TIME DOMAIN COMPENSATION
OF NONLINEAR SYSTEMS

By

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CHAPTER I

INTRODUCTION

Since World War II the study of control systems and dynamical systems in general has received increasing attention. More recently, the design analysis of complex missile and satellite control systems has provided a great impetus to the advancement of studies in these areas. However, the increasing demands for more complex and more reliable systems have led to a realization that some of the frequency domain methods of analysis are inadequate in many cases, especially for nonlinear systems.

The need for more comprehensive methods of system design and analysis that are applicable to linear and nonlinear systems, time-invariant, time-varying and multivariable systems has become evident. More specifically, a particular need exists in the area of system compensation in order to achieve a more optimum response characteristic. For instance, a system designer may have in mind a tentative system design which he wishes to modify or adjust in order to achieve a specified response. However, the questions of just what modifications to make or whether there exist suitable modifications are difficult to answer. Alternatively, one may wish to compensate an existing system in order to improve a certain response characteristic. Again the answers to the questions of whether proper compensation is possible and what form it may take are elusive.

The study reported herein was undertaken to determine a method of system design or compensation that was applicable to nonlinear or time-varying multivariable systems and would aid in achieving a "best" approximation to a desired response characteristic. That is, an analytical technique was sought that would first determine whether an existing system or a proposed model could be modified sufficiently to produce the desired response. If an acceptable response were feasible, then it was desired to know what modifications or additional analytic terms were necessary to achieve this response. Finally, but no less important, it was desired that the designer be able to deduce whether the necessary modifications or additional terms could be realized in the physical system. One is logically led to a study of modern optimal control theories in an investigation of this nature.

A survey of publications pertinent to the topic of this study revealed that the analysis techniques currently available fail to provide the desired capabilities. In most instances, conventional optimal control theories lead to open loop control or, at best, an optimum parameter closed loop system. Although these methods lead to the optimum values of the adjustable parameters, the designer gains no indication of the effect of additional modifications or what other modifications might be desired. Neither have previous works indicated very well how to find fixed parameter closed loop approximations to the optimum open loop control signal. Some techniques do lead to optimum closed loop systems with time-varying feedback gains for linear systems with quadratic performance indices. However, in all but the most sophisticated systems, the physical implementation of the proper time-varying gains can be a most difficult task.

In view of the shortcomings and limitations imposed by current compensation methods, the present study was initiated. It was proposed to develop a technique that would be applicable to nonlinear systems as well as time-invariant and time-varying linear systems and which would determine the proper system modifications and compensation to insure the "best" fit to a preselected system response.

This report documents the results of the ensuing investigation, the development of the technique, and demonstrates its application to several nonlinear systems. In particular, Chapter II presents a brief summary of the statement of the problem and the results of previous investigations. Chapter III states the specific objectives of this study, while the following chapter reviews some basic principles and presents, in detail, the development of the compensation technique. Typical nonlinear hydraulic and electrical systems are utilized to demonstrate the utility of the method in Chapter V. A summary of the application and limitations of the compensation procedure and recommendations for future investigations are given in the final chapter.

CHAPTER II

PROBLEM STATEMENT AND PREVIOUS INVESTIGATIONS

The terms "system" and "control system" are very general and can be used to describe a wide variety of physical, chemical, socio-economical and biological processes. Thus the first section of this chapter is devoted to defining the types of system to which this study is most applicable and some of the primary objectives in system compensation. The second section reviews the works of several investigators who have studied the problem of system modification in order to meet response specifications.

Statement of the Problem

The study reported herein concerns certain aspects of compensating nonlinear mechanical, hydraulic, pneumatic or electrical systems in order to achieve desired response characteristics. "Compensation" is generally taken in this thesis to mean the modification or adjustment of an existing system as opposed to the initial "design" or synthesis of a system. The problem is generally described as determining the proper adjustments to system parameters and selecting the appropriate additional feedback loops and gains.

The general dynamical system model has the form

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

where $\underline{x}$ is an n-vector of state variables, $\underline{u}$ is an m-vector of deterministic inputs, $\underline{f}$ is an n-vector function of $\underline{x}$ and $\underline{u}$, t is the independent variable time and $\dot{\underline{x}}$ is an n-vector of state time derivatives. The inputs $\underline{u}$ are assumed to be known and fixed and the vector function $\underline{f}$ is assumed to be continuous and to have continuous partials with respect to $\underline{x}$ and t.

In order to illustrate some of the problems encountered in system compensation, consider the following. Assume that the following nonlinear differential equation describes the dynamics of some physical system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c_1 x_2 - c_2 x_1 - c_3 x_1 x_2 + u(t) \end{aligned} \tag{2-1}$$

It is desired to alter the system represented by Equation (2-1) in order to obtain a "better" system. The problem, simply stated, is to adjust the coefficients (c_1, c_2, c_3) and/or add additional terms to the equation so that the equation solution will perform in a specified desirable manner and thus, the system performance will respond with a like behavior.

Assume that the system model given in Equation (2-1) has initial coefficient values c_1, c_2 and c_3 such that the solution to the equation is as shown in Figure 1, Curve A. This highly oscillatory response may be undesirable while a response as shown by Curve B would be acceptable. Thus it is desired to add terms as needed to the equation and adjust the gains c_1, c_2 and c_3 and those associated with the new terms so that the response approximates Curve B as closely as possible.

However, the gains or parameter values must be limited to insure physical realizability. Hence any technique developed that would aid in the derivation of the proper parameter values and necessary additional terms must be capable of considering physical realizability requirements also.

In addition to shaping the state response to some desired value while considering realizability requirements, it may also be desirable in some instances to limit extreme values of the state variables. While the above discussion was concerned with shaping the response $x_1(t)$ to some desired function, no control or restriction was placed on $x_2(t)$. Thus it is quite possible that in arriving at the response shown by Curve B in Figure 1, the other state variable is caused to take some undesirable form as shown in Figure 2. Although the exact shape or form of the response of x_2 is not critical, it may be preferable to limit the maximum amplitude of x_2 to $x_2(max)$ as shown. Hence, another problem encountered in system compensation is to limit the maxima or minima of some of the state variables while shaping the response of others.

Previous Investigations

A significant amount of investigation has been accomplished in recent years in the general area of the selection of system parameters or system modifications in order to meet solution specifications. Of consequence though, only a few of these works are generally applicable to nonlinear systems. A brief review of these articles will aid in understanding the technique proposed herein. The following articles are divided into three general categories--(1) parameter optimization only for linear systems, (2) parameter optimization for nonlinear

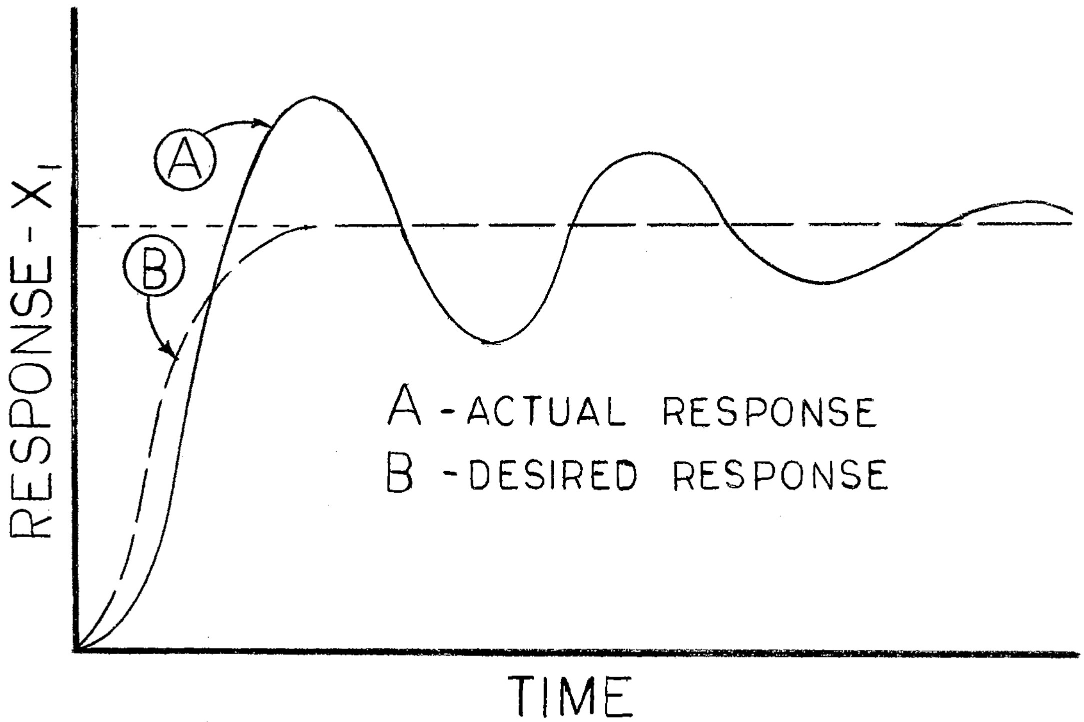

Figure 1. Nonlinear System Response

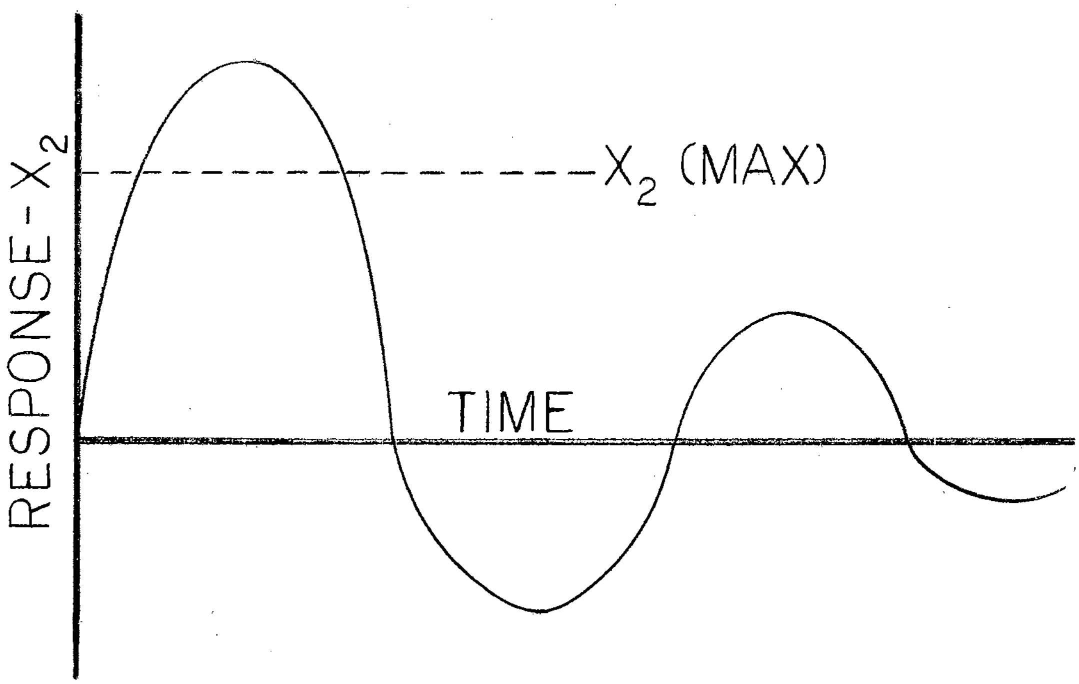

Figure 2. Plot of x_2 Corresponding to Curve B, Figure 1

systems, and (3) parameter optimization and system design of linear and nonlinear systems.

Parameter Optimization of Linear Systems

Perhaps most general in application to linear systems is a method developed by C.M. Bacon in references (1, 2)*. Bacon presents a technique whereby the desired response of a linear dynamical system model is specified as a solution to a linear matrix differential equation. An error criterion is formulated based on the algebraic properties of state-space system models and is then minimized, driving the system parameters to values which minimize weighted differences between coefficients in the desired and optimum solutions.

Bacon discusses systems described in general by

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \\ \underline{y} &= \underline{C}\,\underline{x} + \underline{D}\,\underline{u} \end{aligned} \qquad (2\text{-}2)$$

where $\underline{x}$ = state vector, $\underline{u}$ = input vector, $\underline{y}$ = output vectors, and $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ = coefficient matrices. One particular advantage of Bacon's method over several others is the fact that the necessary coordination of adjustments to elements of the coefficient matrices that are functions of the same system parameter is taken into account. However, as mentioned above, his procedure is limited to linear system models and allows only the specification of linear differential equation solutions as desired responses.

One additional point with regard to Bacon's parameter optimization method will be made here. The error criteria formulated in his

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*Numbers in parentheses refer to references listed in the Bibliography.

paper is not time dependent; it requires the solution of one system of equations to match the solution of another system for all time. Thus the designer is not able to emphasize one portion in time of the system response over another. In some instances, for example, only the transient portion of the system response is of interest to the designer and the response error during this time only would be of interest. The method of parameter selection presented herein allows this desired "weighting" of the error criteria.

A method similar in concept to that presented by Bacon is described by Z.V. Rekasius in (3). Rekasius presents a method generally applicable to linear systems whereby a performance functional is formed from a specified system response. The desired response, expressed by a linear homogeneous differential equation, represents the ideal, toward which the system is optimized. A Lyapunov function is formed to minimize the performance index which in turn drives the system response closer to the ideal response. This method, however, possesses some of the same limitations as the method developed by Bacon; that is, it is applicable to linear systems only and the performance functional is limited in form.

A method of compensation for linear systems is developed by J.G. Mrazek in (4). This method requires that the system state model be written as

$$\underline{\dot{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \,. \tag{2-3}$$

The state equation is then transformed to the normal form so that the system eigenvalues (characteristic values) are displayed in the transformed $\underline{A}$ matrix. The eigenvalues are used to calculate a transient response criterion termed "steadiness factor" which is to a degree, a measure of the total response overshoot to a step input and is also a function of the response rise time. The procedure then is to adjust the eigenvalues to give an acceptable steadiness factor value. The state equation is next transformed back to the standard form (Equation (2-3)) resulting in new coefficient matrices $\underline{A}$ and $\underline{B}$. The response to this new state equation will yield, or approach as closely as possible, the desired steadiness factor.

Mrazek's method offers the advantage of emphasizing the transient portion of the response rather than the complete time history. However, a disadvantage is inherent in that adjustments are made to the coefficients in the $\underline{A}$ matrix without regard for the fact that the same system parameter may be represented in two or more of the coefficients. That is, the possibility exists of the compensation technique adjusting one coefficient upwards and another downwards while they both represent the same system parameter. The method presented in this thesis is not hampered by this difficulty since the physical parameters themselves may be adjusted.

Parameter Optimization of Nonlinear Systems

A.J. Koivuniemi presents an algorithm in (5) for parameter optimization of nonlinear systems in limited cases. He discusses a procedure whereby the elements of a parameter set are adjusted so that the performance index is minimized (via gradient method). However, his method is limited to a particular performance functional, namely

$$J = \tfrac{1}{2}\langle \underline{x}(T), \underline{R}\,\underline{x}(T)\rangle \qquad (2\text{-}4)$$

where $\underline{x}$ is the state vector, T is the (fixed) terminal time of the process, $\underline{R}$ is a constant positive semi-definite matrix, and

J the scalar performance index value. Koivuniemi's method offers the advantage of being applicable to nonlinear systems but is severely limited by the fact that the whole procedure is based on the particular performance index given in Equation (2-4). Since the performance index is evaluated at the terminal time only, the utility of his method is limited to terminal cost problems only.

A method of nonlinear system compensation discussed by D.A. Hullender in Chapter II of reference (6) is a method fairly general in application. This technique makes use of the system sensitivity coefficients to adjust the parameters of the system to obtain a specified response, or essentially, to solve the parameter optimization problem. The sensitivity coefficient is essentially the rate of change of a state variable with respect to a change in a system parameter. By calculating the sensitivity coefficients of a state variable, one is able to compute the required variations in the system parameters in order to drive that state variable to a desired response. A numerical technique may be formulated to minimize an error function (integral square error) by adjusting the system parameters using the gradient method.

This method offers several advantages, not the least of which is its application to nonlinear systems. In addition, the error function is not limited to the integral square error but could be any function of the state variables. Also, this method is not hampered by the problem of coordinating coefficient adjustments with the system parameters since the necessary changes are made directly to the parameters, not the equation coefficients. However one distinct disadvantage is apparent; one that is common to all parameter optimization techniques. At no point in the compensation procedure does the designer have any indication of what the optimum system response might be or whether improvement of the present response is possible. Also, some of the computational difficulties associated with other gradient techniques are shared by this method as well.

Parameter Optimization and Design of Nonlinear Systems

J.E. Bose (7, 8) presents a method of system design that has proven quite effective. He assumes a trial system model given as

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

and then adds a control vector $g(\underline{x})$ that will drive the state response to the desired value $\underline{x}_d$ when properly formulated. That is

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{g}(\underline{x}). \tag{2-5}$$

However, this method requires the system designer to assume the form of $\underline{g}(\underline{x})$ prior to the solution of the problem. That is, the functional form of $\underline{g}$ is first specified, as for example

$$\begin{aligned} g_1 &= k_1 x_1 + k_2 x_2 \\ g_2 &= 0 \\ &\vdots \\ g_n &= k_3 x_1 x_2 + k_4 x_4^2 \end{aligned}$$

and then the technique adjusts the values of k_i to cause $\underline{x}(t)$ to approximate, in a least squares sense, the desired state time history $\underline{x}_d(t)$.

The choice of the form of $\underline{g}(\underline{x})$ is important since once this selection is made, the fitting technique only can adjust the parameters k_i and cannot add new terms if needed. There is little information available to guide the designer in selecting the form of $\underline{g}(\underline{x})$ other than perhaps ingenuity or intuition. This fact represents a fundamental difference between the method of Bose and the technique of this report. It will be shown later that a control vector similar to $\underline{g}(\underline{x})$ in Equation (2-5) will be derived, however the form of it will not be fixed until more information is available or can be determined.

One additional point in regards to Bose's method concerns the specification of the time history of all of the state variables. This, at times, requires the differentiation of the given desired state; a task which can become difficult if the desired state response is not given in analytical form. The method developed in this thesis requires the specification of only the desired state variable time history.

D.R. Unruh (9) discusses a parameter optimization algorithm and associated computer program which is applicable in general to continuous nonlinear systems. This technique, like Bose's, only adjusts constant system parameters and parameters associated with any added terms. Again, like Bose's method, there is little information given as to what additional terms, if any, should be added to properly compensate the system. However, once the system designer is able to determine what additional terms or feedback loops are needed, if any, this program provides an excellent means of determining the optimum set of parameters to yield the desired response.

The method of parametric expansion developed by C.W. Merriam, III (10, 11) leads to the derivation of the necessary linear feedback loops for optimal control of linear or nonlinear systems. The essence of this method involves a minimum error function E which has the assumed form

$$E = k(t) + \sum_{i=1}^{n} k_i(t)x_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}(t)x_i(t)x_j(t)$$

where the $x(t)$'s are the state variables, the $k(t)$'s are time-variable parameters, and n is the order of the system. The elements of the control vector are expressed as functions of the partial derivatives of E with respect to the various state variables and time. The various $k(t)$'s are then determined as the solutions to a set of first-order differential equations. Thus, the control vector may be determined as a linear function of the state variables and a set of time-variable parameters or gains.

Once the control vector has been derived in this manner, however, the designer is still faced with difficulties in implementing the control. The implementation of the time-variable gains would in general be exceedingly difficult without an on-board computer. Furthermore, there is no assurance that all of the state variables required for the generation of the control vector will be observable or available for use. In short, there is no consideration given to physical realizability in the derivation of the control vector through the parametric expansion technique. The method presented herein makes physical realizability a prime consideration.

In view of the original problem statement, the above methods of parameter optimization and system design will now be summarized. The techniques presented by Bacon, Rekasius and Mrazek (1, 2, 3, 4) are limited to linear system models and hence are not applicable to the class of problems under consideration, i.e. nonlinear system models.

Koivumiemi's (5) method is applicable to nonlinear systems but is limited to one particular performance index. A method of parameter optimization generally applicable to nonlinear systems was discussed by Hullender (6) which gave no a priori information as to what system parameters should be allowed to vary. Bose (7, 8) developed a technique applicable to nonlinear systems in which the general form of a control vector is first assumed and then the coefficients associated with this vector determined. The primary difficulty associated with this method and that of Unruh (9) is the fact that the form of the control vector must first be selected by the designer with little guidance as to the proper form to assume. Finally, Merriam's (10, 11) parametric expansion technique provides a more analytical means of deriving the necessary feedback loops yet results in a form that requires time-variable gains. In addition, his technique is restricted to linear feedback loops by the assumed form of the minimum error function.

CHAPTER III

OBJECTIVES OF THE STUDY

Based on the discussion of past work on the problem of system design, the objectives of this research can now be more clearly stated. It is assumed that a trial system model will be available to the system designer. That is, a set of nonlinear differential equations can be determined that adequately model the system. The coefficients of the differential equations will be functions of the physical system parameters or characteristics. The system model is given as

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \tag{3-1}$$

where $\underline{x}$ is an n-vector of state variables, $\underline{u}$ an m-vector of external system inputs and $\underline{f}$ an n-vector function of $\underline{x}$ and $\underline{u}$. A control vector $\underline{q}$ will be added to Equation (3-1) which will drive the state trajectory $\underline{x}(t)$ to the desired trajectory $\underline{x}_d(t)$.

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) + \underline{q} \tag{3-2}$$

The form of $\underline{q}$ will not be specified initially.

A technique was desired that would aid the system designer in selecting the optimum trajectory of the vector $\underline{q}$ to cause $\underline{x}(t)$ to approach as closely as possible the desired system response, and further, to relate the control $\underline{q}$ to the physical system. In an effort to realize this technique, this study was initiated with the following objectives in mind:

1. Determine a method of formulating a performance index to indicate, among other things, how well the system response approximates the desired response.

2. Establish a method of expressing the desired response characteristics in analytical terms, compatable with the procedure for determining the optimum control $\underline{q}$.

3. Develop a compensation procedure that does not require the specification of the time histories of all of the system states. Rather, only the states with a specific desired response should require specification.

4. Demonstrate the feasibility of deriving the time history of the necessary control $\underline{q}$ for the type problems discussed.

5. Relate the time history of the control vector $\underline{q}$ to the state variables in such a way as to generate the control vector q as a function of the state variables.

6. Insure that the compensation technique allows the specification of a desired response trajectory as well as the limiting of state extrema.

7. Insure that the procedure provides a means to maintain physical realizability requirements.

This document reports on a study to develop a method of system compensation that encompasses the objectives stated above.

CHAPTER IV

COMPENSATION TECHNIQUE FOR NONLINEAR SYSTEMS

The primary objectives of this chapter are twofold. Presented first is a general statement of the system compensation method to be developed, a short review of the theory underlying this development and a listing of the necessary assumptions and restrictions that will be imposed. The second objective is the detailed presentation and discussion of a compensation procedure. An attempt has been made throughout the chapter to retain a sense of practicality and realism. That is, many theoretical complexities and difficulties arise in a strictly formal development of any optimal control theory that may be of little consequence in physical systems. Thus some points discussed in this chapter could be belabored further but will not be where it is believed that sufficient development is given for proper application of the method.

Basic Concepts

The general system model that will be studied during this investigation will have the form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$

where t, $\underline{x}$, $\underline{f}$ and $\underline{u}$ are as described in Chapter II. An undetermined control vector $\underline{q}$ will be added to the system model so that

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) + \underline{q}.$$

Then classical optimization theories will be utilized to derive a time history of the optimum $\underline{q}$. That is, a vector control signal will be found that optimizes the system response along some desired trajectory $\underline{x}_d(t)$ or limits the response to some desired maximum or minimum. The optimum control signal will be designated $\underline{q}^*(t)$ and the corresponding optimum response designated $\underline{x}^*(t)$. Although an analytical expression exists for $\underline{q}^*(t)$, it is not easily found for nonlinear systems; generally only computational methods are available which result in a numerical solution for $\underline{q}^*(t)$.

The next step will involve correlating $\underline{q}^*(t)$ with the state variables to determine an approximation to $\underline{q}^*(t)$ that can be implemented in terms of the state variables. That is, it is desired to find a $\hat{\underline{q}}$ such that

$$\hat{\underline{q}} = \hat{\underline{q}}(\underline{x}, \underline{k}) \approx \underline{q}^*(t)$$

where $\underline{k} = \left[k_1 \; k_2 \; . \; . \; . \; k_i\right]^T$ is a constant parameter vector. The objective here is to generate the necessary optimum control $\underline{q}^*$ as a function of the state variables. To do so will require a knowledge of what terms or state variables are required and what can be physically implemented in the system.

Restrictions on the Optimal Control Problem

As mentioned above, classical optimal control theory will be utilized to aid in obtaining the optimum control vector $\underline{q}^*(t)$; hence a statement concerning optimization in general is in order. Briefly, much of the work in optimal controls involves a dynamical system described by a relation such as

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t). \tag{4-1}$$

The optimal control problem is normally to determine an input control vector $\underline{u} = \underline{u}(t)$ that will minimize a performance index or "cost" function

$$J = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L(\underline{x}, \underline{u}, t)\, dt$$

while meeting a set of constraints imposed by

$$\underline{f}(\underline{x}, \underline{u}, t) - \underline{\dot{x}} = \underline{0}.$$

This requires the $\underline{x}$ appearing in the cost function equation to be a solution of Equation (4-1). In addition, a set of inequality constraints

$$\underline{h}(\underline{x}, \underline{u}, t) \geq \underline{0}$$

may also be imposed, depending on the particular problem requirements.

A more precise statement of the control problem will now be given. The dynamical system described by

$$\underline{\dot{x}} = \underline{f}[\underline{x}(t), \underline{u}(t), t]$$

on the closed interval (t_o, t_f), $t_f > t_o$, will be considered. Here $\underline{x}(t)$ and $\underline{f}$ are n-vectors and $\underline{u}(t)$ is an m-vector, with $0 < m \leq n$. At t_o, the initial time,

$$\underline{x}(t_o) = x_o$$

is the initial state and the final state, $\underline{x}(t_f)$, is not fixed. The functions

$$L(\underline{x}, \underline{u}, t) \text{ and } K(\underline{x})$$

are assumed differentiable in $\underline{x}$ and t and describe the performance functional, $J(\underline{u})$, given by

$$J(\underline{u}) = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L[\underline{x}(t), \underline{u}(t), t]\, dt.$$

Here $\underline{x}(t)$ is the trajectory of the state of the system starting from $\underline{x}(t_o) = \underline{x}_o$ and generated by the control $\underline{u}(t)$. The essential problem is to determine the control $\underline{u}(t)$ which minimizes the performance functional $J(\underline{u})$.

Athans and Falb (12), the primary reference for the following statement of the maximum principle of Pontryagin, list additional assumptions. If $f_1(\underline{x}, \underline{u}, t)$, $f_2(\underline{x}, \underline{u}, t)$ . . . $f_n(\underline{x}, \underline{u}, t)$ denote the components of $\underline{f}(\underline{x}, \underline{u}, t)$, then it is assumed that the functions

$$f_i(\underline{x},\underline{u},t), \frac{\partial f_i}{\partial \underline{x}}(\underline{x},\underline{u},t), \frac{\partial f_i}{\partial t}(\underline{x},\underline{u},t), \qquad i = 1, 2, \ldots n$$

and the functions

$$L(\underline{x},\underline{u},t), \frac{\partial L}{\partial \underline{x}}(\underline{x},\underline{u},t), \frac{\partial L}{\partial t}(\underline{x},\underline{u},t)$$

are continuous in the vector space containing the vectors $\underline{x}$, $\underline{u}$, and the scalar t, that is the $(\underline{x}, \underline{u}, t)$ space. The terminal cost function $K[\underline{x}(t_f)]$ must be independent of t and the functions

$$K(\underline{x}), \frac{\partial K}{\partial \underline{x}}(\underline{x}), \frac{\partial^2 K}{\partial x^2}(\underline{x})$$

must be continuous.

It is further assumed in the following discussion that if the function $\underline{f}$ or L depends explicitly on time (i.e. t appears in the equation for $\underline{f}$ or L), then an auxiliary variable

x_{n+1} is introduced so that

$$\dot{x}_{n+1} = 1.$$

The (n + 1)st-order system

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), x_{n+1}(t)]$$

$$\dot{x}_{n+1}(t) = 1$$

$$\underline{x}(t_o) = \underline{x}_o$$

$$x_{n+1}(t_o) = t_o$$

and the performance functional

$$J'(\underline{u}) = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L[\underline{x}(t), \underline{u}(t), x_{n+1}(t)]\, dt$$

is then considered. In the formulation, the state variable x_{n+1} is in reality the independent variable time and the problem has simply been restated in such a manner that $\underline{f}$ and L are functions of the state variables and controls only. Such a formulation will allow the statement of all problems in the form

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t)]$$

$$J(\underline{u}) = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L[\underline{x}(t), \underline{u}(t)]\, dt$$

where it is understood that if t appears explicitly, the necessary auxiliary variable is included in the n state variables.

The Maximum Principle for No Terminal Cost

A statement of the maximum principle of Pontryagin will now be given, under the assumptions listed above, and for the special case of no terminal cost, i.e. $K(\underline{x}) = 0$. Following this discussion, the maximum principle for the case where the performance functional depends upon the terminal state will be given. For the case of no terminal cost, the problem is formed as

$$\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}, \underline{u}) \\ \underline{x}(t_o) &= \underline{x}_o \\ x(t_f) &= \text{unspecified} \\ J(\underline{u}) &= \int_{t_o}^{t_f} L(\underline{x}, \underline{u})\, dt \end{aligned} \qquad (4\text{-}2)$$

where $\underline{x}$ and $\underline{u}$ are understood to be functions of time. The problem is to determine the control $\underline{u}(t)$ which minimizes the performance functional $J(\underline{u})$; the control that does so will be designated $\underline{u}^*(t)$. A set of n "adjoint" or "costate" variables, $\underline{p}(t)$, are introduced that play a role similar to Lagrange multipliers in differential calculus. A scalar function called the Hamiltonian function, or simply the Hamiltonian, is also introduced as

$$H = L(\underline{x}, \underline{u}) + \langle \underline{p}, \underline{f}(\underline{x}, \underline{u}) \rangle .$$

The notation $\langle\ \rangle$ denotes the scalar product of the vectors $\underline{p}$ and $\underline{f}$. The maximum principle of Pontryagin for this problem can now be stated as Theorem 4-1 (12).

Theorem 4-1. Let $\underline{u}^*(t)$ be an admissible control which drives the system of Equation (4-2) from the initial point

$(\underline{x}_o, t_o)$ during the time $t_o - t_f$. Let $\underline{x}^*(t)$ be the state trajectory corresponding to $\underline{u}^*(t)$ originating at $(\underline{x}_o, t_o)$. In order that $\underline{u}^*(t)$ be optimal it is necessary that there exist a function $\underline{p}^*(t)$ such that:

a. $\underline{p}^*(t)$ corresponds to $\underline{u}^*(t)$ and $\underline{x}^*(t)$, so that $\underline{p}^*(t)$ and $\underline{x}^*(t)$ are a solution of the canonical system

$$\dot{\underline{x}}^*(t) = \frac{\partial H}{\partial \underline{p}}(\underline{x}^*, \underline{p}^*, \underline{u}^*)$$

$$\dot{\underline{p}}^*(t) = -\frac{\partial H}{\partial \underline{x}}(\underline{x}^*, \underline{p}^*, \underline{u}^*)$$

$$\underline{x}(t_o) = \underline{x}_o$$

$$\underline{p}(t_f) = \underline{0}.$$

b. The function $H\left[\underline{x}^*(t), \underline{p}^*(t), \underline{u}\right]$ has an absolute minimum as a function of $\underline{u}$ at $\underline{u} = \underline{u}^*(t)$, $t_o \leq t \leq t_f$; that is,

$$\min_{\underline{u}} H[\underline{x}^*, \underline{p}^*, \underline{u}] = H[\underline{x}^*, \underline{p}^*, \underline{u}^*].$$

c. The function $H\left[\underline{x}^*, \underline{p}^*, \underline{u}^*\right]$ is zero for t in (t_o, t_f); that is,

$$H[\underline{x}^*(t), \underline{p}^*(t), \underline{u}^*(t)] = 0, \qquad t_o \leq t \leq t_f.$$

The Maximum Principle for the Terminal Cost Problem

Here, the maximum principle will be given for the problem in which the performance functional is penalized for missing a given point, i.e. the terminal cost problem. For this case, the problem is stated as

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \tag{4-3}$$

$$\underline{x}(t_o) = \underline{x}_o$$

$$\underline{x}(t_f) = \text{unspecified}$$

$$J(\underline{u}) = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L(\underline{x}, \underline{u})\, dt \tag{4-4}$$

where, again, $\underline{x}$ and $\underline{u}$ are understood to be functions of time, but $K(\underline{x})$ does not explicitly contain t. The Hamiltonian is formed as

$$H(\underline{x}, \underline{p}, \underline{u}) = L(\underline{x}, \underline{u}) + \langle \underline{p}, \underline{f}(\underline{x}, \underline{u}) \rangle.$$

Theorem 4-2 gives the maximum principle for this problem.

Theorem 4-2. Let $\underline{u}^*(t)$ be an admissible control which drives the system of Equation (4-3) from the initial point $(\underline{x}_o, t_o)$ during the time $t_o - t_f$. Let $\underline{x}^*(t)$ be the state trajectory corresponding to $\underline{u}^*(t)$ originating at $(\underline{x}_o, t_o)$. In order that $\underline{u}^*(t)$ be optimal for the performance function (4-4), it is necessary that there exist a function $\underline{p}^*(t)$ such that:

a. $\underline{p}^*(t)$ corresponds to $\underline{u}^*(t)$ and $\underline{x}^*(t)$ so that $\underline{p}^*(t)$ and $\underline{x}^*(t)$ are a solution of the canonical system

$$\dot{\underline{x}}^*(t) = \frac{\partial H}{\partial \underline{p}}(\underline{x}^*, \underline{p}^*, \underline{u}^*),$$

$$\dot{\underline{p}}^*(t) = -\frac{\partial H}{\partial \underline{x}}(\underline{x}^*, \underline{p}^*, \underline{u}^*),$$

$$\underline{x}(t_o) = \underline{x}_o,$$

$$\underline{p}(t_f) = \frac{\partial K}{\partial \underline{x}}[\underline{x}^*(t_f)].$$

b. The function $H[\underline{x}^*(t), \underline{p}^*(t), \underline{u}]$ has an absolute minimum as a function of $\underline{u}$ at $\underline{u} = \underline{u}^*(t)$ for t in (t_o, t_f).

c. The function $H[\underline{x}^*(t), \underline{p}^*(t), \underline{u}^*(t)]$ satisfies the relations

$$H[\underline{x}^*(t), \underline{p}^*(t), \underline{u}^*(t)] = -\int_t^{t_f} \frac{\partial H}{\partial t}[\underline{x}^*(\lambda), \underline{p}^*(\lambda), \underline{u}^*(\lambda)]\, d\lambda,$$

$$H[\underline{x}^*(t_f), \underline{p}^*(t_f), \underline{u}^*(t_f)] = 0.$$

Note however that this condition is automatically satisfied and no additional information can be gained from it. This would not be the case if $K(\underline{x})$ were an explicit function of time.

The above statements of Pontryagin's maximum principle have been stated for documentation only. No attempt to prove or justify these theorems will be made. Athans and Falb (12) present a thorough and readable discussion of these principles while Pontryagin, et al. (13) give a rigorous proof for the interested reader. These theorems are well established and presented in several of the current texts on modern control theory.

Use of the Maximum Principle in System Compensation

The basic idea underlying the technique of system compensation presented in this thesis will be outlined in this section. A more complete development will be given in the following sections. As mentioned at the beginning of the chapter, use will be made of optimal control theories to aid in the derivation of the necessary system compensation. The system compensation problem will be formulated so as to meet the requirements necessary to apply the maximum principle of Pontryagin to determine the optimum compensation.

It is assumed that a deterministic mathematical model of the system is available in differential equation form,

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t).$$

The system input $\underline{u}(t)$ is a known, fixed function of time and will not be changed in compensating the system. The notation "$\underline{u}$" may thus be eliminated from the functional notation of $\underline{f}$ without loss of generality. In addition, if it is assumed that if t appears explicitly in the state equation, the necessary auxiliary variable $(\dot{x}_i = 1, x_i(0) = 0)$ has been included in the n state variables, then t may also be eliminated. An unknown compensating control function of time will be added to the system state model. Thus,

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{q}(t).$$

The first objective will be to determine the trajectory $\underline{q}(t)$ that properly compensates the system, while meeting certain restrictions. It is at this point that the use of certain optimal control theories comes into use. If $\underline{q}(t)$ is considered an independent control input to the system, Theorem 4-1 or 4-2 may be utilized to aid in determining an optimum time history for that control, assuming the specifications of the theorems are met. This requires that a performance index be formulated such that minimization of this index will yield the desired optimum control. For this case the performance functional will be written as

$$J(\underline{q}) = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L[\underline{x}(t), \underline{q}(t)]\, dt,$$

where the functional J is a function of $\underline{q}(t)$, the compensating control, rather than $\underline{u}(t)$, the system driving function which will remain fixed. As in the statement of the theorems, $K(\underline{x})$ may equal zero if the problem has no terminal cost.

In most problems in which the system transient response is of primary interest, $K(\underline{x}) = 0$, and the loss function or error index L may be formulated as

$$L[\underline{x}(t), \underline{q}(t)] = L'[\underline{x}(t)] + L''[\underline{q}(t)].$$

The function L' is that portion of the error index which, when minimized, will yield the desired system response characteristics and L'' is that portion which allows restrictions to be placed on the compensating control $\underline{q}(t)$. A detailed discussion of the formulation of L for particular problem requirements will be presented in the following section. For the purpose of the present discussion it will be assumed that the error index may be formulated in terms of $\underline{x}(t)$ and $\underline{q}(t)$ rather than $\underline{u}(t)$ and in keeping with the requirements specified earlier. Formulation of the performance functional represents the first major step in the overall problem solution. For the no terminal cost problem, the performance functional is

$$J(\underline{q}) = \int_{t_o}^{t_f} L(\underline{x}, \underline{q})\, dt.$$

The problem is now to determine the control $\underline{q}(t)$ which minimizes $J(\underline{q})$; this control is designated $\underline{q}^*(t)$. A set of n adjoint variables, $\underline{p}(t)$, are introduced and the Hamiltonian is formed as

$$H = L(\underline{x}, \underline{q}) + \langle \underline{p}, \underline{f} + \underline{q} \rangle .$$

From Theorem 4-1, in order for $\underline{q}^*(t)$ to be an optimal control, it is necessary that $\underline{q}^*(t)$ correspond to $\underline{p}^*(t)$ and $\underline{x}^*(t)$ which are solutions to

$$\dot{\underline{x}}^*(t) = \frac{\partial H}{\partial \underline{p}}(\underline{x}^*, \underline{p}^*, \underline{q}^*), \qquad (4\text{-}5)$$

$$-\dot{\underline{p}}^*(t) = \frac{\partial H}{\partial \underline{x}}(\underline{x}^*, p^*, q^*), \qquad (4\text{-}6)$$

$$\underline{x}(t_o) = \underline{x}_o,$$

$$\underline{p}(t_f) = \underline{0}.$$

Also, it is necessary that

$$\min_{\underline{q}} H[\underline{x}^*, \underline{p}^*, \underline{q}] = H[\underline{x}^*, \underline{p}^*, \underline{q}^*].$$

In most instances this condition can be satisfied by requiring that

$$\frac{\partial H}{\partial \underline{q}} = \underline{0}. \qquad (4\text{-}7)$$

If L is not a linear function of $\underline{q}$, then Equation (4-7) usually can be solved for $\underline{q}$ in terms of $\underline{x}$ and $\underline{p}$.

$$\underline{q}^* = \underline{q}(\underline{x}^*, \underline{p}^*) \qquad (4\text{-}8)$$

This equation can be substituted into Equations (4-5) and (4-6) which become

$$\dot{\underline{x}}^*(t) = \frac{\partial H}{\partial \underline{p}}(\underline{x}^*, \underline{p}^*), \qquad (4\text{-}9)$$

$$\dot{\underline{p}}^*(t) = -\frac{\partial H}{\partial \underline{x}}(\underline{x}^*, \underline{p}^*), \tag{4-10}$$

with the same boundary conditions as above.

It should be noted that it is not essential that Equations (4-7) and (4-8) be calculated; the prime requirement is that the minimum of H with respect to $\underline{q}$ is $H(\underline{q}^*)$. In cases where $\underline{q}$ as a function of $\underline{x}$ and $\underline{p}$ can be determined and substituted into Equations (4-5) and (4-6), little practical information can be gained from the last requirement; that is

$$H^*(t) = 0, \qquad t_o \leq t \leq t_f.$$

This is true in most system compensation problems.

Equations (4-9) and (4-10) represent a 2nth-order, nonlinear, two-point boundary value problem which, when solved along with Equation (4-8), yields the time history of the optimum control $\underline{q}^*(t)$. Methods and requirements for solving two-point boundary value problems will be discussed in the next sub-section.

The solution of the split boundary value problem represents the second major step in the problem solution since it yields the optimum control $\underline{q}^*(t)$ and the resulting optimum response $\underline{x}^*(t)$. At this point the designer may decide whether the optimum response $\underline{x}^*(t)$ is sufficiently improved over the uncompensated response $\underline{x}(t)$ to warrant an attempt to compensate the system. That is, a situation may exist in which $\underline{x}^*(t)$ is little improvement over the original uncompensated $\underline{x}(t)$ or, $\underline{x}^*(t)$ does not meet minimum requirements. The judgment, of course, depends entirely on the particular problem requirements, such as the value of the performance index, rise time, overshoot, terminal conditions, limit cycle characteristics, or whether the response stays within certain bounds. The important point here is the optimum or ideal compensated response which may be achieved under the restrictions imposed. Hence, a measure of how much response improvement can be expected with optimum compensation is given early in the design process.

This fact is one of the principal advantages of the system compensation method described herein; the designer is able to decide whether proper compensation of a system is feasible before actually attempting to perform the compensation, and he has a measure of the maximum response improvement to be expected. Other compensation techniques give no a priori indication of the optimum compensation or the corresponding optimum response and thus the extent to which the actual compensation achieves the optimum is not known.

However, once $\underline{q}^*(t)$ and $\underline{x}^*(t)$ have been obtained and it is determined that $\underline{x}^*(t)$ represents a significant improvement, the next major step becomes that of implementing $\underline{q}^*(t)$. It is the contention of this thesis that to utilize an in-line, real-time computer to generate $\underline{q}^*(t)$ or to record $\underline{q}^*(t)$ on some data recording device (such as magnetic tape) and feed it into the system is unfeasible and unnecessary in many cases. Instead, it is proposed that the system itself be altered in such a way that in effect $\underline{q}^*(t)$ is generated by the system states and thus yields the desired compensation. That is, some functional form of the system state variables is sought which will generate the same time history as $\underline{q}^*(t)$, i.e.

$$\underline{q}\{\underline{x}(t), \underline{k}\} = \underline{q}^*(t),$$

where $\underline{k}$ is a vector of constant parameters. The primary objective at this point is to determine a function of the system states that will generate $\underline{q}^{*}(t)$ and is physically implementable. Normally, a physically realizable state dependent function that generates $\underline{q}^{*}(t)$ exactly cannot be obtained and an approximation

$$\hat{\underline{q}}(\underline{x}, \underline{k}) \approx \underline{q}^{*}(t)$$

must be accepted. However if $\hat{\underline{q}}$ approximates $\underline{q}^{*}$ sufficiently well, the response $\hat{\underline{x}}(t)$ corresponding to $\hat{\underline{q}}$ will closely approximate the optimum response $\underline{x}^{*}(t)$. Although $\hat{\underline{q}}$ will be a sub-optimal control, the designer can assure that it will be one which can be physically implemented.

The assurance of physical realizability is another of the principal advantages of this method of system compensation. Even though some techniques will yield a truly optimum compensation, there is no guarantee that this compensation can be achieved and the designer has no control over what form the compensation takes. On the other hand, through the use of parameter optimization methods the designer may assure that the parameters he selects to optimize are implementable, but he has no a priori knowledge of which parameters to vary for the best results. Furthermore, at no point in the parameter optimization procedure is any indication given of the truly optimum response. Thus the designer does not know just how sub-optimum the system is or what results might be expected with the trial of a different parameter.

A thorough discussion of how the state dependent approximation of $\underline{q}^{*}(t)$ is obtained is presented in a following section. Briefly, the general approach is to study the optimum control $\underline{q}^{*}(t)$ and optimum response $\underline{x}^{*}(t)$ and determine a general form for $\hat{\underline{q}}(\underline{x}, \underline{k})$. For example, in the case of a scalar $\underline{q}$, a typical $\hat{q}$ might be

$$\hat{q}(\underline{x}, \underline{k}) = k_1 + k_2x_1 + k_3x_2^2.$$

Each term in $\hat{q}$ must be a term that can be implemented in the physical system. Once the general form has been selected, the values of the k_i are determined to give a best fit of $\hat{q}(\underline{x}, \underline{k})$ to $q^*(t)$. Various guides to aid in the selection of $\hat{\underline{q}}(\underline{x}, \underline{k})$ and in the determination of the elements of $\underline{k}$ are presented in the following section.

In general, but not always, an increase in the complexity of $\hat{\underline{q}}$ will result in a better fit to $\underline{q}^*$, however more complex forms of $\hat{\underline{q}}$ are usually more difficult to implement. An important advantage of this technique is that at this point in the compensation procedure, the designer can very clearly determine the relative importance of each term in $\hat{\underline{q}}$ in approximating $\underline{q}^*$. He can ascertain the degree of optimality that is sacrificed by not implementing certain terms. That is, the trade-off between the difficulty of implementing a certain term of $\hat{\underline{q}}$ and the loss of optimality by not implementing that term can be examined here with comparative ease. This point will be clarified in the consideration of several example problems in the next chapter.

The final step in the compensation procedure is that of verifying the results of the approximation and implementing the compensation. The approximate control $\hat{\underline{q}}$ is simply added to the system equations and a determination made as to whether the resulting sub-optimal compensated response meets the original problem specifications.

Solution of Two-Point Boundary Value Problems

This section will briefly outline the general procedure for the solution of nonlinear two-point boundary value problems. Although the solution of the split boundary value problem resulting from the necessary conditions of the maximum principle is a key factor in the successful application of this system compensation method, no effort was made to develop solution techniques. Rather, existing methods were relied upon for the solution of the two-point boundary value problems generated in the compensation procedure. A review of "Computational Methods in Optimal Control Problems" is presented by H.R. Sebesta in reference (14) and will be summarized here.

In general the optimal control problem is stated as follows: find the functions

$$\underline{u}(t) = [u_1(t), u_2(t), \ldots u_m(t)]^T$$

which will minimize the performance functional

$$J = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L[\underline{x}(t), \underline{u}(t)]\, dt$$

while satisfying the state equations

$$\dot{x}_1 = f_1(\underline{x}, \underline{u}, t)$$

$$\vdots$$

$$\dot{x}_n = f_n(\underline{x}, \underline{u}, t)$$

and the boundary conditions

$$x_1(0) = x_{10}$$

$$\vdots$$

$$x_n(0) = x_{no}$$

It should be noted that it is not necessary that all n of the initial conditions be specified. Instead, some of the final conditions may be specified or constrained by algebraic relationships. In any case, a properly formulated optimal control problem will result in a set of n system differential equations, n adjoint differential equations and m algebraic equations.

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \quad \text{n-diff. eqns.} \tag{4-11}$$

$$\dot{\underline{p}} = g(\underline{x}, \underline{u}, \underline{p}, t) \quad \text{n-diff. eqns.} \tag{4-12}$$

$$\underline{0} = \frac{\partial \underline{f}}{\partial \underline{u}}\underline{p} + \frac{\partial L}{\partial \underline{u}} \quad \text{m-algebraic eqns.} \tag{4-13}$$

A total of 2n boundary conditions will be specified, some at t_o and the others at t_f. These equations are referred to as the two-point boundary value problem.

In many problems Equation (4-13) can be solved explicitly for the control $\underline{u}$ as a function of $\underline{p}$ and $\underline{x}$, i.e. $\underline{u}(\underline{x}, \underline{p})$. This function can then be substituted into Equations (4-11) and (4-12), thus eliminating the m algebraic equations and leaving only the 2n differential equations for $\underline{x}$ and $\underline{p}$ with 2n boundary conditions. Since the manner in which optimal control theory is utilized in this thesis will yield equations which can be solved for $\underline{u}$, it will be assumed in the following discussion that this has already been accomplished. The differential equations become, upon substitution of $\underline{u}(\underline{x}, \underline{p})$, functions only of $\underline{x}$, $\underline{p}$, and t.

If a simple linear system with linear control and a quadratic performance index is being studied, an analytical solution to the split boundary value problem may be possible. However, since this thesis stresses the compensation of nonlinear systems, the methods of exact analytical solutions will not be discussed. According to Sebesta, there are two alternatives--approximate analytical solutions or computational solutions. Since approximate analytical solutions become exceptionally tedious and difficult for systems of all but the lowest orders and since the use of digital computers is becoming increasingly commonplace, only the computational methods will be considered during the course of this study. These methods resolve the problem into one of determining the proper conditions at t_o for those states that are initially unspecified so that the specified final conditions at t_f are satisfied.

Two computational methods will be presented which may be combined to yield a very workable method of solving two-point boundary value problems. The first, the method of parameter influence coefficients or sensitivity coefficients, proceeds from crude estimates of the unknown initial conditions to a fairly good approximation of the optimum trajectory. However, this method quite often has difficulty converging on the final solution. The second method, that of quasilinearization, will converge to the proper optimum solution given good enough estimates of the initial conditions. The logical combination of these two methods is to use the first method to give a close approximation to the required initial conditions and then use these values as the initial estimates for the second method. The combination of these two methods has been achieved by Dennis Unruh (15) with the resulting algorithm being remarkably efficient and reliable. Unruh's work is summarized in the following paragraphs.*

The method of parameter influence coefficients will be first considered. For simplicity and ease of notation the n adjoint variables, $p_1, p_2, \ldots p_n$, are redefined as $x_{n+1}, x_{n+2}, \ldots x_{2n}$. The two-point boundary value problem can now be stated as the $2n$ vector differential equation

$$\frac{d\underline{x}}{dt} = \underline{F}(\underline{x}, t) \tag{4-14}$$

defined on the interval $t_o \leq t \leq t_f$
with k boundary conditions specified at t_f and $2n-k$ at t_o.
The initial condition vector is written

$$\begin{bmatrix} x_1(t_o) \\ x_2(t_o) \\ \cdot \\ \cdot \\ \cdot \\ x_{2n-k}(t_o) \\ x_1'(t_o) \\ \cdot \\ \cdot \\ \cdot \\ x_k'(t_o) \end{bmatrix} = \begin{bmatrix} x_{1o} \\ x_{2o} \\ \cdot \\ \cdot \\ \cdot \\ x_{2n-k_o} \\ \text{unspecified} \\ \cdot \\ \cdot \\ \cdot \\ \text{unspecified} \end{bmatrix},$$

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*In more recent work, Unruh has developed additional computational algorithms for solving the two-point boundary value problem through an algebraic minimization scheme. This work has not been published as yet.

where the primed states are those with the final conditions specified. The specified final boundary conditions are

$$\begin{bmatrix} x_1'(t_f) \\ x_2'(t_f) \\ \cdot \\ \cdot \\ \cdot \\ x_k'(t_f) \end{bmatrix} = \begin{bmatrix} x_{1f} \\ x_{2f} \\ \cdot \\ \cdot \\ \cdot \\ x_{kf} \end{bmatrix} .$$

A final-condition performance index is formulated that measures the absolute difference between the primed states at time t_f and their specified values.

$$I = \sum_{i=1}^{k} [x_i'(t_f) - x_{if}]^2$$

Note that this is an auxiliary index introduced to facilitate solving the two-point boundary value problem. Since the problem may be simply stated as that of determining the initial conditions of the x_i' that will cause the state trajectory to pass through points x_{if} at $t=t_f$, it is helpful to examine the gradient of the performance index I with respect to $x_i'(t_o)$, $i = 1, 2, \ldots k$.

$$\text{Gradient} = \frac{\partial I}{\partial \underline{x}'(t_o)} = \left[\frac{\partial I}{\partial x_1'(t_o)} \cdots \frac{\partial I}{\partial x_k'(t_o)}\right]$$

This gradient vector gives the direction of steepest ascent of the function I in k-dimensional space. Using the chain rule for partial differentiation, the gradient may be calculated from

$$\frac{\partial I}{\partial \underline{x}'(t_o)} = \frac{\partial I}{\partial \underline{x}'(t_f)} \frac{\partial \underline{x}'(t_f)}{\partial \underline{x}'(t_o)} , \qquad (4\text{-}15)$$

or the transpose of this equation,

$$\begin{bmatrix} \frac{\partial I}{\partial x_1'(t_o)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial I}{\partial x_k'(t_o)} \end{bmatrix} = 2 \begin{bmatrix} \frac{\partial x_1'(t_f)}{\partial x_1'(t_o)} & \cdots & \frac{\partial x_k'(t_f)}{\partial x_1'(t_o)} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial x_1'(t_f)}{\partial x_k'(t_o)} & \cdots & \frac{\partial x_k'(t_f)}{\partial x_k'(t_o)} \end{bmatrix} \begin{bmatrix} x_1'(t_f) - x_{1f} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_k'(t_f) - x_{kf} \end{bmatrix} .$$

In order to determine the gradient vector, the elements in the square matrix above must be evaluated. Since the state trajectory corresponding to Equation (4-14) is in reality a function of $k + 1$ independent variables $x_1'(t_o), \ldots x_k'(t_o)$, t, Equation (4-14) theoretically should be written

$$\frac{\partial \underline{x}}{\partial t} = \underline{F}\left(\underline{x}, t, x_1'(t_o) \ldots x_k'(t_o)\right).$$

The elements of the $\partial \underline{x}'(t_f) / \partial \underline{x}'(t_o)$ matrix may be calculated as in the following. Assuming $\underline{F}$ is continuous in $\underline{x}$ and t,

$$\frac{\partial}{\partial \underline{x}'(t_o)}\left(\frac{\partial \underline{x}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \underline{x}}{\partial \underline{x}'(t_o)}\right) .$$

Thus

$$\frac{\partial}{\partial t}\left(\frac{\partial \underline{x}}{\partial \underline{x}'(t_o)}\right) = \frac{\partial \underline{F}}{\partial \underline{x}'(t_o)} ,$$

and again using the chain rule

$$\frac{\partial}{\partial t}\left(\frac{\partial \underline{x}}{\partial \underline{x}'(t_o)}\right) = \frac{\partial \underline{F}}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{x}'(t_o)} . \qquad (4\text{-}16)$$

Writing this equation out gives

$$\frac{\partial}{\partial t}\begin{bmatrix} \frac{\partial x_1}{\partial x_1'(t_o)} \cdots \frac{\partial x_1}{\partial x_k'(t_o)} \\ \vdots \qquad \vdots \\ \frac{\partial x_{2n}}{\partial x_1'(t_o)} \cdots \frac{\partial x_{2n}}{\partial x_k'(t_o)} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} \cdots \frac{\partial F_1}{\partial x_{2n}} \\ \vdots \qquad \vdots \\ \frac{\partial F_{2n}}{\partial x_1} \cdots \frac{\partial F_{2n}}{\partial x_{2n}} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial x_1'(t_o)} \cdots \frac{\partial x_1}{\partial x_k'(t_o)} \\ \vdots \qquad \vdots \\ \frac{\partial x_{2n}}{\partial x_1'(t_o)} \cdots \frac{\partial x_{2n}}{\partial x_k'(t_o)} \end{bmatrix},$$

a set of $2\cdot n\cdot k$ first order differential equations with zero initial conditions in the first $2n-k$ rows and the identity matrix for the last k rows. If initial conditions are guessed for the k variables $x_1'(t_o), \ldots x_k'(t_o)$, Equations (4-14) and (4-16) may be integrated from t_o to t_f as a coupled set of differential equations. The last k rows of the solution to Equation (4-16) at $t = t_f$ are the values required $\partial \underline{x}'(t_f) / \partial \underline{x}'(t_o)$ to evaluate the gradient of I with respect to $\underline{x}'(t_o)$.

Since the gradient vector points in the direction of steepest ascent, it can be used to determine the proper variation in $\underline{x}'(t_o)$ to reduce I, and thus, cause $\underline{x}'(t_f)$ to approach $\underline{x}_f'$. This variation is calculated in the following equation.

$$[\underline{x}'(t_o)]_{new} = [\underline{x}'(t_o)]_{old} - I\,\frac{\text{grad}(I)}{||\text{grad}(I)||^2}$$

The new value for $\underline{x}'(t_o)$ may be used as initial conditions for a new solution to Equations (4-14) and (4-16) which in turn allows a re-evaluation of the performance index I and the new values for

$\underline{x}'(t_o)$ may be calculated. This process is repeated until I is sufficiently small, that is, $\underline{x}'(t_f)$ is sufficiently close to $\underline{x}_f'$.

One undesirable characteristic of this method of solution of the two-point boundary value problem is that of convergence. In many cases as the minimum of the performance index is approached, the solution will oscillate or "limit cycle" about the true solution and never actually converge to it. This problem becomes especially evident when the minimum performance index is greater than zero. It is at this point that a different method of solution is sought, a method that can start with the best approximation of the parameter influence technique and approach the true solution without encountering the same convergence problems. The method of quasilinearization is one such technique and is discussed in the following paragraphs.

The first step in the method of quasilinearization is to obtain a linear approximation to Equation (4-14). This is accomplished by truncating a Taylor's series expansion of (4-14) about a reference trajectory. The original nonlinear differential equation is

$$\frac{d\underline{x}}{dt} = \underline{F}(\underline{x}, t) \tag{4-14}$$

and the truncated Taylor's series expansion is

$$\frac{d\underline{x}}{dt} = \underline{F}(\underline{x}_r, t) + \left.\frac{\partial \underline{F}}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}_r} (\underline{x} - \underline{x}_r) \quad , \tag{4-17}$$

where $\underline{x}_r(t)$ is the reference trajectory. The reference trajectory is obtained by integrating Equation (4-14) forward from t_o to t_f, using the best available estimate for the unspecified initial conditions $\underline{x}'(t_o)$.

The general philosophy of this technique is briefly outlined. Since Equation (4-17) is linear, the principle of superposition holds and the necessary initial conditions on x_i', $i = 1, 2, \ldots k$, to meet specified final conditions can be easily determined. If the reference trajectory in Equation (4-17) is close to the true solution trajectory, then the initial conditions determined from Equation (4-17) will be the correct initial conditions to cause the solution of Equation (4-14) to pass through the specified final conditions. In general the solution to Equation (4-14) will not satisfy exactly the specified final conditions and the process must be repeated several iterations. With each iteration a new reference trajectory must be generated using as initial conditions the specified conditions at t_o, $x_1(t_o), \ldots x_{2n-k}(t_o)$ and the values for $x_1'(t_o) \ldots x_k'(t_o)$ as determined in the previous iteration.

It is important that the first reference trajectory be close to the correct reference trajectory since Equation (4-17) accurately approximates the nonlinear equation only for small variations from the reference trajectory. Thus if $\underline{x}_r(t)$ is not a good approximation to the true solution, the resulting initial conditions that make the solution to Equation (4-17) meet the specified final conditions will make the solution to Equation (4-14) actually diverge from the correct trajectory, rather than converge.

A computational algorithm developed by Unruh (20) combines the method of parameter influence coefficients and quasilinearization to yield an efficient method of solution of two-point boundary value problems. The algorithm starts with the method of parameter influence coefficients and, if the solution reaches a limit cycle condition, switches to quasilinearization for final convergence. This algorithm was utilized extensively during the preparation of this thesis and was found to be superior to another algorithm developed by Unruh and Sebesta (16) (quasilinearization) and one by Sylvester and Meyer (17) (also quasilinearization).

Compensation Procedure

This section will present in detail the general nonlinear system compensation method developed for this thesis. As mentioned in the previous section the primary items of discussion will include a general procedure for formulating the performance index from a problem statement. Proper utilization of the necessary conditions of Theorems 4-1 and 4-2 to yield a two-point boundary value problem amenable to solution by the methods outlined in the previous section is also discussed. Extensive treatment of the problem of determining the state dependent control $\hat{\underline{q}}(\underline{x}, \underline{k})$ is presented since this is a crucial step in the successful application of this compensation technique. Finally, methods are presented for refining the control $\hat{\underline{q}}(\underline{x}, \underline{k})$ and improving the fit to $\underline{q}^*(t)$ after an initial approximation is made.

Formulation of the Performance Index

As has been indicated previously, the performance functional $J(\underline{q})$ is formulated on the basis of the desired system response and state constraints. The performance functional on the system behavior is a mathematical function of trajectories in a state space which weights various output variables and control parameters in a predetermined fashion. The formulation of this function is an important step in the synthesis or compensation of a control system using optimization theory. The system designer is able to influence the nature of the resulting system by the manner in which he formulates this index.

In general, the requirements considered in performance functional formulation include not only the performance requirements but also restrictions on the optimal control to insure physical realizability. The performance index almost always involves a measure of an error term which represents the difference between some desired response and the actual response. The system designer is interested not so much in an instantaneous value of the error measure as he is in the cumulative effect of this instantaneous measure throughout an interval of time. Hence, the performance functional is usually expressed as the time integral of the error measure over a suitable interval of time, t_o to t_f, throughout which the system performance is of interest. If specific interest in the error value at a particular point in time is indicated, the performance index may be formulated as an integral plus an instantaneous value at that time. Thus, as mentioned in passing earlier, the general form of the performance functional may be given as

$$J = K[\underline{x}(t_f)] + \int_{t_o}^{t_f} L(\underline{x}, \underline{q}, t)\, dt \qquad (4\text{-}18)$$

where $K\left[\underline{x}(t_f)\right]$ is a function evaluated at t_f only, and $L(\underline{x}, \underline{q}, t)$ is an error function integrated from t_o to t_f. The following discussion in this section is devoted to describing how to formulate a performance functional in the general form given above for the particular problem of system compensation as treated in this thesis.

The function $L(\underline{x}, \underline{q}, t)$ is separated into two functions, one describing the response error and the other a function of the control, thus

$$L = L'(\underline{x}, t) + L''(\underline{q}, t).$$

The primary objective in minimizing the performance index is to force the system response to meet the desired characteristics. Thus the total effect of the L'' term on the final value of J must be minimized. This is accomplished through the proper use of weighting coefficients which will be described shortly. First, a discussion of the reasoning in the formulation of the L'' term is presented.

The only manner in which the inclusion of L'' in the performance index assures physical realizability is by restricting the variations of $\underline{q}$. Since the control variable $\underline{q}$ will eventually be implemented with physical hardware, it must be bounded. One means of bounding the control is to place it in the performance functional through the function L''. Another reason for including $\underline{q}$ in the integrand of the performance functional is that by doing so, the singular control problem is circumvented. The singular control problem is termed the case in which the function

$$\frac{\partial H}{\partial \underline{q}} = \underline{0}$$

does not yield $\underline{q}$ as a function of $\underline{x}$ and $\underline{p}$. Although the optimal control problem can still be solved in the case of singular control (the actual necessary condition is that $H(\underline{x}^*, \underline{p}^*, \underline{q})$ be a minimum for $\underline{q} = \underline{q}^*$), in the general case it is much more difficult. The solution does not result in a two-point boundary value problem which can be solved readily on a digital computer. Therefore the function of the control L" is included in the performance index for the system compensation problem.

L" is formulated as

$$k_1 \langle \underline{m}, \underline{m} \rangle$$

where $m_i = q_i / k_{2i}$. For the more common case of a scalar control, this reduces to

$$L'' = k_1 m^2$$

and

$$m = \frac{q}{k_2} \quad .$$

The case of scalar control will be discussed here, but the same reasoning in the formulation of the performance functional for a vector control will apply. The variable m is simply the control variable scaled by $1 / k_2$. The scaling constant k_2 is selected so that the variable m will be within the range $-1 \leq m \leq 1$ as q varies within the desired bounds. The variable m will tend to stay within the range ±1.0 since it is included as a quadratic term in the performance functional. For example, if it is decided that q should be allowed to vary within the bounds ±1000, k_2 would be selected as 10^3.

The initial selection of the bounds on q are based primarily on the manner in which it is added to the system equations. In order for q to influence the trajectory of the solution to the differential equation

$$\dot{x}_i = f_i(\underline{x}, t) + q$$

it can be reasoned that q should take on values at least of the same order of magnitude as $\dot{x}_i$ and f_i. Thus an effective technique for selecting k_2 is to first solve the uncompensated system equations to determine the nominal values of $\dot{x}_i$ or f_i. Then for $m = 1$,

$$k_2 m = k_2 = |q|_{max} \simeq |\dot{x}_i|_{max} .$$

For example, if $|\dot{x}_i|_{max}$ is found to be 4.5×10^2 then the bounds on q may be set at $\pm 10^3$. In this instance, then, $k_2 = 10^3$. A discussion of the selection of k_1 will be deferred until the formulation of L' is presented.

A great deal of flexibility exists in the formulation of that portion of the performance functional concerning the response error. This flexibility makes it difficult to make general statements regarding L', but at the same time, this flexibility represents an advantage since it allows the study of a wide variety of problems. Perhaps the most specific description of L' is that it is simply a means of assessing a penalty to the response when it diverges from the desired trajectory. Any means of achieving this end, while meeting the requirements discussed in the section "Basic Concepts", is satisfactory although the results are not necessarily the same. The effects of various integral performance indices on system response have been studied by several investigators and the results reported in the literature (18, 19). For instance, an integral square error performance index typically results in a slightly underdamped, quick responding system while an integral absolute value index yields a more heavily damped response. In general, it can be stated that the form of the performance functional will have the same characteristic effect on the system response in the case of system compensation as has been found in past studies.

It was not the purpose of this thesis to study the effects of various performance indices but rather to stress the variability of the form of the performance functional and to illustrate its formulation for some of the case problems studied. Perhaps the most commonly used performance measure is the ISE (integral square error) due to its general applicability, its mathematical convenience (i.e. it meets the differentiability and continuity requirements and is relatively easy to manipulate analytically), and the fact that for linear systems it leads to optimal feedback controls which are linear. For the case of an ISE performance functional L' is given in general as

$$L' = k_3 \langle \underline{e}, \underline{R}\, \underline{e} \rangle \tag{4-19}$$

where $\underline{e}$ is the error vector

$\underline{e} = [\underline{x}(t) - \underline{x}_d(t)]$,

$\underline{x}_d(t)$ = any desired trajectory, either data points or a time function,

$\underline{R}$ = a positive semi-definite diagonal weighting matrix,

k_3 = constant.

Carrying out the indicated vector product, L' becomes

$$L' = k_3 (R_1 e_1^2 + R_2 e_2^2 + \ldots + R_n e_n^2)$$

where the R_i's are terms which weight the various errors. These weighting terms may be time-variable if the relative importance of the errors change during the time interval of interest, or they may be zero for those states which are unrestricted. The value of the constant R's and the nominal value of the time-varying R's are selected on a relative basis. That is, these weighting constants set the relative importance of one error with respect to the others. The constant k_3 is then selected (relative to k_1 in L'') to assure that the maximum expected value of L' is large compared to the maximum value of L''.

A good rule of thumb to follow is to select k_3 relative to k_1 so that L' is approximately one order of magnitude larger than L''. Assume that k_1 is selected as 1.0. Since k_2 was selected so that

$$\left(\frac{q}{k_2}\right)^2_{max} = m^2_{max} \simeq 1.0,$$

then

$$(L'')_{max} = (k_1 m^2)_{max} \simeq 1.0.$$

The constant k_3 should then be selected so that

$$[k_3 (R_1 e_1^2 \ . \ . \ . \ R_n e_n^2)]_{max} = 10.0.$$

A more generalized performance functional is the quadratic form in which L' is given as in Equation (4-19) and $\underline{R}$ is simply a positive semi-definite matrix, not necessarily diagonal. Again, the elements of $\underline{R}$ are simply weighting terms which may be constant or time-varying. In this way terms such as

$$R_{12}e_1e_2$$

may be brought into the function if it is deemed desirable for a particular problem.

The use of a quadratic performance functional or the ISE (which is simply a special case of the quadratic) is best suited for problems in which a complete state trajectory is to be optimized. That is, the case in which it is desired that a state or states follow a specific known path from t_o to t_f is usually adequately described with a quadratic or ISE performance functional. Problems in which it is desired to limit state extrema and the terminal cost problem will be discussed following an example illustrating the formulation of an ISE time weighted function. In general, it must be emphasized that whether or not to use a quadratic form, or any other form, must ultimately be decided by the designer based upon the requirements of his particular problem.

Example 4-1. Aircraft Landing System.

This example problem will not be carried through to final solution since it was selected only to illustrate the formulation of a performance functional for a trajectory optimization problem. This problem is a particularly fine example of the flexibility that exists in the development of the performance index. The problem given here was discussed in detail by Ellert and Merrian (11) in an example problem using the parametric expansion technique.

The landing problem described here concerns an automatic control system for the final phase of an aircraft landing; the final phase being the last 100 feet of the aircraft's descent. During this final phase, the elevator deflection controls the longitudinal motion of the aircraft and hence will be the control signal.* The aircraft is subject to other controlling inputs of course, but it is assumed that motion due to these controls is uncoupled from the longitudinal motion which, thus, may be studied separately. The objective of this example is to formulate the performance index necessary to adequately describe the system requirements.

The following aircraft requirements and constraints are considered of primary importance and will be used to formulate the performance index.

1) The desired aircraft altitude $h_d(t)$ during the landing phase is shown in Figure 3 and is given analytically as

$$h_d(t) = \begin{cases} 100e^{-t/5}, & 0 \leq t \leq 15, \\ 20-t, & 15 \leq t \leq 20. \end{cases} \tag{4-20}$$

2) The desired rate of ascent is simply the time derivative of $h_d(t)$ or

$$\dot{h}_d(t) = \begin{cases} -20e^{-t/5}, & 0 \leq t \leq 15, \\ -1, & 15 \leq t \leq 20. \end{cases} \tag{4-21}$$

The rate of ascent is of major importance at touchdown and must be less than zero for the aircraft to avoid floating over the runway and

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*Note that in this problem the control is known a priori to be the elevator deflection and hence the exact limits on its motion can be stated.

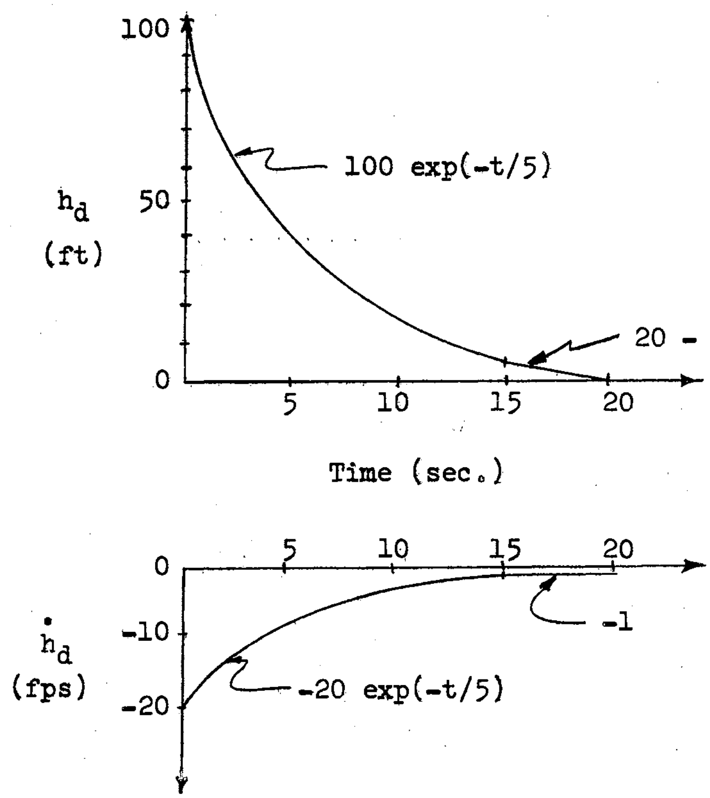

Figure 3. Desired Altitude and Rate of Ascent

perhaps overshooting it. A very large negative value is equally undesirable since it could result in overstressing the landing gear.

3) The aircraft pitch angle $\theta(t)$ at touchdown is desired to be

$$0° \leq \theta(t_f) \leq 10°. \qquad (4\text{-}22)$$

These limits on θ at touchdown are desired since a value less than $0°$ would cause the nose wheel of a tricycle landing gear to contact the ground first and a value greater than $10°$ would result in the tail gear striking the ground.

4) Throughout the landing phase, the aircraft angle of attack $a(t)$ must remain below the stall value, assumed to be $18°$. The aircraft enters the final phase with an angle of attack that is 80% of

the stall value. Hence the angle of attack must not be allowed a positive change of more than $3.6°$.

5) The elevator deflection $e(t)$ is restricted by mechanical stops to the range

$$-35° \leq e(t) \leq 15°.$$

Since the elevator should not operate against the stops, it is desired that $e(t)$ remain within the limits shown above.

For this landing system, the important response characteristics that must be controlled are: the deviation of the aircraft altitude and rate of ascent from the desired trajectories given in Equations (4-20) and (4-21), respectively, deviation of the pitch angle at touchdown from the desired value of $5°$ (which is the midpoint of the prescribed range given in Equation (4-22)), and the deviation of the angle of attack from its initial equilibrium value at t_o. Thus the integrand of the performance functional $L(\underline{x}, \underline{q})$ for this problem may be formed as

$$L = k_1[h(t) - h_d(t)]^2 + k_2[\dot{h}(t) - \dot{h}_d(t)]^2$$

$$+ k_3[\theta(t) - \theta_d(t)]^2 + k_4[a(t) - a_d(t)]^2$$

$$+ k_5[e(t) - e_d(t)]^2.$$

In this equation the k's are weighting factors that indicate the relative importance of the various terms and may be time-varying. The desired values of each of the response terms are designated by the subscript d and have the following values:

$$h_d(t) = 100e^{-t/5}, \quad 0 \leq t \leq 15$$

$$= 20-t \quad , 15 \leq t \leq 20$$

$$\dot{h}_d(t) = -20e^{-t/5}, \quad 0 \leq t \leq 15$$

$$= -1 \quad , 15 \leq t \leq 20$$

$$\theta_d(t_f) = 5°,$$

$$a_d(t) = 14.4°.$$

To maintain the elevator deflection within the prescribed limits it is convenient to select the midpoint of the range as the value of the desired control signal. Thus

$$e_d(t) = -10°.$$

The performance requirements indicate that the altitude and rate of ascent errors should be small at the desired touchdown point $t_f = 20$ to insure actual touchdown very close to this point. Large altitude and rate of ascent errors at $t = 20$ may result in touchdown prior to the start of the runway, or so far down the runway that the aircraft cannot be brought to a stop before the end of the runway. Hence these error terms should be weighted more heavily at or near $t = 20$ to stress their importance at this point. Ellert and Merriam therefore weight the altitude term to accomplish this by making $k_1(t)$ a constant plus an impulse function at $t = 20$.

$$k_1(t) = k_1' + k_1''\delta(t - 20)$$

In weighting the rate of ascent term they also take into consideration the fact that $\ddot{h}(t)$ is not important prior to the start of the runway. Hence,

$$k_2(t) = k_2' + k_2''\delta(t - 20)$$

where $k_2' = 0$ for $t < 15$ and $k_2' =$ constant for $t \geq 15$. However the step change in k_2' at $t = 15$ violates the restraint that $L(\underline{x}, \underline{q}, t)$ be continuous in time. (The impulse function, which also violates this constraint, will eventually be removed from the integral.) In order that L and $\partial L / \partial t$ be continuous as required, k_2' may be formulated as a time function that starts at $t = 15$, i.e.

$$\begin{aligned} k_2'(t) &= k_2'(t - 15)^2, \quad t \geq 15 \\ &= 0 \qquad\qquad\quad\; , \quad t \leq 15. \end{aligned}$$

The pitch angle is important only at touchdown, hence the pitch angle error term should be weighted only at the desired touchdown time, $t = 20$. Ellert and Merriam accomplish this by making the time varying weighting function an impulse at $t = 20$, i.e.

$$k_3(t) = k_3\delta(t - 20).$$

An alternative is to recognize the pitch angle error as a terminal cost problem and to formulate the performance functional as in Equation (4-18). That is,

$$J(e) = K[\theta(t_f)] + \int_{t_o}^{t_f} L(\underline{x}, e, t)\, dt$$

where

$$K[\theta(t_f)] = k_3\left(\theta(t_f) - 5°\right)^2,$$

k_3 is a positive constant and the integrand $L(\underline{x}, e, t)$ does not contain the pitch angle term.

Since the angle of attack and elevator deflection errors are important throughout the entire landing phase, k_4 and k_5 are simply constants. Thus the complete performance functional becomes

$$J = k_3(\theta(20 - 5^\circ)^2 + \int_0^{20} \{[k_1' + k_1'' \, \delta(t - 20)] \, [h(t) - h_d(t)]^2$$

$$+ [k_2'(t) + k_2'' \, \delta(t - 20)] \, [\dot{h}(t) - \dot{h}_d(t)]^2 + k_4[a(t) - 14.4^\circ]^2$$

$$+ k_5[e(t) + 10^\circ]^2\} \, dt.$$

The impulse weighting of the altitude and rate of ascent terms can be treated as additional terminal cost terms and taken from under the integral sign. The performance functional can thus be rewritten

$$J = k_3[\theta(20) - 5^\circ]^2 + k_1'[h(20)]^2 + k_2''[\dot{h}(20) + 1]^2$$

$$+ \int_0^{20} \{k_1'[h(t) - h_d(t)]^2 + k_2(t)[\dot{h}(t) + 1]^2 + k_4[a(t) - 14.4^\circ]^2$$

$$+ k_5[e(t) + 10^\circ]^2\} \, dt. \qquad (4\text{-}23)$$

The problem now becomes that of determining proper values for the constants in the function. It will be assumed that all terms in Equation (4-23) have equal importance, hence they must be weighted so that each contribute equally to the value of the index. The value of k_5 is first selected so that the elevator deflection term approaches unity as e approaches either limit. Thus

$$k_5 = \frac{1}{25^2} = 0.0016.$$

The desired maximum angle of attack is $18°$ which implies that k_4 should be

$$k_4 = \frac{1}{3.6^2} = 0.0772.$$

The authors state in the problem definition that the aircraft is assumed to be waved off and does not attempt to complete the landing if the initial altitude and rate of ascent differ by more than 20% and 25%, respectively, from their desired values. Thus the maximum altitude error is 20 feet, from which k_1 can be determined to be

$$k_1' = \frac{1}{20^2} = .0025.$$

At the point where the rate of ascent is first considered, a 25% error is 0.25 feet/second, thus

$$k_2' = \frac{1}{(.25)^2} = 16.$$

In order to determine the values of the constants associated with the terminal point errors a bit more estimation and ingenuity is required. Since it was assumed that each term was of equal importance, the value of each terminal error term should be approximately the same as the integral from 0 to 20 of the other error terms. Since the terms under the integral were scaled to have a maximum value of approximately 1.0, it is convenient to assume that these terms will average about half this amount throughout the interval of interest. Thus the contribution of the integral error terms to the performance index should be approximately 10.0 each. This in turn implies that each of the terminal errors should be weighted so that their contribution is 10.0. The maximum desired error in pitch angle is 5°, hence

$$k_3(5)^2 = 10,$$

$$k_3 = \frac{10}{5^2} = 0.4.$$

If a 20 feet initial error in altitude error were completely uncorrected during the landing phase, the final altitude error would still be 20 feet and

$$k_1'' = \frac{10}{20^2} = 0.025.$$

By the same reasoning

$$k_2'' = \frac{10}{4^2} = 0.625.$$

Hence the final form of the performance index is

$$J = 0.4[\theta(20) - 5^\circ]^2 + 0.025[h(20)]^2 + 0.625[\mathring{h}(20)]^2$$

$$+ \int_0^{20} \{0.0025[h(t) - h_d(t)]^2 + k_2'(t - 15)^2\ [\mathring{h}(t) + 1]^2$$

$$+ 0.0772[a(t) - 14.4^\circ]^2 + 0.0016[e(t) + 10^\circ]^2\}\ dt, \qquad (4\text{-}24)$$

where

$$k_2' = 0, \quad t < 15$$
$$= 16, \quad t \geq 15.$$

The primary purpose in presenting this example problem was to illustrate the flexibility possible in formulating the performance index. It should be evident that a multitude of diverse performance requirements can be described in the performance functional. Particularly the time weighting of the error terms is important since in this way the particular error that is important at any point in time can be stressed. The particular error terms, the form of the performance functional or the values of the constants are not fixed. Indeed, the constant values selected are only estimates arrived at by the approximate analysis just performed. After solution of the problem, it could be determined that certain of the terms should be weighed more heavily, or less so, which could easily be done and the problem resolved.

One final comment regarding this problem will be made. The performance functional given in Equation (4-24) actually attempts to constrain the angle of attack $a(t)$ to remain within the bounds $10.8° - 18.0°$. The original problem statement was that $a(t)$ should not have a positive change of $3.6°$ from its nominal value of $14.4°$; that is, it should not go above the stall value of $18°$. This in reality, then, is a problem of limiting the positive extreme value of this variable, a problem which will be treated in the following paragraphs.

In many problems, control of an output variable trajectory is not desired, but rather it is desired to limit the maximum or minimum values. That is, the exact shape of the trajectory is not important so long as it remains below some desired maximum level, or vice versa. This was the case with the angle of attack variable in the above problem. The exact value of $a(t)$ was unimportant so long as it remained below $18°$.

Figure 4 illustrates the shape of the error index associated with an error squared term and the desired shape of the index to limit the maximum value of an error. The top curve shows the error index value for a variable x_1 that is desired to be bounded between 9 and 11. That is, the desired range on x_1 is ± 1.0 about the nominal value of 10.0. The expression for this term is $(x_1 - 10)^2$. The bottom curve, however, gives the error index for the variable x_2 which is desired to have a maximum value of 10.0. The index value for any value of x_2 less than 9.0 is zero. At $x_2 = 9.0$, the index becomes a squared expression that is scaled to have a value of 1.0 at $x_2 = 10.0$ and increases as a quadratic for $x_2 \geq 9.0$. This is accomplished by formulating the error expression for x_2 as

$$S(x_2 - 9)^2, \qquad (4\text{-}25)$$

where

$$S = 0, \quad x_2 < 9.0,$$

$$S = 1, \quad x_2 \geq 9.0.$$

This expression accomplishes the task of assessing a penalty or error to x_2 as it approaches or exceeds the desired maximum of 10.0 but does not penalize the variable for values less than 9.0. The error term in Equation (4-25) also satisfies the requirements that $L(\underline{x}, \underline{q}, t)$, $\frac{\partial L}{\partial t}$ and $\frac{\partial L}{\partial \underline{x}}$ be continuous.

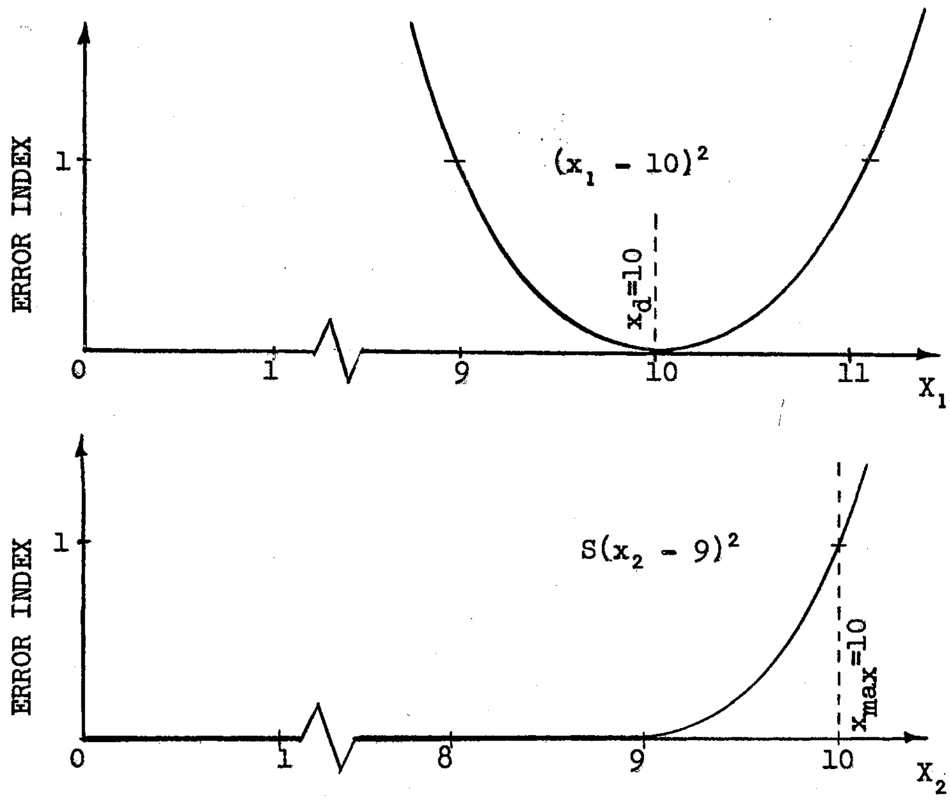

Figure 4. Error Index for Trajectory Fitting and Limiting State Extrema

The general development of the error index for limiting state extrema is discussed in the following. The switch S is set to turn on or change from 0 to 1 at about 90% of the desired maximum and the squared term is then scaled to have a value of 1.0 at the maximum. The switch and scaling could be set to turn on at any point (such as 99% or even 99.99% of the desired maximum) up to the desired limit and still have a value of 1.0 at the limit. However as the turn-on point approaches the desired limit, the error index begins to come closer and closer to approximating a discontinuous index that is zero for values less than the limit and infinity at or beyond the limit. To a computational algorithm used to solve the resulting optimal control problem, this error function does appear discontinuous and serious convergence problems then make a solution almost impossible. Experience has shown that the squared error curve should start at approximately 90-95% of the desired maximum value. The proper scaling of the squared term is shown in general in Equation (4-26).

$$S\left(10\frac{x}{x_{max}} - 9\right)^2 \qquad (4\text{-}26)$$

In this equation x_{max} is the desired maximum value for the variable x, or the minimum if the limit is a negative value.

Recalling the problem of the aircraft landing system, a more appropriate error for the angle of attack would have been the following:

$$k_4 S\left(10\frac{a(t)}{18^\circ} - 9\right)^2 = k_4 S\left(\frac{a(t)}{1.8^\circ} - 9\right)^2$$

where

$$S = 0, \quad a(t) < 16.2^\circ,$$
$$S = 1, \quad a(t) \geq 16.2^\circ.$$

For an angle of attack less than 16.2°, the performance index would not be penalized, but as the angle of attack approaches closer than 16.2° to the stall value, the error is penalized as a quadratic.

One final point will be made regarding the formulation of the performance functional for use in conjunction with the system compensation method of this thesis. There is no restriction on the form of the terms used in the error index $L(\underline{x}, \underline{q}, t)$ other than the differentiability of L and the continuity of L, $\frac{\partial L}{\partial t}$ and $\frac{\partial L}{\partial \underline{x}}$. Thus many varied forms of weighting factors and error terms may appear in the error index. Examples might be an exponential weighting factor e^{-at}, a modified error absolute value $|x - x_d|$, a time times error absolute value $|t(x - x_d)|$, or a fourth power error term $(x - x_d)^4$. In using these error terms in the performance functional for system compensation, the system designer should expect essentially the same response characteristics that would be obtained by using these error terms in any other system design or modification scheme.

The error squared form, however, possesses certain advantages when used in the formulation of the performance functional for this thesis. First, the error squared terms satisfy the continuity and differentiability requirements whereas the absolute value is discontinuous at $x - x_d = 0$ and, hence, must be modified near zero to correct this. Second, the error squared formulation results in a set of differential equations linear in the adjoint variables which is much less likely to have convergence problems. Finally, the error squared form is probably the form most generally applicable to a variety of problems. Therefore, unless specific requirements indicate that another form of error term is required, the error squared form is recommended for at least the initial attempt to compensate a system.

Derivation of the Optimal Control $\underline{q}^*(t)$

The discussion presented within this section has basically two objectives. The first objective is to illustrate the proper application of the necessary conditions given in Theorems 4-1 and 4-2 in formulating the canonical system of system states and adjoint states and the proper boundary conditions on this system. That is, the proper formulation of the two-point boundary value problem, the solution of which yields the desired optimal control. Three simple problems representing the following general cases will be discussed: (1) no terminal cost problem, (2) terminal cost problem, and (3) problem in which the independent variable t appears explicitly. The second objective is to present some suggested techniques for determining a first guess for the initial conditions on the adjoint variables. This problem must be considered since most of the computational techniques for solving two-point boundary value problems require starting guesses for the unspecified initial conditions. These initial guesses must, in general, be fairly accurate.

Example 4-2. No Terminal Cost Problem

This example problem illustrates the proper formulation of the two-point boundary value problem for a simple trajectory optimization problem with no terminal cost. The system to be compensated is defined on the interval $(0, T)$ by the differential equation

$$\dot{x} + x = 0,$$
$$x(0) = x_0.$$

A compensating control q is to be added to the system to minimize the performance functional

$$J = \tfrac{1}{2}\int_0^T [(x - x_d)^2 + m^2]\, dt$$

where $q = km$. The compensated system equation is thus

$$\dot{x} = -x + q.$$

The Hamiltonian is given by

$$H = \tfrac{1}{2}(x - x_d)^2 + \tfrac{1}{2}m^2 + p(-x + km).$$

The necessary conditions of Theorem 4-1 state that for $q = q^*$ to be optimal, the associated x^* and p^* must be solutions to the following system.

$$\dot{x}^* = \left.\frac{\partial H}{\partial p}\right|_* = -x^* + km^*$$

$$\dot{p}^* = -\left.\frac{\partial H}{\partial x}\right|_* = -(x^* - x_d) + p^*$$

$$x^*(0) = x_o$$

$$p^*(T) = 0$$

$$0 = \left.\frac{\partial H}{\partial m}\right|_* = m^* + kp^* \rightarrow m^* = -kp^*$$

If the change of variables $x = x_1$ and $p = x_2$ is made and the equation for m^* is substituted into the equation for $\dot{x}^*$, the two-point boundary value problem becomes

$$\dot{x}^*_1 = -x^*_1 - k^2 x^*_2$$

$$\dot{x}^*_2 = -x^*_1 + x_d + x^*_2$$

$$x^*_1(0) = x_o \qquad (4\text{-}27)$$

$$x^*_2(T) = 0$$

and the relation for q^* is

$$q^* = -k^2 x^*_2.$$

Thus Equation (4-27) represents the results of the application of the necessary conditions for optimal control. The desired optimal control will be obtained upon solution of these equations satisfying the stated boundary conditions.

Example 4-3. Terminal Cost Problem

The application of the necessary conditions of Pontryagin's maximum principle will be illustrated for the case of trajectory optimization with terminal cost. For this problem, the system is given on the interval $(0, T)$ by

$$\dot{x} + x = 0,$$

$$x(0) = x_o.$$

The system is to be compensated to minimize the performance functional

$$J = \tfrac{1}{2}[x(T) - x_d]^2 + \tfrac{1}{2}\int_0^T [(x - x_d)^2 + m^2]\, dt$$

where $q = km$. The compensated system equation is given by

$$\dot{x} = -x + q,$$

and the Hamiltonian is formed as

$$H = \tfrac{1}{2}(x - x_d)^2 + \tfrac{1}{2}m^2 + p(-x + km).$$

Theorem 4-2 states that the necessary conditions for m^* to be an optimal control are that m^* and the associated x^* and p^* be solutions of the following system

$$\dot{x}^* = \left.\frac{\partial H}{\partial p}\right|_* = -x^* + km^* \qquad (4\text{-}28)$$

$$\dot{p}^* = -\left.\frac{\partial H}{\partial x}\right|_* = -(x^* - x_d) + p^*$$

where the boundary conditions are given as

$$x^*(0) = x_o$$

$$p^*(T) = \frac{\partial}{\partial x(T)} \{\tfrac{1}{2}[x^*(T) - x_d]^2\} = x^*(T) - x_d,$$

and

$$0 = \frac{\partial H}{\partial m}\bigg|_* = m^* + kp^* \rightarrow m^* = -kp^*.$$

The final condition on p^* is not easily determined for this case since it is actually a function of the optimum solution itself. The basic method for solution of the two-point boundary value problem must be slightly modified for this case. First, however, the change of variable $x = x_1$ and $p = x_2$ will be made and Equation (4-28) restated as

$$\dot{x}^*_1 = -\dot{x}^*_1 - k^2 x^*_2$$

$$\dot{x}^*_2 = -x^*_1 + x_d + x^*_2$$

$$x^*_1(0) = x_o$$

$$x^*_2(T) = x^*_1(T) - x_d$$

and

$$q^* = -k^2 x^*_2.$$

Recall that the gradient method for solving the two-point boundary value problem began with the formulation of a final-value performance index

$$I = \sum_{i=1}^{k} [x_i'(T) - x_{fi}]^2$$

where the x_i' were the states with final values specified and x_{fi} were the specified final values. Since the basic problem was to determine the necessary initial conditions on x_i' to cause the solution to pass through the specified final points, the gradient of I with respect to $x_i(0)$ was calculated. For the case at hand

$$I = [x_2(T) - (x_1(T) - x_d)]^2,$$

where the asterisks denoting optimal solution have been deleted for convenience in writing. The "gradient" of I with respect to $x_2(0)$ for this scalar case is then

$$\frac{\partial I}{\partial x_2(0)} = 2[x_2(T) - x_1(T) + x_d] \left[\frac{\partial x_2(T)}{\partial x_2(0)} - \frac{\partial x_1(T)}{\partial x_2(0)} \right] .$$

This form of the gradient is basically different from that for the case of known final conditions. The two terms $\partial x_2(T) / \partial x_2(0)$ and $\partial x_1(T) / \partial x_2(0)$ may be determined, however, in the same manner as stated previously, and thus the gradient may be evaluated after an initial guess for $x_2(0)$ is provided and the necessary equations integrated from 0 to T. The method of solution of the two-point boundary value problem is therefore essentially the same with the exception of the form of the gradient of I. This difference must be taken into account in the solution of the two-point boundary value problem for the terminal cost problem.

Example 4-4. Independent Variable Explicit in System Equation

This example problem considers the case of a simple trajectory optimization problem for a system with a time-varying coefficient.

$$\dot{x}_1 + tx_1 = 0$$

$$x_1(0) = x_o$$

$$J = \tfrac{1}{2}\int_0^T [(x_1 - x_d)^2 + m^2]\, dt$$

$$q = km$$

The compensated equation is

$$\dot{x}_1 = -tx_1 + q.$$

At this point an auxiliary state variable defined by $\dot{x}_2 = 1$ is introduced with the initial condition

$$x_2(0) = 0.$$

The augmented system equations then become

$$\dot{x}_1 = -x_1 x_2 + km$$

$$\dot{x}_2 = 1$$

$$x_1(0) = x_o$$

$$x_2(0) = 0.$$

The Hamiltonian is given by

$$H = \tfrac{1}{2}(x_1 - x_d)^2 + \tfrac{1}{2}m^2 + p_1(-x_1 x_2 + km) + p_2.$$

By Theorem 4-1, m^* must satisfy

$$\dot{x}^*_1 = \frac{\partial H}{\partial p_1}\bigg|_* = -x^*_1 x^*_2 + km^*$$

$$\dot{x}^*_2 = \frac{\partial H}{\partial p_2}\bigg|_* = 1$$

$$\dot{p}^*_1 = -\frac{\partial H}{\partial x_1}\bigg|_* = -x^*_1 + x_d + p^*_1 x^*_2$$

$$\dot{p}^*_2 = -\frac{\partial H}{\partial x_2}\bigg|_* = p^*_1 x^*_1$$

$$x^*_1(0) = x_o \qquad (4\text{-}29)$$

$$x^*_2(0) = 0$$

$$x^*_2(T) = T$$

$$p^*_1(T) = 0$$

$$0 = \frac{\partial H}{\partial m}\bigg|_* = m^* + kp^*_1 \rightarrow m^* = -kp^*_1 .$$

Equation (4-29) is the resulting two-point boundary value problem which must be solved while meeting the four boundary conditions shown following the equation. Note that the state x_2 has conditions specified both at $t = 0$ and $t = T$. These conditions must be satisfied since x_2 is in reality the variable t. Note also that the adjoint variable corresponding to the system state with both boundary conditions specified does not have either end point specified.

The remaining paragraphs of this section present three techniques which have proved helpful in determining starting guesses for the initial conditions of the states of the two-point boundary value problem that have unspecified initial conditions. Most of the computational algorithms for solving split boundary value problems require fairly accurate starting conditions to insure convergence to a solution.

The two-point boundary value problem to be solved can be given in general by the 2n set of differential equations

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t) + \underline{q}(\underline{p}), \qquad (4\text{-}30)$$
$$\underline{\dot{p}} = \underline{g}(\underline{x}, \underline{p}, \underline{u}, t),$$

with the appropriate boundary conditions. It is frequently easier to solve the problem

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t), \qquad (4\text{-}31)$$
$$\underline{\dot{p}} = \underline{g}(\underline{x}, \underline{p}, \underline{u}, t),$$

in which the control is not applied to the systems equations. This modified problem certainly does not result in an optimum solution but it will often yield a set of initial conditions for $\underline{p}(t)$ that will be satisfactory for the solution of Equation (4-30). A straightforward technique for solving the equations of (4-31) is to integrate the $\underline{\dot{x}}$ equations from the known initial conditions forward in time from t_o to t_f. The trajectory $\underline{x}(t)$ is stored during this integration. Then the $\underline{\dot{p}}$ equations may be integrated from the known final conditions $p(t_f)$ backwards in time from t_f to t_o, using the previously stored trajectory $x(t)$ as inputs to the $\underline{\dot{p}}$ equations. The values of $\underline{p}(t)$ at t_o then become the initial guesses for the complete solution of Equation (4-30).

The second technique stems from recognition of the fact that through proper weighting in the performance functional the control variable $\underline{q}$ is only slightly constrained. This is done so that $\underline{q}$ may take on values as large as necessary to properly control the system. However in so doing, the sensitivity of $\underline{q}$ with respect to the adjoint variables $\underline{p}$ is greatly increased. If the starting guesses for the initial conditions on the adjoint variables are not very close to the correct values, the adjoint variables (the equations for which are unstable) will become very large on the first iteration of the two-point boundary value solution algorithm. Thus it quite frequently occurs that the large adjoint values and the high sensitivity of $\underline{q}$ with respect to $\underline{p}(t)$ result in an overflow condition in the computer. That is, an arithmetic operation results in a number which is larger than the maximum value allowed in the computer. In order to circumvent this problem, the weighting factor k_1 on the control function in the equation for the error index

$$L = k_3\langle \underline{e}, \underline{r}\ \underline{e}\rangle + k_1\langle \underline{m}, \underline{m}\rangle$$

should be increased. In many cases, an increase in the magnitude of k_1 by one or two orders of magnitude will result in a sufficient decrease in the sensitivity of $\underline{q}$ with respect to $\underline{p}(t)$ so that the resulting two-point boundary value problem can be solved. The initial conditions on the adjoint variables which result from this modified problem can then be used as starting guesses for the original problem with the desired value of k_1. In some instances, k_1 must be decreased in two or three steps, each time solving the two-point boundary value problem using as starting guesses on the adjoint variables the initial conditions resulting from the previous solution.

A final technique is presented for determining starting initial condition guesses. Although sufficient experience has not been gained with this method to fully determine its effectiveness, it is presented as a possible alternative should the other methods discussed fail. The use of this technique is prompted by the fact that the adjoint variable equations are unstable and hence their magnitudes increase with time (for oscillatory adjoint solutions, the peak values increase). Thus in some problems, the $\underline{p}(t)$ trajectories increase to the point of computer overflow. In these cases, the problem final time t_f can be reduced to some point in time less than the value at which the overflow condition occurred, say t_f', where $t_f' < t_f$. The two-point boundary value problem is then solved over the interval t_o to t_f' and the resulting initial conditions on $\underline{p}$ used as starting guesses for the problem over the interval t_o to t_f. As with the previous technique, this method may have to be applied two or three times, each time increasing the final time until t_f is reached.

As should be evident from the preceding discussion, the solution of the two-point boundary value problem can sometimes be very difficult. In fact this problem is currently one of the major limitations in applying optimal control theory to practical nonlinear control problems. In general, the control engineer must bring all of the pertinent information possible into the problem as well as make liberal use of his intuition to solve the problem. A suggestion is made in Chapter VI, SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS for a more refined technique for determining initial starting conditions.

Determination of the Fitted Control $\hat{\underline{q}}(\underline{x}, \underline{k})$

The discussion of this section deals with the problem of determining a function of the optimum system state trajectories that will fit the optimum control $\underline{q}^*(t)$. It will be assumed in this section that the appropriate two-point boundary value problem has been solved to yield $\underline{q}^*(t)$ and that the resulting optimum system response $\underline{x}^*(t)$ is sufficiently improved over $\underline{x}(t)$ to warrant an attempt to implement the optimum control. That is, the decision has been made to try to fit some $\hat{\underline{q}}$ to $\underline{q}^*(t)$. The fitting process is divided basically into two steps, the first being a determination of the general functional form of $\hat{\underline{q}}(\underline{x}, \underline{k})$. This determination is initially based on the general shapes of $\underline{q}^*(t)$ and $\underline{x}^*(t)$, physical realizability requirements and parameters in the original system that can be adjusted. Once the general functional form for $\hat{\underline{q}}(\underline{x}, \underline{k})$ is selected, the second step involves the determination of values of $\underline{k}$ to give the best fit to $\underline{q}^*(t)$. During this step, the k_i's must be limited to physically realizable values.

One of the principal benefits in the method of system compensation discussed in this thesis can be exploited during this phase of the compensation procedure. First, the system designer has the opportunity to study the optimum response and to determine its merits or whether it represents any advantage over the original response. Second, he has available the optimum or ideal control trajectory which produced the optimum response. He thus has a means to measure the success or failure of any of the means he chooses to compensate the system. Finally, the fact that he has the optimum control trajectory available affords some information as to how that control should be physically implemented.

Some of the terms that should be used in $\hat{\underline{q}}(\underline{x}, \underline{k})$ to fit $\underline{q}^*(t)$ can be determined through a close study of plots of $\underline{q}^*(t)$ and $\underline{x}^*(t)$. By comparing the general shapes of the curves for $\underline{q}^*(t)$ and $\underline{x}^*(t)$, correlations between the control and certain state trajectories can be recognized. If a control signal matches fairly closely one of the state trajectories, then a high degree of correlation between these two variables likely exists and that state should be included in the functional form for $\hat{\underline{q}}(\underline{x}, \underline{k})$. Likewise, if the control and some combination of states, such as the product of two states or their sum or difference, appear to be correlated, then this combination of states should be included in $\hat{\underline{q}}(\underline{x}, \underline{k})$.

For example, assume that a second-order system with states x_1 and x_2 is to be compensated with a scalar control q. After $q^*(t)$ has been determined, a plot of $q^*(t)$, $x_1^*(t)$ and $x_2^*(t)$ versus time should be examined. Assume that $x_1^*(0) = x_2^*(0) = 0$, but that $q^*(0) \neq 0$. Obviously, in order to form $\hat{q}$ as a function of x_1^* and x_2^*, $\hat{q}$ must contain a constant bias term to provide a fit at $t = 0$. Thus the first term to be selected in the formulation of $\hat{q}$ will be the constant k_1 so that

$$\hat{q} = k_1 + \text{other terms to be determined.}$$

In addition, if it appears that the general shape of $q^*(t)$ is similar to the shape of $x_2^*(t)$ then the term k_2x_2 should be added to $\hat{q}$. Thus

$$\hat{q} = k_1 + k_2x_2 + \text{other terms.}$$

Many other observations may be considered as well. If, for instance, $q^*(t)$ does not change sign on the interval (t_o, t_f) but $x_1(t)$ and $x_2(t)$ take on both positive and negative values, perhaps some function of $|x_1|$ or x_i^2, $i = 1, 2$, should be considered. A full appreciation for the utility of this technique cannot be gained from a discussion alone. Examples 1 and 2 in Chapter V present excellent demonstrations of the application and utility of this method of determining the form of $\hat{q}$.

Another device for determining the proper correlation between the control and the states involves plotting $\underline{q}^*(t)$ versus the states $\underline{x}^*(t)$ themselves. Consider again the example of the previous paragraph. Since $q^*(t)$, $x_1^*(t)$ and $x_2^*(t)$ are all known at the same points in time, plots of q^* versus x_1^* and q^* versus x_2^* may be obtained. If q^* can be approximated by a linear function of one of the states, then the appropriate plot should be approximately a straight line. If however the plot appears more as a squared or cubic curve, this suggests that $\hat{q}$ be made a function of $k_i x_j^2$ or $k_i x_j^3$. This concept can be extended to plotting q^* versus various combinations of the states such as $x_1 x_2$, $x_1 x_2^2$, $|x_1|$, $x_1 + x_2$, etc. A close correlation between q^* and one of these terms can easily be recognized from these plots.

In most system compensation problems, certain of the system parameters may be adjustable. To determine the proper parameter changes, the terms containing these parameters should be included in $\hat{q}$. Thus if the example system cited previously contains a nonlinear spring modeled by the expression kx_1^3 and if it is determined that the spring rate can be changed, then the term $k_3 x_1^3$ should be added to the others in $\hat{q}$. Thus

$$\hat{q} = k_1 + k_2 x_2 + k_3 x_1^3 .$$

It should be pointed out, however, that the fact that a parameter is not adjustable does not prevent the use of the term associated with that parameter in the expression for $\hat{\underline{q}}$. In this case the system designer simply must realize that this portion of $\hat{\underline{q}}(\underline{x}, \underline{k})$ cannot be implemented by adjusting a parameter in the original system; instead, a new element must be added to the system.

A very important consideration in determining the general form for $\hat{\underline{q}}(\underline{x}, \underline{k})$ is that of physical realizability. If it is determined by one of the techniques described above that a particular term should be included in $\hat{\underline{q}}$, but it is found that the term cannot be implemented in the physical system, then this term should not be included in $\hat{\underline{q}}$. An attempt must be made to fit $\hat{\underline{q}}$ to $\underline{q}^*$ without the use of that particular term. In this manner, the designer can assure that the resulting fitted control $\hat{\underline{q}}(\underline{x}, \underline{k})$ will contain only terms which represent physical elements. If however he learns that a satisfactory fit to $\underline{q}^*(t)$ cannot be accomplished without this term, then additional consideration should be given to determining a means to physically realize the desired expression, perhaps by a more elaborate mechanization. If this cannot be achieved, then the designer must either accept the poorly fitted $\hat{\underline{q}}(\underline{x}, \underline{k})$, attempt to completely redesign the basic system, or abandon the compensation attempt altogether. In any case the system designer has a measure of the degree of optimality he is sacrificing by not implementing the particular term involved.

One final point will be made with regard to physical realizability in connection with determining the general form of $\hat{\underline{q}}(\underline{x}, \underline{k})$. If the system designer recognizes that some physical element could be easily added to the system, the desirability of doing so can be quickly examined, even if no previous indication of the appropriateness of this term has been made. The expression representing the physical element simply can be added to the expression for $\hat{\underline{q}}$ and the resulting improvement, if any, in the fit of $\hat{\underline{q}}$ to $\underline{q}^*$ observed.

Once the general form of $\hat{\underline{q}}(\underline{x}, \underline{k})$ has been selected, the next problem is to determine the proper values of the elements of $\underline{k}$. This can normally be accomplished through the use of a computational curve fitting routine. Several nonlinear least squares curve fitting routines are available for this purpose, two of which have been used successfully in connection with this thesis (23, 24). The requirements of physical realizability must be considered during the fitting process as well as in determining the general form of $\hat{\underline{q}}(\underline{x}, \underline{k})$. Since the terms in $\hat{\underline{q}}$ represent known physical elements, the constant coefficients must in general be limited to insure realizability. The fitting routines then select the proper values of $\underline{k}$ within specified bounds to give the best least squares fit of $\hat{\underline{q}}(\underline{x}, \underline{k})$ to $\underline{q}^*(t)$.

In most physical systems, exact, rigid limits cannot be defined on the realizable parameter values. In many instances reasonable constraints can be selected within which the parameters could be fairly easily implemented, but values outside these bounds might be possible with additional effort if necessary. In problems where this situation exists, it is recommended that first $\hat{\underline{q}}(\underline{x}, \underline{k})$ be fitted to $\underline{q}^*(t)$ without constraints on the parameters $\underline{k}$. If the selected values of $\underline{k}$ fall outside the proposed bounds, then $\hat{\underline{q}}$ should again be fitted to $\underline{q}^*$, this time with constraints on the values of $\underline{k}$. If the fit of $\hat{\underline{q}}$ is significantly deteriorated as a result of constraining $\underline{k}$, then consideration should be given to relaxing some of the bounds. The relative importance of the various k_i in affecting the overall system response can be determined by calculating the sensitivity of the performance index J with respect to the elements of $\underline{k}$. This is accomplished by evaluating

$$\frac{\partial J}{\partial \underline{k}} .$$

A high sensitivity of J with respect to k_i (relative to the other parameters) indicates that the parameter k_i has a significant effect on the performance index and hence should receive top priority in being implemented with the proper value.

It must be pointed out that the procedures for selecting the general form of $\hat{\underline{q}}(\underline{x}, \underline{k})$ and determining the values of $\underline{k}$ discussed in this section cannot be applied in a simple straightforward manner. A trial form for $\hat{\underline{q}}$ must first be selected and the value for $\underline{k}$ determined. If the resulting fit is unsatisfactory or if the k_i's fall outside their bounds, then the form of $\hat{\underline{q}}$ must be modified and the new $\hat{\underline{q}}(\underline{x}, \underline{k})$ fitted to $\underline{q}^*(t)$. A technique for determining the proper modification to $\hat{\underline{q}}$ is presented in the next section.

Verification of the Control

After an initial fit of $\hat{\underline{q}}(\underline{x}, \underline{k})$ to $\underline{q}^*(t)$ has been accomplished, the resulting control trajectory must be examined to determine what modifications may be necessary to improve the fit. One device that is helpful in this respect is a study of the difference in $\hat{\underline{q}}(\underline{x}, \underline{k})$ and $\underline{q}^*(t)$, or the residual $\underline{r}$, defined as

$$\underline{r}(t) = \hat{\underline{q}}[\underline{x}(t), \underline{k}] - \underline{q}^*(t). \qquad (4\text{-}32)$$

Equation (4-32) may be written as

$$\underline{q}^*(t) = \hat{\underline{q}}(\underline{x}, \underline{k}) - \underline{r}(t).$$

This equation states that the optimal control $\underline{q}^*(t)$ is given exactly by $\hat{\underline{q}}(\underline{x}, \underline{k})$ minus $\underline{r}(t)$, where $\underline{r}(t)$ is defined in Equation (4-32). Thus if the trajectory for $\underline{r}(t)$ can be approximated by a function of $\underline{x}(t)$ and $\underline{k}$, $\underline{r}(t) \simeq \hat{\underline{r}}(\underline{x}, \underline{k})$, and then subtracted from $\hat{\underline{q}}$, the resulting function would fit $\underline{q}^*$ more closely. Thus the problem at this point becomes one of determining the general form of $\hat{\underline{r}}(\underline{x}, \underline{k})$. The values of $\underline{k}$ are not evaluated to fit $\hat{\underline{r}}(\underline{x}, \underline{k})$ to $\underline{r}(t)$, instead, $\hat{\underline{r}}$ with the undetermined k_i's is used to modify the general form of $\hat{\underline{q}}(\underline{x}, \underline{k})$.

$$\hat{\underline{q}}'(\underline{x}, \underline{k}) = \hat{\underline{q}}(\underline{x}, \underline{k}) - \hat{r}(\underline{x}, \underline{k})$$

The values of $\underline{k}$ in the modified $\hat{\underline{q}}'(\underline{x}, \underline{k})$ are then evaluated to fit $\hat{\underline{q}}'$ to $\underline{q}^*$.

The same techniques that were used to determine the general form of $\hat{\underline{q}}(\underline{x}, \underline{k})$ can be used to determine the general form of $\hat{\underline{r}}(\underline{x}, \underline{k})$. That is, plots of $\underline{r}(t)$ and $\underline{x}^*(t)$ versus time can be studied to determine possible recognizable correlations between $\underline{r}(t)$ and the state variables. Also, plots of $\underline{r}(t)$ versus the various state variables or combinations of the state variables can be examined for the same purpose. The same restrictions pertaining to physical realizability apply in formulating $\hat{\underline{r}}(\underline{x}, \underline{k})$ as in formulating $\hat{\underline{q}}(\underline{x}, \underline{k})$. Example 2 in Chapter IV illustrates the use of this technique.

Instead of using a least squared error fitting routine to determine the proper elements of $\underline{k}$, an alternate procedure may be utilized. Once the functional form for $\hat{\underline{q}}(\underline{x}, \underline{k})$ has been determined, $\hat{\underline{q}}$ can be added to the original system equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) + \hat{q}(\underline{x}, \underline{k})$$

with $\underline{k}$ as yet unspecified. Then a parameter optimization routine such as described by Unruh (9) can be used to determine the values of $\underline{k}$ that minimize the appropriate performance index. The utilization of this technique possesses certain advantages as well as disadvantages. This method represents a slightly more direct approach since the values of $\underline{k}$ are calculated directly to minimize the performance index and the resulting compensated response is generated in the processes. On the other hand, it is more difficult to constrain the variations of the parameters in a parameter optimization routine than in a least squares fitting program. The computational algorithm developed by Unruh (9) does not allow limiting the parameter values. In addition, the ability to further modify $\hat{\underline{q}}(\underline{x}, \underline{k})$ by examining the residual $\underline{r}(t)$ is lost in using the parameter optimization approach. Perhaps a combination of the two techniques might be utilized in which $\hat{\underline{q}}(\underline{x}, \underline{k})$ is first fitted as closely as possible to $\underline{q}^*(t)$. When the final form for $\hat{\underline{q}}(\underline{x}, \underline{k})$ has been determined and no further modification is desired or possible, then a parameter optimization routine could be used to re-evaluate the $\underline{k}$ parameters. The set of parameters within the desired bounds that resulted in the smallest performance index would be the desired set.

The techniques for determining the fitted control $\hat{\underline{q}}(\underline{x}, \underline{k})$ presented in the last two sections are not meant to be applied in a straightforward manner without regard for the physical implications of each step. In general it is desired to compensate the system in the most economical way possible which in turn usually implies the fewest additions or changes to the original system. Thus the first attempts at approximating $\underline{q}^*(t)$ should be with a simple $\hat{\underline{q}}(\underline{x}, \underline{k})$.

If this attempt fails, then appropriate terms should be added onto $\hat{\underline{q}}(\underline{x}, \underline{k})$.

The entire system compensation procedure discussed in this chapter is designed to give the system designer a maximum insight into the effect of each step in the procedure. A great deal of flexibility exists in the application of this technique which requires the designer to make several decisions. This result is desired, however, because this compensation procedure is meant to be a tool which the designer can use to aid him in first determining whether compensation of a system is feasible and then provide him with information as to how to effect the desired compensation. The overall objective has been to apply some of the sophisticated modern control theories to the practical problem of system design and compensation in such a way that the system designer can direct and be directed by the compensation procedure.

CHAPTER V

APPLICATION OF THE METHOD

Several example problems which illustrate application of the compensation technique developed in Chapter IV are presented and discussed in this chapter. The objectives of this chapter are threefold: first, it is desired to verify the basic concept of using optimal control theory to aid in the derivation of feedback control and compensation elements for a nonlinear system. The second objective is to demonstrate the practicality of this technique in determining physically realizable compensation terms, and finally, the third goal is to clarify certain points discussed in previous chapters by illustrative examples.

The examples presented were chosen primarily to illustrate that the objectives outlined in Chapter III had been achieved and that the technique was not restricted to a particular class of systems. Space would not permit, however, the inclusion of examples of all of the wide variety of problem requirements that could be formulated for study by this method. The problems include the study of a hydraulic spool valve, an electrical circuit and an electro-mechanical liquid level controller. The final example illustrates a case in which the optimum controlled system shows little improvement over the uncompensated system, and thus, little benefit could be gained by attempting to compensate it.

In those problems which are driven by an external forcing function, the forcing function is taken to be a step input at time t_o. The reasoning being that a step input is the most severe input to which a physical system can be subjected and that if adequate compensation can be achieved for the step input, it certainly could be achieved for any lesser input. An impulse input can be treated simply as an initial condition on the state variables, a case which is considered in Example Two.

Example One - Hydraulic Spool Valve

The first example considers the compensation of a hydraulic spool valve shown in Figure 5. This valve was studied by J. Bose (7) in demonstrating a parameter optimization type of compensation procedure. This example will afford an opportunity for comparison of the compensation technique presented in this thesis with that of Bose's.

The dynamic model for this valve is given on page 60 of reference (7) as

$$\ddot{x} + 0.36\dot{x} + 0.24x = f(t),$$

$$x(0) = 0, \quad \dot{x}(0) = 0.$$

For this valve, the response to a step force input $f(t) = 0.24$ is unsatisfactory since the rise time is over 4.0 seconds. In order to compensate the valve response a control variable is added to the system equation.

$$\ddot{x} + 0.36\dot{x} + 0.24x = 0.24 + q$$

To write the equation in state variable form, let $x_1 = x$, $x_2 = \dot{x}$, then

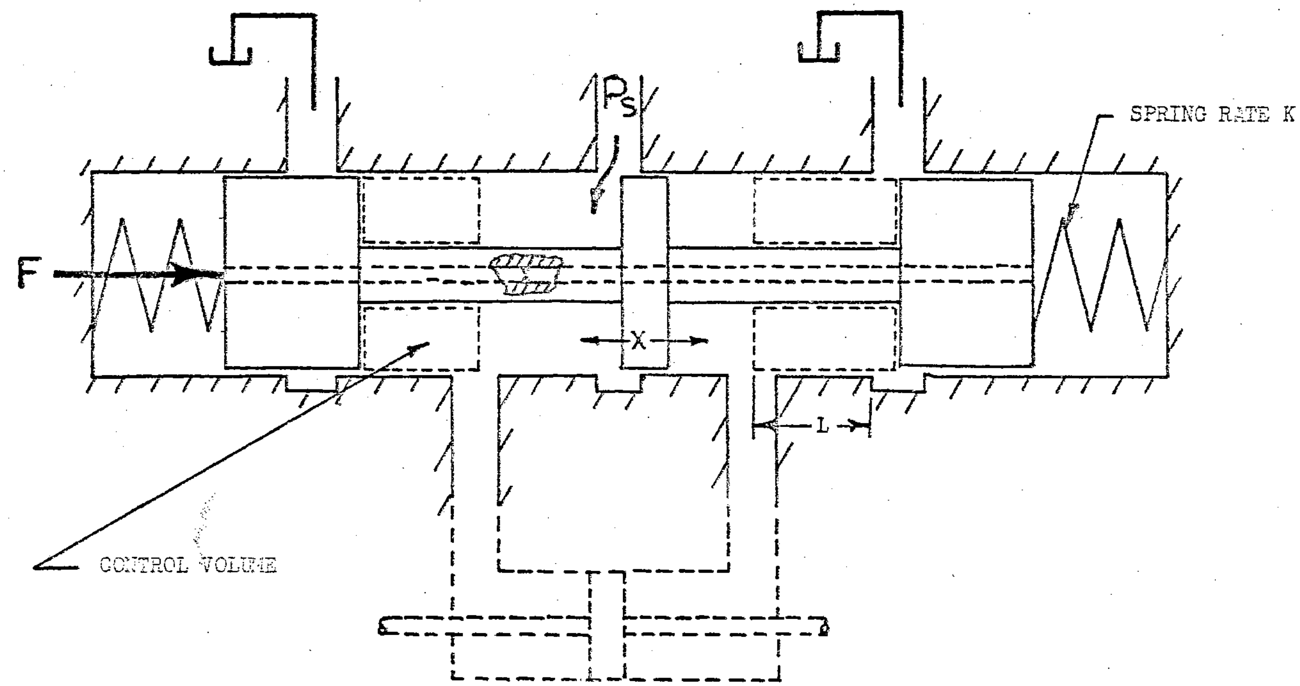

Figure 5. Hydraulic Spool Valve (7, p. 57)

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -.36x_2 - .24x_1 + .24 + q.$$

Also let

$$q = c_1 m.$$

The performance index is formulated as

$$J = \int_0^{5.0} \left[\frac{c_2}{2}(x_1 - 1.0)^2 + \frac{c_3}{2}m^2\right] dt.$$

Since $\dot{x}_2$ is in the range of ± 1.0*, q should be constrained to approximately this magnitude. However since the system is underdamped, the velocity x_2 should be greater in the compensated system. Thus a reasonable assumption is to allow q to vary approximately ± 10.0 in the compensated system. This is accomplished by letting $c_1 = 10.0$, then as $m = \pm 1.0$, $q = \pm 10.0$. Also let $c_3 = 0.1c_2$; therefore $c_2 = 1.0$, $c_3 = 0.1$, and $q = 10m$. The performance index becomes

$$J = \int_0^{5.0} \left[\tfrac{1}{2}(x_1 - 1.0)^2 + \frac{0.1}{2}m^2\right] dt,$$

from which the Hamiltonian is formed as

$$H = \tfrac{1}{2}(x_1 - 1.0)^2 + \frac{0.1}{2}m^2 + p_1 x_2 + p_2(-.36x_2 - .24x_1 + .24 + 10m).$$

Then the differential equations for the state and adjoint variables

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*This information is determined from a solution of the uncompensated system equation.

are found to be

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -.36x_2 - .24x_1 + .24 + 10m$$

$$\dot{p}_1 = 1 - x_1 + .24p_2$$

$$\dot{p}_2 = -p_1 + .36p_2$$

with boundary conditions

$$x_1(0) = 0, \qquad p_1(5) = 0,$$

$$x_2(0) = 0, \qquad p_2(5) = 0,$$

$$0 = .1m + p_2 \rightarrow m = -10p_2.$$

Thus

$$q = 10m = -100p_2.$$

By letting

$$x_3 = p_1 \quad \text{and} \quad x_4 = p_2$$

the two-point boundary value problem to be solved becomes,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -.36x_2 - .24x_1 + .24 - 100x_4 \qquad (5\text{-}1)$$

$$\dot{x}_3 = 1 - x_1 + .24x_4$$

$$\dot{x}_4 = -x_3 + .36x_4$$

with the boundary conditions

$$x_1(0) = 0$$

$$x_2(0) = 0$$

$$x_3(5) = 0$$

$$x_4(5) = 0,$$

and the equation for q is

$$q = -100x_4.$$

Through the use of a two-point boundary value solution program, the initial conditions on the adjoint variables are found to be

$$x_3(0) = -.44260732$$

$$x_4(0) = -.097628796.$$

The results of integrating Equations (5-1) using these initial conditions are shown in Figures 6 and 7. The first figure shows a comparison of the compensated state trajectory and the original uncompensated trajectory. It is obvious that there is a significant improvement in the response and thus the implementation of the necessary control should be attempted.

Figure 7 shows the optimum control $q^*(t)$ and the resulting optimum states. From this figure it is desired to gain some insight as to the terms necessary to implement an approximation to $q^*(t)$. The optimum control $q^*(t)$ has a shape generally similar to that of x_1, inversely of course, except that q^* has a faster initial rise rate and reaches its peak earlier. Hence $\hat{q}$ should be proportional to x_1 and the constant of proportionality should be large initially

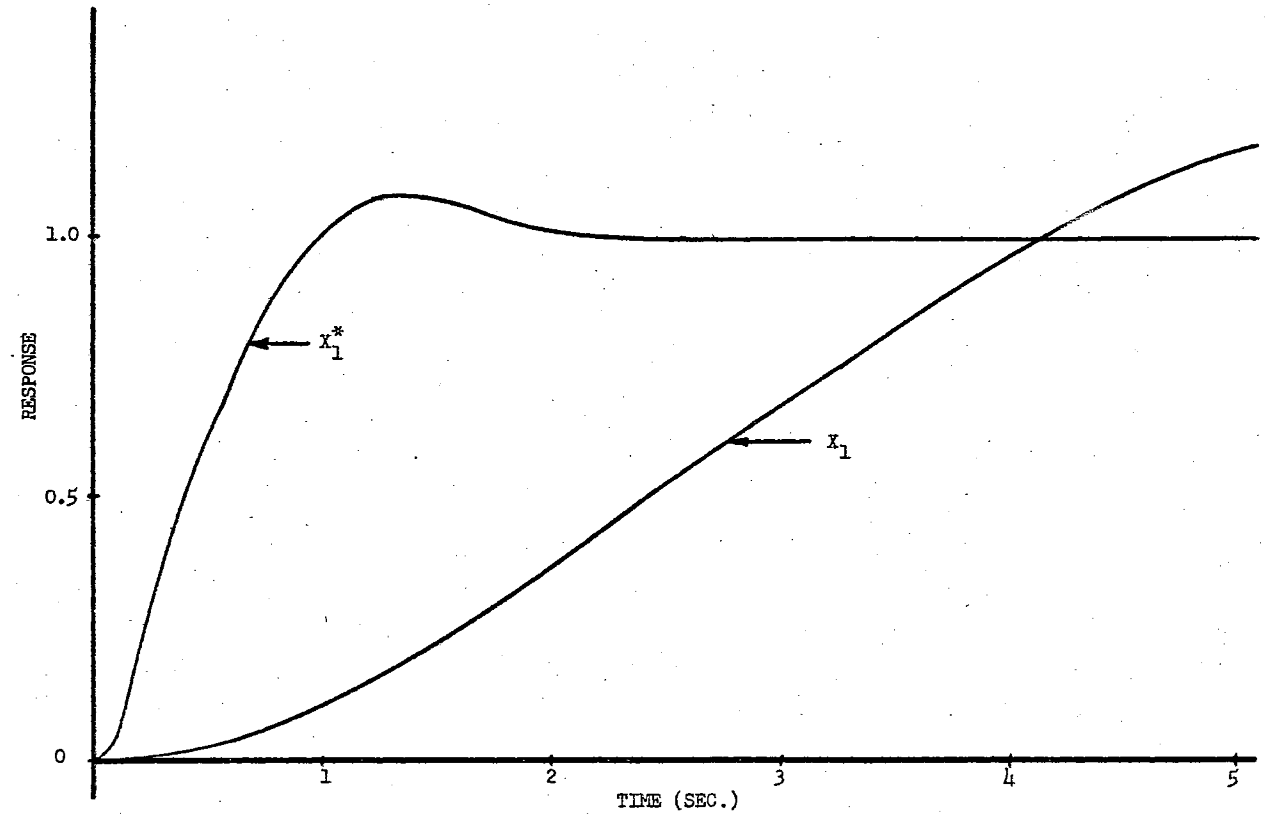


Figure 6. Plot of Optimum Response and Uncompensated Response

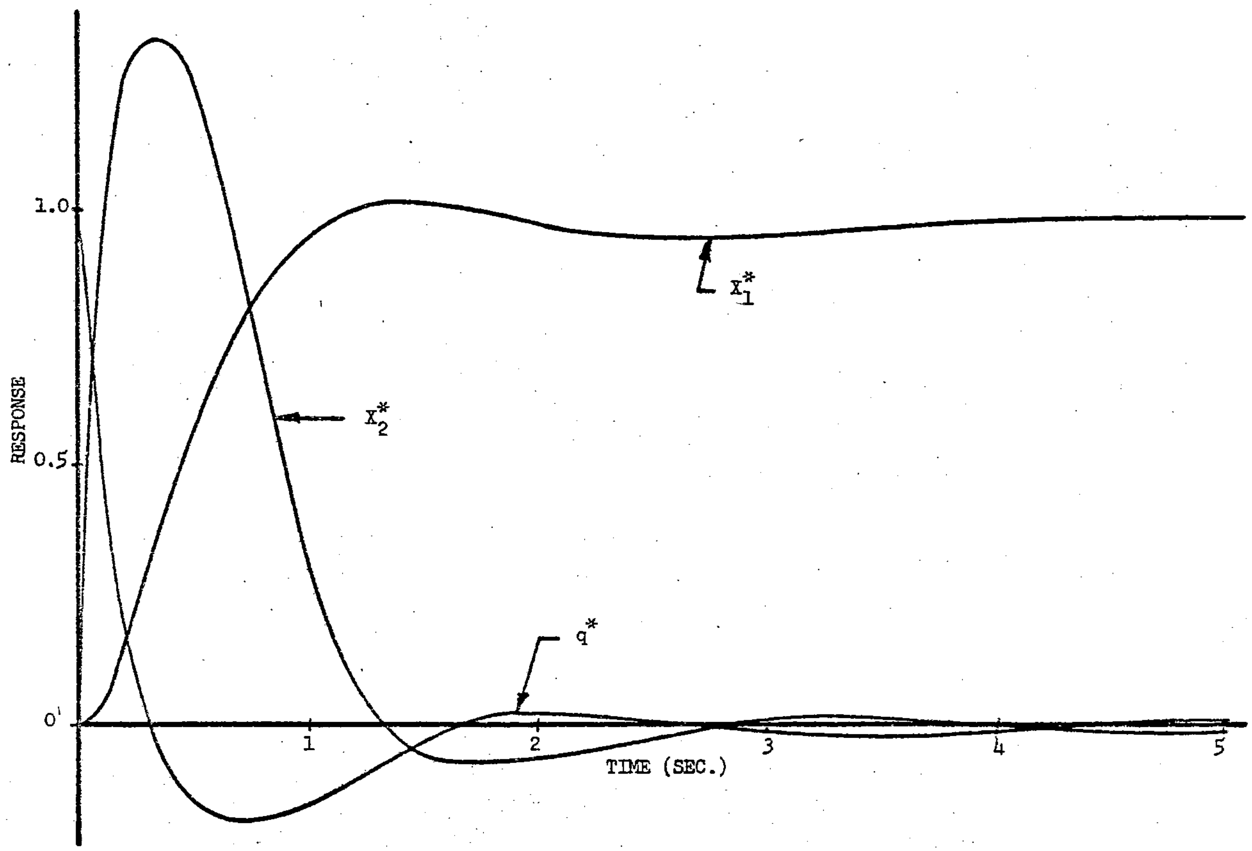


Figure 7. Plot of Optimum Control and State Trajectories

to give $\hat{q}$ a steep slope, but should decrease as x_1 increases to give a good final value. One way of realizing this is to recognize that x_2 is large for a short initial interval, then decreases. Thus if the term $(k_1x_2)x_1$ were used, the coefficient of the x_1 term appears that it might have the desired property.

Another manner in which one might fit q^* is to make $\hat{q}$ a function of x_2 to account for the rapid initial change and then use another function of x_1 to contribute to $\hat{q}$ when x_2 approaches zero. A function of $x_1|x_1|$ or $x_1{}^3$ might serve this purpose since either one is small when x_1 is small but increases rapidly as x_1 increases. Since an $x_1{}^3$ term could be easily physically realized with a nonlinear, "hardening" spring and a term such as $x_1|x_1|$ might be more difficult, the terms $k_2x_2 + k_3x_1{}^3$ are chosen to attempt to approximate q^*.

Thus the total expression used to approximate q^* is now

$$\hat{q} = k_1x_2x_1 + k_2x_2 + k_3x_1{}^3.$$

Since there is obviously a constant bias between x_1, x_2 and q^* (observe at $t = 0$; $x_1 = 0$ and $q^* = 9.76$), a constant term, k_4, must be added. Finally one should note that for this hydraulic valve, it would be a quite simple task to change the spring rate of the valve spring, so the term k_5x_1 is added. Hence

$$\hat{q} = k_1x_2x_1 + k_2x_2 + k_3x_1{}^3 + k_4 + k_5x_1.$$

A least squares curve fit of $\hat{q}$ to q^* determines the constants to be

$$k_1 = -0.00124$$

$$k_2 = -4.07464$$

$$k_3 = -0.00837$$

$$k_4 = 9.76288$$

$$k_5 = -9.75469$$

This approximation to the optimum control, when added to the original system equation, yields

$$\ddot{x} + (4.07 + .36)\dot{x} + (9.75 + .24)x + .00124x(\dot{x}) + .00837x^3 = .24 + 9.76.$$

However, the implementation of this control would require a large change in the coefficient of $\dot{x}$. Since this coefficient is determined largely by the viscous drag between the valve spool and body, a significant change here is difficult to achieve. In addition, a change in spring rate from 0.24 to 9.99 is not impossible but might prove troublesome without major revisions to the valve. Thus a slightly different method of implementing the control should be considered.

In determining a different form for the control vector, some thought must first be given to limiting the values of k_2 and k_5. Since the coefficients these terms affect can be changed, as much compensation as possible should be achieved by their variation, but they must not go beyond physically realizable bounds. Thus k_2 is limited to -0.5 and k_5 to -1.0. Although these restrictions are somewhat arbitrary in view of the limited details of the problem they are certainly within reason.

Since one of the damping terms in q has been restricted, another form of damping should be implemented. Fluid flow through an orifice placed in the by-pass tube in the spool would result in a damping term $k_6 x_2 x_2$. The functional form for $\hat{q}$ now becomes

$$\hat{q} = k_1 x_1 x_2 + k_2 x_2 + k_3 x_1^3 + k_4 + k_5 x_1 + k_6 x_2 |x_2| ,$$

with k_2 and k_5 limited to -0.5 and -1.0 respectively. A least squares fit of $\hat{q}(\underline{x}, \underline{k})$ to $q^*(t)$ yields the following values for $\underline{k}$.

$$k_1 = -4.60$$

$$k_2 = -0.50$$

$$k_3 = -5.996$$

$$k_4 = 6.996$$

$$k_5 = -1.00$$

$$k_6 = -1.24$$

A plot of the optimum and fitted control versus time is shown in Figure 8.

Addition of $\hat{q}$ to the original differential equation results in the following compensated valve equation.

$$\ddot{x} + (0.36 + 0.50)\dot{x} + (0.24 + 1.0)x + 5.996x^3 + 4.60x\dot{x} + 1.24\dot{x}|\dot{x}| = (0.24 + 6.996). \quad (5\text{-}2)$$

The equation response when compensated by the fitted control $\hat{q}$ is shown in Figure 9. This sub-optimum response is still greatly superior to the uncompensated response and hence certainly justifies implementation of $\hat{q}$. An alternative to observing plots of the response is to consider the performance index. For the optimal control the performance index value at $t = t_f$ is 0.1672 and for the fitted control it is 0.1746, an increase of only 4.4%. From this viewpoint too, it appears that $\hat{q}$ should be implemented.

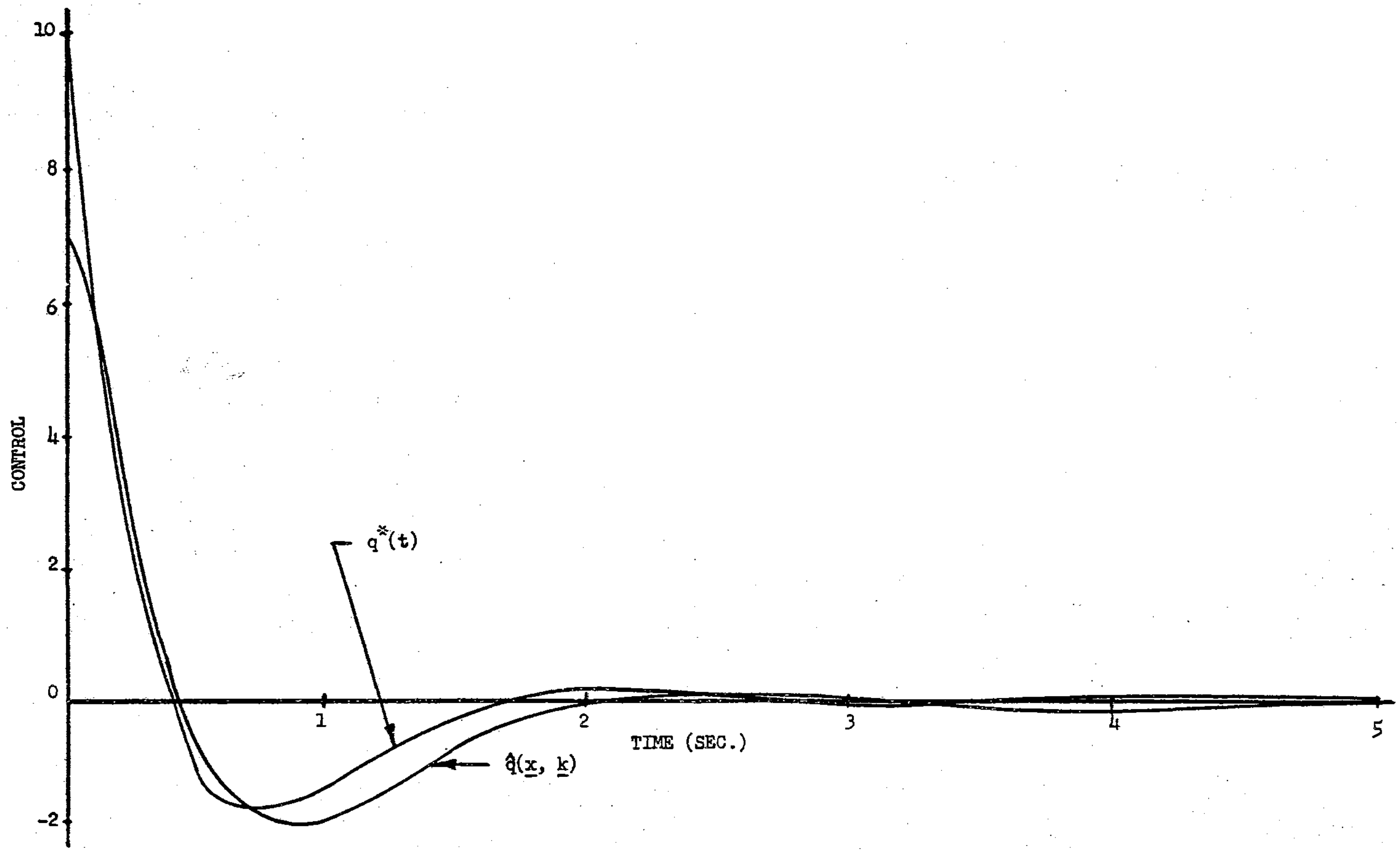


Figure 8. Optimum and Fitted Control Trajectories

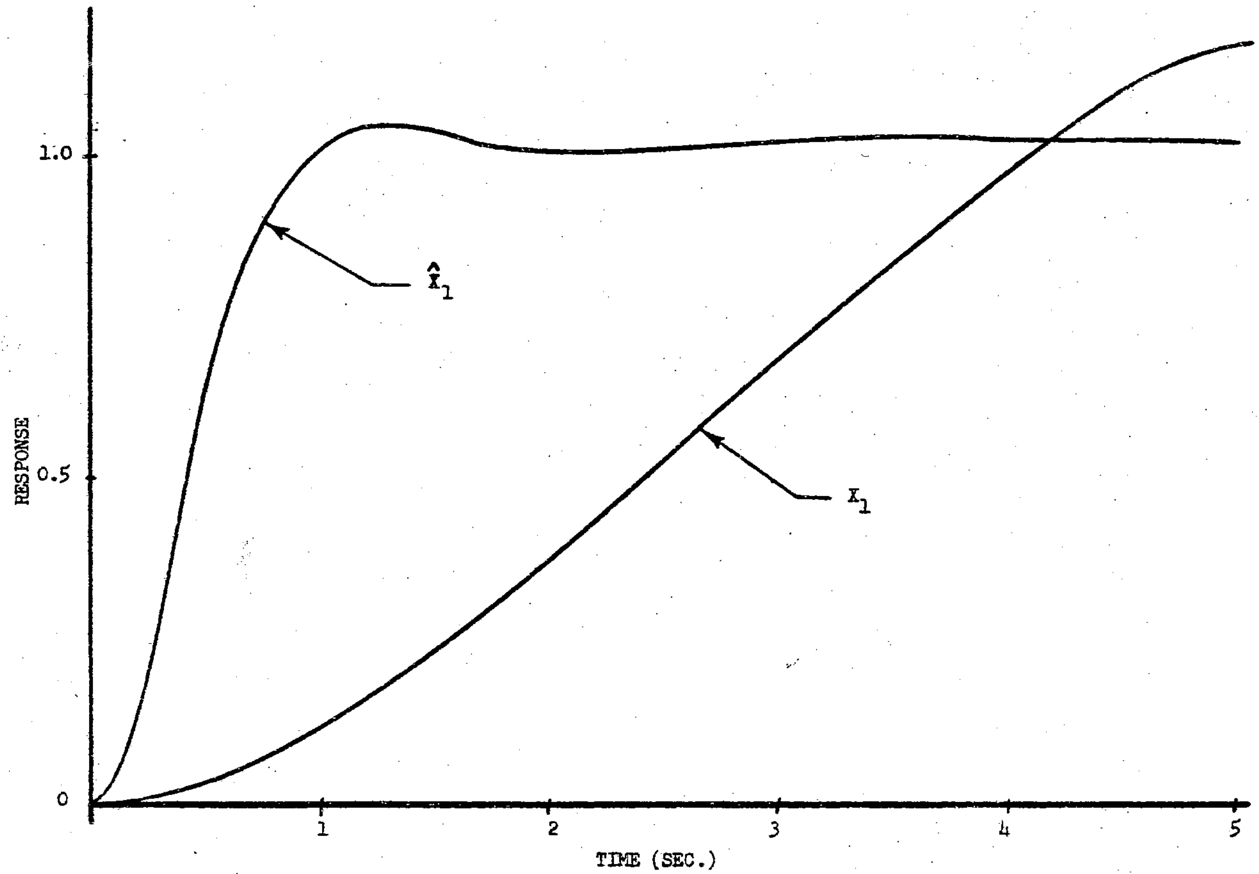


Figure 9. Original and Compensated State Trajectories

In order to interpret Equation (5-2) in physical hardware, two terms will be examined concurrently. The expression

$$(0.36 + 0.50)\dot{x} + 4.60x\dot{x}$$

is rewritten as

$$(0.36 + 0.50)\dot{x} + 4.60x\dot{x} = c_1\dot{x} + c_2(L + x)\dot{x}$$

since the term $c_2(L + x)\dot{x}$ represents an unsteady flow force which is a result of fluid acceleration induced by pressure changes and/or valve displacement. Equating coefficients of like terms yields

$$c_1 + c_2L = 0.86,$$

$$c_2 = 4.6.$$

If c_1 is assumed 0.16, then L, the characteristic length of the control volume, is found to be L = 0.155. Thus the terms under consideration become

$$0.16\dot{x} + 4.60(0.155 + x)\dot{x}.$$

Note that the expression $c_2(L + x)\dot{x}$ does not result in the addition of any physical hardware, but rather, represents a more accurate mathematical model of the valve. It does, however, yield valuable information in that the values of c_2 and L necessary for proper compensation of the valve are known.

The remaining compensating terms can now be implemented. The linear damping term $(c_1\dot{x})$ adjustment can be achieved by making the proper changes in the spool land length, spool diameter, radial clearance and spool mass. The spring force term will require a change in the linear spring rate as well as the addition of a non-linear hardening spring modeled by $5.996x^3$. The addition of an orifice in the spool by-pass tube will yield the $\dot{x}|\dot{x}|$ term with the coefficient determined by proper sizing of the relative diameters of the spool and orifice and the orifice flow coefficient. The step force input to the valve must be 7.236.

The hydraulic valve, when compensated as discussed above, is described by the following differential equation.

$$\ddot{x} + 0.16\dot{x} + 1.24\dot{x}|\dot{x}| + 4.6(0.155 + x)\dot{x} + 1.24x + 5.996x^3 = 7.236 \quad (5\text{-}3)$$

The compensated response is compared with the ideal response in Figure 10 where it can be observed that very little optimality has been sacrificed in implementing the sub-optimal control. Figure 11 shows the optimal control $q^*(t)$, the control fitted to the optimum response, $\hat{q}(\underline{x}^*, \underline{k})$, and the actual control that is generated in the compensated system, $\hat{q}(\hat{\underline{x}}, \underline{k})$. As would be expected from the nearly optimal compensated response, the actual control very closely approximates the optimum control.

Bose (7) used a least squares method to compensate this hydraulic valve and gave the compensated model as

$$\ddot{x} + 0.36\dot{x} + 0.24x + 0.801x^3 + 1.4039x\dot{x} = 1.0.$$

The rise time and overshoot for Bose's compensation were approximately 2.0 seconds and 10% respectively while the compensation applied in Equation (5-3) yields a response with rise time of 1.08 seconds and less than 3% overshoot. The results of this example can be summarized by saying that the compensation procedure described in this thesis has aided in the compensation of a nonlinear system in such a way as to significantly improve its response and to guarantee the physical realizability of the compensating terms.

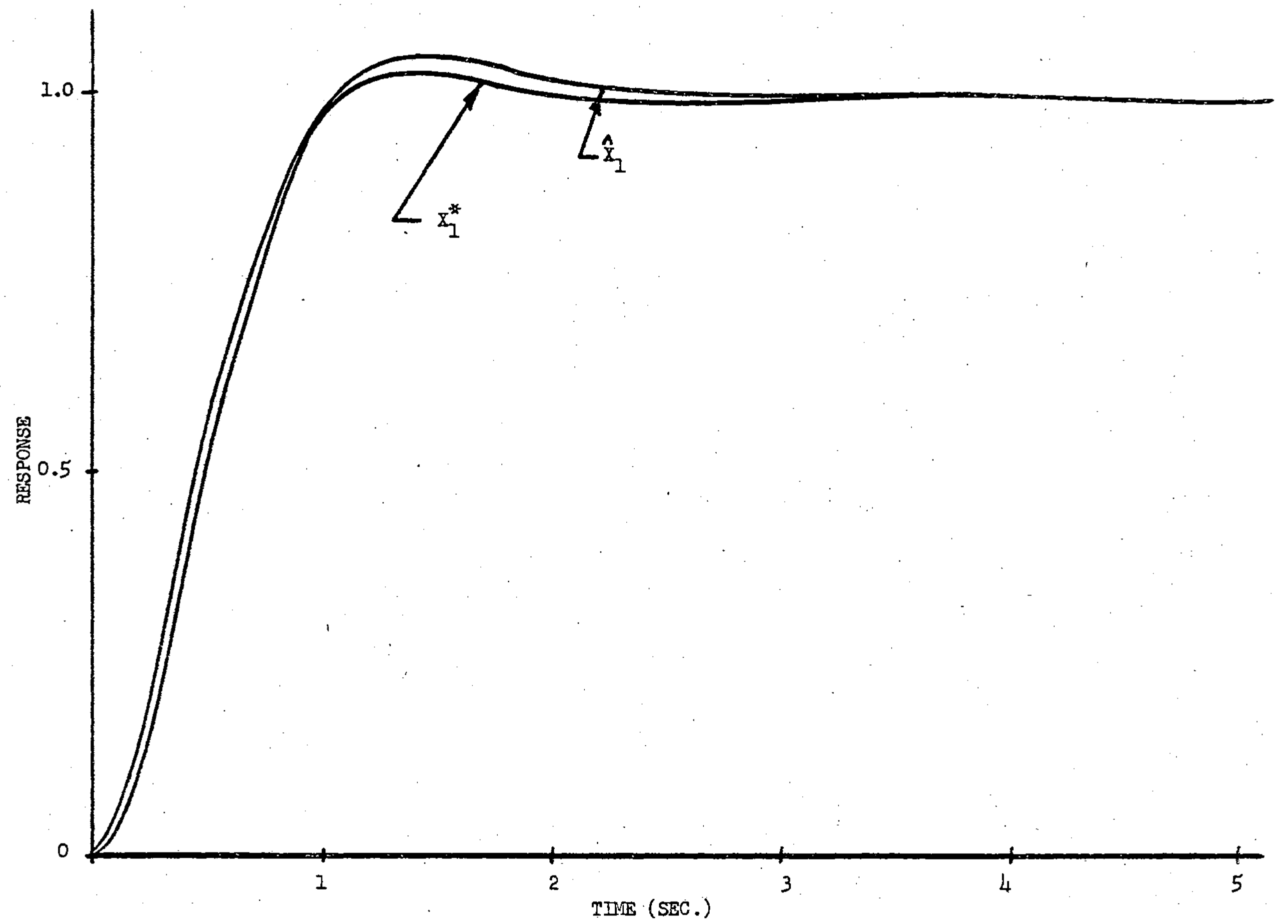


Figure 10. Compensated and Optimum State Trajectories

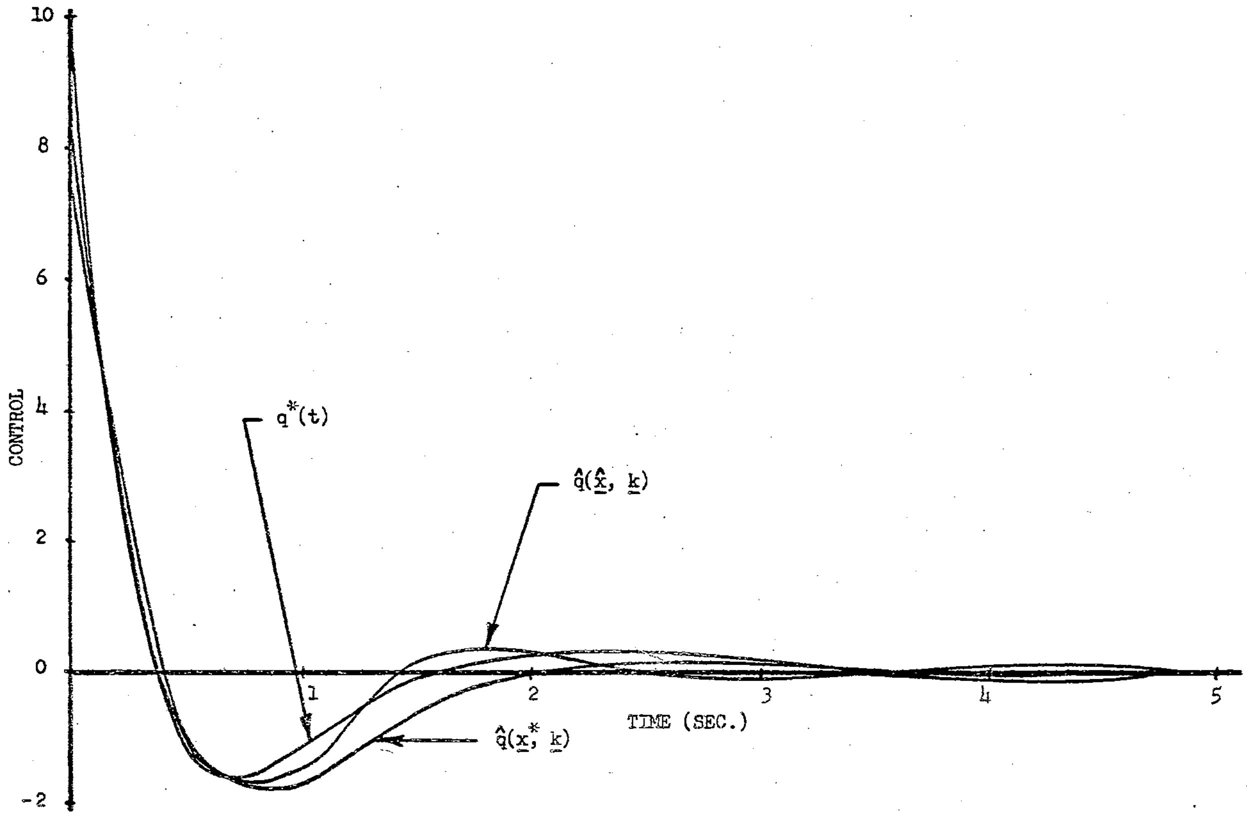


Figure 11. Comparison of Control Trajectories

Example Two - Electrical Circuit

Consider a system utilized as an example by Garrad, N.H. McClamroch, and Clark (20) in "Some Approaches to Suboptimal Feedback Control of Nonlinear Systems". This system consists of an electrical circuit with a nonlinear resistor shown in Figure 12. The differential equation describing the dynamics of this circuit is given as

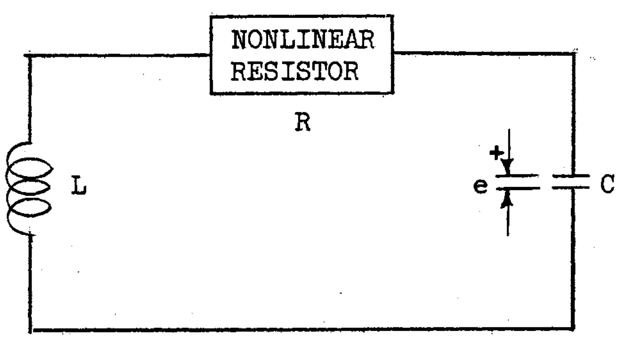
$$\ddot{e} - (1 - e^2)\dot{e} + e = 0 .$$

Consider two sets of initial conditions

$$\begin{bmatrix} e(0) \\ \dot{e}(0) \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} \text{ and } \begin{bmatrix} e(0) \\ \dot{e}(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

The primary purpose for this example problem is to compare the optimization technique described herein with that of Garrard, et al.; hence the same problem will be considered. That problem is, add a control q that will minimize the performance functional

$$J = \tfrac{1}{2} \int_0^{10} (e^2 + \dot{e}^2 + q^2)\, dt.$$



$R = -\sigma(1 - e^2)$ $\sigma = 10.0$

$L = 10.0$ $C = 0.1$

Figure 12. Nonlinear Electrical Oscillatory Circuit

Let $x_1 = e$, $x_2 = \dot{e}$ and write the state equations, to which a control variable q has been added,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1 + q.$$

Since the general state trajectory will be between 1.0 and 0.0, it is desired that the control remain within these limits also, hence let $q = 1.0m$. The Hamiltonian then becomes

$$H = \tfrac{1}{2}(x_1^2 + x_2^2 + m^2) + p_1x_2 + p_2(1 - x_1^2)x_2 - x_1 + m.$$

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = x_2$$

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = (1 - x_1^2)x_2 - x_1 + m$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -x_1 + p_2(-2x_1x_2 - 1.0)$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -x_2 + p_1 + p_2(1 - x_1^2)$$

$$0 = \frac{\partial H}{\partial m} = m + p_2 \rightarrow m = -p_2$$

Now substitute $x_3 = p_1$ and $x_4 = p_2$; the two-point boundary value problem to be solved becomes

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1 - x_4$$

$$\dot{x}_3 = -x_1 + 2x_1x_2x_4 + x_4$$

$$\dot{x}_4 = -x_2 - x_3 -(1 - x_1^2)x_4$$

with the initial conditions for the two cases shown below.

CASE A	CASE B
$x_1(0) = 1.0$	$x_1(0) = 0.5$
$x_2(0) = 1.0$	$x_2(0) = 0.5$
$x_3(10) = 0$	$x_3(10) = 0$
$x_4(10) = 0$	$x_4(10) = 0$

The computational algorithm TPBV is used to obtain the proper initial conditions on the adjoint variables (16).

CASE A	CASE B
$x_3(0) = 2.0766979$	$x_3(0) = 1.6106961$
$x_4(0) = 1.6747117$	$x_4(0) = 1.3330923$

Case A

Now consider the results of Case A shown in Figure 13. This figure, which shows a comparison of the optimum response and the uncompensated response, indicates that a significant improvement can be achieved if the optimum control can be implemented. The desired optimum control $q^*(t)$ and the time histories of x_1 and x_2 are shown in Figure 14.

The problem at this point becomes one of approximating $q^*(t)$ with a function $\hat{q}(\underline{x}, \underline{k})$, that is a function of the state variables. Close examination of the curves in Figure 14 will reveal a close correspondence between the general shape of the $x_2(t)$ plot and the $q^*(t)$ plot. This suggests a term kx_2 in the approximation to $q^*(t)$. However, the magnitude of change in q^* during the first 1.6 seconds is greater with respect to x_2 than during the time 1.6 - 4.0 seconds. This indicates that the coefficient of x_2 should

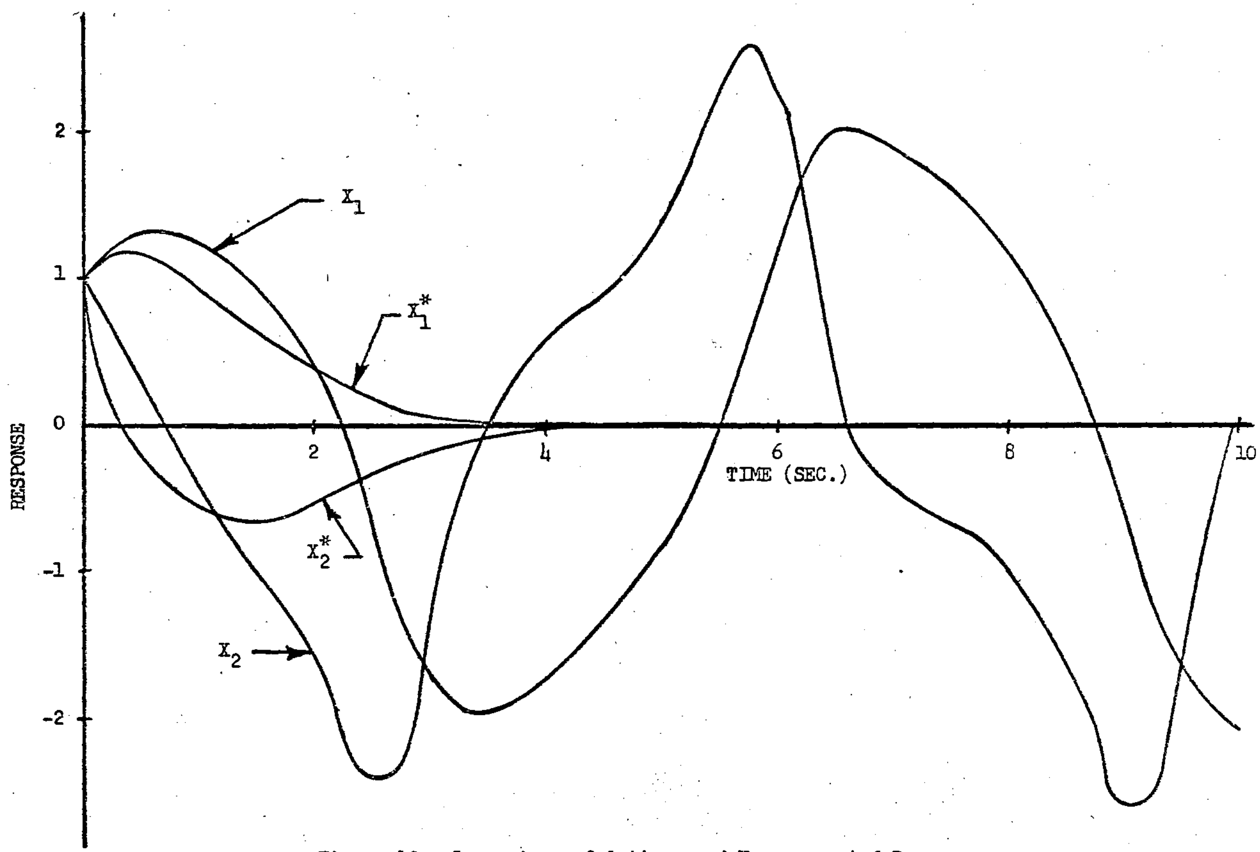


Figure 13. Comparison of Optimum and Uncompensated Responses

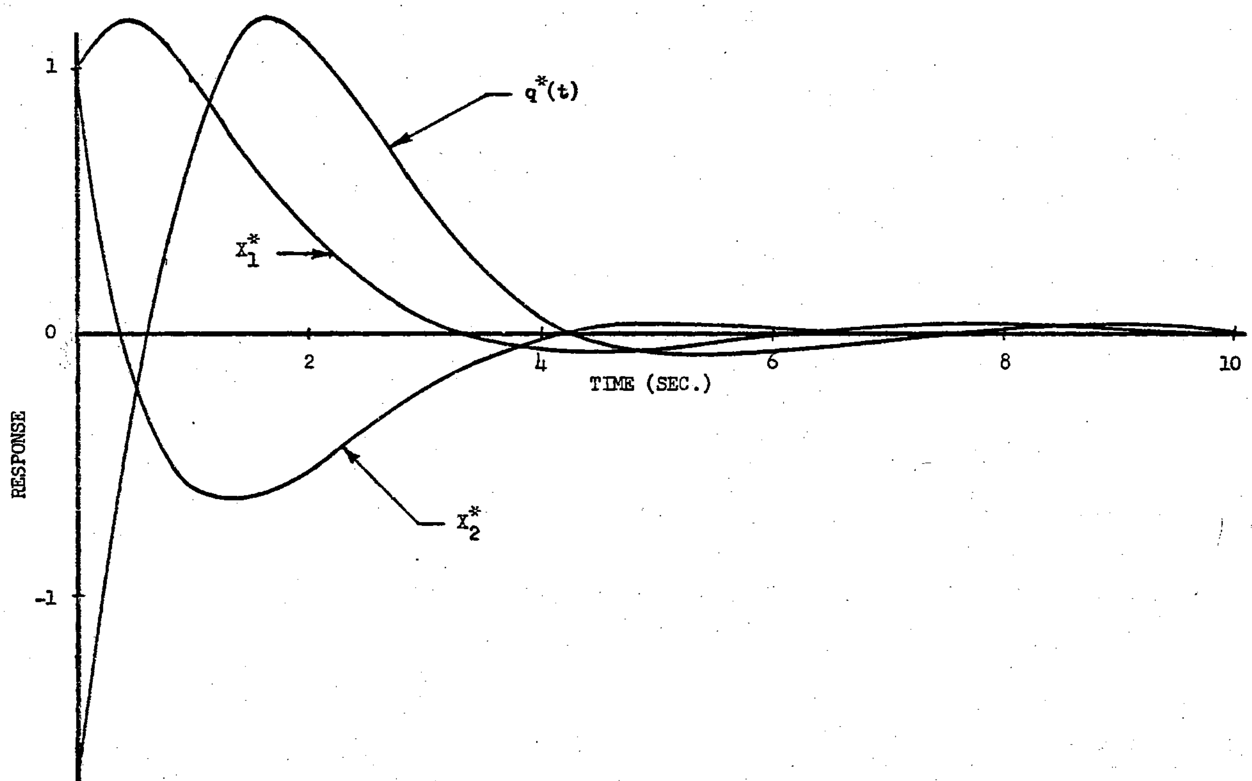


Figure 14. Plot of Optimum Control and State Trajectories

decrease with time. Since $x_1(t)$ follows a generally decreasing path, the coefficient could be made a linear function of x_1 so that $q^*(t)$ would be approximated by

$$\hat{q} = k_1 x_1 x_2 .$$

A selection of $k_1 = -1.65$ to make $\hat{q}(\underline{x}(0), \underline{k}) = q^*(0)$ yields the approximation shown in Figure 15. Although the initial and final portions of the curves coincide, the fit is not close overall. In order to determine what additional terms in $\hat{q}$ are necessary, an examination of plots of the residual $(r = \hat{q} - q^*)$ versus the state variables and various combinations of the state variables will be helpful. Figures 16, 17, 18, and 19 show plots of the residual versus x^*_1, x^*_2, $(x^*_1)^2$ and $(x^*_2)^2$, respectively. Although three of the plots indicate a general nonlinear relationship between r and the state variable, the plot of r versus $(x^*_1)^2$ can be approximated by two straight lines. The dashed line in Figure 18 represents an approximate linear relationship between r and $(x^*_1)^2$ while the remainder of the plot indicates the residual is independent of x_1. Since

$$r = \hat{q} - q^*$$

and

$$q^* = \hat{q} - r,$$

then if an analytical expression for $\hat{r}$ in terms of the state variables can be obtained and subtracted from $\hat{q}$, the new approximate control

$$q' = \hat{q} - \hat{r}$$

should better approximate q^*. If $\hat{r}$ is approximated by the dashed line in Figure 18, the resulting q' should fit q^* better initially

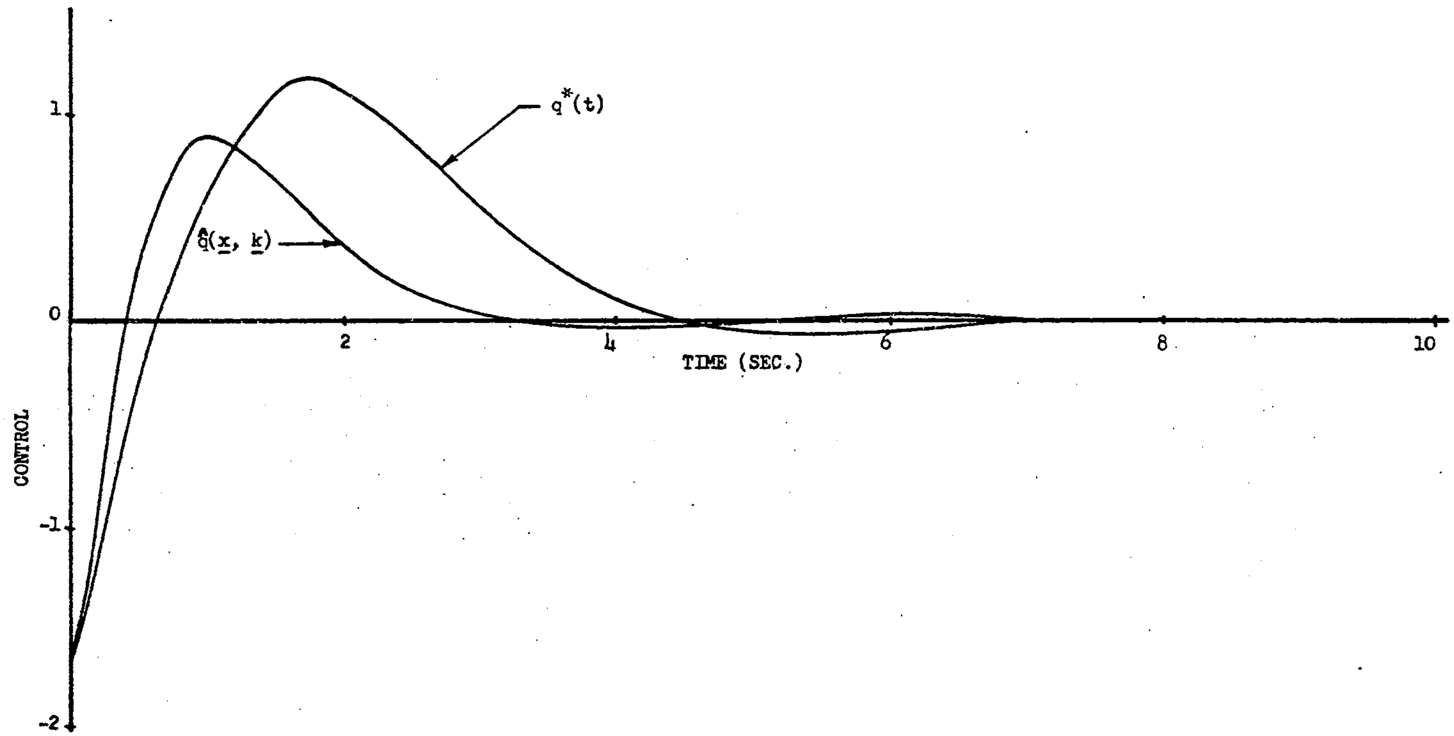


Figure 15. Optimum and Fitted Control Trajectories

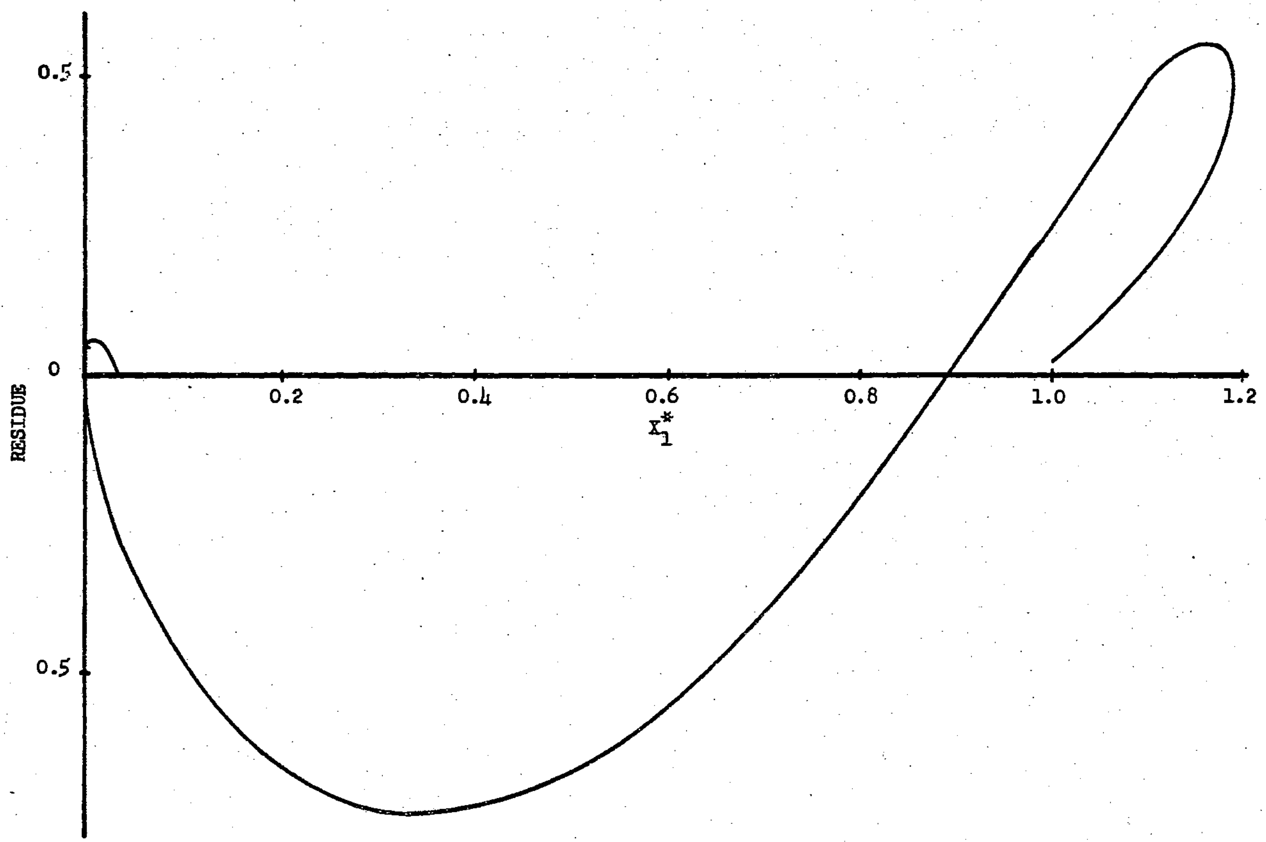


Figure 16. Residue versus $X_1^*(t)$

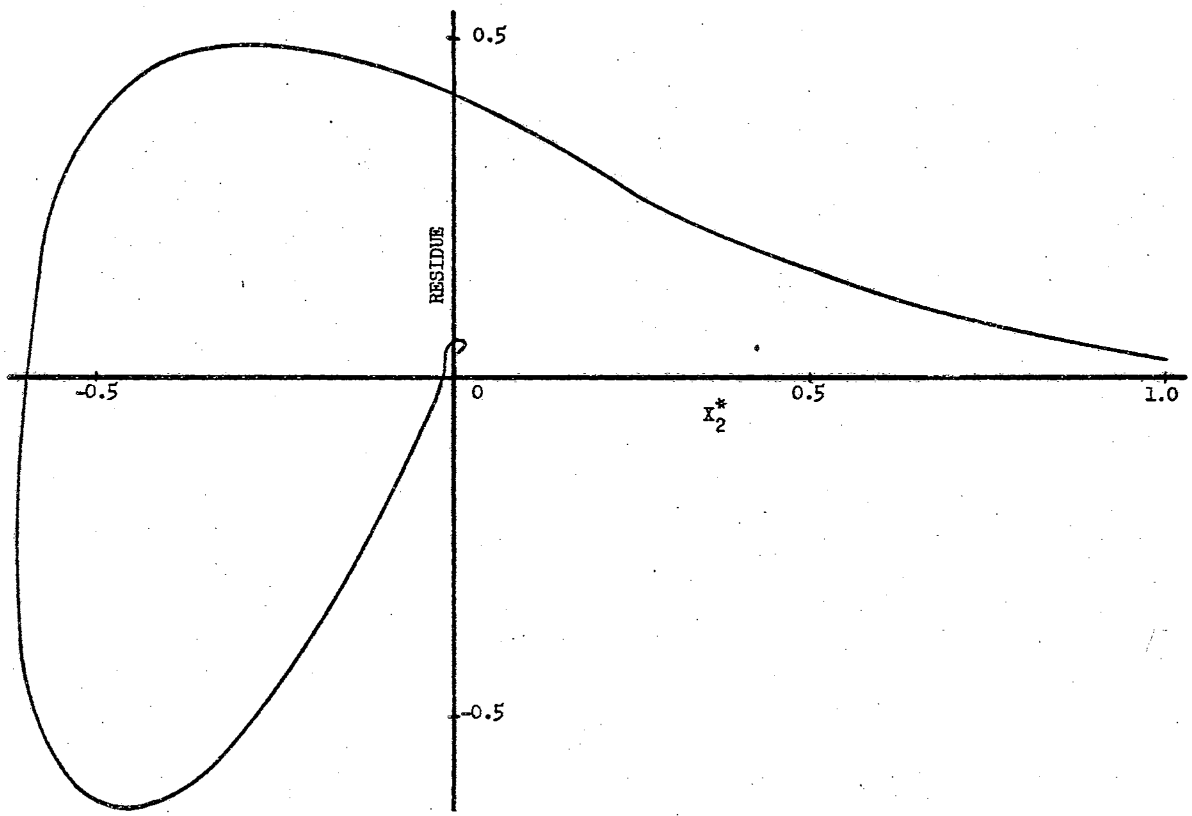


Figure 17. Residue versus $X_2^*(t)$

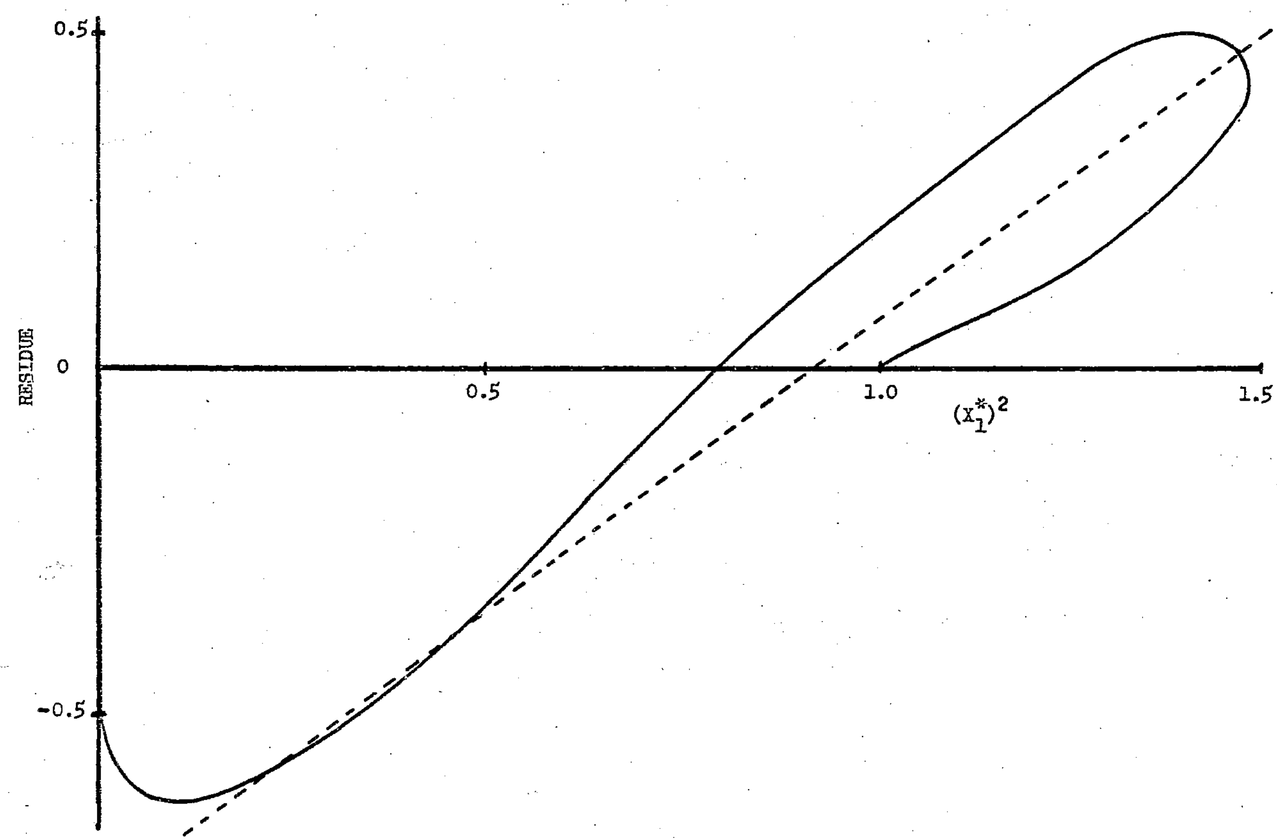


Figure 18. Residue versus $\left[X_1^*(t)\right]^2$

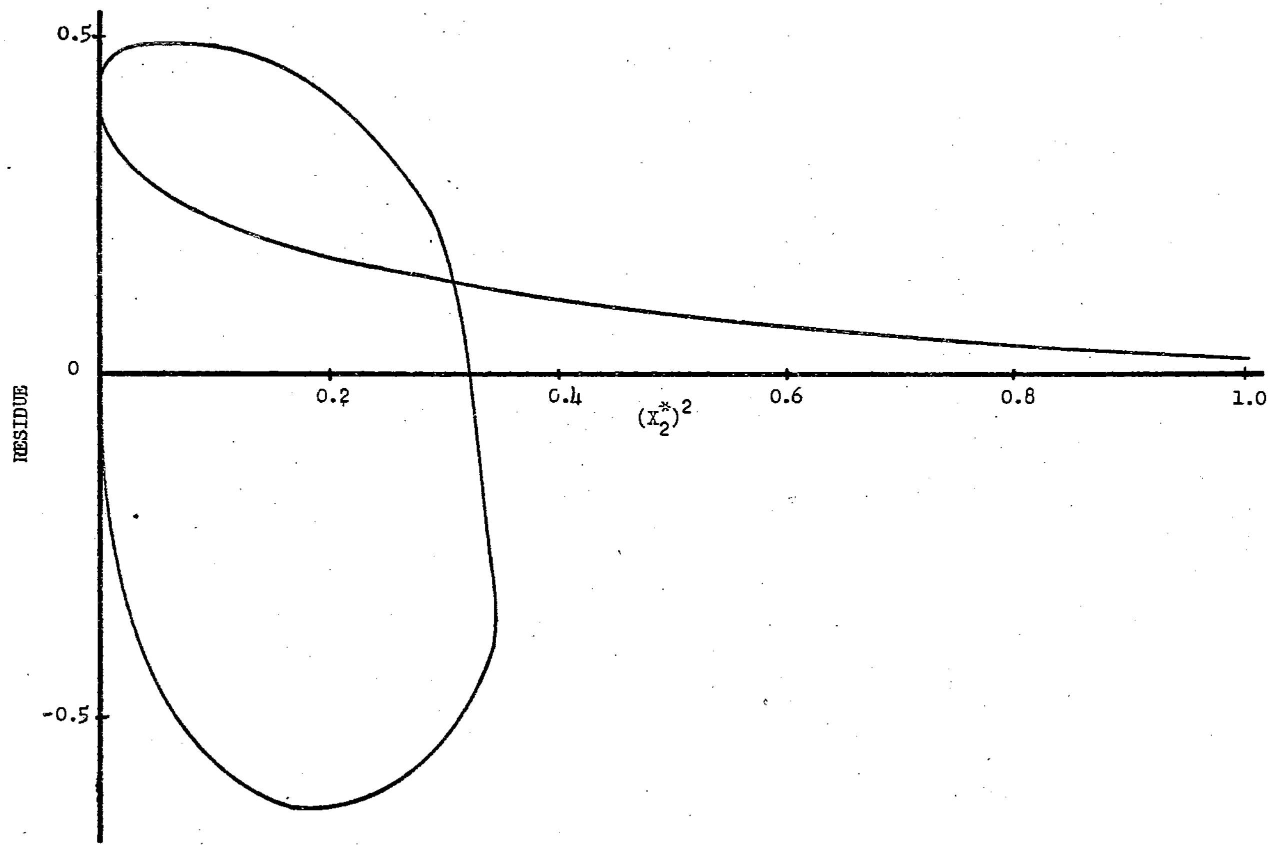


Figure 19. Residue versus $\left[X_2^*(t)\right]^2$

but will have a greater steady state error. The straight line is represented by $\hat{r} = -k_2x_1^2 - k_3$ and thus $\hat{q}$ becomes

$$\hat{q} = k_1x_1x_2 + k_2x_1^2 + k_3 \ .$$

The results of fitting this form of $\hat{q}$ to q^* is shown in Figure 20. Consideration could now be given to reducing the steady state difference between $\hat{q}$ and q^* by adding additional terms. First however, some thought must be given to physical implementation of the desired control. For this circuit the task of generating the term $k_1x_1x_2$ short of adding an analog computer or function multiplier is indeed difficult. Since the term $k_2x_1^2$ can be implemented somewhat more easily, consideration should be given to approximating q^* without the $k_1x_1x_2$ term and with other more easily realized terms. Addition of a linear resistor to the circuit would add a kx_1 term while adjusting the capacitor and nonlinear resistor would allow the inclusion of kx_2 and $k_4(1 - x_1^2)x_2$. Thus a more readily realized control might be given as

$$\hat{q} = k_1 + k_2x_1 + k_3x_2 + k_4(1 - x_1^2)x_2 + k_5x_1^2 \ . \qquad (5\text{-}4)$$

Determination of $\underline{k}$ through the use of a least squared error approximation routine yields the following values (21):

$k_1 = -\ .00109 \qquad k_4 = -.96851$

$k_2 = -\ .42201 \qquad k_5 = \ .37611\ .$

$k_3 = -1.62772$

The use of the form for $\hat{q}$ given in Equation (5-4) with the values for $\underline{k}$ shown above yields a close approximation to q^* as shown by the $\hat{q}(\underline{x}^*, \underline{k})$ curve in Figure 21. With the addition of

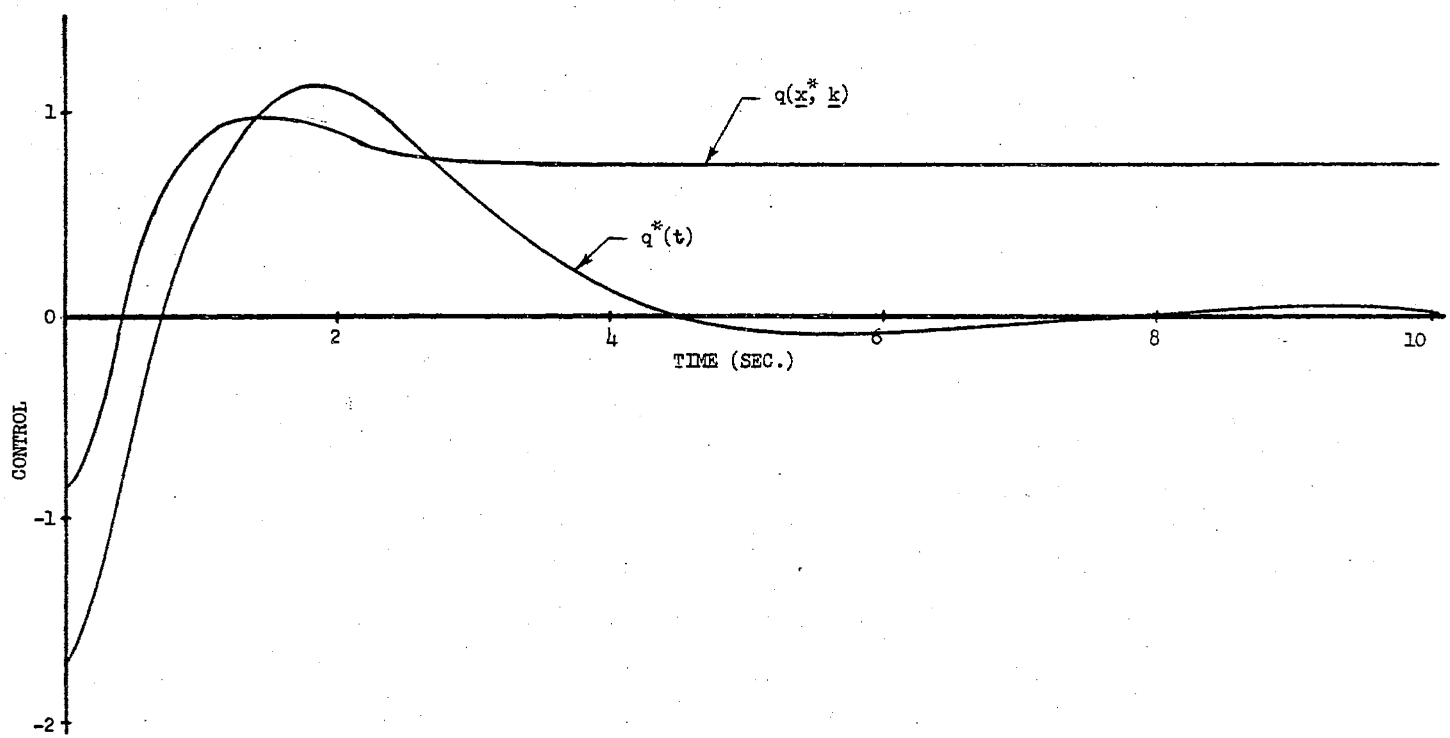


Figure 20. Optimum and Fitted Control Trajectories

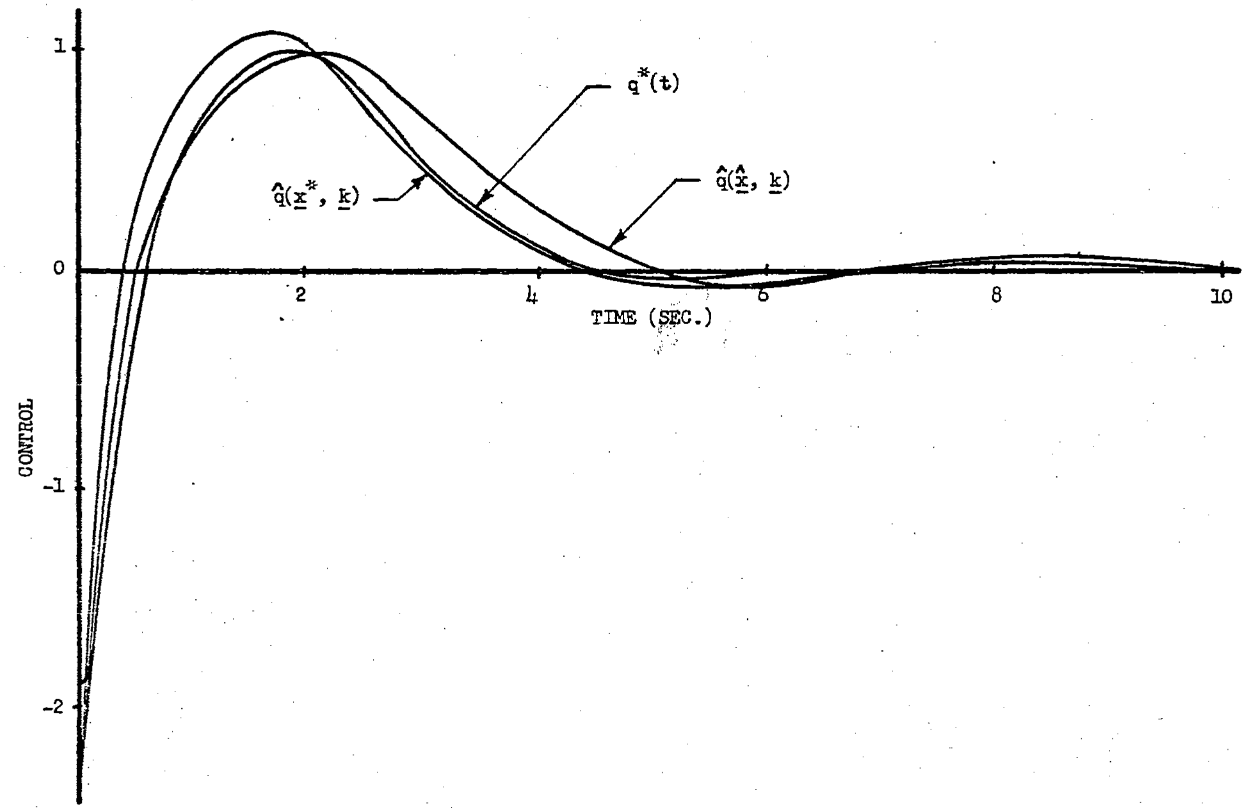


Figure 21. Comparison of Control Trajectories

the control $\hat{q}$, the state equations become

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1 + k_1 + k_2x_1 + k_3x_2 + k_4(1 - x_1^2)x_2 + k_5x_1^2,$$

or expressed in the original second-order form

$$\ddot{e} - (1 - e^2)\dot{e} + e = k_1 + k_2e + k_3\dot{e} + k_4(1 - e^2)\dot{e} + k_5e^2.$$

Grouping like terms and substituting numerical values for $\underline{k}$ yields

$$\ddot{e} - (1 + .96851)(1 - e^2)\dot{e} + 1.62772\dot{e} + (1 + .42201)e - .37611e^2 = -.00109 .$$

The original circuit with the necessary modifications and additions to implement the desired control is shown in Figure 22. The circuit equation is first non-normalized and written as

$$10\ddot{e} - (10 + 9.6851)(1 - e^2)\dot{e} + 16.2772\dot{e} + \frac{1}{.07032}e + (0.0109 - 3.7611e^2) = 0. \qquad (5\text{-}5)$$

The indicated modifications are: (1) increase the constant associated with the nonlinear resistor by 9.6851, (2) add a linear resistor with 16.2772 ohms resistance to the circuit, (3) decrease the capacitance to .07032, (4) add a voltage source with a current dependent resistance to realize the final term on the left-hand side of Equation (5-5). This term can be implemented by realizing that many batteries have an internal resistance that is partially current dependent. This internal resistance R_i can be approximated (in some cases and with certain limits on the current) by

$$R_i = (R_c - R_v i)\text{ohms}$$

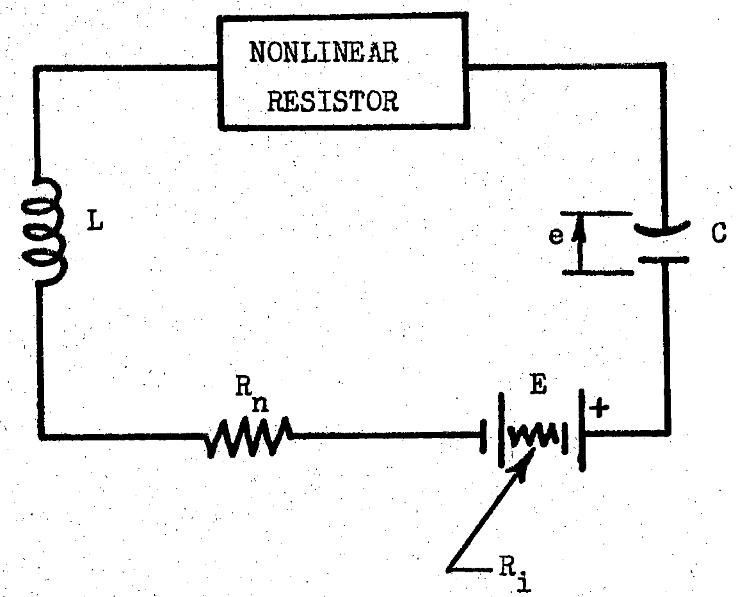


Figure 22. Compensated Electrical Circuit

where R_c and R_v are constants. The inclusion of a battery with voltage E and internal resistance R_i results in the following term in the differential equation:

$$E + R_c\dot{e} - R_v e^2.$$

Since with the addition of this battery, the constant R_c will modify the coefficient of the $\dot{e}$ term, it will be necessary to adjust the linear resistor R_n so that $(R_c + R_n) = 16.2772$. Finally, with a battery for which $E = .0109$ volts and $R_v = 3.7611$ ohms/ampere, the desired control has been implemented.

A comparison of the responses of the compensated and original uncompensated circuit is shown in Figure 23. The performance of the compensated circuit is a significant improvement over the original

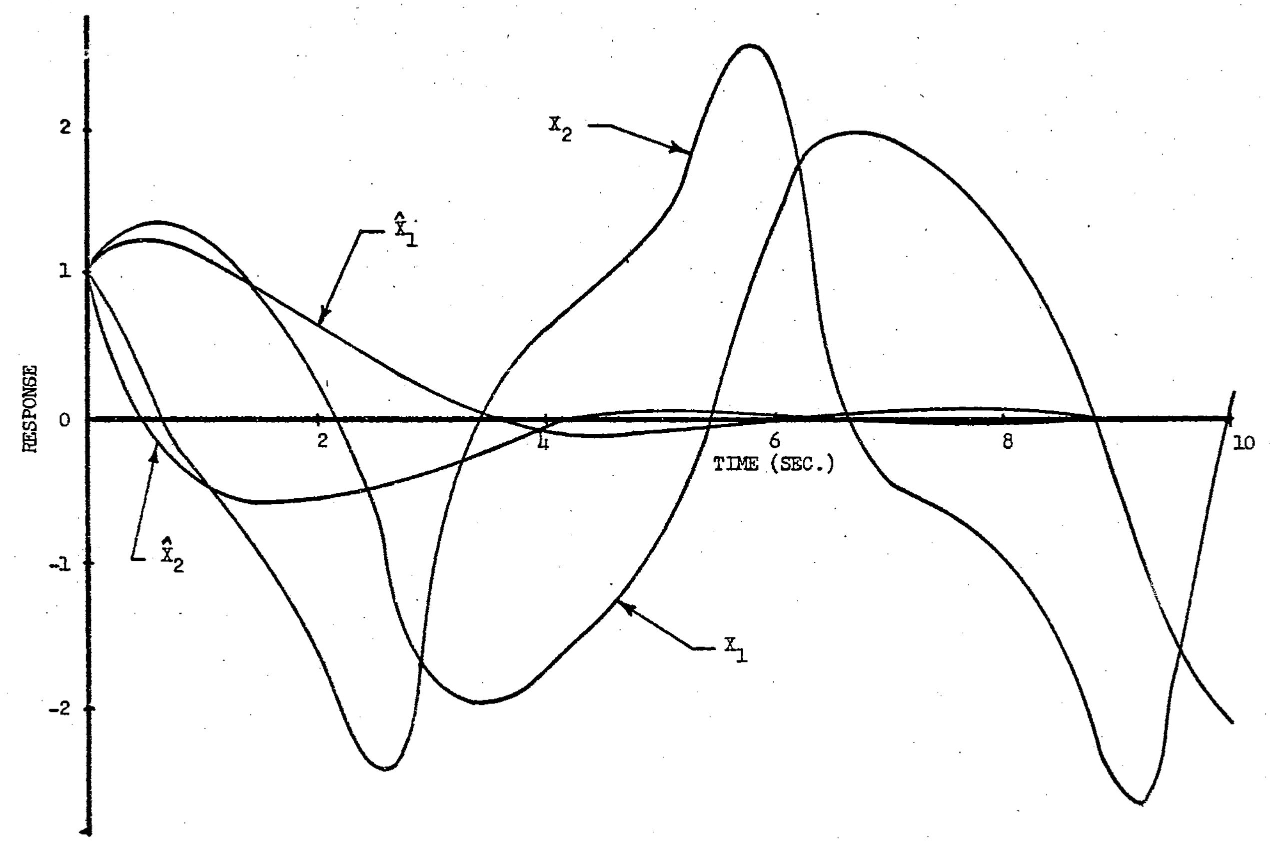


Figure 23. Original and Compensated State Trajectories

response, however, it is not as much of an improvement as the optimum response which is shown in Figure 24. This is due to the fact that the $\hat{q}$ that actually controls the circuit is a function of $\hat{\underline{x}}$ and not $\underline{x}^*$. The difference in $q^*(t)$, $\hat{q}(\underline{x}^*, \underline{k})$ and $\hat{q}(\underline{x}, \underline{k})$ may be seen in Figure 21.

The decision as to whether to accept the compensated system or attempt to further improve its response can be based on two factors. First, the plots shown in Figures 23 and 24 can be studied to compare the uncompensated, the optimum and the compensated responses. If in the judgment of the designer the compensated response meets his requirements, then no further refinement is required. The second factor that can be considered is the performance index values. The optimum performance index value at the final time is 2.563 while the compensated circuit has a performance index of 2.665, approximately 4% less than optimum. For this example problem, no further compensation will be attempted.

Garrard, McClamroch and Clark (20) compare several methods of compensating the circuit of Figure 12. Of the techniques compared, the perturbation method gave the smallest performance index, a value of 2.573. As expected, this value is larger than the optimum performance index, but it is smaller than the performance index given for the compensated system in Figure 22. However the authors had made no consideration of physical realizability and hence were not restricted in the form of compensation used. The control used in the perturbation technique was

$$q = -0.414x_1 - 2.685 + 1.086x_1^2x_2 + 0.583x_1x_2 + 0.072x_2^3$$

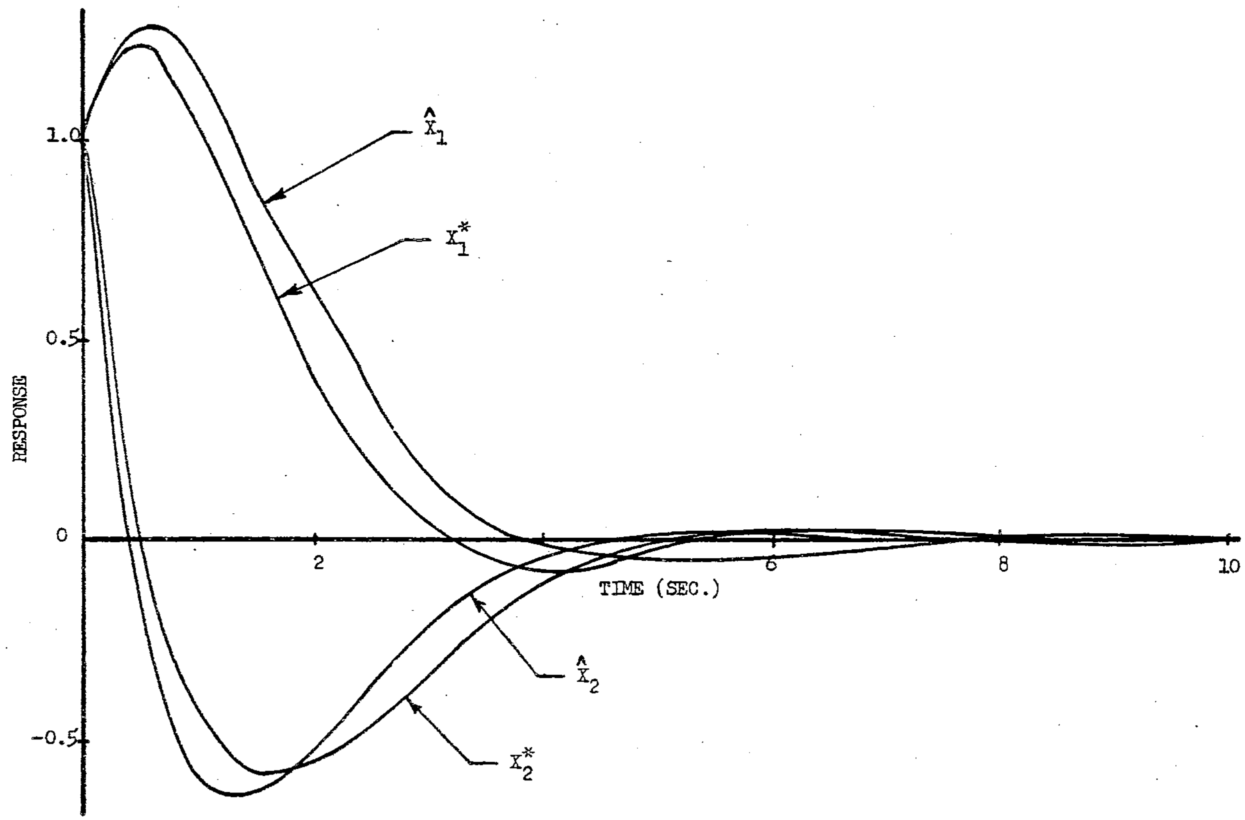


Figure 24. Compensated and Optimum State Trajectories

which would be indeed difficult to physically implement. Thus it might be expected that the performance index be considerably smaller than one for which the compensation must be physically realizable. The important consideration, however, is that the optimum performance index is smaller than either of the sub-optimal cases and that the physically realizable circuit in Figure 22 has a performance index within a few per cent of the optimum.

Case B

The second case for this example problem involves the same circuit starting from a different set of initial conditions. For this case $x_1(0) = 0.5$ and $x_2(0) = 0.5$ and as mentioned previously, $x_3(0) = 1.6106961$ and $x_4(0) = 1.3330923$. The original and optimum circuit responses are shown in Figure 25 followed by a plot of the optimum control and response in Figure 26. Since the only difference between this case and Case A is the change in initial conditions, the same general form for the fitted control will be assumed.

$$\hat{q} = k_1 + k_2 x_1 + k_3 x_2 + k_4 (1 - x_1{}^2) x_2 + k_5 x_1{}^2 \qquad (5\text{-}6)$$

In order to obtain a least square fit of Equation (5-6) to $q^*(t)$, the constants must have the following values:

$k_1 = -0.00029764$

$k_2 = -0.41418$

$k_3 = -1.59657$

$k_4 = -1.04993$

$k_5 = 0.265226$

Figure 27 shows that these values result in an extremely close fit to

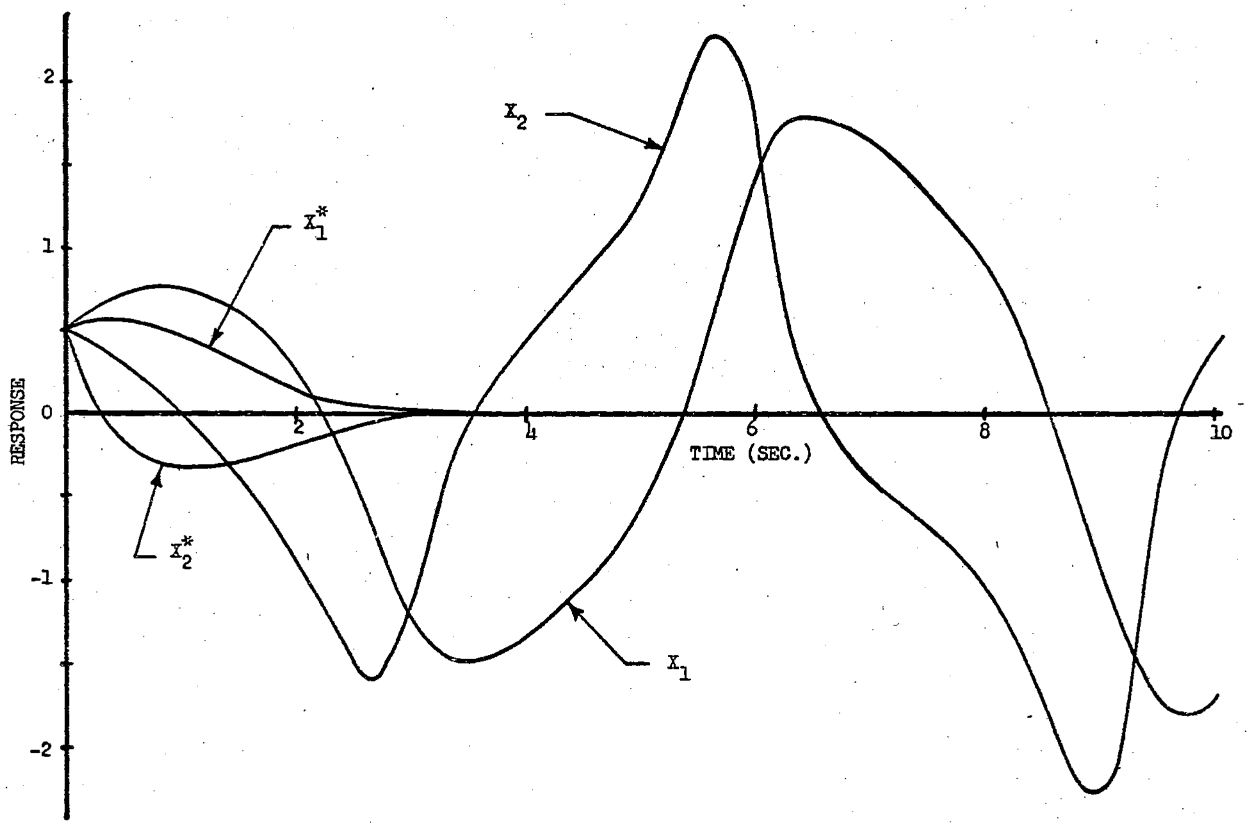


Figure 25. Comparison of Optimum and Uncompensated Responses

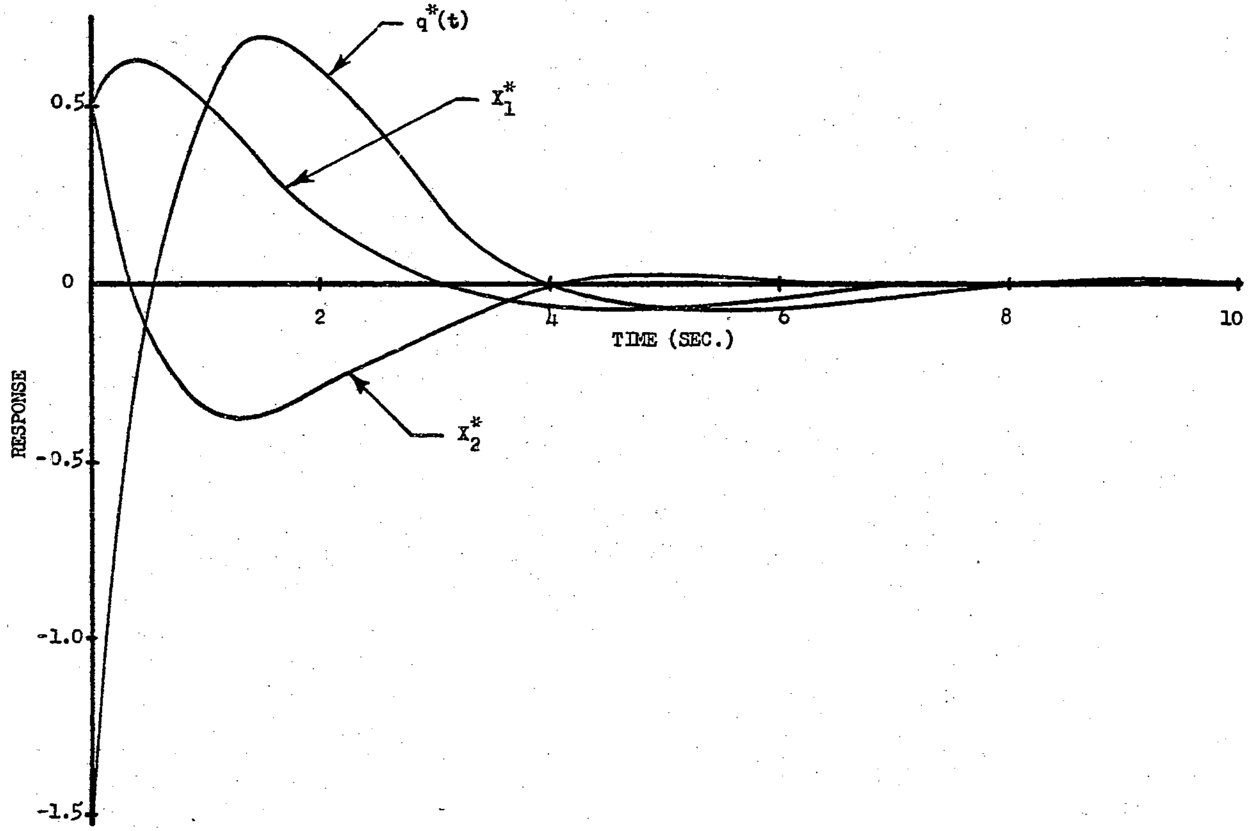


Figure 26. Plot of Optimum Control and State Trajectories

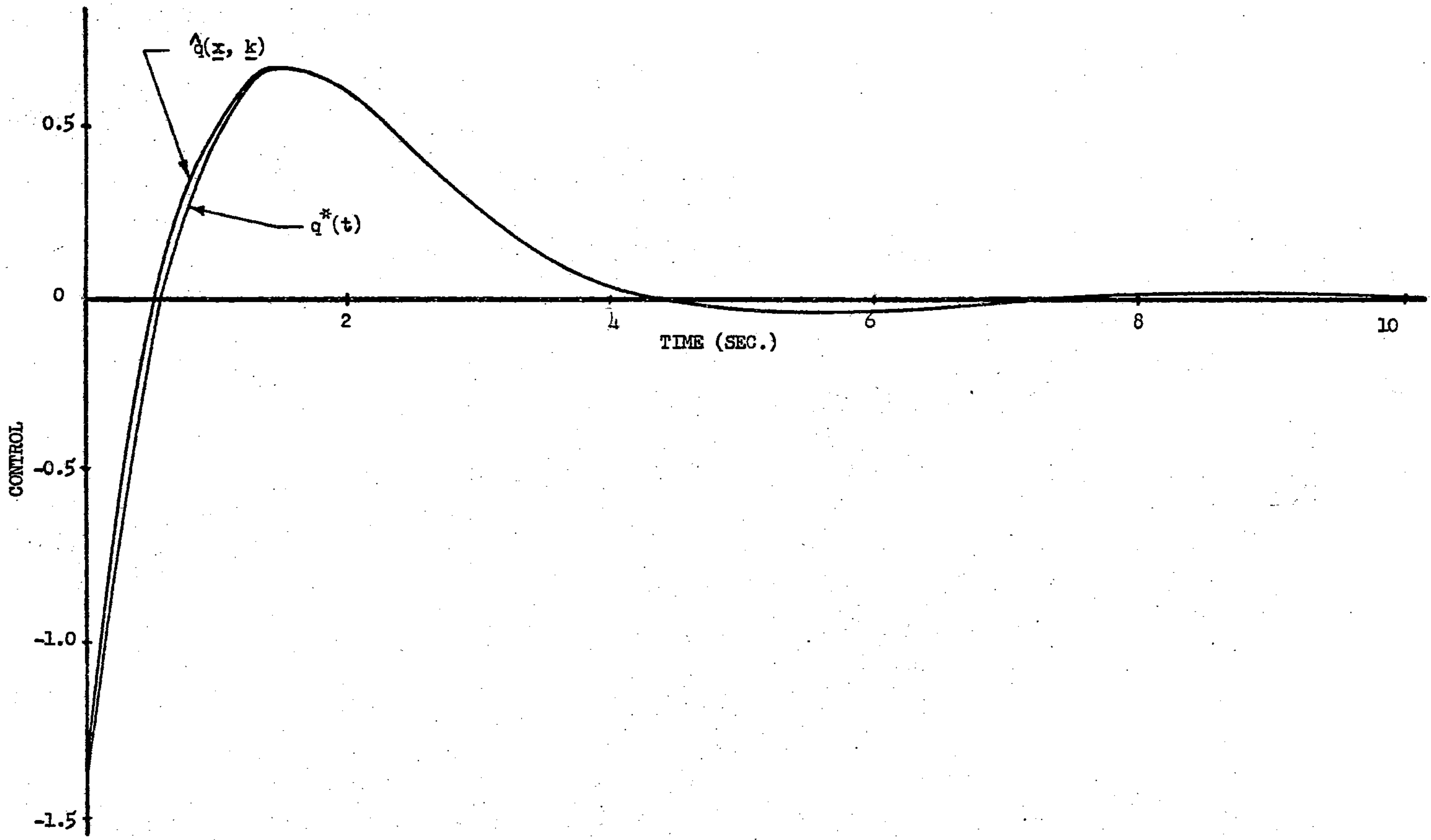


Figure 27. Optimum and Fitted Control Trajectories

the optimum q^*, however, implementation of a k_4 term with absolute value greater than 1.0 is not possible without changing the characteristics of the nonlinear resistor. That is, k_4 less than -1.0 would cause a sign reversal of the nonlinear resistor term and thus cause that element to become an energy sink instead of energy source for small x_1. Although it would be desirable to change the nonlinear resistor in this manner, to do so would not be in keeping with the idea of compensating the circuit. It is assumed that this element cannot be removed or replaced in the circuit, but can be modified somewhat. Therefore k_4 will be restricted to being no less than -0.9.

Restriction of k_4 results in a least squares fit as shown in Figure 28 with the constant parameter values of

$$k_1 = -0.00109799$$

$$k_2 = -0.396004$$

$$k_3 = -3.18588$$

$$k_4 = -0.90000$$

$$k_5 = 0.485762.$$

A plot of the resulting compensated response and the original response is shown in Figure 29 and the compensated and optimum responses are compared in Figure 30. The performance index for the optimal circuit is 0.7970, slightly less than the value of 0.7971 for the method of perturbation discussed by Garrard, et al. (20). The compensated circuit has a performance index value of 0.8795, approximately 10% larger than the optimum value, but again, the compensation can be physically implemented in the electrical circuit.

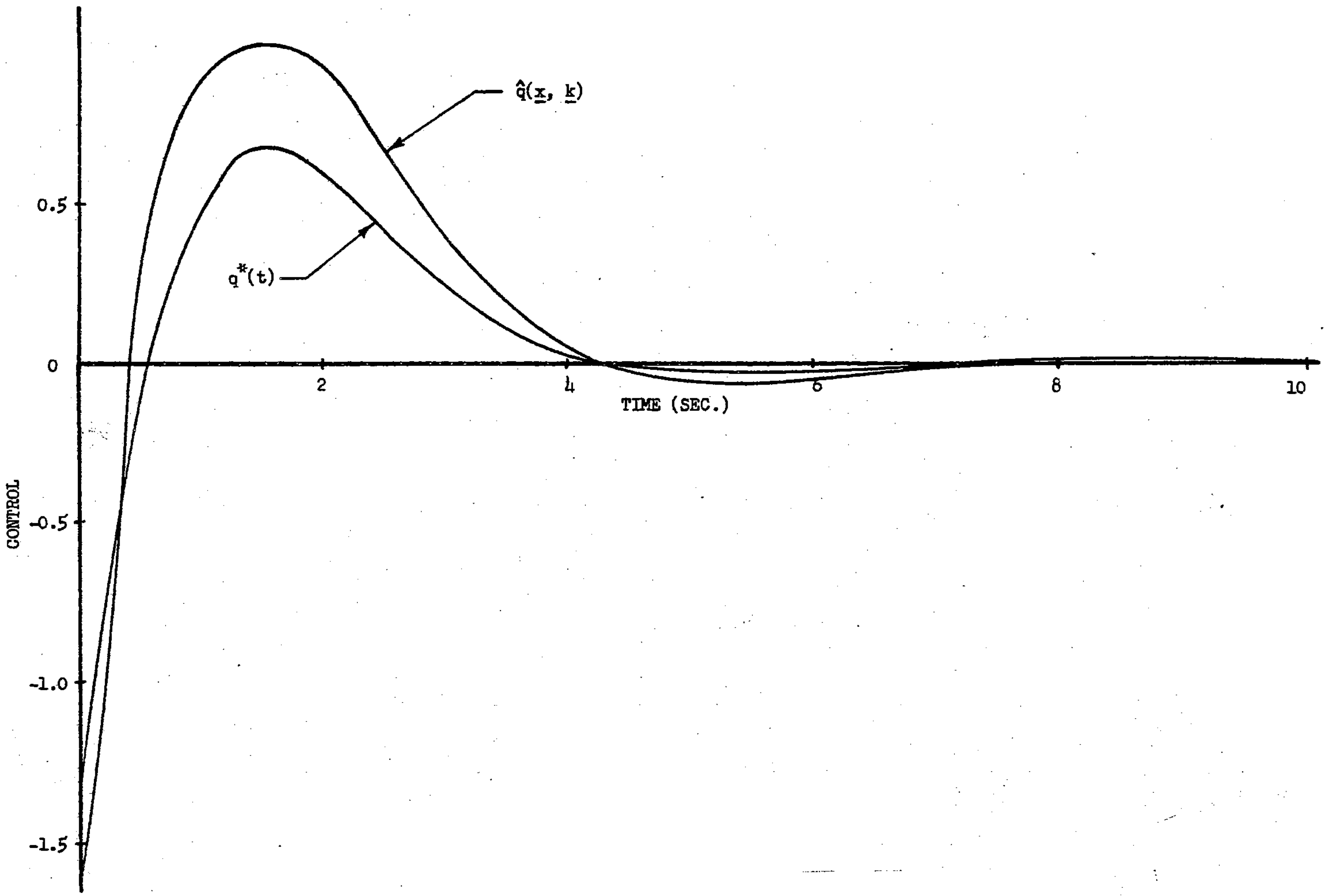


Figure 28. Optimum and Fitted Control Trajectories with Bounds on Physical Parameters

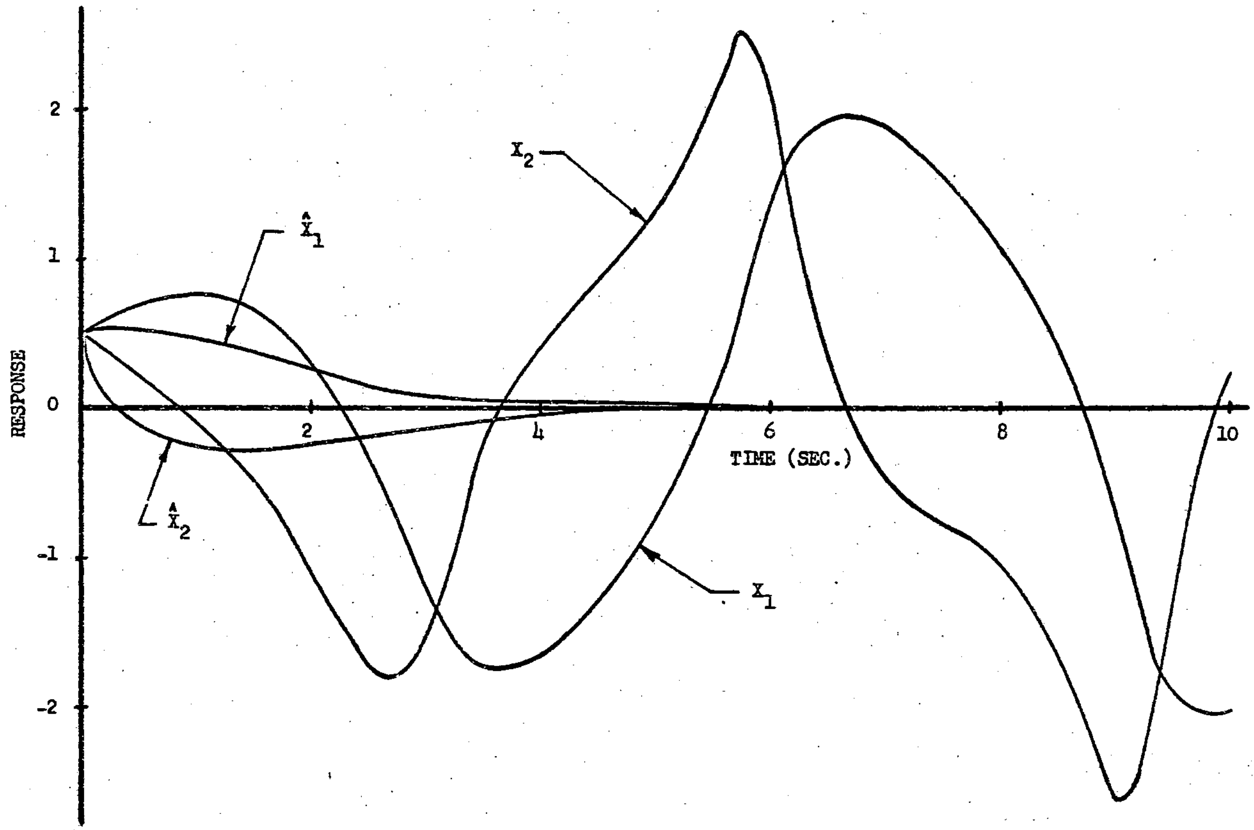


Figure 29. Original and Compensated State Trajectories

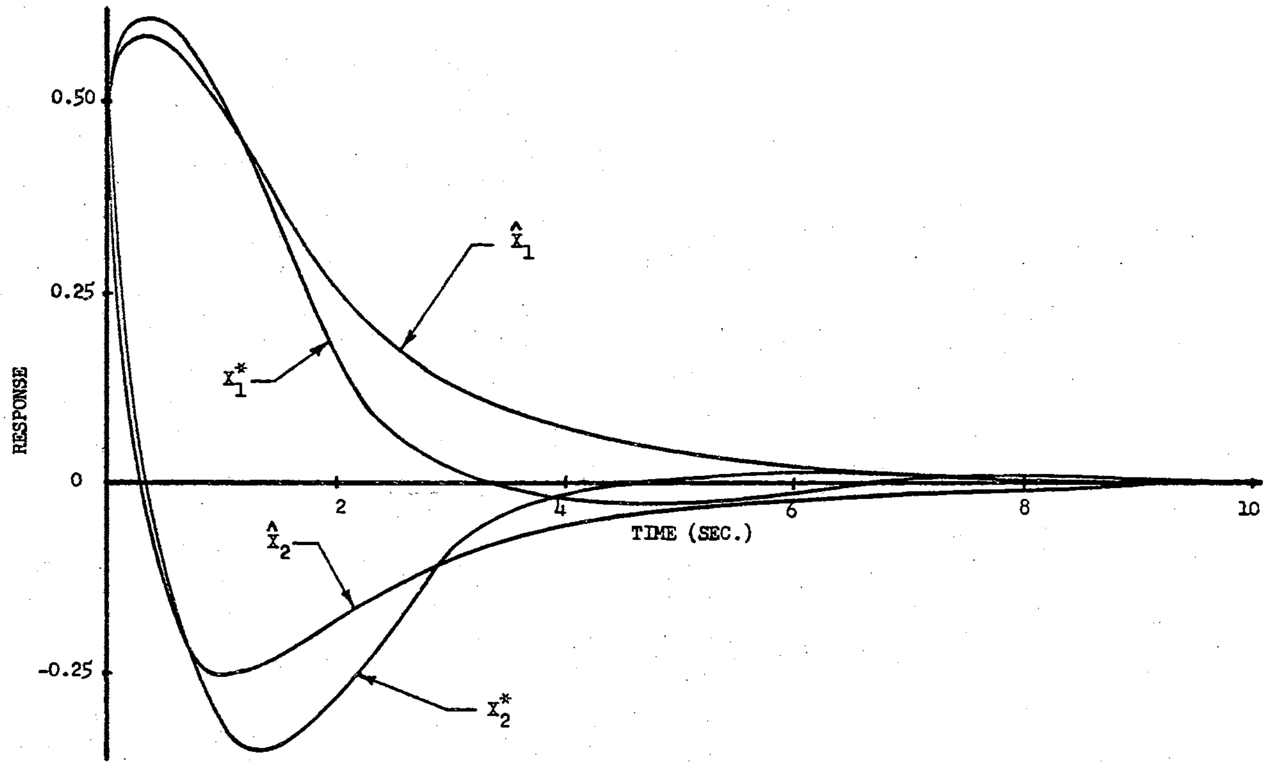


Figure 30. Compensated and Optimum State Trajectories

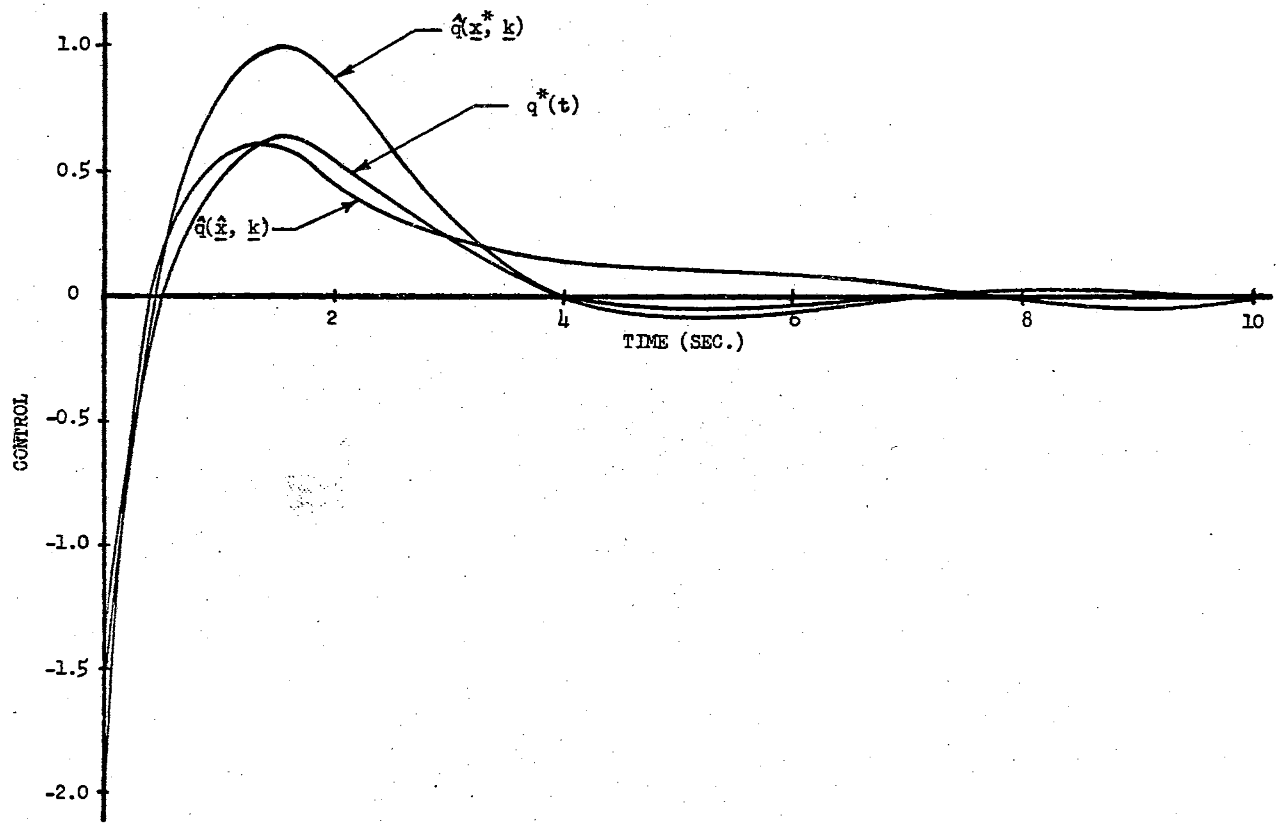


Figure 31. Comparison of Control Trajectories

Example Three - Liquid Level Controller

This example will illustrate the application of the compensation procedure to limiting the state extrema. The system considered in this example is shown in Figure 32. For this system, the equations given below describe the deviations about the initial steady state values of all variables. At $t = 0^-$,

$$q_e = 0$$

$$Q_i(0) = Q_o(0)$$

$$h = 0$$

$$\mathring{h} = 0$$

where q_e represents the maximum external step input to the system, thus at $t = 0^+$, $q_e = 75.0$ inches^3/second. The equations describing the system are

$$q = q_e - q_i - q_o$$

$$q_o = k_3 h$$

$$\tau \frac{dq_i}{dt} + q_i = k_1 e$$

$$e = k_2 h$$

$$q = a \frac{dh}{dt}.$$

The system parameters are

$$k_1 = 10.0 \text{ in}^3/\text{sec-volt} \qquad \tau = 0.2 \text{ sec}$$

$$k_2 = 12.0 \text{ volt/in} \qquad a = 10 \text{ in}^2$$

$$k_3 = 20 \text{ in}^2/\text{sec} \qquad L = 0.65 \text{ in.}$$

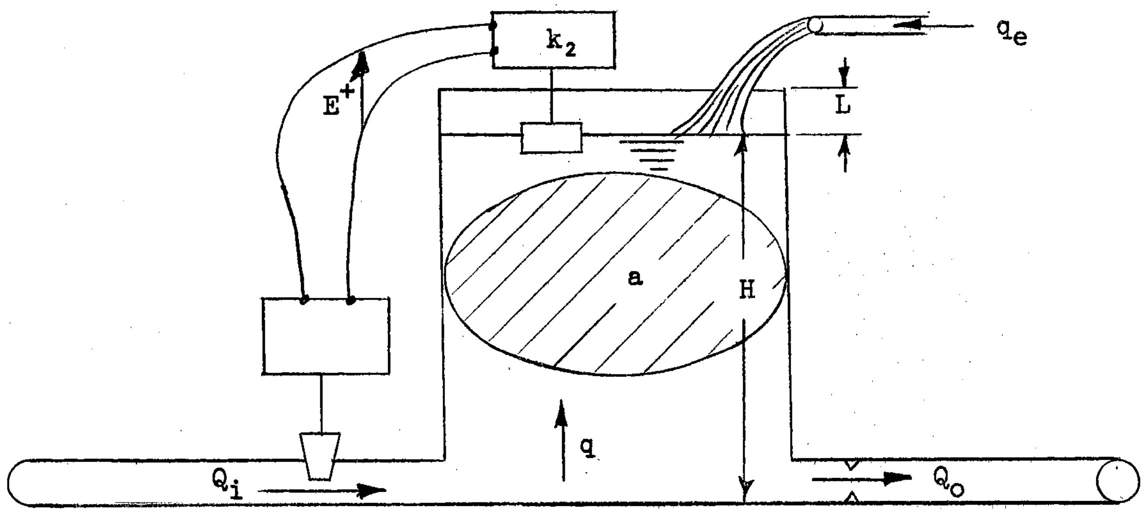


Figure 32. Liquid Level Controller

The overall system differential equation can be shown to be

$$\ddot{h} + A\dot{h} + Bh = Cq_e + D\dot{q}_e.$$

From this equation the state variable diagram in Figure 33 can be constructed from which the state equations are determined as

$$\dot{x}_1 = x_2 + r_1 u$$

$$\dot{x}_2 = r_2 x_1 + r_3 x_2 + r_4 u$$

with the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$. The constants r_i are calculated to be

$$r_1 = 0.1$$

$$r_2 = -75.0$$

$$r_3 = -7.25$$

$$r_4 = -0.2 .$$

A solution to this set of equations shows that for an input of $u = 75.0$ inches^3/second, h theoretically reaches a maximum value of 0.879 inches and would thus overflow the tank. The problem for

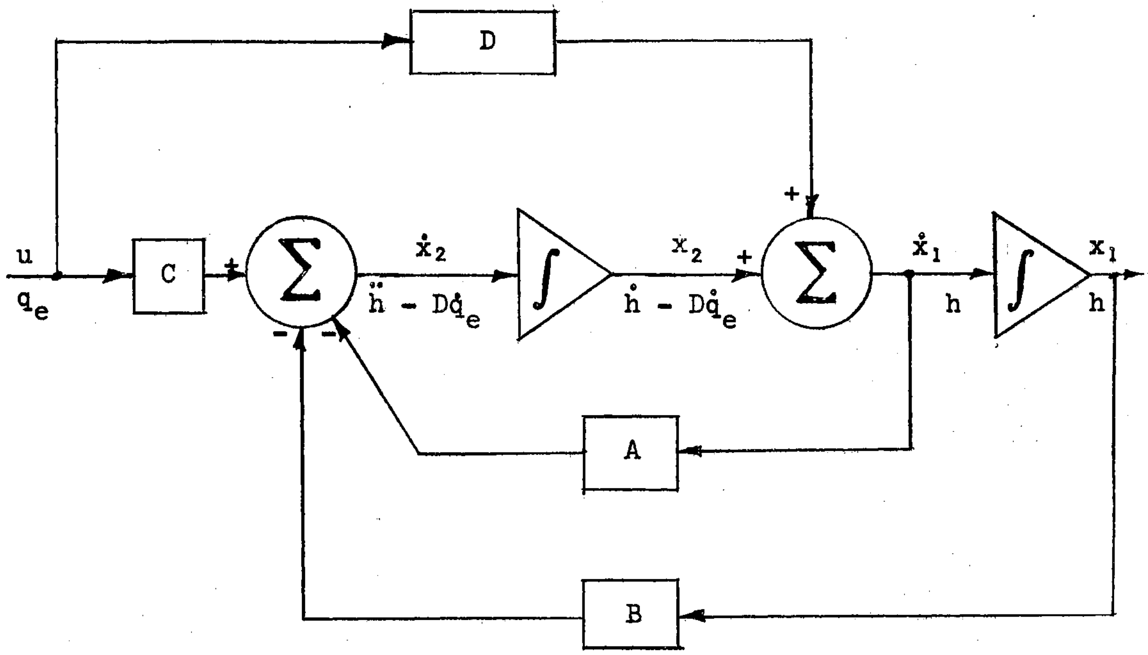


Figure 33. State Variable Diagram for Example Three

this example is to compensate the system in such a manner as to insure that the tank does not overflow, or rather, limit h to less than 0.65 inches.

To limit h(or x_1) to the desired value, a control variable will be added to the state equation for $\dot{x}_2$. Since $\dot{x}_2$ has a maximum amplitude of oscillation of approximately 37 inches/second^2, the control will be given by

$$q = r_5 m = 100m.$$

This scale is chosen so that q may take on values of the same order of magnitude as $\dot{x}_2$ when $|m| < 1.0$. The performance index is formulated as

$$J = \int_0^{1.5} \left[\frac{S}{2}\left(\frac{10x_1}{0.65} - 9\right)^2 + \frac{r_6}{2}m^2 \right] dt$$

where

$$S = 0 \text{ if } x_1 < 0.585,$$

$$S = 1.0 \text{ if } x_1 \geq 0.585,$$

$$r_6 = 0.10.$$

The Hamiltonian is now written

$$H = \frac{S}{2}\left(\frac{10x_1}{0.65} - 9\right)^2 + \frac{r_6}{2}m^2 + p_1(x_2 - r_1u)$$
$$+ p_2(r_2x_1 + r_3x_2 + r_4u + r_5m)$$

from which the following equations are calculated.

$$\dot{x}_1 = x_2 - r_1u$$

$$\dot{x}_2 = r_2x_1 + r_3x_2 + r_4u + r_5m$$

$$\dot{p}_1 = -S\left(\frac{10x_1}{.65} - 9\right) - r_2p_2$$

$$\dot{p}_2 = -p_1 - r_3p_2$$

$$m = -\frac{r_5}{r_6}p_2 \rightarrow q = -\frac{r_5^2}{r_6}p_2 .$$

The boundary conditions for these equations are

$$x_1(\ 0\ ) = 0$$

$$x_2(\ 0\ ) = 0$$

$$p_1(1.5) = 0$$

$$p_2(1.5) = 0 .$$

The computer program TPBVGQ (20) for solving two-point boundary value problem gives

$$p_1(0) = 0.00114714$$

$$p_2(0) = 0.000291052 .$$

A solution to the optimum controlled system is shown compared to the uncompensated system in Figure 34. The optimum response stays satisfactorily below the desired maximum, hence a means to implement the control shown in Figure 35 is sought. However, there appears to be no simple continuous relationship between the control and the state variables. In fact, a relay type control is suggested by the shape of $q^*(t)$ in the figure. Thus the physical system will be studied to determine the feasibility of implementing a relay controller.

For the system shown in Figure 32, a negative constant voltage source that is switched in and out by a relay could be inserted between the liquid level sensing element and the input flow controller. This relay could be activated upon sensing an input q_e and deactivated when h exceeds 0.58 inches.

Figure 36 pictures the system with the compensating relay control installed and Figure 37 shows the resulting system response compared to the uncompensated response. The optimum response and the actual compensated response are compared in Figure 38 while Figure 39 shows $q^*(t)$, $\hat{q}(\underline{x}^*, \underline{k})$ and $\hat{q}(\hat{\underline{x}}, \underline{k})$. This example has illustrated that the compensation procedure may be used to limit state extrema rather than shape the entire response. Also, it has shown that in some cases a relay or discontinuous control may be found desirable from the application of this compensation method.

Example Four - Dynamic System

This example problem, at first glance, appears to be formulated similarly to the other problems, and hence, similar results might be expected. The system equation is

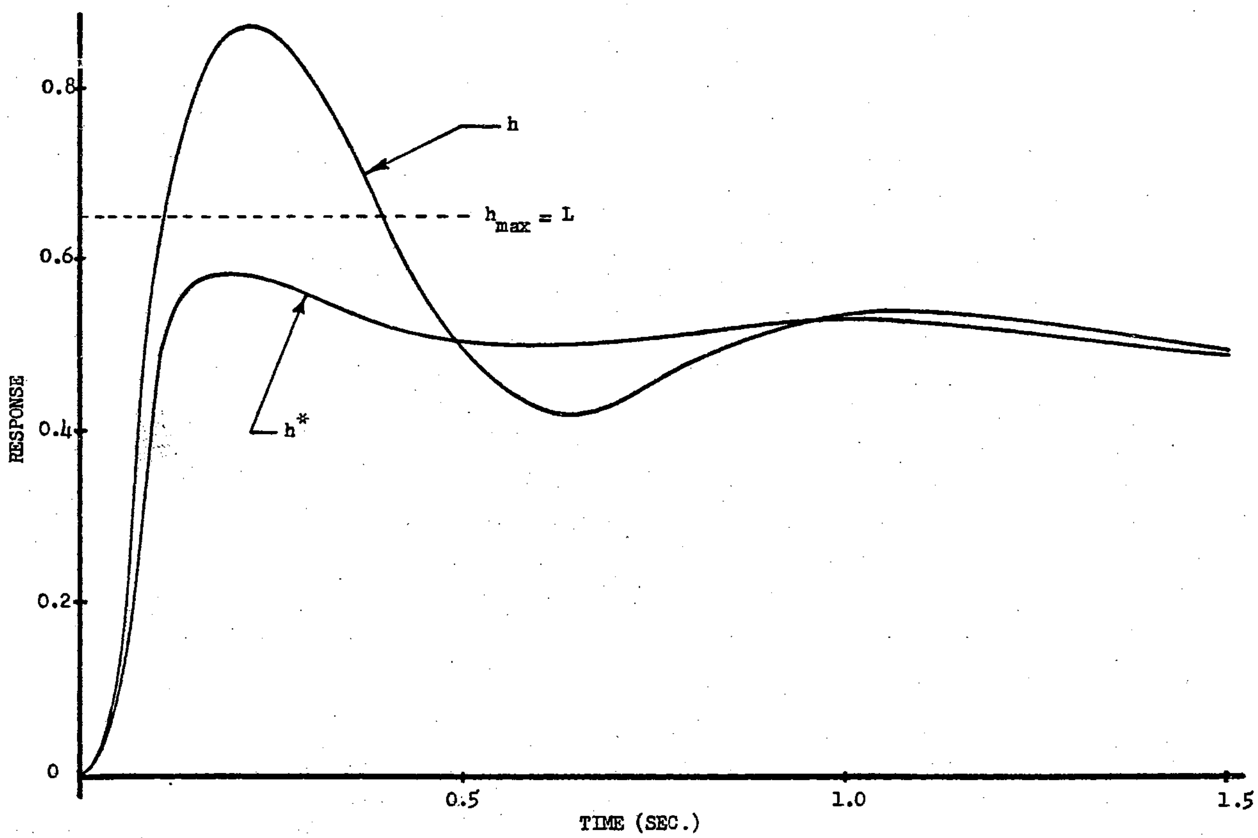


Figure 34. Comparison of Optimum and Uncompensated Responses

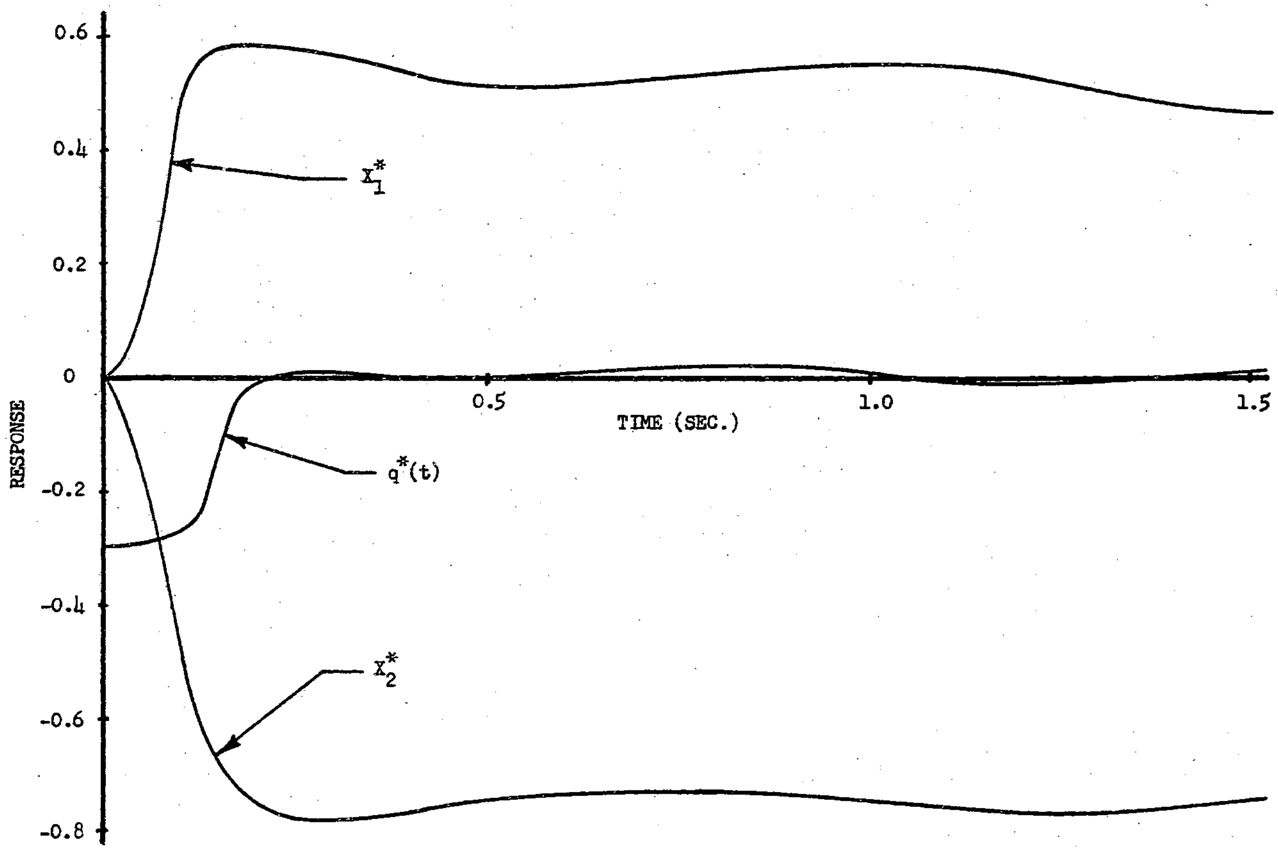


Figure 35. Plot of Optimum Control and State Trajectories

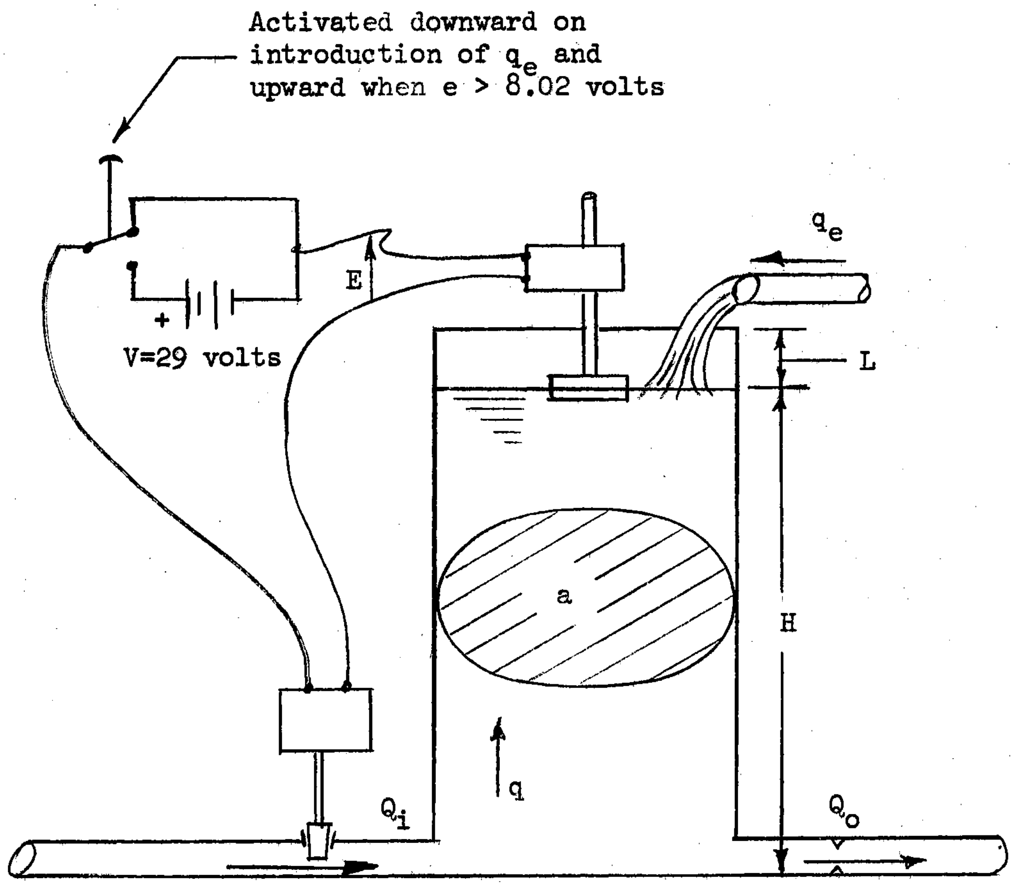


Figure 36. Compensated Liquid Level Controller

$$\ddot{x} + 108\dot{x} - 1238x\dot{x} + 288600x = f(t),$$

$$x(0) = 0,\ \dot{x}(0) = 0,$$

where $f(t)$ is a step input at time zero of -112520. The objective is simply to attempt to decrease the response rise time and to decrease the amount of overshoot. The uncompensated system response reaches the steady state value of approximately -0.39 in slightly less than 0.005 seconds and overshoots to about -0.47 before settling to the final value.

In order to effect the compensation, an undetermined control q is added to the system equation which in state variable form becomes

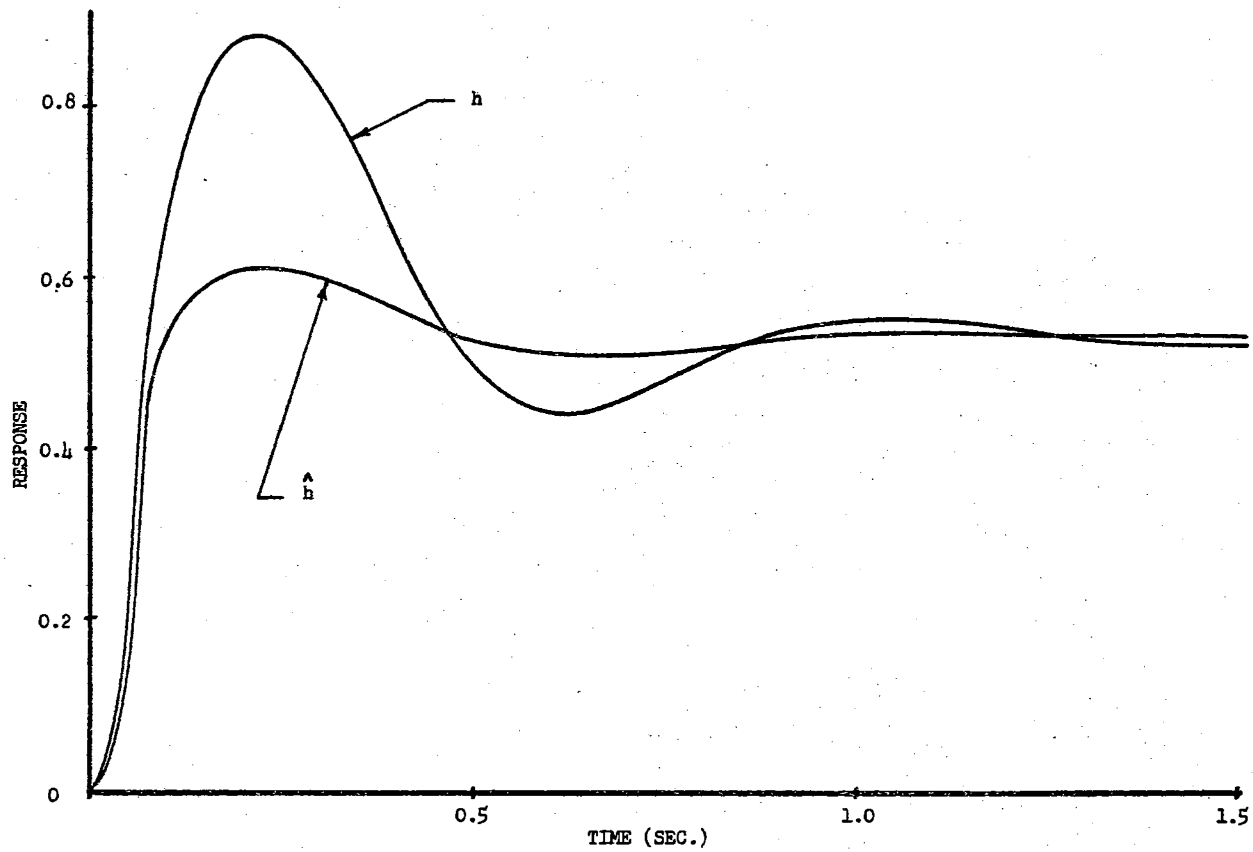


Figure 37. Original and Compensated State Trajectories

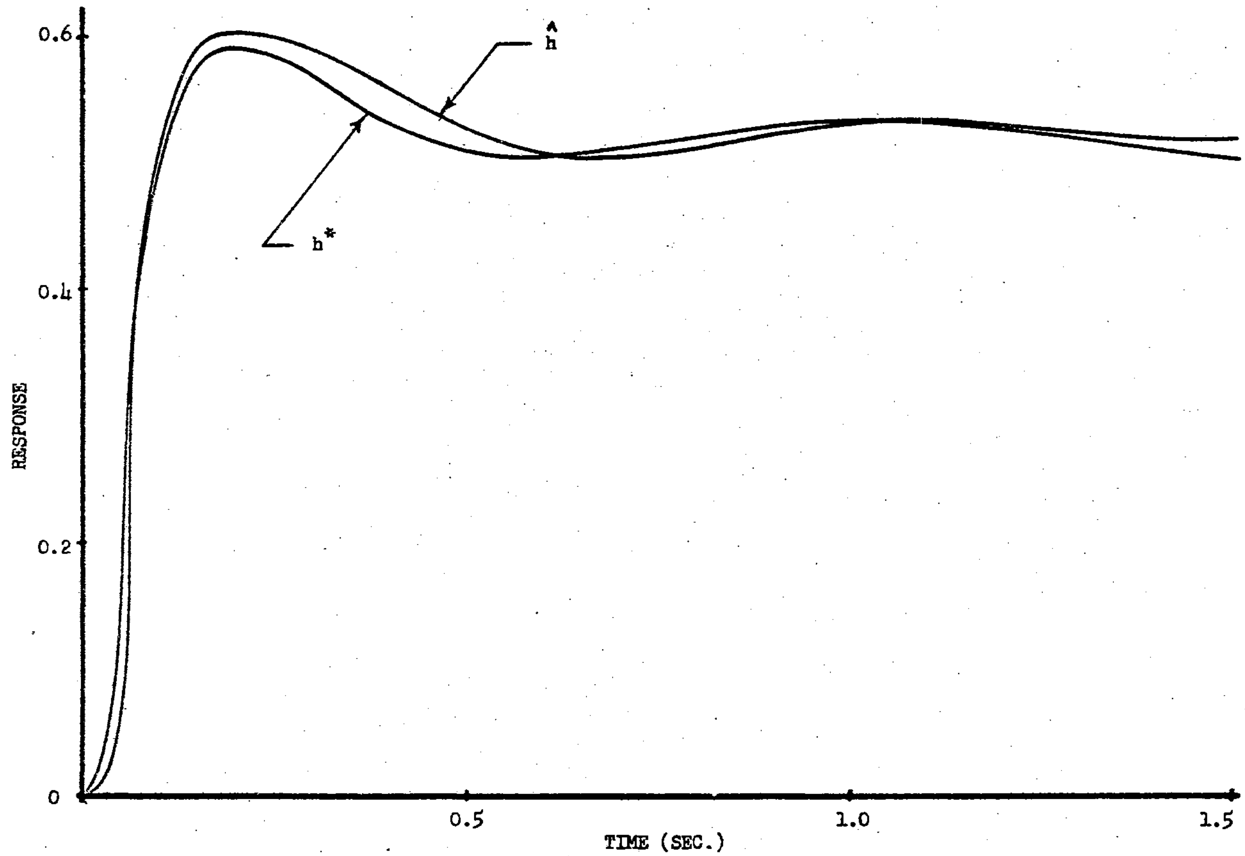


Figure 38. Compensated and Optimum State Trajectories

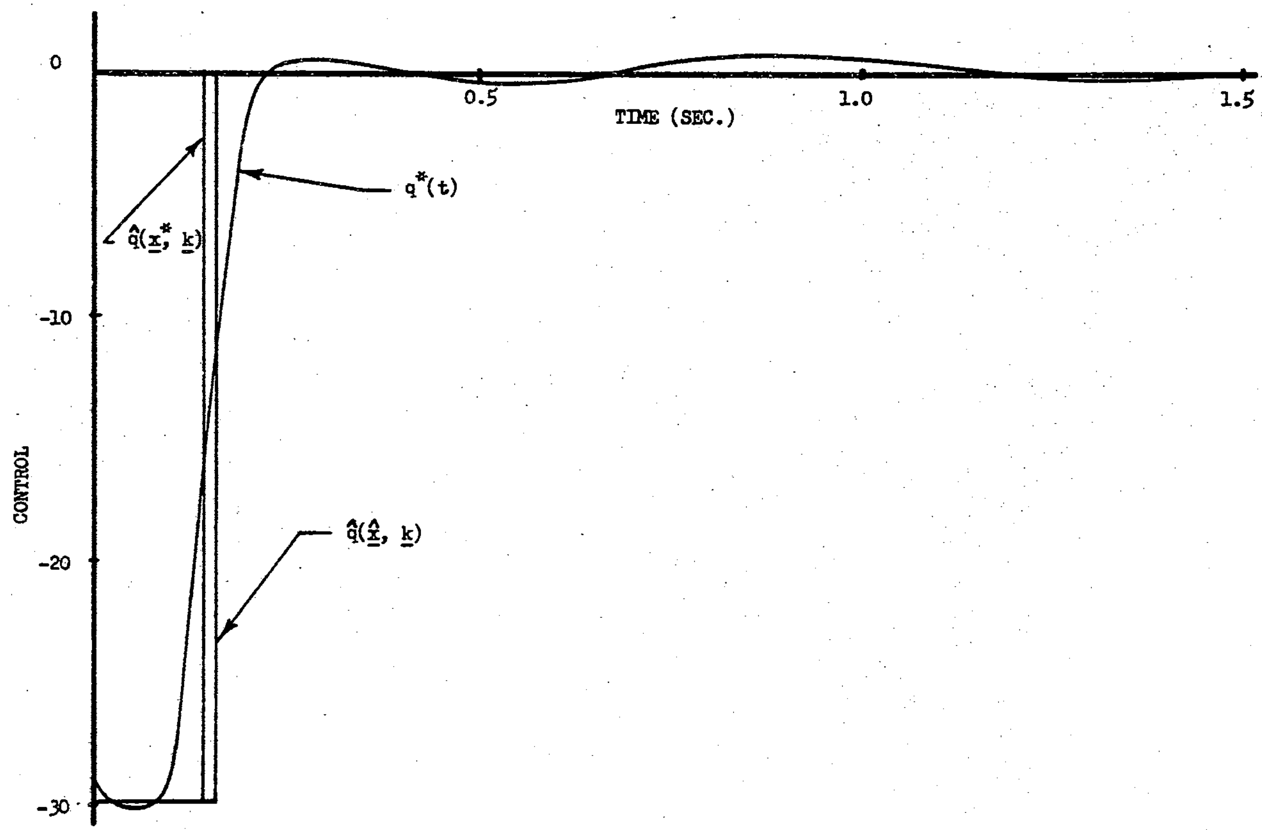


Figure 39. Comparison of Control Trajectories

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -r_1x_1 - r_2x_1x_2 - r_3x_2 + f(t) + q.$$

A performance index for the transient portion of the solution is formulated as

$$J = \int_{t_o}^{t_f} \tfrac{1}{2}[r_4(x_1 - x_d)^2 + r_5(m)^2]\, dt$$

where $t_o = 0$, $t_f = 0.03$, $x_d = -0.39$ and m is the control q scaled by r_6, i.e.

$$m = \frac{q}{r_6}.$$

Since q should be allowed to take on the same order of magnitude as $\dot{x}_2$ which reaches approximately 4×10^4, q is desired to be within the range $|q| \leq 10^5$. In order to maintain $|m| \leq 1.0$, the scaling constant is chosen $r_6 = 10^5$.

The next consideration is that of selecting the relative sizes of r_4 and r_5. The primary objective in minimizing J is to minimize the term $r_4(x_1 - x_d)^2$, hence this term must be weighted more heavily than r_5m^2. With $x_d = -0.39$, the maximum value for $(x_1 - x_d)^2$ is approximately 0.1 and r_6 has already been selected so that m^2 is approximately 1.0. Thus if r_4 is chosen as 1.0 and r_5 as 0.01, then the approximate maximum value of $r_4(x_1 - x_d)^2$ will be ten times that of $r_5(m)^2$. With the constants thus determined the Hamiltonian becomes

$$H = \tfrac{1}{2}(x_1 - x_d)^2 + 0.01m^2 + p_1x_2 + p_2\Big(-r_1x_1 - r_2x_1x_2 - r_3x_2 + f(t) - 10^5m\Big).$$

The necessary conditions for optimality given by Theorem 4-1 require that the following equations be solved.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -r_1x_1 - r_2x_1x_2 - r_3x_2 + f(t) + 10^5m$$

$$\dot{p}_1 = -x_1 + x_d + r_1p_2 + r_2x_2p_2$$

$$\dot{p}_2 = -p_1 + r_3p_2 + r_2x_1p_2$$

The boundary conditions and constants for these equations are

$x_1(0)$	$= 0$	r_1	$= 288600$
$x_2(0)$	$= 0$	r_2	$= -1238$
$p_1(0.03)$	$= 0$	r_3	$= 108$
$p_2(0.03)$	$= 0$	$f(t)$	$= -112520$

and the minimization of $H(q)$ requires

$$m = -p_2 \times 10^7.$$

Solution of the above two-point boundary value problem gives the following values for the necessary initial conditions on p_1 and p_2

$$p_1(0) = 6.2131809 \times 10^{-4},$$

$$p_2(0) = 6.2986326 \times 10^{-7}.$$

A plot of the optimum response and the original response are shown for comparison in Figure 40. The very little difference in these two plots shows that for this formulation of the problem, the optimal

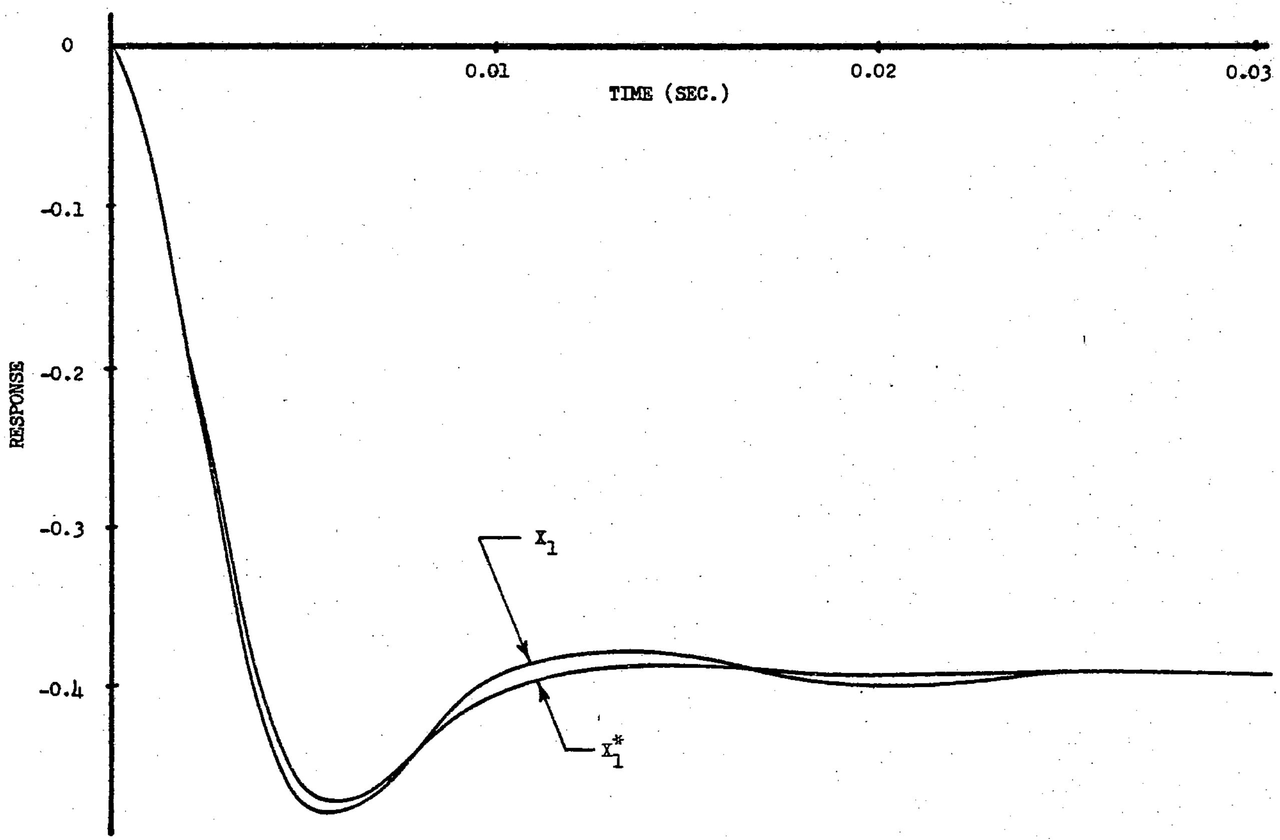


Figure 40. Comparison of Optimum and Uncompensated Responses

response is only slightly better than the original. Thus the compensated system could hardly be expected to show any improvement and would certainly not be worth the effort to attempt to implement this control. The conclusion reached for this problem is that any form of compensation to the basic system would not be expected to improve the response significantly. Any real change in the response would probably require a major change in the system design.

Summary

The illustrative problems used in this chapter were not chosen to demonstrate the breadth of problems for which the compensation technique discussed was applicable. Rather, these examples were selected to illustrate some of the specific characteristics proposed in the statement of the objectives. Indeed, the total scope of different problem formulations, desired characteristics, response trajectory shapes, terminal cost problems and time-varying weighting schemes which could be used cannot be discussed in a few pages. Instead these problems are intended to demonstrate that the compensation technique presented in Chapter IV can be used to achieve the following: (1) the compensation of nonlinear systems, (2) the shaping of the response trajectory, (3) the limiting of state extrema, (4) a determination of whether compensation will result in a significant response improvement and (5) most important, the assurance of physically realizable compensation. Another significant fact is that this technique represents a link between some of the somewhat lofty theoretical developments in modern control theory and some of the actual system design problems.

CHAPTER VI

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

The general problem of compensating nonlinear deterministic systems in the time domain in order to achieve certain response characteristics has been studied in this dissertation. A general procedure for determining if compensation is feasible and what the proper compensation should be has been developed. This chapter presents a brief summary of the advantages and disadvantages of this procedure and a review of the steps to be followed in applying the compensation method to a particular problem. A list of the areas in which it is felt that this thesis represents a contribution to the state of the art is also presented. Finally, several points are discussed which should receive further study in order to extend the capability of the results of this thesis.

Summary and Conclusions

One of the principal advantages of the compensation technique presented in this study is the fact that the entire analysis is conducted in the time domain. That is, the response characteristics may be studied in the real time plane rather than in a complex frequency plane. This fact may at times be thought of as a disadvantage since it does not allow the specification of some frequency response characteristic such as gain margin or phase margin. A far greater advantage that stems from study in the time domain is the ability to apply this compensation method to nonlinear systems. At no point in the development of the theory in Chapter IV was the system model or the control elements restricted to linear form. The examples presented in Chapter V verify this since they all contain nonlinear elements in the system or control vector or both.

Another of the primary advantages is the fact that the performance functional is formulated from the problem requirements and is not some fixed form. The error index may be formulated and weighted as desired to fit the particular problem. The weighting may be time-variable to stress the importance of some state at a particular point in time. The performance functional has the flexibility of being formed to fit the response to some desired trajectory or to restrict the state response from exceeding some maximum or minimum level. Closely connected with the formulation of the performance functional is another advantage which lies in the fact that only the desired state characteristics need be specified. That is if only the trajectory of the first state in a third-order system is of interest, then only that state need be specified in the performance functional.

Once the performance functional has been determined and the two-point boundary value problem formulated, one of the more difficult steps in the compensation procedure must be made. This step involves the solution of the two-point boundary value problem and may be considered as somewhat of a disadvantage for this procedure because of its general difficulty. However once the two-point boundary value problem has been solved, the system designer has the optimum system response available for study. This represents a significant advantage since with this information he can make a decision early in the design process on whether the optimum response represents a significant improvement over the uncompensated response. He further has an indication of just how much improvement stands to be gained if the optimum control is implemented.

The fact that the optimum control trajectory is available for study is also an advantage since guidelines have been established for using the optimum trajectory to aid in determining the general form of the fitted control. In addition, the degree to which the suboptimum physically implemented control approximates the optimum control can easily be studied. The trade-off between optimality and implementability can also be quickly determined by using the optimum trajectory as a standard.

The assurance that the compensating control can be physically realizable is one of the principal advantages of this compensation technique. The system designer controls the general form of the fitted control by selecting only terms that are implementable and assures that the parameters can be realized by limiting their possible values. He is further assured that no time-varying parameters will result. In connection with determining the proper compensating control, both the original system parameters as well as parameters associated with any new terms may be adjusted and the proper values for each set selected.

One final capability will be mentioned that certainly represents an advantage. The ability to adjust the physical system parameters and not just the equation coefficients is very beneficial. Although this point was not made during the discussion of the fitting procedure, either the least squared error fitting routines or a parameter optimization technique may be used to determine proper parameter values as well as coefficient values to fit $\hat{\underline{q}}(\underline{x}, \underline{k})$ to $\underline{q}^*(t)$.

Two requirements on the system state equations limit its application to some extent. The restrictions on the optimal control problem require that $\underline{f}(\underline{x}, t)$ be continuous and differentiable in $\underline{x}$ and t. Thus the many physical systems that have discontinuous state equations cannot utilize this compensation procedure. Another broad category of systems which cannot be studied using the present techniques is stochastic systems or systems with random variables. Recommendations for extending the present compensation method to include both of these types of systems is made in the next section.

A brief step-by-step summary of how to apply the compensation technique to a problem will be given here.

1. Determine the time interval over which the problem is to be studied, $(t_o - t_f)$. Integrate the original uncompensated system equations over the interval $t_o - t_f$.
2. Determine how many control states will be generated and how they will be added to the system equations.
3. From the problem statement of the desired response characteristics, formulate the performance functional.
4. Select the proper values of the weighting coefficients in the error index based on the general form of the performance functional and the information gained in step #1.
5. Form the Hamiltonian function and from this, formulate the two-point boundary value problem.
6. Estimate starting guesses for the states with unspecified

initial conditions.

7. Solve the two-point boundary value problem to obtain the optimum response and the optimum control trajectory.

8. At this point decide whether the optimum response is sufficiently improved over the uncompensated response to warrant an attempt to implement the optimum control.

9. From the optimum control trajectory, knowledge of variable parameters in the original system, and physical realizability considerations, determine the general form of the fitted control $\hat{\underline{q}}(\underline{x}, \underline{k})$.

10. Determine the proper values of the constant parameters $\underline{k}$ that will fit $\hat{\underline{q}}(\underline{x}, \underline{k})$ to the optimum control $\underline{q}^*(t)$.

11. If $\hat{\underline{q}}(\underline{x}, \underline{k})$ does not fit $\underline{q}^*(t)$ sufficiently close, modify the general form of $\hat{\underline{q}}$ by adding or deleting terms to obtain a better fit. Examine the residue, $\underline{r} = \hat{\underline{q}} - \underline{q}^*$, to determine the appropriate modifications to $\hat{\underline{q}}$. Repeat step #10.

12. Add the fitted control $\hat{\underline{q}}(\underline{x}, \underline{k})$ to the original state equations. Integrate the compensated system equations from t_o to t_f to obtain the compensated response.

13. If the compensated response is satisfactory, the problem is finished. If not, return to step #11 and attempt to further modify $\hat{\underline{q}}$ to obtain a still better fit to $\underline{q}^*$. If this action does not result in a satisfactory response, return to step #4 or #3 and change the weighting coefficients or the general form of the performance functional to place added emphasis on the particular characteristic that is

unsatisfactory. Should this again fail to produce acceptable results, return to step #2 and consider the possibility of utilizing additional control states.

The developments presented in this study which should constitute contributions to the general knowledge of dynamical systems analysis and design are listed below.

1. The primary contribution involves development of a general concept of utilizing optimal control theories in the physical compensation of nonlinear dynamical systems.
2. The state of system design and analysis art has been advanced through the development of a compensation technique that incorporates all of the advantages described above into one method. Some of these advantages were available in previous compensation or design procedures, but no one technique combined the several advantages and was as generally applicable as the present method.
3. Another significant development involves the establishment of design guidelines to aid in determining the proper physical elements that should be used to compensate the system.
4. The development of a performance functional to limit state extrema such as in this thesis has not been observed in any published literature. Thus this development represents a further advancement of the state of the art in this area.

Recommendations for Future Investigations

During the development of the compensation procedure presented in this thesis, several areas were recognized as warranting further investigation. In some areas the lack of development hampered the application of this technique, whereas in other instances, it was recognized that further investigations might greatly expand the applicability of the present method. Thus additional research is recommended in the following areas:

1. Efforts should be made to improve existing methods for solving nonlinear two-point boundary value problems. Although three computational routines are referenced which solve this type problem (15, 16, 17), the requirement of providing accurate estimates on the unspecified initial conditions is sometimes difficult to satisfy. The routine developed by Unruh in (15) represents a significant improvement over the other two, but additional development is still desirable.

2. In line with the above requirements, more refined techniques should be developed to aid in determining starting guesses for the unspecified initial conditions in two-point boundary value problems. One possible technique that bears investigation can be outlined briefly by considering the nth-order two-point boundary value problem

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t) \qquad 0 \leq t \leq T$$

with boundary conditions

$$x_1(0) = x_{10}$$

$$\vdots$$

$$x_k(0) = x_{ko}$$

$$x_{k+1}(T) = x_{k+1T}$$

$$\vdots$$

$$x_n(T) = x_{nT}.$$

The solution $\underline{x}(t)$ can be expanded in a Taylor's series expansion about the point T with an expansion interval of $-T$. Thus

$$\underline{x}(T - T) = \underline{x}(0) = \underline{x}(T) + \dot{\underline{x}}(T)T + \ddot{\underline{x}}(T)\frac{T^2}{2!} + \dddot{\underline{x}}(T)\frac{T^3}{3!} + \ldots \ldots \tag{6-1}$$

The entire right-hand side of the above equation can be expressed in terms of the n state variables evaluated at $t = T$. Equation (6-1) is then simply a set of n non-linear algebraic equations in n unknowns $x_1(T), \ldots x_k(T), x_{k+1}(0), \ldots x_n(0)$. If these equations can be solved, the values of $x_{k+1}(0), \ldots x_n(0)$ should be the desired starting conditions, the accuracy of which depends on the number of terms in the series expansion. Due to the general complexity of the original two-point boundary value problem, this technique would of course have to be computerized. However, computer techniques currently exist for taking the derivative of an analytical function. An IBM compiler language FORMAC will perform this function as well as handle other manipulations of equations in analytical form. A computer routine especially for the purpose of taking partial or total derivatives of analytic functions has been written by Stanley Wendt of the School of Mechanical and Aerospace Engineering at Oklahoma State University, but the results of this are as yet unpublished.

3. The extension of this method of compensation to systems with discontinuities or stochastic systems should be invesitgated. Many systems have components which operate against physical limits or stops for periods of time and hence the system model must exhibit these discontinuous constraints. Other systems are described by random variable models in a statistical sense. The inclusion of these system categories would greatly enhance the general applicability of the present compensation technique.

4. A technique for constraining the values of $\underline{k}$ in the fitted control $\hat{\underline{q}}(\underline{x}, \underline{k})$ without placing rigid limits on the parameter variations would be more compatible with physical system design. As discussed in Chapter IV, in actual system design the designer is rarely able to specify exactly the limits to which a parameter can vary, thus the imposed limits are somewhat arbitrary. A more desirable procedure would be to assess an increasing penalty to the parameter as it approaches a tentative limiting point. The technique would be formulated to fit $\hat{\underline{q}}(\underline{x}, \underline{k})$ to $\underline{q}^*(t)$

while minimizing the penalties assessed the parameter variations.

5. The final recommendation concerns an investigation into the possibility of making the fitted control $\hat{\underline{q}}$ a function of the system initial conditions and input. For nonlinear systems, a control which is optimal for one set of initial conditions and driving function is not necessarily optimal for a different input or initial condition. Therefore it would be desirable for a system which is subject to a variety of initial conditions or inputs to be able to formulate $\hat{\underline{q}}$ as a function of these variables as well as the state variable trajectories, i.e.

$$\hat{\underline{q}} = \hat{\underline{q}}(\underline{x}, \underline{k}, \underline{x}_o, \underline{u}).$$

The continued development of the procedures and techniques recommended here combined with the system compensation method presented in this dissertation should provide a comprehensive design tool that makes full use of some of the developments of recent years in the field of modern control theory.

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