in the field of procurement and inventory theory to point out the relationships between the research in this dissertation and the work previously done. It would be neither appropriate nor possible to attempt a complete review here. As Veinott (1966) states in his survey article concerning the status of mathematical inventory theory, "It is naturally impossible to summarize the enormous literature on inventory models."[3] However, a definition of the major areas within inventory theory is necessary so that the reader will have a perspective of the relationship that this research has to the broader field of procurement and inventory theory. Also, since this research is characterized by a new approach to dealing with problems of probabilistic lead time, work in this field will be reviewed to illuminate the differences in approach.

In analyzing the developments in mathematical inventory theory it will be most helpful to refer to the classification of models as used by Starr and Miller (1962). The first distinction made is between static (single-stage) and dynamic (multi-stage) problems. "The distinguishing characteristic of static inventory problems is that only one order is possible."[4] "The defining characteristic of dynamic inventory problems is that more than one order is possible."[5] Within each of these classes, further classification is utilized for models of situations involving certainty, risk (where the random variables can be described with known probability density functions), and uncertainty (where the distributions of the random variables are unknown). Within this initial six class breakdown further classifications are introduced as they appear necessary. Most other authors surveying the inventory field also utilize the static-dynamic dichotomy to classify inventory models, although some of the leading works in the field denote the single order class of models as one-stage models as is done in the well-known work of Arrow, Karlin and Scarf (1958).

The great majority of effort in the field of inventory theory has been directed towards development of dynamic or multi-stage models. For many years following the pioneering article of Arrow, Harris and Marshak (1951), most of the effort was directed toward single product, single installation models. Examples of this type are the well-known economic lot size models and the (S, s) models in which "... if upon review it is discovered that stock on hand plus that on order has fallen to the level $x \leq s$, then the amount $S-x$ is ordered."[6] According to Iglehart (1967), the theory for single product, single installation models is reasonably complete; and since about 1963, "... the dominant theme in inventory research has been the concern with multiproduct and multi-installation models."[7]

The particular problem addressed in this dissertation is of the single-stage or static type since only one order is to be placed for the item or items under consideration. Within this framework, two variables can take on a probability distribution: the demand and the lead time. In an exhaustive search of the literature several single-stage models involving probabilistic demand were found. Typical examples are the Christmas tree problem (Starr and Miller, 1962) and the newsboy problem (Hanssmann, 1962).

Regarding the problem approached in this dissertation, no models dealing with a single-stage, variable lead time model were found although Starr and Miller (1962) go so far as to specifically define the problem within their classification framework. They comment that "If there is a probability distribution for the time lag, we are really dealing with an example of the static inventory problem under risk."[8] A specific model for this class of problem is not developed although they do develop a dynamic model involving a probabilistic lead time. The model they develop deals with the same type of items as the model of this dissertation: "frequently of the sort that are made to order and, hence, often on order. Typically such items are very expensive and, furthermore, the out-of-stock cost on such an item is likely to be very high."[9] The important distinction between the dynamic model under risk of Starr and Miller (1962) and the static model of this dissertation is that the dynamic model assumes the item under consideration has a demand distribution lasting over several periods of time, i.e., that the item will be needed over and over again with a specific probability distribution describing the demand. The model developed in this dissertation assumes a one-time need for the item or group of items under consideration, and a single order is to be placed for these. The model to be developed determines the optimal safety time allowance to minimize costs of lateness and carrying charges. The multi-stage model of Starr and Miller (1962) utilizes queueing theory to find an optimal "reserve" or safety stock of items which minimizes the sum of carrying charges and out-of-stock costs. Other authors that have dealt with the problem of low volume items with probabilistic demand over a constant lead time include Arrow, Harris and Marschak (1951), Heyvaert and Hunt (1956), and Whitin and Youngs (1955). However, no work was found pertaining to the single-order procurement situation with variable lead time.

One article has appeared which dealt specifically with inventory control in a job shop (Walls, 1966). However, this article described a computerized materials management system and did not address the problem approached in this research.

Perhaps the most distinguishing characteristic of the problem under consideration is its assumption of a probabilistic delivery date. A survey of the literature produced very little analytical work dealing with the problem of a probabilistic lead time or delivery date. The only model of this type found within the texts in inventory theory available to the author was the dynamic model just discussed from Starr and Miller (1962). An intensive review was also made of papers published in Management Science, Operations Research, and Production and Inventory Management, and any other papers dealing with lead time that came to the attention of the researcher through bibliography listings. One paper found which dealt with a variable lead time was by Fukuda (1964) in which optimal ordering policies were determined for a dynamic inventory problem where three different lead times could be purchased at different prices through different modes of transportation, the least expensive requiring the longest time for delivery.[10] A dynamic programming approach was used to determine the optimum policy as a function of the cost parameters, future demand, and stock on hand. Earlier papers by Barankin (1961) and others developed dynamic models in which a constant lead time was assumed but with the possibility of an immediate, emergency delivery at a premium cost. In each of these papers the problems approached differ from the problem of this dissertation in that they did not consider lead time as a random variable. Other differences were that they involved dynamic inventory situations and stock levels rather than a single-order job shop situation which is analyzed in this dissertation.

Relationship to Scheduling Algorithms

During the research and preparation of this dissertation the author interviewed several individuals in operations research positions in a wide variety of United States companies concerning the problem approached in this dissertation. One question that often arose concerned the relationship of the model to be developed to scheduling algorithms such as PERT. In the progress of a large job shop project, a PERT model would generally be used to develop the overall project schedule and, in so doing, establish the requirement dates for large parts or subsystems which are to be procured from outside the organization. The procurement model to be developed would then be utilized to determine the proper buffer period between the requirement date and the delivery date to be specified to the vendor or subcontractor. Thus, PERT would establish requirement dates; and then the buffer time calculated with the models developed in this dissertation would be used in determining the delivery date to be specified to vendors.

Summary of Analytical Approach

The intensive review of the literature in the area of procurement and inventory theory resulted in a conclusion that the procurement problem involving a single order for expensive, critical items with a probabilistic lead time or delivery date has not yet been approached from an analytical standpoint. In this dissertation the problem will be approached by finding an optimal safety time allowance, or buffer, which will minimize the expected variable cost of procurement. This variable cost of procurement is to have three components: the cost of inventory value, the cost of storage space, and the cost of lateness. Delivery date will be defined as a random variable with a probability density function. The expected values of each component cost will be found analytically, and the sum of these will give the expected total variable cost of procurement for a given buffer. Either differentiation or a Fibonacci search procedure will be utilized to find that buffer for which the expected total variable cost is a minimum.

The development will begin in Chapter II with definition of the component costs and parameters to represent important variables. Chapters III, IV, and V will develop specific models to deal with the problem for each of the following distributions of delivery date: uniform, chi-square, and Poisson. The sensitivity of the buffer to different parameters will be discussed in Chapter VI, and techniques are presented in Chapter VII which may be helpful to the user in the implementation of the model in practical use. Results will then be summarized in the concluding chapter.

Contributions of Research to Inventory Field

In addition to providing a model to aid in procurement decisions of the type discussed above, this dissertation utilizes a new approach to solving inventory problems which may be helpful to others doing research in the procurement area. Rather than relating cost components to stock levels or economic lot sizes, all component costs are expressed as functions of time. Also, the definition of lateness costs as a continuous, increasing function of time is a new approach that may be utilized in further research.

In addition to these contributions, the research of this dissertation lays the groundwork for further analytical developments which would be of considerable significance. This model determines the total variable cost of procurement as a function of the variance in a vendor's delivery date. Analytical definition of this cost will allow development of a vendor rating system which can reduce each of the important factors in procurement to a dollars and cents ratio. Previously vendor rating systems have been able to quantify differences in bid price and costs of quality but have been inadequate in evaluating costs stemming from differences in on-time delivery capability. The model to be developed should enable materials management people to develop a straightforward vendor rating system which can compare vendors on a strictly quantitative basis in all three important areas.

Other possible extensions of this research include application of the model to evaluating alternative expediting procedures and to the enumeration of specific incentives to be written into construction contracts for on-time completion.

FOOTNOTES

[1]Donald L. Iglehart, "Recent Results in Inventory Theory," The Journal of Industrial Engineering, Vol. XVIII (1967) p. 51.

[2]"Navy and Pratt & Whitney Agree to Make Compromise in Row Over F-111 Engines," Business Week, June 15, 1968, p. 48.

[3]Arthur F. Veinott, Jr., "The Status of Mathematical Inventory Theory," Management Science, Vol. 12 (1966), p. 745.

[4]Martin K. Starr and David W. Miller, Inventory Control: Theory and Practice (Englewood Cliffs, N. J., 1962), p. 19.

[5]Ibid., p. 78.

[6]Harvey M. Wagner, Statistical Management of Inventory Systems O. R. S. A. Publications in Operations Research, No. 6 (New York, 1962), p. 31.

[7]Iglehart, p. 48.

[8]Starr and Miller, p. 20.

[9]Ibid., pp. 146-7.

[10]Yoichiro Fukuda, "Optimal Policies for the Inventory Problem With Negotiable Leadtime," Management Science, Vol. 10 (1964), pp. 690-708.

CHAPTER II

DEVELOPMENT OF GENERAL MODEL

A job shop production system is generally used to manufacture custom-built items or low-demand items in small lot sizes. In the job shop manufacturing facility a large volume, both dollar value and number of components, must be fabricated or otherwise collected and assembled into a relatively small number of finished products. Components that are common to many of the finished products can be economically procured with the aid of models which determine the economic order quantity or other models as discussed in Chapter I. This chapter will outline the development of a model to aid in procurement of the components which are needed only for the manufacture of a given finished product and, thus, must be procured specifically for that particular production run. The development will begin with the definition of the relevant points in the procurement of such components.

Definition of the Procurement Process

The procurement process will be defined as the procedures required to provide necessary material when needed. Certain dates are of significance in the procurement process, such as the requirement date, availability date, delivery date, and order date. These "milestones" in the procurement process and the important variables of lead time and buffer time will be defined in this section and are graphically represented in Figure 2.

Requirement Date

As soon as possible after the decision is made to produce a given finished product, a schedule for the manufacture and assembly of the product is established which will allow for the completion of the product by the desired date. The completion date is usually fixed by a contractual obligation, and the production schedule which allows for completion on this date is typically very tight and may leave little or no time allowance for delays in delivery of components or other delays in manufacturing.

Whether the actual scheduling process is accomplished through PERT or some other scheduling algorithm, the resulting production schedule establishes the requirement date for components or subsystems which must be procured from outside the organization. For this procurement model, the requirement date is defined as that date when a component is needed in order to maintain the production schedule. If the component is not available on the requirement date, then the production schedule is interrupted and a new schedule must be established and/or additional resources must be expended in order to bring the project back on the original schedule.

Availability Date

The availability date for this model is the date at which the procured component has been received, inspected and is ready for the use prescribed for it in the production schedule.

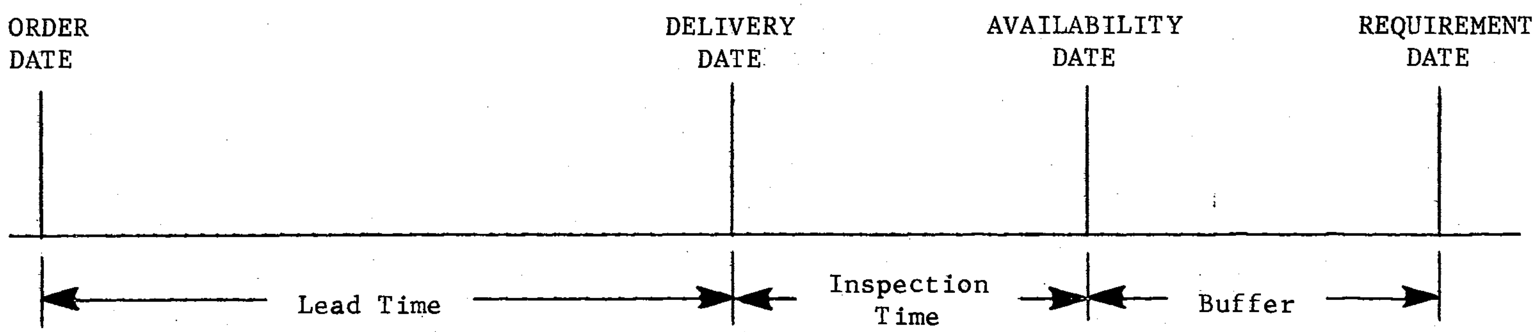


Figure 2. Milestones in the Procurement Process

Delivery Date

The delivery date for this model is the date at which the component is received from the vendor or subcontractor. In order to simplify the development of the model, it will be assumed that delivery and availability occur on the same day as is indicated in Figure 3. This will likely be the case unless lengthy inspection procedures are required. If such procedures are likely to cause availability date to follow delivery date by a certain number of days, then this inspection time should be added to the optimal buffer in specifying the delivery date for the component.

Order Date

The order date is the date when the order is placed for the component being procured, and in this model it should precede the delivery date by the number of days in the expected lead time. In cases when the lead time is known to vary by only one or two days, the maximum lead time may be used to determine the order date since the small average increase in holding costs incurred by this practice would probably be less than the costs of calculations needed to determine the optimal buffer.

In some cases the lead time is not used to specify an order date, especially in the procurement of complex subsystems and made-to-order items. Here the order is generally placed as soon as the requirements are known in order to allow as much lead time as possible. Here the decision is not "when to place the order" but rather "when should delivery be specified." The order is an essential part of the procurement process, but its chief importance in this model concerns its effect on the delivery date as the uncertainty in delivery date is of primary

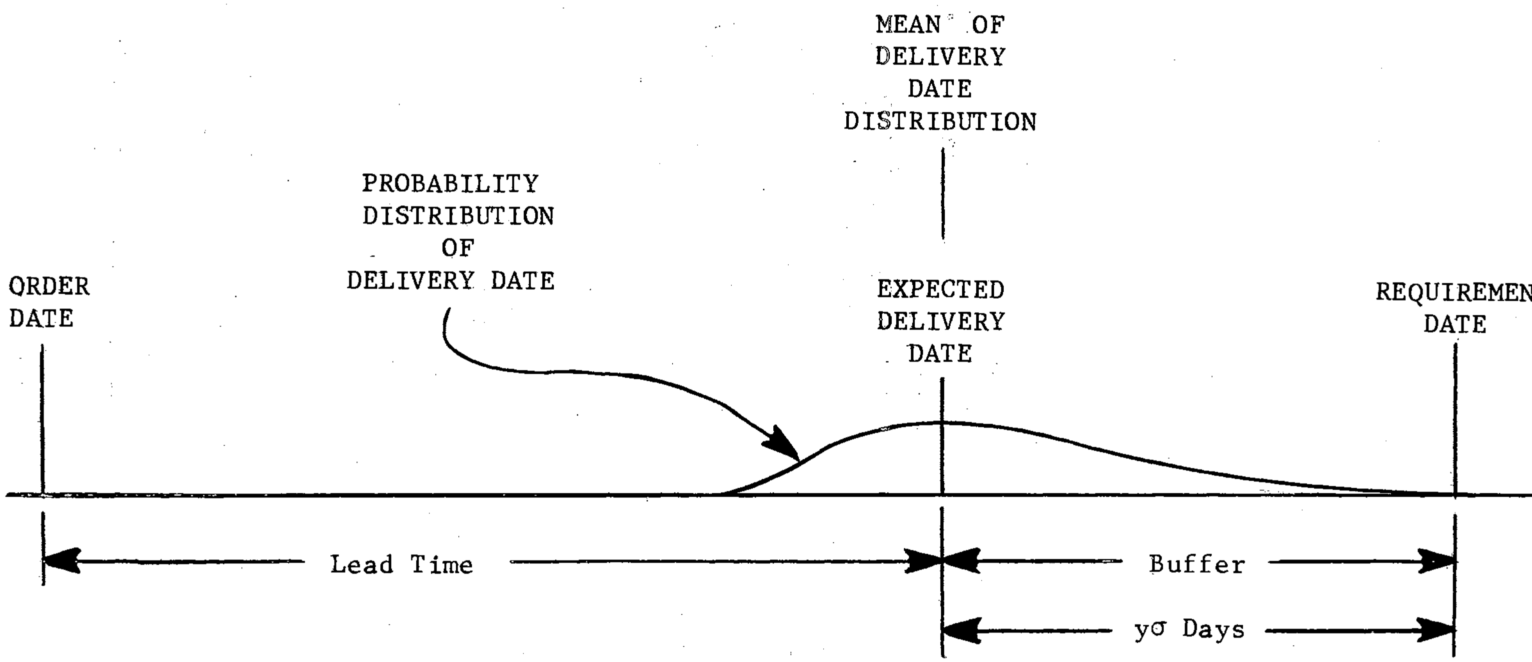


Figure 3. The Procurement Process for Zero Inspection Time and Probabilistic Delivery Date

interest.

Uncertainty in Lead Time or Delivery Date

The lead time is defined as that time between the order and delivery of the component. In many procurement situations the lead time can be considered a constant. When "off-the-shelf items" with no transportation problems are being procured or when components are being procured from a vendor with a near perfect delivery record, lead time and delivery date should be considered a constant and no buffer time is needed.

However, delivery date often varies considerably for a variety of reasons. Delays in transportation often contribute to uncertainty in delivery, and for the type of components under consideration the vendors or subcontractors themselves are often unable to meet delivery schedules. The vendors supplying these expensive or made-to-order parts typically manufacture them on rather tight production schedules of their own. Production delays and bottlenecks incurred by the vendor result in missed delivery dates. In other cases the vendor might have few projects in progress and desire to complete work on a component ahead of schedule in order to keep his facilities in operation, to free machines for possible new contracts, or for other reasons. In some cases, schedules are disrupted by engineering changes which are made while work is in progress. These and other conditions within the vendor's own production facilities can easily result in completion of the component either before or after his scheduled completion date with resulting changes in the delivery date to the prime contractor.

Furthermore, most of the conditions which result in variance in delivery can be considered random in nature. For example, a rail shipment may be "lost" for a period of days before it is missed and expedited. In other instances it may take less time than expected. A vendor may receive a larger number of orders than anticipated during a given period causing missed production schedules. Or an unanticipated cancellation of work may enable the vendor to complete a component ahead of schedule. Engineering changes during production may or may not require extensive rework and liaison with the prime contractor. Any of these may arise without prior notice and affect the delivery date of the component. In fact, so many situations may arise that it would seem on-time delivery more the exception than the rule when procurement of expensive subsystems or made-to-order components is being considered. Whether the situations are felt as acting on lead time or delivery date, the result of both is an uncertain delivery date. Thus, in this model the prime focus will be on the effects of variance in delivery date.

Buffer Time

Because of this general uncertainty in the delivery dates of the parts under consideration, most prime contractors utilize a safety time allowance between the requirement date and the availability date which they regard as desirable. This safety time allowance will be referred to as the buffer. Since availability date is assumed to be the same as delivery date, this model will calculate the optimal buffer between delivery date and requirement date that will minimize the total expected variable cost of procurement. This relationship is illustrated in Figure 3 with the expected delivery date preceding the requirement date by the length of the buffer. As was discussed previously, any difference between delivery and availability dates should be added to the optimal buffer time in specifying the delivery date with respect to the requirement date.

Some general comments should be made concerning the expected behavior of a model finding the optimal buffer time. It is logical that the optimal buffer time should increase as the uncertainty of the delivery date increases. It should also increase as the lateness costs that will be incurred for late delivery increase. The buffer should decrease as the components under consideration incur higher levels of holding costs, i.e., as they become more expensive or require more storage space. The buffer calculated with the aid of any model should agree with these logical considerations.

The buffer in this model will be expressed mathematically as "$y\sigma$" as shown in Figure 3 where

y = a mathematical variable taking on positive real numbers

σ = the standard deviation of the delivery date random variable which will be discussed in the next section.

The buffer is thus a function of the standard deviation of the delivery date distribution. The standard deviation of a distribution increases with the square root of the variance of the distribution and is thus proportional to the uncertainty of the random variable. The use of standard deviation as a measure of dispersion is quite common, e.g., the use of three-sigma control limits in quality control. In this model as the delivery date becomes less certain, the standard deviation increases; and the buffer time also increases for a given component. Thus, the expression of buffer as "$y\sigma$" gives this buffer the desired capacity to vary with the uncertainty of delivery date.

But how many standard deviations should be used in determining the optimal buffer? One would expect this to depend upon the relative magnitudes of holding and lateness costs. The positive number "y" will be found as a function of the cost parameters of the model. The buffer will be a function of the uncertainty in delivery and the holding and lateness cost parameters. For probability distributions that will not allow solution for an analytical expression for y, a search procedure will be applied to find the optimal buffer for a given set of parameters.

Delivery Date as a Random Variable

Although it may be distasteful to some in the procurement field to discuss delivery date as a random experiment, this approach will be used as part of the development of delivery date as a random variable. A random experiment is described by Hogg and Craig (1965) as an experiment whose outcome cannot be predicted with certainty, but such that the collection of every possible outcome of the experiment can be described prior to its performance. In addition,

> If this kind of experiment can be repeated under the same sort of conditions, it is called a random experiment, and the collection of every possible outcome is called the experimental space or the sample space.[1]

The uncertainties inherent in specifying the delivery date for a component have just been discussed, and the range of possible delivery dates can generally be described as occurring within a defined range of dates or sample space. In one sense no two delivery dates will ever be influenced by "the same sort of conditions." However, at the time when orders are placed (and delivery dates specified), the particular set of conditions that will be influencing the vendor during the delivery date period are either unknown or at best known only in general terms, e.g., it may be known that the vendor will be unusually busy during that time. If these factors are unknown then each delivery date is being specified under the same sort of conditions, and the delivery date satisfies the definition of a random experiment. Even if limited information of a general nature is available, all delivery dates specified under a given general condition satisfy the requirements of a random experiment. For example, all delivery dates specified under knowledge that the vendor will be unusually busy are being specified under the same sort of conditions. The difference is that the sample space for this random experiment may differ from those random experiments made under other general conditions. Thus, the delivery date in our problem can be considered a random experiment.

The definition of a random variable is then based on a random experiment as follows.

> Suppose that the outcome of a random experiment can be expressed by a single number. Then the sample space A can be represented by a set of points on a directed line. If we denote the outcome by the symbol X, we call X a <u>random variable</u>.[2]

The outcome of the delivery date "experiment" is the time at which delivery actually occurs, and the sample space of this random experiment is the collection of all possible times at which delivery can occur. As this sample space can be represented by a set of points on a directed line, delivery date can be considered a random variable. The definition of delivery date as a random variable is important, for the probability of a random variable taking on its different possible outcomes can be described by a probability density function (p. d. f.).[3] Also, the mean and variance can be found for most cases where a p. d. f. is known.

The assumption of a p. d. f. to describe the occurrence of the delivery date will make it possible to find the expected values of cost components and to develop a model for the expected cost of procurement.

Definition of Total Variable Cost

In defining the total variable cost of procurement the total cost will first be defined, and the fixed components eliminated to give the total variable cost. The total cost involved in procuring a specific component (or single group of components) for a specific production requirement is made up of the item cost, order cost, holding costs, and lateness costs.

$$\text{Total Cost} = \text{Item} + \text{Order} + \text{Holding} + \text{Lateness.} \tag{2.1}$$

Two of these components may be considered as fixed elements of total cost in this analysis. Item cost is fixed because the basic cost of the components is constant regardless of the delivery date specified. The order cost is fixed also because one and only one order must be placed for the components.

Elimination of the fixed costs in the procurement process will allow the development to concentrate on the costs which can be varied by changes in the buffer which result in different delivery dates. The total variable cost will be composed of holding costs and lateness costs. The traditional holding cost will be broken into two components related to the value of the component and to its storage space requirements. This approach to holding costs, developed by Shamblin and Ferguson (1966), is particularly useful since the traditional definition of holding costs as a function of either space or value would not be valid for many of the items under consideration. For the unique, specially-built items for which the model will be applied, little correlation may exist between size and value. For example, items such as compact electronic gear with high cost and comparatively inexpensive, bulky fuel tanks require such a breakdown of holding costs. Lateness costs are those costs incurred because of late delivery or anticipated late delivery of an item. Some lateness costs are incurred as the result of various expediting procedures taken when the part appears to be arriving late. Other lateness costs may be incurred by production delays, rescheduling and penalties for late completion of the project. These can be expressed most conveniently along with expediting costs in a single cost component. The total variable cost of procurement will be defined as the sum of the inventory value cost, the storage space cost, and the lateness cost.

Total Variable Cost = Inventory Value + Storage Space + Lateness. (2.2)

Although the researcher independently arrived at the need for concentrating analysis on only the holding costs and lateness costs, a search of the literature found that other analysts had used similar approaches when dealing with low-volume items. Heyvaert and Hunt (1956) minimized a total cost function composed of storing costs and costs of non-satisfaction. The storing cost was the "... total of all costs engaged to keep one item in store for a time t."[4] The cost of non-satisfaction was the "... total of all costs resulting from the non-satisfaction of a customer's order."[5] Whitin and Youngs (1955) also neglected the traditional ordering cost in their development.

> The following note is concerned with establishing an inventory control policy for items with extremely low demand. In the event that the expected savings in ordering cost that would result from buying in lots is less than the concomitant increase in carrying charges, it is uneconomical to use a lot size formula. In this

> event it is appropriate to use a system of placing orders as units are demanded.[6]

Whitin and Youngs proceed to develop an expression for a desired reserve of stock by assuming a constant lead time, a Poisson demand for units over the lead time, and minimizing the sum of holding costs and stock-out costs. Although the problem under consideration is different and the mathematical expressions for cost components in each model could not be applied to the other, it is interesting to note the similarity of approaches to definition of total variable cost.

Definition of Cost Components and Parameters

Specific definition of each component of total variable cost in terms of industrial parameters follows. The expected values of these components will be found for the inventory value cost and the storage space cost. The procedure for determining the expected lateness cost will be outlined, and the expected value of this component will be found for three different distributions of delivery date in the succeeding chapters.

Inventory Value Cost

This is the cost of inventory on hand due to tied-up capital, taxes, insurance, and other charges associated with inventory value. The variable portion of this cost is that which is incurred between the delivery date and the requirement date while the item is being stored awaiting use. Since the delivery date is a random variable with a p. d. f., its expected value is the mean of the distribution. Because the mean is positioned $y\sigma$ days before the requirement date, the expected time of delivery is $y\sigma$ days before the requirement date. Therefore, the expected time the component will be in inventory is $y\sigma$ days.

Inventory value cost is a function of the component's value, the expected time in inventory, and a constant representing the cost of capital, taxes, insurance, and other charges that are proportional to the value of inventory. The expected value of inventory value cost can be expressed mathematically as

$$\text{Inventory Value Cost} = V\left[\frac{P}{365}\right] y\sigma \tag{2.3}$$

where

V = the value of the component in dollars.

P = a decimal representing the company's annual cost of capital, the annual tax rate and insurance rate per dollar of inventory, and any other charges that can be expressed as a fraction of the value of inventory.

$y\sigma$ = the expected number of days the component will be in inventory.

<u>Storage Space Cost</u>

This is the cost of providing storage space for the part under consideration. This cost component may or may not be large with respect to the other two components of variable cost depending upon the size of the item and the quality of storage space required. If high cost storage space such as a sterile, dust-free environment is required, the storage costs may be the largest component of total variable cost. The expected cost of storage space for a given delivery date can be determined in a manner similar to the inventory value cost component.

Storage space cost is a function of the size of the component, the cost per unit of time for the space required and the expected time in storage. The expected value of storage space cost will be expressed as

$$\text{Storage Space Cost} = W\left[\frac{C_h}{365}\right] y\sigma \tag{2.4}$$

where

W = the number of units of storage space required by the component.

C_h = the cost per year of providing and maintaining one unit of storage space.

$y\sigma$ = the expected number of days the component will be in inventory.

<u>Lateness Costs</u>

This topic will be covered in greater detail since no analytical work of this nature regarding lateness costs has been published previously. Lateness costs are incurred due to a particular status of a component in the procurement cycle. If a component does not arrive by a certain time, communications with the vendor are initiated at extra cost to determine the status of parts on order. If it appears special transportation and/or handling are needed to assure delivery by the requirement date, these costs are incurred as a means of expediting parts to avoid schedule disruptions. If components are not delivered by the requirement date, as sometimes will be the case, costly delays in production are incurred necessitating rescheduling of the project. These costs will depend primarily upon the amount of "slack" in the production schedule, the degree of urgency of need for the part and the penalties connected with late completion. If the part is so critical to the project schedule that work must stop pending its delivery and large numbers of men are idled, the lateness costs can become of tremendous magnitude.

Because the lateness cost incurred for a given component depends upon the date that it is delivered, lateness costs should be considered as a function over time. An extensive literature search failed to reveal any treatment of lateness costs as a function of time, and this assumption marks a new approach to dealing with problems of lateness in the procurement process. The related problem of costs of obsolescence due to spoilage or cancellation of demand have been defined as functions of time by Grassi and Gradwohl (1959) and others. However, these analyses involve relating costs of obsolescence to economic order quantity and solution for an EOQ. Here the lateness costs, which increase with time, must be related to the length of the buffer in order to determine that optimum buffer for which the expected total variable cost is a minimum.

The form of the lateness cost function should depend upon how early in the procurement process expediting costs are incurred and upon how critical on-time delivery is to the production schedule. One would expect the expediting and procurement policies of various firms to be different. However, it will be assumed that added expediting costs are incurred if delivery is not made by a certain date. This date may either precede or follow the expected delivery date, and for convenience in construction of the model it will be considered as "$d\sigma$" days before the requirement date as shown in Figure 4. The assumption that expediting procedures will start and the first lateness costs will be incurred "$d\sigma$" days before the requirement date also agrees with the logic that firms would start expediting procedures earlier on items with large

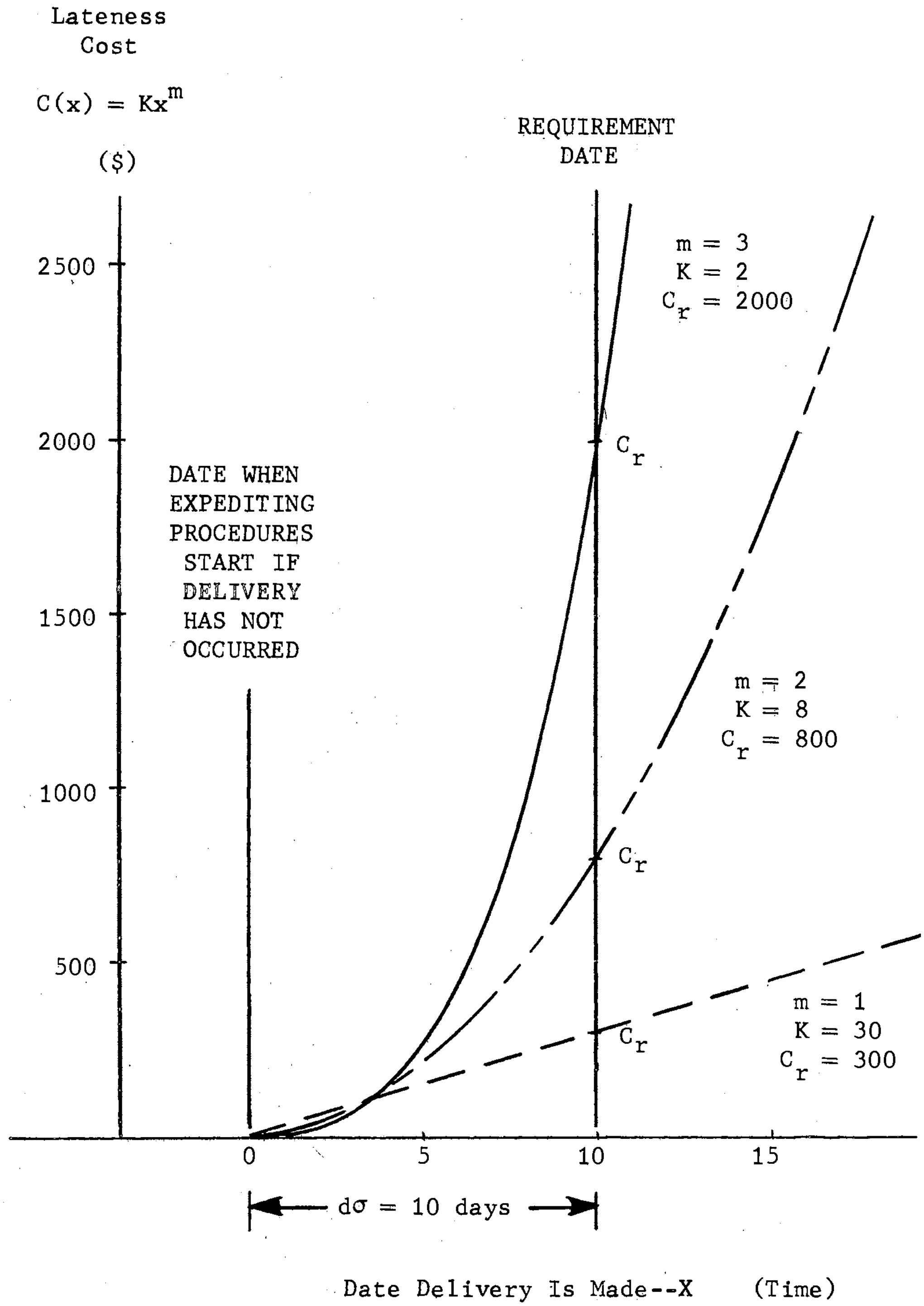


Figure 4. Lateness Cost C(x) as a Function of Delivery Date

uncertainty (σ) in delivery than on items with a more reliable delivery.

After the first delivery date on which lateness costs are incurred, delivery on subsequent dates would incur higher and higher lateness costs. Let C(x) represent the total lateness costs incurred by a part if it arrives at time x (any given delivery date). Thus, C(x) at any point x is the summation of all lateness costs accumulated up to and including time x.

$$C(x) = \int_0^x (\text{All lateness costs})\, dt. \qquad (2.5)$$

If costs of lateness were a constant A dollars per day beginning $d\sigma$ days before the requirement date at a point designated as zero, then

$$C(x) = \int_0^x A\,dt = At \Big|_0^x = Ax, \quad x \geq 0. \qquad (2.6)$$

If each day's lateness costs increased linearly at a rate B dollars per day above the previous day's costs, then

$$C(x) = \int_0^x Bt\ dt = \frac{Bt^2}{2} \Big|_0^x = \frac{B}{2} x^2, \quad x \geq 0. \qquad (2.7)$$

In general, the cumulative costs of lateness can be represented by a function C(x) that represents the costs of lateness incurred if a component is delivered on day x. The general form may be expressed as

$$\begin{aligned} C(x) &= Kx^m \text{ for } x \geq 0 \\ &= 0 \quad \text{for } x < 0. \end{aligned} \qquad (2.8)$$

where

K = a scaling constant in dollars per day

x = the delivery time

m = exponent determining the rate of increase of lateness costs with time; m is allowed to take on integer values 1, 2, and 3.

Note that the point $x = 0$ is defined to be $d\sigma$ days prior to the requirement date. The delivery date or time of delivery X has been defined to be a random variable, and in succeeding chapters different probability distributions will be assumed to describe the behavior of the random variable X. The expected value of $C(x)$ will then be found for each assumption of the delivery date distribution.

Some discussion of the parameters m and K may be helpful for those attempting to apply this model. These parameters determine the shape of the lateness cost function, and manipulation of K and m allows a great deal of flexibility in defining a lateness cost function to approximate the costs of a given procurement situation. A general comment concerning the shape of $C(x)$ is that it is composed of two basic parts: (a) the amount and timing of expediting costs incurred to assure delivery on or prior to the requirement date, and (b) the magnitudes of cost incurred if delivery is late. If there is slack in the production schedule, rescheduling is not of great expense and added costs are expected to be incurred at a linear rate, then a power of $m = 1$ should be used. If expediting costs which enable delivery prior to the requirement date are considerably lower than the costs of rescheduling and delays for late delivery, a power of $m = 2$ might be used. If the costs of lateness become very high when delivery is not made by the requirement date, a power of $m = 3$ would be more appropriate. The exponent m should be chosen such that the lateness cost function "fits" a plot of points of lateness cost incurred at different delivery dates. It must also give a good representation of the costs that will be incurred if delivery takes place following the requirement date. If enough data is available, regression analysis might be used to determine m and other parameters; however, a subjective evaluation will probably be required to determine the value for m. Cost functions for different values of m and K are plotted in Figure 4. The case of $m = 0$ is discussed in Appendix F.

The parameters d and K may be determined more readily. If the number of days prior to the requirement date that expediting procedures are begun is known, then d can be found by dividing σ into this number. Methods for determining σ will be discussed in subsequent chapters. In many cases K may easily be determined from knowing the dollar amount of lateness costs that will be incurred if delivery is made on the requirement date. This dollar amount is associated with a delivery date that is $d\sigma$ days after expediting was first instigated. Since any point on the lateness cost function $C(x)$ is expressed as Kx^m, for $x = d\sigma$ the dollar amount of lateness cost $C(d\sigma)$ would be $K(d\sigma)^m$. The parameter K can thus be expressed as

$$K = \frac{C_r}{(d\sigma)^m} \tag{2.9}$$

where K and m are defined as previously,

C_r = the dollar amount of expediting cost that will be incurred if delivery is not made until requirement date.

$d\sigma$ = the value of x at the requirement date.

The point on $C(x)$ that gives C_r is illustrated graphically in Figure 4.

If PERT is used to determine the overall project schedule, the costs of delays due to delivery date after the requirement date may be found through time-cost trade-off calculations. It would first be necessary to determine the added costs to complete in a shorter time the remaining activities following the event where the component being procured is required. Then, if the component arrives one day past the requirement date, one day will have to be "made up" from the remaining schedule of activities; and the added lateness cost is the time-cost trade-off for a savings of one day. If it is two days late, two days must be made up, etc. Data for these time-cost trade-off calculations are generally available if PERT is being used to schedule the project. This and the other methods discussed for evaluating the parameters are not meant as the only means; they are included as suggestions and to give the reader a feel for the real-world meaning of the parameters.

With the lateness cost function and p. d. f. of delivery date defined, the expected value of lateness cost can be determined. The expediting strategy has been defined such that the first lateness costs are incurred at time $t = 0$ which is $d\sigma$ days before the requirement date as in Figure 4. Now introduce a change of variable such that the requirement date becomes the origin.

$$C(x) = Kx^m \quad \text{for } x \geq 0$$

$$= 0 \quad \text{for } x < 0.$$

Let $\quad t = x - d\sigma \qquad \text{at } x = d\sigma,\ t = 0$

$\quad x = t + d\sigma \qquad \text{at } x = 0,\ t = -d\sigma.$

Then
$$C(t) = K(t + d\sigma)^m \quad \text{for } t \geq -d\sigma \tag{2.10}$$

$$= 0 \quad \text{for } t < -d\sigma.$$

This lateness cost function C(t) gives the cost incurred if a component is delivered at time t. The new origin is illustrated in Figure 5.

Next, assume that a p. d. f. for the delivery date distribution has been specified and adjusted such that its zero reference point is also the requirement date as shown in Figure 5. This can easily be accomplished by letting T be a function of X. Since a function of a random variable is itself a random variable, T is a random variable. The p. d. f. of the random variable T gives the probability that delivery will occur at any time t. The expected lateness cost can be found by multiplying the cost of lateness incurred if delivery is at time t times the probability that delivery occurs at time t and integrating this product over the sample space of t.

$$E(LC) = \int_T C(t) \cdot f(t) \ dt \tag{2.11}$$

where

C(t) = the lateness cost function

f(t) = the p. d. f. of the random variable delivery date.

Substituting C(t) from (2.10) and establishing the proper limits on the integrals gives

$$E(LC) = \int_{-\infty}^{-d\sigma} 0 \cdot f(t) \ dt + \int_{-d\sigma}^{\infty} K(t+d\sigma)^m \cdot f(t) \ dt. \tag{2.12}$$

Since the first part of this expression will be zero for any p. d. f. of delivery date, the expected lateness cost reduces to

$$E(LC) = \int_{-d\sigma}^{\infty} K(t+d\sigma)^m \cdot f(t) \ dt. \tag{2.13}$$

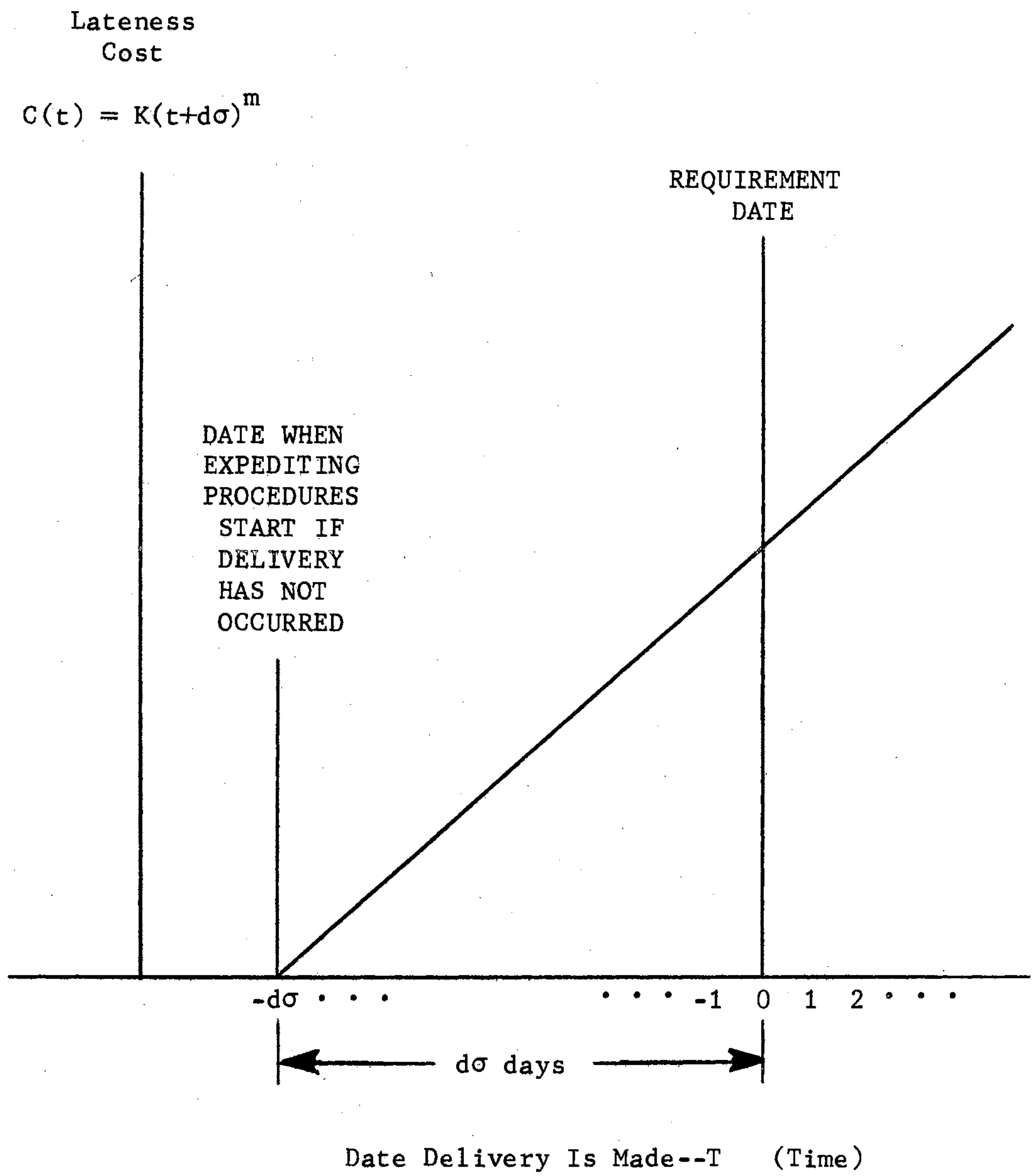


Figure 5. Lateness Cost C(t) With Origin at Requirement Date

Although this development has involved continuous cost functions and assumed a continuous p. d. f. for delivery date, an equivalent expression involving a discrete cost function can be developed for the case of a discrete p. d. f. for the delivery date.

Summary

To summarize this chapter, the expected value of total variable cost, hereafter referred to as TVC, can be found for any distribution of delivery date as

$$TVC = V\left(\frac{P}{365}\right)y\sigma + W\left(\frac{C_h}{365}\right)y\sigma + \int_{-d\sigma}^{\infty} K(t+d\sigma)^m f(t)\,dt \qquad (2.14)$$

where

V = the value of the component in dollars

P = a decimal fraction representing the annual cost of capital, taxes and insurance on inventory value, etc.

W = the number of storage space units required

C_h = the annual cost of one unit of storage space

$y\sigma$ = the expected number of days the component will be in storage prior to the requirement date (the buffer)

$d\sigma$ = the number of days prior to requirement date that expediting procedures begin if the component has not arrived

K = a scaling constant

m = the exponent of lateness cost

t = the time at which the component is delivered

$f(t)$ = the probability of delivery at time t.

In Chapters III, IV, and V, the uniform, chi-square and Poisson distributions respectively will be assumed for delivery date. The expected lateness cost will be derived for each case, and the total variable cost found in terms of the cost parameters. Methods for solving for an optimal buffer using the models developed will be outlined, and example problems formulated to illustrate the solution process.

The uniform, chi-square and Poisson distributions are well-known, and proofs concerning their p. d. f.'s may be found in Hogg and Craig (1965) or any other good text on mathematical statistics. For this reason the p. d. f.'s will be stated without detailed proofs as will be the formulae for their means and variances. Emphasis will be placed on the assumptions necessary to justify use of a particular probability distribution to describe the delivery date random variable.

FOOTNOTES

[1]Robert V. Hogg and Allen T. Craig, Introduction to Mathematical Statistics (New York, 1965), p. 1.

[2]Ibid., p. 13.

[3]Ibid., pp. 16-21.

[4]A. C. Heyvaert and A. Hunt, "Inventory Management of Slow-Moving Parts," Operations Research, Vol. 4 (1956), p. 573.

[5]Ibid.

[6]T. M. Whitin and J. W. T. Youngs, "A Method for Calculating Optimal Inventory Levels and Delivery Time," Naval Research and Logistics Quarterly, Vol. 2 (1955), p. 157.

CHAPTER III

UNIFORM DISTRIBUTION OF DELIVERY DATE

In this chapter a uniform distribution will be assumed to describe the probability of occurrence of the delivery date. The expected value of the lateness cost component of TVC will be derived. An optimal value of the decision variable y will be determined in terms of the cost parameters defined in Chapter II. The results will then be interpreted with the aid of a sample problem.

Assumption of Uniform Distribution

Suppose it is reasonable to assume that the random variable delivery date can take on any value within a certain range from a to b and that the probability of the delivery date occurring at any time within this interval is proportional to the length of the interval. In other words, the probability that the delivery date takes on a value of x within the interval (a, b) is uniform and becomes less as the length of (a, b) is enlarged. The length of (a, b) may be determined from past experience with the vendor under consideration or may be a subjective evaluation of the range of delivery deemed possible in the procurement of a particular component. The greater the uncertainty, the larger should be the interval (a, b).

If a uniform distribution of delivery date is assumed, then the distribution of delivery date could be graphically represented as in

Figure 6. Once the delivery date is specified to the vendor, this date becomes the expected value of the delivery date random variable. The real world interpretation of this statement is that once the delivery date is contractually specified, it is assumed that actual delivery will take place within the time span from a to b, where a and b are equally distant from the contracted delivery date.

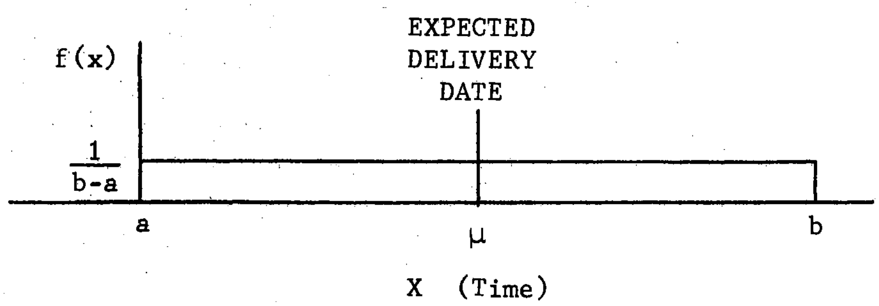


Figure 6. Uniform Distribution of Delivery Date

Under these assumptions the p. d. f. of the random variable delivery date may be written as developed in Hogg and Craig (1965).

$$f(x) = \frac{1}{b-a} \qquad \text{for } a \le x \le b \tag{3.1}$$

$$= 0 \qquad \text{elsewhere.}$$

The mean and standard deviation of this distribution are

$$\mu = \frac{b-a}{2} \text{ and } \sigma = \frac{b-a}{\sqrt{12}} \,. \tag{3.2}$$

Thus, both the mean and the standard deviation are proportional to the length of the interval (a, b) in which the actual delivery date is expected to occur with uniform probability. It is also of interest to note that the length of (a, b) is 2μ and that μ is located at the midpoint of (a, b) due to the symmetry of the distribution.

If $a = 0$, then $b = 2\mu$ where b is now the length of the interval in which delivery is expected to occur. For $a = 0$, equations (3.1) and (3.2) may be written

$$f(x) = \frac{1}{b} \quad \text{for} \quad 0 \leq x \leq 2\mu \tag{3.3}$$

$$\mu = \frac{b}{2} \quad \text{and} \quad \sigma = \frac{b}{\sqrt{12}} \tag{3.4}$$

The buffer ($y\sigma$) has been defined as the time between the expected delivery date and the requirement date. The relationships between the interval (0, b), the expected delivery date μ, the buffer $y\sigma$, and the requirement date can be expressed graphically as in Figure 7.

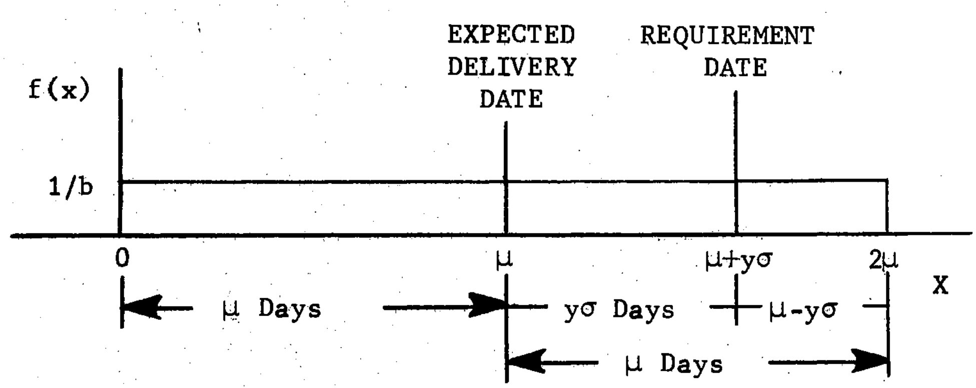


Figure 7. Procurement Milestones and the Uniform Delivery Date Distribution

Development of Expected Lateness Cost

The first step in finding the expected lateness cost component of TVC is to develop expressions for the probability of delivery and cost of delivery in terms of the random variable T that has a value of zero at the requirement date. This will permit multiplication of a lateness cost at time t by the probability of its being incurred, and integration over all possible delivery dates will give the expected value of lateness cost.

First a change of variable will be introduced to move the arbitrary zero reference point on the p. d. f. of delivery date to coincide with the requirement date. Let

$$
\begin{array}{llll}
t = x-(\mu+y\sigma) & \text{at} \quad x = 0, & t = -\mu-y\sigma \\
x = t+\mu+y\sigma & \phantom{\text{at}} \quad x = \mu, & t = -y\sigma \\
dx = dt & \phantom{\text{at}} \quad x = \mu+y\sigma & t = 0 \\
 & \phantom{\text{at}} \quad x = 2\mu & t = \mu-y\sigma
\end{array}
$$

and
$$
\begin{aligned}
f(t) &= \frac{1}{b} && \text{for } -\mu-y\sigma \le t \le \mu - y\sigma \\
&= 0 && \text{elsewhere.}
\end{aligned}
\tag{3.5}
$$

The effect of this change of variable operation is merely to "shift" the origin as is illustrated in Figure 8.

The lateness cost function C(t) has also been defined with the arbitrary zero reference point coinciding with the requirement date in (2.10). Both the probability that delivery date occurs at time t and the lateness costs incurred for delivery at time t have now been defined in terms of functions of the same random variable T with their origins at the requirement date. The compatible equations (2.10) for C(t) and

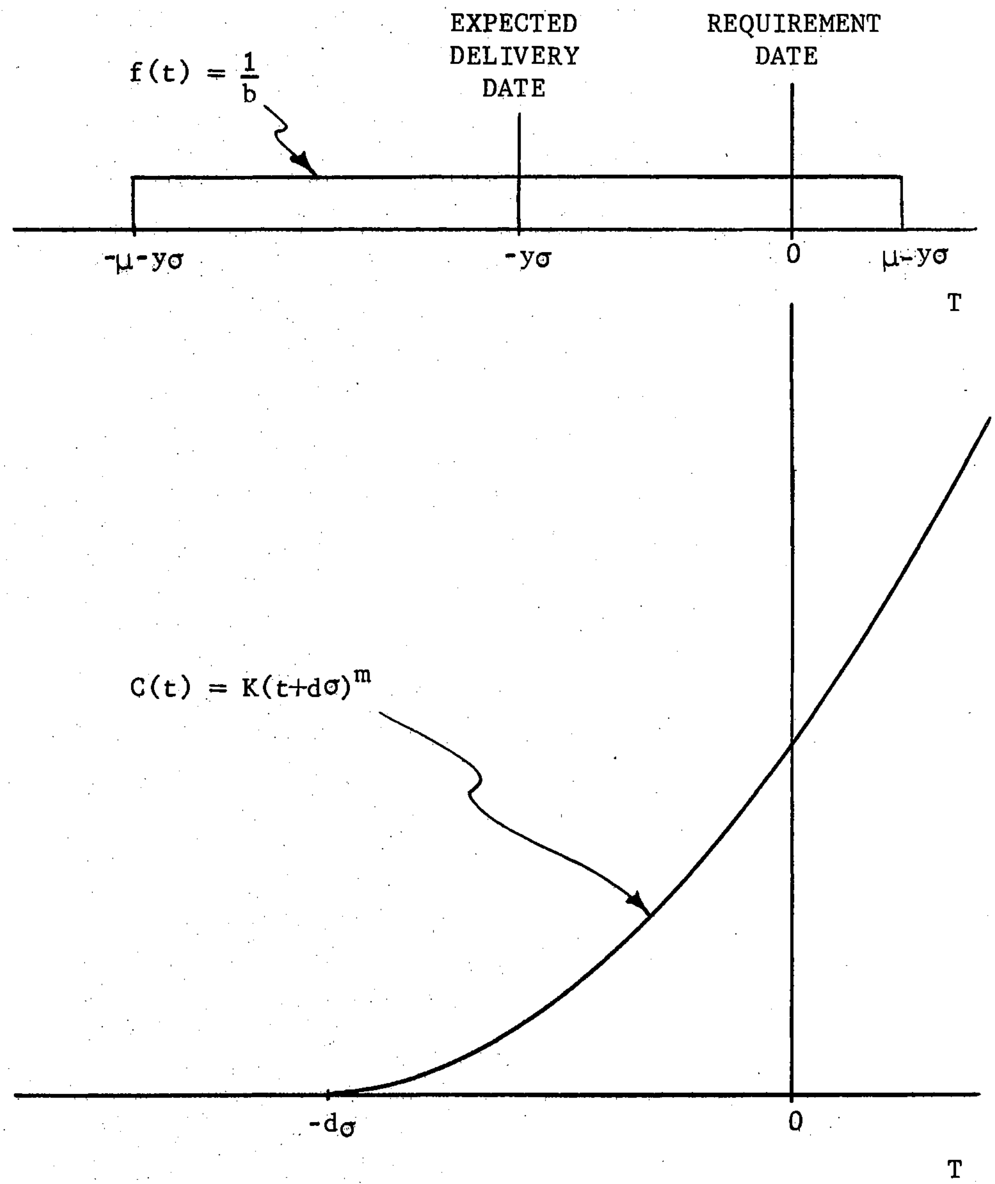


Figure 8. Uniform Delivery Date Distribution and Lateness Cost Function

(3.5) for $f(t)$ are shown in Figure 8 for a value of $m = 2$. Equation (2.12) may now be employed to find the expected value of lateness cost. Since the region of positive probability for $f(t)$ is the interval $(-\mu-y\sigma, \mu-y\sigma)$ equation (2.12) takes the form

$$E(LC) = \int_{-\mu-y\sigma}^{-d\sigma} 0 \cdot \frac{1}{b}\, dt + \int_{-d\sigma}^{\mu-y\sigma} K(t+d\sigma)^m \cdot \frac{1}{b}\, dt.$$

This reduces to

$$E(LC) = \frac{K}{b}\int_{-d\sigma}^{\mu-y\sigma} (t+d\sigma)^m\, dt. \qquad (3.6)$$

Parenthetically it should be noted that the point $-\mu-y\sigma$ must be less than the point $-d\sigma$ in order for equations (3.5) and (3.6) to hold, that is,

$$-\mu-y\sigma < -d\sigma. \qquad (3.7)$$

The definition of $C(t)$ is such that this inequality should always be satisfied. It would not be reasonable to incur lateness costs on a delivery date occurring prior to the interval of positive probability. According to the assumption of a uniform distribution, any delivery prior to the interval of positive probability has a probability of zero. Note also from Figure 8 that y may be greater or less than d, but both are positive real numbers. We shall not allow y to take on negative values, for it is assumed that management would never set a delivery date later than the requirement date for the part.

In order to facilitate the integration of equation (3.6), a change of variable will be introduced. Let

$$w = t+d\sigma \qquad \text{at} \quad t = -d\sigma \,, \quad w = 0$$

$$dw = dt \qquad\qquad t = \mu - y\sigma, \quad w = \mu+\sigma(d-y).$$

Then

$$E(LC) = \frac{K}{b}\int_{0}^{\mu+\sigma(d-y)} w^{m}\, dw = \frac{K}{b}\left[\frac{w^{m+1}}{m+1}\right]\Bigg|_{0}^{\mu+\sigma(d-y)}$$

$$= \frac{K}{b(m+1)}\left[\mu+\sigma(d-y)\right]^{m+1}. \tag{3.8}$$

Utilizing equation (3.4) for μ and σ, equation (3.8) reduces to this expression for the expected lateness cost:

$$E(LC) = \frac{K}{b(m+1)}\left[\frac{-b}{\sqrt{12}}\right]^{m+1}\left[y-(d+\sqrt{3})\right]^{m+1}, \quad y \le (d+\sqrt{3}). \tag{3.9}$$

The requirement $y \le (d+\sqrt{3})$ is established because for $y > d+\sqrt{3}$ the probability of incurring any lateness cost is zero resulting in zero lateness cost.

Derivation of Expressions for y and TVC

Expressions for optimal values of y in terms of the cost parameters will now be derived for lateness cost exponents m = 1, 2, and 3. First, the expected lateness cost component of TVC will be found to complete the expression for TVC developed in Chapter II. As most emphasis in this section will be with lateness costs, the expression for TVC will be simplified by defining one parameter to replace the four holding cost parameters of equation (2.14). Let

$$H = \frac{VP+WC_h}{365} \tag{3.10}$$

where H now represents the total daily holding costs which are composed of an inventory value cost and a storage space cost. The expression for the expected value of TVC may now be written as a function of y using equation (3.9) as the expected lateness cost component and equation (3.4) for σ.

$$TVC(y) = (H)\frac{b}{\sqrt{12}}\,y + \frac{K}{b(m+1)}\left[\frac{-b}{\sqrt{12}}\right]^{m+1}\left[y-(d+\sqrt{3})\right]^{m+1},\ 0 \le y \le d+\sqrt{3}. \tag{3.11}$$

The definition of the interval

$$0 \le y \le d+\sqrt{3} \tag{3.12}$$

includes the feasible values y may assume for the case involving a uniform p. d. f. of delivery date. Values of $y<0$ are strictly prohibited since they give a "negative buffer" meaning that the contracted delivery date would be timed to follow the requirement date by $y\sigma$ days. Values of $y>d+\sqrt{3}$ are not desirable for a uniform distribution of delivery date since they result in buffers so large that there is no probability of incurring lateness costs, as shown in Figure 9. The optimal values of y for m = 1, 2, 3 all satisfy the requirement $y \le d+\sqrt{3}$, but negative values of y are computationally possible as optimum values of y. The reason for this is that a continuous function of TVC must be assumed in order to take the derivative. When the critical point(s) thus found lie outside the required range of y, a special interpretation is necessary. The occurrence of negative values of y will be discussed in detail for

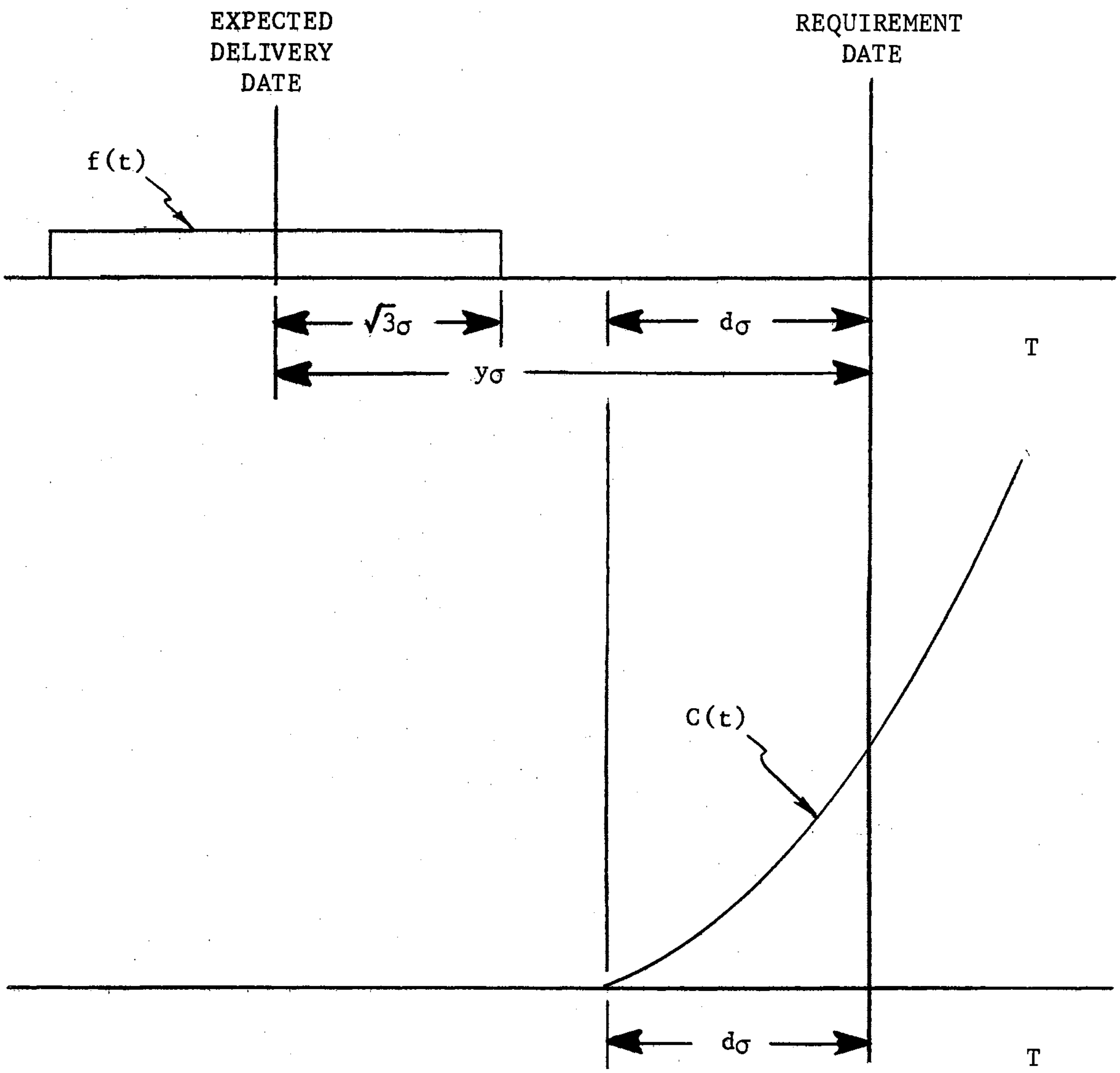


Figure 9. Position of Uniform Delivery Date Distribution and Lateness Cost Function When $y > d+\sqrt{3}$

the case of $m = 1$. Similar discussions for $m = 2$ and 3 are omitted to avoid redundancy.

In solving for optimal (least cost) expressions for y the first derivative TVC'(y) will be employed to determine the critical value or values of TVC. The second derivative TVC"(y) will be used to establish the critical point as a minimum. After deriving an optimal expression for y in terms of the cost parameters, this expression will then be utilized to obtain an equation for optimal TVC. Sample problems follow this section.

Optimal Buffer for m = 1

For lateness cost exponent $m = 1$, equations (3.9) and (3.11) become

$$E(LC) = \frac{Kb}{24}\left[y-(d+\sqrt{3})\right]^2 \tag{3.13}$$

$$TVC(y) = \frac{bH}{\sqrt{12}}\, y + \frac{Kb}{24}\left[y-(d+\sqrt{3})\right]^2, \quad 0 \leq y \leq d+\sqrt{3}. \tag{3.14}$$

Equation (3.14) is valid only for y in the interval $(0, d+\sqrt{3})$. However, as stated previously continuity of TVC over the range $(-\infty, \infty)$ will be assumed to permit the derivative. If the resulting critical points do not lie in the interval defined by equation (3.12), they will still be helpful in indicating the proper optimal value of y. The first and second derivatives of TVC with respect to y are

$$TVC'(y) = \frac{bH}{\sqrt{12}} + \frac{Kb}{12}\left[y-(d+\sqrt{3})\right] \tag{3.15}$$

$$TVC''(y) = \frac{Kb}{12}\,. \tag{3.16}$$

Since TVC"(y) is positive for all values of y, the extremum defined by solving TVC'(y) = 0 for y is a minimum. This value of y resulting in a minimum TVC is found as

$$\frac{bH}{\sqrt{12}} + \frac{Kb}{12}\left[y-(d+\sqrt{3})\right] = 0$$

$$y-(d+\sqrt{3}) = -\frac{\sqrt{12}}{K}H$$

$$y^* = d+\sqrt{3} - \frac{\sqrt{12}}{K}H \quad , \qquad y \geq 0. \qquad (3.17)$$

Equation (3.17) will result in $y \leq d+\sqrt{3}$ for all values of K and H; thus, values above the interval $(0, d+\sqrt{3})$ are of no concern. However, equation (3.17) will give negative values for y when holding costs per day are substantially higher than the daily increase in lateness costs, i.e., when

$$H > K\left(\frac{d+\sqrt{3}}{\sqrt{12}}\right) . \qquad (3.18)$$

Although this condition should not often occur, it is of interest and deserves comment. The situation resulting in negative values for optimal y is illustrated in Figure 10. Holding costs, lateness costs and TVC are plotted as continuous functions of y from equation (3.14) just as they are "seen" in the process of taking the first derivative, setting it equal to zero, and solving for the value of y which minimizes TVC. If inequality (3.18) holds, then the holding costs are so high that the minimum point on the TVC curve lies to the left of the origin. In this situation equation (3.17) gives a negative value of y since it was derived by setting TVC'(y) = 0. The optimal value of y for

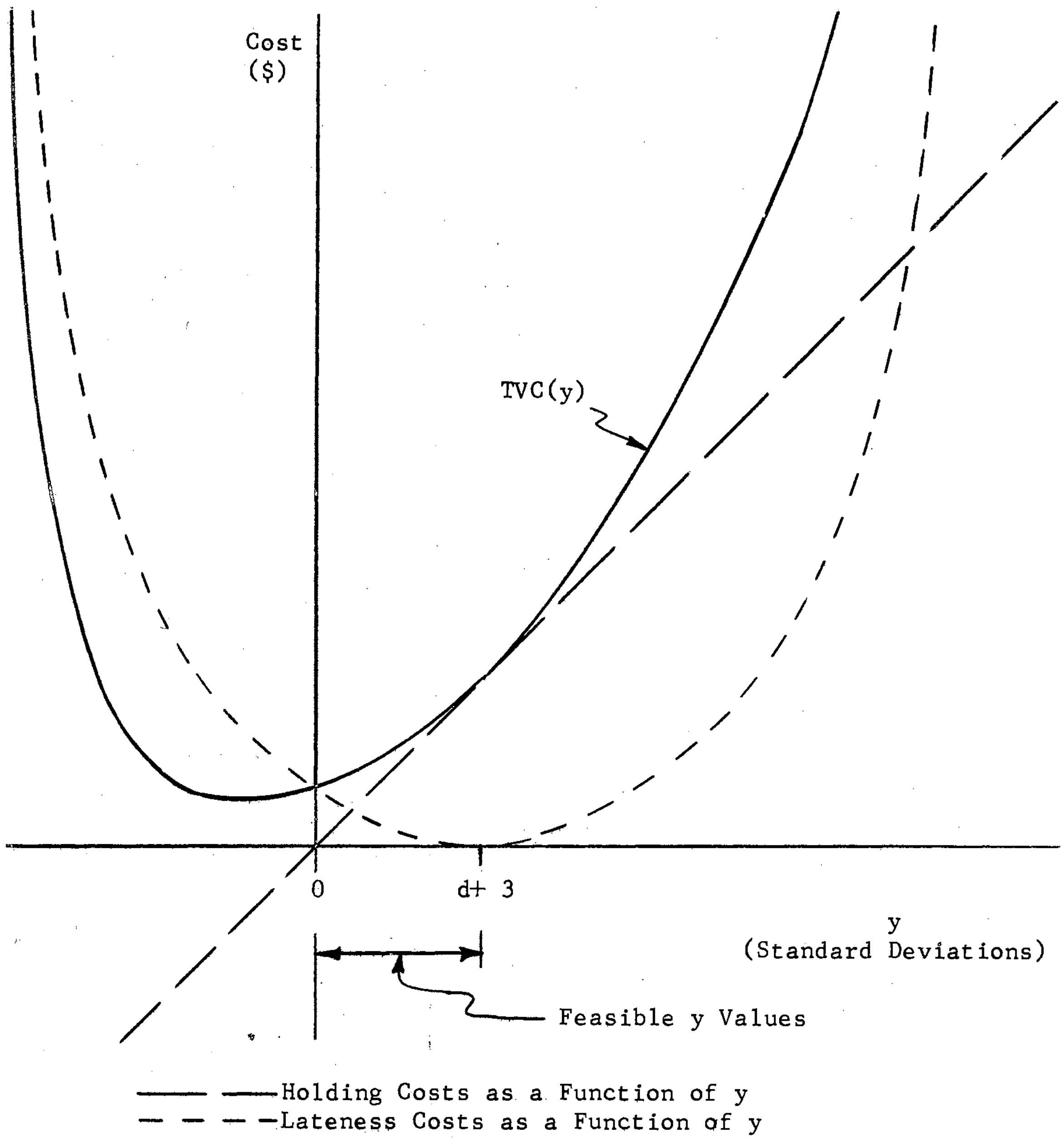


Figure 10. Holding Costs, Lateness Costs, and TVC(y) as Functions of y

situations where inequality (3.18) is satisfied is to set $y = 0$. This rule will always result in a minimum TVC because with the minimum point to the left of the origin TVC(y) will always be increasing in the interval $(0, d+\sqrt{3})$.

If inequality (3.18) is satisfied in a real world situation, it means that costs of inventory value and storage are substantially higher than the expected costs of expediting, rescheduling and production delays. If such is the case it is very possible that the production schedule is too "loose" and a rescheduling of the project might result in considerable savings in work-in-process inventory. Thus, a negative value of y is a warning marker: it may signify a loose production schedule with inflated work-in-process inventory, or it may indicate the lateness cost function being used is disregarding some important cost resulting from production delays. If the former is the problem, the project should be rescheduled and y recalculated using the new requirement date. If the lateness cost function is in error it should be corrected and y recalculated. If neither problem seems to have occurred and the value of y is only slightly negative, then the TVC is very close to $y = 0$; and a zero buffer should be used.

In most real-world cases, the substitution of parameter values into equation (3.17) should result in a positive value of y as is illustrated in Figure 11. This value of y should then be multiplied by the standard deviation of the delivery date distribution to determine the optimal buffer time. The optimal expected delivery date is then $y\sigma$ days prior to the requirement date, and this optimal expected delivery date should be specified to the vendor as the desired delivery date.

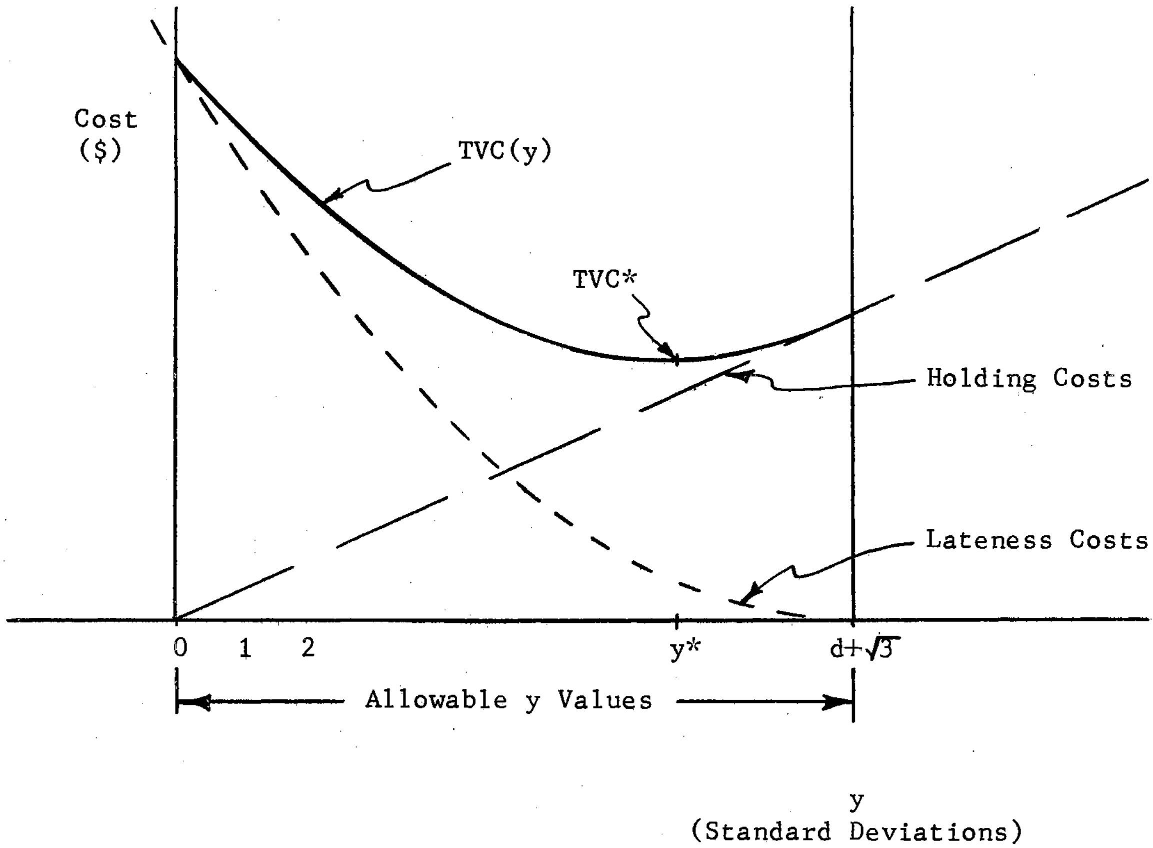


Figure 11. Typical Plot of TVC(y) vs. y Within Allowable Range for y

The expected TVC for positive values of y (non-zero buffers) can be found by substituting equation (3.17) into equation (3.14). The resulting expression for optimal TVC for m = 1 is

$$\mathrm{TVC}^* = (d+\sqrt{3})\left[\frac{bH}{\sqrt{12}}\right] - \left[\frac{b}{2K}\right] H^2. \tag{3.19}$$

Optimal Buffer for m = 2

For lateness cost exponent m = 2, equation (3.9) becomes

$$E(LC) = \frac{-Kb^2}{36\sqrt{12}}\left[y-(d+\sqrt{3})\right]^3, \qquad y \leq d+\sqrt{3}. \tag{3.20}$$

Although it appears the expected lateness cost is negative for m = 2, the quantity in the brackets will also produce a negative number for all feasible values of y. Thus, for every feasible case, the expected lateness cost will be non-negative. Equation (3.14) for TVC and its derivatives are then found as

$$\mathrm{TVC}(y) = \frac{bH}{\sqrt{12}}\, y - \frac{Kb^2}{36\sqrt{12}}\left[y-(d+\ 3)\right]^3, \quad 0 \leq y \leq d+\sqrt{3} \tag{3.21}$$

$$\mathrm{TVC}'(y) = \frac{bH}{\sqrt{12}} - \frac{Kb^2}{12\sqrt{12}}\left[y-(d+\sqrt{3})\right]^2 \tag{3.22}$$

$$\mathrm{TVC}''(y) = \frac{-Kb^2}{6\sqrt{12}}\left[y-(d+\sqrt{3})\right]. \tag{3.23}$$

TVC″(y) will be positive for values of $y < d+\sqrt{3}$. Thus, in order for a critical point defined by equation (3.22) to be a minimum on TVC(y) it must satisfy the condition $y < d+\sqrt{3}$. If the critical point does not satisfy the condition, i.e., if $y > d+\sqrt{3}$, then the extremum it describes is a maximum.

In solving equation (3.22) for an optimum (least cost) value of y, two roots are found as follows.

$$\frac{bH}{\sqrt{12}} - \frac{Kb^2}{12\sqrt{12}}\left[y-(d+\sqrt{3})\right]^2 = 0$$

$$\left[y-(d+\sqrt{3})\right]^2 = \frac{12}{Kb} H .$$

Taking the square root of both sides gives

$$y-(d+\sqrt{3}) = \pm\sqrt{\frac{12}{Kb} H}$$

$$y = d+\sqrt{3} + \sqrt{\frac{12}{Kb} H} \tag{3.24}$$

$$y = d+\sqrt{3} - \sqrt{\frac{12}{Kb} H} . \tag{3.25}$$

Equations (3.24) and (3.25) give the two critical points or extrema of TVC(y). Equation (3.24) gives a value of $y > d+\sqrt{3}$ for every real value of H, K, and b. Thus, for every y determined by equation (3.24), TVC'' (y) is negative. Therefore, equation (3.24) determines a maximum for TVC. In any event equation (3.25) is of primary interest because the critical point it defines satisfies the condition $y < d+\sqrt{3}$ needed to define a critical point as a minimum by the second derivative test for extrema. Thus, the optimal (least cost) value of y is

$$y^* = d+\sqrt{3} - \sqrt{\frac{12}{Kb} H}, \quad 0 \le y \le d+\sqrt{3}. \tag{3.26}$$

As was the case for m = 1, negative values of y are computationally possible. This will result under similar circumstances as previously and a similar interpretation is warranted. Again, small negative values of y should be rounded to zero, while larger negative values ($y < -3$) should be interpreted as warning signals.

For optimal y in the feasible interval, equation (3.26) can be substituted into equation (3.21) to derive the following expression for the minimum expected TVC for the case m = 2.

$$TVC^* = (d+\sqrt{3})\left[\frac{bH}{\sqrt{12}}\right] - \frac{2}{3} H\sqrt{\frac{bH}{K}} . \qquad (3.27)$$

Optimal Buffer for m = 3

For lateness cost component m = 3, equations (3.9) and (3.11) become

$$E(LC) = \frac{Kb^3}{(144)4}\left[y-(d+\sqrt{3})\right]^4 \qquad (3.28)$$

$$TVC(y) = \frac{bH}{\sqrt{12}} y + \frac{Kb^3}{(144)4}\left[y-(d+\sqrt{3})\right]^4, \qquad 0 \le y \le d+\sqrt{3} . \qquad (3.29)$$

$$TVC'(y) = \frac{bH}{\sqrt{12}} + \frac{Kb^3}{144}\left[y-(d+\sqrt{3})\right]^3 \qquad (3.30)$$

$$TVC''(y) = \frac{Kb^3}{48}\left[y-(d+\sqrt{3})\right]^2 . \qquad (3.31)$$

Since TVC"(y) is positive for all values of y, the critical point defined by the first derivative will be a minimum. Solving for an expression for optimal y gives

$$\frac{bH}{\sqrt{12}} + \frac{Kb^3}{144}\left[y-(d+\sqrt{3})\right]^3 = 0$$

$$\left[y-(d+\sqrt{3})\right]^3 = -\frac{12\sqrt{12}}{Kb^2}H$$

$$y-(d+\sqrt{3}) = -\left[\frac{12\sqrt{12}}{Kb^2}H\right]^{1/3}$$

$$y^* = d+\sqrt{3} - \left[\frac{12\sqrt{12}}{Kb^2}H\right]^{1/3} \qquad 0 \le y \le d+3 \ . \tag{3.32}$$

Negative values of y should be treated as discussed in previous cases. For optimal y in the feasible interval, equation (3.32) can be substituted into equation (3.29) to derive the following expression for the minimum expected value of TVC for the case m = 3.

$$TVC^* = (d+\sqrt{3})(\frac{bH}{\sqrt{12}} - \frac{3}{4}H\left[\frac{bH}{K}\right]^{1/3} \ . \tag{3.33}$$

Summary of Expressions for y* amd TVC*

The optimal (least cost) expressions derived for y and TVC for the assumption of a uniform distribution of the delivery date random variable will be summarized in this section. In some cases they will be rewritten to emphasize the common terms and differences between y* and TVC* for the different cases. The expressions have been proven to be optimal in the interval

$$0 \le y \le d+\sqrt{3}. \tag{3.34}$$

For the case m = 1,

$$y^* = (d+\sqrt{3}) - \frac{\sqrt{12}}{b}\left[\frac{bH}{K}\right] \tag{3.35}$$

$$TVC^* = (d+\sqrt{3})\left[\frac{bH}{\sqrt{12}}\right] - \frac{H}{2}\left[\frac{bH}{K}\right]. \tag{3.36}$$

For the case m = 2,

$$y^* = (d+\sqrt{3}) - \frac{\sqrt{12}}{b}\left[\frac{bH}{K}\right]^{1/2} \tag{3.37}$$

$$TVC^* = (d+\sqrt{3})\left[\frac{bH}{\sqrt{12}}\right] - \frac{2H}{3}\left[\frac{bH}{K}\right]^{1/2} \tag{3.38}$$

For the case m = 3,

$$y^* = (d+\sqrt{3}) - \frac{\sqrt{12}}{b}\left[\frac{bH}{K}\right]^{1/3} \tag{3.39}$$

$$TVC^* = (d+\sqrt{3})\left[\frac{bH}{\sqrt{12}}\right] - \frac{3H}{4}\left[\frac{bH}{K}\right]^{1/3} . \tag{3.40}$$

If negative values of y close to zero result from the substitution of parameter values into equations (3.35), (3.37), and (3.39), the value of y = 0 (a zero buffer) should be used. If values of $y < -2$ result, either the project schedule may have too much slack or an improper lateness cost function may have been used as discussed previously.

Sample Problem

A contract has been received by the Least Cost Co. to build an experimental fighter for the Navy. The specifications call for a specially-designed computer to direct fire which will be procured from the Uncertain Delivery Co. The computer will cost \$190,000 and require 10 square feet of high quality storage space that costs \$.50 per day or \$182.50 per year per square foot. The firm's cost of capital, taxes and insurance on inventory items amounts to 15% of the inventory value per year.

PERT has been used to determine a project schedule. The computer will be needed from the vendor to begin testing and assembly on October 1 according to the project schedule. According to past experience with the Uncertain Delivery Co. on this type of component, delivery may be expected to occur with equal probability anywhere in a fourteen day interval which is determined by the week before and the week after the contracted delivery date. If the component is not delivered 8 days prior to the requirement date (October 1), expediting procedures will start. One man will be assigned to "track down" the computer, determine its status, and see that it is delivered as soon as possible. The project schedule is rather "tight," and costs of production delays will be very high; therefore an exponent of $m = 3$ on lateness cost is considered appropriate. The value of $K = 10$ results in a lateness cost function that gives a good representation of the total lateness costs incurred for delivery at any given time. It is desired to find the delivery date which will minimize the variable costs of procurement and to find the expected value of TVC.

The parameters of the problem are

$V = \$190,000$

$P = .15$ per year

$W = 10$ square feet

$C_h = \$182.50$ per square foot per year

$b = 14$ days

$K = 10$

$m = 3$

$d = 8 \text{ days}/\sigma = 8\sqrt{12}/14 \doteq 2$

$H = (VP + WC_h)/365 = \$82.93/\text{day}$ or $\$83.00/\text{day}$

From equation (3.39),

$$y^* = 2+\sqrt{3} - \frac{\sqrt{12}}{14}\left[\frac{(14)(83)}{10}\right]^{1/3}$$

$$= 2.52$$

The optimal buffer is $y\sigma = (2.5)(14/\sqrt{12}) = 10.20$ days.

For a requirement date of October 1, the delivery date specified to the Uncertain Delivery Company should be September 20 in order to minimize the expected variable costs of inventory value, storage space and lateness.

The expected TVC* of procurement for the Least Cost Co. may be found from equation (3.40).

$$TVC^* = (2+\sqrt{3})\left[\frac{(14)(83)}{\sqrt{12}}\right] - \frac{(3)(83)}{4}\left[\frac{(14)(83)}{10}\right]^{1/3}$$

$$= \$948.95.$$

CHAPTER IV

CHI-SQUARE DISTRIBUTION OF DELIVERY DATE

Assumption of a uniform distribution for delivery date resulted in an easily differentiable expression for TVC, and the resulting expressions for y* and TVC* provide simple and easy-to-use tools for procurement personnel. However, many procurement situations will arise in which the assumption of a uniform distribution is not appropriate in that it assumes equal probability of delivery within a given range. In many situations it will be more reasonable to assume that delivery is most likely to occur near the expected delivery date with decreasing probability of delivery as time moves away from the expected delivery date. The chi-square distribution may be used to approximate this situation.

Although the shape of the chi-square distribution makes it appropriate for use in many procurement situations, it is rather difficult to deal with mathematically in the problem being approached here. In fact, the mathematical derivation of the expected lateness cost component of TVC involves an original approach to dealing with the chi-square that may in itself be of interest to some. Unfortunately the resulting expression for expected lateness cost does not allow development of simple expressions for y* and TVC*. In order to facilitate use of the model two FORTRAN programs were written. One calculates cost components and TVC for a given set of parameters and buffer. The second utilizes a Fibonacci search procedure to look at a wide range of possible y values and find the y* and TVC* for the optimal buffer. The sample problem presented in Chapter III will be solved using the assumption of a chi-square distribution for delivery date.

Assumption of Chi-Square Distribution

The chi-square probability distribution is a likely candidate to represent the random variable of delivery date. It is reasonable to assume the probability of delivery before a certain date is zero. Following this earliest possible date the probability of delivery increases slightly for each succeeding day. As the contracted delivery date is approached, the probability of delivery increases to a maximum. The mode of the distribution is reached a short time before the mean or expected delivery date. Following the expected delivery date the probability of delivery occurring tapers off gradually (into a long "tail" of the distribution). Thus, the distribution of probability over x of the chi-square distribution agrees logically with what should be used to describe the behavior of a random variable of delivery date over time. The p. d. f. of the chi-square is plotted in Figure 12 for 8 degrees of freedom.

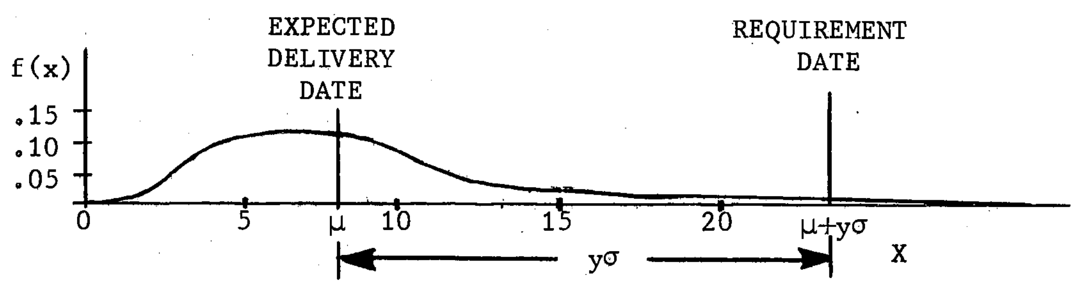


Figure 12. The Chi-Square Probability Distribution

In addition, the dispersion of the chi-square can be changed by changing its degrees of freedom; thus, different levels of uncertainty in delivery can easily be accounted for by changing this parameter. These characteristics make the chi-square distribution a very suitable p. d. f. to assume in describing the random variable delivery date. The probability density function for the chi-square may be written as

$$f(x) = \frac{1}{(r/2)2^{r/2}} x^{(r/2)-1} e^{-x/2},\ 0 < x < \infty, \tag{4.1}$$

$$= 0, \text{ elsewhere}$$

where r is called the number of degrees of freedom of the chi-square p. d. f. The mean and standard deviation of the chi-square are

$$\mu = r \quad \text{and} \quad \sigma = \sqrt{2r}. \tag{4.2}$$

The chi-square distribution will be used to describe the probability of delivery on a given date in the following manner:

a. the mean of the chi-square distribution of delivery date will be defined as being located at the contracted delivery date, which is to be determined by the model,

b. a 90% or 98% range on the delivery of the component will be determined by procurement personnel, and

c. the length of this interval will be used to specify the distribution parameter r.

In step (a) above, the expected delivery date (the chi-square mean) is being defined as the contracted delivery date. Concerning steps (b) and (c), the assumption of the chi-square to describe occurrence of delivery dates in the procurement process is a new application, and some comment on proper use of the extensive chi-square tables is in order. In step (b) above it is necessary that the procurement analyst define "ranges" within which he feels delivery will occur with a given probability. For example, a ".90 range" on delivery date would be an interval of a length such that there is a 90% chance that delivery will occur within this span of time. Table I gives interval lengths for 98% chance of delivery and 90% chance of delivery within the given interval and indicates the procedure in their calculation. If a history of delivery performance is available for a particular vendor, this past record may be used to determine a .90 or .98 confidence interval on his delivery performance. If the order is being placed with a new vendor or under special circumstances, a subjective evaluation of the ".90 range" or ".98 range" on delivery date is necessary.

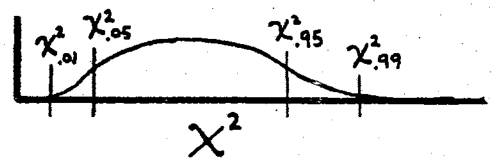
Once the range is determined, step (c) is accomplished through the use of Table I to determine the proper degrees of freedom to use in calculations for TVC and y. The use of even numbered degrees of freedom is a requirement imposed by an essential step in the mathematical formulation of expected lateness cost. This development follows.

Development of Expected Lateness Cost

The development of expected lateness cost involves some interesting mathematical manipulations. A change of variable will be used to "shift" the zero reference point of the chi-square to coincide with the requirement date. This will give the probability of delivery occurring at time t such that this probability can be multiplied by the cost incurred by delivery at time t represented by the lateness cost function of equation (2.10). A second change of variable will then be

TABLE I

PROBABILITY INTERVALS OF 90% AND 98% FOR DETERMINATION OF CHI-SQUARE PARAMETER r



Value of χ^2 for Probability Quantiles				.98 Range on Delivery Date $\chi^2_{.01}-\chi^2_{.99}$	.90 Range on Delivery Date $\chi^2_{.05}-\chi^2_{.95}$	r Value to Use
P = .01	P = .05	P = .95	P = .99			
.02	.10	5.99	9.21	9.2 days	5.9 days	2
.30	.71	9.49	13.28	13.0 days	8.8 days	4
.87	1.63	12.59	16.81	15.9 days	11.0 days	6
1.65	2.73	15.51	20.09	18.4 days	12.8 days	8
2.56	3.94	18.31	23.21	20.7 days	14.4 days	10
3.57	5.23	21.03	26.22	22.7 days	15.8 days	12
4.66	6.57	23.69	29.14	24.5 days	17.1 days	14
5.81	7.96	26.30	32.00	26.2 days	18.3 days	16
7.02	9.39	28.87	34.81	27.8 days	19.5 days	18
8.26	10.85	31.41	37.57	29.3 days	20.6 days	20
9.54	12.34	33.92	40.29	30.8 days	21.6 days	22
10.86	13.85	36.42	42.98	32.1 days	22.6 days	24
12.20	15.38	38.89	45.64	33.4 days	23.5 days	26
13.57	16.93	41.34	48.29	34.7 days	24.4 days	28
14.95	18.49	43.77	50.89	35.9 days	25.3 days	30

Source (χ^2 Values): A. Hald, Statistical Tables and Formulas (New York, 1951), Table V, pp. 40-3. Approximate formula for $r > 30$ is $\chi^2_p = \frac{1}{2}(\sqrt{2r-1} + z_p)^2$ where z_p is the standard normal deviate of probability p.

necessary to adjust the lower bound on the integral to zero enabling integration over the entire range of the chi-square distribution. Next a binomial expansion of the term involving t is necessary. After distributing the integral sign, each term of the binomial expansion will be manipulated into a constant times the integral of a chi-square p. d. f. from zero to infinity. Each of these integrals reduces to a value of 1, and the expected lateness cost can then be expressed in terms of a finite series of terms.

The first change of variable will move the zero reference point of the chi-square delivery date distribution to coincide with the origin defined to be at the requirement date in Figure 13. For the p. d. f. of equation (4.1), let

$$x = t + \mu + y\sigma \quad \text{and at} \quad x = 0,\ t = -(\mu+y\sigma)$$
$$t = x - (\mu+y\sigma) \qquad\qquad x = \infty,\ t = \infty.$$

Now

$$f(t) = \frac{1}{\Gamma(r/2)2^{r/2}}(t+\mu+y\sigma)^{(r/2)-1}e^{-\frac{t+\mu+y\sigma}{2}}, \quad -(\mu+y\sigma) < t < \infty,$$
$$= 0 \text{ elsewhere.} \tag{4.3}$$

Note that the change of variable does not change the shape of the p. d. f. but only shifts the zero reference point to the requirement date. The probability of delivery occurring is still zero until the point $t = -(\mu+y\sigma)$ is reached. For all points $t > -(\mu+y\sigma)$, the value of $(t+\mu+y\sigma)$ is the same as the value of x at the corresponding point. Thus, at any given time prior to the requirement date the probability given by f(t) is identical to the probability given by f(x) at that point in time. The relationship of the chi-square delivery date distribution f(t) and the lateness cost function C(t) from equation (2.10) is illustrated in Figure 13.

The expected value of lateness cost will be found as described in equation (2.12).

$$E(LC) = \int_{-(\mu+y\sigma)}^{-d\sigma} 0 \cdot f(t)\, dt + \int_{-d\sigma}^{\infty} K(t+d\sigma)^{m} \cdot f(t)\, dt$$

$$= \int_{-d\sigma}^{\infty} K(t+d\sigma)^{m} \frac{1}{\Gamma(r/2)2^{r/2}} (t+\mu+y\sigma)^{(r/2)-1} e^{-\frac{t+\mu+y\sigma}{2}}\, dt. \qquad (4.4)$$

The above integral when properly evaluated will yield an expression for the expected lateness cost. First it will be necessary to introduce a change of variable so that the integral is taken over the interval zero to infinity. In order to simplify notation the parameter a will be introduced to represent (r/2). Let

$$a = (r/2). \qquad (4.5)$$

The change of variable will now be performed. Let

$$w = t + d\sigma \quad \text{at} \quad t = -d\sigma,\ w = 0$$

$$dw = dt \qquad\qquad t = \infty\ ,\ w = \infty.$$

$$E(LC) = \int_{0}^{\infty} Kw^{m} \frac{1}{\Gamma(a)2^{a}} (w-d\sigma+\mu+y\sigma)^{a-1}\ e^{-\left[\frac{w-d\sigma+\mu+y\sigma}{2}\right]} dw$$

$$= \frac{Ke^{-\left[\frac{\mu+\sigma(y-d)}{2}\right]}}{\Gamma(a)2^{a}} \int_{0}^{\infty} w^{m} \left[w+\mu+\sigma(y-d)\right]^{a-1} e^{-w/2}\, dw. \qquad (4.6)$$

To simplify notation in following steps, let the following constant terms be reduced to a single parameter.

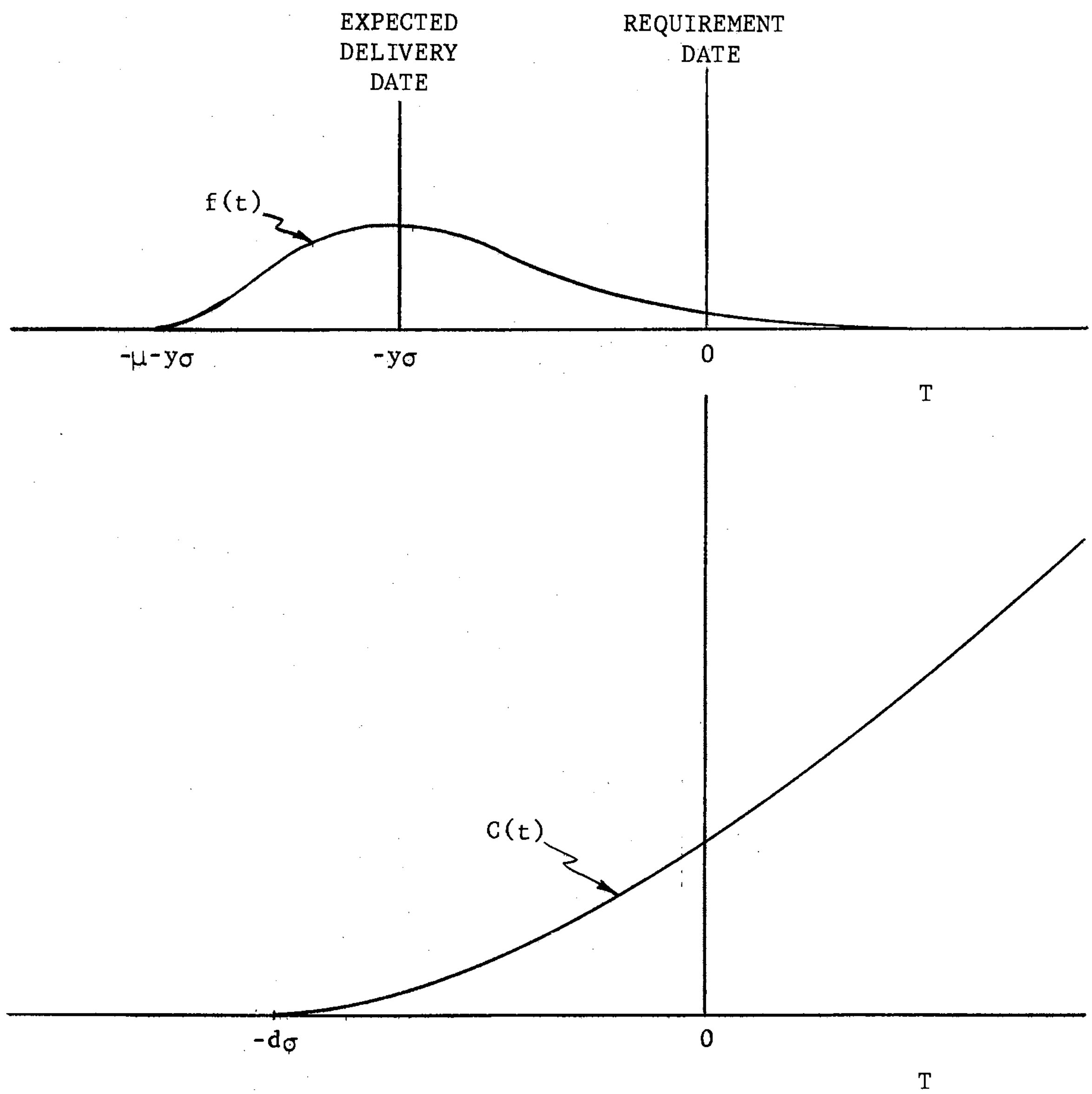


Figure 13. Chi-Square Delivery Date Distribution and Lateness Cost Function

Let

$$C = \frac{Ke^{-\left[\frac{\mu+\sigma(y-d)}{2}\right]}}{\Gamma(a)2^{a}}, \tag{4.7}$$

and

$$s = \mu + \sigma(y-d). \tag{4.8}$$

Equation (4.6) may now be written

$$E(LC) = C\int_0^\infty w^m \; (w+s)^{a-1} \; e^{-w/2} \, dw \tag{4.9}$$

If the requirement is made that (a-1) be an integer, that is, if r is even, then the term $(w+s)^{a-1}$ may be replaced with its binomial expansion.

$$E(LC) = C\int_0^\infty w^m \Big[w^{a-1} + \frac{(a-1)}{1!} w^{a-2} s + \frac{(a-1)(a-2)}{2!} w^{a-3} s^2 + \ldots$$

$$\ldots + \frac{(a-1)(a-2)\ldots(a-1-k+1)}{k!} w^{a-1-k} s^k + \ldots$$

$$\ldots + s^{a-1}\Big] e^{-w/2} \, dw. \tag{4.10}$$

If w^m is now distributed, equation (4.10) becomes

$$E(LC) = C\int_0^\infty \Big\{ w^{m+a-1} + \left[\frac{a-1}{1!}\right] w^{m+a-2} s + \ldots$$

$$\ldots + \left[\frac{(a-1)(a-2)\ldots(a-k)}{k!}\right] w^{m+a-1-k} s^k +$$

$$\ldots + w^m s^{a-1} \Big\} e^{-w/2} \, dw. \tag{4.11}$$

Distributing the integral sign and manipulating coefficients, equation (4.11) becomes

$$E(LC) = C\left\{\Gamma(m+a)2^{m+a}\int_0^\infty \frac{w^{m+a-1}}{\Gamma(m+a)2^{m+a}}\, e^{-w/2}\, dw + \right.$$

$$+ (a-1)s\Gamma(m+a-1)2^{m+a-1}\int_0^\infty \frac{w^{m+a-2}}{\Gamma(m+a-1)2^{m+a-1}}\, e^{-w/2}\, dw +$$

$$\ldots + \frac{(a-1)(a-2)\ldots(a-k)}{k!}\, s^k\Gamma(k')2^{k'}\int_0^\infty \frac{w^{k'-1}}{\Gamma(k')2^{k'}}\, e^{-w/2}\, dw +$$

$$\left.\ldots + s^{a-1}\Gamma(m+1)2^{m+1}\int_0^\infty \frac{w^{m}}{\Gamma(m+1)2^{m+1}}\, e^{-w/2}\, dw\right\} \qquad (4.12)$$

where $k' = m + a - k$ in the kth term. Each of the integrals in equation (4.12) is the integral of a chi-square p. d. f. over its interval of positive probability (zero to infinity). Each of these integrals reduces to a value of one by the definition of a p. d. f., and equation (4.12) reduces to a finite series made up of the terms preceding each of the integral signs. The gamma functions in equation (4.12) may be replaced by factorials according to the identity:

$$\Gamma(n+1) = n! \qquad (4.13)$$

Substitution of the expressions for C, s, and a, equation (4.13), and algebraic manipulation results in the following expression for expected lateness cost.

$$E(LC) = \frac{2^m K e^{-\left[\frac{\mu+\sigma(y-d)}{2}\right]}}{(r/2-1)!}\sum_{i=0}^{r/2-1}(m+r/2-1-i)!\left[\frac{\mu+\sigma(y-d)}{2}\right]^i\frac{(r/2-1)!}{(r/2-1-i)!\, i!}\,. \qquad (4.14)$$

Fibonacci Search for y* and TVC*

Although the binomial expansion and other techniques resulted in successful integration of equation (4.6), the expected lateness cost expression is not easily differentiable with respect to y, which is the decision variable it is desired to optimize. The series in equation (4.14) is a binomial series except for the term $\left[m+(r/2)-1-i\right]!$. This term when combined with the binomial expression is of a nature that eliminates the possibility of finding an easily expressed sum for the series. The expression for the expected value of TVC for the case of a chi-square delivery date must include this series as part of the expression for the expected lateness cost component. Substituting equations (4.14) and (4.2) for μ and σ into equation (2.14) for TVC gives an expression for TVC in terms of the cost parameters of interest.

$$TVC(y) = \left[\frac{VP+WC_h}{365}\right]\sqrt{2r}\cdot y + (\text{COEFFICIENT}) \times (\text{SUM}), \text{ for } y \geq 0. \qquad (4.15)$$

where $$\text{COEFFICIENT} = \frac{2^m K e^{-\left[\frac{r+\sqrt{2r}(y-d)}{2}\right]}}{\left[(r/2)-1\right]!}$$

and $$\text{SUM} = \sum_{i=0}^{r/2-1}\left[m+(r/2)-1-i\right]!\left[\frac{r+\sqrt{2r}(y-d)}{2}\right]^i \frac{\left[(r/2)-1\right]!}{\left[(r/2)-1-i\right]!\,i!}.$$

The relative magnitudes of the series terms were examined to determine whether only one or two of them were significant. A computer program was written which calculated each term separately and its percentage of the sum of the series. Although a few terms of the series were small with respect to others, several terms were of similar magnitude for any set of parameters. Thus the procedure of ignoring all but one or two terms would not result in a valid approximation.

Slide rule calculation of TVC in equation (4.15) is a tedious process, but calculation can easily be accomplished with the aid of a computer. It is reasonable to assume that any large job-shop manufacturer that would have occasion to use this model would also have a digital computer available. Also, the components under consideration in this problem are expensive, critical items and a computer analysis of the variable costs of procurement will in most cases be justified. For this reason a FORTRAN computer program was written to calculate TVC as a function of y, d, the degrees of freedom of the chi-square delivery date distribution, and the cost parameters defined in equation (2.15). This program is included as Appendix A.

The program in Appendix A punches out the input parameters. It calculates and punches the components of TVC and the expected TVC for the value of y used. Values of buffer and the expediting period $d\sigma$ are calculated and punched. This program also was used to investigate the relative magnitudes of terms in the lateness cost series. If sense switch two is turned on, values of the three parts of each term along with the value of their product will be punched. If switch two is off, only the terms and their percentages of the sum are punched.

Since differentiation of the TVC expression in equation (2.15) will not allow for a simple expression for y, it will be necessary to solve for an optimal value of y by other methods. This can be done through an efficient search procedure because of the "U-shaped" nature of the TVC(y) curve as shown in Figure 14. The expected holding costs (the sum of inventory value and storage space) increase linearly as the

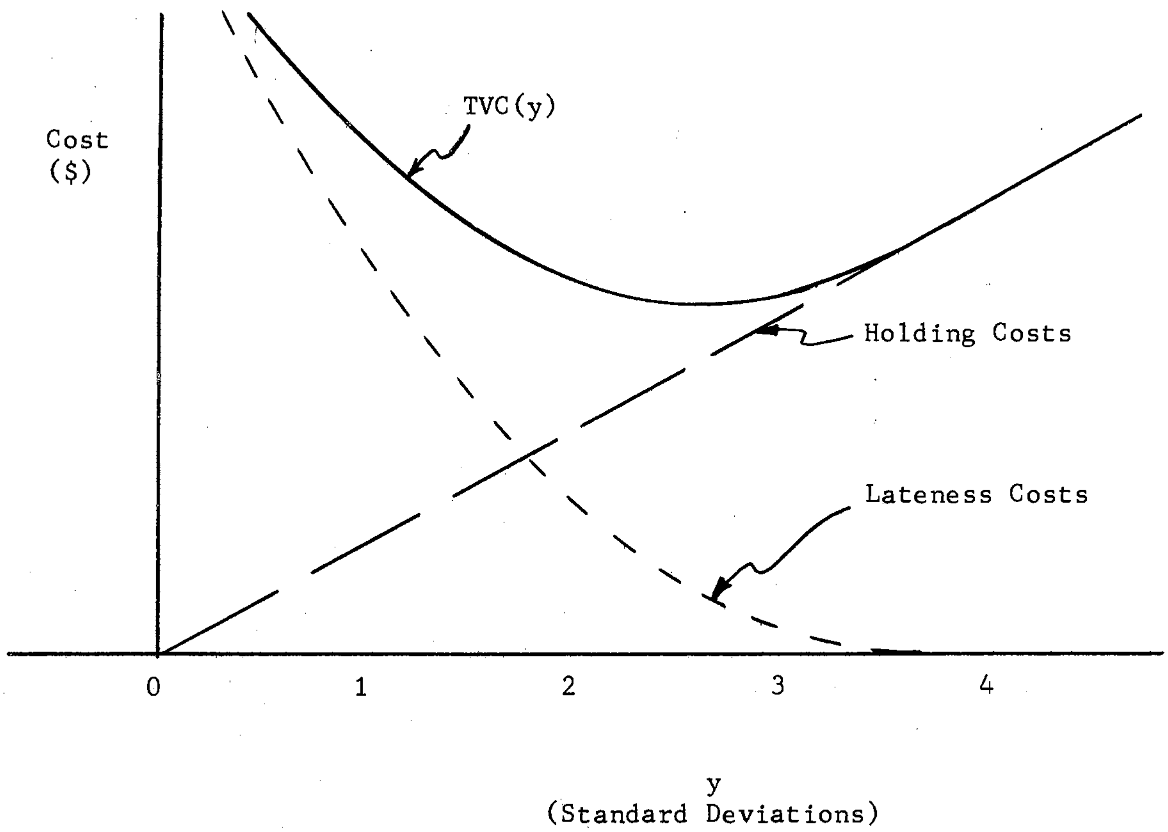


Figure 14. TVC(y) for the Chi-Square Delivery Date Distribution

buffer y increases. As the buffer y is decreased, the probability of incurring higher lateness costs is increased; and the expected lateness cost increases. TVC(y) is thus the sum of one increasing and one decreasing cost component. This would result in a "U-shaped" curve for TVC(y) for positive values of the cost parameters.

If TVC(y) is at a minimum for a positive value of y, it is desirable to locate that value of y to use in calculating the optimal buffer. If TVC(y) reaches a minimum at some negative y, then TVC(y) will be an increasing function for positive values of y, as was illustrated in Figure 10. In this case the optimal buffer is of zero length. Fortunately, several efficient search procedures exist for finding the minimum point within a given interval for a function such as TVC(y).

A particularly useful and interesting procedure for finding the optimum (least cost) value of a function of one variable such as TVC(y) is Fibonacci search. In searching for the minimum TVC(y) it will be necessary to evaluate TVC(y) for different values of y. As this requires considerable computation it would be desirable to minimize the maximum number of evaluations necessary. Under the criterion of minimizing the maximum number of function evaluations required to find the optimum, the Fibonacci search is the best one-dimensional search procedure as discussed in Nemhauser (1966) and in Wilde (1964).

The Fibonacci search procedure is discussed in Appendix E. In order to find the optimum (least cost) values of y and TVC, the range of y from .01 to 9.86 was considered to be 986 discrete points with a minimum occurring at one of these points. Thus, the Fibonacci search procedure finds y* to within .01 and the TVC* associated with y*, and accomplishes this in only fourteen evaluations of TVC(y). A FORTRAN computer program was written to perform this search and is included in Appendix B. This program finds the optimal buffer and corresponding TVC* for a given set of cost parameters and expediting strategy (defined by $d\sigma$, the first day that expediting begins if delivery has not occurred). The program logs each of the y values and TVC evaluations such that a curve of expected TVC vs. y can be plotted to illustrate the sensitivity of y for the given set of parameters. This program will be used to solve a sample problem.

Sample Problem

The same problem presented in Chapter III will be solved using the assumption of a chi-square distribution. All cost parameters will remain the same. But instead of a uniform delivery date distribution, the past performance of the Uncertain Delivery Company indicates that the probability of delivery at different times preceding and following the contracted delivery date is approximated by a chi-square distribution as illustrated in Figure 12. Under similar circumstances in the past, the delivery has been made within a 13 day interval 90% of the time. From Table I, the proper chi-square degrees of freedom to use in calculations is found to be r = 8. For this distribution of delivery date, $\mu = r = 8$ and $\sigma = \sqrt{2r} = \sqrt{16} = 4$ days. The firm's policy regarding expediting is the same as previously, with the first lateness costs being incurred 8 days prior to the requirement date if the part has not yet been received.

In previous dealings with the Uncertain Delivery Co., procurement people for the Least Cost Co. have used a standard buffer of 4 weeks in setting delivery dates. It is desired to calculate the expected value of TVC associated with this buffer and to determine the optimal buffer and TVC*.

The cost parameters are as follows:

V = \$190,000

P = .15 per year

W = 10 square feet

C_h = \$182.50 per square foot per year

K = 10

m = 3

d = 8 days/σ = 8/4 = 2

.90 Range on Delivery Date = 13 days

r from Table I = 8.

From equation (4.15) the expected TVC may be found for a value of y = 28/σ = 28/4 = 7.

$$\mathrm{TVC} = \left[\frac{(190{,}000)(.15)+(10)(182.50)}{365}\right] 7\sqrt{16} +$$

$$+ \left[\frac{(2^3)(10)e^{-\left[\frac{8+4(7-2)}{2}\right]}}{(8/2-1)!}\right] \sum_{i=0}^{8/2-1} \left[\frac{8+4(7-2)}{2}\right]^i \frac{(3+8/2-1-i)!\ (8/2-1\ !}{(8/2-1-i)!\ i!}$$

For y = 7, the components of TVC as calculated by the FORTRAN program in Appendix A are as follows. The printout for this calculation is shown in Figure 15.

Expected Inventory Value Cost	=	\$2186.30
Expected Storage Space Cost	=	140.00
Expected Lateness Cost	=	.40
Expected TVC of Procurement	=	\$2326.70

```
PARAMETERS ARE AS FOLLOWS -- R=      8.0
                             Y=      7.000
                             D=      2.000
                             EXPEDITING COST POWER=      3.000
                             EXPEDITING COST SCALE FACTOR=      10.000
                             SPACE REQUIRED =           10.00
                             COST/SPACE/YEAR=          182.5000
                             VALUE OF PART  =       190000.00
                             COST OF CAPITAL=              .1500

EXPEDITING STARTS D*SIGMA =   8.00  DAYS BEFORE REQUIREMENT DATE.

            NO    FACTORIAL PART     POWER PART    BINOMIAL PART    PRODUCT OF 3
TERM NO.    1    7.2000000E+02  1.0000000E+00  1.0000000E+00  7.2000000E+02
TERM NO.    2    1.2000000E+02  1.3999998E+01  3.0000000E+00  5.0399991E+03
TERM NO.    3    2.4000000E+01  1.9599994E+02  3.0000000E+00  1.4111995E+04
TERM NO.    4    6.0000000E+00  2.7439989E+03  1.0000000E+00  1.6463993E+04

   LATENESS COST           MAGNITUDE        PERCENT
     SERIES                 OF TERM         OF  SUM
 TERM NUMBER      1      .72000000E+03       1.981
 TERM NUMBER      2      .50399991E+04      13.870
 TERM NUMBER      3      .14111995E+05      38.837
 TERM NUMBER      4      .16463993E+05      45.310

 THE LATENESS COST SUM FOR THIS SET OF M,R,Y,D =     .36335987E+05
 THE LATENESS COST CONSTANT COEFF FOR THIS SET =     .11087060E-04

      THE LATENESS COMPONENT OF TOTAL VAR. COST =              .4028

      THE STORAGE SPACE COMPONENT OF TOTAL COST =           139.9999

      THE INVENTORY VALUE COMPONENT OF TOT COST =          2186.3012

      THE TOTAL VARIABLE COST OF PROCUREMENT =             2326.7039

 THE BUFFER FOR THIS TVC CALCULATION IS (Y*SIGMA) =  28.0  DAYS.
```

Figure 15. Sample Problem Output of FORTRAN Program of Appendix A

Calculation of the optimal buffer is done with the aid of the program in Appendix B. After evaluating TVC(y) at the fourteen values of y indicated, the buffer resulting in the least expected TVC was found to be at y = 2.98 or 11.9 days as shown in the output of Figure 16. The expected value of TVC is $1220.58 for this buffer. Rounding the buffer to the nearest whole day, the correct delivery date to specify to the Uncertain Delivery Co. is 12 days before the requirement date. Note that use of the optimal buffer of 12 days rather than the old 28 day buffer results in expected savings of $2327-$1221 or $1106.

```
PARAMETERS ARE AS FOLLOWS -- R=      8.0
                             D=      2.000
                             EXPEDITING COST POWER=      3.000
                             EXPEDITING COST SCALE FACTOR=      10.000
                             SPACE REQUIRED =         10.00
                             COST/SPACE/YEAR=         182.5000
                             VALUE OF PART  =      190000.00
                             COST OF CAPITAL=            .1500
EXPEDITING STARTS D*SIGMA =    7.99  DAYS BEFORE REQUIREMENT DATE.

EVAL NUMBER    14      Y =   6.100      BUFFER =    24.400
     THE LATENESS COMPONENT OF TOTAL VAR. COST =          1.7924
     THE STORAGE SPACE COMPONENT OF TOTAL COST =        121.9999
     THE INVENTORY VALUE COMPONENT OF TOT COST =       1905.2053
     THE TOTAL VARIABLE COST OF PROCUREMENT =          2028.9976

EVAL NUMBER    13      Y =   3.770      BUFFER =    15.080
     THE LATENESS COMPONENT OF TOTAL VAR. COST =         71.5593
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         75.3999
     THE INVENTORY VALUE COMPONENT OF TOT COST =       1177.4793
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1324.4385

EVAL NUMBER    12      Y =   2.330      BUFFER =     9.320
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        576.6073
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         46.5999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        727.7259
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1350.9333

EVAL NUMBER    11      Y =   4.660      BUFFER =    18.640
     THE LATENESS COMPONENT OF TOTAL VAR. COST =         18.1457
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         93.1999
     THE INVENTORY VALUE COMPONENT OF TOT COST =       1455.4519
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1566.7975

EVAL NUMBER    10      Y =   3.220      BUFFER =    12.880
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        162.3447
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         64.3999
     THE INVENTORY VALUE COMPONENT OF TOT COST =       1005.6985
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1232.4431

EVAL NUMBER     9      Y =   2.880      BUFFER =    11.520
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        265.9120
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         57.5999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        899.5067
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1223.0188

EVAL NUMBER     8      Y =   2.670      BUFFER =    10.680
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        358.6776
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         53.3999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        833.9177
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1245.9953

EVAL NUMBER     7      Y =   3.010      BUFFER =    12.040
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        220.4686
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         60.1999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        940.1095
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1220.7780

EVAL NUMBER     6      Y =   3.090      BUFFER =    12.360
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        196.2994
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         61.7999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        965.0958
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1223.1951

EVAL NUMBER     5      Y =   2.960      BUFFER =    11.840
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        236.9918
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         59.1999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        924.4930
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1220.6848

EVAL NUMBER     4      Y =   2.930      BUFFER =    11.720
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        247.4664
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         58.5999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        915.1232
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1221.1896

EVAL NUMBER     3      Y =   2.980      BUFFER =    11.920
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        230.2453
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         59.5999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        930.7396
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1220.5850

EVAL NUMBER     2      Y =   2.990      BUFFER =    11.960
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        226.9413
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         59.7999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        933.8629
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1220.6042

EVAL NUMBER     1      Y =   2.970      BUFFER =    11.880
     THE LATENESS COMPONENT OF TOTAL VAR. COST =        233.5953
     THE STORAGE SPACE COMPONENT OF TOTAL COST =         59.3999
     THE INVENTORY VALUE COMPONENT OF TOT COST =        927.6163
     THE TOTAL VARIABLE COST OF PROCUREMENT =          1220.6117

THE MINIMUM EXPECTED VARIABLE COST OF PROCUREMENT  =     1220.58
THE OPTIMAL BUFFER FOR THIS TVC IS AT Y =  2.98 OR Y*SIGMA =  11.91 DAYS.
```

Figure 16. Sample Problem Output of FORTRAN Program of Appendix B

CHAPTER V

POISSON DISTRIBUTION OF DELIVERY DATE

A third distribution, the Poisson, will be used to describe the probability of occurrence of delivery date. Like the chi-square, the distribution of probability (or shape) of the Poisson is what one would logically assume for a random experiment such as delivery of a component. Many procurement analysts attempting to apply this model will find the Poisson much more familiar and thus easier to deal with than the chi-square. In addition the discrete nature of the Poisson should make it easy to apply since the probabilities of delivery and lateness costs can be specified for given days rather than as continuous functions over time. The Poisson is no stranger to applications of this type as it is used in queueing theory to describe the arrival probabilities of units to be serviced. However, the development of the expected lateness cost and TVC(y) under the assumption of a Poisson distribution of delivery date does not provide an easily differentiable TVC(y) expression. As in the case of the chi-square, computer programs were written to calculate TVC given y and to use Fibonacci search to find y* and TVC*. The sample problem of Chapters III and IV is again solved under the assumption of a Poisson distribution for delivery date.

Assumption of Poisson Distribution

The manner in which the Poisson distributes probability over different possible delivery dates is very similar to that of the chi-square as shown in Figure 17. The greatest difference is that the Poisson is a discrete and the chi-square is a continuous distribution. Both have positive probability for values between zero and infinity and have similar shapes for values of μ and r of four or more. When the Poisson parameter μ is of this magnitude, the probability of occurrence at a particular integer is very small for small integers. In the context of a delivery date distribution, this means the probability of delivery on the very early possible delivery dates is small. The probability of delivery increases for each succeeding day until the mode is reached on or near the expected delivery date μ. Following this date, the probability of delivery tapers off into a long tail as with the chi-square. Another important similarity is that the Poisson distribution is determined by a single parameter μ and the proper value of the parameter to be used in calculations for a given situation can be determined in a manner similar to that used to find r in the case of the chi-square.

The p. d. f. of the Poisson distribution is

$$\begin{aligned} f(t) &= \frac{\mu^t e^{-\mu}}{t!} \quad , \quad t = 0, 1, 2, \ldots \\ &= 0 \quad , \text{ elsewhere.} \end{aligned} \tag{5.1}$$

The mean and standard deviation of the Poisson are both determined by the parameter μ, which is the mean of the distribution.

$$\mu = \mu \quad , \quad \sigma = \sqrt{\mu} \ . \tag{5.2}$$

Poisson distributions for different values of the parameter μ are shown in Figure 17.

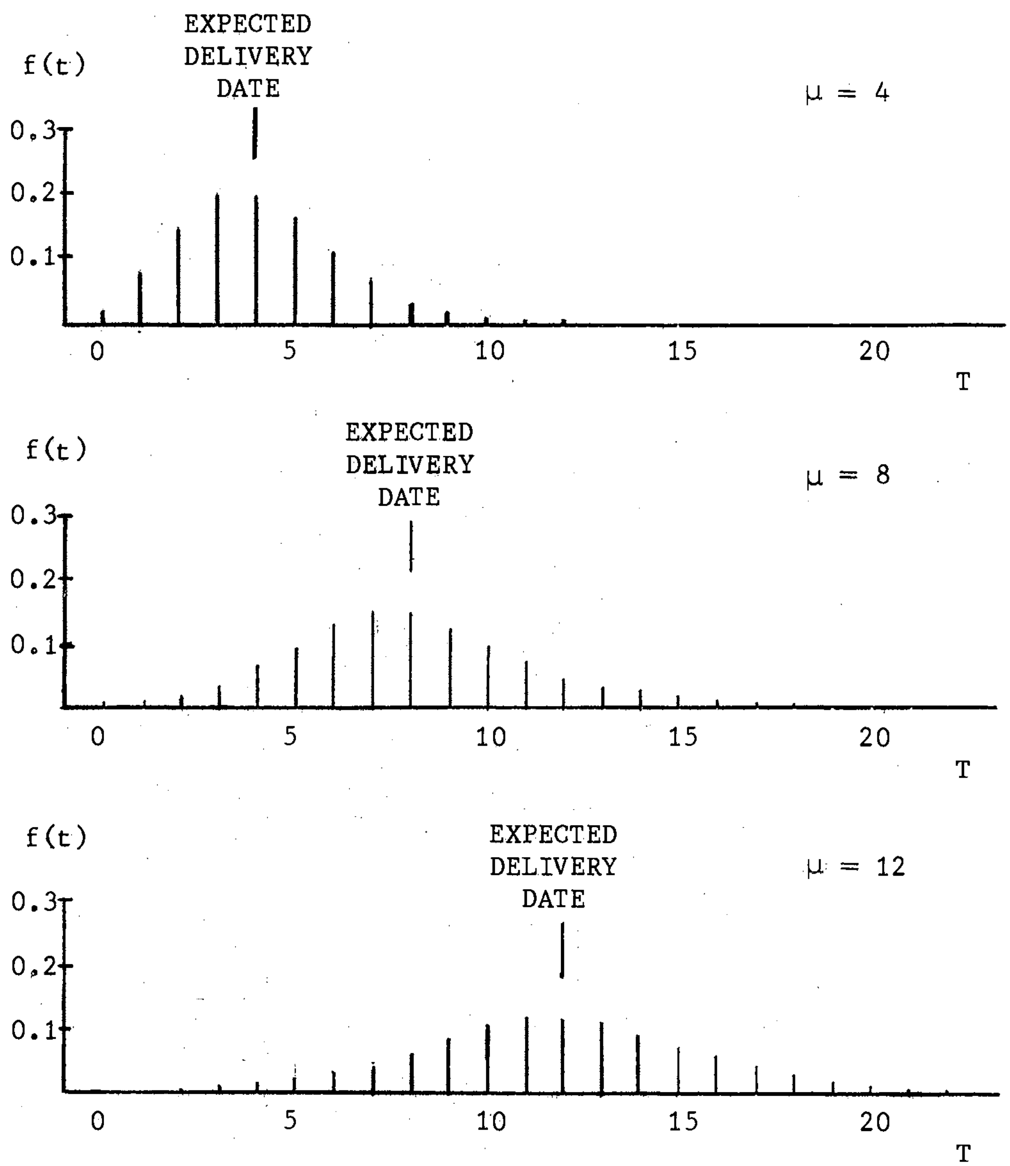


Figure 17. Poisson Distribution of Delivery Date for $\mu = 4$, 8, and 12

The Poisson will be used to describe the probability of delivery on a given date in the same manner as was done for the chi-square in Chapter IV:

a. the mean value of the Poisson distribution of delivery date will be specified at the contracted delivery date which is to be determined by the model,

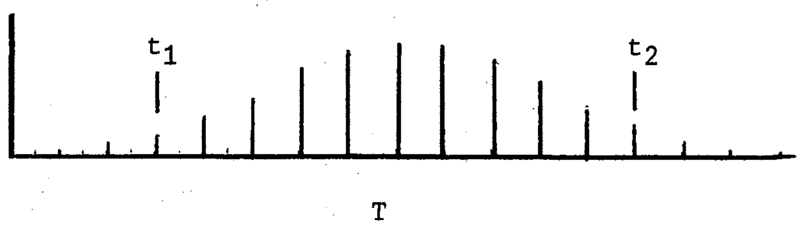
b. a 90% range on the delivery of the component is specified by procurement personnel, and

c. the length of this interval will be used to specify the distribution parameter μ.

The mean of the Poisson will be defined as the expected delivery date, and this will be specified as the delivery date to the vendor when the proper buffer time is calculated. In steps (b) and (c), the process used to find the proper μ for calculations will be similar to that employed to find r in Chapter IV. A .90 range on delivery date will be defined either from a subjective evaluation by the procurement personnel or by establishing a confidence interval on delivery date from past performance of the vendor. The length of this 90% range should then be used to determine the proper value of μ to use in calculations with the aid of Table II. For example, if there is a 90% chance of delivery occurring within a 13 day interval, the proper μ to use in calculations is $\mu = 16$ from Table II. Because of the discrete nature of the Poisson, it is seldom possible to obtain a 90% range with exactly .05 probability on each side for each value of μ. However, this was done when possible and the remaining probability was balanced as evenly as possible on both sides of the .90 range for other values of μ in Table II.

TABLE II

PROBABILITY INTERVALS OF 90% FOR DETERMINATION OF POISSON PARAMETER μ



Cumulative Probability for Values of T				Interval Length	Probability of Delivery Within Interval	Value of μ for This Interval Length
t_1	$P_r\ T \leq t_1$	t_2	$P_r\ T \leq t_2$			
0	.050	6	.960	6 days	.910	3
1	.040	8	.932	7 days	.892	5
2	.062	10	.957	8 days	.894	6
3	.042	12	.936	9 days	.894	8
4	.055	14	.959	10 days	.904	9
5	.038	16	.944	11 days	.906	11
7	.054	19	.957	12 days	.903	13
9	.043	22	.942	13 days	.899	16
11	.055	25	.955	14 days	.900	18
13	.043	28	.944	15 days	.901	21
16	.056	32	.953	16 days	.897	24

Source (Cumulative Probabilities for Poisson): W. J. Fabrycky and Paul E. Torgersen, Operations Economy (Englewood Cliffs, N.J., 1966), Appendix Table A.1, pp. 446-9. Additional cumulative probabilities may be computed from $\sum(\mu^x e^{-\mu}/x!)$ for larger values of μ.

Development of Expected Lateness Cost

In Chapter II the lateness costs were expressed as a continuous function of time in equation (2.10). The development of a discrete function to represent lateness costs would involve the same logic. If delivery has not occurred by a certain number of days (d_σ) prior to the requirement date, expediting and other lateness charges begin to be incurred as shown in Figure 18. As each day passes, higher and higher lateness costs are incurred. Thus, the total lateness cost incurred if delivery occurs on a given day rises as the requirement date approaches. If delivery does not occur as the requirement date is reached and passed, rescheduling and production delays are incurred which result in higher and higher total lateness costs incurred for each succeeding delivery date. The lateness cost incurred by a part if delivery is made on day x will be represented by the following discrete function C(x) where the value of x on the requirement date is d_σ.

$$\begin{aligned} C(x) &= Kx^m \quad , \quad \text{where} \quad x = 0, 1, 2, \ldots \\ &= 0 \quad , \quad x < 0, \end{aligned} \tag{5.3}$$

where K = a scaling constant

x = the delivery time (a random variable)

m = an exponent determining the rate of increase of lateness costs with time.

A change of variable will now be made in equation (5.3) in order to make the origin of the lateness cost function coincide with the Poisson distribution of delivery date. The common origin for this development will be the origin of the Poisson delivery date distribution instead of the requirement date as was the case in previous developments although

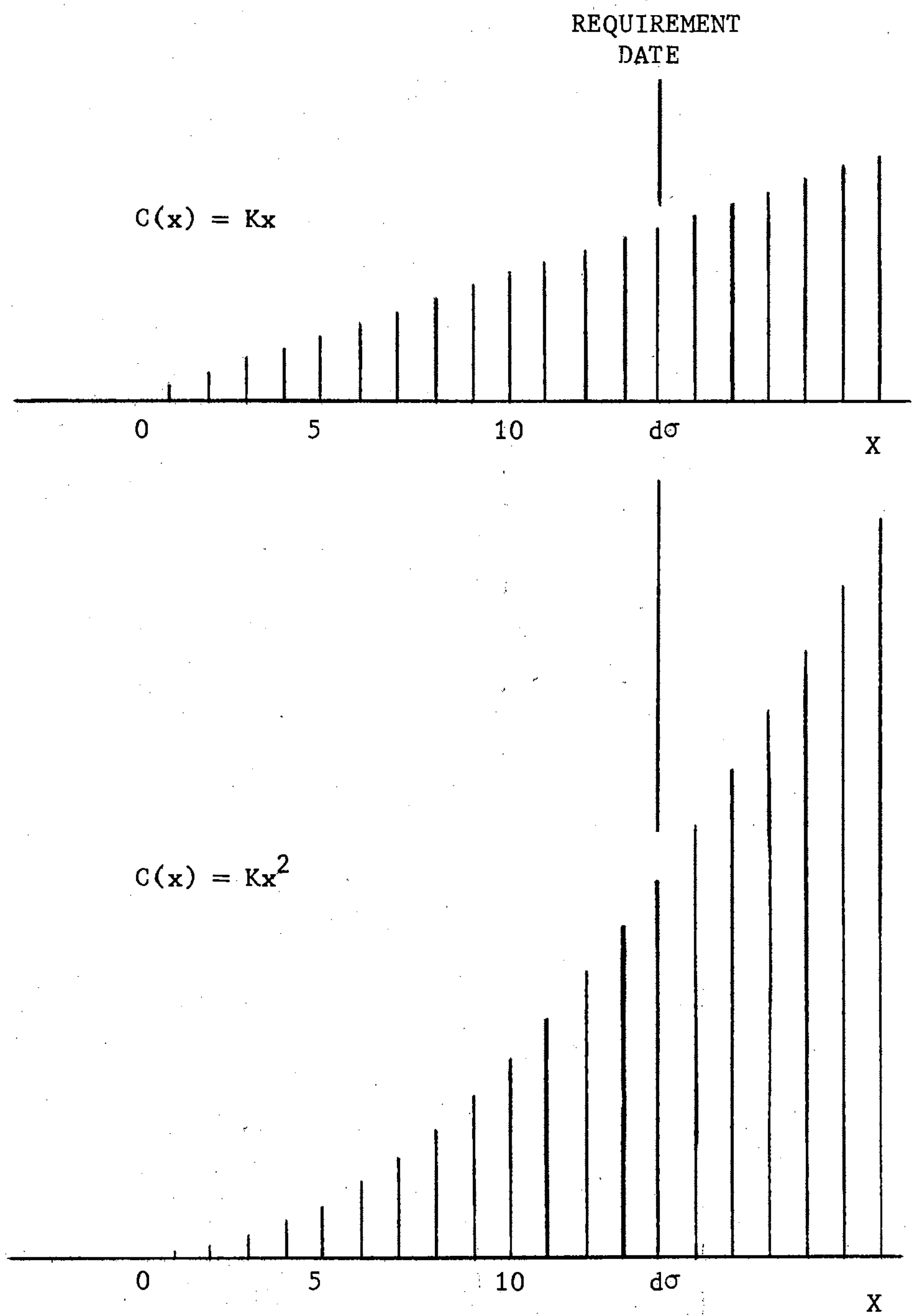


Figure 18. Discrete Lateness Cost Function C(x) for m = 1 and m = 2

the requirement date will be used as the reference point in relating the cost function to the delivery date distribution. As is shown in Figure 19 the value of t for the delivery date distribution is $(\mu+y\sigma)$ at the requirement date. The change of variable must be such that the value of the cost function at the requirement date is $K(d\sigma)^m$, since the present value of x at the requirement date is $d\sigma$. The proper change of variable for the lateness cost function is then to let

$$x = t - (\mu+y\sigma-d\sigma)$$

$$t = x + (\mu+y\sigma-d\sigma)$$

at $$x = 0, \quad t = \mu + y\sigma - d\sigma = \mu + \sigma(y-d)$$

$$x = d\sigma, \quad t = \mu + y\sigma$$

$$\begin{aligned} C(t) &= K\Big[t-(\mu+y\sigma-d\sigma)\Big]^m, \quad & t &= \mu + \sigma(y-d), \\ & & &\mu + \sigma(y-d) + 1, \\ & & &\mu + \sigma(y-d) + 2, \ldots \\ &= 0 \quad , & t &< \mu + \sigma(y-d), \end{aligned} \tag{5.4}$$

for $$\mu = 0, 1, 2, \ldots \tag{5.5}$$

$$y\sigma = 0, 1, 2, \ldots \tag{5.6}$$

$$d\sigma = 0, 1, 2, \ldots \tag{5.7}$$

This change of variable does not change the value of the lateness cost function from its previous value in equation (5.3) at any date before or after the requirement date; it merely changes the arbitrary zero reference point such that lateness cost may be expressed in terms of the random variable T. The requirements of equations (5.5), (5.6), and (5.7) are the most practical way of insuring that the quantity

$$t - (\mu+y\sigma-d\sigma) \tag{5.8}$$

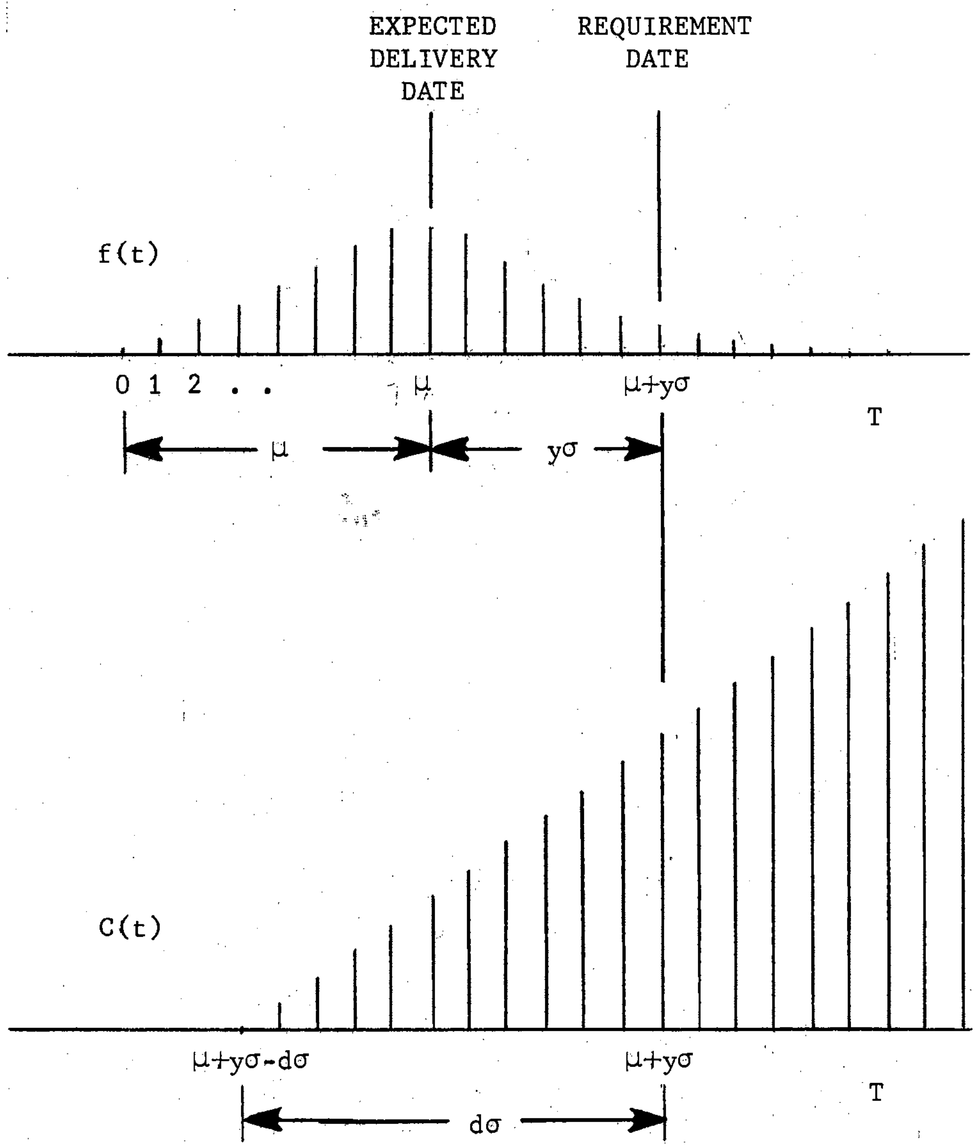


Figure 19. Poisson Delivery Date Distribution and Lateness Cost Function

is an integer for all integer values of t.

Now that the probability of delivery on day t is expressed in the same terms as the lateness cost incurred if delivery is on day t, an equation similar to equation (2.13) may be developed to define the expected lateness cost.

$$E(LC) = \sum_{t=0}^{\mu+\sigma(y-d)-1} 0\cdot f(t) + \sum_{t=\mu+\sigma(y-d)}^{\infty} K\left[t-(\mu+y\sigma-d\sigma)\right]^m \cdot f(t),$$

$$t = 0, 1, 2, \ldots \quad (5.9)$$

In order to simplify notation, let

$$s = \mu + y\sigma - d\sigma = \mu + \sigma(y-d). \quad (5.10)$$

Then equation (5.9) will reduce to

$$E(LC) = \sum_{t=s}^{\infty} K(t-s)^m \frac{\mu^t e^{-\mu}}{t!} . \quad (5.11)$$

It is necessary to manipulate equation (5.11) into an expression involving a finite series. This can be accomplished by expressing equation (5.11) as the difference between an infinite series with a known sum and a finite series as in equation (5.12).

$$E(LC) = \sum_{t=0}^{\infty} K(t-s)^m \frac{\mu^t e^{-\mu}}{t!} - \sum_{t=0}^{s-1} K(t-s)^m \frac{\mu^t e^{-\mu}}{t!}. \quad (5.12)$$

Equation (5.12) will now be manipulated to give expressions for expected lateness cost for the cases of m = 1, 2, and 3.

<u>Expected Lateness Cost for m = 1</u>.

For m = 1 equation (5.12) becomes

$$E(LC) = \sum_{t=0}^{\infty} K(t-s) \frac{\mu^t e^{-\mu}}{t!} - \sum_{t=0}^{s-1} K(t-s) \frac{\mu^t e^{-\mu}}{t!} . \qquad (5.13)$$

If the quantity (t-s) is distributed and f(t) used to represent the p. d. f. of equation (5.1),

$$E(LC) = K\left\{\sum_{0}^{\infty} t\cdot f(t) - s\sum_{0}^{\infty} f(t) - \sum_{0}^{s-1} tf(t) + s\sum_{0}^{s-1} f(t)\right\} . \qquad (5.14)$$

By the definition of a p. d. f., the Poisson p. d. f. summed over its interval of positive probability is equal to 1.0. Thus,

$$\sum_{t=0}^{\infty} f(t) = 1.0. \qquad (5.15)$$

Also, the first moment of the Poisson p. d. f. is found to be its mean; thus,

$$E(t) = \sum_{t=0}^{\infty} t\ f(t) = \mu . \qquad (5.16)$$

Substitution of equations (5.15) and (5.16) into (5.14) gives

$$E(LC) = K\left\{\mu - s(1) - \sum_{t=0}^{s-1} tf(t) + s\sum_{0}^{s-1} f(t)\right\} . \qquad (5.17)$$

Since $s = \mu + \sigma(y-d)$ and $\sigma = \sqrt{u}$, equation (5.17) can be reduced to the following expression for expected lateness cost for a value of m = 1.

$$E(LC) = K\left\{-\sqrt{\mu}(y-d) + s\sum_{t=0}^{s-1} f(t) - \sum_{t=0}^{s-1} t\cdot f(t)\right\}, \text{ for } t = 0, 1, 2, \ldots \tag{5.18}$$

Expected Lateness Cost for m = 2

For m = 2 equation (5.12) becomes

$$E(LC) = \sum_{t=0}^{\infty} K(t-s)^2 f(t) - \sum_{t=0}^{s-1} k(t-s)^2 f(t). \tag{5.19}$$

The first series of equation (5.19) can be reduced from an infinite series to a general expression for the sum.

$$\text{Series } 1 = \sum_{t=0}^{\infty} K(t-s)^2 f(t)$$

$$= \sum_{t=0}^{\infty} K(t^2-2st+s^2)ft$$

$$= K\left\{s^2 \sum_{0}^{\infty} f(t) - 2s\sum_{0}^{\infty} tf(t) + \sum_{0}^{\infty} t^2 f(t)\right\}. \tag{5.20}$$

The second moment of the Poisson is

$$E(t^2) = \sum_{t=0}^{\infty} t^2 f(t) = \mu^2 + \mu. \tag{5.21}$$

Substituting equations (5.15), (5.16), and (5.21) into (5.20) gives

$$\text{Series } 1 = K\left[s^2 - 2s(\mu) + (\mu^2+\mu)\right]. \tag{5.22}$$

Substitution of $s = \mu + \sigma(y-d)$ and $\sigma = \sqrt{\mu}$ allows reduction of the infinite series of equation (5.19) to

$$\text{Series 1} = K\mu\left[1 + (y-d)^2\right]. \tag{5.23}$$

The finite series of equation (5.19) can be manipulated to give

$$\text{Series 2} = K\left\{s^2\sum_0^{s-1} f(t) - 2s\sum_0^{s-1} tf(t) + \sum_0^{s-1} t^2 f(t)\right\}. \tag{5.24}$$

Combining equations (5.23) and (5.24) into equation (5.19) results in the following expression for expected lateness cost for the case $m = 2$.

$$E(LC) = K\left\{\mu\left[1+(y-d)^2\right] - s^2\sum_0^{s-1} f(t) + 2s\sum_0^{s-1} t\cdot f(t) - \sum_0^{s-1} t^2 f(t)\right\}.$$

$$\text{for } t = 0, 1, 2, \ldots \tag{5.25}$$

<u>Expected Lateness Cost for m = 3</u>

For $m = 3$, equation (5.12) becomes

$$E(LC) = \sum_{t=0}^{\infty} K(t-s)^3 f(t) - \sum_{t=0}^{s-1} K(t-s)^3 f(t). \tag{5.26}$$

Proceeding as before, the first series can be reduced to a general expression. Expansion of $(t-s)^3$ and distribution of the summation gives

$$\text{Series 1} = K\left\{-s^3\sum_0^{\infty} f(t) + 3s^2\sum_0^{\infty} tf(t) - 3s\sum_0^{\infty} t^2 f(t) + \sum_0^{\infty} t^3 f(t)\right\}. \tag{5.27}$$

The third moment of the Poisson is

$$E(t^3) = \sum_{t=0}^{\infty} t^3 f(t) = \mu^3 + 3\mu^2 + \mu. \tag{5.28}$$

Substitution of equations (5.15), (5.16), (5.21), and (5.28) into equation (5.27) gives

$$\text{Series } 1 = K\left[-s^3 + 3s^2(\mu) - 3s(\mu^2+\mu) + (\mu^3+3\mu^2+\mu)\right]. \tag{5.29}$$

Substitution of $s = \mu + \sigma(y-d)$ and $\sigma = \sqrt{\mu}$ allows reduction of the infinite series of equation (5.26) to

$$\text{Series } 1 = K\mu\left[1-3\sqrt{\mu}(y-d)-\sqrt{\mu}(y-d)^3\right]. \tag{5.30}$$

The finite series of equation (5.26) can be manipulated to give

$$\text{Series } 2 = K\left\{-s^3\sum_0^{s-1} f(t) + 3s^2\sum_0^{s-1} tf(t) - 3s\sum_0^{s-1} t^2 f(t) + \sum_0^{s-1} t^3 f(t)\right\}. \tag{5.31}$$

These can be combined according to equation (5.26) into the following expression for expected lateness cost for the case m = 3.

$$E(LC) = K\left\{\mu\left[1-3\sqrt{\mu}(y-d) - \sqrt{\mu}(y-d)^3\right] + s^3\sum_0^{s-1} f(t)\right.$$

$$\left. -3s^2\sum_0^{s-1} tf(t) + 3s\sum_0^{s-1} t^2 f(t) - \sum_0^{s-1} t^3 f(t)\right\},$$

$$\text{for } t = 0, 1, 2, \ldots \tag{5.32}$$

Fibonacci Search for y* and TVC*

Expressions for TVC(y) may now be written utilizing the expected cost components of inventory value and storage space from equation (2.14) and the expected lateness cost components just developed. For the case of m = 1,

$$TVC(y) = \left[\frac{VP+WC_h}{365}\right] y\sqrt{\mu} + K\left\{- \sqrt{\mu}(y-d) + s\sum_{t=0}^{s-1} f(t) - \sum_{t=0}^{s-1} tf(t)\right\},$$

$$\text{for } t = 0, 1, 2, \ldots \qquad (5.33)$$

For the case of m = 2,

$$TVC(y) = \left[\frac{VP+WC_h}{365}\right] y\sqrt{\mu}+K\left\{\mu\left[1+(y-d)^2\right] - s^2\sum_{t=0}^{s-1} f(t) + 2s\sum_{t=0}^{s-1} tf(t)\right.$$

$$\left. - \sum_{t=0}^{s-1} t^2 f(t)\right\}, \quad t = 0, 1, 2, \ldots \qquad (5.34)$$

For the case of m = 3,

$$TVC(y) = \left[\frac{VP+WC_h}{365}\right] y\sqrt{\mu}+K\left\{\mu\left[1-3\sqrt{\mu}(y-d)-\sqrt{\mu}(y-d)^3\right] + s^3\sum_{t=0}^{s-1} f(t)\right.$$

$$\left. -3s^2\sum_{t=0}^{s-1} t\cdot f(t) + 3s\sum_{t=0}^{s-1} t^2 f(t) - \sum_{t=0}^{s-1} t^3 f(t)\right\},$$

$$t = 0, 1, 2, \ldots \qquad (5.35)$$

Also, the restrictions of equations (5.5), (5.6), and (5.7) must be met for the TVC expressions to be valid.

$$\mu = 0, 1, 2, \ldots \tag{5.5}$$

$$y\sigma = 0, 1, 2, \ldots \tag{5.6}$$

$$d\sigma = 0, 1, 2, \ldots \tag{5.7}$$

Equation (5.6) restricts the buffer to a whole number of days, and equation (5.7) requires that expediting procedures are started a whole number of days prior to the requirement date.

The parameter s appears in both the summand and as the terminal value of the index of summation in each of the expressions for TVC(y). Recalling equation (5.10) it is important to note that s is a function of y.

$$s = \mu + \sqrt{\mu}(y-d) \tag{5.10}$$

Thus, TVC(y) is not easily differentiable with respect to y. As a result the optimal TVC* and y* must be found utilizing a search procedure as was the case with the chi-square distribution of delivery date. Again, a Fibonacci search can be employed; and a computer program was written to facilitate the calculations for an optimal buffer and TVC* for the assumption of a Poisson distribution for delivery date.

A program to calculate TVC for a given set of cost parameters is included in Appendix C. This program, written in FORTRAN, will calculate TVC based on the expressions stated in equations (5.33), (5.34), and (5.35) depending on which value of m is specified in the input data. This program will be most helpful in calculating the expected TVC of procurement for a given buffer time, expediting strategy, and set of cost parameters.

A program utilizing Fibonacci search to find the optimal buffer and TVC* is included in Appendix D. This program searches TVC(y) over a range of buffer lengths from 0 to 88 days. It converts the integer value of the buffer to the corresponding value of y, calculates TVC(y), and continues further evaluations of buffer and TVC(y) until the optimal buffer is found. Searching values of buffer from 1 to 88 days requires 9 evaluations of TVC, and the different values of buffer evaluated are logged as the search progresses. If the optimal buffer is found to be one day after the nine evaluations of TVC, a tenth evaluation will be made for a zero buffer to determine if it is the minimum TVC. Each evaluation of TVC(y) is recorded in the output, and a curve can be plotted from these to determine the sensitivity of TVC to buffer for any set of cost parameters.

Sample Problem

The same problem presented in Chapter III will be solved under the assumption of a Poisson distribution of delivery date to compare the results obtained under the three distribution assumptions. All cost parameters will remain the same. This time it is decided to use a Poisson distribution to describe the delivery date random variable of the Uncertain Delivery Company. Since delivery can be expected within a 13 day interval with probability of .90, the distribution parameter μ is found to be $\mu = 16$ from Table II. The Least Cost Company's policy concerning expediting procedures is the same as previously with expediting inquiries and other procedures beginning eight days prior to the requirement date if the part has not been received.

It is desired to calculate the expected TVC of procurement for the buffer time of four weeks that is generally used in dealings with the Uncertain Delivery Company. Calculation of the optimal buffer and TVC* is also desired.

The cost parameters are as follows:

$V = \$190,000$

$P = .15$ per year

$W = 10$ square feet

$C_h = \$182.50$ per square foot per year

$K = 10$

$m = 3$

$d = 8$ days$/\sqrt{\mu} = 2$

.90 range on Delivery Date = 13 days

μ from Table II = 16

$s = \mu + \sqrt{\mu}(y-d) = 16 + 4(7-2) = 36$

For a buffer of 28 days, $y = 28/4 = 7.0$.

From equation (5.35) the expected TVC of procurement may be calculated.

$$TVC = \left[\frac{(190,000)(.15) + (10)(182.50)}{365}\right](7)(4)$$

$$+ 10 \left\{16\left[1-(3)(4)(7-2)-4(7-2)^3\right] + 36^3 \sum_{t=0}^{35} \frac{16^t e^{-16}}{t!}\right.$$

$$-(3)(36)^2 \sum_{t=0}^{35} t \frac{16^t e^{-16}}{t!} + (3)(36) \sum_{t=0}^{35} t^2 \frac{16^t e^{-16}}{t!}$$

$$\left. - \sum_{t=0}^{35} t^3 \frac{16^t e^{-16}}{t!}\right\}. \tag{5.36}$$

From the computer program in Appendix C, the cost components and expected TVC for the 28 day buffer are calculated as follows.

Expected Inventory Value Cost	=	\$2186.30
Expected Storage Space Cost	=	140.00
Expected Lateness Cost	=	.35
Expected TVC of Procurement	=	\$2326.65

As in the case of the chi-square the expected lateness cost for this buffer is negligible. The expected TVC of procurement under the assumption of a Poisson delivery date differs by only \$.05 from the TVC calculated under the assumption of a chi-square distribution. The output of the program in Appendix C for this problem is given in Figure 20.

Calculation of the optimal buffer is done through a Fibonacci search of TVC with the aid of the computer program in Appendix D. The optimal buffer is found to be 11 days and the minimum TVC* is \$1,059.07. This compares with an optimal buffer of 12 days and TVC* of \$1,220.58 calculated under the assumption of a chi-square distribution of delivery date. The two solutions for the optimal buffer vary by less than 10% and the expected TVC* by slightly over 10%. Thus, the assumption of the Poisson and the chi-square to describe the delivery date produce very similar results.

The output of the FORTRAN program of Appendix D is shown in Figure 21. It should be noted that use of an 11 day (optimal) buffer instead of the 28 day buffer would result in expected savings in the TVC of procurement of \$2,376.70-\$1,059.07 = \$1317.63. Although \$145 additional lateness cost can be expected with the shorter buffer, substantial savings can be expected from lower holding costs.

Exponential Lateness Cost With Poisson Delivery

All developments of TVC in Chapters III, IV, and V have used

```
    PARAMETERS ARE AS FOLLOWS -- U=MEAN OF POISSON =  16.0
                                 Y=      7.000
                                 D=      2.000
                                 EXPEDITING COST EXPONENT (M)=     3.000
                                 EXPEDITING COST SCALE FACTOR=    10.000
                                 SPACE REQUIRED=             10.00
                                 COST/SPACE/YEAR=           182.50000
                                 VALUE OF PART =         190000.00
                                 COST OF CAPITAL=              .1500
                                 NUMBER OF TERMS IN SERIES =   35
EXPEDITING STARTS D*SIGMA =    8  DAYS BEFORE REQUIREMENT DATE.

     THE LATENESS COMPONENT OF TOTAL VAR. COST =             .3460

     THE STORAGE SPACE COMPONENT OF TOTAL COST =          139.9999

     THE TIED-UP CAPITAL COMPONENT OF TOT COST =         2186.3011

     THE TOTAL VARIABLE COST OF PROCUREMENT =            2326.6470

THE BUFFER FOR THIS Y =    7.00  IS Y*SIGMA OR  28  DAYS.
```

Figure 20. Sample Problem Output of FORTRAN Program of Appendix C

```
PARAMETERS ARE AS FOLLOWS-- U= MEAN OF POISSON =  16.0
                            D=      2.000
                            EXPEDITING COST EXPONENT (M)=      3.000
                            EXPEDITING COST SCALE FACTOR=     10.000
                            SPACE REQUIRED=                 10.00
                            COST/SPACE/YEAR=              182.50000
                            VALUE OF PART =             190000.00
                            COST OF CAPITAL=                 .1500

FIBONACCI SEARCH NO.   9, BUFFER FOR THIS CALCULATION OF T.V.C. IS     55.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =              .5430
      THE STORAGE SPACE COMPONENT OF TOTAL COST =           274.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =          4294.5204
      THE TOTAL VARIABLE COST OF PROCUREMENT =             4570.0633

FIBONACCI SEARCH NO.   8, BUFFER FOR THIS CALCULATION OF T.V.C. IS     34.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =              .4570
      THE STORAGE SPACE COMPONENT OF TOTAL COST =           169.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =          2654.7944
      THE TOTAL VARIABLE COST OF PROCUREMENT =             2825.2513

FIBONACCI SEARCH NO.   7, BUFFER FOR THIS CALCULATION OF T.V.C. IS     21.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =              .3410
      THE STORAGE SPACE COMPONENT OF TOTAL COST =           104.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =          1639.7259
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1745.0668

FIBONACCI SEARCH NO.   6, BUFFER FOR THIS CALCULATION OF T.V.C. IS     13.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =            49.9080
      THE STORAGE SPACE COMPONENT OF TOTAL COST =            64.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =          1015.0684
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1129.9763

FIBONACCI SEARCH NO.   5, BUFFER FOR THIS CALCULATION OF T.V.C. IS      8.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =           593.3150
      THE STORAGE SPACE COMPONENT OF TOTAL COST =            39.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =           624.6574
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1257.9724

FIBONACCI SEARCH NO.   4, BUFFER FOR THIS CALCULATION OF T.V.C. IS     16.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =             8.2910
      THE STORAGE SPACE COMPONENT OF TOTAL COST =            79.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =          1249.3150
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1337.6059

FIBONACCI SEARCH NO.   3, BUFFER FOR THIS CALCULATION OF T.V.C. IS     11.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =           145.1730
      THE STORAGE SPACE COMPONENT OF TOTAL COST =            54.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =           858.9040
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1059.0770

FIBONACCI SEARCH NO.   2, BUFFER FOR THIS CALCULATION OF T.V.C. IS     10.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =           238.1040
      THE STORAGE SPACE COMPONENT OF TOTAL COST =            49.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =           780.8218
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1068.9258

FIBONACCI SEARCH NO.   1, BUFFER FOR THIS CALCULATION OF T.V.C. IS     12.0 DAYS
      THE LATENESS COMPONENT OF TOTAL VAR. COST =            86.2350
      THE STORAGE SPACE COMPONENT OF TOTAL COST =            59.9999
      THE TIED-UP CAPITAL COMPONENT OF TOT COST =           936.9862
      THE TOTAL VARIABLE COST OF PROCUREMENT =             1083.2212


THE OPTIMAL TOTAL EXPECTED VARIABLE COST OF PROCUREMENT IS          1059.07
THE BUFFER TIME RESULTING IN THIS MINIMUM TVC IS     11.0 DAYS.
```

Figure 21. Sample Problem Output of FORTRAN Program of Appendix D

equation (2.8) to represent the costs of lateness over time. In equation (2.8), $C(x) = Kx^m$ for $x \geq 0$, integer values of 1, 2 and 3 were allowed for m. This is because integer values for m were required as a part of each of the mathematical developments of expected lateness cost presented thus far. Manipulation of K for different integer values of m should allow sufficient flexibility in defining lateness costs so that there should be no difficulty in applying the model to real-world procurement situations. The consistent use of equation (2.8) also resulted in models with common parameters which will facilitate comparisons in later chapters.

However, an even more flexible expression of lateness cost is possible for the assumption of a Poisson distribution of delivery date. The Poisson p. d. f. is such that exponential terms are easily manipulated in conjunction with it. This section will develop a TVC equation for an exponential lateness cost function for those who may prefer its use.

Assume that lateness costs are incurred in the same manner as previously, but now assume that they are approximated by the function

$$\begin{aligned} C(x) &= Ke^{mx} \quad , \quad x = 0, 1, 2, \ldots \\ &= 0 \qquad\quad , \quad x < 0, \end{aligned} \tag{5.37}$$

where K and m are both non-negative scaling constants. Since there is no requirement for m to be an integer, equation (5.37) is a more flexible cost function than equation (2.8).

A change of variable is now introduced to put the lateness cost function in the same terms of the Poisson p. d. f. of equation (5.1). The change of variable will be the same as that used to obtain equation

(5.4). Let

$$x = t-(\mu+y\sigma-d\sigma)$$

$$t = x+(\mu+y\sigma-d\sigma)$$

at $$x = 0, \quad t = \mu+y\sigma-d\sigma = \mu+\sigma(y-d)$$

$$x = d\sigma, \ t = \mu+y\sigma$$

$$\begin{aligned} C(t) &= Ke^{m[t-(\mu+y\sigma-d\sigma)]}, \quad t = \mu+\sigma(y-d), \\ &\qquad\qquad \mu+\sigma(y-d)+1, \\ &\qquad\qquad \mu+\sigma(y-d)+2, \ldots \\ &= 0, \quad t<\mu+\sigma(y-d). \end{aligned} \tag{5.38}$$

As was the case previously the change of variable does not alter the value of the lateness cost function at any point with respect to the requirement date; it merely shifts the origin. The relationship of C(t) and the p. d. f. of delivery date is shown in Figure 22.

The expected lateness cost can now be represented as

$$E(LC) = \sum_{t=0}^{\mu+\sigma(y-d)-1} (0) \frac{\mu^t e^{-\mu}}{t!} + \sum_{t=\mu+\sigma(y-d)}^{\infty} Ke^{m[t-(\mu+y\sigma-d\sigma)]} \frac{\mu^t e^{-\mu}}{t!}. \tag{5.39}$$

In order to simplify notation, equation (5.10) will be employed as before.

$$s = \mu+y\sigma-d\sigma = \mu+\sigma(y-d). \tag{5.10}$$

Equation (5.39) can be manipulated to give a finite series expression for expected lateness cost as follows.

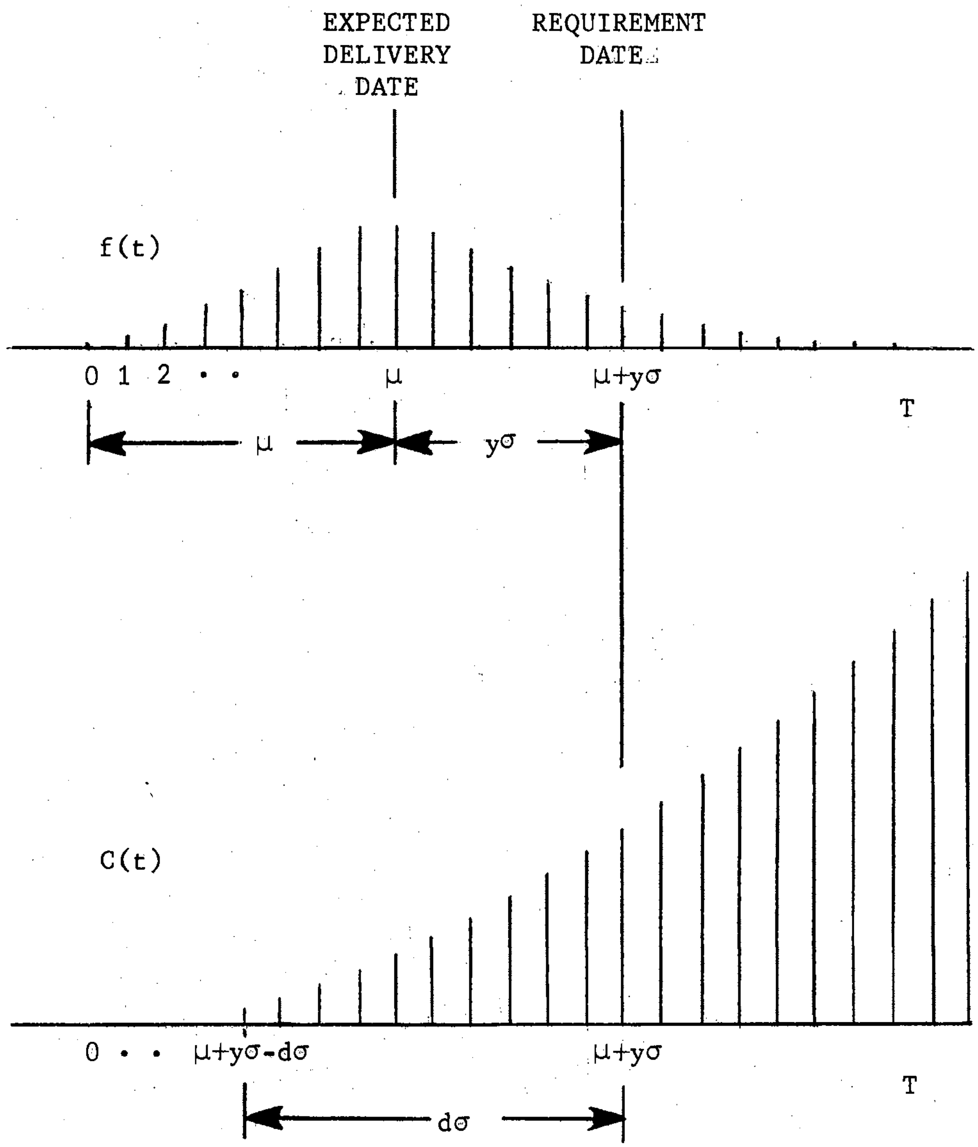


Figure 22. Poisson Delivery Date Distribution and Exponential Lateness Cost Function

$$E(LC) = \sum_{t=s}^{\infty} Ke^{m(t-s)} \frac{\mu^t e^{-\mu}}{t!} \tag{5.40}$$

$$= \sum_{t=0}^{\infty} Ke^{m(t-s)} \frac{\mu^t e^{-\mu}}{t!} - \sum_{t=0}^{s-1} Ke^{m(t-s)} \frac{\mu^t e^{-\mu}}{t!}$$

$$= Ke^{-ms-\mu+\mu e^m} \sum_{t=0}^{\infty} \frac{(\mu e^m)^t e^{-\mu e^m}}{t!} - \sum_{t=0}^{s-1} Ke^{m(t-s)} \frac{\mu^t e^{-\mu}}{t!} \tag{5.41}$$

$$= Ke^{\mu(e^m-1-m)-m\sqrt{\mu}(y-d)} \left[1 \right] - \sum_{t=0}^{s-1} Ke^{m(t-s)} \frac{\mu^t e^{-\mu}}{t!} \tag{5.42}$$

Similar manipulations can be employed to the finite series to give the following expression for lateness cost.

$$E(LC) = Ke^{\mu(e^m-1-m)-m\sqrt{\mu}(y-d)} \left[1 - \sum_{t=0}^{\mu+\sqrt{\mu}(y-d)-1} \frac{(\mu e^m)^t e^{-\mu e^m}}{t!} \right]. \tag{5.43}$$

This expression may be readily evaluated for given values of K, μ, m, y, and d. It is the product of a constant times a probability. The term in the brackets is 1.0 minus the cumulative probability of a Poisson random variable with mean = μe^m from zero to $\left[\mu+\sqrt{\mu}(y-d)\right]$. Because the tables available on the Poisson are quite extensive, this quantity may be evaluated easily. This will facilitate the calculation of expected lateness cost.

The expected TVC of procurement may be expressed as follows for the case of exponential lateness cost.

$$TVC(y) = \left[\frac{VP+WC_h}{365}\right] y\sqrt{\mu}$$
$$+ Ke^{\mu(e^m-m-1)-m\sqrt{\mu}(y-d)} \left[1 - \sum_{t=0}^{\mu+\sqrt{\mu}(y-d)-1} \frac{(\mu e^m)^t e^{-\mu e^m}}{t!} \right]. \tag{5.44}$$

Integer values are required for μ, $y\sqrt{\mu}$ (the buffer), and $d\sqrt{\mu}$ (the number of days prior to requirement date that expediting starts if delivery is not made). The optimal buffer may be found through a Fibonacci search of non-zero buffer times as was done for the case of the Poisson distribution of delivery with m = 1, 2, and 3 in previous sections of this chapter.

CHAPTER VI

SENSITIVITY ANALYSIS OF PARAMETERS

In the preceding chapters models were developed to aid in the solution of a difficult procurement problem: how large a buffer should be allowed when the requirement date is firm and the delivery date for an item is uncertain. This chapter will discuss the sensitivity of responsiveness of the models to changes in parameters. The emphasis in Chapter VI will center on answering questions of the following type: if the estimate of a certain parameter is in error by a certain amount, by how much will the buffer be in error and what increase in the total variable cost can be expected? In addition to the sensitivity of the cost parameters, this chapter will discuss differences in the buffer and TVC* calculated under the models developed in Chapters III, IV, and V for different assumptions for the delivery date distribution.

In many instances sensitivity analyses such as these can be accomplished quite readily with the introduction of error ratios and algebraic manipulation of TVC equations. Unfortunately the TVC expressions of the TVC models in Chapters III, IV, and V do not allow for this type of analysis. Rather, computer simulation was used to find optimal buffers and TVC* for several different combinations of parameters representing a variety of procurement situations. This computer simulation provided a very efficient method of analysis for the models developed under the assumptions of the Poisson and chi-square distributions of delivery date since the Fibonacci search procedure provided TVC evaluations for a wide range of buffers around the optimum. For calculations involving the assumption of a uniform delivery date distribution, a desk-top logrithmic computer proved quite satisfactory.

Analysis of the sensitivity of the models is further complicated by the large number of parameters involved. A complete analysis of all possible combinations of delivery date time intervals, lateness cost parameters, holding cost parameters, and the different delivery date distributions is beyond the scope of this paper. Thus, the discussion will be limited to specific examples which illustrate the degree of responsiveness of the models to changes in one particular parameter of the models while holding other parameters constant.

Sensitivity of Buffer to Assumptions in Delivery Date Distribution

If the wrong assumption is made concerning the proper p. d. f. to be used to describe the delivery date random variable, how does this affect the optimum buffer and the expected TVC of procurement? The same problems that necessitated a Fibonacci search for optimum TVC also prevent a general mathematical formulation of sensitivity. However, considerable insight into this question can be gained through comparisons of optimal buffers and TVC* calculated under the three models developed for similar procurement situations. For a valid comparison the calculations should involve the same cost parameters and level of uncertainty in delivery. As discussed previously the standard deviation of a probability distribution is a well-accepted measure of uncertainty, and the sample problems in Chapters III, IV, and V were formulated such that the standard deviations in each case were very similar (σ = 2.0). Thus, the question concerning sensitivity of the buffer to errors in the distribution assumption will be discussed with respect to the sample problem presented in earlier chapters.

Comparison of the calculations for the sample problem shows the three different assumptions give very similar results for the optimal buffer and TVC*. The results are summarized in Table III along with the averages of results for the three distributional assumptions and the per cent deviation from the average for each case. The three optimal buffers deviate from the average by less than 10% while the expected TVC* varies from the average by slightly more than that for the assumptions of a uniform and a chi-square distribution for delivery date.

TABLE III

OPTIMAL BUFFER AND TVC* IN SAMPLE PROBLEM FOR THREE DISTRIBUTION ASSUMPTIONS

	Uniform	Chi-Square	Poisson	Average
Buffer (days)	10.2	11.9	11.0	11.03
% Difference	7.5%	7.9%	0.3%	--
TVC* ($)	949	1221	1059	1076
% Difference	11.8%	13.5%	1.6%	--

Calculations involving an assumption of uniform probability of delivery over an interval would certainly be expected to give a slightly smaller buffer and lower TVC than those calculated under an assumption of the chi-square or Poisson at the same level of uncertainty. This is because the uniform distribution has equal probability of incurring lateness charges at all delivery dates within a given interval, and thus has more of its probability weighted in the very early possible delivery dates where lateness costs are negligible than do the Poisson and chi-square. Also, the uniform distribution does not have a long "tail" which allows probability of delivery at times well past the requirement date when lateness costs are extremely high. Thus, one would expect optimal buffers calculated under a uniform distribution to be slightly less than those for the chi-square and Poisson as is shown in the results of Table III.

One might ask just how much of an increase in cost might be expected if the wrong assumption is made in the choice of a p. d. f. for the delivery date random variable with all other things equal. The comparison of all combinations of erroneous assumptions in delivery date p. d. f. are given in Table IV for the data of the sample problem solved earlier. For example if a uniform distribution is assumed, calculations will yield an optimal buffer of 10.2 days. If, in fact, the random variable delivery date has a chi-square distribution with equal variance, the 10.2 day buffer used because of the erroneous assumption gives an expected TVC of $1272 which is $51 or 4.2% above the TVC* that would be expected if the chi-square distribution had been correctly assumed and the optimal buffer of 11.9 days used. If the true distribution were a Poisson, the 10.2 day buffer used would result in an expected TVC of $1069 which is $40 or 0.9% above the TVC* of $1059 for the Poisson which would be expected if the optimal buffer of 11 days had been used. Note that the 10.2 day buffer specified by the uniform must be "rounded" to

10 days in order to calculate the expected TVC which would result under the true Poisson distribution since only integer values of buffer are allowed in the Poisson calculations.

TABLE IV

BUFFER, TVC, AND PER CENT ERROR RESULTING FROM INCORRECT ASSUMPTIONS FOR DELIVERY DATE DISTRIBUTION IN SAMPLE PROBLEM

Assumed Distribution	True Distribution		
	Uniform	Chi-Square	Poisson
Uniform	10.2 days $949 Optimum	10.2 days $1272 4.2%	10 days $1069 0.9%
Chi-Square	11.9 days $1007 6.1%	11.9 days $1221 Optimum	12 days $1083 2.3%
Poisson	11 days $963 1.5%	11 days $1234 1.1%	11 days $1059 Optimum

Examination of Table IV shows that errors in assumption of the delivery date p. d. f. result in very small increases in the expected TVC. The largest error combination of 6.1% results if the chi-square is assumed when the delivery date actually has a uniform distribution. If there is no information which would indicate the true form of the delivery date p. d. f., Table IV would indicate the Poisson should be assumed to calculate the optimal buffer to use in establishing delivery date. This is because the optimal buffer found under the Poisson assumption lies between the two specified by the uniform and chi-square assumptions, and expected increases in TVC are only 1.5% and 1.1% respectively if the Poisson is being assumed erroneously.

The true distribution of delivery date in a given situation may be of a form other than those considered in this dissertation, even though the distributions used in mathematical developments were chosen both for their appropriateness to describe the delivery date random variable and to facilitate real world application of the models. If it is desired to examine the effects of assumptions involving other distributions, models to find TVC and optimal buffers for these distributions must be derived.

Not only do the models derived present similar results for the optimal buffer decision and TVC, but they also produce very similar results for expected TVC over a complete range of buffer lengths. Table V presents a tabular comparison of the expected TVC for each distributional assumption for buffer lengths ranging from 28 days, where lateness costs are negligible, down to a zero buffer. Note that the expected TVC's are of the same magnitude at each buffer length and that they follow a similar pattern. Note also that the expected TVC calculated under the assumption of a uniform distribution is less than the expected TVC calculated under the assumption of the chi-square and Poisson distributions for large buffers as was discussed earlier. TVC calculations are possible for the uniform case only in the feasible region $0 \le y \le (d + \sqrt{3})$ which for the sample problem becomes the interval $(0, 3.77)$.

TABLE V

COMPARISON OF TVC CALCULATIONS FOR DIFFERENT BUFFER LENGTHS IN SAMPLE PROBLEM

y	Buffer	Total Variable Cost for Each Distribution		
	(Days)	Chi-square	Poisson	Uniform
7.00	28	$2327	$2327	-
6.00	24	1996	1994	-
5.00	20	1672	1662	-
4.00	16	1380	1338	-
3.50	14	1270	1191	$1175
3.00	12	1221	1083	1021
2.75	11	1234	1059	968
2.50	10	1286	1069	949
2.25	9	1391	1128	985
2.00	8	1567	1258	1100
1.50	6	2231	1838	1686
1.00	4	3537	3088	2990
0.75	3	4540	4075	4019
0.50	2	5846	5373	5366
0.25	1	7513	7036	7087
0.00	0	9600	9121	9242

This concludes the discussion of the sensitivity of the models of this dissertation to differences and errors in the assumption concerning the form of the delivery date distribution. The sensitivity of different assumption combinations will probably vary slightly with different cost parameters and levels of uncertainty. However, the in-depth analysis presented for this case should give the reader some feel for the magnitude of errors that might be encountered in this assumption. The analysis of Table IV indicates that these errors are slight, especially if the Poisson is chosen in situations where there is complete uncertainty concerning the true form of the distribution. The methodology presented here should also aid those applying the models in determining the sensitivity of a buffer decision for a particular procurement situation.

Sensitivity of Buffer to Uncertainty in Delivery Date

One of the most important decisions in the application of the models derived is the determination of the time interval within which delivery will most likely occur. In the case of the uniform delivery date distribution the length of the interval which should bracket the true delivery date is denoted by the parameter "b." In the case of the chi-square delivery date distribution, a 90% or 98% range on delivery date must be specified in order to determine the degrees of freedom "r" to use in TVC calculations. A 90% range may also be used in the case of the Poisson distribution to determine the proper value of "μ" for TVC calculations. Each of these distribution parameters is proportional to the standard deviation of the delivery date distribution associated with it. The greater the uncertainty concerning the actual time when delivery might occur, the longer the time interval necessary in order to bracket the actual delivery date at a given probability level.

If the wrong time interval is used in calculations, to what degree does this affect the optimal buffer and expected TVC of procurement? This question will be examined in detail for the case of the Poisson delivery date distribution only since in the previous section it was shown that calculations under the three different assumptions for delivery date distribution produced very similar results. The TVC for different buffer lengths was calculated for values of μ = 8, 11 and 16 corresponding to 90% ranges on delivery date of 9, 11, and 13 days, respectively. The curves of TVC vs. buffer length were plotted as shown in Figure 23. The error resulting from incorrect assumptions in the 90% range on delivery date are shown in Table VI.

TABLE VI

BUFFER, TVC, AND PER CENT ERROR RESULTING FROM INCORRECT ASSUMPTIONS OF DELIVERY DATE RANGE

Assumed Value of Range (μ)	True Value of Range (μ)		
	9 days (μ = 8)	11 days (μ = 11)	13 days (μ = 16)
9 days (μ = 8)	7 days $678 Optimum	7 days $870 4.6%	7 days $1484 40.1%
11 days (μ = 11)	8 days $714 5.3%	8 days $832 Optimum	8 days $1258 18.8%
13 days (μ = 16)	11 days $919 35.5%	11 days $940 11.5%	11 days $1059 Optimum

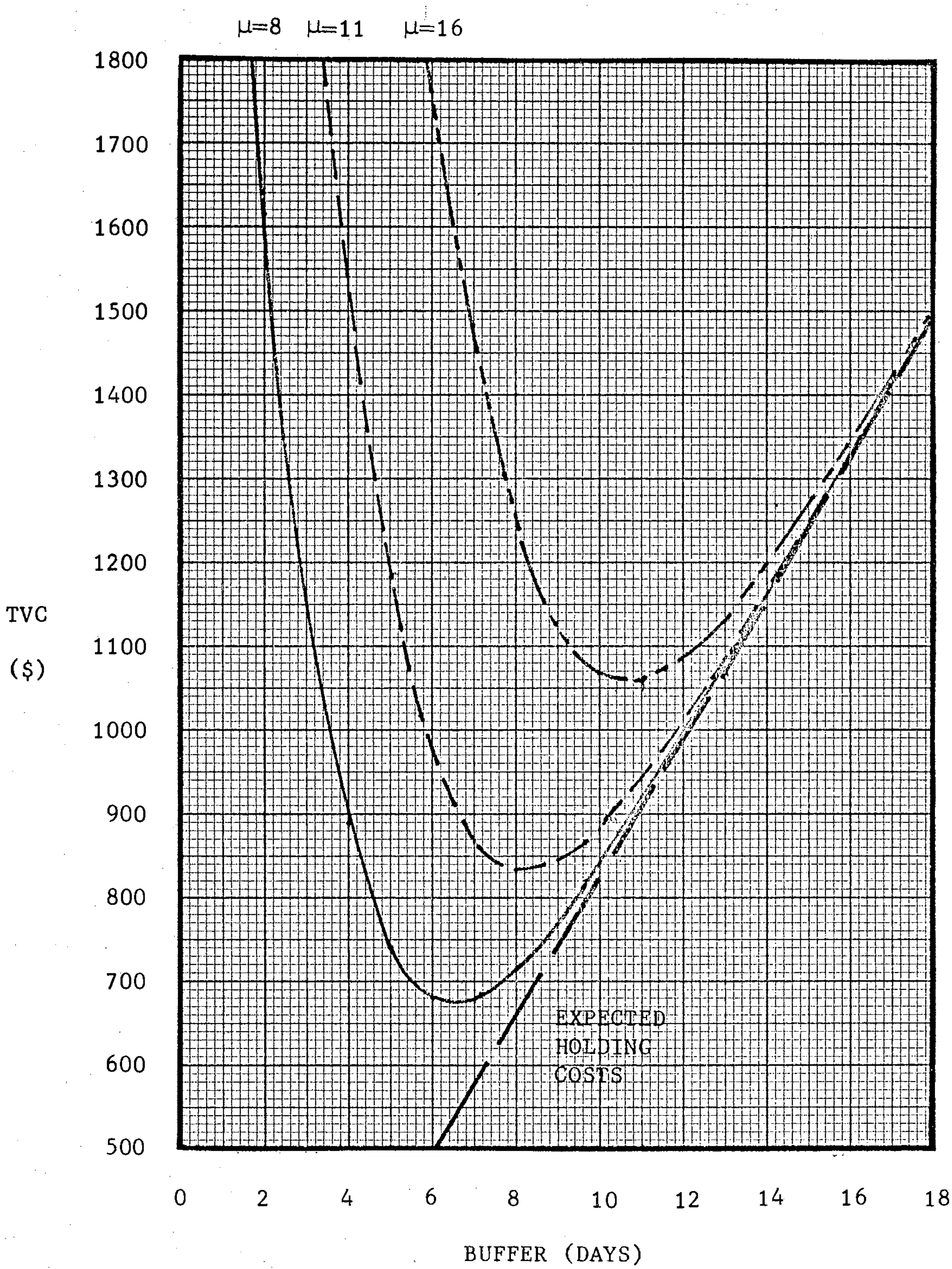


Figure 23. TVC vs. Buffer for μ = 8, 11, and 16

Note in Figure 23 that the curves of TVC vs. buffer all approach a straight line as the buffer gets large. This line is the expected holding cost and can be drawn as shown in the figure. The difference between the expected TVC curve and the expected holding costs is the expected lateness cost which approaches zero as the buffer becomes large (15 days or more). In the following sections other examples may be observed in which the curves of TVC vs. buffer approach the same line for large buffers, and the same reasoning can be applied in those cases.

In Table VI the buffer, expected TVC, and per cent error resulting from errors in estimating the delivery date range are given for the sample problem. If a 90% range on delivery of 11 days is assumed when the actual 90% range is 9 days, the 8 day buffer specified by the range of 11 days results in an expected TVC that is 5.3% above the optimum for the 9 day range. Thus a 22% error results in a 5.3% increase in TVC. If the true value of 90% range is 11 days and a range of 13 days is erroneously assumed, this 18% error results in an increase in expected TVC of only 11.5%. If a range of 11 days is erroneously assumed for a true range of 13 days, this 15% error results in an increase in expected TVC of 18.8%. Thus, the sensitivity of the models to errors in choosing the 90% range of delivery date varies according to the magnitude of the range. The larger the range, the greater the per cent error in TVC.

Part of this increase in sensitivity with larger ranges may be attributed to the fact that the parameter "d" is being held constant in the analysis. As the range increases, the Poisson parameter μ and the standard deviation σ also increase. If d is held constant, then $d\sigma$ (the number of days before requirement date that expediting starts) also increases as σ increases. This is in keeping with the assumption that as the uncertainty increases, expediting would begin earlier. Earlier timing of expediting tends to increase the optimal buffer and TVC* and, thus, sensitivity. If this assumption does not hold and the quantity $d\sigma$ were held constant, the sensitivity of TVC to errors in delivery date range would probably not be as great.

It should also be noted in sensitivity comparisons involving the Poisson that the use of the Poisson requires integer values for the buffer, and the resulting error comparisons are not as accurate as would be the case if the chi-square model were being used. Even though only integer buffers (whole days) would be used in a real world application of the models, the decimal fractions are helpful in gaining a perspective of the trends in sensitivity.

Sensitivity of Buffer to Errors in Lateness Cost Parameters

In the development of the models of previous chapters, three parameters were used to define the costs of lateness incurred for delivery at any given time. The lateness cost function was defined in equation (2.8) as

$$
\begin{aligned}
C(x) &= Kx^m \quad \text{for} \quad x \geq 0 \\
&= 0 \quad \text{for} \quad x < 0
\end{aligned}
\tag{6.1}
$$

where K = a scaling constant

x = the delivery date (a random variable)

m = an exponent determining the rate of increase of lateness costs with time.

Lateness costs were defined to start at a point $x = 0$ defined to be $d\sigma$ days prior to the requirement date. Thus, another important parameter, "d," may be defined as

> d = the number of standard deviations of the delivery date distribution before the requirement date that lateness costs are first incurred.

As the parameter "d" determines the origin of the lateness cost function with respect to the requirement date, it indicates the "timing" of the first of the costs of lateness incurred. For this reason, "d" will be referred to as the "timing parameter" of lateness costs. The rate of increase of lateness costs with time for different possible delivery dates depends upon the urgency with which the part is needed and the costs of delays stemming from late delivery. The rate of increase of lateness costs with time, "m," will thus be referred to as the "urgency parameter" of lateness cost. The parameter "K" will be referred to as the "scaling parameter." The sensitivity of TVC to each of these lateness cost parameters will be discussed separately.

Sensitivity of Model to Scaling Parameter "K"

If the wrong value of K is used in calculations, how does this affect the optimal buffer and expected TVC of procurement? This question will be examined for the procurement situation involved in the sample problem discussed previously under the assumption of a chi-square distribution of delivery date. The curves of TVC vs. buffer are plotted in Figure 24 for values of K = 8, 10, 12, 20, and 30. Note that the curves are much farther apart for K = 8, 10, and 12 than they would be for values of K = 28, 30, and 32. Thus the optimal buffer is not a

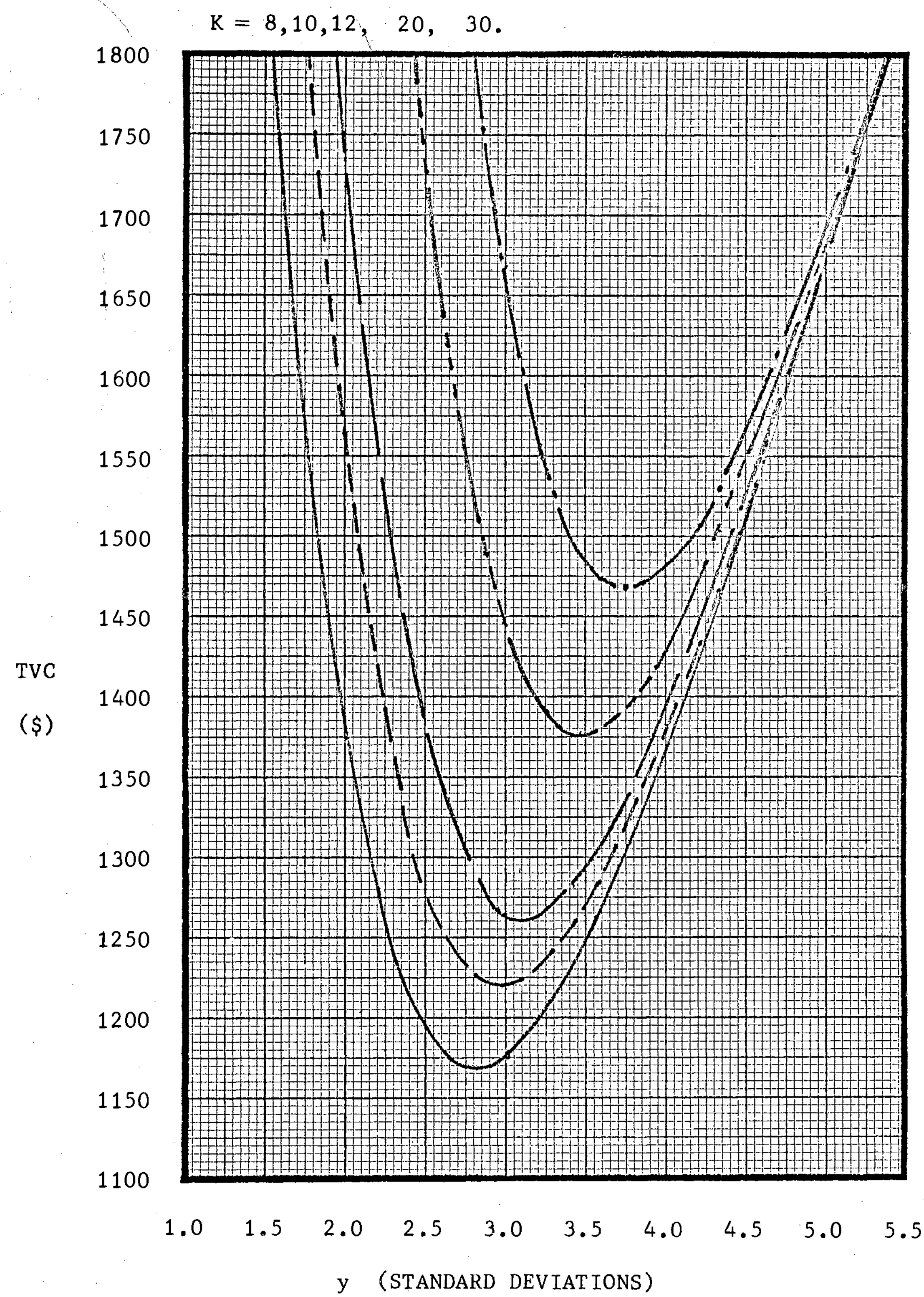


Figure 24. TVC vs. y for K = 8, 10, 12, 20, and 30

linear function of K. However, it is also of interest to note that the locus of minimum points of the TVC curves is a straight line which would indicate that the optimal buffer is directly proportional to some function of K. The effect of errors in the estimation of the scaling parameter K for the sample problem are summarized in Table VII.

In the procurement situation of Table VII, the buffer and expected TVC of procurement are not greatly sensitive to errors in K. If the true value of K is 8.0 and a value of 12.0 is assumed (an error of 50%), the resulting increase in expected TVC is only 1.4%. It is noticeable, however, that errors on the low side of the true value of K are more costly than errors on the high side. For example, if the true value of K is 20.0 and a lower value of K = 8.0 is assumed and used in calculations (an error of 60%), the increase in expected TVC is 10.1%. However, if the true value of K is 8.0 and a higher value of K = 20.0 is assumed (an error of 150%), the increase in expected TVC is only 6.3%. Also, for a true value of K = 10 as in the sample problem, a 20% error on the low side (K = 8) gives a TVC increase of 0.4% while the same 20% error (K = 12) on the high side gives a TVC increase of 0.3%.

Although this point is of some interest, the fact that errors of 20% result in less than 1% increase in TVC is of much greater importance. This relative insensitivity of the model to the parameter K suggests that procurement analysts need not go to great expense in determining a highly precise value for K, but should concentrate more effort on other parameters where errors result in a greater increase in the expected TVC.

TABLE VII

BUFFER, TVC, AND PER CENT ERROR RESULTING FROM INCORRECT ASSUMPTIONS FOR SCALING PARAMETER "K"

Assumed Value of K	True Value of K			
	8.0	10.0	12.0	20.0
8.0	11.3 days $1169 Optimum	11.3 days $1227 0.4%	11.3 days $1285 1.8%	11.3 days $1517 10.1%
10.0	11.9 days $1175 0.5%	11.9 days $1221 Optimum	11.9 days $1267 0.4%	11.9 days $1451 5.3%
12.0	12.4 days $1186 1.4%	12.4 days $1224 0.3%	12.4 days $1262 Optimum	12.4 days $1415 2.7%
20.0	13.9 days $1243 6.3%	13.9 days $1265 3.7%	13.9 days $1288 2.1%	13.9 days $1378 Optimum

<u>Sensitivity of Model to Timing Parameter "d"</u>

In this section the responsiveness of the model to changes in the timing parameter is discussed. Curves of TVC vs. buffer were computed and plotted for values of d = 2, 2.5, 3, and 3.5 for the same procurement situation as in the last section with the exception that the urgency parameter m = 2 instead of 3. These curves are shown in Figure 25. Examination of the results of these curves produces a very interesting observation: the optimal value of y is exactly 1.00 less than the value of d in each case. As d increases from 2.0 to 3.0, y* increases from 1.0 to 2.0 in this particular procurement situation. A very similar result is found in procurement situations represented by other combinations of parameters. For example, in the sample problem in which m = 3, as the parameter d was increased from 2.0 to 3.0, y* increased from 2.98 to 3.98. This would suggest a relationship between y and d for the chi-square assumption of delivery date that is identical to that found for the case of the uniform. Recalling equation (3.35) for the case of m = 1,

$$y^* = d + \sqrt{3} - \frac{\sqrt{12}}{b}\left[\frac{bH}{K}\right] . \tag{6.2}$$

The curves of Figure 25 indicate that an intensive analysis of different procurement situations as represented by a variety of combinations of cost parameters might yield a similar equation for y* for the assumption of the chi-square and/or Poisson distributions for delivery date. Such an expression would greatly facilitate application of these models as it would make the computerized Fibonacci search unncessary.

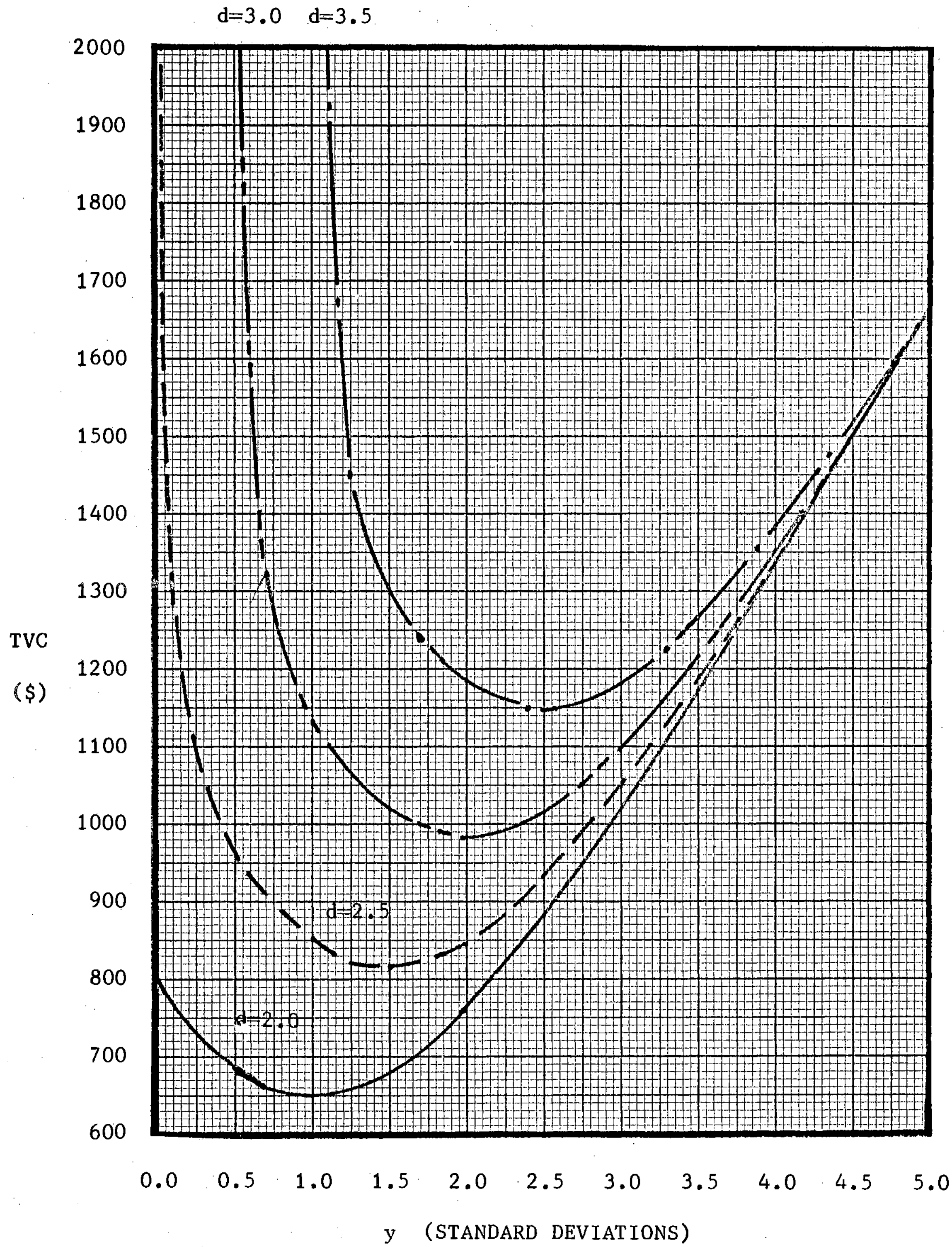


Figure 25. TVC vs. y for d = 2.0, 2.5, 3.0, and 3.5

The responsiveness of y* and buffer length to changes in d presents an interesting insight into the determination of the optimal buffer length. One might feel that the sole purpose of the buffer is to insure delivery by the requirement date, and that buffer lengths should be specified solely to insure on-time delivery. However, a more efficient purpose of the buffer is to minimize the total variable cost of procurement, including the costs of lateness. Under this latter approach to defining buffer length, which is the foundation of this dissertation, the optimal buffer bears a more direct relationship to the timing of the expediting and other lateness costs than to the uncertainty of the delivery date. The responsiveness of optimal buffer length to changes in buffer length can also be observed in Table VIII.

If the wrong value of d is chosen for calculations, how does this affect the optimal buffer and TVC of procurement? An analysis of this question is presented in Table VIII. The timing parameter d is more sensitive than the scaling parameter K, but less sensitive than the assumption concerning the delivery date range. If a value of d = 2.0 is used when the true value is 2.5, the resulting increase in TVC is only 4.4%. In general, an error of 20% in estimating d results in an increase in the expected TVC of only about 5% in Table VIII. It should also be noted that it is slightly more costly to underestimate d than to overestimate d by the same amount. For example, if the true value of d is 2.5, a "low estimate" of d = 2.0 results in an increase in TVC of 4.4%, while a "high estimate" of d = 3.0 results in a 3.9% increase. However, both of these errors in d of 20% produced increases in expected TVC of less than 5%.

TABLE VIII

BUFFER, TVC, AND PER CENT ERROR RESULTING FROM INCORRECT ASSUMPTIONS FOR TIMING PARAMETER "d"

Assumed Value of d	True Value of d			
	2.0	2.5	3.0	3.5
2.0	4 days $650 Optimum	4 days $852 4.4%	4 days $1132 15.3%	4 days $4174 264 %
2.5	6 days $682 4.9%	6 days $816 Optimum	6 days $1018 3.7%	6 days $1298 13.1%
3.0	8 days $765 17.7%	8 days $848 3.9%	8 days $ 982 Optimum	8 days $1185 3.2%
3.5	10 days $884 36 %	10 days $931 14.1%	10 days $1014 3.3%	10 days $1148 Optimum

Estimation of d should present few problems as a company should have a definite expediting strategy concerning the timing of certain expediting procedures as a part of their overall policies concerning materials management. After determining the standard deviation of the delivery date distribution, the value of d can be easily determined.

Sensitivity of Model to Urgency Parameter "m"

In this section the responsiveness of the model to changes in the urgency parameter m will be examined for the sample problem under the assumption of a Poisson distribution of delivery date. The plots of TVC vs. buffer for m = 1, 2, and 3 is shown in Figure 26. These curves indicate that buffer and TVC are more sensitive to changes in m than to changes in any other parameter with all other parameters held constant. Note also that changes in m have a dramatic effect on the shape of the TVC curves where changes in the other parameters usually altered the position of the minimum but otherwise did not change the shape of the curve greatly. It must be remembered that the urgency parameter m is the exponent of the delivery time in the lateness cost function and determines the rate of increase of lateness costs with time. Since m has such an important role in determining the shape of the lateness cost function, it is easy to understand the high sensitivity of buffer and TVC to changes in m. In addition to Figure 26 the high sensitivity of buffer and TVC to changes in the urgency parameter m can be observed in Table IX.

If the wrong value of m is used in calculations, how does this affect the buffer and expected TVC of procurement? This question is answered in Table IX for the sample problem discussed earlier under the

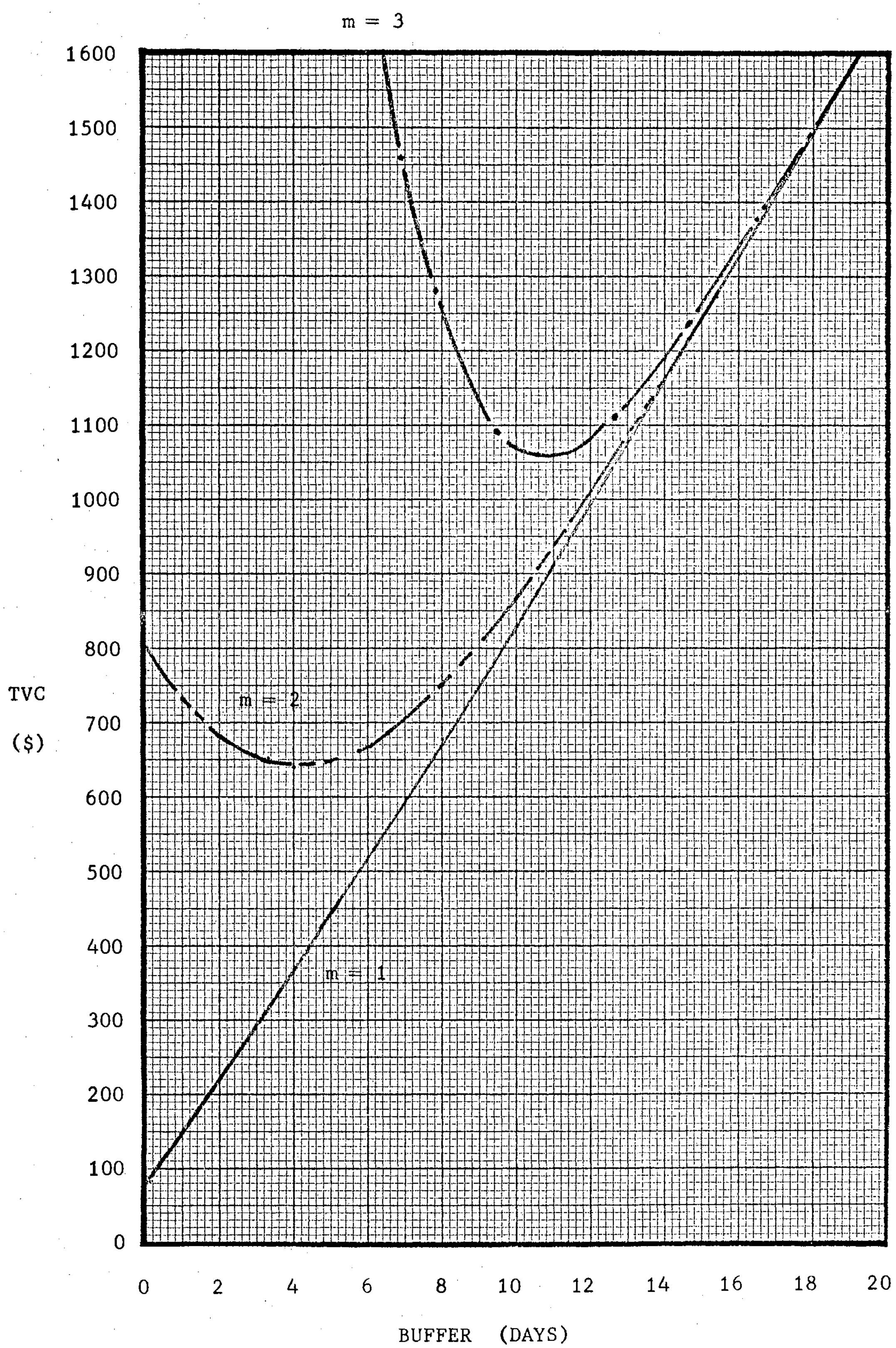


Figure 26. TVC vs. Buffer for m = 1, 2, and 3

assumption of a Poisson distribution of delivery date. If the true value of m is 1.0, the assumption of m = 2.0 (a 100% error) increases the expected TVC by 37% while the assumption of m = 3.0 (a 200% error) increases the expected TVC by 1050%. For a true value of m = 2.0, 50% errors in m result in increases in expected TVC of 25% and 46%. However, for a true value of m = 3, errors incurred by assuming m = 2.0 and m = 1.0 result in expected TVC increases of 192% and 762% respectively.

TABLE IX

BUFFER, TVC, AND PER CENT ERROR RESULTING FROM INCORRECT ASSUMPTIONS FOR URGENCY PARAMETER "m"

Assumed Value of m	True Value of m		
	1.0	2.0	3.0
1.0	0 days $80 Optimum	0 days $800 25%	0 days $9121 762%
2.0	4 days $375 37%	4 days $643 Optimum	4 days $3088 192%
3.0	11 days $919 1050%	11 days $938 46%	11 days $1059 Optimum

Although errors in the selection of m result in increases in expected TVC of 25% and more, the selection of a value of m should not be of too great difficulty as only three possible values are permitted in the models. The urgency of a part and the magnitude of lateness costs incurred for different delivery dates should be such that m can be readily determined. However, if there is complete uncertainty about the shape of the lateness cost function, Table IX would indicate the use of a value of m = 2 would result in a smaller increase in expected TVC if the true value of m were in fact one or three.

Sensitivity of Buffer to Errors in Holding Cost Parameters

The models developed earlier each have four holding cost parameters as defined in equation (2.14):

V = the value of the component in dollars

P = a decimal fraction representing the annual cost of capital, taxes, and insurance on inventory value

W = the number of storage space units required

C_h = the annual cost of one unit of storage space.

Of these four parameters, two should be known with certainty in most procurement situations. At the time the order is being placed, the value or price of the component and the amount and quality of the storage space needed should be known. Errors may arise in the evaluation of P and C_h although the accounting department should be able to give very good estimates. It should be noted that these parameters have probably already been evaluated several times as they are essential parameters in the well-known and widely used economic order quantity models.

In order to evaluate the sensitivity of the models to changes or errors in holding costs, it will be most convenient to lump all parameters into a single parameter "H" as was done in equation (3.10):

$$H = \frac{VP+WC_h}{365}. \tag{6.3}$$

H represents the total holding costs per day for a component or group of components which include both the inventory value cost and the storage space cost. Evaluation of response of models to changes in H will give more general results than could be obtained from examining either P or C_h. The value of P would be important only if the value of V were large, and the value of C_h only if W were large. But by lumping all holding cost parameters into the total daily holding cost H, the response of the models to different general levels of holding costs can be determined.

Curves of TVC vs. buffer are plotted in Figure 27 for values of H = \$40, \$60, and \$80 per day for the procurement situation of the sample problem under an assumption of a chi-square delivery date distribution. Changes in H alter the shape of the TVC curve somewhat, especially to the right of the minimum point where the largest components in the expected TVC are the expected holding costs. The changes in the optimal buffer are indicated in Table X.

If the estimate of the daily holding cost H is in error, how does this affect the optimal buffer and expected TVC of procurement? Table X shows that the buffer and expected TVC have very low sensitivity to errors in H for the procurement situation of sample problem discussed previously under a chi-square distribution of delivery date. For a true value of H = \$60/day, errors of 33% result in increases in expected TVC of less than 2%. If the true value of H is \$40/day and H = \$80/day is assumed, this 100% error results in an increase in

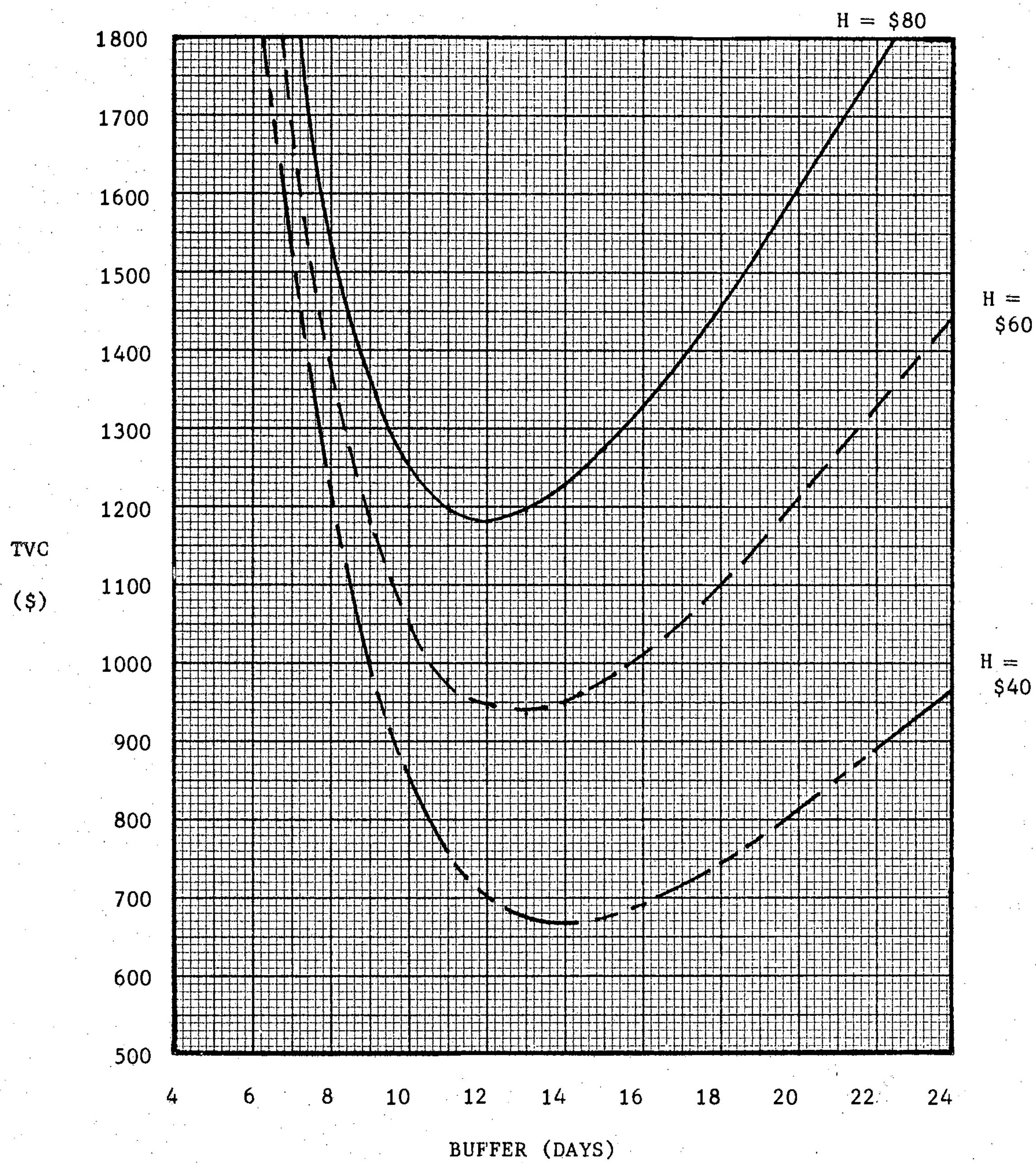


Figure 27. TVC vs. Buffer for H = $40, $60, and $80/Day

expected TVC of only 5.2%. Although the sensitivity of the model to changes in H is very low in the sample problem (where the urgency parameter m = 3), procurement situations in which m = 2 or 1 might result in higher sensitivity to changes in H. This possibility should be investigated in the context of the particular procurement situation.

TABLE X

BUFFER, TVC, AND PER CENT ERROR RESULTING FROM INCORRECT ASSUMPTIONS FOR DAILY HOLDING COSTS "H"

Assumed Value of H	True Value of H		
	$40/day	$60/day	$80/day
$40/day	14.0 days $667 Optimum	14.0 days $947 1.3%	14.0 days $1227 3.6%
$60/day	12.8 days $678 2.8%	12.8 days $935 Optimum	12.8 days $1192 0.5%
$80/day	12.0 days $702 5.2%	12.0 days $943 0.9%	12.0 days $1184 Optimum

Summary

In this chapter the responsiveness of the models developed earlier to changes in values of the parameters has been investigated. Curves of expected TVC have been calculated and plotted for different values of each parameter while holding the other parameters constant. Although no mathematical derivations to indicate the general sensitivity for the individual parameters were possible, this examination of changes in expected TVC for particular situations should be of some help to the reader in evaluating the models.

The sensitivity of the optimal buffer and expected TVC of procurement to errors in parameters was also analyzed for the procurement situation of the sample problem of Chapters III, IV, and V. The relative sensitivity of the different cost parameters can be compared for errors of about 20% in the estimation of their values in the sample problem. The parameters are ranked as follows in order of sensitivity with the most sensitive parameter listed first.

1.	Urgency parameter "m":	discussed below
2.	90% Range on delivery:	error of 18% low increases TVC by 4.6%
		error of 18% high increases TVC by 11.5%
3.	Timing parameter "d":	error of 20% low increases TVC by 4.4%
		error of 20% high increases TVC by 3.9%
4.	Scaling parameter "K":	error of 20% low increases TVC by 0.4%
		error of 20% high increases TVC by 0.3%
5.	Daily holding cost "H":	error of 25% low increases TVC by 0.5%
		error of 38% high increases TVC by 0.9%

The expected TVC of procurement was highly sensitive to errors in the urgency parameter of lateness cost "m" because this parameter can only take on values of one, two, and three. Errors in specifying m can result in increases in expected TVC of from 25% to 1050% for extreme errors. However, the procurement analyst should be able to select the proper value of m from the three choices quite readily in most situations. Except for the choice of a value for m, the models were not extremely sensitive to errors in estimating parameters in the sample problem with errors in parameter values of 20% resulting in TVC increases of only 5% or less. However, it should be noted that in other procurement situations the sensitivities might be somewhat higher or lower, especially for different values of m.

The sensitivity of the buffer and expected TVC to errors in the assumption of a p. d. f. to represent the delivery date distribution was also analyzed. It was found that most errors resulted in an increase in TVC of less than 5% for the sample problem. The only exception was the case where a chi-square distribution was erroneously assumed when the true distribution of delivery date was a uniform distribution, and this error resulted in an increase of only 6.1% in the expected TVC. Thus, the sensitivity of the buffer and expected TVC to an error in this assumption is not substantial. It should be remembered, however, that each of the sensitivity analyses examined the effects of errors in parameters for the sample problem only. In other particular situations, the sensitivity of the parameters may be different. Sensitivity analyses on a particular situation should be performed whenever the sums of money involved justify the added information concerning sensitivity of parameters and the distribution assumption.

In the sensitivity analyses of some parameters, very interesting relationships between the parameters and the variable y were observed. In particular, the locus of minima of the TVC curves was a straight line in most cases. This would suggest that empirical relationships between y and the parameters might be found for the assumptions of the chi-square and Poisson distributions of the delivery date random variable. If such relationships could be found that are as simple and easy to use as the y* and TVC* expressions for the assumption of a uniform delivery date distribution, these expressions for the chi-square and Poisson assumptions would preclude the computerized search for y* and TVC* and greatly facilitate the application of these models.

CHAPTER VII

AIDS IN APPLICATION OF MODELS

In some instances a great deal of analytical effort is spent in the mathematical development of highly sophisticated operations research models whose maximum potential will probably never be realized. One reason for this in the opinion of the author is that when the individuals faced with the real-world decisions attempt to apply these models, they either apply them erroneously or do not apply them at all because of difficulty in understanding the articles in which the models are presented. It has been the objective of the author to write this dissertation in a manner such that a procurement analyst might read and understand the development of the models and the meaning of the parameters used. As each parameter was introduced, methods were discussed for evaluation of the parameter in specific procurement situations. The parameters and their evaluation were discussed somewhat further in the sensitivity discussions of Chapter VI. Before evaluating the parameters of holding costs and lateness costs, those sections of Chapters II and VI dealing with those parameters should be studied thoroughly. The distribution parameters for each distribution of the delivery date random variable are discussed in detail in the sections of Chapters III, IV, and V dealing with the assumption of the particular distribution to describe the delivery date random variable.

In the case of the assumption of a uniform distribution of the delivery date random variable, the expressions developed for y* and TVC* provide a simple and readily applied decision aid for determining the optimal buffer in situations involving uncertain delivery. However, if the chi-square or Poisson distribution is chosen to describe the delivery date random variable, a computerized search of the TVC equation is necessary to determine the optimal buffer. In some situations such a procedure might be inconvenient or infeasible because of other demands on the computer. In any event the procurement analyst's time will be consumed in preparing the computer input for the analysis. The time consumed in this procedure will undoubtedly reduce the number of applications of the models in a given firm.

In order to insure easy applicability of the models, especially under the chi-square or Poisson assumptions for the delivery date distribution, a set of decision tables may be calculated which would provide a decision concerning the optimal buffer for a wide range of procurement situations. These precalculated tables giving the optimal buffer and expected TVC* for a given procurement situation would simplify greatly the use of these models in that procurement analysts can refer to them and readily determine the proper buffer and expected TVC* of procurement. Examples of such tables are presented as Tables XI, XII, XIII, XIV, XV, and XVI. Five variables are needed to specify the optimal buffer and expected TVC* for any given procurement situation: H, K, m, d, and a distribution parameter designated by the range on delivery date. On these tables the three latter parameters are held constant, and the optimal buffer and TVC* are given for different

TABLE XI

TABLE FOR DETERMINING OPTIMAL BUFFER AND TVC* FOR 90% RANGE ON DELIVERY DATE OF 12.8 DAYS AND URGENCY PARAMETER m = 3

Value of Scaling Parameter K	Value of H (Daily Holding Cost)			
	$20/day	$50/day	$80/day	$110/day
5	14.0 days $334	11.4 days $709	10.0 days $1028	9.0 days $1312
10	15.9 days $370	13.4 days $804	12.0 days $1184	11.1 days $1530
20	17.8 days $406	15.3 days $897	14.0 days $1335	13.1 days $1740
30	18.8 days $427	16.4 days $950	15.1 days $1421	14.2 days $1861
50	20.1 days $453	17.8 days $1016	16.5 days $1528	15.6 days $2010

Note:

Calculations involve the assumption of a chi-square distribution with 8 degrees of freedom to describe the delivery date random variable. Expediting is assumed to begin 8 days prior to the requirement date ($d = 2.0$, $\sigma = 4.0$, $d\sigma = 8.0$ days).

TABLE XII

TABLE FOR DETERMINING OPTIMAL BUFFER AND TVC* FOR 90% RANGE ON DELIVERY DATE OF 12.8 DAYS AND URGENCY PARAMETER m = 2

Value of Scaling Parameter K	Value of H (Daily Holding Cost)			
	\$20/day	\$50/day	\$80/day	\$110/day
5	7.1 days \$208	3.0 days \$355	0.0 days \$400	0.0 days \$400
10	9.6 days \$252	6.2 days \$483	4.2 days \$637	2.5 days \$737
20	11.8 days \$294	8.8 days \$596	7.1 days \$833	5.8 days \$1026
30	13.0 days \$317	10.1 days \$659	8.6 days \$938	7.4 days \$1177
50	14.5 days \$346	11.8 days \$734	10.3 days \$1064	9.2 days \$1356

Note:

Calculations involve the assumption of a chi-square distribution with 8 degrees of freedom to describe the delivery date random variable. Expediting is assumed to begin 8 days prior to the requirement date ($d = 2.0$, $\sigma = 4.0$, $d\sigma = 8.0$ days).

TABLE XIII

TABLE FOR DETERMINING OPTIMAL BUFFER AND TVC* FOR 90% RANGE ON DELIVERY DATE OF 12.8 DAYS AND URGENCY PARAMETER m = 1

Value of Scaling Parameter K	Value of H (Daily Holding Cost)			
	\$20/day	\$50/day	\$80/day	\$110/day
20	0.0 days \$160	0.0 days \$160	0.0 days \$160	0.0 days \$160
30	5.8 days \$199	0.0 days \$240	0.0 days \$240	0.0 days \$240
50	8.4 days \$238	0.2 days \$400	0.0 days \$400	0.0 days \$400
70	9.7 days \$261	5.4 days \$483	0.0 days \$560	0.0 days \$560
100	11.0 days \$284	7.4 days \$554	4.6 days \$733	0.0 days \$800

Note:

Calculations involve the assumption of a chi-square distribution with 8 degrees of freedom to describe the delivery date random variable. Expediting is assumed to begin 8 days prior to the requirement date ($d = 2.0$, $\sigma = 4.0$, $d\sigma = 8.0$ days).

TABLE XIV

TABLE FOR DETERMINING OPTIMAL BUFFER AND TVC* FOR 90% RANGE ON DELIVERY DATE OF 19.5 DAYS AND URGENCY PARAMETER m = 3

Value of Scaling Parameter K	Value of H (Daily Holding Cost)			
	$20/day	$50/day	$80/day	$110/day
5	22.7 days $520	19.4 days $1144	17.6 days $1698	16.4 days $2209
10	25.0 days $565	21.9 days $1262	20.2 days $1892	19.1 days $2481
20	27.3 days $608	24.3 days $1376	22.7 days $2079	21.5 days $2741
30	28.6 days $633	25.6 days $1441	24.1 days $2185	23.0 days $2889
50	30.2 days $665	27.3 days $1521	25.8 days $2315	24.7 days $3072

Note:

Calculations involve the assumption of a chi-square distribution with 18 degrees of freedom to describe the delivery date random variable. Expediting is assumed to begin 12 days prior to the requirement date (d = 2.0, σ = 6.0, $d\sigma$ = 12.0 days).

TABLE XV

TABLE FOR DETERMINING OPTIMAL BUFFER AND TVC* FOR 90% RANGE ON DELIVERY DATE OF 19.5 DAYS AND URGENCY PARAMETER m = 2

Value of Scaling Parameter K	Value of H (Daily Holding Cost)			
	$20/day	$50/day	$80/day	$110/day
5	12.8 days $344	7.7 days $644	4.1 days $819	1.0 days $895
10	16.1 days $402	11.7 days $811	9.1 days $1121	7.0 days $1361
20	19.0 days $455	15.1 days $959	12.8 days $1377	11.2 days $1737
30	20.6 days $484	16.9 days $1040	14.8 days $1513	13.3 days $1934
50	22.6 days $520	19.0 days $1137	17.1 days $1676	15.7 days $2167

Note:

Calculations involve the assumption of a chi-square distribution with 18 degrees of freedom to describe the delivery date random variable. Expediting is assumed to begin 12 days prior to the requirement date ($d = 2.0$, $\sigma = 6.0$, $d\sigma = 12.0$ days).

TABLE XVI

TABLE FOR DETERMINING OPTIMAL BUFFER AND TVC* FOR 90% RANGE ON DELIVERY DATE OF 19.5 DAYS AND URGENCY PARAMETER m = 1

Value of Scaling Parameter K	Value of H (Daily Holding Cost)			
	$20/day	$50/day	$80/day	$110/day
20	0.0 days $240	0.0 days $240	0.0 days $240	0.0 days $240
30	8.9 days $301	0.0 days $360	0.0 days $360	0.0 days $360
50	12.8 days $357	0.0 days $600	0.0 days $600	0.0 days $600
70	14.9 days $390	8.2 days $732	0.0 days $840	0.0 days $840
100	16.7 days $421	11.3 days $836	6.8 days $1110	0.0 days $1200

Note:

Calculations involve the assumption of a chi-square distribution with 18 degrees of freedom to describe the delivery date random variable. Expediting is assumed to begin 12 days prior to the requirement date ($d = 2.0$, $\sigma = 6.0$, $d\sigma = 12.0$ days).

combinations of H and K. The timing parameter d should be fixed by the firm's expediting policies for different levels of uncertainty and/or urgency. For example, the value of d = 2.0 was used in the calculation of all six tables. If the firm has a well-defined expediting strategy, only three tables are necessary for each level of uncertainty as represented by the different ranges on delivery date. In these examples Tables XI, XII, and XIII are calculated for a 90% range on delivery date of 12.8 day under the assumption of a chi-square distribution for the delivery date random variable. Tables XIV, XV, and XVI are also calculated under the chi-square assumption but for a 90% range on delivery date 19.5 days. Because of the low sensitivity of optimal buffer and expected TVC* to errors in K and H, interpolation can be used to determine values for the optimal buffer and expected TVC* for values of K and H between those listed in the table. After determining the proper values of the parameters in a given procurement situation, quick reference to a set of these decision tables would give the optimal buffer to use in specifying the delivery date and the expected total variable cost of procurement.

A second method for increasing the applicability of the models is to construct a set of nomographs which would give the optimal buffer and expected TVC* of procurement for a given set of parameters. The nomographs could be constructed from the tables calculated by the firm for its particular expediting policies. An example of such a nomograph was constructed from the optimal buffer lengths given in Table XI and is included in Figure 28. The advantage of using nomographs is that interpolation, when parameters lie between those used in calculations, is much easier with a nomograph than with a table. However, the tables

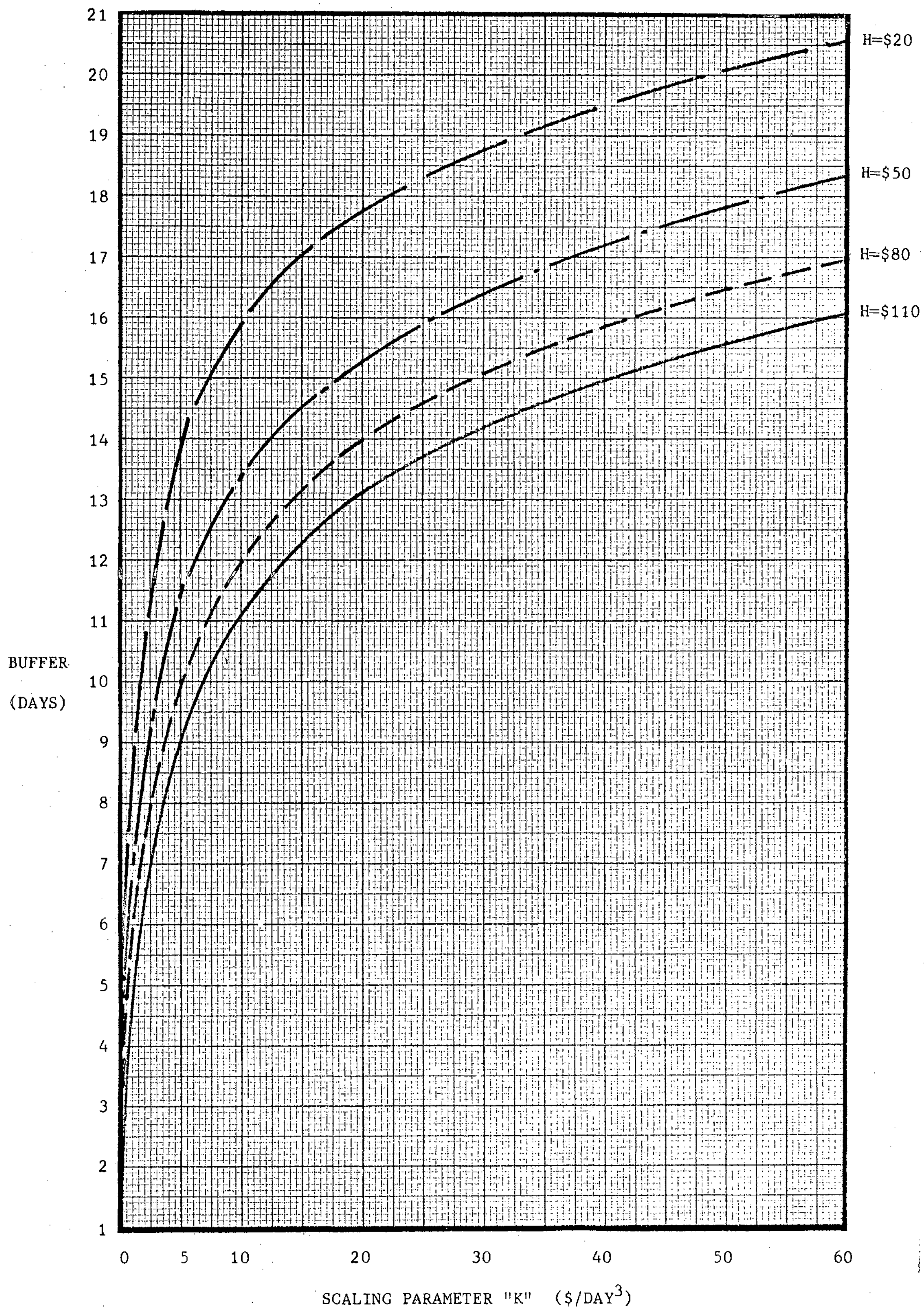


Figure 28. Nomograph Constructed From Table XI

list both the optimal buffer and expected TVC* of procurement together where separate nomographs would be needed to give both the optimal buffer and the expected TVC* of procurement.

It is hoped that these suggestions can be utilized by procurement analysts to facilitate the application of the models developed. Even in the case of the uniform assumption of delivery date, precalculated tables and/or nomographs might improve the applicability of the models.

CHAPTER VIII

CONCLUSION

The problem approached in this dissertation is that of determining the optimal safety time allowance, or buffer, to be used in procurement situations where the delivery date is uncertain and may be defined as a random variable. Although developed for a single-stage procurement situation, the models can be applied in any situation in which the number of items to be ordered has been determined and it is desired to determine the proper buffer to use in specifying delivery date that will minimize the expected total variable cost of procurement. The type of procurement situation where these models will be most useful is the procurement of large or expensive items or subsystems needed in a job shop manufacturing situation. Many situations also arise in the construction industry in which an expensive, critical item with uncertain delivery time must be procured to meet a fixed construction schedule. The models developed in this dissertation provide a useful decision aid for determining the optimal buffer and delivery date for these procurement situations.

Summary of Approach

In dealing with the problem of uncertain delivery time, models were developed to find that buffer length which minimizes the total variable cost of procurement, composed of holding costs and lateness costs. Ordering costs were not a factor in the analysis as only one order is being placed and economic lot size is not a factor. The holding costs were broken down into two components: the cost of providing storage space for the part while in inventory and the costs of tied-up capital, taxes, and insurance that are associated with the inventory value. The expected total variable cost of procurement was thus defined to have the following components:

$$\text{TVC} = \text{Inventory Value Cost} + \text{Storage Space Cost} + \text{Lateness Cost} \quad (8.1)$$

This dissertation utilizes a new approach to dealing with the costs of lateness in procurement situations that may have applicability in the development of other procurement models. This approach is based on the assumption that the total lateness costs incurred in a given procurement situation depend upon the time at which the part is delivered. This assumption recognizes that if delivery is made several days before the requirement date, no lateness charges are incurred. However, if delivery is not made by a certain number of days prior to the requirement date, expediting procedures are begun in an effort to locate the item and insure its delivery by the requirement date. As the requirement date approaches, higher and higher expediting costs are incurred if the part has not arrived. If the part does not arrive by the requirement date, rescheduling of the project is necessary. Depending upon the urgency of the part, production facilities may be idled or substantial penalties for late completion of the project may be incurred. Each of these would increase the total costs of lateness incurred by the delivery of the part at different times after the requirement date with costs increasing significantly for every day that delivery is not made. Thus, the costs of lateness $C(x)$ should be defined as a mathematical function of the delivery date as in equation (2.8):

$$C(x) = Kx^m \quad \text{for } x \geq 0$$
$$\qquad = 0 \quad \text{for } x < 0 \tag{8.2}$$

where K = a scaling constant

x = the delivery date (a random variable)

m = an exponent determining the rate of increase of lateness costs with time.

Lateness costs were defined to start at the time $x = 0$ defined to be $d\sigma$ days prior to the requirement date, where d is a positive number and σ is the standard deviation of the distribution of the delivery date random variable.

In summarizing the development of expressions for TVC, it will be convenient to lump the inventory value cost and storage space cost into a single parameter representing the total daily holding costs:

$$H = \frac{VP+WC_h}{365} \tag{8.3}$$

where H = the total inventory holding costs/day

V = the value of item or items ordered

P = a decimal fraction representing the annual cost of capital, taxes, and insurance on inventory value

W = the number of storage space units required

C_h = the annual cost of providing and maintaining one unit of storage space.

The expected TVC of procurement may now be expressed as the sum of expected holding costs and expected lateness costs:

$$\text{TVC} = H(y\sigma) + \int_X C(x)\cdot f(x)\,dx \tag{8.4}$$

where $y\sigma$ = the expected number of days that the item will be in inventory (the buffer length)

X = the delivery date random variable

$C(x)$ = the lateness cost as a function of delivery date

$f(x)$ = the p. d. f. of the random variable X.

The mathematical variable y is the number of standard deviations of delivery date distribution in the buffer length and is required to be a non-negative real number. The variable y determines the buffer length $y\sigma$ for any given distribution of the delivery date random variable once the mean and standard deviation of the random variable are specified for the procurement situation.

After a change of variable which moves the origin of the delivery date random variable to coincide with the requirement date, the expected TVC can be expressed as

$$\text{TVC} = H(y\sigma) + \int_{-d\sigma}^{\infty} K(t+d\sigma)^m\, f(t)\,dt. \tag{8.5}$$

where $d\sigma$ is the number of days prior to the requirement date that expediting procedures start which indicate the beginning of lateness costs.

The assumption of three different probability distributions is made to describe the behavior of the delivery date random variable, and expressions for the expected value of TVC are derived for each case. The mean of the delivery date distribution is the expected delivery date, and this expected delivery date is specified to be $y\sigma$ days prior to the requirement date, $y\sigma$ being the length of the buffer. Much of the research of this dissertation involves the evaluation of equation (8.5) for different probability distribution assumptions for f(t) in order to find expressions for TVC that can be manipulated to determine the least cost buffer for a given procurement situation.

Summary of TVC Models and Sensitivity

In Chapter III a uniform distribution was assumed for the delivery date random variable. The expected TVC of procurement as a function of y was found to be

$$TVC(y) = \left[\frac{Hb}{\sqrt{12}}\right] y + \frac{K}{b(m+1)}\left[\frac{-b}{\sqrt{12}}\right]^{m+1} y-(d+\sqrt{3})^{m+1}$$

$$\text{for} \quad 0 \leq y \leq d+\sqrt{3} \qquad (8.6)$$

where b is the interval in days in which delivery will occur. Differentiation of TVC(y) for m = 1, 2, and 3 provided the following expression for the optimal y for a given set of cost parameters:

$$y^* = (d+\sqrt{3}) - \frac{\sqrt{12}}{b}\left[\frac{bH}{K}\right]^{1/m} \qquad (8.7)$$

for $0 \leq y \leq d+\sqrt{3}$. The value found for y^* should be multiplied by the standard deviation ($\sigma = b/\sqrt{12}$) to determine the optimal buffer. A delivery date which is $y^*\sigma$ days before the requirement date should then be specified to the vendor or sub-subcontractor. If the value of y given by this equation is negative, a value of $y = 0$ should be used indicating that the optimal decision is to have no buffer, i. e., the delivery date should be specified on the requirement date. If the optimal buffer is used, the optimal expected TVC of procurement is found to be

$$TVC^* = (d+\sqrt{3})\left[\frac{bH}{\sqrt{12}}\right] - \frac{mH}{m+1}\left[\frac{bH}{K}\right]^{1/m} \tag{8.8}$$

for values of y^* in the interval $(0, d+\sqrt{3})$. The equations for y^* and TVC^* developed for the assumption of a uniform distribution of delivery date thus provide a convenient tool for use in determining the optimal buffer in procurement situations involving a probabilistic delivery date.

In Chapter IV a chi-square distribution is assumed for the delivery date random variable. The shape of the chi-square distribution is such that it describes very well the behavior of random variable such as the delivery date. The proper chi-square degrees of freedom r to use in calculations can be determined by defining a 90% or 98% range or confidence interval on delivery date. The length of this interval can then be compared with the lengths of confidence intervals of the chi-square distribution for different degrees of freedom with the aid of Table I in order to determine r. For the case of the chi-square assumption for the delivery date distribution, the expected TVC of procurement as a function of y is found to be

$$TVC(y) = (H\sqrt{2r})y + (\text{COEFFICIENT}) \times (\text{SUM}), \text{ for } y \geq 0, \tag{8.9}$$

where $\text{COEFFICIENT} = \dfrac{2^m K e^{-\left[\frac{r+\sqrt{2r}(y-d)}{2}\right]}}{\left[(r/2)-1\right]!}$

and $\text{SUM} = \sum_{i=0}^{(r/2)-1} \left[m+(r/2)-1-i\right]!\left[\dfrac{r+\sqrt{2r}(y-d)}{2}\right]^i \dfrac{\left[(r/2)-1\right]!}{\left[(r/2)-1-i\right]!\,i!}$.

A requirement that the degrees of freedom r be an even number is imposed by one of the steps in the development of expected TVC.

Calculation of TVC(y) is very difficult without the aid of an electronic computer. Since most firms having a use for the models developed would also have a digital computer at their disposal, FORTRAN programs were written to calculate TVC(y) for a given set of parameters and buffer length and to find the optimal buffer and minimum expected TVC of procurement. The optimal buffer is found through a Fibonacci search for the minimum point on the TVC(y) curve. These computer programs are included in Appendices A and B, and the Fibonacci search procedure is outlined in Appendix E.

In Chapter V a Poisson distribution is assumed for the delivery date random variable, and the method suggested for evaluating the distribution parameter μ is similar to that employed to find r for the case of the chi-square distribution in Chapter IV. Table II was constructed to aid in evaluating μ after defining a 90% range on delivery date. The discrete nature of the Poisson results in a slightly different development for the expected TVC of procurement, and essential steps in the development require that μ, the buffer $y\sigma$, and the expediting time $d\sigma$ all be non-negative integers. The expected TVC of procurement as a function of y was found for the cases m = 1, 2, and 3. For the case of

$m = 1$,

$$TVC(y) = (H\sqrt{\mu})y + K\left\{-\sqrt{\mu}(y-d) + s\sum_{t=0}^{s-1} f(t) - \sum_{t=0}^{s-1} tf(t)\right\}. \quad (8.10)$$

For the case of $m = 2$,

$$TVC(y) = (H\sqrt{\mu})y + K\Bigg\{\mu\left[1+(y-d)^2\right] - s^2\sum_{t=0}^{s-1} f(t)$$

$$+ 2s\sum_{t=0}^{s-1} tf(t) - \sum_{t=0}^{s-1} t^2 f(t) \quad . \qquad (8.11)$$

For the case of $m = 3$,

$$TVC(y) = (H\sqrt{\mu})y + K\Bigg\{\mu\left[1-3\sqrt{\mu}(y-d)-\sqrt{\mu}(y-d)^3\right] + s^3\sum_{t=0}^{s-1} f(t)$$

$$- 3s^2\sum_{t=0}^{s-1} t\cdot f(t) + 3s\sum_{t=0}^{s-1} t^2 f(t) - \sum_{t=0}^{s-1} t^3 f(t)\Bigg\} \quad (8.12)$$

where $s = \mu + \sqrt{\mu}(y-d)$. A FORTRAN program was written to calculate TVC(y) with these equations for a given procurement situation and buffer length. This program is included in Appendix C. A second FORTRAN program was written utilizing a Fibonacci search procedure to find the minimum expected TVC and buffer length and is included in Appendix D.

A sample problem demonstrating the use of the models was formulated and solved in each of the three chapters. In Chapters IV and V the computer outputs for the sample problem calculated using the programs in the appendices are included.

In Chapter VI the responsiveness of the models to changes in parameters was analyzed. Curves of expected TVC were calculated and plotted for different values of each parameter while holding the other parameters constant. The sensitivity of the optimal buffer and expected TVC of procurement to errors in parameters was also analyzed for the sample problem of Chapters III, IV, and V. Tables were constructed to show the difference in buffer length and per cent increase in expected TVC for different errors in the parameters. It was found that the expected TVC increased only about 5% or less for errors in parameter values of 20% for all of the parameters except the urgency parameter of lateness cost "m." This parameter may take on only the values 1, 2, and 3; and few errors should be made in evaluating it. If there were complete uncertainty about m, however, a value of m = 2 would result in an increase in expected TVC of 25% or 46% if the correct value of m were 1 or 3 for the date of the sample problem.

The sensitivity of the buffer and expected TVC to errors in the assumption of a p. d. f. to represent the delivery date was also analyzed for the procurement situation of the sample problem. It was found in Table IV that these errors in assuming the wrong distribution resulted in an increase of only 6% or less in the expected TVC of procurement. If the Poisson were assumed and used in calculations, the increase in expected TVC would be only about 1.5% if the true distribution were the uniform or the chi-square.

The expressions for y* and TVC* developed under the assumption of a uniform distribution of the delivery date random variable provide an easily used tool to aid procurement analysts in decisions where they are applicable. The need for computer solution for the optimal buffer in the case of the chi-square and Poisson assumptions may hamper the application of these models. It is suggested in Chapter VII that firms calculate a set of decision tables for ready decisions concerning optimal buffer length and TVC* for a wide range of procurement situations. These tables would be specifically computed to embody the timing of expediting procedures of that particular firm as recognized in the value of the timing parameter "d" used in the calculations. Nomographs could be constructed from these decision tables which would also be of value. Examples of each are included in Chapter VII.

Areas of Further Study

During the progress of the research and writing of this dissertation certain areas worthy of further study have been discovered. In the sensitivity analyses of some parameters, very interesting relationships between the parameters and the variable y were observed. In particular, the locus of the minimum points on the TVC curves was a straight line in most cases. This would suggest that empirical relationships between y and the parameters might be found for the assumptions of the chi-square and Poisson distributions of the delivery date random variable. If simple expressions for y* and TVC* could be found for the cases of these distribution assumptions, the computerized search of TVC(y) for the minimum TVC and optimal buffer would no longer be necessary. This would greatly facilitate the application of these models.

In addition to the additional research proposed above, many procurement situations involve the problem approached in this dissertation but with the additional complication that the requirement date is not firm. The models developed in this dissertation assume that the requirement date is fixed and known, and lateness costs before and after this date are a function of the time at which the item is delivered. If the requirement date is also a random variable the models developed would not directly apply. Further research might be undertaken to develop models for the case where both the delivery date and the requirement date are random variables. This might be accomplished through defining a joint probability density function to describe the behavior of the two random variables. However, stochastic independence of the two variables may not be justified. If some degree of correlation exists between the two random variables, the bivariate normal distribution might be used to develop models for the optimal buffer. The question of a variable requirement date presents an interesting problem worthy of further research.

The new approach to dealing with lateness costs embodied in this research also provides a method for evaluating alternative expediting policies. The costs and timing of the various alternative policies could be used to determine a lateness cost function for each particular set of expediting procedures. The different lateness cost functions could then be used to find the optimal buffer and TVC* that would result from use of each of the alternative expediting policies. If other factors influencing the decision were equal, the policy resulting in the lowest expected TVC* of procurement should be used. If other factors were also being considered, this evaluation of alternative policies would at least eliminate those policies resulting in a substantially higher expected total variable cost of procurement than the others.

The models of this dissertation also provide a tool which may be of substantial value in vendor rating systems. It has long been recognized that procurement should not be based on the bid price quoted by the vendor alone. Because of poor quality and/or uncertain delivery time, the vendor quoting the lowest bid price may not be the best source of procurement for a component. The costs incurred as the result of poor quality can readily be evaluated; these are the increased costs of inspection and rejects. No methods for determining the cost of uncertain or unreliable delivery performance have been presented prior to this dissertation, and attempts to quantify this factor have included such methods as finding the ratio of promises made to promises kept as suggested in Feigenbaum (1961) on page 512.

The TVC models of this dissertation provide not only a method for determining the optimal buffer for uncertain delivery, but also provide a means for evaluating the total variable cost of procurement as a function of the uncertainty in a vendor's delivery capability. In comparing the alternative bids of two vendors for a large subsystem, the prime contractor could evaluate the two vendors on the basis of the total cost of procurement. This total cost would be composed of the vendor's bid price for the item, the added quality costs, and the total variable cost of procurement as a function of the uncertainty of delivery.

$$\text{Total Cost} = \text{Bid Price} + \text{Quality Costs} + \text{TVC}. \tag{8.13}$$

The bid price is known, and the probable quality costs can be determined from past experience with the vendor. Past experience can also be used to determine a 90% range of delivery for each contractor, and this value used to determine the expected TVC of procurement for the given subsystem if procured from each vendor.

For example, assume that a given item must be procured from either subcontractor A or B. In terms of the models developed earlier, assume that this item has an urgency parameter $m = 3$, scaling constant $K = 10$, and daily holding costs $H = \$50/\text{day}$. From experience with each vendor, the 90% range on delivery for each is estimated to be 13 days for vendor A and 20 days for vendor B. Assuming the delivery date for this item is approximated by a chi-square p. d. f. in both cases, Tables XI and XIV of Chapter VII give an expected TVC* of $804 and $1262 for vendors A and B, respectively. In the case of vendor B a buffer of 22 days is needed whereas in the case of vendor A only 13 days are needed. Thus the greater uncertainty in the delivery time of vendor B results in a longer buffer and a higher expected TVC of procurement. The values of expected TVC can now be inserted into equation (8.13) and the total cost of procurement that can be expected for each vendor has been quantified. If the quality costs are equal, then vendor B's bid price must be lower than A's by at least $1262-$804 or $458 in order to justify awarding the contract to vendor B.

The capability to evaluate the uncertainty of delivery in terms of dollars and cents enables the factor of on-time delivery capability to be brought directly into the award of contracts. This factor is usually a qualitative or subjective factor in contract awards, but the models of this dissertation provide a means to define and evaluate this factor, and thus give quantitative cost data to justify a decision. Further research and development work should be undertaken to facilitate the application of the models in vendor rating systems.

Whereas much remains to be done, it is hoped that this research and dissertation have contributed in some small measure to the body of knowledge in the field of procurement and inventory control and that the models produced by the research will be of some aid to those who must make the very difficult procurement decisions of industry.

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APPENDIXES

FOREWORD TO THE APPENDICES

The programs presented in this Appendix were written in FORTRAN II-D for use on an IBM 1620-1311 Model I with 20K storage and 1622 Card Read-Punch. The programs can easily be adapted for use on other systems.

Appendices A, B, C, and D each include a brief summary of the calculations performed by the program, a description of input format for data, and a program listing. An example of the output of each program is presented in the text as part of the solution to the sample problems. Documentation of the programs is accomplished by the use of "COMMENT" statements within the program listing rather than with the aid of a detailed flow chart. The "COMMENT" statements are used to describe the calculations, the variables used, and/or the steps being performed in different sections of the program. It is felt that this type of documentation will facilitate the adaptation of the programs to run on system configurations other than that for which they were written.

Appendix E is a summary of the Fibonacci search procedure which is used in the programs of Appendices B and D to find the TVC* of procurement and the buffer associated with this minimum cost. The Fibonacci search is presented in many different texts, and the purpose of the discussion in Appendix E is to briefly describe the procedure and show how it is applied in the search for the minimum TVC within this dissertation. Examples illustrating the logic of the search are taken from the context of the sample problem solution under the chi-square assumption of delivery date.

APPENDIX A

CALCULATION OF TVC FOR A GIVEN BUFFER LENGTH UNDER CHI-SQUARE DELIVERY DATE DISTRIBUTION

The program listed in this appendix will calculate the expected TVC of procurement for a given set of cost parameters and a given buffer length under the assumption of a chi-square delivery date distribution. The expression used to calculate TVC is that given in equation (4.14). However, since the DO loop used to calculate the lateness cost series require the index of summation to begin with the integer one instead of zero, the initial and terminal values of the index of summation were adjusted to one and r/2. The series of the expected lateness cost is thus calculated in the program as

$$\mathrm{SUM} = \sum_{i=1}^{r/2} \left[m+(r/2)-i\right]! \left[\frac{r+\sqrt{2r}(y-d)}{2}\right]^{i-1} \frac{\left[(r/2)-1\right]!}{\left[(r/2)-1\right]!\,(i-1)!}$$

in program statements 41 through 62.

This program was also used to study the relative magnitudes of terms in the lateness cost series. If Sense Switch 2 is on, the three basic parts of each term in the lateness cost series above will be punched out as well as the value of each term and its per cent of the sum of the series. If sense switch 2 is off, only the series terms and their per cent of the sum will be punched.

The input data for this program should be placed on seven cards organized as follows with the parameter values punched in each card as indicated:

Card 1: Degrees of Freedom, r; col. 1-3 with no decimal places

Card 2: No. of Std. Deviations in Buffer, y; col. 6-10 with 3 decimal places

Card 3: Timing Parameter, d; col. 11-15 with 3 decimal places

Card 4: Urgency Parameter, m; col. 16-20 with 3 decimal places

Card 5: Scaling Parameter, K; col. 21-30 with 3 decimal places

Card 6: Space Units Required, W; col. 31-40 with 5 decimal places, and Annual Cost of Space Unit, C_h; col. 41-50 with 5 decimal places

Card 7: Value of Item, V; col. 51-60 with 5 decimal places, and Annual Cost of Capital, etc., P; col. 61-70 with 5 decimal places.

If the data does not "fit" the prescribed format, decimal points punched in the cards take precedence over those specified in the format statements. If it is desired to change the order or format of input data, this can easily be accomplished by changing statements 1-7 and 10-16. However, the same variable names should be used in the new READ statements so that the variables will be defined.

A sample of output from this program is included in the solution to the sample problem of Chapter IV in Figure 15, page 76.

A listing of the statements of this program follows on the next page.

```
34000320070136000320070249024025119636113000102
ZZJOB
ZZFORX
C  PROGRAM TO CALCULATE THE EXPECTED TOTAL VARIABLE COST OF PROCUREMENT FOR THE
C  ASSUMPTION OF A CHI-SQUARE DISTRIBUTION OF DELIVERY DATE FOR A GIVEN BUFFER.
C  READ IN DATA
    1 READ 10, R
    2 READ 11, Y
    3 READ 12, D
    4 READ 13, POWER
    5 READ 14, SCALE
    6 READ 15, W,CSPACE
    7 READ 16, V,COFCAP
C  CONVERT M FROM FLOATING POINT TO FIXED POINT
    8 MPOWER = POWER + 0.1
C  CALCULATE THE NUMBER OF DAYS BEFORE REQUIREMENT DATE EXPEDITING
C  SHOULD START.  THIS IS D*STANDARD DEVIATION.
    9 DSIGMA = D*((2.0*R)**0.5)+0.00001
C  INPUT FORMAT STATEMENTS
   10 FORMAT(F3.0)
   11 FORMAT(5X,F5.3)
   12 FORMAT(10X,F5.3)
   13 FORMAT(15X,F5.3)
   14 FORMAT(20X,F10.3)
   15 FORMAT(30X,2(F10.5))
   16 FORMAT(50X,2F10.5)
C  CALCULATION OF OFTEN USED TERMS AND INITIALIZATION
C  RPART IS R+SIGMA(Y-D) OVER 2
   17 RPART=(R+((2.0*R)**0.5)*(Y-D))/2.0
C  N IS THE TERMINAL VALUE OF INDEX OF SUMMATION--(R/2)
   18 N=R/2.0 + 0.1
   19 SUM=0.0
C  PUNCH OUT INPUT DATA
   20 PUNCH 101, R
   21 PUNCH 102, Y
   22 PUNCH 103, D
   23 PUNCH 104, POWER
   24 PUNCH 105, SCALE
   25 PUNCH 132, W
   26 PUNCH 133, CSPACE
   27 PUNCH 130, V
   28 PUNCH 131, COFCAP
   29 PUNCH 109, DSIGMA
   30 DIMENSION A(22), PERCNT(22), FACTOR(22)
C  IF SENSE SWITCH 2 IS ON, THE 3 PARTS OF EACH TERM IN THE LATENESS COST SERIES
C  WILL BE PUNCHED OUT.  IF SS2 OFF, PROCEED.
   31 IF (SENSE SWITCH 2) 32,33
   32 PUNCH 122
C  CALCULATE FACTORIALS TO BE USED IN LATENESS COST SERIES.
   33 L = N-1+MPOWER
   34 FACTOR(1) = 1.0
   35 DO 37 I = 2,L
   36 S = I
   37 FACTOR(I) = FACTOR(I-1)*S
   38 J = L+1
   39 DO 40 I = J,20
   40 FACTOR(I) = 0.0
C  CALCULATE AND SUM THE LATENESS COST SERIES.  EACH TERM OF THE SERIES IS
C  COMPOSED OF 3 PARTS--PARTA, PARTB, PARTC.
   41 DO 62 I=1,N
   42 U=I
C  PARTA = (M+N-I) FACTORIAL

   43 J = MPOWER+N-I
   44 PARTA = FACTOR(J)
C  PARTB = R+ SIGMA(Y-D) OVER 2 RAISED TO THE (I-1)TH POWER
   45 PARTB=RPART**(U-1.0)
C  PARTC = BINOMIAL TERM-- (N-1) FACTORIAL/(N-I)FACTORIAL*(I-1)FACTORIAL.
   46 PARTC = FACTOR(N-1)
```

```
C  IF (N-I) IS ZERO, DEFINE 0 FACTORIAL AS 1.0
   47 IF (N-I) 48,48,50
   48 FACT = 1.0
   49 GO TO 52
   50 J =N-I
   51 FACT =FACTOR(J)
   52 PARTC = PARTC/FACT
C  IF (I-1) IS ZERO, DEFINE 0 FACTORIAL AS 1.0
   53 IF (I-1) 54,54,56
   54 FACT = 1.0
   55 GO TO 57
   56 FACT = FACTOR(I-1)
   57 PARTC = PARTC/FACT
C  A(I) IS THE ITH TERM OF THE LATENESS COST SERIES.
   58 A(I)= PARTA*PARTB*PARTC
C  SUM ACCUMULATES EACH TERM  A(I) AS IT IS CALCULATED IN THE DO LOOP
   59 SUM= SUM+A(I)
C  IF SENSE SWITCH 2 IS ON, PUNCH PARTA, PARTB, PARTC, AND THEIR PRODUCT--A(I)
   60 IF (SENSE SWITCH 2) 61,62
   61 PUNCH 123,I,PARTA,PARTB,PARTC,A(I)
   62 CONTINUE
C  END OF SERIES CALCULATIONS.  PUNCH OUT HEADINGS FOR LISTING OF EACH SERIES
C  TERM AND ITS PERCENTAGE OF TOTAL SUM. USED TO EVALUATE SIGNIFICANCE OF TERMS.
   65 PUNCH 100
   67 PUNCH 110
   68 PUNCH 111
   70 DO 73 I=1,N
   71 PERCNT(I)=(A(I)/SUM)*100.0
   73 PUNCH 112, I, A(I), PERCNT(I)
C  CALCULATE CONSTANT COEFFICIENT OF SERIES SUM IN EXPECTED LATENESS COST.
   75 CONST=(SCALE*(2.0**POWER))/(FACTOR(N-1)*EXPF(RPART))
C  CALCULATE EXPECTED LATENESS COST
   76 COSTLC = CONST*SUM
   77 PUNCH 100
C  PUNCH OUT LATENESS COST COEFFICIENT AND SUM OF SERIES,THEN LATENESS COST.
   78 PUNCH 115, SUM
   80 PUNCH 116, CONST
   81 PUNCH 100
   82 PUNCH 117,COSTLC
C  CALCULATE EXPECTED STORAGE SPACE COST
   85 COSTSS= W*CSPACE*Y*((2.0*R)**0.5)/365.0
C  CALCULATE EXPECTED INVENTORY VALUE COST
   86 COSTTC= V*COFCAP*Y*((2.0*R)**0.5)/365.0
C  CALCULATE EXPECTED T.V.C. FOR THIS SET OF COST PARAMETERS.
   87 TOTC= COSTSS+COSTTC+COSTLC
C  CALCULATE THE BUFFER (Y*SIGMA) FOR THIS EVALUATION OF T.V.C.
   88 BUFFER = Y*((2.0*R)**0.5) + 0.00001
C  PUNCH OUT REMAINING COST COMPONENTS AND EXPECTED T.V.C. OF PROCUREMENT.
   89 PUNCH 100
   90 PUNCH 125,COSTSS
   91 PUNCH100
   92 PUNCH 126, COSTTC
   93 PUNCH 100
   94 PUNCH 127, TOTC
C  PUNCH THE BUFFER USED IN THIS EVALUATION OF T.V.C.
   95 PUNCH 100

   96 PUNCH 128, BUFFER
C  TYPES--ENTER NEW DATA AND PUSH START--ON CONSOLE TYPEWRITER.
   97 TYPE 129
C  COMPUTER THEN PAUSES WHILE NEW DATA IS READIED.  PUSH START TO PRODEED.
   98 PAUSE
   99 GO TO 1
C  OUTPUT FORMAT STATEMENTS.
  100 FORMAT(1X)
  101 FORMAT(5X,34H PARAMETERS ARE AS FOLLOWS -- R=   ,F6.1)
  102 FORMAT(35X,4HY=   ,F8.3)
  103 FORMAT(35X,4HD=   ,F8.3)
  104 FORMAT(35X,24HEXPEDITING COST POWER=   ,F8.3)
```

```
105 FORMAT(35X,31HEXPEDITING COST SCALE FACTOR=  ,F8.3)
107 FORMAT(2F16.5)
108 FORMAT(2F16.5)
109 FORMAT(29H EXPEDITING STARTS D*SIGMA = ,F6.2,31H  DAYS BEFORE REQU
1091IREMENT DATE.)
110 FORMAT(2X,15H  LATENESS COST,10X,25H MAGNITUDE        PERCENT)
111 FORMAT(3X,6HSERIES,16X,25H  OF TERM           OF  SUM)
112 FORMAT(16H TERM NUMBER    ,I3,5X,E16.8,5X,F7.3)
115 FORMAT(49H THE LATENESS COST SUM FOR THIS SET OF M,R,Y,D = ,E16.8)
116 FORMAT(49H THE LATENESS COST CONSTANT COEFF FOR THIS SET = ,E16.8)
117 FORMAT (5X,45H THE LATENESS COMPONENT OF TOTAL VAR. COST = ,F16.4)
1220FORMAT(10X,65HNO    FACTORIAL PART     POWER PART  BINOMIAL PART
1221 PRODUCT OF 3)
123 FORMAT(9HTERM NO. ,I3,3X,4(E15.7))
125 FORMAT(5X,45H THE STORAGE SPACE COMPONENT OF TOTAL COST = ,F16.4)
126 FORMAT(5X,45H THE INVENTORY VALUE COMPONENT OF TOT COST = ,F16.4)
127 FORMAT(5X,42H THE TOTAL VARIABLE COST OF PROCUREMENT = ,F19.4)
1280FORMAT(51H THE BUFFER FOR THIS TVC CALCULATION IS (Y*SIGMA) =,F6.1
1281,7H  DAYS.)
129 FORMAT(31H ENTER NEW DATA AND PUSH START.)
130 FORMAT(35X,18HVALUE OF PART   =   ,F12.2)
131 FORMAT(35X,18HCOST OF CAPITAL=   ,F14.4)
132 FORMAT(35X,18HSPACE REQUIRED =   ,F12.2)
133 FORMAT(35X,18HCOST/SPACE/YEAR=   ,F14.4)
210 END
```

APPENDIX B

SOLUTION FOR OPTIMAL BUFFER AND TVC* UNDER CHI-SQUARE DELIVERY DATE DISTRIBUTION

The program listed in this appendix will find the minimum expected TVC of procurement and the y* and optimal buffer associated with this minimum expected TVC for a given set of cost parameters under the assumption of a chi-square delivery date distribution. In order to find the optimum values of y and TVC*, the interval of y from .01 to 9.86 was considered to be 986 discrete points. The program evaluates TVC at 14 of these points in finding that value of y which results in the lowest TVC. Thus, the optimal y is found to within .01 and the optimal TVC to within a few cents. The logic of the Fibonacci search procedure is outlined in Appendix E. The evaluation of TVC for a given y value is accomplished essentially with the program of Appendix A that has been adapted for use with the Fibonacci search; thus the programs of Appendices A and B have many common statements and variable names.

The input data and format for this program are the same as those for the program in Appendix A with the exception that no card for the parameter y is needed. The input for this program should be placed on six cards organized as follows with the parameter values punched as indicated:

Card 1: Degrees of Freedom, r; col. 1-3 with no decimal places

Card 2: Timing Parameter, d; col. 11-15 with 3 decimal places

Card 3: Urgency Parameter, m; col. 16-20 with 3 decimal places

Card 4: Scaling Parameter, K; col. 21-30 with 3 decimal places

Card 5: Space Units Required, W; col. 31-40 with 5 decimal places, and Annual Cost of Space Unit, C_h; col. 41-50 with 5 decimal places

Card 6: Value of Item, V; col. 51-60 with 5 decimal places, and Annual Cost of Capital, etc., P; col. 61-70 with 5 decimal places.

The order or format of input data can be changed by altering statements 1-7 and 10-16. If it is desired to use this program to calculate decision tables as suggested in Chapter VII, it will be advisable to change the program to replace the four holding cost parameters with the daily holding costs H. This can be accomplished by changing the following statements: numbers 6, 7, 24, 25, 26, 27, 85, 86, 87, 90, 92, and removing format statements 15, 16, 125, 126, 130, 131, 132, and 133. Essentially the changes required are

(1) Provision of a READ statement for H

(2) Provision of an output listing for H with the other parameters

(3) Elimination of statements 85 and 86 which calculate storage space and inventory value costs separately and replacing them with a statement to calculate total expected holding costs as done in equation (8.9)

(4) Change TVC equation to include total holding costs

(5) Modify output statements and format.

In the calculation of tables it may also be desired to have the computer generate values of certain parameters. This can easily be done by elimination of READ statements for those parameters and inclusion of logic statements which will accomplish the desired sequence of cost parameter values.

It may also be desired to change the number of evaluations of Fibonacci search, and thus the number of y values searched. For example, 12 evaluations search for minimum TVC within the interval (.01, 3.77) rather than the interval (.01, 9.86) that is searched with 14 evaluations. In the Fibonacci procedure, additional evaluations greatly increase the size of the interval of y values searched. But as most optimal buffers will be associated with values of y less than 5, additional evaluations are unnecessary. However, if the optimal value of y for a given set of parameters is the upper limit of the interval in which evaluations were made, the number of evaluations should be increased so as to evaluate higher values of y. The number of evaluations can easily be changed by changing the following statements; numbers 617, 619, 623, and 629. If the number of evaluations desired is "n," these statements should read as follows:

```
617  K = "n"
619  MIN = "n"
623  K = "n-1"
629  IF (K-"n"), 630, 623, 623
```

where n is a positive integer. Note that any number of evaluations may be performed up to and including n = 16.

A sample of output from this program is included in the solution to the sample problem of Chapter IV in Figure 16, page 78. Note that the program logs each of the y values and TVC evaluations such that a curve of expected TVC vs. y can be plotted to illustrate the sensitivity of TVC to different buffer lengths for the given set of parameters. In the calculation of decision tables as suggested in Chapter VII, the output of successive TVC evaluations can be eliminated if desired by removing statements 77, 79, 82, 90, 92, and 94 and their corresponding format statements.

A listing of the statements of this program follows on the next page.

```
34000320070136000320070249024025119636113OO1O2
ZZJOB
ZZFORX
C  FIBONACCI SEARCH OF TVC(Y) TO FIND THE OPTIMAL BUFFER FOR CHI-SQUARE DEL.DIST
C  READ IN DATA (DISTRIBUTION PARAMETERS, LATENESS AND OTHER COST PARAMETERS)
    1 READ 10, R
    3 READ 12, D
    4 READ 13, POWER
    5 READ 14, SCALE
    6 READ 15, W,CSPACE
    7 READ 16, V,COFCAP
C  CALCULATE THE NUMBER OF DAYS BEFORE REQUIREMENT DATE EXPEDITING
C  SHOULD START.  THIS IS D*STANDARD DEVIATION.
    9 DSIGMA = D*((2.0*R)**0.5)
C  FORMAT STATEMENTS FOR INPUT OF DATA
   10 FORMAT(F3.0)
   12 FORMAT(10X,F5.3)
   13 FORMAT(15X,F5.3)
   14 FORMAT(20X,F10.3)
   15 FORMAT(30X,2(F10.5))
   16 FORMAT(50X,2F10.5)
   17 DIMENSION A(22), FACTOR(22)
C  PUNCH OUT INPUT DATA
   20 PUNCH 101, R
   21 PUNCH 103, D
   22 PUNCH 104, POWER
   23 PUNCH 105, SCALE
   24 PUNCH 132, W
   25 PUNCH 133, CSPACE
   26 PUNCH 130, V
   27 PUNCH 131, COFCAP
   28 PUNCH 109, DSIGMA
C  CALCULATION OF OFTEN USED TERMS AND INITIALIZATION
C  CONVERT M FROM FLOATING POINT TO FIXED POINT
   29 MPOWER = POWER + 0.1
C  N IS THE TERMINAL VALUE OF INDEX OF SUMMATION--(R/2)
   30 N=(R/2.0)+0.1
C  L IS THE NUMBER OF FACTORIALS THAT WILL BE NEEDED.
   31 L = N-1+MPOWER
   32 FACTOR(1) = 1.0
C  DO LOOP 33 TO 35 CALCULATES FACTORIALS. NOTE MAX PERMITTED VALUE OF L IS 22.
   33 DO 35 I = 2,L
   34 S = I
   35 FACTOR(I) = FACTOR(I-1)*S
C  DIMENSION STATEMENT FOR VARIABLES INVOLVED IN FIBONACCI SEARCH
  600 DIMENSION M(16), Y(16), TOTC(16)
C  FIBONACCI NUMBERS
  601 M(1)=1
  602 M(2)=2
  603 M(3)=3
  604 M(4)=5
  605 M(5)=8
  606 M(6)=13
  607 M(7)=21
  608 M(8)=34
  609 M(9)=55
  610 M(10)=89
  611 M(11)=144
  612 M(12)=233
  613 M(13)=377
  614 M(14)=610
  615 M(15)=987
  616 M(16)=1597
C  FIBONACCI INITIALIZATION FOR FIRST EVALUATION OF TVC
  617 K= 14
  618 BOUND = 0.0
  619 MIN = 14
  620 B=M(K)
  621 Y(K)=B*0.01
  622 GO TO 39
```

```
C  FIBONACCI INITIALIZATION OF Y AND BUFFER FOR SECOND EVALUATION OF TVC
  623 K = 13
  624 B=M(K)
  625 Y(K)=(B*0.01)+BOUND
C  RPART IS R+SIGMA(Y-D) OVER 2. A NEW RPART MUST BE CALCULATED FOR EACH Y VALUE
   39 RPART=(R+((2.0*R)**0.5)*(Y(K)-D))/2.0
C  REINITIALIZE SUM FOR EACH TVC EVALUATION
   40 SUM=0.0
C  CALCULATE AND SUM THE LATENESS COST SERIES.  EACH TERM OF THE SERIES IS
C  COMPOSED OF 3 PARTS--PARTA, PARTB, PARTC.
   41 DO 62 I=1,N
   42 U=I
C  PARTA = (M+N-I) FACTORIAL
   43 J = MPOWER+N-I
   44 PARTA = FACTOR(J)
C  PARTB = R+ SIGMA(Y-D) OVER 2 RAISED TO THE (I-1)TH POWER
   45 PARTB=RPART**(U-1.0)
C  PARTC = BINOMIAL TERM-- (N-1) FACTORIAL/(N-I)FACTORIAL*(I-1)FACTORIAL.
   46 PARTC = FACTOR(N-1)
C  IF (N-I) IS ZERO, DEFINE 0 FACTORIAL AS 1.0
   47 IF (N-I) 48,48,50
   48 FACT = 1.0
   49 GO TO 52
   50 J =N-I
   51 FACT =FACTOR(J)
   52 PARTC = PARTC/FACT
C  IF (I-1) IS ZERO, DEFINE 0 FACTORIAL AS 1.0
   53 IF (I-1) 54,54,56
   54 FACT = 1.0
   55 GO TO 57
   56 FACT = FACTOR(I-1)
   57 PARTC = PARTC/FACT
C  A(I) IS THE ITH TERM OF THE LATENESS COST SERIES.
   58 A(I)= PARTA*PARTB*PARTC
C  SUM ACCUMULATES EACH TERM  A(I) AS IT IS CALCULATED IN THE DO LOOP
   59 SUM= SUM+A(I)
   62 CONTINUE
C  END OF LATENESS COST SERIES CALCULATIONS
C  CALCULATE CONSTANT COEFFICIENT OF SERIES SUM IN EXPECTED LATENESS COST.
   75 CONST=(SCALE*(2.0**POWER))/(FACTOR(N-1)*EXPF(RPART))
C  CALCULATE EXPECTED LATENESS COST
   76 COSTLC = CONST*SUM
   77 PUNCH 100
C  CALCULATE BUFFER FOR THIS TVC EVALUATION.  +0.00001 IS FOR ROUND OFF ERROR
   78 BUFFER = Y(K)*((2.0*R)**0.5)+0.00001
C  PUNCH K, BUFFER, COST COMPONENTS, AND TVC
   79 PUNCH 112, K, Y(K), BUFFER
   82 PUNCH 117,COSTLC
C  CALCULATE EXPECTED STORAGE SPACE COST
   85 COSTSS= W*CSPACE*Y(K)*((2.0*R)**0.5)/365.0
C  CALCULATE EXPECTED INVENTORY VALUE COST
   86 COSTTC= V*COFCAP*Y(K)*((2.0*R)**0.5)/365.0
C  CALCULATE EXPECTED T.V.C. FOR THIS SET OF COST PARAMETERS AND Y.
   87 TOTC(K)=COSTSS+COSTTC+COSTLC
   90 PUNCH 125,COSTSS
   92 PUNCH 126, COSTTC
   94 PUNCH 127, TOTC(K)
C  IF THIS IS THE FIRST TVC EVALUATION (K=14), BRANCH BACK TO CALC K=13
  629 IF(K-14) 630,623,623
C  IS TVC(K) JUST CALCULATED LESS THAN THE PRESENT TVC MINIMUM
  630 IF (TOTC(K)-TOTC(MIN)) 640,640,631
C  IF NOT, BRANCH TO CALCULATE THE NEW BUFFER AND TVC DEPENDING ON WHETHER
C  THE VALUE OF Y JUST USED IN CALCULATING TVC(K) IS LESS THAN Y FOR TVC(MIN)
  631 IF (Y(MIN)-Y(K)) 651,651,635
  635 BOUND=Y(K)
  636 K=K-1
  637 B=M(K+1)
  638 IF (K-1) 700,625,625
```

```
C  IF NEW TVC(MIN), STORE NEW MIN AND BRANCH BACK TO ANOTHER EVALUATION
  640 IF (Y(MIN)-Y(K)) 645,650,650
  645 BOUND=Y(MIN)
  646 MIN= K
  647 K=K-1
  648 B=M(K+1)
  649 IF (K-1) 700,625,625
  650 MIN=K
  651 K=K-1
  652 IF (K-1) 700,624,624
C  PUNCH OPTIMAL BUFFER AND MINIMUM TOTAL COST IN INTERVAL
  700 PUNCH 100
  701 PUNCH 100
  702 PUNCH 134, TOTC(MIN)
  703 BUFFER = Y(MIN)*((R*2.0)**0.5)
  704 PUNCH 135, Y(MIN), BUFFER
  705 TYPE 129
  706 PAUSE
  707 GO TO 1
C  FORMAT STATEMENTS FOR LISTING OF PARAMETERS AND FOR OUTPUT OF EXPECTED COSTS
  100 FORMAT(1X)
  101 FORMAT(5X,34H PARAMETERS ARE AS FOLLOWS -- R=  ,F6.1)
  102 FORMAT(35X,4HY=   ,F8.3)
  103 FORMAT(35X,4HD=   ,F8.3)
  104 FORMAT(35X,24HEXPEDITING COST POWER=  ,F8.3)
  105 FORMAT(35X,31HEXPEDITING COST SCALE FACTOR=  ,F8.3)
  107 FORMAT(2F16.5)
  108 FORMAT(2F16.5)
  109 FORMAT(29H EXPEDITING STARTS D*SIGMA = ,F6.2,31H  DAYS BEFORE REQU
  1091IREMENT DATE.)
  112 FORMAT(15H EVAL NUMBER   ,I3,5X,3HY =,F8.3,5X,8HBUFFER =,F10.3)
  117 FORMAT (5X,45H THE LATENESS COMPONENT OF TOTAL VAR. COST = ,F16.4)
  125 FORMAT(5X,45H THE STORAGE SPACE COMPONENT OF TOTAL COST = ,F16.4)
  126 FORMAT(5X,45H THE INVENTORY VALUE COMPONENT OF TOT COST = ,F16.4)
  127 FORMAT(5X,42H THE TOTAL VARIABLE COST OF PROCUREMENT = ,F19.4)
  129 FORMAT(31H ENTER NEW DATA AND PUSH START.)
  130 FORMAT(35X,18HVALUE OF PART  =  ,F12.2)
  131 FORMAT(35X,18HCOST OF CAPITAL=  ,F14.4)
  132 FORMAT(35X,18HSPACE REQUIRED =  ,F12.2)
  133 FORMAT(35X,18HCOST/SPACE/YEAR=  ,F14.4)
  134 FORMAT(54H THE MINIMUM EXPECTED VARIABLE COST OF PROCUREMENT  = ,F
  134110.2)
  135 FORMAT(42H THE OPTIMAL BUFFER FOR THIS TVC IS AT Y =,F6.2,13H OR Y
  1351*SIGMA =,F7.2,6H DAYS.)
  210 END
```

APPENDIX C

CALCULATION OF TVC FOR A GIVEN BUFFER LENGTH UNDER POISSON DELIVERY DATE DISTRIBUTION

The program listed in this appendix will calculate the expected TVC of procurement for a given set of cost parameters and a given buffer length under the assumption of a Poisson delivery date distribution. The expressions used to calculate TVC are those given in equations (5.33), (5.34), and (5.35). Note that all necessary moments of the Poisson are calculated initially, then the program branches to calculate TVC depending upon the value of m specified.

The input data and format for this program are the same as that for the program in Appendix A with the exception that the first card gives the value of the Poisson parameter μ. The seven input cards should be organized as follows with the parameter values punched as indicated:

Card 1: Mean of Poisson, μ; col. 1-3 with no decimal places

Card 2: No. of Std. Deviations in buffer, y; col. 6-10 with 3 decimal places

Card 3: Timing Parameter, d; col. 11-15 with 3 decimal places

Card 4: Urgency Parameter, m; col. 16-20 with 3 decimal places

Card 5: Scaling Parameter, K; col. 21-30 with 3 decimal places

Card 6: Space Units Required, W; col. 31-40 with 5 decimal places, and Annual Cost of Space Unit, C_h; col. 41-50 with 5 decimal places

Card 7: Value of Item, V; col. 51-60 with 5 decimal places, and Annual Cost of Capital, etc., P; col. 61-70 with 5 decimal places.

Note that integer values are required for the mean μ, the buffer $y\sigma$, and the expediting time $d\sigma$. Thus the proper values of y and d to use in a particular situation must first be calculated from the buffer length and expediting time after the proper range on delivery date has been determined. When the 90% range on delivery date is established, Table II can be used to determine μ. The values of y and d can then be found as

$$y = \text{Buffer}/\sqrt{\mu}$$

$$d = \text{Expediting Time}/\sqrt{\mu}$$

where $\sqrt{\mu}$ is the standard deviation of the delivery date distribution and Buffer and Expediting Time are expressed as positive integers in days. The order or format of input data can be changed by altering statements 1-7 and 11-17.

A sample of output from this program is included in the solution to the sample problem of Chapter V in Figure 20, page 98.

A listing of the statements of this program follows on the next page.

```
34000320070133000320070249024025119636113001O2
ZZJOB
ZZFORX
C   CALCULATION OF EXPECTED TVC OF PROCUREMENT FOR GIVEN VALUES OF Y, D, AND COST
C   PARAMETERS UNDER AN ASSUMPTION OF A POISSON DELIVERY DATE DISTRIBUTION.
C   READ IN DATA (DISTRIBUTION PARAMETERS, LATENESS AND OTHER COST PARAMETERS)
     1 READ 11, U
     2 READ 12, Y
     3 READ 13, D
     4 READ 14, EXPON
     5 READ 15, SCALE
     6 READ 16, W, CSPACE
     7 READ 17, V, COFCAP
C   FORMAT STATEMENTS FOR INPUT OF DATA
    11 FORMAT(F3.0)
    12 FORMAT(5X,F5.3)
    13 FORMAT(10X,F5.3)
    14 FORMAT(15X,F5.3)
    15 FORMAT(20X,F10.3)
    16 FORMAT(30X,2(F10.5))
    17 FORMAT(50X,2F10.5)
C   CALCULATION OF GENERAL PARAMETERS
    61 S=U+((U**0.5)*(Y-D))
    62 ETOU=EXPF(-U)
    63 N=S-0.99
C   INITIALIZE SUMS TO O TH TERM SINCE DO LOOPS START WITH T = 1
C   SUMO WILL BE USED TO ACCUMULATE THE SUMMATION OF F(T)
C   SUM1 WILL BE USED TO ACCUMULATE THE SUMMATION OF T*F(T),  ETC.
    64 SUMO = ETOU
    65 SUM1 = 0.0
    66 SUM2 = 0.0
    67 SUM3 = 0.0
       KBUFF = Y*(U**0.5)+0.1
       KEXPED = D*(U**0.5)+0.1
C   PUNCH OUT INITIAL PARAMETERS
    21 PUNCH 501, U
    22 PUNCH 502, Y
    23 PUNCH 503, D
    24 PUNCH 504, EXPON
    25 PUNCH 505, SCALE
    26 PUNCH 506, W
    27 PUNCH 507, CSPACE
    28 PUNCH 508, V
    29 PUNCH 509, COFCAP
    69 PUNCH 510, N
       PUNCH 515, KEXPED
C   CALCULATE SUMS OF SERIES (1 TO  S-1) FOR DIFFERENT MOMENTS OF T
    80 DO 95 I=1,N
C   CALCULATE FACTORIAL PART OF POISSON PROBABILITY DENSITY FUNCTION
    81 FACTOR = 1.0
    82 L = I
    83 DO 85 J=1,L
    84 X=J
    85 FACTOR = FACTOR*X
C   CALCULATE THE ITH TERM OF THE PDF FOR EACH SERIES
    86 T=I
    87 PROB=(U**T)*ETOU/FACTOR
    88 TERM1=T*PROB
    89 TERM2=(T**2.0)*PROB
    90 TERM3=(T**3.0)*PROB
C   ACCUMULATE SUMS FOR EACH SERIES
    91 SUMO=SUMO+PROB
    92 SUM1=SUM1+TERM1
    93 SUM2=SUM2+TERM2
    94 SUM3=SUM3+TERM3
    95 CONTINUE
C   BRANCH TO CALCULATE LATENESS COST DEPENDING ON EXPONENT OF LC FUNCTION(1,2,3)
    96 M=EXPON+0.01
    97 GO TO (100,200,300), M
```

```
C  CALCULATION OF LATENESS COST FOR LC FUNCTION EXPONENT = 1
 100 PARTA = (U**0.5)*(Y-D)
 101 PARTB = S*SUM0
 102 PARTC = SUM1
 104 COSTLC = SCALE*(-PARTA+PARTB-PARTC)
 110 GO TO 400
C  CALCULATION OF LATENESS COST FOR LC FUNCTION EXPONENT = 2
 200 PARTA = U*(1.0+((Y-D)**2))
 201 PARTB = (S**2)*SUM0
 202 PARTC = 2.0*S*SUM1
 203 PARTD = SUM2
 204 COSTLC = SCALE*(PARTA-PARTB+PARTC-PARTD)
 210 GO TO 400
C  CALCULATION OF LATENESS COST FOR LC FUNCTION EXPONENT = 3
 300 PARTA = U*(1.0-(3.0*(U**0.5)*(Y-D))-((U**0.5)*((Y-D)**3)))
 301 PARTB = (S**3)*SUM0
 302 PARTC = 3.0*(S**2.0)*SUM1
 303 PARTD = 3.0*S*SUM2
 304 PARTE = SUM3
 305 COSTLC = SCALE*(PARTA+PARTB-PARTC+PARTD-PARTE)
 310 GO TO 400
 400 CONTINUE
C  CALCULATION OF STORAGE SPACE COMPONENT OF TOTAL VARIABLE COST
 401 COSTSS = W*CSPACE*Y*(U**0.5)/365.0
C  CALCULATION OF TIED-UP CAPITAL COMPONENT OF TOTAL VARIABLE COST
 402 COSTTC = V*(COFCAP/365.0)*Y*(U**0.5)
C  CALCULATION OF THE TOTAL VARIABLE COST OF PROCUREMENT
 403 TOTC = COSTLC+COSTSS+COSTTC
C  PUNCH COST COMPONENTS AND TOTAL VARIABLE COST
 410 PUNCH 500
 411 PUNCH 511, COSTLC
     PUNCH 500
 412 PUNCH 512, COSTSS
     PUNCH 500
 413 PUNCH 513, COSTTC
     PUNCH 500
 414 PUNCH 514, TOTC
     PUNCH 500
 416 PUNCH 516, Y,KBUFF
 425 TYPE 599
 426 GO TO 1
C  FORMAT STATEMENTS FOR LISTING OF PARAMETERS AND FOR OUTPUT OF EXPECTED COSTS
 500 FORMAT(1X)
 501 FORMAT(5X,50H PARAMETERS ARE AS FOLLOWS -- U=MEAN OF POISSON = ,F5
 5011.1)
 502 FORMAT(35X,4HY=  ,F8.3)
 503 FORMAT(35X,4HD=  ,F8.3)
 504 FORMAT(35X,31HEXPEDITING COST EXPONENT (M)=  ,F8.3)
 505 FORMAT(35X,31HEXPEDITING COST SCALE FACTOR=  ,F8.3)
 506 FORMAT(35X,17HSPACE REQUIRED=  ,F15.2)
 507 FORMAT(35X,18HCOST/SPACE/YEAR=  ,F17.5)
 508 FORMAT(35X,17HVALUE OF PART =  ,F15.2)
 509 FORMAT(35X,18HCOST OF CAPITAL=  ,F16.4)
 510 FORMAT(35X,29HNUMBER OF TERMS IN SERIES =  ,I3)
 511 FORMAT (5X,45H THE LATENESS COMPONENT OF TOTAL VAR. COST = ,F16.4)
 512 FORMAT(5X,45H THE STORAGE SPACE COMPONENT OF TOTAL COST = ,F16.4)
 513 FORMAT(5X,45H THE TIED-UP CAPITAL COMPONENT OF TOT COST = ,F16.4)
 514 FORMAT(5X,42H THE TOTAL VARIABLE COST OF PROCUREMENT = ,F19.4)
 515 FORMAT(29H EXPEDITING STARTS D*SIGMA = ,I3,31H  DAYS BEFORE REQUIR
 5151EMENT DATE.)
 5160FORMAT(25H THE BUFFER FOR THIS Y = ,F6.2,16H  IS Y*SIGMA OR ,I3,7H
 5161  DAYS.)
 599 FORMAT(31H ENTER NEW DATA AND PUSH START.)
 600 END
```

APPENDIX D

SOLUTION FOR OPTIMAL BUFFER AND TVC* UNDER POISSON DELIVERY DATE DISTRIBUTION

The program listed in this appendix will find the minimum TVC of procurement and the optimal buffer associated with this minimum expected TVC for a given set of cost parameters under the assumption of a Poisson delivery date distribution. In order to find the optimum buffer and TVC*, integer values of buffer from 1 to 88 days were searched using a Fibonacci search procedure to find that buffer length yielding the minimum TVC. If after the 9 evaluations needed to search the interval of buffer lengths from 1 to 88 it is found that the minimum TVC is at a buffer length of one day, the TVC for a buffer length of zero days is evaluated to determine if this is the minimum point. The optimal buffer is thus found from all possible integer buffer lengths from 0 to 88 days, and the minimum expected TVC associated with this buffer length. The Fibonacci search logic is the same as that used in the program of Appendix B with the exception that integer values of buffer are represented by the Fibonacci numbers. The value of y used in the TVC calculations is then calculated as

$$y = \text{Buffer}/\sqrt{\mu}. \tag{D.1}$$

The TVC calculations are performed in the same manner as done in the program of Appendix C. Thus the program of Appendix D has many statements common to the programs of Appendices B and C.

The input data and format for this program are the same as these for the program in Appendix C with the exception that no card for the parameter y is needed. The input for this program should be placed on six cards organized as follows with the parameter values punched as indicated:

Card 1: Mean of Poisson, μ; col. 1-3 with no decimal places

Card 2: Timing Parameter, d; col. 11-15 with 3 decimal places

Card 3: Urgency Parameter, m; col. 16-20 with 3 decimal places

Card 4: Scaling Parameter, K; col. 21-30 with 3 decimal places

Card 5: Space Units Required, W; col. 31-40 with 5 decimal places, and Annual Cost of Space Unit, C_h; col. 41-50 with 5 decimal places

Card 6: Value of Item, V; col. 51-60 with 5 decimal places, and Annual Cost of Capital, etc., P; col. 61-70 with 5 decimal places.

The program is written such that only integer values of buffer are used in TVC evaluations, but the procedure outlined in Appendix C must be followed to calculate a value for the parameter d that will guarantee an integer value of $d\sigma$. The order or format of input data can be changed by altering statements 1-7 and 11-17. If it is desired to use this program to calculate decision tables as described in Chapter VII, changes similar to those described in the discussion of Appendix B may be utilized to facilitate calculation of the tables.

If it is desired to change the range of buffer values searched, this can be accomplished in the same manner described in Appendix B. For "n" evaluations of TVC, statement numbers 45, 46, 63, and 425 should read

```
 45  K = "n"
 46  MIN = "n"
 63  K = "n-1"
425  IF (K-"n") 430, 63, 63
```

where n is a positive integer. Note that any number of evaluations may be performed up to and including n = 11. However, the time needed to perform a single evaluation of TVC increases greatly as the integers used become large. Thus, more than 9 evaluations should be used only when the optimum buffer lies at the top of the range searched for 9 evaluations, which is 88 days.

A sample of output from this program is included in the solution to the sample problem of Chapter V in Figure 21, page 99.

A listing of the statements of this program follows on the next page.

```
3400032007013600032007024902402511963611300102
7ZJOB
ZZFORX
C   FIBONACCI SEARCH OF TVC(Y) TO FIND THE OPTIMAL BUFFER FOR POISSON DEL. DIST.
C   READ IN DATA (DISTRIBUTION PARAMETERS, LATENESS AND OTHER COST PARAMETERS)
     1 READ 11, U
     3 READ 13, D
     4 READ 14, EXPON
     5 READ 15, SCALE
     6 READ 16, W, CSPACE
     7 READ 17, V, COFCAP
C   LAST WILL BE USED IN CALCULATING TVC FOR ZERO BUFFER IF MIN BUFFER IS 1 DAY.
     8 LAST = 1
C   FORMAT STATEMENTS FOR INPUT OF DATA
    11 FORMAT(F3.0)
    13 FORMAT(10X,F5.3)
    14 FORMAT(15X,F5.3)
    15 FORMAT(20X,F10.3)
    16 FORMAT(30X,2(F10.5))
    17 FORMAT(50X,2F10.5)
C   PUNCH OUT INITIAL PARAMETERS
    21 PUNCH 501, U
    23 PUNCH 503, D
    24 PUNCH 504, EXPON
    25 PUNCH 505, SCALE
    26 PUNCH 506, W
    27 PUNCH 507, CSPACE
    28 PUNCH 508, V
    29 PUNCH 509, COFCAP
C   DIMENSION STATEMENT FOR VARIABLES INVOLVED IN FIBONACCI SEARCH
    30 DIMENSION M(11), BUFFER(11), Y(11), TOTC(11)
C   FIBONACCI NUMBERS
    31 M(1)=1
    32 M(2)=2
    33 M(3)=3
    34 M(4)=5
    35 M(5)=8
    36 M(6)=13
    37 M(7)=21
    38 M(8)=34
    39 M(9)=55
    40 M(10)=89
    41 M(11)=144
C   FIBONACCI INITIALIZATION FOR FIRST EVALUATION OF TVC
    45 K = 9
    46 MIN = 9
    47 BOUND = 0.0
    48 B = M(K)
    49 BUFFER(K) = B
    50 Y(K) = BUFFER(K)/(U**0.5)
    51 GO TO 71
C   FIBONACCI INITIALIZATION OF Y AND BUFFER FOR SECOND EVALUATION OF TVC
    63 K = 8
    64 B = M(K)
    65 BUFFER(K) = B+BOUND
    66 Y(K) = BUFFER(K)/(U**0.5)
C   CALCULATION OF GENERAL PARAMETERS
    71 S=U+((U**0.5)*(Y(K)-D))
    72 ETOU=EXPF(-U)
    73 N=S-0.99
C   INITIALIZE SUMS TO 0 TH TERM SINCE DO LOOPS START WITH T = 1
C   SUMO WILL BE USED TO ACCUMULATE THE SUMMATION OF F(T)
C   SUM1 WILL BE USED TO ACCUMULATE THE SUMMATION OF T*F(T),  ETC.
    74 SUMO = ETOU
    75 SUM1 = 0.0
    76 SUM2 = 0.0
    77 SUM3 = 0.0
    78 FACTOR = 1.0
```

```
C  CALCULATE SUMS OF SERIES (1 TO  S-1) FOR DIFFERENT MOMENTS OF T
   80 DO 95 I=1,N
C  CALCULATE FACTORIAL PART OF POISSON PROBABILITY DENSITY FUNCTION
   81 T = I
   82 FACTOR = FACTOR*T
C  CALCULATE THE ITH TERM OF THE PDF FOR EACH SERIES
   87 PROB=(U**T)*ETOU/FACTOR
   88 TERM1=T*PROB
   89 TERM2=(T**2.0)*PROB
   90 TERM3=(T**3.0)*PROB
C  ACCUMULATE SUMS FOR EACH SERIES
   91 SUM0=SUM0+PROB
   92 SUM1=SUM1+TERM1
   93 SUM2=SUM2+TERM2
   94 SUM3=SUM3+TERM3
   95 CONTINUE
C  BRANCH TO CALCULATE LATENESS COST DEPENDING ON EXPONENT OF LC FUNCTION(1,2,3)
   96 I=EXPON+0.01
   97 GO TO (100,200,300), I
C  CALCULATION OF LATENESS COST FOR LC FUNCTION EXPONENT = 1
  100 PARTA = (U**0.5)*(Y(K)-D)
  101 PARTB = S*SUM0
  102 PARTC = SUM1
  104 COSTLC = SCALE*(-PARTA+PARTB-PARTC)
  110 GO TO  400
C  CALCULATION OF LATENESS COST FOR LC FUNCTION EXPONENT = 2
  200 PARTA = U*(1.0+((Y(K)-D)**2))
  201 PARTB = (S**2)*SUM0
  202 PARTC = 2.0*S*SUM1
  203 PARTD = SUM2
  204 COSTLC = SCALE*(PARTA-PARTB+PARTC-PARTD)
  210 GO TO 400
C  CALCULATION OF LATENESS COST FOR LC FUNCTION EXPONENT = 3
  300 PARTA = U*(1.0-(3.0*(U**0.5)*(Y(K)-D))-((U**0.5)*((Y(K)-D)**3)))
  301 PARTB = (S**3)*SUM0
  302 PARTC = 3.0*(S**2.0)*SUM1
  303 PARTD = 3.0*S*SUM2
  304 PARTE = SUM3
  305 COSTLC = SCALE*(PARTA+PARTB-PARTC+PARTD-PARTE)
  310 GO TO 400
  400 CONTINUE
C  CALCULATION OF STORAGE SPACE COMPONENT OF TOTAL VARIABLE COST
  401 COSTSS =(W*CSPACE*Y(K)*(U**0.5))/365.0
C  CALCULATION OF TIED-UP CAPITAL COMPONENT OF TOTAL VARIABLE COST
  402 COSTTC = V*(COFCAP/365.0)*Y(K)*(U**0.5)
C  CALCULATION OF THE TOTAL VARIABLE COST OF PROCUREMENT
  403 TOTC(K) = COSTLC+COSTSS+COSTTC
C  PUNCH K, BUFFER, COST COMPONENTS, AND TVC
  408 PUNCH 500
  409 PUNCH 510, K, BUFFER(K)
  411 PUNCH 511, COSTLC
  412 PUNCH 512, COSTSS
  413 PUNCH 513, COSTTC
  414 PUNCH 514, TOTC(K)
C  IF THIS IS THE FIRST TVC EVALUATION (K= 9), BRANCH BACK TO CALC K= 8
  425 IF (K-9) 430,63,63
C  IS TVC(K) JUST CALCULATED LESS THAN THE PRESENT TVC MINIMUM
  430 IF (TOTC(K)-TOTC(MIN)) 440,440,431
C  IF NOT, BRANCH TO CALCULATE THE NEW BUFFER AND TVC DEPENDING ON WHETHER
C  THE VALUE OF Y JUST USED IN CALCULATING TVC(K) IS LESS THAN Y FOR TVC(MIN)
  431 IF (Y(MIN)-Y(K)) 453,453,435
  435 BOUND = BUFFER(K)
  436 K = K-1
  437 B=M(K+1)
  438 IF (K-1) 460,65,65
```

```
C  IF NEW TVC(MIN), STORE NEW MIN AND BRANCH BACK TO ANOTHER EVALUATION
 440 IF (Y(MIN)-Y(K)) 445,450,450
 445 BOUND = BUFFER(MIN)
 446 MIN = K
 447 K = K-1
 448 B = M(K+1)
 449 IF (K-1) 460,65,65
 450 MIN = K
 451 IF (K-0) 452,452,453
 452 K=1
 453 K=K-1
 454 IF (K-1) 460,64,64
C  TEST TO SEE IF PRESENT BUFFER IS 1 DAY. IF SO, CALCULATE TVC FOR BUFFER = 0
 460 IF (LAST) 490,490,461
 461 J = BUFFER(K+1) + 0.1
 462 IF (J-1) 463,463,490
 463 K = 0
 464 BUFFER(K) = 0.0
 465 LAST = 0
 466 GO TO 66
C  PUNCH OPTIMAL BUFFER AND MINIMUM TOTAL COST IN INTERVAL
 490 PUNCH 500
 491 PUNCH 521, TOTC(MIN)
 492 PUNCH 522, BUFFER(MIN)
 495 TYPE 599
 496 PAUSE
 497 GO TO 1
C  IF TOTC(MIN) IS AT BUFFER=1, THEN IT SHOULD BE COMPARED WITH TVC FOR BUFFER=0
C  FORMAT STATEMENTS FOR LISTING OF PARAMETERS AND FOR OUTPUT OF EXPECTED COSTS
 500 FORMAT(1X)
 501 FORMAT(5X,50H PARAMETERS ARE AS FOLLOWS-- U= MEAN OF POISSON = ,F5
 5011.1)
 503 FORMAT(35X,4HD=  ,F8.3)
 504 FORMAT(35X,31HEXPEDITING COST EXPONENT (M)=  ,F8.3)
 505 FORMAT(35X,31HEXPEDITING COST SCALE FACTOR=  ,F8.3)
 506 FORMAT(35X,18HSPACE REQUIRED =  ,F12.2)
 507 FORMAT(35X,18HCOST/SPACE/YEAR=  ,F14.4)
 508 FORMAT(35X,18HVALUE OF PART  =  ,F12.2)
 509 FORMAT(35X,18HCOST OF CAPITAL=  ,F14.4)
 510 FORMAT(21HFIBONACCI SEARCH NO. ,I3,43H, BUFFER FOR THIS CALCULATIO
 5101N OF T.V.C. IS ,F8.1,5H DAYS)
 511 FORMAT (5X,45H THE LATENESS COMPONENT OF TOTAL VAR. COST = ,F16.4)
 512 FORMAT(5X,45H THE STORAGE SPACE COMPONENT OF TOTAL COST = ,F16.4)
 513 FORMAT(5X,45H THE TIED-UP CAPITAL COMPONENT OF TOT COST = ,F16.4)
 514 FORMAT(5X,42H THE TOTAL VARIABLE COST OF PROCUREMENT = ,F19.4)
 521 FORMAT(58HTHE OPTIMAL TOTAL EXPECTED VARIABLE COST OF PROCUREMENT
 5211IS,F15.2)
 522 FORMAT(49HTHE BUFFER TIME RESULTING IN THIS MINIMUM TVC IS ,F8.1,6
 5221H DAYS.)
 599 FORMAT(31H ENTER NEW DATA AND PUSH START.)
 600 END
```

APPENDIX E

OUTLINE OF FIBONACCI SEARCH PROCEDURE

Fibonacci search is used in the computer programs of this dissertation to find the minimum expected variable cost of procurement. A brief summary of the search procedure and its use in this application is included for those who may not be familiar with it. For examples of the logic involved in the Fibonacci search, specific reference is made to the search for the minimum TVC(y) under the chi-square delivery date distribution.

More detailed discussion of the Fibonacci search and its comparisons to other techniques may be found in Nemhauser (1966) or Wilde (1964). The following summary was adapted from these sources.

Fibonacci search can be applied to any unimodal function of one variable, and it guarantees that the optimal solution may be found after no more than a fixed number of evaluations of the function are made. If the variable is discrete as in the case of the Poisson delivery distribution, the number of points searched depends only on the total number of feasible points. When the variable is continuous as in the case of the chi-square delivery date distribution, the number of points that must be searched depends only on the size of the interval and the degree of accuracy required. For the search of Appendix B, the degree of accuracy specified was such that the optimal y was found to within .01 standard deviation. The minimum TVC was thus found to within a few cents. This method can be considered as an optimal search procedure since it minimizes the maximum number of points that must be searched for an arbitrary unimodal function of one variable. TVC(y) was shown to be a strictly convex (U-shaped) or a constantly increasing curve in the region of feasible y, and thus TVC is always unimodal with respect to finding a minimum. It is called Fibonacci search because the number of points examined and the strategy for placing the points are closely related to the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n$$

where $F_1 = 1$ and $F_2 = 1$. The Fibonacci search can best be described by reference to Table XVII adapted from page 98 of Nemhauser (1966).

For any given number of evaluations to be performed, column two in Table XVII gives the number of points that will be searched. In Appendix B, 14 evaluations are used to search 986 points on the TVC(y) curve. These points are assumed to be values of y that are .01 apart from y = .01 to y = 9.86. Columns d_1 and d_2 give the first two points where TVC(y) is to be evaluated which are points number 377 and 610. In translating the points d_1 and d_2 into values of y to be used in TVC(y) evaluations they were multiplied by 0.01. Thus the first two evaluations of TVC(y) were made at y = 6.10 and y = 3.77.

After the first two evaluations are performed, the values of TVC(y) that have been calculated are compared. If TVC(6.10) > TVC(3.77) this indicates that the minimum TVC(y) must occur at some value of y < 6.10, since TVC(y) is unimodal in the interval (.01, 9.86) with respect to finding a minimum. Thus, all points y ≥ 6.10 are eliminated as possibly resulting in the minimum TVC(y). If TVC(6.10) had been less than

TABLE XVII

EVALUATION POINTS IN FIBONACCI SEARCH

Number of Evaluations	No. of Points Searched		d_1	d_2
n = 1	total points = 1	Evaluations at	-	-
2	2		1	2
3	4		2	3
4	7		3	5
5	12		5	8
6	20		8	13
7	33		13	21
8	54		21	34
9	88		34	55
10	143		55	89
11	232		89	144
12	376		144	233
13	609		233	377
14	986		377	610
15	1586		610	987
16	2583		987	1597

TVC(3.77), all points $y < 3.77$ would have been eliminated.

Once the Fibonacci search has begun, the procedure at each stage is basically very simple. Somewhere in the remaining interval for y will be a value of y for which TVC(y) has been previously evaluated. To continue the search, the next value of y should be located symmetrically with respect to the one already in the interval. In the case where TVC(6.10) > TVC(3.77), the evaluation of TVC(y) already within the remaining interval is that for y = 3.77. The point y = 3.77 is 6.10-3.77 = 2.33 from the upper bound of the interval. Thus, the next evaluation of TVC(y) should be 2.33 from the lower bound of the interval which is at y = 2.33.

At each stage the two points within the remaining interval where evaluations are needed for the next comparison are given as d_1 and d_2 in Table XVII. For example if the search has progressed to the point where only 5 evaluations remain, the two evaluations of TVC(y) needed for the next comparison are the fifth and eighth points from the lower bound. One of these will have been evaluated previously. After the second is evaluated, the two are compared; and all points between the point yielding the higher TVC(y) evaluation and its nearest bound are eliminated. The search then proceeds to the next stage until the final three points are evaluated and the point yielding the minimum TVC(y) is found.

Examples of the Fibonacci search procedure are included in Figure 16 on page 78 and Figure 29 on page 99. The reader can gain a better understanding of the logic involved in choosing the next value of y to use in evaluation of TVC(y) by plotting the values of TVC(y) vs. y as they are evaluated. At each stage the reader should perform the comparison of the TVC(y) making note of the points eliminated by the comparison and make reference to Table XVII to find the point within the remaining interval where the next TVC(y) evaluation should be made. For further study pages 94 through 99 of Nemhauser (1966) and pages 24 through 32 of Wilde (1964) offer excellent presentations of the Fibonacci search.

APPENDIX F

DISCUSSION OF EXPECTED LATENESS COST FOR $m = 0$

If the urgency parameter of lateness cost "m" is allowed to equal zero, an interesting situation develops. If $m = 0$ and $d = 0$ the lateness cost term in the model reduces to the method often described in texts on inventory theory for dealing with the probability of late delivery. This method might be termed the "out-of-stock cost" method. One example where this method is used is the model presented on pages 146-150 of Starr and Miller (1962) for a dynamic inventory situation. On page 149 the cost of lateness is defined as the out-of-stock cost times the probability that delivery is late. In this appendix the lateness cost function of this dissertation will be analyzed for the case of $m = 0$. This will illustrate how the "out-of-stock cost" method can be considered as a special case of the model developed in this dissertation for dealing with one-stage procurement situations.

The lateness cost function is defined in equation (2.10) as follows

$$C(t) = K(t + d\sigma)^m \qquad \text{for} \quad t \geq - d\sigma$$

$$= 0 \qquad \text{for} \quad t < - d_\sigma$$

where

K = a scaling constant

t = the delivery date (a random variable)

d_σ = the number of days prior to the requirement date that expediting begins

$m =$ the rate at which lateness costs increase with time

($m = 1$, 2, or 3 in this dissertation).

If m is allowed to equal zero, then equation (2.10) reduces to

$$
\begin{aligned}
C(t) &= K(t + d\sigma)^0 \\
&= K \qquad \text{for} \quad t \geq - d\sigma \\
&= 0 \qquad \text{for} \quad t < - d\sigma \ .
\end{aligned}
$$

If $m = 0$, $C(t)$ is equal to zero until time $t = - d\sigma$ is reached and is equal to a constant sum K for all points after the time $t = - d\sigma$. In other words no lateness cost is incurred if delivery is made prior to a time $d\sigma$ days before requirement date, and a fixed amount of lateness cost is incurred for delivery at any time after $t = - d\sigma$.

If d is set equal to zero, then no lateness cost is incurred if delivery is made prior to the requirement date; and if delivery is made after the requirement date, the cost K is incurred. This situation is identical to the method of dealing with lateness costs that has previously been used. Generally when developing procurement models allowing for late delivery it is assumed that no lateness charges are incurred if delivery is made prior to the requirement date. However, it may actually be the case that substantial expediting costs were incurred in the effort to obtain delivery by the requirement date. Also in previous developments it is generally assumed that an "out-of-stock cost" is incurred if delivery is made at any time after the requirement date. This assumption is also unrealistic in many cases. For example, delivery only one day past the requirement date would in most cases be much less costly than the additional delay caused by delivery three weeks late.

The expected value of lateness cost in this model for the case of $m = 0$ is also consistent with the "out-of-stock cost" method of dealing with lateness cost. The expected lateness cost used in this dissertation is given in equation (2.13) as

$$E(LC) = \int_{-d\sigma}^{\infty} K(t + d\sigma)^m \cdot f(t)\, dt$$

where $f(t)$ is the p.d.f. of the delivery date distribution. For $m = 0$ and $d = 0$ this reduces to

$$E(LC) = \int_{0}^{\infty} K(t + d\sigma)^m \cdot f(t)\, dt$$

$$= K \int_{0}^{\infty} f(t)\, dt \ .$$

Because the requirement date is defined at $t = 0$, the above expression for expected lateness cost is a constant times the probability that delivery is after the requirement date. This is the way that lateness costs are defined on page 149 of Starr and Miller (1962).

Thus, the method of defining an "out-of-stock cost" for items delivered late may be considered a special case of the model developed in this dissertation for one stage procurement situations. The models developed herein allow much greater flexibility in dealing with the costs of lateness. Models are developed for integer values of $m = 1$, 2, and 3. In addition a model is developed under the assumption of a Poisson p.d.f. for delivery date that allows for any positive value of m to be used. If the use of $m = 1$, 2, or 3 is too restrictive for a particular situation, the Poisson p.d.f. with exponential lateness cost may be used as described on pages 97-104 of this dissertation.

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