STABILITY AND BLOWUP PROBLEMS ON THE MAGNETOHYDRODYNAMIC EQUATIONS AND BOUSSINESQ EQUATIONS

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Fluid mechanics is the study of the behavior of fluids and deformation of the fluid under the influence of shearing forces. One of the most widely used and studied system of equations in fluid mechanics are the Navier-Stokes equations. We study two closely related systems, the magnetohydrodynamics (MHD) equations and the Boussinesq equations. For the MHD equations, we discuss the stabilization effect of a background magnetic field on electrically conducting fluids as well as the construction of a conditional finite time blowup for a 1D transformation of the 2D ideal MHD equations. Additionally, we present the existence and uniqueness of weak solutions to the d-dimensional Boussinesq equations with fractional dissipation and no thermal diffusion.

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CHAPTER 1

INTRODUCTION

Fluid mechanics is the study of the behavior of fluids and deformation of the fluid under the influence of shearing forces. One of the most widely used and studied systems of equations in fluid mechanics is the Navier-Stokes equations. The Navier-Stokes equations play an important role in physical applications from modeling hurricane paths to blood flow patterns (see, e.g. [40, 23]). For incompressible fluids, the Navier-Stokes equations can be written as

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where u denotes the velocity field, p denotes the pressure, ν represents the kinematic fluid viscosity, and f represents the external force. In the special case when $\nu = 0$, the Navier Stokes equations become the Euler equations.

Here, and throughout this paper, we write for convenience

$$\partial_t := \frac{\partial}{\partial t}, \quad \partial_i := \frac{\partial}{\partial x_i}.$$

The first equation of (1.1) comes from the conservation of momentum stated in Newton's second law. The second equation comes from the conservation of mass for incompressible fluids. The equations describe the fluid velocity which is greatly influenced by the fluid's viscosity.

Viscosity is a measure of the resistance of a fluid to deformation under shear stress. Although most fluids have some viscosity, there are instances where viscosity is very small. Modeling large vortices shed by jumbo jets or predicting hurricane trajectories often use viscosity values in the range of $\nu \sim 10^{-6}$ [40, p. 3]. Such applications highlight the need to the study ideal fluids, also known as inviscid fluids, which have no viscosity. Much of the research presented here investigates the behavior of fluid with no viscosity or where the dissipation term is fractional in two particular fluid equations related to the Navier Stokes equations.

There are many systems of partial differential equations that are very closely related to the Navier-Stokes equations which have arisen from modeling physical phenomena of different fluids. Two such systems are the Boussinesq equations and the magnetohydrodynamics (MHD) equations. The Boussinesq equations model the velocity of a fluid where the temperature or density of the fluid differs within the fluid. The MHD equations model the velocity and the magnetic field of electrically conducting fluids, i.e. plasmas. Although these equations model different physical phenomena, the mathematical tools used to analyze the effects of the fluid's viscosity and dissipation on the behavior of solutions are closely related. This work highlights the author's analysis of these two fluid dynamical systems. In the remaining sections of this first chapter, we highlight the physical applications and provide justification for the analytic study of both the MHD and Boussinesq systems. We begin with an introduction to the MHD equations and then move on to the Boussinesq equations.

Note that throughout this paper, analyses will be performed using Sobolev spaces and some commonly used results in calculus and functional analysis. The statements of these inequalities and definitions of these spaces can be found in the appendix.

1.1 Introduction to the MHD equations

Magnetohydrodynamics is the study of the mutual interaction of magnetic fields and flows of electrically conducting fluids. Many fields including geophysics, metallurgy, and astrophysics are concerned with magnetohydrodynamics. Magnetohydrodynamics arise in many natural and man-made instances including solar magnetic fields generating solar flares, dampening motion of poured liquid metal in casting, and electromagnetic pumps used in nuclear reactors (see, e.g. [23, 15, 26]).

The MHD equations are the main feature of the study of magnetohydrodynamics. The MHD equations were initially derived by the Nobel Laureate Hannes Alfvén [2]. The standard incompressible MHD equations can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \, \nabla \cdot b = 0, \end{cases} \tag{1.2}$$

where u represents the velocity field, p the pressure, b the magnetic field, $\nu \geq 0$ the fluid viscosity, and $\eta \geq 0$ the magnetic diffusivity (resistivity).

These are nonlinear coupled equations where the first equation of (1.2) is the Navier-Stokes equation with the Lorentz force generated by the magnetic field and the second equation is the induction equation for the magnetic field.

The MHD equations are of great interest in mathematics and the nonlinear coupling of the equations poses quite a challenge. Even in 2D, these are particularly difficult equations to analyze. Fundamental questions regarding the behavior of solutions to the MHD equations such as stability and existence have attracted considerable interest in recent years, but many issues have yet to be resolved. In this work, results will be presented that completely solve or provide significant insight to a few of these questions for two-dimensional MHD flow.

One of the fundamental questions is whether physically relevant solutions can develop singularities in finite time or if the solutions remain smooth for all time [49]. In particular, if given smooth initial data

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x),$$

which satisfies $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$, will the solution remain smooth? The answer to this question depends greatly on the fluid viscosity, ν, and the magnetic resistivity, η. We provide a brief summary of results in this area.

1. When $\nu > 0$ and $\eta > 0$, (1.2) is the fully dissipative case which has a large number of physical applications. In the fully dissipative case, any initial data $(u_0, b_0) \in L^2(\mathbb{R}^2)$ leads to the existence of a unique global solution which becomes instantaneously smooth for all time, i.e. $(u, b) \in C^\infty(\mathbb{R}^2 \times (T, \infty))$ for any $T > 0$ (see e.g. [49]). This result shows that even without the initial conditions being smooth and just L^2 the dissipation and diffusion terms $\nu\Delta u$ and $\eta\Delta b$, respectively, effectively control the energy within the system and make it instantaneously smooth even if not initially smooth.

2. When $\nu > 0$ and $\eta = 0$, (1.2) is said to have only dissipation and no diffusion. The global regularity problem remains open and even the global existence of weak solutions remains open. There have been results obtained by F. Lin, L. Xu, and P. Zhang [38] and results obtained by X. Ren, J. Wu, Z. Xiang, and Z. Zhang [43] which show global regularity of solutions with smooth initial data where the solution is near a non-trivial steady state solution.

3. When $\nu = 0$ and $\eta > 0$, (1.2) is said to have only diffusion and no dissipation. This model is often used in instances where magnetic resistance plays an important physical role such as magnetic turbulence and magnetic reconnection [49]. The global regularity problem remains open, but there has been substantial progress in recent years. A

breakthrough in the area came from the work of Q. Jiu, D. Niu, J. Wu, X. Xu, and H. Yu [31] where global *a prior* bounds were found. However, without a global L^∞ bound for the vorticity, the global regularity problem still remains open.

4. When $\nu = \eta = 0$, (1.2) is said to be ideal and can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0. \end{cases}$$

   The ideal MHD models the behavior of a perfectly conducting fluid under the influence of magnetic field. Due to the lack of velocity dissipation and magnetic diffusion in the ideal equations, the global well-posedness issue is extremely difficult and remains open. The work presented in chapter 3 details the author's contribution to the understanding of the behavior of these systems. In particular, a conditional blowup result for the 1D transformation of the 2D ideal MHD equations will be presented.

In addition to the question of global regularity, this work presents stability results for a particular 2D MHD flow with no dissipation and only damping in the vertical component. Attention is focused on the following 2D incompressible MHD flow

$$\begin{cases} \partial_t u_1 + (u \cdot \nabla) u_1 = -\partial_1 P + (B \cdot \nabla) B_1, \quad x \in \mathbb{R}^2, \ t > 0, \\ \partial_t u_2 + (u \cdot \nabla) u_2 + \gamma u_2 = -\partial_2 P + (B \cdot \nabla) B_2, \\ \partial_t B + u \cdot \nabla B = \eta \Delta B + B \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot B = 0. \end{cases} \tag{1.3}$$

Here we have written the velocity equation in terms of a horizontal component u_1 and a vertical component u_2. These equations are just (1.2) without the dissipation term $\nu \Delta u$ and with the addition of a damping term γu_2 in the vertical component.

When there is no magnetic field, $B \equiv 0$, the global regularity of (1.3) as well as the stability near the trivial solution remains open. However, when the velocity is coupled with a magnetic field in the MHD system above, the background magnetic field smooths and stabilizes the fluid even without the dissipation term $\nu\Delta u$. In fact, if velocity and vorticity are initially small they remain small and decay algebraically in time. Numerical simulations and experiments have shown such behavior of magnetic fields influencing electrically conducting fluids. In this paper, we establish these results as mathematical facts in chapter 2 and provide the explicit decay rates.

1.2 Introduction to the Boussinesq equations

The Boussinesq equations play an important role in the study of many physical fluid phenomena where there is convection or a buoyancy driven flow such as atmospheric and oceanic fronts along with Rayleigh-Bérnard convection. The Boussinesq equations are derived from the Navier-Stokes equations while additionally taking temperature or density into account. Let $d \geq 2$ be an integer. The standard d-dimensional incompressible Boussinesq equations can be written as written as

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u + \theta e_d, \\ \partial_t \theta + (u \cdot \nabla)\theta = \kappa\Delta\theta, \\ \nabla \cdot u = 0, \end{cases} \tag{1.4}$$

where u denotes the velocity field, p the pressure, and θ represents the density/temperature. Here $\nu > 0$ denotes the kinematic viscosity, κ represents the thermal diffusivity constant, and $e_d = (0, 0, \ldots, 0, 1)$. For natural convection applications, θ represents the temperature; whereas, in geophysical applications, θ represents density.

The first equation in (1.4) reflects the Navier-Stokes equations with an additional buoyancy term in the direction of the gravitational force, θe_d. The second equation in (1.4) is the heat flow in a temperature/density gradient. Finally, the last equation $\nabla \cdot u = 0$ states that the fluid is incompressible.

An important question for systems of equations in fluid mechanics is whether or not global in time solutions exist for sufficiently smooth initial data. In the Boussinesq equations, the dissipation terms $\nu\Delta u$ and $\kappa\Delta\theta$ play a important part in controlling the long time behavior of the system. For the inviscid Boussinesq system, where $\nu = \kappa = 0$, the global regularity remains an open problem. In recent years, the Boussinesq equations have attracted considerable attention in the mathematical community and significant progress has been made, but there are still open problems in addition to the global regularity for the inviscid case. One such area where open problems arise is the case when there is fractional dissipation.

Typically, the Bousinessq equations are written using a standard Laplacian operator, but there are instances in geophysics where a need for a fractional Laplacian, $(-\Delta)^\alpha$, arise. In modeling atmospheric flows, changes in atmospheric properties occur as the middle atmosphere travels upward. As the atmosphere thins, the effect of thermal diffusion is attenuated resulting in a need for a fractional Laplacian in the Boussinesq model ([7],[24]).

The d-dimensional Boussinesq equations with fractional dissipation can be written as

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nu(-\Delta)^\alpha u - \nabla p + \theta e_d, \quad x \in \mathbb{R}^d,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta = 0, \quad x \in \mathbb{R}^d,\ t > 0, \\ \nabla \cdot u = 0, \\ (u, \theta)|_{t=0} = (u_0, \theta_0), \end{cases} \tag{1.5}$$

where $\alpha > 0$ is a real parameter. The fractional Laplacian $(-\Delta)^\alpha$ used here is defined via the Fourier transform,

$$\mathcal{F}((-\Delta)^\alpha f)(\xi) = (4\pi^2|\xi|^2)^\alpha\, \mathcal{F}(f)(\xi),$$

where $\mathcal{F}(f)(\xi)$ denotes the Fourier transform of f,

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx.$$

Occasionally, we will also use the notation $\Lambda = (-\Delta)^{\frac{1}{2}}$.

Many researchers have investigated the global existence and uniqueness of solutions to (1.5) for $d = 3$ and $\alpha \geq \frac{5}{4}$ with initial data $(u_0, \theta_0) \in H^s(\mathbb{R}^3)$ where $s > \frac{5}{2}$, (see [32, 42, 52, 53, 60]) and where $s > \frac{5}{4}$ (see [32]). Less has been investigated for this problem in weak settings. One result was the paper of Larios, Lunasin, and Titi [35]. They were able to show, among many other results, that $u_0 \in H^1(\mathbb{T}^2)$ and $\theta_0 \in L^2(\mathbb{T}^2)$ lead to a unique and global strong solution of (1.5) where $\mathbb{T}^2$ denotes the 2D periodic box.

In our research [4] that will be discussed in Chapter 4, we looked to further these results by finding the weakest possible setting where uniqueness of solutions occurs for the partially dissipated Boussinesq equations. We obtained two main results. The first result established the global existence of weak solutions of (1.5) for any $\alpha > 0$ with initial data $(u_0, \theta_0) \in L^2(\mathbb{R}^d)$. Uniqueness of weak solutions then occurs for $\alpha \geq \frac{1}{2} + \frac{d}{4}$ with initial data $u_0 \in L^2(\mathbb{R}^d), \theta_0 \in L^2(\mathbb{R}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$. This established uniqueness is in, what appears to be, the weakest functional setting possible for the partially dissipated Boussinesq equations. The second main result established the zero thermal diffusion limit of the fully dissipative Boussinesq equations

$$\begin{cases} \partial_t u^{(\eta)} + u^{(\eta)} \cdot \nabla u^{(\eta)} = -\nu(-\Delta)^\alpha u^{(\eta)} - \nabla P^{(\eta)} + \theta^{(\eta)} \mathbf{e}_d, \quad x \in \mathbb{R}^d, \ t > 0, \\ \partial_t \theta^{(\eta)} + u^{(\eta)} \cdot \nabla \theta^{(\eta)} = \eta \Delta \theta^{(\eta)}, \quad x \in \mathbb{R}^d, \ t > 0, \\ \nabla \cdot u^{(\eta)} = 0, \\ (u^{(\eta)}, \theta^{(\eta)})|_{t=0} = (u_0^{(\eta)}, \theta_0^{(\eta)}) \end{cases} \tag{1.6}$$

and showed that the solution of (1.6) converges strongly to the corresponding solution of (1.4) with an explicit convergence rate as $\eta \to 0$. The Yudovich approach and lower regularity quantities were used due to the weak initial setup $u_0^{(\eta)} \in L^2(\mathbb{R}^d), \theta_0^{(\eta)} \in L^2(\mathbb{R}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$. This is an interesting result as there does not appear to be much research on the zero thermal diffusion limit, particularly when the functional setting is weak. The work from [4] will be discussed in further detail in Chapter 4.

CHAPTER 2

STABILIZATION OF 2D MHD FLOWS

2.1 Stabilization Effects of a Background Magnetic Field

In this chapter, we study the stabilizing and smoothing effects of a background magnetic field on electrically conducting fluids. These effects have been observed in physical and numerical simulations and we now establish the observations as rigorous mathematical facts. As outlined in the introduction, we focus on following the MHD flow

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 = -\partial_1 P + (B\cdot\nabla)B_1, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \gamma u_2 = -\partial_2 P + (B\cdot\nabla)B_2, \\ \partial_t B + u\cdot\nabla B = \eta\Delta B + B\cdot\nabla u, \\ \nabla\cdot u = 0,\ \nabla\cdot B = 0, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^2$ and $t > 0$. Consider the following steady state solution

$$u^{(0)} \equiv 0, \quad B^{(0)} \equiv (0,1). \tag{2.2}$$

This steady state solution solves (2.1). We are interested in the behavior if we perturb this steady state slightly. Let (u,b) be the perturbation near this particular steady state solution

(2.2) with $b = B - B^{(0)}$. Then (u, b) solves the MHD equations

$$\begin{cases} \partial_t u_1 + (u \cdot \nabla) u_1 = -\partial_1 P + (b \cdot \nabla) b_1 + \partial_2 b_1, \\ \partial_t u_2 + (u \cdot \nabla) u_2 + \gamma u_2 = -\partial_2 P + (b \cdot \nabla) b_2 + \partial_2 b_2, \\ \partial_t b + (u \cdot \nabla) b = \eta \Delta b + (b \cdot \nabla) u + \partial_2 u, \\ \nabla \cdot u = 0, \, \nabla \cdot b = 0, \end{cases} \tag{2.3}$$

with $x \in \mathbb{R}^2$, $t > 0$. In particular, we are interested in initial conditions

$$u(x, 0) = u_0(x), \quad b(x, 0) = b_0(x),$$

such that (2.3) possesses a unique global solution and if the velocity and vorticity are initially small then they remain small for all time and actually decay algebraically in time.

The stability problem for the MHD equations with only magnetic diffusion is still a major open problem. The results presented here advance the progress toward that problem. The difference between our results and the still open problem regarding the MHD equations with only magnetic diffusion is that we have included the damping term γu_2 in the vertical direction in order to obtain the desired stability. These results are completely new and advance the techniques and understanding that will be needed to solve the stability problem without the vertical damping term γu_2.

In order to analyze the stability, we consider the vorticity, $\omega = \nabla \times u$, and the current density, $j = \nabla \times b$. By taking the curl of (2.3) we obtain

$$\begin{cases} \partial_t \omega + (u \cdot \nabla) \omega = \gamma \mathcal{R}_1^2 \omega + (b \cdot \nabla) j + \partial_2 j, \\ \partial_t j + (u \cdot \nabla) j = \eta \Delta j + (b \cdot \nabla) \omega + \partial_2 \omega + Q, \end{cases} \tag{2.4}$$

where

$$Q := -2\partial_1 b_1 (\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1 (\partial_2 b_1 + \partial_1 b_2).$$

Here the term $\mathcal{R}_1^2 = \partial_1^2(-\Delta)^{-1}$ denotes the squared Riesz transform.

Yudovich theory has been very successful in dealing with the global regularity problem on the 2D Euler and related equations. However, Yudovich's approach relies crucially on the boundedness of the vorticity. The Riesz transform makes the situation more difficult and the Yudovich approach is no longer applicable. The term $\mathcal{R}_1^2$ is not bounded on L^∞ as it involves singular integral operators and this Reisz transform term can even increase the vorticity's L^∞ norm as shown by T. Elgindi [19].

Since the Yudovich theory is not applicable in this case, a different approach must be used. Our approach takes advantage of a special wave-type structure present in all the physical quantities. As it turns out, u, b, ω, and j all satisfy the exact same wave equation with differing nonhomogeneous terms. This wave structure is particularly useful for obtaining some of the bounds needed including bounds for

$$\int_0^t \|\partial_2 u\|_{L^2(\mathbb{R}^2)}^2 \, d\tau, \quad \text{and} \quad \int_0^t \|\partial_2 \omega(\tau)\|_{H^1(\mathbb{R}^2)}^2 \, d\tau,$$

which are not a consequence of the damping or dissipation of the MHD flow. The main strategic approach to obtain the desired results is to define an energy functional, show that it satisfies an appropriate bound, and then use the bootstrapping argument to show that the bounds hold for all time giving the desired stability result.

Before we state the main results, we first eliminate the pressure terms from (2.3) and (2.4) and provide the explicit wave-type equations that the physical quantities satisfy. Here we apply the Leray-Helmholtz projection operator $\mathbb{P} := I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (2.3) to obtain

$$\begin{cases} \partial_t u = \gamma \mathcal{R}_1^2 u + \partial_2 b + N_1, \\ \partial_t b = \eta \Delta b + \partial_2 u + N_2, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0, \end{cases} \tag{2.5}$$

where $x \in \mathbb{R}^2$, $t > 0$, and where N_1 and N_2 are the nonlinear terms

$$N_1 = \mathbb{P}((b \cdot \nabla)b - (u \cdot \nabla)u),$$
$$N_2 = (b \cdot \nabla)u - (u \cdot \nabla)b.$$

Differentiating (2.5) with respect to t we obtain

$$\begin{cases} \partial_{tt}u = \gamma \mathcal{R}_1^2 \partial_t u + \partial_2 \partial_t b + \partial_t N_1, \\ \partial_{tt}b = \eta \Delta \partial_t b + \partial_2 \partial_t u + \partial_t N_2. \end{cases}$$

Substituting in (2.5) for $\partial_t b$ and $\partial_t u$ we obtain

$$\begin{cases} \partial_{tt}u = \gamma \mathcal{R}_1^2 \partial_t u + \partial_2(\eta \Delta b + \partial_2 u + N_2) + \partial_t N_1, \\ \partial_{tt}b = \eta \Delta \partial_t b + \partial_2(\gamma \mathcal{R}_1^2 u + \partial_2 b + N_1) + \partial_t N_2. \end{cases}$$

Rearranging, we obtain

$$\begin{cases} \partial_{tt}u = \gamma \mathcal{R}_1^2 \partial_t u + \eta \Delta \partial_2 b + \partial_{22} u + \partial_2 N_2 + \partial_t N_1, \\ \partial_{tt}b = \eta \Delta \partial_t b + \gamma \mathcal{R}_1^2 \partial_2 u + \partial_{22} b + \partial_2 N_1 + \partial_t N_2. \end{cases}$$

Again, using (2.5) to make a substitution for $\partial_2 u$ and $\partial_2 b$ yields

$$\begin{cases} \partial_{tt}u = \gamma \mathcal{R}_1^2 \partial_t u + \eta \Delta(\partial_t u - \gamma \mathcal{R}_1^2 u - N_1) + \partial_{22} u + \partial_2 N_2 + \partial_t N_1, \\ \partial_{tt}b = \eta \Delta \partial_t b + \gamma \mathcal{R}_1^2(\partial_t b - \eta \Delta b - N_2) + \partial_{22} b + \partial_2 N_1 + \partial_t N_2. \end{cases}$$

Rearranging one final time and using the fact that $\Delta \mathcal{R}_1^2 = -\partial_{11}$ then yields the desired wave type equations for the velocity and magnetic field equations

$$\begin{cases} \partial_{tt} u - (\eta\Delta + \gamma\mathcal{R}_1^2)\partial_t u - (\eta\gamma\partial_{11} u + \partial_{22} u) = N_3, \\ \partial_{tt} b - (\eta\Delta + \gamma\mathcal{R}_1^2)\partial_t b - (\eta\gamma\partial_{11} b + \partial_{22} b) = N_4, \end{cases}$$

where

$$N_3 = (\partial_t - \eta\Delta)N_1 + \partial_2 N_2,$$
$$N_4 = (\partial_t - \gamma\mathcal{R}_1^2)N_2 + \partial_2 N_1.$$

Similarly, we can rewrite (2.4) and we find that the vorticity, ω, and current density, j, satisfy the same wave equation but with different nonlinear terms

$$\begin{cases} \partial_{tt}\omega - (\eta\Delta + \gamma\mathcal{R}_1^2)\partial_t\omega - (\eta\gamma\partial_{11}\omega + \partial_{22}\omega) = N_5, \\ \partial_{tt} j - (\eta\Delta + \gamma\mathcal{R}_1^2)\partial_t j - (\eta\gamma\partial_{11} j + \partial_{22} j) = N_6, \end{cases}$$

where

$$N_5 = (\partial_t - \eta\Delta)(b \cdot \nabla j - u \cdot \nabla\omega) + \partial_2(b \cdot \nabla\omega - u \cdot \nabla j + Q),$$
$$N_6 = (\partial_t - \gamma\mathcal{R}_1^2)(b \cdot \nabla\omega - u \cdot j + Q) + \partial_2(b \cdot \nabla j - u \cdot \nabla\omega).$$

The smoothing and stabilization properties exhibited by the wave equations above are far greater than the original vorticity and current density system (2.4). These properties combined with the dissipation and damping components allow us to establish the desired stability.

We now state the main results found in [5].

Theorem 2.1.1. *Let* $(u_0, b_0) \in H^3(\mathbb{R}^2)$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then there exists* *sufficiently small $\delta = \delta(\gamma, \eta) > 0$ such that, if*

$$\|\nabla u_0\|_{H^2(\mathbb{R}^2)} + \|\nabla b_0\|_{H^2(\mathbb{R}^2)} \leq \delta,$$

then (2.3) possesses a unique global solution $(u, b) \in C([0, \infty); H^3(\mathbb{R}^2))$ satisfying

$$\|(u,b)(t)\|^2_{H^1(\mathbb{R}^2)} + \int_0^t \left(\|u_2(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla b(\tau)\|^2_{H^1(\mathbb{R}^2)} \right) d\tau \leq C \left(\|u_0\|^2_{H^1(\mathbb{R}^2)} + \|b_0\|^2_{H^1(\mathbb{R}^2)} \right), \tag{2.6}$$

$$\|(\nabla u, \nabla b)(t)\|^2_{H^2(\mathbb{R}^2)} + \int_0^t \left(\|\partial_1 u(\tau)\|^2_{L^2(\mathbb{R}^2)} + \|\Delta u(\tau)\|^2_{H^1(\mathbb{R}^2)} + \|\Delta b(\tau)\|^2_{H^2(\mathbb{R}^2)} \right) d\tau \leq C\delta^2, \tag{2.7}$$

for any $t > 0$ and some constant $C > 0$. Furthermore, the following time decay estimate holds

$$\|\nabla u(t)\|_{H^2(\mathbb{R}^2)} + \|\nabla b(t)\|_{H^2(\mathbb{R}^2)} \leq C \left(\|(u_0, b_0)\|_{L^2(\mathbb{R}^2)} + \delta \right) (1+t)^{-\frac{1}{2}}, \tag{2.8}$$

when δ is small enough. In particular, for any $2 \leq q < \infty$, as $t \to \infty$,

$$\|(u,b)(t)\|_{L^q(\mathbb{R}^2)} \to 0, \tag{2.9}$$

$$\|(u,b)(t)\|_{W^{1,\infty}(\mathbb{R}^2)} \to 0, \quad \text{and} \tag{2.10}$$

$$\|(\nabla u, \nabla b)(t)\|_{W^{1,q}(\mathbb{R}^2)} \to 0. \tag{2.11}$$

Additionally, we obtain the sharp decay rates as stated in the following theorem.

Theorem 2.1.2. *Assume $(u_0, b_0) \in L^1(\mathbb{R}^2) \cap H^3(\mathbb{R}^2)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$ satisfying*

$$\|(u_0, b_0)\|_{L^1(\mathbb{R}^2)} + \|(u_0, b_0)\|_{H^3(\mathbb{R}^2)} \leq \delta,$$

for some δ small enough. Then for $m = 0, 1, 2$, the small global solution (u, b) of the system

(2.3) *obeys*

$$\|D^m u(t)\|_{L^2(\mathbb{R}^2)} + \|D^m b(t)\|_{L^2(\mathbb{R}^2)} \le C\delta(1+t)^{-\frac{1+m}{2}},$$

where $C > 0$ is a constant independent of δ and t.

The proof of Theorem 2.1.1 is very tedious and long, and is thus not included here. Details can be found in a manuscript submitted for publication [5]. The remainder of this chapter will be devoted to proving Theorem 2.1.2.

2.2 Preliminaries

The sharp decay rates in Theorem 2.1.2 cannot be shown using energy estimates. Instead, an integral representation must be used in conjunction with the bootstrapping argument to obtain the desired decay rates. We need two lemmas in order to obtain the rates.

The first lemma provides an explicit decay rate for the heat kernel associated with the fractional Laplacian, Λ^α, for $\alpha \in \mathbb{R}$. The fractional Laplacian operator is defined using the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).$$

Lemma 2.2.1. *Let $\alpha \ge 0$, $\beta > 0$, and $1 \le q \le p \le \infty$. Then there exists a constant C such that, for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^p(\mathbb{R}^d)} \le C t^{-\frac{\alpha}{\beta} - \frac{d}{\beta}\left(\frac{1}{q} - \frac{1}{p}\right)} \|f\|_{L^q(\mathbb{R}^d)}. \tag{2.12}$$

The proof of the above lemma can be found in [17].

Lemma 2.2.2. *Assume* $0 < s_1 \leq s_2$. *Then, for some constant* $C > 0$,

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}\, d\tau \leq \begin{cases} C(1+t)^{-s_1}, & \text{if } s_2 > 1, \\ C(1+t)^{-s_1}\ln(1+t), & \text{if } s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & \text{if } s_2 < 1. \end{cases}$$

These two lemmas will be used in the proof of the decay rates.

Now that we have stated the preliminary lemmas, we must convert (2.5) into an integral representation. We then obtain upper bounds for the kernels of the integral representation. Finally, we apply the bootstrapping argument to show that the decay rate holds for all time. These steps are detailed in the following sections culminating in the proof of Theorem 2.1.2.

2.3 Integral Representation of Solutions

This section details the derivation of the integral representation of (2.5). By taking the Fourier transform of (2.5) we obtain

$$\begin{cases} \partial_t \widehat{u} = -\gamma\xi_1^2|\xi|^{-2}\widehat{u} + i\xi_2\widehat{b} + \widehat{N_1}, \\ \partial_t \widehat{b} = -\eta|\xi|^2\widehat{b} + i\xi_2\widehat{u} + \widehat{N_1}. \end{cases}$$

We write this as

$$\partial_t \widehat{U} = A\widehat{U} + \widehat{N} \tag{2.13}$$

where

$$\widehat{U} = \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix}, \quad A = \begin{pmatrix} -\gamma\xi_1^2|\xi|^{-2} & i\xi_2 \\ i\xi_2 & -\eta|\xi|^2 \end{pmatrix}, \quad \widehat{N} = \begin{pmatrix} \widehat{N_1} \\ \widehat{N_2} \end{pmatrix}.$$

The solution of (2.13) is given by

$$\widehat{U}(t) = e^{At}\widehat{U_0} + \int_0^t e^{A(t-\tau)}\widehat{N(\tau)}\,d\tau,$$

where the characteristic polynomial associated with A is given by

$$\lambda^2 + (\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2)\lambda + (\gamma\eta\xi_1^2 + \xi_2^2) = 0.$$

We then find the eigenvalues of the matrix A to be

$$\lambda_1 = \frac{-(\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2) - \sqrt{\Gamma}}{2},$$
$$\lambda_2 = \frac{-(\gamma\xi_1^2|\xi|^{2} + \eta|\xi|^{-2}) + \sqrt{\Gamma}}{2},$$

where

$$\Gamma = (\gamma\xi_1^2|\xi|^{-1} + \eta|\xi|^2)^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2).$$

The corresponding eigenvectors are given by

$$v_1 = \begin{pmatrix} \lambda_1 + \eta|\xi|^2 \\ i\xi_2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \lambda_2 + \eta|\xi|^2 \\ i\xi_2 \end{pmatrix}.$$

Then the diagonalization for matrix A is given by

$$A = (v_1, v_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(v_1, v_2)^{-1}.$$

From this, we can now write a more explicit formula for e^{At}

$$
\begin{aligned}
e^{At} &= (v_1, v_2) \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} (v_1, v_2)^{-1} \\
&= \frac{1}{(\lambda_1 - \lambda_2) i\xi_2} \begin{pmatrix} \lambda_1 + \eta|\xi|^2 & \lambda_2 + \eta|\xi|^2 \\ i\xi_2 & i\xi_2 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} i\xi_2 & -(\lambda_2 + \eta|\xi|^2) \\ -i\xi_2 & \lambda_2 + \eta|\xi|^2 \end{pmatrix} \\
&= \begin{pmatrix} \eta|\xi|^2 G_1(t) + G_2(t) & G_1(t) i\xi_2 \\ G_1(t) i\xi_2 & -\eta|\xi|^2 G_1(t) + G_3(t) \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(t) &= \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \\
G_2(t) &= \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_2 t} + \lambda_1 G_1(t), \\
G_3(t) &= \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1(t).
\end{aligned}
$$

We may then write

$$
\begin{aligned}
&\widehat{M_1}(\xi, t) = \eta|\xi|^2 G_1(t) + G_2(t), \\
&\widehat{M_2}(\xi, t) = i\xi_2 G_1(t), \\
&\widehat{M_3}(\xi, t) = -\eta|\xi|^2 G_1(t) + G_3(t).
\end{aligned}
$$

These kernels influence the decay rates of both u and b. We may then represent (u, b) as

$$
\widehat{u}(\xi, t) = \widehat{M_1}(\xi, t)\widehat{u_0} + \widehat{M_2}(\xi, t)\widehat{b_0} + \int_0^t \left(\widehat{M_1}(\xi, t-\tau)\widehat{N_1}(\tau) + \widehat{M_2}(\xi, t-\tau)\widehat{N_2}(\tau) \right) d\tau, \quad (2.14)
$$

$$
\widehat{b}(\xi, t) = \widehat{M_2}(\xi, t)\widehat{u_0} + \widehat{M_3}(\xi, t)\widehat{b_0} + \int_0^t \left(\widehat{M_2}(\xi, t-\tau)\widehat{N_1}(\tau) + \widehat{M_3}(\xi, t-\tau)\widehat{N_2}(\tau) \right) d\tau. \quad (2.15)
$$

Note that in the case when $\lambda_1 = \lambda_2$, our representation for (u, b) above remains valid as both $G_2(t)$ and $G_3(t)$ are well defined as long as we use the limiting form of G_1

$$G_1(t) = \lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = te^{\lambda_1 t}.$$

We now have an integral representation for (u, b) in (2.14) and (2.15). In the following section we will find bounds for the kernels $\widehat{M_i}(\xi, t)$ for $m = 1, 2, 3$, and with these bounds we will be able to apply the bootstrapping argument to the integral representation to complete the proof of Theorem 2.1.2.

2.4 Upper Bounds for the Kernels of the Integral Representation

In this section, we find upper bounds for the kernels $\widehat{M_i}(\xi, t)$. In order to obtain the desired results, we much subdivide the frequency space into three subdomains, S_1, S_{21}, S_{22}, and analyze the behavior of the kernels in each of these subdomains. We state this result as a proposition that we will then use to complete the proof of Theorem 2.1.2.

Proposition 2.4.1. *We divide $\mathbb{R}^2$ into two subdomains, $\mathbb{R}^2 = S_1 \cup S_2$ with*

$$S_1 := \left\{ \xi \in \mathbb{R}^2 : \sqrt{\Gamma} \leq \frac{\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2}{2} \ \text{or}\ 3(\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2)^2 \leq 16(\gamma\eta\xi_1^2 + \xi_2^2) \right\},$$
$$S_2 := \left\{ \xi \in \mathbb{R}^2 : \sqrt{\Gamma} > \frac{\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2}{2} \ \text{or}\ 3(\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2)^2 > 16(\gamma\eta\xi_1^2 + \xi_2^2) \right\}.$$

Then we have the following two results.

1. *There are two constants $C > 0$ and $c_0 > 0$ such that, for any $\xi \in S_1$,*

$$Re\,\lambda_1 \leq -\frac{\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2}{2},$$
$$Re\,\lambda_2 \leq -\frac{\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2}{4},$$
$$|G_1(t)| \leq te^{-\frac{-\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2}{4}t},$$

$$|\widehat{M}_i(\xi,t)| \le Ce^{-c_0|\xi|^2 t}, \quad i=1,2,3.$$

2. There is a constant $C>0$ such that, for any $\xi \in S_2$,

$$\begin{aligned}
\lambda_1 &< \frac{3(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)}{4}, \\
\lambda_2 &\le -\frac{\gamma\eta\xi_1^2+\xi_2^2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}, \\
|G_1(t)| &< \frac{2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}\left(e^{-\frac{3}{4}(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)t}+e^{-\frac{C|\xi|^2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}t}\right), \\
|\widehat{M}_i(t)| &< C\left(e^{-\frac{3}{4}(\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2)t}+e^{-\frac{C|\xi|^2}{\gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2}t}\right), \quad i=1,2,3.
\end{aligned}$$

If we further write $S_2 = S_{21}\cup S_{22}$ with

$$S_{21} := \{\xi \in S_2 : \, |\xi| \le 1\},$$
$$S_{22} := \{\xi \in S_2 : \, |\xi| > 1\},$$

then for $i=1,2,3$, and some constants $C>0$, $c_1>0$, $c_2>0$,

$$|\widehat{M}_i(\xi,t)| < Ce^{-c_1|\xi|^2 t}, \quad \text{if } \xi \in S_{21},$$
$$|\widehat{M}_i(\xi,t)| < Ce^{-c_1|\xi|^2 t} + Ce^{-c_2 t}, \quad \text{if } \xi \in S_{22}.$$

We now prove the above proposition. For convenience, let us write

$$B = \gamma\xi_1^2|\xi|^{-2}+\eta|\xi|^2.$$

Then we can write the eigenvalues λ_1 and λ_2 along with Γ as

$$\lambda_1 = \frac{-B-\sqrt{\Gamma}}{2}, \qquad \lambda_2 = \frac{-B+\sqrt{\Gamma}}{2}, \qquad \Gamma = B^2-(\gamma\eta\xi_1^2+\xi_2^2).$$

We first consider the case when $\xi \in S_1$, i.e. $\Gamma \leq \frac{B}{2}$. Then we have

$$-\frac{3B}{4} \leq Re\,\lambda_1 \leq -\frac{B}{2}, \qquad Re\,\lambda_2 \leq -\frac{B}{4}.$$

By the Mean Value Theorem

$$|G_1(t)| = \left|\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}\right| \leq te^{\lambda_2 t} \leq te^{-\frac{B}{4}t}.$$

Using the fact that $xe^{-x} \leq C$ for $x \geq 0$ then if λ_1 is a real number we have

$$|\widehat{M_1}(t)| = \left|\eta|\xi|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right| \leq Bte^{-\frac{B}{4}t} + CBte^{-\frac{B}{4}t} + e^{-\frac{B}{4}t} \leq Ce^{-c_0|\xi|^2 t},$$

for some constant c_0 dependent of γ and η. On the other hand, if λ_1 is imaginary, i.e. $\Gamma = B^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2) < 0$ we must consider two possible subcases

(i) $|\sqrt{\Gamma}| \geq \sqrt{\gamma\eta\xi_1^2 + \xi_2^2}$,

(ii) $|\sqrt{\Gamma}| \leq \sqrt{\gamma\eta\xi_1^2 + \xi_2^2}$.

In case (i), we have by the definition of G_1 that

$$|\lambda_1 G_1(t)| = \frac{\sqrt{\gamma\eta\xi_1^2 + \xi_2^2}}{|\sqrt{\Gamma}|}\left|e^{\lambda_1 t} - e^{\lambda_2 t}\right| \leq Ce^{-\frac{B}{4}t}.$$

In case (ii), we have

$$\gamma\eta\xi_1^2 + \xi_2^2 \geq 4(\gamma\eta\xi_1^2 + \xi_2^2) - B^2,$$

or equivalently,

$$3(\gamma\eta\xi_1^2 + \xi_2^2) \leq B^2.$$

Then we have

$$|\lambda_1 G_1(t)| = \sqrt{\gamma\eta\xi_1^2 + \xi_2^2}|G_1(t)| \leq CBte^{-\frac{B}{4}t} \leq Ce^{-\frac{B}{4}t}.$$

Therefore, if λ_1 is imaginary, then

$$|\widehat{M_1}(\xi,t)| = \left|\eta|\xi|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right| \leq Bte^{-\frac{B}{4}t} + Ce^{-\frac{B}{4}t} \leq Ce^{-c_0|\xi|^2 t}.$$

Hence, for $\xi \in S_1$, the upper bound for the kernel $\widehat{M_1}(\xi,t)$ is

$$|\widehat{M_1}(\xi,t)| \leq Ce^{-c_0|\xi|^2 t}. \tag{2.16}$$

Similarly, we obtain the same bound for $\widehat{M_3}(\xi,t)$

$$|\widehat{M_3}(\xi,t)| \leq Ce^{-c_0|\xi|^2 t}. \tag{2.17}$$

We will, in fact, obtain the same bound for $\widehat{M_2}(\xi,t)$ as well. But in order to prove this, we must consider the two cases

(i) $|\sqrt{\Gamma}| \geq |\xi_2|$,

(ii) $|\sqrt{\Gamma}| \leq |\xi_2|$.

In the case of (i), using the fact that $xe^{-x} \leq C$ for $x \geq 0$, we have

$$|\widehat{M_2(t)}| = |\xi_2 G_1(t)| = \left|\frac{\xi_2}{\sqrt{\Gamma}}\right| \left|e^{\lambda_1 t} - e^{\lambda_2 t}\right| \leq Ce^{-c_0|\xi|^2 t}.$$

In the case of (ii), we have that $|\sqrt{\Gamma}| \leq |\xi_2|$ which is equivalent to

$$-\xi_2^2 \leq B^2 - 4(\gamma\eta\xi_1^2 + \xi_2^2) \leq \xi_2^2.$$

Thus

$$B^2 \geq 4(\gamma\eta\xi_1^2 + \xi_2^2) - \xi_2^2 \geq \xi_2^2.$$

Hence,

$$|\widehat{M_2}(\xi,t)| = |\xi_2 G_1(t)| \leq B|G_1(t)| \leq Bte^{-\frac{B}{4}t} \leq Ce^{-c_0|\xi|^2 t}.$$

This completes the upper bounds for $\widehat{M_i}(\xi,t)$ in the subdomain $\xi \in S_1$.

Now for the other subdomain, we assume $\xi \in S_2$. Then $\frac{B}{2} \le \sqrt{\Gamma} \le B$. We then have the following for λ_1, λ_2 and G_1

$$
\begin{aligned}
& -B \le \lambda_1 < -\frac{3}{4}B, \\
& \lambda_2 = \frac{\Gamma - B^2}{2(B+\sqrt{\Gamma})} \le -\frac{\gamma\eta\xi_1^2 + \xi^2 + 2}{B} \le -\frac{C|\xi|^2}{B}, \\
& |G_1(t)| \le \frac{1}{\lambda_2 - \lambda_1}\left(e^{\lambda_1 t} + e^{\lambda_2 t}\right) < \frac{2}{B}\left(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\right).
\end{aligned}
$$

Consequently, we have the following upper bounds for $\widehat{M_1}(\xi,t)$ and $\widehat{M_3}(\xi,t)$

$$
\begin{aligned}
& |\widehat{M_1}(\xi,t)| = \left|\eta|\xi|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right| \le 2B|G_1(t)| + e^{\lambda_2 t} < C\left(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\right), \\
& |\widehat{M_3}(\xi,t)| = \left|-\eta|\xi|^2 G_1(t) - \lambda_1 G_1(t) + e^{\lambda_1 t}\right| < C\left(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\right).
\end{aligned}
$$

For the bound of $\widehat{M_2}(\xi,t)$, notice that since $\sqrt{\Gamma} > \frac{B}{2}$, we have

$$
\frac{3}{4}B^2 > 4(\gamma\eta\xi_1^2 + \xi_2^2) \ge \xi_2^2.
$$

Therefore,

$$
|\widehat{M_2}(\xi,t)| = |\xi_2 G_1(t)| < CB|G_1(t)| < C\left(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}t}\right).
$$

It remains to show the improved upper bounds for $\widehat{M_i}(\xi,t)$ which is accomplished by further subdividing the subdomain $S_2 = S_{21} \cup S_{22}$. Observe, for $\xi \in S_2$,

$$
\frac{|\xi|^2}{B} = \frac{|\xi|^2}{\gamma\xi_1^2|\xi|^{-2} + \eta|\xi|^2} \ge \frac{|\xi|^2}{\gamma + \eta|\xi|^2} \ge \begin{cases} C|\xi|^2, & \text{if } \xi \in S_{21}, \text{ i.e. } |\xi| \le 1, \\ C, & \text{if } \xi \in S_{22}, \text{ i.e. } |\xi| > 1. \end{cases}
$$

Then, in the case when $\xi \in S_{21}$ the upper bounds for $\widehat{M}_i(\xi,t)$ can be improved to

$$|\widehat{M}_i(\xi,t)| < C\left(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}}\right) \le Ce^{-c_1|\xi|^2 t}.$$

Similarly, in the case when $\xi \in S_{22}$ the upper bounds for $\widehat{M}_i(\xi,t)$ can be improved to

$$|\widehat{M}_i(\xi,t)| < C\left(e^{-\frac{3}{4}Bt} + e^{-\frac{C|\xi|^2}{B}}\right) \le Ce^{-c_1|\xi|^2 t} + Ce^{-c_2 t}.$$

This completes the proof of the proposition. The upper bounds for $\widehat{M}_i(\xi,t)$ along with the integral representation (2.14) and (2.15) will be used to complete the proof of Theorem 2.1.2 in the following section.

2.5 Proof of Decay Rates

We now detail the proof of Theorem 2.1.2. Here we are assuming that the initial data (u_0, b_0) satisfies

$$\|(u_0,b_0)\|_{H^3} \le \delta, \qquad \|(u_0,b_0)\|_{L^1} \le \delta,$$

for sufficiently small $\delta > 0$ and that (u,b) is the corresponding global solution established by Theorem 2.1.1 which has the properties

$$\|(u,b)(t)\|_{H^3}^2 + \int_0^t \left(\|u_2(\tau)\|_{L^2}^2 + \|\nabla u(\tau)\|_{H^2}^2 + \|\nabla b(\tau)\|_{H^3}^2\right) d\tau \le C\delta^2, \tag{2.18}$$

and

$$\|\nabla u(t)\|_{H^2} + \|\nabla b(t)\|_{H^2} \le C\delta(1+t)^{-\frac{1}{2}}, \tag{2.19}$$

where C are constants independent of δ.

We begin the proof by differentiating the integral representation in (2.14) and (2.15) to obtain

$$\widehat{\partial_k^m u}(\xi,t) = \widehat{M_1}(\xi,t)\widehat{\partial_k^m u_0} + \widehat{M_2}(\xi,t)\widehat{\partial_k^m b_0} + \int_0^t \left(\widehat{M_1}(\xi,t-\tau)\widehat{\partial_k^m N_1}(\tau) + \widehat{M_2}(\xi,t-\tau)\widehat{\partial_k^m N_2}(\tau)\right) d\tau, \tag{2.20}$$

$$\widehat{\partial_k^m b}(\xi,t) = \widehat{M_2}(\xi,t)\widehat{\partial_k^m u_0} + \widehat{M_3}(\xi,t)\widehat{\partial_k^m b_0} + \int_0^t \left(\widehat{M_2}(\xi,t-\tau)\widehat{\partial_k^m N_1}(\tau) + \widehat{M_3}(\xi,t-\tau)\widehat{\partial_k^m N_2}(\tau)\right) d\tau, \tag{2.21}$$

for $k = 1, 2$ and $m = 0, 1, 2$.

We will complete the proof using the bootstrapping argument. We make the assumption that, for $t \leq T$,

$$\|u(t)\|_{L^2(\mathbb{R}^2)} + \|b(t)\|_{L^2(\mathbb{R}^2)} \leq C_0\delta(1+t)^{-\frac{1}{2}}, \tag{2.22}$$

$$\|Du(t)\|_{L^2(\mathbb{R}^2)} + \|Db(t)\|_{L^2(\mathbb{R}^2)} \leq C_1\delta(1+t)^{-1}, \tag{2.23}$$

$$\|D^2u(t)\|_{L^2(\mathbb{R}^2)} + \|D^2b(t)\|_{L^2(\mathbb{R}^2)} \leq C_2\delta(1+t)^{-\frac{3}{2}}, \tag{2.24}$$

where C_m $(m = 0, 1, 2)$ will be specified later. Using the assumptions (2.22), (2.23), (2.24), we must then show that $(D^m u(t), D^m b(t))$ actually admits smaller upper bounds

$$\|u(t)\|_{L^2(\mathbb{R}^2)} + \|b(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_0}{2}\delta(1+t)^{-\frac{1}{2}}, \tag{2.25}$$

$$\|Du(t)\|_{L^2(\mathbb{R}^2)} + \|Db(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_1}{2}\delta(1+t)^{-1}, \tag{2.26}$$

$$\|D^2u(t)\|_{L^2(\mathbb{R}^2)} + \|D^2b(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{C_2}{2}\delta(1+t)^{-\frac{3}{2}}, \tag{2.27}$$

for $t \leq T$. By showing that $(D^m u(t), D^m b(t))$ admits these smaller upper bounds for $t \leq T$, then the bootstrapping argument gives the desired result that these smaller upper bounds hold for all $t \leq \infty$. Therefore, all that remains is to prove that (2.25), (2.26), (2.27) actually hold for $t \leq T$.

We start with the estimate of $\|\partial_k^m u\|_{L^2(\mathbb{R}^2)}$. Taking the L^2 norm on both sides of (2.14) and using Plancherel's Theorem, we have

$$\|\partial_k^m u\|_{L^2(\mathbb{R}^2)} = \|\widehat{\partial_k^m u}(t)\|_{L^2(\mathbb{R}^2)} \leq \|\widehat{M}_1(t)\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} + \|\widehat{M}_2(t)\widehat{\partial_k^m b_0}\|_{L^2(\mathbb{R}^2)} + \int_0^t \|\widehat{M}_1(t-\tau)\widehat{\partial_k^m N_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau + \int_0^t \|\widehat{M}_2(t-\tau)\widehat{\partial_k^m N_2}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau. \quad (2.28)$$

Due to similarity of terms, we only provide the estimates for the first and third term. Without loss of generality, assume $t > 1$. By Proposition 2.4.1 and Lemma 2.12, we can bound the first term on the right hand side of (2.28) as follows

$$\begin{aligned}\|\widehat{M}_1(t)\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} &\leq C\|e^{-\widetilde{c}_0|\xi|^2 t}\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} + \|e^{-c_2 t}\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} \\ &= \||\xi|^m e^{-\widetilde{c}_0|xi|^2 t}\widehat{\Lambda^{-m}\partial_k^m} u_0\|_{L^2(\mathbb{R}^2)} + e^{-c_2 t}\|\widehat{\partial_k^m u_0}\|_{L^2(\mathbb{R}^2)} \\ &\leq C(1+t)^{-\frac{1+m}{2}}\|u_0\|_{L^1(\mathbb{R}^2)} + C(1+t)^{-\frac{1+m}{2}}\|u_0\|_{L^2(\mathbb{R}^2)} \\ &\leq C(1+t)^{-\frac{1+m}{2}}\delta, \end{aligned} \quad (2.29)$$

where $\widetilde{c}_0 = \min\{c_0, c_1\}$ and we have used the simple fact that $e^{-c_2 t}(1+t)^s \leq C(c_2, s)$ for any $s \geq 0$ since exponential decay negates algebraic growth. This bound holds for $m = 0, 1, 2$. Now that we have bounded the first term of the right hand side of (2.28), it is easy to see that the second term will share the same bound.

We move to the third term of (2.28). Using the fact that the projection operator $\mathbb{P}$ is bounded in L^2 and invoking Proposition 2.4.1, we have

$$\begin{aligned}&\int_0^t \|\widehat{M}_1(t-\tau)\widehat{\partial_k^m N_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\quad\leq \int_0^t \|\widehat{M}_1(t-\tau)\widehat{\partial_k^m Q_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau \\ &\quad\leq C\int_0^t \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{\partial_k^m Q_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau + C\int_0^t e^{-c_2(t-\tau)}\|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2(\mathbb{R}^2)}\, d\tau, \end{aligned} \quad (2.30)$$

where $Q_1 = u \cdot \nabla u - b \cdot \nabla b$.

From here, we must bound (2.30) separately for each case when $m = 0, 1, 2$. We begin with the case when $m = 0$. Here we must split the first integral of (2.30) into two parts

$$\int_0^t \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q_1}(\tau)\|_{L^2}\,d\tau = \int_0^{\frac{t}{2}} \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q_1}(\tau)\|_{L^2}\,d\tau + \int_{\frac{t}{2}}^t \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q_1}(\tau)\|_{L^2}\,d\tau. \tag{2.31}$$

By the assumption (2.22), (2.18), and Lemma 2.12, we have

$$\begin{aligned}
\int_0^{\frac{t}{2}} \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widetilde{Q}_1(\tau)\|_{L^2}\,d\tau &= \int_0^{\frac{t}{2}} \||\xi|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{\Lambda^{-1}Q_1(\tau)}(\tau)\|_{L^2}\,d\tau \\
&\le C\int_0^{\frac{t}{2}} (t-\tau)^{-1}\left(\|u(\tau)\otimes u(\tau)\|_{L^1} + \|b(\tau)\otimes b(\tau)\|_{L^1}\right)d\tau \\
&\le C\left(\frac{t}{2}\right)^{-1} \sup_{0\le\tau\le t}\left(\|u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2}\right)\int_0^{\frac{t}{2}}\left(\|u(\tau)\|_{L^2} + \|b(\tau)\|_L^2\right)d\tau \\
&\le CC_0\left(\frac{t}{2}\right)^{-1}\delta^2\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{1}{2}}\,d\tau \\
&\le CC_0\delta^2(1+t)^{-\frac{1}{2}},
\end{aligned}$$

where we have used $u\cdot\nabla u = \nabla\cdot(u\otimes u)$ and $b\cdot\nabla b = \nabla\cdot(b\otimes b)$. We must estimate the second integral of (2.31) in a slightly different manner. Using the property 2.19 we have

$$\begin{aligned}
\int_{\frac{t}{2}}^t \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{Q_1}(\tau)\|_{L^2}\,d\tau &\le C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\|u\cdot\nabla u - b\cdot\nabla b\|_{L^1}\,d\tau \\
&\le C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\left(\|u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2}\|\nabla b(\tau)\|_{L^2}\right)d\tau \\
&\le CC_0\delta^2\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}\,d\tau \\
&\le CC_0\delta^2\left(1+\frac{t}{2}\right)^{-1}\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\,d\tau \\
&\le CC_0\delta^2(1+t)^{-\frac{1}{2}}.
\end{aligned}$$

This completes the bound for the first term of (2.30).

We now move on to the second term of (2.30). Using the fact that $e^{-c_2t}(1+t)^s \le C(c_2, s)$

for any $s > 0$ and using (2.25) along with (2.19) we obtain the following bound

$$\begin{aligned}\int_0^t e^{-c_2(t-\tau)}\|\widehat{Q_1}(\tau)\|_{L^2}\,d\tau &\leq C\int_0^t (1+t-\tau)^{-s}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u(\tau)\|_{L^2}\|\Delta u(\tau)\|_{L^2}^{\frac{1}{2}}\,d\tau\\ &\leq CC_0^{\frac{1}{2}}\delta^2\int_0^t (1+t-\tau)^{-s}(1+\tau)^{-\frac{1}{2}}\,d\tau\\ &\leq CC_0^{\frac{1}{2}}\delta^2(1+t)^{-\frac{1}{2}},\end{aligned}$$

where $s > 1$. This completes both bounds for (2.30). Therefore, when $m = 0$, the third term of (2.28) is bounded by

$$\int_0^t \|\widehat{M_1}(t-\tau)\widehat{N_1}(\tau)\|_{L^2}\,d\tau \leq C\left(C_0 + C_0^{\frac{1}{2}}\right)\delta^2(1+t)^{-\frac{1}{2}}.$$

The fourth term of (2.28) shares the same bound as that of the third term bound we just obtained when $m = 0$. Thus, we have shown that there exist $C_3 > 0$ and $C_4 > 0$ such that

$$\|u(t)\|_{L^2} \leq C_3\delta(1+t)^{-\frac{1}{2}} + C_4(1+C_0)\delta^2(1+t)^{-\frac{1}{2}}.$$

If C_0 and $\delta > 0$ sufficiently small satisfy

$$C_3 \leq \frac{C_0}{8}, \qquad C_4(1+C_0)\delta \leq \frac{C_0}{8},$$

and thus

$$\|u(t)\|_{L^2} \leq \frac{C_0}{4}\delta(1+t)^{-\frac{1}{2}}. \tag{2.32}$$

Similarly, we obtain the same bound for $\|b(t)\|_{L^2}$ from (2.21). Combining these we have obtained the desired bound (2.25)

$$\|u(t)\|_{L^2} + \|b(t)\|_{L^2} \leq \frac{C_0}{2}\delta(1+t)^{-\frac{1}{2}}.$$

We now move on to the cases when $m = 1$ and $m = 2$. Recall that we have already bounded the first term of (2.28) for $m = 0, 1, 2$, so we focus our attention on the third term of (2.28) which we have only shown for $m = 0$ thus far. We split the first time integral of (2.30) into two terms. Observe

$$\begin{aligned}
&\int_0^t \|e^{-\widetilde{c}_0|\xi|^2(t-\tau)}\widehat{\partial_k^m Q_1}(\tau)\|_{L^2}\,d\tau \\
&\quad\le \int_0^{t-1} \||\xi|^{m+1}e^{-\widehat{c}_0|\xi|^2(t-\tau)}\widehat{\Lambda^{-(m+1)}\partial_k^m} Q_1(\tau)\|_{L^2}\,d\tau + \int_{t-1}^t \|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2}\,d\tau \\
&\quad\le C\int_0^{t-1} (1+t-\tau)^{-\frac{m+1}{2}}\|\Lambda^{-1}Q_1(\tau)\|_{L^2}\,d\tau + C(m)\int_{t-1}^t (1+t-\tau)^{-\frac{m+1}{2}}\|\widehat{\partial_k^m Q_1(\tau)}\|_{L^2}\,d\tau \\
&\quad\le C\int_0^t (1+t-\tau)^{-\frac{m+1}{2}}\|\Lambda^{-1}Q_1(\tau)\|_{L^2}\,d\tau + C(m)\int_0^t (1+t-\tau)^{-\frac{m+1}{2}}\|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2}\,d\tau,
\end{aligned}$$

where we have use Lemma 2.2.2 and the fact that

$$(t-\tau)^{-\frac{m+1}{2}} \le C(1+t-\tau)^{-\frac{m+1}{2}} \qquad \text{for any } \tau \in [0, t-1],$$

to bound the first term above. We also used the fact that $(1+t-\tau) \le 2$ for $\tau \in [t-1, t]$ to bound the second term above.

The second time integral of (2.30) can be bounded by

$$\int_0^t e^{-c_2(t-\tau)}\|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2}\,d\tau \le C(m)\int_0^t (1+t-\tau)^{-\frac{m+1}{2}}\|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2}\,d\tau,$$

thanks to the fact $e^{-c_2 t}(1+t)^s \le C(c_2, s)$ for any $t > 0$ and any constant $s > 0$. Therefore, combining these to bounds, the third term of (2.28) is bounded by

$$\begin{aligned}
\int_0^t \|\widehat{M}_1(t-\tau)\widehat{\partial_k^m N_1}(\tau)\|_{L^2}\,d\tau &\le C\int_0^t (1+t-\tau)^{-\frac{m+1}{2}}\|\Lambda^{-1}Q_1(\tau)\|_{L^2}\,d\tau \\
&\quad + C(m)\int_0^t (1+t-\tau)^{-\frac{m+1}{2}}\|\widehat{\partial_k^m Q_1}(\tau)\|_{L^2}\,d\tau. \qquad (2.33)
\end{aligned}$$

In order to continue to bound this, we must consider the cases $m = 1$ and $m = 2$ separately.

For $m = 1$, by Hölder's inequality and the Sobolev inequality, we bound the first term of (2.33) as follows

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-1} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau \\
&\qquad \le C \int_0^t (1+t-\tau)^{-1} \left(\|u(\tau)\|_{L^4}^2 + \|b(\tau)\|_{L^4}^2 \right) d\tau \\
&\qquad \le C \int_0^t (1+t-\tau)^{-1} \left(\|u(\tau)\|_{L^2} \|\nabla u(\tau)\|_{L^2} + \|b(\tau)\|_{L^2} \|\nabla b(\tau)\|_{L^2} \right) d\tau.
\end{aligned}$$

Then by (2.32), assumption (2.23), and Lemma 2.2.2,

$$\begin{aligned}
\int_0^t (1+t-\tau)^{-1} \|\Lambda^{-1} Q_1(\tau)\|_{L^2}\, d\tau &\le C C_0 C_1 \delta^2 \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-\frac{3}{2}}\, d\tau \\
&\le C C_1 \delta^2 (1+t)^{-1}.
\end{aligned}$$

Similarly, for $m = 1$ we bound the second term of (2.33) using (2.32), the assumption (2.23), and the decay estimate $\|(\Delta u(t), \Delta b(t))\|_{H^1} \le C\delta(1+t)^{-\frac{1}{2}}$ to obtain

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-1} \|\partial_k Q_1(\tau)\|_{L^2}\, d\tau \\
&\quad \le \int_0^t (1+t-\tau)^{-1} \left(\|\nabla u(\tau)\|_{L^4}^2 + \|u(\tau)\|_{L^4} \|\Delta u(\tau)\|_{L^4} \right. \\
&\qquad\qquad \left. + \|\nabla b(\tau)\|_{L^4}^2 + \|b(\tau)\|_{L^4} \|\Delta b(\tau)\|_{L^4} \right) d\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-1} \Big(\|\nabla u(\tau)\|_{L^2} \|\Delta u(\tau)\|_{L^2} + \|u(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}} \|\Delta u(\tau)\|_{H^1} \\
&\qquad\qquad + \|\nabla b(\tau)\|_{L^2} \|\Delta b(\tau)\|_{L^2} + \|b(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla b(\tau)\|_{L^2}^{\frac{1}{2}} \|\Delta b(\tau)\|_{H^1} \Big) d\tau \\
&\quad \le C(C_1 + C_0^{\frac{1}{2}} C_1^{\frac{1}{2}} \delta^2 \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-\frac{5}{4}}\, d\tau \\
&\quad \le C(C_1 + C_1^{\frac{1}{2}}) \delta^2 (1+t)^{-1}.
\end{aligned}$$

Therefore, in the case when $m = 1$, the third term of (2.30) can be bounded by

$$\int_0^t \|\widehat{M_1}(t-\tau)\widehat{\partial_k N_1}(\tau)\|_{L^2}\, d\tau \le C(1+C_1)\delta^2(1+t)^{-1}. \tag{2.34}$$

Combining the estimates (2.29) and (2.34) yields

$$\|\nabla u\|_{L^2} \le C_t\delta(1+t)^{-1} + C_6(1+C_1)\delta^2(1+t)^{-1},$$

for some constants $C_5 > 0$ and $C_6 > 0$. Therefore, if C_1 and $\delta > 0$ sufficiently small satisfy

$$C_5 \le \frac{C_1}{8}, \quad C_6(1+C_1)\delta \le \frac{C_1}{8},$$

then

$$\|\nabla u\|_{L^2} \le \frac{C_1}{4}\delta(1+t)^{-1}. \tag{2.35}$$

Similarly, we obtain the same bound for $\|\nabla b(t)\|_{L^2}$ from (2.21). Combining these we obtain the desired bound (2.26)

$$\|\nabla u(t)\|_{L^2} + \|\nabla b(t)\|_{L^2} \le \frac{C_1}{2}\delta(1+t)^{-\frac{1}{2}}.$$

Finally, we bound (2.33) for the case when $m = 2$. Using a similar argument to that used in the case when $m = 1$ we get

$$\begin{aligned}\int_0^t (1+t-\tau)^{-\frac{3}{2}}\|\Lambda^{-1}Q_1(\tau)\|_{L^2}\, d\tau &\le CC_0C_1\delta^2\int_0^t (1_t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{3}{2}}\, d\tau \\ &\le C\delta^2(1+t)^{-\frac{3}{2}}.\end{aligned}$$

Then by Hölder's inequality and Sobolev's inequality,

$$
\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_k^2 Q_1(\tau)\|_{L^2}\, d\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \big(\|\nabla u(\tau)\|_{L^4} \|\Delta u(\tau)\|_{L^4} + \|u(\tau)\|_{L^\infty} \|\nabla^3 u(\tau)\|_{L^2} \\
&\qquad + \|\nabla b(\tau)\|_{L^4} \|\Delta b(\tau)\|_{L^4} + \|b(\tau)\|_{L^\infty} \|\nabla^3 b(\tau)\|_{L^2} \big)\, d\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-\frac{3}{2}} \Big(\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}} \|\Delta u(\tau)\|_{L^2} \|\nabla^3 u(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad + \|u(\tau)\|_{L^2}^{\frac{1}{2}} \|\Delta u(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla^3 u(\tau)\|_{L^2} + \|\nabla b(\tau)\|^{\frac{1}{2}} \|\Delta b(\tau)\|_{L^2} \|\nabla^3 b(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad + \|b(\tau)\|_{L^2}^{\frac{1}{2}} \|\Delta b(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla^3 b(\tau)\|_{L^2} \Big)\, d\tau.
\end{aligned}
$$

Using (2.32), (2.36), the assumption (2.24), and the decay rate $\|(\nabla^3 u, \nabla^3 b)\|_{L^2} \le C\delta(1+t)^{-\frac{1}{2}}$, we then have

$$
\begin{aligned}
\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_k^2 Q_1(\tau)\|_{L^2}\, d\tau &\le C \left(C_1^{\frac{1}{2}} C_2 + C_0^{\frac{1}{2}} C_2^{\frac{1}{2}} \right) \delta^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{3}{2}}\, d\tau \\
&\le C \left(C_2 + C_2^{\frac{1}{2}} \right) \delta^2 (1+t)^{-\frac{3}{2}}.
\end{aligned}
$$

Hence

$$
\int_0^t \|\widehat{M_1}(t-\tau)\widehat{\partial_k^2 N_1}(\tau)\|_{L^2}\, d\tau \le C(1+C_2)\delta^2(1+t)^{-\frac{3}{2}}.
$$

Therefore,

$$
\|\Delta u\|_{L^2} \le C_7 \delta (1+t)^{-\frac{3}{2}} + C_8 (1+C_2)\delta^2 (1+t)^{-\frac{3}{2}},
$$

for constants $C_7 > 0$ and $C_8 > 0$. Therefore, if C_2 and $\delta > 0$ sufficiently small satisfy

$$
C_7 \le \frac{C_2}{8}, \quad C_8(1+C_2)\delta \le \frac{C_2}{8},
$$

then

$$
\|\Delta u\|_{L^2} \le \frac{C_2}{4} \delta (1+t)^{-1}. \tag{2.36}
$$

Similarly, we obtain the same bound for $\|\Delta b(t)\|_{L^2}$ from (2.21). Combining these we obtain the desired bound (2.27)

$$\|\Delta u(t)\|_{L^2} + \|\Delta b(t)\|_{L^2} \leq \frac{C_2}{2}\delta(1+t)^{-\frac{1}{2}}.$$

Then the bootstrapping argument implies that the decay rates (2.25), (2.26), (2.27) hold for all $t \leq T$ with $T = \infty$. This completes the proof of Theorem 2.1.2 and our discussion for the MHD stability problem.

CHAPTER 3

GROWING AND SINGULAR SOLUTIONS OF THE 2D MHD EQUATIONS

Recall the incompressible ideal MHD equations

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \; \nabla \cdot b = 0. \end{cases} \tag{3.1}$$

The ideal MHD equations are particularly interesting and difficult equations to analyze as they lack both dissipation, $\nu\Delta u$, and diffusion, $\eta\Delta b$, terms that can typically help control the behavior of the plasma flow. The absence of the dissipative type terms presents the possibility for growing solutions and singular solutions. It remains an outstanding open questions whether the solution to the ideal MHD equations preserves the smoothness of initial data globally in time. Based on the behavior of other hydrodynamical systems and the fact that growing solutions to the 2D ideal MHD equations exist, we believe it is possible to construct initial conditions that result in a finite time blowup. Although there exist a few results that show growing solutions or solutions with a finite time blow-up to the incompressible 3D ideal MHD equations, all known results have solutions with infinite energy which is not physically meaningful.

As with many equations that describe physical phenomena, results are often restricted to solutions with a physical meaning. In particular, with the MHD equations, scientists are interested in finite energy solutions. We are particularly interested in finite energy solutions, but infinite energy solutions are still worth studying given the difficulty of the ideal MHD equations. In this section, we provide singular solutions and double exponential growth solutions with infinite energy. These solutions provide insight to the behavior of the ideal MHD equations and lend support to the search of a singular solution with finite energy.

We begin with a singular solution to the 2D ideal MHD equation.

Lemma 3.0.1. *Let C be a constant vector field. Then, for $x \in \mathbb{R}^2$ and $t \geq 0$*

$$\begin{cases} u = \dfrac{C}{1-t}, \\ p = -\dfrac{C \cdot x}{(1-t)^2}, \\ b = 0, \end{cases} \tag{3.2}$$

is a singular solution to the 2D ideal MHD equation (3.1).

To see this, observe the equations above satisfy $\nabla \cdot u = \nabla \cdot b = 0$ and trivially satisfy the magnetic equation

$$\partial_t b + u \cdot \nabla b = b \cdot \nabla u.$$

Additionally,

$$\partial_t u = \frac{C}{(1-t)^2},$$
$$\nabla p = -\frac{C}{(1-t)^2}.$$

Then (u, p, b) also satisfies the velocity equation

$$\partial_t u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b.$$

This shows that (u, p, b) is a solution to (3.1). Clearly $u \to \infty$ as $t \to 1^-$. Thus, (3.2) is singular solution to (3.1). This solution, however, has infinite energy because $u(x,t)$ is constant in the spacial component

$$\|u(t)\|_{L^2(\mathbb{R}^2)} = \left\| \frac{C}{1-t} \right\|_{L^2(\mathbb{R}^2)} = \infty.$$

In addition to singular solutions, double exponentially growing solutions of the 2D ideal MHD equations (3.5) with infinite energy also exist.

Lemma 3.0.2. *Assume* $\psi_0 = \psi_0(x_1)$. *Then*

$$\begin{cases} u = (-e^t x_1, e^t x_2), \\ p = -\frac{1}{2}x_1^2(e^{2t} - e^t) - \frac{1}{2}x_2^2(e^{2t} + e^t), \\ \psi = \psi_0(e^{e^t - 1} x_1), \end{cases}$$

solves the MHD equation (3.5). *In particular*

$$b = (0, e^{e^t-1}\partial_1\psi_0(e^{e^t-1}x_1)),$$

grows exponentially.

First note that

$$b \cdot \nabla b = (b_1\partial_1 + b_2\partial_2)(0, e^{e^t-1}\partial_1\psi_0(e^{e^t-1}x_1)) = (0, 0).$$

To see that (u, p, ψ) satisfies the u equation of (3.5) observe

$$\begin{aligned} \partial_t u &= \partial_t(-e^t x_1, e^t x_2), \\ &= (-e^t x_1, e^t x_2) \end{aligned}$$

$$
\begin{aligned}
u \cdot \nabla u &= (-e^t x_1 \partial_1 + e^t x_2 \partial_2)(-e^t x_1, e^t x_2) \\
&= (e^{2t} x_1, e^{2t} x_2), \\
\nabla p &= (\partial_1 p, \partial_2 p) \\
&= (e^t x_1(-e^t + 1), e^t x_2(-e^t - 1)), \\
b \cdot \nabla b &= (0, 0).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p - b \cdot \nabla b &= (-e^t x_1, e^t x_2) + (e^{2t} x_1, e^{2t} x_2) \\
&\quad + (e^t x_1(-e^t + 1), e^t x_2(-e^t - 1)) \\
&= (0, 0).
\end{aligned}
$$

In order to verify (u, p, ψ) satisfy the ψ equation

$$
\partial_t \psi + u \cdot \nabla \psi = 0,
$$

notice that

$$
\begin{aligned}
\partial_t \psi &= \partial_1 \psi_0(e^{e^t - 1}) e^t e^{e^t - 1} x_1, \\
u \cdot \nabla \psi &= -e^t x_1 \partial_1 \psi_0(e^{e^t - 1} x_1) e^{e^t - 1}.
\end{aligned}
$$

Thus

$$
\partial_t \psi + u \cdot \nabla \psi = \partial_1 \psi_0(e^{e^t - 1}) e^t e^{e^t - 1} x_1 - e^t x_1 \partial_1 \psi_0(e^{e^t - 1} x_1) e^{e^t - 1} = 0,
$$

and

$$
\psi(x_1, x_2, 0) = \psi_0(e^{e^0 - 1} x_1) = \psi_0(x_1).
$$

This completes the proof.

The double exponentially growing solution above as not been presented in any publication known to the author at this time. Even though both solutions presented above have infinite energy on the whole space, they may be physically relevant locally. In fact the second example represents the strain flow with fluids compressed in one direction and stretched in the other. In addition, their construction may help provide insight into finding growing solutions with finite energy.

3.1 Transformation of the 2D Ideal MHD Equations to a 1D System

We now turn our attention to the search for finite energy solutions to the incompressible 2D ideal MHD equations which blow up in finite time. Given the difficulty in analyzing these equations, we make use of the fact that solutions to the ideal system are scale invariant. This scale invariance allows the reduction from a two-dimensional spacial domain to a one-dimensional spacial domain and the possibility for more fruitful analysis.

In the case of (3.1), scale invariance of solutions means that whenever $(u(x,t), b(x,t))$ is a solution to (3.1) then for any $\lambda > 0$ then

$$\begin{cases} u_\lambda(x,t) = \frac{1}{\lambda} u(\lambda x, t), \\ b_\lambda(x,t) = \frac{1}{\lambda} b(\lambda x, t), \\ p_\lambda(x,t) = \frac{1}{\lambda^2} p(\lambda x, t), \end{cases}$$

also solves (3.1).

Our analysis will focus on the vorticity, $\omega = \nabla \times u$, and the current density, $j = \nabla \times b$, equations that are found by applying $\nabla \times$ to (3.1) to obtain

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j = b \cdot \nabla \omega + Q(u,b), \end{cases} \tag{3.3}$$

where

$$Q(u, b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2).$$

The vorticity and current density are also scale invariant in the ideal MHD case. If (ω, j) are solutions to (3.3), then for all $\lambda > 0$

$$\begin{cases} \omega_\lambda(x,t) = \omega(\lambda x, t), \\ j_\lambda(x,t) = j(\lambda x, t), \end{cases} \tag{3.4}$$

also solves (3.3).

We will also consider the vorticity and stream function formulation of the 2D ideal MHD equations

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j, \\ \partial_t \psi + u \cdot \nabla \psi = 0, \end{cases} \tag{3.5}$$

where ψ is the stream function given by $b = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$.

Using a polar coordinate transformation, we are able to transform both the 2D vorticity and current density formulation (3.3) along with the 2D vorticity and stream function formulation (3.5) into one-dimensional systems due to the scale invariant property of solutions. Let ϕ and ψ be the stream functions associated with u and b, respectively, with $u = \nabla^\perp \phi$ and $b = \nabla^\perp \psi$. We consider a special class of scale invariant solutions

$$\begin{cases} \omega(x,t) = g(\theta, t), \\ j(x,t) = h(\theta, t), \\ \phi(x,t) = r^2 G(\theta, t), \\ \psi(x,t) = r^2 H(\theta, t), \end{cases} \tag{3.6}$$

where (r, θ) is the associated polar coordinates for $x = (x_1, x_2)$

$$x_1 = r\cos\theta,$$

$$x_2 = r\sin\theta.$$

Using these scale invariant solutions and a polar coordinate transformation, (3.3) transforms into the one-dimensional system

$$\begin{cases} \partial_t g + 2G\partial_\theta g = 2H\partial_\theta h, \\ \partial_t h + 2G\partial_\theta h = 2H\partial_\theta g - 2\partial_\theta G \partial_{\theta\theta} H + 2\partial_\theta H \partial_{\theta\theta} G, \end{cases} \tag{3.7}$$

and (3.5) transforms into the one-dimensional system

$$\begin{cases} \partial_t g + 2G\partial_\theta g = 2H\partial_\theta h, \\ \partial_t H + 2G\partial_\theta H = 2H\partial_\theta G, \end{cases} \tag{3.8}$$

where G and H are defined by

$$\partial_{\theta\theta} G + 4G = g, \tag{3.9}$$

$$\partial_{\theta\theta} H + 4H = h. \tag{3.10}$$

The explicit construction of (3.7) and (3.8) is tedious and the full details can be found in Appendix A.3.

The goal of transforming the 2D ideal MHD equations into a 1D system is to discover a finite-energy solution which blows up in finite time. Once this is complete, the finite time blowup for the 1D system would allow for the construction of a finite time blowup in the 2D system thus solving a long outstanding open problem.

This goal, however, has not yet been fully achieved. The results presented here provide a conditional result for a finite time blowup. While this falls short of the goal of fully solving this open problem, the conditional result provides great insight into behavior of the system.

There are two main considerations for constructing the blow up which is to either have the 1D transformation of vorticity, g, blow up or to have the 1D transformation of the current density stream function, H, blow up.

At first glance it appears that having the blowup occur in H is viable as its equation has the structure of vortex stretching. However, the more fruitful progress has been made searching for a blowup to occur in the 1D transformation of the vorticity, g.

Before we state the conditional blowup result, we discuss the known properties of the 1D systems (3.7) and (3.8).

3.2 Properties of the 1D Systems

Here we state the known results for the 1D Systems (3.7) and (3.8). We begin with the explicit formulation for G and H.

Lemma 3.2.1. *Let $-\pi \leq a < b \leq \pi$, where $b - a \neq \frac{\pi}{2}$. Assume $g \in C([a,b])$. Then*

$$\begin{cases} \partial_{\theta\theta} G + 4G = g, \quad \theta \in [a,b], \\ G(a) = G(b) = 0, \end{cases} \tag{3.11}$$

has a unique solution $G \in C^2([a,b])$.

Furthermore,

$$\begin{aligned} G(\theta) = &-\frac{1}{A}\cos(2\theta + \tfrac{\pi}{2} - 2b)\int_a^\theta g(\rho)\sin(2\rho - 2a)\,d\rho \\ &-\frac{1}{A}\sin(2\theta - 2a)\int_\theta^b g(\rho)\cos(2\rho + \tfrac{\pi}{2} - 2b)\,d\rho. \end{aligned} \tag{3.12}$$

and

$$\partial_\theta G(\theta) = \frac{2}{A}\sin(2\theta + \tfrac{\pi}{2} - 2b)\int_a^\theta g(\rho)\sin(2\rho - 2a)\,d\rho$$
$$- \frac{2}{A}\cos(2\theta - 2a)\int_\theta^b g(\rho)\cos(2\rho + \tfrac{\pi}{2} - 2b)\,d\rho. \tag{3.13}$$

where $A = 2\sin(2b - 2a)$*. Similarly, if* $h \in C([a,b])$ *then* H *has the same explicit equation as* G *with* g *replaced by* h*.*

This lemma is proven using the Sturm-Liouville method. To do this, a fundamental solution must be constructed. Solving the homogeneous problems

$$\begin{cases} G'' + 4G = 0, \\ G(a) = 0, \end{cases}$$

and

$$\begin{cases} G'' + 4G = 0, \\ G(b) = 0, \end{cases}$$

yields

$$G_1(\theta) = \sin(2\theta - 2a),$$
$$G_2(\theta) = \cos(2\theta + \tfrac{\pi}{2} - 2b).$$

We now construct the fundamental solution constant, A,

$$A = (G_1G_2' - G_1'G_2) = -2\sin(2b - 2a).$$

Therefore, the fundamental solution to the homogeneous part of(3.11) is

$$\Gamma(\theta,\rho) = -\frac{1}{2\sin(2b-2a)}\begin{cases}\sin(2\rho-2a)\cos(2\theta+\frac{\pi}{2}-2b), & \text{if } a\le\rho\le\theta\le b,\\ \sin(2\theta-2a)\cos(2\rho+\frac{\pi}{2}-2b), & \text{if } a\le\theta\le\rho\le b.\end{cases}$$

In particular, the solution to the inhomogeneous part of (3.11) is

$$\begin{aligned}\int_a^b \Gamma(\theta,\rho)g(\rho)\,d\rho &= \int_a^\theta \Gamma(\theta,\rho)g(\rho)\,d\rho + \int_\theta^b \Gamma(\theta,\rho)g(\rho)\,d\rho\\ &= -\frac{1}{A}\cos(2\theta+\tfrac{\pi}{2}-2b)\int_a^\theta g(\rho)\sin(2\rho-2a)\,d\rho\\ &\quad -\frac{1}{A}\sin(2\theta-2a)\int_\theta^b g(\rho)\cos(2\rho+\tfrac{\pi}{2}-2b)\,d\rho.\end{aligned}$$

Thus, the general solution to (3.11) without boundary conditions is given by

$$\begin{aligned}G(\theta) &= c_1\sin(2\theta-2a)+c_2\cos(2\theta+\tfrac{\pi}{2}-2b)\\ &\quad -\frac{1}{A}\cos(2\theta+\tfrac{\pi}{2}-2b)\int_a^\theta g(\rho)\sin(2\rho-2a)\,d\rho\\ &\quad -\frac{1}{A}\sin(2\theta-2a)\int_\theta^b g(\rho)\cos(2\rho+\tfrac{\pi}{2}-2b)\,d\rho.\end{aligned}$$

Using the given boundary conditions $G(a)=G(b)=0$, we find that $c_1=c_2=0$. Therefore,

$$\begin{aligned}G(\theta) &= -\frac{1}{A}\cos(2\theta+\tfrac{\pi}{2}-2b)\int_a^\theta g(\rho)\sin(2\rho-2a)\,d\rho\\ &\quad -\frac{1}{A}\sin(2\theta-2a)\int_\theta^b g(\rho)\cos(2\rho+\tfrac{\pi}{2}-2b)\,d\rho.\end{aligned}$$

Differentiating this result with respect to θ yields the desired equation for $\partial_\theta G(\theta)$. This completes the proof.

These explicit equations for G and H will be used to investigate the behavior of solutions. Additionally, it can be shown that H maintains its initial sign for all time.

Lemma 3.2.2. *Suppose g, H are solutions to* (3.8)*. Let $X(a,t)$ be the particle trajectory defined by*

$$\begin{cases} \dfrac{dX(a,t)}{dt} = 2G(X(a,t),t), \\ X(a,0) = a. \end{cases}$$

Then

$$H(X(a,t),t) = H(a,0)e^{\int_0^t 2\partial_\theta G(X(a,\tau),\tau)\, d\tau}.$$

To see this, let $X(a,t)$ be the particle trajectory defined in the lemma. Since H is a solution to (3.8) then it satisfies the equation

$$\partial_t H + 2G\partial_\theta H = 2H\partial_\theta G.$$

Then

$$\partial_t H(X(a,t),t) + 2G(X(a,t),t)\partial_\theta H(X(a,t),t) = 2H(X(a,t),t)\partial_\theta G(X(a,t),t).$$

Thus

$$\frac{d}{dt}H(X(a,t),t) = 2H(X(a,t),t)\partial_\theta G(X(a,t),t).$$

Hence

$$H(X(a,t),t) = H(a,0)e^{\int_0^t 2\partial_\theta G(X(a,\tau),\tau)\, d\tau}.$$

Using these properties, we can now show the local well-posedness of the 1D system (3.7).

Proposition 3.2.3 (Local Well-Posedness of the 1D System)**.** *Let $\Omega = [a,b]$ for some $-\pi \leq$*

$a < b \le \pi$ with $b - a \neq \frac{\pi}{2}$. Let g_0, $h_0 \in H^1(\Omega)$. Then there exist $T > 0$ and a unique solution (g, h) to (3.7) with boundary conditions

$$G(a) = G(b) = H(a) = H(b) = 0,$$

satisfying

$$(g, h) \in C([0, T); H^1(\Omega)).$$

Furthermore, if, for $T^* > T$

$$\int_0^{T^*} \|g(\cdot, t)\|_\infty \, dt < \infty \quad \text{and} \quad \int_0^{T^*} \|\partial_\theta g(\cdot, t)\|_\infty \, dt < \infty,$$

then (g, h) can be extended to $[0, T^*)$.

The proof for local existence relies on the local *a priori* bounds on $\|g\|_{H^1}$ and $\|h\|_{H^1}$. We start with the L^2 estimate. Dotting the (3.7) with g and h, respectively, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|g\|_{L^2}^2 + \|h\|_{L^2}^2\right) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= -2\int_\Omega Gg\partial_\theta g \, d\theta - 2\int_\Omega Gh\partial_\theta h \, d\theta, \\
I_2 &= 2\int_\Omega Hg\partial_\theta h \, d\theta + 2\int_\Omega Hh\partial_\theta g \, d\theta + 2\int_\Omega h\partial_\theta H \partial_{\theta\theta} G \, d\theta, \\
I_3 &= -2\int_\Omega h\partial_\theta G \partial_{\theta\theta} H \, d\theta.
\end{aligned}$$

We estimate the terms I_1, I_2, I_3. Recall that $G = 0$ and $H = 0$ on $\partial\Omega$. Then, using integration by parts, we obtain

$$I_1 = -2\int_\Omega (Gg\partial_\theta g + Gh\partial_\theta h) \, d\theta$$

$$
\begin{aligned}
&= -\int_\Omega G\partial_\theta(g^2+h^2)\,d\theta \\
&= \int_\Omega \partial_\theta G(g^2+h^2)\,d\theta.
\end{aligned}
$$

By (3.13), $\|\partial_\theta G\|_{L^\infty(\Omega)} \le C\|g\|_{L^2(\Omega)}$. Then

$$
\begin{aligned}
|I_1| &\le \|\partial_\theta G\|_{L^\infty(\Omega)} \left(\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)}\right) \\
&\le C\|g\|_{L^2(\Omega)} \left(\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

To estimate I_2 we make use of the fact that g and G satisfy the relationship

$$\partial_{\theta\theta}G + 4G = g.$$

Using the relationship above and integration by parts, we estimate I_2 as follows

$$
\begin{aligned}
I_2 &= 2\int_\Omega (Hg\partial_\theta h + Hh\partial_\theta g)\,d\theta + 2\int_\Omega h\partial_\theta H\partial_{\theta\theta}G\,d\theta \\
&= 2\int_\Omega H\partial_\theta(gh)\,d\theta + 2\int_\Omega h\partial_\theta H(g-4G)\,d\theta \\
&= 2\int_\Omega H\partial_\theta(gh)\,d\theta + 2\int_\Omega \partial_\theta H(gh)\,d\theta - 8\int_\Omega \partial_\theta HGh\,d\theta \\
&= -2\int_\Omega \partial_\theta H(gh)\,d\theta + 2\int_\Omega \partial_\theta H(gh)\,d\theta - 8\int_\Omega \partial_\theta HGh\,d\theta \\
&= -8\int_\Omega \partial_\theta HGh\,d\theta.
\end{aligned}
$$

By (3.12) and (3.13), $\|G\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Omega)}$ and $\|\partial_\theta H\|_{L^\infty(\Omega)} \le C\|h\|_{L^2(\Omega)}$. Then

$$
\begin{aligned}
|I_2| &\le 8\|\partial_\theta H\|_{L^\infty(\Omega)}\|G\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)} \\
&\le C\|g\|_{L^2(\Omega)}\|h\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Now we estimate I_3 to complete the L^2 estimates. Again, by (3.12) and (3.13), $\|\partial_\theta G\|_{L^\infty(\Omega)} \le$

$C\|g\|_{L^2}$ and $\|\partial_{\theta\theta}H\|_{L^2(\Omega)}\| \le C\|h\|_{L^2(\Omega)}$. So term I_3 can be estimated by

$$
\begin{aligned}
|I_3| &= \left|2\int_\Omega h\partial_\theta G\partial_{\theta\theta}H\,d\theta\right| \\
&\le 2\|\partial_\theta G\|_{L^\infty(\Omega)}\|h\|_{L^2(\Omega)}\|\partial_{\theta\theta}H\|_{L^2(\Omega)} \\
&\le C\|g\|_{L^2(\Omega)}\|h\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Therefore, g and h satisfy

$$
\frac{1}{2}\frac{d}{dt}\left(\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)}\right) \le C\|g\|_{L^2(\Omega)}\left(\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)}\right).
$$

We now move on to the H^1 estimate. Differentiating (3.7) with respect to θ, dotting with $\partial_\theta g$ and $\partial_\theta h$, respectively, and integrating in space we obtain

$$
\frac{1}{2}\frac{d}{dt}\left(\|\partial_\theta g\|^2_{L^2(\Omega)} + \|\partial_\theta h\|^2_{L^2(\Omega)}\right) = K_1 + K_2 + K_3,
$$

where

$$
\begin{aligned}
K_1 &= -2\int_\Omega G\partial_{\theta\theta}g\partial_\theta g\,d\theta - 2\int_\Omega G\partial_{\theta\theta}h\partial_\theta h\,d\theta - 2\int_\Omega \partial_\theta g\partial_\theta G\partial_\theta g\,d\theta - 2\int_\Omega \partial_\theta h\partial_\theta G\partial_\theta h\,d\theta, \\
K_2 &= 2\int_\Omega H\partial_{\theta\theta}h\partial_\theta g\,d\theta + 2\int_\Omega H\partial_{\theta\theta}g\partial_\theta h\,d\theta + 2\int_\Omega \partial_\theta h\partial_\theta H\partial_\theta g\,d\theta + 2\int_\Omega \partial_\theta g\partial_\theta H\partial_\theta h\,d\theta, \\
K_3 &= 2\int_\Omega \partial_\theta H\partial_{\theta\theta\theta}G\partial_\theta h\,d\theta - 2\int_\Omega \partial_\theta G\partial_{\theta\theta\theta}H\partial_\theta h\,d\theta.
\end{aligned}
$$

We begin with K_1. Using the fact that $G = 0$ on $\partial\Omega$ we find that $K_1 = 0$. To see this, observe

$$
\begin{aligned}
K_1 &= -2\int_\Omega G\partial_{\theta\theta}g\partial_\theta g\,d\theta - 2\int_\Omega G\partial_{\theta\theta}h\partial_\theta h\,d\theta - 2\int_\Omega \partial_\theta g\partial_\theta G\partial_\theta g\,d\theta - 2\int_\Omega \partial_\theta h\partial_\theta G\partial_\theta h\,d\theta, \\
&= -\int_\Omega \partial_\theta\left(G((\partial_\theta g)^2 + (\partial_\theta h)^2)\right)\,d\theta
\end{aligned}
$$

$$= -\, G((\partial_\theta g)^2 + (\partial_\theta h)^2)\big|_{\partial\Omega}$$
$$= 0.$$

Similarly, since $H = 0$ on $\partial\Omega$ we find that $K_2 = 0$ as shown below

$$\begin{aligned} K_2 &= 2\int_\Omega H\partial_{\theta\theta}h\partial_\theta g\, d\theta + 2\int_\Omega H\partial_{\theta\theta}g\partial_\theta h\, d\theta + 2\int_\Omega \partial_\theta h\partial_\theta H\partial_\theta g\, d\theta + 2\int_\Omega \partial_\theta g\partial_\theta H\partial_\theta h\, d\theta \\ &= 2\int_\Omega \partial_\theta\left(H\partial_\theta g\partial_\theta h\right)\, d\theta \\ &= 2\; H\partial_\theta g\partial_\theta h|_{\partial\Omega} \\ &= 0. \end{aligned}$$

To bound K_3 we will make use of the relations

$$\partial_{\theta\theta}G + 4G = g,$$
$$\partial_{\theta\theta}H + 4H = h.$$

Then we have

$$\begin{aligned} K_3 &= 2\int_\Omega \partial_\theta H\partial_{\theta\theta\theta}G\partial_\theta h\, d\theta - 2\int_\Omega \partial_\theta G\partial_{\theta\theta\theta}H\partial_\theta h\, d\theta \\ &= 2\int_\Omega \partial_\theta H(\partial_\theta g - 4\partial_\theta G)\partial_\theta h\, d\theta - 2\int_\Omega \partial_\theta G(\partial_\theta h - 4\partial_\theta H)\partial_\theta h\, d\theta \\ &= 2\int_\Omega \partial_\theta H\partial_\theta g\partial_\theta h\, d\theta - 8\int_\Omega \partial_\theta H\partial_\theta G\partial_\theta h\, d\theta - 2\int_\Omega \partial_\theta G\partial_\theta h\partial_\theta h\, d\theta + 8\int_\Omega \partial_\theta G\partial_\theta H\partial_\theta h\, d\theta \\ &= 2\int_\Omega \partial_\theta H\partial_\theta g\partial_\theta h\, d\theta - 2\int_\Omega \partial_\theta G\partial_\theta h\partial_\theta h\, d\theta. \end{aligned}$$

By (3.13), $\|\partial_\theta G\|_{L^\infty(\Omega)} \le C\|g\|_{L^2(\Omega)}$ and $\|\partial_\theta H\|_{L^\infty(\Omega)} \le C\|h\|_{L^2(\Omega)}$. From this, we have the following bound for K_3

$$|K_3| \le 2\|\partial_\theta H\|_{L^\infty(\Omega)}\|\partial_\theta g\|_{L^2(\Omega)}\|\partial_\theta h\|_{L^2(\Omega)} + 2\|\partial_\theta G\|_{L^\infty(\Omega)}\|\partial_\theta h\|^2_{L^2(\Omega)}$$

$$\leq C\|h\|_{L^2(\Omega)}\|\partial_\theta g\|_{L^2(\Omega)}\|\partial_\theta h\|_{L^2(\Omega)} + C\|g\|_{L^2(\Omega)}\|\partial_\theta h\|^2_{L^2(\Omega)}.$$

Therefore, $Y(t) = \|g\|^2_{H^1(\Omega)} + \|h\|^2_{H^1(\Omega)}$ satisfies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Y(t) &\leq C\|g\|_{L^2(\Omega)}\left(\|g\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\Omega)}\right) \\ &\quad + C\|h\|_{L^2(\Omega)}\|\partial_\theta g\|_{L^2(\Omega)}\|\partial_\theta h\|_{L^2(\Omega)} + C\|g\|_{L^2(\Omega)}\|\partial_\theta h\|^2_{L^2(\Omega)} \\ &\leq C\|g\|_{H^1(\Omega)}Y(t) \\ &\leq C\sqrt{Y(t)}Y(t).\end{aligned}$$

This inequality implies that there exists a $T > 0$ such that $Y(t) \leq C$ for $t < T$. This completes the local existence portion. It remains to be shown that local solutions are unique.

We will show that if $(g_1, h_1) \in H^1(\Omega)$ and $(g_2, h_2) \in H^1(\Omega)$ are two solutions to (3.7) then they must be identical. Let G_i, H_i be the solutions to

$$\begin{aligned}&\partial_{\theta\theta}G_i + 4G_i = g_i, \\ &\partial_{\theta\theta}H_i + 4H_i = h_i, \\ &G_i(a) = G_i(b) = 0, \\ &H_i(a) = H_i(b) = 0,\end{aligned}$$

for $i = 1, 2$. Taking the difference of the equation for $\partial_t g_1$ and $\partial_t g_2$ from (3.7) yields

$$\begin{aligned}\partial_t(g_1 - g_2) = &-2G_1\partial_\theta(g_1 - g_2) - 2(G_1 - G_2)\partial_\theta g_2 \\ &+ 2H_1\partial_\theta(h_1 - h_2) + 2(H_1 - H_2)\partial_\theta h_2.\end{aligned}$$

Dotting with $(g_1 - g_2)$ and integrating in space results in

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g_1 - g_2\|_{L^2}^2 &= -\int_\Omega 2G_1(g_1-g_2)\partial_\theta(g_1-g_2)\,d\theta - \int_\Omega 2(G_1-G_2)\partial_\theta g_2(g_1-g_2)\,d\theta \\ &\quad + \int_\Omega 2H_1\partial_\theta(h_1-h_2)(g_1-g_2)\,d\theta + \int_\Omega 2(H_1-H_2)\partial_\theta h_2(g_1-g_2)\,d\theta \\ &= \int_\Omega \partial_\theta G_1(g_1-g_2)^2\,d\theta - \int_\Omega 2(G_1-G_2)\partial_\theta g_2(g_1-g_2)\,d\theta \\ &\quad + \int_\Omega 2H_1\partial_\theta(h_1-h_2)\,d\theta + \int_\Omega 2(H_1-H_2)\partial_\theta h_2(g_1-g_2)\,d\theta.\end{aligned}$$

Similarly, for the difference of h_1 and h_2 we obtain

$$\begin{aligned}\partial_t(h_1-h_2) &= -2G_1\partial_\theta(h_1-h_2) - 2(G_1-G_2)\partial_\theta h_2 \\ &\quad + 2H_1\partial_\theta(g_1-g_2) + 2(H_1-H_2)\partial_\theta g_2 \\ &\quad + 2\partial_\theta H_1\partial_{\theta\theta}(G_1-G_2) + 2\partial_\theta(H_1-H_2)\partial_{\theta\theta}G_2 \\ &\quad - 2\partial_\theta G_1\partial_{\theta\theta}(H_1-H_2) - 2\partial_\theta(G_1-G_2)\partial_{\theta\theta}H_2.\end{aligned}$$

Dotting with $(h_1 - h_2)$, integrating in space, and using integration by parts yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|h_1 - h_2\|_{L^2}^2 &= \int_\Omega \partial_\theta G_1(h_1-h_2)^2\,d\theta - \int_\Omega 2(G_1-G_2)\partial_\theta h_2(h_1-h_2)\,d\theta \\ &\quad + \int_\Omega 2H_1\partial_\theta(g_1-g_2)(h_1-h_2)\,d\theta + \int_\Omega 2(H_1-H_2)\partial_\theta g_2(h_1-h_2)\,d\theta \\ &\quad + \int_\Omega 2\partial_\theta H_1\partial_{\theta\theta}(G_1-G_2)(h_1-h_2)\,d\theta + \int_\Omega 2\partial_\theta(H_1-H_2)\partial_{\theta\theta}G_2(h_1-h_2)\,d\theta \\ &\quad - \int_\Omega 2\partial_\theta G_1\partial_{\theta\theta}(H_1-H_2)(h_1-h_2)\,d\theta - \int_\Omega 2\partial_\theta(G_1-G_2)\partial_{\theta\theta}H_2(h_1-h_2)\,d\theta.\end{aligned}$$

Adding, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|g_1 - g_2\|_{L^2}^2 + \|h_1 - h_2\|_{L^2}^2\right) &= \int_\Omega \partial_\theta G_1((g_1-g_2)^2 + (h_1-h_2)^2)\,d\theta \\ &\quad - \int_\Omega 2(G_1-G_2)\partial_\theta g_2(g_1-g_2)\,d\theta\end{aligned}$$

$$+\int_\Omega 2H_1\partial_\theta(h_1-h_2)(g_1-g_2)\,d\theta$$
$$+\int_\Omega 2(H_1-H_2)\partial_\theta h_2(g_1-g_2)\,d\theta$$
$$-\int_\Omega 2(G_1-G_2)\partial_\theta h_2(h_1-h_2)\,d\theta$$
$$+\int_\Omega 2H_1\partial_\theta(g_1-g_2)(h_1-h_2)\,d\theta$$
$$+\int_\Omega 2(H_1-H_2)\partial_\theta g_2(h_1-h_2)\,d\theta$$
$$+\int_\Omega 2\partial_\theta H_1\partial_{\theta\theta}(G_1-G_2)(h_1-h_2)\,d\theta$$
$$+\int_\Omega 2\partial_\theta(H_1-H_2)\partial_{\theta\theta}G_2(h_1-h_2)\,d\theta$$
$$-\int_\Omega 2\partial_\theta G_1\partial_{\theta\theta}(H_1-H_2)(h_1-h_2)\,d\theta$$
$$-\int_\Omega 2\partial_\theta(G_1-G_2)\partial_{\theta\theta}H_2(h_1-h_2)\,d\theta.$$

Then we have the following bound

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|g_1-g_2\|_{L^2}^2+\|h_1-h_2\|_{L^2}^2\right) \le{}& \|\partial_\theta G_1\|_{L^\infty}\left(\|g_1-g_2\|_{L^2}^2+\|h_1-h_2\|^2\right)\\
&+2\|G_1-G_2\|_{L^\infty}\|\partial_\theta g_2\|_{L^2}\|g_1-g_2\|_{L^2}\\
&+2\|H_1\|_{L^\infty}\|\partial_\theta h_1-\partial_\theta h_2\|_{L^2}\|g_1-g_2\|_{L^2}\\
&+2\|H_1-H_2\|_{L^\infty}\|\partial_\theta h_2\|_{L^2}\|g_1-g_2\|_{L^2}\\
&+2\|G_1-G_2\|_{L^\infty}\|\partial_\theta h_2\|_{L^2}\|h_1-h_2\|_{L^2}\\
&+2\|H_1\|_{L^\infty}\|\partial_\theta g_1-\partial_\theta g_2\|_{L^2}\|h_1-h_2\|_{L^2}\\
&+2\|H_1-H_2\|_{L^\infty}\|\partial_\theta g_2\|_{L^2}\|h_1-h_2\|_{L^2}\\
&+2\|\partial_\theta H_1\|_{L^\infty}\|\partial_{\theta\theta}G_1-\partial_{\theta\theta}G_2\|_{L^2}\|h_1-h_2\|_{L^2}\\
&+2\|\partial_\theta H_1-\partial_\theta H_2\|_{L^\infty}\|\partial_{\theta\theta}G_2\|_{L^2}\|h_1-h_2\|_{L^2}\\
&+2\|\partial_\theta G_1\|_{L^\infty}\|\partial_{\theta\theta}H_1-\partial_{\theta\theta}H_2\|_{L^2}\|h_1-h_2\|_{L^2}\\
&+2\|\partial_\theta G_1-\partial_\theta G_2\|_{L^\infty}\|\partial_{\theta\theta}H_2\|_{L^2}\|h_1-h_2\|_{L^2}.
\end{aligned}$$

Using the explicit equations for G and H in Lemma 3.11 we have the following bounds for $i = 1, 2$

$$\|\partial_\theta G_i\|_{L^\infty} \leq \|g_i\|_{L^2},$$

$$\|\partial_{\theta\theta} G_i\|_{L^2} \leq \|g_i\|_{L^2},$$

$$\|\partial_\theta H_i\|_{L^\infty} \leq \|h_i\|_{L^2},$$

$$\|\partial_{\theta\theta} H_i\|_{L^2} \leq \|h_i\|_{L^2}.$$

Therefore, using the bounds above, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|g_1 - g_2\|_{L^2}^2 + \|h_1 - h_2\|_{L^2}^2\right) \leq C\left(\|g_1\|_{H^1}, \|g_2\|_{H^1}, \|h_1\|_{H^1}, \|h_2\|_{H^1}\right) \cdot \left(\|g_1 - g_2\|_{L^2}^2 + \|h_1 - h_2\|_{L^2}^2\right).$$

This completes the proof for local existence and uniqueness of solutions to the 1D formulation (3.7).

3.3 Conditional Blow Up Result

This section states and proves the conditional blowup result for the 1D system (3.8). The final construction resembles a basic differential equation with a known finite time blowup. Consider the differential equation

$$\begin{cases} \frac{d}{dt}F = BF^2, \\ F(0) = F_0, \end{cases}$$

with $B, F_0 > 0$. This is a simple separable equation with solution

$$F(t) = \frac{F_0}{1 - BF_0 t}.$$

Observe that

$$F(t) \to \infty \quad \text{as} \quad t \to \frac{1}{BF_0}.$$

The conditional result models this simple example as we will show

$$\frac{d}{dt}\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta = B\left(\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta\right)^2,$$

for some $B > 0$. We state the conditional result as a proposition followed by its proof. We then discuss attempts to remove the conditional aspect of the result.

Proposition 3.3.1. *Let $\Omega = [0, \frac{\pi}{4}]$. Let $g_0, h_0 \in H^1(\Omega)$ with $g_0(\theta) \geq 0$ not identically zero. If $g \geq 0$ and $\partial_\theta g \geq 0$, not identically zero, then the unique smooth solution (g, h) to (3.7) blows up in finite time.*

We now prove Proposition 3.3.1. We begin by integrating the first equation of (3.8)

$$\partial_t g + 2G\partial_\theta g = 2H\partial_\theta h$$

from $\theta = 0$ to $\theta = \frac{\pi}{4}$. Integrating by parts, and using the boundary conditions

$$G(0,t) = H(0,t) = G(\tfrac{\pi}{4},t) = H(\tfrac{\pi}{4},t) = 0$$

we obtain

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \partial_t g(\theta,t)\,d\theta &= -\int_0^{\frac{\pi}{4}} 2G(\theta,t)\partial_\theta g(\theta,t)\,d\theta + \int_0^{\frac{\pi}{4}} 2H(\theta,t)\partial_\theta h(\theta,t)\,d\theta \\
&= \int_0^{\frac{\pi}{4}} -2G\partial_\theta g\,d\theta + \int_0^{\frac{\pi}{4}} 2H\partial_\theta h\,d\theta \\
&= \int_0^{\frac{\pi}{4}} -2G(\partial_{\theta\theta\theta}G + 4\partial_\theta G)\,d\theta + \int_0^{\frac{\pi}{4}} 2H(\partial_{\theta\theta\theta}H + 4\partial_\theta H)\,d\theta \\
&= -2G\partial_{\theta\theta}G\big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 2\partial_\theta G\partial_{\theta\theta}G\,d\theta - 4G^2\big|_0^{\frac{\pi}{4}}
\end{aligned}$$

$$+\, 2H\partial_{\theta\theta}H\big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\partial_\theta H \partial_{\theta\theta}H\, d\theta + 4H^2\big|_0^{\frac{\pi}{4}}$$

$$= (\partial_\theta G)^2(\tfrac{\pi}{4}, t) - (\partial_\theta G)^2(0,t) - (\partial_\theta H)^2(\tfrac{\pi}{4}, t) + (\partial_\theta H)^2(0,t).$$

Thus

$$\frac{d}{dt}\int_0^{\frac{\pi}{4}} g(\theta,t)\, d\theta = (\partial_\theta G)^2(\tfrac{\pi}{4}, t) - (\partial_\theta G)^2(0,t) - (\partial_\theta H)^2(\tfrac{\pi}{4}, t) + (\partial_\theta H)^2(0,t). \tag{3.14}$$

Recall the one-dimensional system we are working with states

$$\partial_t H + 2G\partial_\theta H = 2H\partial_\theta G.$$

Differentiating with respect to θ we have, after simplifying,

$$\partial_t(\partial_\theta H) + 2G\partial_{\theta\theta}H = 2H\partial_{\theta\theta}G. \tag{3.15}$$

Evaluating the above equation at $\theta = 0$ and $\theta = \frac{\pi}{4}$ we obtain

$$\partial_t(\partial_\theta H)(0,t) = -2G(0,t)\partial_{\theta\theta}H(0,t) + 2H(0,t)\partial_{\theta\theta}G(0,t) = 0, \tag{3.16}$$

$$\partial_t(\partial_\theta H)(\tfrac{\pi}{4},t) = -2G(\tfrac{\pi}{4},t)\partial_{\theta\theta}H(\tfrac{\pi}{4},t) + 2H(\tfrac{\pi}{4},t)\partial_{\theta\theta}G(\tfrac{\pi}{4},t) = 0. \tag{3.17}$$

Therefore, (3.14) reduces to

$$\frac{d}{dt}\int_0^{\frac{\pi}{4}} g(\theta,t)\, d\theta = (\partial_\theta G)^2(\tfrac{\pi}{4}, t) - (\partial_\theta G)^2(0,t). \tag{3.18}$$

Recall that, from (3.13)

$$\partial_\theta G = \sin(2\theta)\int_0^\theta g(\rho,t)\sin(2\rho)\, d\rho - \cos(2\theta)\int_\theta^{\frac{\pi}{4}} g(\rho,t)\cos(2\rho)\, d\rho.$$

Therefore

$$\partial_\theta G(0,t) = -\int_0^{\frac{\pi}{4}} g(\theta,t)\cos(2\theta)\,d\theta,$$
$$\partial_\theta G(\tfrac{\pi}{4},t) = \int_0^{\frac{\pi}{4}} g(\theta,t)\sin(2\theta)\,d\theta.$$

From this, we may write (3.18) as

$$\begin{aligned}
\frac{d}{dt}\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta &= \left(\int_0^{\frac{\pi}{4}} g(\theta,t)\sin(2\theta)\,d\theta\right)^2 - \left(\int_0^{\frac{\pi}{4}} g(\theta,t)\cos(2\theta)\,d\theta\right)^2 \\
&= \left(\int_0^{\frac{\pi}{4}} g(\theta,t)\sin(2\theta)\,d\theta + \int_0^{\frac{\pi}{4}} g(\theta,t)\cos(2\theta)\,d\theta\right) \\
&\quad\cdot \left(\int_0^{\frac{\pi}{4}} g(\theta,t)\sin(2\theta)\,d\theta - \int_0^{\frac{\pi}{4}} g(\theta,t)\cos(2\theta))\,d\theta\right) \\
&= \left(\int_0^{\frac{\pi}{4}} g(\theta,t)(\sin(2\theta)+\cos(2\theta)\,d\theta\right) \\
&\quad\cdot \left(\int_0^{\frac{\pi}{4}} g(\theta,t)(\sin(2\theta)-\cos(2\theta))\,d\theta\right).
\end{aligned}$$

If $g(\theta,t) \geq 0$, $\partial_\theta g(\theta,t) \geq 0$ not identically zero for $\theta \in [0,\frac{\pi}{4}]$ then by the Mean Value Theorem, there exist ξ_1, $\xi_2 \in [0,\frac{\pi}{4}]$ such that

$$\int_0^{\frac{\pi}{4}} g(\theta,t)(\sin(2\theta)+\cos(2\theta))\,d\theta = (\sin(2\xi_1)+\cos(2\xi_1))\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta, \qquad \text{and}$$
$$\int_0^{\frac{\pi}{4}} g(\theta,t)(\sin(2\theta)-\cos(2\theta))\,d\theta = (\sin(2\xi_2)-\cos(2\xi_2))\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta.$$

Thus

$$\frac{d}{dt}\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta = B\left(\int_0^{\frac{\pi}{4}} g(\theta,t)\,d\theta\right)^2, \tag{3.19}$$

where

$$B = (\sin(2\xi_1) + \cos(2\xi_1))(\sin(2\xi_2) - \cos(2\xi_2)).$$

In order to use the Mean Value Theorem above, $g(\theta, t) \geq 0$ for $\theta \in [0, \frac{\pi}{4}]$. Also, the additional condition $\partial_\theta g(\theta, t) \geq 0$ not identically zero for $\theta \in [0, \frac{\pi}{4}]$ ensures that

$$(\sin(2\xi_2) - \cos(2\xi_2)) \int_0^{\frac{\pi}{4}} g(\theta, t)\, d\theta > 0.$$

Thus, if $g(\theta, t) \geq 0$ and $\partial_\theta g(\theta, t) \geq 0$ not identically zero for $\theta \in [0, \frac{\pi}{4}]$ then $B > 0$. Therefore (3.19) shows a finite time blowup as desired.

In this result, we have assumed that $g(\theta, t) \geq 0$ and $\partial_\theta g(\theta, t) \geq 0$ not identically zero for $\theta \in [0, \frac{\pi}{4}]$. The ultimate goal is to find suitable initial conditions such that $g(\theta, t) \geq 0$ and $\partial_\theta g(\theta, t) \geq 0$ not identically zero for $\theta \in [0, \frac{\pi}{4}]$. This goal has not yet been accomplished. The following section provides a brief discussion on attempts to prove these assumed conditions.

3.4 Conditional Result Discussions

In this section we briefly discuss the nonnegativity conditions for $g(\theta, t)$ and $\partial_\theta g(\theta, t)$. Since g satisfies (3.7)

$$\partial_t g + 2G\partial_\theta g = 2H\partial_\theta h,$$

and H maintains the sign of H_0 for all time, then it suffices to show that $\partial_\theta h \geq 0$ for all time. This then shows $g \geq 0$.

Due to the similar and coupled structure of both the g and h equations for (3.7) we can combine the equations for analysis. In particular, we wish to analyze $\partial_\theta h$ and $\partial_\theta g$ simultaneously. Taking the derivative of (3.7) with respect to θ yields

$$\partial_t(\partial_\theta g) + 2\partial_\theta G \partial_\theta g + 2G\partial_{\theta\theta} g = 2\partial_\theta H \partial_\theta h + 2H\partial_{\theta\theta} h, \tag{3.20}$$

$$\partial_t(\partial_\theta h) + 2\partial_\theta G \partial_\theta h + 2G\partial_{\theta\theta} h = 2\partial_\theta H \partial_\theta g + 2H\partial_{\theta\theta} g + 2\partial_\theta G \partial_{\theta\theta\theta} H + 2\partial_\theta H \partial_{\theta\theta\theta} G. \tag{3.21}$$

Adding and subtracting the equations above yields

$$\partial_t(\partial_\theta g + \partial_\theta h) + 2(\partial_\theta G - \partial_\theta H)\partial_\theta(g+h) + 2(G-H)\partial_{\theta\theta}(g+h) = M, \tag{3.22}$$

$$\partial_t(\partial_\theta h - \partial_\theta g) + 2(\partial_\theta G + \partial_\theta H)\partial_\theta(h-g) + 2(G+H)\partial_{\theta\theta}(h-g) = M, \tag{3.23}$$

where $M = 2\partial_\theta H \partial_{\theta\theta\theta} G - 2\partial_\theta G \partial_{\theta\theta\theta} H$. At this point, it appears as though no progress has been made because in order to determine the behavior of $\partial_\theta g$ and $\partial_\theta h$ we must know of the behavior of the third derivatives of H and G. However, we may use the stream function relationship (3.9) between g and G and similarly with h and H to rewrite M in terms of only first derivatives with respect to θ. By taking the derivative of (3.9) with respect to θ we obtain

$$\partial_{\theta\theta\theta} G + 4\partial_\theta G = \partial_\theta g,$$
$$\partial_{\theta\theta\theta} H + 4\partial_\theta H = \partial_\theta h.$$

Using the equations above, we can rewrite M as

$$\begin{aligned} M = 2\partial_\theta H \partial_{\theta\theta\theta} G - 2\partial_\theta G \partial_{\theta\theta\theta} H &= 2\partial_\theta H(\partial_\theta g - 4\partial_\theta G) - 2\partial_\theta G(\partial_\theta h - \partial_\theta H) \\ &= 2\partial_\theta H \partial_\theta g - 8\partial_\theta H \partial_\theta G - 2\partial_\theta G \partial_\theta h + 8\partial_\theta G \partial_\theta H \\ &= 2\partial_\theta H \partial_\theta g - 2\partial_\theta G \partial_\theta h. \end{aligned}$$

We may then rewrite (3.22) as

$$\begin{aligned} &\partial_t(\partial_\theta g + \partial_\theta h) + 3(\partial_\theta G - \partial_\theta H)(\partial_\theta g + \partial_\theta h) + 2(G-H)\partial_\theta(\partial_\theta g + \partial_\theta h) \\ &\qquad = (\partial_\theta H + \partial_\theta G)(\partial_\theta g - \partial_\theta h), \qquad (3.24) \\ &\partial_t(\partial_\theta h - \partial_\theta g) + 3(\partial_\theta G + \partial_\theta H)(\partial_\theta h - \partial_\theta g) + 2(G+H)\partial_\theta(\partial_\theta h - \partial_\theta g) \\ &\qquad = (\partial_\theta H - \partial_\theta G)(\partial_\theta g + \partial_\theta h). \qquad (3.25) \end{aligned}$$

These are written such a way so the only terms that affect the sign of the equation are the terms on the right hand side. Therefore, if

$$\partial_\theta H + \partial_\theta G \leq 0 \qquad \text{and} \qquad \partial_\theta H - \partial_\theta G \geq 0,$$

then the desired result $\partial_\theta h \geq 0$ is obtained.

Although, in some sense, it does not appear that much progress has been made because through this manipulation the required condition went from needing to show $g \geq 0$ to needing to show

$$\partial_\theta H + \partial_\theta G \leq 0 \qquad \text{and} \qquad \partial_\theta H - \partial_\theta G \geq 0.$$

This, however, is an improvement in the fact that $\partial_\theta G$ and $\partial_\theta H$ have explicit constructions (3.13) that can be analyzed.

At the time of this writing, the above conditions have yet to be shown, but the author continues pursuit of this problem.

This completes our discussion for the magnetohydrodynamics equations and attention will now be turned to the Boussinesq equations in the following chapter.

CHAPTER 4

FRACTIONALLY DISSIPATIVE BOUSSINESQ EQUATIONS WITHOUT THERMAL DIFFUSION

4.1 Global Existence and Uniqueness of Weak Solutions

Recall the d-dimensional Boussinesq equations with fractional dissipation and no thermal diffusion

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nu(-\Delta)^\alpha u - \nabla p + \theta e_d, \quad x\in\mathbb{R}^d,\ t>0, \\ \partial_t\theta + u\cdot\nabla\theta = 0, \quad x\in\mathbb{R}^d,\ t>0, \\ \nabla\cdot u = 0, \\ (u,\theta)|_{t=0} = (u_0,\theta_0). \end{cases} \tag{4.1}$$

There has been much investigation regarding the global existence and uniqueness of solutions to (4.1) for $d=3$ and $\alpha\ge\frac{5}{4}$ with initial data $(u_0,\theta_0)\in H^s(\mathbb{R}^3)$ where $s>\frac{5}{2}$, (see [32, 42, 52, 53, 60]) and where $s>\frac{5}{4}$ (see [32]). There has been significantly less investigation in the weak setting, and in particular, little was known in search of the weakest possible functional setting where solutions were unique. The results of [4] proved uniqueness in what appears to be the weakest functional setting known, with initial data $u_0\in L^2(\mathbb{R}^d), \theta_0\in L^2(\mathbb{R}^d)\cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$, for the partially dissipated Boussinesq equations.

In this chapter, we present results from the author's joint work in [4] establishing global existence of weak solutions to the d-dimensional Boussinesq equations, for $d\ge 2$ with fractional dissipation and no thermal diffusion along with a uniqueness. In particular, the author's result regarding the existence of weak solutions will be detailed while the remainder of results will be summarized.

We begin by defining the meaning of weak solutions to (4.1) with any $\alpha > 0$.

Definition 4.1.1. *Consider (4.1) with $\alpha > 0$ and $(u_0, \theta_0) \in L^2(\mathbb{R}^d)$ and $\nabla \cdot u_0 = 0$. Let $T > 0$ be arbitrarily fixed. The pair (u, θ) satisfying*

$$u \in C_w([0,T]; L^2) \cap L^2(0,T; \mathring{H}^\alpha), \theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2)$$

with $\nabla \cdot u = 0$ is a weak solution of (4.1) on $[0,T]$ if the following two conditions hold.

1. *For any $\phi \in C_0^\infty(\mathbb{R}^d \times [0,T))$ with $\nabla \cdot \phi = 0$,*

$$-\int_0^T \int_{\mathbb{R}^d} u \cdot \partial_t \phi \, dx \, dt - \int_{\mathbb{R}^d} u_0(x) \cdot \phi(x,0) \, dx - \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla \phi u \, dx \, dt \tag{4.2}$$

$$+ \int_0^T \int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} u \cdot (-\Delta)^{\alpha/2} \phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} \theta \boldsymbol{e}_d \cdot \phi \, dx \, dt. \tag{4.3}$$

2. *For any $\psi \in C_0^\infty(\mathbb{R}^d \times [0,T))$*

$$-\int_0^T \int_{\mathbb{R}^d} \partial_t \psi \theta \, dx \, dt - \int_{\mathbb{R}^d} \theta_0(x) \psi(x,0) \, dx = \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla \psi \theta \, dx \, dt. \tag{4.4}$$

The author's contribution to the work in [4] is the following proposition stating the existence of global weak solutions which will be proven in detail in the following section.

Proposition 4.1.2. *Consider (4.1) with $\alpha > 0$ and $(u_0, \theta_0) \in L^2(\mathbb{R}^d)$ and $\nabla \cdot u_0 = 0$. Let $T > 0$ be arbitrarily fixed. Then (4.1) has a global weak solution (u, θ) as given in Definition 4.1.1 satisfying*

$$\|\theta(t)\|_{L^2} \le \|\theta_0\|_{L^2},$$

$$\|u(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \le \left(\|u_0\|_{L^2} + t\|\theta_0\|_{L^2}\right)^2 .$$

This existence result was then combinated with a smoothing result to obtain the desired existence and uniqueness result of weak solutions for the d-dimenional fractionally dissipative Boussineq equations as stated in [4].

Theorem 4.1.3. *Consider the d-dimensional equations in* (4.1)*.*

1. *Let $\alpha > 0$ and $(u_0, \theta_0) \in L^2(\mathbb{R}^d)$ with $\nabla \cdot u_0 = 0$. Let $T > 0$ be arbitrarily fixed. Then* (4.1) *has a global weak solution (u, θ) on $[0, T]$ satisfying*

$$u \in C_w([0,T]; L^2) \cap L^2(0,T; H^\alpha), \qquad \theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2).$$

2. *Let $\alpha \ge \frac{1}{2} + \frac{d}{4}$. Assume $u_0 \in L^2(\mathbb{R}^d)$ and $\theta_0 \in L^2(\mathbb{R}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{R}^d)$ with $\nabla \cdot u_0 = 0$. Then* (4.1) *has a unique and global weak solution (u, θ) satisfying*

$$\begin{aligned} &u \in C([0,T]; L^2) \cap L^2(0,T; H^\alpha), \\ &u \in \widetilde{L}^1(0,T; \mathring{H}^{1+\frac{d}{2}}), \\ &\theta \in C_w([0,T]; L^2) \cap L^\infty(0,T; L^2 \cap L^{\frac{4d}{d+2}}), \end{aligned}$$

In particular, u satisfies

$$\sup_{q \ge 2} \frac{1}{\sqrt{q}} \int_0^T \|\nabla u(t)\|_{L^q} \, dt < \infty.$$

The space-time space $\widetilde{L}^1(0,T; \mathring{H}^{1+\frac{d}{2}})$ is defined in the appendix.

The proof of Theorem 4.1.3 relies on global in time bounds on the weak solution. In order to show existence of weak solutions, a sequence of approximate systems is constructed and shown to have global smooth solutions $(u^{(n)}, \theta^{(n)})$. Global uniform bounds are then established on this sequence to obtain a strongly convergent subsequence of $u^{(n)}$ and finally the limit of the convergent subsequences is shown to be the weak solution. The strong convergence of $u^{(n)}$ will allow us to overcome the difficulty of having no strong convergence in $\theta^{(n)}$. This is detailed in the following section. Uniqueness of solutions for $\alpha \geq \frac{1}{2} + \frac{d}{4}$ must also be shown. We outline the strategy to establish the desired uniqueness here. For full detail of the uniqueness result, see [4].

In order to establish uniqueness, we consider the the difference $(\widetilde{u}, \widetilde{\theta})$ with

$$\widetilde{u} := u^{(1)} - u^{(2)}, \quad \widetilde{\theta} := \theta^{(1)} - \theta^{(2)}.$$

Let $P^{(1)}$ and $P^{(2)}$ be the corresponding pressure terms and $\widetilde{P} := P^{(1)} - P^{(2)}$. In order to obtain necessary bounds, we introduce the lower regularity quantities $h^{(1)}$ and $h^{(2)}$ as solutions to the respective Poisson equations

$$-\Delta h^{(1)} = \theta^{(1)}, \quad -\Delta h^{(2)} = \theta^{(2)},$$

and set

$$\widetilde{h} = h^{(1)} - h^{(2)}.$$

It follows from (4.1) that $(\widetilde{u}, \widetilde{\theta})$ satisfies

$$\begin{cases} \partial_t \widetilde{u} + u^{(1)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(2)} + \nu(-\Delta)^\alpha \widetilde{u} + \nabla \widetilde{P} = \widetilde{\theta} \mathbf{e}_d, \\ \partial_t \widetilde{\theta} + u^{(1)} \cdot \nabla \widetilde{\theta} + \widetilde{u} \cdot \nabla \theta^{(2)} = 0, \\ \nabla \cdot \tilde{u} = 0, \\ \widetilde{u}_0 = 0, \widetilde{\theta} = 0. \end{cases} \tag{4.5}$$

To obtain uniqueness, it must be shown that

$$\|u^{(2)}(t) - u^{(1)}(t)\|_{L^2}^2 + \|\theta^{(2)}(t) - \theta^{(1)}(t)\|_{L^2}^2 = \|\tilde{u}(t)\|_{L^2}^2 + \|\Delta \tilde{h}(t)\|_{L^2}^2 = 0.$$

Dotting the first equation of (4.5) by $\widetilde{u}$ and dotting the second equation by $\widetilde{h}$, integrating by parts, and adding the results, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|\widetilde{u}\|_{L^2}^2 + \|\nabla\widetilde{h}\|_{L^2}^2\right) + \nu\|\Lambda^\alpha\widetilde{u}\|_{L^2}^2 &= -\int_{\mathbb{R}^d} \widetilde{u}\cdot\nabla u^{(2)}\cdot\widetilde{u}\,dx + \int_{\mathbb{R}^d}\widetilde{\theta}\cdot(\mathbf{e}_d\cdot\widetilde{u})\,dx \\
&\quad + \int_{\mathbb{R}^d} u^{(1)}\cdot\nabla\widetilde{\theta}\widetilde{h}\,dx + \int_{\mathbb{R}^d}\widetilde{u}\cdot\nabla\theta^{(2)}\widetilde{h}\,dx \\
&:= K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

We then bound each K_i using integration by parts, Hölder's inequality, Sobolev inequality and interpolation inequalities. We find that, for $\delta > 0$,

$$
G_\delta(t) := \|\widetilde{u}(t)\|_{L^2}^2 + \|\nabla\widetilde{h}(t)\|_{L^2}^2 + \delta
$$

obeys the inequality

$$
\begin{aligned}
\frac{d}{dt}G_\delta(t) &\le \left(1 + \|\Lambda u^{(2)}\|_{L^2}^2\right)G_\delta(t) + C\left(1 + \frac{\|\nabla^{(1)}\|_{L^p}}{p}\right)pM^{\frac{1}{p}}G_\delta(t)^{1-\frac{1}{p}}, \quad \text{for } d = 2, \\
\frac{d}{dt}G_\delta(t) &\le \left(1 + \|\Lambda^{\frac{1}{2}+\frac{d}{4}}u^{(2)}\|_{L^2}^2\right)G_\delta(t) + C\frac{\|\nabla u^{(1)}\|_{L^p}}{p}pM^{\frac{d}{2p}}G_\delta(t)^{1-\frac{d}{2p}}, \quad \text{for } d \ge 3,
\end{aligned}
$$

where $M = \|\theta_0\|_{L^2}^2$.

We are able to reduce the bounds above to

$$
G_\delta(t) \le G_\delta(0) + C\int_0^t \gamma(s)\phi(G_\delta(s))\,ds, \tag{4.6}
$$

where

$$
\gamma(t) = C + C\|\Lambda^{\frac{1}{2}+\frac{d}{4}}u^{(2)}\|_{L^2}^2 + C\frac{\|\nabla u^{(1)}\|_{L^p}}{p}, \tag{4.7}
$$

$$
\phi(r) = r + r(\ln M - \ln r). \tag{4.8}
$$

This step highlights a difficulty of this problem. Since a weak functional setting is used for solutions, u is not Lipschitz resulting in the corresponding vorticity not necessarily being bounded. So it appears as though $\gamma(t)$ cannot be bounded. This is overcome with the following smoothing property

$$\|u\|_{\widetilde{L}^1\left(0,T;\mathring{B}^{1+\frac{d}{2}}_{2,2}\right)} \leq C\left(T, \|u_0\|_{L^2}, \|\theta_0\|_{L^2\cap L^{\frac{4d}{d+2}}}\right)$$

along with a special case of this property. This smoothing property is derived using the Littlewood-Paley decomposition and Besov space techniques. We state this smoothing estimate result as Proposition 4.1.4.

Proposition 4.1.4. *Let $d \geq 2$. Consider 4.1 with $\alpha \geq \frac{1}{2} + \frac{d}{4}$. Assume (u_0, θ_0) satisfies $\nabla \cdot u_0 = 0$, with*

$$\begin{aligned} &u_0 \in L^2(\mathbb{R}^d),\\ &\theta_0 \in L^2(\mathbb{R}^d) \cap L^{\frac{4d}{d+2}}(\mathbb{R}^d). \end{aligned}$$

Let (u, θ) be the corresponding global weak solution of 4.1. Then for any $0 < t \leq T$,

$$\|u\|_{\widetilde{L}^1\left(0,t;B^{1+\frac{d}{2}}_{2,2}\right)} \leq C\left(t, \|u_0\|_{L^2}, \|\theta_0\|_{L^2}\right).$$

As a special consequence,

$$\sup_{q\geq 2} \int_0^t \frac{\|\nabla u(\tau)\|_{L^q}}{\sqrt{q}}\, d\tau \leq C\left(t, \|u_0\|_{L^2}, \|\theta_0\|_{L^2}\right).$$

Proposition 4.1.4 allows us to bound the terms of $\gamma(t)$ in (4.7). Using Proposition 4.1.4 to bound the terms of $\gamma(t)$ we are able to bound (4.6) as

$$G_\delta(t) \leq (eM)^{1-e^{-\int_0^t \gamma(s)\,ds}} G_\delta(0)^{e^{-\int_0^t \gamma(s)\,ds}}.$$

Since $G_0(0) = 0$, then letting $\delta \to 0$, yields the desired result

$$\|u^{(2)}(t) - u^{(1)}(t)\|_{L^2}^2 + \|\theta^{(2)} - \theta^{(1)}\|_{L^2}^2 = 0.$$

This completes the sketch of the proof of uniqueness for $\alpha \geq \frac{1}{2} + \frac{d}{4}$. We now prove, in detail, the existence of weak solutions for $\alpha > 0$.

4.2 Proof of Existence for Global Weak Solutions

The proof of Proposition 4.1.2 is divided into three main steps. To begin, global existence of smooth solutions must be established for a sequence of approximate systems using Picard's theorem. The second step extracts a strongly convergent subsequence using Aubin-Lions method once uniform bounds have been established for the sequence of approximate systems. Finally, the limit of this strongly convergent subsequence must then be shown to be the actual weak solution thus completing the proof.

Step 1: Establishing global existence of smooth solutions to an approximate system. Let $n \in \mathbb{N}$. Consider the following approximate system

$$\begin{cases} \partial_t u^{(n)} + \mathbb{P} J_n(u^{(n)} \cdot \nabla u^{(n)}) + \nu(-\Delta)^\alpha u^{(n)} = \mathbb{P} J_n(\theta^{(n)} \mathbf{e}_d), \\ \partial_t \theta^{(n)} + J_n(u^{(n)} \cdot \nabla \theta^{(n)}) = 0, \\ \nabla \cdot u^{(n)} = 0, \\ u^{(n)}(x,0) = J_n u_0, \;\; \theta^{(n)}(x,0) = J_n \theta_0. \end{cases} \tag{4.9}$$

We seek a solution $(u^{(n)}, \theta^{(n)}) \in L_n^2$ satisfying (4.9) Note that the functions in $L_n^2(\mathbb{R}^d)$ are smooth with

$$L_n^2 \subseteq \bigcap_{m=0}^{\infty} \mathring{H}^m.$$

In particular, if $f \in L^2_n$,

$$\|f\|^2_{\mathring{H}^m} = \sum_{|\beta|=m} \|D^\beta f\|^2_{L^2} = \sum_{|\beta|=m} \|\widehat{D^\beta f}\|^2_{L^2} = \sum_{|\beta|=m} \|(2\pi i\xi)^\beta \widehat{f}\|^2_{L^2} \leq (2\pi n)^{2\beta}\|f\|^2_{L^2}.$$

Picard's Theorem will be used to show that (4.9) has a unique global solution in L^2_n. We begin by applying Lemma A.2.4 to show (4.9) has a local-in-time solution. First write (4.9) as

$$\frac{dy}{dt} = F(y),$$

with

$$\begin{aligned} Y &= (u^{(n)}, \theta^{(n)})^T, \\ F(Y) &= (F_1(Y), F_2(Y))^T \\ &= (-\mathbb{P}J_n(u^{(n)} \cdot \nabla u^{(n)}) - \nu(-\Delta)^\alpha u^{(n)} + \mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d), -J_n(u^{(n)} \cdot \nabla \theta^{(n)}))^T. \end{aligned}$$

It must be shown that $F : L^2_n \longrightarrow L^2_n$ is locally Lipschitz. Set $E = L^2_n$ and $O = E$. Let $Y \in L^2_n$.

$$\begin{aligned} \|F_1(Y)\|_{L^2} &\leq \|u^{(n)} \cdot \nabla u^{(n)}\|_{L^2} + \|\nu(-\Delta)^\alpha u^{(n)}\|_{L^2} + \|\theta^{(n)}\|_{L^2} \\ &\leq \|u^{(n)}\|_{L^4}\|\nabla u^{(n)}\|_{L^4} + \nu\|u^{(n)}\|_{\mathring{H}^{2\alpha}} + \|\theta^{(n)}\|_{L^2} \\ &\leq \|u^{(n)}\|_{\mathring{H}^{\frac{d}{4}}}\|u^{(n)}\|_{\mathring{H}^{1+\frac{d}{4}}} + \nu\|u^{(n)}\|_{\mathring{H}^{2\alpha}} + \|\theta^{(n)}\|_{L^2} \\ &\leq (2\pi n)^{2(1+\frac{d}{4})}\|u^{(n)}\|^2_{L^2} + \nu(2\pi n)^{2\alpha}\|u^{(n)}\|_{L^2} + \|\theta^{(n)}\|_{L^2}. \end{aligned}$$

That is $F_1(Y) \in L^2(\mathbb{R}^d)$. Similarly, $F_2(Y) \in L^2(\mathbb{R}^d)$. We also have,

$$\text{supp } \widehat{F_1(Y)}, \quad \text{supp } \widehat{F_2(Y)} \subseteq B(0, n).$$

Therefore, $F(Y) \in L_n^2(\mathbb{R}^d)$. In order to show $F(Y)$ is locally Lipschitiz, let $Y = (u^{(n)}, \theta^{(n)})^T \in L_n^2$ and $Z = (v^{(n)}, \rho^{(n)})^T \in L_n^2$. Then

$$\begin{aligned}
&\|F_2(Y) - F_2(Z)\|_{L^2} \\
&\qquad = \| - J_n(u^{(n)} \cdot \nabla\theta^{(n)}) + J_n(v^{(n)} \cdot \nabla\rho^{(n)})\|_{L^2} \\
&\qquad = \| - J_n((u^{(n)} - v^{(n)}) \cdot \nabla\theta^{(n)}) - J_n(v^{(n)} \cdot \nabla(\theta^{(n)} - \rho^{(n)}))\|_{L^2} \\
&\qquad \leq \|(u^{(n)} - v^{(n)}) \cdot \nabla\theta^{(n)}\|_{L^2} + \|v^{(n)} \cdot \nabla(\theta^{(n)} - \rho^{(n)})\|_{L^2} \\
&\qquad \leq \|u^{(n)} - v^{(n)}\|_{L^2}\|\nabla\theta^{(n)}\|_{L^\infty} + \|v^{(n)}\|_{L^\infty}\|\nabla(\theta^{(n)} - \rho^{(n)})\|_{L^2} \\
&\qquad \leq \|u^{(n)} - v^{(n)}\|_{L^2}\|\theta^{(n)}\|_{\mathring{H}^{1+\frac{d}{2}+\epsilon}} + \|v^{(n)}\|_{\mathring{H}^{\frac{d}{2}+\epsilon}}\|\theta^{(n)} - \rho^{(n)}\|_{\mathring{H}^1} \\
&\qquad \leq (2\pi n)^{1+\frac{d}{2}+\epsilon}\|\theta^{(n)}\|_{L^2}\|u^{(n)} - v^{(n)}\|_{L^2} + (2\pi n)^{1+\frac{d}{2}+\epsilon}\|v^{(n)}\|_{L^2}\|\theta^{(n)} - \rho^{(n)}\|_{L^2} \\
&\qquad \leq L\|Y - Z\|_{L^2},
\end{aligned}$$

where $\epsilon > 0$ is a small parameter and $L = (2\pi n)^{1+\frac{d}{2}+\epsilon}(\|Y\|_{L^2} + r)$ for $\|Z - Y\| \leq r$. Therefore $F_2(Y)$ is locally Lipschitz. Similarly, $F_1(Y)$ is locally Lipschitz and hence $F(Y)$ is locally Lipchitz. Then by Picard's Existence and Uniqueness Theorem A.2.4, the sequence of approximate systems (4.9) have unique local-in-time solutions in L_n^2.

Next we use Picard's Extension Theorem (Lemma A.2.5) to show the solutions to the sequence of approximate systems are not just local, but actually global in time. Using the energy method, it can be shown that for any $t \leq T$, $\|(u^{(n)}, \theta^{(n)})\|_{L^2} < +\infty$. This can be seen by dotting (4.9) with $(u^{(n)}, \theta^{(n)})$ which yields

$$\frac{1}{2}\frac{d}{dt}(\|u^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2) + \nu\|\Lambda^\alpha u^{(n)}\|_{L^2}^2 = M_1 + M_2 + M_3,$$

where $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)\,dx$ and

$$\begin{aligned}
M_1 &= -\int_{\mathbb{R}^d} \mathbb{P}J_n(u^{(n)} \cdot \nabla u^{(n)}) \cdot u^{(n)}\,dx, \\
M_2 &= \int_{\mathbb{R}^d} \mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d) \cdot u^{(n)}\,dx,
\end{aligned}$$

$$M_3 = -\int_{\mathbb{R}^d} J_n(u^{(n)} \cdot \nabla\theta^{(n)}) \cdot \theta^{(n)}\, dx.$$

Observe

$$\begin{aligned} M_1 &= -\int \mathbb{P} J_n(u^{(n)} \cdot \nabla u^{(n)}) \cdot u^{(n)} dx \\ &= -\int J_n(u^{(n)} \cdot \nabla u^{(n)}) \cdot \mathbb{P} u^{(n)} dx \\ &= -\int J_n(u^{(n)} \cdot \nabla u^{(n)}) \cdot u^{(n)} dx \\ &= -\int (u^{(n)} \cdot \nabla u^{(n)}) \cdot u^{(n)} dx = 0. \end{aligned}$$

Similarly, $M_3 = 0$. Also, since the projection $\mathbb{P}$ is bounded in L^2 we have

$$|M_2| \leq \|u^{(n)}\|_{L^2} \|\theta^{(n)}\|_{L^2}.$$

Hence,

$$\frac{d}{dt}\left(\|u^{(n)}\|_{L^2}^2 + \|\theta^{(n)}\|_{L^2}^2\right) + 2\nu \|\Lambda^\alpha u^{(n)}\|_{L^2}^2 \leq \|u^{(n)}\|_{L^2} \|\theta^{(n)}\|_{L^2}.$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\|\theta^{(n)}\|_{L^2} = 0,$$

which can also be written as

$$\|\theta^{(n)}(t)\|_{L^2} = \|J_n \theta_0\|_{L^2}.$$

Therefore,

$$\|u^{(n)}(t)\|_{L^2} \leq \|J_n u_0\|_{L^2} + t\|J_n \theta_0\|_{L^2} \leq \|u_0\|_{L^2} + t\|\theta_0\|_{L^2},$$

and

$$\|u^{(n)}(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda^\alpha u^{(n)}\|_{L^2}^2 d\tau \leq (\|u_0\|_{L^2} + t\|\theta_0\|_{L^2})^2.$$

Hence, $(u^{(n)}, \theta^{(n)}) \in L_n^2$ for all time $t \leq T$. By Picard's Extension Theorem A.2.5, $(u^{(n)}, \theta^{(n)})$ is global in time. Therefore, the sequence of approximate systems (4.9) have global in time solutions.

Step 2. Extraction of a strongly convergent subsequence.

The goal of Step 2 is to extract a subsequence of $u^{(n)}$ from the solutions to the sequence of approximate solutions in which the extracted subsequence converges strongly in $L^2(0,T;L^2(\mathbb{R}^d))$. Aubin-Lions lemma will be used. In order to use the Aubin-Lions method we must show that

$$\partial_t u^{(n)} \in L^2(0,T;H^{-s}), \tag{4.10}$$

where $s = \max\{\alpha, 1+\frac{d}{2}-\alpha\}$. Let $\phi \in H^s$. Taking the L^2-inner product of ϕ and the velocity equation in (4.9) produces

$$\int_{\mathbb{R}^d} \phi \cdot \partial_t u^{(n)}\ dx = Q_1 + Q_2 + Q_3,$$

with

$$\begin{aligned}
Q_1 &= -\int \phi \cdot \mathbb{P}J_n(u^{(n)} \cdot \nabla u^{(n)})\ dx,\\
Q_2 &= -\nu \int \phi \cdot (-\Delta)^\alpha u^{(n)}\ dx,\\
Q_3 &= \int \phi \cdot \mathbb{P}J_n(\theta^{(n)}\mathbf{e}_d)\ dx.
\end{aligned}$$

Using integration by parts and applying Hölder's and Sobolev's inequalities yields

$$\begin{aligned}
|Q_1| &\le \|u^{(n)}\|^2_{L^{\frac{2d}{d-\alpha}}} \|\nabla \mathbb{P}J_n\phi\|_{L^{\frac{d}{\alpha}}}\\
&\le C\,\|u^{(n)}\|^{\frac{1}{2}}_{L^2}\|\Lambda^\alpha u^{(n)}\|^{\frac{1}{2}}_{L^2}\,\|\mathbb{P}J_n\phi\|_{H^{1+\frac{d}{2}-\alpha}}\\
&\le C\,\|u^{(n)}\|^{\frac{1}{2}}_{L^2}\|\Lambda^\alpha u^{(n)}\|^{\frac{1}{2}}_{L^2}\,\|\phi\|_{H^{1+\frac{d}{2}-\alpha}}.
\end{aligned}$$

Again, by integration by parts and Hölder's inequality, we have

$$|Q_2| \leq \nu \|\Lambda^\alpha \phi\|_{L^2} \|\Lambda^\alpha u^{(n)}\|_{L^2} \leq \nu \|\phi\|_{H^s} \|\Lambda^\alpha u^{(n)}\|_{L^2}.$$

Thus

$$|Q_3| \leq \|\phi\|_{H^s} \|\theta^{(n)}\|_{L^2}.$$

Hence,

$$\left| \int \phi \cdot \partial_t u^{(n)} \, dx \right| \leq C \|\phi\|_{H^s} \bigg(\|\Lambda^\alpha u^{(n)}\|_{L^2} (1 + \|u^{(n)}\|_{L^2}) + \|\theta^{(n)}\|_{L^2} \bigg).$$

Therefore,

$$\|\partial_t u^{(n)}\|_{H^{-s}} \leq C \bigg(\|\Lambda^\alpha u^{(n)}\|_{L^2} (1 + \|u^{(n)}\|_{L^2}) + \|\theta^{(n)}\|_{L^2} \bigg).$$

Squaring both sides of the above equation and integrating in time yields

$$\begin{aligned}
&\int_0^T \|\partial_t u^{(n)}\|_{H^{-s}}^2 \, dt \\
&\quad \leq C \int_0^T \left(1 + \|u^{(n)}\|_{L^2}\right)^2 \|\Lambda^\alpha u^{(n)}\|_{L^2}^2 \, dt + C \int_0^T \|\theta^{(n)}\|_{L^2}^2 \, dt \\
&\quad + C \int_0^T \left(1 + \|u^{(n)}\|_{L^2}\right) \|\Lambda^\alpha u^{(n)}\|_{L^2} \|\theta^{(n)}\|_{L^2}^2 \, dt \\
&\quad \leq C \sup_{0 \leq t \leq T} (1 + \|u^{(n)}\|_{L^2}^2) \int_0^T \|\Lambda^\alpha u^{(n)}\|_{L^2}^2 \, dt + CT \sup_{0 \leq t \leq T} \|\theta^{(n)}\|_{L^2} \\
&\quad + C \left(T \sup_{0 \leq t \leq T} \|\theta^{(n)}\|_{L^2} \right) \cdot \left(\sup_{0 \leq t \leq T} (1 + \|u^{(n)}\|_{L^2}) \right) \int_0^T \|\Lambda^\alpha u^{(n)}\|_{L^2}^2 \, dt \\
&\quad < +\infty.
\end{aligned}$$

Therefore we have shown (4.10). We have that

$$u^{(n)} \in L^2(0, T; H^\alpha(\mathbb{R}^d)), \quad \partial_t u^{(n)} \in L^2(0, T; H^{-s}(\mathbb{R}^d)),$$

Additionally, we have that $H^\alpha(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ is locally compact and $L^2(\mathbb{R}^d) \hookrightarrow H^{-(1+d/2-\alpha)}$ is continuous. Thus, we can apply the Aubin-Lions Lemma to extract a convergent subsequence from $u^{(n)}$ in $L^2(0,T;L^2(\mathbb{R}^d))$.

Step 3. Showing the limit of the subsequence is the weak solution.

Now that we have extracted a convergent subsequence from $u^{(n)}$ in $L^2(0,T;L^2(\mathbb{R}^d)$, it must be shown that the limit of this convergent subsequence is, in fact, the weak solution. Let u be the limit of $u^{(n)}$ and θ be the weak limit of $\theta^{(n)}$. Then

$$\theta \in L^\infty(0,T;L^2(\mathbb{R}^d)), \quad u \in L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^\alpha(\mathbb{R}^d)).$$

We then wish to show that (u,θ) is the weak solution.

Note that from (4.9) we have that, for any $\phi \in C_0^\infty(\mathbb{R}^d \times [0,T))$ with $\nabla \cdot \phi = 0$, and for any $\psi \in C_0^\infty(\mathbb{R}^d \times [0,T))$,

$$\begin{aligned}
-\int_0^T \int_{\mathbb{R}^d} u^{(n)} \cdot \partial_t \phi \, dx\, dt - \int_{\mathbb{R}^d} u_0^{(n)} \cdot \phi(x,0)\, dx - \int_0^T \int_{\mathbb{R}^d} u^{(n)} \cdot \nabla (J_n \phi) u^{(n)} \, dx\, dt \\
+ \int_0^T \int_{\mathbb{R}^d} \Lambda^\alpha u^{(n)} \cdot \Lambda^\alpha \phi \, dx\, dt = \int_0^T \int_{\mathbb{R}^d} \theta^{(n)} \mathbf{e}_d \cdot J_n \phi \, dx\, dt, \\
-\int_0^T \int_{\mathbb{R}^d} \partial_t \psi \theta^{(n)} \, dx\, dt + \int_{\mathbb{R}^d} \theta_0^{(n)} \psi(x,0)\, dx = \int_0^T \int_{\mathbb{R}^d} u^{(n)} \cdot \nabla (J_n \psi) \theta^{(n)} \, dx\, dt.
\end{aligned}$$

We must verify that as $n \to \infty$, the terms above converge to the corresponding terms in the definition of the weak solution given in Definition 4.1.1. In particular, we need the strong convergence $u^{(n)} \to u$ in $L^2(0,T;L^2)$. It suffices to consider the convergence of the nonlinear terms. Let

$$\begin{aligned}
A &:= -\int_0^T \int_{\mathbb{R}^d} u \cdot \nabla (J_n \phi) u \, dx\, dt, \\
A^{(n)} &:= -\int_0^T \int_{\mathbb{R}^d} u^{(n)} \cdot \nabla (J_n \phi) u^{(n)} \, dx\, dt,
\end{aligned}$$

and consider the difference

$$\begin{aligned}A^{(n)} - A =& -\int_0^T \int_{\mathbb{R}^d} (u^{(n)} - u) \cdot \nabla (J_n \phi) u^{(n)} \, dx \, dt \\ &+ \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla (J_n \phi - \phi) u^{(n)} \, dx \, dt \\ &+ \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla \phi \cdot (u^{(n)} - u) \, dx \, dt \\ =& R_1 + R_2 + R_3.\end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned}|R_1| &\le \|u^{(n)} - u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\nabla J_n \phi\|_{L^\infty(\mathbb{R}^d \times [0,T])} \|u^{(n)}\|_{L^2(\mathbb{R}^d \times [0,T])} \\ &\le C \|u^{(n)} - u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\phi\|_{H^{2+\frac{d}{2}}} \|u_0\|_{L^2(\mathbb{R}^d \times [0,T])} \to 0 \text{ as } n \to \infty.\end{aligned}$$

Similarly,

$$\begin{aligned}|R_2| &\le \|u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\nabla (J_n \phi - \phi)\|_{L^\infty(\mathbb{R}^d \times [0,T])} \|u^{(n)}\|_{L^2(\mathbb{R}^d \times [0,T])} \\ &\le C \|u_0\|_{L^2} \|J_n \phi - \phi\|_{H^{2+\frac{d}{2}}} \|u_0\|_{L^2} \to 0 \text{ as } n \to \infty\end{aligned}$$

and, as $n \to \infty$,

$$|R_3| \le \|u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\nabla \phi\|_{L^\infty(\mathbb{R}^d \times [0,T])} \|u^{(n)} - u\|_{L^2(\mathbb{R}^d \times [0,T])} \to 0.$$

Therefore $|A^{(n)} - A| \to 0$ as $n \to \infty$. This shows that the first nonlinear term of the weak formulation of the approximate systems converges to the first nonlinear term of the weak solution formulation of (4.1).

The convergence of the other nonlinear term is slightly difference. We do not have strong convergence in $\theta^{(n)}$. Define

$$B := -\int_0^T \int_{\mathbb{R}^d} u \cdot \nabla(J_n\psi)\theta \, dx \, dt,$$
$$B^{(n)} := -\int_0^T \int_{\mathbb{R}^d} u^{(n)} \cdot \nabla(J_n\psi)\theta^{(n)} \, dx \, dt$$

and consider the difference

$$\begin{aligned} B^{(n)} - B = & -\int_0^T \int_{\mathbb{R}^d} (u^{(n)} - u) \cdot \nabla(J_n\psi)\theta^{(n)} \, dx \, dt \\ & + \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla(J_n\psi - \psi)\theta^{(n)} \, dx \, dt \\ & + \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla\psi \cdot (\theta^{(n)} - \theta) \, dx \, dt \\ = & W_1 + W_2 + W_3. \end{aligned}$$

Using Hölder's inequality, we have

$$\begin{aligned} |W_1| &\leq \|u^{(n)} - u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\nabla J_n\psi\|_{L^\infty(\mathbb{R}^d \times [0,T])} \|\theta^{(n)}\|_{L^2(\mathbb{R}^d \times [0,T])} \\ &\leq C\|u^{(n)} - u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\psi\|_{H^{2+\frac{d}{2}}} \|\theta_0\|_{L^2(\mathbb{R}^d \times [0,T])} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} |W_2| &\leq \|u\|_{L^2(\mathbb{R}^d \times [0,T])} \|\nabla(J_n\psi - \psi)\|_{L^\infty(\mathbb{R}^d \times [0,T])} \|\theta^{(n)}\|_{L^2(\mathbb{R}^d \times [0,T])} \\ &\leq C\|u_0\|_{L^2} \|J_n\psi - \psi\|_{H^{2+\frac{d}{2}}} \|\theta_0\|_{L^2} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Here we must estimate W_3 differently from R_3 since we do not have strong convergence in $\theta^{(n)}$. We can treat $u \cdot \nabla\psi$ as a test function since L^2 functions can be approximated by smooth functions with compact support. Since $\theta^{(n)}$ converges weakly to θ, we then have

$$W_3 \to 0 \quad \text{as } n \to \infty.$$

This shows that $|B^{(n)} - B| \to 0$ as $n \to \infty$. Therefore, the limit (u, θ) of the extracted subsequence of solutions to the approximate systems is indeed a weak solution. This completes the proof of Proposition 4.1.2.

APPENDICES

A.1 Sobolev Spaces

This appendix provides background information for readers unfamiliar with some or all of the definitions or notations for Sobolev Spaces and Besov Spaces.

Definition A.1.1. *We call the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index if each α_i is a non-negative integer. We denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ by x^α. We say $|\alpha|$ is the degree of α where*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Finally, the differential operator, D^α, of order $|\alpha|$ is just

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The notation $\beta \leq \alpha$ is used often which just means that if α, β are multi-indices then $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ for all $1 \leq j \leq n$.

Definition A.1.2. *Suppose $u, D^\alpha u \in L^1(\Omega)$. If*

$$\int_\Omega u(x) D^\alpha \phi(x)\ dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x) \phi(x)\ dx,$$

for all test functions $\phi \in C_0^\infty$ then we say $D^\alpha u$ is the weak partial derivative of u of order α.

Definition A.1.3. *We say X is embedded in Y, written $X \hookrightarrow Y$, if*

- $X \subset Y$
- *There exists an $M > 0$ such that $\|x\|_Y \leq M\|x\|_X$.*

Definition A.1.4 (Sobolev Space). *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. For $1 \leq p \leq \infty$ and every $m \in \mathbb{N}$ with $m \geq 1$ then the Sobolev space $W^{m,p}$ is defined as*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } 0 \leq |\alpha| \leq m\}$$

which is equipped with the norm

$$\|u\|_{W^{m,p}} = \left\{ \sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_p^p \right\}^{\frac{1}{p}}.$$

In particular, when $p = 2$ then $W^{m,p}(\Omega)$ is written as $H^m(\Omega)$. More generally, for any $s \geq 0$,

$$\|f\|_{H^s} = \left(\int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty.$$

H^s is called the inhomogeneous Sobolev space. We also consider the homogeneous Sobolev space $\mathring{H}^s$.

Definition A.1.5. *Let $s \geq 0$ and $\Omega \subset \mathbb{R}^n$. Then the homogeneous Sobolev space $\mathring{H}^s$ is defined as*

$$\mathring{H}^s(\Omega) = \left\{ f \in \mathcal{S}' : \widehat{f} \in L^1_{loc}(\Omega) \text{ and } \|f\|_{\mathring{H}^s} = \left(\int_\Omega |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty \right\},$$

where $\mathcal{S}'$ is the set of tempered distributions.

We may also define H^s in terms of localization operators which will allow us to define the space-time space $\widetilde{L}^1(0, T; \mathring{H}^{1+\frac{d}{2}})$ found in 4.1.3. In order to define $\mathring{H}^{1+\frac{d}{2}}$ this way, we must define a partition of unity and the Littlewood-Paley decomposition.

Let $\mathcal{S}$ denote the Schwartz space and $\mathcal{S}'$ its dual which is the space of tempered distributions. Let $\mathcal{S}_0$ denote the subspace of $\mathcal{S}$ defined by

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S} : \int_{R^d} x^\beta \phi(x)\, dx = 0, |\beta| = 0, 1, 2, \dots \right\}$$

and $\mathcal{S}_0'$ denotes the dual of $\mathcal{S}_0$.

Definition A.1.6 (Partition of unity). *There exist two functions ψ and ϕ where $\psi, \phi \in C_0^\infty(\mathbb{R}^d)$ with*

$$\psi(\xi) + \sum_{j=0}^{\infty} \phi(2^{-j}\xi) = 1$$

for $\xi \in \mathbb{R}^d$ and

$$\begin{aligned} supp\, \psi &\subset B(0, \tfrac{4}{3}), \\ supp\, \psi &\subset B(\tfrac{3}{4}, \tfrac{8}{3}), \end{aligned}$$

with $\psi \equiv 1$ on $B(0, \frac{3}{4})$. Here $B(x, r)$ denotes the ball of radius r centered at x.

Multiplying the partition of unity by a Fourier transform yields

$$\widehat{f}(\xi)\psi(\xi) + \sum_{j=0}^{\infty} \phi(2^{-j}\xi)\widehat{f}(\xi) = \widehat{f}(\xi), \quad \xi \in \mathbb{R}^d.$$

Define

$$\begin{aligned} \widehat{\Delta_{-1} f}(\xi) &= \psi(\xi)\widehat{f}(\xi), \\ \widehat{\Delta_j f}(\xi) &= \phi(2^{-j}\xi)\widehat{f}(\xi). \end{aligned}$$

So

$$\Delta_{-1} f + \sum_{j=0}^{\infty} \Delta_j f = f.$$

Thus the inhomogeneous Littlewood-Paley decomposition can be written as

$$f = \sum_{j=-1}^{\infty} \Delta_j f, \quad f \in \mathcal{S}'.$$

Definition A.1.7. *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^d)$ consists of $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{sj}\|\Delta_j f\|_{L^p(\mathbb{R}^d)}\|_{l^q} < \infty.$$

For the homogeneous Besov space, we use the homogeneous Littlewood-Paley decomposition.

$$\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

Then

$$\widehat{f}(\xi) = \sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi)\widehat{f}(\xi) \quad \text{for } \widehat{f}(0) = 0.$$

Define $\widehat{\mathring{\Delta}_j f} = \phi(2^{-j}\xi)\widehat{f}(\xi)$. Then

$$\widehat{f}(\xi) = \sum_{j=-\infty}^{\infty} \widehat{\mathring{\Delta}_j f}.$$

So the homogeneous Littlewood-Paley decomposition can be written as

$$f = \sum_{j=-\infty}^{\infty} \mathring{\Delta}_j f, \quad f \in \mathcal{S}'_0.$$

Definition A.1.8. *For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}(\mathbb{R}^d)$ consists of $f \in \mathcal{S}'_0$ satisfying*

$$\|f\|_{\mathring{B}^s_{p,q}} \equiv \|2^{sj}\|\mathring{\Delta}_j f\|_{L^p(\mathbb{R}^d)}\|_{l^q} < \infty.$$

In particular, $B^s_{2,2} = H^s$ *and* $\mathring{B}^2_{2,2} = \mathring{H}^s$.

This allows us to now define the space-time space $\widetilde{L}^1(0,T;\mathring{H}^{1+\frac{d}{2}})$ used in Theorem 4.1.3. $\widetilde{L}^1(0,T;\mathring{H}^{1+\frac{d}{2}})$ is defined through the norm

$$\|f\|_{\widetilde{L}^1(0,T;\mathring{H}^{1+\frac{d}{2}})} \equiv \|2^{j(1+\frac{d}{2})}\|\mathring{\Delta}_j f\|_{L^1(0,T;L^p)}\|_{l^q}.$$

A.2 Basic Calculus and Functional Analysis Results

Lemma A.2.1 (Hölder's inequality). *Suppose* $f, g : \mathbb{R}^n \to \mathbb{R}$ *are Lebesgue measurable. For* $1 \le p \le q \le \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|fg\|_{L^1} \le \|f\|_{L^p}\|g\|_{L^q}.$$

Lemma A.2.2 (Minkowski's Inequality for Integrals). *Suppose* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is Lebesgue measurable. For* $1 \le p \le \infty$,

$$\left(\int \left|\int f(x,y)\,dy\right|\,dx\right)^{\frac{1}{p}} \le \int \left(\int |f(x,y)|^p\,dx\right)^{\frac{1}{p}} dy.$$

Lemma A.2.3 (Gagliardo-Nirenberg-Sobolev interpolation inequality). *Assume* $\Omega = \mathbb{R}^d$ *with* $1 \le p, q, r \le \infty$ *and* $l < m$ *where* $l, m \in \mathbb{N}$. *Then*

$$\|D^l f\|_{L^p} \le C\|f\|^a_{L^q}\|D^m f\|^{1-a}_{L^r},$$

where

$$\frac{1}{p} - \frac{l}{d} = a \cdot \frac{1}{q} + (1-a)\left(\frac{1}{r} - \frac{m}{d}\right),$$

for $a \in [0,1]$ *and* $l \le m(1-a)$. *In the case when* $q = \infty$ *and* $l = 0$, *then* $f \to 0$ *as* $|x| \to \infty$ *or* $f \in L^b(\mathbb{R}^d)$ *for some* $b \ge 1$. *In the case where* $\frac{d}{r}$ *is an integer, then* $a \ne 0$.

Lemma A.2.4 (Picard Existence and Uniqueness Theorem). *Let* E *be a Banach* *space. Let $O \subseteq E$ be an open subset. Let $F : O \longrightarrow E$ be a locally Lipschitz map. More precisely, for any $y \in O$, there is a neighborhood of y (denoted by $U(y)$) and $L = L(y, U)$ such that*

$$\|F(y) - F(z)\|_E \leq L\|y - z\|_E, \quad \forall z \in U(y).$$

Then, for any $y_0 \in O$, the ODE

$$\begin{cases} \frac{dy}{dt} = F(y), \\ y|_{t=0} = y_0 \in O. \end{cases} \tag{11}$$

has a unique local solution, namely, there is $T > 0$ and a unique solution $y = y(t)$ satisfying $y \in C^1(0, T; O)$.

Lemma A.2.5 (**Picard Extension Theorem**). *Assume the conditions in Lemma A.2.4 hold and Let $y = y(t)$ be the local solution. Then either $y(t)$ is global in time, namely, $T = \infty$, or for a finite $T_0 > 0$, $\lim_{t \to T_0} y(t) \notin O$.*

Lemma A.2.6 (**Hodge decomposition in $\mathbb{R}^d$**). *For every $v \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, there exist a unique w and p satisfying*

$$v = w + \nabla p, \quad \nabla \cdot w = 0,$$

and $w \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, $\nabla p \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$, and $\|v\|_{L^2}^2 = \|w\|_{L^2}^2 + \|\nabla p\|_{L^2}^2$.

There is a special consequence of Lemma A.2.6, which in order to state, we much introduce the following notation. $\widehat{f}(\xi)$ represents the Fourier transform of f,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx.$$

For a positive integer n, we denote by $B(0, n)$ the ball centered at the origin with radius n, and define

$$\widehat{J_n f}(\xi) = \chi_{B(0,n)}(\xi)\, \widehat{f}(\xi)$$

In addition, we write

$$L^2_n = \{f \in L^2(\mathbb{R}^d) : supp\ \widehat{f} \subset B(0,n)\},$$
$$L^2_{n,\sigma} = \{f \in L^2_n(\mathbb{R}^d) : \nabla \cdot f = 0\}.$$

Now we can state the special consequence of Lemma A.2.6.

Corollary A.2.7. *There exists a linear bounded operator* $\mathbb{P} : L^2_n \to L^2_{n,\sigma}$ *satisfying:*

- *For any* $f \in L^2_n$, $\|\mathbb{P}f\|_{L^2} \leq \|f\|_{L^2}$.
- *For any* $f \in L^2_{n,\sigma}$, $\mathbb{P}f = f$. *Especially, for any* $f \in L^2_n$, $\mathbb{P}^2 f = \mathbb{P}f$.

In addition, we will also need the following Aubin-Lions compactness Lemma.

Lemma A.2.8 (Aubin-Lions). *Let* $X_1 \hookrightarrow X_2 \hookrightarrow X_3$ *be three Banach spaces with the first embedding being compact and the second being continuous. Let* $T > 0$. *For* $1 \leq p, q \leq +\infty$, *let*

$$W = \{u \in L^p(0,T;X_1), \partial_t u \in L^q(0,T;X_3)\}.$$

Then,

(i). If $p < +\infty$, *then the embedding of* W *into* $L^p(0,T;X_2)$ *is compact;*

(ii). If $p = +\infty$ *and* $q > 1$, *then the embedding of* W *into* $C(0,T;X_2)$ *is compact.*

Lemma A.2.8 states that any bounded sequence in W has a convergent subsequence in $L^p(0,T;X_2)$.

A.3 Derivation of 1D Transformations of the 2D Ideal MHD

Recall (3.1)

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = b \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \cdot b = 0. \end{cases} \tag{12}$$

Applying $\nabla\times$ to the equations above gives the equations for vorticity $\omega = \nabla \times u$ and current density $j = \nabla \times b$

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j = b \cdot \nabla w + Q(u,b), \end{cases} \tag{13}$$

where

$$Q(u,b) = 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2). \tag{14}$$

Here we consider a special class of scale-invariant solutions. We consider

$$\begin{cases} \omega(x,t) = g(\theta,t), \\ j(x,t) = h(\theta,t), \\ \phi(x,t) = r^2 G(\theta,t), \\ \psi(x,t) = r^2 H(\theta,t), \end{cases} \tag{15}$$

where (r,θ) is the associated polar coordinates of x

$$x_1 = r\cos\theta,$$

$$x_2 = r\sin\theta,$$

or

$$r^2 = x_1^2 + x_2^2,$$
$$\tan\theta = \frac{x_2}{x_1},$$

and ϕ and ψ are the stream functions associated with u and b, respectively, with $u = \nabla^{\perp}\phi$ and $b = \nabla^{\perp}\psi$.

We now justify the use of the ansatz (15). Consider that if (u, b) solves (3.1) then

$$\begin{cases} u_\lambda(x,t) = \frac{1}{\lambda}u(t,\lambda x), \\ b_\lambda(x,t) = \frac{1}{\lambda}b(t,\lambda x), \\ p_\lambda(x,t) = \frac{1}{\lambda^2}p(t,\lambda x), \end{cases}$$

will also solve (3.1).

Similarly, if (ω, j) solves (3.3) then

$$\begin{cases} \omega_\lambda(x,t) = \omega(\lambda x, t), \\ j_\lambda(x,t) = j(\lambda x, t), \end{cases}$$

will also solve (3.3).

We make the ansatz that the vorticity and the current density are radially homogeneous with degree zero, i.e. using polar coordinates,

$$\begin{cases} \omega(r,\theta) = g(\theta), \\ j(r,\theta) = h(\theta). \end{cases}$$

Writing the stream functions ϕ and ψ associated with ω and j, respectively, we have that

$$\Delta\phi = \omega, \quad \Delta\psi = j.$$

So if

$$\phi_\lambda = \frac{1}{\lambda^2}\phi(\lambda x, t), \text{ and } \psi_\lambda = \frac{1}{\lambda^2}\psi(\lambda x, t),$$

then

$$\Delta\phi_\lambda = \omega_\lambda, \text{ and } \Delta\psi_\lambda = j_\lambda.$$

Now we take $\lambda = \frac{1}{r}$. Then

$$\lambda x = \frac{1}{r}x = (\cos\theta, \sin\theta).$$

Thus

$$\begin{cases} \omega_\lambda(x,t) = \omega(\cos\theta, \sin\theta, t) = g(\theta, t), \\ j_\lambda(x,t) = j(\cos\theta, \sin\theta, t) = h(\theta, t), \\ \phi_\lambda(x,t) = r^2\phi(\cos\theta, \sin\theta, t) = r^2 G(\theta, t), \\ \psi_\lambda(x,t) = r^2\psi(\cos\theta, \sin\theta, t) = r^2 H(\theta, t). \end{cases}$$

This justifies our use of the ansatz of the special class of scale invariant solutions (15).

Next, we derive the equations for g, h, G and H. Recall

$$u = \nabla^\perp\phi = (-\partial_2\phi, \partial_1\phi),$$
$$b = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi),$$

From this we have that

$$\partial_1\theta = -\frac{\sin\theta}{r}, \qquad \partial_1 r = \cos\theta,$$
$$\partial_2\theta = \frac{\cos\theta}{r}, \qquad \partial_2 r = \sin\theta.$$

Using the ansatz and differentiating ϕ with respect to x, we have

$$\begin{aligned}\partial_2\phi &= \partial_2(r^2G(\theta,t)) \\ &= 2r\partial_2 rG(\theta,t) + r^2\partial_\theta G\partial_2\theta \\ &= 2r\sin\theta G(\theta,t) + r^2\partial_\theta G\cdot\left(\frac{\cos\theta}{r}\right) \\ &= 2r\sin\theta G(\theta,t) + r\cos\theta\partial_\theta G(\theta,t),\end{aligned}$$

and,

$$\begin{aligned}\partial_1\phi &= \partial_1(r^2G(\theta,t)) \\ &= 2r\partial_1 rG(\theta,t) + r^2\partial_\theta G\partial_1\theta \\ &= 2r\cos\theta G(\theta,t) + r^2\partial_\theta G\cdot\left(\frac{-\sin\theta}{r}\right) \\ &= 2r\cos\theta G(\theta,t) - r\sin\theta\partial_\theta G(\theta,t).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\partial_2\psi &= 2r\sin\theta H(\theta,t) + r\cos\theta\partial_\theta H(\theta,t), \\ \partial_1\psi &= 2r\cos\theta H(\theta,t) - r\sin\theta\partial_\theta H(\theta,t).\end{aligned}$$

From this, we may write

$$u = \left(-2r\sin\theta G(\theta,t) - r\cos\theta\partial_\theta G(\theta,t), 2r\cos\theta G(\theta,t) - r\sin\theta\partial_\theta G(\theta,t)\right), \tag{16}$$

$$b = \left(-2r\sin\theta H(\theta,t) - r\cos\theta\partial_\theta H(\theta,t), 2r\cos\theta H(\theta,t) - r\sin\theta\partial_\theta H(\theta,t)\right). \tag{17}$$

Differentiating the vorticity, ω, in space we have

$$\partial_1\omega = \partial_1(g(\theta,t)) = \partial_\theta g\partial_1\theta = \partial_\theta g\cdot\left(\frac{-\sin\theta}{r}\right),$$

and

$$\partial_2 \omega = \partial_2(g(\theta, t)) = \partial_\theta g \partial_2 \theta = \partial_\theta g \cdot \left(\frac{\cos\theta}{r}\right).$$

Similarly, differentiating the current density, j, in space we have

$$\partial_1 j = \partial_1(h(\theta, t)) = \partial_\theta h \partial_1 \theta = \partial_\theta h \cdot \left(\frac{-\sin\theta}{r}\right),$$

and

$$\partial_2 j = \partial_2(h(\theta, t)) = \partial_\theta h \partial_2 \theta = \partial_\theta h \cdot \left(\frac{\cos\theta}{r}\right).$$

Then the vorticity equation of (3.3) becomes

$$\begin{aligned}
\partial_t g + (-2r\sin\theta G(\theta,t) - r\cos\theta\partial_\theta G(\theta,t))\partial_\theta g \cdot \left(\frac{-\sin\theta}{r}\right) \\
+ \left(2r\cos\theta G(\theta,t) - r\sin\theta\partial_\theta G(\theta,t)\right)\partial_\theta g \cdot \left(\frac{\cos\theta}{r}\right) \\
= (-2r\sin\theta H(\theta,t) - r\cos\theta\partial_\theta H(\theta,t))\partial_\theta h \cdot \left(\frac{-\sin\theta}{r}\right) \\
+ (2r\cos\theta H(\theta,t) - r\sin\theta\partial_\theta H(\theta,t))\partial_\theta h \cdot \left(\frac{\cos\theta}{r}\right)
\end{aligned}$$

After simplifying, we obtain

$$\partial_t g + 2G\partial_\theta g = 2H\partial_\theta h. \tag{18}$$

In order to rewrite the current density equation of (3.3), we must rewrite $Q(u, b)$ using our ansatz. We begin by finding the partial derivatives of b found in $Q(u, b)$ which are

$$\begin{aligned}
\partial_1 b_1 =& \partial_1(-2r\sin\theta H(\theta,t) - r\cos\theta\partial_\theta H(\theta,t)) \\
=& -2\partial_1 r\sin\theta H - 2r\cos\theta\partial_1\theta H - 2r\sin\theta\partial_\theta H\partial_1\theta \\
& - \partial_1 r\cos\theta\partial_\theta H + r\sin\theta\partial_1\theta\partial_\theta H - r\cos\theta\partial_{\theta\theta}H\partial_1\theta
\end{aligned}$$

$$
\begin{aligned}
&= -2\cos\theta\sin\theta H - 2r\cos\theta\left(-\frac{\sin\theta}{r}\right)H - 2r\sin\theta\partial_\theta H\left(-\frac{\sin\theta}{r}\right)\\
&\quad -\cos\theta\cos\theta\partial_\theta H + r\sin\theta\left(-\frac{\sin\theta}{r}\right)\partial_\theta H - r\cos\theta\partial_{\theta\theta}H\left(-\frac{\sin\theta}{r}\right)\\
&=\sin^2\theta\partial_\theta H - \cos^2\theta\partial_\theta H + \sin\theta\cos\theta\partial_{\theta\theta}H,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_1 b_2 =&\partial_1(2r\cos\theta H(\theta,t) - r\sin\theta\partial_\theta H(\theta,t))\\
=&2\partial_1 r\cos\theta H - 2r\sin\theta\partial_1\theta H + 2r\cos\theta\partial_\theta H\partial_1\theta\\
&-\partial_1 r\sin\theta\partial_\theta H - r\cos\theta\partial_1\theta\partial_\theta H - r\sin\theta\partial_{\theta\theta}H\partial_1\theta\\
=&2\cos^2\theta H - 2r\sin\theta\left(-\frac{\sin\theta}{r}\right)H + 2r\cos\theta\partial_\theta H\left(-\frac{\sin\theta}{r}\right)\\
&-\sin\theta\cos\theta\partial_\theta H - r\cos\theta\left(-\frac{\sin\theta}{r}\right)\partial_\theta H - r\sin\theta\partial_{\theta\theta}H\left(-\frac{\sin\theta}{r}\right)\\
=&2H - 2\sin\theta\cos\theta\partial_\theta H + \sin^2\theta\partial_{\theta\theta}H,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_2 b_1 =&\partial_2(-2r\sin\theta H(\theta,t) - r\cos\theta\partial_\theta H(\theta,t))\\
=&-2\partial_2 r\sin\theta H - 2r\cos\theta\partial_2\theta H - 2r\sin\theta\partial_\theta H\partial_2\theta\\
&-\partial_2 r\cos\theta\partial_\theta H + r\sin\theta\partial_2\theta\partial_\theta H - r\cos\theta\partial_{\theta\theta}H\partial_2\theta\\
=&-2\sin^2\theta H - 2r\cos\theta\left(\frac{\cos\theta}{r}\right)H - 2r\sin\theta\partial_\theta H\left(\frac{\cos\theta}{r}\right)\\
&-\sin\theta\cos\theta\partial_\theta H + r\sin\theta\left(\frac{\cos\theta}{r}\right)\partial_\theta H - r\cos\theta\partial_{\theta\theta}H\left(\frac{\cos\theta}{r}\right)\\
=&-2H - 2\sin\theta\cos\theta\partial_\theta H - \cos^2\theta\partial_{\theta\theta}H.
\end{aligned}
$$

Similarly, we find the partial derivatives of u found in $Q(u,b)$ which are,

$$\begin{aligned}
\partial_1 u_1 =& \partial_1(-2r\sin\theta G(\theta,t) - r\cos\theta\partial_\theta G(\theta,t)) \\
=& -2\partial_1 r\sin\theta G - 2r\cos\theta\partial_1\theta G - 2r\sin\theta\partial_\theta G\partial_1\theta \\
& -\partial_1 r\cos\theta\partial_\theta G + r\sin\theta\partial_1\theta\partial_\theta G - r\cos\theta\partial_{\theta\theta}G\partial_1\theta \\
=& -2\cos\theta\sin\theta G - 2r\cos\theta\left(-\frac{\sin\theta}{r}\right)G - 2r\sin\theta\partial_\theta G\left(-\frac{\sin\theta}{r}\right) \\
& -\cos^2\theta\partial_\theta G + r\sin\theta\left(-\frac{\sin\theta}{r}\right)\partial_\theta G - r\cos\theta\partial_{\theta\theta}G\left(-\frac{\sin\theta}{r}\right) \\
=& \sin^2\theta\partial_\theta G - \cos^2\theta\partial_\theta G + \sin\theta\cos\theta\partial_{\theta\theta}G,
\end{aligned}$$

and

$$\begin{aligned}
\partial_1 u_2 =& \partial_1(2r\cos\theta G(\theta,t) - r\sin\theta\partial_\theta G(\theta,t)) \\
=& 2\partial_1 r\cos\theta G - 2r\sin\theta\partial_1\theta G + 2r\cos\theta\partial_\theta G\partial_1\theta \\
& -\partial_1 r\sin\theta\partial_\theta G - r\cos\theta\partial_1\theta\partial_\theta G - r\sin\theta\partial_{\theta\theta}G\partial_1\theta \\
=& 2\cos^2\theta G - 2r\sin\theta\left(-\frac{\sin\theta}{r}\right)G + 2r\cos\theta\partial_\theta G\left(-\frac{\sin\theta}{r}\right) \\
& -\sin\theta\cos\theta\partial_\theta G - r\cos\theta\left(-\frac{\sin\theta}{r}\right)\partial_\theta G - r\sin\theta\partial_{\theta\theta}G\left(-\frac{\sin\theta}{r}\right) \\
=& 2G - 2\sin\theta\cos\theta\partial_\theta G + \sin^2\theta\partial_{\theta\theta}G,
\end{aligned}$$

and

$$\begin{aligned}
\partial_2 u_1 =& \partial_2(-2r\sin\theta G(\theta,t) - r\cos\theta\partial_\theta G(\theta,t)) \\
=& -2\partial_2 r\sin\theta G - 2r\cos\theta\partial_2\theta G - 2r\sin\theta\partial_\theta G\partial_2\theta \\
& -\partial_2 r\cos\theta\partial_\theta G + r\sin\theta\partial_2\theta\partial_\theta G - r\cos\theta\partial_{\theta\theta}G\partial_2\theta \\
=& -2\sin^2\theta G - 2r\cos\theta\left(\frac{\cos\theta}{r}\right)G - 2r\sin\theta\partial_\theta G\left(\frac{\cos\theta}{r}\right) \\
& -\sin\theta\cos\theta\partial_\theta G + r\sin\theta\left(\frac{\cos\theta}{r}\right)\partial_\theta G - r\cos\theta\partial_{\theta\theta}G\left(\frac{\cos\theta}{r}\right)
\end{aligned}$$

$$= -2G - 2\sin\theta\cos\theta\partial_\theta G - \cos^2\theta\partial_{\theta\theta}G.$$

Using these, we find that

$$\begin{aligned} Q(u,b) &= 2\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - 2\partial_1 u_1(\partial_2 b_1 + \partial_1 b_2) \\ &= -2\partial_\theta G\partial_{\theta\theta}H + 2\partial_\theta H\partial_{\theta\theta}G. \end{aligned}$$

So the current density equation of (3.3) becomes

$$\begin{aligned} \partial_t h &+ (-2r\sin\theta G(\theta,t) - r\cos\theta\partial_\theta G(\theta,t))\partial_\theta h\cdot\left(\frac{-\sin\theta}{r}\right) \\ &+ (2r\cos\theta G(\theta,t) - r\sin\theta\partial_\theta G(\theta,t))\,\partial_\theta h\cdot\left(\frac{\cos\theta}{r}\right) \\ &= (-2r\sin\theta H(\theta,t) - r\cos\theta\partial_\theta H(\theta,t))\partial_\theta g\cdot\left(\frac{-\sin\theta}{r}\right) \\ &+ (2r\cos\theta H(\theta,t) - r\sin\theta\partial_\theta H(\theta,t))\partial_\theta g\cdot\left(\frac{\cos\theta}{r}\right) \\ &- 2\partial_\theta G\partial_{\theta\theta}H + 2\partial_\theta H\partial_{\theta\theta}G. \end{aligned}$$

This reduces to

$$\partial_t h + 2G\partial_\theta h = 2H\partial_\theta g + -2\partial_\theta G\partial_{\theta\theta}H + 2\partial_\theta H\partial_{\theta\theta}G.$$

Then, writing out the equation $\Delta\psi = \omega$ using our ansatz we have

$$\Delta(r^2 G(\theta,t)) = g(\theta,t).$$

From this we have

$$\begin{aligned} \Delta(r^2G) &= (\partial_r^2(r^2) + \frac{1}{r}\partial_r(r^2) + \frac{1}{r^2}(0))G + (\partial_r^2(0) + \frac{1}{4}\partial_r(0) + \frac{1}{r^2}\partial_{\theta\theta}G)r^2 \\ &= 4G + \partial_{\theta\theta}G. \end{aligned}$$

Similarly, writing out the equation $\Delta\phi = j$ using our ansatz we have

$$\Delta(r^2 H) = 4H + \partial_{\theta\theta} H.$$

In summary, (g, h, G, H) satisfies

$$\begin{cases} \partial_t g + 2G\partial_\theta g = 2H\partial_\theta h, \\ \partial_t h + 2G\partial_\theta h = 2H\partial_\theta g + -2\partial_\theta G\partial_{\theta\theta} H + 2\partial_\theta H\partial_{\theta\theta} G, \\ \partial_{\theta\theta} G + 4G = g, \\ \partial_{\theta\theta} H + 4H = h, \\ g(\theta, 0) = g_0(\theta), \quad h(\theta, 0) = h_0(\theta). \end{cases} \tag{19}$$

We can also consider the vorticity and stream function formulation of the 2D ideal MHD

$$\begin{cases} \partial_t \omega + u \cdot \nabla\omega = b \cdot \nabla j, \\ \partial_t \psi + u \cdot \nabla\psi = 0, \end{cases} \tag{20}$$

where ψ is the stream function given by $b = \nabla^\perp \psi$. Then the 1D system corresponding to this system would be

$$\begin{cases} \partial_t g + 2G\partial_\theta g = 2H\partial_\theta h, \\ \partial_t H + 2G\partial_\theta H = 2H\partial_\theta G, \\ \partial_{\theta\theta} G + 4G = g, \\ \partial_{\theta\theta} H + 4H = h, \\ g(\theta, 0) = g_0(\theta), \quad H(\theta, 0) = H_0(\theta). \end{cases} \tag{21}$$

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