CERTAIN UPPER SEMI-CONTINUOUS DECOMPOSITION SPACES OF E_2 AND E_3

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PREFACE

Research in the area of upper semi-continuous decomposition spaces of E_2 and E_3 has been conducted by many of the well-known mathematicians of today. Among those who have made significant contributions are R. L. Moore, R. H. Bing, G. T. Whyburn, L. F. McAuley, and J. H. Roberts. Their results have been published in scientific journals spanning a period of about 45 years.

The purpose of this study is to present in one paper the results which have been obtained relative to those upper semi-continuous decomposition spaces in E_2 and E_3 which are topologically equivalent to E_2 and E_3 respectively. An effort has been made to unify and modernize the notation, definitions and terminology used in the various papers.

In many places in the text of this paper it was difficult to find a notation which would properly distinguish between points in the original space and elements in the decomposition. In most cases lower case letters were used to denote both, however, care was taken to always refer to these by using the words "point" and "element" in conjunction with the symbol. Although the elements in a decomposition are treated as points in the decomposition space, they are always referred to by the word "element". Notation such as [20, p. 3] refers the reader to page 3 in reference number 20 in the bibliography.

I am deeply indebted to Dr. John Jobe for his guidance and assistance during the preparation of this thesis. I also wish to express my appreciation to Dr. E. K. McLachlan for serving as the chairman of my advisory committee, and to Dr. W. Ware Marsden for serving as a member of my advisory committee. In addition, I am grateful for the encouragement given me by many friends and relatives.

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CHAPTER I

INTRODUCTION AND BASIC CONCEPTS

One of the more useful theorems relative to E_2 is one of R. L. Moore's which states that if G is an upper semi-continuous decomposition of E_2 such that the elements of G are bounded continua which do not separate E_2, then the decomposition space is topologically equivalent to E_2. The purpose of this paper is to exhibit the work of Moore pertaining to this and to discuss what has been done in extending this theorem to E_3.

There are examples to indicate that the theorem does not generalize to E_3 unless additional restrictions are placed on the elements making up the decomposition. For certain restrictions it has been proved that the resulting decomposition space is equivalent to E_3. For others, the question has not yet been answered.

Work on this topic began in the 1920's and Moore presented his conclusions relative to E_2 in 1924. Much of the advancement with respect to E_3 has taken place in the late 1950's and early 1960's. Some of the people associated with this work are R. H. Bing, L. F. McAuley, E. Dyer, M.-E. Hamstrom, M. K. Fort, and G. T. Whyburn. This is certainly only a partial listing for many people have published papers related either directly or indirectly to this topic.

Basic Concepts and Assumptions

In this paper there are certain basic concepts which will be assumed to be true for both E_2 and E_3. Some of these will be given here and others will be introduced in later sections when they are needed.

First of all, a list of the axioms which are assumed for E_2 are:

Axiom 1: There exists a sequence M_1, M_2, ..., such that (a) for every n, M_n is a collection covering E_2 such that each element of M_n is a region, (b) for every n, M_{n+1} is a subcollection of M_n and (c) if R is a region, x and y are points of R, then there exists a natural number m such that if A is any region belonging to M_m and containing x then $\overline{A} \subset R$ and, unless $x=y$, $\overline{A}$ does not contain y.

Axiom 2: Every region is a connected set of points.

Axiom 3: If R is a region, $E_2 - \overline{R}$ is a connected set of points.

Axiom 4: If R is a region, $\overline{R}$ satisfies the Borel-Lebesgue property.

Axiom 5: There exists an infinite set of points with no limit point.

Axiom 6: If R is a region and ab is an arc such that ab - $\{a\}$ is a subset of R then $(R \cup \{a\})$ - ab is connected.

Axiom 7: Every boundary point of a region is a limit point of the exterior of that region.

Axiom 8: Every simple closed curve is the boundary of at least one region.

It will be shown that an upper semi-continuous decomposition of E_2 whose elements are bounded continua in E_2 which do not separate E_2 will yield a decomposition space which also satisfied these axioms. Moore [12] has proved that any space satisfying these axioms is topologically equivalent to E_2.

All of the work in this paper is in a metric space and two concepts which will be useful here are those of lower distance and upper distance between sets.

Definition: Let x and g be two sets and let P denote any point of x. Let d (P,Q) denote the distance between two points, P and Q. Let $\ell(P,g) = \text{glb}\{d(P,Q) | Q \in g\}$. Then the lower distance from the set x to the set g is denoted by $\ell(x,g)$ where $\ell(x,g) = \text{glb}\{\ell(P,g) | P \in x\}$. The upper distance, $u(x,g)$, is defined to be equal to $\text{lub}\{\ell(P,g) | P \in x\}$.

There are several definitions of an upper semi-continuous collection. Two of them are stated here.

Definition A: [16, p. 416] A collection G is said to be upper semi-continuous if for each element $g \in G$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in G$ and $\ell(x,g) < \delta$ then $u(x,g) < \epsilon$.

Definition B: [22, p. 122] A collection of sets G is said to be upper semi-continuous provided that if $g \in G$ and U is any neighborhood of g then there exists a neighborhood V of g such that if $h \in G$ and $h \cap V \neq \emptyset$ then $h \subset U$.

The second of these, Definition B, is the more common of the two, however, Definition A seems to be the most convenient for this paper. Before adopting Definition A it would be advisable to show that the two definitions are actually equivalent. This is done in the following theorem.

Theorem 1: Let G be a collection of sets. Then for each element $g \in G$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in G$ and $\ell(x,g) < \delta$ then $u(x,g) < \epsilon$ if and only if when U is any neighborhood of g there exists a neighborhood V of g such that if $h \in G$ and $h \cap V \neq \emptyset$ then $h \subset U$.

Proof: Let G be a collection such that for every $g \in G$ and every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in G$ and $\ell(x,g) < \delta$ then $u(x,g) < \epsilon$. Then for every $\epsilon > 0$ let U_ϵ be a neighborhood of g such that every point of U_ϵ is at a distance less than ϵ from g. If U is any neighborhood of g there exists an $\epsilon > 0$ such that $U_\epsilon \subset U$. Then there exists a $\delta > 0$ such that if $x \in G$ and $\ell(x,g) < \delta$ then $u(x,g) < \epsilon$. Let V_δ be a neighborhood of g for which every point is at a distance less than δ from g. If $x \in G$ and $x \cap V_\delta \neq \emptyset$ then $\ell(x,g) < \delta$ and therefore $u(x,g) < \epsilon$. But if $u(x,g) < \epsilon$ then $x \subset U_\epsilon$ and therefore $x \subset U$.

Conversely, let G be a collection of sets, $g \in G$, such that when U is any neighborhood of g there exists a neighborhood V of g such that if $h \in G$ and $h \cap V \neq \emptyset$ then $h \subset U$. Then for every $\epsilon > 0$ let U_ϵ be a neighborhood of G such that for every point p of U the distance from p to some point of g is less than ϵ. Then there exists a neighborhood V of g such that if $h \in G$ and $h \cap V \neq \emptyset$ then $h \subset U_\epsilon$. Choose δ such that $0 < \delta < \epsilon$. Then V_δ is a neighborhood of g such that every point of V_δ is at a distance less than δ from some point of g. But $V_\delta \cap V$ is a neighborhood of g and $(V_\delta \cap V) \subset U$. If $h \in G$ and $h \cap (V_\delta \cap V) \neq \emptyset$ then $\ell(h,g) < \delta$, but also $h \subset U_\epsilon$ and therefore $u(h,g) < \epsilon$.

Thus Definitions A and B are equivalent and may be used interchangeably.

Procedure

A survey of the published results concerning upper semi-continuous decompositions of E_2 and E_3 was made. The principal sources were research articles published in mathematical and scientific journals. The material was analyzed and is presented here in expository form.

In Chapter II of this paper the results of Moore pertaining to E_2 will be exhibited in detail. Using a general upper semi-continuous decomposition of E_2, a space will be formed and it will be shown that this space satisfies the eight axioms which were stated previously. Chapter III will consist of examples of upper semi-continuous decompositions of E_2. Particular emphasis will be given to an example by J. H. Roberts of a decomposition of E_2 into nondegenerate continua, no one of which separates the plane.

Many people have made attempts toward extending the theorem of Moore's to E_3. In Chapter IV the results of their efforts will be discussed along with some examples and counterexamples related to this work. In addition, some problems which are as yet unsolved will be mentioned. Perhaps someone reading this paper will be able to find the solution to some of these.

CHAPTER II

R. L. MOORE'S THEOREM

The study of decomposition spaces was begun in the 1920's. The first important result was the following theorem published by R. L. Moore [16] in 1925.

Theorem: If G is an upper semi-continuous decomposition of E_2 into continua which do not separate the plane, then the decomposition space of elements of G is topologically equivalent to E_2.

In his proof of this theorem, Moore [12] used a previous result which showed that a space satisfying Axioms 1-8 was topologically equivalent to E_2. If G is any upper semi-continuous collection satisfying the hypothesis of this theorem, then following Moore's method of proof, it will be shown that if each continuum of G is considered as a point, and if a suitable definition of region is chosen, then all the axioms previously stated for E_2 will hold when the space is the collection G. The space of elements of G will be topologically equivalent to the space of points in E_2. A detailed development of Moore's work will be given in this chapter.

Suppose that some definite upper semi-continuous collection G of bounded continua of E_2 has been selected in such a way that no element of G separates E_2 and such that every point of E_2 belongs to some element of G. The letter G will be used throughout this chapter to refer to this particular upper semi-continuous collection.

The following definitions will be used in connection with the collection G.

Definition 1: If K is some subcollection of the collection G and if p is an element of G, then p is said to be a limit element of the set K provided that for every real number $\varepsilon > 0$ there exists some element g of K, $g \neq p$, such that $u(g,p) < \varepsilon$.

Definition 2: If $K \subset G$ then $K = K_1 \cup K_2$ provided every element of K belongs to either K_1 or K_2 and every element of either K_1 or K_2 also belongs to K.

Definition 3: If $A \subset G$ and $B \subset G$, then A and B are said to be mutually exclusive provided no element of G belongs to both A and B. In addition, if A and B are mutually exclusive and neither contains a limit element of the other then A and B are said to be mutually separated.

Definition 4: If $A \subset G$ then A is said to be connected in G if it cannot be written as the union of two mutually separated sets.

Definition 5: A subset A of G is closed in G provided it contains all of its limit elements.

Definition 6: A continuum of elements of G is any set which is both closed and connected in G.

Definition 7: A set K of elements of G is said to be bounded in G provided the set K* is bounded in E_2. The notation K* is used to denote the set of points obtained by taking the union of the points of all the elements of K.

Definition 8: A closed, connected and bounded subset H of G is a simple closed curve in G provided H is disconnected by the omission of any two of its elements.

Definition 9: If h_1 and h_2 are elements of a bounded continuum H in G, then H is said to be an arc in G from h_1 to h_2 provided H is disconnected by the omission of any element other than h_1 and h_2. The elements h_1 and h_2 are called end-elements.

Definition 10: A domain D of elements of G is a connected subset of G such that for every element $d \in D$ there exists a real number $\delta > 0$ such that if $g \in G$ and $u(g,x) < \delta$ then $g \in D$.

Definition 11: An element g of G belongs to the boundary of a set H of elements of G if and only if x either belongs to H and is a limit element of G - H or x belongs to G - H and is a limit element of H.

Definition 12: A domain D of elements of G is a complementary domain of a closed set H in G provided $(\overline{D} - D) \subset H$.

Definition 13: If D is a bounded domain then the outer boundary of D is the boundary of the unbounded complementary domain of the boundary of D.

Lemma: If H is a finite subcollection of elements of G then H has no limit element.

Proof: Let $H = \{g_1, g_2, \ldots, g_k\}$ and suppose that H has a limit element g. We would like to show that this leads to a contradiction. Let ϵ_n represent the upper distance of g_n from g for every n = 1, 2, ..., k, such that $g_n \neq g$. Choose $\epsilon = \frac{1}{2}(\min\{\epsilon_i \mid i = 1, 2, \ldots, k, g_i \neq g\})$. Then $\epsilon > 0$ and therefore by the definition of a limit element there must exist an element g_o of H, distinct from g, for which $u(g_o,g) < \epsilon$. But no such element exists since ϵ was chosen to be less than $u(g_i,g)$ for every i = 1, 2, ..., k. Therefore, the original supposition must be false and the set H has no limit element.

Theorem 1: If K is a set of points and H is the set of all elements $g \in G$ such that g contains at least one point of K, then H is closed in G if K is closed and H is connected in G if K is connected.

Proof: If H is a finite collection, then, by the lemma, H is closed. Therefore suppose H is an infinite set. Then assuming K is closed, let us show that H is also closed. Let p be a limit element of H. Then for every integer n there exists an element h_n of H such that $u(h_n,p) < 1/n$, and if $i \neq j$, $h_i \neq h_j$. Thus for every point $x_n \in h_n$, $\ell(x_n,p) < 1/n$. For every n, h_n contains a point k_n belonging to K. Thus for every n there exists a point $y_n \in p$ such that the distance from k_n to y_n is less than 1/n. Since $p \in G$, p is a bounded continuum and therefore the sequence y_1, y_2, ... has a sequential limit point $y \in p$. Then y is a limit point of the sequence

k_1, k_2, ..., and, since K is closed, $y \in K$. Therefore $p \in H$ since p contains a point of K, and H is closed.

If K is connected, let us show that the supposition H is not connected in G leads to a contradiction. If H is not connected in G then H can be written as the union of mutually separated sets H_1 and H_2 in G. Let $K_1 = K \cap H_1{}^*$ and $K_2 = K \cap H_2{}^*$. Since K is connected, either K_1 contains a limit point of K_2 or vice versa. Without loss of generality suppose there exists a point $k \in K_1$ such that k is a limit point of K_2. Let $p \in H_1$ such that $k \in p$. Since G is an upper semi-continuous collection, if $\epsilon > 0$ there exists a $\delta > 0$ such that whenever there exists an element p_o for which $\ell(p_o,p) < \delta$ then $u(p_o,p) < \epsilon$. But, since k is a limit point of K_2, there exists a point $k_\delta \in K_2$ such that the distance from k_δ to k is less than δ. Let h_δ be an element of H_2 which contains k. Then $\ell(h_\delta,p) < \delta$ and this implies that $u(h_\delta,p) < \epsilon$, and, since this is true for every $\epsilon > 0$, p is a limit element of H_2. But this contradicts the assumption that H_1 and H_2 were mutually separated sets since $p \in H_1$ and therefore it must be false that H can be written as the union of mutually separated sets. Thus H is connected in G.

Lemma: If K is a continuum in G then K* is a continuum in E_2.

Proof: Let p be a limit point of K*. Then every region in E_2 containing p contains infinitely many points of K*. Consider the collection of open disks with center at p and radii $1/n$, $n = 1, 2, \ldots$. Then for every integer $n > 0$ there exists a point p_n belonging to the open disk with radius $1/n$ such that $p_n \in K^*$ and $p_n \neq p$. Then the distance from p_n to p is less than 1/n. Let $k \in G$ such that $p \in k$ and suppose that k_1, k_2, ... are elements of K such that $p_i \in k_i$ for every i = 1, 2, ... Then for every n, $\ell(k_n,k) < 1/n$ and, since G is an upper semi-continuous collection, this implies that for every $\epsilon > 0$ there exists a k_i such that $u(k_i,k) < \epsilon$. Therefore $k \in K$ and $p \in K^*$. Thus K^* is closed.

Suppose, however, that K^* is not connected. Then $K^* = A \cup B$ where A and B are mutually separated closed sets, closed because K^* is closed. Let K_1 and K_2 be subcollections of elements of K which contain points of A and B respectively. If $K_1 \cap K_2 \neq \emptyset$ then there exists a $k \in K$ such that $k \cap A \neq \emptyset$ and $k \cap B \neq \emptyset$. Therefore $k = (k \cap A) \cup (k \cap B)$. But since A and B are mutually separated sets, so also are $(k \cap A)$ and $(k \cap B)$. This contradicts the fact that k is a continuum in E_2 and therefore it must be true that $K_1 \cap K_2 = \emptyset$. Then because K is connected, there exists a $k \in K_1$ such that k is a limit element of K_2 or vice versa. Without loss of generality suppose $k \in K_1$ and k is a limit element of K_2. Then for every $n > 0$ there exists $k_n \in K_2$ such that $u(k_n,k) < 1/n$. Then for each n there is an $x_n \in K_n$ such that the distance from x_n to some point $y_n \in k$ is less than 1/n. Since k is a bounded continuum the sequence y_1, y_2, ... has a limit point in k which is also a limit point of the sequence x_1, x_2, ... This implies that k contains a limit point of B and therefore A contains a limit point of B. But this contradicts the assumption that A and B were mutually separated. Therefore K^* is connected.

Theorem 2: If D is a bounded complementary domain of a bounded continuum of elements of G, and K is the outer boundary of D, and p is an element of K, then K is a continuum of elements of G and $K - \{p\}$ is connected.

Proof: Let E denote the unbounded complementary domain of the boundary of D and let B denote the boundary of E*. According to the definition of outer boundary, since K is the outer boundary of D, K is the boundary of E. In order to make use of Theorem 1, let us show that every point of B belongs to some element of K and every element of K contains a point of B.

Let $k \in K$. Then k is a boundary element of E and therefore either $k \in E$ and is a limit element of $G - E$ or $k \in G - E$ and is a limit element of E. If the former is true then for every $n > 0$ there exists $g_n \in G - E$ such that $u(g_n,k) < 1/n$. But then for each g_n there exists a point $x_n \in g_n$ such that $\ell(x_n,k) < 1/n$. This implies that for every n there exists a point $y_n \in k$ such that the distance from x_n to y_n is less than $1/n$. Since k is a bounded continuum belonging to G, the sequence $y_1, y_2, \ldots,$ has a limit point $y \in k$ which is also a limit point of the sequence $x_1, x_2, \ldots$. Then $y \in E^*$ because $y \in k$ and $k \in E$. Therefore, y is a boundary point of E*. By a similar method, if $k \in G - E$ then k contains a point $y \in E^*$ which is a limit element of E*. Thus every element of K contains a point of B.

Let $b \in B$. Since B is the boundary of E*, either $b \in E^*$ and is a limit point of $(G - E)^*$ or $b \in (G - E)^*$ and is a limit point of E*. If $b \in E^*$, then for every $n > 0$ there exists a point

$x_n \in (G - E)^*$ such that the distance from x_n to b is less than 1/n and such that $x_i \neq x_j$ if $i \neq j$. Let $e \in E$ such that $b \in e$. If $e_1, e_2, \ldots$ represent the elements of $G - E$ which contain $x_1, x_2, \ldots$ respectively, then $\ell(e_i,e) < 1/i$ for every $i = 1, 2, \ldots$. Since G is an upper semi-continuous collection this implies that for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\ell(e_i,e) < \delta$ then $u(e_i,e) < \epsilon$ and thus e is a limit element of $G - E$. Therefore $e \in K$ and $b \in e$. Similarly, if $b \in (G - E)^*$ then there exists an $e \in K$ such that $b \in e$ and therefore every point of B belongs to an element of K.

Now, it has been shown that every element of K contains a point of B and every point of B is contained in some element of K, and B is closed. A corollary of the Phragmen-Brouwer Theorem [22, p. 106] states that if a compact set is the common boundary of two domains then it is a continuum. The set B satisfies these hypotheses and hence B is a continuum. Thus Theorem 1 implies that K is both closed and connected. Therefore K is a continuum of elements of G.

In order to show that for any element $p \in K$, $K - \{p\}$ is connected, suppose on the contrary that $K - \{p\}$ can be written as the union of mutually separated sets M and N. Then $M \cup \{p\}$ and $N \cup \{p\}$ are closed and connected and their only common element is p. Let $x \in D$ and $y \in E$, and suppose that $d \in x$ and $e \in y$. According to the preceding lemma, the sets $(M \cup \{p\})^*$ and $(N \cup \{p\})^*$ are continua in E_2 and their intersection is the continuum p. Then their union is a continuum in E_2 which separates the point d from the point e. Thus, either $(M \cup \{p\})^*$ or $(N \cup \{p\})^*$ separates d from e, for suppose this is not true. Then neither $(M \cup \{p\})^*$ nor $(N \cup \{p\})^*$ separates d from e. But in a 2-sphere, S^2, if two points are not separated by either of two closed sets whose intersection is connected, then they are not separated by the union of the two sets [25, p. 65]. Since E_2 is homeomorphic to $S^2 - \{x\}$ where x is any point of S^2, and because $(M \cup \{p\})^*$ and $(N \cup \{p\})^*$ are compact sets in E_2, for this case, the theorem would also hold in E_2. But this yields a contradiction since K separates the points d and e in E_2. Therefore, one of $(M \cup \{p\})^*$ and $(N \cup \{p\})^*$ separates d from e. Without loss of generality suppose $(M \cup \{p\})^*$ does this. Then $M \cup \{p\}$ separates x from y in G; i.e. $G - (M \cup \{p\}) = W \cup Z$ where W and Z are mutually separated sets of elements with $x \in W$ and $y \in Z$. The set D is a domain containing x and, since by the definition a domain is connected, $D \subset W$. For the same reason $E \subset Z$. Let $q \in N$. Then q is a limit element of D and therefore of W. Thus, since $q \notin (M \cup \{p\})$, $q \in W$. But q is also a limit element of E and therefore of Z. This contradicts the assumption that W and Z are mutually separated. Therefore, the assumption that $K - \{p\}$ was not connected was false and $K - \{p\}$ is connected.

It is now possible to make a definition of a region of elements of G. It will be shown that the space G with regions defined in the following way will properly satisfy the desired axioms.

Definition 14: A region of elements of G is a bounded domain of elements of G which has a connected boundary.

The following theorem will be useful in future proofs. It is somewhat less general than a theorem of Moore's [15, p 469] in which he shows that if M is a closed point set in E_2 and K is a bounded maximal connected subset of M which does not separate E_2, then, for every $\epsilon > 0$, there exists a simple closed curve which encloses K and contains no point of M and which is such that every point within it is at a distance less than ϵ from some point of K. The more restricted form of this theorem will be sufficient for this paper.

Theorem 3: If K is a continuum in E_2 which does not separate E_2 then, for every $\epsilon > 0$, there exists a simple closed curve which encloses K such that for every point x contained within the closed curve, the distance from x to some point of K is less than ϵ.

Proof: Let C denote any circle which encloses K, let r denote the radius of C and let $d = \ell(C,K)$. For every integer $n > 0$, let T_n be the set of points x such that x can be joined to C by a simple continuous arc every point of which is at a distance greater than or equal to d/2n from every point of K and at a distance less than or equal to r from the center of C. Then by the way T_n is defined, for every n, T_n is a bounded connected point set with $T_n \subset T_{n+1}$. Since T_n is a bounded subset of E_2, $\overline{T}_n$ is compact. Now, consider the collection of all open disks in E_2 with radii d/3n. Since for every n, this collection covers E_2, obviously it covers $\overline{T}_n$. But $\overline{T}_n$ being closed and compact implies that there exists a finite subcollection, call it G_n, which covers $\overline{T}_n$. Let H_n be the circles of radius d/3n, each of which is the boundary of a disk in G_n. Then every

$x \in T_n$ is contained within a circle belonging to H_n with radius equal to $d/3n$, and without loss of generality we may assume that every element of G_n contains a point of T_n. Let $F_n = H_n^* \cup G_n^*$. The set F_n is closed because it is the union of a finite collection of closed disks. Suppose F_n is not connected. Then $F_n = X \cup Z$ where X and Z are mutually separated sets. But then $T_n = (T_n \cap X) \cup (T_n \cap Z)$ and since these are mutually separated sets this implies that T_n is not connected. This is a contradiction and therefore the assumption that F_n is not connected is false. Let J_n denote the boundary of the complementary domain D_n of F_n which contains K. Then J_n is a simple closed curve enclosing K. Then if $\varepsilon > 0$, there exists an $n > 0$ such that every point of D_n is at a distance less than ε from some point of K, for, if not, there exists an ε, such that for every n, there is a point $p_n \in D_n$ such that p_n is at a distance greater than or equal to ε from every point of K. Then there exists a point p which is a sequential limit point of some subsequence of $p_1, p_2, \ldots$, and such that, for every n, the distance from p to every point of K is greater than or equal to ε. Since, by hypothesis, K does not separate E_2, there exists an arc from p to some point of C which does not intersect K. Let h be the minimum distance from this arc to K and let k be the smallest positive integer such that $k > d/2h$. Then $p \notin J_k \cup D_k$. But since p is a sequential limit point of some subsequence of p_1, $p_2, \ldots$, there is an integer $m > k$ such that $p_m \notin J_k \cup D_k$. But $p_m \in J_m \cup D_m \subset J_k \cup D_k$ according to the way these are defined. Therefore, a contradiction has been reached and hence it is true that for every ε there exists an $n > 0$ for which every point of D_n

is at a distance less than ϵ from some point of K. Therefore the theorem is proved.

Theorem 4: If p is an element of G and $\epsilon > 0$, there exists a region R of elements of G such that for every element r belonging to R, $u(r,p) < \epsilon$.

Proof: Because p is a continuum which does not separate E_2, Theorem 3 implies there exists a simple closed curve J of points of E_2 such that p is enclosed by J and such that every point on or within J is at a distance less than ϵ from some point of p. Let H be the set of all elements of G such that if $h \in H$ then h contains at least one point of J. Since J is a continuum in E_2, H is a continuum of elements of G and H* is a continuum in E_2. Let D denote the complementary domain of H* which contains the point set p, and let B denote the boundary of D. According to the second Phragmen-Brouwer property [25, p. 47], B is a closed and connected set of points. Let R denote the set of all elements of G which are subsets of D. Then the boundary of R is the collection of elements of G which contain points of B, and it follows that the boundary of R is connected since B is both closed and connected. Then R is a domain and therefore it is a region. Every element r which belongs to R is at an upper distance less than ϵ from p.

Theorem 5: If p is an element of G and K is a set of elements of G then p is a limit element of K if and only if every region of elements of G which contains p contains at least one element of K which is distinct from p.

Proof: Let p be a limit element of K and suppose there exists a region R of elements of G which contains p but contains no element of K. Let B be the boundary of R* and let k be the smallest distance from B to some point of p. Choose $\epsilon < k$. Then Theorem 4 implies that there exists a region R_ϵ of elements of G such that every element of R_ϵ is at an upper distance less than ϵ from p and furthermore, $R_\epsilon \subset R$. Therefore R_ϵ contains no element of K. But this implies that there does not exist an element of K, distinct from p, whose upper distance from p is less than ϵ, and this is a contradiction of the hypothesis that p is a limit element of K. Thus every region of elements of G which contains p also contains an element of K distinct from p.

If every region which contains p contains at least one element of K distinct from p, then for every $\epsilon > 0$, let R_ϵ be a region containing p and such that for each element r belonging to R_ϵ, $u(r,p) < \epsilon$. By hypothesis then, for every ϵ, R_ϵ contains an element of K distinct from p. Therefore, for every ϵ, there exists some element of K whose upper distance from p is less than ϵ. Thus p is a limit element of K.

At this time it is possible to show that if the word "point" in Axioms 1, 2, 4, and 5 is reinterpreted to mean "element of G", then the space of elements of G, with regions in G defined as bounded domains in G whose boundaries are connected, will satisfy these axioms. In the material that follows each axiom will be restated in terms of the space G and accompanying it will be the necessary proof that it has been satisfied. It is assumed that the regions of E_2 are open spheres.

Axiom 1: There exists a sequence $M_1, M_2, \ldots,$ such that (1) for every n, M_n is a collection covering G such that each element of M_n is a region, (2) for every n, M_{n+1} is a subcollection of M_n and (3) if R is a region of elements of G, x and y are elements of R, then there exists a natural number m such that if A is any region belonging to M_m and containing x then $\overline{A} \subset R$ and, unless $x = y$, $\overline{A}$ does not contain y.

Proof: If g is an element of G then g is a bounded continuum in E_2. The space E_2 satisfies Axiom 1 as it was originally stated and since g is a closed and bounded point set in E_2, for every integer n there exists at least one finite subcollection of regions belonging to the collection G_n which properly cover g. Let $\{F_\alpha | \alpha \in \pi\}$ be all finite subcollections of G_n such that for each j, F_j properly covers g. Let $R_{gnF_j} = \{x | x \in G \text{ and } x \subset F_j^*\}$. Then for each n let $M_n = \{R_{gnF_j} | g \in G, F_j \in G_n\}$. Obviously, for each n, M_n is a collection of regions covering G, and M_{n+1} is a subcollection of M_n. Therefore conditions (1) and (2) of the axiom have been satisfied.

In order to show that condition (3) is also satisfied, let R be a region in G, and let x and y be elements of R. Suppose $x \neq y$. Because x and y are mutually exclusive closed point sets in E_2, each of which is compact, there exists a bounded domain D in E_2 which contains x and such that $\overline{D}$ contains no point of y. Furthermore it is possible to choose D in such a way that $\overline{D} \subset R^*$. Then there exists a domain D_1 which contains x and $\overline{D}_1 \subset D$. Note that D and D_1 are domains in E_2 rather than domains with respect to G. There exists a region

K with respect to G whose elements are the elements of G which are subsets of $R^* - y$ and such that $x \in K$. Now suppose that for every n, there exists a region R_{tnF} belonging to the collection M_n, such that R_{tnF} contains x and $\overline{R}_{tnF}$ is not a subset of K. There exists an m such that no region of the set G_m intersects both $\overline{D}_1$ and $E_2 - D$. Then for each $n > m$, some region of F, the finite subcollection of G_n which determines R_{tnF}, intersects $E_2 - D$ and therefore lies in $E_2 - \overline{D}_1$. Since t is covered by F and since it may be assumed that every region contained in F contains some point of t, t contains a point P_{nt} of $E_2 - \overline{D}_1$. Since x is also covered by F, there exists a region belonging to F which contains both a point of x and a point of t. Then for every $\delta > 0$ there exists an n such that when $t \in R_{tnF}$ then $\ell(t,x) < \delta$. Then because G is an upper semi-continuous collection, for every $\epsilon > 0$, there exists an n such that when $t \in R_{tnF}$, $u(t,x) < \epsilon$. Therefore the sequence of points P_{1t}, P_{2t}, ..., has a subsequence which converges to a point $X \in x$. But this is a contradiction since x is contained in the domain D_1 and, for each i, P_{it} belongs to $E_2 - \overline{D}_1$. Therefore there exists a number m such that if A is a region with respect to G belonging to M_m and containing x, then $\overline{A} \subset K \subset R - \{y\}$. Therefore the collection M_1, M_2, ... satisfies the conditions of part (3) of the axiom.

Axiom 2: Every region is a connected set of elements of G.

Proof: By definition, a region of elements of G is a bounded domain of elements of G. But a domain of elements of G is defined to be connected. Therefore every region is a connected set of elements of G.

Axiom 4: If R is a region in G, then $\overline{R}$ satisfies the Borel-Lebesgue property.

Proof: If R is a region in G then $\overline{R}$ is closed and bounded. Let H be any collection of regions in G such that H covers $\overline{R}$. Then $\overline{R}^*$ is closed and bounded in E_2 and therefore $\overline{R}^*$ has the Borel-Lebesgue property. If h is a region belonging to the collection H then h^* is a domain in E_2. Therefore there exists a finite sub-collection $\{h_1, h_2, \ldots, h_n\}$ of elements of H such that $\{h_1^*, \ldots, h_n^*\}$ covers $\overline{R}^*$. Thus $\{h_1, \ldots, h_n\}$ covers $\overline{R}$. Hence $\overline{R}$ satisfies the Borel-Lebesgue property.

Axiom 5: There exists an infinite set of elements of G with no limit element.

Proof: Let us suppose G does not satisfy this axiom; i.e. suppose every infinite set of elements of G has a limit element. In E_2 there exists an infinite set of points $P_1, P_2, \ldots$ with no limit point. Let $g_1, g_2, \ldots$ be the elements of G such that g_n contains the point P_n. At most a finite number of the g_i are equal since, for every i, g_i is compact and if g_i contains infinitely many of the points then they have a limit point in g_i. Thus, without loss of generality, suppose that all the g_i are distinct. By our supposition, the sequence $g_1, g_2, \ldots$ has a limit element g belonging to G. Then for every $\epsilon > 0$ there exists a g_n such that $u(g_n, g) < \epsilon$. Thus for every ϵ and for every point P_n, there exists a point X_n belonging to g such that the distance from P_n to X_n is less than ϵ. Now, because g is compact, there exists an $X \in g$ such that X is a limit point of the sequence $X_1, X_2, \ldots$. But this implies that X is also a limit point of the sequence $P_1, P_2, \ldots$, and this contradicts the fact that they have no limit point. Therefore there exists an infinite set of elements of G having no limit element.

As a consequence of these four axioms the following theorems may now be proved for the space G. Unless otherwise indicated, in the future material, the word region will be used to mean a region with respect to G.

Theorem 6: No element of a region is a boundary element of that region.

Proof: Let R be a region and x an element of R such that x belongs to the boundary of R. Then x is a limit element of G - R. Theorem 5 implies that every region containing x contains an element of G - R. This implies that R contains an element of G - R, but this is impossible. Therefore if $x \in R$ then x does not belong to the boundary of R.

Theorem 7: If p is a limit element of M then every region containing p contains infinitely many elements of M.

Proof: Let R be a region containing p. According to Theorem 5, R contains at least one element p_1 of M distinct from p. By Axiom 1 there exists a region R_2 such that $p \in R_2$, $\overline{R}_2 \subset R$ and $p_1 \notin \overline{R}_2$. But R_2 contains an element p_2 of M distinct from p. This process may be continued indefinitely. Therefore, it follows that R contains infinitely many elements of M.

Theorem 8: No element of G is a limit element of a finite collection of elements of G.

Proof: Let $p \in G$ and suppose M is a subset of G which contains only finitely many elements. Suppose p is a limit element of M. Then Theorem 7 implies that every region containing p contains infinitely many elements of M. This is a contradiction since M contains only finitely many elements. Thus the theorem is true.

Theorem 9: If p is an element of G then there exists an infinite sequence of regions R_1, R_2, ... such that, (1) p is the only element common to all the regions, (2) for every n, $\overline{R}_{n+1} \subset R_n$, and (3) if R is a region about p then there exists an n such that $\overline{R}_n$ is a subset of R.

Proof: There exists a region belonging to the collection M_1 which contains p. Let $R_1 \in M_1$ such that $p \in R_1$. According to Axiom 1 there exists an integer m_1 such that if $R \in M_{m_1}$ and $p \in R$ then $\overline{R} \subset R_1$. Let $R_2 \in M_{m_1}$ such that $p \in R_2$. Then there exists an integer m_2 such that if $R \in M_{m_2}$ and $p \in R$ then $\overline{R} \subset R_2$. Continuing this process we get a sequence of regions R_1, R_2, ..., such that $p \in R_i$ and $\overline{R}_{i+1} \subset R_i$. Since M_k is a subcollection of M_j whenever $k > j$, it is possible to assume without any loss of generality that $m_i \geq m_j$

whenever $i > j$. Suppose $p \neq \bigcap_{i=1}^{\infty} R_i$. Certainly $p \in \bigcap_{i=1}^{\infty} R_i$. Then there exists a $q \in \bigcap_{i=1}^{\infty} R_i$, $q \neq p$. Thus for every i, $p \in R_i$, $q \in R_i$. By Axiom 1 there exists a k such that if $n > k$ and $R \in M_n$, $p \in R$, then $\overline{R} \subset R_i$ and $q \notin \overline{R}$. But there exists a j such that $R_j \in M_n$ and therefore $q \notin \overline{R}_j$. This is a contradiction of the assumption that $q \in \bigcap_{i=1}^{\infty} R_i$ and therefore $\bigcap_{i=1}^{\infty} R_i = p$. If R is any region about p then according to Axiom 1 there exists an integer n such that if K is a region of M_n containing p then $\overline{K} \subset R$. There exists an $m \geq n$ such that $R_j \in M_m$. But then $R_j \in M_n$ since M_m is a subcollection of M_n and therefore $\overline{R}_j \subset R$.

Theorem 10: If two regions H and K have an element p in common, then there exists a region R which contains p and such that $R \subset H \cap K$.

Proof: Let H and K be distinct regions such that $p \in H$ and $p \in K$. According to Theorem 9, there exists integers m and n such that R_m contains p and $\overline{R}_m \subset H$ and R_n contains p and $\overline{R}_n \subset K$. But $p \in R_{m+n}$, $R_{m+n} \subset R_m$ and $R_{m+n} \subset R_n$, and thus $R_{m+n} \subset (R_m \cap R_n) \subset (H \cap K)$.

Theorem 11: If p is a limit element of $M \cup N$, where M and N are subsets of the space G, then p is a limit element of either M or N.

Proof: Suppose p is a limit element of $M \cup N$ but p is not a limit element of either M or N separately. Then there exists regions R_m and R_n containing p such that R_m contains no element of M different from p and R_n contains no other element of N. Theorem 10 implies there is a region R containing p such that $R \subset R_n \cap R_m$. But then R contains no element of $M \cap N$ different from p and this is a contradiction of the hypothesis that p is a limit element of $M \cap N$. Therefore, it is true that p is a limit element of either M or N.

Definition 15: An element p will be called a sequential limit element of the sequence of elements $p_1, p_2, \ldots$ if for every region R containing p there exists an integer m such that if $n > m$ then p_n lies in R.

Theorem 12: If p is a sequential limit element of the sequence of elements $p_1, p_2, \ldots$, then the set $\{p_1, p_2, \ldots\}$ has no other limit element.

Proof: Suppose both p and x are limit elements of the set $\{p_1, p_2, \ldots\}$. Theorem 8 implies that there exists a region R containing p but not x. According to Theorem 9 there exists a region K containing p such that $\overline{K} \subset R$. Then there is an integer m such that when $n > m$, $p_n \in K$. But $x \notin \overline{K}$ and thus x is not a limit element of $\{p_{m+1}, p_{m+2}, \ldots\}$. But neither is x a limit element of $\{p_1, p_2, \ldots, p_m\}$ since, by Theorem 8, no element is a limit element of a finite set of elements. Therefore, by Theorem 11, x is not a limit element of $\{p_1, p_2, \ldots\}$. Therefore, the conclusion is that p is the only limit element of $\{p_1, p_2, \ldots\}$.

Theorem 13: If p is a limit element of the set M then there exists an infinite sequence of elements of M all distinct from p such that p is the sequential limit element of this sequence.

Proof: Let R_1, R_2, ... be a sequence of regions containing p and satisfying the conditions of Theorem 9. For every n, R_n contains an element of M distinct from p. Let $q_1 \in R_1$ such that $q_1 \in M$ and $q_1 \neq p$. Since $q_1 \notin \bigcap_{i=1}^{\infty} R_i$, there exists an integer n_2 such that $q_1 \notin R_{n2}$. Let $q_2 \in R_{n2} \cap M$, $q_2 \neq p$. Continuing in this manner, there will be determined a subsequence of the sequence of regions, R_{n1}, R_{n2}, R_{n3}, ... such that for each i, $q_i \in R_{ni} \cap M$, $q_i \neq p$. Then the sequence q_1, q_2, ... is an infinite sequence of elements of M. Furthermore p is a sequential limit element of this sequence. In order to show this, let R be any region containing p. By Theorem 9, there exists an integer n such that $\overline{R}_n \subset R$. If $m > n$ then $R_m \subset R_n$, therefore $R_m \subset R$ for every $m \geq n$. Thus there exists an n_k such that $R_{nj} \subset R$ for $n_j \geq n_k$. Then $q_j \in R$ for every $j \geq k$. Therefore, p is a sequential limit element of the sequence q_1, q_2, ...

Definition 16: If p_1 and p_2 are distinct elements of G, then a simple chain from p_1 to p_2 is a finite sequence of regions R_1, R_2, ..., R_n, such that (1) $p_1 \in R_i$ if and only if $i = 1$, (2) $p_2 \in R_i$ if and only if $i = n$, and (3) if $1 \leq i \leq n$, $1 \leq j \leq n$, $i < j$, then $R_i \cap R_j \neq \emptyset$ if and only if $j = i + 1$. Each region will be called a link of the chain.

Theorem 14: If M is a connected set of elements, p and q are distinct elements of M, and H is a set of regions covering M then there exists a simple chain from p to q, every link of which is a region of H.

Proof: Suppose the theorem is false. If there is no such chain from p to q then M can be written as the union of two sets, X_p and X_q where every element belonging to X_p can be joined to p by a simple chain of regions of H and X_q is all other elements of M. Because M is connected, either X_p contains a limit element of X_q or vice versa. Suppose $x \in X_p$ and x is a limit element of X_q. There is at least one region of H, say h_x, containing x. Then h_x also contains an element $y \in X_q$. The element x can be joined to p by a simple chain $h_1, h_2, \ldots, h_n$ of regions belonging to H. Let h_k be the first link of this chain which intersects h_x. Then $h_1, h_2, \ldots, h_k, h_x$ is a simple chain of regions of H from p to y. But this is a contradiction since $y \in X_q$. In the second case, suppose $x \in X_q$ and x is a limit element of X_p. Let h_x be a region of H containing x. Then $h_x \cap X_p \neq \emptyset$. Let $y \in h_x \cap X_p$. Then there is a simple chain of regions of H, $h_1, \ldots, h_n$, from p to y. Let h_k be the first link of this chain which intersects h_x. Then $h_1, h_2, \ldots, h_k, h_x$ is a simple chain of regions of H from p to x. Again this is a contradiction since $x \in X_q$. Therefore, since both cases lead to contradictions, it must be true that there is a simple chain of regions of H from p to q.

Theorem 15: If $R_1, R_2, R_3, \ldots, R_n$ is a finite set of regions, the set $\overline{\bigcup_{i=1}^{n} R_i}$ possesses the Borel-Lebesque property.

Proof: Let H be any collection of regions covering $\overline{\bigcup_{i=1}^{n} R_i}$. But $\overline{\bigcup_{i=1}^{n} R_i} = \bigcup_{i=1}^{n} \overline{R}_i$ and therefore the collection H covers $\overline{R}_i$ for every $i = 1, \ldots, n$. According to Axiom 4, there is a finite subcollection of elements of H which covers $\overline{R}_i$. Then let H_i be a finite subcollection of H which covers $\overline{R}_i$. Then $\bigcup_{i=1}^{n} H_i$ is a finite subcollection of H which covers $\overline{\bigcup_{i=1}^{n} R_i}$. Therefore, $\overline{\bigcup_{i=1}^{n} R_i}$ possesses the Borel-Lebesgue property.

Theorem 16: Every closed and bounded set of elements possesses the Borel-Lebesgue property.

Proof: Let A be a closed and bounded set of elements of G. By definition if A is bounded then A* is bounded in E_2 and since A is closed A* is closed in E_2. Then A* has the Borel-Lebesgue property in E_2. This implies there exists a collection of regions of E_2, $R_1, \ldots, R_k$, such that $R_i \in G_n$ and the finite sub-collection of G_n covers A*. Then $R = \{x | x \in G \text{ and } x \subset \bigcup_{i=1}^{k} R_i\}$ is a region in G and $A \subset R \subset \overline{R}$.

Let H be any collection of regions of G which covers A. Because A is closed, if $q \in \overline{R} - A$ then q is not a limit element of A and hence there exists a region R_q containing q such that $R_q \cap A = \emptyset$. Then $H \cup \{R_q | q \in \overline{R} - A\}$ is a collection of regions covering $\overline{R}$. By Axiom 4, $\overline{R}$ satisfies the Borel-Lebesgue property and thus a finite subcollection of $H \cup \{R_q | q \in \overline{R} - A\}$ covers $\overline{R}$. This finite subcollection also covers A however, and since no R_q contains an element of A this implies that a finite subcollection of H covers A. Therefore, A has the Borel-Lebesgue property.

Theorem 17: Every infinite, bounded set of elements has at least one limit element.

Proof: Let $p_1, p_2, \ldots$ be an infinite, bounded set of elements of G. Then $\bigcup_1^{\infty} \{p_i\}$ is bounded in E_2. Every bounded infinite sequence of points of E_2 has a limit point. Thus let $x_1, x_2, \ldots$ be a sequence of points of E_2 such that for each i, $x_i \in p_i$. The point set $\{x_1, x_2, \ldots\} \subset \bigcup_1^{\infty} \{p_i\}$ and therefore forms a bounded sequence. Let x be a limit point of $\{x_1, x_2, \ldots\}$ and let $p \in G$ such that $x \in p$. But then, because G is an upper semi-continuous collection this implies that p is a limit element of the sequence $p_1, p_2, \ldots$. Therefore, every infinite, bounded set of elements has at least one limit element.

Theorem 18: If $B_1, B_2, \ldots$ is an infinite sequence of bounded sets of elements of G such that, for each n, $\overline{B}_{n+1} \subset B_n$, then $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ and $\bigcap_{i=1}^{\infty} B_i$ is closed.

Proof: By the Axiom of Choice, choose $p_1, p_2, \ldots$ such that, for every i, $p_i \in B_i$. If there exists an integer j such that for every $k > j$, $p_k = p_j$, then $p_j \in B_i$ for every i and hence $p_j \in \bigcap_{i=1}^{\infty} B_i$. Otherwise the sequence $p_1, p_2, \ldots$ is bounded and hence, by Theorem 17, it has a limit element p. Then p is a limit element of $\{p_{n+1}, p_{n+2}, \ldots\}$ which is a subset of B_{n+1} and $\overline{B}_{n+1} \subset B_n$. Therefore $p \in B_n$. But since this is true for any value of n, $p \in \bigcap_{i=1}^{\infty} B_i$. Thus in either case $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$.

If q is a limit element of $\bigcap_{i=1}^{\infty} B_i$ then for every n, $q \in \overline{B}_{n+1} \subset B_n$. Therefore, $q \in \bigcap_{i=1}^{\infty} B_i$ and $\bigcap_{i=1}^{\infty} B_i$ is closed.

Theorem 19: If D is a domain of elements of G then for every element p of D there exists a region R_p, containing p, such that $R_p \subset D$.

Proof: By definition, D is a connected subset of G such that if $p \in D$, there exists a $\delta > 0$ such that if $g \in G$ and $u(g,p) < \delta$ then $g \in D$. But by Theorem 4, there exists a region R_p containing p such that $u(g,p) < \delta$ for every $g \in R_p$. Thus $R_p \subset D$.

Theorem 20: If p and q are distinct elements of a domain D, there exists an arc from p to q lying wholly in D.

Proof: If $x \in D$, then there exists a region R containing x and lying wholly in D. Let H_1 be the collection of all such regions such that each element of H_1 is contained in a region belonging to M_1. Then H_1 covers D and by Theorem 14 there exists a simple chain from p to q, every link of which belongs to H_1. Call this chain C_1 and denote the links by R_{11}, R_{12}, ..., R_{1m_1}. Let $p_{10} = p$, $p_{1m_1} = q$, and, if $0 < i < m_1$, let p_{1i} be an element common to R_{1i} and $R_{1(i+1)}$. The elements p_{10} and p_{11} can be connected by a simple chain C_{11} each link of which along with its limit elements lies completely in R_{11} and in some region of M_2. Let C'_{11} be the chain which remains after the deletion of all the links of C_{11} after the first link which intersects R_{12}. Some point belonging to the intersection of R_{12} and the last link of C'_{11} can be joined to the point p_{12} by a chain C_{12}, each link of which, along with its limit elements, is contained in R_{12} and in some region of M_2. Let C'_{12} be the chain which is left after deleting all the links of C_{12} which either precede the last link which has a point in common with the last link of C'_{11} or follow the first link which intersects R_{13}. Continuing this process, a finite set of simple chains C'_{11}, C'_{12}, ..., C'_{1m_1} will be generated with the properties that (1) for each n, $1 \leq n \leq m_1$, each link of C'_{1n} is a region whose closure is contained in R_{1n} and in some region belonging to M_2, (2) for each $n < m_1$, the last link of C'_{1n} is the only one that intersects $R_{1(n+1)}$, (3) for each n, $1 < n \leq m_1$, the first link of C'_{1n} is the only one that intersects the last link of $C'_{1(n-1)}$, and (4) the first link of C'_{11} contains p and the last link of C'_{1m_1} contains q. Then the links of these chains form a simple chain C_2 from p to q. It should be observed that each link of the chain C_2 lies completely in one of the links of C_1 and if the mth link of C_2 lies in the nth link of C_1 then if $j > m$ the jth link of C_2 lies in the kth link of C_1 where $k \geq n$. By the same method there exists a chain C_3 which bears the same relation to C_2 as that of C_2 to C_1. Continuing this process will generate an infinite sequence of chains $C_1, C_2, \ldots$, with the properties that (1) if x is a link of C_{n+1} then $\overline{x}$ is contained in some link of C_n, (2) if the mth link of C_{n+1} lies in the kth link of C_n, then for every $i > m$, the ith link of C_{n+1} lies in the jth link of C_n for some $j \geq k$, and (3) every link of C_n is a region whose closure is contained in some region of M_n.

Let C_n' be the set of all elements which belong to some link of C_n. Then it will be shown that the set $C = \bigcap_{n=1}^{\infty} C_n'$ is a simple arc from p to q.

First, C is closed. Each C_n' is a bounded set such that $\overline{C}_{n+1}' \subset C_n'$ and thus Theorem 18 implies that C is closed.

Next, to show that C is connected suppose that on the contrary C can be written as $S_1 \cup S_2$ where S_1 and S_2 are mutually separated sets. Because C is closed, both S_1 and S_2 are closed. About each element p of S_1 there exists a region R containing no element of S_2. There exists an integer n such that if $R_p \in M_n$ and contains p then $\overline{R}_p \subset R$. Then Theorem 16 implies there exists a finite collection $R_{1p}, R_{2p}, \ldots, R_{np}$ of regions which covers S_1 and, for each i, $\overline{R}_{ip} \cap S_2 = \emptyset$. Similarly, there exists a finite collection of regions $H_{1q}, H_{2q}, \ldots, H_{mq}$ which covers S_2, and, such that for every j, $\overline{H}_{jq} \cap \bigcup_{i=1}^{n} R_{ip} = \emptyset$. For each n, C_n' is connected and intersects both $\bigcup_{i=1}^{n} R_{ip}$ and $\bigcup_{j=1}^{m} H_{jq}$. Therefore C_n' contains a boundary element of

$\bigcup_{i=1}^{n} R_{ip}$. Let B be the boundary of $\bigcup_{i=1}^{n} R_{ip}$. Then the sets $B \cap C_1'$, $B \cap C_2'$, ... satisfy the conditions of Theorem 18, which implies there exists an element p_o belonging to $\bigcap_{i=1}^{\infty}(B \cap C_i')$. Then $p_o \in C_i'$ for every i but $p_o \notin C$. This is a contradiction and therefore C is connected.

Finally, it is to be shown that if any element of C other than p and q is deleted the remaining set is no longer connected. Let x and y be any two elements of C. By Axiom 1, there exists an integer n such that no region of M_n contains both x and y. But every link of the chain C_n is contained in some region of M_n, therefore x and y belong to different links of C_n. Furthermore, if x lies in a link that precedes the link containing y in C_n, then for every $m \geq n$, every link of C_m containing x precedes every link of C_m that contains y. The element x will be said to precede y if there is an n such that every link of C_n containing x precedes every one which contains y. The relation "precedes" is a linear order on C.

Suppose now that $x \in C$ and $x \neq p$, $x \neq q$. Then $C - \{x\} = S_p \cup S_q$ where S_p is the set of elements of C which precede x, S_q is the set of those which follow x. Clearly, because p precedes every other element of C and every other element precedes q, $p \in S_p$ and $q \in S_q$. Also, $S_p \cap S_q = \emptyset$. Let $y \in S_p$ and suppose y is a limit element of S_q. Then there exists an integer n such that every link of C_n which contains y precedes every one that contains x, and no link of C_n which contains y intersects any link which contains x. But if y is a limit element of S_q then every link containing y also contains an element z belonging to S_q. Then z precedes x. But this is a contradiction and hence y is not a limit element of S_q. Similarly no element of S_q is a limit element of S_p. Therefore $C - \{x\}$ is not connected since it is the union of mutually separated sets.

Thus C is an arc from p to q and the theorem has been established.

Theorem 21: No arc of elements of G is disconnected by the omission of either of its extremities.

Proof: Let a and b be distinct elements of G and let A be an arc from a to b. Suppose $A - \{a\} = X \cup Y$ where X and Y are mutually separated sets and suppose $b \in Y$. Let $x \in X$. Then it was shown in the proof of Theorem 20 that since $x \neq a$, $x \neq b$, $A - \{x\}$ can be written as the union of mutually separated sets P and Q where $a \in P$ and $b \in Q$. Then

$$A = \{a\} \cup X \cup Y = \{a\} \cup (X \cap P) \cup (X \cap Q) \cup (Y \cap P) \cup (Y \cap Q) \cup \{x\}$$
$$= [\{a\} \cup (X \cap P) \cup (X \cap Q) \cup (Y \cap P) \cup \{x\}] \cup (Y \cap Q)$$

where the sets $[\{a\} \cup (X \cap P) \cup (X \cap Q) \cup (Y \cap P) \cup \{x\}]$ and $(Y \cap Q)$ are mutually separated and non-empty. This is a contradiction of the fact that an arc is connected, therefore $A - \{a\}$ is connected. Similarly, it can be shown that $A - \{b\}$ is also connected.

A convenient notation for an arc with extremities p and q is pq. If x is an element distinct from p and q and belonging to the arc pq then it will be said that x is between p and q on the arc.

Theorem 22: If x is between p and q on the arc pq then pq is the union of arcs px and xq having only the element x in common.

Proof: It has been shown that $pq - \{x\} = A_p \cup B_q$ where A_p and B_q are mutually separated sets, A_p contains those elements which precede x and B_q contains those which follow x. Then $A_p \cup \{x\}$ and $\{x\} \cup B_q$ are closed.

Suppose $A_p \cup \{x\}$ is not connected. Then $A_p \cup \{x\} = C \cup D$, C and D mutually separated, and without loss of generality suppose $x \in D$. Then $C \subset A$ and therefore C and B_q are mutually separated. Hence $pq = C \cup (D \cup B_q)$ and since C and $D \cup B_q$ are mutually separated this contradicts the fact that pq is connected. Therefore $A_p \cup \{x\}$ is connected. Similarly $\{x\} \cup B_q$ is connected. Let y be an element of $A_p \cup \{x\}$ different from p and x. Then it is known that $pq - \{y\} = X_p \cup Y_q$ where X_p and Y_q are mutually separated. The set $(A_p \cup \{x\}) - \{y\}$ is contained in $pq - \{y\}$ and has an element in common with X_p. Because $y \in (A_p \cup \{x\})$ this implies that y precedes x in order from p to q and hence $x \in Y_q$. Therefore $(A_p \cup \{x\}) - \{y\}$ is the union of mutually separated sets contained in X_p and Y_q. Thus y separates $A_p \cup \{x\}$ and $A_p \cup \{x\}$ is an arc from x to p. Similarly $\{x\} \cup B_q$ is an arc from x to q. Since A_p and B_q are mutually separated sets the only element common to the arcs px and xq is x.

Theorem 23: If x and y are elements of arc pq then pq contains an arc with x and y as its extremities.

Proof: The theorem is obviously true if either $x = p$ or $y = q$, hence assume that both x and y are distinct from p and q and that x precedes y in order from p to q. Then according to Theorem 22, py is an arc from p to y and because x precedes y, $x \in py$. Then applying

Theorem 22 again, xy is an arc from x to y.

Theorem 24: If K is an arc of elements of G, then every closed and connected subset of K which contains more than one element is itself an arc of elements of G.

Proof: Let K be the arc pq and let H be a closed and connected subset of K. If $p \in H$ then p will be the first element of H. Otherwise let S_1 be the set of elements which precede every element of H. Let $h \in H$. Then ph is an arc from p to h and $S_1 \subset ph$. Let $S_2 = ph - S_1 = ph \cap H$ and hence S_2 is closed since it is the intersection of two closed sets. Then because $S_1 \cap S_2 = \emptyset$ and S_2 is closed, S_2 must contain a limit element s of S_1. Then s belongs to H and every element which precedes s belongs to S_1 and therefore s will be the first element of H. Similarly, a last element of H, call it t, can be found and then H is the arc st.

Theorem 25: Let pq denote an arc of elements of G with p and q as its extremities, let $x \in pq - \{p,q\}$, and let $H \subset pq$. Then x is a limit element of the set H if and only if for every arc A of elements of G with the properties that (1) $x \in A$, (2) $A \subset pq$, and (3) x is not an extremity of A, it is true that A contains at least one element of H which is distinct from x.

Proof: Suppose first that every arc which contains x, is a subset of pq, and does not have x as an extremity also contains at least one element of H distinct from x. Now x is a limit element of the arc px, for otherwise px is not connected. By the same reasoning x is also a limit element of the arc xq. Theorem 13 implies that there exists an infinite sequence of elements of px, p_1, p_2, ..., such that x is the sequential limit element of the sequence, and, if $i < j$ then p_i precedes p_j in order from p to x. Similarly, there exists an infinite sequence q_1, q_2, ... belonging to xq such that x is the sequential limit element of the sequence and if $i < j$ then q_i is preceded by q_j in order from x to q. Then the arc p_1q_1 contains x and thus contains an element h_1 of H. Let j and k be integers such that h_1 does not belong to the arc p_jq_k. Then there exists an element h_2 of H such that h_2 is contained in p_jq_k. Continuing in this manner we acquire an infinite sequence h_1, h_2, ..., of elements of H. Suppose an infinite number of these, h_{n_1}, h_{n_2}, ..., precede x in order from p to q. Then there exists a sequence p_{m_1}, h_{n_1}, p_{m_2}, h_{n_2}, ..., such that h_{n_i} was chosen from the arc $p_{m_i}q_j$ for some value of j. According to Theorem 17 this sequence has a limit element. Since x is a sequential limit element of the subsequence p_{m_1}, p_{m_2}, ..., suppose that x is not a limit element of the subsequence h_{n_1}, h_{n_2}, ... Then there exists a y different from x such that y is a limit element of this subsequence. Either x precedes y or y precedes x. If the first is true then y belongs to the set xq - $\{x\}$ while all the elements of the sequence belong to the set px - $\{x\}$. But these are mutually separated sets and hence p is not a limit element of the sequence. If y precedes x, then there exists an n_k such that y precedes p_{n_k}. Then y belongs to the arc pp_{n_k} and for infinitely many of the values of m_j, h_{m_j} follows p_{n_k} and thus y is not a limit element of the sequence. Therefore x is a limit element of H.

Now suppose x is a limit element of H and that there exists an arc ab, $a \neq x$, $b \neq x$, such that $x \in ab$ and $ab \subset pq$ but $ab \cap H = \emptyset$. Then every element of H either precedes a or follows b in order from p to q. Then pb is an arc and $pb - \{a\} = X \cup Y$ where X is the set of elements which precede a and Y is the collection of elements which follow a in order from p to b. Since $x \in Y$, x is not a limit element of X and thus not a limit element of $H \cap X$. Similarly, if $aq - \{b\} = W \cup Z$ where W is the set of elements which precede b and z is those which follow b in order from a to q, then $x \in W$ and is not a limit element of $H \cap Z$. But $H = (H \cap X) \cup (H \cap Z)$ and therefore x is not a limit element of H. Therefore, the contradiction implies that the assumption was false. Thus every arc containing x also contains an element of H.

Definition 17: If p is an element belonging to a connected set C of elements of G then p will be called a cut element of C if $C - \{p\}$ is disconnected.

Theorem 26: Every closed, connected and bounded set has at least two non-cut elements.

Proof: Suppose M is a closed, connected and bounded set of elements of G. Let $p \in M$ and suppose that if $q \in M - \{p\}$ then q is a cut element of M. Then let $M - \{q_\alpha\} = P_\alpha \cup Q_\alpha$, where q_α is any element of $M - \{p\}$, α belongs to some index set π, and P_α and Q_α are mutually separated sets, and suppose that $p \in P_\alpha$ for each α. Let $H = \{Q_\alpha \cup \{q_\alpha\} | \alpha \in \pi\}$. For every α, $Q_\alpha \cup \{q_\alpha\}$ is closed since M is closed and P_α contains no point or limit point of Q. Suppose $Q_\alpha \cup \{q_\alpha\}$ is not connected. Then $Q_\alpha \cup \{q_\alpha\} = A \cup B$ where A and B are mutually separated sets. Without loss of generality suppose that $q_\alpha \in A$. Then B contains no point or limit point of P_α and hence M is the union of the sets B and $A \cup P_\alpha$ which are mutually separated. This contradiction then implies that $Q_\alpha \cup \{q_\alpha\}$ is connected and, since it is contained in a bounded set, it is also bounded. Let K be a monotonic subcollection of sets of H. Then Theorem 18 implies the sets of the collection K have an element k in common. Then $M - \{k\} = P_k \cup Q_k$ such that P_k and Q_k are mutually separated and, since $p \neq k$, suppose $p \in P_k$. Let $Q_\beta \cup \{q_\beta\} = K_\beta$ be a set belonging to the collection K, $q_\beta \neq k$. Then $M - \{q_\beta\} = P_\beta \cup Q_\beta$ where these are mutually separated sets and $p \in P_\beta$. Then K_β contains k and because $k \neq q_\beta$, $k \in Q_\beta$. Then since $k \notin P_\beta \cup \{q_\beta\}$, and $P_\beta \cup \{q_\beta\}$ is closed and connected, either $P_\beta \cup \{q_\beta\} \subset P_k$ or $P_\beta \cup \{q_\beta\} \subset Q_k$. But p belongs to both P_k and P_β and thus $P_\beta \cup \{q_\beta\} \subset P_k$. Therefore $Q_k \cup \{k\} \subset K_\beta$. But $Q_k \cup \{k\}$ belongs to the collection H and therefore it is contained in every other set of the collection K. But according to a theorem of Moore's [16, p. 14] when for any monotonic subcollection K there exists a set belonging to K which is a subset of every other set belonging to K then there exists a set H_o in H such that H_o contains no other set of H. There is an element h_o such that $M - \{h_o\} = (H_o - \{h_o\}) \cup P_o$ where $H_o - \{h_o\}$ and P_o are mutually separated sets and P_o contains p. Let $h_1 \in (H_o - \{h_o\})$. Then $M - \{h_1\} = Q_1 \cup P_1$,

Q_1 and P_1 mutually separated with $p \in P_1$. But $Q_1 \cup \{h_1\}$ belongs to the collection H and since $P_o \cup \{h_o\} \subset P_1 \cup \{h_1\}$, $Q_1 \cup \{h_1\} \subset H_o$. This is a contradiction of the assumption that H_o contains no other element of H, therefore, for every element $p \in M$, there is a non-cut element of M distinct from p. Thus M has at least two non-cut elements.

Theorem 27: If K is a closed, connected and bounded set of elements of G, and H is a connected proper subset of K, then the set K - H contains an element of G whose omission does not disconnect K.

Proof: Suppose, on the contrary, if $x \in K - H$ then $K - \{x\}$ is disconnected. Therefore $K - \{x\}$ is the union of mutually separated sets A and B. Since H is connected and $x \notin H$, $H \subset A$ or $H \subset B$. Suppose, without loss of generality, that $H \subset A$. Then $B \cup \{x\}$ is closed and connected and according to Theorem 26, there exists an element $b \in B$ such that b is a non-cut element of $B \cup \{x\}$. Thus, $(B \cup \{x\}) - \{b\}$ is connected. Therefore $(A \cup \{x\}) \cup [(B \cup \{x\}) - \{b\}] = K - \{b\}$ is connected. Thus K - H contains a non-cut element of K.

Theorem 28: If pq is an arc of elements of G, then G - pq is connected.

Proof: Let pq = A. Since A is closed, connected and bounded in G, A* is closed, connected and bounded in E_2. Suppose G - pq is the union of the mutually separated sets X and Y. Then $E_2 - A^* = X^* \cup Y^*$, where X* and Y* are mutually separated sets. Because A* is closed and bounded, there exists a circle C in E_2 such that A* is contained in the interior of C. Let I be the interior of C. Then $E_2 - I$ is closed and connected and furthermore, according to Theorem 1, the set $P = \{g | g \in G \text{ and } g \cap (E_2 - I) \neq \emptyset\}$ is both closed and connected. Then $P \subset X$ or $P \subset Y$. Without loss of generality suppose it is the case that $P \subset Y$. Then $X^* \subset I$ and thus X^* is bounded. Let D be a component of X^*. Then D is a domain in E_2, D is connected, and the boundary of D is contained in A^*.

If $E = \{g | g \in G \text{ and } g \subset D\}$ then it is asserted that $E^* = D$, for suppose this is not the case. Then there exists a $g \in G$ such that $g \cap D \neq \emptyset$ but $g \not\subset D$. But $g \subset X^*$ and since g and D are both connected and their intersection is not empty, then $g \cup D$ is a connected subset of X^* which contains D. This is a contradiction since D is a component of X^*. Thus E is a maximal connected subset of X and E is a bounded complementary domain with respect to A.

Let B be the outer boundary of E. According to Theorem 2, B is closed, connected, and if $b \in B$ then $B - \{b\}$ is still connected. But B is also a connected subset of pq, and Theorem 24 implies that B is an arc. This contradicts the fact that $B - \{b\}$ is connected for every $b \in B$. Therefore G is not separated by any arc pq.

Theorem 29: If M is a simple closed curve of elements of G and if p and q are distinct elements belonging to M, then M is the union of two arcs which have in common only their terminal elements p and q.

Proof: By definition $M - \{p,q\} = C \cup D$ where C and D are mutually separated sets.

Suppose M has a cut element x. Then $M - \{x\} = H \cup K$, where H and K are mutually separated sets. Then $H \cup \{x\}$ and $K \cup \{x\}$ are closed and connected and therefore each of them is a nondegenerate bounded continuum. According to Theorem 26, $H \cup \{x\}$ has at least two non-cut elements and $K \cup \{x\}$ has at least two non-cut elements. Therefore, there exist elements h and k such that h and k are non-cut elements distinct from x belonging to H and K respectively. Then $M - \{h,k\} = (H \cup \{x\} - \{h\}) \cup (K \cup \{x\} - \{k\})$ and because these are connected sets with the element x in common, $M - \{h,k\}$ is connected. This is in contradiction with the definition of a simple closed curve and thus no element of M is a cut element.

Let $c \in C$ and $d \in D$. The sets $C \cup \{p,q\}$ and $D \cup \{p,q\}$ are each closed, connected, and bounded. Thus each is a continuum. Suppose $C \cup \{p,q\}$ and $D \cup \{p,q\}$ each has more than two non-cut elements. Then there exist elements x and y, distinct from p and q, such that $x \in C$ and $y \in D$ and x and y are non-cut elements of $C \cup \{p,q\}$ and $D \cup \{p,q\}$ respectively. Then $M - \{x,y\} = (C \cup \{p,q\} - \{x\}) \cup (D \cup \{p,q\} - \{y\})$ and because these connected sets have elements p and q in common, $M - \{x,y\}$ is connected. This is a contradiction of the definition and therefore one of $C \cup \{p,q\}$ and $D \cup \{p,q\}$ does not contain a non-cut element distinct from p and q. Without loss of generality suppose $C \cup \{p,q\}$ is disconnected by the omission of any element other than p and q. Then $C \cup \{p,q\}$ is, by definition, an arc from p to q.

Suppose $y \in D$ and $D \cup \{p,q\} - \{y\}$ is connected. Then $C \cup \{p,q\} - \{c\} = A \cup B$ where A and B are mutually separated and $p \in A$, $q \in B$. Thus $M - \{y,c\} = A \cup B \cup (D \cup \{p,q\} - \{y\})$. But both A and B have an element in common with $D \cup \{p,q\} - \{y\}$, and since each of the sets is connected, their union is connected. This again contradicts the definition of M. Therefore $D \cup \{p,q\}$ is disconnected by the omission of any element other than p and q.

Thus $C \cup \{p,q\}$ and $D \cup \{p,q\}$ are arcs, their union is M, and, obviously, their only common elements are p and q.

Theorem 30: If J is a simple closed curve of elements of G, then G - J is the union of two domains of elements of G. Only one of these domains is bounded and J is the boundary of each of them.

(In the proof of this theorem use will be made of the following Theorems A and B. These are theorems which have been proved for the plane by Anna M. Mullikin [19].

Theorem A: If M_1 and M_2 are two closed, connected, bounded point sets, neither of which disconnects a plane S, a necessary and sufficient condition that their union, M shall disconnect S is that $M_1 \cap M_2$ be not connected.

Theorem B: If M_1 and M_2 are two closed, bounded, connected point sets in a plane S, such that neither M_1 nor M_2 disconnects S and such that M_1 and M_2 have in common only K_1 and K_2, where K_1 and K_2 are mutually exclusive connected sets, then $S - (M_1 \cup M_2)$ is the union of exactly two mutually exclusive, connected domains.)

Proof: Let p and q denote distinct elements of J. According to Theorem 29, J is the union of two arcs A and B which have p and q as their extremities. By Theorem 28, neither A nor B separates G and therefore neither A* nor B* separates E_2. Also $A^* \cap B^* = p \cup q$ where p and q are mutually exclusive continua. Then according to Theorems A and B, $E_2 - J^*$ is the union of mutually exclusive connected domains D_1 and D_2 of points of E_2. Obviously then, since J^* is bounded, one of the domains is bounded and the other unbounded. Suppose D_1 is the bounded domain. Let H_1 be the set of elements of G which are contained in D_1 and let H_2 be those elements of G contained in D_2. Then it is clear that H_1 and H_2 are mutually exclusive domains of elements of G and that $G - J = H_1 \cup H_2$. Let B denote the boundary of H_1. If $B \neq J$ then it must be a proper subset of J. Since B is closed and connected, Theorems 29 and 24 imply that B is an arc of elements of G. But $G - B = H_1 \cup [H_2 \cup (J - B)]$ and the sets H_1 and $[H_2 \cup (J - B)]$ are mutually separated. Thus the assumption that $B \neq J$ implies a contradiction and therefore $B = J$. Similarly it can be shown that J is also the boundary of H_2.

Definition 18: Of the two domains complementary to a simple closed curve of elements of G, the bounded one will be called the interior of the curve, while the unbounded domain will be called the exterior.

Theorem 31: If D_1 and D_2 are bounded domains of elements of G, and D_1 has a connected boundary, and the boundary of D_2 is a subset of D_1, then D_2 is a subset of D_1.

Proof: Since D_2 is bounded, it has at least one boundary element. Each of its boundary elements belongs to D_1, therefore D_1 contains at least one element of D_2. Suppose D_2 is not a subset of D_1. Then D_2

contains an element of $G - D_1$. Because D_2 is connected it follows that D_2 contains an element of the boundary B_1 of D_1. The boundary B_2 of D_2 is a subset of D_1 and thus does not intersect B_1. The set B_1 is connected, therefore B_1 is necessarily contained in D_2. Let E be the unbounded complementary domain of B_1. Since B_1 is the boundary of E and D_2 contains B_1, then D_2 contains an element of E. But E is connected and contains no element of B_2, therefore E is contained in D_2. But this is contrary to the hypothesis that D_2 is bounded. Therefore D_2 is a subset of D_1.

Theorem 32: If R is a region of elements of G and K is either a single element or an arc of elements of G every element of which (except possibly a terminal element, in the case K is an arc) belongs to R, then R - K is a domain of elements of G.

Proof: Let B denote the boundary of R. By definition, B is connected. If K is a single element of G then K does not separate E_2 and hence does not separate G. Otherwise, Theorem 28, implies K does not separate G. Thus K^* does not separate E_2. If x and y are elements of R - K then x and y are not separated in E_2 by B^*. Also, either $K^* \cap B^* = \emptyset$ or the common part of K^* and B^* is a closed, bounded, and connected point set making up a single element of G. It follows then, that $K^* \cup B^*$ does not separate x and y in E_2. Thus, $K \cup B$ does not separate x from y in G and therefore R - K is connected.

Suppose R - K is not a domain. Then there exists an element $r \in R - K$ such that every region containing r contains an element which does not belong to R - K. Evidently then, every region containing r contains an element of K. But this implies that r is a limit element of K which is in contradiction with the fact that K is closed. Therefore R - K is a domain.

Theorem 33: If R is a region of elements of G there exists a simple closed curve of elements of G such that every element of G which belongs to this curve is an element of R.

Proof: Let p and q be distinct elements of R. According to Theorem 20 there exists an arc pq of elements of G such that every element of pq belongs to R. Let r be some element of pq distinct from p and q. By Theorem 32, R - $\{r\}$ is a domain. Therefore, there exists a simple continuous arc pyq, having p and q as terminal elements and containing the element y, such that pyq is contained in R - $\{r\}$. Then it is easily seen that the arcs pxq and pyq either form a simple closed curve of elements belonging to R or contain one as a proper subset of their union.

Theorem 34: If K and R are regions of elements of G and the boundary of R is a subset of $\overline{K}$, then R is a subset of K.

Proof: Suppose R is not contained in K. If $R \cap (S - \overline{K}) \neq \emptyset$, then $S - \overline{K} = S_1 \cup S_2$ where S_1 is a subset of R and no element of S_2 belongs to R. The set S_1 can contain no limit element of the set S_2, thus, since $S - \overline{K}$ is connected, S_2 must contain a limit element of S_1. But this implies that S_2 contains a boundary element of R which is contrary to the hypothesis. Therefore R is a subset of $\overline{K}$. If R contains a boundary element of K then R contains an element of $S - \overline{K}$.

Since it has been shown that this is not possible, it follows that R is a subset of K.

Theorem 35: If pxq and pyq are arcs of elements of G which have p and q as their terminal elements but have no other elements in common, and J is the simple closed curve formed by these two arcs, pzq is an arc, every element of which, except for p and q, belongs to the interior of J, and J_1 is the simple closed curve formed by arcs pxq and pzq and J_2 denotes the one formed by pyq and pzq, then (1) the interior of J_1 is a subset of the interior of J, (2) except for p and q, pyq lies in the exterior of J_1, (3) the interior of J_1 does not intersect the interior of J_2, and (4) the interior of J is the union of the sets which are the interiors of J_1 and J_2, and pzq - $\{p,q\}$.

Proof: Let R be the interior of J, R_1 the interior of J_1 and R_2 the interior of J_2. Each of R, R_1, and R_2 is a bounded domain with a connected boundary and therefore each is a region. Then according to Theorem 34, R_1 is a subset of R.

Because pyq is a part of the boundary of R, Theorem 6 implies that no element of pyq is contained in R_1. The only elements that pyq has in common with the boundary of R_1 are p and q. Therefore, except for p and q, pyq lies in the exterior of J_1.

Suppose $R_1 \cap R_2 \neq \emptyset$. Since the boundary of R_2 contains elements that are not contained in R_1 nor the boundary of R_1, R_2 is not a subset of R_1. Therefore $R_2 = S_1 \cup S_2$ where S_1 is a subset of R_1 but no element of S_2 belongs to R_1. Because S_1 cannot contain a limit element of S_2 it follows that S_2 contains a limit element of S_1. Clearly, this element must be a boundary element of R_1. Therefore R_2 contains an element of pyq or pzq. This is a contradiction however, and, thus, $R_1 \cap R_2 = \emptyset$.

By hypothesis, pzq - $\{p,q\}$ is contained in R. In part (1) it was shown that R_1 is a subset of R. Similarly, R_2 is a subset of R. Now, suppose $R \neq R_1 \cup R_2 \cup (pzq - \{p,q\})$. Let Y be the set of all other elements of R so that $R = R_1 \cup R_2 \cup (pzq - \{p,q\}) \cup Y$. The sets R_1, R_2, and Y are mutually separated. Let w denote an element of Y. There exists an arc wz lying entirely in R. Let j denote the first element of wz, in order from w to z, such that j belongs to pzq. Then wj is an arc lying entirely in R. Now wj - $\{j\}$ is connected and lies entirely in $R_1 \cup R_2 \cup Y$. Since wj - $\{j\}$ has an element in common with Y, it is contained in Y. The element j divides the arc pzq into two arcs, pz_1j and jz_2q. Let r be an element of R_1. There exists a region T about the element z_1 which contains no element belonging to the closed set $\{r\} \cup jz_2q \cup J$. Since T contains a boundary element of R_1, T contains an element g belonging to R_1. There exist arcs rg and gz_1 lying in R_1 and T respectively. Then $rg \cup gz_1$ contains an arc rh_1 such that h_1 is an element of T and of pz_1j, and such that $rh_1 - \{h_1\}$ is a subset of R_1. Similarly, there exists an arc rk_1 such that k_1 belongs to jz_2q and $rk_1 - \{k_1\}$ is contained in R_1. There exists an arc $h_1s_1k_1$ which is a subset of $rh_1 \cup rk_1$, and $h_1s_1k_1 - \{h_1,k_1\}$ is contained in R_1. Similarly, there exist elements h_2 and k_2 on the arcs ph_1 and k_1q respectively but distinct from p and q, and an arc $h_2s_2k_2$ such that $h_2s_2k_2 - \{h_2,k_2\}$

is contained in R_2. Let J_3 denote the simple closed curve formed by $h_1s_1k_1 \cup h_1h_2 \cup h_2s_2k_2 \cup k_1k_2$. By Theorem 34, R_3, the interior of J_3, is a subset of R. Since s_1 belongs to the domain R_1 and is a boundary element of R_3, it follows that $R_3 \cap R_1 \neq \emptyset$. Similarly, $R_3 \cap R_2 \neq \emptyset$. Since w and y are elements lying entirely without R_1, there exists an arc wy which contains no element of R_1. Let a be the first element that wy and pyq have in common. Then aw - $\{a\}$ is obviously a subset of Y. Thus there exists an arc ja which is a subset of jw $\cup$ aw and such that ja - $\{j,a\}$ is contained in Y. Hence ja contains no element of J_3. But since a is exterior to J_3, j is also exterior to J_3 and consequently the arc h_1jk_1 is exterior to J_3 with the exception of the terminal elements h_1 and k_1. Thus every element of R_3 belongs to R_1, R_2, or Y. Since these are mutually separated sets and R_3 contains elements belonging to both R_1 and R_2, this implies that R_3 is not connected. This is a contradiction, therefore it follows that $R = R_1 \cup R_2 \cup (pzq - \{p,q\})$.

Theorem 36: If p and q are two distinct elements of G and pxq, pyq, and pzq are arcs no two of which have in common any element other than p and q, and J_1, J_2, and J_3 are the simple closed curves formed by these arcs taken in pairs, then the interiors of J_1, J_2, and J_3 are not mutually exclusive.

Proof: Suppose J_1 = pxq $\cup$ pyq, J_2 = pxq $\cup$ pzq, and J_3 = pyq $\cup$ pzq. Let R_1, R_2, and R_3 denote the interiors of J_1, J_2, and J_3 respectively, and, in the same sense, let E_1, E_2, and E_3 denote their exteriors. Suppose R_1, R_2, and R_3 are mutually exclusive. Then E_1 contains R_2,

R_3, and $pzq - \{p,q\}$. Since E_1 is unbounded, E_1 contains at least one element which does not belong to R_2, R_3, or $pzq - \{p,q\}$. Therefore let Y be the collection of all such elements of E_1. Clearly, $Y = E_1 \cap E_2 \cap E_3$. Let a be an element of Y. Then there exists an arc az which lies entirely in E. Let b be the first element that az has in common with pzq. Then $ab - \{b\}$ is a subset of Y. Similarly, there exists an arc ac, such that c belongs to pxq and $ac - \{c\}$ is contained in Y. In the same way as it was shown in the proof of Theorem 35, it can be shown that there exist elements h_1, h_2, k_1, and k_2 such that $h_1 \in pb$, $h_2 \in h_1b$, $k_1 \in bq$, $k_2 \in bk_1$, and such that in each case these elements are distinct from p, b, and q. It can also be shown that there exist arcs $h_1s_1k_1$ and $h_2s_2k_2$ which, except for their terminal elements, lie entirely in R_2 and R_3 respectively. Let K_1, K_2, and K_3 denote the simple closed curves formed by $h_1s_1k_1 \cup k_1k_2 \cup k_2s_2h_2 \cup h_2h_1$, $h_1s_1k_1 \cup h_1bk_1$, and $h_2s_2k_2 \cup k_2bh_2$ respectively. Let L_1, L_2, and L_3 denote their respective interiors. By Theorem 34, L_2 and L_3 are subsets of R_2 and R_3 respectively.

There are now three cases to be considered.

Case 1. Suppose $b \in L_1$. Then by Theorem 35, $L_1 = L_2 \cup L_3 \cup (h_2bk_2 - \{h_2,k_2\})$. But since b is a limit element of $ab - \{b\}$, it follows that L_1 must contain an element of $ab - \{b\}$ and, hence, of Y. Thus the supposition that $b \in L_1$ leads to a contradiction, since $Y \cap L_1 = \emptyset$.

Case 2. Suppose $p \in L_1$. Then J_1 is contained in L_1. But c is an element of J_1 and ac does not intersect K_1. Therefore $a \in L_1$. Thus every element of Y belongs to L_1. But this is a contradiction since L_1 is bounded and Y is not bounded. Therefore $p \notin L_1$.

Case 3. Suppose neither b nor p belongs to L_1. Then $J_1 \cap L_1 = \emptyset$ and $pzq \cap L_1 = \emptyset$. Thus L_1 is contained in $R_1 \cup R_2 \cup R_3 \cup Y$ and these are mutually separated sets. But L_1 is connected and hence L_1 cannot intersect more than one of the sets R_1, R_2, R_3, and Y. But s_1 belongs to R_2 and to the boundary of L_1 and s_2 belongs to R_3 and to the boundary of L_1, and this implies that $R_2 \cap L_1$ and $R_3 \cap L_1$ are not empty. Thus this supposition also leads to a contradiction.

It follows then that R_1, R_2, and R_3 are not mutually exclusive.

Theorem 37: If pxq and pyq are arcs of elements of G which have p and q as their only common elements, J is the simple closed curve formed by these arcs, and pzq is an arc which, except for p and q, lies in the exterior of J, then (1) either y is in the exterior of J_1, the simple closed curve formed by pxq and pzq, or x is in the exterior of J_2, the simple closed curve formed by pyq and pzq, (2) if y is in the exterior of J_1 then x is in the interior of J_2 and the interior of J_2 is the union of the interiors of J and J_1, and the set $pxq - \{p,q\}$.

Proof: Suppose y is not in the exterior of J_1. Then $pyq - \{p,q\}$ is contained in the interior of J_1 and Theorem 35 implies that x is in the exterior of J_2.

Secondly, if y belongs to the exterior of J_1, suppose x is not in the interior of J_2. Then since x does not belong to J_2, x belongs to the exterior of J_2 and, hence, $pxq - \{p,q\}$ belongs to the exterior of J_2. Let R and R_1 denote the interiors of J and J_1 respectively.

Suppose R and R_1 have a common element. Then $R_1 = M \cup M_1$ where M is contained in R and M_1 does not contain an element of R nor of its boundary. It follows then that one of the sets M and M_1 does not exist. Thus either R_1 and R have no common elements or R_1 is contained in R. If R_1 is contained in R then $\overline{R}_1$ is contained in $\overline{R}$ and since $z \in \overline{R}_1$ and z does not belong to the boundary of R, then $z \in R$. But this is in contradiction with the hypothesis, therefore R_1 and R have no elements in common.

Similarly, since y belongs to the exterior of J_1, it can be shown that, if R_2 is the interior of J_2, the sets R, R_1, and R_2 are mutually exclusive. But this contradicts Theorem 36. Therefore, if y is in the exterior of J_1 then x is in the interior of J_2. Then Theorem 35 implies that $R_2 = R \cup R_1 \cup (pxq - \{p,q\})$.

Theorem 38: If R is a region and p is an element of R, then there exists a simple closed curve of elements of G which lies in R and whose interior contains p and is a subset of R.

Proof: Let x be an element of R distinct from p. By Axiom 1, there exists a region K which contains x and is a subset of R but does not contain p. By Theorem 33, K contains a simple closed curve J. Theorem 34 implies that p is exterior to J. Let a_1 be an element of J. Then in R there exists an arc pa_1. Let a be the first element of J which belongs to pa_1. Then pa is an arc whose only common element with J is a. Let I be the interior of J. Choose c to be an element of the boundary of R. Since $J \cup I \cup pa$ is a closed set contained in R there exists a region H about c which does not intersect $J \cup I \cup pa$.

The set H contains at least one element d of R. Let b_1 be an element of J distinct from a. By Theorems 32 and 20, there exists an arc b_1d in R - pa. Let b denote the last element that b_1d has in common with J so that bd is a subarc whose only common element with J is b. In H, there exists an arc cd. Let e be the first element that cd has in common with bd. Then ce $\cup$ be is an arc cb, every element of which, except c, belongs to R, and such that b is the only element it has in common with pa $\cup$ J $\cup$ I. The elements a and b separate J into two arcs, ax_1b and ay_1b. There exists about x_1 a region R_1 which is contained in R and does not intersect the closed set pa $\cup$ ay_1b $\cup$ bc. There exists an element h in R_1 such that $h \notin J \cup I$. Then there exists an arc hx_1 in R. Let x be the first element hx_1 has in common with J. By Theorem 32, there exists an arc hy_1 in R - (pa $\cup$ ax_1b $\cup$ bc). Let y be the first element hy_1 has in common with J. Then hy $\cup$ hx contains as a subset an arc yzx which, except for its terminal elements, is contained in R - (J $\cup$ I). Then yzx and xay are arcs whose union is a simple closed curve J_1. The interior of J_1 belongs to R, according to Theorem 34. Therefore c is in the exterior of J_1. Since bc has no element in common with J_1, it follows that b is exterior to J_1. Then according to Theorem 37, the interior of the closed curve J_2 formed by yzx and ybx contains a. But the arc pa contains no element of J_2 and hence p is in the interior of J_2.

Definition 19: A set R of elements of G will be said to be a region in the restricted sense if and only if it is the interior of some simple closed curve of elements of G.

Theorem 39: If p is an element of G and H is a set of elements of G then p is a limit element of H if and only if every region in the restricted sense that contains p contains an element of H distinct from p.

Proof: First, suppose p is a limit element of H and let R be a region in the restricted sense which contains p. Since R is also a region in the original sense, R contains an element of H distinct from p.

Conversely, if every region in the restricted sense which contains p also contains an element of H distinct from p, let R be a region in the usual sense which contains p. Then Theorem 38 implies that R contains a region in the restricted sense about p and hence R contains an element of H distinct from p. Therefore p is a limit element of H.

Finally now, it is possible to show that the other axioms are satisfied by the space of elements of G if the word region now is assumed to mean "region in the restricted sense". It has been established that Axioms 1, 2, 4, and 5 hold true for the set of elements of G with the original interpretation for regions. It is clear, by Theorems 38 and 39 that these axioms continue to hold if region is interpreted in the restricted sense. Thus in the following material, the word region should be taken to mean region in the restricted sense.

Axiom 3: If R is a region, $G - \overline{R}$ is a connected set of elements.

Proof: If R is a region then R is the interior of a simple closed curve J and $\overline{R} = R \cup J$. By Theorem 30, G - J is the union of two domains, one of which is R. Thus $G - \overline{R}$ is the unbounded domain and, by definition, a domain is connected.

Axiom 6: If R is a region and ab is an arc such that ab - $\{a\}$ is a subset of R then $(R \cup \{a\})$ - ab is connected.

This axiom follows directly from Theorem 32.

Axiom 7: Every boundary element of a region is a limit element of the exterior of that region.

Axiom 7 is a direct consequence of Theorem 30.

Axiom 8: Every simple closed curve is the boundary of at least one region.

This axiom is obviously satisfied by the way in which regions in the restricted sense have been defined.

It has been shown that the space of elements of G with regions defined to be the interiors of simple closed curves of elements of G satisfies all the axioms set down for the space E_2. Moore [12] has shown that for every space S satisfying Axioms 1-8, there exists a one-to-one correspondence between the elements of S and the points of E_2 which preserves limits. Thus there exists a one-to-one correspondence between the elements of the space G and the points of E_2 such that the space of elements of G is topologically equivalent to the space of elements of E_2.

CHAPTER III

EXAMPLES OF UPPER SEMI-CONTINUOUS DECOMPOSITIONS

In this chapter there will be exhibited examples of upper semi-continuous decompositions of E_2, beginning with the very simple ones and concluding with an example in which all of E_2 is decomposed into non-degenerate elements, none of which separate E_2. The obvious and trivial example is that in which each point of E_2 is an element in the decomposition. This is no different from the space E_2 however and therefore it is not of interest here.

A decomposition may have a finite number of nondegenerate elements or it may have infinitely many. The only upper semi-continuous collections to be considered here are those whose nondegenerate elements are continua which do not separate E_2. In the examples the nondegenerate elements will be described in detail. It should be understood then that the elements making up the decomposition space will be the nondegenerate elements together with all other points of E_2.

Example 1: Let R be the unit square with vertices (0,0), (1,0), (1,1), (0,1). Let $K = \{L | L$ is a vertical line segment of unit length contained within or on $R\}$. Then the segments belonging to K together with all points of $E_2 - K^*$ form an upper semi-continuous decomposition of E_2.

This example might be expanded by considering a collection of disjoint unit squares R_1, R_2, R_3, ... and then filling each square with segments of unit length.

Example 2: Let

$$A_n = \{(x,y) | x = -2^{-n},\ 0 \geq y \geq -2^{-n}\} \cup \{(x,y) | -2^{-n} \leq x \leq 2^{-n},\ y = -2^{-n}\}$$
$$\cup \{(x,y) | x = 2^{-n},\ -2^{-n} \leq y \leq 0\}.$$

Let $K = \{A_n | n = 0, 1, 2, \ldots\}$. Then the elements of K are the nondegenerate elements of a decomposition of E_2.

Example 3: Suppose

$$K = \{(x,y) | y = \sin \frac{1}{x},\ 0 < x \leq 1\} \cup \{(x,y) | x = 0,\ -1 \leq y \leq 1\}$$

is the only nondegenerate element in a decomposition. Then the decomposition space which is formed will satisfy all the conditions of this paper.

Example 4: Let C_n be a closed disk with center at (1,n) and radius 1/4. Then $K = \{C_n | n \text{ is an integer}\}$ is the collection of nondegenerate elements of a decomposition of E_2.

One might exhibit many examples of upper semi-continuous decompositions of E_2 similar to these. The most interesting example, however, is the decomposition of E_2 which has no element which is a single point and no element which separates E_2.

Roberts' Example

The question was raised by C. Kuratowski as to whether there exists an upper semi-continuous collection of mutually exclusive continua no one of which is a point such that (1) the union of the continua of the collection fills a square plus its interior and (2) if each continuum of the collection is regarded as a point the space so obtained is in continuous one-to-one correspondence with a square plus its interior.

Kuratowski posed this question in a letter to R. L. Moore in 1927. It was answered by J. H. Roberts [20] in 1928. The material which follows is chiefly the work of Roberts in which he exhibited an upper semi-continuous collection of continua filling the plane, such that each continuum is bounded, nondegenerate, and does not separate the plane.

Let $M = \{(x,y) | y = \sin \frac{1}{x(1-x)}, 0 < x < 1\}$. Clearly, $\overline{M}$ is a continuum. Let C denote the Cantor set on the interval $0 \leq x \leq 1$ of the x-axis and let s_1, s_2, s_3, ... denote the complementary segments so that $s_1 = \{x | 1/3 < x < 2/3\}$, $s_2 = \{x | 1/9 < x < 2/9\}$, $s_3 = \{x | 7/9 < x < 8/9\}$, ... For each point p of C which is not an endpoint of any complementary segment of C let V_p denote the vertical interval of length 2 with p as center, and let H be the collection of all such intervals. Let M_n be a point set equivalent to M, and whose limit sets are the vertical intervals two units in length which have the endpoints of the segment s_n as midpoints, and such that if (x_1,y_1) and (x_2,y_2) are distinct points of M_n then $x_1 \neq x_2$, and for every point (x,y) belonging to M_n, $|y| \leq 1$ (see Figure 1).

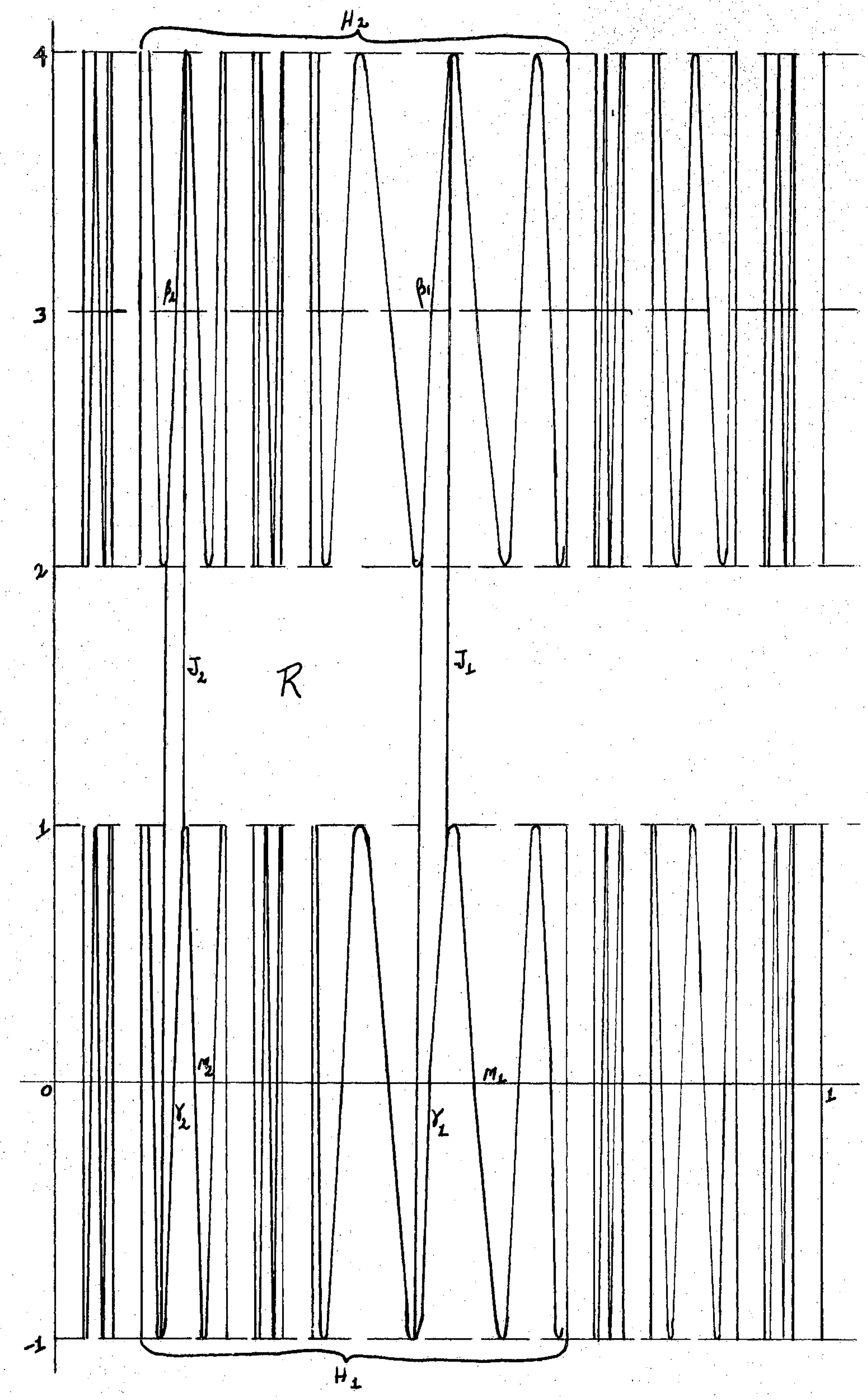

Figure 1.

Let K denote the union of the sets $\overline{M}_1$, $\overline{M}_2$, ..., together with the intervals of H. Then K is the union of a collection α_K of mutually exclusive continua, the elements of α_K being the intervals of H and the continua of the sequence $\overline{M}_1$, $\overline{M}_2$, ... It will be shown that the collection α_K is upper semi-continuous and that it is an arc with respect to its elements. A continuum N in E_2 will be equivalent to K if and only if there exists a continuous transformation T_N of E_2 into itself such that $T_N(K) = N$. Let α_N denote the collection of all point sets $T_N(g)$ where g is a continuum of the set α_K.

Lemma: The set K is both closed and connected.

Proof: Let p denote a point of $E_2 - K$. Then the object is to prove that p is not a limit point of K and therefore that K is closed.

The set K is contained within and on the rectangle R whose vertices are (0,1), (1,1), (1,-1), and (0,-1). Suppose first that p belongs to the exterior of R. Then let d denote the greatest lower bound of the set of distances from p to x where x is any element of R. Let U be a region containing p such that if $y \in U$ the distance from p to y is less than d/2. Then U is a region containing p but no point of K and hence p is not a limit point of K.

If $p \in R$ or to the interior of R then p lies on a vertical line which intersects the x-axis in a point q lying in the interval $0 \leq x \leq 1$. Since p does not belong to K, q cannot be an element of C. Therefore q belongs to one of the complementary intervals s_i. But p does not belong to M_i, nor to the vertical lines V_1 and V_2

which are the limiting sets of M_i. Therefore, choose d_1 to be the greatest lower bound of the distances from p to points of M_i, d_2 to be the greatest lower bound of the distances from p to points of V_1 and d_3 to be the greatest lower bound of the distances from p to points of V_2. Let $d = \min\{d_1, d_2, d_3\}$. If U denotes a region about p such that, for every $x \in U$, the distance from p to x is less than d/2, then U is a region containing p but no point of K. Therefore, p is not a limit point of K.

Therefore the set K is closed.

In order to show that K is connected, suppose, on the contrary, that it is not. Then $K = A \cup B$, where A and B are mutually separated sets. Since each element of α_K is connected, it is contained in either A or B.

Suppose the point (0,0) belongs to A and (1,0) to B. Then consider the point $p = \sup\{A \cap [(0,0), (1,0)]\}$. Necessarily, p is a point of C and if p is an endpoint of some interval s_i, then it must be the right endpoint, for otherwise, it would imply that some element of α_K intersected both A and B. But then p is a limit point from the right of the set C which implies that p is not the $\sup\{A \cap [(0,0), (1,0)]\}$.

If both (0,0) and (1,0) belong to A then consider the point $q = \sup\{B \cap [(0,0), (1,0)]\}$. As in the first case, q must belong to C and if q is an endpoint of an interval s_j, then it must be the right endpoint. Thus q is a limit point of C from the right and therefore q is not the $\sup\{B \cap [(0,0), (1,0)]\}$.

Hence both cases lead to contradictions and therefore it is implied that K is connected.

Before continuing to show that the collection α_K is upper semi-continuous and an arc with respect to its elements, it is necessary to state some useful definitions and theorems.

Definition: A transformation $f(A) = B$ is said to be monotone provided that for each point $y \in B$, $f^{-1}(y)$ is a continuum.

If $f(A) = B$ is monotone, the decomposition of A associated with f, i.e., into the sets $[f^{-1}(x)]$, $x \in B$, is an upper semi-continuous decomposition into continua [22, p. 127].

Theorem: [22, p. 127] Any monotone transformation $f(A) = B$ on a compact space A is equivalent to an upper semi-continuous decomposition of A into continua. Conversely, any upper semi-continuous decomposition of A into continua with decomposition space A' is equivalent to a monotone transformation $f(A) = A'$.

This theorem, stated without proof, is a well-known theorem related to transformations. It implies that if a monotone transformation f is defined which maps K onto an arc A, such that $\{f^{-1}(a) | a \in A\} = \alpha_K$, then the collection α_K would be upper semi-continuous and an arc with respect to its elements. It is the intent here to define such a transformation.

Let A be a closed segment with endpoints denoted by 0 and 1. Any point of A will be denoted by some real number x, $0 \leq x \leq 1$. Since K is the union of elements of α_K the transformation $f: K \longrightarrow A$

will be defined in terms of the elements of α_K (see Figure 2). It should be recalled that $\overline{M}_1$ is defined over $\overline{s}_1$, $\overline{M}_2$ over $\overline{s}_2$, etc., where $\overline{s}_1 = \{x | 1/3 \leq x \leq 2/3\}$, $\overline{s}_2 = \{x | 1/9 \leq x \leq 2/9\}$, ... Then let $f(\overline{M}_1) = 1/2$, $f(\overline{M}_2) = 1/4$, $f(\overline{M}_3) = 3/4$, $f(\overline{M}_4) = 1/8$, ... For the elements $V_{(0,0)}$ and $V_{(1,0)}$, let $f[V_{(0,0)}] = 0$ and $f[V_{(1,0)}] = 1$. Observe that for every $V_p \in H$, $p \neq (0,0)$, $p \neq (1,0)$, $p = \inf \{q | q \in C,$ $q > p$, and q is an endpoint of a complementary interval$\}$. Then let $f(V_p) = \inf \{f(q) | q \in C, q > p,$ and q is an endpoint of a complementary interval$\}$. Clearly, f is a monotone transformation and $\{f^{-1}(q) | q \in f(K)\} = \alpha_K$. It is necessary to show that $f(K) = A$.

Let p be any element of A. Obviously, if p is an element of the set $D = \{\frac{k}{2^n} | 0 \leq k \leq 2^n, n = 0, 1, \ldots\}$ then p is the image of some element of α_K. Suppose then that $p \in (A - D)$. Then $p = \inf \{q | q > p,$ $q \in D\}$, and hence $p = f(V_j)$ where $j = \inf \{m | m \in C$, m is an endpoint of a complementary segment, and $m > j\}$. Hence $f(K) = A$. It remains to show that f is continuous.

Let Q be any open interval in A. Then there exist points a and b in A such that a and b are the endpoints of Q, $a < b$. Then $f^{-1}(a)$ and $f^{-1}(b)$ are elements of α_K and furthermore they belong to the closure of $f^{-1}(Q)$. For, if not, there exists an open interval contained in $[(0,0), (1,0)]$ which contains $f^{-1}(a) \cap [(0,0), (1,0)]$ but contains no point of $f^{-1}(Q)$. But it contains points of at least one element g of α_K which lies to the right of $f^{-1}(a)$. Then f(g) is greater than a and less than every point of Q and hence a is not an endpoint of Q. Similarly, the same contradiction can be reached

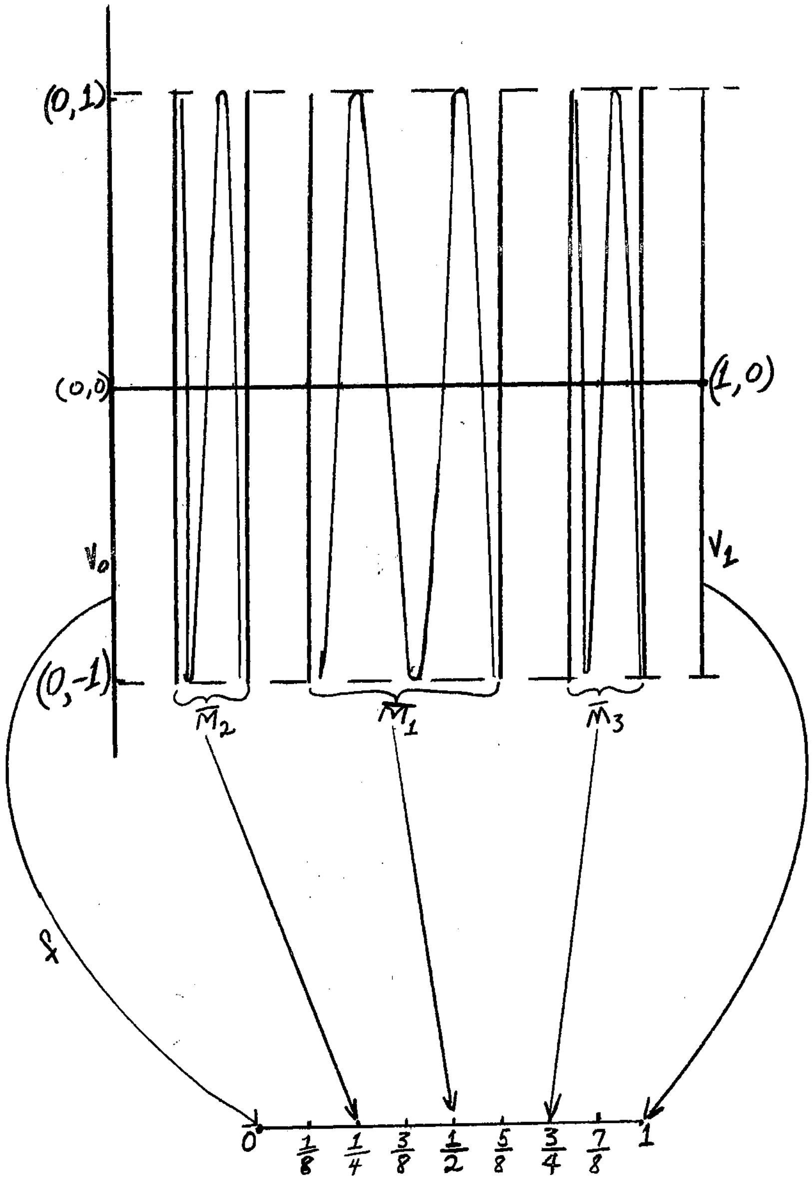

Figure 2.

if it is assumed that $f^{-1}(b)$ does not belong to the closure of $f^{-1}(Q)$. Hence $f^{-1}(Q)$ is open with respect to K and therefore f is continuous.

Thus f is a continuous one-to-one mapping of the elements of α_K onto the arc A. It is clear that f maps open sets of elements of α_K onto open sets. Therefore f is a homeomorphism from α_K to A and α_K is an upper semi-continuous collection which is an arc with respect to its elements.

Lemma. If J is a simple closed curve axbcyda such that the arcs axb and cyd of J are of diameter greater than 1, then there exists a continuum N equivalent to K, containing axb and cyd and lying wholly within or on J, and such that the arcs axy and cyd correspond, under the transformation T_N, to the end elements of α_K, and every element of α_N is of diameter greater than 1.

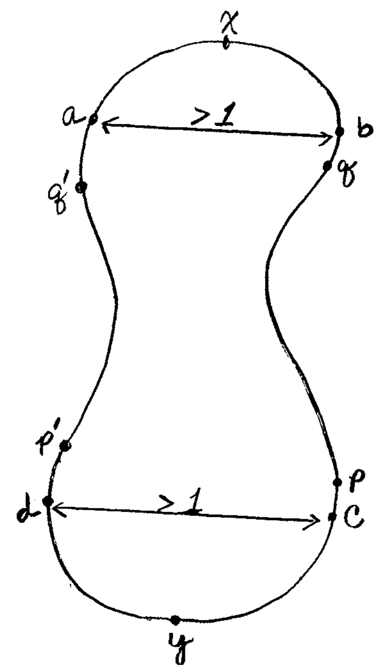

Figure 3.

Proof: Since the diameters of axb and cyd are each greater than 1, there exist points p and q on the arc cb and distinct from c and b and points p' and q' on arc ad but distinct from a and d such that the distances from p to p' and from q to q' are both greater than 1 (see Figure 3). Furthermore p precedes q in the order from c to b and p' precedes q' in order from d to a, and if w and z are points of da and cb respectively such that the distance from w to z is less than or equal to 1 then w is between p' and q' on da and z is between p and q on cb.

Clearly then, there exists a homeomorphic mapping h which maps the rectangle R onto J so that $h[(0,-1)] = c$, $h[(0,1)] = d$, $h[(1/3,1)] = p'$, $h[(2/3,1)] = q'$, $h[(1,1)] = a$, $h[(1,-1)] = b$, $h[(2/3,-1)] = q$, and $h[(1/3,-1)] = p$, and such that h maps the interior of R onto the interior of J. Then $h(K) = N$ is a continuum homeomorphic to K, containing axb and cyd and lying wholly within or on J, and such that the arcs axb and cyd correspond under the transformation to the end elements of α_K. In addition, every element of α_N is of diameter greater than 1. The homeomorphism h is the transformation T_N of the lemma.

Theorem: If k is any positive number, there exists an upper semi-continuous collection of continua filling the plane, all bounded, all of diameter greater than k, and no one separating the plane.

Proof: Let γ_1 and γ_2 be arcs of diameter greater than 1 which are subsets of M_1 and M_2 respectively. Let T denote a transformation of the plane into itself which translates every point vertically upward through a distance of 3 units. Let $T(K) = K_1$, $T(\gamma_1) = \beta_1$, and $T(\gamma_2) = \beta_2$. Let J_1 denote the simple closed curve composed of the arcs γ_1 and β_1 and the two vertical intervals whose endpoints are the endpoints of γ_1 and β_1, and let J_2 denote the simple closed curve formed in the same way using γ_2 and β_2. Let N_i $(i = 1, 2)$ denote a continuum equivalent to K such that the end elements of N_i

are γ_i and β_i, and such that each element of α_{N_i} is of diameter greater than 1, and all points of N_i except the points on γ_i and β_i are within J_i. Let H_1 denote the union of the continua $\overline{M}_1$ and $\overline{M}_2$ and all elements of α_K between $\overline{M}_1$ and $\overline{M}_2$. Let H_2 be the image of H_1 under the translation T. Let V_i $(i = 1, 2)$ be the union of the continuum N_i and the elements of α_K and α_{K_1} which contain γ_i and β_i. Let R denote the bounded complementary domain of the continuum $V_1 \cup V_2 \cup H_1 \cup H_2$.

Suppose a collection of continua $H_1, H_2, \ldots, H_n, V_1, V_2, \ldots, V_n$ has been defined having the following properties:

Property 1: For each k, $(2 < k \leq n)$, $H_k \subset R \cup V_1 \cup V_2$ and $V_k \subset R \cup H_1 \cup H_2$.

Property 2: For each $1 \leq i \leq n$, H_i is the union of the elements of an upper semi-continuous collection, F_{H_i}, such that (1) each element of F_{H_i} is of diameter greater than 1 and is either a simple continuous arc or a continuum equivalent to $\overline{M}$, (2) F_{H_i} is an arc with respect to its elements, and (3) the end elements of F_{H_i} are elements of F_{V_1} and F_{V_2}. For each $1 \leq i \leq n$, V_i is the union of the elements of an upper semi-continuous collection, F_{V_i}, such that (1) each element of F_{V_i} is of diameter greater than 1 and is either an arc or a continuum equivalent to $\overline{M}$, (2) F_{V_i} is an arc with respect to its elements, and (3) the end elements of F_{V_i} are elements of F_{H_1} and F_{H_2}.

Property 3: For each pair of values of i and j ($1 \leq i \leq n$, $1 \leq j \leq n$), $H_i \cap V_j$ is an element of F_{H_i} and F_{V_j}. If $i \neq j$ then $H_i \cap H_j = \emptyset$ and $V_i \cap V_j = \emptyset$.

Property 4: (a) For each bounded complementary domain D of the continuum $X_{k-1} = V_1 \cup V_2 \cup [\bigcup_{i=1}^{k-1} H_i]$, $k > 2$, there exist positive integers i_D and j_D ($i_D < j_D < k$) such that the boundary of D is a subset of $V_1 \cup V_2 \cup H_{i_D} \cup H_{j_D}$. If $1 \leq k \leq n$ and D_{k-1} and D are complementary domains of X_{k-1} such that $\overline{D}_{k-1}$ contains H_k, then $i_{D_{k-1}} + j_{D_{k-1}} \leq i_D + j_D$, and each point p of D_{k-1} at a distance greater than $1/(k - 1)$ from every point of $H_{i_{D_{k-1}}}$ is separated from this continuum in D_{k-1} by the continuum H_k.

(b) For each bounded complementary domain D of the continuum $Y_{k-1} = H_1 \cup H_2 \cup [\bigcup_{i=1}^{k-1} V_i]$, $k > 2$, there exist two positive integers i_D and j_D, ($i_D < j_D < k$), such that the boundary of D is a subset of the point set $H_1 \cup H_2 \cup V_{i_D} \cup V_{j_D}$. If $1 \leq k \leq n$ and D_{k-1} and D are complementary domains of Y_{k-1} such that $\overline{D}_{k-1}$ contains V_k, then $i_{D_{k-1}} + j_{D_{k-1}} \leq i_D + j_D$, and each point p of D_{k-1} at a distance greater than $1/(k - 1)$ from every point of $V_{i_{D_{k-1}}}$ is separated from this continuum in the domain D_{k-1} by the continuum V_k.

Property 5: For every $1 \leq i \leq n$ every component of $H_i - \bigcup_{k=1}^{n} V_k$, and every component of $V_i - \bigcup_{k=1}^{n} H_k$ is equivalent to $H_1 - (\overline{M}_1 \cup \overline{M}_2)$.

It is clear that the set H_1, H_2, V_1, V_2 satisfies these five properties. In order to show inductively the existence of an infinite collection $H_1, H_2, \ldots; V_1, V_2, \ldots$ having these properties, it is necessary to exhibit an H_{n+1} and a V_{n+1}.

Let D_n denote a bounded complementary domain of $X_n = V_1 \cup V_2 \cup [\bigcup_{i=1}^{n} H_i]$ such that if D is any other bounded complementary domain of X_n then $i_{D_n} + j_{D_n} \leq i_D + j_D$. Let $e = i_{D_n}$. By Theorem 3, Chapter II, there exists a simple closed curve J_n enclosing H_e but not containing or enclosing any point of any other continuum H_j, $1 \leq j \leq n$, and, in addition, such that every point within J_n is at a distance less than $1/n$ from some point of H_e. For every $1 \leq t \leq n$ the arc of elements F_{V_t} contains an element $\overline{M}_{tn}$ such that (1) $\overline{M}_{tn}$ is equivalent to $\overline{M}$ and (2) if Q_{tn} denotes the element common to F_{V_t} and F_{H_e}, then $\overline{M}_{tn}$ and all elements of F_{V_t} between $\overline{M}_{tn}$ and Q_{tn} belong to D_n and to the interior of J_n. The continuum $\overline{M}_{tn}$ contains an arc γ_{tn} of diameter greater than 1 which, under a continuous transformation of $\overline{M}_{tn}$ into $\overline{M}$, goes into a subset of M. Let S be the set of $n - 1$ components of $D_n - \bigcup_{k=1}^{n} V_k$. For each domain G of the set S there exist just two integers r_G and s_G, $r_G < s_G \leq n$, such that $\gamma_{r_G n}$ and $\gamma_{s_G n}$ are on the boundary of G. Clearly there exist in $\overline{G}$ and within J_n two mutually exclusive arcs which together with $\gamma_{r_G n}$ and $\gamma_{s_G n}$ form a simple closed curve lying, except for the arcs $\gamma_{r_G n}$ and $\gamma_{s_G n}$, wholly in G. Let N_G denote a continuum equivalent to K, such that every element of α_{N_G} except the end elements $\gamma_{r_G n}$ and $\gamma_{s_G n}$ is a point set of diameter greater than 1 lying wholly within J_G. Let H_{n+1} be the union of all the continua N_G for each domain G of the set S, together with $\bigcup_{i=1}^{n} \overline{M}_{in}$.

In an analogous manner V_{n+1} may be defined. Clearly, H_{n+1} and V_{n+1} are defined in such a way as to satisfy properties 1-5. Thus there exists an infinite collection of continua H_1, H_2, ...; V_1, V_2, ..., such that for every positive integer n, $n \geq 2$, the subcollection H_1, H_2, ..., H_n, V_1, V_2, ..., V_n has properties 1-5.

Property 4 implies that if p is any point of R not belonging to the continuum H_n, there exists an integer t such that H_t separates p from H_n in R. To see that this is true, let $p \in R$ such that $p \notin H_m$ and suppose $m > 2$. If the elements of F_{H_i}, i = 1, 2, m, and the elements of F_{V_j}, j = 1, 2, are considered as the nondegenerate elements in a decomposition of E_2, then in the decomposition space formed of these elements and all points of E_2 not contained in one of them, F_{H_i} and F_{V_j} are arcs with respect to their elements. No two of the arcs F_{H_i} have an element in common, and $F_{V_1} \cap F_{V_2} = \emptyset$ For every pair of values of i and j, $F_{H_i} \cap F_{V_j}$ is a single element. Let g_1, g_2, g_3, and g_4, denote respectively the elements which belong to $F_{V_1} \cap F_{H_1}$, $F_{V_2} \cap F_{H_1}$, $F_{V_2} \cap F_{H_2}$, and $F_{V_1} \cap F_{H_2}$. Let g_5 and g_6 denote the elements which belong to $F_{H_m} \cap F_{V_1}$ and $F_{H_m} \cap F_{V_2}$ respectively.

Then $g_6g_3g_4g_5$, $g_5g_1g_2g_6$, and g_5g_6 are arcs with only their terminal elements in common and, except for its terminal elements, g_5g_6 is contained in the interior of the simple closed curve J formed by $g_6g_3g_4g_5 \cup g_5g_1g_2g_6$. Theorem 35, Chapter II, implies that the interior of J, which is the region R, is the union of disjoint sets D_1, D_2, and $g_5g_6 - \{g_5, g_6\}$, where D_1 denotes the interior of the simple closed curve $g_5g_1g_2g_6 \cup g_5g_6$ and D_2 is the interior of the simple closed curve $g_6g_3g_4g_5 \cup g_5g_6$. Hence if the point p of R does not belong to H_m then either $p \in D_1$ or $p \in D_2$.

Without loss of generality suppose $p \in D_1$. Let k_o be an integer such that $m < k_o$. Let $X_{k_o-1} = V_1 \cup V_2 \cup (\bigcup_{i=1}^{k_o-1} H_i)$ and suppose D_m is the complementary domain of X_{k_o-1} such that $D_m \subset D_1$ and the boundary of D_m is a subset of $H_m \cup H_j \cup V_1 \cup V_2$. If $p \notin \overline{D}_m$ then p is separated from H_m in R by the continuum H_j. Suppose then that $p \in \overline{D}_m$. This implies that either $p \in D_m$ or $p \in H_j$.

If $p \in H_j$ then by the way in which the elements H_i are defined in property 4 (a), there exists an integer k, $k > m$, $k > j$, such that if D is any complementary domain of X_{k-1} then $m + j < i_D + j_D$ and $H_k \subset \overline{D}_m$. Then by definition of H_k, H_k separates p from H_m in R.

If $p \in D_m$ then for some integer k, $1/(k - 1) < d$, where $d = g.l.b.\{d_x | x \in H_m, d_x$ denotes the distance from x to p$\}$, the complementary domain D_{k-1} of X_{k-1} which is a subset of D_m and whose boundary is contained in $H_m \cup H_s \cup V_1 \cup V_2$, is determined by the unique integers m and s where $m < s$ and, if D is any other complementary domain of X_{k-1} then $m + s < i_D + j_D$. Then by property 4 (a),

$H_k \subset \overline{D}_{k-1}$ and every point of D_{k-1} whose distance from every point of H_m is greater than $1/(k - 1)$ is separated from H_m by H_k. Therefore, if $p \in \overline{D}_{k-1}$ then it is separated from H_m by H_k and if $p \notin \overline{D}_{k-1}$ then it is separated from H_m by both H_k and H_s.

Thus, in the preceding paragraphs two things have been shown: (1) if p is any point of R and $p \notin H_m$, $m > 2$, then p is separated from H_m in R by H_t for some integer t, and (2) if H_j and H_k are distinct continua then for some integer t, H_t separates H_j from H_k in R. In a similar way it can be shown that if $m = 1$ or 2 and $p \notin H_m$ then p is separated from H_m in R by H_t, for some integer t. In an analogous manner these same properties can be shown to hold for the continua V_i. In particular it can be shown that if $p \in R$ and $p \notin V_n$ then there exists an integer t such that V_t separates p from V_n in R.

Let p denote any point whatsoever of R. Consider the collection $A_n = \{S | S$ is a continuum which is a finite union of continua of the form H_i or V_i, $p \notin S$, $S \subset \bigcup_{i=1}^{n} (H_i \cup V_i)$, $n \geq 2\}$. For each $n \geq 2$, there exists a continuum $S_n \in A_n$ such that if $S \in A_n$ then $S \subset S_n$. Let $A = \{S_n | n = 2, 3, \ldots, S_n \in A_n\}$. For each S_n belonging to A let G_{np} denote the complementary domain of S_n which contains p.

For any integer $n_o \geq 2$, the complementary domain $G_{n_o p}$ is bounded by $H_i \cup H_j \cup V_k \cup V_m$, for integers i, j, k, m, each of which is less than or equal to n_o. The truth of this can be shown in the following way. Let $Q = \{H_i | H_i \subset S_{n_o}$ and H_i separates p from H_1 in R$\}$. If $Q \neq \emptyset$ let H_q denote the element of Q such that if $H_i \in Q$, $i \neq q$, then H_q separates H_i from p. If $Q = \emptyset$ then let $H_q = H_1$. Let $T = \{H_i | H_i \subset S_{n_o}$ and H_i separates p from H_2 in R$\}$. If $T \neq \emptyset$ let H_t denote the element of T such that if $H_i \in T$, $i \neq t$, then H_t separates H_i from p. Otherwise, if $T = \emptyset$, let $H_t = H_2$. Let $W = \{V_i | V_i \subset S_{n_o}$ and V_i separates p from V_1 in R$\}$. If $W \neq \emptyset$ let V_w denote the element of W such that if $V_i \in W$, $i \neq w$, then V_w separates V_i from p. If $W = \emptyset$, let $V_w = V_1$. Let $X = \{V_i | V_i \subset S_{n_o}$ and V_i separates p from V_2 in R$\}$. If $X \neq \emptyset$, let V_x denote the element of X such that if $V_i \in X$, $i \neq x$, V_x separates V_i from p. Otherwise, let $V_x = V_2$. For the continua H_q, H_t, V_w, and V_x consider the collections F_{H_q}, F_{H_t}, F_{V_w}, and F_{V_x}. Each of these collections is an arc with respect to its elements and, if $g_1 = F_{H_q} \cap F_{V_w}$, $g_2 = F_{H_q} \cap F_{V_x}$, $g_3 = F_{H_t} \cap F_{V_x}$, and $g_4 = F_{H_t} \cap F_{V_w}$, then $g_1 g_2 g_3 g_4 g_1$ forms a simple closed curve J with respect to its elements. No point of S_{n_o} is contained in the interior of J for if there were then it would contradict the conditions on H_q, H_t, V_w, and V_x.. In addition, q, t, w, and x are less than or equal to n_o for otherwise $H_q \cup H_t \cup V_w \cup V_x$ would not be contained in S_{n_o}. Hence the interior of J is a complementary domain of S_{n_o}. Clearly, p is contained in the interior of J since any assumption that it does not will contradict one of the conditions placed upon H_q, H_t, V_w, or V_x. Therefore $G_{n_o p}$ is the interior of J.

Then applying property 4, there exist integers q', t', w', and x' such that $H_{q'}$ separates p from H_q, $H_{t'}$ separates p from H_t, $V_{w'}$

separates p from V_w, and $V_{x'}$ separates p from V_x. Let n_1 be the maximum of the integers q', t', w', and x'. Then the complementary domain G_{n_1p} of S_{n_1} is bounded by a subset of $H_{q'} \cup H_{t'} \cup V_{w'} \cup V_{x'}$ and $\overline{G}_{n_1p} \subset G_{n_op}$. Furthermore if n is an integer such that $n_o < n < n_1$ then $G_{n_1p} \subset G_{np} \subset G_{n_op}$. Hence there exists a subsequence G_1, G_2, ... of the set of domains of the form G_{np} such that $\overline{G}_{k+1} \subset G_k$.

Let $T_p = \bigcap_{k=1}^{\infty} G_k = \bigcap_{n=2}^{\infty} G_{np}$. Since T_p is the intersection of a countable collection of compact sets with the above properties, T_p is a continuum. For every integer k, the boundary of G_k is a subset of $H_i \cup H_j \cup V_m \cup V_n$ for positive integers i, j, m, and n, and the boundary contains at least one element of $F_{H_i} \cup F_{H_j} \cup F_{V_m} \cup F_{V_n}$. Hence its boundary contains a subset of diameter greater than 1. Therefore the domain G_k is of diameter greater than 1. Thus T_p is of diameter greater than or equal to 1.

Let $X = \{T_p | p \in R\}$. In order for the collection X to be the desired decomposition of the domain R, it is necessary to show that the continua are disjoint and that the collection is upper semi-continuous.

To show that the continua are disjoint, let p and q be distinct points of R and suppose $T_p \cap T_q \neq \emptyset$ and $T_p \neq T_q$. Then either T_p contains a point x such that $x \notin T_q$ or there exists a point $y \in T_q$ such that $y \notin T_p$. If $x \in T_p$ and $x \notin T_q$ then there exists a domain G_{nq} which contains T_q but such that $x \notin \overline{G}_{nq}$. Then the complementary domain G_{nx} which contains x is distinct from G_{nq} and, by definition,

$T_x \subset G_{nx}$. Thus either $x \notin T_p$ or $T_p \cap T_q = \emptyset$. In either case a contradiction of the hypothesis results. Similarly the assumption that $y \in T_q$ and $y \notin T_p$ yields a contradiction. Therefore if p and q are distinct points of R, either $T_p = T_q$ or $T_p \cap T_q = \emptyset$.

Let $h \in X$ and let M be any region containing h. By definition $h = \bigcap_{k=1}^{\infty} G_n$ where the domains G_n have the properties defined above. Then there exists a domain G_k of this collection such that $\overline{G}_k \subset M$, for, if not, one could exhibit a sequence of points belonging to $E_2 - M$ and having a limit point in M. If p is any point of G_k, then by the way in which the elements of X are defined, the continuum of X which contains p is a subset of G_k. Therefore, if $k \in X$, $k \neq h$, and $k \cap D \neq \emptyset$, then $k \subset D \subset M$. Thus the collection X is upper semi-continuous.

Suppose there exists an element $h \in X$ such that $R - h = R_1 \cup R_2$ where R_1 and R_2 are mutually separated. Then this implies that each of the domains which determine h separates R. Hence, according to the way these domains are defined, every domain containing h either intersects every V_i, $i > 2$, or every one intersects every H_i, $i > 2$. But this is contrary again to the way in which these sets are defined since no point can belong to more than one H_i or to more than one V_i and since in each successive step of selecting the complementary domain which contains h, certain of the H_i and V_i were excluded from intersecting that domain. Thus no continuum belonging to X separates R.

Therefore X is an upper semi-continuous collection of continua filling up the domain R, each element of X is of diameter greater than or equal to 1, and no element of X separates R. The domain R is bounded and its boundary is connected, hence R is homeomorphic with the interior of the unit circle [22, p. 161]. The interior of the unit circle is homeomorphic to E_2, therefore R is homeomorphic to E_2. In particular, if k is any positive number there exists a continuous one-to-one mapping f between the points of R and E_2 such that if x and y are distinct points of R, the distance between f(x) and f(y) is greater than k times the distance from x to y in R. Obviously the image under f of a continuum in R will be a continuum in E_2 and the collection of continua corresponding to X under the mapping will be an upper semi-continuous decomposition of E_2 into continua no one of which separates the plane. Furthermore every element of the decomposition will be of diameter greater than k.

It is of interest here to note that in a later paper (Duke Mathematical Journal, Vol. II, 1936, pp. 10-17) Roberts proved that there does not exist an upper semi-continuous collection G of arcs filling the plane. Prior to that publication some believed that the previous example implied the existence of a decomposition of E_2 into arcs.

CHAPTER IV

RESULTS OBTAINED FOR E_3

It was in 1925 that Moore proved that an upper semi-continuous decomposition of E_2 into continua which did not separate E_2 formed a decomposition space homeomorphic with E_2. During the next ten years several people were doing significant work on the theory of upper semi-continuous decompositions but none of it pertained directly to E_3. In an address before the American Mathematical Society in 1935, G. T. Whyburn suggested that there was a need to study what conditions on an upper semi-continuous decomposition of E_3 were sufficient for the associated decomposition space to be topologically E_3.

It is known that Moore's theorem cannot be extended to E_3 without some additional restrictions. The investigations are continuing and many question remain unanswered. It is the aim here to point out what has been accomplished and to list some of the questions which have yet to be answered.

A simple example will show that not every decomposition of E_3 will yield E_3. Consider the decomposition whose only nondegenerate element is a circle. Certainly this is a decomposition of E_3 into continua which do not separate E_3. It is known, however, that not only is E_3 simply connected but it will remain so if a single point is removed. The decomposition space which has been formed is also simply connected but it fails to remain so when the point corresponding to the circle is removed [5]. It can be seen that this is the case since an open disk containing the circle is disconnected when the circle is removed.

To show that a decomposition space is not topologically E_3 it is best to find some simple property that is known for E_3 which the space lacks. The alternate technique is to prove that there does not exist a homeomorphism between the two spaces. This second method was used by Bing [3] to show that the space known as the "dogbone space" was topologically different from E_3. It seems useful to give a brief description of the dogbone space here since it has been a favorite counterexample for several theories on decompositions of E_3.

Definition: An arc J in E_3 is tame if it has the following properties at each point $p \in J$. (1) For every $\epsilon > 0$, there exists a 2-sphere K of diameter less than ϵ such that p lies in the bounded complementary domain of K and $J \cap K$ is a set containing exactly one point when p is an endpoint and exactly two points when p is not an endpoint. (2) An open subset of J containing p lies on a disk in E_3.

An arc which is not tame is said to be wild. Figure 4 is an illustration of an arc in E_3 which is not tame. It fails to satisfy the first property at the points p and q.

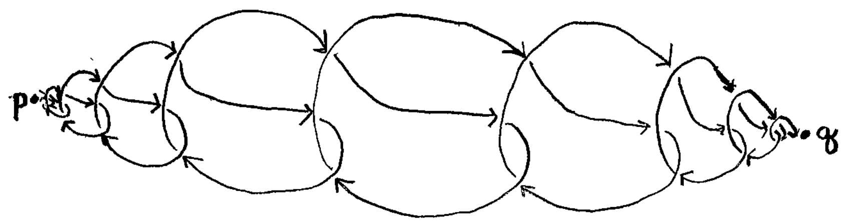

Figure 4.

The dogbone space is a decomposition of E_3 into points and tame arcs. The way in which the tame arcs are formed is probably best described by use of a picture. Let T be a double solid torus, as shown in Figure 5, and in the interior of T place four double tori T_1, T_2, T_3, T_4 so that T_i is linked with T_j through their corresponding loops as indicated in the figure, and so that if $i \neq j$, $T_i \cap T_j = \emptyset$. In each T_i are placed four double tori T_{i1}, T_{i2}, T_{i3}, T_{i4} in the same way and the process is continued in this fashion. Then each component of $T \cap (\bigcup_{i=1}^{4} T_i) \cap (\bigcup_{i=1}^{4} \bigcup_{j=1}^{4} T_{ij}) \cap \ldots$ is a tame arc and Bing [3] has indicated that there are uncountably many of these.

Once it was observed that the theorem of Moore's did not generalize to E_3, inquiries were begun to find what restrictions were necessary in order that an upper semi-continuous decomposition of E_3 into continua which do not separate E_3 will form a decomposition space topologically equivalent to E_3. Several theorems have been

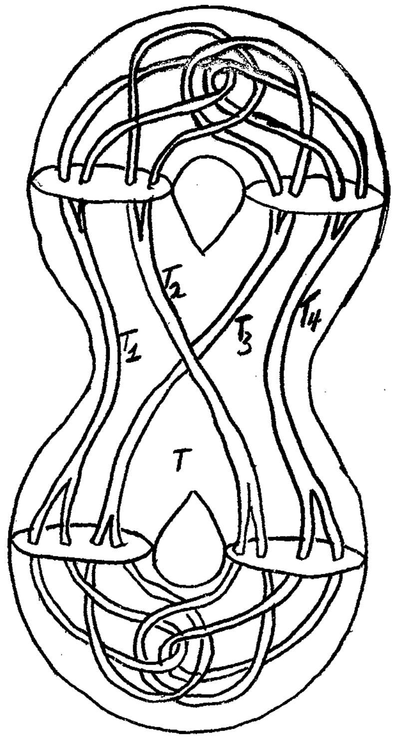

Figure 5.

proved in this area and many unresolved questions remain to be explored. Some of the results which have been obtained will be stated and discussed and some of the unsolved problems will be noted.

Definition: A compact continuum g is starlike if it contains a point p such that for every line L through p, $L \cap g$ is a line segment. Then the set g is said to be starlike with respect to p.

In order for a continuum to be starlike in E_3 it must be three dimensional and contain an interior point. The drawing in Figure 6 (a) represents an ordinary cylinder plus its interior. This set of points is starlike since for any point p in the interior, any line through p will intersect the set in a line segment. In Figure 6 (b), the solid cube is starlike with respect to any point except

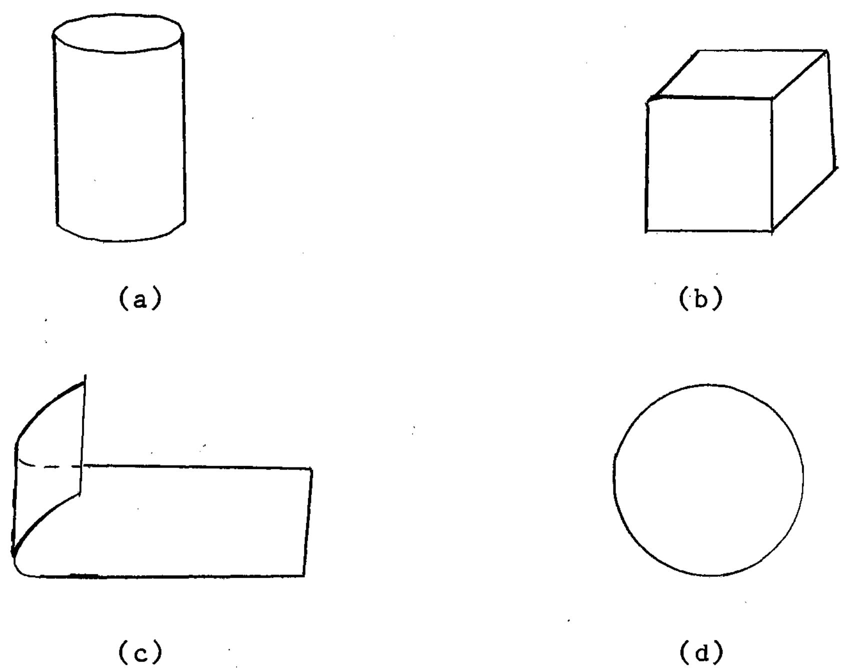

Figure 6.

those points which lie on an edge. Figures 6 (c) and 6 (d) represent surfaces in E_3 which are not starlike. The set of starlike continua in E_3 includes as a subset the set of all convex bodies in E_3, i.e., the set of convex sets which contain an interior point. Bing [8] has proved the following theorem relative to this.

Theorem 1: Suppose G is an upper semi-continuous decomposition of E_3 such that G has only a countable number of nondegenerate elements and each is starlike. Then the decomposition space G is topologically equivalent to E_3.

A simple example of a decomposition such as this can easily be defined. It is known that the set of points (x,y,z) in E_3, where x, y, and z are integers, is a countable collection. Suppose S_{xyz} is defined to be the set of points of E_3 whose distance from (x,y,z) is less than or equal to 1/4. Then let $K = \{S_{xyz} | (x,y,z) \in E_3,$ x, y, z are integers$\}$ be the collection of nondegenerate elements in a decomposition of E_3. Every element of K is a convex body and hence the decomposition space so formed is topologically E_3.

In a similar theorem, W. R. Smythe, Jr. [21], has proved that if G is an upper semi-continuous decomposition of E_n whose nondegenerate elements are compact and strictly convex then the decomposition space is homeomorphic to E_n. A set C in E_n is strictly convex if every segment joining two points of C is contained, except possibly for its endpoints, in the interior of the set. As far as E_3 is concerned, Smythe's theorem is a special case of Theorem 1. His theorem is more general in that it can be applied to E_n for $n \geq 3$.

From Theorem 1, one is led to inquire whether the theorem would also hold if the nondegenerate elements were merely convex continua. This question has been partially answered in a theorem proved by Louis F. McAuley [11]. He has considered the case where G is an upper semi-continuous collection of straight line intervals and points filling up E_3 and has proved the following theorem.

Theorem 2: Suppose that G is an upper semi-continuous collection of straight line intervals and points such that each member of the collection H of all nondegenerate elements of G is parallel to at least one of a countable number of fixed lines L_1, L_2, L_3, ... Then the decomposition space is topologically E_3.

This theorem can be illustrated with an example such as the following. Let L_1, L_2, ..., L_{180} be a collection of lines in the yz-plane such that L_n forms an angle of n degrees with the positive y-axis and contains the point (0,0,0). Let C be a cylinder defined by the equation $y^2 + z^2 = 1$. Let L'_n, $1 \leq n \leq 180$, be a collection of lines on the cylinder C such that L'_n is perpendicular to L_n at its point of intersection with C for which the z coordinate is greater than or equal to 0 and $y \neq 1$. Each pair L_n and L'_n of intersecting lines determines a plane p_n. For each n, let $S_n = \{s_x | s_x$ is a segment of unit length which does not intersect the interior of C, $s_x \subset P_n$, s_x is perpendicular to L'_n at the point (x, sin n, cos n), $x = 0, 1, 1/2, \ldots, 1/2^n, \ldots\}$ (see Figure 7). Then $H = \bigcup_{n=1}^{180} S_n$ is the set of nondegenerate elements of a decomposition of E_3. Each

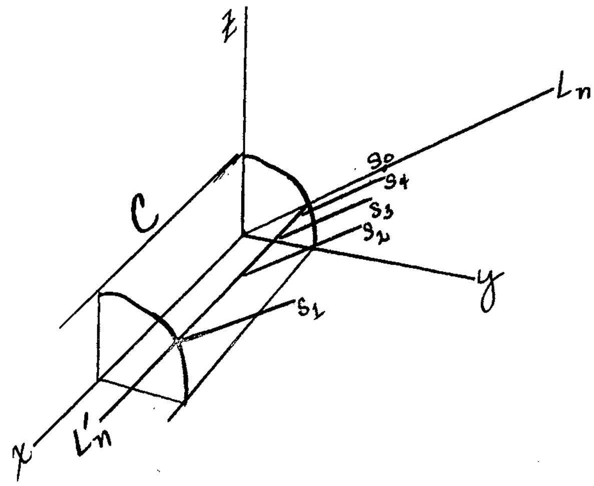

Figure 7.

element of H is parallel to one of the lines L_n and no two of the elements have a point in common. For any element of S_i there exists a domain containing it which does not intersect an element of S_j for $j \neq i$. Furthermore, for any element of S_i except s_o, there exists a domain containing it which intersects no other element of S_i. For the element s_o, every domain containing it contains infinitely many of the elements of S_i. Let D be any such domain and let $A = \{s_x | s_x \cap D \neq \emptyset, s_x \not\subset D\}$. The set A contains at most a finite number of elements of H. There exists a domain D' containing s_o which does not intersect an element of A. Then $D' \cap D$ is a domain containing s_o, $(D' \cap D) \subset D$, and if $s_x \cap (D' \cap D) \neq \emptyset$, then $s_x \subset D$. Hence the collection of elements of H is upper semi-continuous and according to Theorem 2, the resulting decomposition space is topologically E_3.

Four years prior to the publication of McAuley's theorem Bing [8] published the proof of a theorem which could be treated as a corollary to Theorem 2. In it the nondegenerate elements of the decomposition were all vertical intervals.

E. Dyer and M.-E. Hamstrom [9, p. 116] have proved a theorem having to do with a decomposition whose nondegenerate elements are compact continua in E_3. These are not restricted to being convex but as a special case the theorem may also be applied to convex continua and will partially answer the question regarding convex continua. Their theorem is the following one.

Theorem 3: If G is a decomposition of E_3 into points and compact continua such that each continuum lies in a horizontal plane and does not separate that plane, then the decomposition space is topologically equivalent to E_3.

This theorem makes it possible to consider decompositions in which the nondegenerate elements are closed disks, curves, arcs, and other continua so long as they each lie in a horizontal plane. It has contributed toward varying the types of decomposition elements that can be used, but the restriction of each element to a horizontal plane remains a hindrance.

Two additional theorems of Bing's [8] cover some of the decompositions one might wish to consider where the elements are not confined to a horizontal plane. They each impose another restriction which is equally limiting, however. That is, to satisfy these theorems the collection of nondegenerate elements must be countable.

The theorems are these.

Theorem 4: Let G be an upper semi-continuous decomposition of E_3 into continua with the following properties: (a) the complement of each element of G is topologically equivalent to the complement of a point, (b) G has only a countable number of nondegenerate elements, and (c) the union of the nondegenerate elements is the intersection of a countable collection of open sets. Then the decomposition space G is topologically equivalent to E_3.

Theorem 5: Suppose G is an upper semi-continuous decomposition of E_3 such that G has only a countable number of nondegenerate elements and each is a tame arc. Then the decomposition space is topologically equivalent to E_3.

The example following Theorem 1 can also be used as an example to illustrate Theorem 4. The dogbone space can be used to show the necessity for the restriction to countable collections in Theorem 5. The set of nondegenerate elements in the dogbone space is uncountable and the decomposition space is not homeomorphic to E_3. The following example does satisfy the conditions of Theorem 5.

Let $S = \{s | s \text{ is rational}, 1 \leq s \leq 2\}$. For each element s in S let C_s be a right circular cylinder having its base on the xy-plane, the center of its base at (0,0,0), and the radius of its base s, and let the height of C_s be two. Then for every $s \in S$, let H_s be a circular helix lying on C_s and described by the parametric equations $x = s(\cot t)$, $y = s(\sin t)$, $z = \frac{1}{2\pi} t$. Then if the set $H = \{H_s | s \in S\}$ is the set of nondegenerate elements of a decomposition of E_3, the conditions of Theorem 5 are met and therefore the space is homeomorphic to E_3.

Another theorem regarding decompositions with only a countable number of nondegenerate elements has been proved by Steve Armentrout [2]. His theorem places conditions on the decomposition space rather than on the decomposition elements which cause the space to be homeomorphic to E_3.

Theorem 6: Suppose G is an upper semi-continuous decomposition of E_3 into compact sets and that G has only countably many nondegenerate elements. If the decomposition space S associated with G is a separable metric space such that each point of S has an open neighborhood V in S such that V is homeomorphic to E_3, then S is homeomorphic to E_3.

Known Counterexamples

In addition to the positive results that have been mentioned, some negative results have also been obtained. That is, for some of the conjectures on upper semi-continuous decompositions, counterexamples have been found. The first of these is, of course, the dogbone space. It disproved the theory that a decomposition for which the complement of any element was equivalent to the complement of a point in E_3 would form a space homeomorphic to E_3.

Bing [7] has described another decomposition of E_3 which shows that having only a countable number of nondegenerate elements is not sufficient for the space to be homeomorphic to E_3. Each nondegenerate element in this example is an indecomposable continuum formed by the intersection of a countable collection of solid tori in E_3. The nondegenerate elements are formed in the following way. Let T_0 be a solid round torus. In the interior of T_0 are placed two solid tori T_{00} and T_{01}, linked as shown in Figure 8. These are constructed so that the diameter of T_{01} is less than half that of T_0. The center axis of T_{00} lies in the same plane as the axis of T_0, while the axis of T_{01} is in a plane perpendicular to this one. In the same manner as T_{01} and T_{00} were constructed in T_0, construct T_{0i0} and T_{0i1} in T_{0i}, $i = 1, 2$, and continue in this way.

Let

$$Y = T_0 \cap \left(\bigcup_{i=0,1} T_{0i} \right) \cap \left(\bigcup_{i=0,1} \bigcup_{j=0,1} T_{0ij} \right) \cap \left(\bigcup_{i=0,1} \bigcup_{j=0,1} \bigcup_{k=0,1} T_{0ijk} \right) \cap \cdots$$

Then the components of Y together with the points of $E_3 - Y$ are the elements of the decomposition G. Using the ternary representation of the numbers of the Cantor set, $.0a_1a_2\ldots$, where $a_i = 0$ or 1, can be used to represent the component $T_0 \cap T_{0a_1} \cap T_{0a_2} \cap \ldots$. If the ternary representation contains infinitely many 1's, then for some integer k, if $j > k$, $a_j = 1$, and the tori in this sequence are defined in such a way that their diameters form a decreasing sequence of numbers approaching 0. Hence their intersection is a point. Therefore there exists only a countable number of nondegenerate elements in G.

Bing proved that the space formed by the elements of G was different from E_3 by showing that there is an element in the decomposition space which is not contained in a small neighborhood bounded

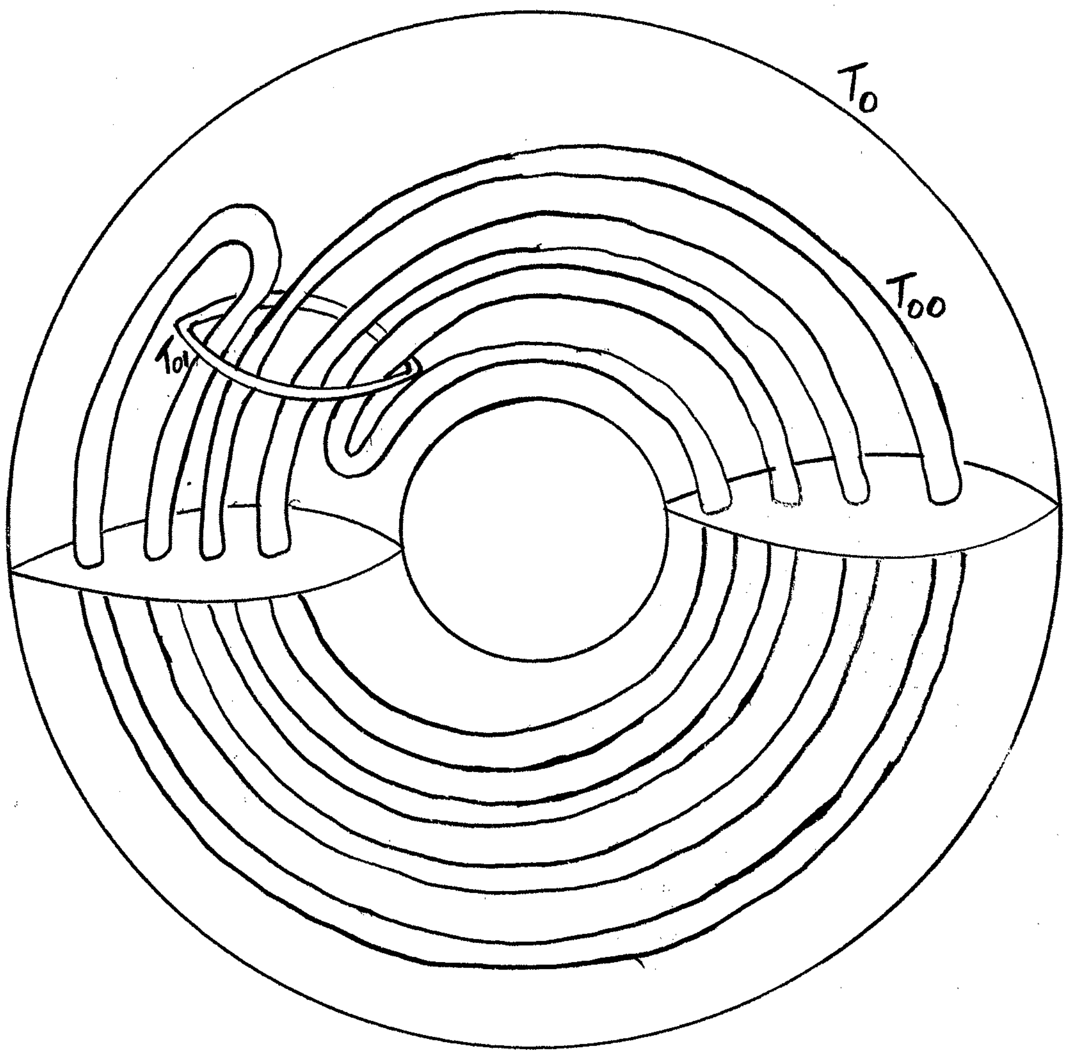

Figure 8.

by a 2-sphere.

Bing [5] and McAuley [10] have each published examples of decompositions of E_3 whose nondegenerate elements are straight line intervals. In each case it has been conjectured that the resulting space is different from E_3. The examples are similar in that each consists of an uncountable collection of line segments formed by the intersection of a collection of tubular neighborhoods and contained between a pair of horizontal planes. Whether or not a decomposition of E_3 into points and straight line segments must necessarily yield a space equivalent to E_3 seems to remain an open question.

Some Unanswered Questions

Many questions have been raised in regard to decompositions of E_3 for which no published answer has been found. Some of these will be noted here.

Question 1: Does there exist an upper semi-continuous decomposition of E_3 into, at most, countably many disks and one-point sets such that the decomposition space is not homeomorphic to E_3?

Question 2: Is it true that if G is an upper semi-continuous decomposition of E_3 into straight-line intervals and one-point sets, then the decomposition space is equivalent to E_3?

If the conjectures on the examples of Bing and McAuley mentioned above are correct then a negative answer can be given to Question 2. In both examples there are uncountably many nondegenerate elements. It may be that in order to have an affirmative answer to this question the set of nondegenerate elements will have to be countable.

J. H. Roberts showed that there was no upper semi-continuous decomposition of E_2 into arcs. In connection with this one might ask the following question.

Question 3: Is there an upper semi-continuous decomposition of E_3 into arcs?

Questions have also been raised in related areas. Some work has been done on embedding decompositions of E_3 in E_4 or E_5, and on the cross product of certain of the decomposition spaces with E_1. It would appear than an expository paper on the work that has been done along this line would be of value.

The notion of equivalent decompositions has also been studied. The equivalence used in this area is more restrictive than topological equivalence. A definition of it and a survey of the work that has been done in the area can be found in "Equivalent Decompositions of E_3" by Steve Armentrout, Lloyd Lininger, and Donald Myer, Annals of Mathematics Studies, No. 60, Princeton University Press, 1966, pp. 27-31.

Conclusion

The study of decompositions and decomposition spaces is valuable in furthering the study of topology in general. Many properties of a space can be more easily revealed by using a decomposition of the space. Once two spaces are known to be topologically equivalent then topological properties which hold in one space will also hold in the other. Perhaps when more is known about the decomposition of Euclidean spaces of dimension higher than 3, more properties of these spaces will be revealed.

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VITA

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