SYNTHESIS OF DISTRIBUTED NETWORKS

BY

LEWIS GILDART MINOR

Bachelor of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia
1964

Master of Science
Louisiana Polytechnic Institute
Ruston, Louisiana
1965

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
August, 1969

SYNTHESIS OF DISTRIBUTED NETWORKS

Thesis Approved:

Thesis Adviser

Dean of the Graduate College

ACKNOWLEDGEMENTS

This thesis would not have been possible without the patient guidance and contributions of Dr. Rao Yarlagadda, my Thesis Adviser, and I would like to express my appreciation for the many hours of assistance that were so freely given by Dr. Yarlagadda.

I would like to thank the National Science Foundation for the financial support under Project GK-1722 that made this thesis, and the completion of my studies, possible.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Figure		Page

CHAPTER I

INTRODUCTION

<u>1.1 General Discussion</u>. Recently, there has been an increased interest in integrated and thin-film circuits because of possible reductions in size, cost, and power requirements and because of improved reliability (1). One of the major problems in thin-film and integrated circuits is that the well developed lumped parameter theory cannot be used directly since the elements of the thin-film and integrated networks must in general be treated as distributed networks. However, many of the distributed components used in thin-film and integrated networks can be modeled accurately by the uniform transmission line of finite length ($\overline{\text{URC}}$ elements) (2). Figures 1.1.1a and 1.1.1b show two possible forms of the $\overline{\text{URC}}$ element used in thin-film and integrated circuits respectively (2).

The $\overline{\text{URC}}$ element of Figures 1.1.1a and 1.1.1b is normally represented by the symbol in Figure 1.1.2 with the terminals as shown. If the $\overline{\text{URC}}$ element in Figure 1.1.2 is considered to be a two-port network, the open circuit impedance matrix has the form

$$[Z] = \begin{bmatrix} \dfrac{R}{\sqrt{pRC}\tanh\sqrt{pRC}} & \dfrac{R}{\sqrt{pRC}\sinh\sqrt{pRC}} \\ \\ \dfrac{R}{\sqrt{pRC}\sinh\sqrt{pRC}} & \dfrac{R}{\sqrt{pRC}\tanh\sqrt{pRC}} \end{bmatrix} \qquad (1.1.1)$$

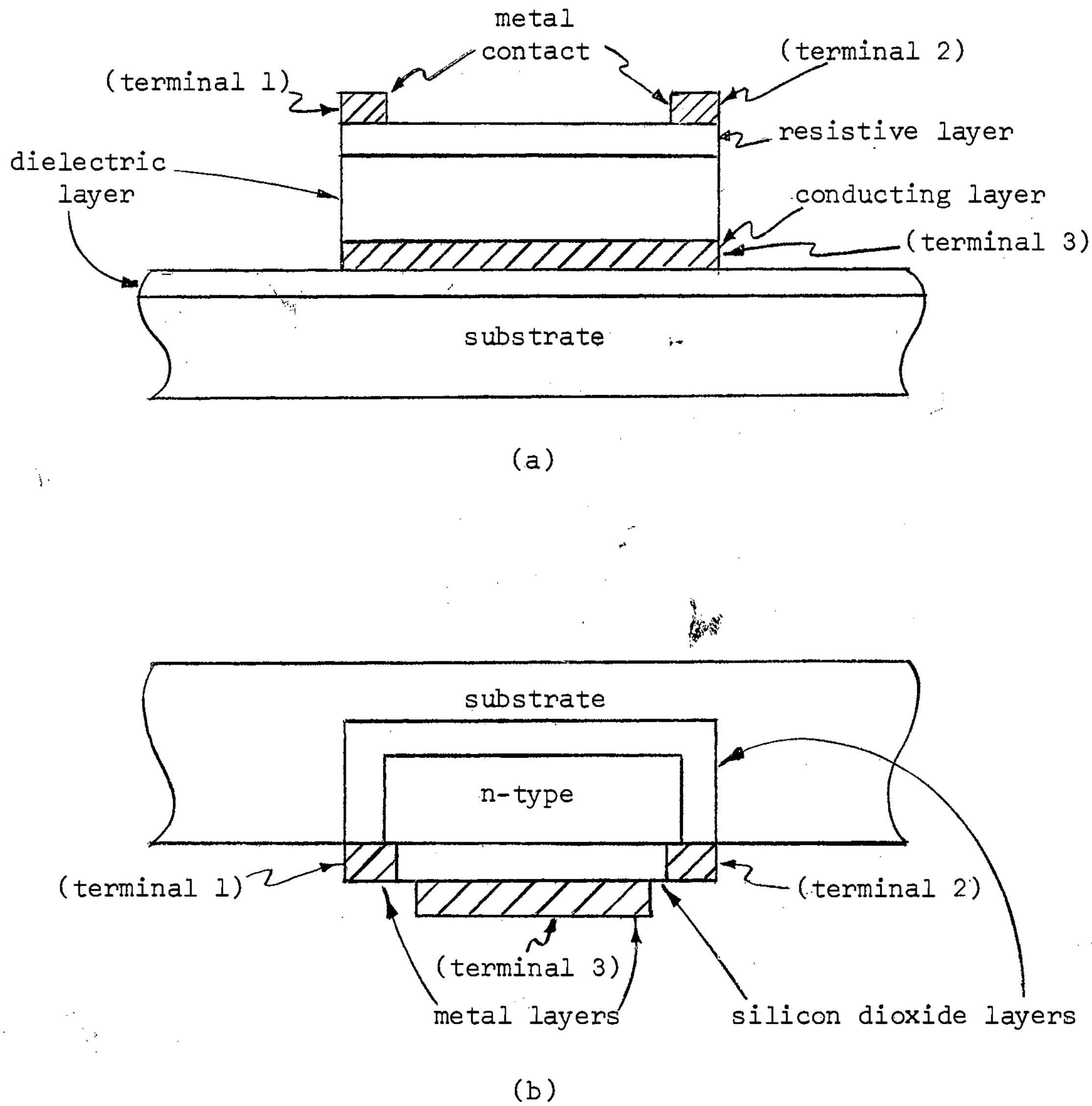


Figure 1.1.1. Examples of Distributed RC Networks Which Can Be Modeled by the $\overline{\text{URC}}$ Element

Where C is the total capacitance of the line, R is the total resistance of the line (2). The irrational hyperbolic forms that appear in Equation 1.1.1 cause great difficulty in the analysis, synthesis, and particularly in the synthesis approximation problem for networks with $\overline{\text{URC}}$ elements ($\overline{\text{URC}}$ network). The problem becomes even more complex if each of the RC products (product of R and C from Equation 1.1.1) has a different value for each $\overline{\text{URC}}$ element.

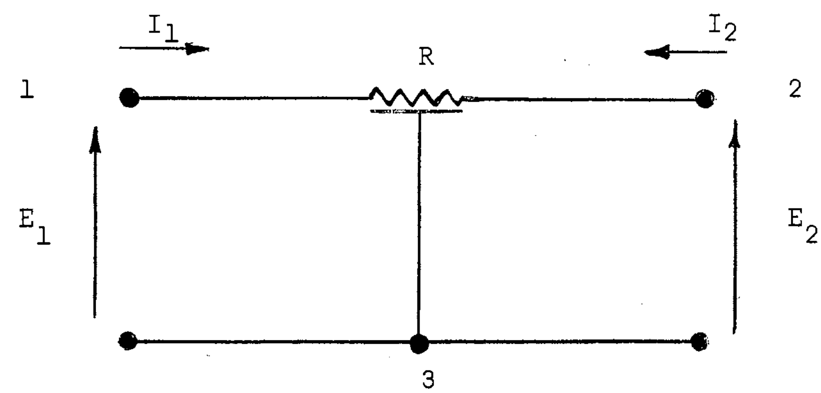


Figure 1.1.2. Symbol for $\overline{\text{URC}}$ Element

<u>1.2 Review of the Literature.</u> If terminals 2 and 3 of the $\overline{\text{URC}}$ element in Figure 1.1.2 are open-circuited (no load), the driving point impedance looking into terminals 1 and 3 is

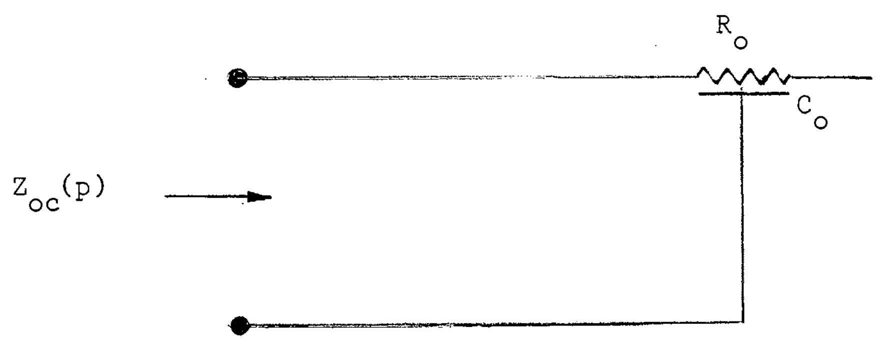
$$Z_{oc}(p) = \frac{R_o}{\sqrt{pR_oC_o}\ \tanh \sqrt{pR_oC_o}} \tag{1.2.1}$$

where R_o is the total resistance of the line, and C_o is the total capacitance of the line (1). Similarly, if terminals 2 and 3 are short-circuited the driving point impedance looking into terminals 1 and 3 is

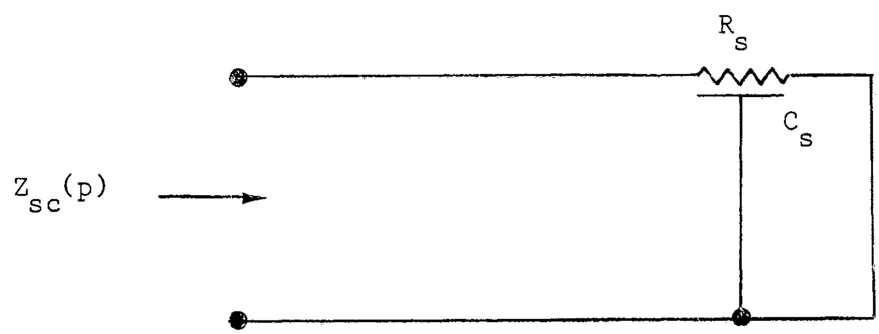
$$Z_{sc}(p) = \frac{R_s \tanh \sqrt{pR_sC_s}}{\sqrt{pR_sC_s}} \tag{1.2.2}$$

where R_s is the total resistance of the line, and C_s is the total capacitance of the line.

The symbolic representation for these two cases is given in Figures 1.2.3a and 1.2.3b respectively.



(a)



(b)

Figure 1.2.3. Networks for $Z_{oc}(p)$ and $Z_{sc}(p)$

The following definitions are given to aid in the subsequent discussion.

<u>Definition 1.1.1 Z_{OC}-Element</u>: A Z_{oc}-element is a distributed element having the driving point impedance given in Equation 1.2.1 for Z_{oc}.

<u>Definition 1.1.2 Z_{SC}-Element</u>: A Z_{sc}-element is a distributed network

element having the driving point impedance given in Equation 1.2.2 for Z_{sc}.

To further simplify notation, a $\overline{URC}$ element will now be defined to be either a Z_{oc}-element or a Z_{sc}-element, and a $\overline{URC}$ network will be defined to be a network of $\overline{URC}$ elements.

Wyndrum (3) has given a synthesis procedure for $\overline{URC}$ networks. He approached the problem by the use of the positive real transformations

$$Z_{LC}(p) = (Z_{RC}(p^2))\ p \tag{1.2.3}$$

$$s = \tanh\ a\ p \tag{1.2.4}$$

where $a = \sqrt{R_s C_s}$ (or $a = \sqrt{R_o C_o}$), Z_{RC} is the impedance of the $\overline{URC}$ element ($Z_{oc}(p)$ or $Z_{sc}(p)$, Z_{LC} is the impedance of Z_{RC} under the transformation (in Equation 1.2.3). Using these transformations in Equations 1.2.1 and 1.2.2 $Z_{oc}(p)$ and $Z_{sc}(p)$ become

$$Z_{oc}(s) = \frac{R_o}{\sqrt{R_o C_o}\ s} = \frac{1}{C\ s} \tag{1.2.5}$$

$$Z_{sc}(s) = \frac{sR_s}{\sqrt{R_s C_s}} = L\ s \tag{1.2.6}$$

where $C = \sqrt{R_o C_o}/R_o$ and $L = R_s/\sqrt{R_s C_s}$. Thus the transformations reduce $Z_{oc}(p)$ and $Z_{sc}(p)$ to the impedance of a capacitor and an inductor (respectively) in the s-domain when each RC product is the <u>same</u> for each $\overline{URC}$ element. Wyndrum gave a sufficient condition for the driving point impedance function Z(p) of a $\overline{URC}$ network with elements having the same RC product to be realizable.

The sufficient condition is that Z(p) be transformed to a realizable lumped LC function by the transformations of Equation 1.2.3 and 1.2.4.

O'Shea (4) gave a set of necessary and sufficient conditions -- using a different transformation -- for Z(p) to be realizable (RC product is the same for each element). Giguere, Swamy, and Bhattacharyya (5) later showed that the two classes of impedances realized by Wyndrum and O'Shea are identical. Further, they have shown that any realizable $\overline{\text{URC}}$ impedance function can be synthesized by a cascade synthesis procedure given by Wyndrum.

Wyndrum has given a rpocedure to realize the driving point impedance given as

$$Z = \frac{K_o}{\sqrt{p}\tanh\sqrt{pRC}} + \frac{K_\infty}{\sqrt{p}}\tanh\sqrt{pRC} + \sum_{i=1}^{n} \frac{K_i \tanh\sqrt{pRC}}{\sqrt{p}\,(\tanh^2(\sqrt{pRC})+\beta_i^{\,2})} \qquad (1.2.7)$$

where K_o, K_∞, K_i, β_i are positive constants and n is an integer (3). Very little has been reported on the use of Equation 1.2.7 or the impedance of other distributed networks to approximate a rational impedance function or an impedance specified in a Bode plot. Wyndrum has given a series of Bode plots which can be used to approximate a given impedance function, but the procedure is a graphical one and inherently inaccurate. The approximations are further limited by the assumption that each element has the same RC product.

Heizer (6) has approached the problems of synthesis with distributed elements by constructing a distributed network with a rational driving point impedance. Unfortunately, the networks are very difficult to fabricate and offer almost no freedom of choice in the location of the poles for the driving point impedance.

Still another approach is taken by Rohrer, Resh, Hoyt (7). Here the given impedance function is approximated by using a single distributed RC network, with an arbitrary taper. The taper is adjusted to minimize the difference between the impedance of the distributed network and the given impedance function. The procedure also applies to transfer functions. Although the method appears to hold great promise, the error function that is minimized in the method is expressed in integral form which may not have a practical form in some applications. In those cases where the error can be minimized, the taper may be too complex for practical fabrication.

Recently there has been a great deal of interest in multivariable impedance functions (8), (9), The most recent paper, by Koga (8), gave the necessary and sufficient conditions for the synthesis of finite passive n-ports with prescribed positive real matrices of several variables. However, in general the method requires transformers in the realization. In this thesis multivariable functions will be used extensively in connection with the realization of $\overline{\text{URC}}$ networks (without transformers) with elements having different RC products. Therefore, some of the important definitions concerning the multivariable functions are given in Appendix B.

1.3 Motivation and Objective. It is evident from the previous section that very little has been done with $\overline{\text{URC}}$ networks with elements having different RC products. A synthesis procedure does not exist for this type of network, and no practical and accurate procedure has been given which can be used in the approximation problem (even in the case where the RC products are assumed to be equal). Further, it is evident that $\overline{URC}$ networks with elements having the same RC products is a subclass of $\overline{URC}$ networks with elements having different RC products. This wider class of networks should provide more flexibility in the approximation problem and yield more accurate results. A natural approach to the synthesis of these networks appeared to be the use of multivariable impedance functions. Therefore, a study of $\overline{URC}$ networks with different RC products and their relationship with multivariable functions appeared to be an excellent topic for research.

In the analysis problem, rational approximations exist for $Z_{oc}(p)$ and $Z_{sc}(p)$ which are based on infinite product expansions and could be used to study $\overline{URC}$ networks having elements with different RC products, but apparently the possibility of improving these approximations has not been considered, and therefore, is another topic for research.

The primary objectives in this thesis will be to:

(1) Improve existing rational approximations for $Z_{oc}(p)$ and $Z_{sc}(p)$.

(2) Develop a synthesis procedure for the driving point impedance $Z(p)$ of $\overline{URC}$ networks with elements having different RC products, and to find some of the important properties of $Z(p)$.

(3) Develop procedures for the approximation of rational impedance functions and impedance functions (rational or irrational) which are specified in a magnitude plot (Bode plot) with $\overline{URC}$ networks with elements having different RC products.

<u>1.4 Organization of the Thesis</u>. The analysis problem is considered in Chapter II and new rational approximations for $Z_{oc}(p)$ and $Z_{sc}(p)$

are derived that are valid over a wider range of frequencies than conventional approximations for the same number of terms. A lumped RC network is derived from these approximations and can be used to model each $\overline{\text{URC}}$ element.

Chapter III gives a relatively simple procedure which can be used to approximate a given rational driving point impedance with a network of $\overline{\text{URC}}$ elements. A method to realize a given rational transfer function using operational amplifiers, $\overline{\text{URC}}$ elements, and gyrators is also given.

Chapter IV deals with a method to remove the restriction that each $\overline{\text{URC}}$ element have the same RC product. Wyndrum's transformations are generalized and the result is a class of multivariable driving point impedance functions which are useful in analysis, synthesis, and in the approximation problem. Some basic properties of the multivariable impedance functions are derived, and some necessary conditions for realizability are given.

Chapter V gives a new method for the synthesis of any realizable driving point function of a $\overline{\text{URC}}$ network with elements having different RC products. In the realization transformers and gyrators are not used. The method can also be used to find the graph for the classical topological formula for the driving point admittance (10), (11). Therefore, its application is not necessarily restricted to the synthesis of $\overline{\text{URC}}$ networks.

Chapter VI considers the general problem of approximating an impedance function specified in a magnitude plot (Bode plot) with a $\overline{\text{URC}}$ network having elements with different RC products. The results of Chapter IV and V are used as a tool to develop the general form of the impedance function Z(p) for a $\overline{\text{URC}}$ network having elements with different RC products. Several properties given by Wyndrum for the impedance of $\overline{URC}$ networks are extended for the case of different RC products. A computer program is given which can be used as a tool in the approximation problem where a least squares approach is used. The program is a modification of a method given by Fletcher and Powell (12) for the minimization of nonlinear functions.

CHAPTER II

OPTIMAL MODELS FOR THE $\overline{\text{URC}}$ NETWORK

2.1 Introduction. In Chapter I the driving point admittance of a $\overline{\text{URC}}$ network with the output short-circuited and with the output open-circuited were given as

$$Z_{sc} = \frac{R_s \tanh \sqrt{R_s C_s p}}{\sqrt{R_s C_s p}} \tag{2.1.1}$$

$$Z_{oc} = \frac{R_o}{\sqrt{R_o C_o p} \tanh \sqrt{R_o C_o p}} \tag{2.1.2}$$

respectively. The irrational, hyperbolic functions in Z_{sc} and Z_{oc} make analysis of $\overline{\text{URC}}$ networks rather difficult and provide very little insight. Therefore, it is desirable to find approximations for Z_{sc} and Z_{oc} that are simple, rational functions. In this chapter new approximations are found for Z_{oc} and Z_{sc} that are rational functions and are valid over a wide range of frequencies. The approximations are useful in the analysis problem and can be used to find RC networks which approximate the corresponding distributed networks.

2.2 Simple Rational Approximations for Z_{OC} and Z_{SC}. One way of obtaining a rational approximation is by expanding Z_{oc} and Z_{sc} into a ratio of infinite products (2), (13). These expansions have the form

$$Z_{sc} = \frac{R_s \prod_{n=1}^{\infty} \left(1 + \frac{R_s C_s p}{n^2 \pi^2}\right)}{\prod_{n=1}^{\infty} \left(1 + \frac{4 R_s C_s p}{(2n-1)^2 \pi^2}\right)} \tag{2.2.1}$$

$$Z_{oc} = \frac{R_o \prod_{n=1}^{\infty} \left(1 + \frac{4 R_o C_o p}{(2n-1)^2 \pi^2}\right)}{R_o C_o p \prod_{n=1}^{\infty} \left(1 + \frac{R_o C_o p}{n^2 \pi^2}\right)} \tag{2.2.2}$$

In most applications, these functions are approximated by $\bar{Z}'_{oc}$ and $\bar{Z}'_{sc}$ where each of these functions are obtained by truncating the products such that they have a finite number of terms. If the approximations $\bar{Z}'_{oc}$ and $\bar{Z}'_{sc}$ are derived for Z_{oc} and Z_{sc} in this manner, they can be made as accurate as desired by including a sufficient number of terms in the products. Unfortunately, the driving point impedance (transfer function) of a $\overline{\text{URC}}$-network with k elements where each element is approximated by $\bar{Z}'_{sc}$ and $\bar{Z}'_{oc}$ has a complexity which grows rapidly with the number of terms used in the approximations $\bar{Z}'_{oc}$ and $\bar{Z}'_{sc}$. Therefore, it is desirable to minimize the number of terms used to approximate Z_{oc} and Z_{sc} such that the approximations meet some specified standard for accuracy. To study this problem Bode plots of the approximations given by Equations 2.2.1 and 2.2.2 can be made (truncated products) for various numbers of terms with RC = 1 (normalized).

A given approximation $\bar{Z}'_{oc}$ ($\bar{Z}'_{sc}$) for Z_{oc} (Z_{sc}) is very good for low frequencies. However, for high frequencies the accuracy of the approximation depends on the number of terms used. This follows since the high frequency asymptote of the Bode plot of Z_{oc} (Z_{sc}) has a slope of

-10db/decade while the high frequency asymptote of the Bode plot for the rational function $\bar{Z}'_{oc}$ ($\bar{Z}'_{sc}$) must have a slope of n(20)db/decade where n=0 or n=-1. This point is illustrated by the Bode plot for $\bar{Z}'_{sc}$ and Z_{sc} in Figure 2.2.1 and the Bode plot for $\bar{Z}'_{oc}$ and Z_{oc} in Figure 2.2.2 where the dashed line in each plot corresponding to Z_{sc} and Z_{oc} respectively and the solid line corresponds to the asymptotic magnitude characteristic of the approximations $\bar{Z}'_{sc}$ and $\bar{Z}'_{oc}$ respectively. The plots are given for $R_s=R_o=1$ and the frequency axis scaled so that the plots apply for Z_{oc} and Z_{sc} with arbitrary RC products (see Figures 2.2.1 and 2.2.2).

The number of break-points is equal to the number of terms in $\bar{Z}'_{sc}$ ($\bar{Z}'_{oc}$). Thus, if $\bar{Z}'_{sc}$ and $\bar{Z}'_{oc}$ are required to meet some standard of accuracy specified in terms of error in db of the Bode plot over the specified range of frequencies $0 < \omega RC < \omega_{max}$ where ω_{max} is the maximum frequency, then the number of terms necessary for the required accuracy can be found by trial and error.

In the next section new approximations $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$ will be found that have fewer terms than $\bar{Z}'_{oc}$ and $\bar{Z}'_{sc}$ respectively and meet the assumed standard of accuracy.

<u>2.3 Optimal Rational Approximations for Z_{oc} and Z_{sc}.</u> In this section the form of the approximations $\bar{Z}'_{sc}$ and $\bar{Z}'_{oc}$ will be used to obtain new approximations $\bar{Z}_{sc}$ and $\bar{Z}_{oc}$ except that the poles and zeros of $\bar{Z}'_{sc}$ and $\bar{Z}'_{oc}$ will be adjusted from their original values to give a higher degree of accuracy for the same number of terms, for a given range of frequencies.

At this point a criterion must be selected that can be used in judging the merits of the approximations. One frequently used criterion

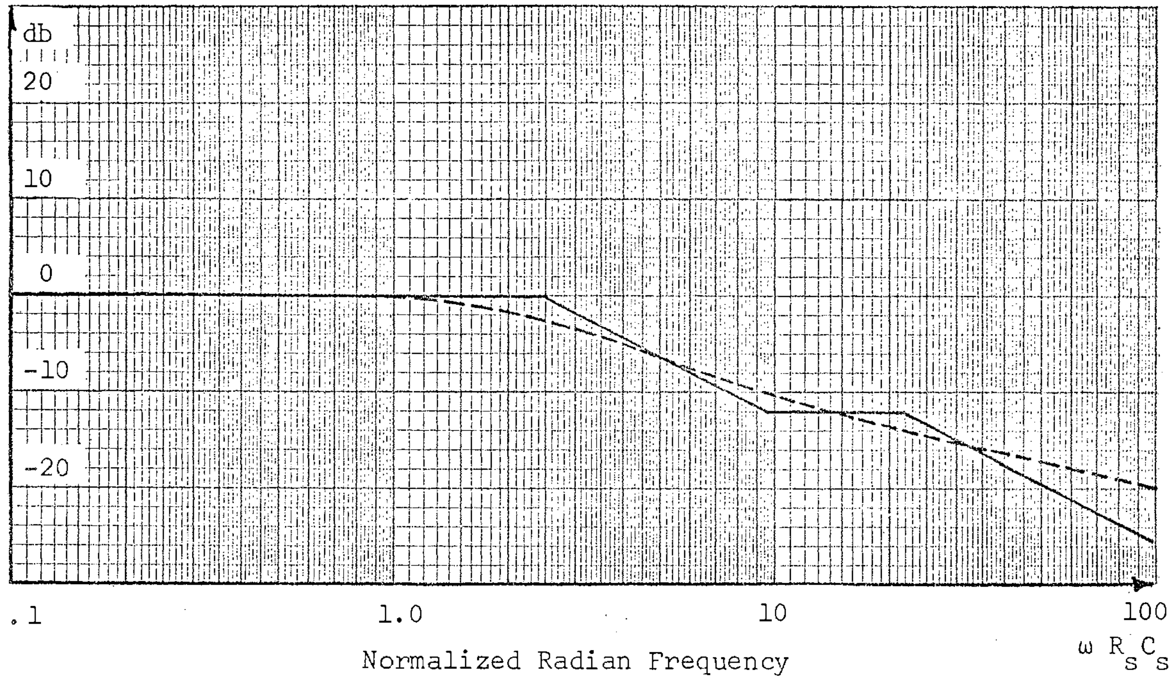


Figure 2.2.1. Bode Plot for Z_{sc} and $\bar{Z}'_{sc}$

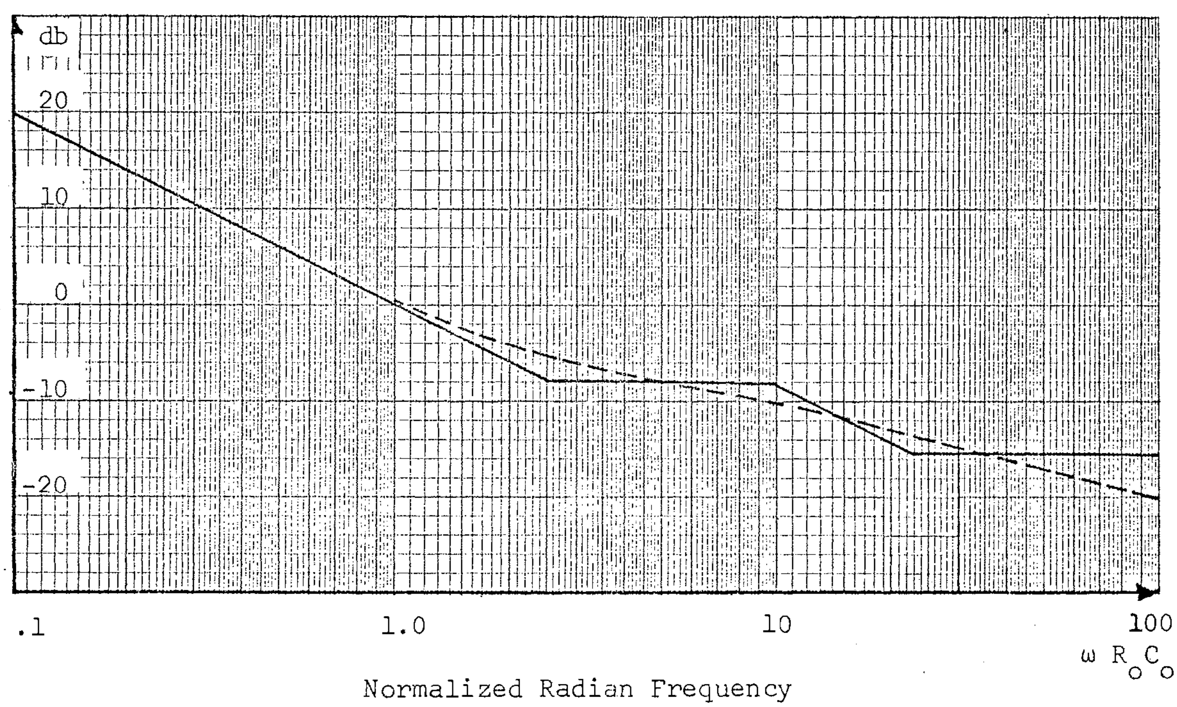


Figure 2.2.2. Bode Plot for Z_{oc} and $\bar{Z}'_{oc}$

For error in magnitude approximation problems is to require that the magnitude of the error in db be less than ±.1db for any frequency in some specified range of frequencies. This criterion is adequate for most applications in network synthesis and specifically for the approximation of Z_{oc} and Z_{sc}. The range of normalized frequencies to be used here given in radian per second is $0 < \omega RC < 100$. At the present state of the art the RC product is very small for most thin-film and integrated $\overline{URC}$ elements. It is difficult to give an upper bound that applies in every case, but usually $RC << 10^{-4}$ ohm-farads and $RC << 10^{-6}$ ohm-farads for thin-film and integrated networks respectively. Therefore each $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$ satisfying the given criterion for accuracy are valid for a wide band of frequencies.

Since Z_{oc} and Z_{sc} are minimum phase functions the close approximation of the magnitude functions (see Equations 2.2.1 and 2.2.2) is sufficient to guarantee the phase of Z_{oc} and Z_{sc} will be closely approximated by the phase of $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$.

Since the criterion for judging the relative merits of the approximations $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$ has been selected, a method of adjusting the poles and zeros of $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$ must also be selected that is compatible with the criterion. The least squares method of fitting curves (see Appendix A) was found to be an effective way to adjust the poles and zeros of $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$ such that the approximations give minimum error. As a first step in the least squares analysis an expression that gives the real and imaginary parts of $Z_{sc}(j\omega)$ and $Z_{oc}(j\omega)$ must be found. The derivations for the expressions are lengthy but are straight forward. The expressions are -

$$\text{Re } \{Z_{oc} (j\omega)\} = \frac{R_s \, W \, [\cosh W \sinh W - \cos W \sin W]}{2 \, W^2 \, [\sinh^2 W \cos^2 W + \cosh^2 W \sin^2 W]} \qquad (2.3.1)$$

$$\text{Im } \{Z_{oc} (j\omega)\} = \frac{-R_s \, W \, [\cosh W \sinh W + \cos W \sin W]}{2 \, W^2 \, [\sinh^2 W \cos^2 W + \cosh^2 W \sin^2 W]} \qquad (2.3.2)$$

$$\text{Re } \{Z_{sc} (j\omega)\} = \frac{R_o \, W \, [\sinh W \cosh W + \cos W \sin W]}{2 \, W^2 \, [\cosh^2 W \cos^2 W + \sinh^2 W \sin^2 W]} \qquad (2.3.3)$$

$$\text{Im } \{Z_{sc} (j\omega)\} = \frac{R_o \, W \, [\cos W \sin W - \sinh W \cosh W]}{2 \, W^2 \, [\cosh^2 W \cos^2 W + \sinh^2 W \sin^2 W]} \qquad (2.3.4)$$

where $W = (\sin \Pi/4)\sqrt{RC\omega}$

Then $|Z_{oc} (j\omega)|$ and $|Z_{sc} (j\omega)|$ can be found from Equations 2.3.1-2.3.4. Since $|Z_{oc} (j\omega)|$ and $|Z_{sc} (j\omega)|$ are functions of ω, 41 equally spaced points $\log \omega_i$, $i=1,2,\ldots,41$ were selected on the $\log \omega$ axis where $.01 \leq \omega_i \leq 100$. Then if ω_i are the frequency values of the normalized frequency plot of $Z_{oc} (j\omega)$ and $Z_{sc} (j\omega)$ ($\omega_i = \omega RC$), a squared error function for the least square analysis F_{oc} can be defined as

$$F_{oc} = \sum_{i=1}^{41} \left(|Z_{oc} (j\omega_i)| - |\bar{Z}_{oc} (j\omega_i)| \right)^2 \qquad (2.3.5)$$

Similarly, error function F_{sc} can be defined as

$$F_{sc} = \sum_{i=1}^{41} \left(|Z_{sc} (j\omega_i)| - |\bar{Z}_{sc} (j\omega_i)| \right)^2 \qquad (2.3.6)$$

Now the squared error functions F_{sc} and F_{oc} can be minimized by adjusting the parameters (the poles and zeros of $\bar{Z}_{sc}$ and $\bar{Z}_{oc}$) and a computer program was written to do this minimization. The program is a modification of some of the more recent techniques to minimize nonlinear functions (12). The modifications were necessary to solve convergence problems caused by the nature of F_{sc} and F_{oc}. The program and its description are given in Appendix A.

To find the functions $\bar{Z}_{sc}$ and $\bar{Z}_{oc}$ that satisfy the criterion given above, the number of terms in $\bar{Z}_{sc}$ and $\bar{Z}_{oc}$ was increased after each computer run until the error criterion was satisfied. The end result of this work is given in Equations 2.3.7 and 2.3.8.

$$Z_{oc} = \frac{R_o}{\sqrt{pR_oC_o}\ \tanh \sqrt{pR_oC_o}} \approx \frac{R_o\ (\tau_1 p + 1)(\tau_2 p + 1)}{pR_oC_o\ (\tau_3 p + 1)(\tau_4 p + 1)} = \bar{Z}_{oc} \qquad (2.3.7)$$

where

$$\tau_1 = R_oC_o\ (.40006)$$

$$\tau_2 = R_oC_o\ (.03267)$$

$$\tau_3 = R_oC_o\ (.09253)$$

$$\tau_4 = R_oC_o\ (.01098)$$

$$Z_{sc} = \frac{R_s\ \tanh \sqrt{pR_sC_s}}{\sqrt{p\ R_sC_s}} \approx \frac{R_s\ (\tau_1 p + 1)(\tau_2 p + 1)}{(\tau_3 p + 1)(\tau_4 p + 1)} = \bar{Z}_{sc} \qquad (2.3.8)$$

where

$$\tau_1 = R_sC_s\ (.09253)$$

$$\tau_2 = R_sC_s\ (.01098)$$

$$\tau_3 = R_sC_s\ (.40006)$$

$$\tau_4 = R_sC_s\ (.03267)$$

Equation 2.3.7 and 2.3.8 can be written in a different form by using partial fraction expansions and are given below.

$$\bar{Z}_{oc} = \frac{1}{C_o p} + \frac{R_o\ (.22568)}{R_o C_o\ (.09253)p+1} + \frac{R_o\ (.10340)}{R_o C_o\ (.01098)p+1} \qquad (2.3.9)$$

$$\bar{Z}_{sc} = R_s\ (.10814) + \frac{R_s\ (.81408)}{R_s C_s\ (.40006)p+1} + \frac{R_s\ (.07777)}{R_s C_s\ (.03267)p+1} \qquad (2.3.10)$$

Equations 2.3.9 and 2.3.10 give insight into the behavior of Z_{oc} and Z_{sc} and these functions can be synthesized by lumped RC networks and are given in Figures 2.2.3a and 2.2.3b respectively.

The approximations $\bar{Z}_{oc}$ and $\bar{Z}_{sc}$ are compared to the approximations $\bar{Z}'_{oc}$ and $\bar{Z}'_{sc}$ given by Equations 2.2.1 and 2.2.2 for the same number of terms in Tables 2.3.1 and 2.3.2. In Tables 2.3.1 and 2.3.2 the frequency ωRC is given in the first column, and the remaining columns are given in db. The errors in the approximations are defined by

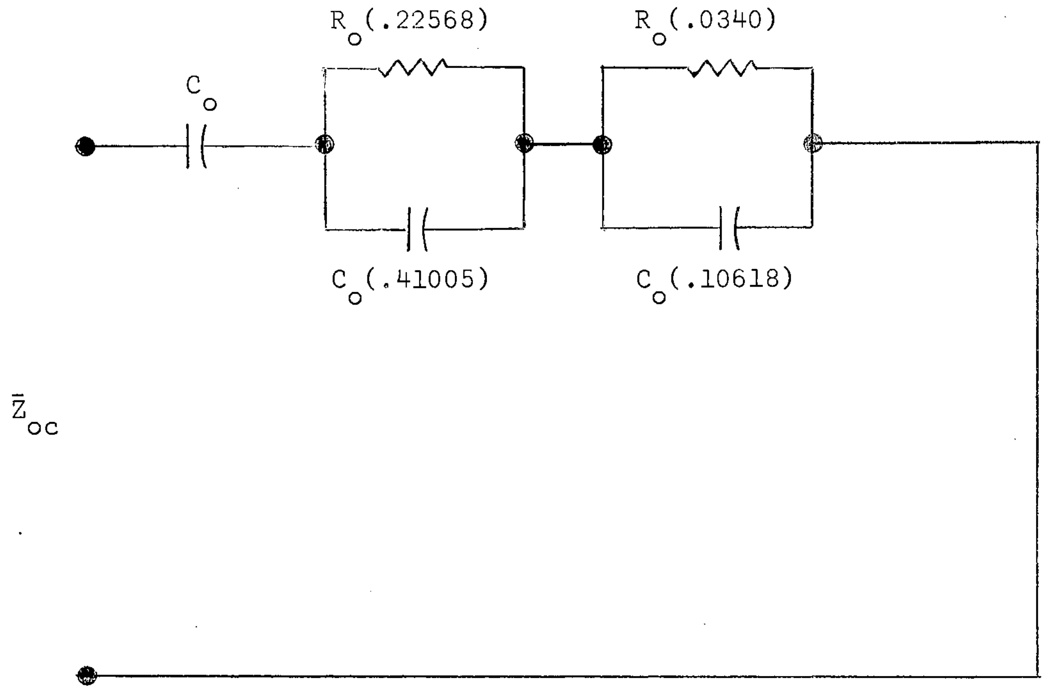
$$\text{Error } |\bar{Z}'_{oc}\ (j\omega)| = 20 \text{ Log}_{10} |Z_{oc}\ (j\omega)| - 20 \text{ Log}_{10} |\bar{Z}'_{oc}\ (j\omega)| \qquad (2.3.11)$$

$$\text{Error } |\bar{Z}_{oc}\ (j\omega)| = 20 \text{ Log}_{10} |Z_{oc}\ (j\omega)| - 20 \text{ Log}_{10} |\bar{Z}_{oc}\ (j\omega)| \qquad (2.3.12)$$

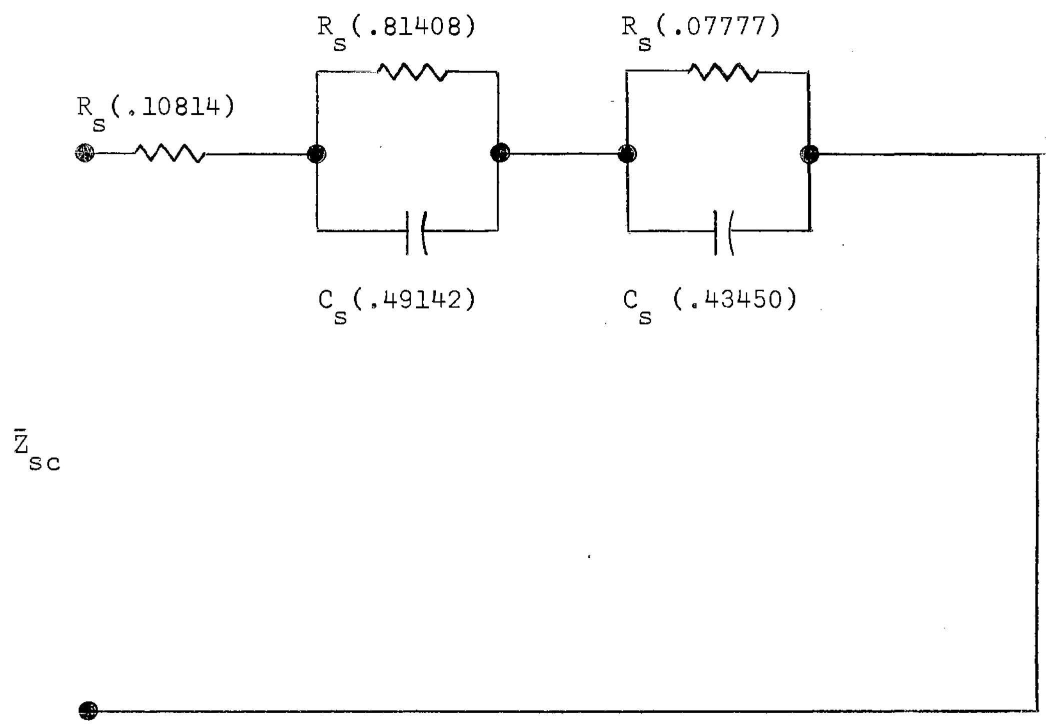
$$\text{Error } |\bar{Z}'_{sc}\ (j\omega)| = 20 \text{ Log}_{10} |Z_{sc}\ (j\omega)| - 20 \text{ Log}_{10} |\bar{Z}'_{sc}\ (j\omega)| \qquad (2.3.13)$$

$$\text{Error } |\bar{Z}_{sc}\ (j\omega)| = 20 \text{ Log}_{10} |Z_{sc}\ (j\omega)| - 20 \text{ Log}_{10} |\bar{Z}_{sc}\ (j\omega)| \qquad (2.3.14)$$

It can be seen by Table 2.3.1 that the largest error in the magnitude of $\bar{Z}_{oc}$ over the range $.01 < \omega R_o C_o < 100$ is $-.096$db at $\omega R_o C_o = 64$.



(a)



(b)

Figure 2.2.3. Equivalent Networks for the Z_{OC}-Element and Z_{SC}-Element Which Are Valid Over the Range $0 \leq p\ R_oC_o \leq 100$ and $0 \leq p\ R_sC_s \leq 100$ Respectively

TABLE 2.3.1

COMPARISON OF TWO APPROXIMATIONS FOR Z_{oc}

$\omega R_s C_s$	$\lvert Z_{sc}(j\omega)\rvert$ (db)	$\lvert \bar{Z}'_{sc}(j\omega)\rvert$ (db)	ERROR $\lvert \bar{Z}'_{sc}(j\omega)\rvert$ (db)	$\lvert \bar{Z}_{sc}(j\omega)\rvert$ (db)	ERROR $\lvert \bar{Z}_{sc}(j\omega)\rvert$ (db)
.0100	-.00006	-.00007	.00000	-.00006	-.00000
.0126	-.00010	-.00011	.00000	-.00010	-.00000
.0160	-.00017	-.00017	.00000	-.00016	-.00000
.0200	-.00027	-.00027	.00000	-.00026	-.00000
.0250	-.00042	-.00042	.00000	-.00041	-.00000
.0320	-.00069	-.00069	.00000	-.00067	-.00001
.0400	-.00108	-.00108	.00000	-.00105	-.00002
.0500	-.00169	-.00169	.00000	-.00165	-.00003
.0640	-.00276	-.00276	.00000	-.00271	-.00004
.0800	-.00432	-.00432	.00000	-.00423	-.00008
.1000	-.00675	-.00674	-.00001	-.00661	-.00013
.1260	-.01071	-.01070	-.00001	-.01049	-.00021
.1600	-.01725	-.01724	-.00002	-.01691	-.00033
.2000	-.02693	-.02690	-.00003	-.02639	-.00053
.2500	-.04199	-.04195	-.00005	-.04116	-.00082
.3200	-.06857	-.06849	-.00008	-.06721	-.00135
.4000	-.10662	-.10650	-.00012	-.10452	-.00209
.5000	-.16534	-.16515	-.00019	-.16213	-.00320
.6400	-.26734	-.26702	-.00032	-.26225	-.00508
.8000	-.41001	-.40952	-.00050	-.40242	-.00758
1.000	-.62292	-.62214	-.00078	-.61186	-.01056
1.260	-.94693	-.94569	-.00124	-.93120	-.01520
1.600	-1.4290	-1.4270	-.00199	-1.4077	-.02136
2.000	-2.0455	-2.0424	-.00312	-2.0191	-.02641
2.500	-2.8397	-2.8349	-.00487	-2.8102	-.02955
3.200	-3.9192	-3.9112	-.00797	-3.8911	-.02808
4.000	-5.0430	-5.0306	-.01245	-5.0231	-.01994
5.000	-6.2540	-6.2346	-.01942	-6.2497	-.00433
6.400	-7.6218	-7.5900	-.03174	-7.6416	+.01976
8.000	-8.8229	-8.7735	-.04942	-8.8864	+.04193
10.00	-9.9527	-9.8759	-.07679	-10.007	+.05513
12.60	-11.0384	-10.917	-.12083	-11.087	+.04884
16.00	-12.090	-11.898	-.19201	-12.106	+.01601
20.00	-13.041	-12.747	-.29376	-13.011	-.03025
25.00	-13.989	-13.545	-.44468	-13.920	-.06961
32.00	-15.050	-14.359	-.69104	-14.976	-.07432
40.00	-16.018	-15.011	-1.0069	-15.987	-.03079
50.00	-16.989	-15.560	-1.4281	-17.029	-.04064
64.00	-18.061	-16.036	-2.0254	-18.158	-.09695
80.00	-19.030	-16.353	-2.6778	-19.095	-.06474
100.00	-20.000	-16.579	-3.4206	-19.903	-.09621

TABLE 2.3.2

COMPARISON OF TWO APPROXIMATIONS FOR Z_{sc}

$\omega R_o C_o$	$\lvert Z_{oc}(j\omega)\rvert$ (db)	$\lvert \bar{Z}'_{oc}(j\omega)\rvert$ (db)	ERROR	$\lvert \tilde{Z}_{oc}(j\omega)\rvert$ (db)	ERROR
			$\lvert \bar{Z}'_{oc}(j\omega)\rvert$ (db)		$\lvert \tilde{Z}_{oc}(j\omega)\rvert$ (db)
.01000	40.00006	40.00006	0.00000	40.000	-.00000
.01260	37.99269	37.99269	0.00000	37.992	-.00000
.01600	35.91776	35.91777	0.00000	35.917	+.00000
.02000	33.97966	33.97966	0.00000	33.979	+.00000
.02500	32.04161	32.04162	0.00000	32.041	+.00001
.03200	29.89768	29.89768	0.00000	29.897	+.00001
.04000	27.95987	27.95987	0.00000	27.959	+.00002
.05000	26.02228	26.02228	0.00000	26.002	+.00003
.06400	23.87916	23.87916	0.00000	23.879	+.00005
.08000	21.94251	21.94251	0.00000	21.942	+.00008
.10000	20.00674	20.00674	0.00000	20.006	+.00013
.12600	18.00329	18.00328	.00001	18.003	+.00021
.16000	15.93485	15.93483	.00001	15.934	+.00034
.20000	14.00632	14.00629	.00002	14.005	+.00053
.25000	12.08319	12.08314	.00004	12.082	+.00083
.32000	9.96556	9.96549	.00007	9.9642	+.00135
.40000	8.06541	8.06529	.00012	8.0633	+.00209
.50000	6.18593	6.18574	.00019	6.1827	+.00321
.64000	4.14373	4.14342	.00031	4.1386	+.00509
.80000	2.34821	2.34771	.00049	2.3406	+.00759
1.00000	.62291	.62213	.00077	.61186	+.01106
1.26000	-1.06048	-1.06171	.00123	-1.0762	+.01572
1.60000	-2.65331	-2.65530	.00199	-2.6746	+.02137
2.00000	-3.97501	-3.97813	.00311	-4.0014	+.02643
2.50000	-5.11901	-5.12388	.00487	-5.1485	+.02957
3.20000	-6.18376	-6.19173	.00797	-6.21185	+.02809
4.00000	-6.99813	-7.01058	.01244	-7.0180	+.01995
5.00000	-7.72531	-7.74473	.01942	-7.7296	+.00433
6.40000	-8.50176	-8.53350	.03174	-8.4820	-.01977
8.00000	-9.23884	-9.28826	.04942	-9.1969	-.04195
10.00000	-10.04721	-10.12400	.07679	-9.9920	-.05516
12.60000	-10.96897	-11.08979	.12082	-10.920	-.04887
16.00000	-11.99203	-12.18403	.19200	-11.976	-.01603
20.00000	-12.97920	-13.27295	.29375	-13.009	+.03024
25.00000	-13.96899	-14.41366	.44467	-14.038	+.06961
32.00000	-15.05234	-15.74338	.69104	-15.126	+.07434
40.00000	-16.02260	-17.02952	1.0069	-16.053	+.03081
50.00000	-16.99036	-18.41853	1.4281	-16.949	-.04063
64.00000	-18.06173	-20.08714	2.0254	-17.964	-.09695
80.00000	-19.03084	-21.70869	2.6778	-18.966	-.06475
100.00000	-19.99999	-23.42068	3.4206	-20.096	+.09618

The largest error of $\bar{Z}'_{oc}$ can be seen to be + 3.42 db at $\omega R_oC_o = 100$ over the range $.01 < \omega R_oC_o < 100$. Similarly, from Table 2.3.2 the largest error in the magnitude of $\bar{Z}_{sc}$ in Equation 2.3.8 over the range $.01 < \omega R_sC_s < 100$ is +.096 db at $\omega R_sC_s = 64$. The largest error in the magnitude of the corresponding approximation $\bar{Z}'_{sc}$ over the range $.01 < \omega R_sC_s < 100$ is -3.4 db at $\omega R_sC_s = 100$. The values for error in the approximations are not given in Tables 2.3.1 and 2.3.2 for $\omega R_sC_s < .01$ (or $\omega R_oC_o < .01$) but tests showed that there was no significant error for $\omega R_sC_s < .01$ (or $\omega R_oC_o < .01$) in any of the approximations. Also note that the absolute value of the error in the approximations $\bar{Z}'_{oc}$ (or $\bar{Z}'_{sc}$) is greater than .1 db for $10 < \omega R_oC_o$ (or $\omega R_sC_s) \leq 100$. Therefore, with the same number of terms, the new approximations are valid over a wider range of frequencies than $\bar{Z}'_{oc}$ and $\bar{Z}'_{sc}$.

The methods of this section can also be applied to find an optimal approximation for the open circuit voltage transfer function for a $\overline{\text{URC}}$ network. The approximation and details are given in the next section.

<u>2.4 Optimal Rational Approximation for the Open Circuit Voltage Transfer Function of a $\overline{\text{URC}}$ Element.</u> Consider the $\overline{\text{URC}}$ element in Figure 2.4.1. The open circuit voltage transfer function of the $\overline{\text{URC}}$ element in Figure 2.4.1 is

$$G(p) = \frac{E_2}{E_1} = \frac{1}{\cosh \sqrt{RC\,p}} \qquad (2.4.1)$$

Then using the methods identical to those in Section 2.3 for Z_{oc} and Z_{sc}, Equation 2.4.1 can be approximated by the rational function

$$G(p) \approx \bar{G}(p) = \frac{1}{(\tau_1 p + 1)(\tau_2 p + 1)(\tau_3 p + 1)^2} \qquad (2.4.2)$$

where

$$\tau_1 = RC\,(.40753)$$

$$\tau_2 = RC\,(.04260)$$

$$\tau_3 = RC\,(.01496)$$

The error defined by

$$e = \left|\, |G(j\omega)| - |\bar{G}(j\omega)| \,\right|$$

is less than .16 db for $.01 \leq \omega RC \leq 100$. The error is not significant for $0 \leq \omega RC \leq .01$. Even though .16 db is more than the assumed standard of .1 db, it is felt that the approximation is good enough using four terms in the approximation.

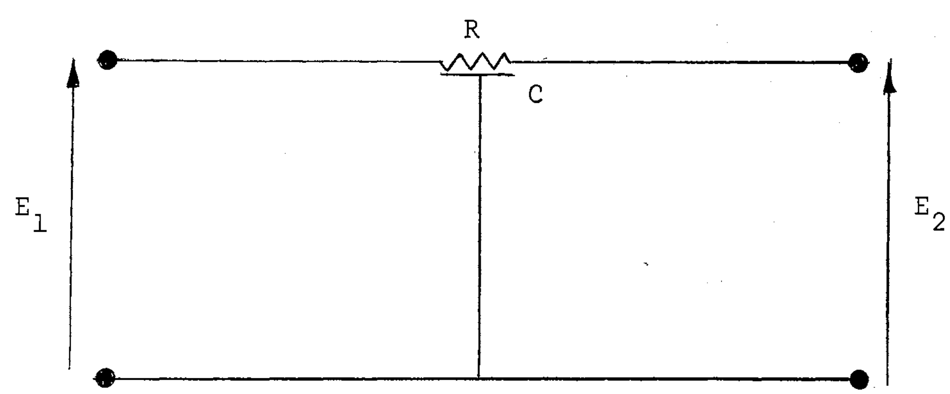


Figure 2.4.1 $\overline{\text{URC}}$ Element

CHAPTER III

APPROXIMATE SYNTHESIS OF RATIONAL DRIVING POINT IMPEDANCES WITH $\overline{URC}$ NETWORKS

<u>3.1 Introduction.</u> In Chapter I three methods were given that can be used to approximate rational driving point impedances. However, all of these methods are difficult to apply for the reasons given in Section 1.2. In this section a new method will be derived that is rather simple to apply and gives an accurate approximation for a given rational impedance.

<u>3.2 Synthesis of Rational RC Driving Point Impedances.</u> The two driving point impedances of the $\overline{URC}$ network Z_{oc} and Z_{sc} were given in Equation 2.1.1 and 2.1.2 respectively, and the infinite product expansions for Z_{sc} and Z_{oc} were given in Equations 2.2.1 and 2.2.2, respectively. In the discussion to follow <u>infinite</u> product expansions for Z_{oc} and Z_{sc} are used instead of the optimal approximations since they are exact for all frequencies and no computation is required in the discussion. An examination of the finite product expansions show that Z_{sc} behaves as a resistor of value R_s when C_s becomes very small and similarly, Z_{oc} behaves as a capacitor of value R_oC_o/R_o (or C_o) when R_o becomes very small. Thus, it is clear that lumped RC functions can be approximated if C_o and R_o can be made very small. In general, however, R_o and C_s cannot be made arbitrarily small for applications where

$\overline{URC}$ networks would be applicable. This is especially true for integrated circuit applications where resistors cannot be made much smaller than 2 ohms/square (1). A more conservative estimate is 5 ohms/square. There is also a limit on how large resistors can be made. They can be made with a resistance as high as 300 ohms/square and be connected in series to form a resistor as large as 30 K ohms (1). Thus it is reasonable to restrict R_s and R_o to be in the range 5 ohms $\leq R_o$ (or R_s) $\leq$ 20 K ohms. The resistors in thin-film $\overline{URC}$ elements can be made smaller and also larger than they can be in integrated circuits. However, for the work in this section the value of R_s and R_o will be restricted to the range 5 ohms $\leq R_o$ (or R_s) $\leq$ 20 K ohms. The capacitance C_s can be controlled by reducing the width of the $\overline{URC}$-element, but there is also a practical limit to how small the width can be made (3).

Now again consider the infinite product expansions for Z_{oc} and Z_{sc}. An examination of the expansions shows that they have the same properties as RC impedance functions except that they have an infinite number of poles and zeros. Therefore, it is reasonable to restrict this work to the approximation of rational RC impedance functions. In general a rational RC impedance function Z(p) can be expanded in partial fraction form as

$$Z(p) = K_o + \frac{K_\infty}{p} + \sum_{i=1}^{n} \frac{K_i}{a_i\, p + 1} \qquad (3.2.1)$$

where each K_i and a_i are positive and real constants, and n is a positive integer. The synthesis of Z(p) in Equation 3.2.1 by a lumped RC network is classical (14). As explained in Section 3.1, each of the resistors (capacitors) of a lumped RC network can be approximated by a Z_{sc}-element (Z_{oc}-element). The synthesis of Z(p) (First Foster form) in terms of these $\overline{URC}$ elements is shown in Figure 3.2.1 where the capacitances C_s for the Z_{sc}-elements and the resistances R_o for the Z_{oc}-elements are set to the smallest possible practical values.

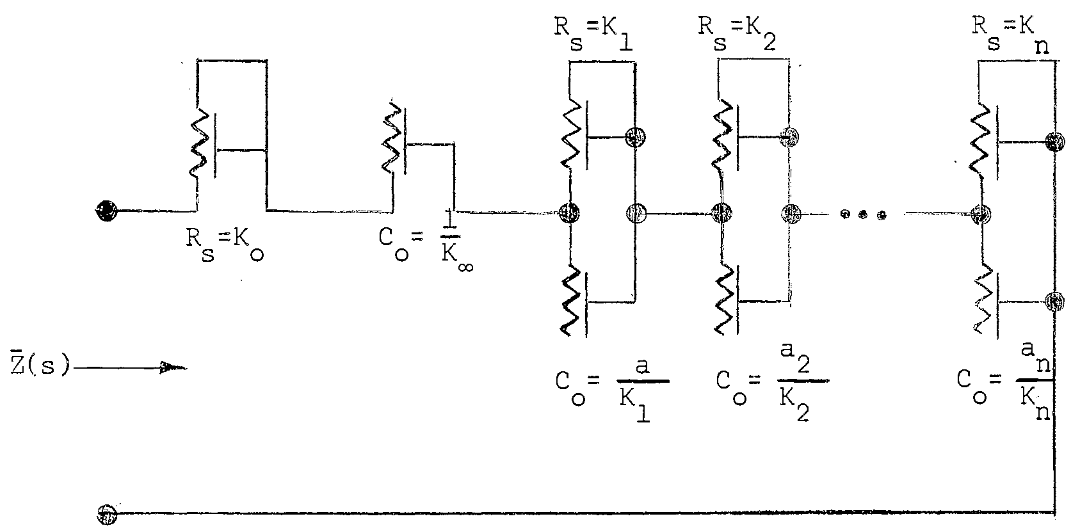


Figure 3.2.1. Approximation for Z(p)

The network of Figure 3.2.1 can be used in many practical applications where the high frequency behavior is not important. However, the parameters of the network in Figure 3.2.1 can be adjusted to give minimum error over some assumed range of frequencies and significantly improve the approximation by the same methods used in Chapter II. This will be the approach taken here except that optimal approximations will be found only for terms of the form

$$Z_i(p) = \frac{K_i}{a_i\, p + 1} \qquad (3.2.2)$$

The terms K_o and K_∞/p will be approximated as shown in Figure 3.2.1. Note that the approximations are given term by term instead of for the entire function Z(p) (see Chapter VI) so that the results can be applied to an arbitrary function Z(p) which can be written in the form of Equation 3.2.1.

In Chapter VI it will be shown that the magnitude plate (Bode plot) of an impedance function for a network with $\overline{\text{URC}}$ elements having different RC products has a slope of -10 db/decade for high frequencies. Then it is clear that any approximation found will be valid only in a band of frequencies less than some finite maximum frequency. For the work here the maximum frequency will be $\omega RC = 100$ where RC is the largest RC product in the approximating network. The justification for this assumption is the same as for the similar assumption made in Section 2.3.

The least square approach used in Section 2.3 will be used here with the exception that some of the parameters will be constrained. The method used to constrain the parameters in the computer program is given in Appendix A. The impedance of a parallel circuit consisting of a Z_{oc}-element and a Z_{sc}-element is

$$\bar{Z}_i(p) = \frac{Z_{oc}(p)\ Z_{sc}(p)}{Z_{oc}(p) + Z_{sc}(p)} \qquad (3.2.3)$$

where R_o, R_s, R_oC_o, and R_sC_s are the parameters in the impedance $\bar{Z}_i(p)$ of the parallel circuit. Note that if the parameters of $\bar{Z}_i(p)$ are adjusted to give an optimal approximation for

$$Z_i'(p) = \frac{K_i}{10\ p + 1} \tag{3.2.4}$$

then $Z_i'(p)$ can be made to approximate $Z_i(p)$ with an arbitrary value of a_i by scaling all RC products in the approximation $\bar{Z}_i(p)$. Then the squared error function F, to be used in least squares analysis is defined by

$$F = \sum_{k=1}^{41} \left(|Z_i'\ (j\omega_k)| - |\bar{Z}_i\ (j\omega_k)| \right)^2 \tag{3.2.5}$$

where the constraint for R_o and R_s is given by
5 ohms $\leq R_o$ (or R_s) $\leq$ 20 K ohms, ω_k, k=1,2,...,k is such that $\text{Log}_{10}\ \omega_i$, i=1,2,...,41 are 41 equally spaced points on the $\text{Log}_{10}\omega$ axis, and $.01 \leq \omega_i \leq 100$. The program in Appendix A can be used to minimize F subject to the constraints for a given value of K_i.

The results of the computer analysis for several values of K indicate that F takes a minimum value when

$$\left.\begin{aligned} R_s &= K_i \text{ ohms} \\ R_o &= 5 \text{ ohms} \\ R_o C_o &= \frac{50}{R_s} \text{ ohm-farads} \end{aligned}\right\} \tag{3.2.6}$$

and R_sC_s is selected from the design curve given in Figure 3.2.2. The design curve for R_sC_s was determined empirically as a function of K_i from the data obtained in computer runs for a range of values of K_i $1 \times 10^3 \leq K_i \leq 15 \times 10^3$. When $K_i > 15 \times 10^3$, a value of R_sC_s = .02 ohm-farads is an optimum value for R_sC_s. The plots of $\bar{Z}_i'\ (j\omega)$ (in db) for a wide range of values of K_i are given in Figure 3.2.3.

Now let E_i be the error defined by

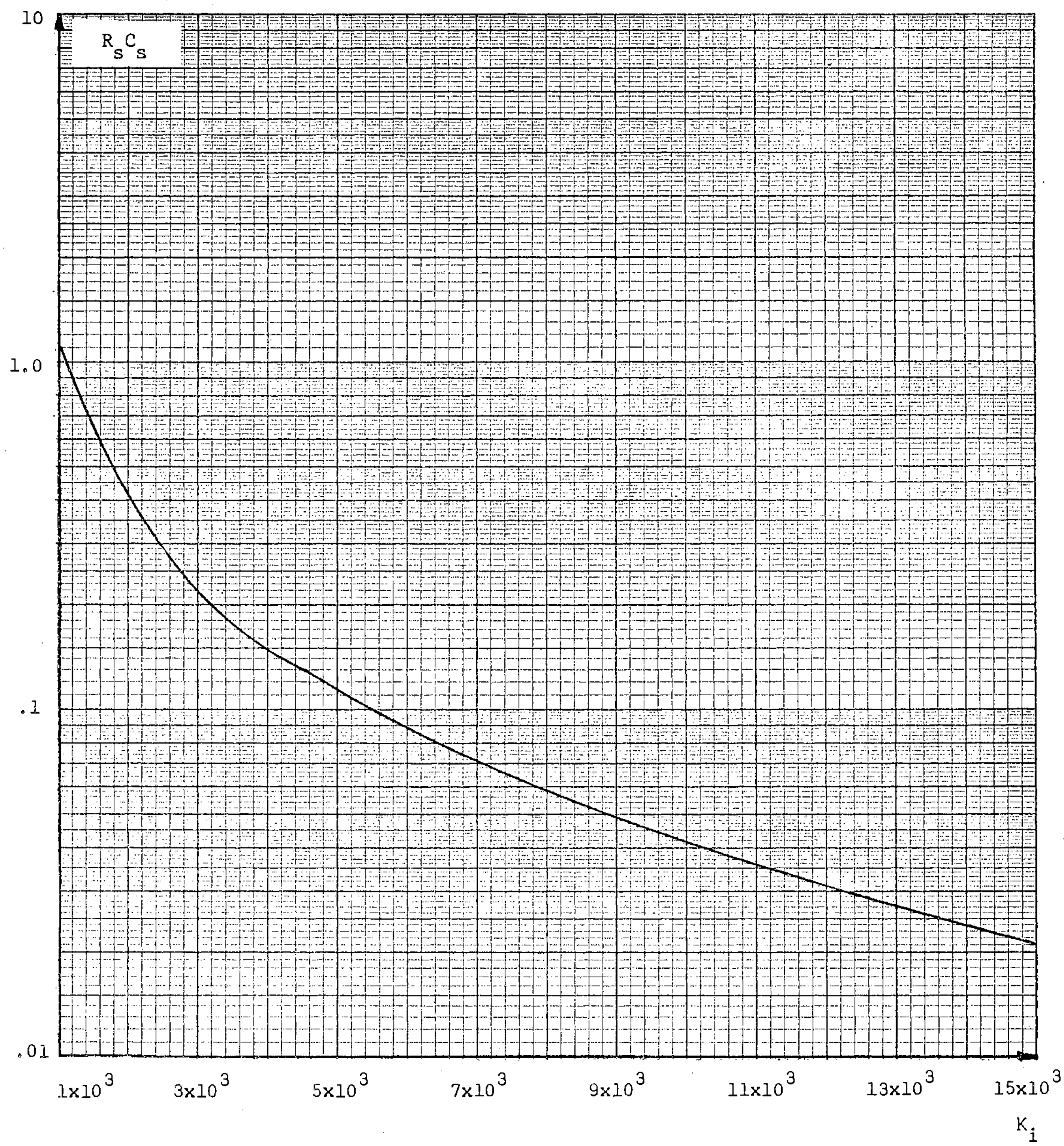


Figure 3.2.2. Optimal Value of R_sC_s as a Function of K_i

$$E_i = 20 \text{ Log}_{10} |\bar{Z}' (j\omega)| - 20 \text{ Log}_{10} \left|\frac{K_i}{10\ j\omega + 1}\right| \qquad (3.2.7)$$

Plots of error E_i, for a range of values of K_i are given in Figures 3.2.4 - 3.2.7.

It is clear from these plots that the error E_i is small for large values of $R_s = K_i$ for $.01 \leq \omega R_o C_o \leq 100$, and E_i becomes larger for the higher frequencies ($10 \leq \omega R_o C_o \leq 100$) as K_i becomes smaller. The error plots in Figures 3.2.4 - 3.2.7 cannot be used to find the error of the total approximation $\bar{Z}(p)$ for $Z(p)$, but do provide useful data on each term $\bar{Z}_i(p)$. If the error E_i for some Z (p) of Z(p) is too large for a particular application, it may be necessary to use a hybrid of thin-film and integrated circuit devices where the parameters can be adjusted over a wider range of values (1). The more general procedure for approximation given in Chapter VI may also give better results when the methods of this section are not adequate. The method of Chapter VI may in general use less elements and has the added advantage of avoiding cumulative error inherent in this method. However, the simplicity of the method in this section, where an optimal approximation for Equation 3.2.4 is obtained makes its use particularly attractive when the errors can be kept below the acceptable level.

The results of this section can now be illustrated by an example.

Example 3.2.1: Consider the function Z(p) given in the partial fraction form

$$Z(p) = \frac{5000}{1\text{x}10^{-3}p + 1} + \frac{8000}{2\text{x}10^{-4}p + 1} = Z_1(p) + Z_2(p)$$

First consider $Z_1(p)$ where $K_1 = 5000$. Then the parameters for $\bar{Z}_1'(p)$ can be found from Equation 3.2.6, and are: $R_s = 5000$ ohms, $R_o = 5$ ohms,

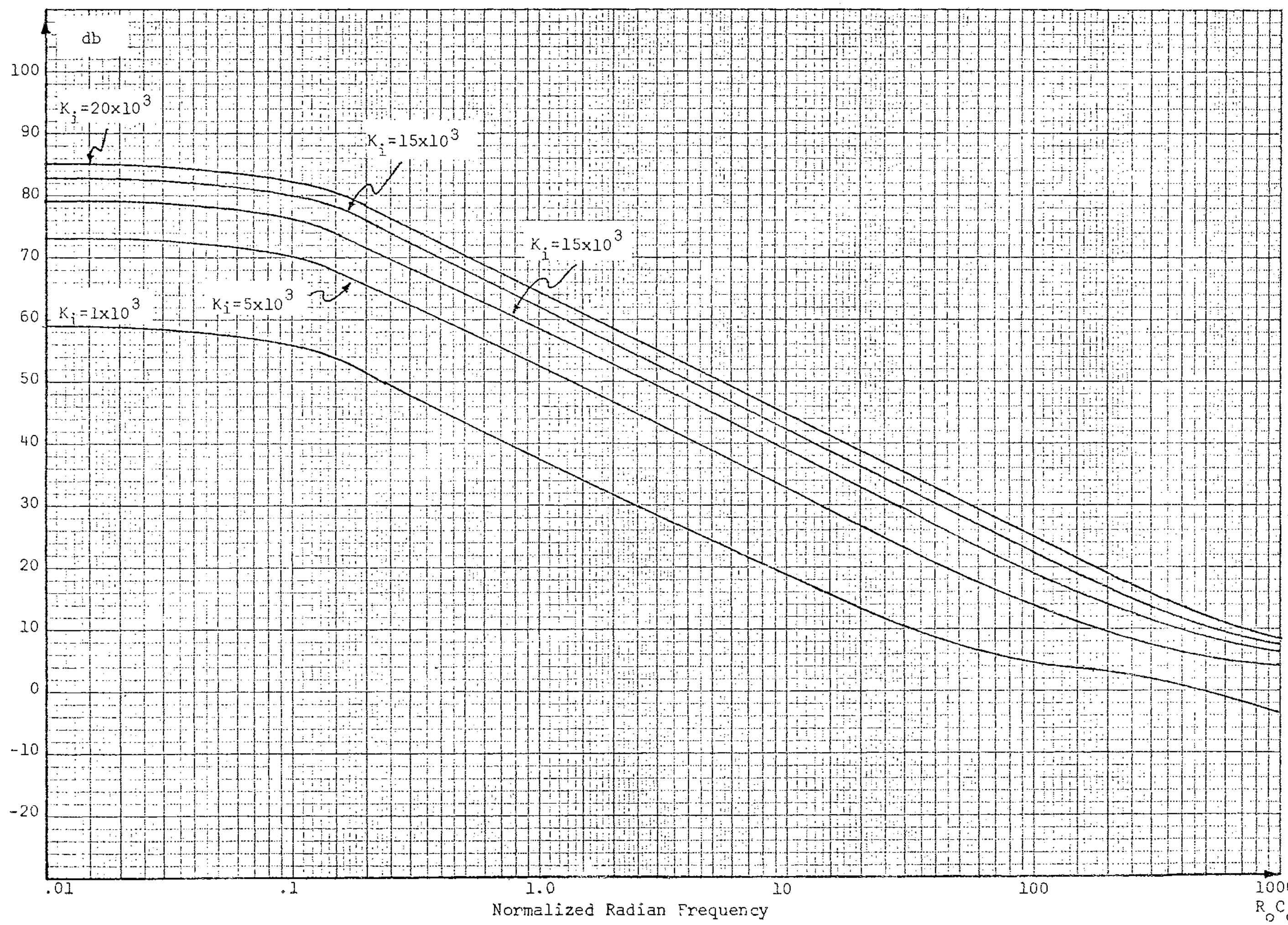


Figure 3.2.3. Plot of $|\bar{Z}'(j\omega)|$ for Different Values of K_i

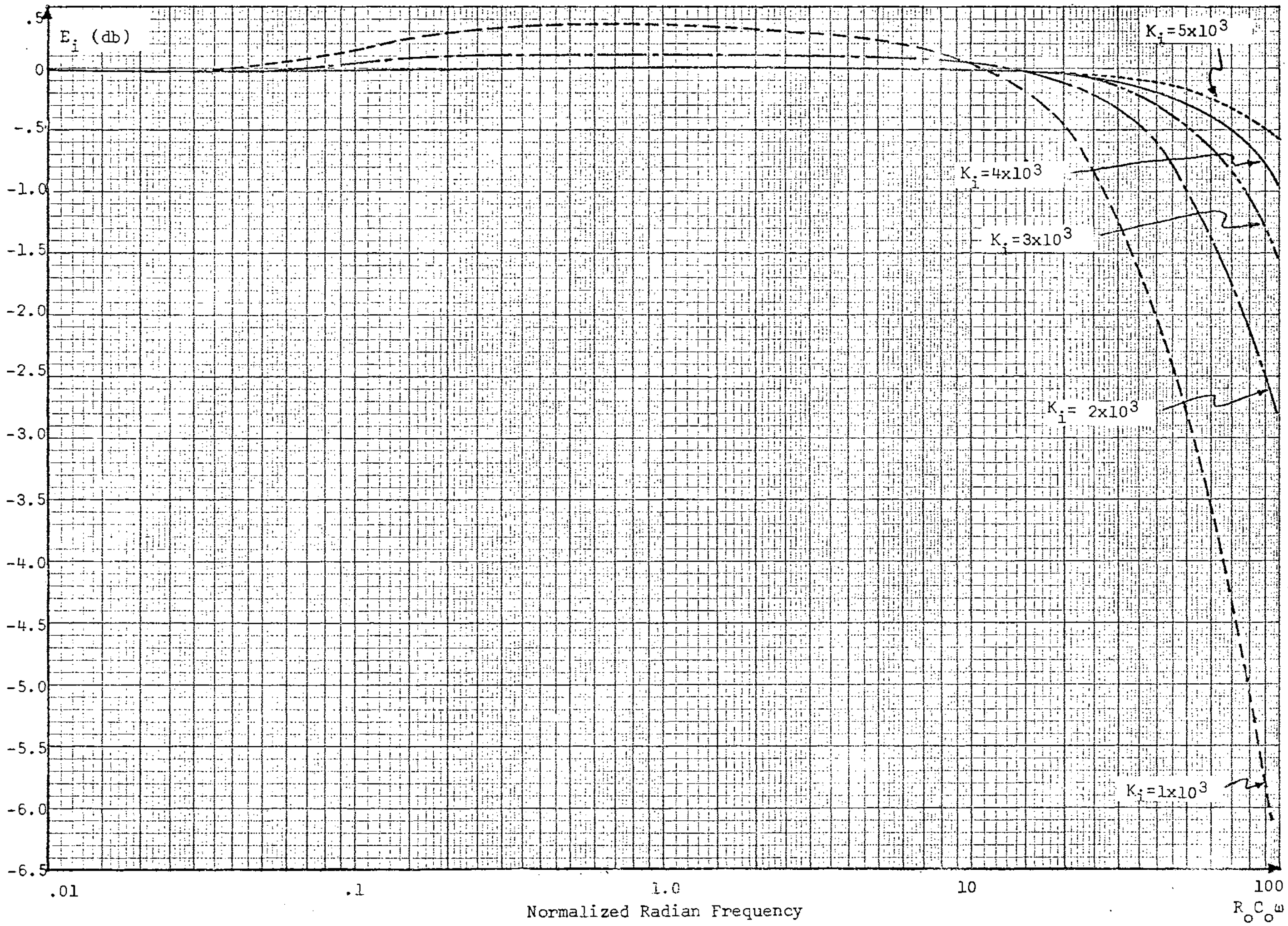


Figure 3.2.4. Plot of Error E_i

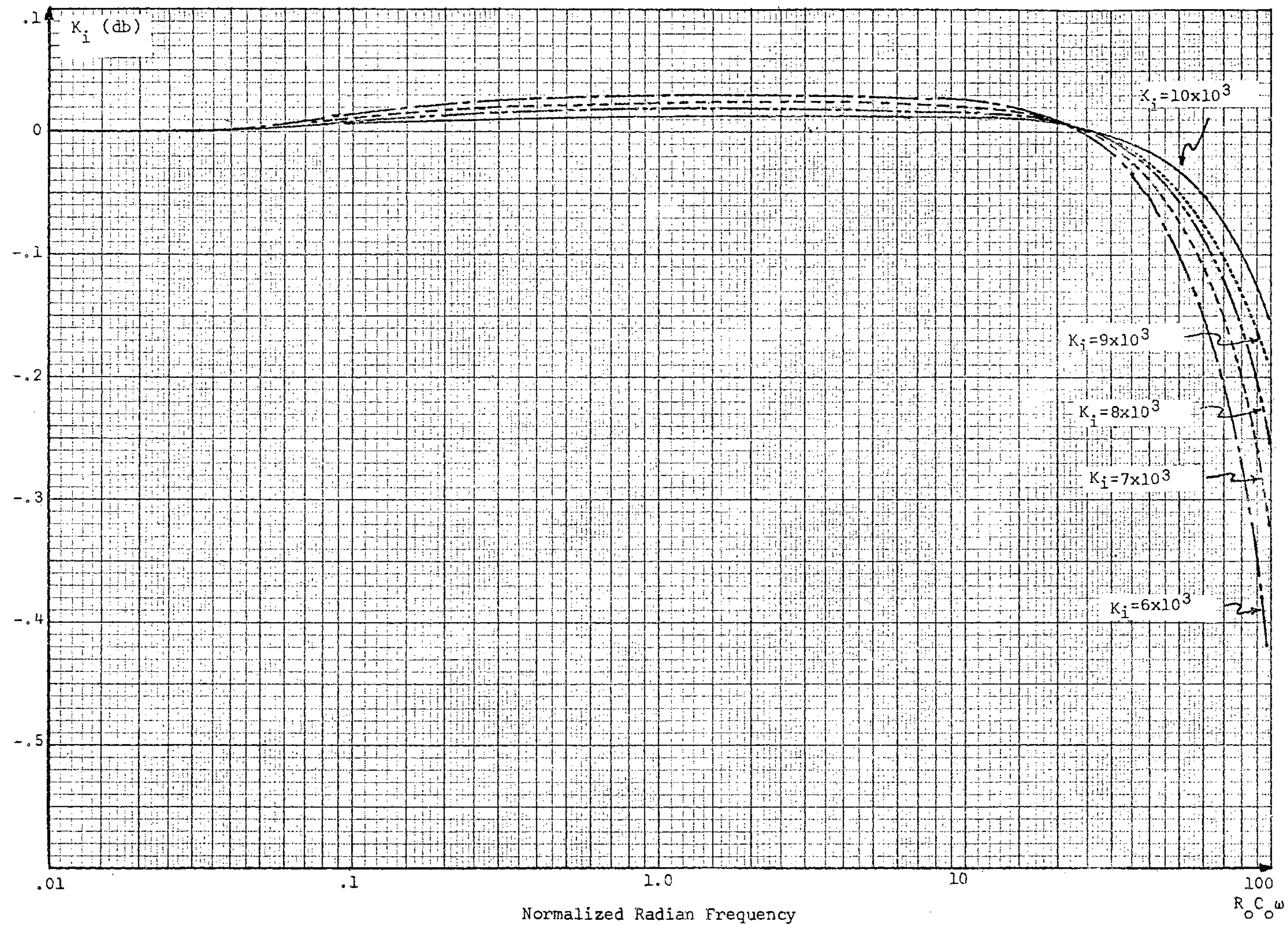


Figure 3.2.5. Plot of Error E_i

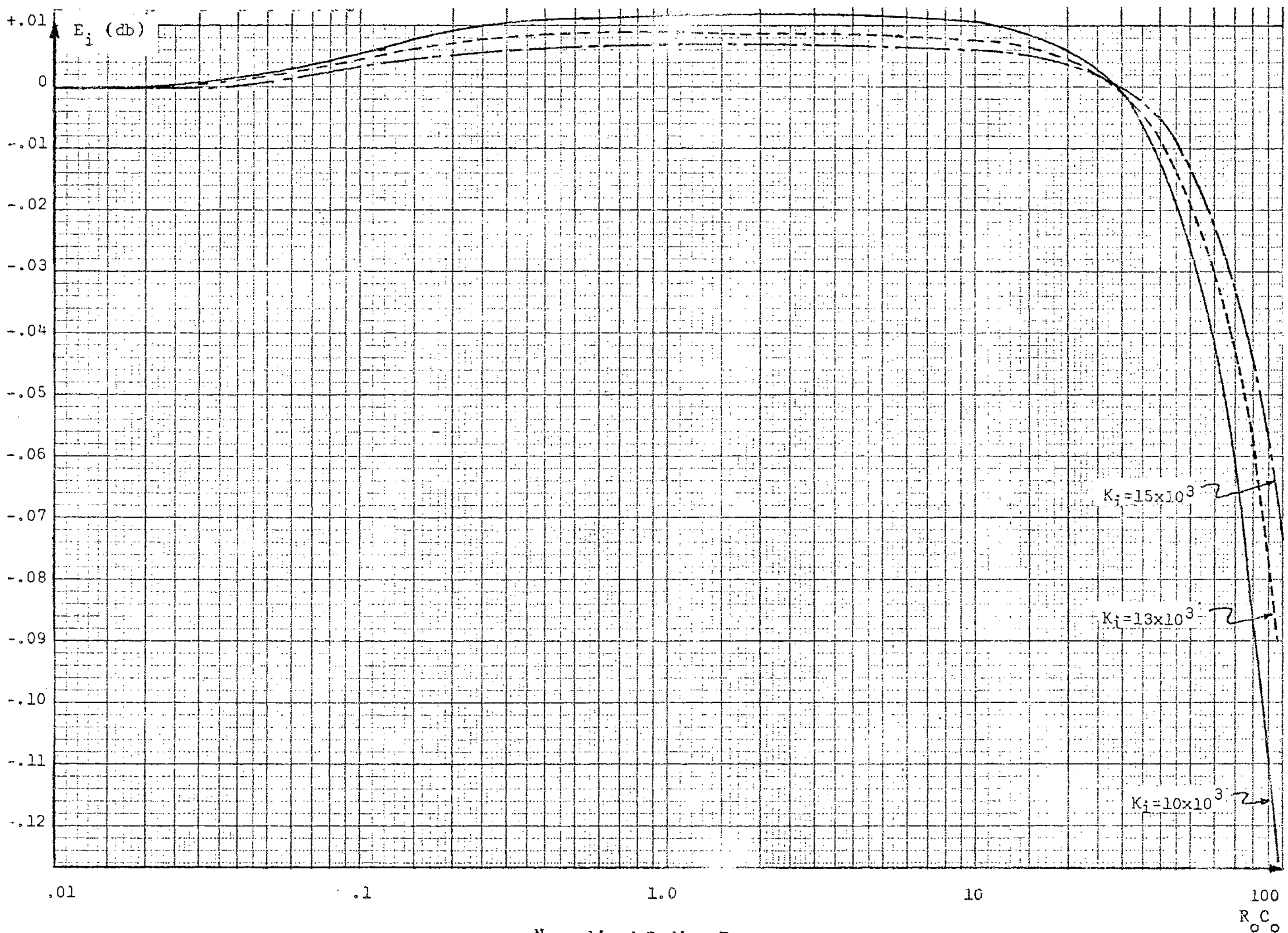


Figure 3.2.6. Plot of Error E_i

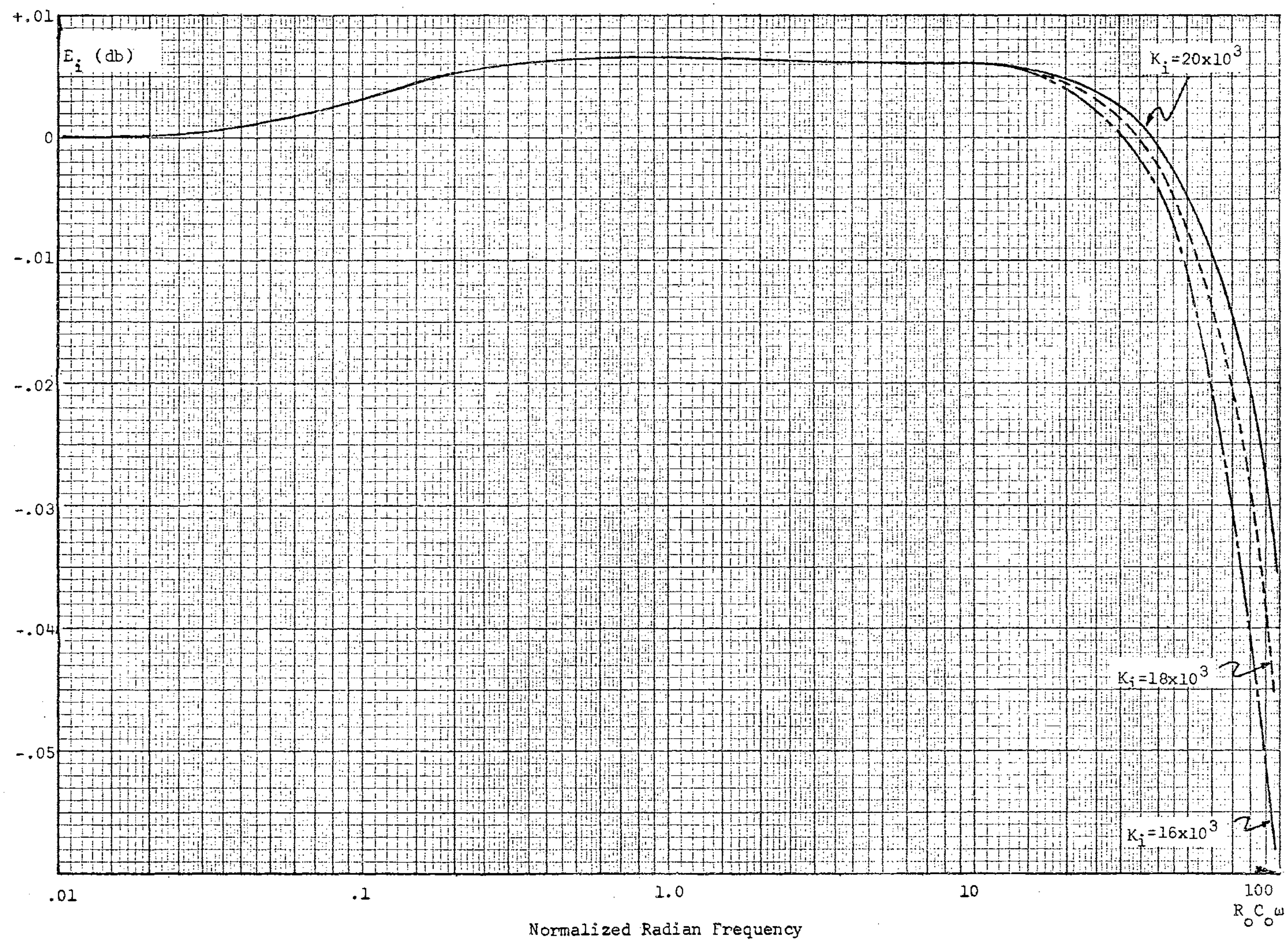


Figure 3.2.7. Plot of Error E_i

$R_oC_o = 50/R_s = .01$. Figure 3.2.2 can be used to find the value of $R_sC_s = .118$ ohm-farads. Now since the approximation $\bar{Z}_1'(p)$ is for the function

$$Z'(p) = \frac{5000}{10\,p + 1}$$

the frequency has to be scaled by some factor T such that $T \cdot 10 = 10^{-3}$. Then $T = 10^{-4}$ and R_oC_o and R_sC_s have to be scaled by the same amount. Then $R_o'C_o' = R_oC_o \cdot T = (.01)(10^{-4}) = 10^{-6}$ ohms-farads, and $R_s'C_s' = R_sC_s \cdot T = .118\text{x}10^{-4} = .118\text{x}10^{-4}$ ohm-farads. In the same way the parameters for $\bar{Z}_{2(s)}$ can be found as $R_o = 5$ ohms, $R_s = 8000$ ohms, $R_o'C_o' = (.625\text{x}10^{-2})(10^{-4}) = .625\text{x}10^{-6}$, $R'C' = (.624\text{x}10^{-1})(10^{-4}) = .624\text{x}10^{-5}$. The network is given in Figure 3.2.8.

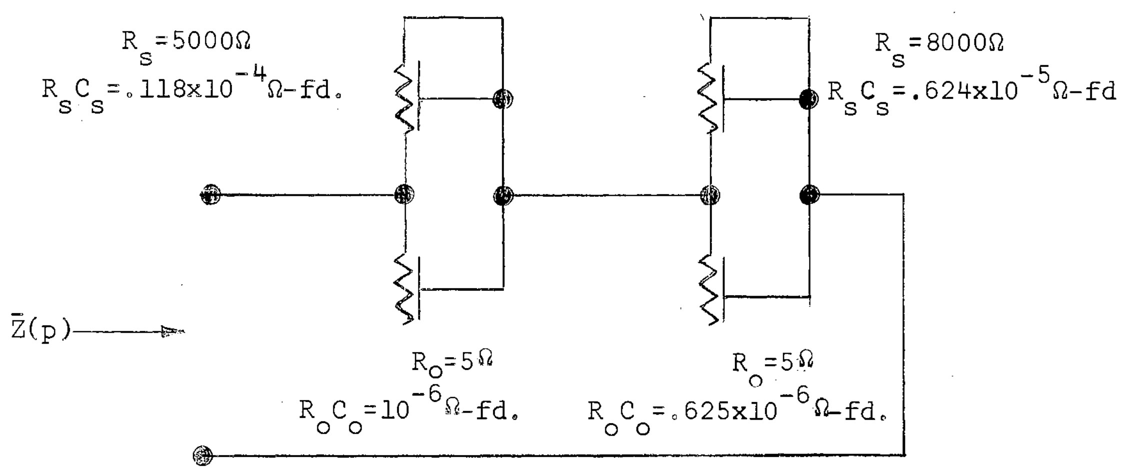


Figure 3.2.8. Network for Example 3.2.1

The largest error in the approximation for the terms $\bar{Z}_{1(s)}$ and $\bar{Z}_{2(s)}$ is $E_1 = -.6$db and $E_2 = .24$db respectively, at $R_oC_o\omega = 100$ (see Figure

3.2.5). Note that $|E_1|$ (or $|E_2|$) < .1db for $\omega R_o C_o < 40$.

3.3 Synthesis of Rational Transfer Functions With Distributed Elements Using Operational Amplifiers. Recently there has been a large amount of interest in synthesis using operational amplifiers (15). In this section operational amplifiers will be used with Z_{oc}-elements and Z_{sc}-elements to obtain a realization procedure for any rational transfer function with constant coefficients.

First consider the network frequently used in analog computation and shown in Figure 3.3.1

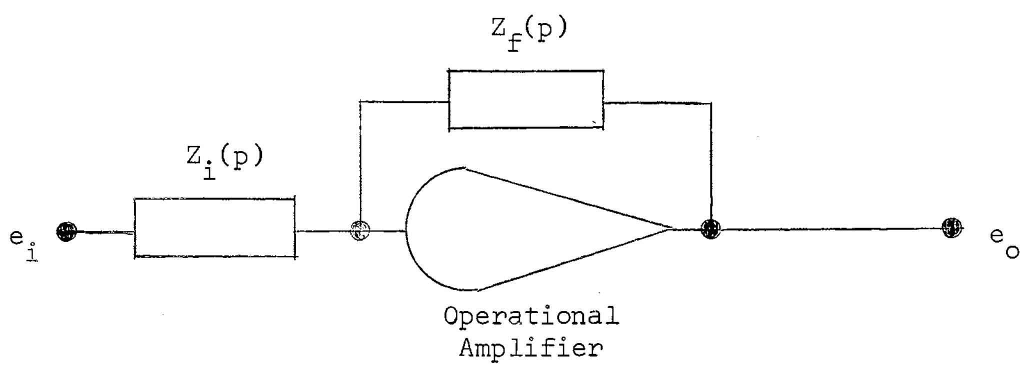


Figure 3.3.1. Network Used in Analog Computation

where $Z_f(p)$ and $Z_i(p)$ are the impedance functions of the elements shown and the operational amplifier has a very high gain. It is well known that the transfer function for the network shown in Figure 3.3.1 is

$$\frac{e_o}{e_i} \approx \frac{Z_f(p)}{Z_i(p)} \tag{3.3.1}$$

Now let $Z_f(p) = Z_{sc}(p)$ and $Z_i(p) = Z'_{sc}(p)$ where $Z_{sc}(p)$ and $Z'_{sc}(p)$ are impedances of the form given in Equation 2.1.1 with different parameters R_s^0, $R_s^0 C_s^0$ and R_s^1, $R_s^1 C_s^1$ respectively. Then

$$\frac{e_o}{e_i} \approx \frac{\dfrac{-R_s^0 \tanh \sqrt{p\, R_s^0 C_s^0}}{\sqrt{p\, R_s^0 C_s^0}}}{\dfrac{R_s^1 \tanh \sqrt{p\, R_s^1 C_s^1}}{\sqrt{p\, R_s^1 C_s^1}}} \tag{3.3.2}$$

When $R_s^0 C_s^0 = R_s^1 C_s^1$ (same RC products), Equation 3.3.2 reduces to

$$\frac{e_o}{e_i} \approx -\frac{R_s^0}{R_s^1} \tag{3.3.3}$$

Thus, it is possible to build the summing amplifier network shown in Figure 3.3.2 where $R_s^0 C_s^0 = R_s^1 C_s^1 = \ldots = R_s^n C_s^n$ and each block shown in the figure corresponds to a Z_{sc}-element.

The output voltage e_o for Figure 3.3.2 is

$$e_o \approx -R_s^0 \sum_{i=1}^{n} \frac{e_i}{R_s^i} \tag{3.3.4}$$

In the following, an integrating amplifier will be constructed using Z_{oc}-elements, Z_{sc}-elements and a gyrator (16). The ideal gyrator is shown symbolically in Figure 3.3.3, terminated by Z_L.

The open circuit parameter equations of a gyrator are

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

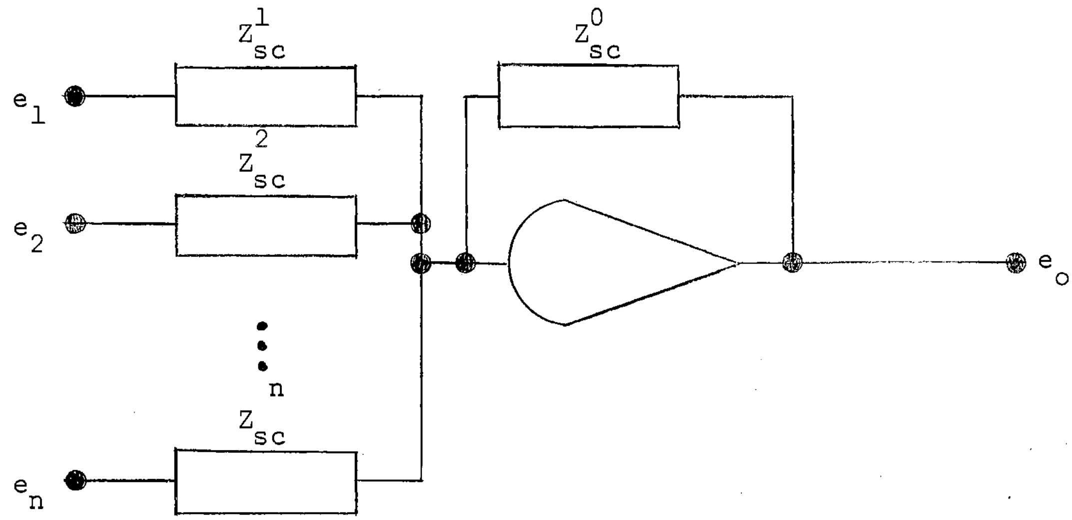


Figure 3.3.2. Summing Amplifier

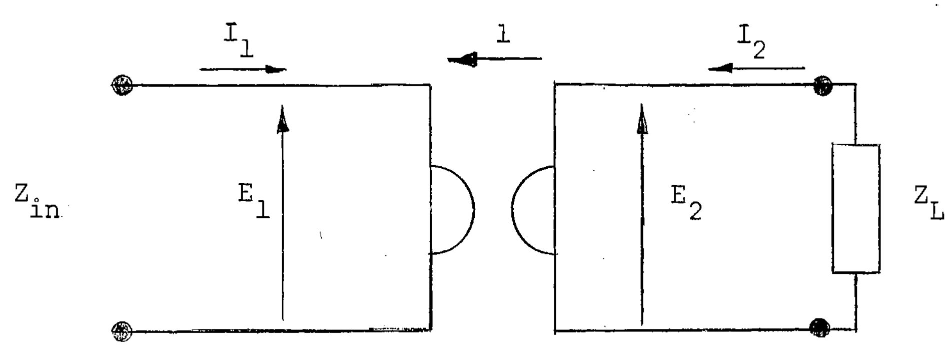


Figure 3.3.3. Gyrator Terminated by Z_L.

Note that $Z_{in} = 1/Z_L$, for Figure 3.3.3. Now consider the network shown in Figure 3.3.4 where the Z_{oc}-element and Z_{sc}-element are labeled. The transfer function for the network in Figure 3.3.4 is

$$\frac{e_o}{e_i} = \frac{-Z_{sc}}{\frac{1}{Z_{oc}}} = \frac{\dfrac{-R_s \tanh \sqrt{p\, R_s C_s}}{\sqrt{p\, R_s C_s}}}{\dfrac{\sqrt{p\, R_o C_o} \tanh \sqrt{p\, R_o C_o}}{R_o}} = \frac{-R_s}{p\, C_o} \tag{3.3.5}$$

where $R_o C_o = R_s C_s$.

Then since an integrating amplifier and a summing amplifier can be built using distributed elements, the methods used in analog computation can be used to realize any given rational transfer function with constant coefficients.

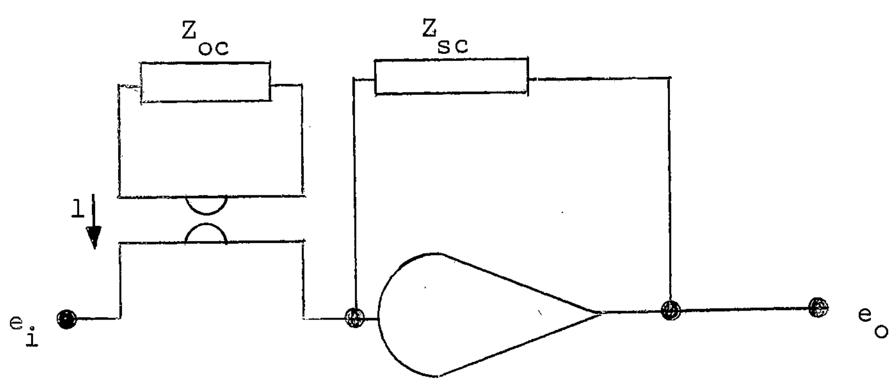


Figure 3.3.4. Integrating Amplifier

CHAPTER IV

MULTIVARIABLE IMPEDANCE FUNCTIONS FOR $\overline{\text{URC}}$ NETWORKS WITH ELEMENTS HAVING DIFFERENT RC PRODUCTS

4.1 Introduction. This chapter deals with the definition of a multivariable impedance function which can be used for $\overline{\text{URC}}$ networks with elements having different RC products. Some new properties are derived for the multivariable impedance function which are useful in the synthesis problem.

4.2 Multivariable Impedance Functions. In Chapter I Wyndrum's method of synthesis was briefly discussed where all the $\overline{\text{URC}}$ netowrk elements were restricted to have the same RC products. This restriction can be removed by using the theory recently developed by Koga (8) on the synthesis of impedance functions of several variables. Impedance functions of several variables are obtained for $\overline{\text{URC}}$ networks when the transformations used by Wyndrum are generalized (3). The transformations for $\overline{\text{URC}}$ networks when the RC product is the same for each element are

$$Z_{LC}(p) = (Z_{RC}(p^2))p \qquad (4.2.1)$$

$$s(p) = \tanh (ap) \qquad (4.2.2)$$

where a = RC, Z_{RC} is the impedance of a $\overline{\text{URC}}$ network, $Z_{LC}$is the impedance of Z_{RC} under the transformation in Equation 4.2.1 p is the frequency variable of a $\overline{\text{URC}}$ element, and s is the transformed domain.

The open-circuit and short-circuit impedance of a $\overline{\text{URC}}$ element are

$$Z_{oc} = \frac{R_o}{\sqrt{R_o C_o p} \tanh \sqrt{R_o C_o p}} \quad (4.2.3)$$

$$Z_{sc} = \frac{R_s \tanh \sqrt{R_s C_s p}}{\sqrt{R_s C_s p}} \quad (4.2.4)$$

Now assume that a $\overline{\text{URC}}$ network with k elements has a different RC product for each element. Each of the RC products is denoted by $B_j^2 = R_j C_j$ where $j=1,2,\ldots,k$. Note that some of the RC products may be equal in magnitude but for simplicity each RC product B_j^2 will carry a different subscript. It is evident that no generality is lost by this assumption. Now, the open-circuit and short-circuit impedances of Equations 4.2.3 and 4.2.4 become

$$Z_{oc} = \frac{R_i}{\sqrt{B_i^2 p} \tanh \sqrt{B_i^2 p}} \quad (4.2.5)$$

$$Z_{sc} = \frac{R_j \tanh \sqrt{B_j^2 p}}{\sqrt{B_j^2 p}} \quad (4.2.6)$$

Using the transformations given in Equation 4.2.1 in Equations 4.2.5 and 4.2.6

$$Z_{oc} = \frac{R_i}{\sqrt{B_i^2} \tanh p \sqrt{B_i^2}} \quad (4.2.7)$$

$$Z_{sc} = \frac{R_j \tanh p \sqrt{B_j^2}}{\sqrt{B_j^2}} \quad (4.2.8)$$

Since each RC product is different, the transformation given in Equation 4.2.2 must be changed to

$$s_i(p) = \tanh (B_i p) \qquad (4.2.9)$$

Using this transformation in Equations 4.2.7 and 4.2.8, the result is

$$Z_{oc} = \frac{R_i}{B_i s_i} \qquad (4.2.10)$$

$$Z_{sc} = \frac{R_j s_j}{B_j} \qquad (4.2.11)$$

Therefore, the driving point impedance of a circuit with k $\overline{\text{URC}}$ elements with different RC products is transformed into a multivariable driving point function of k variables s_j, $j=1,2,\ldots,k$.

4.3 Properties of $\overline{\text{URC}}$ Multivariable Driving Point Functions. In this section the notation $\overline{\text{URC}}$MVDPF will be used to denote a $\overline{\text{URC}}$ multivariable driving point function. Existing theorems and definitions dealing with multivariable driving point functions which relate to this section are given in Appendix B.

Theorem 4.3.1 Topological Formula Reactance Property: A necessary condition that the topological formula for the driving point admittance (see Appendix B)

$$Y = \frac{\Delta}{\Delta_{11}}$$

to correspond to a network with k elements without transformers is that Y be a reactance function of k variables.

Proof: Consider an arbitrary graph G with k elements and let each branch have an admittance y_i, $i=1,2,\ldots,k$. Now using Theorem B.2.3 the driving point admittance $Y(y_1,y_2,\ldots,y_k)$ can be computed for the graph G. It follows from Theorem B.1.1 that $Y(y_1,y_2,\ldots,y_k)$ is a positive real function. This can be seen by setting $y_i = A_i s + B_i/s$, where A_i and B_i are positive, real, and arbitrary constants. Now the corresponding network only has inductors and capacitors. From conventional network theory, Y must be a positive real function of s for every set of positive constants A_i and B_i. Y can be shown to satisfy

$$Y(y_1,y_2,\ldots y_k) \equiv -Y(-y_1,-y_2,\ldots,-y_k) \qquad (4.3.1)$$

by considering Theorem B.2.4. It follows that Y is a reactance function of k variables.

Lemma 4.3.1 $\overline{\text{URC}}$MVDPF Reactance Property: A necessary condition that a $\overline{\text{URC}}$MVDPF W with k variables be realizable as a network with elements of the form $L_i s_i$ or $1/C_i s_i$ where L_i and C_i are positive and real constants is that W be a reactance function of k variables s_i, $i=1,2,\ldots,k$.

Proof: To be realizable, the $\overline{\text{URC}}$MVDPF must correspond to some graph G, and it follows from Theorem B.2.3 and 4.3.1 that the driving point function can be obtained in terms of the branch admittances and is a reactance function of k variables. If $y_i = L_i s_i$ or $y_i = 1/C_i s_i$, depending on the admittance of the branch, for $i=1,2,\ldots,k$ the $\overline{\text{URC}}$MVDPF is obtained. Since $L_i s_i$ and $1/C_i s_i$ are reactance functions when L_i and C_i are positive and real and since the reactance function of a reactance function is again a reactance function, it follows that the $\overline{\text{URC}}$MVDPF must be a reactance function.

In the previous theorem the $\overline{\text{URC}}$MVDPF was obtained from the topological formula $Y = \frac{\Delta}{\Delta_{11}}$ where each $y_i = C_i s_i$ of $y_i = 1/L_i s_i$. It follows from Theorem B.2.4 that if the numerator and denominator of Y are multiplied by $L_i s_i$ for every term of the form $1/L_i s_i$ in the numerator or denominator that the result will be of the form

$$Y(s_1,s_2,\ldots,s_k) = \frac{\sum a_i PN_i}{\sum b_j PD_j} \qquad (4.3.2)$$

where a_i and b_j are positive and real constants and PN_i and PD_j are products of the elements from a subset of the set $\{s_1,s_2,\ldots,s_k,1\}$.

Definition 4.3.1 Normal Form of the $\overline{\text{URC}}$MVDPF: A $\overline{\text{URC}}$MVDPF is said to be the normal form if it has the form of Equation (4.3.2).

Example 4.3.1: Consider the graph shown in the Figure 4.3.1.

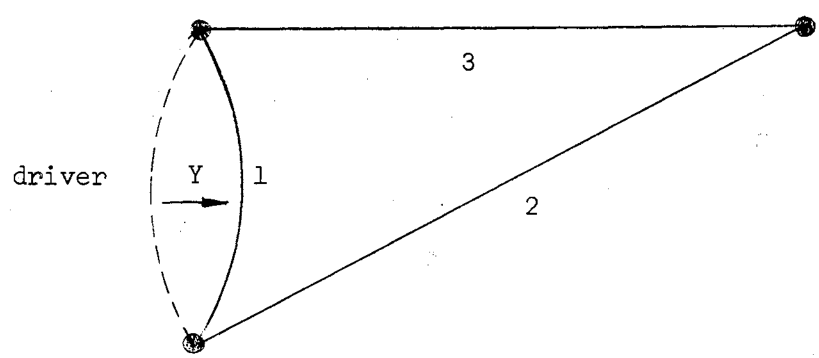


Figure 4.3.1. Graph for Example 4.3.1

The graph corresponds to the topological formula for the driving point admittance given below

$$Y = \frac{y_1y_2 + y_1y_3 + y_2y_3}{y_2 + y_3} \qquad (4.3.3)$$

If $y_1 = 1/L_1s_1$, $y_2 = 1/L_2s_2$, and $y_3 = C_3s_3$, the result is

$$Y = \frac{\dfrac{C_3s_3}{L_1s_1} + \dfrac{1}{L_1s_1L_2s_2} + \dfrac{C_3s_3}{L_2s_2}}{\dfrac{1}{L_2s_2} + C_3s_3} \tag{4.3.4}$$

Rearranging in to the normal form

$$Y = \frac{C_3L_2s_2s_3 + L_1C_3s_1s_3 + 1}{L_1L_2C_3s_1s_2s_3 + L_1s_1} \tag{4.3.5}$$

To aid in the proof of subsequent theorems and discussion, the following definitions are given.

Definition 4.3.2 $\overline{\text{URC}}$-Product: A $\overline{\text{URC}}$-product is defined to be a product of elements from a subset of the set $\{s_1, s_2, \ldots, s_k; 1\}$.

Definition 4.3.3 Degree of a $\overline{\text{URC}}$-Product: The degree of a $\overline{\text{URC}}$-product is defined to be the integer equal to the number of variables s_i in the $\overline{\text{URC}}$-product. If there are no variables s_i in the product, the degree is defined to be zero.

Definition 4.3.4 Irreducible Function: A function $F = N/D$ is said to be irreducible if the numerator and the denominator have no common factors.

Theorem 4.3.2 Reducibility of the $\overline{\text{URC}}$MVDPF and Δ/Δ_{11}: Given a graph G, the topological formula $Y = \Delta/\Delta_{11}$ and the $\overline{\text{URC}}$MVDPF for graph G are reducible functions if and only if the graph G is separable.

Proof: Assume that Δ/Δ_{11} (or $\overline{\text{URC}}$MVDPF) corresponds to some graph G and there are common factors in the numerator and the denominator. The common factors may be canceled and Δ/Δ_{11} (or the $\overline{\text{URC}}$MVDPF) is no longer a function of at least one variable y_i (or s_i). This is true since the maximum degree of each variable in the numerator or denominator is equal to one. Since the driving point function $Y = \Delta/\Delta_{11}$ ($\overline{\text{URC}}$MVDPF) does not depend on some of the variables, G must be a separable graph.

Now assume that G corresponds to a separable graph with components $G_1, G_2, \ldots, G_n$ where G_i is a nonseparable graph. Then Theorem B.2.7 implies the determinant of the node-admittance matrix for graph G can be written as

$$\Delta = \Delta_1 \cdot \Delta_2 \cdots \Delta_n \tag{4.3.6}$$

where Δ_i is the determinant of the node-admittance matrix corresponding to graph G_i. Let G_1 be the graph that becomes nonseparable when the input vertices are identified. Now a 2-tree for G must have each of the input vertices in a separate component part of the graph by definition. Since G_1 is the component containing the input vertices, every node in G_i, i=2,3,...,n and the node common to G_1 and G_i must be joined by a path of elements from any 2-tree of G. Hence, every 2-tree of G has the elements of a tree from each graph G_i, i=2,3,...,n. It follows that

$$\Delta_{11} = \Delta'_{11} \cdot \Delta_2 \cdot \Delta_3 \cdots \Delta_n \tag{4.3.7}$$

where Δ'_{11} is the 2-tree for graph G_1 and

$$Y = \frac{\Delta_1 \, (\Delta_2 \cdot \Delta_3 \cdots \Delta_n)}{\Delta'_{11} (\Delta_2 \cdot \Delta_3 \cdots \Delta_n)} = \frac{\Delta_1}{\Delta'_{11}} \tag{4.3.8}$$

and Y is a reducible function. The $\overline{\text{URC}}$MVDPF corresponding to Y must also be reducible.

Since a graph is either a separable or a nonseparable graph, Theorem 4.3.2 implies Δ/Δ_{11}($\overline{\text{URC}}$MVDPF) is irreducible if and only if the graph corresponding to Δ/Δ_{11} ($\overline{\text{URC}}$MVDPF) is nonseparable. The example given below illustrates Theorem 4.3.2.

Example 4.3.2: Consider the separable graph shown in the figure below.

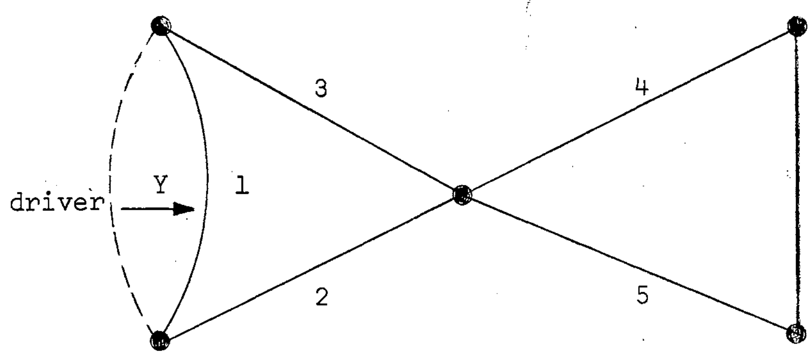


Figure 4.3.2. Graph for Example 4.3.2

Δ for the graph in the Figure 4.3.2 has the form

$$\Delta = (y_1y_2 + y_2y_3 + y_1y_3)(y_4y_5 + y_5y_6 + y_4y_6) \qquad (4.3.9)$$

and Δ_{11} for the graph has the form

$$\Delta_{11} = (y_2 + y_3)(y_4y_5 + y_5y_6 + y_4y_6) \qquad (4.3.10)$$

Then

$$Y = \frac{\Delta}{\Delta_{11}} = \frac{y_1y_2 + y_2y_3 + y_1y_3}{y_2 + y_3} \qquad (4.3.11)$$

Theorem 4.3.3 Uniqueness Property for Δ/Δ_{11}: If a given graph is nonseparable when the input vertices are identified, the topological formula $Y = \Delta/\Delta_{11}$ corresponding to the graph is unique. If a given realizable topological formula $Y = \Delta/\Delta_{11}$ is irreducible, the graph corresponding to Y has a form unique within a 2-isomorphism (see Definition B.2.4).

Proof: Assume that G is a nonseparable graph when the input vertices are identified. It is known (Appendix B) that Δ is equal to the sum of all possible tree-admittance products for graph G where a tree-admittance product is the product of the branches of a tree. It is obvious that Δ must be unique in form. Similarly, Δ_{11} is unique since it is computed as the sum of all possible 2-tree products where the input vertices are in different connected parts of the graph. Theorem 4.3.2 states that $Y = \Delta/\Delta_{11}$ is an irreducible function when the graph is nonseparable. Hence, $Y = \Delta/\Delta_{11}$ is unique.

Assume that the topological formula for the DPF (driving point function) is realizable and irreducible. Then Y must correspond to a nonseparable graph (Theorem 4.3.2) and no terms can be canceled from Y. Since Y is realizable, the set of all possible trees can be obtained from Δ. When all of the elements of each tree are known, the graph is determined to within a 2-isomorphism (10), (11).

Lemma 4.3.2 Uniqueness Property for the $\overline{\text{URC}}$MVDPF: If a given graph is nonseparable when the input vertices are identified, the corresponding $\overline{\text{URC}}$MVDPF in normal form is unique. If a given $\overline{\text{URC}}$MVDPF is in the normal form, is irreducible, and is realizable, the graph corresponding to the $\overline{\text{URC}}$MVDPF is unique to within a 2-isomorphism.

Proof: That the graph is unique for a given nonseparable graph follows from Theorem 4.3.3 since there is only one normal form of the $\overline{\text{URC}}$MVDPF which can be obtained from the topological formula $Y = \Delta/\Delta_{11}$ (corresponding to the given graph).

Assume that a realizable $\overline{\text{URC}}$MVDPF is given in the normal form and is irreducible. The normal form of the $\overline{\text{URC}}$MVDPF can be obtained from Δ/Δ_{11} for a given graph by setting y_i equal to the DPF for each element

e_i and by multiplying numerator and denominator by $L_i s_i$ when an element of the form $1/L_i s_i$ is in the numerator or denominator, it is evident that given the normal form, the form of the topological equation can be obtained by dividing the numerator and denominator by the proper variables s_i such that the result has the correct form when y_i is set equal to s_i or $1/s_i$ (Theorem B.2.4). The constants associated with the variables s_i are not needed to find the form of the topological formula and may be considered to have a value of one. The variables which are used in the division of numerator and denominator cannot be identified by inspection, but there are only a finite number of possible divisions that could be made ($\sum_{i=1}^{k} \binom{k}{i}$ for k variables in the $\overline{\text{URC}}$MVDPF). Now since the topological formula has a numerator with products all having a degree of (v-2) (see Theorem B.2.4), and each product has variables y_i of degree one, there can be only one possible set of divisors that give the correct form for Δ/Δ_{11}. This implies there is only one possible topological formula $Y = \Delta/\Delta_{11}$ for the given $\overline{\text{URC}}$MVDPF in normal form. Hence, it follows from Theorem 4.3.3 that every realization of the $\overline{\text{URC}}$MVDPF must correspond to graphs unique to within a 2-isomorphism.

It is apparent from Lemma 4.3.2 that there is no single network which realizes all possible $\overline{\text{URC}}$MVDPFs. A given graph with k arbitrary elements of the form s_i (or $1/s_i$) gives only 2^k possible normal forms for the $\overline{\text{URC}}$MVDPF. The constants of the $\overline{\text{URC}}$-products can be varied by changing element values (in the $\overline{\text{URC}}$MVDPF) but are interdependent.

<u>Theorem 4.3.4 Degree and Ordering of the $\overline{\text{URC}}$-Products of a $\overline{\text{URC}}$MVDPF:</u> Every realizable $\overline{\text{URC}}$MVDPF in the normal form must have the following properties:

i) The numerator or denominator have $\overline{\text{URC}}$-products of only even or only odd degree, and further, if the denominator has only $\overline{\text{URC}}$-products of even degree, then the numerator always has $\overline{\text{URC}}$-products of only odd degree and vice versa.

ii) The highest degree of the numerator $\overline{\text{URC}}$-products and highest degree of the denominator $\overline{\text{URC}}$-products must differ by one.

iii) The numerator (or denominator) with $\overline{\text{URC}}$-products of even degree or odd degree (odd or even degree) has all $\overline{\text{URC}}$-products of degree r, $m \leq r \leq u$ and no others where r is either even or odd, m is the degree of the $\overline{\text{URC}}$-product with maximum degree, and u is the degree of the $\overline{\text{URC}}$-product with minimum degree. The integer u for the numerator must differ by one from the integer u of the denominator.

<u>Proof:</u> It has been shown that the normal form of a realizable $\overline{\text{URC}}$MVDPF W with k variables can be written in the general form as

$$W = \frac{\sum a_i PN_i}{\sum b_j PD_j}$$

where PN_i and PD_j are $\overline{\text{URC}}$-products and a_i and b_j are positive constants. A necessary condition for W to be realizable is that W be a reactance function (Lemma 4.3.1). Then by the definition of a multivariable reactance function (see Appendix B)

$$W(s_1,s_2,\ldots,s_k) \equiv -W(-s_1,-s_2,\ldots,-s_k)$$

If N denotes the numerator of W and D denotes the denominator of W, then <u>either</u> $N(s_1,s_2,\ldots,s_k) \equiv N(-s_1,-s_2,\ldots,-s_k)$ and $D(s_1,s_2,\ldots,s_k) \equiv -D(-s_1,-s_2,\ldots,s_k)$ <u>or</u> $N(s_1,s_2,\ldots,s_k) \equiv -N(-s_1,-s_2,\ldots,-s_k)$ and $D(s_1,s_2,\ldots,s_k) \equiv D(-s_1,-s_2,\ldots,-s_k)$. Now if $D \equiv D^*$ (*indicates each variable s_i is replaced by $-s_i$), then $D = \sum b_j PD_j$ implies each PD_j is of even degree. Similarly if $N \equiv N^*$, then each PN_i is of even degree.

Further if $D \equiv -D^*$, then each PD_j is of odd degree, and if $N \equiv -N^*$, then each PN_i is of odd degree. This gives property (i).

Now let $s_i = p$ for $i=1,2,\ldots,k$ in W. Since W is a reactance function of k variables, it follows from the definition of a multivariable reactance function that the function obtained is a reactance function of one variable (an LC fucntion). If there are no cancellations of some f(p) when $s_i = p$ for every i, properties (ii) and (iii) follow directly. If a function f(p) can be canceled from W when $s_i = p$ $i=1,2,\ldots,k$, then W can be written

$$W = \frac{n\ f(p)}{d\ f(p)}$$

where n/d is an LC function of p. Let m_1 be the highest power of p in n f(p), and let m_2 be the highest power of p in d f(p), then from the properties of LC functions it follows that $|m_1-m_2| = 1$, and therefore property (ii) follows. Now using property (i) and using the properties of LC functions it follows that f(p) is a polynomial with only odd or even powers of p. Let the highest and lowest powers of p in f(p) be q_1 and q_2 respectively. Note that n has only odd (or even) powers of p, and d has only even (or odd) powers of p. Now let u_1 be the lowest power of p in n and u_2 be the lowest power of p in d. For n (or d) with even powers of p, all even powers must be present between m_1 and $u_1 = 0$ (m_2 and $u_2 = 0$). Similarly, for n (or d) with odd powers of p all odd powers of p must be present between m_1 and $u_1 = 1$ (m_2 and $u_2 = 1$). Now the highest power of p in n f(p) must be $m' = m_1 + q_1$ and the lowest power must be $u' = u_1 + q_2$. Similarly, the highest power of p in d f(p) must be $m'' = m_2 + q_1$ and the lowest must be $u'' = u_2 + q_2$. All even or odd powers (which ever is applicable) are present between m' and u' (m" and u"). That $|u' - u''| = 1$ can be shown in the same manner used to establish (ii). Therefore (iii) follows.

Example 4.3.3 below is used to illustrate Theorem 4.3.4. This example illustrates some of the differences between multivariable reactance functions and one variable reactance functions.

<u>Example 4.3.3:</u> Consider the realizable $\overline{\text{URC}}$MVDPF function (reactance function) given below in normal form

$$W = \frac{s_1 s_2 s_3}{s_2 s_3 + s_1 s_3 + s_1 s_2} \tag{4.3.12}$$

Note that there is no $\overline{\text{URC}}$-product of first degree in the numerator or of zero degree (a constant) in the denominator. Let $s_i = p$. Then

$$\frac{ppp}{pp + pp + pp} = p$$

Note that the equation above does not have LC function form until after the cancellation of p^2, and Equation 4.3.12 satisfies (i), (ii), (iii) of Theorem 4.3.4.

The conditions in Theorem 4.3.4 may be thought of as necessary conditions for a $\overline{\text{URC}}$MVDPF in normal form to be a reactance function or to be realizable. However, the conditions are not sufficient for a function to be a reactance function as can be shown by the following example.

<u>Example 4.3.4:</u> Consider the multivariable function

$$W = \frac{s_1 s_2 + s_1 s_3}{s_1 s_2 s_3 + s_3} \tag{4.3.13}$$

which satisfies Theorem 4.3.4. It will now be shown that W is not positive real (see Appendix B for a definition). Let $s_1 = .01 \angle 60°$, $s_2 = .01 \angle -20°$, and $s_3 = 10^{-6} \angle -89°$. Then rearranging Equation 4.3.13.

$$W = \frac{\frac{s_1 s_2}{s_3} + s_1}{s_1 s_2 + 1} \qquad (4.3.14)$$

Substituting the values for s_1, s_2, and s_3

$$W = \frac{\frac{(10^{-2} \angle 60°)(10^{-2} \angle -20°)}{10^{-6} \angle -89°} + 10^{-2} \angle 60°}{(10^{-2} \angle 60°)(10^{-2} \angle -20°) + 1} \qquad (4.3.15)$$

In Equation 4.3.15

$$|(10^{-2} \angle 60°)(10^{-2} \angle -20°)| << 1$$

and

$$|(10^{-2} \angle 60°)| << \left| \frac{(10^{-2} \angle 60°)(10^{-2} \angle -20°)}{(10^{-6} \angle -89°)} \right|$$

Therefore W can be approximated by

$$W \quad \frac{(10^{-2} \angle 60°)(10^{-2} \angle -20°)}{10^{-6} \angle -89°} = 100 \angle 129°$$

which has a negative real part. This implies W is not positive real and cannot be a reactance function.

CHAPTER V

SYNTHESIS OF DRIVING POINT FUNCTIONS OF $\overline{\text{URC}}$ NETWORK WITH ELEMENTS HAVING DIFFERENT RC PRODUCTS

5.1 Introduction. This chapter deals with a new method for the synthesis of the driving point function of $\overline{\text{URC}}$ networks having elements with different RC products. In the realization, transformers and gyrators are not used. One by-product of this method can also be used in finding the graph corresponding to the classical topological formula for the driving point admittance (11).

5.2 Basis for the Synthesis of $\overline{\text{URC}}$MVDPFs. Koga (8) has given a general synthesis procedure to realize multivariable functions. Further, he has given the necessary and sufficient conditions for the realization. Unfortunately, the synthesis procedure in general requires transformers. Since transformers cannot be used in most applications where $\overline{\text{URC}}$MVDPFs are applicalbe, a method that does not require transformers is desirable. Further, it is desirable that the method be easily programmable on the digital computer. To develop a procedure having these properties, it is necessary to consider some fundamental properties of circuits of a graph since the synthesis method to be developed consists of finding a circuit matrix corresponding to the $\overline{\text{URC}}$MVDPF.

Theorem 5.2.1 Placement of Elements in a Circuit: A necessary and sufficient condition that any two elements in a graph can be placed in a circuit is that the graph be nonseparable.

Proof: Assume any two elements of a connected graph G can be placed in some circuit. Then no cut vertex can exist in G since any two vertices can be joined by at least two different paths containing different vertices, and G is nonseparable by definition (10).

Assume graph G is nonseparable. Then no cut vertex exists by definition. Since the graph is connected, any two vertices v_a and v_b can be joined by a path of elements e_a, e_1, e_2,...,e_n, e_b (e_a incident with v_a, e_b incident with v_b). Let v be a vertex incident with e_a and different from v_a. Then since v cannot be a cut vertex there must be another path connecting v_a and v_b not containing v (10). Therefore, a circuit exists that contains any two elements e_a and e_b.

The importance of Theorem 5.2.1 is that it insures that any element of a nonseparable one port network N with graph G can be placed in a circuit with the driver of the network. It follows that each element e_i can be seen as the driving point function of the network N_1 obtained from N by taking a circuit which has a set of elements including e_i and the driver, by short-circuiting each element in this circuit except the driver and e_i and by open-circuiting the remaining elements of N.

Assume that W is a $\overline{\text{URC}}$MVDPF which can be realized by a network N. Then $\overline{\text{URC}}$MVDPF W can be written in the form (see Section 4.3)

$$Y = \frac{\sum a_i PN_i}{\sum b_j PD_j} \tag{5.2.1}$$

where a_i and b_j are positive constants and PN_i and PD_j are $\overline{\text{URC}}$-products. Since each s_i in each of the $\overline{\text{URC}}$-products has a degree of one, it follows that every $\overline{\text{URC}}$MVDPF with k elements can be written in the form

$$W = \frac{A\, s_i + C}{B\, s_i + D} \tag{5.2.2}$$

where A, C, B, and D are polynomials with (k-1) variables s_j, $j=1,2,\ldots i-1,i+1,\ldots,k$. Some of the polynomials A, B, C. and D may be identically zero. The possible cases are: $A, B, C, D \not\equiv 0$; $A \equiv 0$, $B, C, D \not\equiv 0$; $A \equiv 0, D \equiv 0, B, C \not\equiv 0$; $C \equiv 0, A, B, D \not\equiv 0$; $C \equiv 0, B \equiv 0$, $A, C, D \not\equiv 0$; $B \equiv 0, A, C, D \not\equiv 0$; $D \equiv 0, A, B, C \not\equiv 0$. All other cases either give an undefined W, a $W \equiv 0$, or a W that is not a function of all k variables.

Since it is known that W corresponds to a network with elements having the impedances of the form s_i or $1/s_i$, each of the limits

$$\lim_{s_i \to 0} W = \frac{C}{D} \tag{5.2.3}$$

$$\lim_{s_i \to \infty} W = \frac{A}{B} \tag{5.2.4}$$

must correspond to either short-circuiting (open-circuiting) or open-circuiting (short-circuiting) the element corresponding to the variable s_i respectively. At this point the type of element ($L_i s_i$ or $1/C_i s_i$) is unknown. This information must be obtained before the limits of Equations 5.2.3 and 5.2.4 can be related directly to an open circuit or to a short circuit operation on the network. It is important to note that when a $\overline{\text{URC}}$MVDPF is written in the form of Equation 5.2.2 the limit as s_i goes to zero or infinity can be obtained by inspection, and if the $\overline{\text{URC}}$MVDPF W has k elements, the limit of W as s_i goes to zero or infinity is equal to either zero (a short circuit), or infinity (an open circuit) or a function of (k-1) variables.

Previously it was shown that each element e_i can be placed in a circuit with the driver, and as a result, the impedance of each e_i is given when the proper elements are open-circuited or short-circuited. Then if the limit is taken of W as s_i goes to zero for (k-1) of the k variables, and if the proper limits are chosen ($s_i \rightarrow 0$ or $s_i \rightarrow \infty$); the result will be a function equal to the driving point function of any chosen element e_j of the network. It will be evident from the example given below that there may be more than one set of limit operations that give a result equal to the driving point function of an element even when there is only one circuit containing the driver and the element. The following definition will be made as an aid in the example and subsequent work.

Definition 5.2.1 Set of Open-Circuits and Short-Circuits S_{ij}: Let G be a graph with k elements e_i. Then let $\bar{e}_i$ denote that e_i is short-circuited, $\underline{e}_i$ denote that e_i is open-circuited, (e_i) denote that e_i is not open-circuited or short-circuited. Then S_{ij} is defined to be a set of operations $\bar{e}_m$ (or $\underline{e}_m$) where i identifies the element e_i given as the driving point function (DPF) and j identifies one such set.

Example 5.2.1: Consider the graph shown in Figure 5.2.1.

Then $S_{31} = \{\bar{e}_1, \underline{e}_2, (e_3), \underline{e}_4, \underline{e}_5, (e_d)\}$, $S_{32} = \{\underline{e}_1, \bar{e}_2, (e_3), \underline{e}_4, \bar{e}_5, (e_d)\}$, $S_{33} = \{\underline{e}_1, \underline{e}_2, (e_3), \underline{e}_4, \underline{e}_5, (e_d)\}$, $S_{34} = \{\underline{e}_1, \bar{e}_2, (e_3), \underline{e}_4, \underline{e}_5, (e_d)\}$, $S_{35} = \{\underline{e}_1, \underline{e}_2, (e_3), \underline{e}_4, \bar{e}_5, (e_d)\}$, give a network having the driving point function (DPF) of e_3. Similarly, $S_{11} = \{(e_1), \underline{e}_2, \underline{e}_3, \underline{e}_4, \bar{e}_5, (e_d)\}$, $S_{12} = \{(e_1), \bar{e}_2, \underline{e}_3, \underline{e}_4, \bar{e}_5, (e_d)\}$, and $S_{13} = \{(e_1), \bar{e}_2, \underline{e}_3, \underline{e}_4, \underline{e}_5, (e_d)\}$ give a network having the DPF of e_1.

All possible sets S_{3j} (or S_{1j}) have been given in Example 5.2.1 (for elements e_3 and e_1). Therefore, any circuit containing the driver and e_3 (or e_1) must correspond to at least one of the sets S_{3j} (or S_{1j}). Further, the sets S_{33}, S_{13}, and S_{11} correspond to the rows of the c-circuit matrix (see Appendix B) for the graph of Figure 5.2.1 given in the matrix

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & d \end{array}$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Some of the S_{ij}'s have short circuits which may not be needed to obtain the DPF of e_i. In Example 4.4.1 the set $S_{32} = \{\underline{e}_1, \bar{e}_2, (e_3), \underline{e}_4, \bar{e}_5, (e_d)\}$ gives the DPF of e_3, but elements e_2 and e_5 need not be shorted to give the DPF of e_3 since $S_{33} = \{\underline{e}_1, \underline{e}_2, (e_3), \underline{e}_4, \underline{e}_5, (e_d)\}$ also gives the DPF of e_3. Figure 5.2.2 illustrates this particular point.

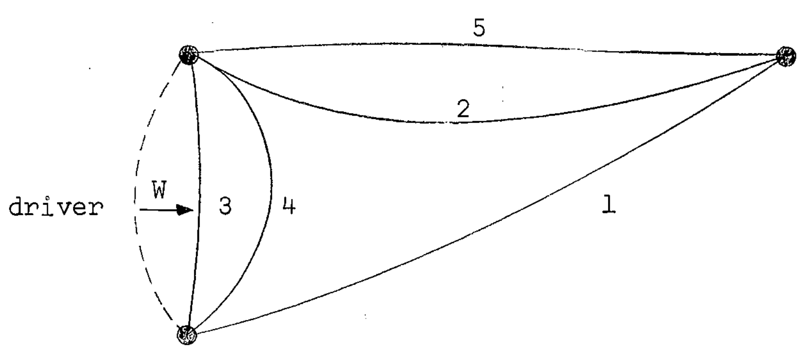


Figure 5.2.1. Graph for Example 5.2.1

As can be seen, the graph shown in Figure 5.2.2 is separable when the element e_1 is open-circuited. The DPF W' is not a function of e_2 and e_5. and therefore, S_{32} and S_{33} give the same result.

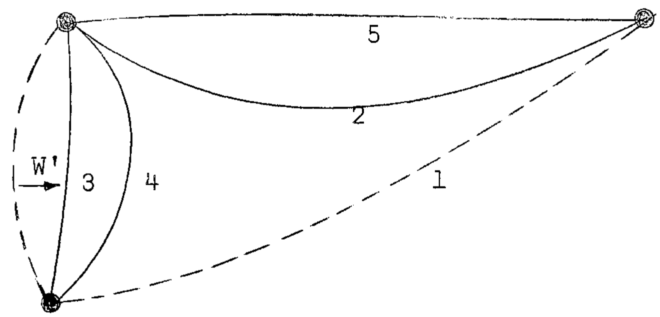


Figure 5.2.2. Graph of Figure 5.2.1 With e_1 Open-Circuited

Example 5.2.1 and the related discussion were developed in terms of open circuits and short circuits, but the same results can be obtained in terms of sets of limit operations ($s_i \to o$ or $s_i \to \infty$) for (k-1) variables. The principle difference is that the limit operations pertain to a DPF W instead of directly to the graph. Now the relation between a set of (k-1) limit operations and a circuit will be derived by considering a one-port network (having a driver and k elements) with graph G and corresponding to the $\overline{\text{URC}}$MVDPF W. First select the set of circuits $\{S_{ij}\}$ of graph G having element e_i and e_d in each circuit. Then select the circuits from $\{S_{ij}\}$ having a minimum number of elements (there may be more than one circuit having the number of elements equal to the minimum). Each of these circuits can be denoted c_{ij} where i corresponds to one of the circuits having e_i and the driver and a minimum number of elements ($j=0,1,2,\ldots,n_i$ where n_i is the number of such circuits for each i). Now a subset of the set of circuits $\{c_{ij}, i=1,2,\ldots,k; j=1,2,\ldots,n_i\}$ must be the set of all circuits having the driver as an element since each element e_i is included in some circuit c_{ij}. This subset will be denoted by F_c for use in the subsequent discussion in this chapter. Note that the word subset is used because some of the circuits e_{ij} may be identical (be the same circuit). Now a subset of F_c containing e-v+1 circuits will be shown to be an independent set of circuits where e is the number of elements and v is the number of vertices (of G).

Theorem 5.2.2 Independence of Circuits: Let G be the nonseparable graph of a one-port network, and let F_c be the set of all circuits for graph G having the driver as an element. Then a subset of the circuits of F_c is a set of e-v+1 independent circuits (the subset of F_c corresponds to a circuit matrix of maximum rank).

Proof: Let F_c be the set of all circuits containing the driver for graph G (corresponding to a one-port network). Now assume there exists a circuit c_x independent of the set of circuits F_c not containing the driver. Then there exists a circuit c_1 of the set F_c having at least one of the elements contained in c_x (any element can be placed in a circuit with the driver). Let P_{x1} be a path of elements that c_1 and c_x have in common, and let v_1 and v_2 denote the vertices incident at each end of path P_{x1}. Then there exists a path P_1 connecting v_1 and v_2 and containing the driver (using elements from c_1 which are not contained in P_{x1}). Since c_x is a circuit, there exists a different path from P_{x1}, P_{x2}, connecting v_1 and v_2 and containing the elements of c_x not contained in P_{x1}. Then the driver, and elements from P_1 and P_{x2} form a circuit c_2 (which is one of the circuits of F_c) and the driver and elements from P_1 and P_{x1} form the circuit c_1. Now it follows that the ring sum (mod 2 sum) of c_1 and c_2 is c_x (10). Therefore, c_x is not independent of the set of circuits F_c, and F_c has e-v+1 independent circuits. Although F_c has e-v+1 independent circuits, the number of circuits in F_c

may be greater than e-v+1. Thus a <u>subset</u> of F_c has e-v+1 circuits which are independent and correspond to a circuit matrix of maximum rank.

It will be shown that the set of circuits F_c (defined earlier) can be found from the sets of (k-1) limit operations that give the DPF of each element e_i. Earlier it was shown that the DPF of each element e_i can be found for a given realizable $\overline{\text{URC}}$MVDPF W of k variables by taking some set of (k-1) successive limits ($s_j \to o$ or $s_j \to \infty$ for each j, $j \neq i$) of W. Therefore, the type $L_i s_i$ (or $1/C_i s_i$) and value L_i (or C_i) can be found for each element e_i and each of the (k-1) limits can be identified as either corresponding to a short circuit or an open circuit in the network corresponding to W. From Example 4.4.1 one can observe that it is possible to obtain the DPF of some element e_q from several different sets of (k-1) limit operations. Now consider all possible sets of (k-1) limit operations that might give the DPF for some e_q. There are $2^{(k-1)}$ possible ways to take (k-1) limits ($s_i \to o$ or $s_i \to \infty$ for each $i \neq q$ can be represented by a binary bit 0 or 1 and the set of (k-1) limit operations can be represented by a binary number) and each set must be tested to see if it will give the DPF of the element e_q. Some of these $2^{(k-1)}$ sets of (k-1) limit operations may give a result equal to zero (a short circuit), or infinity (an open circuit) and therefore, do not give the DPF of e_q. Further, some of the $2^{(k-1)}$ sets of (k-1) limit operations may give the DPF of e_q (see Example 4.4.1). However, when the sets of (k-1) limit operations having a <u>minimum</u> number of limits corresponding to shorts and giving the DPF of e_q are selected from the $2^{(k-1)}$ possible sets of (k-1) limit operations for e_q, each of the limits corresponding to a short circuit corresponds to an element in one of the circuits c_{qj}, $j=1,2,\ldots,n_q$ (c_{qj} is defined above). Thus, the set of circuits $\{c_{ij}, i=1,2,\ldots,k; j=1,2,\ldots,n_i\}$ can be found by considering the k $2^{(k-1)}$ sets of (k-1) limit operations ($2^{(k-1)}$ for each element).

Now let F_c again be defined as the subset of $\{c_{ij}$ $i=1,2,\ldots,k;$ $j=1,2,\ldots,n_i\}$ such that each circuit is not identical to any other circuit and all circuits having the driver are in the set. Then the circuit matrix B of e-v+1 rows with maximum rank where e, the number of elements and v, the number of vertices, can be found from F_c by finding e-v+1 independent circuits of F_c. Note that if the given $\overline{\text{URC}}$MVDPF has k elements and a driver, then e = k+1. Also note that the number of vertices v can be found by using Theorem B.2.4. The e-v+1 independent circuits can be found by finding the largest nonsingular determinant (mod 2) of the matrix corresponding to F_c. Without losing any generality the circuit matrix B given earlier can be written in the form

$$B = \left[B_1 \mid B_2\right] \qquad (5.2.5)$$

where B_1 is a nonsingular matrix (mod 2) of order (e-v+1) x (e-v+1) and B_2 is a matrix of order (e-v+1) x (v-1). Since B_1 is a nonsingular matrix, B can be premultiplied by B_1^{-1} (mod 2) to give the fundamental c-circuit matrix (see Definition B.2.6). Then premultiplying by B_1^{-1}, the result is

$$B_c = B_1^{-1}\,[B] = \left[U \mid B_1^{-1}\,B_2\right] = [U \mid E] \qquad (5.2.6)$$

where U is a unit matrix of order (e-v+1) x (e-v+1) and $E = B_1^{-1}\,B_2$ is a matrix of order (e-v+1) x (v-1). Each column of U corresponds to a chord of some tree T and each column of E corresponds to a tree branch of T (10).

Finally, the graph corresponding to B_c can be found by using the well-known methods (17). Therefore, the procedure given above is one method of realizing a realizable $\overline{\text{URC}}$MVDPF and is summarized in the following steps where W is assumed to be a realizable $\overline{\text{URC}}$MVDPF:

1. Find the element values and types by using all k $e^{(k-1)}$ sets of (k-1) limit operations on W.
2. Find the set of circuits $\{c_{ij}, i=1,2,\ldots,k; j=1,2,\ldots,n_i\}$ by locating the sets of (k-1) limit operations having limits corresponding to a minimum number of short circuits and giving the DPF of e_i for every i.
3. Find F_c from $\{c_{ij}\ i=1,2,\ldots,k; j=1,2,\ldots,n_i\}$ by inspection.
4. Find (e-v+1) independent circuits in F_c and write in matrix form $B = [B_1 \mid B_2]$ where b_1 is of rank (e-v+1).
5. Find B_1^{-1} and then find $B_c = \left[U \mid B_1^{-1} B_2\right]$.
6. Realize B_c as a graph of k elements with a driver.

The synthesis method above can be used to find the network to realize any given realizable $\overline{\text{URC}}$MVDPF, but it is clear that the method would not be practical for a $\overline{\text{URC}}$MVDPF having a large number of variables s_i since k $2^{(k-1)}$ sets of (k-1) limit operations need to be found. A much more efficient method will be derived in the next section. However, the work here does form a basis for all subsequent work and the above synthesis procedure is illustrated by the following example.

<u>Example 5.2.2</u>: Consider the realizable $\overline{\text{URC}}$MVDPF Z given below

$$Z = \frac{6\ s_1 s_2 s_3 + s_1}{6\ s_2 s_3 + 2\ s_1 s_3 + 1} \tag{5.2.7}$$

Since there are three variables, there are 12 ($2^{(k-1)} = 3 \cdot 2^2$) sets of (k-1) limit operations. The notation illustrated by the equations below will be used to simplify the notations.

$$L_Z(\bar{s}_i\bar{s}_j) = \lim_{s_i \to \infty} (\lim_{s_j \to \infty} Z)$$

$$L_Z(\underline{s}_i\underline{s}_j) = \lim_{s_i \to o} (\lim_{s_j \to o} Z)$$

$$L_Z(\bar{s}_i\underline{s}_j) = \lim_{s_i \to \infty} (\lim_{s_j \to o} Z)$$

$$L_Z(\underline{s}_i\bar{s}_j) = \lim_{s_i \to o} (\lim_{s_j \to \infty} Z)$$

Then the 12 sets of (k-1) limits are

$$L_Z(\bar{s}_1,\bar{s}_2) = \infty$$

$$L_Z(\bar{s}_1,\underline{s}_2) = 1/2\ s_3$$

$$L_Z(\underline{s}_1,\bar{s}_2) = 0$$

$$L_Z(\underline{s}_1,\underline{s}_2) = 0$$

$$L_Z(\bar{s}_2,\bar{s}_3) = 0$$

$$L_Z(\bar{s}_2,\underline{s}_3) = s_1$$

$$L_Z(\underline{s}_2,\bar{s}_3) = 0$$

$$L_Z(\underline{s}_2,\underline{s}_3) = s_1$$

$$L_Z(\bar{s}_1,\bar{s}_3) = 3\ s_2$$

$$L_Z(\bar{s}_1,\underline{s}_3) = \infty$$

$$L_Z(\underline{s}_1,\bar{s}_3) = 0$$

$$L_Z(\underline{s}_1,\underline{s}_3) = 0$$

The sets that give the DPF of e_1, e_2, and e_3 are

$$L_Z(\bar{s}_2,\bar{s}_3) = s_1$$

$$L_Z(\bar{s}_2,\underline{s}_3) = s_1$$

$$L_Z(\underline{s}_2,\underline{s}_3) = s_1$$

$$L_Z(\bar{s}_1,\bar{s}_3) = 3\ s_2$$

$$L_Z(\bar{s}_1,\underline{s}_2) = 1/2\ s_3$$

Since the type and value of each element is now known, the limits in the above equations can be identified as corresponding to either open circuits or short circuits. The equations giving the DPF of e_i and having a <u>minimum</u> number of short circuits can be found for each e_i to be

$$L_Z(\underline{s}_2,\underline{s}_3) = s_1$$

$$L_Z(\bar{s}_1,\bar{s}_3) = 3\ s_2$$

$$L_Z(\bar{s}_1,\underline{s}_2) = 1/2\ s_3$$

Then the set of circuits corresponding to Equation 5.2.8 written in matrix form is

$$\begin{array}{c} \\ \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad (5.2.9)$$

Note that c_{21} and c_{31} are the same circuits. The duplication occurs becuase the DPF of each element in a circuit can be given by short-circuiting all other elements in the circuit and open-circuiting the remaining elements in the network. So for a circuit with n elements in some graph (excluding the driver), there will be n duplications. Now from Equation 5.2.9 the matrix F_c corresponding to the set F_c (eliminate duplications) can be written as

$$F_c = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & d \end{matrix} \\ \left|\begin{array}{cc:cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right| \end{array}$$

Note that F_c in the above equation has two independent circuits which can be found by inspection and further, F_c already has the form of the fundamental c-circuit matrix $B_c = [U \,\vdots\, E]$. B_c can now be realized as a graph G and the result is given in Figure 5.2.3a. The network corresponding to G is given in Figure 5.2.3b. Where the schematic symbols for an inductor and capacitor are used to distinguish between the two types of elements.

Figure 5.2.3. Realization for Z

5.3 Synthesis of Realizable $\overline{\text{URCMVDPFs}}$. The synthesis method developed in the previous section can be simplified by the next theorem which eliminates much of the unnecessary checking of limits. The theorem applies to a general class of functions including the $\overline{\text{URC}}$MVDPF.

For simplicity the following notation will be used in the theorem. Let L_W be a function of n variables x_i, i=1,2,...,k where $x_i = \bar{s}_i$ or $\underline{s}_i$. Further, let

$$L_W(\bar{s}_i) = \lim_{s_i \to \infty} W$$

$$L_W(\underline{s}_i) = \lim_{s_i \to 0} W$$

and let

$$L_W(\bar{s}_i, \underline{s}_j) = \lim_{s_i \to \infty} \{ \lim_{s_j \to 0} W \}$$

Note the order in which the limits are taken.

<u>Theorem 5.3.1 $\overline{\text{URC}}$MVDPF Limit Theorem:</u> Let W be an irreducible function of k variables which can be written in the form

$$W = \frac{\sum a_i PN_i}{\sum b_j PD_j} \tag{5.3.1}$$

where a_i and b_j are positive and real constants and PN_i and PD_j ($PN_i \neq PD_j$) are $\overline{\text{URC}}$-products. Then:

i) W can be reduced to one of the forms $a_i/b_j s_u$ or $a_i s_u/b_j$ for each u=1,2,...,k by a set of (k-1) limit operations if and only if there exists at least one pair PN_i and PD_j with their respective coefficients such that

$$\frac{a_i PN_i}{b_j PD_j} = \frac{a_i}{b_j} s_u \quad \text{or} \quad \frac{a_i}{b_j s_u} \tag{5.3.2}$$

ii) When a pair PN_i and PD_j exists which satisfies Equation 5.3.2, a set of (k-1) limit operations that gives $a_i s_u/b_j$ or $a_i/b_j s_u$ is defined by

$$L_W\ (x_1, x_2, \ldots x_{u-1}, x_{u+1}, \ldots, x_k)$$

where $s_i = \bar{s}_i$ if s_i is in <u>both</u> $\overline{\text{URC}}$-products PN_i and PD_j, and $x_i = \underline{s}_i$ if s_i is not in both $\overline{\text{URC}}$-products PN_i or PD_j for $i=1,2,\ldots u-1,u+1,\ldots,k$.

<u>Proof</u>: Assume there exists a pair of $\overline{\text{URC}}$-products PN_i and PD_j with their respective coefficients a_i and b_j in Equation 5.3.1 that satisfy Equation 5.3.2. A limit of (k-1) variables will now be shown to exist such that $L_W = a_i s_u/b_j$ or $a_i/b_j s_u$. Now let Y be the $\overline{\text{URC}}$-product of elements which are common to PN_i and PD_j ($PN_i = Y$ and $PD_j = Ys_u$ <u>or</u> $PN_i = Ys_u$ and $PD_j = Y$). Since some of the other $\overline{\text{URC}}$-products PN_q and PD_r in Equation 5.3.1 may have Y as a factor, then Equation 5.3.1 can be written in <u>one</u> of the two forms

$$W = \frac{a_i Y + \sum a_q\ PN'_q\ Y + \sum a_n\ Pn_n}{b_j Ys_u + \sum b_r\ PD'_r\ Y + \sum b_m\ PD_m} \qquad (5.3.3)$$

$$W = \frac{a_i Ys_u + \sum a_q\ PN'_q\ Y + \sum a_n\ PN_n}{b_j Y + \sum b_r\ PD'_r\ Y + \sum b_m\ PD_m}$$

where $PN_q = PN'_q\ Y$ and $PD_r = PD'_r\ Y$ and PN_n and PD_m are $\overline{\text{URC}}$-products not having Y as a factor in Equation 5.3.1. Now considering the unit of W as $Y \to \infty$, there are two possible cases corresponding to Equation 5.3.3 and 5.3.4.

$$W' = \lim_{Y \to \infty} W = \frac{a_i + \sum a_q\ PN'_q}{b_j s_u + \sum b_r\ PD'_r} \qquad (5.3.5)$$

$$W' = \lim_{Y} W = \frac{a_i s_u + a_q PN'_q}{b_j + b_r PD'_r} \quad (5.3.6)$$

Let Z be the set of elements which are not in the $\overline{URC}$-products PN_i and PD_j and take the limit of W' in Equations 5.3.5 and 5.3.6 as each of the s_i in Z go to zero. (Note that the set of elements in PN'_q and PD'_r is a subset of the set Z). Then W' in Equation 5.3.5 and 5.3.6 reduces to one of the forms indicated by W" below.

$$W'' = \frac{a_i}{b_j s_u} \quad (5.3.7)$$

$$W'' = \frac{a_i s_u}{b_j} \quad (5.3.8)$$

Therefore, it follows from Equations 5.3.5 - 5.3.8 that the limit L_W of (k-1) variables where $x_i = \bar{s}_i$ for each element s_i in the set Y and $x_j = \underline{s}_j$ for each element in the set Z is equal to either $a_i s_u/b_j$ (or $a_i/s_u b_j$). Thus, part (ii) of the theorem is proven (and the "sufficiency" in part (i)).

Now assume that a limit L_W of (k-1) variables is equal to either $a_i s_u/b_j$ (or $a_i/s_u b_j$). It will now be shown that there exists a pair of $\overline{URC}$-products PN_i and PD_j with their respective coefficients a_i and b_j such that they satisfy Equation 5.3.2. First consider Equation 5.3.1. Since each variable s_i is of degree one, W can be written in the form

$$W = W_o = \frac{A_o s_m + C_o}{B_o s_m + D_o} \quad (5.3.9)$$

where A_o, B_o, C_o, and D_o have at most (k-1) variables and are not functions of s_m. Now assume x_m is a variable corresponding to the first limit taken in the sequence of limits in the assumed limit L_W of (k-1) variables ($m \neq u$). Then note that $L_W(\bar{s}_m) = A_o/B_o$ and $L_W(\underline{s}_m) = C_o/D_o$ where A_o/B_o (C_o/D_o) must be defined, finite, and nonzero if $x_m = \bar{s}_m$ ($x_m = \underline{s}_m$) is the form of the variable x_m of the assumed limit L_W of (k-1) variables since L_W of (k-1) variables is defined, finite and nonzero. Now $W_1 = A_o/B_o$ (C_o/D_o) can be written in the form of Equation 5.3.9 as

$$W_1 = \frac{A_o}{B_o} \left(\text{or } \frac{C_o}{D_o}\right) = \frac{A_1 s_r + C_1}{B_1 s_r + D_1} \tag{5.3.10}$$

where $r \neq m$ and $r \neq u$ and A_1, B_1, C_1, and D_1 are a function of at most (k-2) variables. Now since the assumed limit L_W is defined, finite, and nonzero at each step of the sequence of limits it represents, the process can be repeated until the function W_{k-1} is given as

$$W_{k-1} = \frac{a_i s_u}{b_j} \quad \text{or} \quad \frac{a_i}{b_j s_u} \tag{5.3.11}$$

where the assumed limit of (k-1) variables L_W is equal to W_{k-1}.

In the above discussion, it was shown that the form given in Equation 5.3.9 can be used in each step to obtain W_{k-1} from $W = W_o$ and that W_{k-1} is equivalent to the assumed limit of L_W of (k-1) variables. It will now be shown that the process can be reversed and the <u>form</u> of each W_i ($i=0,1,2,\ldots,k-1$) can be reconstructed starting from the assumed limit L_W of (k-1) variables.

First consider the function W_{k-1} which can be written in <u>one</u> of the two forms

$$W_{k-1} = \frac{a_i s_u}{b_j} = \frac{E_{k-1} + G_{k-1}}{F_{k-1} + H_{k-1}} \tag{5.3.12}$$

$$W_{k-1} = \frac{a_i}{b_j s_u} = \frac{E_{k-1} + G_{k-1}}{F_{k-1} + H_{k-1}} \quad (5.3.13)$$

where $E_{k-1} = a_i s_u$ (or a_i), $F_{k-1} = b_j$ (or $b_j s_u$), $G_{k-1} = H_{k-1} = 0$ (Note that $E_{k-1}/F_{k-1} = a_i s_u/b_j$ or $a_i/s_u b_j$). Then let x_r, $r \neq u$, correspond to the last limit taken in the sequence of limits L_W or (k-1) variables. Then if $x_r = \bar{s}_r$, W_{k-2} where $Wk-1 = \lim_{s_r \to \infty} W_{k-2}$ can be written

$$W_{k-2} = \frac{A_{k-2}s_r + C_{k-2}}{B_{k-2}s_r + D_{k-2}} = \frac{(E_{k-1} + G_{k-1})s_r + G_{k-2}}{(F_{k-1} + H_{k-1})s_r \quad H_{k-2}} = \frac{E_{k-2} + G_{k-2}}{F_{k-2} + H_{k-2}} \quad (5.3.14)$$

where $A_{k-2} = E_{k-1} + G_{k-1}$, $B_{k-2} = F_{k-1} + H_{k-1}$, $C_{k-2} = G_{k-2}$, $D_{k-2} = H_{k-2}$, $E_{k-2} = E_{k-1} \cdot s_r$, and $F_{k-2} = F_{k-1} \cdot s_r$. Further G_{k-2} and H_{k-2} are not functions of s_r and may be zero. If $x_r = \underline{s}_r$, W_{k-2} where $W_{k-1} = \lim_{s_r \to 0} W_{k-2}$ can be written in one of the forms given below

$$W_{k-2} = \frac{A_{k-2}s_r + C_{k-2}}{B_{1-2}s_r + D_{k-2}} = \frac{A_{k-2}s_r + E_{k-1} + G_{k-1}}{B_{k-2}s_r + F_{k-1} + H_{k-1}} = \frac{E_{k-2} + G_{k-2}}{F_{K-2} + H_{K-2}} \quad (5.3.15)$$

where A_{k-2} and B_{k-2} are equal to the sum of $\overline{\text{URC}}$-products, $C_{k-2} = E_{k-1} + G_{k-1}$, $D_{k-2} = F_{k-1} + H_{k-1}$, $E_{k-2} = E_{k-1}$, $F_{k-2} = F_{k-1}$, $G_{k-2} = A_{k-2}s_r + G_{k-1}$ and $H_{k-2} = B_{k-2}s_r + H_{k-1}$

$$W_{k-2} = \frac{A_{k-2}s_r + C_{k-2}}{B_{k-2}s_r + D_{k-2}} = \frac{(E_{k-1} + G_{k-1})s_r + G_{k-2}}{(F_{k-1} + H_{k-1})s_r + H_{k-2}} = \frac{E_{k-2} + G_{k-2}}{F_{k-2} + H_{k-2}} \quad (5.3.16)$$

where $A_{k-2} = E_{k-1} + G_{k-1}$, $B_{k-2} = F_{k-1} + H_{k-1}$, $C_{k-2} = G_{k-2} = 0$, $D_{k-2} = H_{k-2} = 0$, $E_{k-2} = E_{k-1} \cdot s_r$, and $F_{k-2} = F_{k-1} \cdot s_r$. Note that E_{k-1}, G_{k-1}, F_{k-1}, and H_{k-1} are not functions of s_r and therefore

$\lim_{s_r \to o} W_{k-2} = E_{k-2}/F_{k-2}$. Also note that E_{k-2} and F_{k-2} of Equation 5.3.14-5.3.16 are such that $E_{k-2}/F_{k-2} = a_i s_u/b_j$ (or $a_i/s_u b_j$). Since Equation 5.3.12 has the same form as Equations 5.3.14 - 5.3.16, the process can be repeated and every W_i, i=0,1,2,...,k-1 has the form

$$W_i = \frac{E_i + G_i}{F_i + H_i} \tag{5.3.17}$$

where E_i and F_i are $\overline{\text{URC}}$-products (with coefficients a_i and b_j) such that $E_i/F_i = a_i s_u/b_j$ (or $a_i/b_j s_u$), and G_i and H_i are equal to the sum of $\overline{\text{URC}}$-products. Since $W = W_o$, the "necessity" of part (i) is proven.

To simplify subsequent work the following definition is given.

Definition 5.3.1 $\overline{\text{URC}}$-Product Ratio P_{ij}: If W is a function of the form given in Equation 5.3.1, a $\overline{\text{URC}}$ product ratio P_{ij} will be defined to be the ratio $P_{ij} = a_i PN_i/b_j PD_j$ where PN_i and PD_j are $\overline{\text{URC}}$-products of W with coefficients a_i and b_j respectively.

Theorem 5.3.1 will now be illustrated by following examples.

Example 5.3.1: Consider the function W which has the form of Equation 5.3.1.

$$W = \frac{a_1 s_1}{b_1 s_1 s_2 + b_2} = \frac{a_1\ PN_1}{b_1\ PD_1 + b_2\ PD_2} \tag{5.3.18}$$

Then the $\overline{\text{URC}}$-product ratios satisfying part (i) of Theorem 5.3.1 are

$$P_{11} = \frac{a_1\ PN_1}{b_1\ PD_2} = \frac{a_1 s_1}{b_1 s_1 s_2} = \frac{a_1}{b_1 s_2} \tag{5.3.19}$$

$$P_{12} = \frac{a_1\ PN_1}{b_2\ PD_2} = \frac{a_1 s_1}{b_2} \tag{5.3.20}$$

Part (ii) of Theorem 5.3.1 gives the limits which correspond to Equations 5.3.19 and 5.3.20 as

$$L_W\,(\bar{s}_1) = \frac{a_1}{b_1 s_2} \tag{5.3.21}$$

$$L_W\,(\underline{s}_2) = \frac{a_1 s_1}{b_2} \tag{5.3.22}$$

respectively.

<u>Example 5.3.2:</u> Consider the function W which has the form of Equation 5.3.1.

$$W = \frac{a_1\overbrace{s_1s_2s_3s_4}^{PN_1} + a_2\overbrace{s_1s_2}^{PN_2} + a_3\overbrace{s_2s_3}^{PN_3} + a_4\overbrace{s_3s_4}^{PN_4} + a_5\overbrace{\;}^{PN_5}}{b_1\underbrace{s_1s_2s_3}_{PD_1} + b_2\underbrace{s_1s_3s_4}_{PD_2} + b_3\underbrace{s_1}_{PD_3}} \tag{5.3.23}$$

The $\overline{\text{URC}}$-product ratios that satisfy part (i) of Theorem 5.3.1 are given with their corresponding limits L_W of (k-1) variables in Table 5.3.1. Note that the $\overline{\text{URC}}$-product ratios that correspond to limits L_W of (k-1) variables which are equal to zero or infinity are not shown.

<u>Lemma 5.3.1 $\overline{\text{URC}}$-Product Ratio Rule for $\overline{\text{URC}}$MVDPFs:</u> Let W be the realizable $\overline{\text{URC}}$MVDPF of k elements

$$W = \frac{\sum a_i PN_i}{\sum b_j PD_j}$$

where a_i and b_j are positive constants and PN_i and PD_j ($PN_i \neq PD_j$) are $\overline{\text{URC}}$-products. Then there exists at least one $\overline{\text{URC}}$-product ratio such that

$$P_{ij} = \frac{a_i PN_i}{b_j PD_j} = \frac{a_i s_u}{b_j} \text{ or } \frac{a_i}{b_j s_u}$$

for every u, u=1,2,...,k. Further, if there are more than one such P_{ij} giving $a_i/b_j s_u$ or $a_i s_u/b_j$ for a particular u, then each of these P_{ij} gives the same function.

<u>Proof</u>: Earlier it was established in Section 4.4 that a realizable $\overline{\text{URC}}$MVDPF can be reduced to the DPF of each element e_i by some set of (k-1) limit operations.

If there are more than one such set of limits for e_i, each set must give the same DPF for e_i. Therefore, the lemma follows directly from Theorem 5.3.1.

The following example illustrates Lemma 5.3.1.

<u>Example 5.3.3:</u> The $\overline{\text{URC}}$MVDPF Y given below is known to be realizable for some set of positive constants: a_1, a_2, a_3, b_1, and b_2.

$$Y = \frac{a_1 s_2 s_3 + a_2 s_1 s_3 + a_3}{b_1 s_1 s_2 s_3 + b_2 s_1}$$

Consider the set of all possible $\overline{\text{URC}}$-product ratios which give a function of one variable.

$$P_{11} = \frac{a_1 s_2 s_3}{b_1 s_1 s_2 s_3} = \frac{a_1}{b_1 s_1}$$

$$P_{21} = \frac{a_2 s_1 s_3}{b_1 s_1 s_2 s_3} = \frac{a_2}{b_1 s_2}$$

$$P_{22} = \frac{a_2 s_1 s_3}{b_2 s_1} = \frac{a_2 s_3}{b_2}$$

$$P_{32} = \frac{a_3}{b_2 s_2} = \frac{a_3}{b_2 s_2}$$

TABLE 5.3.1

$\overline{\text{URC}}$-PRODUCT RATIOS FOR W

$\overline{\text{URC}}$-PRODUCT RATIO P_{ij}	LIMIT OPERATION L_W
$P_{31} = \frac{a_3 s_2 s_3}{b_1 s_1 s_2 s_3} = \frac{a_3}{b_1 s_1}$	$L_W (\bar{s}_2, \bar{s}_3, \underline{s}_4) = \frac{a_3}{b_1 s_1}$
$P_{41} = \frac{a_4 s_3 s_4}{b_2 s_1 s_3 s_4} = \frac{a_4}{b_2 s_1}$	$L_W (\bar{s}_3, \bar{s}_4, \underline{s}_2) = \frac{a_4}{b_2 s_1}$
$P_{53} = \frac{a_5}{b_3 s_1}$	$L_W (\underline{s}_2, \underline{s}_3, \underline{s}_4) = \frac{a_5}{b_3 s_1}$
$P_{12} = \frac{a_1 s_1 s_2 s_3 s_4}{b_2 s_1 s_3 s_4} = \frac{a_1 s_2}{b_2}$	$L_W (\bar{s}_1, \bar{s}_3, \bar{s}_4) = \frac{a_1 s_2}{b_2}$
$P_{23} = \frac{a_2 s_1 s_2}{b_3 s_1} = \frac{a_2 s_2}{b_3}$	$L_W (\bar{s}_1, \underline{s}_3, \underline{s}_4) = \frac{a_2 s_2}{b_3}$
$P_{21} = \frac{a_2 s_1 s_2}{b_1 s_1 s_2 s_3} = \frac{a_2}{b_1 s_3}$	$L_W (\bar{s}_1, \bar{s}_2, \underline{s}_4) = \frac{a_2}{b_1 s_3}$
$P_{11} = \frac{a_1 s_1 s_2 s_3 s_4}{b_1 s_1 s_2 s_3} = \frac{a_1 s_4}{b_1}$	$L_W (\bar{s}_1, \bar{s}_2, \bar{s}_3) = \frac{a_1 s_4}{b_1}$

Note that if $a_3/b_2 = a_2/b_2$ the equation satisfies Lemma 5.3.1.

Theorem 5.3.1 can also be used efficiently in finding the circuits in a graph corresponding to a realizable $\overline{\text{URC}}$MVDPF and therefore will form the basis for a new synthesis procedure. Theorem 5.3.1 gives the necessary and sufficient conditions that a function W, having the form of a $\overline{\text{URC}}$MVDPF, be reducible to the form $L_i s_i$ or $1/C_i s_i$ (where L_i and C_i are positive constants). Now let W be a realizable $\overline{\text{URC}}$MVDPF. Then the theorem also gives a direct way to determine sets of (k-1) limit operations that give the DPF of each element e_i. Then since the type of element ($L_i s_i$ or $1/C_i s_i$) for each element is known, each of the sets of (k-1) limit operations can be related to a corresponding set S_{ij} of (k-1) elements which are either open-circuited or short-circuited, where i denotes the element whose DPF is given and j denotes one such set (see Section 4.4). Now the element e_i whose DPF is obtained from W and some but not all of the short-circuited elements of S_{ij} (for some j) form a circuit with the driver (see Section 4.4) in the graph corresponding to W. The unnecessary shorts in S_{ij} occur because Theorem 5.3.1 does not take into account any cancellations of variables. The cancellations can be detected when the set of (k-1) limit operations given by the theorem are computed in the conventional way (not using Theorem 5.3.1). To further explain this problem, consider Theorem 4.3.2 which states that a $\overline{\text{URC}}$MVDPF W' is reducible if and only if the graph corresponding to W' is a separable graph where the element corresponding to each canceled variable cannot form a circuit including the driver. Let W' be a DPF--which is a function of at least two variables--obtained from W when less than (k-1) limits are taken (the limits of W correspond to open-circuiting or short-circuiting some elements of the graph). Let e_q be an element in the component part (see Appendix B) of the grapn G' (corresponding to W') not containing the driver. Then the limit of W' as s_q goes to zero is equal to the limit of W' as s_q goes to infinity (e_q can be open-circuited or short-circuited with no effect on the DPF W'). Thus, it is clear that Theorem 5.3.1 cannot be used to find the circuits unless a procedure is found to eliminate the unnecessary short circuits that might be given in each S_{ij}. This problem can be completely solved without resorting to finding all $2^{(k-1)}$ sets of (k-1) limit operations for a given $\overline{\text{URC}}$MVDPF W with k elements as was done in Section 4.4.

First let W be a given realizable $\overline{\text{URC}}$MVDPF with k elements. W has sets of (k-1) limit operations giving the DPF of each e_i which are given by Theorem 5.3.1, and these sets correspond to the sets S_{ij}. Since W is a realizable function there is a corresponding topological formula $Y = \Delta/\Delta_{11}$ of k variables where Δ and Δ_{11} are defined in Appendix B. Note that the multivariable function Y has the form of Equation 5.3.1, and Theorem 5.3.1 can be applied to find the sets of (k-1) limit operations giving the DPF of each y_i. These sets correspond to the <u>same</u> sets S_{ij} given by W. W and Y have the same sets S_{ij} because there is a one-to-one relationship between W and Y (see Section 4.3). Now let T_1 be a tree of graph G (note that the driver is not included in G), and let Q be the set of 2-trees in Δ_{11} which can be found from T_1 be deleting one element at a time from the tree such that the input vertices of G are in different component parts of the graph corresponding to tree T_1. Then if y_i is an element in T_1 and is given as the DPF by a $\overline{\text{URC}}$-product ratio PN_1/PD_j satisfying Theorem 5.3.1 (where PN_1 is equal to the tree product for T_1), then y_i corresponds to the element deleted from T_1 to give the 2-tree product PD_j. Also if y_j is <u>not</u> in tree T_1, then for each $\overline{URC}$-product ratio $PN_1/PD_q = y_i$, $i \neq j$, $q=1,2,\ldots,n$ where n is the number of 2-trees in Q, y_j must be open-circuited (limit as $y_j \to o$) to give the DPF of a single element y_i. If the elements of G not in tree T_1 are open-circuited, then the result is a graph $G' \equiv T_1$. Let G" be the graph consisting of G' and the driver. There is only one circuit in G" and it contains the driver and some of the branches of the tree T_1. Since all branches of T_1 may not be included in the circuit, G" may be a separable graph and each element in the component part of G" containing the driver is a circuit element. Now consider the set of $\overline{URC}$-product ratios $P = PN_1/PD_q$, $q=1,2,\ldots,n$. There exists a 2-tree product PD_q from the set Q that does not have element y_i if y_i is circuit element and therefore the DPF of each circuit element y_i is given by Theorem 5.3.1. Let T_1' consist of the set of elements which is a subset of the elements in T_1 such that the elements are not the circuit elements in G". Then the elements of T_1'' are in the component of G" not containing the driver. Now each element of the set T_1' is an element of PN_1 and an element of every 2-tree product corresponding to the set Q. Therefore, every $\overline{URC}$-product ratio of the set P gives the DPF of some element of the circuit and a set of unnecessary shorts corresponding to all the elements of T_1'. Then the set of short circuits and open circuits S_{ij} found by Theorem 5.3.1 for G and giving the DPF of each element of the circuit can be written in matrix form as

$$F_{T_1} = \begin{array}{c} \begin{array}{cccccc|cccc|ccc} d & 1 & 2 & 3 & \ldots & n & n+1 & n+2 & \ldots & n+m & n+n+1 & \ldots & n+n+k \end{array} \\ \left[\begin{array}{cccccc|cccc|ccc} 1 & (1) & 1 & & \ldots & 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 1 & 1 & (1) & & \ldots & 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & & \ldots & 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ & & \vdots & & & & & & \vdots & & & \vdots & \\ 1 & 1 & 1 & & \ldots & (1) & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \end{array}\right] \end{array} \qquad (5.3.24)$$

where columns d, 1, 2,...,n correspond to the elements in the circuit for T_1, the parenthesis identify the element y_i given as the DPF for a $\overline{\text{URC}}$-product ratio columns, n+1, n+2,...,n+m are the m elements of the set T_1', n+m+1, n+m+2...,n+m+k are the elements of G which are not in the tree T_1, and each row corresponds to a $\overline{\text{URC}}$-product ratio of the set P. A set of equations similar to Equation 5.3.24 can be found for every tree T_i and its corresponding circuit formed when the driver is added to the input vertices of graph G. Therefore, if all the sets S_{ij} given by Theorem 5.3.1 for the topological formula $Y = \Delta/\Delta_{11}$ are written in matrix form (like Equation 5.3.24), the sets S_{ij} for each tree can be found. Then since the form of the matrix corresponding to the sets S_{ij} is known, all unnecessary shorts can be identified as corresponding to columns (n+1), (n+2),...,(n+m) of Equation 5.3.24.

<u>Example 5.3.4:</u> Consider the realizable topological formula

$$Y = \frac{\overbrace{y_1y_3y_5}^{T_1}+\overbrace{y_1y_3y_4}^{T_2}+\overbrace{y_1y_2y_5}^{T_3}+\overbrace{y_1y_2y_4}^{T_4}+\overbrace{y_2y_3y_5}^{T_5}+\overbrace{y_2y_3y_4}^{T_6}+\overbrace{y_3y_4y_5}^{T_7}+\overbrace{y_2y_4y_5}^{T_8}}{y_3y_5+y_3y_4+y_2y_5+y_2y_4}$$

where T_i identifies the set of elements in each tree. Then Theorem 5.3.1 can be used to find the set of $\overline{\text{URC}}$-product ratios that give the DPF of each y_i. These URC-product ratios are

$$\frac{y_1y_3y_5}{y_3y_5} = y_1$$

$$\frac{y_1y_3y_4}{y_3y_4} = y_1$$

$$\frac{y_1 y_2 y_5}{y_2 y_5} = y_1$$

$$\frac{y_1 y_2 y_4}{y_2 y_4} = y_1$$

$$\frac{y_2 y_3 y_5}{y_3 y_5} = y_2$$

$$\frac{y_2 y_3 y_5}{y_2 y_5} = y_3$$

$$\frac{y_2 y_3 y_4}{y_3 y_4} = y_2$$

$$\frac{y_2 y_3 y_4}{y_2 y_4} = y_3$$

$$\frac{y_3 y_4 y_5}{y_3 y_5} = y_4$$

$$\frac{y_3 y_4 y_5}{y_3 y_4} = y_5$$

$$\frac{y_2 y_4 y_5}{y_2 y_5} = y_4$$

$$\frac{y_2 y_4 y_5}{y_2 y_4} = y_5$$

These $\overline{\text{URC}}$-product ratios give the sets of short circuits and open circuits S_{ij} which give the DPF of each element y_i, and are expressed in the matrix given previously as

$$
S = \begin{array}{c|cccccc|c}
 & d & 1 & 2 & 3 & 4 & 5 & \\
\hline
T_1 & 1 & (1) & 0 & \boxed{1} & 0 & \boxed{1} & 1 \\
\hline
T_2 & 1 & (1) & 0 & \boxed{1} & \boxed{1} & 0 & 2 \\
\hline
T_3 & 1 & (1) & \boxed{1} & 0 & 0 & \boxed{1} & 3 \\
\hline
T_4 & 1 & (1) & \boxed{1} & 0 & \boxed{1} & 0 & 4 \\
\hline
T_5 & 1 & 0 & (1) & 1 & 0 & \boxed{1} & 5 \\
 & 1 & 0 & 1 & (1) & 0 & \boxed{1} & 6 \\
\hline
T_6 & 1 & 0 & (1) & 1 & \boxed{1} & 0 & 7 \\
 & 1 & 0 & 1 & (1) & \boxed{1} & 0 & 8 \\
\hline
T_7 & 1 & 0 & 0 & \boxed{1} & (1) & 1 & 9 \\
 & 1 & 0 & 0 & \boxed{1} & 1 & (1) & 10 \\
\hline
T_8 & 1 & 0 & \boxed{1} & 0 & (1) & 1 & 11 \\
 & 1 & 0 & \boxed{1} & 0 & 1 & (1) & 12 \\
\hline
\end{array}
\qquad (5.3.25)
$$

Now the unnecessary shorts (circled by dotted lines) can be eliminated from the above matrix and be written as the matrix of circuits

$$
C = \begin{array}{c|cccccc|c}
 & d & 1 & 2 & 3 & 4 & 5 & \\
\hline
T_1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
T_2 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\
\hline
T_3 & 1 & 1 & 0 & 0 & 0 & 0 & 3 \\
\hline
T_4 & 1 & 1 & 0 & 0 & 0 & 0 & 4 \\
\hline
T_5 & 1 & 0 & 1 & 1 & 0 & 0 & 5 \\
 & 1 & 0 & 1 & 1 & 0 & 0 & 6 \\
\hline
T_6 & 1 & 0 & 1 & 1 & 0 & 0 & 7 \\
 & 1 & 0 & 1 & 1 & 0 & 0 & 8 \\
\hline
T_7 & 1 & 0 & 0 & 0 & 1 & 1 & 9 \\
 & 1 & 0 & 0 & 0 & 1 & 1 & 10 \\
\hline
T_8 & 1 & 0 & 0 & 0 & 1 & 1 & 11 \\
 & 1 & 0 & 0 & 0 & 1 & 1 & 12 \\
\hline
\end{array}
\qquad (5.3.26)
$$

The rows of the above matrix which are identical to some other row can be eliminated to give the matrix of circuits F_c defined in Section 4.4.

$$F_c = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 4 & 3 & 4 & d \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \qquad (5.3.27)$$

Note that F_c has full rank and therefore the circuits are an independent set of circuits. Further, F_c has the form of the fundamental c-circuit matrix B_c and therefore the non-oriented graph corresponding to B_c can be found (see Section 4.4). The graph corresponding to $B_c = F_c$ of Equation 5.3.27 is given in Figure 5.3.1 where v_1 and v_1' are the input vertices and the admittance for each element e_i is given by y_i.

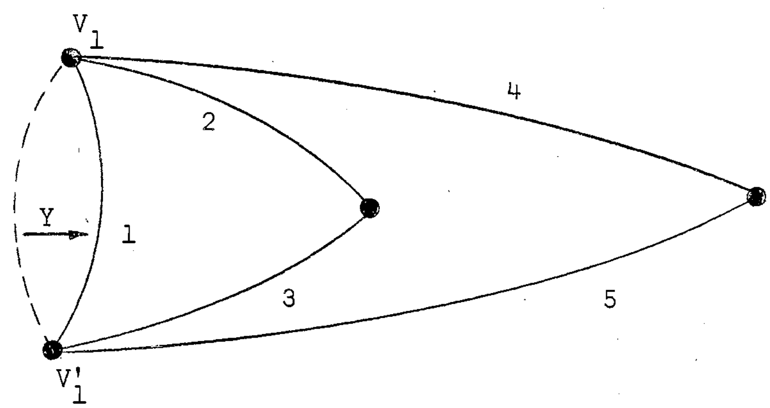


Figure 5.3.1. Realization for Y

In the development of this synthesis method, a given $\overline{\text{URC}}$MVDPF W of k variables is assumed to be realizable and the topological formula $Y = \Delta/\Delta_{11}$ corresponding to W is found. Since W and Y have the same sets S_{ij} giving the DPF of each element, the method also applies to any realizable $\overline{\text{URC}}$MVDPF. In the following example a $\overline{\text{URC}}$MVDPF is realized by this method.

<u>Example 5.3.5:</u> Let Z be the realizable $\overline{\text{URC}}$MVDPF (impedance) given as

$$Z = \frac{a_1s_1s_2s_4s_5+a_2s_1s_2s_3s_5+a_3s_1s_4+a_4s_1s_5+a_5s_1s_3+a_6s_2s_5+a_7}{b_1s_1s_2s_3s_4s_5+b_2s_1s_3s_4+b_3s_1s_3s_5+b_4s_2s_4s_5+b_5s_4+b_6s_5}$$

The $\overline{\text{URC}}$-product ratios are

$$P_{11} = \frac{a_1s_1s_2s_4s_5}{b_1s_1s_2s_3s_4s_5} = \frac{a_1}{b_1s_3}$$

$$P_{21} = \frac{a_2s_1s_2s_3s_5}{b_1s_1s_2s_3s_4s_5} = \frac{a_2}{b_1s_4}$$

$$P_{32} = \frac{a_3s_1s_4}{b_2s_1s_3s_4} = \frac{a_3}{b_2s_3}$$

$$P_{52} = \frac{a_5s_1s_3}{b_2s_1s_3s_4} = \frac{a_5}{b_2s_4}$$

$$P_{23} = \frac{a_2s_1s_2s_3s_5}{b_3s_1s_3s_5} = \frac{a_2s_2}{b_3}$$

$$P_{43} = \frac{a_4s_1s_5}{b_3s_1s_3s_5} = \frac{a_4}{b_3s_3}$$

$$P_{53} = \frac{a_5s_1s_3}{b_3s_1s_3s_5} = \frac{a_5}{b_3s_5}$$

$$P_{14} = \frac{a_1 s_1 s_2 s_4 s_5}{b_4 s_2 s_4 s_5} = \frac{a_1 s_1}{b_4}$$

$$P_{64} = \frac{a_6 s_2 s_5}{b_4 s_2 s_4 s_5} = \frac{a_6}{b_4 s_4}$$

$$P_{35} = \frac{a_3 s_1 s_4}{b_5 s_4} = \frac{a_3 s_1}{b_5}$$

$$P_{75} = \frac{a_7}{b_5 s_4} = \frac{a_7}{b_5 s_4}$$

$$P_{46} = \frac{a_4 s_1 s_5}{b_6 s_5} = \frac{a_4 s_1}{b_6}$$

$$P_{66} = \frac{a_6 s_2 s_5}{b_6 s_5} = \frac{a_6 s_2}{b_6}$$

$$P_{76} = \frac{a_7}{b_6 s_5} = \frac{a_7}{b_6 s_5}$$

Note that Lemma 5.3.1 implies the impedance of each element must satisfy the following conditions.

$$L_1 s_1 = \frac{a_1}{b_4} s_1 = \frac{a_4}{b_6} s_1 = \frac{a_3}{b_5} s_1$$

$$L_2 s_2 = \frac{a_2}{b_3} s_2 = \frac{a_6}{b_6} s_2$$

$$\frac{1}{C_3 s_3} = \frac{a_1}{b_1 s_3} = \frac{a_3}{b_2 s_3} = \frac{a_4}{b_3 s_3}$$

$$\frac{1}{C_4 s_4} = \frac{a_2}{b_1 s_4} = \frac{a_5}{b_2 s_4} = \frac{a_6}{b_4 s_4} = \frac{a_7}{b_5 s_4}$$

$$\frac{1}{C_5 s_5} = \frac{a_5}{b_3 s_5} = \frac{a_7}{b_6 s_5}$$

Then the sets S_{ij} given by Theorem 5.3.1 are

$$S = \begin{array}{c} \\ P_{11} \rightarrow \\ P_{21} \rightarrow \\ P_{32} \rightarrow \\ P_{52} \rightarrow \\ P_{23} \rightarrow \\ P_{43} \rightarrow \\ P_{53} \rightarrow \\ P_{14} \rightarrow \\ P_{64} \rightarrow \\ P_{35} \rightarrow \\ P_{75} \rightarrow \\ P_{46} \rightarrow \\ P_{66} \rightarrow \\ P_{76} \rightarrow \end{array} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & d \\ \hline 0 & 0 & (1) & 1 & 1 & 1 \\ 0 & 0 & 1 & (1) & 1 & 1 \\ \hline 0 & 1 & (1) & 1 & 0 & 1 \\ 0 & 1 & 1 & (1) & 0 & 1 \\ \hline 0 & (1) & 1 & 0 & 1 & 1 \\ 0 & 1 & (1) & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & (1) & 1 \\ \hline (1) & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & (1) & 1 & 1 \\ \hline (1) & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & (1) & 0 & 1 \\ \hline (1) & 1 & 0 & 0 & 1 & 1 \\ 1 & (1) & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & (1) & 1 \end{array} \qquad (5.3.28)$$

Eliminating the duplicate row and unnecessary short circuits the matrix S reduces to

$$F_c = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & d \\ \hline 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{array} \qquad (5.3.29)$$

There are four vertices in the graph G corresponding to Z as can be seen by finding the number of elements in one tree product of the topological formula $Y = \Delta/\Delta_{11}$ corresponding to Z (see Theorem B.2.4). Then there are e-v+1 = 3 indipendent circuits in graph G where e equals the number of variables in Z (corresponding to the element e_i) plus one (corresponding to the driver) and v equals the number of vertices. It has been established (see Theorem 4.4.2) that F_c has e-v+1=3 independent rows. The three independent rows in F_c can be found by finding a submatrix B_1 of order three from F_c such that the determinant of B_1 (mod 2) is not zero. Consider the determinant of the submatrix B_1 of F_c given as

$$|B_1| = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \end{array} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

Then the adjoint matrix of B_1 (mod 2) can be defined to be

$$\text{ADJ } B_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then $B^{-1} = \text{ADJ } B/_1|B_1| = \text{ADJ } B_1$. Now consider the matrix B which results when row four is deleted from the matrix F_c. B can be partitioned as

$$B = \left[B_1 \,\vdots\, B_2\right]$$

Then the c-circuit matrix can be obtained from B be premultiplying B by B_1^{-1} and is given below.

$$B_c = B_1^{-1} [B_1 \,\vdots\, B_2] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & d \end{array} \\ \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \end{array}$$

and

$$B_c = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & d \end{matrix} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

Now B_c can be realized as the graph G shown in Figure 5.3.2a and the network corresponding Z is shown in Figure 5.3.2b.

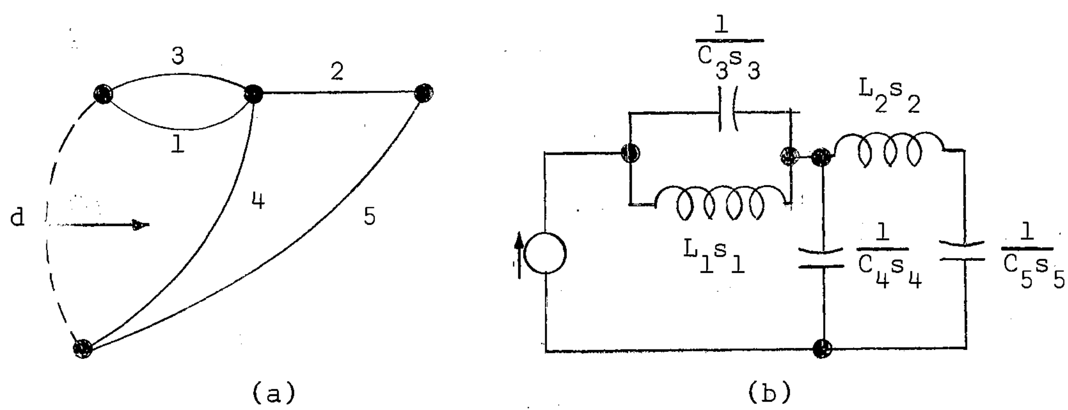


Figure 5.3.2. Realization for Z.

In the development of the synthesis method of this section the $\overline{\text{URC}}$MVDPF is assumed to be realizable. It should be emphasized that if a multivariable function W has the form of a $\overline{\text{URC}}$MVDPF but is not known to be realizable then the synthesis method itself can be used as a test for realizability. If no graph can be found (by the synthesis method) that corresponds to the multivariable function W, then it is evident that W is not realizable. There is also another important point which has not been emphasized. It has been shown that a $\overline{\text{URC}}$MVDPF exists for <u>every</u> $\overline{\text{URC}}$ network with elements having different RC products. It can be recalled from Chapter IV that when the RC products for any two elements of the same type (Z_{oc} element or Z_{sc} element) are equal, the RC products are <u>assumed</u> to be different so that a new variable s_i is introduced by the transformations (see Equation 4.2.9) for <u>every</u> element. The result is the $\overline{\text{URC}}$MVDPF for a given network of $\overline{\text{URC}}$ elements. Now consider the case where there are elements of the same type having the same RC product in the $\overline{\text{URC}}$ network. If the same transformation is used for each of these elements ($s_i = \tanh a_i p$), then the result is a multivariable function not having the form of the $\overline{\text{URC}}$MVDPF. In particular each of the variables s_i may not be of order one. The synthesis procedure in this section is not applicable to this type of multivariable function and appears to be an excellent topic for further research. Finally, note that the synthesis method applies to a $\overline{\text{URC}}$MVDPF which corresponds to either an admittance or an impedance (see Example 5.3.4 and Example 5.3.5).

<u>5.4 Ladder Synthesis and Reduction of a Class of $\overline{\text{URC}}$MVDPFs</u>. The method developed in Section 4.5 can be used to realize any realizable $\overline{\text{URC}}$MVDPF. In this section a simple method of reducing a given $\overline{\text{URC}}$MVDPF to a simpler function will be given for a certain class of $\overline{\text{URC}}$MVDPFs.

Assume that a realizable $\overline{\text{URC}}$MVDPF Z is given and that the type of each element has been found from the set of $\overline{\text{URC}}$-product ratios. Now assume that a limit is taken of Z with respect to some variable s_i ($s_i \to 0$ or $s_i \to \infty$). When the limit corresponds to <u>open-circuiting</u> the element e_i in the network N corresponding to Z and further, the limit of Z is equal to infinity (open-circuit), then it is evident that the network can be drawn in the form given in Figure 5.4.1.

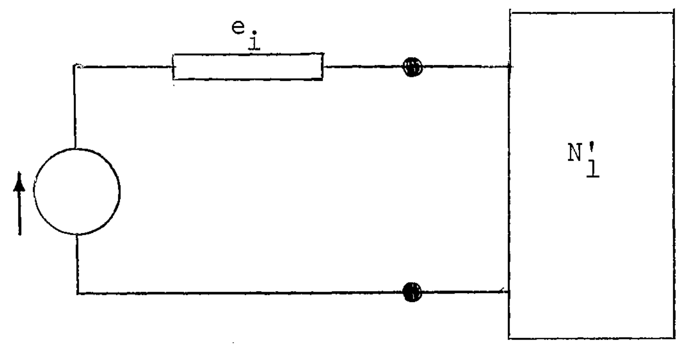


Figure 5.4.1. Network N

where N_1' is the network corresponding to Z_1' given when the limit is taken of Z with respect to the variable s_i such that the limit corresponds to <u>short-circuiting</u> e_i. Since N_1' corresponds to a realizable $\overline{\text{URC}}$MVDPF Z_1' with (k-1) elements, the process can be repeated if there exists an element e_j satisfying the conditions given for e_i. Therefore, in general the network N has the form given in Figure 5.4.2.

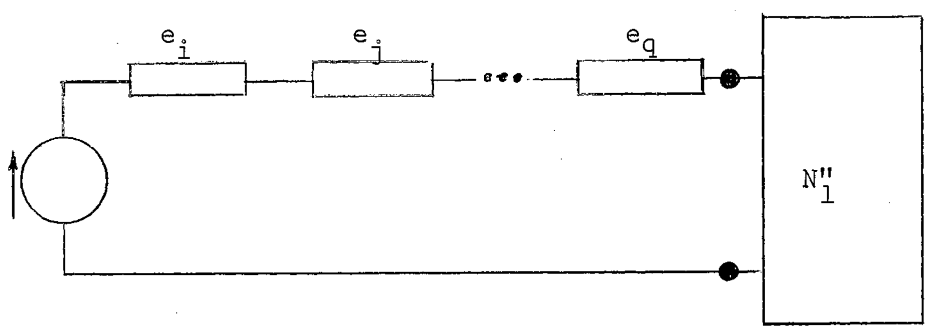


Figure 5.4.2. Network N

where N''_1 is the network corresponding to N when the elements $e_i, e_j, \ldots, e_q$ are short-circuited. Now assume that the limit is taken of Z''_1 ($\overline{URC}$MVDPF corresponding to N''_1) with respect to some variable s_u ($u \neq i,j,\ldots,q$; indicated in Figure 5.4.2) ($s_u \to o$ or $s_u \to \infty$) such that the limit correspoind to <u>short-circuiting</u> e_u in the network N''_1, and the limit of Z''_1 goes to zero. Then the network N''_1 must have the form given in Figure 5.4.3. Note that unlike classical ladder synthesis e_u of Figure 5.4.3 corresponds to an impedance.

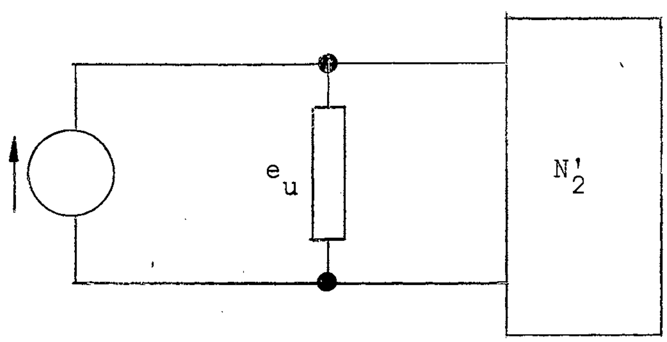


Figure 5.4.3. Network N''_1

Then if the assumed conditions are met in each step, then in general N''_1 has the form of Figure 5.4.4. Note unlike classical ladder synthesis e_i, $i=i',j',\ldots,q'$ all correspond to impedances in Figure 5.4.4. Therefore, in general a realizable $\overline{URC}$MVDPF W has a network N of the form given in Figure 5.4.5, where NS_i and NP_i are defined in Figure 5.4.6a and Figure 5.4.6b and N' is a network with less than k elements.

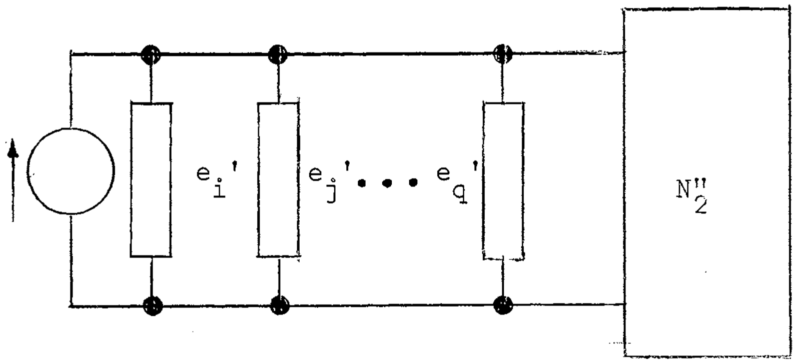


Figure 5.4.4. Network N_1''

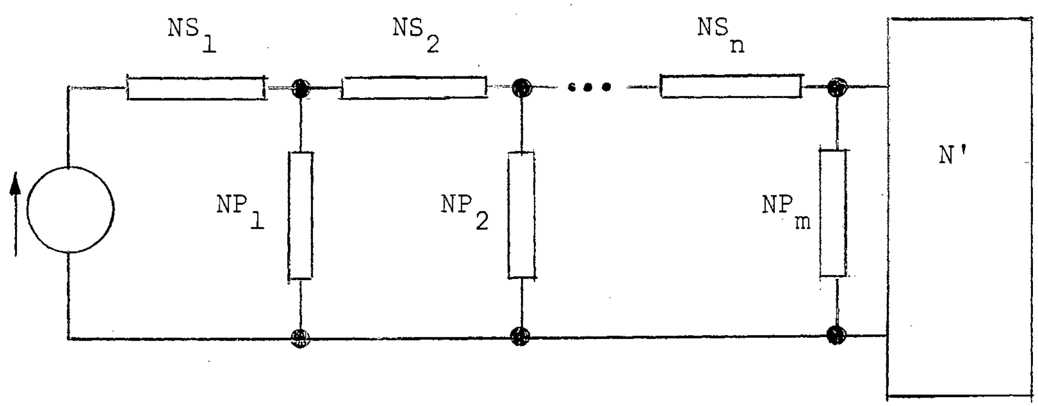


Figure 5.4.5. Network N

The conditions necessary for a given $\overline{\text{URC}}$MVDPF Z to have a network of the form given in Figure 5.4.5 can be checked by inspection. If no ladder elements can be obtained (conditions for removal are not met), then m = n = 0 (see Figure 5.4.5) and the given $\overline{\text{URC}}$MVDPF W corresponds to network N' (Figure 5.4.5). It is important to note that the $\overline{\text{URC}}$MVDPF corresponding to N' of Figure 5.4.5 is always realizable.

<u>Example 5.4.1</u>: Consider the $\overline{\text{URC}}$MVDPF Z which is realizable by a network N. Let

$$Z = \frac{s_1s_2s_3s_4+s_1s_2+s_2s_3+s_3s_4+1}{s_1s_2s_3+s_1s_3s_4+s_1}$$

The element types can be found from the $\overline{\text{URC}}$-product ratios as $1/s_1$, s_2, $1/s_3$, s_4. Now note that

$$\lim_{s_1 \to 0} Z = \infty$$

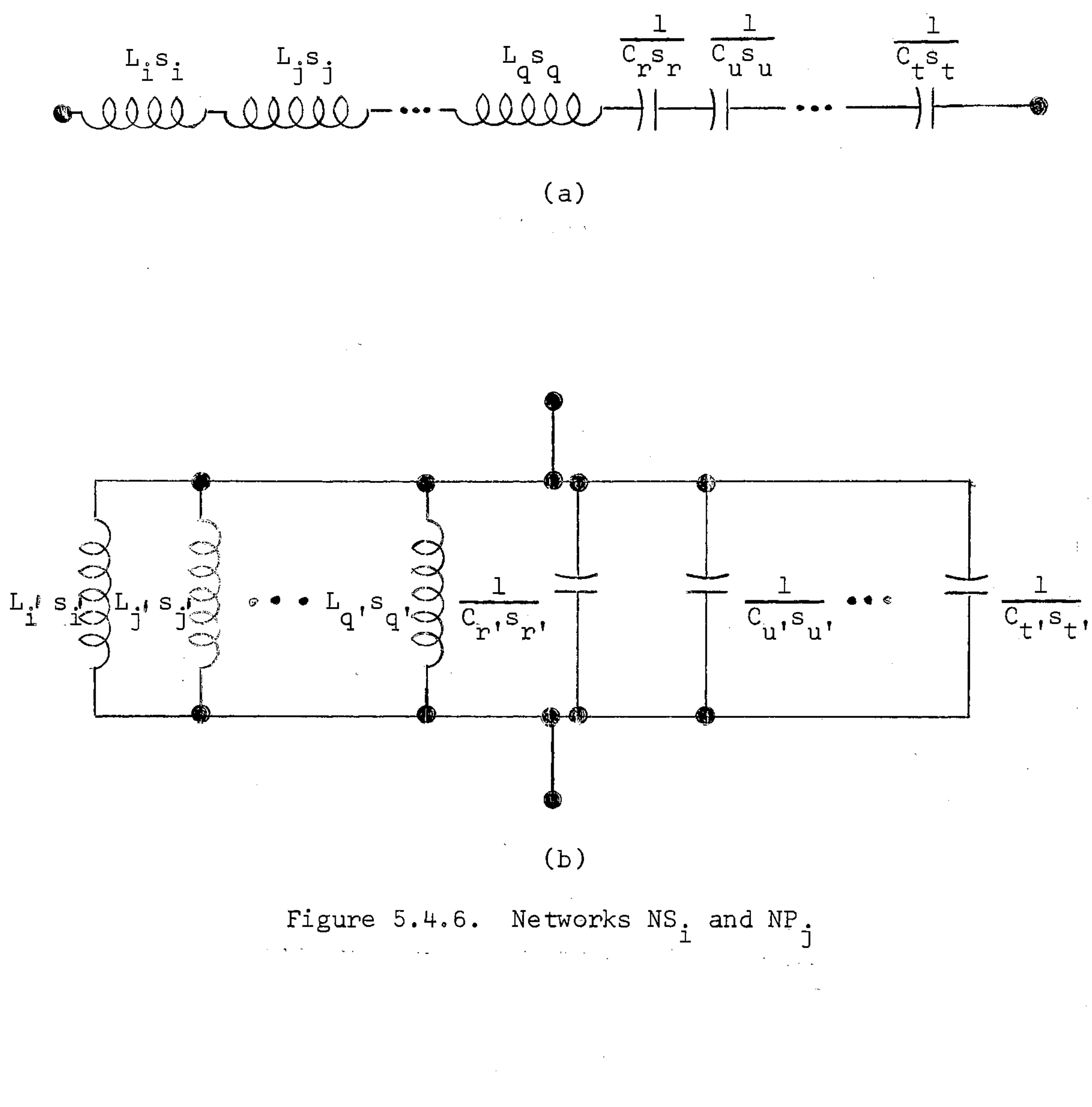


Figure 5.4.6. Networks NS_i and NP_j

Then N has the form given in Figure 5.4.7.

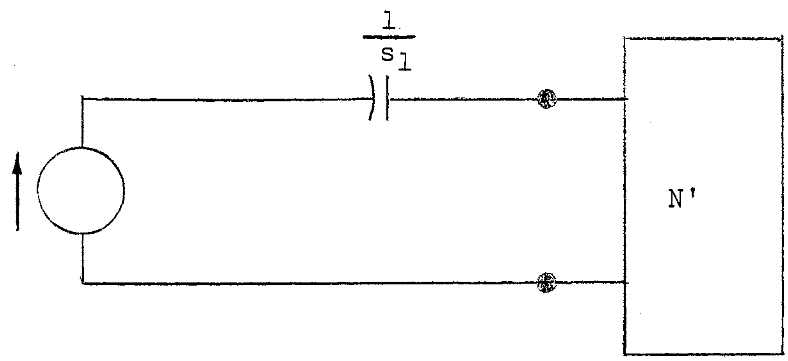


Figure 5.4.7. Network N

The $\overline{\text{URC}}$MVDPF Z' corresponding to N' can be found from Z by taking the limit

$$Z' = \lim_{s_1 \to \infty} Z = \frac{s_2 s_3 s_4 + s_2}{s_2 s_3 + s_3 s_4 + 1}$$

Now note that

$$\lim_{s_2 \to 0} Z' = 0$$

Therefore the network N' has the form given in Figure 5.4.8.

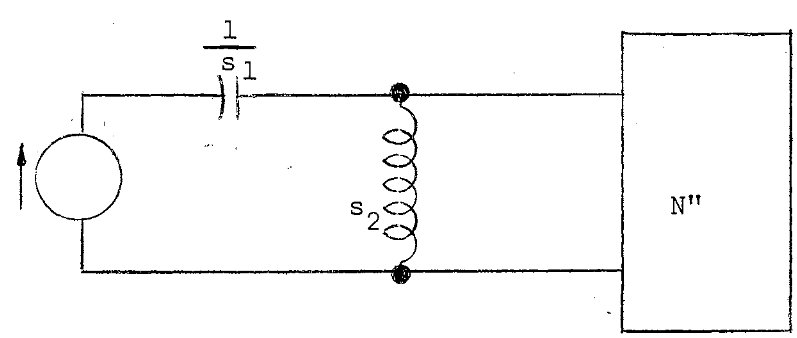


Figure 5.4.8. Network N

where the $\overline{\text{URCM}}$VEPF Z" corresponding to N" is given by

$$\lim_{s_2 \to \infty} Z' = Z'' = \frac{s_3 s_4 + 1}{s_3}$$

Similarly,

$$\lim_{s_3 \to 0} Z'' = \infty$$

and the process can be continued to give the network in Figure 5.4.9.

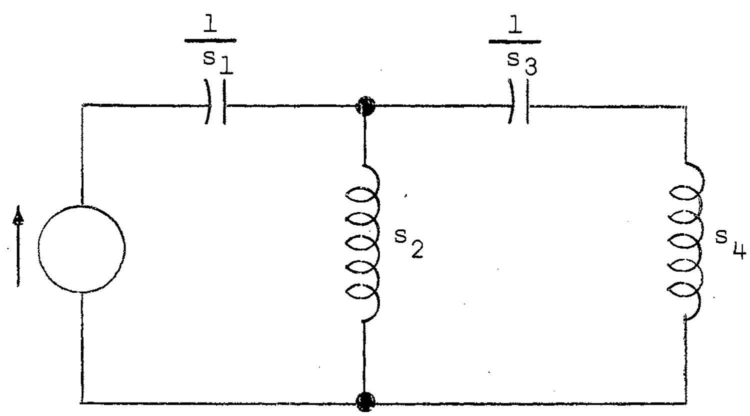


Figure 5.4.9. Network N

The method given here can be used to simplify a certain class of $\overline{\text{URCM}}$VDPFs before applying the synthesis methods of Section 4.5, but it should be emphasized that in general the network given as N' in Figure 5.1.1 cannot be reduced further by applying these methods.

CHAPTER VI

APPROXIMATION OF DRIVING POINT IMPEDANCES WITH $\overline{\text{URC}}$ NETWORKS HAVING ELEMENTS WITH DIFFERENT RC PRODUCTS

6.1 Introduction. In an earlier chapter the driving point function of a $\overline{\text{URC}}$ network with elements having different RC products was discussed in terms of a $\overline{\text{URC}}$MVDPF, W. It is important to note that W can be related to the p-domain by the transformations that were made to obtain W (see Equations 4.2.1 and 4.2.9). The properties that were derived for W in Chapter IV will be used to obtain properties for W in the p-domain.

After some properties are derived, a general method will be given which can be used to approximate an impedance specified in a Bode plot.

6.2 General Form of the Driving Point Impedance for a $\overline{\text{URC}}$ Network. In Chapter IV it is shown that a $\overline{\text{URC}}$MVDPF W of K elements can be written in the form given below

$$Z = \frac{\sum a_i PN_i}{\sum b_j PD_j} \tag{6.2.1}$$

where a_i and b_j are positive constants and PN_i and PD_j are $\overline{\text{URC}}$-products that satisfy the necessary conditions given in Section 4.3. Therefore, using Equation 6.2.1 and the transformations given in Equations 4.2.1 and 4.2.9, the general form of the driving point impedance of a $\overline{\text{URC}}$ network having elements with different RC products can be given as

$$Z(p) = \frac{\sum a_i \, TN_i}{\sqrt{p} \sum b_j \, TD_j} \tag{6.2.2a}$$

or

$$\sqrt{p}\, Z(p) = \frac{\sum a_i \, TN_i}{\sum b_j \, TD_j} \tag{6.2.2b}$$

where TN_i (TD_j) are equal to the product of elements from a subset of the set $\{1;\ \tanh \sqrt{\tau_1 p},\ \tanh \sqrt{\tau_2 p}, \ldots, \tanh \sqrt{\tau_k p}\ \}$ ($\tau_i = R_i C_i$). Since Equations 5.3.1 and $\sqrt{p}\, Z(p)$ in Equation 6.2.2b are very similar in form, many of the properties developed for the $\overline{\text{URC}}$MVDPFs can be applied directly to $\sqrt{p}\, Z(p)$ in Equation 6.2.2b. The most important properties which are directly applicable are given by Lemma 4.3.1 and 5.3.1 and Theorem 4.3.4.

6.3 Properties of $|Z(j\omega)|$. Before the function Z(p) given in Equation 6.2.2a can be used to approximate an impedance function Z'(p) specified in a Bode plot, it is necessary to consider some general properties of $|Z(j\omega)|$ to insure that the approximation will be successful. Some of these properties are given by Wyndrum (3) for a network of $\overline{\text{URC}}$ elements with elements having the same RC product, and the properties are given below for networks consisting of elements with different RC products.

Theorem 6.3.1 Asymptotic Behavior of $d\,|Z(j\omega)|/d\omega$: The asymptotic slope $d\,|Z(j\omega)|/d\omega$ for the driving point impedance Z(p) for any $\overline{\text{URC}}$ network with elements having different RC products as $\omega \to \infty$ is -10db/decade.

Proof: It has been shown that the asymptotic slope $d|Z(j\omega)|/d\omega$ of the driving point impedance for any $\overline{URC}$ network where each element has the same RC product as $\omega \to \infty$ is -10 db/decade (3). This result can be extended to the case of $\overline{URC}$ networks with elements having different RC products, by considering the driving point impedance of two Z_{oc}-elements with different RC products τ_1 and τ_2 ($\tau_1 \neq \tau_2$). Let $p = j\omega$. Then for the Z_{oc}-elements it follows that

$$\lim_{\omega \to \infty} \frac{R_o}{\sqrt{j\omega\tau_1} \tanh \sqrt{j\omega\tau_1}} = \lim_{\omega \to \infty} \frac{R_o}{\sqrt{j\omega\tau_2} \tanh \sqrt{j\omega\tau_2}} \qquad (6.3.1)$$

since τ_1 and τ_2 are finite positive constants. Similarly, for two Z_{sc}-elements having different RC products τ_1' and τ_2' ($\tau_1' \neq \tau_2'$) it follows that

$$\lim_{\omega \to \infty} \frac{R_s \tanh \sqrt{j\omega\tau_1'}}{\sqrt{j\omega\tau_1'}} = \lim_{\omega \to \infty} \frac{R_s \tanh \sqrt{j\omega\tau_2'}}{\sqrt{j\omega\tau_2'}} \qquad (6.3.2)$$

since τ_1' and τ_2' are finite positive constants. Therefore, the asymptotic slope $d|Z(j\omega)|/d\omega$ as $\omega \to \infty$ for the driving point impedance of any $\overline{URC}$ network with elements having different RC products is the same as a $\overline{URC}$ network with elements having the same RC products and therefore the theorem follows.

It is also important to note that for finite frequencies the Z_{oc}-element and the Z_{sc}-element can be approximated to any desired degree of accuracy by a finite lumped RC network obtained from the truncated infinite product expansions for $Z_{oc}(p)$ and $Z_{sc}(p)$ respectively. The number of terms in the truncated infinite product expansions can be increased until the desired accuracy is achieved. Then any $\overline{URC}$ network having elements with different RC products can be approximated to any degree of accuracy for finite frequencies by a finite lumped RC network. Thus the properties of $|Z(j\omega)|$ are known for $0 \leq \omega \leq \omega_1$ where ω_1 is a finite frequency.

6.4 Approximation Problem. In this section a general method will be given that can be used to approximate driving point impedance $Z'(p)$ specified by a magnitude plot for a band of frequencies. Note that the asymptotic approximation of $Z'(p)$ must have the properties given in the previous section.

In an earlier section several necessary conditions for a multivariable impedance function to be realizable as a $\overline{URC}$ network were given, and sufficient conditions were given in the form of a synthesis procedure (see Section 5.3).

There are two general approaches that can be used in the approximation problem. The first approach is to assume a function $\bar{Z}'(p)$ having the form of Equation 6.2.2a with arbitrary constants a_i, b_j, and τ_i, such that all necessary conditions for realizability are satisfied. Lemma 5.3.1 can be used to find the relationship that must exist between the coefficients a_i and b_j for realizability (see Example 5.3.3). Then $\bar{Z}'(j\omega)$ can be found by using

$$\sqrt{j\omega\tau_i} = \pm\sqrt{\omega\tau_i}\,(\cos \Pi/4 + j \sin \Pi/4) \qquad (6.4.1)$$

and

$$\tanh \sqrt{j\omega\tau_i} = \pm \frac{\sinh\phi\cosh\phi + j\sin\phi\cos}{\cosh^2\phi\cos^2\phi + j\sinh^2\phi\sin^2} \qquad (6.4.2)$$

where $\phi = +\sqrt{\omega\tau_i}\cos \Pi/4$. Note that part (i) of Theorem 4.3.4 guarantees that $\bar{Z}'(j\omega)$ is single valued for a particular value of ω and that the plus sign may be used in the right hand sides of Equations 6.4.1 and 6.4.2 without loss of generality. Since most computers use languages (such as FORTRAN IV) that have built in complex number subroutines, $|\bar{Z}'(j\omega)|$ can easily be found by using Equations 6.4.3 and 6.4.4 for any set of parameters $\{a_i\}$, $\{b_j\}$, and $\{\tau_i\}$. A least squares approach can now be used which is identical to the one used in Chapter II and III.

A set of frequencies ω_i, $i=1,2,\ldots,n$ need to be selected so that they cover the band of frequencies over which the approximation $\bar{Z}'(p)$ is to be valid. Then a squared error function can be defined as

$$S = \sum_{i=1}^{n} \left(|Z'(j\omega_i)| - |Z'(j\omega_i)| \right)^2 \qquad (6.4.3)$$

The function S can be minimized with respect to the parameters $\{a_i\}$, $\{b_j\}$, $\{\tau_i\}$, and constraints can be imposed on these parameters so that Lemma 5.3.1 is satisfied. Note that the constraint for τ_i is $\tau_i > 0$ for every i. If some of the parameters converge to a value such that the impedance of some of the elements ($L_i s_i$ or $1/C_i s_i$) are very large or very small compared with the other elements, these elements can be open-circuited or short-circuited respectively. The resulting network has fewer elements and has a driving point impedance which can be found by inspection from $\bar{Z}'(p)$ using the corresponding $\overline{\text{URC}}$MVDPF. Note also that additional constraints can be imposed on the range of values for the element values (and RC products τ_i) so that the network for $\bar{Z}(p)$ is practical. However, in this case when an element value (or RC product τ_i) is driven into a constraint, the constraint must be relaxed to see if it will become very large or very small when it is desirable to minimize the number of elements in the approximation by the procedure given above where no constraints are used other than those to insure realizability. In the approximation procedure above the realizability of the chosen $\overline{\text{URC}}$MVDPF must be tested, and if it is not realizable, a new function must be found.

One way to avoid the trial and error method is to approach the problem by assuming some topology and finding the corresponding $\overline{\text{URC}}$MVDPF. In this way the $\overline{\text{URC}}$MVDPF is known to be realizable. The realizable $\overline{\text{URC}}$MVDPF can be given arbitrary coefficients a_i, b_j satisfying Lemma 5.3.1. The methods used in the first approach can then be used to find the proper element values and RC products. Examples of the procedures for using the least squares program have already been given in Chapter II and III.

In conclusion it should be noted that a $\overline{\text{URC}}$ network may be found, in some cases, that has fewer elements than a corresponding lumped element RC network which approximates a given function. A good example of this is illustrated by the RC networks approximating the <u>single</u> Z_{oc}-element and Z_{sc}-element shown in Figure 2.2.3.

CHAPTER VII

SUMMARY, CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

7.1 Summary and Conclusions. This thesis deals with the analysis, synthesis, and the approximation of driving point impedances of $\overline{\text{URC}}$ networks Z(p) where each element has a different RC product. The rational approximations for the elements of a $\overline{\text{URC}}$ network, which are based on the infinite product expansions, are improved by finding new approximations which are valid over a wider band of frequencies. The synthesis of $\overline{\text{URC}}$ networks with elements having different RC products is solved by generalizing Wyndrum's transformations which transform Z(p) into a multivariable impednace function ($\overline{\text{URC}}$MVDPF). Some necessary conditions for the realizability of Z(p) ($\overline{\text{URC}}$MVDPF) are given. Sufficient conditions are given in terms of a new synthesis procedure which applies to any realizable driving point impedance Z(p). In the realization transformers and gyrators are not used. The impedance dunctions of lumped RC networks are approximated by $\overline{\text{URC}}$ networks and a rather simple method is developed which minimizes the error in the approximations. Design curves are given to aid in the approximations. Finally, the general problem of approximating a driving point impedance function specified in a magnitude plot with $\overline{\text{URC}}$ networks having elements with different RC products is approached by using the necessary conditions derived for $\overline{\text{URC}}$MVDPFs. The procedure is basically one of a least squares approach and a program is given that is well suited to the nonlinearities that occur from the application of least squares methods.

In conclusion, it is felt that the $\overline{\text{URC}}$MVDPF is an effective tool for analysis, synthesis, and approximation with $\overline{\text{URC}}$ networks, and may have application in other areas in network theory.

7.2 Suggestions for Further Study. The use of the $\overline{\text{URC}}$MVDPF has produced some new and interesting problems. The most important problem is one of finding the sufficient conditions to realize a given $\overline{\text{URC}}$MVDPF. Perhaps the assumption that the $\overline{\text{URC}}$MVDPF be a reactance function might be a sufficient condition. This author could not find a counter-example to disprove this statement. If this condition is sufficient, it would also be applicable to synthesis of the classical topological formula for the driving point admittance.

Another interesting problem is the synthesis of multivariable impedance functions which result when some, but not all of the $\overline{\text{URC}}$ elements have RC products which are equal. The introduction of the generalized transformations will produce a multivariable impedance function which does not have the form of the $\overline{\text{URC}}$MVDPF, but appears to be related to the $\overline{\text{URC}}$MVDPF. It may be possible to find the existing relationships. If this synthesis problem can be solved, the problem of finding a realizable topological formula for the driving point admittance from a given realizable lumped RC (LC) driving point admittance might also be solved.

BIBLIOGRAPHY

(1) Motorola Handbook. Analysis and Design of Integrated Circuits. New York: McGraw-Hill, 1967.

(2) Chirlian, P. M. Integrated and Active Network Analysis and Synthesis. Englewood Cliffs, New Jersey: Prentice-Hall, 1967.

(3) Wyndrum, R. W. "The Exact Synthesis of Distributed RC Networks." Technical Report 400-76, New York University, May, 1963.

(4) O'Shea, R. P. "Synthesis Using Distributed RC Networks." IEEE International Convention Record, Vol. 13, Part 7 (1965), 18-29.

(5) Giguere, J. C., M. N. S. Swamy, and B. B. Bhattacharyya. "Driving Point Synthesis Using Uniform Transmission Lines." IEEE Proceedings, (May, 1968).

(6) Hiezer, K. W. "Distributed RC Networks With Rational Transfer Functions." IRE Transactions on Circuit Theory, Vol. CT-9 (December, 1962), 356-362.

(7) Rohrer, R. A., J. A. Resh, and R. A. Hoyt. "Distributed Network Synthesis for a Class of Integrated Circuits." IEEE Convention Record, Vol. 13, Part 7 (1965), 100-112.

(8) Koga, T. "Synthesis of Finite Passive n-Ports With Prescribed Positive Real Matrices of Several Variables." IEEE Transactions on Circuit Theory, Vol. CT-115, No. 1 (March, 1968).

(9) Ozaki, H. and T. Kasami. "Positive Real Functions of Several Variables and Their Application to Variable Networks," IRE Transactions on Circuit Theory, Vol. CT-7 (September, 1960), 251-260.

(10) Seshu, S. Linear Graphs and Electrical Networks. London: Addison-Wesley, 1961.

(11) Seshu, S. "Topological Considerations in the Design of Driving-Point Functions." IRE Transactions on Circuit Theory, Vol. CT-2, No. 4 (December, 1955), 356-367.

(12) Fletcher, R. and M. J. D. Powell. "A Rapidly Convergent Descent Method for Minimization." The Computer Journal, Vol. 6 (1964), 163-168.

(13) Ghausi, M. S. and J. J. Kelly. Introduction to Distributed Parameter Networks. New York: Holt, Rinehart, and Winston, 1968.

(14) Volkenburg, M. E. V. Network Analysis. Englewood Cliffs, New Jersey: Prentice-Hall, 1955.

(15) Huelsman, L. P. Theory and Design of Active RC Circuits. New York: 1968.

(16) Newcomb, R. W. Active Integrated Circuit Synthesis. Englewood Cliffs, New Jersey: Prentice-Hall, 1968.

(17) Fu, Y. "Realization of Circuit Matrices." IEEE Transactions on Circuit Theory, Vol. CT-12 (December, 1965), 604-607.

(18) McCracken, D. D. and W. S. Dorn. Numerical Methods and Fortran Programming. New York: John Wiley and Sons, 1964.

(19) Rosenbrock, H. H. "An Automatic Method for Finding the Greatest or Least Value of a Function." The Computer Journal, Vol. 3 (1960), 175-184.

APPENDIX A

PROGRAM FOR THE LEAST SQUARES ESTIMATION OF NONLINEAR PARAMETERS

A.1 Introduction. It is frequently necessary to represent by some functional relationship data that is given as a set of order pairs (Y_i, ω_i), $i=1,2,\ldots,n$. One very powerful method of finding a functional relationship is by the method of least squares curve fitting (18). The method of least squares consists of minimizing S, defined as

$$S\ (b_1, b_2, \ldots, b_k) = \sum_{i=1}^{n} \left(\bar{Y}\ (b_1, b_2, \ldots, b_k;\ \omega_i) - Y_1 \right)^2 \qquad (A.1.1)$$

where $\bar{Y}\ (b_1, b_2, \ldots, b_k;\ \omega_i)$ is some function with parameters b_i, $i=1,2,\ldots,k$ and n is some integer. Let $\{\hat{b}_i\}$ be the set of parameters that gives a minimum value for S. Then

$$\bar{Y}\ (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_k;\ \omega_i) \approx Y_i$$

for every $i=1,2,\ldots,n$. In this thesis, S is a nonlinear function and the numerical method given in the next section can be used to minimize S.

A.2 Minimization of Nonlinear Functions. Fletcher and Powell (12) have given a powerful method to minimize nonlinear functions which has quadratic convergence, but does not require the computation of second order partials. This method was used with some modifications in the minimization of the squared error function S in Equation A.1.1 for distributed network synthesis problems to insure convergence and practicality. Fletcher and Powell assumed that the first order partial derivatives of the function S $(b_1,b_2,\ldots,b_k)$ to be minimized are defined analytically at each point. Since it is not practical to find analytic expressions for the partial derivatives the applications of this thesis, the needed partials were computed by the well-known methods of finite differences (18). In general the Fletcher-Powell method converges faster than the method of steepest descent, whenever the method converges and this is especially true near the minimum value of the function being minimized (12).

When the Fletcher-Powell method was found to diverge at any iteration, the steepest descent method was used for one or more iterations (since the gradient is computed in each iteration as part of the Fletcher-Powell method).

The advantage of the quadratic convergence of the Fletcher-Powell method is not lost by this modification since the method is reinstated as soon as there is convergence. In the Fletcher-Powell method each iteration is defined by

$$\vec{b}^{i+1} = \vec{b}^{i} + \lambda^{i}\,\vec{g}^{i} \qquad (A.2.1)$$

where $\vec{g}^{i} = (g_1^i, g_2^i, \ldots, g_k^i)$ is a vector computed by the method, λ^i is a scalar to be determined, and $\vec{b}^{i} = (b_1^i, b_2^i, \ldots, b_k^i)$ is the previous value of the iteration. The scalar λ^i is determined such that S $(b_1^{i+1}, b_2^{i+2}, \ldots, b_k^{k+2})$ is a minimum. In practice it was found that convergence of the method depends on the accurate determination of λ^i. λ^i can be found by combination of systematic searching and cubic interpalation, and the method is given in the flow chart in Figure A.2.2, where the variables in the flow chart are defined in Figure A.2.1, M is a positive constant, and the standard mathematical symbols for union and intersection are used (Note $S(\vec{b}^{i+1})$ is a function of λ^i since $\vec{b}^{i+1}$ is a function of λ^i in Equation A.2.1).

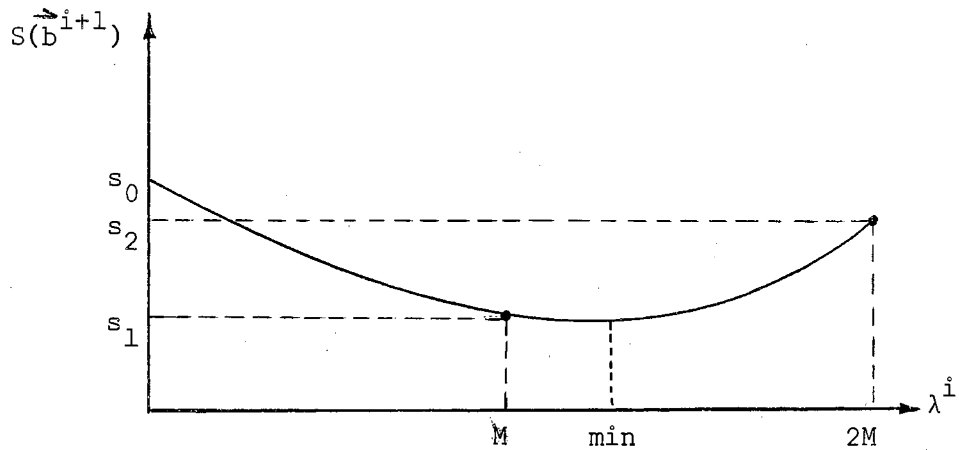


Figure A.2.1. Definition of Variables in the Flow Chart of Figure A.2.2

<u>A.3 Constraints.</u> This section gives an effective method that has been used frequently to constrain variables in least squares curve fitting problems. Consider Equation A.1.1 and let each parameter b_i be constrained by $L_i \leq B_i \leq U_i$ where L_i and U_i are constants and $i=1,2,\ldots,k$. Now let S be redefined such that

$$S = \sum_{i=1}^{n} \left(\bar{Y}\,(b_1,b_2,\ldots,b_k;\ \omega_i) - Y_i\right)^2 + \sum_{i=1}^{k} \phi_i^{\,2} \qquad \text{(A.3.1)}$$

where

$$\phi_i = 0 \quad \text{if} \quad L_i \leq b_i \leq U_i \qquad \text{(A.3.2)}$$

$$\phi_i = (b_i - U_i)^8 \text{ if } b_i > U_i \qquad \text{(A.3.3)}$$

$$\phi_i = (b_i - L_i)^8 \text{ if } b_i < L_i \qquad \text{(A.3.4)}$$

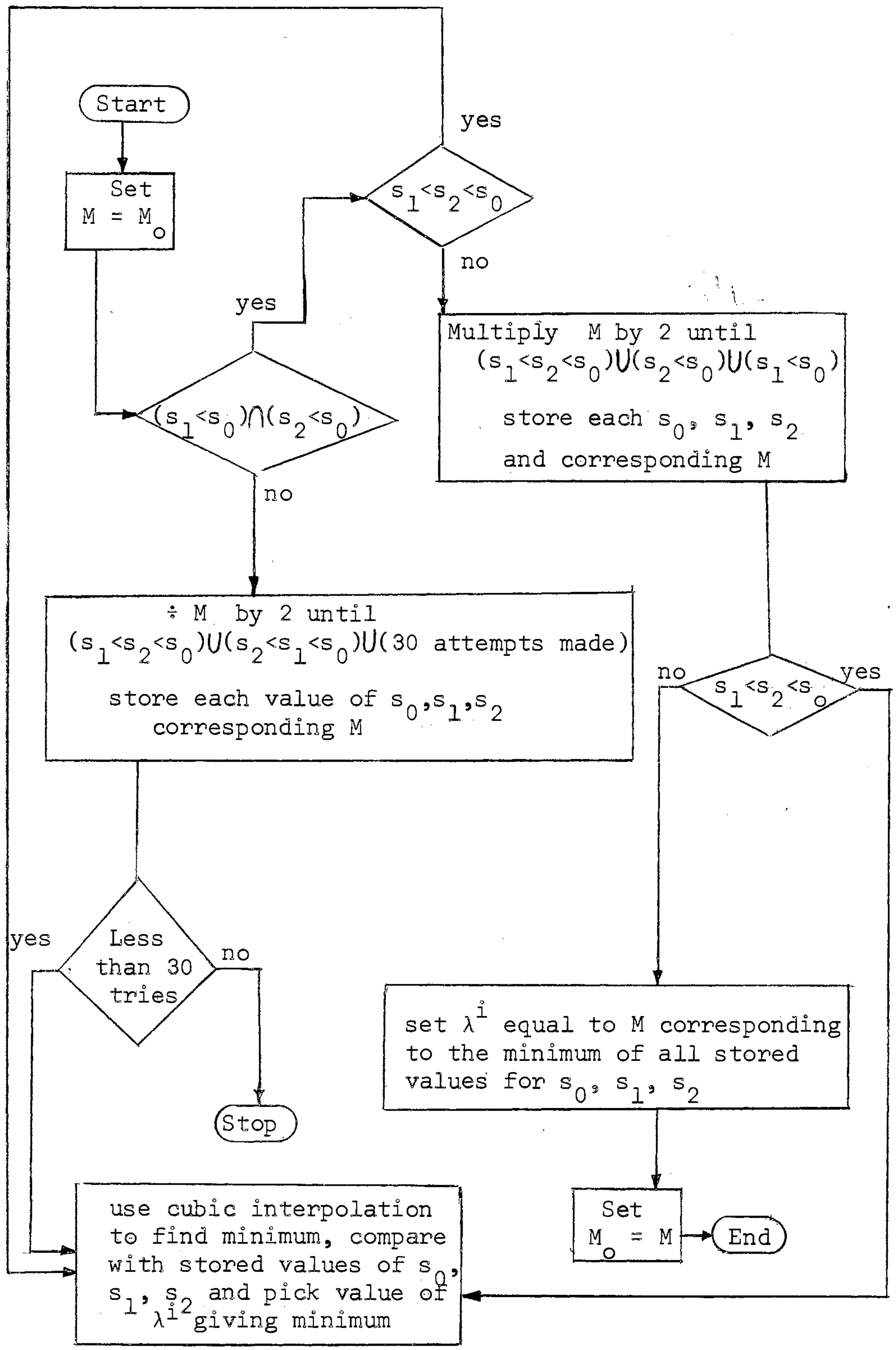


Figure A.2.2. Flow Chart for Computation of λ^i

for i=1,2,...,k. The use of the power of eight is arbitrary in Equations A.3.3 and A.3.4, but in any case when the constraints are not satisfied, S in Equation A.3.1 becomes large. There are also other ways of defining each ϕ_i which may be better suited for a given problem (19).

A.4 Least Squares Program. The program used for the least squares problems in this thesis is given in Table A.4.1.

The user must supply the subroutine FCODE (Y,X,B,F,I,RES) with the dimensioned variables being Y(200), S(200), B(50). The relationship between Equation A.1.1 and the variables in FCODE is given in Table A.4.2. The user must also supply the subroutine SUBZ(Y,X,B,N) and GENF(N,K,NCON,X,Y). SUBZ may be used to alter the input data Y(I) and X(I) before beginning computation. GENF may be used to generate Y(I) and X(I) if they are not read into the program (IOPTL = 1). If either one or both of these subroutines are not needed, they still must be supplied since they will be called by the main program. In this case they will consist of only a DIMENSION, a RETURN, and an END statement.

An example of subroutines SUBZ, GENF, AND FCODE is given in Table A.4.4. Note for FCODE in Table A.4.5 there are three constrained parameters. In this case the number of data points is 43 and each variable RES corresponding to I = 44, 45, 46 are the constrain squares corresponding to ϕ_i (the square of RES = ϕ_i) in Equation A.3.1 for B(1), B(2), and B(3) respectively.

TABLE A.4.1

PROGRAM FOR THE LEAST SQUARES ESTIMATION OF NONLINEAR PARAMETERS

```
C      PROGRAM FOR THE LEAST SQUARES ESTIMATION OF NONLINEAR PARAMETERS
       DIMENSION DBG(50),DFG(50)
       DIMENSION X(200),Y(200),B(50),SB(50),GD(50)        ,G(50),SPH(50),
      1SM(50)
       DIMENSION GG(50,50),SG(50)
       IDAM=0
       KICK=0
       READ(5,900) IOPT1
       READ (5,900) N,K,MIN,MAX,NCON,ZETA,DEL
       IF(IOPT3.EQ.0) GO TO 151
       CALL GENF(N,K,NCON,X,Y)
       GO TO 152
151    READ(5,901) (Y(I),X(I)   ,I=1,N)
152    CONTINUE
       READ(5,902)  (B(I),I=1,K)
       WRITE(6,903)  N,K,MIN,MAX,NCON,ZETA,DEL
       PHI=0.
       DO 1 I=1,N
       CALL FCODE(Y,X,B,F,I,RES)
1      PHI=PHI+RES**2
       SO=PHI
       IT=0
       DO 41  I=1,K
       DO 41  J=1,K
       IF(I.EQ.J)  GOTO 42
       GG(I,J)=0.
       GO TO 41
42     GG(I,J)=1.
41     CONTINUE
       CALL SUBZ(Y,X,B,N)
130    WRITE(6,915)
       WRITE(6,904)  IT,(B(I),I=1,K)
       WRITE(6,908)
       DO 999 I=1,N
       CALL FCODE(Y,X,B,F,I,RES)
999    WRITE(6,911) Y(I),F,RES,X(I)
       WRITE(6,907)  PHI
       NTIL=N+NCON
       NN=N+1
150    IF(NCON.EQ.0) GO TO 440
       WRITE(6,910)
       DO 301 I=NN,NTIL
       CALL FCODE(Y,X,B,F,I,RES)
       III=I-N
301    WRITE(6,909) III,RES
440    IF(KICK.NE.1) GO TO 140
300    CALL EXIT
C      BEGIN COMPUTATION OF GRADIENT.......................................
140    IF(IT.EQ.0)  GO TO 101
       DO 29 J=1,K
29     SG(J)=G(J)
101    CONTINUE
       DO 100 J=1,K
100    SB(J)=B(J)
       PHIN=0.
       DO 26 J=1,K
       B(J)=SB(J)+ABS(SB(J))*DEL
       DO 700 JJ=1,NTIL
       CALL FCODE (Y,X,B,F,JJ,RES)
700    PHIN=RES**2+PHIN
       G(J)=(PHIN-PHI)/(ABS(SB(J))*DEL)
```

A.4.1 (Continued)

```
      PHIN=0.
26    B(J)=SB(J)
      WRITE(6,912)  (G(I),I=1,K)
C     END GRADIENT COMPUTATION ................................................
C     BEGIN FLETCHER-POWELL ITERATION.........................................
      IF(IT.EQ.0)  GO TO 111
      DENA=0.
      DO 28 I=1,K
      DFG(I)=G(I)-SG(I)
28    DENA=DENA+DFG(I)*SVXM*GD(I)
      DO 30  I=1,K
      DBG(I)=0.
      DO 30  J=1,K
30    DBG(I)=GG(I,J)*DFG(J)+DBG(I)
      DENB=0.
      DO 31  I=1,K
31    DENB=DENB+DBG(I)*DFG(I)
      DO 32  I=1,K
      DO 32  J=1,K
32    GG(I,J)=GG(I,J)+GD(I)*GD(J)*SVXM**2/DENA-DBG(I)*DBG(J)/DENB
111   WRITE(6,955)
      DO 202  I=1,K
202   WRITE(6,911) (GG(I,J),J=1,K)
556   DO 27  I=1,K
      GD(I)=0.
      DO 27  J=1,K
27    GD(I)=GD(I)-GG(I,J)*G(J)
C     COMPUTE STEP SIZE.....................................................
      XNU=0.
      DO 550 I=1,K
550   XNU=XNU+G(I)*GD(I)
      XNU=ABS(2.*PHI/XNU)
      XM=AMIN1(XNU,1.)
      IN=0
      IB=1
      IK=0
      DO 2 J=1,K
2     B(J)=SB(J)+XM*GD(J)
      IBK=1
      GO TO 4
5     S1=SS
      DO 6  J=1,K
6     B(J)=SB(J)+2.*XM*GD(J)
      IBK=2
      GO TO 4
7     S2=SS
      IF(S2.GE.S0) GO TO 12
      IK=IK+1
      SM(IK)=2.*XM
      SPH(IK)=S2
12    IF(S1.LT.S0.AND.S2.LT.S0)  GO TO 13
11    IN=IN+1
      IF(S1.GE.S0)  GO TO 9
      IK=IK+1
      SM(IK)=XM
      SPH(IK)=S1
9     IF(S1.LT.S2.AND.S2.LT.S0)  GO TO 19
      IF(S2.LT.S1.AND.S1.LT.S0) GO TO 200
      IF(IN.NE.20) GO TO 302
      WRITE(6,914)
      IDAM=IDAM+1
      IF(IDAM.EQ.2) GO TO 555
      DO 551 J=1,K
```

A.4.1 (Continued)

```
      DO 551 I=1,K
      IF(I.NE.J) GO TO 552
      GG(I,J)=1.
      GO TO 551
552   GG(I,J)=0.
551   CONTINUE
      GO TO 556
555   WRITE(6,913)
      KICK=1
      GO TO 130
302   XM=XM/2.
      DO 23  J=1,K
23    B(J)=SB(J)+XM*GD(J)
      IBK=3
      GO TO 4
24    S2=S1
      S1=SS
      GO TO 11
13    IK=IK+1
      SM(IK)=XM
      SPH(IK)=S1
      IK=IK+1
      SM(IK)=XM*2.
      SPH(IK)=S2
      IF(S1.LT.S2.AND.S2.LT.S0)  GOTO 19
16    XM=XM*2.
      DO 14  J=1,K
14    B(J)=SB(J)+2.*XM*GD(J)
      IBK=4
      GO TO 4
15    S1=S2
      S2=SS
      IF(S2.GE.S0) GO TO 200
      IK=IK+1
      SM(IK)=XM*2.
      SPH(IK)=S2
      IF(S1.LT.S2.AND.S2.LT.S0) GO TO 19
      GO TO 16
C     CUBIC INTERPOLATION...................................................
19    C=(3.*S0-4.*S1+S2)/(2.*S0-4.*S1+2.*S2)
C     END CUBIC INTERPOLATION...............................................
      IK=IK+1
      C=C*XM
      DO 17  J=1,K
17    B(J)=SB(J)+C*GD(J)
      IBK=5
      GO TO 4
18    SPH(IK)=SS
      SM(IK)=C
200   PHMIN=SPH(1)
      IMIN=1
      DO 21 J=2,IK
      IF(SPH(J).GE.PHMIN)  GO TO 21
      PHMIN=SPH(J)
      IMIN=J
21    CONTINUE
      PHI=PHMIN
      S0=PHI
      XM=SM(IMIN)
      SVXM=XM
C     END STEP SIZE COMPUTATION.............................................
C     COMPUTE NEW VALUES FOR B(I)...........................................
      DO 22 J=1,K
```

A.4.1 (Continued)

```
22      B(J)=SB(J)+SM(IMIN)*GD(J)
C       END COMPUTATION FOR NEW VALUES OF B(I)..................................
C       END OF FLETCHER-POWELL ITERATION........................................
        GO TO 66
C       BEGIN COMPUTATION OF SUM OF SQUARES.....................................
4       SS=0.
        DO 3  J=1,NTIL
        CALL FCODE(Y,X,B,F,J,RES)
3       SS=RES**2+SS
C       END OF SUM OF SQUARES COMPUTATIONS......................................
        GO TO (5,7,24,15,18),IBK
66      IT=IT+1
        IDAM=0
        IF(IT.LE.MIN)  GO TO 130
        IF(IT.GE.MAX)  GO TO 20
        WRITE(6,915)
        WRITE(6,904)  IT,(B(I),I=1,K)
        WRITE(6,907) PHI
        GO TO 150
20      KICK=1
        GO TO 130
900     FORMAT(5I5,E15.8,E15.8)
901     FORMAT(2F10.6)
902     FORMAT(8F10.6)
903     FORMAT(2X,4HN = ,I5,5X,4HK = ,I5,5X,6HMIN = ,I5,5X,6HMAX = ,I5,5X,
       16HNCON = ,I5,5X/2X,48HMINIMUM PERCENT IMPROVEMENT IN SUM OF SQUARE
       1S = ,E15.8,5X,6HDEL = ,E15.8)
904     FORMAT (/2H (I3,13H) PARAMETERS 5E18.8/(18X,5E18.8))
906     FORMAT(4(5X,E15.8))
907     FORMAT(/2X,17HSUM OF SQUARES = ,E15.8)
908     FORMAT(8X,3HOBS,16X,4HPRED,16X,4HDIFF,16X,4HFREQ)
909     FORMAT(8X,I5,4X,E15.8)
910     FORMAT(/2X,10HCONSTRAINT    ,3X,7HSQUARES   )
911     FORMAT(6(5X,E15.8))
955     FORMAT(/2X,17HG MATRIX BY ROWS        )
912     FORMAT(/2X,16HGRADIENT BY ROWS  /6(5X,E15.8))
913     FORMAT(/2X,36HSCALED 30 TIMES WITH NO IMPROVEMENT     /)
914     FORMAT(/2X,58HQUADRATIC METHOD FAILED, RESETTING G MATRIX TO UNIT
       1MATRIX/)
915     FORMAT(/120H ...................................................
       1............................................................
       1)
        END
```

TABLE A.4.2

VARIABLES OF SUBROUTINE FCODE

Mathematical Symbol	FORTRAN Variable
Y_i	Y(I)
ω_i	X(I)
b_i, $i=1,2,\ldots,k$	B(I)
$\bar{Y}_i$, $(b_1,b_2,\ldots,b_k;\ \omega_i)$	F
i	I
$\bar{Y}_i\ (b_1,b_2,\ldots,b_k;\ \omega_i) - Y_i$	RES

TABLE A.4.3

INPUT DATA

Input Item No.	Mathematical Symbol	FORTRAN Label	Format	Card Columns	Comments
1	-	IOPTI	I5	1-5	= 0 Read in Y(I), X(I) = 1 Compute Y(I), X(I) by subroutine GENF
2	n	N	I5	1-5	No. of data points Y(I)
	k	K	I5	6-10	Total number of parameters B(I)
	-	MIN	I5	11-15	No. of detailed print outs desired (MIN ≤ MAX)
	-	MAX	I5	21-25	No. of constrained parameters
	-	ZETA	E15.8	26-40	Minimul allowable percent improvement in squared error function
	M_o	DEL	E15.8	41-55	Initial value for M_o in Table A.2.2 (DEL = 1.E-5 is adequate in most cases.)
3	Y_i	Y(I)	F10.6	1-10	Omit if IOPT = 1. Use as many cards as needed (one pair of data points per card).
	ω_i	X(I)	F10.6	11-20	
4	b_i	B(I)	8F10.6	1-80	Initial values of B(I), eight per card

TABLE A.4.4

EXAMPLES OF SUBROUTINES USED IN PROGRAM

```
      SUBROUTINE FCODE(Y,X,B,F,I,RES)
      DIMENSION Y(200),X(200)    ,B(50),PRNT(5)
      IF(I.GT.45)  GO TO 12
      IF(I.GT.44)  GO TO 11
      IF(I.GT.43) GO TO 10
      COMPLEX ZOC,ZS,Z
      RO=B(1)
      RS=B(2)
      XKS=B(3)
      WS=ABS(X(I))
      WO=ABS(X(I)*XKS)
      SK=SQRT(WS)*.70710678
      RNS=RS*SK*(SINH(SK)*COSH(SK)+COS(SK)*SIN(SK))
      XIS=RS*SK*(COS(SK)*SIN(SK)-SINH(SK)*COSH(SK))
      DS=2.*WS*(.70710678**2)*((COSH(SK)*COS(SK))**2+(SINH(SK)*SIN(SK))
     1**2)
      RNS=RNS/DS
      XIS=XIS/DS
      SK=SQRT(WO)*.70710678
      RNO=RO*SK*(COSH(SK)*SINH(SK)-COS(SK)*SIN(SK))
      XIO=-RO*SK*(COSH(SK)*SINH(SK)+COS(SK)*SIN(SK))
      DO=2.*WO*(.70710678**2)*((SINH(SK)*COS(SK))**2+(COSH(SK)*SIN(SK))
     1**2)
      RNO=RNO/DO
      XIO=XIO/DO
      ZS=CMPLX(RNS,XIS)
      ZOC=CMPLX(RNO,XIO)
300   Z=ZS*ZOC/(ZS+ZOC)
      XMAG=CABS(Z)
         F=20.*ALOG10(XMAG)
      RES=Y(I)-F
      RETURN
10    IF(B(1).LT.5.)  GOTO 1
      IF(B(1).GT.20000.)  GO TO 2
      RES=0.
      RETURN
1     RES=(B(1)-5.)**4
      RETURN
2     RES=(B(1)-20000.)**4
      RETURN
11    IF(B(2).LT.50.)  GO TO 3
      IF(B(2).GT.20000.) GO TO 4
      RES=0.
       RETURN
3     RES=(B(2)-50.)**6
      RETURN
4     RES=(B(2)-20000.)**4
      RETURN
12    IF(B(3).LT..005)  GO TO 7
      IF(B(3).GT.100.)  GO TO 8
      RES=0.
      RETURN
7     RES=(1000.*(B(3)-.005))**6
      RETURN
8     RES=(B(3)-100.)**6
      RETURN
      END
```

A.4.4 (Continued)

```
SUBROUTINE SUBZ(Y,X,B,N)
DIMENSION Y(200),X(200),B(50)
RETURN
END

SUBROUTINE GENF(N,K,NCON,X,Y)
DIMENSION X(200),Y(200)
RETURN
END
```

APPENDIX B

MULTIVARIABLE IMPEDANCE FUNCTIONS

B.1 Introduction. Positive real functions of several variables were introduced in the problem of designing a passive network having variable parameters (9). The theory has recently been developed by Koga (8). This appendix is a collection of theorems and definitions which relate directly to this thesis.

Definition B.1.1 Complex Plane C^k: If a complex plane is denoted by C then $C^k = CXCX... C$ is the Cartesian product of k copies of the complex plane.

Definition B.1.2 Open Polydomain $D_r \subset C^k$: If $D_i \subset C^k$ (i=1,2,...,k) is any connected open subset of the complex plane, the product set $D = D_1 \times D_2 \times ... D_k \quad C^k$ will be called an open polydomain. If an open polydomain is defined by $D_{1r} \times D_{2r} \times ... D_{kr}$ where $D_{ir} = \{\lambda_i \varepsilon C;$ $Re(\lambda_i) > 0$, then it will be denoted by D_r.

Definition B.1.3 Positive Function of k Variables: If a rational function f of k variables satisfied $Re(f) \geq 0$ in the open polydomain $D_r \subset C^k$, then f is called a positive function of k variables.

Definition B.1.4 Positive Real Function of k Variables: If a positive function of k variables $W(\lambda_1,\lambda_2,...,\lambda_k)$ is real for λ_i $(1 \leq i \leq k)$ real, then W is a positive real function of k variables.

Definition B.1.5 Reactance Function of k Variables: If a positive real function of k variables W satisfies $W(\lambda_1,\lambda_2,...,\lambda_k)+W(-\lambda_1,-\lambda_2,...,-\lambda_k)\equiv 0$

then W is called a reactance function of k variables.

Theorem B.1.1 Positive Real Function Test: A function of k variables $W(\lambda_1,\lambda_2,\ldots,\lambda_k)$ is a positive real function if and only if W is a positive real function of p after substitution of $\lambda_i = \alpha_i p + \beta_i p^{-1}$ for every real, positive, value of the constants α_i and β_i $(1 \leq i \leq k)$.

Proof: See reference (8).

Theorem B.1.2 Right Half Plane Properties: The numerator and denominator of a positive function of k variables prescribed in the irreducible form have no zeros in the open polydomain $D_r \subset C^k$.

Proof: See (8).

Theorem B.1.3 Decomposition Theorem: If a positive real function $W(\lambda_1,\lambda_2,\ldots,\lambda_k)$ has poles on the imaginary λ_i-axis including infinity on each complex plaine $i=1,2,\ldots,k$ independently of the other variables, then W can be decomposed as

$$W(\lambda_1,\lambda_2,\ldots,\lambda_k) = \sum_{i=1}^{k} Z_i(\lambda_i) + W_1(\lambda_1,\lambda_2,\ldots,\lambda_k)$$

where $Z_i(\lambda_i)$ is a reactance function of λ_i alone which has the above mentioned poles and W_1 is a positive real function of k variables.

Proof: See (8).

Theorem B.1.4 Necessary and Sufficient Conditions for W to be a Reactance Matrix: Let an n x n matrix $W(\lambda_1,\lambda_2,\ldots,\lambda_k)$ be prescribed as

$$W = \frac{A\lambda_i + C}{B\lambda_i + D}$$

where $B\lambda_i + D$ is the least common denominator of W, $B \not\equiv 0$ and $D \not\equiv 0$ being polynomials in $\lambda_i (1 \leq i \leq k)$, and A, C are polynomial matrices of $\lambda_i (1 \leq i \leq k)$. Then the necessary and sufficient conditions for W to be a reactance matrix of (k+1) variables are:

i) D/B is a reactance function of $\lambda_i (1 \leq i \leq k)$,

ii) A/B, C/D are reactance matrices of $\lambda_i (1 \leq i \leq k)$,

iii) $(BD-AD)/B^2$ is non-negative Hermitian for $Re(\lambda_i) = 0$ $(1 \leq i \leq k)$ except at singularities.

Proof: See (8).

B.2 Topological Formulas for the Driving Point Function. Material on topological formulas and the synthesis of topological formulas can be found in works by Seshu (10), (11).

Definition B.2.1 Tree-Admittance Product: The tree-admittance product is the product of the admittances of the branches of a tree for some network.

Theorem B.2.1 Determinant Δ: The determinant Δ of the node-admittance matrix Y of a passive network N without mutual inductance is

$$\Delta = \sum_{\substack{\text{all} \\ \text{trees}}} (\text{tree-admittance product of tree } t_i \text{ of } N)$$

Proof: See (10).

Definition B.2.2 2-Tree $T_{2i,j}$: A 2-tree is a pair of unconnected, circuitless subgraph, each subgraph being connected, which together include all the vertices of the graph. One (or in trivial graphs, both) of the subgraphs may consist of an isolated vertex. The symbol $T_{2i,j}$ denotes a 2-tree with vertices i and j in different connected parts.

Definition B.2.3 2-Tree Product: A 2-tree product is the product of the admittances of the branches of a 2-tree. The product for an isolated vertex is defined to be 1.

Theorem B.2.2 Co-factor Δ_{ii}: If r is the reference vertex of node equations, the co-factor of an element in the (i,i)-position position is given by

$$\Delta_{ii} = \sum_{\substack{\text{all} \\ \text{2-trees}}} (T_{2i,r} \text{ products})$$

Proof: See (10).

Theorem B.2.3 Topological Formula for the Driving Point Admittance: The driving point impedance for a network which contains no magnetic coupling is given by

$$Y\ (y_1, y_2, \ldots, y_k) = \frac{\Delta}{\Delta_{11}}$$

Proof: See (10).

Theorem B.2.4 Form of $\Delta/\Delta ii$: The driving point admittance given in Theorem B.2.3 as $Y = \Delta/\Delta_{ii}$ will have Δ as a homogeneous polynomial of degree (v-1) and Δ_{ii} is a homogeneous polynomial of degree (v-2) in the variables $y_1, y_2, \ldots, y_e$ where v is the number of vertices of the graph corresponding to Y and each y_i is of degree one.

Proof: See (11).

Theorem B.2.5 Parallel Element Removal: If the elements y_i $(1 \leq i \leq m)$ have the two input vertices of a one-port as endpoints then

$$Y(s) = \frac{\Delta}{\Delta_{11}} = \sum_{j=1}^{m} y_j + \frac{\Delta'}{\Delta'_{11}}$$

where Δ' and Δ'_{11} are not functions of y_j $(1 \leq j \leq m)$.

Proof: See (11).

Theorem B.2.6 Parallel Element Condition: Every element y_i appears in Δ , but an element y_i appears in Δ_{11} if and only if y_i does not have the two input vertices of the one-port as endpoints.

Proof: See (11).

Definition B.2.5 2-Isomorphism: Two graphs G_1 and G_2 are 2-isomorphic if they become isomorphic under (repeated application of) either or both of the following operations:

1. Separation into components.
2. If the graph consists of two subgraphs H_1 and H_2 which have only two vertices in common, the interchange of their names in one graph.

Definition B.2.6 c-Circuit Matrix: The c-circuit matrix B_c for a given tree of a connected graph G is the matrix corresponding the set of e-v+1 circuits formed by each chord and its unique tree path where e is the number of elements and v is the number of vertices in G.

Theorem B.2.7 Δ for Separable Graphs: If a graph G is separable into nonseparable graphs $G_1, G_2, \ldots, G_n$ then $\Delta = \Delta_1 \cdot \Delta_2, \ldots, \Delta_n$ where Δ is for graph G and Δ_i is the Δ for graph G_i for every i.

Definition B.2.8 Component Parts of a Graph: If a separable graph G is separated into maximal connected subgraphs which are nonseparable, then each subgraph G_i is known as a component part or component of the graph G.

VITA

Lewis Gildart Minor

Candidate for the Degree of

Doctor of Philosophy

Thesis: SYNTHESIS OF DISTRIBUTED NETWORKS

Major Field: Electrical Engineering

Biographical:

Personal Data: Born on April 28, 1941, in Shreveport, Louisiana, the son of Clyde R. and May M. Minor, Jr.

Education: Attended primary and secondary schools in Shreveport, Louisiana; graduated from Sewanee Military Academy, Sewanee, Tennessee in May, 1959; received the Bachelor of Electrical Engineering degree from the Georgia Institute of Technology in June, 1964; received a Master of Science degree from the Louisiana Polytechnic Institute in August, 1965; with a major in Electrical Engineering; completed requirements for the Doctor of Philosophy degree in August, 1969.

Professional Experience: Employed by Oklahoma State University as a graduate assistant from September, 1965 to June, 1969, specific assignments included: the computer evaluation of bistable flip-flops, statistical evaluation of relay contact data, electrolytic tank studies of aircraft antennas, and distributed network analysis and synthesis.

Professional Organizations: Member of Institute of Electrical and Electronics Engineers.