AN INTRODUCTION TO ANALYSIS
INFINITE SEQUENCES
AND SERIES

By

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CHAPTER I

INTRODUCTION

Much discussion has taken place concerning the content of the senior year of high school mathematics. It has been strongly argued that this is the time for a course in calculus. Whether or not calculus belongs in the high school mathematics curriculum is a subject of wide debate. A good calculus course might be the proper culmination to the elementary mathematics studied in high school. But again, there is disagreement in the mathematical community as to what constitutes a "good calculus course". Many people assume that calculus is chiefly concerned with differentiation and integration, but this is a restricted point of view. The essential idea in calculus is that of a limit, and without a clear understanding of a limit any calculus course is a failure. Too often a course in calculus is begun with an off-hand reference to limits as something too hard for the students to understand. Then on the other hand, some courses begin with a full epsilon-delta treatment. Either of these approaches is avoiding the problem, and that is to teach the concept of a limit with real meaning.

The author firmly believes that the proper culmination to the elementary mathematics studied in high school is an introduction to analysis rather than a manipulative approach to calculus. Some of the techniques of calculus are easy to teach without developing understanding. The student gets a false idea about calculus and then has more difficulty in a formal course attempting to develop understanding. It therefore seems desirable to introduce the ideas of analysis in some setting in which manipulation plays a subordinate role.

The author has developed in this paper an introduction to analysis based on a discussion of infinite sequences and series. The student is already familar with the concept of a finite sequence and has worked with arithmetic progressions and geometric progressions. The study of infinite sequences is a natural extension of these basic concepts already studied in algebra. The variety of behavior which can be studied in examples of sequences is interesting and challenging to the student.

Since a sequence is a function with domain the positive integers, the study of sequences will reinforce and strengthen the student's concepts of a function, graph of a function, and image values of a function plotted on a number line. At the same time new ideas are introduced such as a bounded function, monotone function, least upper bound and greatest lower bound of a set of function values.

Since the metric on the real line is the absolute value, the student is immediately involved with absolute values and inequalities. These he has already studied, but he has had little opportunity to use them. In addition the student comes in contact for the first time with inequalities that are subject to restrictions which must always be stated. The student who can use such inequalities meaningfully is well on the way to understanding the basic ideas of analysis.

In this development the null sequence plays a basic role. This type of sequence is studied by itself and then is used to define the limit of a sequence. The purpose of using this approach is to keep the algebraic operations as simple as possible. Thus the idea of a limit is clarified and not camouflaged with manipulations. The basic limit theorems are proved first for null sequences and then extended to the general case. This repetition reinforces the ideas involved and incidentally simplifies the technical aspects of the proofs.

In Chapter IV, sequences are represented as series. The author has chosen to begin with a sequence and to use differences to obtain the general term of the series. This way the major emphasis is placed as it should be on the sequence. Sometimes this concept is approached through addition. This makes the student feel that he is simply adding infinitely many terms which gives him a false idea about the binary operation addition.

The student is familar with the fact that every rational number can be represented by a terminating or by a repeating decimal, and conversely. Justifications are given for the algorithm used in elementary mathematics to find a rational number from a given repeating decimal.

The student has studied about theorems, the converse of a theorem, the inverse of a theorem, and the contrapositive of a theorem. This information is applied in the study of tests for convergence discussed in this paper. Here also are theorems which state necessary and sufficient conditions, some which state necessary conditions, and some which state only sufficient conditions. In this context the student gains useful experience in logical distinctions and has reinforced an appreciation of the importance of careful analysis of the exact information contained in a theorem.

In the final chapter the student is introduced to power series as a generalization of the idea of a polynomial with which the student is already familar. Some elementary functions are expressed as power series such as sin x and cos x. The student is familar with these functions but has associated them with angle measures either in degrees or radians, and has derived their properties by geometric arguments. In this discussion several identities involving these functions are obtained by using the power series representations. It is important for the student to realize that these properties and identities are inherent in the nature of the function and independent of the method by which it is defined.

Proofs in analysis are generally written in paragraph form with many algebraic steps omitted. The author has chosen to use the double column type proof in this introduction to analysis to emphasize the individual steps involved and the reasons for each step. This is the same method used in high school when the student begins the study of a new topic. Once the student thoroughly understands the basic concepts of a new idea, material written in paragraph form will be easier to read and comprehend. Many of the basic definitions have been written symbolically as well as verbally. This serves a two-fold purpose. It acquaints the student with the usual shorthand notation of analysis and, more important, it requires the student to distinguish precisely between the situation in which the truth set is nonempty and the situation in which the truth set is the universal set.

The minimum requirement for understanding the ideas presented in this paper is the completion of the nine-point program as established by the Commission on Mathematics of the College Entrance Examination

Board. A student who has completed the above minimum requirement and understands the ideas presented in this paper should have very little difficulty with any course in calculus, even the most formal and rigorous.

In order to maintain contact with the high school point of view, the author enlisted the help of Ted Tatchio during the summer between his junior and senior year at C. E. Donart High School. The discussions with him helped to clarify some of the aims of the paper and to point out some useful directions. It also encouraged the author in the belief that this material is both accessible and interesting to the good high school student.

Although the author has written this paper with the high school student specifically in mind, the approach would be equally good and the material just as useful for a pre-calculus course at a junior college or at a university. It would also prove useful for outside reading for students who are having difficulty with proofs in a course in advanced calculus or elementary functions. Another use for this material might be a summer institute for accelerated high school juniors.

CHAPTER II

SEQUENCES

A function is a correspondence between the elements of two sets, its domain and its range, such that to each element of the domain there corresponds one and only one element of the range. Sometimes a more formal definition of a function is used as follows: a function is a nonempty set of ordered pairs no two of which have the same first coordinate. If the domain of a function is the positive integers, or a subset of the integers, the function is called a sequence. If f denotes the function, then the function value f(n) is called the nth term of the sequence. Sometimes if the function is a sequence the usual functional notation is departed from and the value of the function at n is denoted by a_n, i.e. $f(n) = a_n$. The subscript n of a_n is called the index of the term a_n.

Definition 2.1: (Finite Sequence) A finite sequence, or n-tuple, is a function whose domain is the finite set of positive integers $\{1, 2, 3, \ldots, n\}$.

Notation: Ordered n-tuples or finite sequences will be denoted by

$$< a_i >_{i=1}^{n}$$

or $a_1, a_2, a_3, \ldots, a_n$.

Example 2.1: $f(n) = 1/n$ for $n = 1, 2, 3, 4, 5$ or

$$< 1/n >_{n=1}^{5}$$

The domain is $\{1, 2, 3, 4, 5\}$, and the range is $\{1, 1/2, 1/3, 1/4, 1/5\}$. The function $f = \{(1, 1), (2, 1/2), (3, 1/3), (4, 1/4), (5, 1/5)\}$. Sometimes the finite sequence is written as $1, 1/2, 1/3, 1/4, 1/5$.

Definition 2.2: (Infinite sequence) An infinite sequence is a function whose domain is the set of all positive integers $\{1, 2, 3, \ldots, n, \ldots\}$.

Notation: Infinite sequences will be denoted by

$$< a_n >_{n=1}^{\infty},$$

or simply $< a_n >$ or $a_1, a_2, a_3, \ldots, a_n, \ldots$. The range of the sequence $< a_n >$ will be denoted by $\{a_n\}$.

Example 2.2: $f(n) = 1/n$, $n = 1, 2, 3, \ldots, n, \ldots$ or $< 1/n >$ The domain is $\{1, 2, 3, \ldots, n, \ldots\}$, and the range is $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$. The function

$$f = \{ (1, 1), (2, 1/2), (3, 1/3), \ldots, (n, 1/n), \ldots\}.$$

Thus this function is a sequence, and can be written as

$$< 1/n >_{n=1}^{\infty}$$

or $<1/n>$ or $1, 1/2, 1/3, \ldots, 1/n, \ldots$ where the order in which the terms are written implies the element of the domain with which each term is paired. Since it is impossible to write all the terms, the ellipsis ... is used to indicate "continue in like manner".

The range of a function may be a quite general set. It is useful to consider sequences where the range is a set of complex numbers, or a set of n-tuples, or even a set of functions. Unless otherwise specified only infinite sequences of real numbers will be considered in this paper. . This restriction of the range to real numbers seems desirable since the reader is already familiar with the real number system. The general properties of sequences can be developed when the range consists of real numbers. The techniques used in this development can be applied to more general situations.

The graph of a sequence is simply the graph of a function and is found by plotting $\{(n, a_n) \mid n \text{ is a positive integer}\}$. Consider Example 2. 2 as graphed in Figure 2. 1.

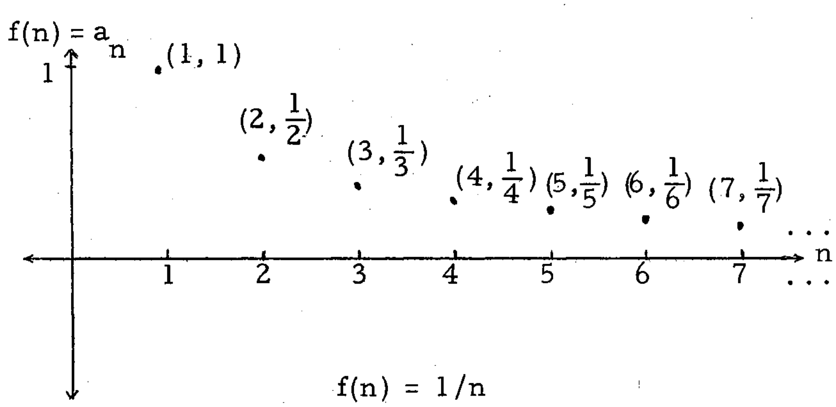
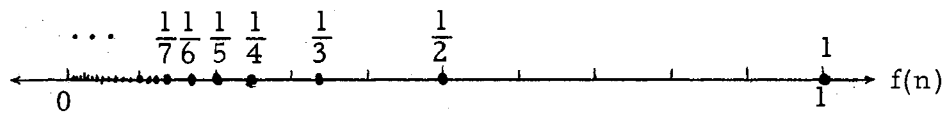


Figure 2. 1

The geometric properties of a particular sequence can sometimes be exhibited more easily by plotting the range or image of the sequence rather than the graph. In this case the function values appear as points on a real number line. Consider Example 2.2 as presented in Figure 2.2.



$f(n) = 1/n$

Figure 2.2

Two functions f and g are equal if and only if they have the same domain D and $f(x) = g(x)$ for all x belonging to D. Since the domain for all infinite sequences is the same, the definition for equality can be rephrased in a slightly simpler form as follows:

<u>Definition 2.3:</u> (Equality) Let $< a_n >$ and $< b_n >$ be two sequences. These sequences are equal if and only if $a_n = b_n$ for every positive integer n.

Certain mathematical concepts can be written more clearly with symbols used for a word or a group of words. The use of symbols also encourages precise language. Thus one goal in the early stages of the study of mathematics is to learn these symbols and their usages in mathematical sentences. Mathematical vocabulary and mathematical grammar become very important steppingstones to a mature approach to the study of mathematics. As the theory of sequences and series is developed, notation will be introduced and explained. From that point on this notation will be used without further discussion. Appendix I contains a list of the notation symbols and the page number on which the notation is introduced.

Notation: iff means "if and only if" and $\forall$ means "for every".

The words "if and only if" occur quite frequently in mathematics. A definition is an "if and only if" statement. Some theorems are "if and only if" statements. Hence it is convenient to write "iff" for these four words. It is important to keep clearly in mind whether a statement is true in all cases or in some cases. For example, $n^2 - 4 = 0$ is true only if $n = 2$ or $n = -2$, but $n^2 - 4 = (n+2)(n-2)$ is true for every n. Hence the symbol "$\forall$" is convenient to use when a statement is true in all cases.

Definition 2.3': (Equality) Let $< a_n >$ and $< b_n >$ be two sequences. These sequences are equal iff $a_n = b_n$, $\forall$ n, such that n is a positive integer.

Example 2.3: On intelligence tests, one might find a problem such as the following: Find the next term of the sequence 1, 1/2, 1/3, ... The answer most people will give is 1/4 since it seems natural that the function should be $f(n) = 1/n$ for each positive integer n. The above question, in a sense, is not a fair one since $f(n) = 1/(n^3 - 6n^2 + 12n - 6)$ is also a possible function.

For $n = 1$, $1/(1^3 - 6 \cdot 1^2 + 12 \cdot 1 - 6) = 1$.

For $n = 2$, $1/(2^3 - 6 \cdot 2^2 + 12 \cdot 2 - 6) = 1/2$.

For $n = 3$, $1/(3^3 - 6 \cdot 3^2 + 12 \cdot 3 - 6) = 1/3$.

Thus the first three terms are 1, 1/2, 1/3; but the fourth term is not 1/4 since for n = 4, $1/(4^3 - 6 \cdot 4^2 + 12 \cdot 4 - 6) = 1/10$.

A third possible function is as follows:

$$f(n) = \begin{cases} 1 & \text{if there is a remainder of 1 when n is divided by 3} \\ 1/2 & \text{if there is a remainder of 2 when n is divided by 3} \\ 1/3 & \text{if there is a remainder of 0 when n is divided by 3} \end{cases}$$

Let n = 4. There is a remainder of 1 when 4 is divided by 3. Thus f(4) = 1, and the fourth term in this sequence is 1.

A fourth possible function is as follows:

Let $a_1 = 1/1$, $a_2 = 1/2$, $a_3 = 1/3$ and for $n > 3$,

$$a_n = \frac{1}{\frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}}.$$

Thus,

$$a_4 = \frac{1}{\frac{1}{a_3} + \frac{1}{a_2}} = \frac{1}{\frac{1}{1/3} + \frac{1}{1/2}} = \frac{1}{5}.$$

A definition which assigns the value of the nth term by a formula involving one or more preceding terms is called a recurrence relation and is frequently used to determine the terms of a sequence.

This example is given to show that a sequence is not determined by listing a finite number of terms and then using the ellipsis . . . . More information is needed in order to determine the remaining terms in the sequence.

<u>Definition 2.4</u>: (Bounded, Unbounded) If a real number K exists

such that $|a_n| \leq K$ for every n, i.e. $-K \leq a_n \leq K$, then the sequence $< a_n >$ is said to be <u>bounded</u> by K. If for every real number K, there exists at least one n, say n_0, such that $|a_{n_0}| > K$, then the sequence $< a_n >$ is said to be <u>unbounded</u>.

<u>Notation</u>: $\exists$ means "there exists" and $\ni$ means "such that".

As mentioned before some statements are true in all cases; some are true for particular cases; and some are true for at least one case. The existence of at least one case is very important for some mathematical ideas. This concept occurs so frequently that a symbol is helpful, and hence the symbol "$\exists$" is used to mean "there exists". Notice that nothing is implied about the uniqueness. The case in question may be unique or may not be unique. The words "such that" occur so frequently in mathematical sentences that the symbol "$\ni$" is used for these words.

<u>Definition 2.4'</u>: (Bounded, Unbounded) If $\exists\ K > 0 \ni |a_n| \leq K$, $\forall$ n; i.e. $-K \leq a_n \leq K$, then the sequence $< a_n >$ is said to be <u>bounded</u> by K. If $\forall$ real number K, $\exists\ n_0 \ni |a_{n_0}| > K$, then the sequence $< a_n >$ is said to be <u>unbounded</u>.

<u>Example 2.4</u>: $f(n) = 1/n$

Since $|1/n| = 1/n \leq 1$, $\forall n$, the sequence is bounded. See Figure 2.2 for the image of this sequence on the number line. Observe that the range of the sequence is contained in a finite interval when the sequence is bounded.

<u>Definition 2.5</u>: (Bounded above, or bounded to the right, and upper bound) If $\exists$ K $\ni a_n \leq K$, $\forall n$, the sequence $< a_n >$ is <u>bounded</u>

<u>above</u>, or <u>bounded to the right</u>, and K is called an <u>upper bound</u> for the sequence.

<u>Example 2.5</u>: $f(n) = -n$

This sequence $-1, -2, -3, \ldots, -n, \ldots$ is bounded above, but is not bounded in the sense of Definition 2.4. Observe that $\{k \mid k \geq -1\}$ is the set of all upper bounds for this sequence. Since -1 is the least element of this set, -1 is called the least upper bound of the sequence. A formal definition for the least upper bound will be given in Definition 2.7.

<u>Definition 2.6</u>: (Bounded below, or bounded to the left, and lower bound) If $\exists\, k \ni k \leq a_n, \ \forall\, n$, the sequence $< a_n >$ is <u>bounded below</u>, or <u>bounded to the left</u>, and k is called a <u>lower bound</u> for the sequence.

<u>Example 2.6</u>: $f(n) = n$

The sequence $1, 2, 3, \ldots, n, \ldots$ is bounded below, but is not bounded in the sense of Definition 2.4. Observe that $\{k \mid k \leq 1\}$ is the set of all lower bounds for this sequence. Since 1 is the largest element of this set, 1 is called the greatest lower bound of the sequence. A formal definition for the greatest lower bound will be given in Definition 2.8.

<u>Example 2.7</u>:

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases}$$

This sequence is neither bounded below nor above.

Theorem 2.1: A sequence is bounded above and bounded below if and only if the sequence is bounded.

Proof: (a) Assume the sequence is bounded and prove that it is bounded above and bounded below.

1. Let $< a_n >$ be a sequence which is bounded	1. Hypothesis
2. $\exists\ K \geq 0 \ni \lvert a_n \rvert \leq K, \quad \forall n$	2. Definition 2.4'
3. $-K \leq a_n \leq K, \quad \forall n$	3. Step 2 and absolute value written as an inequality
4. $-K \leq a_n, \quad \forall n$, implies that the sequence is bounded below	4. Definition 2.6
5. $a_n \leq K, \quad \forall n$, implies that the sequence is bounded above	5. Definition 2.5
6. Therefore $< a_n >$ is bounded above and bounded below	6. Steps 4 and 5

(b) Assume the sequence is bounded above and bounded below and prove that it is bounded.

1. Let $< a_n >$ be a sequence which is bounded above and bounded below	1. Hypothesis
2. $\exists\ K_1 \ni a_n \leq K_1, \quad \forall n$	2. Definition 2.5
3. $\exists\ k_1 \ni k_1 \leq a_n, \quad \forall n$	3. Definition 2.6
4. $k_1 \leq a_n \leq K_1, \quad \forall n$	4. Steps 2 and 3

5. Let $K_2 = \|K_1\|$ and $k_2 = \|k_1\|$	5. Assumption and notation
6. Pick the maximum of K_2 and k_2, denoted by $\max(K_2, k_2) = K$	6. Real numbers can be compared by the Trichotomy Axiom
7. $a_n \leq K_1 \leq K_2 \leq K, \ \forall n$	7. Steps 4, 5, 6
8. $-K \leq -k_2 \leq k_1 \leq a_n, \ \forall n$	8. Steps 4, 5, 6
9. $-K \leq a_n \leq K, \ \forall n$	9. Steps 7, 8, and transitive property of inequalities
10. K is a real number $\ni \|a_n\| \leq K, \ \forall n$	10. Step 9 and inequality written as an absolute value statement
11. $< a_n >$ is bounded	11. Definition 2.4'

In Example 2.5, -1 is the least element of the set of upper bounds for the given sequence and hence is called the least upper bound of the sequence. This is an example in which it is fairly easy to see the least element of the set of upper bounds. It might be a more difficult problem to find the least upper bound (l. u. b.) of some other sequence. The question may arise as to whether a sequence has a l. u. b. or not. If the sequence is bounded, the completeness property of the real number system assures us of the existence of the l. u. b.. What special characteristics does the l. u. b. have that distinguish it? The l. u. b. is an element of the set of upper bounds, and it is less than or equal to each element in the set of upper bounds. The l. u. b. is a real number M which is an upper bound and which has the property that no real number smaller than M is also an upper bound. A frequent device in mathematical arguments is to represent a real number smaller than a number M by $M - \epsilon$ where ϵ is used to mean a positive real number. With the help of this device the characteristics of the l. u. b. can be expressed precisely in mathematical language as follows:

Definition 2. 7: (Least Upper Bound, l. u. b.) A number M is called the least upper bound (l. u. b.) of a sequence $< a_n >$ if $a_n \leq M$, $n = 1, 2, 3, \ldots$ and if for every $\epsilon > 0$ (i. e. ϵ is a positive real number), there exists at least one term of the sequence which is greater than $M - \epsilon$.

Definition 2. 7': A number M is the l. u. b. of a sequence $<a_n>$ if

(i) $a_n \leq M, \ \forall n$, and

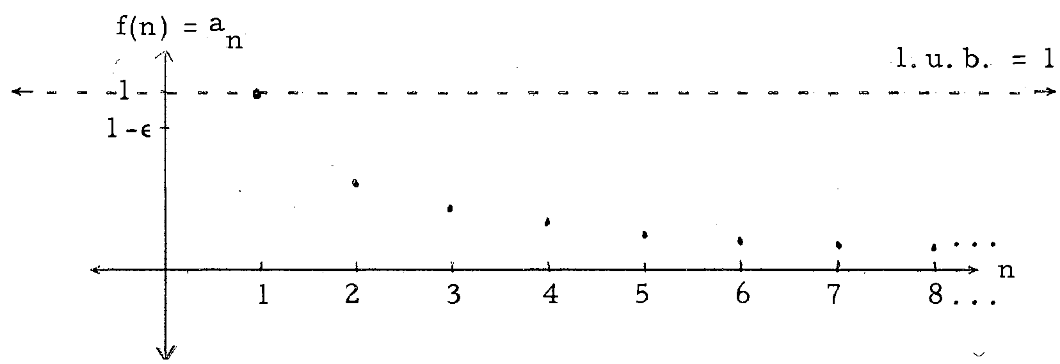
(ii) $\forall \epsilon > 0, \exists n_0 \ni a_{n_0} > M - \epsilon$.

The first part of this definition, $a_n \leq M$, $n = 1, 2, 3, \ldots$, means $a_1 \leq M$, $a_2 \leq M$, $a_3 \leq M, \ldots, a_n \leq M, \ldots$, that is, every term of the sequence is less than or equal to M. This establishes the fact that M is an upper bound of the sequence. The second part of this definition, for every $\epsilon > 0$, there exists at least one term of the sequence which is greater than $M - \epsilon$, makes M the least element in the set of upper bounds. No matter how small ϵ is chosen to be, there is always a term of the sequence greater than $M - \epsilon$. If this were not the case then some number smaller than M would be an upper bound.

The l. u. b. of a sequence may be a term of the sequence or it may not be a term of the sequence. The following examples will illustrate this fact.

Example 2.8: $f(n) = 1/n$

Since every term of the sequence is less than or equal to 1 and there exists at least one term of the sequence (namely 1) which exceeds $1 - \epsilon$ for every positive ϵ, 1 is the l.u.b. of the sequence. In this case 1 is also a term of the sequence. See Figure 2.3.



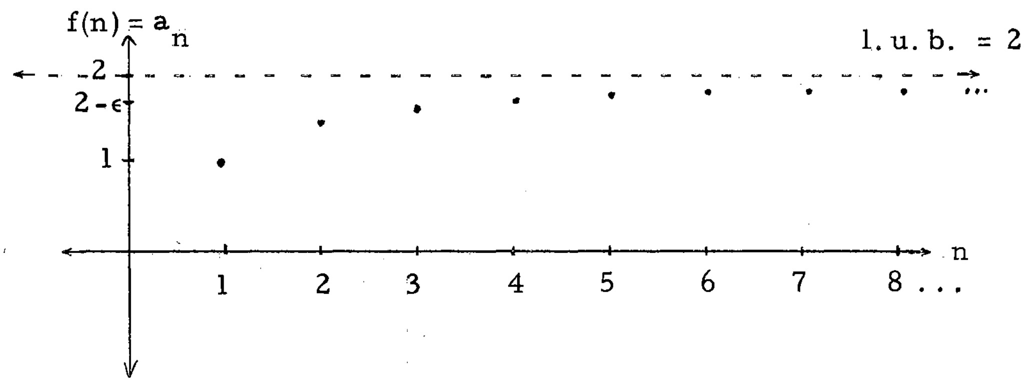
$1, 1/2, 1/3, \ldots, 1/n, \ldots$

Figure 2.3

Example 2.9: $f(n) = 2 - 1/n$

Every term of the sequence is less than or equal to 2. For any positive ϵ it is possible to choose an integer n_0 such that $n_0 > 1/\epsilon$ which means $1/n_0 < \epsilon$ and $2 - 1/n_0 > 2 - \epsilon$. Thus for any positive ϵ there is at least one term of the sequence (namely $2 - 1/n_0$) which exceeds $2 - \epsilon$. Hence 2 is the l.u.b. of the sequence. In this case 2 is not a term of the sequence. See Figure 2.4.

The concept of greatest lower bound (g.l.b.) is defined in a similar manner as stated in Definition 2.8.



1, 3/2, 5/3, . . . , 2 - 1/n, . . .

Figure 2. 4

<u>Definition 2. 8</u>: (Greatest Lower Bound, g. l. b.) A number m is called the <u>greatest</u> <u>lower</u> <u>bound</u> (g. l. b.) of a sequence $< a_n >$ if $a_n \geq m$, n = 1, 2, 3, . . . and if for every $\epsilon > 0$, there exists at least one term of the sequence which is less than $m + \epsilon$.

<u>Definition 2. 8'</u>: A number m is the g. l. b. of a sequence $< a_n >$ if

(i) $a_n \geq m$, $\forall$ n, and

(ii) $\forall\ \epsilon > 0$, $\exists\ n_0 \ni a_{n_0} < m + \epsilon$.

<u>Example 2. 10</u>: f(n) = 1/n

The number 0 is the g. l. b. of this sequence. Observe that 0 is not a term of the sequence. See Figure 2. 3 in Example 2. 8 for the graph of this sequence.

<u>Example 2. 11</u>: f(n) = 2 - 1/n

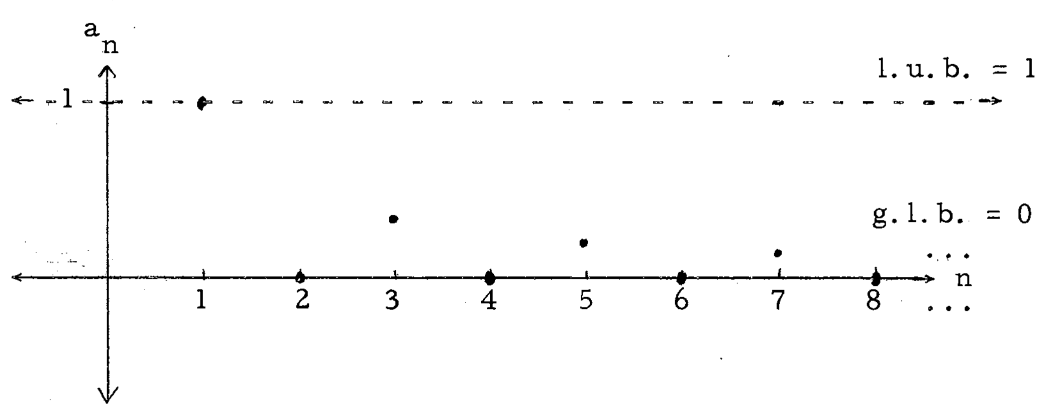
The number 1 is the g. l. b. of this sequence. Observe that 1 is a term of the sequence. See Figure 2.4 in Example 2.9 for the graph of this sequence.

The four Examples 2.8, 2.9, 2.10, and 2.11 have illustrated that the l.u.b. and g.l.b. may or may not be a term of the sequence. In Examples 2.8 and 2.10 the l.u.b. is a term of the sequence while the g.l.b. is not. In Examples 2.9 and 2.11 the g.l.b. is a term of the sequence while the l.u.b. is not. It is possible to have sequences such that both the l.u.b. and g.l.b. are terms of the sequence or neither the l.u.b. nor g.l.b. is a term of the sequence. The following examples illustrate this fact.

Example 2.12:

$$f(n) = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

The number 1 is the l.u.b; and the number 0 is the g.l.b.; and 0, 1 are terms of the sequence. See Figure 2.5.



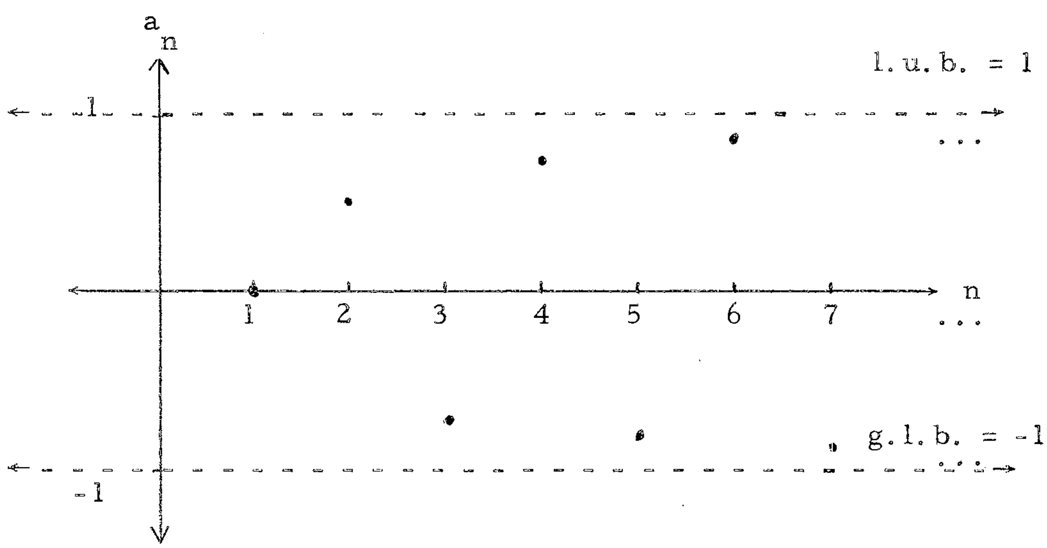
$1, 0, 1/3, 0, 1/5, 0, \ldots, 1/n, 0, \ldots$

Figure 2.5

Example 2.13:

$$f(n) = \begin{cases} 1 - 1/n & \text{if } n \text{ is even} \\ -1 + 1/n & \text{if } n \text{ is odd} \end{cases}$$

This sequence is bounded, but neither the l.u.b. nor g.l.b. is a term of the sequence. See Figure 2.6.



0, 1/2, -2/3, 3/4, -4/5, 5/6, . . . , -1 + 1/n, 1 - 1/n, . . .

Figure 2.6

In Example 2.2, $f(n) = 1/n$, $a_n = 1/n$ and $a_{n+1} = 1/(n+1)$. Since $n + 1 > n$, $\forall n$, $1/n > 1/(n+1)$, $\forall n$. Observe that this inequality is true for all pairs of consecutive terms of the sequence. In fact, each term of the sequence is larger than its successor. In Example 2.9,

$$f(n) = 2 - 1/n = \frac{2n - 1}{n},$$

$$a_n = \frac{2n - 1}{n},$$

and

$$a_{n+1} = \frac{2n + 1}{n + 1}.$$

Since $2n^2 + n - 1 < 2n^2 + n, \quad \forall n$, then

$$\frac{2n - 1}{n} < \frac{2n + 1}{n + 1}, \quad \forall n.$$

Each term of the sequence is smaller than its successor. Sequences occur frequently with behavior similar to one or the other of these examples. The following definition will characterize sequences of this type.

Definition 2.9: (Monotone Sequence) The sequence $< a_n >$ is monotone if and only if it satisfies one of the following conditions:

(i) $a_n \geq a_{n+1}, \quad \forall n$

(ii) $a_n \leq a_{n+1}, \quad \forall n$.

The sequence $< a_n >$ is said to be monotone non-increasing if and only if (i) holds, and monotone non-decreasing if and only if (ii) holds. The sequence is said to be monotone if and only if it is either monotone non-increasing or monotone non-decreasing.

Notation: $a_n \downarrow$ means the sequence $< a_n >$ is non-increasing

$a_n \uparrow$ means the sequence $< a_n >$ is non-decreasing

The inequality in part (i) was stated in its weaker form

$a_n \geq a_{n+1}$, but in Example 2.2 the strict inequality $a_n > a_{n+1}$ holds. If $a_n > a_{n+1}$ in part (i) of Definition 2.9, the sequence $< a_n >$ is said to be strictly decreasing; and if $a_n < a_{n+1}$ in part (ii) of Definition 2.9, the sequence $< a_n >$ is said to be strictly increasing.

Example 2.14:

$$f(n) = 1 - 1/n = \frac{n-1}{n}$$

Since

$$a_n = \frac{n-1}{n},$$

$$a_{n+1} = \frac{n}{n+1},$$

and $n^2 - 1 < n^2$, $\forall n$, then

$$\frac{n-1}{n} < \frac{n}{n+1}, \quad \forall n;$$

i.e. $a_n < a_{n+1}$, $\forall n$. Hence this sequence is strictly increasing.

Example 2.15:

$$f(n) = \begin{cases} 1 - 1/n & \text{if } n \text{ is odd} \\ \dfrac{n-2}{n-1} & \text{if } n \text{ is even} \end{cases}$$

This sequence is non-decreasing, but not strictly increasing. Observe the terms of the sequence,

$$0, 0, 2/3, 2/3, 4/5, 4/5, \ldots, \frac{n-1}{n}, \frac{n-2}{n-1}, \ldots .$$

Example 2.16:

$$f(n) = 1 + 1/n = \frac{n+1}{n}$$

Since

$$a_n = \frac{n+1}{n},$$

$$a_{n+1} = \frac{n+2}{n+1},$$

and $n^2 + 2n + 1 > n^2 + 2n, \quad \forall\, n$, then

$$\frac{n+1}{n} > \frac{n+2}{n+1}, \quad \forall\, n;$$

i.e. $a_n > a_{n+1}, \quad \forall\, n$. Hence this sequence is strictly decreasing.

Example 2.17:

$$f(n) = \begin{cases} 1 + 1/n & \text{if } n \text{ is odd} \\ \dfrac{n}{n-1} & \text{if } n \text{ is even} \end{cases}$$

This sequence is non-increasing, but not strictly decreasing. Observe the terms of the sequence,

$$2, 2, 4/3, 4/3, 6/5, 6/5, \ldots, \frac{n+1}{n}, \frac{n}{n-1}, \ldots .$$

Many interesting theorems can be established concerning the concepts just defined and illustrated. The following theorems are left for the reader to prove.

Theorem 2.2: A monotone non-decreasing sequence that is bounded above is bounded.

Theorem 2.3: A monotone non-increasing sequence that is bounded below is bounded.

The property of boundedness places a restriction on the range of the sequence. Monotonicity is a property depending on the relation of each term to the succeeding one. The following definition characterizes a sequence by examining the relation between the number zero and the terms of the sequence with special attention to those terms with index greater than or equal to n_0. The reader should study this definition very carefully and attempt to master its meaning and all that it implies.

<u>Definition 2.10</u>: (Null Sequence) A sequence $< a_n >$ is called a <u>null sequence</u> if it possesses the following property: given any arbitrary positive number ϵ, the inequality $|a_n| < \epsilon$ is satisfied by all the terms, with the exception of at most a finite number of them. In other words: choose an arbitrary positive number ϵ, and a number n_0 can always be found, such that $|a_n| < \epsilon$ for every $n > n_0$.

The arbitrarily chosen positive number is usually denoted by ϵ. Sometimes it is convenient to denote it by $\epsilon/2$ or ϵ^2, ϵ/K, $(K > 0)$, etc. The form of the arbitrarily chosen positive number is determined by its use in a specific argument. The place in a given sequence beyond which the terms remain numerically less than ϵ, will depend in general on the magnitude of ϵ. In general, it will lie further and further to the right (i.e. n_0 will be larger and larger) the smaller the given ϵ. This dependence of the number n_0 on ϵ is often emphasized by saying explicitly: "To each given ϵ corresponds a number $n_0 = n_0(\epsilon)$ such that ...". Using the notation $n_0(\epsilon)$ emphasizes the fact that n_0 is a function of ϵ. Observe that n_0 need not be an integer, and also that n_0 is not unique. Any $n_1 > n_0$ can be used just as well. In a null sequence, no term need be equal to zero. The definition is satisfied provided that whatever choice of ϵ is made all terms whose index n is greater than some n_0 are in absolute value less than ϵ.

The geometric interpretation of this definition is as follows. If a two dimensional graph of a sequence is used, $< a_n >$ is a null sequence if the whole of its graph, with the exception of at most a finite number of points, lies in an ϵ-strip about the axis of abscissa. The ϵ-strip is defined by drawing parallels to the axis of abscissa through the two points $(0, \epsilon)$ and $(0, -\epsilon)$. No matter what ϵ-strip about the axis is chosen, the graph of a null sequence can have at most a finite number of points outside this strip.

If the image of the sequence on a real number line is used, $<a_n>$ is a null sequence if its terms ultimately (for $n > n_0$) all belong to the interval $(-\epsilon, \epsilon)$. Call such an interval for brevity an ϵ-neighborhood of the origin. Then $< a_n >$ is a null sequence if every ϵ-neighborhood of the origin contains all but a finite number, at most, of the terms of the sequence.

In Definition 2. 10, no essential modification is produced by interchanging "$< \epsilon$" and "$\leq \epsilon$". If for every $n > n_0$, $|a_n| < \epsilon$, then $|a_n| \leq \epsilon$. Conversely, if given any ϵ, n_0 can be determined so that $|a_n| \leq \epsilon$ for every $n > n_0$, then choosing any positive number $\epsilon_1 < \epsilon$ there exists an n_1 such that $|a_n| \leq \epsilon_1$, for every $n > n_1$, and consequently $|a_n| < \epsilon$ for every $n > n_1$. Precisely analogous considerations show that in Definition 2. 10 "$> n_0$" and "$\geq n_0$" are interchangeable alternatives.

<u>Example 2. 18</u>: $f(n) = 1/n$

This sequence is a null sequence. The sequence is a monotone decreasing sequence. $|1/n| < \epsilon$ provided $n > 1/\epsilon$. It is sufficient to put $n_0 = 1/\epsilon$. If $n > n_0$, $n > 1/\epsilon$ and $1/n < \epsilon$. Thus for every ϵ if the choice $n_0 = 1/\epsilon$ is made then $|1/n| < \epsilon$ for every $n > n_0$ and $< 1/n >$ is seen to satisfy the definition of a null sequence. In this example the dependence of n_0 on ϵ is seen explicitly. Actually the choice of n_0 is not unique. Any number greater than $1/\epsilon$ will do just as well.

To prove that a given sequence is a null sequence, it must be shown that for a prescribed $\epsilon > 0$, the corresponding n_0 can actually be determined. On the other hand the reader should be sure to understand clearly what is meant by a sequence not being a null sequence.

If the definition is to be negated, this means that the statement is not true that for every positive number ϵ, beyond a certain point $|a_n|$ is always less than ϵ. That is, there exists at least one positive number ϵ_0, such that no matter what n_0 is chosen, $|a_n|$ is not for $n > n_0$ always less than ϵ_0. In other words, after every n_0 there is some larger index n for which $|a_n| \geq \epsilon_0$.

Symbolically, this is stated as follows: A sequence $< a_n >$ is not a null sequence if

$$\exists\, \epsilon_0 > 0 \ni \forall\, n_0 \quad \exists\, n_1 > n_0 \ni |a_{n_1}| \geq \epsilon_0.$$

It is natural to ask whether there is any relationship between the property of being a null sequence and the properties of boundedness and monotonicity already defined. In Example 2.18 the sequence $< 1/n >$ was shown to be a null sequence. It was earlier shown that this sequence is monotone. A little reflection however shows that a null sequence need not be monotone. Exactly the same argument used in

Example 2.18 would show that the sequence $< (-1)^n \, 1/n >$ is also a null sequence. Since the terms of this sequence are alternately positive and negative this sequence is clearly not monotone.

It is necessary however that a null sequence be bounded. Geometrically boundedness means that the range of the sequence is contained in a finite interval (see Example 2.4). In the case of a null sequence all but at most a finite number of points of the range lie in the interval $(-\epsilon, \epsilon)$. It is necessary only to enlarge this interval to include the finite number of omitted terms in order to find an interval containing the whole range of the sequence. This geometric argument is stated in formal terms in the following theorem.

<u>Theorem 2.4</u>: A null sequence is a bounded sequence.

Proof:

1. Let $< a_n >$ be a null sequence	1. Assumption
2. $\forall \epsilon > 0 \; \exists \; n_0 \ni |a_n| < \epsilon, \; \forall \; n > n_0$	2. Definition 2.10
3. Choose $\epsilon = 1$, then $\exists \; n_1 \ni |a_n| < 1, \; \forall \; n > n_1$	3. Step 2 is true for all $\epsilon > 0$, therefore true if $\epsilon = 1$
4. Consider $|a_1|, |a_2|, |a_3|, \ldots, |a_{n_1}|$	4. Assumption
5. Let $K = \max(1, |a_1|, |a_2|, \ldots, |a_{n_1}|)$	5. In a finite set of real numbers, there exists a largest number
6. $|a_1| \leq K, \; |a_2| \leq K, \; |a_3| \leq K, \ldots, |a_{n_1}| \leq K, \ldots$	6. Steps 3 and 5

7. $\lvert a_n \rvert \leq K, \quad \forall n$	7. Step 6
8. $< a_n >$ is bounded	8. Definition 2.4'

A large part of the following discussion will be concerned with showing that a sequence appearing in the course of an investigation is a null sequence. This can be accomplished by actually specifying the $n_0 = n_0(\epsilon)$ which corresponds to the chosen $\epsilon > 0$, as in Example 2.8. Very often, however, it will be accomplished by comparing the sequence to be investigated with a known null sequence. The following theorems serve as a basis for this.

Theorem 2.5: Let $< a_n >$ be a null sequence. Suppose that for a fixed positive number K the terms of a sequence $< a_n' >$ under investigation satisfy the condition that, for all $n > n_0'$, $\lvert a_n' \rvert \leq K \lvert a_n \rvert$. Then $< a_n' >$ is also a null sequence.

Proof:

1. $< a_n >$ is a null sequence	1. Hypothesis
2. Choose $\epsilon > 0$, then $\epsilon / K > 0$	2. $K > 0$ and $\epsilon > 0$
3. $\exists\, n_0 \ni \lvert a_n \rvert < \epsilon / K, \quad \forall n > n_0$	3. Definition 2.10
4. $\exists\, n_0' \ni \lvert a_n' \rvert \leq K \lvert a_n \rvert, \quad \forall n > n_0'$	4. Hypothesis
5. Let $n_1 = \max(n_0, n_0')$	5. Assumption
6. $\lvert a_n' \rvert \leq K \lvert a_n \rvert < \epsilon, \quad \forall n > n_1$	6. Steps 3, 4, 5
7. $< a_n' >$ is a null sequence	7. Definition 2.10

Study carefully the proof of Theorem 2.5. Observe that every mathematical sentence concerning the terms a_n is qualified by stating for which indices n the sentence is true. This is very important since the sentence may be nonsense without the proper restrictions. When the reader is attempting to prove theorems, keep this fact in mind and check each sentence to be sure which indices n make the sentence true.

Theorem 2.6: Let $< a_n >$ be a null sequence and $< b_n >$ be a bounded sequence. Then the sequence $< c_n >$ with the terms $c_n = a_n b_n$, $\forall$ n, is also a null sequence.

Proof:

1. $< b_n >$ is a bounded sequence	1. Hypothesis
2. $\exists\ K > 0 \ni |b_n| \leq K, \ \forall n$	2. Definition 2.4'
3. $< a_n >$ is a null sequence	3. Hypothesis
4. $|c_n| = |a_n b_n| = |a_n|\ |b_n| \leq K\ |a_n|, \ \forall n$	4. Hypothesis, absolute value theorems, step 2
5. Therefore $< c_n >$ is a null sequence	5. Steps 3 and 4, Theorem 2.5

Corollary 2.6: Let $< a_n >$ be a null sequence and $< b_n >$ be a null sequence. Then the sequence $< c_n >$ with the terms $c_n = a_n b_n$, $\forall$ n is also a null sequence.

Proof: Left for the reader.

Theorem 2.7: Let $< a_n >$ be a null sequence with positive terms, and α be an arbitrary positive real number. Then $< a_n^\alpha >$ is also a null sequence.

Proof: Left for the reader.

Definition 2.11: (Subsequence) If $v_1, v_2, v_3, \ldots, v_n, \ldots$ is an arbitrary sequence of natural numbers such that $v_1 < v_2 < v_3 < \ldots < v_n < \ldots$ and if $a_{v_n} = a'_n$, then $< a'_n >$ is called a subsequence of the sequence $< a_n >$.

Example 2.19: Let $< 1/n >$ be a given sequence. The even integers $2, 4, 6, \ldots, 2n, \ldots$ is an ordered sequence of natural numbers, and therefore is a valid choice for $v_1, v_2, v_3, \ldots, v_n, \ldots$ in Definition 2.11. Thus $1/2, 1/4, 1/6, \ldots, 1/2n, \ldots$ is a subsequence of $1, 1/2, 1/3, \ldots, 1/n, \ldots$.

Example 2.20: Let $< a_n >$ be a given sequence. For any fixed integer k, the sequence $k + 1,\ k + 2,\ k + 3, \ldots,\ k + n, \ldots$ is an ordered sequence of natural numbers and a valid choice for $v_1, v_2, v_3, \ldots, v_n, \ldots$ in Definition 2.11. Thus, $a_{k+1}, a_{k+2}, a_{k+3}, \ldots, a_{k+n}, \ldots$ is a subsequence of $< a_n >$. This particular subsequence is important and is referred to as a terminal segment of the sequence $< a_n >$ since $k+1,\ k+2,\ k+3, \ldots,\ k+n, \ldots$ is a terminal seqment of the natural numbers. In order to see the importance of a terminal segment the reader should look again at the definition of a null sequence. The terms of the sequence for which $n > n_0$ constitute a terminal segment of the sequence beginning with the term a_{n_0+1}. The definition of a null sequence could be rephrased in the following way.

Definition 2.10': (Null Sequence) A sequence $< a_n >$ is called a null sequence if given any arbitrary positive number ϵ, there exists a terminal segment of the sequence which is bounded by ϵ.

Theorem 2.8: Let $< a_n >$ be a null sequence. Then every subsequence $< a_n' >$ of $< a_n >$ is also a null sequence.

Proof: Left for the reader.

Theorem 2.9: Let $< a_n >$ and $< b_n >$ be two null sequences. Then the sequence $< c_n >$, with $c_n = a_n + b_n$, $\forall n$, is also a null sequence.

Proof: Left for the reader. Hint: (Remember that $|x + y| \leq |x| + |y|$ for all real numbers x and y.)

Theorem 2.10: Let $< a_n >$ and $< b_n >$ be two null sequences. Suppose that $< c_n >$ is such that $a_n \leq c_n \leq b_n$, after a certain stage. Then $< c_n >$ is also a null sequence.

Proof:

1. $< a_n >$ and $< b_n >$ are null sequences	1. Hypothesis
2. $\forall \epsilon > 0 \ \exists \ n_0 \ni |a_n| < \epsilon, \ \forall n > n_0$ and $\exists n_1 \ni |b_n| < \epsilon, \ \forall n > n_1$	2. Definition 2.10
3. $\exists \ n_2 \ni a_n \leq c_n \leq b_n, \ \forall n > n_2$	3. Hypothesis and meaning of after a certain stage
4. Let $N = \max (n_0, n_1, n_2)$	4. Trichotomy Axiom
5. $\forall \epsilon > 0 \ \exists \ N \ni |a_n| < \epsilon, \ \forall n > N$ and $|b_n| < \epsilon, \ \forall n > N$ and $a_n \leq c_n \leq b_n, \ \forall n > N$	5. Steps 2, 3, 4

6.	$\forall\, n > N,\ -\epsilon < a_n < \epsilon$ $-\epsilon < b_n < \epsilon$ $a_n \leq c_n \leq b_n$	6. Step 5 and absolute value written in inequality form
7.	$\forall\, n > N, -\epsilon < a_n \leq c_n \leq b_n < \epsilon$ or $-\epsilon < c_n < \epsilon$ or $\lvert c_n \rvert < \epsilon$	7. Step 6, Transitive property, inequality written as an absolute value statement
8.	$< c_n >$ is a null sequence	8. Steps 5 and 7, Definition 2.10

<u>Definition 2.12</u>: (Rearrangement) If $V_1, V_2, V_3, \ldots, V_n, \ldots$ is a sequence of natural numbers in which every natural number appears exactly once, then $< V_n >$ is called a <u>rearrangement</u> of the sequence of natural numbers, and more generally, $< x_n' >$, with $x_n' = x_{V_n}$, is called a <u>rearrangement</u> of the sequence $< x_n >$.

<u>Theorem 2.11</u>: If $< x_n >$ is a null sequence, then every one of its rearrangements $< x_n' >$ is also a null sequence.

Proof:

1.	$< x_n >$ is a null sequence	1. Hypothesis
2.	$\forall\, \epsilon > 0\ \exists\ n_0 \ni \lvert x_n \rvert < \epsilon,\ \ \forall\, n > n_0$	2. Definition 2.10
3.	Let $< x_n' >$ be a rearrangement of $< x_n >$, then $x_n' = x_{V_n}$, $\forall\, n$, where $< V_n >$ is a rearrangement of the sequence of natural numbers	3. Assumption and Definition 2.12
4.	There exists an identity mapping which establishes a 1-1 corres-	4. Step 3, Definition 2.12, and meaning of "every

pondence between the indices $1, 2, 3, \ldots, n, \ldots$ of the sequence $< x_n >$ and the indices $V_1, V_2, V_3, \ldots, V_n, \ldots$ of the sequence $< x'_n >$ such that the corresponding terms of the sequences are the same	natural number appears exactly once"
5. Each of the terms $x_1, x_2, x_3, \ldots, x_{n_0}$ has exactly one term which corresponds to it in the sequence $< x'_n >$ by the above 1-1 correspondence	5. Step 4
6. Consider the set of x'_n which correspond with $x_1, x_2, x_3, \ldots, x_n$ and let N be the maximum of the indices of the terms in the set of x'_n. Then $N \geq n_0$.	6. Trichotomy Axiom and the nature of the rearrangement
7. Hence $\exists\, N \ni \lvert x'_n \rvert < \epsilon, \ \forall n' > N$	7. Step 2 and 6
8. $< x'_n >$ is a null sequence	8. Step 7, Definition 2.10

It will be helpful if the reader has some examples of null sequences to use in the later chapters, so consider the following:

<u>Example 2.21</u>: $0, 0, 0, \ldots, 0, \ldots$

$< 0 >$ is certainly a null sequence since $|0| = 0 < \epsilon$, $\forall n$ and $\forall \epsilon$, and hence satisfies the definition of a null sequence.

<u>Example 2.22</u>: For an arbitrary $\alpha > 0$, $< 1/n^\alpha >$ is a null sequence. $< 1/n >$ is a null sequence in Example 2.18. Theorem 2.7

implies that $< (1/n)^{\alpha} >$ is a null sequence also. Since $(1/n)^{\alpha} = 1/n^{\alpha}$, then $< 1/n^{\alpha} >$ is a null sequence.

The next example is quite useful in our future work. The proof involves some properties that should be familar from high school algebra. One of these is the following property of inequalities. If two positive numbers are ordered, i.e. $0 < a < b$, then their reciprocals are inversely ordered, i.e. $0 < 1/b < 1/a$. Another property that is used is a result of the binomial theorem. It can be proved that $(1+p)^n > np$ if $p > 0$ and n is a positive integer. The remainder of the proof is application of the definition of a null sequence.

Example 2.23: Prove that $< r^n >$ is a null sequence, if $|r| < 1$.

Proof:

1. If $r = 0$, then $r^n = 0, \ \forall n$	1. $0^n = 0, \ \forall n$, law of exponents
2. $< r^n >$ is a null sequence if $r = 0$	2. Example 2.21
3. Suppose $\lvert r \rvert < 1, \ r \neq 0$	3. Assumption
4. $\lvert 1/r \rvert > 1, \ r \neq 0$	4. If $0 < a < b$, then $0 < 1/b < 1/a$ for positive real numbers
5. Let $\lvert 1/r \rvert = 1 + p, \ p > 0$	5. A number greater than 1 can be written as 1 plus some positive number

6. For $n \geq 1$, $\lvert r^n \rvert = \frac{1}{(1+p)^n} < \frac{1}{np}$	6. According to the binomial theorem, $(1+p)^n > np$ and Step 5
7. Hence $\lvert r^n \rvert < \epsilon$, $\forall n > \frac{1}{\epsilon p}$	7. Since $\frac{1}{np} < \epsilon$ if $n > \frac{1}{\epsilon p}$ and Step 6
8. $\forall \epsilon > 0 \ \exists \ n_0 = \frac{1}{\epsilon p} \ni \lvert r^n \rvert < \epsilon$, $\forall n > \frac{1}{\epsilon p}$	8. Step 7
9. $< r^n >$ is a null sequence if $\lvert r \rvert < 1$, $r \neq 0$	9. Definition 2.10 and Step 8
10. $< r^n >$ is a null sequence if $\lvert r \rvert < 1$	10. Steps 2 and 9

CHAPTER III

LIMITS OF SEQUENCES

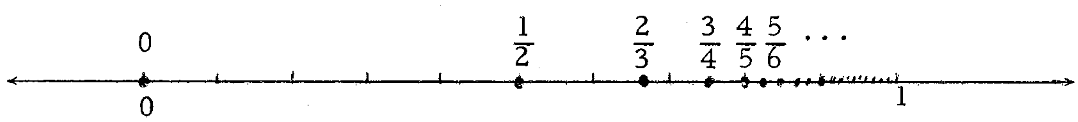
Before the concept of a limit of a sequence is formally defined, consider the following example.

<u>Example 3.1</u>: $f(n) = a_n = 1 - 1/n = \frac{n-1}{n}$

The terms of this sequence are

$$0, 1/2, 2/3, 3/4, \ldots, \frac{n-1}{n}, \ldots .$$

Plot the range of the sequence on a real number line. See Figure 3.1.



$$0, 1/2, 2/3, 3/4, \ldots, \frac{n-1}{n}, \ldots$$

Figure 3.1

This sequence is monotone increasing. Observe that the larger the index n, the closer the point of the sequence is to the point 1. The distance between two points is found by the use of absolute value; i.e. $|0 - 1| = 1$, $|1/2 - 1| = 1/2$, $|2/3 - 1| = 1/3$, and so on. In general the distance between a point of the sequence and the point 1 is

$$\left|(1 - \frac{1}{n}) - 1\right| = \left|-\frac{1}{n}\right| = \frac{1}{n}.$$

Let $d_n = 1/n$. The distance sequence $< d_n >$ is a null sequence. Therefore $\forall \epsilon > 0 \ \exists \ n_0 \ \ni |(1 - 1/n) - 1| < \epsilon, \quad \forall n > n_0$. This means that if ϵ is chosen, then all but a finite number of points of the sequence lie between the point $1 + \epsilon$ and the point $1 - \epsilon$. The number 1 is called the limit of the sequence. The sequence is said to converge to the number 1.

Definition 3.1: (Convergent and Divergent Sequences and Limit of a Sequence) If $< a_n >$ is a given sequence of numbers, and if it is related to a certain number A in such a way that $< a_n - A >$ is a null sequence, then the sequence $< a_n >$ converges to A; i.e. it is convergent, with the limit A. A sequence $< a_n >$ which is not convergent is said to be divergent.

Notation: $a_n \to A$ means that "a_n converges to A"; $\lim a_n = A$ means that "$< a_n >$ is convergent and the limit of the sequence $< a_n >$ is A"; $a_n \uparrow A$ means that "$< a_n >$ is monotone non-decreasing and converges to A"; and $a_n \downarrow A$ means that "$< a_n >$ is monotone non-increasing and converges to A."

If Definition 3.1 and the definition of a null sequence are combined, the idea of the existence of a limit of a sequence can be written in a more formalized manner as follows:

Definition 3.2: (Convergent and Divergent Sequences and Limit of a Sequence) Let $< a_n >$ be a given sequence. Then $\lim a_n = A$ iff

$\forall \epsilon > 0 \; \exists \; n_0 \ni |a_n - A| < \epsilon, \; \forall n > n_0$. The sequence $< a_n >$ is <u>convergent</u> if there exists a real number A such that $\lim a_n = A$. Otherwise the sequence $< a_n >$ is <u>divergent</u>. Hence the limit of the sequence $< a_n >$ does not exist. Then $\forall$ real number A,

$\exists \; \epsilon_0 > 0 \ni \; \forall n_0 \; \exists \; n_1 > n_0 \ni |a_{n_1} - A| \geq \epsilon_0$.

Observe that null sequences are sequences which converge to 0 since $\forall \epsilon > 0 \; \exists \; n_0 \ni |a_n - 0| < \epsilon, \; \forall n > n_0$. Henceforth, the statement "$< a_n >$ is a null sequence" can be written $a_n \to 0$ or $\lim a_n = 0$.

<u>Example 3.2</u>: $\quad a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

Is this sequence convergent? If so, what is its limit? The terms of the sequence are

$$2, 3/2, 4/3, \ldots, \frac{n+1}{n}, \ldots .$$

The limit of the sequence appears to be the number 1. It is easy to apply Definition 3.1 and use the idea of a null sequence. Since the sequence $< (1 + 1/n) - 1 >$ is simply the sequence $< 1/n >$ and $< 1/n >$ is shown to be a null sequence in Example 2.18, $< 1 + 1/n >$ is convergent with the limit 1.

Definition 3.2 enables us to combine the work that was done in Example 2.18 with Definition 3.1. If Definition 3.2 is applied directly to the problem, the following analysis is required.

Consider $|(1 + 1/n) - 1| = |1/n|$. Can an n_0 be found for every $\epsilon > 0$ such that $|1/n| < \epsilon, \; \forall n > n_0$? Consider $|1/n| = \epsilon$. This means that $n = 1/\epsilon$. In order for $1/n$ to be smaller than ϵ, n must be greater than $1/\epsilon$. Also if $n > n_0$, then $1/n < 1/n_0$. Therefore, let $n_0 = 1/\epsilon$. Now for every $n > n_0$, $1/n < 1/n_0 = \epsilon$. Thus the choice $n_0 = 1/\epsilon$

satisfies the requirement that $|(1+1/n) - 1| < \epsilon, \ \forall n > n_0$. By the preceding argument it has been established that

$$\forall \epsilon > 0 \ \exists n_0 \ni |(1+1/n) - 1| < \epsilon, \ \forall n > n_0,$$

and this sequence is convergent with limit 1. As pointed out in Chapter II, the choice of n_0 is not unique. Any $n_1 > n_0$ will satisfy the conditions. In general, the choice of n_0 depends on ϵ.

<u>Example 3.3</u>: $a_n = (-1)^{n+1}$

The terms of this sequence are $1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots$. Plot the sequence on a rectangular coordinate system. See Figure 3.2.

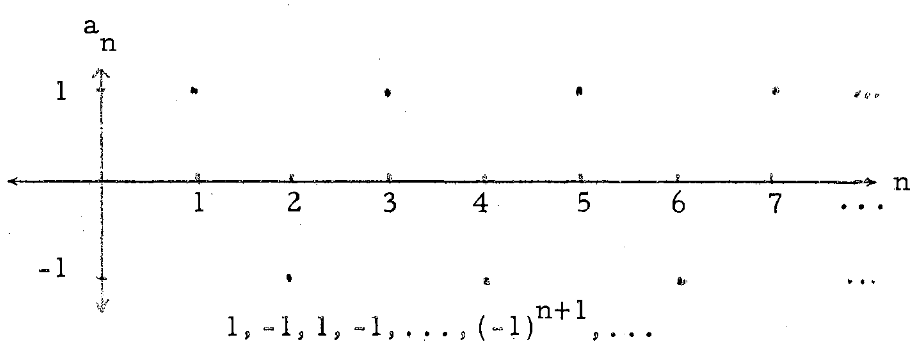


Figure 3.2

This sequence is not related to a number A in such a way that $< (-1)^{n+1} - A >$ is a null sequence. The number A cannot be 1 since all the even terms are less than 1/2 and a distance from 1 greater than 1/2. Thus $< a_n - 1 >$ is not a null sequence. The number A cannot be - 1, since for all odd n, $(-1)^{n+1} > 1/2$, and hence a distance from -1 greater than 3/2. Thus $< a_n -(-1) >$ is not a null sequence.

The number A cannot be any other number, say A_0. Consider $|1 - A_0| = p$ and $|-1 - A_0| = q$. Now $q > p$, $q = p$, or $q < p$. If $q \geq p$, pick $\epsilon = p/2$ and none of the terms lie between $A_0 - p/2$ and $A_0 + p/2$. If $q < p$, pick $\epsilon = q/2$ and none of the terms lie between $A_0 - q/2$ and $A_0 + q/2$. Thus in general,

$$\forall A \; \exists \; \epsilon_0 > 0 \ni \forall n_0 \; \exists \; n_1 > n_0 \ni |a_{n_1} - A| \geq \epsilon_0.$$

Hence the sequence $< a_n >$ is divergent by Definition 3.2.

Example 3.4: $a_n = n$

The terms of this sequence are $1, 2, 3, \ldots, n, \ldots$. There is no number $A \ni -\epsilon < a_n - A < \epsilon$. Suppose such an A exists. This would require $a_n < A + \epsilon$ for $n \geq$ some n_0. Then choose n to be an integer $\ni n > A + \epsilon$, and the terms of the sequence with index greater than n cannot satisfy $a_n < A + \epsilon$. Hence $< a_n - A >$ is not a null sequence for any number A. The sequence $< a_n >$ is divergent by Definition 3.1.

The divergent sequence in Example 3.4 is not a bounded sequence, while the divergent sequence in Example 3.3 is. Thus a divergent sequence may or may not be bounded. A very important property of a convergent sequence is that it must be bounded as proved in the following theorem.

Theorem 3.1: A convergent sequence is a bounded sequence.

Proof: (This can be proved directly from the definition as in Theorem 2.4, but proof is simpler as follows.)

1. Let $< a_n >$ be a sequence $\ni a_n \to A$	1. Hypothesis
2. $< a_n - A >$ is a null sequence	2. Definition 3.1

3. $< a_n - A >$ is bounded	3. Theorem 2.4
4. $\exists\ M > 0 \ni \lvert a_n - A\rvert \leq M, \quad \forall n$	4. Definition 2.4'
5. $\lvert a_n - A\rvert \geq \lvert a_n\rvert - \lvert A\rvert, \ \forall n$ or $\lvert a_n\rvert \leq \lvert a_n - A\rvert + \lvert A\rvert, \ \forall n$	5. Absolute value theorem and property of inequalities
6. $\lvert a_n\rvert \leq M + \lvert A\rvert, \ \forall n$	6. Steps 4 and 5, Transitive property
7. Let $K = M + \lvert A\rvert$ and $\exists\ K \ni \lvert a_n\rvert \leq K, \ \forall n$ and $< a_n >$ is bounded.	7. Definition 2.4'

Since the contrapositive of a theorem is a statement equivalent to the theorem, the contrapositive of Theorem 3.1 gives the result that an unbounded sequence is a divergent sequence.

Notation: It is convenient to say that an unbounded monotone non-decreasing sequence diverges to $+\infty$, and an unbounded monotone non-increasing sequence diverges to $-\infty$. Keep in mind that the symbols $+\infty$ and $-\infty$ are not real numbers and cannot be treated as such.

Observe that the sequence $< a_n >$ in Example 3.4 is unbounded and monotone increasing. Hence the sequence diverges to $+\infty$.

The next three theorems concerning convergent sequences can be proved with the help of corresponding theorems about null sequences in Chapter II and hence are left for the reader to prove.

Theorem 3.2: Let $< a_n >$ be a convergent sequence such that $a_n \rightarrow A$. Then every subsequence $< a'_n >$ of $< a_n >$ also converges to A.

Proof: Left for the reader.

Corollary 3.2: If $\lim a_n = A$, then for any fixed integer k, $\lim a_{n+k} = A$.

Proof: Left for the reader. Hint: (Every terminal segment of a sequence is a subsequence.)

Theorem 3.3: Let $< a_n >$ and $< b_n >$ be two convergent sequences which converge to the same limit A. Suppose that $< c_n >$ is such that $a_n \leq c_n \leq b_n$, after a certain stage. Then $c_n \to A$ also.

Proof: Similar to Theorem 2.10.

Theorem 3.4: If $a_n \to A$ and if $< a'_n >$ is a rearrangement of $< a_n >$, then also $a'_n \to A$.

Proof: Similar to Theorem 2.11.

It has already been pointed out that the converse of Theorem 3.1 which states "a bounded sequence is a convergent sequence" is not necessarily true. However, if the bounded sequence is also monotone, then the sequence is convergent as proved in the next theorem.

Theorem 3.5: A bounded monotone sequence is convergent. If the sequence is monotone non-increasing, then its limit is the g.l.b. of the sequence. If the sequence is monotone non-decreasing, then its limit is the l.u.b. of the sequence.

Proof: (a) Suppose $< a_n >$ is monotone non-increasing.

1. Let $< a_n >$ be monotone non-increasing and bounded, and denote the greatest lower bound of the sequence $< a_n >$ by α	1. Assumption and notation
2. $\forall \epsilon > 0 \ \exists \ n_0 \ni a_{n_0} < \alpha + \epsilon$, and hence $a_n < \alpha + \epsilon, \ \forall n > n_0$	2. Definition 2.8' (ii), $< a_n >$ is non-increasing sequence
3. $a_n \geq \alpha, \quad \forall n$	3. Definition 2.8' (i)
4. Thus $\alpha \leq a_n < \alpha + \epsilon, \ \forall n > n_0$, or $\alpha - \epsilon < \alpha \leq a_n < \alpha + \epsilon, \ \forall n > n_0$, or $\alpha - \epsilon < a_n < \alpha + \epsilon, \ \forall n > n_0$, or $-\epsilon < a_n - \alpha < \epsilon, \ \forall n > n_0$, or $\lvert a_n - \alpha \rvert < \epsilon, \ \forall n > n_0$	4. Steps 2 and 3, $\alpha - \epsilon < \alpha$, transitive property of "less than," add $-\alpha$ to each term, and inequality written in absolute value form
5. Hence $\forall \epsilon > 0 \ \exists \ n_0 \ni \lvert a_n - \alpha \rvert < \epsilon, \ \forall n > n_0$	5. Steps 2 and 4
6. $\lim a_n = \alpha$ or $< a_n >$ is convergent with limit α	6. Step 5 and Definition 3.2

(b) It is left for the reader to prove that a bounded monotone non-decreasing sequence is convergent.

Definition 3.3: (Constant Sequence) If the sequence $< c_n >$ is such that $c_n = c, \ \forall n$; i.e. $c, c, c, \ldots, c, \ldots$, then the sequence is called a constant sequence.

Observe that a constant sequence is monotone and bounded, and hence the following corollary to Theorem 3.5.

Corollary 3.5: If $< c_n >$ is a constant sequence such that $c_n = c, \quad \forall n$, then $\lim c_n = c$.

Proof: Left for the reader.

It is sometimes convenient to have a relationship between the limit of a monotone non-decreasing sequence and a specified upper bound, hence a modified version of Theorem 3.5 is as follows:

Theorem 3.6: If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a sequence for which $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$, and if $\exists$ a number $K \ni a_n \leq K, \ \forall n$, then the given sequence $< a_n >$ has a limit and $\lim a_n \leq K$.

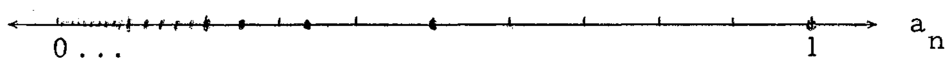
Proof:

1. $<a_n>$ is bounded and monotone non-decreasing, hence $\lim a_n = A$ where A is l.u.b. of $<a_n>$	1. Hypothesis and Theorem 3.5
2. K is an upper bound of $<a_n>$	2. Hypothesis and Definition 2.4'
3. $A \leq K$	3. Definition 2.7'
4. $\lim a_n \leq K$	4. Steps 1 and 3 and transitive property

The arguments used in Examples 3.3 and 3.4 to show that these sequences diverge are difficult and cumbersome. Part of the problem was that it was necessary to consider all possible candidates for the limit and prove that no one of them satisfies the definition. An easier way to show that a sequence is divergent is established after the next example. It depends on characterizing convergence by considering the relation between the terms of the sequence from some point on. Thus instead of investigating the distance from a point of the sequence to a fixed point, the points of the sequence are compared to each other.

<u>Example 3.5</u>: $a_n = 1/n$

The terms of this sequence are $1, 1/2, 1/3, \ldots, 1/n, \ldots$. Plot the terms on a number line. See Figure 3.3.



$1, 1/2, 1/3, \ldots, 1/n, \ldots$

Figure 3.3

Observe the following inequalities:

$\forall\, n > 1,\ |1 - 1/n| < 1;\ \forall\, n > 2,\ |1/2 - 1/n| < 1/2;\ \forall\, n > 3,\ |1/3 - 1/n| < 1/3;\ \ldots,\ \forall\, n > m,\ |1/m - 1/n| < 1/m;\ \forall\, n > m+1,\ |1/(m+1) - 1/n| < 1/(m+1),\ \ldots,$.

Suppose ϵ is chosen to be 2/5, then $|1/m - 1/n| < \epsilon,\ \forall\, n, m > 2$. Suppose ϵ is chosen to be 3/10, then $|1/m - 1/n| < \epsilon,\ \forall\, n, m > 3$. In general, if ϵ is chosen there will exist an $n_0(\epsilon)$ such that $|1/m - 1/n| < \epsilon,\ \forall\, n, m > n_0$. In examples above, if $\epsilon = 2/5$, then $n_0 = 2$. If $\epsilon = 3/10$, then $n_0 = 3$.

All convergent sequences behave in this manner and hence it is possible to show that a sequence is convergent without knowing its limit. This property of convergent sequences is called the Cauchy condition and is stated in the next theorem.

<u>Theorem 3.7</u>: (Cauchy's condition) If a sequence $< a_n >$ is convergent, then $\forall \epsilon > 0 \; \exists \; n_0 = n_0(\epsilon) \ni |a_n - a_m| < \epsilon$ for all pairs of indices $n, m > n_0$.

Proof:

1. $a_n \to A$ $\forall \epsilon > 0 \; \exists \; n_0 \ni |a_n - A| < \epsilon/2$, $\forall n > n_0$ and $|a_m - A| < \epsilon/2$, $\forall m > n_0$	1. Hypothesis and notation, and Definition 3.2
2. $\forall n, m > n_0$ $|a_n - a_m| = |(a_n - A) + (A - a_m)|$ $\leq |a_n - A| + |A - a_m|$ $< \epsilon/2 + \epsilon/2 = \epsilon$ or simply $\forall n, m > n_0 \;\; |a_n - a_m| < \epsilon$	2. Add and subtract the number A, Triangle inequality, step 1 and $|a_m - A| = |A - a_m|$, transitive property of $=, \leq$, and $<$
3. Hence $\forall \epsilon > 0 \; \exists \; n_0 \ni |a_n - a_m| < \epsilon$ $\forall n, m > n_0$	3. Steps 1 and 2

The converse of this theorem is true for real numbers. The proof of this depends on the completeness of the real numbers and hence is omitted from this paper.

The contrapositive of Theorem 3.7 gives the result that "if $< a_n >$ is a given sequence and

$$\exists\, \epsilon_0 > 0 \ni \forall n \exists n_0, m_0 > n \ni |a_{n_0} - a_{m_0}| \geq \epsilon_0,$$

then the sequence $< a_n >$ is divergent."

The following example will illustrate this fact.

Example 3.6: $a_n = (-1)^{n+1}$

It is easier to show that this sequence is divergent by using the contrapositive of the Cauchy condition rather than the method used in Example 3.3. Let $\epsilon = 2$ and show that $\forall n \exists n_0, m_0 > n \ni |a_{n_0} - a_{m_0}| \geq 2$. For every n chosen let n_0 be an odd integer greater than n, then $a_{n_0} = 1$ and let m_0 be an even integer greater than n, then $a_{m_0} = -1$, then $|a_{n_0} - a_{m_0}| = |1 - (-1)| = 2$. Hence by the contrapositive of the Cauchy condition, this sequence is divergent.

If in the sequence $< a_n >$ all the terms are different from zero, it is possible to construct a sequence $< 1/a_n >$ whose terms are multiplicative inverses of the terms of the sequence $< a_n >$. The following theorem establishes a useful property of the sequence $< 1/a_n >$ in the event that $< a_n >$ converges to a limit A which is not zero. In particular it is shown that $< 1/a_n >$ is a bounded sequence. This is quite useful in proving theorems about sequences whose terms are written as quotients.

Theorem 3.8: If a convergent sequence $< a_n >$ has all its terms different from zero, and if its limit A is also not equal to 0, then the numbers $|a_n|$ possess a positive lower bound, and the sequence $< 1/a_n >$ is bounded.

Proof:

1. Let $\epsilon = 1/2\ |A| > 0$	1. $A \neq 0$ in Hypothesis

2. $\exists\, n_0 \ni \lvert a_n - A\rvert < 1/2\, \lvert A\rvert, \quad \forall n > n_0$	2. $< a_n >$ is convergent, Definition 3.2
3. $\forall n > n_0$ $\lvert A\rvert - \lvert a_n\rvert \le \lvert A - a_n\rvert$ $= \lvert a_n - A\rvert$ $< 1/2\, \lvert A\rvert$ or $\lvert A\rvert - \lvert a_n\rvert < 1/2\, \lvert A\rvert$ or $\lvert A\rvert - 1/2\lvert A\rvert < \lvert a_n\rvert$ or $\lvert a_n\rvert > 1/2\, \lvert A\rvert$	3. Step 2, absolute value theorems, Transitive property of $\le$, $=$ and $<$, properties of inequalities
4. Consider the positive numbers $\lvert a_1\rvert, \lvert a_2\rvert, \lvert a_3\rvert, \ldots, \lvert a_{n_0}\rvert$, and $1/2\lvert A\rvert$	4. Assumption
5. Let k be the smallest of the positive numbers above, denoted by $k = \min(1/2\, \lvert A\rvert, \lvert a_1\rvert, \lvert a_2\rvert, \lvert a_3\rvert, \ldots, \lvert a_{n_0}\rvert)$ and $k > 0$	5. There is a smallest positive number in a finite set of positive numbers
6. $\lvert a_n\rvert \ge k > 0, \quad \forall n$ or $0 < k \le \lvert a_n\rvert, \quad \forall n$	6. Steps 3 and 5
7. $0 < \lvert \frac{1}{a_n} \rvert \le k, \quad \forall n$	7. If $0 < a \le b$, then $0 < 1/b \le 1/a$
8. $< 1/a_n >$ is bounded	8. Step 7, Definition 2.4'

<u>Example 3.7</u>: $a_n = 1 + \frac{1}{n} = \frac{n+1}{n}$

The terms of this sequence are

$$2, 3/2, 4/3, \ldots, \frac{n+1}{n}, \ldots,$$

and the limit was shown to be the number 1 in Example 3.2. All the terms are different from zero, and the limit is 1 which is not equal to zero. By Theorem 3.8, the sequence $< 1/a_n >$ which is $< n/(n+1) >$ is bounded. The terms of this new sequence are

$$1/2, 2/3, 3/4, \ldots, \frac{n}{n+1}, \ldots$$

and obviously

$$\frac{n}{n+1} < 1, \quad \forall n.$$

Calculations with convergent sequences are based on the next four theorems. They will allow the reader to find some very simple limits without using the definition.

<u>Theorem 3.9</u>: If $< a_n >$ and $< b_n >$ are sequences such that $a_n \to A$ and $b_n \to B$, then $< a_n + b_n >$ converges to $A + B$.

Proof:

1. $a_n \to A$ implies $< a_n - A >$ is a null sequence	1. Hypothesis and Definition 3.1
2. $b_n \to B$ implies $< b_n - B >$ is a null sequence	2. Hypothesis and Definition 3.1
3. Then $< c_n >$ with $c_n = (a_n - A) + (b_n - B) = (a_n + b_n) - (A + B)$ is a null sequence	3. Theorem 2.9 and algebra

4. $<(a_n + b_n) - (A + B)>$ is a null sequence implies that $<a_n + b_n>$ converges to $A + B$	4. Step 3, and Definition 3.1

<u>Theorem 3.10:</u> If $<a_n>$ and $<b_n>$ are sequences such that $a_n \rightarrow A$ and $b_n \rightarrow B$, then $<a_n - b_n>$ converges to $A - B$.

Proof: Similar to Theorem 3.9. Hint: Use Theorem 2.6 to show that $<-(b_n - B)>$ is a null sequence.

<u>Theorem 3.11:</u> If $<a_n>$ and $<b_n>$ are sequences such that $a_n \rightarrow A$ and $b_n \rightarrow B$, then $<a_n b_n>$ converges to AB.

Proof:

1. $a_n \rightarrow A$ and $<a_n - A>$ is a null sequence	1. Hypothesis and Definition 3.1
2. $b_n \rightarrow B$ and $<b_n - B>$ is a null sequence	2. Hypothesis and Definition 3.1
3. $a_n b_n - AB = a_n b_n - Ab_n + Ab_n - AB$ $= (a_n - A)b_n + (b_n - B)A, \ \forall n$	3. Add and Subtract the term Ab_n and group terms
4. $<b_n>$ is bounded	4. Step 2 and Theorem 3.1
5. $<(a_n - A)b_n>$ is a null sequence	5. Step 1 and 4, Theorem 2.6
6. $<A>$ is bounded	6. Constant sequence is convergent by Corollary 3.5 and bounded by Theorem 3.1

7. $< (b_n - B)A >$ is a null sequence — 7. Steps 2 and 6, Theorem 2.6

8. $< a_n b_n - AB >$ is a null sequence — 8. Steps 3, 5, and 7, Theorem 2.9

9. $< a_n b_n >$ converges to AB — 9. Step 8 and Definition 3.1

<u>Corollary 3.11</u>: If $\lim a_n = A$, then $\lim ka_n = kA$ where k is a real number such that $k \neq 0$.

Proof: Left for the reader.

<u>Theorem 3.12</u>: If $< a_n >$ and $< b_n >$ are sequences such that $a_n \to A$, $b_n \to B$, every $b_n \neq 0$ and $B \neq 0$, then $< a_n/b_n >$ converges to A/B.

Proof:

1. $a_n \to A$ implies that $< a_n - A >$ is a null sequence — 1. Hypothesis and Definition 3.1

2. $b_n \to B$ implies that $< b_n - B >$ is a null sequence — 2. Hypothesis and Definition 3.1

3. $$\frac{a_n}{b_n} - \frac{A}{B} = \frac{a_nB - b_nA}{b_nB}$$
$$= \frac{a_nB - AB + AB - b_nA}{b_nB}$$
$$= \frac{(a_n - A)B - (b_n - B)A}{b_nB}$$
— 3. Common denominator, Add and subtract AB in the numerator and group terms

4. $< B >$ and $< A >$ are bounded sequences	4. Constant sequence is convergent by Corollary 3.5 and bounded by Theorem 3.1
5. $< (a_n - A) B >$ is a null sequence	5. Steps 1 and 4 and Theorem 2.6
6. $< (b_n - B) A >$ is a null sequence	6. Steps 2 and 4 and Theorem 2.6
7. $< -1 >$ is a bounded sequence, hence $< -(b_n - B)A >$ is a null sequence	7. Constant sequence is convergent by Corollary 3.5 and bounded by Theorem 3.1, and use Theorem 2.6 on Step 6
8. $< (a_n - A)B + [-(b_n - B)A] >$ is a null sequence	8. Steps 5 and 7, Theorem 2.9
9. $< Bb_n >$ is convergent, $b_n \neq 0$, $\forall n$, $B \neq 0$	9. Step 2, Corollary 3.11, Hypothesis
10. $< 1/Bb_n >$ is bounded	10. Theorem 3.8
11. Therefore $< \frac{a_n}{b_n} - \frac{A}{B} >$ is a null sequence	11. Steps 3, 8, and 10, Theorem 2.6
12. $< \frac{a_n}{b_n} >$ converges to $\frac{A}{B}$	12. Step 11, and Definition 3.1

<u>Corollary 3.12</u>: If $< a_n >$ is a sequence such that $a_n \to A$, $a_n \neq 0$, $\forall n$ and $A \neq 0$, then $< 1/a_n >$ converges to $1/A$.

Proof: Left for the reader.

Example 3.8: Find the lim $\frac{2n - 2}{3n + 7}$.

$$\frac{2n - 2}{3n + 7} = \frac{2 - 2/n}{3 + 7/n}$$ Multiply numerator and denominator by $1/n$

Then

$$\lim \frac{2n - 2}{3n + 7} = \lim \frac{2 - 2/n}{3 + 7/n}$$

$$= \frac{\lim (2 - 2/n)}{\lim (3 + 7/n)}$$ Theorem 3.12

$$= \frac{\lim 2 - \lim 2/n}{\lim 3 + \lim 7/n}$$ Theorem 3.10 and Theorem 3.9

$$= \frac{\lim 2 - \lim (2 \cdot 1/n)}{\lim 3 + \lim (7 \cdot 1/n)}$$ Factor $2/n$ and $7/n$

$$= \frac{2 - \lim (2 \cdot 1/n)}{3 + \lim (7 \cdot 1/n)}$$ Corollary 3.5

$$= \frac{2 - 0}{3 + 0}$$ $\lim 1/n = 0$ and Corollary 3.11

$$= \frac{2}{3}$$ Calculations

Therefore,

$$\lim \frac{2n - 2}{3n + 7} = \frac{2}{3}$$ Transitive property of equality

If a sequence $< a_n >$ is convergent to the real number A, then any ϵ - neighborhood of A contains all the terms of the sequence with the possible exception of a finite number at most. There is contained in any neighborhood of A certainly an infinite number of terms of the sequence. It is possible for a sequence $< b_n >$ to have the property that any ϵ -neighborhood of some real number B contains an infinite number of terms of the sequence, but an infinite number of the terms are outside the neighborhood. The condition that an ϵ-neighborhood contain an infinite number of terms of the sequence is a weaker condition than the condition that a sequence have a limit but important enough to be considered and given a special name.

Definition 3.4: (Cluster point of a sequence) A number A is called a cluster point of a given sequence $< a_n >$ if for every $\epsilon > 0$, there is always an infinite number of indices n for which $|a_n - A| < \epsilon$; i.e. $\forall \epsilon > 0$ and $\forall$ n, $\exists\, n_0 \geq n \ni |a_{n_0} - A| < \epsilon$.

Observe that if an $n_1 \ni n_1 \geq n$ cannot be found for every n then there would be only a finite number of indices n $\ni |a_n - A| < \epsilon$. The distinction between this definition and the definition of a limit is the fact that $|a_n - A| < \epsilon$ needs to be fulfilled not for every n after a certain stage, but only for any infinite number of n's, and therefore in particular for at least one n_0 beyond every n. This is a good example to illustrate the importance of specifying the indices n for which a mathematical sentence is true. Observe that the limit of a convergent sequence $< a_n >$ is always a cluster point of the sequence, but a cluster point of the sequence is not necessarily the limit of the sequence. The definition of a limit states a stronger condition than the definition of a cluster point, or the definition of a cluster point states a weaker condition than the definition of a limit. The reader has experienced similar situations in algebra. If $x < y$, then $x \leq y$; but if $x \leq y$, then $x < y$ or $x = y$. So $x \leq y$ is a weaker condition than

$x < y$.

Example 3.9: $a_n = (-1)^n (1 + 1/n)$

The terms of this sequence are

$$-2, 3/2, -4/3, 5/4, \ldots, (-1)^n (1 + 1/n), \ldots .$$

Plot the terms on a rectangular coordinate system. See Figure 3.4.

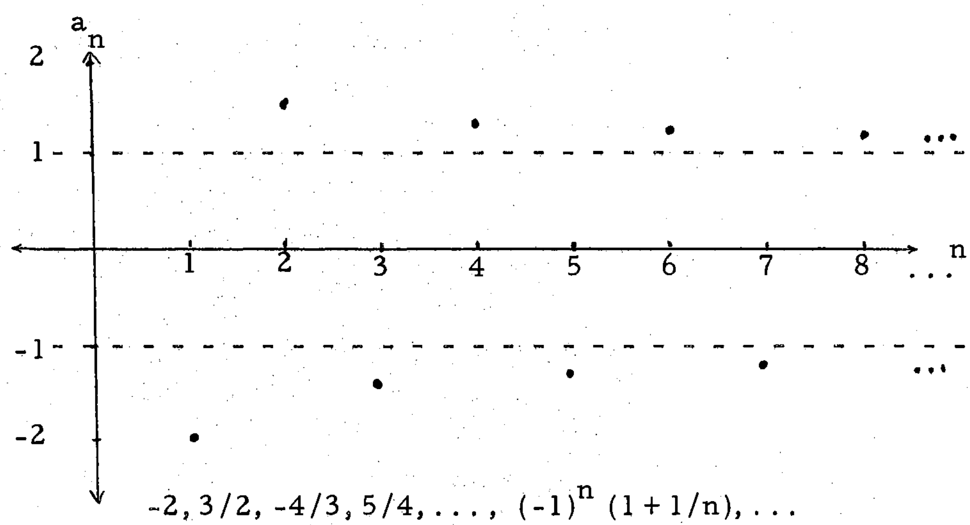


Figure 3.4

When n is even,

$$|(-1)^n(1 + 1/n) - 1| = |1 + 1/n - 1| = |1/n| = 1/n$$

and $1/n < \epsilon$ if $n > 1/\epsilon$. Hence

$$\forall \epsilon > 0 \ \exists \ n_0 = 1/\epsilon \ \ni |(-1)^n(1 + 1/n) - 1| < \epsilon$$

for infinitely many even indices $n > 1/\epsilon$. Thus the number 1 is a cluster point of the sequence. When n is odd,

$$|(-1)^n(1 + 1/n) - (-1)| = |-1 - 1/n + 1| = |-1/n| = 1/n$$

and $1/n < \epsilon$ if $n > 1/\epsilon$. Hence

$$\forall \epsilon > 0 \ \exists \ n_0 = 1/\epsilon \ \ni |(-1)^n(1 + 1/n) - (-1)| < \epsilon$$

for infinitely many odd indices $n > 1/\epsilon$. Thus the number -1 is a cluster point of the sequence.

It is easy to show by the contrapositive of Cauchy's condition that this sequence has no limit and hence is divergent. This is a particular case of the following more general theorem. If a sequence has more than one cluster point, it cannot have a limit. This can be proved directly by using the contrapositive of Cauchy's condition. In this discussion it is obtained as a direct consequence of Theorem 3.17.

Example 3.10: $a_n = 1 + \sin \frac{n\pi}{2}$

The terms of this sequence are

$$2, 1, 0, 1, 2, 1, 0, 1, \ldots, 1 + \sin \frac{n\pi}{2}, \ldots .$$

The cluster points for this sequence are 0, 1, and 2, but there is no limit of the sequence.

Example 3.11:

$$a_n = \begin{cases} 1/2 & \text{if } n = 1, 4, 7, 10, \ldots \\ 1/3 & \text{if } n = 2, 5, 8, 11, \ldots \\ 1/4 & \text{if } n = 3, 6, 9, 12, \ldots \end{cases}$$

The cluster points are 1/2, 1/3, and 1/4, but there is no limit of the sequence.

It can be proved that if an infinite number of terms of a sequence lie in a bounded interval $[a, b]$, then there exists at least one cluster point of the sequence in that interval. If a sequence is bounded above and has one or more cluster points, there is a least upper bound of the set consisting of all the cluster points. This least upper bound of cluster points is also a cluster point of the sequence as the next theorem will prove.

Theorem 3.13: If a sequence is bounded above and has one or more cluster points, then the least upper bound of all cluster points is a cluster point.

Proof:

1. Let $< a_n >$ be a sequence which is bounded above and has one or more cluster points	1. Hypothesis
2. If there is a finite number of cluster points, the l.u.b. is the largest one, and hence it is a cluster point.	2. Trichotomy Axiom
3. If there is an infinite number of cluster points, let A = l.u.b. of the cluster points and let C be an arbitrary cluster point.	3. Notation
4. $C \leq A, \ \forall C$ and $\forall \epsilon > 0 \ \exists C_0 \ni C_0 > A - \epsilon/2$	4. Definition 2.7'

5. This means $A - \epsilon/2 < C_0 \leq A$
or $A - \epsilon/2 < C_0 \leq A < A + \epsilon/2$
or $A - \epsilon/2 < C_0 < A + \epsilon/2$
or $-\epsilon/2 < C_0 - A < \epsilon/2$
or $|C_0 - A| < \epsilon/2$

5. Step 4, $A < A + \epsilon/2 \ \forall \epsilon$, Transitive property, add $-A$ to each term, inequality written in absolute value form

6. $|a_n - C_0| < \epsilon/2$ for infinitely many a_n

6. C_0 is a cluster point and Definition 3.4

7. $\forall n, \ |a_n - A| = |(a_n - C_0) + (C_0 - A)| \leq |a_n - C_0| + |C_0 - A|$

7. Add and subtract C_0, triangle inequality

8. $|a_n - A| \leq |a_n - C_0| + |C_0 - A| < \epsilon/2 + \epsilon/2 = \epsilon$
or
$|a_n - A| < \epsilon$ for infinitely many a_n

8. Steps 5, 6, 7, transitive property

9. Thus $\forall \epsilon > 0 \ |a_n - A| < \epsilon$ for infinitely many n and hence A is a cluster point.

9. Steps 4, 8 and Definition 3.4

Definition 3.5: (Limit Superior, Greatest limit, or Upper limit) Let $< a_n >$ be a sequence which is bounded above. The least upper bound of its cluster points is called the limit superior, greatest limit or upper limit.

Notation: $\overline{\lim}\, a_n$ or lim sup a_n means "the limit superior of a sequence $< a_n >$".

Example 3.12: In Example 3.9, the lim sup $a_n = 1$. In Example 3.10, the lim sup $a_n = 2$. In Example 3.11, the lim sup $a_n = 1/2$.

In a similar manner, if a sequence is bounded below and has one or more cluster points, there is a greatest lower bound of all cluster points. It is also a cluster point as the next theorem states.

Theorem 3.14: If a sequence is bounded below and has one or more cluster points, then the greatest lower bound of all cluster points is a cluster point.

Proof: Left for the reader.

Definition 3.6: (Limit inferior, Least limit, or Lower limit) Let $< a_n >$ be a sequence which is bounded below. The greatest lower bound of its cluster points is called the limit inferior, least limit, or lower limit.

Notation: $\underline{\lim}\, a_n$ or lim inf a_n means "the limit inferior of a sequence $< a_n >$".

Example 3.13: In Example 3.9, the lim inf $a_n = -1$. In Example 3.10, the lim inf $a_n = 0$. In Example 3.11, the lim inf $a_n = 1/4$.

Theorem 3.15: Let $< a_n >$ be a given sequence. Then $\overline{\lim}\, a_n = A$ iff for each $\epsilon > 0$ it is true that $a_n < A + \epsilon$ for all but a finite number of terms, and $A - \epsilon < a_n$ for infinitely many terms.

Proof: (a) Assume that $\forall \epsilon > 0$ it is true that $a_n < A + \epsilon$ for all but a finite number of terms, and $A - \epsilon < a_n$ for infinitely many terms, then prove that $\overline{\lim}\, a_n = A$.

1. $\forall \epsilon > 0 \ \exists n_0 \ni a_n < A + \epsilon, \ \forall n > n_0$ and $A - \epsilon < a_n$ for infinitely many terms	1. Hypothesis and meaning of all but a finite number of terms
2. $A - \epsilon < a_n < A + \epsilon$ or $-\epsilon < a_n - A < \epsilon$ or $\lvert a_n - A \rvert < \epsilon$ for infinitely many terms. Hence A is a cluster point, and it remains to be shown that A is the l.u.b. of all cluster points.	2. Step 1, add -A to each term, inequality written in absolute value form, Definition 3.4
3. Suppose B is a cluster point $\ni$ $A < B$, then $\lvert A - B \rvert = k > 0$	3. Assumption and difference between two real numbers
4. $\forall \epsilon > 0 \ \lvert a_n - B \rvert < \epsilon$ for infinitely many terms	4. Step 3 and Definition 3.4
5. Let $\epsilon = k/3$ and there exists only a finite number of terms $a_n \ni a_n > A + k/3$	5. A is a cluster point and $\lvert a_n - A \rvert < k/3$ for all but a finite number of terms
6. Hence $\lvert a_n - B \rvert < k/3$ is true for at most a finite number of terms	6. All $a_n \ni \lvert a_n - B \rvert < k/3$ must be terms greater than $A + k/3$
7. Hence the assumption that B is a cluster point $\ni A < B$ leads to a contradiction and A is the l.u.b of the cluster points or $\overline{\lim}\, a_n = A$	7. Steps 4 and 6, Definition 3.5

(b) Assume that $\overline{\lim}\ a_n = A$, then prove that $\forall\ \epsilon > 0$ it is true that $a_n < A + \epsilon$ for all but a finite number of terms, and $A - \epsilon < a_n$ for infinitely many terms.

1. $\overline{\lim}\ a_n = A$ implies A is the l. u. b. of all cluster points and A is a cluster point.	1. Hypothesis, Definition 3. 5, Theorem 3. 13
2. $\forall\ \epsilon > 0\ \lvert a_n - A\rvert < \epsilon$ for infinitely many indices or $A - \epsilon < a_n < A + \epsilon$ or simply $A - \epsilon < a_n$ for infinitely many indices	2. Definition 3. 4, and if infinitely many terms between two numbers then infinitely many greater than the smaller one
3. Suppose that $\forall\ \epsilon > 0\ \exists$ infinitely many indices $\ni a_n \geq A + \epsilon$. Then some number larger than A would be a cluster point	3. Assumption and a bounded interval with infinite number of points has at least one cluster point.
4. This contradicts the fact that A is the l. u. b. of all cluster points. Hence a finite number of indices such that $a_n > A + \epsilon$	4. Steps 1 and 3
5. $\forall \epsilon > 0\ \ a_n < A + \epsilon$ for all but a finite number of terms and $A - \epsilon < a_n$ for infinitely many terms	5. Steps 2 and 4

Theorem 3.16: Let $< a_n >$ be a given sequence. Then $\underline{\lim}\, a_n = A$ iff for each $\epsilon > 0$ it is true that $A - \epsilon < a_n$ for all but a finite number of terms, and $a_n < A + \epsilon$ for infinitely many terms.

Proof: Similar to Theorem 3.15.

The lim sup a_n and lim inf a_n have been defined, and Theorems 3.15 and 3.16 proved independently of any considerations of convergence. It may be shown that specific knowledge of the lim sup a_n and lim inf a_n suffices to decide whether the sequence converges. The following theorem will demonstrate this.

Theorem 3.17: The sequence $< a_n >$ is convergent iff its lower and upper limits are equal. If A is their common value, then $a_n \to A$; i.e. $\lim\inf a_n = \lim a_n = \lim\sup a_n$.

Proof: (a) Assume $\lim\inf a_n = \lim\sup a_n = A$, then show that $a_n \to A$.

1. $\lim\inf a_n = A = \lim\sup a_n$	1. Hypothesis
2. $\forall\, \epsilon > 0 \quad A - \epsilon < a_n$ for all but a finite number of terms or $\exists\, n_0 \ni A - \epsilon < a_n,\ \forall\, n > n_0$	2. Theorem 3.16 and meaning of all but a finite number
3. $\forall\, \epsilon > 0 \quad a_n < A + \epsilon$ for all but a finite number of terms or $\exists\, n_1 \ni a_n < A + \epsilon,\ \forall\, n > n_1$	3. Theorem 3.15 and meaning of all but a finite number
4. Let $N = \max(n_0, n_1)$	4. Trichotomy Axiom

5. $\forall \epsilon > 0 \ \exists \ N \ni A - \epsilon < a_n < A + \epsilon$, $\forall n > N$, or $\|a_n - A\| < \epsilon$, $\forall n > N$	5. Steps 2, 3, 4, and inequality written as an absolute value statement
6. $a_n \to A$	6. Definition 3.2

(b) Assume that $a_n \to A$, show that lim inf a_n = lim sup a_n. This part of the proof is left for the reader to prove.

Corollary 3.17: The sequence $< a_n >$ has a limit iff there exists exactly one cluster point of the sequence.

If a sequence has no upper bound, it is convenient to write lim sup $a_n = +\infty$, and if there is no lower bound, it is convenient to write lim inf $a_n = -\infty$.

Example 3.14: $a_n = (-1)^n n$

The terms of this sequence are $-1, 2, -3, 4, \ldots, (-1)^n n, \ldots$. The lim sup $a_n = +\infty$, and the lim inf $a_n = -\infty$.

Example 3.15: $3, -3, 2, -2, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots$

Find the (a) l. u. b., (b) g. l. b., (c) lim sup and (d) lim inf for this sequence.

CHAPTER IV

SERIES

In Chapter III it was shown that all convergent sequences satisfy the Cauchy condition as stated in Theorem 3.7. If a sequence $< a_n >$ is convergent, then

$$\forall \epsilon > 0 \ \exists \ n_0 = n_0 (\epsilon) \ni |a_n - a_m| < \epsilon$$

for all pairs of indices $n, m > n_0$. Without loss of generality the numbers n, m can be thought of in the following way. Let $n_0 < m < n$. Then $m = n - k$ for some positive integer k. In particular, if $k = 1$, then $m = n - 1$, and

$$\forall \epsilon > 0 \ \exists \ n_0 = n_0 (\epsilon) \ni |a_n - a_{n-1}| < \epsilon, \ \forall n > n_0.$$

This implies that if $< a_n >$ is convergent then $< a_n - a_{n-1} >$ is a null sequence. On the basis of the ideas above it is convenient and useful to think of a sequence as follows.

Let $< s_n >$ be a sequence. Form a new sequence considering the difference between each pair of terms as follows:

$$x_1 = s_1, \ x_2 = s_2 - s_1, \ x_3 = s_3 - s_2, \ldots, \ x_n = s_n - s_{n-1}, \ldots .$$

The sequence $< x_n >$ is called the difference sequence.

Observe that since $s_1 = x_1$, $s_2 = s_1 + x_2$, and thus $s_2 = x_1 + x_2$, $s_3 = s_2 + x_3$, and thus $s_3 = x_1 + x_2 + x_3$. In general, for each n, $s_n = s_{n-1} + x_n$ and thus $s_n = x_1 + x_2 + x_3 + \ldots + x_n$.

Notation:

$$\sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \ldots + x_n = s_n$$

where Σ means summation and k = 1 to n is referred to as the index on the summation.

The sequence $< s_n >$ can be exhibited in terms of its differences by writing

$$< s_n > = < x_1 + x_2 + x_3 + \ldots + x_n >$$

or

$$< s_n > = < \sum_{k=1}^{n} x_k >.$$

When a sequence is written in terms of its differences it is said to be written in infinite series form or as an infinite series.

Thus for any given sequence $< s_n >$ its sequence of differences $< x_n >$ can be formed by calculating $x_n = s_n - s_{n-1}$. On the other hand if a sequence $< x_n >$ is given the sequence for which $< x_n >$ is the sequence of differences can be determined by calculating

$$s_n = x_1 + x_2 + x_3 + \ldots + x_n.$$

If the sequence s_n converges, the lim s_n exists and is equal to a real number s. That is,

$$\lim \sum_{k=1}^{n} x_k$$

exists and its value is s. Since

$$s_n = \sum_{k=1}^{n} x_k,$$

it seems logical to use the notation

$$\lim s_n = \sum_{k=1}^{\infty} x_k = s.$$

Notation: The notation

$$\sum_{k=1}^{\infty} x_k$$

is used to represent the sequence $< s_n >$ in addition to the $\lim s_n$. The context in which the notation is used will make clear which meaning is intended. Sometimes the notation

$$\sum_{k=1}^{\infty} x_k$$

is written as follows:

$$\sum_{1}^{\infty} x_k, \quad \sum_{n} x_k, \quad \sum x_k,$$

or

$$x_1 + x_2 + x_3 + \ldots + x_n + \ldots .$$

The preceding discussion can be summarized formally in the following definitions.

Definition 4.1: (Infinite Series) The notation

$$x_1 + x_2 + x_3 + \ldots + x_n + \ldots$$

is used to represent the sequence $< s_n >$ where

$$s_n = x_1 + x_2 + x_3 + \ldots + x_n.$$

A sequence is called an infinite series when written in this form.

Definition 4.2: (Terms of a series, Partial sums of a series)

In the infinite series $x_1 + x_2 + x_3 + \ldots + x_n + \ldots$, the numbers x_n are called the terms of the series, and the numbers s_n are called the partial sums of the series.

Definition 4.3: (Convergent series, Divergent series) The infinite series $x_1 + x_2 + x_3 + \ldots + x_n + \ldots$ is a convergent series iff $\lim s_n$ exists. If the $\lim s_n$ does not exist, the series is divergent.

Definition 4.4: (Value of a series) If the infinite series $x_1 + x_2 + x_3 + \ldots + x_n + \ldots$ converges, then $\lim s_n = s$ where s is a real number. The real number s is called the value of the series.

Observe that the value of a series is not a sum as defined by the binary operation of addition. The calculation of

$$\sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \ldots + x_n$$

is an accurate use of the binary operation of addition performed (n-1) times. On the other hand the evaluation of

$$\sum_{k=1}^{\infty} x_k$$

involves the calculation of the limit of a sequence as discussed in Chapter III.

Consider Theorem 3.7 and its converse in terms of the preceding definitions above about infinite series. This yields the following statement: The necessary and sufficient condition for the convergence of the series Σx_n is that for every $\epsilon > 0$, there exists a number $n_0 = n_0(\epsilon)$ such that for every $n, m > n_0$, $|s_m - s_n| < \epsilon$. Without loss generality, m can be considered to be greater than n, i.e. $n_0 < n < m$.

Then $m = n + p$ for $p \geq 1$ and p is an integer. Hence Theorem 3.7 and its converse can be expressed as follows:

Let Σx_n be an infinite series and let s_n represent the nth partial sum. Σx_n converges iff

$$\forall \epsilon > 0 \ \exists n_0 = n_0(\epsilon) \ni \forall n > n_0$$

and

$$p \geq 1, \quad |s_{n+p} - s_n| < \epsilon .$$

Since

$$s_n = x_1 + x_2 + x_3 + \ldots + x_n$$

and

$$s_{n+p} = x_1 + x_2 + x_3 + \ldots + x_n + \ldots + x_{n+p},$$

then

$$s_{n+p} - s_n = x_{n+1} + x_{n+2} + x_{n+3} + \ldots + x_{n+p}.$$

A useful way to express the definition of a convergent series is to use the above information as follows:

Definition 4.3': (Convergent Series) Σx_n converges iff

$$\forall \epsilon > 0 \ \exists n_0 = n_0(\epsilon) \ni \forall n > n_0$$

and

$$p \geq 1, \quad |x_{n+1} + x_{n+2} + x_{n+3} + \ldots + x_{n+p}| < \epsilon .$$

The first theorem in this chapter is a result of Definition 4.3' using $p = 1$. A detailed proof is given to reinforce the idea that the difference sequence $< x_n >$ is a null sequence if the sequence $< s_n >$ converges.

Theorem 4.1: If the series Σx_n converges, then the sequence

$< x_n >$ is a null sequence.

Proof:

1. Σx_n converges	1. Hypothesis
2. $\forall\ \epsilon > 0\ \exists\ n_0 = n_0(\epsilon) \ni \forall\ n > n_0$ and $\forall\ p \geq 1$ $\lvert x_{n+1} + x_{n+2} + \ldots + x_{n+p} \rvert < \epsilon$	2. Definition 4.3'
3. Let p = 1, then $\forall\ \epsilon > 0$ $\exists\ n_0 = n_0(\epsilon) \ni \forall n > n_0\ \lvert x_{n+1} \rvert < \epsilon$	3. Step 2 is true $\forall\ p \geq 1$, hence is true if p = 1
4. $< x_{n+1} >$ is a null sequence and $\lim x_{n+1} = 0$	4. Step 3 and Definition 2.10
5. $\lim x_n = 0$ and $< x_n >$ is a null sequence	5. Step 4 and Corollary 3.2

Theorem 4.2: If $\lim x_n \neq 0$, then the series Σx_n diverges. (Contrapositive of Theorem 4.1)

In checking a series for convergence, Theorem 4.2 is quite useful if the $\lim x_n \neq 0$. If $\lim x_n = 0$, no information is obtained from this. The converse of Theorem 4.1 is not true. The condition $\lim x_n = 0$ is only a necessary condition for convergence. It is not a sufficient condition as the following example of the harmonic series will show.

Example 4.1: (Harmonic Series) The series

$$1 + 1/2 + 1/3 + \ldots + 1/n + \ldots$$

is called the harmonic series. Observe that $x_n = 1/n$ and $\lim 1/n = 0$, but this series is divergent. This is seen by showing that the sequence of partial sums does not satisfy the Cauchy condition.

$$s_{2n} = \underbrace{1 + 1/2 + 1/3 + \ldots + 1/n + \ldots + 1/2n}_{2n \text{ terms}}$$

$$s_n = \underbrace{1 + 1/2 + 1/3 + \ldots + 1/n}_{n \text{ terms}}$$

$$s_{2n} - s_n = \underbrace{1/(n+1) + 1/(n+2) + 1/(n+3) + \ldots + 1/2n}_{n \text{ terms}}$$

The above statements are true for all n.

$1/(n+1) + 1/(n+2) + 1/(n+3) + \ldots + 1/2n \geq 1/2n + 1/2n + 1/2n + \ldots + 1/2n$

since

$$1/(n+1) \geq 1/2n, \ 1/(n+2) > 1/2n, \ \ldots, \ 1/2n = 1/2n$$

in the expression for $s_{2n} - s_n$.

$$\underbrace{1/2n + 1/2n + 1/2n + \ldots + 1/2n}_{n \text{ terms}} = n(1/2n) = 1/2 \ .$$

Hence $s_{2n} - s_n \geq 1/2$ for all n. Let $\epsilon = 1/2$ and

$$\forall n \ \exists \ 2n, \ n \geq n \ \ni |s_{2n} - s_n| \geq 1/2.$$

Therefore the harmonic series is divergent by the contrapositive of Theorem 3.7.

Observe that the representation of a series as a summation is not unique. The series $1 + 1/2 + 1/3 + \ldots + 1/n + \ldots$ can be written

$$\sum_{n=1}^{\infty} 1/n, \quad \sum_{k=1}^{\infty} 1/k,$$

or

$$1 + 1/2 + \sum_{n=3}^{\infty} 1/n.$$

It can also be expressed as

$$\sum_{n=0}^{\infty} 1/(n+1)$$

or

$$\sum_{n=3}^{\infty} 1/(n-2).$$

How a series is expressed will depend upon the situation in which it occurs. In general,

$$\sum_{n=0}^{\infty} a_n, \quad a_0 + \sum_{n=1}^{\infty} a_n,$$

or

$$a_0 + a_1 + a_2 + \ldots + a_p + \sum_{n=p+1}^{\infty} a_n$$

represent the same series.

The following examples will be about series with partial sums that are easy to examine. This is in general not the case. More sophisticated methods for handling series in general will be developed later.

<u>Example 4.2</u>: (Geometric Series)

$$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

From the identity

$$1 - r^n = (1 - r)(1 + r + r^2 + \ldots + r^{n-1})$$

it is easy to deduce the nth partial sum s_n of the geometric series.

$$s_n = \sum_{k=1}^{n} ar^{k-1}$$

$$= a\sum_{k=1}^{n} r^{k-1}$$

$$= a(1 + r + r^2 + \ldots + r^{n-1})$$

$$= \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{a}{1 - r} - \frac{ar^n}{1 - r} \quad (r \neq 1)$$

$$s_n = a + a + a + a + \ldots + a = na$$

if $r = 1$. The $\lim r^n = 0$ if $|r| < 1$, since Example 2.23 proves that $<r^n>$ is a null sequence if $|r| < 1$. So consider the nth partial sum s_n of the geometric series when a is a real number such that $a \neq 0$ and r is a real number such that $|r| < 1$.

$$\lim s_n = \lim \left(\frac{a}{1 - r} - \frac{ar^n}{1 - r}\right)$$

$$= \lim \frac{a}{1 - r} - \lim\left(\frac{a}{1 - r} \cdot r^n\right) \quad \text{by Theorem 3.10}$$

$$= \frac{a}{1 - r} - \lim\left(\frac{a}{1 - r} \cdot r^n\right) \quad \text{by Corollary 3.5}$$

$$= \frac{a}{1 - r} - 0 \quad \text{by Corollary 3.11 and Example 2.23}$$

$$= \frac{a}{1 - r} \quad \text{by subtraction}$$

Thus the geometric series

$$\sum_{k=1}^{\infty} ar^{k-1}$$

converges and has the value $\frac{a}{1-r}$ if $|r| < 1$.

Suppose $a = 3$, $r = 1/2$, then

$$\sum_{k=1}^{\infty} 3(1/2)^{k-1} = \frac{3}{1 - 1/2} = \frac{3}{1/2} = 6.$$

If $a = 1$ and $r = 1/2$, then

$$1 + 1/2 + 1/4 + \ldots + (1/2)^{k-1} + \ldots = \frac{1}{1 - 1/2} = \frac{1}{1/2} = 2.$$

If $a = 1$ and $r = -1/2$, then

$$1 - 1/2 + 1/4 + \ldots + (-1/2)^{k-1} + \ldots = \frac{1}{1 - (-1/2)} = \frac{1}{3/2} = 2/3.$$

Observe that not only is convergence known for particular values of r, but the value of the series is also known. The geometric series is a very important series to know and use in more complicated situations as will be revealed in the following chapters. Since $< r^n >$ is unbounded when $|r| > 1$, the sequence $< ar^n >$ is unbounded and hence the sequence $< s_n >$ is unbounded. By the contrapositive of Theorem 3. 1, the sequence $< s_n >$ is divergent, and the geometric series diverges if $|r| > 1$. If $r = 1$, $s_n = na$ and $< s_n >$ is unbounded, hence series diverges. If $r = -1$, s_n is bounded but $|s_n - s_{n-1}| = a$, $\forall n$. Hence Cauchy's condition is not satisfied and the series diverges.

<u>Example 4. 3</u>: Every rational number has a decimal representation which is terminating or repeating, and conversely. A decimal representation is a series of the form

$$.a_1a_2a_3 \ldots = a_1\left(\frac{1}{10}\right) + a_2\left(\frac{1}{100}\right) + a_3\left(\frac{1}{1000}\right) + \ldots$$

$$= a_1(10^{-1}) + a_2(10^{-2}) + a_3(10^{-3}) + \ldots$$

$$= \sum_{k=1}^{\infty} a_k(10^{-1})^k$$

If a rational number has a repeating decimal representation it can be written as a geometric series, i.e.

$$.333 \ldots = 3(10^{-1}) + 3(10^{-2}) + \ldots$$

$$= \sum_{k=1}^{\infty} \frac{3}{10}(10^{-1})^{k-1}$$

where $a = 3/10$ and $r = 10^{-1} = 1/10$, and

$$.181818\ldots = 1(10^{-1}) + 8(10^{-2}) + 1(10^{-3}) + 8(10^{-4}) + 1(10^{-5}) + 8(10^{-6}) + \ldots$$

$$= 10(10^{-2}) + 8(10^{-2}) + 10(10^{-4}) + 8(10^{-4}) + 10(10^{-6}) + 8(10^{-6}) + \ldots$$

$$= 18(10^{-2}) + 18(10^{-2})^2 + 18(10^{-2})^3 + \ldots + 18(10^{-2})^k + \ldots$$

$$= \sum_{k=1}^{\infty} \frac{18}{100}(10^{-2})^{k-1}$$

where $a = 18/100$ and $r = 10^{-2} = 1/100$, and

$$.123123123\ldots = 1(10^{-1}) + 2(10^{-2}) + 3(10^{-3}) + 1(10^{-4}) + 2(10^{-5}) + 3(10^{-6}) + \ldots$$

$$= 100(10^{-3}) + 20(10^{-3}) + 3(10^{-3}) + 100(10^{-6}) + 20(10^{-6}) + 3(10^{-6}) + \ldots$$

$$= 123(10^{-3}) + 123(10^{-3})^2 + 123(10^{-3})^3 + \ldots + 123(10^{-3})^k + \ldots$$

$$= \sum_{k=1}^{\infty} \frac{123}{1000}(10^{-3})^{k-1}$$

where $a = 123/1000$ and $r = 10^{-3} = 1/1000$.

Consider the following problem. Find the rational numbers whose decimal representations are (1) .333..., (2) .181818..., and (3) .123123123... . In Example 4.2, the value of a geometric series is given by $\frac{a}{1-r}$ if $|r| < 1$. Hence,

$$.333\ldots = \sum_{k=1}^{\infty} \frac{3}{10}(10^{-1})^{k-1} = \frac{3/10}{1-1/10} = \frac{3}{10}\cdot\frac{10}{9} = \frac{1}{3}$$

and

$$.181818\ldots = \sum_{k=1}^{\infty} \frac{18}{100}(10^{-2})^{k-1} = \frac{18/100}{1-1/100} = \frac{18}{100}\cdot\frac{100}{99} = \frac{2}{11}$$

and

$$.123123123\ldots = \sum_{k=1}^{\infty} \frac{123}{1000}(10^{-3})^{k-1} = \frac{123/1000}{1-1/1000} = \frac{123}{1000}\cdot\frac{1000}{999} = \frac{41}{333}.$$

The reader should be able to generalize this procedure for any repeating decimal and should attempt to do this.

<u>Example 4.4:</u> Consider an algebraic expression such as $\frac{1}{2-x}$. Many algebraic expressions such as this one are in the form of the value of a geometric series. By algebraic manipulations, a geometric series can be found such that $\frac{1}{2-x}$, or a similar expression, is the value of the series.

$$\frac{1}{2-x} = \frac{1}{2(1-x/2)} = \frac{1/2}{1-x/2}$$

$$= \sum_{k=1}^{\infty} (1/2)(x/2)^{k-1}$$

where $a = 1/2$ and $r = x/2$. Since the series converges if $|r| < 1$, this implies that the series converges if $|x/2| < 1$ or $|x| < 2$.

Observe that the expression $\frac{1}{2-x}$ defines a function with domain of all the real numbers except 2, but the geometric series has a value only on the set $\{x \mid |x| < 2\}$. Functions defined by series will be discussed more thoroughly in Chapter VII.

Example 4.5: Consider the following finite sum.

$$\sum_{k=1}^{n} (b_k - b_{k+1}) = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots + (b_n - b_{n+1})$$

$$= b_1 - b_{n+1}$$

This sum is said to be a telescoping sum. To extend this idea to infinite series, consideration is given to those series $\Sigma\, a_n$ for which each term a_n may be expressed as a difference of the form $a_n = b_n - b_{n+1}$. The following argument shows that $\Sigma\, a_n$ converges iff the sequence $< b_n >$ converges, in which case $\Sigma\, a_n = b_1 - \lim b_n$. If sequence $< b_n >$ converges, then $\lim b_n = \lim b_{n+1}$ exists. The partial sum $s_n = b_1 - b_{n+1}$ and hence $\lim s_n = \lim b_1 - \lim b_{n+1} = b_1 - \lim b_n$. Hence $\Sigma\, a_n$ converges and $\Sigma\, a_n = b_1 - \lim b_n$. If $\Sigma\, a_n$ converges, then $\lim s_n$ exists and $b_{n+1} = b_1 - s_n$. Thus $\lim b_{n+1} = \lim b_n$ exists. Hence the sequence $< b_n >$ converges and $\Sigma\, a_n = b_1 - \lim b_n$. Let

$$a_n = \frac{1}{n(n+1)} .$$

Then

$$a_n = \frac{1}{n} - \frac{1}{n+1} .$$

In this example, $b_1 = 1$, $b_n = 1/n$ and $\lim 1/n = 0$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 - 0 = 1.$$

Example 4.6: The following series is a telescoping series. Prove that the series converges and has the sum indicated.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$$

Example 4.7:

$$\log \frac{n}{n+1} = \log n - \log (n+1)$$

Since $\log n \to +\infty$, the series

$$\sum_{n=1}^{\infty} \log \frac{n}{n+1}$$

diverges.

Example 4.8:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$$

Let s_n represent the partial sums, then $s_0 = 1$, $s_1 = 2$, $s_2 = 2\frac{1}{2}$, and for $n \geq 3$,

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{2 \cdot 3 \cdot 4 \ldots n}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \ldots + \frac{1}{2 \cdot 2 \cdot 2 \ldots 2}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - (1/2)^n}{1 - 1/2}$$

$$= 1 + 2 - \frac{1}{2^{n-1}}$$

$$= 3 - \frac{1}{2^{n-1}}$$

$$< 3$$

The partial sums can be written as follows:

$s_0 = 1$, $s_1 = 2$, $s_2 = 2 + 1/2!, \ldots,$ $s_n = 2 + 1/2! + 1/3! + \ldots + 1/n!,$

$s_{n+1} = s_n + 1/(n+1)!, \ldots$.

Since $s_{n+1} = s_n + 1/(n+1)!$ and $1/(n+1)! > 0$ for all n, $s_{n+1} > s_n$ for all n; i.e. $s_0 < s_1 < s_2 < \ldots < s_n < \ldots$ and it was shown above that $s_n \leq 3$ for all n. Thus by Theorem 3.6 the series

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

is convergent with value $s \leq 3$. The value of this series can be shown to be the number e which is strictly less than 3.

The main problems in the study of infinite series are to represent functions by series as in Example 4.4; to determine whether or not a given series converges as in Examples 4.1, 4.2, 4.5, 4.6, 4.7, and 4.8; if the series converges, to determine the value of the series as in Examples 4.2, 4.5, and 4.6. Observe that in Example 4.8, the series is convergent and the value $s \leq 3$, but the value of the series is not obtained by the method used to check its convergence.

The next two theorems are quite useful and are left for the reader to prove. Observe that the word diverges appears in parentheses after the word converges. This means that both theorems are true for divergence as well as for convergence. Hence each theorem being of an "iff" variety is really two theorems on convergence and two theorems on divergence. The reader might list these to make clear the understanding of each theorem before attempting a proof.

Theorem 4.3: Let m be a positive integer. The series

$$\sum_{k=m}^{\infty} x_k$$

converges (diverges) iff

$$\sum_{k=1}^{\infty} x_k$$

converges (diverges).

Theorem 4.4: Let c be a nonzero real number. The series

$$\sum_{k=1}^{\infty} x_k$$

converges (diverges) iff

$$\sum_{k=1}^{\infty} cx_k$$

converges (diverges).

The following example illustrates the last two theorems and shows the relationship between the values of the series involved in the case of convergence.

Example 4.9: In Example 4.5, the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is shown to be convergent with value 1. Consider

$$\sum_{n=5}^{\infty} \frac{1}{n(n+1)} = \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7} + \frac{1}{7\cdot 8} + \ldots + \frac{1}{n(n+1)} + \ldots .$$

$$s_n = \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{5} - \frac{1}{n+1}$$

$s_n < 1/5$ for all n, and $s_n < s_{n+1}$ for all n. Thus the series is convergent by Theorem 3.6. Since

$$\lim s_n = \lim \left(\frac{1}{5} - \frac{1}{n+1}\right) = \frac{1}{5},$$

the value of the series is the number 1/5. Observe that the value of

$$\sum_{k=5}^{\infty} \frac{1}{n(n+1)}$$

is the value of

$$\sum_{k=1}^{\infty} \frac{1}{n(n+1)}$$

minus the sum of the first four terms.

$$1 - \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5}\right) = 1 - \frac{4}{5} = \frac{1}{5}.$$

Now consider

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = \frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \ldots + \frac{5}{n(n+1)} + \ldots$$

$$s_n = 5\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= 5\left(1 - \frac{1}{n+1}\right)$$

$$= 5 - \frac{5}{n+1},$$

$s_n \leq 5$ for all n, and $s_n < s_{n+1}$ for all n. Thus the series is convergent by Theorem 3.6.

$$\lim s_n = \lim \left(5 - \frac{5}{n+1}\right) = 5.$$

Observe that

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} = 5 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} .$$

In dealing with finite sums the commutative and associative properties of addition make it possible to perform the addition of two finite sums by adding term by term and then summing, i.e.

$$\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$$

or

$$(a_1+a_2+a_3+\ldots+a_n) + (b_1+b_2+b_3+\ldots+b_n) = (a_1+b_1)+(a_2+b_2)+\ldots+(a_n+b_n).$$

The right member of the above equation can be obtained by applying the commutative and associative properties a finite number of times to the left member. The distributive property applies also to finite sums, i.e.

$$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$$

since

$$ca_1+ ca_2+ ca_3+ \ldots + ca_n = c(a_1+ a_2+ a_3+ \ldots + a_n).$$

Since

$$\sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} a_k + (-1) \sum_{k=1}^{n} b_k$$

$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} (-b_k)$$

$$= \sum_{k=1}^{n} (a_k - b_k),$$

subtraction can be performed term by term for finite sums.

The next three theorems provide a natural extension of these properties of finite sums to convergent infinite series and thereby justifies many algebraic manipulations with convergent series. These theorems are important as they not only provide the convergence of new series, but also set up a relation between their values and those of the old series.

<u>Theorem 4.5:</u> Convergent series may be added term by term. If

$$\sum_{k=0}^{\infty} a_k = s \quad \text{and} \quad \sum_{k=0}^{\infty} b_k = t,$$

then

$$\sum_{k=0}^{\infty} (a_k + b_k) = s + t.$$

Proof:

1. Let S_n, T_n, and U_n be the nth partial sums of $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$, $\sum_{k=0}^{\infty} (a_k+b_k)$ respectively.	1. Notation
2. $S_n = \sum_{k=0}^{n} a_k$, $T_n = \sum_{k=0}^{n} b_k$, and $U_n = \sum_{k=0}^{n} (a_k+b_k)$	2. Step 1 and Definition 4.1
3. $U_n = \sum_{k=0}^{n} (a_k+b_k) = \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = S_n + T_n$	3. Step 2 and addition of finite sums

4. $S_n \to s$ and $T_n \to t$	4. Convergent series and hypothesis
5. $U_n = (S_n + T_n) \to s + t$	5. Steps 3, 4, Theorem 3.9
6. $\sum_{k=0}^{\infty} (a_k + b_k) = s + t$	6. Steps 1, 5, Definition 4.3

Theorem 4.6: Convergent series may be subtracted term by term. If

$$\sum_{k=0}^{\infty} a_k = s \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k = t,$$

then

$$\sum_{k=0}^{\infty} (a_k - b_k) = s - t.$$

Proof: Similar to Theorem 4.5.

Theorem 4.7: Convergent series may be multiplied by a constant term by term. If

$$\sum_{k=0}^{\infty} a_k = s$$

and c is an arbitrary real number, then

$$\sum_{k=0}^{\infty} ca_k = cs.$$

Proof: Use Theorem 4.4.

Example 4.10: Consider Example 4.3 again. Find the rational number whose decimal representation is .181818... . The reader is probably already familiar with the following technique for solving this problem.

Let $x = .181818\ldots$, then $100x = 18.181818\ldots$. Now subtract as follows and solve for x.

$$\begin{array}{rcl} 100\,x & = & 18.181818\ldots \\ -\,x & = & .181818\ldots \\ \hline 99\,x & = & 18 \end{array}$$

$$x = \frac{18}{99} = \frac{2}{11}$$

The justification for this procedure is found by using the previous two theorems in the following argument.

$x = .181818\ldots$

$= 18(10^{-2})+18(10^{-2})^2+\ldots+18(10^{-2})^k+\ldots$ — Expand by powers of 10^{-2}

$= \sum_{k=1}^{\infty} 18(10^{-2})(10^{-2})^{k-1}$ — Write geometric series in summation form

$= \sum_{k=1}^{\infty} 18(10^{-2})^k$ — Simplify general term

$100x = \sum_{k=1}^{\infty} (100)18(10^{-2})^k$ — By Theorem 4.7

$= \sum_{k=1}^{\infty} (10^{-2})^{-1}\, 18(10^{-2})^k$ — Substitute $(10^{-2})^{-1}$ for 100

$= \sum_{k=1}^{\infty} 18(10^{-2})^{k-1}$ — Combine powers of 10^{-2}

$= \sum_{k=0}^{\infty} 18(10^{-2})^k$ — Change index on summation

$= 18 + \sum_{k=1}^{\infty} 18(10^{-2})^k$ — Equivalent form of the series

Now by Theorem 4.6,

$$100x - x = 18 + \sum_{k=1}^{\infty} 18(10^{-2})^k - \sum_{k=1}^{\infty} 18(10^{-2})^k$$

$$99x = 18 + \sum_{k=1}^{\infty} [18(10^{-2})^k - 18(10^{-2})^k]$$

$$= 18 + \sum_{k=1}^{\infty} 0$$

$$= 18 + 0$$

$$= 18$$

$$x = \frac{18}{99}$$

$$= \frac{2}{11}$$

It might appear to the reader at first glance that one may insert parentheses in infinite series exactly as in finite sums. Consider the next example.

Example 4.11:

$$\sum_{k=1}^{\infty} (-1)^{k+1} = 1 - 1 + 1 - 1 + \ldots + (-1)^{n+1} + \ldots$$

If allowed to "insert parentheses" as in finite sums, then its sum could be written $(1-1) + (1-1) + (1-1) + \ldots$ and hence certainly would equal 0. It could, however, also be written $1 - (1-1) - (1-1) - \ldots$, and hence certainly would equal 1. Consider the partial sums, $s_1 = 1$, $s_2 = 0$, $s_3 = 1$, $s_4 = 0$, ... The sequence of partial sums is divergent. Therefore the series is divergent by Definition 4.3.

The next question that might be asked is as follows. Can parentheses ever be inserted in infinite series without altering the value of the series? This is the same thing as asking when is the associative property applicable to infinite series. The next theorem states the necessary conditions for an associative property to apply to infinite series.

Theorem 4.8: An associative property holds unrestrictedly for convergent infinite series; that is to say, $a_0 + a_1 + a_2 + \ldots = s$ implies

$$(a_0 + a_1 + \ldots + a_{v_1}) + (a_{v_1+1} + a_{v_1+2} + \ldots + a_{v_2}) + \ldots = s$$

if $v_1, v_2, v_3, \ldots$ denotes any increasing sequence of distinct nonnegative integers and the sum of the terms enclosed in each bracket is considered as one term of a new series $A_0 + A_1 + A_2 + \ldots + A_k + \ldots$ where, therefore, for $k = 0, 1, 2, \ldots,$

$$A_k = a_{v_k+1} + a_{v_k+2} + \ldots + a_{v_{k+1}} \quad (v_0 = -1).$$

Proof:

1. Let S_n be the nth partial sum of ΣA_k and s_n be the nth partial sum of Σa_k — 1. Notation

2. $S_0 = A_0 = a_0 + a_1 + \ldots + a_{v_1} = s_{v_1}$ — 2. Relationship between the partial sums of the two series

$S_1 = A_0 + A_1 = s_{v_2}$

$S_2 = A_0 + A_1 + A_2 = s_{v_3}$

and, in general,

$S_n = A_0 + A_1 + A_2 + \ldots + A_n = s_{v_{n+1}}$

3. $s_{v_1}, s_{v_2}, \ldots, s_{v_n}, \ldots$ is a subsequence of $< s_n >$ — 3. Definition 2.11

4. $S_n = s_{v_{n+1}} \to s$ — 4. Theorem 3.2

5. ΣA_k converges to s — 5. Step 4

It is useful to consider along with the series Σa_n an associated series $\Sigma |a_n|$ where $\Sigma |a_n| = |a_1| + |a_2| + |a_3| + \ldots + |a_n| + \ldots$. Convergent series can be divided into two categories depending upon whether or not the series $\Sigma |a_n|$ converges. These two categories are described in the next definition.

<u>Definition 4.5</u>: (Absolute and Conditional convergence) The series Σa_k is said to be <u>absolutely convergent</u> if $\Sigma |a_k|$ converges, where $\Sigma |a_k| = |a_1| + |a_2| + |a_3| + \ldots + |a_n| + \ldots$. If Σa_k converges but $\Sigma |a_k|$ diverges, then Σa_k is said to be <u>conditionally convergent.</u>

There exists many tests for determining whether a series is absolutely convergent. All the tests in the next chapter on positive term series are available for use on absolutely convergent series. Since $\Sigma |a_k| = |a_1| + |a_2| + |a_3| + \ldots |a_n| + \ldots$, $\Sigma |a_k|$ is always a positive term series. Operations on absolutely convergent series, on the whole, are precisely the same as on finite sums, whereas this is in general no longer the case for conditionally convergent series. Tests for conditionally convergent series are discussed in Chapter VI.

In general, convergence of Σa_k does not imply convergence of $\Sigma |a_k|$, but the convergence of $\Sigma |a_k|$ does not imply the convergence of Σa_k as the next theorem will prove.

Theorem 4.9: If $\Sigma\,|a_k|$ converges, then $\Sigma\, a_k$ converges and $|\Sigma\, a_k| \leq \Sigma\,|a_k|$.

Proof:

1. Let S_n, T_n, and U_n be the nth partial sums of the infinite series $\Sigma\, a_k$, $\Sigma\,|a_k|$ and $\Sigma\,(a_k + |a_k|)$ respectively and $S_n + T_n = U_n$ or $S_n = U_n - T_n$	1. Notation and addition of finite sums
2. $0 \leq a_k + |a_k| \leq 2\,|a_k|, \quad \forall\, k$	2. $a_k \leq |a_k|, \quad \forall\, k$
3. Let T represent the value of $\Sigma\,|a_k|$ and $0 \leq U_n \leq 2T_n \leq 2T$ for every n	3. $\Sigma\,|a_k|$ converges by hypothesis, Step 2 and T_n is non-decreasing
4. $0 \leq U_1 \leq U_2 \leq \ldots \leq U_n \leq \ldots \leq 2T$	4. $a_k + |a_k| \geq 0$ if $a_k \geq 0$ $a_k + |a_k| = 0$ if $a_k < 0$ and Step 3
5. $\lim U_n = U \leq 2T$	5. Step 4 and Theorem 3.6
6. $\lim S_n = \lim (U_n - T_n)$ $= \lim U_n - \lim T_n$ $= U - T$	6. Step 1, $U_n \to U$, $T_n \to T$ and Theorem 3.10
7. Therefore $\Sigma\, a_k$ is convergent with value $S = U - T \leq T$	7. Steps 5 and 6
8. The series $\Sigma\,(-a_k)$ has value $-S$	8. $\Sigma\, a_k$ converges to S in Step 7 and Theorem 4.7.

9. $T = \Sigma |a_k| = \Sigma |-a_k|$ — 9. $|a_k| = |-a_k|, \quad \forall k$

10. A similar argument yields $-S \leq T$ — 10. Steps 1 - 7

10. $|S| \leq T$ or $|\Sigma a_k| \leq \Sigma |a_k|$ — 11. S and -S are less than or equal to T, and substitution

Suppose a given series Σa_k has infinitely many positive terms and infinitely many negative terms. Two new series can be formed by considering the positive terms alone, and also the negative terms alone. These two series are related to the given series in a manner which is described in the next theorem. Before proving this theorem, consider how these two new series are to be formed.

Let

$$\Sigma a_k = a_1 + a_2 + a_3 + \dots + a_n + \dots .$$

If $a_k \geq 0$,

$$\frac{|a_k| + a_k}{2} = \frac{a_k + a_k}{2} = a_k \geq 0$$

and

$$\frac{|a_k| - a_k}{2} = \frac{a_k - a_k}{2} = 0 .$$

If $a_k < 0$,

$$\frac{|a_k| + a_k}{2} = \frac{-a_k + a_k}{2} = 0$$

and

$$\frac{|a_k| - a_k}{2} = \frac{-a_k - a_k}{2} = -a_k \geq 0 .$$

Therefore if

$$p_k = \frac{|a_k| + a_k}{2} \quad \text{and} \quad q_k = \frac{|a_k| - a_k}{2},$$

$(k = 1, 2, 3, \ldots)$ the terms of p_k will either be a_k or 0, and the terms of q_k will either be 0 or $-a_k$. Observe that $2p_k = |a_k| + a_k$ and $2q_k = |a_k| - a_k$ and solving these equations simultaneously, $a_k = p_k - q_k$ and $|a_k| = p_k + q_k$. Hence $\Sigma\, p_k$ is a new series where each term is related to the terms in $\Sigma\, a_k$ as follows: $p_k = a_k$ if $a_k \geq 0$ and $p_k = 0$ if $a_k < 0$, and $\Sigma\, q_k$ is a new series where each term is related to the terms in $\Sigma\, a_k$ as follows: $q_k = 0$ if $a_k \geq 0$ and $q_k = -a_k$ if $a_k < 0$. Observe that the terms of both series are nonnegative, and that there exist infinitely many terms which are 0 in each series.

Let

$$\Sigma\, a_k = 1 - 1/2 + 1/3 - 1/4 + \ldots + (-1)^{n+1}\, 1/n + \ldots,$$

then

$$\Sigma\, p_k = 1 + 0 + \frac{1}{3} + 0 + \ldots + \frac{1 + (-1)^{n+1}}{2}\left(\frac{1}{n}\right) + \ldots$$

and

$$\Sigma\, q_k = 0 + \frac{1}{2} + 0 + \frac{1}{4} + \ldots + \frac{1 + (-1)^{n}}{2}\left(\frac{1}{n}\right) + \ldots, \quad .$$

<u>Theorem 4.10</u>: Let $\Sigma\, a_k$ be a given series and define

$$p_k = \frac{|a_k| + a_k}{2} \quad \text{and} \quad q_k = \frac{|a_k| - a_k}{2} \quad (k = 1, 2, 3, \ldots)$$

Then:

(i) If $\Sigma\, a_k$ is conditionally convergent, both $\Sigma\, p_k$ and $\Sigma\, q_k$ diverge.

(ii) If $\Sigma |a_k|$ converges, both $\Sigma\, p_k$ and $\Sigma\, q_k$ converge and $\Sigma\, a_k = \Sigma\, p_k - \Sigma\, q_k$

Proof: (i)

1. Assume Σa_k converges and $\Sigma \lvert a_k \rvert$ diverges	1. Hypothesis and Definition 4.5
2. $p_k = a_k + q_k$ or $q_k = p_k - a_k$	2. Calculations
3. If Σq_k converges, then Σp_k converges	3. Steps 1, 2 and Theorem 4.5
4. If Σp_k converges, then Σq_k converges	4. Steps 1, 2 and Theorem 4.6
5. Hence if either Σp_k or Σq_k converges, both must converge and then $\Sigma \lvert a_k \rvert$ converges since $\lvert a_k \rvert = p_k + q_k$	5. Steps 3, 4 and Theorem 4.5
6. The assumption in Step 5 contradicts the hypothesis	6. $\Sigma \lvert a_k \rvert$ diverges
7. Therefore both Σp_k and Σq_k diverge	7. Steps 1 - 6

Proof: (ii)

1. Assume Σa_k converges and $\Sigma \lvert a_k \rvert$ converges	1. Hypothesis, Definition 4.5, and Theorem 4.9
2. $\Sigma (p_k - q_k)$ converges and $\Sigma (p_k + q_k)$ converges	2. Step 1, $a_k = p_k - q_k$ and $\lvert a_k \rvert = p_k + q_k$, substitution

3. $\Sigma\ [(p_k - q_k) + (p_k + q_k)] = \Sigma\ 2p_k$ converges and $1/2\ \Sigma\ 2p_k = \Sigma\ p_k$ converges	3. Step 2, Theorem 4.5, Theorem 4.7
4. $\Sigma\ [(p_k + q_k) - (p_k - q_k)] = \Sigma\ 2q_k$ converges and $1/2\ \Sigma\ 2q_k = \Sigma\ q_k$ converges	4. Step 2, Theorem 4.6, Theorem 4.7
5. Therefore both $\Sigma\ p_k$ and $\Sigma\ q_k$ converge and $\Sigma\ a_k = \Sigma\ p_k - \Sigma\ q_k$	5. Steps 3, 4, and $a_k = p_k - q_k$ and Theorem 4.6

The commutative property a + b = b + a does not in general hold for infinite series. If $v_0, v_1, v_2, \ldots$ is any rearrangement of the sequence 0, 1, 2, . . . , then the series

$$\sum_{n=0}^{\infty} a_n' = \sum_{n=0}^{\infty} a_{v_n}$$

with $a_n' = a_{v_n}$ for n = 0, 1, 2, . . . will be said to result from the given series

$$\sum_{n=0}^{\infty} a_n$$

by rearrangement of the latter. The value of actual sums of a finite number of terms remains unaltered however the terms may be arranged. For infinite series this is no longer the case. Hence the order of terms must be taken into account. The following theorem states conditions under which it is possible to rearrange a series without altering the value of the series.

<u>Theorem 4.11</u>: Let the series $\Sigma\ u_k$ be absolutely convergent, with value s. Let $\Sigma\ v_k$ be any series obtained by a rearrangement of

the terms of Σu_k (i.e. every v_i is some u_j and every u_m is some v_n). Then Σv_k is convergent, with value s.

Proof:

1. First, let all u's (and hence all v's) be nonnegative and $s = \Sigma u_k$	1. Assumption and hypothesis
2. The partial sums of Σv_k cannot exceed s. Thus the series Σv_k must be convergent and its value s' must satisfy the inequality $s' \leq s$	2. Each v_i is some u_j and partial sums are non-decreasing, Theorem 3.6
3. Reverse the roles of Σu_k and Σv_k and $s \leq s'$	3. Steps 1 - 2 and hypothesis
4. $s' = s$	4. Steps 2 and 3
5. In the general case of an absolutely convergent series, $\Sigma u_k = \Sigma p_k - \Sigma q_k$ and both Σp_k and Σq_k converge	5. Theorem 4.10
6. In the rearranged series Σv_k, the separation into positive and negative terms yields $\Sigma v_k = \Sigma p_k' - \Sigma q_k'$ where $\Sigma p_k'$ is a rearrangement of Σp_k and $\Sigma q_k'$ is a rearrangement of Σq_k	6. Theorem 4.10 and rearrangement

7. $\Sigma p_k' = \Sigma p_k$ — 7. Steps 1 - 4, series with non-negative terms

$\Sigma q_k' = \Sigma q_k$

8. Hence Σv_k is convergent with same value as Σu_k which is s — 8. Steps 5, 6, 7, and hypothesis

If the terms of a series are all positive, then the series if it converges must converge absolutely. Any rearrangement of such a series will therefore converge to the same value as the original series. In this case also the converse of Theorem 4.8 is true.

Theorem 4.12: If the terms of a convergent infinite series

$$\sum_{k=0}^{\infty} A_k$$

are sums of positive real numbers; i.e.

$$A_k = a_{v_k+1} + a_{v_k+2} + \ldots + a_{v_{k+1}},$$

for $k = 0, 1, 2, \ldots$ and $v_0 = -1$, then the parentheses may be removed from

$$\sum_{k=0}^{\infty} A_k$$

and the new series

$$\sum_{k=0}^{\infty} a_k$$

converges to the same value.

Proof:

1. Let S_n and s_n be the nth partial sums of ΣA_k and Σa_k, respectively — 1. Notation

2. $S_0 = A_0 = s_{v_1}$, $S_1 = A_0 + A_1 = s_{v_2}$, and, in general $S_n = s_{v_{n+1}}$	2. Relationship between the partial sums of the two series
3. $\Sigma A_k = s$ implies $S_n = s_{v_{n+1}} \to s$	3. Hypothesis, Definition 4.3, Step 2
4. $s_n \leq s_{n+1} \leq s, \ \forall n$	4. Hypothesis, $a_k \geq 0, \ \forall k$
5. $\forall \epsilon > 0 \ \exists \ v_{n_0} \ni \lvert s_n - s \rvert < \epsilon, \ \forall n > v_{n_0}$	5. Step 3, Definition 3.2
6. $s_n \to s$	6. Steps 3, 4, and 5
7. $\Sigma a_k = s$	7. Step 6, Definition 4.3

CHAPTER V

CONVERGENCE TESTS FOR POSITIVE TERM SERIES

In theory, the convergence or divergence of a particular series $\Sigma\, a_k$ is decided by examining the sequence of partial sums $<s_n>$ to see whether or not it converges. In Example 4.2, it was easy to write the nth partial sum and hence to find the limit. It was observed that the geometric series converges and has the value $\frac{a}{1-r}$ if $|r| < 1$. In Example 4.1, the harmonic series is shown to be divergent because the sequence of partial sums is unbounded since it does not satisfy the Cauchy condition. In Example 4.8,

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

is shown to be convergent since $S_n \leq 3$ for all n, and S_n is monotone increasing. In the majority of cases there is no nice formula for simplifying the nth partial sum, and the convergence or divergence may be rather difficult to establish in a straight-forward manner. Early investigators in the subject, notably Cauchy and his contemporaries, realized this difficulty, and they developed a number of "convergence tests" that bypassed the need for an explicit knowledge of the partial sums.

Convergence tests may be broadly classified into three categories: (i) sufficient conditions, (ii) necessary conditions, and (iii) necessary and sufficient conditions. Let C represent some condition in question. A test of the form, "If C is satisfied, then $\Sigma\ x_n$ converges," is a test which states sufficient conditions for convergence. This means that the condition C is enough information for the convergence of $\Sigma\ x_n$. A test of the form, "If $\Sigma\ x_n$ converges, then C is satisfied," is a test which states necessary conditions for convergence. This means that the convergence of $\Sigma\ x_n$ implies that the condition C is true also. A test of the form, "$\Sigma\ x_n$ converges iff C is satisfied," is a test which states necessary and sufficient conditions. "If C is satisfied, then $\Sigma\ x_n$ converges," is the sufficient part of the test, and "If $\Sigma\ x_n$ converges, then C is satisfied," is the necessary part of the test. Observe that "iff" theorems always state "necessary and sufficient" conditions.

Beginners often use such tests incorrectly by failing to realize the difference between a necessary condition and a sufficient condition. Theorems 4.1 and 4.2 are simple examples of the types above. Theorem 4.1 states a necessary condition for convergence, while Theorem 4.2 states a sufficient condition for divergence.

In establishing criteria for convergence, the first type of series to be considered is the series of positive terms, that is, series of the form $\Sigma\ a_k$, where each $a_k \geq 0$. Since the sequence of partial sums of such a series is non-decreasing, it is possible to state a simple necessary and sufficient condition for convergence. This is done in the following theorem.

Theorem 5.1: Let $\Sigma\ a_k$ be a positive term series. Series $\Sigma\ a_k$ converges if and only if the sequence of partial sums $< u_n >$ is bounded above.

Proof: (a) Prove the "sufficient" part of the theorem. Assume the sequence of partial sums $< u_n >$ is bounded above, and prove that $\Sigma\ a_k$ converges.

1. $< u_n >$ is bounded above	1. Hypothesis
2. $< u_n >$ is monotone non-decreasing	2. $a_k \geq 0$ for all k
3. $< u_n >$ is bounded	3. Theorem 2.2
4. $< u_n >$ converges	4. Steps 2, 3 and Theorem 3.5
5. $\Sigma\ a_k$ converges	5. Definition 4.3

(b) Prove the "necessary" part of the theorem. Assume $\Sigma\ a_k$ converges, and prove that the sequence of partial sums $< u_n >$ is bounded above.

1. $\Sigma\ a_k$ converges	1. Hypothesis
2. $< u_n >$ converges	2. Step 1, and Definition 4.3
3. $< u_n >$ is bounded	3. Theorem 3.1
4. $< u_n >$ is bounded above and bounded below	4. Theorem 2.1
5. $< u_n >$ is bounded above	5. Step 4

Observe that Theorem 5.1 requires knowledge of the partial sums. The next theorem will establish a test based on comparing a series with one known to converge or diverge.

<u>Theorem 5.2</u>: (1st Comparison Test) Let $\Sigma\, c_k$ be a known convergent series of positive terms and $\Sigma\, d_k$ a known divergent series of positive terms. If $\Sigma\, a_k$ is a positive term series such that for all k greater than some k_0, $a_k \leq c_k$, then $\Sigma\, a_k$ converges. If for all k greater than some k_0, $a_k \geq d_k$, then $\Sigma\, a_k$ diverges.

Proof: (a) Assume $\Sigma\, c_k$ is a convergent series of positive terms and $\Sigma\, a_k$ is a positive term series such that for all k greater than some k_0, $a_k \leq c_k$, then prove that $\Sigma\, a_k$ converges.

1. Let A_n, C_n denote the nth partial sums of $\sum_{k=k_0+1}^{\infty} a_k$ and $\sum_{k=k_0+1}^{\infty} c_k$, respectively	1. Notation
2. $A_n \leq C_n$	2. $a_k \leq c_k$ for all $k > k_0$
3. $\sum_{k=1}^{\infty} c_k$ converges	3. Hypothesis
4. $\sum_{k=k_0+1}^{\infty} c_k$ converges	4. Theorem 4.3
5. $< C_n >$ is bounded above	5. Theorem 5.1
6. $< A_n >$ is bounded above	6. Steps 2 and 5
7. $\sum_{k=k_0+1}^{\infty} a_k$ converges	7. Theorem 5.1
8. $\sum_{k=1}^{\infty} a_k$ converges	8. Theorem 4.3

(b) Assume $\Sigma\, d_k$ is a divergent series of positive terms and $\Sigma\, a_k$ is a positive term series such that for all k greater than some k_0,

$a_k \geq d_k$, then prove that $\Sigma\, a_k$ diverges.

1. Let A_n, D_n denote the nth partial sums of $\sum_{k=k_0+1}^{\infty} a_k$ and $\sum_{k=k_0+1}^{\infty} d_k$ respectively	1. Notation
2. $A_n \geq D_n$	2. $a_k \geq d_k$ for all $k > k_0$
3. $\sum_{k=1}^{\infty} d_k$ diverges	3. Hypothesis
4. $\sum_{k=k_0+1}^{\infty} d_k$ diverges	4. Theorem 4.3
5. $< D_n >$ is not bounded above	5. Contrapositive of Theorem 5.1
6. $< A_n >$ is not bounded above	6. Steps 2 and 5
7. $\sum_{k=k_0+1}^{\infty} a_k$ diverges	7. Contrapositive of Theorem 5.1
8. $\sum_{k=1}^{\infty} a_k$ diverges	8. Theorem 4.3

Definition 5.1: (Dominating Series) If $0 \leq b_k \leq a_k$ for every integer k, then the series $\Sigma\, a_k$ is said to dominate the series $\Sigma\, b_k$.

According to Theorem 5.2 and Definition 5.1, every infinite series dominated by a convergent series is also convergent, and every infinite series which dominates a divergent series is also divergent.

Example 5.1: (Hyperharmonic or p series)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

The series is the divergent harmonic series $\Sigma\, 1/k$ if $p = 1$. If $p < 1$, then $k^p \leq k$, $\forall\, k$ and $1/k^p \geq 1/k$, $\forall\, k$. Hence the series $\Sigma\, 1/k^p$ diverges by Theorem 5.2, if $p < 1$. Therefore the p series diverges if $p \leq 1$. If $p > 1$, the previous theorem does not help so it is necessary to return to a direct examination of the partial sums. The following argument will show that the partial sums are bounded above, and hence the series converges by Theorem 5.1.

Let m be an integer such that $2^m > n$. Then if S_n denotes the nth partial sum of

$$\sum_{k=1}^{\infty} \frac{1}{k^p},$$

then

$$S_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} .$$

If S_{2^m} denotes the 2^mth partial sum of

$$\sum_{k=1}^{\infty} \frac{1}{k^p},$$

then

$$S_{2^m} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{(2^m)^p}$$

since $2^m > n$. The numbers 2^m and n are both positive integers and since $2^m > n$, then $2^m - 1 \geq n$ or $n \leq 2^m - 1$. Hence $S_n \leq S_{2^m-1}$, i.e.

$$S_n \leq S_{2^m-1} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{(2^m-1)^p} .$$

Insert parentheses by powers of 2; i. e. group the 1st term all alone, then the next two terms, then the next four terms, and so on until the last group will have 2^{m-1} terms in it.

$$S_{2^m-1} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \dots + \frac{1}{7^p}\right) + \dots + \left(\frac{1}{(2^{m-1})^p} + \dots + \frac{1}{(2^m-1)^p}\right)$$

Consider each group of terms in a parenthesis in the expression for S_{2^m-1}. The first term in each parenthesis is the largest. Substitute this term for each of the smaller terms grouped with it, and the following inequality will be the result.

$$S_{2^m-1} \leq 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \dots + \frac{1}{4^p}\right) + \dots + \left(\frac{1}{(2^{m-1})^p} + \dots + \frac{1}{(2^{m-1})^p}\right)$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \dots + \frac{2^{m-1}}{(2^{m-1})^p}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \dots + \frac{1}{(2^{m-1})^{p-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \dots + \frac{1}{(2^{p-1})^{m-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \dots + \frac{1}{(2^{p-1})^{m-1}}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \dots + \left(\frac{1}{2^{p-1}}\right)^{m-1}$$

Since $p > 1$,

$$0 < \frac{1}{2^{p-1}} < 1.$$

The value of a geometric series with

$$r = \frac{1}{2^{p-1}}$$

and a = 1 is

$$\frac{1}{1 - \frac{1}{2^{p-1}}} .$$

$$1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots + \left(\frac{1}{2^{p-1}}\right)^{m-1}$$

is the mth partial sum of the geometric series above and since the partial sums are increasing,

$$1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \ldots + \left(\frac{1}{2^{p-1}}\right)^{m-1} < \frac{1}{1 - \frac{1}{2^{p-1}}} , \quad \forall m.$$

Hence,

$$S_n \leq S_{2^m - 1} < \frac{1}{1 - \frac{1}{2^{p-1}}} , \quad \forall n .$$

Thus the partial sums are bounded above and by Theorem 5.1 the series is convergent. So the p series is convergent for $p > 1$. The method used here is similar to the technique used in Example 4.1. In Example 4.1, by proper substitutions of unequal terms, it was easy to show that the partial sums were unbounded. In this example, the proper substitutions of unequal terms lead to an upper bound of the partial sums. The fact that the series is convergent is all the information gained from this analysis. Compare this example with Example 4.8.

Example 5.2:

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \ldots + \frac{1}{n^2+1} + \ldots$$

Since $k^2 < k^2 + 1$, $\forall k$, then

$$\frac{1}{k^2+1} < \frac{1}{k^2}, \quad \forall k.$$

It is natural to compare the given series with

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

In Example 5.1, the p series converges if $p > 1$. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a p series with $p = 2$, therefore

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges. The given series is dominated by

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1}$$

converges by Theorem 5.2.

Example 5.3:

$$\sum_{k=2}^{\infty} \frac{1}{\ln k} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \ldots + \frac{1}{\ln n} + \ldots$$

Since $k \geq \ln k$ for all $k \geq 2$, then

$$\frac{1}{\ln k} \geq \frac{1}{k}$$

for all $k \geq 2$. Since

$$\sum_{k=2}^{\infty} \frac{1}{k}$$

diverges,

$$\sum_{k=2}^{\infty} \frac{1}{\ln k}$$

diverges by Theorem 5.2.

The direct comparison tests of Theorem 5.2 are simple in concept, but it is often tedious work to demonstrate the necessary inequality between the general terms of the series being compared. A slightly more sophisticated comparison test is developed in Theorem 5.3 and is fairly easy to use.

Theorem 5.3: (2nd Comparison Test) If $a_k > 0$ and $b_k > 0$ and $\lim a_k/b_k = L > 0$, then $\Sigma\, a_k$ and $\Sigma\, b_k$ are either both convergent or both divergent.

Proof:

1. $\lim a_k/b_k = L > 0$	1. Hypothesis
2. $\forall\ \epsilon > 0,\ \exists\ k_0 \ni \left\lvert \frac{a_k}{b_k} - L \right\rvert < \epsilon,$ $\forall\ k > k_0$	2. Definition 3.2
3. Let $\epsilon = L/2 > 0$ and $\exists\ k_1 \ni$ $\left\lvert \frac{a_k}{b_k} - L \right\rvert < \frac{L}{2},\ \forall\, k > k_1$	3. Step 2 is true for all $\epsilon > 0$, hence is true if $\epsilon = L/2 > 0$

4. $L/2 < a_k/b_k < 3L/2, \quad \forall k > k_1$ or $(L/2)\, b_k < a_k < (3L/2) b_k, \quad \forall k > k_1$	4. Step 3 written as an inequality
5. Hence the following four inequalities are true $\forall k > k_1$ (i) $a_k > (L/2)\, b_k$ (ii) $a_k < (3L/2)\, b_k$ (iii) $b_k < (2/L)\, a_k$ (iv) $b_k > (2/3L)\, a_k$	5. Write the second inequality in Step 4 in simple sentences
6. If $\Sigma\, a_k$ converges, then $\Sigma\, (L/2)\, b_k$ converges	6. Step 5(i) and Theorem 5.2
7. $2/L \;\; \Sigma\, (L/2) b_k = \Sigma\, b_k$ converges	7. Step 6 and Theorem 4.4
8. If $\Sigma\, b_k$ converges, then $\Sigma\; 2a_k/3L$ converges	8. Step 5(iv) and Theorem 5.2
9. $3L/2 \;\; \Sigma\;\; 2a_k/3L \;=\; \Sigma\, a_k$ converges.	9. Step 8 and Theorem 4.4
10. If one series converges, then the other one does also	10. Steps 6 - 9
11. If $\Sigma\, a_k$ diverges, then $\Sigma\, 3Lb_k/2$ diverges	11. Step 5(ii) and Theorem 5.2

12. $2/3L \ \Sigma\ (3L/2)\ b_k = \Sigma\ b_k$ diverges — 12. Step 11 and Theorem 4.4

13. If $\Sigma\ b_k$ diverges, then $\Sigma\ (2/L)a_k$ diverges — 13. Step 5(iii) and Theorem 5.2

14. $L/2\ \Sigma\ (2/L)\ a_k = \Sigma\ a_k$ diverges — 14. Step 13 and Theorem 4.4

15. If one series diverges, then the other one does also — 15. Steps 11 - 14

Example 5.4: Determine whether the following infinite series is convergent or divergent.

$$1 + 1/16 + 1/49 + \ldots + 1/(3n-2)^2 + \ldots$$

It is natural to compare this series with the convergent series $\Sigma\ 1/k^2$.

$$\lim \frac{1/(3k-2)^2}{1/k^2} = \lim \frac{k^2}{(3k-2)^2} = \lim \left(\frac{k}{3k-2}\right)^2 = \lim \left(\frac{1}{3-2/k}\right)^2 = \frac{1}{9}$$

Hence by Theorem 5.3, the series is convergent since $\Sigma\ 1/k^2$ is convergent.

Example 5.5: Determine whether the following infinite series is convergent or divergent.

$$1 + 1/3 + 1/5 + \ldots + 1/(2n-1) + \ldots$$

It is natural to compare this series with the divergent series $\Sigma\ 1/k$.

$$\lim \frac{1/(2k-1)}{1/k} = \lim \frac{k}{2k-1} = \lim \frac{1}{2-1/k} = \frac{1}{2}$$

Hence by Theorem 5.3, the series is divergent since $\Sigma\ 1/k$ is divergent.

The next test is a special case of the 2nd Comparison Test, and the proof is left for the reader.

Theorem 5.4: (3rd Comparision Test) If $c_1, c_2, c_3, \ldots, c_n, \ldots$ is a sequence of positive numbers such that $\lim c_n = c (c > 0)$, then the two positive series $\Sigma\, a_k$, $\Sigma\, c_k a_k$ either both converge or both diverge.

The following examples use positive term sequences with limits that are positive.

Example 5.6: Use the 3rd Comparision Test to test the series

$$\sum_{k=1}^{\infty} \frac{k+2}{k(3k-1)} .$$

$$\frac{k+2}{k(3k-1)} = \frac{1}{k} \cdot \frac{k+2}{3k-1} = \frac{1+2/k}{3-1/k} \cdot \frac{1}{k}$$

Since

$$\lim \frac{1+2/k}{3-1/k} = \frac{1}{3}$$

and $\Sigma\, 1/k$ diverges, the series

$$\sum_{k=1}^{\infty} \frac{k+2}{k(3k-1)}$$

diverges by Theorem 5.4.

Example 5.7: Use the 3rd Comparison Test to test the series

$$\sum_{k=1}^{\infty} \frac{k+4}{k^3-2k+1} .$$

$$\frac{k+4}{k^3-2k+1} = \frac{1+4/k}{k^2-2+1/k} = \frac{1+4/k}{k^2(1-2/k^2+1/k^3)} = \frac{1+4/k}{1-2/k^2+1/k^3} \cdot \frac{1}{k^2}$$

Since

$$\lim \frac{1+4/k}{1-2/k^2+1/k^3} = 1$$

and $\Sigma\ 1/k^2$ converges because it is a p series with p = 2, the series

$$\sum_{k=1}^{\infty} \frac{k+4}{k^3-2k+1}$$

converges by Theorem 5.4.

Observe that the next two tests, the ratio and the root test, in reality are special cases of the Comparison Tests. In both tests convergence is deduced from the fact that the series in question can be dominated by a suitable geometric series $\Sigma\ ar^k$. The usefulness of these tests in practice is that a choice of a particular comparison series is not explicitly required.

Theorem 5.5: (Ratio Test) If $a_k > 0$, $\forall k$, let

$$L = \lim\sup \frac{a_{k+1}}{a_k}$$

and

$$\ell = \lim\inf \frac{a_{k+1}}{a_k} .$$

Then $\Sigma\ a_k$ converges if $L < 1$, and diverges if $\ell > 1$; if $\ell \leq 1 \leq L$, no conclusion can be reached about the behavior of $\Sigma\ a_k$.

Proof: (a) Assume $L < 1$ and prove that $\Sigma\ a_k$ converges.

1. $L = \lim\sup\ a_{k+1}/a_k < 1$	1. Hypothesis
2. Choose $B \ni L < B < 1$	2. Real numbers are dense

3. Let $\epsilon = (B - L) > 0$, then $a_{k+1}/a_k < L + \epsilon = L + (B - L) = B$ for all but a finite number of terms	3. Theorem 3.15
4. For $\epsilon = (B - L) > 0 \ \exists\ N \ni a_{k+1}/a_k < B$ for all $k \geq N$	4. Step 3 and meaning of all but a finite number of terms
5. In particular, $a_{N+1} < Ba_N$; $a_{N+2} < Ba_{N+1} < B^2 a_N$; $a_{N+3} < Ba_{N+2} < B^3 a_N$; and in general, $a_{N+p} < B^p a_N$ where p is a positive integer	5. Step 4
6. Let $k = N + p$, then $p = k - N$ and $a_k < B^{k-N} a_N = a_N B^{-N} \cdot B^k$ for all $k \geq N$	6. Steps 4 and 5
7. ΣB^k converges	7. Step 2, geometric series with r less than 1
8. N is a fixed positive integer, $B > 0$, $B^{-N} > 0$, $a_N > 0$. Hence $a_N B^{-N}$ is a positive constant, so $\Sigma (a_N B^{-N}) B^k$ converges	8. Step 2 and 7, Hypothesis, and Theorem 4.4

9. $\Sigma\ a_k$ converges — 9. Steps 6, 8 and Theorem 5.2

(b) Assume $\ell > 1$ and prove that $\Sigma\ a_k$ diverges.

Proof: Left for the reader.

(c) If $\ell \leq 1 \leq L$, no conclusion can be reached about the behavior of $\Sigma\ a_k$.

1. $\Sigma\ 1/k$ diverges and — 1. Calculation of limit

$$\lim \frac{1/(k+1)}{1/k} = \lim \frac{k}{k+1} = 1$$

2. $\Sigma\ 1/k^2$ converges and — 2. Calculation of limit

$$\lim \frac{1/(k+1)^2}{1/k^2} = \lim \frac{k^2}{(k+1)^2} = 1$$

3. No conclusion can be drawn when $\ell \leq 1 \leq L$ — 3. In Steps 1 and 2 examples have been given such that one converges and the other diverges, but in both cases $\ell = 1 = L$.

<u>Example 5.8</u>: Use the ratio test to test the following series, $\Sigma\ a_k$ where a_k is given by

$$a_k = \begin{cases} \dfrac{1}{3^m} & \text{if } k \text{ is even, i.e. } k = 2m \\[2ex] \dfrac{2}{3^{m+1}} & \text{if } k \text{ is odd, i.e. } k = 2m + 1 \end{cases}$$

If k is even, then

$$\frac{a_{k+1}}{a_k} = \frac{2}{3^{m+1}} \cdot \frac{3^m}{1} = \frac{2}{3} .$$

If k is odd, then

$$\frac{a_{k+1}}{a_k} = \frac{1}{3^{m+1}} \cdot \frac{3^{m+1}}{2} = \frac{1}{2} .$$

Hence,

$$\lim\sup \frac{a_{k+1}}{a_k} = \frac{2}{3} < 1 .$$

Therefore the series converges by Theorem 5.5.

Example 5.9: Use the ratio test to test the following series, $\Sigma\, a_k$ where a_k is given by

$$a_k = \begin{cases} 3^m & \text{if k is even, } \quad \text{i.e. } k = 2m \\ \frac{3^{m+1}}{2} & \text{if k is odd, } \quad \text{i.e. } k = 2m+1 \end{cases}$$

If k is even, then

$$\frac{a_{k+1}}{a_k} = \frac{3^{m+1}}{2} \cdot \frac{1}{3^m} = \frac{3}{2} .$$

If k is odd, then

$$\frac{a_{k+1}}{a_k} = 3^{m+1} \cdot \frac{2}{3^{m+1}} = 2 .$$

Hence,

$$\lim\inf \frac{a_{k+1}}{a_k} = \frac{3}{2} > 1.$$

Therefore the series diverges by Theorem 5.5.

In many cases, the sequence of ratios $< a_{k+1}/a_k >$ is convergent. When this happens, the statement of the ratio test is much simpler. Since $\lim\inf a_n = \lim a_n = \lim\sup a_n$ when the limit exists, the ratio test is simply stated in the next theorem.

Theorem 5.6: If $\lim a_{k+1}/a_k = r$, then Σa_k is convergent if $r < 1$ and is divergent if $r > 1$; if $r = 1$, Σa_k may be either.

Example 5.10: Use Theorem 5.6 to test the convergence or divergence of

$$\sum_{k=0}^{\infty} \frac{1}{k!} .$$

Since

$$a_{k+1} = \frac{1}{(k+1)!} \quad \text{and} \quad a_k = \frac{1}{k!} ,$$

then

$$\frac{a_{k+1}}{a_k} = \frac{1/(k+1)!}{1/k!} = \frac{k!}{(k+1)!} = \frac{k!}{(k+1)k!} = \frac{1}{k+1}$$

and

$$\lim \frac{1}{k+1} = 0 .$$

By Theorem 5.6, $\lim a_{k+1}/a_k = 0$ which is less than 1, hence the series is convergent.

Example 5.11: Test the series

$$\sum_{k=1}^{\infty} \frac{2^k}{k^2}$$

for convergence or divergence.

$$a_{k+1} = \frac{2^{k+1}}{(k+1)^2} \quad \text{and} \quad a_k = \frac{2^k}{k^2}$$

and

$$\lim \left(\frac{2^{k+1}}{(k+1)^2} \cdot \frac{k^2}{2^k} \right) = \lim \frac{2k^2}{(k+1)^2}$$

$$= 2 \lim \left(\frac{k}{k+1} \right)^2$$

$$= 2 \lim \left(\frac{1}{1+1/k} \right)^2$$

$$= 2 \ ,$$

By Theorem 5.6, $\lim a_{k+1}/a_k = 2$ which is greater than 1, hence the series diverges.

Theorem 5.7: (Root Test) Let $a_k > 0$, $\forall k$, and lim sup $(a_k)^{1/k} = r$. Then $\Sigma\, a_k$ converges if $r < 1$ and diverges if $r > 1$; when $r = 1$, no conclusion can be reached.

Proof: (a) Assume $r < 1$ and prove that $\Sigma\, a_k$ converges.

1. $r = \limsup (a_k)^{1/k} < 1$	1. Hypothesis
2. Choose $B \ni r < B < 1$	2. Real numbers are dense
3. Let $\epsilon = (B-r) > 0$ and $(a_k)^{1/k} < r + \epsilon = r + (B-r) = B$ for all but a finite number of terms	3. Theorem 3.15
4. For $\epsilon = B - r \ \exists\ N \ni (a_k)^{1/k} < B$, $\forall\ k > N$	4. Step 3 and meaning of all but a finite number of terms

5. $a_k < B^k, \quad \forall k > N$	5. Raise each term to the kth power in Step 4
6. ΣB^k converges	6. Step 2 and geometric series with r less than 1
7. Σa_k converges	7. Steps 5, 6 and Theorem 5.2

(b) Assume $r > 1$ and prove that Σa_k diverges.

1. $r = \lim\sup (a_k)^{1/k} > 1$	1. Hypothesis
2. Choose $B \ni 1 < B < r$	2. Real numbers are dense
3. Let $\epsilon = (r - B) > 0$ and $(a_k)^{1/k} > r - \epsilon = r - (r - B) = B > 1$ for infinitely many terms	3. Theorem 3.15 and Step 2
4. $a_k > 1$ for infinitely many terms	4. $(a_k)^{1/k} > 1$ in Step 3 and raise each term to the kth power
5. It is impossible for $a_k \to 0$	5. Step 4 and $a_k > 0, \quad \forall k$
6. Σa_k diverges	6. Theorem 4.2

(c) If $r = 1$, no conclusion

1. $\Sigma 1/k$ diverges and $\lim (1/k)^{1/k} = 1$	1. The calculation of this limit is beyond the scope of this paper, hence is simply stated

2. $\Sigma\ 1/k^2$ converges and $\lim\ (1/k^2)^{1/k} = 1$	2. The calculation of this limit is beyond the scope of this paper, hence is simply stated
3. Therefore no conclusion can be drawn when $r = 1$	3. Steps 1 and 2

Although the ratio test is easier to apply than the root test, the root test is more powerful than the ratio test; i.e., the root test can be applied, with conclusive results, to a larger class of series than the ratio test. If the ratio test yields a definite conclusion about the convergence or divergence of a given series, the root test will also yield a definite conclusion; however, there exist series for which the ratio test does not yield a definite conclusion, but the root test does.

Example 5.12:

$$1/4 + 1/2 + 1/8 + 1/4 + 1/16 + 1/8 + 1/32 + \ldots$$

where the terms are given by the following:

$$a_k = \begin{cases} \dfrac{1}{2^m} & \text{if k is even; i.e. } k = 2m \\[2ex] \dfrac{1}{2^{m+1}} & \text{if k is odd; i.e. } k = 2m - 1 \end{cases}$$

The sequence of term ratios is

$$\frac{1/2}{1/4} = 2, \quad \frac{1/8}{1/2} = \frac{1}{4}, \quad \frac{1/4}{1/8} = 2$$

and in general,

$$\frac{a_{k+1}}{a_k} = \begin{cases} 2 & \text{when k is odd} \\ 1/4 & \text{when k is even} \end{cases}$$

The lim sup $a_{k+1}/a_k = 2 > 1$ and the lim inf $a_{k+1}/a_k = 1/4 < 1$, therefore the ratio test is inconclusive. However the root test may be used as follows: when $a_k = a_{2m} = 1/2^m$, the lim $(1/2^m)^{1/2m}$ = lim $(1/2)^{m/2m}$ = lim $(1/2)^{1/2} = (1/2)^{1/2}$, when

$$a_k = a_{2m-1} = \frac{1}{2^{m+1}},$$

the

$$\lim \left(\frac{1}{2^{m+1}}\right)^{1/(2m-1)} = \lim \left(\frac{1}{2}\right)^{(m+1)/(2m-1)}$$

$$= \lim \left(\frac{1}{2}\right)^{(1+1/m)/(2-1/m)}$$

$$= \left(\frac{1}{2}\right)^{1/2}.$$

The lim sup of the sequence $< a_k^{1/k} >$ is $(1/2)^{1/2}$ since the limit is $(1/2)^{1/2}$. Since $(1/2)^{1/2}$ is less than 1, the series converges by the root test.

This chapter is a very elementary introduction to the study of convergence tests for positive term series. The tests included here are the best known. There are many refinements of the ratio and comparison tests which one would want to have available for use if the convergence or divergence of the series is not strong enough to be determined by the tests given in this chapter. The reader is referred to the following books for additional tests: Infinite Sequences and Series by Konrad Knopp and Infinite Series by Earl D. Rainville.

CHAPTER VI

TESTS FOR CONDITIONAL CONVERGENCE

In the last chapter several tests for convergence were discussed that can be used with series of nonnegative terms. Since

$$\Sigma\ |a_k| = |a_1| + |a_2| + |a_3| + \ldots + |a_n| + \ldots,\ \Sigma |a_k|$$

is always a positive term series. Hence all the tests in Chapter V can be used to test absolute convergence of any series. The next discussion will be about series whose terms may be positive or negative, and the tests presented are most interesting in case the series does not converge absolutely. The simplest examples occur when the terms alternate in sign. These are called alternating series as the next definition will characterize.

Definition 6.1: (Alternating Series) An infinite series of the form

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k = (a_1) + (-a_2) + (a_3) + (-a_4) + \ldots + (-1)^{n-1} a_n + \ldots$$

where each $a_k > 0$ is called an alternating series.

This is frequently written

$$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1} a_n + \ldots$$

although the even numbered terms are actually the negative numbers $-a_2$, $-a_4$, $-a_6$, ... $-a_{2n}$, ...

<u>Example 6.1</u>: A classical example of an alternating series is the alternating harmonic series

$$1 - 1/2 + 1/3 - 1/4 + \ldots + (-1)^{k-1}(1/k) + \ldots$$

which can be shown to be convergent with the value ln 2.

It is convenient to have a simple test which state sufficient conditions for the convergence of an alternating series. This theorem will be proved by using the Cauchy condition to show that the sequence $< s_n >$ of partial sums converges.

Let

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

be an alternating series and let s_n represent the nth partial sum, then

$$s_{n+p} - s_n = (-1)^n a_{n+1} + (-1)^{n+1} a_{n+2} + (-1)^{n+2} a_{n+3} + \ldots + (-1)^{n+p-1} a_{n+p}$$

$$= (-1)^n (a_{n+1} + (-1)^1 a_{n+2} + (-1)^2 a_{n+3} + \ldots + (-1)^{p-1} a_{n+p})$$

$$= (-1)^n (a_{n+1} - a_{n+2} + a_{n+3} - \ldots + (-1)^{p-1} a_{n+p}) .$$

The difference $s_{n+p} - s_n$ has exactly p terms with alternating signs as described above. With this information consider now the following theorem.

<u>Theorem 6.1</u>: (Alternating Series Test) If (1) $a_{k+1} \leq a_k, \quad \forall k,$ and (2) $\lim a_k = 0$, then the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

is convergent.

Discussion of the proof: Let $< s_n >$ be the sequence of partial sums for

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

and show that this sequence is convergent by the Cauchy condition. In applying the Cauchy condition inequalities are obtained for $s_{n+p} - s_n$ by appropriate grouping of the terms. Condition (1) in the hypothesis plays an important part of these inequalities.

Proof:

1. Let $s_n = a_1 - a_2 + a_3 - \ldots + (-1)^{n-1} a_n$ — 1. Assumption and notation

2. $|s_{n+p} - s_n|$
$= |(-1)^n (a_{n+1} - a_{n+2} + \ldots + (-1)^{p-1} a_{n+p})|$
$= |(a_{n+1} - a_{n+2} + \ldots + (-1)^{p-1} a_{n+p})|$ — 2. Step 1, calculations, properties of absolute value

3. $a_k - a_{k+1} \geq 0, \quad \forall k$ — 3. Hypothesis (1), $a_{k+1} \leq a_k$, $\forall$ k

4. If p is odd, then
$a_{n+1} - a_{n+2} + a_{n+3} - \ldots + (-1)^{p-1} a_{n+p}$
$= (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \ldots + a_{n+p}$
> 0 — 4. Associative property, Step 3, Definition 6.1, $a_k > 0, \quad \forall k$

5. If p is odd, then

$a_{n+1}-a_{n+2}+a_{n+3}-\ldots+(-1)^{p-1}a_{n+p}$

$= a_{n+1}-(a_{n+2}-a_{n+3})-\ldots-(a_{n+p-1}-a_{n+p})$

$\leq a_{n+1}$

5. Associative property, Step 3, Definition 6.1, $a_k > 0$, $\forall k$

6. If p is even, then

$a_{n+1}-a_{n+2}+a_{n+3}-\ldots+(-1)^{p-1}a_{n+p}$

$= (a_{n+1}-a_{n+2})+\ldots+(a_{n+p-1}-a_{n+p})$

≥ 0

6. Associative property, Step 3

7. If p is even, then

$a_{n+1}-a_{n+2}+a_{n+3}-\ldots+(-1)^{p-1}a_{n+p}$

$= a_{n+1}-(a_{n+2}-a_{n+3})-\ldots\ -a_{n+p}$

$\leq a_{n+1}$

7. Associative property, Step 3, Definition 6.1, $a_k > 0$, $\forall k$

8. Hence $|s_{n+p}-s_n| \leq a_{n+1} = |a_{n+1}|$ $\forall$ n and $\forall$ p

8. Steps 2, 4, 5, 6, 7, Definition 6.1, $a_k > 0$, $\forall k$, Definition of absolute value

9. $< a_{n+1} >$ is a null sequence

9. Hypothesis (2), Corollary 3.2

10. $< s_{n+p}-s_n >$ is a null sequence

10. Steps 8, 9, Theorem 2.5

11. $< s_n >$ converges and $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges

11. Step 10 and Cauchy's condition, Definition 4.3

Example 6.2: The test in Theorem 6.1 shows the convergence of the alternating harmonic series which was defined in Example 6.1. Since $a_k = 1/k$ and $a_{k+1} = 1/(k+1)$ and $1/(k+1) \leq 1/k, \ \forall k$, then $a_{k+1} \leq a_k, \ \forall k$ and part (1) of the hypothesis in Theorem 6.1 is satisfied. Since $\lim a_k = \lim 1/k = 0$, part (2) of the hypothesis in Theorem 6.1 is satisfied. By Theorem 6.1, the alternating harmonic series is convergent. Observe that this test tells nothing about the value of the series.

Example 6.3: Test the following alternating series for convergence.

$$\frac{2}{3} - \frac{1}{2} + \frac{4}{9} - \ldots + \frac{(-1)^{k-1}(k+1)}{3k} + \ldots$$

$$a_k = \frac{k+1}{3k} \quad \text{and} \quad a_{k+1} = \frac{k+2}{(3k+1)}$$

Check to see if $a_{k+1} \leq a_k$. It is if

$$\frac{k+2}{3(k+1)} \leq \frac{k+1}{3k} .$$

Since $3k^2 + 6k \leq 3k^2 + 6k + 3, \forall k$, then $(k+2)3k \leq 3(k+1)(k+1), \ \forall k$, and hence

$$\frac{k+2}{3(k+1)} \leq \frac{k+1}{3k}, \quad \forall k.$$

Therefore, $a_{k+1} \leq a_k, \ \forall k$ and part (1) of Theorem 6.1 is satisfied. Check to see if $\lim a_k = 0$. The

$$\lim a_k = \lim \frac{k+1}{3k} = \lim \frac{1 + 1/k}{3} = \frac{1}{3} .$$

Therefore $\lim a_k \neq 0$. Hence the above series is divergent by Theorem 4.2.

In Example 6.3, part (1) of the hypothesis in Theorem 6.1 is satisfied. However, part (2) is not satisfied. In practice, it will save time to check part (2) of the hypothesis first since Theorem 4.2 states that $\lim a_k \neq 0$ is a sufficient condition for divergence.

Even though a test is available for checking an alternating series, it is possible to have an alternating series which converges but the test in Theorem 6.1 cannot be applied since the hypothesis is not satisfied. Theorem 6.1 states sufficient conditions for convergence. To show that these conditions are not necessary, consider the next example.

Example 6.4: Consider the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

where a_k is given as follows:

$$a_k = \begin{cases} \frac{1}{3^k} & \text{if k is odd} \\ \frac{1}{2^k} & \text{if k is even} \end{cases}$$

Since $a_k \leq 1/2^k$, $\forall$ k, then $\lim a_k = 0$. Now check to see if $a_{k+1} \leq a_k$, $\forall$ k. Consider the first four values of a_k, i.e. $a_1 = 1/3$, $a_k = 1/4$, $a_3 = 1/27$, and $a_4 = 1/16$. Since $27 > 16$, then $1/16 > 1/27$, or $a_4 > a_3$. Therefore $a_{k+1} \not\leq a_k$, $\forall$ k. Since the hypothesis of Theorem 6.1 is not satisfied, no information is gained about the convergence or divergence of this series. This can be shown to converge absolutely by using the comparison test as the following argument shows.

Since $2^k < 3^k$, $\forall\, k \geq 1$, then $(1/3)^k < (1/2)^k$, $\forall\, k \geq 1$. Therefore $a_k \leq (1/2)^k$, $\forall\, k \geq 1$. The series $\Sigma(1/2)^k$ is convergent since it is a geometric series such that $|r| < 1$. The series $\Sigma\, a_k$ is convergent since it is dominated by the series $\Sigma\, (1/2)^k$. Since $|(-1)^{k-1}a_k| = a_k$, $\Sigma\,|(-1)^{k-1}a_k| = \Sigma\, a_k$ and $\Sigma\, a_k$ converges, then $\Sigma\,|(-1)^{k-1}a_k|$ converges. This implies that $\Sigma(-1)^{k-1}a_k$ is absolutely convergent by Definition 4.5 and hence convergent by Theorem 4.9.

It is important to have tests for determining convergence for more general series that are conditionally convergent. The next tests are particularly useful for this purpose. They depend on the partial summation formula of Abel which is developed in the next theorem.

Theorem 6.2: (Partial summation formula of Abel) Let $< a_n >$ and $< b_n >$ be two sequences of real numbers, and

$$A_n = a_1 + a_2 + a_3 + \ldots + a_n \,.$$

The following identity is true:

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k) \,.$$

Discussion of the proof: The following proof is obtained by making the proper substitution for a_k in terms of A_k and A_{k-1} in the product $a_k b_k$. Then by use of properties for finite sums, change of index on summation and algebraic manipulations, the desired identity is obtained.

Proof:

1. Let $A_0 = 0$	1. Simplify notation

2. $A_k = a_1 + a_2 + a_3 + \ldots + a_k$ $A_{k-1} = a_1 + a_2 + a_3 + \ldots + a_{k-1}$ Therefore $A_k - A_{k-1} = a_k$	2. Hypothesis and calculation
3. $\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k$ $= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k$	3. Step 2 and substitution, subtraction of finite sums
4. $\sum_{k=1}^{n} A_{k-1} b_k = A_0 b_1 + \ldots + A_{n-1} b_n$ $= 0 + A_1 b_2 + \ldots + A_{n-1} b_n$ $= A_1 b_1 + \ldots + A_{n-1} b_n$ $= \sum_{k=1}^{n-1} A_k b_{k+1}$	4. Expand finite sum, $A_0 b_1 = 0$ since $A_0 = 0$ in Step 1, calculation, change of index on summation
5. $\sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k$ $= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n-1} A_k b_{k+1} - A_n b_{n+1} + A_n b_{n+1}$ $= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}$ $= A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$	5. Step 4 and substitution, add and subtract $A_n b_{n+1}$, combine 2nd and 3rd terms, commutative property and combine two finite sums with the same index
6. Therefore $\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k)$	6. Steps 3, 5 and transitive property of equality

Observe that

$$\sum_{k=1}^{n} a_k b_k$$

is the nth partial sum for the series

$$\sum_{k=1}^{\infty} a_k b_k \; .$$

Therefore

$$\sum_{k=1}^{\infty} a_k b_k$$

converges if both the series

$$\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$$

and the sequence $< A_n b_{n+1} >$ converge.

<u>Theorem 6.3</u>: (Dirichlet's Test) Let $\Sigma\, a_k$ be a series of real terms whose partial sums form a bounded sequence. Let $< b_n >$ be a decreasing sequence which converges to 0. Then $\Sigma\, a_k b_k$ converges.

Discussion of the proof: In order to use the partial summation formula of Abel, it is necessary to consider the behavior of the sequence $< A_n b_{n+1} >$ and the behavior of the series

$$\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k).$$

Since the partial sum sequence $< A_n >$ is bounded and $< b_n >$ is a null sequence, it is fairly easy to show that $< A_n b_{n+1} >$ is a null sequence, hence convergent. The absolute convergence of

$$\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$$

is established by showing that it is dominated by the convergent positive term series

$$\sum_{k=1}^{\infty} M(b_k - b_{k+1}).$$

Proof:

1. Let $A_n = a_1+a_2+a_3+\ldots+a_n$	1. Notation
2. $\exists M \ni \lvert A_n \rvert \leq M, \ \forall n$	2. Hypothesis, partial sums are bounded, Definition 2.4'
3. $\lim b_n = 0$	3. Hypothesis, $b_n \downarrow 0$
4. $\lim b_{n+1} = 0$ or $< b_{n+1} >$ is a null sequence	4. Step 3 and Corollary 3.2, null sequence has a limit of 0
5. $< A_n b_{n+1} >$ is a null sequence or $\lim A_n b_{n+1} = 0$	5. Steps 2, 5 and Theorem 2.6, null sequence has a limit of 0
6. Therefore $< A_n b_{n+1} >$ converges	6. Step 5 and Definition 3.1
7. $b_k - b_{k+1} > 0, \ \forall k$	7. Hypothesis, $b_n \downarrow 0$
8. $\lvert b_{k+1} - b_k \rvert = b_k - b_{k+1}, \ \forall k$	8. Definition of absolute value, Step 7
9. $\lvert A_k(b_{k+1}-b_k) \rvert \leq M(b_k-b_{k+1}), \ \forall k$	9. Steps 2 and 8, theorem about absolute values

10. $\sum_{k=1}^{n} (b_k - b_{k+1}) = b_1 - b_{n+1}$ — 10. Calculation of telescoping sum

11. $\lim (b_1 - b_{n+1}) = \lim b_1 - \lim b_{n+1}$ $= b_1 - 0 = b_1$ — 11. Theorem 3.10, Corollary 3.5 and Step 4, calculation

12. Therefore $\sum_{k=1}^{\infty} (b_k - b_{k+1})$ converges — 12. Steps 10 and 11, limit of partial sums exist and is equal to b_1

13. $\sum_{k=1}^{\infty} M(b_k - b_{k+1})$ converges — 13. Step 12 and Theorem 4.4

14. Hence $\Sigma A_k(b_{k+1} - b_k)$ is absolutely convergent, hence convergent — 14. Steps 9, 13 and Theorem 5.2, Definition 4.5, Theorem 4.9

15. Therefore $\Sigma a_k b_k$ converges — 15. Steps 6, 14 and Theorem 6.2

Example 6.5: Test the following series for convergence.

$$\sum_{k=1}^{\infty} c_k = 1 + 1/2 - 2/3 + 1/4 + 1/5 - 2/6 + 1/7 + 1/8 - 2/9 + \dots$$

where each c_k is given as follows:

$$c_k = \begin{cases} 1/k \text{ if } k \text{ has a remainder of 1 when divided by 3} \\ 1/k \text{ if } k \text{ has a remainder of 2 when divided by 3} \\ -2/k \text{ if } k \text{ has a remainder of 0 when divided by 3} \end{cases}$$

Factor each c_k as follows:

$$c_k = \begin{cases} 1 \cdot 1/k \text{ if } k = 1, 4, 7, \ldots \\ 1 \cdot 1/k \text{ if } k = 2, 5, 8, \ldots \\ -2 \cdot 1/k \text{ if } k = 3, 6, 9, \ldots \end{cases}$$

Let $c_k = a_k b_k$,

$$\sum_{k=1}^{\infty} a_k = 1 + 1 - 2 + 1 + 1 - 2 + 1 + 1 - 2 + \ldots$$

and $< b_n > = < 1/n >$. The following argument will show that

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges. The sequence $< b_n > = < 1/n >$ is a monotone decreasing null sequence. Let A_n represent the nth partial sum of the series

$$\sum_{k=1}^{\infty} a_k .$$

Then $A_1 = 1$, $A_2 = 2$, $A_3 = 0$, $A_4 = 1$, $A_5 = 2$, $A_6 = 0$, ... Hence $0 \leq A_n \leq 2$, $\forall n$, and the sequence $< A_n >$ of partial sums of the series

$$\sum_{k=1}^{\infty} a_k$$

is bounded. Therefore

$$\sum_{k=1}^{\infty} a_k$$

as defined above and $< b_n > = < 1/n >$ satisfy the hypothesis of Theorem 6.3, hence

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges.

Example 6.6: Test the following series for convergence.

$$\sum_{k=1}^{\infty} c_k = 1 + 1/2 - 1/3 - 1/4 + 1/5 + 1/6 - 1/7 - 1/8 + \ldots$$

where each c_k is given as follows:

$$c_k = \begin{cases} 1/k \text{ if } k \text{ has a remainder of 1 when divided by 4} \\ 1/k \text{ if } k \text{ has a remainder of 2 when divided by 4} \\ -1/k \text{ if } k \text{ has a remainder of 3 when divided by 4} \\ -1/k \text{ if } k \text{ has a remainder of 0 when divided by 4} \end{cases}$$

Factor each c_k as follows:

$$c_k = \begin{cases} 1 \cdot 1/k \text{ if } k = 1, 5, 9, \ldots \\ 1 \cdot 1/k \text{ if } k = 2, 6, 10, \ldots \\ -1 \cdot 1/k \text{ if } k = 3, 7, 11, \ldots \\ -1 \cdot 1/k \text{ if } k = 4, 8, 12, \ldots \end{cases}$$

Let $c_k = a_k b_k$,

$$\sum_{k=1}^{\infty} a_k = 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 + \ldots$$

and $< b_n > = < 1/n >$. Here as in the preceding example $0 \leq A_n \leq 2$, $\forall n$, and $< b_n >$ is a monotone decreasing null sequence. Hence

$$\sum_{k=1}^{\infty} c_k$$

converges.

The following theorem gives a test which is slightly different but just as important in its application.

Theorem 6.4: (Abel's Test) The series

$$\sum_{k=1}^{\infty} a_k b_k$$

converges if

$$\sum_{k=1}^{\infty} a_k$$

converges and if $< b_n >$ is a monotone convergent sequence.

Discussion of the proof: Consider the behavior of the sequence $< A_n b_{n+1} >$ and the behavior of the series

$$\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k).$$

The sequence $< A_n b_{n+1} >$ is no longer a null sequence as in Theorem 6.3, but it does converge. This time the absolute convergence of

$$\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$$

is established by showing that it is dominated by the convergent series

$$\sum_{k=1}^{\infty} M|b_{k+1} - b_k|.$$

Proof:

1. Let $A_n = a_1 + a_2 + a_3 + \ldots + a_n$	1. Notation
2. $< A_n >$ converges	2. Step 1, Σa_k converges, Definition 4.3
3. $\lim b_n$ exists	3. Hypothesis, $< b_n >$ is convergent sequence, Definition 3.2

4. $\lim b_{n+1}$ exists and $< b_{n+1} >$ is convergent sequence	4. Step 3, Corollary 3.2
5. $< A_n b_{n+1} >$ converges	5. Steps 2, 4, Theorem 3.11
6. $< A_n >$ is bounded	6. Step 2 and Theorem 3.1
7. $\exists M \ni \lvert A_n \rvert \leq M, \ \forall n$	7. Step 6, Definition 2.4'
8. $\lvert A_k(b_{k+1} - b_k) \rvert \leq M \lvert b_{k+1} - b_k \rvert, \ \forall k$	8. Multiply each term in Step 7 by $\lvert b_{k+1} - b_k \rvert$, absolute value theorem
9. $< b_n >$ is monotone implies that $b_{k+1} - b_k \geq 0, \ \forall k$ or $b_k - b_{k+1} \geq 0, \ \forall k$	9. Hypothesis and Definition 2.9
10. $\sum_{k=1}^{\infty} (b_k - b_{k+1})$ converges	10. Steps 10, 11, 12 in Theorem 6.3
11. $\sum_{k=1}^{\infty} (-1)(b_k - b_{k+1}) = \sum_{k=1}^{\infty} (b_{k+1} - b_k)$ converges	11. Step 10, Theorem 4.4
12. Therefore $\sum_{k=1}^{\infty} \lvert b_{k+1} - b_k \rvert$ converges	12. Since $\lvert b_{k+1} - b_k \rvert = b_{k+1} - b_k, \ \forall k$, or $\lvert b_{k+1} - b_k \rvert = b_k - b_{k+1}, \ \forall k$ and Steps 9, 10, 11

13. $\sum_{k=1}^{\infty} M\lvert b_{k+1} - b_k\rvert$ converges	13. Step 12 and Theorem 4.4
14. $\sum_{k=1}^{\infty} \lvert A_k(b_{k+1} - b_k)\rvert$ converges	14. Steps 8, 13 and Theorem 5.2
15. $\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$ is absolutely convergent, hence convergent	15. Step 14 and Definition 4.5, Theorem 4.9
16. Hence $\Sigma\, a_k b_k$ converges	16. Steps 5, 15, and Theorem 6.2

Example 6.7: Test the following series for convergence.

$$\sum_{k=1}^{\infty} c_k = 2 + 3/4 - 4/9 - 5/16 + 6/25 + 7/36 - 8/49 - 9/64 + \ldots$$

where each c_k is given as follows:

$$c_k = \begin{cases} (k+1)/k^2 & \text{if } k \text{ has a remainder of 1 when divided by 4} \\ (k+1)/k^2 & \text{if } k \text{ has a remainder of 2 when divided by 4} \\ -(k+1)/k^2 & \text{if } k \text{ has a remainder of 3 when divided by 4} \\ -(k+1)/k^2 & \text{if } k \text{ has a remainder of 0 when divided by 4} \end{cases}$$

Let $c_k = a_k b_k$. Consider $\lvert c_k\rvert = (k+1)/k^2 = 1/k \cdot (k+1)/k = 1/k(1 + 1/k)$. Factor each c_k as follows:

$$c_k = \begin{cases} (1 \cdot 1/k)(1 + 1/k) & \text{if } k = 1, 5, 9, \ldots \\ (1 \cdot 1/k)(1 + 1/k) & \text{if } k = 2, 6, 10, \ldots \\ (-1 \cdot 1/k)(1 + 1/k) & \text{if } k = 3, 7, 11, \ldots \\ (-1 \cdot 1/k)(1 + 1/k) & \text{if } k = 4, 8, 12, \ldots \end{cases}$$

Let

$$\sum_{k=1}^{\infty} a_k = 1 + 1/2 - 1/3 - 1/4 + 1/5 + 1/6 - 1/7 - 1/8 + \ldots$$

and $< b_n > = < 1 + 1/n >$. The following argument will show that

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges. The sequence $< b_n > = < 1 + 1/n >$ is a monotone decreasing sequence which converges to the number 1. The series

$$\sum_{k=1}^{\infty} a_k$$

as defined above was shown to be convergent in Example 6.6. Therefore,

$$\sum_{k=1}^{\infty} a_k$$

as defined above and $< b_n > = < 1 + 1/n >$ satisfy the hypothesis of Theorem 6.4, hence

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges. Note that $\Sigma\, c_k$ is conditionally convergent, since

$$2 + 3/4 + 4/9 + 5/16 + 6/25 + 7/36 + 8/49 + 9/64 + \ldots$$

diverges.

Example 6.8: Test the following series for convergence.

$$\sum_{k=1}^{\infty} c_k = 1 + 3/\sqrt{2} - 4/\sqrt{3} + 1/\sqrt{4} + 3/\sqrt{5} - 4/\sqrt{6} + \ldots$$

where each c_k is given by the following:

$$c_k = \begin{cases} 1/\sqrt{k} & \text{if } k \text{ has a remainder of 1 when divided by 3} \\ 3/\sqrt{k} & \text{if } k \text{ has a remainder of 2 when divided by 3} \\ -4/\sqrt{k} & \text{if } k \text{ has a remainder of 0 when divided by 3} \end{cases}$$

Let $c_k = a_k b_k$ and factor each c_k as follows:

$$c_k = \begin{cases} 1(1/\sqrt{k}) & \text{if } k = 1, 4, 7, \ldots \\ 3(1/\sqrt{k}) & \text{if } k = 2, 5, 8, \ldots \\ -4(1/\sqrt{k}) & \text{if } k = 3, 6, 9, \ldots \end{cases}$$

Let

$$\sum_{k=1}^{\infty} a_k = 1 + 3 - 4 + 1 + 3 - 4 + 1 + 3 - 4 + \ldots$$

and $< b_n > = < 1/\sqrt{n} >$. The following argument will show that

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges. The sequence $< b_n > = < 1/\sqrt{n} >$ is a monotone decreasing null sequence. Let A_n represent the nth partial sum of the series

$$\sum_{k=1}^{\infty} a_k .$$

Then $A_1 = 1$, $A_2 = 4$, $A_3 = 0$, $A_4 = 1$, $A_5 = 4$, $A_6 = 0$, ... Hence $0 \leq A_n \leq 4$, $\forall n$, and the sequence $< A_n >$ of partial sums of the series

$$\sum_{k=1}^{\infty} a_k$$

is bounded. Therefore

$$\sum_{k=1}^{\infty} a_k$$

as defined above and $< b_n > = < 1/\sqrt{n} >$ satisfy the hypothesis of Theorem 6.3, hence

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges.

Example 6.9: Test the following series for convergence.

$$\sum_{k=1}^{\infty} c_k = 1 + 3/4 - 5/9 - 7/16 + 9/25 + 11/36 - 13/49 - 15/64 + \ldots$$

where each c_k is given by the following:

$$c_k = \begin{cases} (2k-1)/k^2 & \text{if k has a remainder of 1 when divided by 4} \\ (2k-1)/k^2 & \text{if k has a remainder of 2 when divided by 4} \\ -(2k-1)/k^2 & \text{if k has a remainder of 3 when divided by 4} \\ -(2k-1)/k^2 & \text{if k has a remainder of 0 when divided by 4} \end{cases}$$

Let $c_k = a_k b_k$ and factor each c_k as follows:

$$c_k = \begin{cases} (1/k)(2-1/k) & \text{if } k = 1, 5, 9, \ldots \\ (1/k)(2-1/k) & \text{if } k = 2, 6, 10, \ldots \\ (-1/k)(2-1/k) & \text{if } k = 3, 7, 11, \ldots \\ (-1/k)(2-1/k) & \text{if } k = 4, 8, 12, \ldots \end{cases}$$

Let

$$\sum_{k=1}^{\infty} a_k = 1 + 1/2 - 1/3 - 1/4 + 1/5 + 1/6 - 1/7 - 1/8 + \ldots$$

and $< b_n > = < 2-1/n >$. The following argument will show that

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges. The sequence $< b_n > = < 2-1/n >$ is a monotone increasing sequence which converges to the number 2. The series

$$\sum_{k=1}^{\infty} a_k$$

as defined above was shown to be convergent in Example 6.6. Therefore

$$\sum_{k=1}^{\infty} a_k$$

as defined above and $< b_n > = < 2 - 1/n >$ satisfy the hypothesis of Theorem 6.4. Hence

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} a_k b_k$$

converges.

<u>Example 6.10</u>: Test the following series for convergence.

$$\sum_{k=1}^{\infty} (-1)^{k-1} c_k = 1 - 1/4 + 1/3 - 1/16 + 1/5 - 1/36 + \ldots$$

where each c_k is given as follows:

$$c_k = \begin{cases} 1/k & \text{if } k = 1, 3, 5, \ldots \\ 1/k^2 & \text{if } k = 2, 4, 6, \ldots \end{cases}$$

Since

$$\sum_{k=1}^{\infty} (-1)^{k-1} c_k$$

is an alternating series, it is natural to try the alternating series test as stated in Theorem 6.1. Since $1/k^2 \leq 1/k$, $\forall k$, then $c_k \leq 1/k$, $\forall k$. The sequence $< 1/k >$ is a null sequence and by Theorem 2.5, $< c_k >$ is a null sequence. Thus $\lim c_k = 0$ and part (2) of the hypothesis in Theorem 6.1 is satisfied. Since $c_1 = 1$, $c_2 = 1/4$, $c_3 = 1/3$ and $1/4 < 1/3$, then $c_3 \not\leq c_2$. In general, $c_{k+1} \not\leq c_k$ when k is even. Hence part (1) of the hypothesis in Theorem 6.1 is not satisfied. So Theorem 6.1 cannot be applied to this series. No information is gained thus far as to the behavior of the series.

The next logical thing to try is Dirichlets's Test or Abel's Test. In order to use either of these tests, $(-1)^{k-1}c_k$ must be factored appropriately to satisfy the hypothesis. Factor $(-1)^{k-1}c_k$ as follows:

$$(-1)^{k-1}c_k = \begin{cases} 1 \cdot 1/k & \text{if } k = 1, 3, 5, \ldots \\ (-1/k)1/k & \text{if } k = 2, 4, 6, \ldots \end{cases}$$

Let $< b_n > = < 1/n >$ and $< b_n >$ is a monotone decreasing null sequence which satisfies the hypothesis of both tests. Let

$$\Sigma a_k = 1 - 1/2 + 1 - 1/4 + 1 - 1/6 + \ldots .$$

Let A_n represent the nth partial sum of Σa_k. Then $A_1 = 1$, $A_2 = 1-1/2$, $A_3 = (1-1/2) + 1$, $A_4 = (1-1/2) + (1-1/4)$, ... and in general,

$$\begin{aligned} A_{2n} &= (1-1/2) + (1-1/4) + \ldots + (1-1/2n) \\ &= 1/2 + 3/4 + \ldots + (2n-1)/2n \\ &\geq 1/2 + 1/2 + \ldots + 1/2 \\ &= n/2 \end{aligned}$$

Therefore $\forall K$, pick $n_0 > 2K$ and $|A_{2n_0}| > K$ and $< A_n >$ is unbounded by Definition 2.4'. By the contrapositive of Theorem 3.1, $< A_n >$ diverges. By Definition 4.3, Σa_k diverges. So neither Dirichlet's Test nor Abel's Test can be applied with this factorization. Consider another factorization as follows:

$$(-1)^{k-1}c_k = \begin{cases} 1 \cdot 1/k & \text{if } k = 1, 3, 5, \ldots \\ (-1) \cdot 1/k^2 & \text{if } k = 2, 4, 6, \ldots \end{cases}$$

Let

$$\Sigma a_k = 1 - 1 + 1 - 1 + 1 - 1 + \ldots .$$

In Example 4.9, the sequence of partial sums is shown to be bounded but the series $\Sigma\, a_k$ is divergent. This choice of $\Sigma\, a_k$ is a candidate for Dirichlet's Test but not for Abel's Test. Let $< b_n >$ be described as follows:

$$b_n = \begin{cases} 1/n & \text{if } n = 1, 3, 5, \ldots \\ 1/n^2 & \text{if } n = 2, 4, 6, \ldots \end{cases}$$

Consider some terms in this sequence, $b_1 = 1$, $b_2 = 1/4$, $b_3 = 1/3$. Since $1/3 > 1/4$, this is not a decreasing sequence even though it converges to 0. So neither Dirichlet's Test nor Abel's Test can be applied with this factorization. It is impossible to consider all the possible factorizations and at this stage of the analysis, it is natural to wonder whether the series might diverge. Since $\lim (-1)^{k-1} c_k = 0$, Theorem 4.2 cannot be applied.

Another possibility is to investigate directly the behavior of the sequence of partial sums. However, in this case, Theorem 4.10 may give some useful information. Let

$$\Sigma\, p_k = 1 + 0 + 1/3 + 0 + 1/5 + 0 + 1/7 + 0 + \ldots$$

and

$$\Sigma\, q_k = 0 + 1/2^2 + 0 + 1/4^2 + 0 + 1/6^2 + \ldots .$$

The contrapositive of part (i) of Theorem 4.10 is stated as follows: If $\Sigma\, p_k$ converges or $\Sigma\, q_k$ converges, then $\Sigma(-1)^{k-1} c_k$ either diverges or is absolutely convergent.

In $\Sigma\, q_k$, $q_k \leq 1/k^2$, $\forall k$. Since $\Sigma\, 1/k^2$ converges, $\Sigma\, q_k$ converges by Theorem 5.2. Therefore $\Sigma\, (-1)^{k-1} c_k$ either diverges or is absolutely convergent.

The contrapositive of part (ii) of Theorem 4.10 is stated as follows: If $\Sigma\ p_k$ diverges or $\Sigma\ q_k$ diverges, then $\Sigma\ (-1)^{k-1}c_k$ is not absolutely convergent.

Let T_n be the nth partial sum of $\Sigma\ p_k$. Let n be even.

$$T_{2n} = 1 + 0 + 1/3 + 0 + 1/5 + \ldots + 1/(n-1) + 0 + \ldots + 1/(2n-1) + 0$$

$$T_n = 1 + 0 + 1/3 + 0 + 1/5 + \ldots + 1/(n-1) + 0$$

$$\begin{aligned}
|T_{2n} - T_n| &= 1/(n+1) + 0 + 1/(n+3) + 0 + 1/(n+5) + \ldots + 1/(2n-1) + 0 \\
&\geq 1/(2n-1) + 1/(2n-1) + 1/(2n-1) + \ldots + 1/(2n-1) \\
&= (1/(2n-1))(n/2) \\
&= n/2(2n-1) \\
&\geq n/2(2n) \\
&= 1/4
\end{aligned}$$

By the contrapositive of Theorem 3.7, $\Sigma\ p_k$ diverges. Hence $\Sigma\ (-1)^{k-1}c_k$ diverges.

CHAPTER VII

POWER SERIES

From the study of algebra, the reader is familar with mathematical sentences such as the following: (i) $3 + 2 = 7$, (ii) $5 + 1 > 3$, (iii) $x + 5 = 11$, and (iv) $x + 7 > 15$. The reader can readily determine whether the sentences in (i) and (ii) are true or false since the sentences contain only real numbers. The sentence in (i) is false since $3 + 2 = 5$ and $5 \neq 7$, and the sentence in (ii) is true since $5 + 1 = 6$ and $6 > 3$. The reader cannot determine whether the sentences in (iii) and (iv) are true or false until a real number is substituted for the symbol x. When the choice of a real number for x is made, then a decision can be made as to whether the sentence is true or false. Sentences such as (iii) and (iv) are referred to as open sentences in algebra. The symbol x is a placeholder for a real number. Considering the set of real numbers as the universal set, an open sentence divides the universe into two sets. One set contains all substitutions for x that make the open sentence true; the other set contains all substitutions for x that make the open sentence false. The first of these two sets is called the solution set or truth set of the open sentence. The truth set of the sentence in (iii) is $\{6\}$. The truth set of the sentence in (iv) is $\{x | x > 8\}$.

An analogous situation occurs with infinite series except that the question of concern is as follows: Is the series convergent or divergent? If the series is composed of real numbers, this question can be answered. If the series contains a placeholder, then a real number must be substituted for the placeholder before the above question can be answered. For example, (i) $1 + 1/2 + 1/3 + \ldots + 1/n + \ldots$ and (ii) $1 + 1 + 1/2! + 1/3! + \ldots + 1/n! + \ldots$ are series whose terms are real numbers. The series in (i) is divergent,and the series in (ii) is convergent. However, the following series are not composed of only real numbers: (i) $1 + 1/2^p + 1/3^p + \ldots + 1/n^p + \ldots$ and (ii) $1 + r + r^2 + \ldots + r^{n-1} + \ldots$. The symbol p is a placeholder for a real number in statement (i), and the series was shown in Example 5.1 to be convergent if $p > 1$ and divergent if $p \leq 1$. The symbol r is a placeholder for a real number in statement (ii), and in Example 4.2 the series was shown to be convergent if $|r| < 1$ and divergent if $|r| \geq 1$.

In this chapter, the series will be considered with terms of the form $a_n x^n$ where a_n is a real number and x is a placeholder for a real number. The reader is already familar with expressions which are sums of terms of the form $a_n x^n$. These expressions are called polynomials and are written as follows: $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$. If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, then for every choice of a real number for x, f(x) is a real number. Thus a polynomial can be used to define a function f whose domain is the real numbers. Series with terms of the form $a_n x^n$ can also be used to define functions. They are called power series. If the series converges for a particular choice of a real number x, then the function value is defined and is a real number. If the series diverges, this is not the case. Thus the domain of a function defined by a power series may be a proper subset of the real numbers. Early users of power series treated them very much like polynomials. They did not worry about whether or not their power series converged, and they did not distinguish between formal manipulations with power series and operations on the functions whose values are given by power series. However, they soon discovered that certain precautions had to be taken. These precautions will be pointed out in this chapter. Consider now the formal definition for a power series.

Definition 7.1: (Power Series) An infinite series of the form

$$\sum_{k=0}^{\infty} a_k(x - b)^k = a_0 + a_1(x - b) + a_2(x - b)^2 + \ldots + a_n(x - b)^n + \ldots$$

where each a_k is a real number and b is a real number, is called a power series in (x - b). If b = 0, then the series takes the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

and is called a power series in x. The numbers $a_0, a_1, a_2, \ldots$ are called coefficients.

For each value of x for which the power series converges the value of the series is a real number. Thus the power series gives the function values of a function f whose domain is the set of x-values for which the series converges. Then it is written

$$f(x) = \sum_{k=0}^{\infty} a_k(x - b)^k.$$

The geometric series is the power series

$$\sum_{k=0}^{\infty} ax^k,$$

since

$$\sum_{k=1}^{\infty} ax^{k-1} = \sum_{k=0}^{\infty} ax^k$$

by making a change on the index of summation. If

$$f(x) = \sum_{k=0}^{\infty} ax^k,$$

then the domain of f is (-1, 1), and the function values are given by $a/(1-x)$. This function f is a restriction of the function g where $g(x) = a/(1-x)$, and the domain of g is the set of all real numbers except the number $x = 1$. A restriction of a function is a new function which is a proper subset of the original function. In other words, the domain of the restriction is a proper subset of the domain of the original function, and the function values are assigned in the same way as in the original function.

If

$$f(x) = \sum_{k=0}^{\infty} a_k(x - b)^k,$$

then no matter what real numbers are represented by a_k, there exists at least one value in the domain of f, i. e. $x = b$. The power series in $(x - b)$ converges to a_0 at b, since $S_n(b) = a_0$, $\forall n$, and $\lim S_n(b) = a_0$.

In general, power series converge for some values of x, and diverge for others. In special instances, the two extreme cases may occur, in which the series converges for every x, or for none except when $x = b$. In the first of these special cases the power series is said to be everywhere convergent, and in the second, leaving out the self-evident point of convergence $x = b$, it is said to be nowhere convergent.

In studying power series it is sufficient to consider $\Sigma\, a_k x^k$ since the general case $\Sigma\, a_k(x-b)^k$ can be reduced to this by a translation of the origin along the x-axis.

Definition 7.2: (Region of Convergence) The totality of points x for which the given series $\Sigma\, a_k x^k$ converges is called the region of convergence. This set of real numbers is the domain for the function f such that $f(x) = \Sigma\, a_k x^k$.

The first important fact about a power series is expressed in the following theorem:

Theorem 7.1: If the power series

$$\sum_{k=0}^{\infty} a_k x^k$$

is convergent at x_0, where $x_0 \neq 0$, then the series is absolutely convergent at any number x_1 for which $|x_1| < |x_0|$.

Discussion of the proof: Since $|x_1| < |x_0|$, then $\left|\frac{x_1}{x_0}\right| < 1$. The geometric series with $r = \left|\frac{x_1}{x_0}\right|$ is convergent. It is fairly easy to show that $\Sigma\, a_k(x_1)^k$ is absolutely convergent by dominating it with an appropriate multiple of the geometric series with $r = \left|\frac{x_1}{x_0}\right|$.

Proof:

1. $\sum_{k=0}^{\infty} a_k(x_0)^k$ is convergent	1. Hypothesis, power series is convergent at x_0
2. $< a_k(x_0)^k >$ is a null sequence, hence convergent to 0	2. Step 1 and Theorem 4.1
3. $\exists\ A > 0 \ni |a_k(x_0)^k| \leq A$ for all integers $k \geq 0$	3. Convergent sequence in Step 2 is bounded by Theorem 3.1, Definition 2.4'

4. $a_k(x_1)^k = a_k(x_0)^k \ (x_1/x_0)^k, \ \forall k$	4. Multiply and divide by $(x_0)^k$ since $x_0 \neq 0$
5. $\lvert a_k(x_1)^k \rvert = \lvert a_k(x_0)^k \rvert \ \lvert x_1/x_0 \rvert^k$ $\leq A \ \lvert x_1/x_0 \rvert^k, \ \forall k$	5. Step 4 and properties of absolute value, Step 3
6. $\lvert x_1/x_0 \rvert < 1$	6. $\lvert x_1 \rvert < \lvert x_0 \rvert$ by hypothesis
7. $\sum_{k=0}^{\infty} A \ \lvert x_1/x_0 \rvert^k$ is convergent	7. Geometric series with $\lvert r \rvert < 1$ is convergent
8. $\sum_{k=0}^{\infty} a_k(x_1)^k$ is absolutely convergent for $x_1 \ni \lvert x_1 \rvert < \lvert x_0 \rvert$	8. Steps 5, 7, Theorem 5.2 and Definition 4.5

Suppose the series $\Sigma\, a_k x^k$ is convergent for at least one nonzero value of x, say x_0, but is also divergent for some value of x. Consider the set of numbers x for which $\Sigma\, a_k x^k$ is convergent. Denote this set of numbers by A. The set A is not empty for $x_0 \in A$. The set A is bounded, because of the assumption that the series is not convergent for all values of x. To say that A is not bounded means that for every real number x there exists an x_1 in A such that $|x| < x_1$. By Theorem 7.1, this would imply that $\Sigma\, a_k x^k$ is convergent for all x which contradicts the assumption that the series is not convergent for all values of x.

Since A is not empty and has an upper bound, it has a least upper bound r, i.e. r is the least upper bound of the numbers for which $\Sigma\, a_k x^k$ is convergent. All x_1 such that $|x_1| < |x_0|$ belong to A by Theorem 7.1. Hence A contains some positive numbers and r must be positive. The following discussion shows that the series $\Sigma a_k x^k$ is absolutely convergent if $|x| < r$ and is divergent if $|x| > r$. Choose y so that y is in set A and $|x| < y \leq r$. This is possible, since r is the least upper bound of A. Then the series $\Sigma a_k x^k$ converges absolutely by Theorem 7.1. Suppose $|x| > r$. Then the series cannot converge for if it did, Theorem 7.1 implies that $\Sigma a_k y^k$ converges if $r < y < |x|$, so that y would be in A, contrary to the fact that r is the least upper bound of A.

The preceding discussion shows that there are three possibilities for a power series $\Sigma a_k x^k$.

(1) It is absolutely convergent for all values of x. (Everywhere convergent)

(2) It diverges for every x such that $x \neq 0$. (Nowhere convergent)

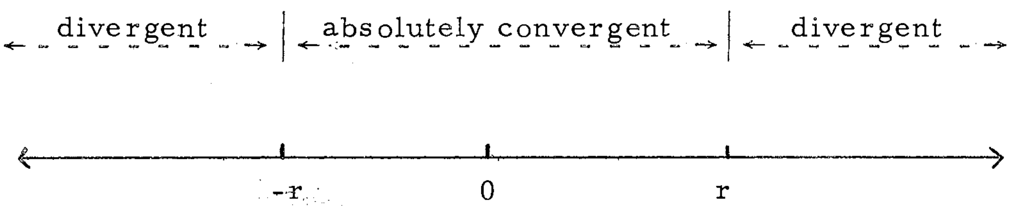
(3) There is a positive number r such that the series converges absolutely if $|x| < r$ and diverges if $|x| > r$.

Definition 7.3: (Radius of convergence, Interval of convergence) Let $\Sigma a_k x^k$ be a power series. Let r be the least upper bound of the positive numbers x for which $\Sigma a_k x^k$ is convergent. Then r is called the radius of convergence. The interval $-r < x < r$ is called the interval of convergence.

Consider the three possibilities for a power series $\Sigma a_k x^k$ as described preceding Definition 7.3. In case (1), the radius of convergence is infinite, and the interval of convergence is the entire x-axis. It is convenient symbolism to write $r = \infty$ in this case. In case (2), $r = 0$, and there is no interval of convergence. In case (3) the series may or may not converge at the end points $x = r$ and $x = -r$. It may converge at both, at just one, or at neither.

Therefore, the region of convergence is either a single point; the entire real line; or a finite interval which may be closed, open, half open-half closed, or half closed-half open.

If the region of convergence S is a finite interval, and the series is of the form $\Sigma a_k x^k$, then the midpoint of S is 0. See Figure 7.1.



$-r < x < r$

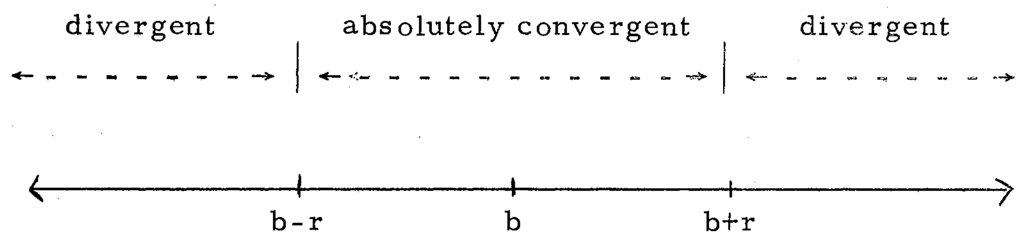
Figure 7.1

The midpoint of S is b if the series is of the form $\Sigma a_k(x-b)^k$. See Figure 7.2.

It is natural now to look for a method of calculating the magnitude of the radius of convergence. Theorem 7.2 establishes the formula

$$r = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} .$$

Since for any sequence the $\overline{\lim}$ is either a real number or infinity, this formula always gives in theory a value for r. Examples showing how



$$-r < x - b < r$$

or

$$b - r < x < b + r$$

Figure 7.2

to calculate r in particular cases follow the theorem.

<u>Theorem 7.2:</u> Consider the power series $\Sigma\, a_k x^k$. Let u denote the limit superior of the (positive) sequence of numbers

$$|a_1|,\ \sqrt{|a_2|},\ \sqrt[3]{|a_3|},\ \ldots,\ \sqrt[n]{|a_n|},\ \ldots ;$$

i.e.

$$u = \overline{\lim}\ \sqrt[n]{|a_n|}.$$

(a) If $u = 0$, the power series is everywhere convergent and $r = \infty$.

(b) If $u = \infty$, the power series is nowhere convergent and $r = 0$.

(c) If $0 < u < \infty$, the power series converges absolutely for every x such that $|x| < 1/u$ and diverges for every x such that $|x| > 1/u$. Thus

$$r = \frac{1}{u} = \frac{1}{\overline{\lim}\ \sqrt[n]{|a_n|}}$$

is the radius of convergence of the given power series.

Proof: (a)

1. $u = \overline{\lim} \sqrt[n]{\lvert a_n \rvert} = 0$	1. Hypothesis
2. $\forall \epsilon > 0 \; \exists \, n_0 \ni \sqrt[n]{\lvert a_n \rvert} < \epsilon, \; \forall n > n_0$	2. Step 1, Theorem 3.15 and meaning of all but a finite number of terms
3. Let x_0 be an arbitrary real number not equal to 0, then $1/(2\lvert x_0 \rvert) > 0$	3. Assumption and a positive number is greater than 0
4. $\exists \, n_1 \ni \sqrt[n]{\lvert a_n \rvert} < 1/(2\lvert x_0 \rvert), \; \forall n > n_1$	4. Step 2 is true for all $\epsilon > 0$, hence true for $\epsilon = 1/(2\lvert x_0 \rvert)$
5. $\lvert a_n \rvert < \dfrac{1}{2^n \lvert x_0 \rvert^n}, \; \forall n > n_1$	5. Step 4 and raise each term to the nth power
6. $\lvert a_n \rvert \, \lvert x_0^n \rvert < 1/2^n, \; \forall n > n_1$ or $\lvert a_n x_0^n \rvert < 1/2^n, \; \forall n > n_1$	6. Multiply each term in Step 5 by $\lvert x_0^n \rvert$, absolute value theorem
7. $\Sigma \; 1/2^n$ is convergent	7. Geometric series with $\lvert r \rvert < 1$
8. Therefore $\Sigma \; a_k(x_0)^k$ is absolutely convergent, hence convergent	8. Steps 6, 7 and Theorem 5.2, Theorem 4.9

Proof: (b)

1. $u = \overline{\lim} \sqrt[n]{\lvert a_n \rvert} = \infty$ implies that $< \sqrt[n]{\lvert a_n \rvert} >$ is unbounded	1. Hypothesis and meaning of $u = \infty$
2. Suppose $\Sigma a_k x^k$ converges for $x = x_1 \neq 0$	2. Assumption
3. $< a_n x_1^n >$ is a null sequence, hence convergent to 0 and bounded	3. Step 2 and Theorem 4.1, Theorem 2.4
4. $< \sqrt[n]{\lvert a_n x_1^n \rvert} >$ is a null sequence, hence convergent to 0 and bounded	4. Step 3, Theorem 2.7, Theorem 2.4
5. $\exists K_1 \ni \sqrt[n]{\lvert a_n x_1^n \rvert} < K_1, \forall n$	5. Step 4 and Definition 2.4'
6. $\sqrt[n]{\lvert a_n \rvert} < K_1 / \lvert x_1 \rvert = K, \forall n$	6. Calculations on Step 5
7. $< \sqrt[n]{\lvert a_n \rvert} >$ is bounded	7. Step 6 and Definition 2.4'
8. Hence the assumption in Step 2 leads to a contradiction of the hypothesis in Step 1	8. Steps 3-7
9. The series cannot converge for any $x \neq 0$ and is said to be nowhere convergent	9. Steps 2-8

Proof: (c)

1. Let x_1 be an arbitrary number not equal to 0 for which $\|x_1\| < 1/u$	1. Assumption, $0 < u < \infty$ implies that $0 < 1/u < \infty$, real numbers are dense
2. Choose a positive number $q \ni \|x_1\| < q < 1/u$	2. Real numbers are dense
3. $1/q > u$	3. Since $1/u > q$ in Step 2
4. $\exists\, n_0 \ni \sqrt[n]{\|a_n\|} < 1/q, \ \forall n > n_0$	4. Step 3, Theorem 3.15
5. $\sqrt[n]{\|a_n x_1^n\|} < \|x_1\|/q < 1, \ \forall n > n_0$	5. Multiply by $\|x_1\|$, $\|x_1\| < q$ in Step 2
6. $\|a_n x_1^n\| < (\|x_1\|/q)^n < 1, \ \forall n > n_0$	6. Raise each term to the nth power
7. $\sum_{n=0}^{\infty} (\|x_1\|/q)^n$ converges	7. Geometric series with $\|r\| < 1$
8. $\Sigma \|a_k x_1^k\|$ converges	8. Steps 6, 7, Theorem 5.2
9. $\Sigma a_k x_1^k$ is absolutely convergent	9. Step 8, Definition 4.5
10. On the other hand, let $\|x_2\| > 1/u$ or $\|1/x_2\| > u$	10. Assumption

11. $\sqrt[n]{\lvert a_n \rvert} > 1/\lvert x_2 \rvert$ for infinitely many indices n	11. Step 10, Theorem 3.15
12. $\sqrt[n]{\lvert a_n x_2^n \rvert} > 1$ for infinitely many indices n	12. Multiply each term in Step 11 by $\lvert x_2 \rvert$
13. $\lvert a_n x_2^n \rvert > 1$ for infinitely many indices n	13. Raise each term to nth power in Step 12
14. It is impossible for $a_n x_2^n \to 0$	14. Step 13
15. $\Sigma\, a_k x_2^k$ diverges	15. Step 14, Theorem 4.2

Example 7.1: Use Theorem 7.2 to find the radius of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^k x^k}{(2k+1)^k} .$$

Since

$$a_n = \frac{(-1)^n n^n}{(2n+1)^n}$$

and

$$\sqrt[n]{\lvert a_n \rvert} = \sqrt[n]{\left| \frac{(-1)^n n^n}{(2n+1)^n} \right|} = \left| \frac{(-1)n}{2n+1} \right| = \frac{n}{2n+1} ,$$

then

$$\lim \sqrt[n]{\lvert a_n \rvert} = \lim \frac{n}{2n+1} = \lim \frac{1}{2+1/n} = \frac{1}{2} .$$

Since the limit exists in this example and $\underline{\lim} = \lim = \overline{\lim}$ when the limit exists, then

$$\overline{\lim} \sqrt[n]{|a_n|} = 1/2.$$

Therefore

$$r = \frac{1}{1/2} = 2.$$

Example 7.2: Use Theorem 7.2 to find the radius of convergence for the power series

$$\sum_{k=0}^{\infty} [4 + (-1)^k]^{-k} x^k.$$

Since $a_n = [4 + (-1)^n]^{-n}$, then

$$\sqrt[n]{|a_n|} = \sqrt[n]{|[4 + (-1)^n]^{-n}|} = |4 + (-1)^n|^{-1} = \frac{1}{4 + (-1)^n}.$$

If n is even,

$$\frac{1}{4 + (-1)^n} = \frac{1}{4 + 1} = \frac{1}{5}.$$

If n is odd,

$$\frac{1}{4 + (-1)^n} = \frac{1}{4 - 1} = \frac{1}{3}.$$

Therefore, the

$$\overline{\lim} \sqrt[n]{|a_n|} = 1/3.$$

Hence

$$r = \frac{1}{1/3} = 3.$$

In many cases the radius of convergence of a power series is found by using d'Alembert's ratio test. It is sometimes easier to calculate with quotients than with nth roots. The following statement is simply an extension of the ratio test, Theorem 5.6.

If the $\lim |a_{n+1}/a_n| = L$ exists, and $u_n = a_n x^n$, then

$$\lim |u_{n+1}/u_n| = \lim |a_{n+1}/a_n| \; |x| = L \; |x|.$$

The series converges if $L\,|x| < 1$, and diverges if $L\,|x| > 1$. Thus $r = 1/L$ if $L \neq 0$, $r = \infty$ if $L = 0$, and $r = 0$ if $L = \infty$.

The following examples illustrate the technique for finding the radius of convergence by the ratio test.

<u>Example 7.3:</u> $1 + x + x^2/2! + \ldots + x^n/n! + \ldots$

Consider

$$\lim \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim \frac{1}{n+1} \; |x| = 0$$

for every number x. This series converges for every number x. Its interval of convergence is $(-\infty, \infty)$, and its radius of convergence is $r = \infty$. The same result is obtained by calculating L directly:

$$L = \lim \left| \frac{1/(n+1)!}{1/n!} \right| = \lim \frac{1}{n+1} = 0.$$

Hence $r = \infty$.

<u>Example 7.4:</u>

$$1 - x + \frac{x^2}{2} - \ldots + \frac{(-1)^n x^n}{n} + \ldots$$

$$\lim \left| \frac{(-1)^{n+1} x^{n+1}/(n+1)}{(-1)^n x^n/n} \right| = \lim \frac{n}{n+1} \; |x| = |x|.$$

Therefore the series converges if $|x| < 1$ and diverges if $|x| > 1$ and $r = 1$. Again if L is calculated directly,

$$L = \lim \left| \frac{(-1)^{n+1}/(n+1)}{(-1)^n/n} \right| = \lim \frac{n}{n+1} = 1.$$

Hence $r = 1/L = 1$. If $x = 1$, the series is $1 - 1 + 1/2 - \ldots + (-1)^n/n + \ldots$ which is a convergent series. If $x = -1$, the series $1 + 1 + 1/2 + \ldots + 1/n + \ldots$, since

$$\frac{(-1)^n(-1)^n}{n} = \frac{1}{n}.$$

The series is divergent. Hence $(-1, 1]$ is its region of convergence.

Example 7. 5:

$$\frac{1}{3} + \frac{(x-2)}{3} + \frac{(x-2)^2}{36} + \ldots + \frac{(x-2)^n}{3^n n^2} + \ldots$$

$$\lim \left| \frac{(x-2)^{n+1}}{3^{n+1}(n+1)^2} \cdot \frac{3^n n^2}{(x-2)^n} \right| = \lim \frac{n^2}{3(n+1)^2} |x-2| = \frac{|x-2|}{3}.$$

Hence the series converges if $|x-2|/3 < 1$ or $|x-2| < 3$. If $|x-2| = 3$, the series is the p series or alternating p series for $p = 2$. Since these series also converge, the region of convergence is $\{x \mid |x-2| \leq 3\}$, i.e. the closed interval $[-1, 5]$ and $r = 3$. Direct calculation of L gives

$$L = \lim \left| \frac{1/3^{n+1}(n+1)^2}{1/3^n n^2} \right| = \frac{1}{3}$$

and $r = 1/(1/3) = 3$. Note that the center of the region of convergence is 2 in this example.

In the following example L cannot be calculated since $\lim |a_{n+1}/a_n|$ does not exist. The ratio test does not determine the radius of convergence, but r can be calculated by Theorem 7. 2.

Example 7.6:

$$\sum_{n=1}^{\infty} (2^{(-1)^n - n}) x^n = \frac{1}{2^2} x + \frac{1}{2^1} x^2 + \frac{1}{2^4} x^3 + \frac{1}{2^3} x^4 + \ldots$$

If n is odd,

$$\frac{2^{(-1)^{n+1} - (n+1)}}{2^{(-1)^n - n}} = \frac{2^{1-n-1}}{2^{-1-n}} = \frac{2^{-n}}{2^{-1-n}} = \frac{1}{2^{-1}} = 2.$$

If n is even,

$$\frac{2^{(-1)^{n+1} - (n+1)}}{2^{(-1)^n - n}} = \frac{2^{-1-n-1}}{2^{1-n}} = \frac{2^{-n-2}}{2^{1-n}} = \frac{1}{2^3} = \frac{1}{8}.$$

Thus $\overline{\lim}\ |a_{n+1}/a_n| = 2$ and $\underline{\lim}\ |a_{n+1}/a_n| = 1/8$, and hence $\lim |a_{n+1}/a_n|$ does not exist. For the ratio

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right|$$

in a similar way it follows that

$$\overline{\lim} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = 2\ |x|$$

and

$$\underline{\lim} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \frac{|x|}{8} .$$

Theorem 5.5 gives the result that the series converges if $2|x| < 1$ and diverges if $|x|/8 > 1$. That is, the series is known to converge if $|x| < 1/2$ and diverge if $|x| > 8$. Thus some information is given but not enough to determine r.

Now

$$\sqrt[n]{|a_n|} = \left|2^{\frac{(-1)^n - n}{n}}\right| = \left|2^{\frac{(-1)^n}{n} - 1}\right|$$

and

$$\lim \left(\frac{(-1)^n}{n} - 1\right) = -1.$$

Therefore

$$\sqrt[n]{|a_n|} = 1/2.$$

Hence $r = 1/(1/2) = 2$, and the series converges for all x such that $|x| < 2$ and diverges for all x such that $|x| > 2$. The reader can check the values $x = 2$ and $x = -2$ to see what happens at the endpoints.

In Chapter IV, Theorem 4.5 states that convergent series may be added term by term. The next theorem is a natural extension of this theorem to power series.

Theorem 7.3: If $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $|x| < r_1$, and

$$g(x) = \sum_{k=0}^{\infty} b_k x^k,$$

$|x| < r_2$, then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k,$$

$|x| < r$ where $r = \min(r_1, r_2)$.

Proof:

1. Let x_1 be a real number such that $|x_1| < r$ — 1. Assumption

2. $f(x_1) = \sum_{k=0}^{\infty} a_k(x_1)^k$

2. $\sum_{k=0}^{\infty} a_k(x_1)^k$ converges since $|x_1| < r \leq r_1$ and $f(x_1)$ is the value of the series

3. $g(x_1) = \sum_{k=0}^{\infty} b_k(x_1)^k$

3. $\sum_{k=0}^{\infty} b_k(x_1)^k$ converges since $|x_1| < r \leq r_2$ and $g(x_1)$ is the value of the series

4. $f(x_1)+g(x_1) = \sum_{k=0}^{\infty} [a_k(x_1)^k + b_k(x_1)^k]$

$$= \sum_{k=0}^{\infty} (a_k+b_k)(x_1)^k$$

4. Steps 2, 3, Theorem 4.5, distributive property

5. Hence

$$f(x)+g(x) = \sum_{k=0}^{\infty} (a_k+b_k)x^k, \quad |x| < r$$

5. Steps 1-4

<u>Corollary 7.3</u>: If $f(x) = \sum_{k=0}^{\infty} a_k x^k$,

$|x| < r_1$ and

$$g(x) = \sum_{k=0}^{\infty} b_k x^k,$$

$|x| < r_2$, then

$$f(x) - g(x) = \sum_{k=0}^{\infty} (a_k - b_k)x^k,$$

$|x| < r$ where $r = \min(r_1, r_2)$.

Proof: Left for the reader. Hint: $a_k - b_k = a_k + (-b_k)$.

A suitable definition for a series which is the product of convergent series is not as obvious as the corresponding definition for a sum or a difference. Because power series can be used to suggest one meaningful way to define a product series, a discussion of multiplication has been postponed until this chapter. Consider the multiplication of finite sums as follows: Let

$$C_n = \sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} b_k .$$

If $n = 0$, then $C_0 = a_0 b_0$. If $n = 1$, then

$$C_1 = (a_0 + a_1)(b_0 + b_1) = a_0 b_0 + a_0 b_1 + a_1 b_0 + a_1 b_1 .$$

If $n = 2$, then

$$\begin{aligned} C_2 &= (a_0 + a_1 + a_2)(b_0 + b_1 + b_2) \\ &= a_0 b_0 + a_0 b_1 + a_0 b_2 + a_1 b_0 + a_1 b_1 \\ &\quad + a_1 b_2 + a_2 b_0 + a_2 b_1 + a_2 b_2 \end{aligned}$$

and in general,

$$\begin{aligned} \sum_{k=0}^{n} a_k \cdot \sum_{k=0}^{n} b_k &= (a_0 + a_1 + a_2 + \ldots + a_n)(b_0 + b_1 + b_2 + \ldots + b_n) \\ &= a_0 b_0 + a_0 b_1 + a_0 b_2 + \ldots + a_0 b_n \\ &\quad + a_1 b_0 + a_1 b_1 + a_1 b_2 + \ldots + a_1 b_n \\ &\quad + a_2 b_0 + a_2 b_1 + a_2 b_2 + \ldots + a_2 b_n \\ &\quad + \ldots \\ &\quad + a_n b_0 + a_n b_1 + a_n b_2 + \ldots + a_n b_n \end{aligned}$$

Observe that each term of the first sum is multiplied by each term of the second sum. Since there is a finite number of terms in the product, the commutative property and the associative property are applicable, and the terms can be arranged in any order without affecting the result. In other words, the product of two finite sums is unique. Can these ideas be extended to infinite series and a suitable definition obtained for multiplying two infinite series and obtaining a product series?

Let

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k$$

be two infinite series. It seems logical that the series which is the product of

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k$$

should contain in some way all the terms in the following array.

$$\begin{array}{cccccc}
a_0b_0 & a_0b_1 & a_0b_2 & \cdots & a_0b_n & \cdots \\
a_1b_0 & a_1b_1 & a_1b_2 & \cdots & a_1b_n & \cdots \\
a_2b_0 & a_2b_1 & a_2b_2 & \cdots & a_2b_n & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_nb_0 & a_nb_1 & a_nb_2 & \cdots & a_nb_n & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

I

Let

$$A_n = \sum_{k=0}^{n} a_k \quad \text{and} \quad B_n = \sum_{k=0}^{n} b_k$$

be the nth partial sums of

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k,$$

respectively. Construct the sequence $< A_nB_n >$. The terms of this sequence are as follows:

$$A_0B_0 = a_0b_0$$

$$A_1B_1 = a_0b_0 + a_0b_1 + a_1b_0 + a_1b_1$$

$$A_2B_2 = a_0b_0 + a_0b_1 + a_0b_2 + a_1b_0 + a_1b_1 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2$$

and in general,

$$A_nB_n = (a_0 + a_1 + a_2 + \ldots + a_n)(b_0 + b_1 + b_2 + \ldots + b_n).$$

Now construct the difference sequence as follows. Let

$$d_0 = A_0B_0 = a_0b_0$$

$$d_1 = A_1B_1 - A_0B_0 = a_0b_1 + a_1b_0 + a_1b_1$$

$$d_2 = A_2B_2 - A_1B_1 = a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2$$

and in general,

$$d_n = A_nB_n - A_{n-1}B_{n-1}.$$

Then the series

$$\sum_{k=0}^{\infty} d_k = A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1})$$

contains each term of the array I once and only once. The term a_0b_0 in the array I is actually a term of the series, while each of the other terms in the array I is only part of a finite sum which is a term of the series. Keep in mind that the nth partial sum of the above product series is A_nB_n by construction. If two convergent series are multiplied, the product series as defined above is convergent, and its value is the product of the values of the two original series as is proved in the next theorem.

Theorem 7.4: If

$$\sum_{k=0}^{\infty} a_k = A, \quad \sum_{k=0}^{\infty} b_k = B, \quad A_n = \sum_{k=0}^{n} a_k \quad \text{and} \quad B_n = \sum_{k=0}^{n} b_k,$$

then

$$\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k = A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1}) = AB.$$

Proof:

1. $A_n \to A$ and $B_n \to B$ — 1. Hypothesis, Definition 4.3

2. $A_nB_n \to AB$ — 2. Step 1, Theorem 3.11

3. Therefore $A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1})$ converges to AB — 3. Step 2, Definition 4.3

The general term $A_nB_n - A_{n-1}B_{n-1}$ can be expressed in a slightly different form which is useful in later work. So consider the following theorem.

Theorem 7.5: If

$$A_n = \sum_{k=0}^{n} a_k \quad \text{and} \quad B_n = \sum_{k=0}^{n} b_k,$$

then

$$A_nB_n - A_{n-1}B_{n-1} = a_nB_n + b_nA_{n-1}, \quad n \geq 1.$$

Proof:

1. $A_n = a_n + A_{n-1}$ and $B_n = b_n + B_{n-1}$, $n \geq 1$	1. Hypothesis and calculations
2. For $n \geq 1$, $A_nB_n = (a_n + A_{n-1})(b_n + B_{n-1})$ $= a_nb_n + a_nB_{n-1} + b_nA_{n-1} + A_{n-1}B_{n-1}$	2. Step 1 and multiplication
3. For $n \geq 1$, $a_nb_n + a_nB_{n-1} = a_nB_n$	3. Calculations
4. For $n \geq 1$, $A_nB_n - A_{n-1}B_{n-1} = a_nB_n + b_nA_{n-1}$	4. Substitute Step 3 in Step 2, add $(-A_{n-1}B_{n-1})$ to both sides of the equation in Step 2

Therefore the product series as constructed can be expressed in another way, i.e.

$$A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1}) = A_0B_0 + \sum_{k=1}^{\infty} (a_nB_n + b_nA_{n-1}) .$$

If two infinite series are both absolutely convergent, then the product series as constructed above is absolutely convergent and has for its value the product of the two original series. The next theorem will establish this fact.

<u>Theorem 7.6</u>: If

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k$$

are both absolutely convergent and have values A and B, respectively, then the product series

$$A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1})$$

is absolutely convergent and has value AB.

Proof:

1. $\sum_{k=0}^{\infty} a_k = A,\quad \sum_{k=0}^{\infty} b_k = B,$ $A_n = \sum_{k=0}^{n} a_k,\quad B_n = \sum_{k=0}^{n} b_k,$ $A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1}) = AB$	1. Hypothesis, Theorem 7.4
2. Let C_n and D_n be the nth partial sums for $\sum_{k=0}^{\infty} \lvert a_k \rvert$ and $\sum_{k=0}^{\infty} \lvert b_k \rvert$, i.e. $C_n = \sum_{k=0}^{n} \lvert a_k \rvert$ and $D_n = \sum_{k=0}^{n} \lvert b_k \rvert$	2. Notation
3. Construct the sequence $< C_nD_n >$ and the series $C_0D_0 + \sum_{k=1}^{\infty} (C_kD_k - C_{k-1}D_{k-1})$	3. Construction is the same as with the sequence $< A_nB_n >$
4. C_n converges and D_n converges, hence $< C_nD_n >$ is convergent and $C_0D_0 + \sum_{k=1}^{\infty} (C_kD_k - C_{k-1}D_{k-1})$ converges	4. Hypothesis, Theorem 3.11, Definition 4.3

5. $C_nD_n - C_{n-1}D_{n-1}$ $= \|a_n\| D_n + \|b_n\| C_{n-1}, \ n \geq 1$	5. Theorem 7.5
6. For $n \geq 1$, $\|A_nB_n - A_{n-1}B_{n-1}\| = \|a_nB_n + b_nA_{n-1}\|$ $\leq \|a_nB_n\| + \|b_nA_{n-1}\|$ $= \|a_n\|\,\|B_n\| + \|b_n\|\,\|A_{n-1}\|$ $\leq \|a_n\| D_n + \|b_n\| C_{n-1}$ $= C_nD_n - C_{n-1}D_{n-1}$	6. Theorem 7.5, theorem about absolute value, $\|A_{n-1}\| \leq C_{n-1}$, $\|B_n\| \leq D_n$, substitution from Step 5
7. $\sum_{k=1}^{\infty} \|A_kB_k - A_{k-1}B_{k-1}\|$ converges	7. Steps 4 and 6, $\sum_{k=1}^{\infty} (C_kD_k - C_{k-1}D_{k-1})$ converges, Theorem 5.2
8. $\|A_0B_0\| + \sum_{k=1}^{\infty} \|A_kB_k - A_{k-1}B_{k-1}\|$ converges	8. Step 7, Theorem 4.3
9. $A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1})$ is absolutely convergent and has value AB	9. Step 8, Definition 4.5, Step 1

<u>Theorem 7.7</u>: If

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k$$

are both absolutely convergent and have values A and B, respectively, and if the terms in the array I are arranged in any sequence $< u_k >$ such that every element of the array is included, the series

$$\sum_{k=0}^{\infty} u_k$$

converges absolutely and has value AB.

Proof:

1. Let A_n, B_n, C_n, and D_n be defined as in Theorem 7.6	1. Notation
2. $C_0D_0 + \sum_{k=1}^{\infty} (C_kD_k - C_{k-1}D_{k-1})$ converges	2. Theorem 7.6
3. Remove parentheses in Step 2. The resulting series is of the form $\Sigma \|a_k\|\|b_j\|$, and converges. The terms are the absolute values of the terms in array I	3. Theorem 4.12, Definition of C_kD_k
4. Since the series in Step 3 is a positive term series, it can be rearranged and the series thus formed is convergent	4. Theorem 4.11
5. Let $< u_k >$ be some arrangement of the array I	5. Assumption
6. $\Sigma \|u_k\|$ converges	6. Steps 3, 4, 5

7. Σu_k converges absolutely — 7. Step 6, Definition 4.5

8. Rearrange Σu_k and insert parentheses such that the series formed is $A_0B_0 + \sum_{k=1}^{\infty} (A_kB_k - A_{k-1}B_{k-1})$ which converges to AB. Hence Σu_k converges to AB — 8. Step 7, Theorem 7.6, Theorem 4.11, Theorem 4.8

The product series as defined in Theorem 7.6 is not the most useful arrangement of the terms in the array I. A consideration of the multiplication of power series suggests an arrangement called the Cauchy product.

Again some useful information can be obtained from analogy with the finite case. Consider the multiplication of some polynomials as follows:

$$(a_0 + a_1x)(b_0 + b_1x) = a_0b_0 + (a_1b_0 + a_0b_1)x + a_1b_1\,x^2$$

$$(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2) =$$

$$\begin{array}{cccccc} & a_0b_0 & + & a_0b_1\,x & + & a_0b_2\,x^2 \\ + & a_1b_0x & + & a_1b_1\,x^2 & + & a_1b_2\,x^3 \\ + & a_2b_0x^2 & + & a_2b_1\,x^3 & + & a_2b_2\,x^4 \end{array}$$

$$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_1b_2 + a_2b_1)x^3 + a_2b_2x^4$$

Observe that the coefficient of each term is the sum of the coefficients of the diagonal elements in the above finite array.

Let

$$\sum_{k=0}^{\infty} a_kx^k \text{ and } \sum_{k=0}^{\infty} b_kx^k$$

be two power series. Consider the following array which contains the product of each term of one series by each term of the other.

$$
\text{II} \qquad
\begin{array}{cccccc}
a_0b_0 & a_0b_1\,x & a_0b_2\,x^2 & \dots & a_0b_n\,x^n & \dots \\
a_1b_0\,x & a_1b_1\,x^2 & a_1b_2\,x^3 & \dots & a_1b_n\,x^{n+1} & \dots \\
a_2b_0\,x^2 & a_2b_1\,x^3 & a_2b_2\,x^4 & \dots & a_2b_n\,x^{n+2} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
a_nb_0\,x^n & a_nb_1\,x^{n+1} & a_nb_2\,x^{n+2} & \dots & a_nb_n\,x^{2n} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

Observe that in array II, all the terms of a given power of x, say x^n, lie on a diagonal, and the coefficients are of the form a_ib_j such that $i + j = n$. So a grouping by triangles instead of by rectangles appears to be much more useful in multiplying power series. Observe that the sums of the diagonal coefficients are as follows.

$$
\begin{aligned}
\text{If } n &= 0, \; a_0b_0 \\
n &= 1, \; a_0b_1 + a_1b_0 \\
n &= 2, \; a_0b_2 + a_1b_1 + a_2b_0 \\
n &= 3, \; a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0,
\end{aligned}
$$

and in general the coefficient of x^n is

$$a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \dots + a_{n-2}b_2 + a_{n-1}b_1 + a_nb_0.$$

This can be simplified by simply writing

$$\sum_{k=0}^{n} a_k b_{n-k} .$$

Using these finite sums as terms of a series, then the series

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right)$$

is an arrangement of the terms in array I. The following theorem concerning multiplication is quite useful in application and especially in multiplying power series.

Theorem 7.8: (Cauchy Product) If the series

$$\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k$$

are both absolutely convergent and have values A and B, respectively, then the product series defined as follows,

$$\sum_{k=0}^{\infty} a_k \cdot \sum_{k=0}^{\infty} b_k = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right),$$

is absolutely convergent and has value AB.

Proof:

1. Let $< u_k >$ be a sequence such that every element of array I is included	1. Assumption
2. $\Sigma\, u_k$ converges absolutely and has value AB	2. Step 1, Theorem 7.7

3. Rearrange Σu_k and insert parentheses such that the series formed is $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$. Then $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ converges absolutely and has value AB.	3. Step 2, Theorem 4.11, Theorem 4.8

The next theorem is a natural extension of multiplication to power series. The proof is left for the reader.

Theorem 7.9: If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad |x| < r_1$$

and

$$g(x) = \sum_{k=0}^{\infty} b_k x^k, \quad |x| < r_2,$$

then

$$f(x)g(x) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n, \quad |x| < r,$$

where $r = \min(r_1, r_2)$.

Example 7.7: Show that

$$\frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$$

by using power series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1-x} \cdot \frac{1}{1+x} = \sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} 1^k (-1)^{n-k} \right) x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \right) x^n$$

$$\sum_{k=0}^{n} (-1)^{n-k} = (-1)^n + (-1)^{n-1} + (-1)^{n-2} + \ldots + (-1)^0$$

$$= (-1)^n + (-1)^{n-1} + (-1)^{n-2} + \ldots + 1$$

If n is even, then

$$\sum_{k=0}^{n} (-1)^{n-k} = 1 - 1 + 1 - 1 + \ldots + 1 = 1.$$

If n is odd, then

$$\sum_{k=0}^{n} (-1)^{n-k} = -1 + 1 - 1 + 1 - 1 + \ldots + 1 = 0.$$

Therefore, let $n = 2k$ to represent an even integer and the conclusions are as follows:

$$\frac{1}{1-x} \cdot \frac{1}{1+x} = \sum_{k=0}^{\infty} x^{2k}$$

But

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k}.$$

Therefore

$$\frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$$

The following theorem gives the power series related to certain elementary functions with which the reader is already familar. Since the identification of power series with known functions is beyond the scope of this paper the proof of the theorem is not complete.

Theorem 7.10:

(1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all values of x

(2) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all values of x

(3) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all values of x

(4) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for x such that $-1 < x < 1$

(5) $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$ for x such that $-1 < x \leq 1$.

Proof:

(1) This series was discussed in Example 7.3.

(2) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots$

$$\lim \left| \frac{(-1)^{n+1} x^{2n+2}/(2n+2)!}{(-1)^n x^{2n}/(2n)!} \right| = \lim \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right|$$

$$= \lim \left| \frac{(-1)x^2}{(2n+2)(2n+1)} \right|$$

$$= \lim \frac{1}{(2n+2)(2n+1)} \cdot x^2 = 0.$$

The series converges for every number x by the ratio test. Its interval of convergence is $(-\infty, \infty)$.

(3) Left for the reader. Similar to (2).

(4) This series is the geometric series and was discussed in Example 4.2.

(5) A series very similar to this one was discussed in Example 7.4.

Example 7.8: Find a power series for cosh x. cosh x is one of the hyperbolic functions, and is defined as follows:

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \ \forall x$$

and

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}, \ \forall x,$$

then

$$\cosh x = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2n!} x^n, \ \forall x,$$

by Theorem 7.3. If n is odd,

$$\frac{1 + (-1)^n}{2} = \frac{1 - 1}{2} = 0.$$

Hence all the odd terms are zero. The remaining terms are those in which n is even. Let n = 2k and simplify the above expression as follows:

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \forall x$$

Example 7.9: Find a power series for sinh x. sinh x is another hyperbolic function defined as follows:

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

The reader should study Example 7.8 carefully and should be able to find a power series for sinh x.

Example 7.10: Find a power series for $\frac{e^{2x}}{1-3x}$.

$$\frac{e^{2x}}{1-3x} = e^{2x} \cdot \frac{1}{1-3x}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}, \quad \forall x$$

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n, \quad |3x| < 1 \text{ or } |x| < 1/3$$

Therefore,

$$\frac{e^{2x}}{1-3x} = \left(\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}\right)\left(\sum_{n=0}^{\infty} 3^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{2^k}{k!} \cdot 3^{n-k}\right) x^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{2^k}{3^k k!}\right) (3x)^n, \quad |x| < 1/3$$

by Theorem 7.5.

The next two examples are identities which are usually obtained by other methods. It is interesting to observe that these identities are obtained from the power series representations of the functions involved. Thus the geometric interpretation which is usually used to derive them, though convenient, is not an essential element in the truth of the properties. The reader will need to recall the binomial expansion and the notation for a combination.

Binomial Theorem: If n is a positive integer, then

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{n} y^n,$$

where

$$\binom{n}{0} = \binom{n}{n} = 1$$

and in general

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} .$$

<u>Example 7.11</u>: Show that $\sin^2 x + \cos^2 x = 1$ by using power series.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$-\frac{x^2}{2!} + \frac{x^4}{2!\,2!} - \frac{x^6}{2!\,4!} + \frac{x^8}{2!\,6!} - \frac{x^{10}}{2!\,8!} + \ldots$$

$$\frac{x^4}{4!} - \frac{x^6}{4!\,2!} + \frac{x^8}{4!\,4!} - \frac{x^{10}}{4!\,6!} + \frac{x^{12}}{4!\,8!} - \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!} - \cdots$$

$$- \frac{x^4}{3!} + \frac{x^6}{3!\,3!} - \frac{x^8}{3!\,5!} + \frac{x^{10}}{3!\,7!} - \frac{x^{12}}{3!\,9!} + \cdots$$

$$\frac{x^6}{5!} - \frac{x^8}{5!\,3!} + \frac{x^{10}}{5!\,5!} - \frac{x^{12}}{5!\,7!} + \frac{x^{14}}{5!\,9!} - \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

Form the Cauchy products as follows, and since only even powers of x occur, let n be an even integer.

$$\cos^2 x = 1 - \left(\frac{x^2}{2!} + \frac{x^2}{2!}\right) + \left(\frac{x^4}{4!} + \frac{x^4}{2!\,2!} + \frac{x^4}{4!}\right) + \cdots$$

$$+ (-1)^{\frac{n}{2}}\left(\frac{1}{n!} + \frac{1}{2!(n-2)!} + \frac{1}{4!(n-4)!} + \cdots + \frac{1}{(n-4)!4!} + \frac{1}{(n-2)!2!} + \frac{1}{n!}\right)x^n$$

$$+ \cdots$$

and

$$\sin^2 x = x^2 - \left(\frac{x^4}{3!} + \frac{x^4}{3!}\right) + \left(\frac{x^6}{5!} + \frac{x^6}{3!\,3!} + \frac{x^6}{5!}\right) + \cdots$$

$$+ (-1)^{\frac{n+2}{2}}\left(\frac{1}{(n-1)!} + \frac{1}{3!(n-3)!} + \frac{1}{5!(n-5)!} + \cdots + \frac{1}{(n-3)!3!} + \frac{1}{(n-1)!}\right) x^n$$

$$+ \cdots$$

$$\sin^2 x + \cos^2 x = 1 - \left(\frac{x^2}{2!} - \frac{2x^2}{2!} + \frac{x^2}{2!}\right) + \left(\frac{x^4}{4!} + \frac{4x^4}{4!} + \frac{6x^4}{4!} + \frac{4x^4}{4!} + \frac{x^4}{4!}\right) + \dots$$

$$+ (-1)^{\frac{n}{2}}\left(\frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{2!(n-2)!} + \dots + \frac{1}{(n-2)!2!} + \frac{1}{(n-1)!} + \frac{1}{n!}\right) x^n$$

$$+ \dots$$

Consider the coefficient of x^n for $n \geq 2$.

$$(-1)^{\frac{n}{2}}\left(\frac{\binom{n}{0}}{n!} - \frac{\binom{n}{1}}{n!} + \frac{\binom{n}{2}}{n!} - \dots + \frac{\binom{n}{2}}{n!} - \frac{\binom{n}{1}}{n!} + \frac{\binom{n}{0}}{n!}\right)$$

$$= \frac{(-1)^{\frac{n}{2}}}{n!}\left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + \binom{n}{2} - \binom{n}{1} + \binom{n}{0}\right)$$

$$= \frac{(-1)^{\frac{n}{2}}}{n!} (1-1)^n$$

$$= 0 .$$

Therefore all terms of $\sin^2 x + \cos^2 x$ are 0 except the first one. Hence $\sin^2 x + \cos^2 x = 1$.

<u>Example 7.12</u>: Show that $\sin(x + y) = \sin x \cos y + \cos x \sin y$ by using power series. Let n be an odd integer and form the Cauchy products as follows:

$$\sin x \cos y = x - \left(\frac{x^3}{3!} + \frac{xy^2}{2!}\right) + \left(\frac{x^5}{5!} + \frac{x^3 y^2}{3!\,2!} + \frac{xy^4}{4!}\right) - \dots$$

$$+ (-1)^{\frac{n+3}{2}}\left(\frac{x^n}{n!} + \frac{x^{n-2}y^2}{(n-2)!2!} + \dots + \frac{x^3 y^{n-3}}{3!(n-3)!} + \frac{xy^{n-1}}{(n-1)!}\right) + \dots$$

and

$$\cos x \sin y = y - \left(\frac{x^2 y}{2!} + \frac{y^3}{3!}\right) + \left(\frac{x^4 y}{4!} + \frac{x^2 y^3}{2!\,3!} + \frac{y^5}{5!}\right) - \dots$$

$$+ (-1)^{\frac{n+3}{2}}\left(\frac{x^{n-1} y}{(n-1)!} + \frac{x^{n-3} y^3}{(n-3)!3!} + \dots + \frac{x^2 y^{n-2}}{2!(n-2)!} + \frac{y^n}{n!}\right) + \dots$$

Adding term by term gives the following result:

$\sin x \cos y + \cos x \sin y$

$$= (x + y) - \left(\frac{x^3}{3!} + \frac{x^2 y}{2!} + \frac{xy^2}{2!} + \frac{y^3}{3!}\right)$$

$$+ \left(\frac{x^5}{5!} + \frac{x^4 y}{4!} + \frac{x^3 y^2}{3!\,2!} + \frac{x^2 y^3}{2!\,3!} + \frac{xy^4}{4!} + \frac{y^5}{5!}\right) + \dots$$

$$+ (-1)^{\frac{n+3}{2}}\left(\frac{\binom{n}{0} x^n}{n!} + \frac{\binom{n}{1} x^{n-1} y}{n!} + \frac{\binom{n}{2} x^{n-2} y^2}{n!} + \dots + \frac{\binom{n}{1} xy^{n-1}}{n!} + \frac{\binom{n}{0} y^n}{n!}\right)$$

$$+ \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (x+y)^{2k+1}}{(2k+1)!}$$

$$= \sin (x + y).$$

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APPENDIX

SYMBOLS USED

Symbol	Meaning	Page
$< a_i >_{i=1}^{n}$	Ordered n-tuple or finite sequence	6
$a_1, a_2, a_3, \ldots, a_n$	Ordered n-tuple of finite sequence	6
$< a_n >_{n=1}^{\infty}$	Infinite sequence	7
$< a_n >$	Infinite sequence	7
$a_1, a_2, a_3, \ldots, a_n, \ldots$	Infinite sequence	7
$\{a_n\}$	Range of the sequence $< a_n >$	7
iff	If and only if	10
$\forall$	For every	10
$\exists$	There exists	12
$\ni$	Such that	12
$a_n \downarrow$	The sequence $< a_n >$ is non-increasing	21
$a_n \uparrow$	The sequence $< a_n >$ is non-decreasing	21
$a_n \rightarrow A$	In the sequence $< a_n >$, a_n converges to A	37
$\lim a_n = A$	$< a_n >$ is convergent, and the limit of the sequence $< a_n >$ is A	37

$a_n \uparrow A$	$< a_n >$ is monotone non-decreasing and converges to A	37
$a_n \downarrow A$	$< a_n >$ is monotone non-increasing and converges to A	37
$+ \infty$	Unbounded monotone non-decreasing sequence diverges to $+ \infty$	41
$- \infty$	Unbounded monotone non-increasing sequence diverges to $- \infty$	41
$\overline{\lim}\ a_n$	The limit superior of a sequence $< a_n >$	58
$\limsup a_n$	The limit superior of a sequence $< a_n >$	58
$\underline{\lim}\ a_n$	The limit inferior of a sequence $< a_n >$	59
$\liminf a_n$	The limit inferior of a sequence $< a_n >$	59
$\sum_{k=1}^{n} x_k$	Summation from $k = 1$ to $k = n$	65
$\sum_{k=1}^{\infty} x_k$	Represents the sequence $< s_n >$ in addition to the $\lim s_n$	66

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