THE PERMANENT FUNCTION

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CHAPTER I

INTRODUCTION

Definition of the Permanent

There are many scalar functions which can be associated with a square matrix such as the determinant, permanent, rank, etc. Of these by far the most important seems to be the determinant function. Much is known about the determinant, and a great many papers and books have been published stating the properties of this function and its applications to practical problems. Very little has been done in developing the theory of other scalar functions except in specific cases where just enough is done to make them useful in a particular area of study. One reason for this is the fact that most of the scalar functions do not lend themselves to such rich application as does the determinant function. Thus, the other functions have been pushed to the background and will make an appearance only occasionally. One such function which has received renewed interest in the last few years has been the permanent function.

Let A be an n-square matrix with elements a_{ij}, i, j=1, . . . , n, belonging to the complex field. The following definition can now be stated.

<u>Definition 1.1</u>. The permanent of an n-square matrix A, denoted as p(A), is defined as

$$p(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all n! permutations σ of the numbers $1, \ldots, n$, and $\sigma(i)$ denotes the i-th number in a given permutation.

For example, if $n = 4$, the expression $a_{13}a_{22}a_{34}a_{41}$ is an addend in the expansion of p(A). There would be 24 such addends, that is, one addend for each permutation of the numbers 1, 2, 3, 4. The sum of the 24 addends would be p(A) for this case.

Another way of stating Definition 1.1 is the following. Write down all possible products, each of n factors, that can be obtained by picking one and only one element from each row and from each column. There will be n! such products. The algebraic sum of these addends is the value of or the expansion of p(A). Thus, it is easy to see that

$$p(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\,\sigma(i)} = \sum_{\sigma} \prod_{i=1}^{n} a_{\sigma(i)i} \, . \tag{1}$$

It will be advantageous to use both forms of (1).

Notation

An old notation for p(A) given by Thomas Muir [34] in 1882 is the following. If A is an n-square matrix then

$$p(A) = \overset{+}{\left|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\right|}^{+} .$$

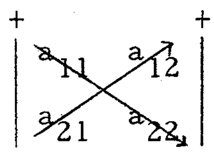
Thus,

$$\left|\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\right|^{+} = a_{11}a_{22} + a_{21}a_{12},$$

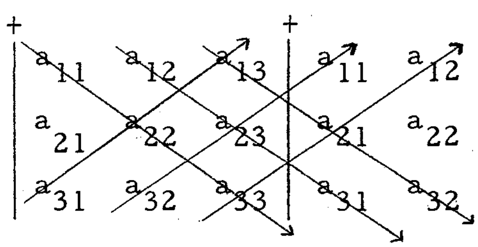
and

$$\left|\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right|^{+} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} .$$

A simple computation rule can be used to evaluate both of the above second and third order permanents. For the second order permanent we have

p(A) =  .

That is, the permanent can be found by taking the sum of the products of the elements lying on the arrows. For the third order permanent the first two columns of A are written again and then the value of p(A) can be found by summing the products of the elements lying on the arrows as given below:

p(A) =  .

This type of computational device is not valid for $n > 3$, since all of the addends of p(A) cannot be obtained in this manner.

Some additional notation which will be used throughout this paper will now be stated.

Define $M_{m,n}$ to be the class of all mxn matrices with elements in the complex field. Elements of $M_{m,n}$ will be denoted with capital letters such as A, B, X, etc., while elements of the respected matrices will be denoted with small letters such as a_{ij}, b_{ij}, x_{ij}, etc. In statements involving the matrix A we will sometimes write $A = (a_{ij})$ where a_{ij} denotes the element of A in the i-th row and j-th column. We will reserve the notation E_{ij} to be the element of $M_{m,n}$ with zeros in every position except the ij-th position in which case $e_{ij} = 1$. Thus, if $M_{m,n}$ is restricted to matrices with real elements, then E_{ij}, $i=1,\ldots,m$, $j=1,\ldots,n$, forms a basis for $M_{m,n}$ and $M_{m,n}$ is isomorphic to euclidean mn-space. That is, $M_{m,n}$ with the operations of matrix addition and scalar multiplication is isomorphic to E^{mn}. Also the topology of $M_{m,n}$ is induced from E^{mn}, namely, a set is open in $M_{m,n}$ if it is the image of an open set in E^{mn}.

If $E_{ij} \in M_{n,n}$ then a matrix of the form

$$P = \sum_{i=1}^{n} E_{i\sigma(i)}$$

for some permutation σ of $1,\ldots,n$ is called an n-square permutation matrix. Multiplying on the left of an mxn matrix A by an m-square permutation matrix rearranges the rows of A according to the permutation σ of $1,\ldots,m$. Multiplying A on the right by an n-square permutation matrix rearranges the columns of A according to the permutation σ of $1,\ldots,n$.

The notation for the determinant of an n-square matrix A will be d(A) and

$$d(A) = \sum_{\sigma} \epsilon(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} ,$$

where $\epsilon(\sigma) = \pm 1$ according to whether the permutation σ is even or odd.

The usual notation for rank of a matrix will be used. That is, $\rho(A)$ will denote the rank of A.

History of the Permanent

The name for the permanent function seems to have originated in a publication by Cauchy in 1812. In that article he considered functions which are not changed by permutations of variables calling them symmetric functions. For example, the functions

$$f(a_1, a_2, b_1, b_2) = a_1 b_2 + a_2 b_1,$$

$$g(a_1, a_2, a_3, b_1, b_2, b_3) = a_1 b_2 + a_2 b_3 + a_3 b_1 + a_2 b_1 + a_3 b_2 + a_1 b_3,$$

are not changed if the variables a_i and b_i undergo the same permutations. That is, if (a_1, a_2, a_3) is permuted to (a_3, a_1, a_2) and (b_1, b_2, b_3) is permuted to (b_3, b_1, b_2) then

$$g(a_1, a_2, a_3, b_1, b_2, b_3) = g(a_3, a_1, a_2, b_3, b_1, b_2).$$

Cauchy also noted that functions such as

$$h(a_1, a_2, b_1, b_2) = a_1 b_2 - a_2 b_1,$$

may differ by a $\pm$ sign whenever a permutation is given for the variables a_1, a_2 and b_1, b_2. Thus, if it were not for this alternation in sign these functions would also be symmetric. He then decides to extend his definition of symmetric functions to include those which may change sign after a given permutation. Thus, this calls for some way of distinguishing between the two basic types of symmetric functions.

The ones which may change sign are called "fonctions symetriques alternees", while those which do not change sign are called "fonctions symetriques permanentes". His notation for the functions f and g are:

$$f(a_1, a_2, b_1, b_2) = S^2(a_1 b_2),$$

$$g(a_1, a_2, a_3, b_1, b_2, b_3) = S^3(a_1 b_2).$$

His definition of the permanent is more general than what is used today. Using his notation the only symmetric functions which are regarded as permanents at the present time are $S^2(a_1 b_2), S^3(a_1 b_2 c_3), S^4(a_1 b_2 c_3 d_4), \ldots,$ where every term involves the full number of letters.

From 1812 to 1882 very little was done in developing any theory directly connected with the study of the permanent function. Mostly, the results in this period involve very special identities between permanents and determinants. Various notation and names for the permanent were used. We have already discussed Cauchy's notation. In 1857, A. Cayley gave the notation for p(A) as

$$\left\{\begin{matrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{matrix}\right\}$$

and proved a very special identity for the product of a determinant and permanent, (see problem 2, Chapter IV).

In 1865, J. Horner used the notation

$$p(A) = \left|\overline{\begin{matrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{matrix}}\right.$$

and proved an identity expressing the product of two determinants as a sum of "conterminants" (permanents). This identity can be found in Theorem 2.20.

J. Hammond in 1879, published an article in which he proposed a problem about permanents, calling them "alternate determinants". This seemed to prompt Thomas Muir [34] in 1882 to prove part of Hammond's problem (see Theorem 2.17) and to give a summary of results about permanents to that date. In Muir's article the name and notation were standardized, and Definition 1.1 was introduced.

From 1882 to 1913 a few interesting identities relating permanents to determinants were proved, (see problem 3, Chapter IV and Theorem 2.15).

In 1913, a problem proposed by G. Polya [39] in which the question of whether or not - signs can be affixed to elements of a square matrix so that the determinant of this matrix has the same value as the permanent of the original matrix has led to an interesting generalization by Marvin Marcus and Henryk Minc, (see Theorem 2.22 and results which follow).

The next important event came in the form of another problem proposed in 1926 by B. L. Van der Waerden [44]. He wanted to find the minimum value of the permanent over a certain class of matrices, (see Conjecture 3.2). His question has resulted in several present day research papers but still remains unanswered.

From 1926 to sometime in the mid-fifties the permanent function seemed to be in hibernation. The awakening came with the permanent assuming a more important role in certain combinatorial applications and a general concern on the part of several men to solve the Van der Waerden conjecture.

Synopsis

This dissertation is an exposition of the properties of the permanent function. As we have seen from the section on the history of the permanent it is a subject which was first viewed as something akin to the determinant but not nearly as useful. This attitude has largely been the result of a lack of application of the permanent to any practical problems. With applications being found in combinatorial theory it has caused the subject to assume an importance which heretofore it did not have. These applications also bring out a concern among mathematicians as to the amount of knowledge that is known about the permanent. Thus, in the last ten years the knowledge of the permanent has increased tremendously and approximately 80 to 90% of all publications have occurred in this period. This increase in knowledge in such a short time brings with it many problems. First, there is a notable lack of organization in the published material as a whole. In some cases this involves articles which give some interesting results but failed to prove the main theorem which the author was really after. Then a later article actually proves this theorem and leads then to some confusion as to how the first results fit into the overall picture. This type of situation seems to be especially true in many papers involving inequalities for the permanent. Also another factor which contributes to the lack of organization is the fact that the results are widely scattered and appear in many different mathematical periodicals. As should be expected though, research does not come in organized form but in bits and pieces with some of the pieces seemingly having little to do with each other. Thus, this dissertation is an attempt to organize the known results so that study in this area will be made easier and can be continued at perhaps an even faster rate.

Secondly, there has been no attempt made in present literature to give a fairly complete list of the statements and proofs of many of the elementary properties concerning the permanent function. Most of the elementary properties which are known usually occur as they are needed in the proofs of some of the more complicated results. Of course the easier theorems are the logical place to begin a study of the permanent so a publication is needed to give the proper beginning for this subject. This dissertation is an attempt to fulfill this need since it contains a good many of the elementary theorems concerning permanents and gives the student a background from which he can prove or at least understand the proofs of most results about permanents in the literature.

Thirdly, many articles about permanents begin with the statement that permanents have certain combinatorial applications and never give an example of how they are used. With the one exception of Ryser's book [40] almost no examples are given which illustrate the usefulness of the permanent. Thus, this dissertation fulfills a need to show the student some interesting examples which should leave no doubt as to what types of problems the permanent can be of maximal use. These examples can be found in Chapter IV.

The material covered in Chapter II is especially designed for the student beginning a study of the permanent function for the first time. It begins with elementary theorems concerning the permanent and leads to the more difficult question as to how the permanent and determinant are related. Several theorems are proved which involve some special identities between permanents and determinants, and then it is shown that the permanent cannot be transformed in a systematic way into the determinant and vice versa. The concluding part of this chapter characterizes the types of transformations which leave the permanent unaltered.

Chapter III deals with inequalities concerning the permanent. The first part of the chapter gives some elementary inequalities concerning non-negative matrices, and then we become more restrictive and consider only the class of doubly stochastic matrices. There is a great deal known about this class of matrices, and the discussion of them is quite lengthy. The outstanding problem connected with permanents is the unresolved conjecture of Van der Waerden concerning the minimum of the permanent over the set of doubly stochastic matrices. A list of the known results concerning this conjecture is given. From here it is shown how the permanent can be thought of as an inner product in a certain type of vector space. This expression of the permanent as an inner product along with the Cauchy-Schwarz inequality enables us to prove certain inequalities of importance for the permanent. The last part of the chapter concerns some general theorems along with a summary of the better known miscellaneous inequalities.

Chapter IV is concerned with the application of the permanent to combinatorial problems, and several examples are given. Also the existing methods of computing the permanent are discussed along with some inequalities for (0, 1) matrices which give some upper and lower bounds for the permanent in many cases involving problems of application. The chapter concludes with a list of problems concerning permanents of two types. The first type consists of solved problems which are usually difficult and interesting but in most cases not important enough to be included in the main body of this dissertation. The second type gives some unsolved problems and indicates the direction of current research.

CHAPTER II

PROPERTIES OF THE PERMANENT

Introduction

The definition of the permanent function suggests that there are a number of properties of the determinant which carry over and give similar results for the permanent. In fact, this similarity seems to account for the reason that so little has been published concerning the more elementary properties of the permanent. Most of the publications deal with some rather sophisticated results and assume a knowledge of the simpler properties. Thus, we shall begin with a fairly comprehensive list of the elementary results of the permanent along with the proofs of these results. While many of these properties are direct analogs of results of the determinant function, there are also a good many results which are changed considerably or have no analog in determinant theory.

Elementary Properties of the Permanent

Theorem 2.1. If A is an n-square matrix and A' is its transpose then $p(A) = p(A')$.

Proof: Since the products involved in the permanent of A are exactly the same products involved when all the rows and columns are interchanged this gives $p(A) = p(A')$.

This theorem is important in that for every theorem in permanents concerning the rows of a matrix there is a corresponding theorem concerning the columns of the matrix, and vice versa.

Theorem 2.2. If A is an n-square matrix and A* is its conjugate transpose then $\overline{p(A)} = p(A^*)$ where $\overline{p(A)}$ denotes the conjugate of p(A).

Proof: By Theorem 2.1 the transpose does not affect the value of the permanent. Hence, $p(A^*)$ consists of sums and products of the conjugates of elements of p(A). From complex analysis the conjugate of the sum of two or more complex numbers is equal to the sum of the conjugates, and the conjugate of the product is equal to the product of the conjugates. Therefore, $p(A^*) = \overline{p(A)}$.

Theorem 2.3. If A is an n-square matrix such that A has a row (or column) of zeros then p(A) = 0.

Proof: By the definition of the permanent it consists of the sum of the products of elements one from each row and from each column. Since one row is zero then each of these products contains a factor of zero and thus is zero. Hence the sum of the products is zero and the theorem follows.

Theorem 2.4. If in the square matrix A one row (or column) is multiplied by the constant c then p(A) is multiplied by c.

Proof: Let B be the matrix identical to A except the i-th row, which is the i-th row of A multiplied by c. In the expansion of p(B) every addend must contain a factor from the i-th row. Thus, every addend contains the factor c. If the factor c is left out then the resulting addends are just addends from p(A). Hence, p(B) = cp(A).

Theorem 2.5. If A is an n-square matrix and c is a constant then $p(cA) = c^n p(A)$.

Proof: Since cA multiplies every row by c and there are n rows it follows from Theorem 2.4 that $p(cA) = c^n p(A)$ and the proof is complete.

We note here that the first few theorems we have proved are direct analogs of theorems in determinant theory. The next theorem and its corollary are examples of results which are distinctly different from those obtained in determinants.

Theorem 2.6. If A is an n-square matrix such that every row (or column) is the same then

$$p(A) = n! \prod_{i=1}^{n} a_{ii} .$$

Proof: Since every row is the same, A can be written as

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} .$$

In the expansion of p(A) we must have an element from each row and column. This implies that every addend must be $a_1 a_2 \ldots a_n$. Since there are n! addends,

$$p(A) = n! \prod_{i=1}^{n} a_i .$$

But $a_{ii} = a_i$ for every i and the theorem follows.

<u>Corollary 2.7.</u> If A is an n-square matrix such that every entry is a, then $p(A) = n!a^n$.

Proof: This is a direct consequence of Theorem 2.6.

We now state a theorem which gives us the value of the permanent of certain special types of matrices. The proof of this theorem is just a direct consequence of the definition of the permanent.

<u>Theorem 2.8.</u> Let D be an n-square diagonal matrix, P an n-square permutation matrix, and T an n-square triangular matrix. Then

(a) $p(D) = \prod_{i=1}^{n} d_{ii}$,

(b) $p(P) = 1$,

(c) $p(T) = \prod_{i=1}^{n} t_{ii}$.

<u>Theorem 2.9.</u> If A is an n-square matrix, D and L are n-square diagonal matrices then

$$p(DAL) = p(D)p(A)p(L) = p(A) \prod_{i=1}^{n} d_{ii} l_{ii} .$$

Proof: Consider the matrix DAL. Since D and L are diagonal matrices, the i-th row of A is multiplied by d_{ii} of D while the i-th column of A is multiplied by l_{ii} of L. Therefore, using Theorem 2.4 we have

$$p(DAL) = p(A) \prod_{i=1}^{n} d_{ii} l_{ii} .$$

Now using Theorem 2. 8,

$$p(A) \prod_{i=1}^{n} d_{ii} l_{ii} = \left(\prod_{i=1}^{n} d_{ii} \right) p(A) \left(\prod_{i=1}^{n} l_{ii} \right) = p(D)p(A)p(L)$$

and the proof is complete.

<u>Theorem 2. 10.</u> If P and Q are n-square permutation matrices and A is an n-square matrix then $p(PAQ) = p(A)$.

Proof: Multiplication by permutation matrices P and Q rearranges the rows and columns of A but leaves the addends in the expansion of p(A) unchanged. Therefore,

$$p(PAQ) = p(A).$$

Theorem 2. 10 gives us an important result about permanents. Namely, that interchanging rows or columns of a matrix A does not change the value of the permanent of A. This result is not true for determinants and gives a good example of how these two scalar functions differ.

We now introduce some notation which will be used in the theorems that follow. If A is an mxn matrix then let the p-th row of A be denoted by $A_{(p)}$. Let $Q_{r,n}$ be the totality of strictly increasing sequences of r integers chosen from $1, \ldots, n$. Thus if n=3 and r = 2 then $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$. Next, let $G_{r,n}$ be the totality of non-decreasing sequences of r integers chosen from $1, \ldots, n$. Then $G_{2,3} = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\}$. It is also useful to assign an ordering to the sequences in $Q_{r,n}$ and $G_{r,n}$. This is called the lexicographic ordering and is defined as follows.

Definition 2.11. If α and β are sequences in $Q_{r,n}$ (or $G_{r,n}$), $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ then α is said to precede β or $\alpha < \beta$, if there exists an integer t, $(1 \leq t \leq r)$, for which

$$\alpha_1 = \beta_1, \ldots, \alpha_{t-1} = \beta_{t-1}, \quad \alpha_t < \beta_t .$$

For example, (1, 1, 2, 5) precedes (1, 2, 2, 5) in $G_{4,5}$.

Now if $\alpha \in Q_{r,m}$ and $\beta \in Q_{r,n}$ then let $A[\alpha/\beta]$ be the submatrix obtained from A by using the rows in the sequence α and the columns in the sequence β. Also let $A(\alpha/\beta)$ be the submatrix obtained from A by deleting the rows in the sequence α and columns in the sequence β. For example, if A is a 3x4 matrix and $\alpha = (1, 3)$, $\beta = (1, 2, 4)$ then $A[\alpha/\beta]$ is the 2x3 matrix obtained from A by using rows 1 and 3 and columns 1, 2, and 4. We can also extend the definition of $A[\alpha/\beta]$ to the case where α and β are sequences in $G_{r,n}$ by simply allowing repetitions in the choices for the rows and columns of A.

Definition 2.12. If from an n-square matrix A we delete the i-th row and the j-th column the permanent of the (n-1)-square matrix will be called the major of the element a_{ij}, and is denoted by $p(A(i/j))$.

Theorem 2.13. If A is an n-square matrix then p(A) is equal to the sum of the products of the elements of any row (or column) of A, each by its major; that is,

$$p(A) = \sum_{j=1}^{n} a_{ij} p(A(i/j)) = \sum_{i=1}^{n} a_{ij} p(A(i/j)).$$

Proof: To prove p(A) can be expanded by rows, consider the addends of p(A). Each of these addends contains an element from every row of the matrix A. Hence, for the i-th row each addend of the permanent of A contains one of the elements $a_{i1}, a_{i2}, \ldots, a_{in}$. Therefore the permanent of A is a linear combination of these elements; that is,

$$p(A) = \sum_{j=1}^{n} C_j a_{ij} .$$

Consider C_j for some $j = 1, \ldots, n$. Then C_j is the coefficient of a_{ij} and there are exactly (n - 1)! terms which have a_{ij} in them. This is true since the definition of the permanent allows only one element from each row and column to appear in each term. Thus the terms involving a_{ij} are

$$\sum^{(n-1)!} \prod_{\substack{k=1 \\ k \neq i}}^{n} a_{k\sigma(k)} a_{ij}$$

where σ is some permutation of the numbers $1, \ldots, j-1, j+1, \ldots, n$, and the summation extends over all (n-1)! permutations. But this is the same as

$$a_{ij} \sum^{(n-1)!} \prod_{\substack{k=1 \\ k \neq i}}^{n} a_{k\sigma(k)}$$

which is just the permanent of the matrix A with the i-th row and j-th column deleted times the element a_{ij}. Hence

$$C_j = p(A(i/j))$$

and

$$p(A) = \sum_{j=1}^{n} a_{ij} p(A(i/j)) .$$

The above expression is called the expansion of the permanent of A according to its i-th row. The expansion according to the i-th column is established in the same way.

Theorem 2.14. If A is an n-square matrix such that

$$A_{(p)} = a_{p1} + b_{p1}, a_{p2} + b_{p2}, \ldots, a_{pn} + b_{pn}$$

then

$$p(A) = p(B) + p(C)$$

where

$$B_{(p)} = a_{p1}, a_{p2}, \ldots, a_{pn}$$

and

$$C_{(p)} = b_{p1}, b_{p2}, \ldots, b_{pn},$$

and the remaining rows of B and C are the same as the corresponding rows of A.

Proof: Expanding all three permanents by the p-th row we see that

$$p(A) = \sum_{i=1}^{n} (a_{pi} + b_{pi})\, p(A(p/i))$$

$$= \sum_{i=1}^{n} a_{pi} p(A(p/i)) + \sum_{i=1}^{n} b_{pi}\, p(A(p/i))$$

$$= p(B) + p(C).$$

Theorems 2.13 and 2.14 are both direct analogs of theorems in determinant theory. We now show by example that one of the more important properties of determinants, that is, invariance of the determinant under addition of a multiple of a row (column) to another row (column) fails in the theory of permanents. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}.$$

Then B is the matrix A except that the second row of B is the sum of the first and second rows of A. Now $p(A) = 13$, but $p(B) = 17$.

The reason for the failure of the above mentioned theorem in permanent theory is the fact that a permanent of a matrix with two or more rows the same is not necessarily zero (see Theorem 2.6). This prompts the following theorem which is a variation of an early result given by F. Ferber in 1899.

<u>Theorem 2.15.</u> If A is an n-square matrix such that A has m rows, $m \leq n$, (or columns) which are the same, then

$$p(A) = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}$$

has at most $n!/m!$ distinct addends.

Proof: If $m = n$ then by Theorem 2.6, p(A) has only one addend. Thus, assume $m < n$. If the first m rows of A are not the same then multiplication by a suitable permutation matrix P will rearrange A such that its first m rows are the same. This multiplication has no effect on the addends of p(A) since by Theorem 2.10, $p(PA) = p(A)$. Now expanding p(A) by the last row gives n addends each consisting of an element in the last row times an $(n-1)$-th degree permanent made up from the first $n-1$ rows of A. Again expand each of these $(n-1)$-th degree permanents by their last row. This will give n(n-1) addends consisting of an element which is a product of an element in the n-th and (n-1)-th rows and a (n-2)-th degree permanent made up from the first n-2 rows. Continue this expansion process until there are n(n-1)...(m+1) addends each consisting of an element which is a product of elements from the last n-m rows of A and a m-th degree permanent which is made up from the first m rows of A. Since the first m rows of A are the same, then the rows of these m-th degree permanents are the same. Now applying Theorem 2.6, each m-th degree permanent has one distinct addend. Therefore, p(A) has at most $n(n-1)\ldots(m+1) = n!/m!$ distinct addends and the proof is complete.

We now prove a theorem which is attributed to the French mathematician Laplace since it is analogous to the Laplace expansion theorem concerning determinants.

Theorem 2.16. (Laplace expansion theorem for permanents) Let A be an n-square matrix and r an integer such that $1 \leq r \leq n$. Let $\alpha \in Q_{r,n}$. Then

$$p(A) = \sum_{\beta \in Q_{r,n}} p(A[\alpha/\beta])\, p(A(\alpha/\beta)).$$

Proof: Let r be given and suppose $1 \leq r \leq n$. Let $\alpha = (1, \ldots, r)$. Now consider the permanent $p(A[\alpha/\alpha])$ in the upper left hand corner of A. The addends of this permanent are of the form $a_{1i_1}\ldots a_{ri_r}$ where $i_1, \ldots, i_r$ is a permutation of the numbers $1, \ldots, r$. For the permanent $p(A(\alpha/\alpha))$ in the lower right hand corner of A the addends are of the form $a_{(r+1)i_{(r+1)}}, \ldots, a_{ni_n}$ where $i_{r+1}, \ldots, i_n$ is a permutation of the numbers r+1, . . . , n. Consider the product, $p(A[\alpha/\alpha])p(A(\alpha/\alpha))$. Since all the r! addends of the first element of the product as well as the (n-r)! addends from the second element of the product are formally distinct, r!(n-r)! distinct addends of p(A) are obtained.

Next consider $\beta \in Q_{r,n}$ such that $\beta \neq \alpha$. Then by multiplying on the right by the proper permutation matrix the columns associated with the sequence β can be made to be the first r columns of the matrix A. Thus, $A[\alpha/\beta]$ is brought to the upper left hand corner. Also at the same time the matrix $A(\alpha/\beta)$ is brought to the lower right hand corner. By the same process as above, r!(n-r)! formally distinct addends of p(A) are obtained. These addends are also distinct from the above addends since different columns are used. Since there are n!/r!(n-r)! ways of forming r!(n-r)! distinct products of p(A), all of the n! addends of p(A) are obtained. Therefore, Laplace's theorem is established whenever $\alpha = (1, \ldots, r)$.

Next, suppose $\alpha = (k_1, \ldots, k_r)$ where $\alpha \in Q_{r,n}$. By multiplying on the left by a permutation matrix the rows associated with the sequence α again represents the first r rows of A. In the same way as in the preceding paragraph we have

$$p(A) = \sum_{\beta \in Q_{r,n}} p(A[\alpha/\beta])p(A(\alpha/\beta)).$$

Thus, Laplace's theorem is completely established except for the case r = n. In this case the term on the right in the conclusion of the theorem becomes p(A), and the theorem is true.

Consider the following example of the Laplace expansion for permanents. Let

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ -1 & 1 & 4 & -1 \\ 3 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 \end{bmatrix}$$

and r = 2. Let $\alpha = (1, 2)$. Then

$$Q_{2,4} = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\},$$

and

$$\begin{aligned} p(A) = {} & \overset{+\qquad +}{\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}} \overset{+\qquad +}{\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}} + \overset{+\qquad +}{\begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix}} \overset{+\qquad +}{\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}} + \overset{+\qquad +}{\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}} \overset{+\qquad +}{\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix}} \\ & + \overset{+\qquad +}{\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix}} \overset{+\qquad +}{\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}} + \overset{+\qquad +}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} \overset{+\qquad +}{\begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix}} + \overset{+\qquad +}{\begin{vmatrix} 0 & 1 \\ 4 & -1 \end{vmatrix}} \overset{+\qquad +}{\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix}} \\ = {} & (1)(8) + (8)(1) + (-3)(-1) + (4)(8) + (0)(12) + (4)(-1) \\ = {} & 47. \end{aligned}$$

Laplace's theorem and the expansion of the permanent by rows or columns gives two ways to find the value of any permanent besides that of using the definition. Another way of thinking of the value of a permanent was given by Thomas Muir [34] in 1882. It is the following theorem.

<u>Theorem 2.17.</u> If A is an n-square matrix then p(A) is equal to the coefficient of $X_1 X_2 \ldots X_n$ in the expansion of

$$\prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} X_j .$$

Proof: The proof is by induction. For $n=1$, $A=(a_{11})$ and $p(A) = a_{11}$. Also

$$\prod_{i=1}^{1} \sum_{j=1}^{1} a_{ij}X_j = a_{11}X_1$$

and the coefficient of X_1 is the permanent of A. Hence the theorem is true for $n=1$.

Now assume the theorem true for $n=k$. Consider the following product,

$$B = \prod_{i=1}^{k} \sum_{j=1}^{k+1} a_{ij}X_j .$$

By grouping and using the assumption that the theorem is true for $n=k$ the coefficients of

$$\begin{array}{l} X_1X_2 \cdots X_k \\ X_1X_2 \cdots X_{k-1}X_{k+1} \\ \qquad\vdots \\ X_2X_3 \cdots X_{k+1} \end{array} \tag{2}$$

can be obtained. These are $p(A(k+1/k+1))$, $p(A(k+1/k))$, ..., $p(A(k+1/1))$ respectively. Thus

$$\prod_{i=1}^{k+1} \sum_{j=1}^{k+1} a_{ij}X_j = (a_{k+1\,1}X_1 + a_{k+1\,2}X_2 + \cdots + a_{k+1\,k+1}X_{k+1})B.$$

To find the coefficient of $X_1, X_2 \ldots X_{k+1}$ the first term of (2) must be multiplied by $a_{k+1\,k+1}X_{k+1}$. The second term of (2) must be multiplied by $a_{k+1\,k}X_k$. Finally, the last term of (2) must be multiplied by $a_{k+1\,1}X_1$. This gives the following coefficient of $X_1X_2 \ldots X_{k+1}$,

$$\sum_{i=1}^{k+1} a_{k+1i}p(A(k+1/i)).$$

But this is just the expansion of the permanent of A by the (k+1)-th row. Therefore, the theorem is true for k+1. Thus, by induction it is true for all positive integers.

A clever way of obtaining the coefficient in Theorem 2.17 is given by Herbert S. Wilf [46]. Let

$$f(X_1, \ldots, X_n) = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}X_j .$$

Then

$$\frac{\delta^n f}{\delta X_1 \delta X_2 \ldots \delta X_n}$$

is the coefficient of $X_1X_2 \ldots X_n$, and thus the permanent of A by the above theorem.

Product Theorems

Let us now prove a theorem concerning the permanent of the product of two matrices. This theorem is an analog of the Binet-Cauchy theorem for the determinant of the product of two matrices which states that if $A \in M_{m,n}$ and $B \in M_{n,m}$ with $1 \leq m \leq n$, then

$$d(AB) = \sum_{\alpha \in G_{m,n}} d(A[1, \ldots, m/\alpha])d(B[\alpha/1, \ldots, m]).$$

We shall need this theorem later.

<u>Theorem 2.18.</u> (Binet-Cauchy theorem for permanents) Let $A \in M_{m,n}$ and $B \in M_{n,m}$ with $1 \leq m \leq n$. Then $C = AB \in M_{m,m}$ and

$$p(C) = \sum_{\alpha \in G_{m,n}} \frac{p(A[1, \ldots, m/\alpha])p(B[\alpha/1, \ldots, m])}{u(\alpha)}$$

where $u(\alpha)$ is the product of the factorials of the multiplicities of the distinct integers appearing in the sequence α.

Proof: The product matrix C can be written as

$$\begin{bmatrix} \Sigma\, a_{1t_1} b_{t_1 1} & \cdots & \Sigma\, a_{1t_m} b_{t_m m} \\ \vdots & & \vdots \\ \Sigma\, a_{mt_1} b_{t_1 1} & \cdots & \Sigma\, a_{mt_m} b_{t_m m} \end{bmatrix}$$

where each t_i, $i=1, \ldots, m$, is summed from 1 to n. Since each of the columns is a sum of n elements, Theorem 2.14 says that the permanent of C can be rewritten as a sum of n^m permanents, each of the following form:

$$\left|\begin{matrix} a_{1t_1} b_{t_1 1} & \cdots & a_{1t_m} b_{t_m m} \\ \vdots & & \vdots \\ a_{mt_1} b_{t_1 1} & \cdots & a_{mt_m} b_{t_m m} \end{matrix}\right|^{+} \qquad (3)$$

After the common elements have been factored out of the respective columns, (3) can be written as

$$\left|\begin{matrix} a_{1t_1} & \cdots & a_{1t_m} \\ \vdots & & \vdots \\ a_{mt_1} & \cdots & a_{mt_m} \end{matrix}\right|^{+} (b_{t_1 1} \cdots b_{t_m m}), \qquad (4)$$

or

$$p(A[1, \ldots, m/t_1, \ldots, t_m])b_{t_1 1} \cdots b_{t_m m}.$$

Now let $\alpha = t_1 \leq t_2 \leq \ldots \leq t_m$ be a particular selection of m of the numbers $1, \ldots, n$. Then consider the sum

$$\Sigma \quad p(A[1, \ldots, m/\alpha])b_{t_1 1} \cdots b_{t_m m} \tag{5}$$

where the summation extends over the m! permutations of the numbers $t_1, \ldots, t_m$. Since the permanent is unaltered by interchanging columns, (5) can be rewritten as

$$p(A[1, \ldots, m/\alpha]) \;\; \Sigma\, b_{t_1 1} \cdots b_{t_m m}.$$

But $\Sigma\, b_{t_1 1} \cdots b_{t_m m}$ is just the value of the permanent given by $p(B[\alpha/1, \ldots, m])$.

Now consider the terms of the product matrix C like those of (4) which contain the permanent $p(A[1, \ldots, m/\alpha])$. If the numbers $t_1, \ldots, t_m$ are distinct then the m! permutations of these numbers are distinct. But if $t_i = t_j$ for some i and j then there are not m! distinct permutations. Thus the $\Sigma\, b_{t_1 1} \cdots b_{t_m m}$ counts some factors more than once. For example, if $\alpha = (1, 1, 2, 2, 2, 3)$ then there are 6! permutations of α of which only $60 = 6!/2!3!$ are distinct. Therefore, define $u(\alpha)$ to be the product of the factorials of the multiplicities of the distinct integers appearing in the sequence α; e.g., $u(1, 1, 2, 2, 2, 3) = 2!3!$. Then the sum

$$\frac{\Sigma\, b_{t_1 1} \cdots b_{t_m m}}{u(\alpha)}$$

is precisely the coefficient of $p(A[1, \ldots, m/\alpha])$ in the product matrix C.

Thus, if $G_{m,n}$ is the totality of non-decreasing sequences of m integers chosen from 1, . . . , n we have

$$p(C) = \sum_{\alpha \in G_{m,n}} \frac{p(A[1, \ldots, m/\alpha])p(B[\alpha/1, \ldots, m])}{u(\alpha)} .$$

As an example of the Binet-Cauchy theorem let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \\ 4 & -1 \end{bmatrix} .$$

Then n = 3 and m = 2 while

$$G_{2,3} = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3)\} .$$

Therefore,

$$\begin{aligned} p(AB) = {} & p(A[1,2/1,1])\, p(B[1,1/1,2]) + p(A[1,2/2,2])\, p(B[2,2/1,2]) \\ & + p(A[1,2/3,3])p(B[3,3/1,2]) + p(A[1,2/1,2])\, p(B[1,2/1,2]) \\ & + p(A[1,2/1,3])p(B[1,3/1,2]) + p(A[1,2/2,3])\, p(B[2,3/1,2]), \end{aligned}$$

and

$$\begin{aligned} p(AB) &= (-2)(0)/2 + (6)(2)/2 + (0)(-8)/2 + (-2)(-2) + (-2)(2) + (2)(3) \\ &= 12. \end{aligned}$$

Theorem 2.19. The product of two permanents of the n-th order is expressible as the sum of n! permanents of the same order. That is,

$$p(A)p(B) = \sum \begin{vmatrix} a_{11}b_{i_1 1} & \cdots & a_{1n}b_{i_1 n} \\ \vdots & & \vdots \\ a_{n1}b_{i_n 1} & \cdots & a_{nn}b_{i_n n} \end{vmatrix}^{+}$$

where the summation extends over the n! permutations $i_1, \ldots, i_n$ of the numbers $1, \ldots, n$.

Proof: Consider a given addend of the permanent on the right hand side. This gives

$$a_{j_1 1} b_{i_1 1} \quad \cdots \quad a_{j_n n} b_{i_n n}$$

where $j_1, \ldots, j_n$ is a particular permutation of $1, \ldots, n$. The summation then of $i_1, \ldots, i_n$ over the n! permutations of $1, \ldots, n$ gives the following product:

$$a_{j_1 1} \cdots a_{j_n n} \; \Sigma \; b_{i_1 1} \cdots b_{i_n n}.$$

Thus we see that p(B) is the coefficient of

$$a_{j_1 1} \cdots a_{j_n n}$$

for the given permutation $j_1, \ldots, j_n$. To find the other addends of the permanent on the right, the n! permutations of $j_1, \ldots, j_n$ over $1, \ldots, n$ are taken. This gives

$$\Sigma \; a_{j_1 1} \cdots a_{j_n n} p(B),$$

or p(A)p(B). Thus the theorem is established.

If A and B are m-square matrices then the results of the Binet-Cauchy theorem and Theorem 2.19 implies that the permanent of the product is not equal to the product of the permanents, that is, $p(AB) \neq p(A)p(B)$. Since this result is true for determinants this again shows that even though many analogous results from determinant theory are true for permanents many other properties do not follow. In general, any property from determinant theory which depends on the change of sign inherent in the definition of the determinant will not be true for permanents.

Theorem 2.20. (Horner, 1865). The product of two determinants of the n-th order is expressible as the sum of n! permanents of the same order. Thus,

$$d(A)d(B) = \Sigma\, (-1)^q \begin{vmatrix} a_{11}b_{i_1 1} & \cdots & a_{1n}b_{i_1 n} \\ \vdots & & \vdots \\ a_{n1}b_{i_n 1} & \cdots & a_{nn}b_{i_n n} \end{vmatrix}^{+}$$

where the summation extends over the n! permutations $i_1, \ldots, i_n$ of the numbers $1, \ldots, n$ and q is the number of inversions of $i_1, \ldots, i_n$ from the normal order $1, \ldots, n$.

Proof: Consider the principal addend of the permanent on the right hand side, that is,

$$(-1)^q a_{11}b_{i_1 1} \cdots a_{nn}b_{i_n n} = (-1)^q a_{11} \cdots a_{nn} b_{i_1 1} \cdots b_{i_n n},$$

where q is the number of inversions of $i_1, \ldots, i_n$ from the normal order $1, \ldots, n$. Taking the sum over the remaining permutations of $i_1, \ldots, i_n$ gives

$$a_{11} \cdots a_{nn} \sum^{n!} (-1)^q b_{i_1 1} \cdots b_{i_n n} = a_{11} \cdots a_{nn} d(B).$$

This is just the main diagonal addend of the determinant of A times the determinant of B.

In like manner consider any other addend of the permanent on the right hand side.

$$(-1)^q a_{j_1 1} b_{i_{j_1} 1} \cdots a_{j_n n} b_{i_{j_n} n} = (-1)^q a_{j_1 1} \cdots a_{j_n n} b_{i_{j_1} 1} \cdots b_{i_{j_n} n} . \tag{6}$$

The part of the product in (6) which involves the a's is an addend of the determinant of A if the factor $(-1)^p$ where p is the number of inversions of $j_1, \ldots, j_n$ from the normal order $1, \ldots, n$ is placed with the product. Now to put $i_{j_1}, \ldots, i_{j_n}$ into the order $i_1, \ldots, i_n$ requires another p inversions. Since $i_1, \ldots, i_n$ is then some permutation of $1, \ldots, n$, and it requires q inversions to place the i's in normal order, this means

$$(-1)^p a_{j_1 1} \cdots a_{j_n n} (-1)^{p+q} b_{i_{j_1} 1} \cdots b_{i_{j_n} n} \tag{7}$$

is an addend of the determinant of A multiplied by an addend of the determinant of B. When the sum is then taken over all possible permutations of the i's with respect to the j's an addend of d(A) is multiplied by d(B). The sum then taken over all possible permutations of the j's yields d(A)d(B). Since

$$(-1)^{2p+q} = (-1)^q,$$

the expression given by (7) is the same as the expression given by (6), and the proof is complete.

<u>Theorem 2.21.</u> The product of a permanent and a determinant both of the n-th order is expressible as the sum of n! determinants of the same order. Thus,

$$p(A)d(B) = \Sigma (-1)^q \begin{vmatrix} a_{11}b_{i_1 1} & \cdots & a_{1n}b_{i_1 n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1}b_{i_n 1} & \cdots & a_{nn}b_{i_n n} \end{vmatrix}^{+}$$

where the summation extends over the n! permutations $i_1, \ldots, i_n$ of the numbers $1, \ldots, n$, and q is the number of inversions of $i_1, \ldots, i_n$ from the normal order $1, \ldots, n$.

Proof: Consider an addend of the determinant on the right hand side, that is,

$$(-1)^q(-1)^p a_{j_1 1} \ \cdots \ a_{j_n n} b_{i_{j_1} 1} \ \cdots \ b_{i_{j_n} n}$$

where p is the number of inversions of $j_1, \ldots, j_n$ from the normal order $1, \ldots, n$, and q is the number of inversions of $i_1, \ldots, i_n$ from the normal order $1, \ldots, n$. But this can be written as

$$a_{j_1 1} \ \cdots \ a_{j_n n}(-1)^{p+q} b_{i_{j_1} 1} \ \cdots \ b_{i_{j_n} n},$$

where the a's constitute an addend of the permanent of A while the b's together with the element $(-1)^{p+q}$ constitute an addend of the determinant of B. To show the last statement observe that there are p inversions required to place $i_{j_1}, \ldots, i_{j_n}$ into the order $i_1, \ldots, i_n$ and then q inversion to place $i_1, \ldots, i_n$ into the order $1, \ldots, n$. Therefore when the summation is taken over all n! permutations of the i's for a given permutation of the j's, the following is obtained:

$$a_{j_1 1} \ \cdots \ a_{j_n n} d(B).$$

Considering the remaining addends of the determinant, that is, the permutations of the j's, then p(A)d(B) is obtained, and the proof is complete.

Notice that in Theorems 2.19 and 2.20 the conclusions are very similar except for the respective signs. Also Theorems 2.20 and 2.21 differ only in the fact that permanents are used in one case whereas determinants are used in the other. For example, if

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -3 \\ 2 & 1 \end{bmatrix}$$

then

$$p(A)p(B) = {}^{+}\begin{vmatrix} 3(-1) & 2(-3) \\ -1(2) & 4(1) \end{vmatrix}^{+} + {}^{+}\begin{vmatrix} 3(2) & 2(1) \\ -1(-1) & 4(-3) \end{vmatrix}^{+},$$

$$= -12 + 12 - 72 + 2$$

$$= -70$$

whereas,

$$d(A)d(B) = {}^{+}\begin{vmatrix} 3(-1) & 2(-3) \\ -1(2) & 4(1) \end{vmatrix}^{+} + (-1)\, {}^{+}\begin{vmatrix} 3(2) & 2(1) \\ -1(-1) & 4(-3) \end{vmatrix}^{+}$$

$$= -12 + 12 - [-72 + 2]$$

$$= 70.$$

Relationship Between the Permanent and the Determinant

At this point several similarities between the permanent and the determinant have been mentioned as well as some properties which are not common to both. Let us now examine this relationship in more detail. At first appearance the relationship seems to be a close one. In fact, the question might be asked, and rightly so, if there exist any transformations on n-square matrices that change the permanent into the determinant or vice versa. That is, does there exist an onto function γ: $M_{n,n} \to M_{n,n}$ such that for every $A \in M_{n,n}$, $p(A) = d(\gamma(A))$? We shall say the function γ changes the permanent into the determinant. If such a function γ exists, then all the theorems which hold for determinants would hold for permanents under this transformation, and there would be no need to examine permanents as a separate entity. For example, let γ be a ring homomorphism on n-square matrices such that for $A \in M_{n,n}$, $p(A) = d(\gamma(A))$. Then from determinant theory it is known that the product of the determinants is equal to the determinant of the product. Thus, if $B \in M_{n,n}$ then

$$p(AB) = d(\gamma(AB)) = d(\gamma(A)\gamma(B)) = d(\gamma(A))d(\gamma(B)) = p(A)p(B),$$

and the product of the permanents is equal to the permanent of the product. Thus, we have a theorem concerning permanents which parallels a theorem of determinant theory. Since we already know this is not true (see remarks after Theorem 2.19), there must not exist such a ring homomorphism. Now it cannot only be shown that no such ring homomorphism exists such as γ, but also no linear transformation of $M_{n,n}$ into itself exists which changes the permanent into the determinant. This is a fairly recent discovery of Marcus and Minc [18].

As early as 1913, G. Polya [39] was concerned with the problem of converting the permanent into the determinant by affixing $\pm$ signs to the elements. He found that this could be done in the 2-square case by considering the following mapping: let $\gamma : M_{2,2} \to M_{2,2}$ be such that if $A \in M_{2,2}$ then

$$\gamma(A) = \begin{bmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{bmatrix}. \tag{8}$$

Thus,

$$p(A) = d(\gamma(A)) \quad \text{or} \quad p(\gamma(A)) = d(A).$$

The answer to Polya's problem was given by Gabor Szego [40]. He proved that if $n > 2$, then there exists no uniform way of affixing $\pm$ signs to the elements of an n-square matrix so as to change the permanent into the determinant. This is stated in the following theorem.

Theorem 2.22. If $n > 2$, then there does not exist a function F of $M_{n,n}$ into itself such that for $A \in M_{n,n}$, $B = F(A)$ with $b_{ij} = \pm a_{ij}$ and $p(F(A)) = d(A)$.

Proof: The proof is by contradiction. Suppose the theorem is false and that by affixing the signs $(-1)^{e_{ij}}$ to the element a_{ij} of A the permanent of the matrix thus formed is the determinant of A. For this to be true the addend

$$(-1)^{e_{1i_1} + \cdots + e_{ni_n}} a_{1i_1} \cdots a_{ni_n}$$

must be an addend of the determinant of A. Thus

$$(-1)^{e_{1i_1} + \cdots + e_{ni_n}} = (-1)^q$$

where q is the number of inversions of $i_1, \ldots, i_n$ from the natural order $1, \ldots, n$. That is

$$e_{1i_1} + \ldots + e_{ni_n} \equiv q \bmod 2. \qquad (9)$$

There are n! such equations as (9), and the contradiction comes from showing the incompatibility of this system. Consider

$$\sum_{k=1}^{n-3} e_{kk} + e_{(n-2)i_\alpha} + e_{(n-1)i_\beta} + e_{ni_\gamma} \equiv q \bmod 2, \qquad (10)$$

where i_α, i_β, i_γ is some permutation of the numbers n-2, n-1, n, and q is the number of inversions required to put $1, \ldots, n-3, i_\alpha, i_\beta, i_\gamma$ into natural order. There are six such equations like (10), and their sum is as follows:

$$6 \sum_{k=1}^{n-3} e_{kk} + 2 \sum_{i,j=n-2}^{n} e_{ij} \equiv 3 \bmod 2.$$

But this is impossible, since an even number can never be congruent to an odd number modulo 2. Hence the theorem is true.

Before showing the result mentioned by Marcus and Minc, some background definitions and theorems are needed.

Definition 2.23. Let A be an mxn matrix and r be an integer such that $1 \leq r \leq \min(m,n)$, then the r-th permanental compound matrix, denoted as $P_r(A)$, is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose entries are $p(A[\alpha/\beta])$, $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$ arranged lexicographically in α and β.

Note, that $P_n(X) = p(X)$ and $P_1(X) = X$ if $X \in M_{n,n}$. As an example, let A be a 2x3 matrix and r=2, then

$$P_2(A) = (p(A[1,2/1,2]),\ p(A[1,2/1,3]),\ p(A[1,2/2,3])).$$

It is useful to have a notation for the entry of $P_r(A)$ in the (α, β) position. Define $A_{\alpha\beta}$ to be this element. Then $A_{\alpha\beta} = p(A[\alpha/\beta])$.

Some results about permanental compounds that will prove to be useful are contained in the following theorems.

Theorem 2.24. Let A be an mxn matrix and r an integer such that $1 \leq r \leq \min(m, n)$. Then $P_r(A') = (P_r(A))'$.

Proof: Let $\alpha \in Q_{r,n}$ and $\beta \in Q_{r,m}$. For the given sequences α and β consider the term $A'_{\alpha\beta}$ of $P_r(A')$. Thus $A'_{\alpha\beta} = p(A'[\alpha/\beta])$. But by the definition of transpose,

$$p(A'[\alpha/\beta]) = p(A[\beta/\alpha]) = A_{\beta\alpha} .$$

But $A_{\beta\alpha}$ is the entry of $P_r(A)$ in the position given by β and α. Thus the entry given by α and β in $(P_r(A))'$ is $A_{\beta\alpha}$, and the theorem is proved.

Theorem 2.25. Let A be an mxn matrix and r an integer such that $1 \leq r \leq \min(m, n)$. Then if D is a diagonal m-square matrix then $P_r(DA) = P_r(D)P_r(A)$.

Proof: Notice that for an m-square diagonal matrix D the entries of $P_r(D)$ are

$$D_{\alpha\beta} = 0 \qquad \text{if } \alpha \neq \beta,$$

$$D_{\alpha\beta} = d_{\alpha_1} \cdots d_{\alpha_r} \qquad \text{if } \alpha = \beta,$$

where $\alpha = (\alpha_1, \ldots, \alpha_r)$. Thus, consider an entry corresponding to a fixed $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$ of DA. Then

$$(DA)_{\alpha\beta} = p[(DA)[\alpha/\beta]] = d_{\alpha_1} \cdots d_{\alpha_r} p(A[\alpha/\beta]) = D_{\alpha\alpha} A_{\alpha\beta} .$$

Hence $P_r(DA) = P_r(D)P_r(A)$, and the proof is complete.

<u>Theorem 2.26</u>. If Q is an m-square permutation matrix, A is an mxn matrix, and r is an integer such that $1 \leq r \leq \min(m, n)$ then $P_r(QA) = P_r(Q)P_r(A)$.

Proof: Let Q be an m-square permutation matrix such that

$$q_{1i_1} \cdots q_{mi_m} = 1$$

where $i_1, \ldots, i_m$ is some permutation of $1, \ldots, m$. Then QA is the matrix whose k-th row, $k=1, \ldots, m$ is the i_k-th row of A. Thus $P_r(QA)$ is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose α-th row for some $\alpha \in Q_{r,m}$, $\alpha = (\alpha_1, \ldots, \alpha_r)$ is just

$$A_{(j_{\alpha_1}, \ldots, j_{\alpha_r})\beta}, \quad \beta \in Q_{r,n}$$

where $j_{\alpha_1}, \ldots, j_{\alpha_r}$ is a rearrangement of $i_{\alpha_1}, \ldots, i_{\alpha_r}$ such that $j_{\alpha_1} < \ldots < j_{\alpha_r}$. Now $P_r(Q)$ is the $\binom{m}{r} \times \binom{m}{r}$ matrix whose α-th row is

$$Q_{\alpha(j_{\alpha_1}, \ldots, j_{\alpha_r})} = 1$$

$$Q_{\alpha\beta} = 0, \quad \beta \neq (j_{\alpha_1}, \ldots, j_{\alpha_r}),$$

that is, $P_r(Q)$ is a permutation matrix. Consider $P_r(Q)P_r(A)$. Since $P_r(Q)$ interchanges rows of $P_r(A)$, the α-th row of $P_r(Q)P_r(A)$ becomes

$$A_{(j_{\alpha_1}, \ldots, j_{\alpha_r})\beta}, \quad \beta \in Q_{r,n}$$

and $P_r(QA) = P_r(Q)P_r(A)$ as was to be shown.

In order to help understand the proof of Theorem 2.26 consider the following example where $m = n = 4$, $r = 2$, and Q is a permutation matrix such that $q_{12}q_{23}q_{31}q_{44} = 1$. Then $i_1 = 2$, $i_2 = 3$, $i_3 = 1$, $i_4 = 4$ is a permutation of 1, 2, 3, 4. Thus QA is the matrix whose 1st, 2nd, 3rd, and 4th rows are just the 2nd, 3rd, 1st, and 4th rows of A. If $\alpha = (\alpha_1, \alpha_2) = (2, 3)$ then the α-th row of $P_2(QA)$ is just $A_{(1,3)\beta}$, $\beta \in Q_{2,4}$ where $(j_{\alpha_1}, j_{\alpha_2}) = (1,3)$ is a rearrangement of $(i_{\alpha_1}, i_{\alpha_2}) = (3, 1)$ so that $j_{\alpha_1} < j_{\alpha_2}$ which then is a term of $Q_{2,4}$. Now $P_2(Q)$ is the matrix whose (2, 3) row is

$$Q_{(2,3)(1,3)} = 1$$

and

$$Q_{(2,3)\beta} = 0$$

if $\beta \neq (1,3)$, $\beta \in Q_{2,4}$. Thus $P_2(Q)$ is a permutation matrix. Consider $P_2(Q)P_2(A)$. Under this product the (2, 3) row becomes

$$A_{(1,3)\beta}, \quad B \in Q_{2,4}$$

and

$$P_2(QA) = P_2(Q)P_2(A).$$

<u>Theorem 2.27</u>. Let A be an mxn matrix, Q an n-square permutation matrix, and D an n-square diagonal matrix. Let r be an integer such that $1 \leq r \leq \min(m, n)$. Then

(a) $P_r(AQ) = P_r(A)P_r(Q)$,

(b) $P_r(AD) = P_r(A)P_r(D)$.

Proof:

$$P_r(AQ) = (P_r((AQ)'))' = (P_r(Q'A'))' = (P_r(Q')P_r(A'))'$$
$$= (P_r(A'))'(P_r(Q'))' = P_r(A)P_r(Q),$$

by Theorems 2.24, 2.26, and properties of the transpose. The proof of (b) is similar.

Definition 2.28. Let A be an mxn matrix and r be an integer such that $1 \leq r \leq \min(m, n)$, then the r-th determinantal compound matrix, denoted as $C_r(A)$, is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose entries are $d(A[\alpha/\beta])$, $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$ arranged lexicographically in α and β.

Define $|A|_{\alpha\beta}$ to be the element in the (α,β) position of $C_r(A)$.

Theorem 2.29. Let A be an mxn matrix, B an nxk matrix and r an integer such that $1 \leq r \leq \min(m, n, k)$. Then $C_r(AB) = C_r(A)C_r(B)$.

Proof: Let D = AB and let α and β be sequences such that $\alpha \in Q_{r,m}$, $\beta \in Q_{r,k}$. Then the element in the (α,β) position of $C_r(D)$, is $|D|_{\alpha\beta}$. But by the Binet-Cauchy theorem for determinants

$$d(D[\alpha/\beta]) = \sum_{\delta \in Q_{r,n}} d(A[\alpha/\delta])d(B[\delta/\beta]).$$

Consider the α-th row of $C_r(A)$, $d(A[\alpha/\delta])$ where $\delta \in Q_{r,n}$, and the β-th column of $C_r(B)$, $d(B[\delta/\beta])$, where $\delta \in Q_{r,n}$. Thus the (α,β) element in the product $C_r(A)C_r(B)$ is

$$\sum_{\delta \in Q_{r,n}} d(A[\alpha/\delta])d(B[\delta/\beta])$$

and $C_r(AB) = C_r(A)C_r(B)$, and the proof is complete.

Definition 2.30. The mapping T which takes the set of mxn matrices into mxn matrices is said to be linear if

$$T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$$

for all mxn matrices X and Y and complex scalars α and β .

Thus we see that γ defined in (8) is linear since

$$\begin{bmatrix} \alpha x_{11} + \beta y_{11} & \alpha x_{12} + \beta y_{12} \\ -\alpha x_{21} - \beta y_{21} & \alpha x_{22} + \beta y_{22} \end{bmatrix} = \alpha \begin{bmatrix} x_{11} & x_{12} \\ -x_{21} & x_{22} \end{bmatrix} + \beta \begin{bmatrix} y_{11} & y_{12} \\ -y_{21} & y_{22} \end{bmatrix}$$

or

$$\gamma(\alpha X + \beta Y) = \alpha\gamma(X) + \beta\gamma(Y).$$

In the proofs that follow some special notation will be used. We now define this notation. If $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$ then the symbol $E_{\alpha\beta}$ will denote the $\binom{m}{r} \times \binom{n}{r}$ unit matrix with 1 in the position (α, β) and zero elsewhere. If $u_1, \ldots, u_n$ are vectors in some vector space V, then $< u_1, \ldots, u_n >$ will denote the subspace of V spanned by these vectors. If $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are members of an arbitrary n-dimensional vector space V then $x \perp y$ will denote that

$$\sum_{i=1}^{n} x_i y_i = 0.$$

That is, the vectors are orthogonal. Also if B is an sxt matrix and A is an kxq matrix then

$$B \dotplus A = \begin{bmatrix} B & 0_{s,q} \\ 0_{k,t} & A \end{bmatrix}$$

will denote the direct sum of A and B. If A is the kxq matrix all of whose entries are zero, then $A = 0_{k,q}$. If $k = 0$ or $q = 0$, then

$$B \dotplus 0_{k,q} = [B \quad 0_{s,q}] \quad \text{or} \quad \begin{bmatrix} B \\ 0_{k,t} \end{bmatrix}.$$

The proof that there is no linear transformation which converts the permanent into the determinant will be given as a corollary of the following more general theorem.

Theorem 2.31. There is no linear transformation T of $M_{m,n}$ into itself such that

$$P_r(T(X)) = C_r(X) \tag{11}$$

for all $X \in M_{m,n}$, where r is an integer such that $2 \le r \le \min(m, n)$ and $m + n > 4$.

The proof of Theorem 2.31 is by contradiction. Assume $m + n > 4$ and T is a linear transformation satisfying (11). Then the proof will be accomplished in a series of lemmas.

Lemma 2.32. T is nonsingular.

Proof: Assume $T(A) = 0$. Then for every $X \in M_{m,n}$

$$C_r(A + X) = P_r(T(A + X)) = P_r(T(A) + T(X))$$
$$= P_r(T(X)) = C_r(X).$$

Let Y be the r-square matrix

$$Y = \begin{bmatrix} -t & -a_{12} & \dots & -a_{1r} \\ 0 & -t & \dots & -a_{2r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -t \end{bmatrix}$$

and consider the mxn matrix X such that

$$X = Y \dotplus 0_{m-r,n-r}.$$

Then in the (1, 1) position of $C_r(A+X)$ one finds

$$\prod_{i=1}^{r} (a_{ii} - t)$$

whereas in the (1, 1) position of $C_r(X)$ is found $(-1)^r t^r$. Since these two polynomials must be the same, this implies $a_{ii} = 0$, $i = 1, \ldots, r$. Now for any permutation matrices P and Q Theorem 2.29 implies

$$C_r(PAQ + PXQ) = C_r(P(A+X)Q) = C_r(P)C_r(A+X)C_r(Q)$$

$$= C_r(P)C_r(X)C_r(Q) = C_r(PXQ).$$

Now as X varies over $M_{m,n}$, so does PXQ. Therefore any element can be brought to be in the (i, i), i=1, . . . , r, position by appropriate permutation matrices, and we can conclude in the same way as above that $a_{ij} = 0$ for every i and j. Hence A = 0 and T is nonsingular.

Lemma 2.33. If (11) holds, then there exists a nonsingular linear transformation S_{r-1} of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ into itself such that

$$P_{r-1}(T(X)) = S_{r-1}(C_{r-1}(X)). \tag{12}$$

Proof: Let Y = T(X). Since T is nonsingular this implies $X = T^{-1}(Y)$. Thus there exists constants $g^{u,v}_{p,q}$ and $s^{p,q}_{i,j}$, i, u, p=1,. . .m and j, v, q = 1, . . . , n, such that

$$x_{uv} = \sum_{p=1}^{m} \sum_{q=1}^{n} g^{u,v}_{p,q} \, y_{pq} \quad \text{and} \quad y_{pq} = \sum_{i=1}^{m} \sum_{j=1}^{n} s^{p,q}_{i,j} \, x_{ij} \tag{13}$$

where the scalars $g^{u,v}_{p,q}$ and $s^{p,q}_{i,j}$ are the entries in the matrix representation of T^{-1} and T with respect to the natural basis in $M_{m,n}$. Also (11) asserts that

$$Y_{\alpha\beta} = |X|_{\alpha\beta} \tag{14}$$

where $\alpha \in Q_{r,m}$, $\beta \in Q_{r,n}$.

Now (14) can be regarded as a polynomial identity in the variables y_{ij} by using equation (13).

If in (14), s is an integer in the sequence α, t is an integer in the sequence β, then denote α'_s as the sequence of $Q_{r,m}$ with s deleted and β'_t as the sequence in $Q_{r,n}$ with t deleted. Thus

$$Y_{\alpha'_s\beta'_t} = \frac{\delta Y_{\alpha\beta}}{\delta y_{st}} = \frac{\delta|X|_{\alpha\beta}}{\delta y_{st}} .$$

But (13) implies

$$\frac{\delta|X|_{\alpha\beta}}{\delta y_{st}} = \sum_{u=1}^{m} \sum_{v=1}^{n} \frac{\delta|X|_{\alpha\beta}}{\delta x_{uv}} \frac{\delta x_{uv}}{\delta y_{st}}$$

while

$$\frac{\delta|X|_{\alpha\beta}}{\delta x_{uv}} = X_{\alpha'_u\beta'_v} \quad \text{and} \quad \frac{\delta x_{uv}}{\delta y_{st}} = g^{u,v}_{s,t} .$$

Thus,

$$Y_{\alpha'_s\beta'_t} = \sum_{u=1}^{m} \sum_{v=1}^{n} X_{\alpha'_u\beta'_v} g^{u,v}_{s,t} ,$$

and the (r-1)-th order permanental majors of Y= T(X) are fixed linear homogeneous functions of the (r-1)-th order determinantal minors of X. That is, there exists a linear mapping S_{r-1} of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ into itself such that (12) holds.

To complete the proof S_{r-1} must be shown to be nonsingular. By (14) we have

$$|X|_{\alpha'_p\beta'_q} = \frac{\delta|X|_{\alpha\beta}}{\delta x_{pq}} = \frac{\delta Y_{\alpha\beta}}{\delta x_{pq}} = \sum_{u=1}^{m}\sum_{v=1}^{n} \frac{\delta Y_{\alpha\beta}}{\delta y_{uv}} \frac{\delta y_{uv}}{\delta x_{pq}}$$

$$= \sum_{u=1}^{m}\sum_{v=1}^{n} s_{p,q}^{u,v} \, Y_{\alpha'_u\beta'_v} \; .$$

Thus, we obtain $C_{r-1}(X) = R_{r-1}(P_{r-1}(Y))$ where R_{r-1} is a mapping of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ into itself. Since $P_{r-1}(Y) = S_{r-1}(C_{r-1}(X))$, we have

$$C_{r-1}(X) = R_{r-1}S_{r-1}(C_{r-1}(X)). \tag{15}$$

Now there exists a basis in $M_{\binom{m}{r-1},\binom{n}{r-1}}$ of the form $C_{r-1}(X)$, $X \in M_{m,n}$. For let

$$X = \sum_{t=1}^{r-1} E_{i_t j_t}$$

where $\pi = (i_1, \ldots, i_{r-1}) \in Q_{r-1,m}$ and $\gamma = (j_1, \ldots, j_{r-1}) \in Q_{r-1,n}$. Then $C_{r-1}(X) = \pm E_{\pi\gamma}$. Hence by (15), $R_{r-1}S_{r-1}$ is the identity map on the basis of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ and S_{r-1} is nonsingular.

We observe that Lemma 2.33 implies that $P_2(T(X)) = S_2(C_2(X))$ for every $X \in M_{m,n}$. Thus we have the following lemma.

<u>Lemma 2.34</u>. If $\rho(X) = 1$ then $P_2(T(X)) = 0$.

Proof: If $\rho(X) = 1$ this implies that all second order determinants are zero. Hence $C_2(X) = 0$ and $S_2(C_2(X)) = 0$ since any linear transformation always takes the zero element to the zero element. Therefore,

$$P_2(T(X)) = S_2(C_2(X)) = 0,$$

and the lemma is proved.

Lemma 2.35. If $P_2(Y) = 0$ and $Y \neq 0$ then

(a) Y has exactly one non-zero row,

(b) Y has exactly one non-zero column,

(c) by permutation of rows and columns Y may be brought to the form

$$\alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22}, \ \alpha\beta\gamma\delta \neq 0, \ \alpha\delta + \beta\gamma = 0.$$

Proof: If there is just one row or column of Y which is not zero then (a) or (b) holds respectively.

Now assume there exists elements y_{rs} and y_{uv}, $r < u$, $s < v$ such that $y_{rs}y_{uv} \neq 0$. Since $P_2(Y) = 0$ this implies $y_{rv}y_{us} \neq 0$. By permuting rows and columns these four entries may be taken to be in the top left 2-square submatrix of Y. Let $y_{11} = \alpha$, $y_{12} = \beta$, $y_{21} = \gamma$, and $y_{22} = \delta$. Then since $P_2(Y) = 0$ this implies for $j > 2$ that

$$\begin{aligned} \alpha y_{2j} + \gamma y_{1j} &= 0 \\ \beta y_{2j} + \delta y_{1j} &= 0. \end{aligned} \qquad (16)$$

Since $\alpha\delta + \beta\gamma = 0$ and these entries are non-zero, we have $\alpha\delta - \beta\gamma \neq 0$. Hence the solution of (16) must be $y_{2j} = y_{1j} = 0$. Similarly $y_{i1} = y_{i2} = 0$, if $i > 2$. If both $i > 2$ and $j > 2$, then $\delta y_{ij} + y_{2j}y_{i2} = 0$ since $P_2(Y) = 0$. But $y_{2j} = y_{i2} = 0$ implies $y_{ij} = 0$. Hence

$$Y = \alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22}.$$

Lemma 2.36. Let X and Z be rank 1 matrices such that $\rho(X + Z) = 1$. Also assume T(X) is of the form (c) of Lemma 2.35. If

$$G_1 = \alpha E_{11} + \gamma E_{21}, \quad G_2 = \beta E_{12} + \delta E_{22}$$

$$G_3 = \alpha E_{11} + \beta E_{12}, \quad G_4 = \gamma E_{21} + \delta E_{22}$$

then $W = T(Z)$ is spanned by the vectors G_1, G_2, G_3.

Proof: By Lemmas 2.34 and 2.35, T(Z) is a row, column, or 2-square matrix. Suppose $W = T(Z)$ is a row matrix and the i-th row is non-zero. Then since

$$P_2(T(X + Z)) = P_2(T(X) + T(Z)) = 0$$

this means $w_{ij} = 0$, $j > 2$. Thus

$$W = w_{i1}E_{i1} + w_{i2}E_{i2}, \quad i=1, \ldots, m,$$

while the other entries are zero. Also if $i \neq 1, 2$ then $P_2(T(X + Z)) = 0$ implies

$$\gamma w_{i2} + \delta w_{i1} = 0$$

and

$$\gamma\beta + \delta\alpha = 0.$$

In order for these equations to be consistent since $\delta, \gamma \neq 0$ is for $\alpha w_{i2} - \beta w_{i1} = 0$. But also $\alpha w_{i2} + \beta w_{i1} = 0$. Thus $w_{i1} = w_{i2} = 0$. That is, $W = 0$. This contradicts the fact that W was a row matrix. Hence

$$W = w_{i1}E_{i1} + w_{i2}E_{i2}, \quad i=1 \text{ or } 2.$$

Suppose then that $W = w_{11}E_{11} + w_{12}E_{12}$. Then $(\alpha + w_{11})\delta + (\beta + w_{12})\gamma = 0$, implies $w_{11}\delta + w_{12}\gamma = 0$ since $\alpha\delta + \beta\gamma = 0$. But these equations imply that $\beta w_{11} - \alpha w_{12} = 0$ or $w_{11} = m\alpha$ and $w_{12} = m\beta$ for some number m. Hence W is a multiple of G_3.

By similar arguments if W is either a row or column matrix, it is in the space generated by G_1, G_2, and G_3.

Next assume W = T(Z) has the form given in (c) of Lemma 2.35 to within permutation. Since $P_2(T(X) + T(Z)) = 0$ this means

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \dotplus 0_{m-2,n-2}$$

where $w_{11}w_{22} + w_{21}w_{12} = 0$ and $w_{11}w_{12}w_{21}w_{22} \neq 0$. Thus

$$T(X) + T(Z) = \begin{bmatrix} \alpha + w_{11} & \beta + w_{12} \\ \gamma + w_{21} & \delta + w_{22} \end{bmatrix} \dotplus 0_{m-2,n-2}$$

and

$$(\alpha + w_{11})(\delta + w_{22}) + (\beta + w_{12})(\gamma + w_{21}) = 0$$

implies

$$\alpha w_{22} + \delta w_{11} + \beta w_{21} + \gamma w_{12} = 0. \qquad (17)$$

Now let $\gamma/\alpha = c$ and $w_{21}/w_{11} = d$. Then $\gamma = \alpha c$ implies $\delta = -c\beta$ since $\alpha\delta + \beta\gamma = 0$, and $w_{21} = w_{11}d$ implies $w_{22} = -dw_{12}$, since

$$w_{11}w_{22} + w_{21}w_{12} = 0.$$

Substituting in (17) and factoring gives

$$(c - d)(\alpha w_{12} - \beta w_{11}) = 0. \qquad (18)$$

Hence if c = d, then

$$w_{11}/w_{21} = \alpha/\gamma = -\beta/\delta = -w_{12}/w_{22},$$

implies the vectors (w_{11}, w_{21}) and (α, γ) are linearly dependent as well as (w_{12}, w_{22}) and (β, δ). Thus $W \in \langle G_1, G_2 \rangle \subseteq \langle G_1, G_2, G_3 \rangle$. Next if $\alpha w_{12} = \beta w_{11}$ in (18) then (w_{11}, w_{12}) and (α, β) are linearly dependent. Also since

$$w_{11}/w_{12} = \alpha/\beta \quad \text{and} \quad w_{11}w_{22} + w_{21}w_{12} = 0$$

this implies $-w_{21}/w_{22} = \alpha/\beta$ or $\alpha w_{22} + \beta w_{21} = 0$. That is, $(w_{21}, w_{22}) \perp (\beta, \alpha)$. Also $(\beta, \alpha) \perp (\gamma, \delta)$ which implies (w_{21}, w_{22}) and (γ, δ) are linearly dependent. Hence $W \in \langle G_3, G_4 \rangle \subseteq \langle G_1, G_2, G_3 \rangle$ since $G_4 = G_1 + G_2 - G_3$, and the proof is complete.

Lemma 2.37. If $\rho(X) = 1$, then $\rho(T(X)) = 1$.

Proof: By Lemma 2.34, $P_2(T(X)) = 0$. Since X and T are non-singular Lemma 2.35 implies T(X) is a row, column, or 2-square matrix. Thus if T(X) is a row or column matrix then $\rho(T(X)) = 1$, and the proof is complete. Now assume T(X) is a 2-square matrix. Since the concern is for the rank of T(X) no generality is lost by assuming

$$T(X) = \alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22},$$

where $\alpha\delta + \beta\gamma = 0$ and $\alpha\beta\gamma\delta \neq 0$.

Now if $\rho(T(X)) = 1$ then the proof is complete. Thus assume $\rho(T(X)) = 2$. Next let $X_1, \ldots, X_{n-1}$ and $Z_1, \ldots, Z_{m-1}$ be matrices such that

$$V_1 = \langle X, X_1, \ldots, X_{n-1} \rangle \quad \text{and} \quad V_2 = \langle X, Z_1, \ldots, Z_{m-1} \rangle$$

are of dimension n and m respectively, consist of rank 1 matrices,

$$\dim(V_1 + V_2) = \dim \langle X, X_1, \ldots, X_{n-1}, Z_1, \ldots, Z_{m-1} \rangle = n + m - 1,$$

and moreover,

$$\rho(X + X_i) = 1, \; i=1,\ldots,n-1$$

$$\rho(X + Z_i) = 1, \; i=1,\ldots,m-1.$$

Therefore if $A \in V_1$, $B \in V_2$ then

$$A = c_1 X + c_2 X_1 + \ldots + c_n X_{n-1}$$

and

$$B = b_1 X + b_2 Z_1 + \ldots + b_m Z_{m-1}.$$

Thus,

$$T(A + B) = (c_1 + b_1)T(X) + c_2 T(X_1) + \ldots + b_m T(Z_{m-1})$$

where by Lemma 2.36,

$$T(X), \; T(X_1), \; \ldots, \; T(Z_{m-1}) \subseteq G_1, G_2, G_3 >.$$

Hence $\dim(T(V_1 + V_2)) = n+m-1 < 4$ or $n+m < 5$. This contradicts the fact that $m + n \geq 5$. Therefore $\rho(T(X)) = 1$, and the lemma is proved.

We now state a lemma proved by Marcus and Moyls [27] in 1959. The proof is omitted.

<u>Lemma 2.38</u>. Let T be a linear transformation on the space of mxn matrices. If the set of rank 1 matrices is invariant under T then for every $X \in M_{m,n}$ if $m \neq n$, $T(X) = AXB$ where $A \in M_{m,m}$, $B \in M_{n,n}$, and $d(A)d(B) \neq 0$. If $m = n$ then $T(X) = AXB$ or $T(X) = AX'B$ where A and B are nonsingular n-square matrices.

We are now ready to apply the above lemmas and prove Theorem 2.31.

Proof of Theorem 2.31. By Lemma 2.37, T is a linear transformation which preserves rank 1. Therefore applying Lemma 2.38

$$P_r(AXB) = C_r(X)$$

or

$$P_r(AX'B) = C_r(X)$$

for every $X \in M_{m,n}$. Since $d(A)d(B) \neq 0$ we can choose X_o so that $AX_oB = L$ where L is the mxn matrix with 1 in every position. Then $\rho(L) = 1$ and $X_o = A^{-1}LB^{-1}$. Since

$$1 \leq \rho(X_o) = \rho(A^{-1}LB^{-1}) \leq \rho(L) = 1,$$

we have $\rho(X_o) = 1$ and $C_r(X_o) = 0$. But

$$P_r(AX_oB) = r! \sum_{\alpha \in Q_{r,m}} \sum_{\beta \in Q_{r,n}} E_{\alpha\beta} \neq 0.$$

This contradiction tells us no such T exists which satisfies (11), and thus the theorem is proved.

For the case $m = n = r > 2$, Theorem 2.31 states that there is no linear transformation on n-square matrices that converts the determinant into the permanent. This is proved in the following corollary.

Corollary 2.39. There is no linear map T on $M_{n,n}$, $n > 2$, into itself such that for all $X \in M_{n,n}$ $p(T(X)) = d(X)$.

Proof: Suppose there exists such a mapping. Then

$$p(T(X)) = P_n(T(X)) = d(X) = C_n(X),$$

and this contradicts the theorem.

Theorem 2.31 is a special case of the more general theorem proved by Marcus and Minc [18]. It is as follows.

Theorem 2.40. There is no linear transformation T of $M_{m,n}$ into itself such that $P_r(T(X)) = S_r(C_r(X))$ for all $X \in M_{m,n}$ where S_r is a linear nonsingular map of $M_{\binom{m}{r},\binom{n}{r}}$ into itself if $m + n > 4$ and r is an integer such that $2 \leq r \leq \min(m, n)$.

Considering the linear transformation γ of $M_{2,2}$ into itself defined in (8), Theorem 2.40 takes on a more positive tone. Thus, we have the following theorem.

Theorem 2.41. If T is a linear transformation of $M_{2,2}$ into itself such that $P_2(T(X)) = S_2(C_2(X))$ for every $X \in M_{2,2}$ and S_2 is a nonsingular linear map of $M_{1,1}$ into itself then $\gamma(T(X)) = AXB$ or $\gamma(T(X)) = AX'B$ where $A, B \in M_{2,2}$ and $d(AB) \neq 0$.

Proof: From the definition of γ we see that $d(\gamma(T(X)))=p(T(X))$. Also since $X \in M_{2,2}$, $P_2(T(X)) = p(T(X))$ and $C_2(X) = d(X)$. Thus $P_2(T(X)) = S_2(C_2(X))$ implies that

$$p(T(X)) = S_2(d(X)). \tag{19}$$

Since S_2 is a nonsingular linear map from the complex field to the complex field, $S_2(x) = ax$ for some non-zero constant a. Hence $p(T(X)) = ad(X)$. Now if $\rho(X) = 1$ then $d(X) = 0$ and (16) implies $d(\gamma(T(X))) = 0$. That is, the rank of $\gamma(T(X))$ is 0 or 1. From Lemma 2.32, T is nonsingular and thus $\gamma(T)$ is nonsingular. By Lemma 2.37, $\rho(\gamma(T(X))) = 1$. Since $\gamma(T)$ preserves rank 1 matrices we have by Lemma 2.38 that $\gamma(T(X)) = AXB$ or $\gamma(T(X)) = AX'B$ where $A, B \in M_{2,2}$ and $d(AB) \neq 0$, as was to be shown.

Consider the following example: Let T be the linear map of $M_{2,2}$ into itself such that if $X \in M_{2,2}$ then

$$T(X) = \begin{bmatrix} -x_{22} & x_{12} \\ x_{21} & x_{11} \end{bmatrix}.$$

Now $P_2(T(X)) = S_2(C_2(X))$ where $S_2(y) = -y$. Thus,

$$\gamma(T(X)) = \begin{bmatrix} -x_{22} & x_{12} \\ -x_{21} & x_{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= AX'B$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } d(AB) \neq 0.$$

Characterization of Invariant Transformations

The results of Theorem 2.31 and Corollary 2.39 might bring up the following question to the interested student: What types of linear transformations leave the permanent unaltered? That is, if T is a linear transformation such that for all $X \in M_{n,n}$ $p(T(X)) = p(X)$ then how can T be characterized? We know such transformations exist from the results of theorems like 2.1 and 2.10. The answer to this question will be given after the following important theorem.

Theorem 2.42. Let T be a linear transformation of $M_{m,n}$ into itself and r be an integer such that $2 \leq r \leq \min(m,n)$. Assume that

$$P_r(T(X)) = P_r(X) \tag{20}$$

for all $X \in M_{m,n}$ and $m + n > 4$. Then if $m \neq n$, there exist permutation matrices $P \in M_{m,m}$, $Q \in M_{n,n}$ and diagonal matrices $D \in M_{m,m}$, $L \in M_{n,n}$ such that

$$T(X) = DPXQL \tag{21}$$

for all $X \in M_{m,n}$. If $m = n$ then T has the form (21) or $T(X) = DPX'QL$ for all $X \in M_{m,n}$.

The proof of this theorem will be done in a series of lemmas. The first part of the proof is similar to the proof of Theorem 2.31 and we will be able to use some of those results.

Assume $m + n > 4$ and T is a linear transformation satisfying (20).

Lemma 2.43. T is nonsingular.

Proof: By changing C_r to P_r in the proof of Lemma 2.32 and using the results of Theorems 2.26 and 2.27 the proof is complete.

Lemma 2.44. If (20) holds then there exists a nonsingular linear transformation S_{r-1} of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ into itself such that for every $X \in M_{m,n}$

$$P_{r-1}(T(X)) = S_{r-1}(P_{r-1}(X)). \tag{22}$$

Proof: By changing $|X|_{\alpha\beta}$ to $X_{\alpha\beta}$, C_s to P_s, $s = r-1$ or r, and the words determinantal minors to permanental majors in the proof of Lemma 2.33, we have the proof of Lemma 2.44.

By using (22) we proceed to reduce $r-1$ to $r-2$, etc., finally arriving at

$$P_2(T(X)) = S_2(P_2(X)). \tag{23}$$

Lemma 2.45. If $X \in M_{m,n}$ such that $\rho(X) = 1$ and $P_2(T(X)) = 0$ then $\rho(T(X)) = 1$.

Proof: The proof of Lemma 2.37 did not depend directly on the fact that T was required to be a linear transformation satisfying $P_r(T(X)) = C_r(X)$ but rather on the fact that $P_2(T(X)) = 0$ and T was nonsingular. By Lemma 2.43, T is nonsingular and the hypothesis tells us that $P_2(T(X)) = 0$. Hence using the proof of Lemma 2.37 we have $\rho(T(X)) = 1$ as was to be shown.

Corollary 2.46. If $F_{ij} = T(E_{ij})$ then $\rho(F_{ij}) = 1$.

Proof: Since $\rho(E_{ij}) = 1$ and

$$P_2(T(E_{ij})) = S_2(P_2(E_{ij})) = S_2(0) = 0$$

then Lemma 2.45 implies $\rho(F_{ij}) = 1$.

Lemma 2.47. If $P_2(Y) = 0$ and $\rho(Y) = 1$, then Y is a row or column matrix.

Proof: From Lemma 2.35, Y is a row or column matrix or Y can be brought to the form $\alpha E_{11} + \beta E_{12} + \gamma E_{21} + \delta E_{22}$ where $\alpha\delta + \beta\gamma = 0$ and $\alpha\delta\beta\gamma \neq 0$. If Y has this last form and $\rho(Y) = 1$ this implies that $\alpha\delta - \beta\gamma = 0$. These equations $\alpha\delta + \beta\gamma = 0$, $\alpha\delta - \beta\gamma = 0$, and $\alpha\delta\beta\gamma \neq 0$ are all three impossible, thus Y is a row or column matrix.

Corollary 2.46 and Lemma 2.47 tells us that F_{ij} is either a row or column matrix. If A is a row or column matrix then let h(A) denote the number of non-zero elements of A.

Lemma 2.48. If $F_{ij} = T(E_{ij})$ then $h(F_{ij}) = 1$.

Proof: No generality is lost if we assume $i = j = 1$ and F_{11} is a row matrix with its non-zero row in row 1. Also since we are interested only in the number of non-zero elements, no generality is lost by assuming that the first row of F_{11} is $(a_1, a_2, \ldots, a_h, 0, \ldots, 0)$. That is, $h(F_{11}) = h$. Suppose $h \geq 2$. Consider F_{1t}, $t=1, \ldots, n$. Then $\rho(E_{11} + E_{1t}) = 1$ and

$$P_2(T(E_{11} + E_{1t})) = S_2(P_2(E_{11} + E_{1t})) = S_2(0) = 0.$$

Thus, by Lemma 2.45,

$$\rho(T(E_{11} + E_{1t})) = \rho(F_{11} + F_{1t}) = 1.$$

Hence, by Lemma 2.47, $F_{11} + F_{1t}$ is a row or column matrix.

Suppose first that $F_{11} + F_{1t}$ is a row matrix. Then the non-zero row is the first row since F_{11} is a row matrix with non-zero first row and F_{1t} is a row or column matrix. For suppose the k-th row, $k \neq 1$, contains a non-zero element. Then this implies $a_i = 0$ for $i = 1$ or 2 or F_{1t} is not a row or column matrix both of which gives a contradiction. Thus,

$$F_{1t} = (F_{11} + F_{1t}) - F_{11}, \quad t=2, \ldots, n$$

is a row matrix with a non-zero first row.

Next suppose $F_{11} + F_{1t}$ is a column matrix and the k-th column contains at least one non-zero element in the j-th row, $j \neq 1$. This implies either $a_1 = 0$ or $a_2 = 0$ which is a contradiction since $h \geq 2$ or F_{1t} is not a row or column matrix which contradicts Lemma 2.47. Thus $F_{11} + F_{1t}$ is a column matrix with the only non-zero element in the first row. That is, $F_{11} + F_{1t}$ is a row matrix with non-zero first row. Thus F_{1t} is again a row matrix with a non-zero first row.

Next consider F_{s1}, $s=2,\ldots,m$. Then $\rho(F_{11}+F_{s1}) = 1$, and $P_2(F_{11}+F_{s1}) = 0$ implies $F_{11}+F_{s1}$ is a row or column matrix. By arguments similar to the above $F_{11}+F_{s1}$ is a row matrix with non-zero first row. Therefore $F_{s1} = (F_{11}+F_{s1}) - F_{11}$ is also a row matrix with non-zero first row. Now the vector space spanned by the matrices E_{1t}, $t=1,\ldots,n$ and E_{s1}, $s=1,\ldots,m$ has dimension $m+n-1$. Under the transformation T the vector space spanned by F_{1t}, $t=1,\ldots,n$ and F_{s1}, $s=1,\ldots,m$ has at most dimension n. Since a nonsingular linear transformation preserves linear independence and $m+n-1 > n$, this leads to an impossible situation. Thus, $h(F_{11}) = 1$ or $h(F_{ij}) = 1$ which was to be proved.

Notice that Lemma 2.48 tells us $T(E_{ij}) = c_{ij}E_{i'j'}$ for some constant c_{ij}. Moreover $c_{ij} \neq 0$ since T is nonsingular. If $i' = \pi(i,j)$ and $j' = \delta(i,j)$ then

$$T(E_{ij}) = c_{ij}E_{\pi(i,j)\delta(i,j)} .$$

<u>Lemma 2.49</u>. If $(i,j) \neq (s,t)$ then $T(E_{ij}) \neq T(E_{st})$.

Proof: The result follows since T is nonsingular.

<u>Definition 2.50</u>. If $C, X \in M_{m,n}$, then let $C*X$ be the Hadamard product of C and X defined as $C*X = Y \in M_{m,n}$ such that $y_{ij} = c_{ij}x_{ij}$, $i=1,\ldots,m$, $j=1,\ldots,n$.

We are now ready to prove an important lemma.

<u>Lemma 2.51</u>. There exists an $H \in M_{m,n}$, $h_{ij} \neq 0$ for every i and j, and permutation matrices $P \in M_{m,m}$, $Q \in M_{n,n}$, such that if

$m \neq n$ then for all $X \in M_{m,n}$

$$T(X) = H^*(PXQ). \tag{24}$$

If $m = n$ then T has the form (24) or else

$$T(X) = H^*(PX'Q)$$

for all $X \in M_{m,m}$.

Proof: To fix the notation let us assume $m \leq n$. Let $X \in M_{m,n}$. Then

$$X = x_{11}E_{11} + x_{12}E_{12} + \ldots + x_{mn}E_{mn}.$$

$$T(X) = x_{11}T(E_{11}) + x_{12}T(E_{12}) + \ldots + x_{mn}T(E_{mn})$$

$$= x_{11}c_{11}E_{\pi(1,1)\delta(1,1)} + \ldots + x_{mn}c_{mn}E_{\pi(m,n)\delta(m,n)}.$$

Now by a suitable permutation of $T(X)$ we can make $\pi(1,1) = 1 = \delta(1,1)$. That is, there exists permutation matrices P_1 and Q_1 such that

$$P_1T(X)Q_1 = x_{11}c_{11}E_{11} + x_{12}c_{12}P_1E_{\pi(1,2)\delta(1,2)}Q_1 + \ldots$$
$$+ x_{mn}c_{mn}P_1E_{\pi(m,n)\delta(m,n)}Q_1.$$

Now $P_2(E_{11} + E_{1t}) = 0$ for $t = 2, \ldots, n$ implies

$$P_2(T(E_{11} + E_{1t})) = S_2(P_2(E_{11} + E_{1t})) = 0.$$

Hence

$$P_2(F_{11} + F_{1t}) = P_2(T(E_{11} + E_{1t})) = 0.$$

Thus F_{11} and F_{1t} must have their non-zero elements in the same row or column. That is, $\pi(1,1) = \pi(1,t)$ or $\delta(1,1) = \delta(1,t)$. Under the permutations P_1 and Q_1 they end up in the first row or column. Thus

$\pi(1,t) = 1$ or $\delta(1,t) = 1$. Also $P_2(E_{11} + E_{s1}) = 0$, $s=2,\ldots,m$ implies $F_{11} + F_{s1}$ must have their non-zero elements in the same row or column. That is,

$$\pi(s,1) = \pi(1,1) = 1 \quad \text{or} \quad \delta(s,1) = \delta(1,1) = 1.$$

Next, $P_2(E_{11} + E_{22}) \neq 0$ implies $P_2(F_{11} + F_{22}) \neq 0$. This means F_{11} and F_{22} do not have elements in the same row or column. That is,

$$\pi(1,1) \neq \pi(2,2) \quad \text{and} \quad \delta(1,1) \neq \delta(2,2).$$

Under the permutations P_1 and Q_1 these results remain true which means $\pi(2,2) > 1$ and $\delta(2,2) > 1$. Now there exist permutation matrices P_2 and Q_2 of the last $m-1$ rows and $n-1$ columns such that $\pi(2,2) = 2$ and $\delta(2,2) = 2$. These permutations have no effect on the first row or column. Thus,

$$\begin{aligned} P_2P_1T(X)Q_1Q_2 = {} & x_{11}c_{11}E_{11} + x_{12}c_{12}P_1E_{\pi(1,2)\delta(1,2)}Q_1 + \cdots \\ & + x_{22}c_{22}E_{22} + x_{23}c_{23}P_2P_1E_{\pi(2,3)\delta(2,3)}Q_1Q_2 + \cdots \\ & + x_{mn}c_{mn}P_2P_1E_{\pi(m,n)\delta(m,n)}Q_1Q_2. \end{aligned}$$

Again $P_2(E_{22} + E_{2t}) = 0$ for $t=3,\ldots,n$ and $P_2(E_{22} + E_{s2}) = 0$ for $s=3,\ldots,m$ implies that $\pi(2,t) = 2$ or $\delta(2,t) = 2$, $t=2,\ldots,n$ and $\pi(s,2) = 2$ or $\delta(s,2) = 2$, $s=2,\ldots,m$.

Now $P_2(E_{11} + E_{22}) \neq 0$ and $P_2(E_{22} + E_{33}) \neq 0$ implies that $\pi(3,3) \neq \pi(2,2)$ and $\pi(3,3) \neq \pi(1,1)$. Also $\delta(3,3) \neq \delta(2,2)$ and $\delta(3,3) \neq \delta(1,1)$. Thus under the permutations P_1, P_2, Q_1, Q_2 we have $\pi(3,3) > 2$ and $\delta(3,3) > 2$. Continuing in the same way as above we have the following conclusions:

(a) $\pi(i,i) = \delta(i,i) = i$ for $i=1,\ldots,m$,

(b) $\pi(\alpha, t) = \alpha$ or $\delta(\alpha, t) = \alpha$, $\alpha \leq t \leq n$, $\alpha \leq m$,

(c) $\pi(s, \beta) = \beta$ or $\delta(s, \beta) = \beta$, $\beta \leq s \leq m$.

Therefore for $\alpha \leq m$, $\beta \leq m$, $\alpha \neq \beta$ we have

$$\pi(\alpha, \beta) = \alpha \quad \text{and} \quad \delta(\alpha, \beta) = \beta \tag{25}$$

or

$$\pi(\alpha, \beta) = \beta \quad \text{and} \quad \delta(\alpha, \beta) = \alpha \tag{26}$$

since $\pi(\alpha, \beta) \neq \delta(\alpha, \beta)$ by Lemma 2.49.

Suppose (25) holds and $m < n$. Then the p-th row of

$$P_m \ldots P_1 T(X) Q_1 \ldots Q_m$$

is

$$x_{p1} c_{p1} E_{p1} + \ldots + x_{pm} c_{pm} E_{pm} + x_{p(m+1)} c_{p(m+1)} E_{p\delta(p, m+1)} + \ldots$$
$$+ x_{pn} c_{pn} E_{p\delta(p, n)}.$$

If $k > m$ then $\delta(p, k) > m$ by Lemma 2.49. Also if $p \neq q$ then $P_2(E_{pk} + E_{qk}) = 0$ implies $\delta(p, k) = \delta(q, k)$. Thus by a suitable permutation Q_o of the last n-m columns we can make $\delta(p, k) = k$, $m < k \leq n$. Thus

$$P_m \ldots P_1 T(X) Q_1 \ldots Q_m Q_o = C * X.$$

By letting $P = (P_m \ldots P_1)^{-1}$ and $Q = (Q_1 \ldots Q_m Q_o)^{-1}$ we have

$$T(X) = P(C*X)Q = (PCQ)*(PXQ) = H*(PXQ)$$

where $H = PCQ$ and $h_{ij} \neq 0$ since $c_{ij} \neq 0$ for every i and j.

Next suppose (25) holds and $m = n$. Then

$$P_m \ldots P_1 T(X) Q_1 \ldots Q_m = C*X$$

and letting $P = (P_m \ldots P_1)^{-1}$, $Q = (Q_1 \ldots Q_m)^{-1}$, and $H = PCQ$ we have

$$T(X) = H*(PXQ)$$

where $h_{ij} \neq 0$ for every i and j.

The case where (26) holds and $m < n$ is impossible since $\pi(\alpha,\beta) \leq m$. That is, $\pi(\alpha,\beta) \neq k$, $k > m$ since there are only m rows in the matrix.

For the remaining case suppose (26) holds and $m = n$. Then the p-th row of

$$P_m \ldots P_1 T(X) Q_1 \ldots Q_m$$

is

$$x_{p1}c_{p1}E_{1p} + \ldots + x_{pm}c_{pm}E_{mp}.$$

Thus,

$$P_m \ldots P_1 T(X) Q_1 \ldots Q_m = C*X' .$$

By letting $(P_m \ldots P_1)^{-1} = P$, $(Q_1 \ldots Q_m)^{-1} = Q$ and $H = PCQ$, we have

$$T(X) = H*PX'Q,$$

where $h_{ij} \neq 0$ for every i and j. Thus, the lemma is proved.

Lemma 2.52. The matrix H defined in Lemma 2.51 is such that $\rho(H) = 1$.

Proof: Let $1 \leq i < s \leq m$, $1 \leq j < t \leq n$. If $T(X) = H*(PXQ)$ then choose X so that

$$PXQ = E_{ij} + E_{it} - E_{sj} + E_{st}.$$

If $T(X) = H*(PX'Q)$ then choose X so that

$$PX'Q = E_{ij} + E_{it} - E_{sj} + E_{st}.$$

In either case,

$$P_2(PXQ) = P_2(PX'Q) = 0.$$

Thus by Theorems 2.10, 2.26, and 2.27 we have $P_2(X) = P_2(X') = 0$. Since S_2 is a linear transformation and must map 0 to 0 then

$$P_2(T(X)) = P_2(h_{ij}E_{ij} + h_{it}E_{it} - h_{sj}E_{sj} + h_{st}E_{st})$$
$$= S_2(P_2(X)) = S_2(0) = 0.$$

Thus $h_{ij}h_{st} - h_{it}h_{sj} = 0$. That is each second order determinant of H is zero which implies $\rho(H) = 1$ since $H \neq 0$ and the lemma is proved.

Proof of Theorem 2.42. Lemma 2.51 tells us that T(X) has the form (24) if $m \neq n$ and (24) or else $T(X) = H*(PX'Q)$ if $m = n$. Hence consider the matrix H. Since the rank of H is 1 and the elements are non-zero then the rows are multiples of the first row. If $(q_1, \ldots, q_n)$ denotes the first row of H then let the p-th row be denoted as $(d_p q_1, \ldots, d_p q_n)$. Note that if $d_1 = 1$, then $h_{ij} = d_i q_j$. If we let D be the diagonal matrix with main diagonal $(d_1, d_2, \ldots, d_m)$ and L the diagonal matrix with main diagonal $(q_1, q_2, \ldots, q_n)$ where $D \in M_{m,m}$ and $L \in M_{n,n}$ then

$$T(X) = H*(PXQ) = DPXQL$$

or

$$T(X) = H*(PX'Q) = DPX'QL$$

as was to be shown.

Theorem 2.42 is a special case of a theorem given by Marcus and May [17] where the equation (20) is changed to read

$$P_r(T(X)) = S_r(P_r(X))$$

for a nonsingular linear map S_r of $M_{\binom{m}{r},\binom{n}{r}}$ into itself.

We are now able to characterize the type of linear transformation which leaves the permanent unaltered. This characterization is given in the following theorem.

Theorem 2.53. Let T be a mapping of $M_{n,n}$, $n > 2$, into itself. Then T is linear and $p(T(X)) = p(X)$ for every $X \in M_{n,n}$ if and only if

$$T(X) = DPXQL$$

or

$$T(X) = DPX'QL$$

where P and Q are n-square permutation matrices and D and L are n-square diagonal matrices such that $p(D)p(L) = 1$.

Proof: First suppose T is linear and $p(T(X)) = p(X)$. Then by letting $m = n = r$ in Theorem 2.42 we have

$$p(T(X)) = P_n(T(X)) = P_n(X) = p(X).$$

Thus

$$T(X) = DPXQL$$

or

$$T(X) = DPX'QL.$$

But Theorems 2.25, 2.26, and 2.27 imply that

$$P_n(T(X)) = P_n(DPXQL) = P_n(D)P_n(P)P_n(X)P_n(Q)P_n(L)$$

$$= p(D)p(P)p(X)p(Q)p(L) = p(X).$$

Using Theorem 2.10 we have $p(D)p(X)p(L) = p(X)$ which implies $p(D)p(L) = 1$.

Next, suppose $T(X) = DPXQL$. Then

$$T(\alpha X + \beta Y) = DP(\alpha X + \beta Y)QL = \alpha DPXQL + \beta DPYQL = \alpha T(X) + \beta T(Y).$$

Thus, T is linear. Also

$$p(T(X)) = p(DPXQL) = p(D)p(P)p(X)p(Q)p(L) = p(D)p(X)p(L) = p(X)$$

since $p(L)p(D) = 1$.

Finally if $T(X) = DPX'QL$ then because of Theorem 2.1 the same result as above follows, and the proof is complete.

Consider the following example of Theorem 2.53. Let $n = 3$ and T be the nonsingular transformation such that if $X \epsilon M_{3,3}$ then

$$T(X) = \begin{bmatrix} .5x_{21} & x_{22} & .5x_{23} \\ x_{11} & 2x_{12} & x_{13} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}.$$

Then it is easy to see that

$$T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$$

and

$$p(T(X)) = p(X).$$

Hence $T(X) = DPXQL$ or $T(X) = DPX'QL$. Finding D, P, Q, and L we have

$$T(X) = \begin{bmatrix} .5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $Q = I$ and $p(D)p(L) = 1$.

We notice that Theorem 2.42 when restricted to n-square matrices gives a much stronger result (Theorem 2.53). Thus, the question might be asked as to what are the possibilities of making

Theorem 2.42 into an if and only if statement. This situation seems rather hopeless because it is generally not true that

$$P_r(DPXQL) = P_r(X)$$

even when D and L are identity matrices.

There is one piece of unfinished business before we are through with the characterization of linear transformations which leave the permanent unaltered. This concerns the case when n = 2. Since Theorem 2.53 depended heavily on Theorem 2.42 and Theorem 2.42 did not cover the case when m = n = 2 we will not be able to achieve the same type of result as Theorem 2.53. We can however, prove some results for the linear transformation $\gamma T \gamma$ where γ is the mapping defined in (8). To simplify the notation let $ST(X) = S(T(X))$. We thus have the following theorem.

<u>Theorem 2.54.</u> If T is a linear transformation of $M_{2,2}$ into itself such that $p(T(X)) = \alpha p(X)$ for every $X \in M_{2,2}$ and α is a non-zero constant then

$$\gamma T \gamma(X) = AXB$$

or

$$\gamma T \gamma(X) = AX'B$$

where $A, B \in M_{2,2}$, $d(AB) \neq 0$.

Proof: From the definition of γ we know that $p(\gamma(X)) = d(X)$. Thus

$$p(\gamma^2 T\gamma(X)) = d(\gamma T\gamma(X)) = p(T\gamma(X)).$$

But

$$P_2(T\gamma(X)) = p(T\gamma(X)) = \alpha p(\gamma(X)) = \alpha d(X) = \alpha C_2(X).$$

Since $T\gamma$ is a linear transformation and $\alpha \neq 0$ then the hypothesis of Theorem 2.41 is satisfied for $T\gamma$. Thus,

$$\gamma T\gamma(X) = AXB$$

or

$$\gamma T\gamma(X) = AX'B$$

where $A, B \in M_{2,2}$, $d(AB) \neq 0$, and the proof is complete.

Again this cannot be an if and only if theorem by the following example. Suppose

$$\gamma T\gamma(X) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for all $X \in M_{2,2}$. Then

$$T(X) = \begin{bmatrix} 2x_{11} & 2x_{12} \\ -x_{11} + x_{21} & x_{12} + x_{22} \end{bmatrix}$$

and $p(T(X)) \neq \alpha p(X)$ for any non-zero constant α.

CHAPTER III

INEQUALITIES FOR THE PERMANENT FUNCTION

Introduction

Much of the modern interest in the permanent function has been due to two questions proposed in the first part of this century. The first of these consisted of trying to relate the permanent function in some simple fashion or otherwise to the more widely known determinant function. This question has been answered in a fairly conclusive form in Chapter II.

The second of these questions concerned a matrix inequality. It is known as the Van der Waerden conjecture since it has not been proven true or false. It was proposed by Van der Waerden [44] in 1926. It is concerned with the minimum value of the permanent function over the set of doubly stochastic matrices.

Definition 3.1. An n-square matrix A with elements in the real field is said to be doubly stochastic if $a_{ij} \geq 0$ for every i and j, and the row and column sums are 1.

For example the n-square matrix J_n with every entry $1/n$ is a doubly stochastic matrix. Also an n-square permutation matrix is doubly stochastic.

We are now able to state the conjecture given by Van der Waerden.

Conjecture 3.2. If A is an n-square doubly stochastic matrix, then

$$p(A) \geq \frac{n!}{n^n} \tag{27}$$

with equality if and only if $A = J_n$.

It is easy to see that if $A = J_n$ then $p(A) = \frac{n!}{n^n}$, and therefore at least one matrix can be found for which the conjectured lower bound is attained. That is, there cannot be a "better" lower bound. Thus it is surprising that this rather simple looking conjecture would remain in doubt. This seems to be basically the story behind many matrix inequalities involving the permanent function. Even though the inequalities themselves may portray rather simple results, the proofs in many cases involve very complicated mathematics and are not restricted to any particular area of mathematics. In the theorems that involve matrix inequalities we shall follow the course of introducing definitions and theorems which fall outside the basic results of linear algebra as they are needed to prove various inequalities concerning the permanent function. Also in most cases we will not deal with general mxn matrices but rather with square matrices with some special properties such as being doubly stochastic.

Inequalities for Non-negative Matrices

We shall begin this chapter with some simple inequalities involving non-negative matrices. That is, matrices with non-negative elements.

Theorem 3.3. If A and B are n-square matrices with non-negative elements then

$$p(A + B) \geq p(A) + p(B).$$

Proof: Every addend of p(A) and p(B) occurs as an addend in the expansion of p(A + B). In addition, the expansion of p(A + B) contains some non-negative terms not in the expansion of p(A) or p(B). Thus

$$p(A + B) \geq p(A) + p(B)$$

as was to be shown.

Theorem 3.4. If A is an n-square matrix with non-negative elements then for any permutation σ of the numbers $1, \ldots, n$,

$$p(A) \geq \prod_{i=1}^{n} a_{i\sigma(i)} \geq 0.$$

Proof: For every σ,

$$\prod_{i=1}^{n} a_{i\sigma(i)}$$

is non-negative and occurs in the expansion of p(A). Thus

$$p(A) \geq \prod_{i=1}^{n} a_{i\sigma(i)} \geq 0.$$

Theorem 3.5. If A and B are n-square non-negative matrices then

$$p(AB) \geq p(A)p(B).$$

Proof: Consider the addends of p(AB). They are of the form

$$a_{1j_1} b_{j_1\sigma(1)} a_{2j_2} b_{j_2\sigma(2)} \cdots a_{nj_n} b_{j_n\sigma(n)}$$

where j_k, $k=1, \ldots, n$ can take on any value between 1 and n and σ

denotes a permutation of $1, \ldots, n$.

Now consider an addend of p(A)p(B). They are of the form

$$a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}b_{1\overline{\sigma}(1)}b_{2\overline{\sigma}(2)} \cdots b_{n\overline{\sigma}(n)}$$

where σ and $\overline{\sigma}$ denote permutations of $1, \ldots, n$ respectively. Thus, all the addends of p(A)p(B) are included in the addends of p(AB). Since they are all positive we have

$$p(AB) \geq p(A)p(B),$$

as was to be shown.

Inequalities for Doubly Stochastic Matrices

Let us now look at some inequalities involving the permanents of doubly stochastic matrices. In order to do this some background material will be needed. We proceed with these results.

Definition 3.6. A set $S \subset E^n$ is said to be convex if for each pair of points $x, y \in S$ the line segment joining x and y is a subset of S. That is,

$$Z = \{z : z = \alpha x + (1 - \alpha)y, \ 0 \leq \alpha \leq 1\}$$

is such that $Z \subset S$.

For example the set $S \subset E^2$ such that $S = \{(x, y) : x^2 + y^2 \leq 1\}$ is convex since for each pair of points (x_1, y_1) and (x_2, y_2) in S the line segment joining these two points is a subset of S.

The set of matrices $M_{n,n}$ with real elements is isomorphic to E^{n^2}. If K_n denotes the set of n-square doubly stochastic matrices then K_n is a subset of E^{n^2}. Thinking of the set K_n in this way has considerable merit as we shall see in the theorems that follow.

Theorem 3.7. The set K_n is a convex subset of E^{n^2}.

Proof: Let A, B ϵ K_n and suppose α is a real number such that $0 \leq \alpha \leq 1$. Then consider $\alpha A + (1-\alpha)B$. The elements of this matrix are non-negative since the sum and product of non-negative numbers are non-negative. Thus consider the row sum of the i-th row:

$$\sum_{j=1}^{n} (\alpha a_{ij} + (1-\alpha)b_{ij}) = \alpha \sum_{j=1}^{n} a_{ij} + (1-\alpha) \sum_{j=1}^{n} b_{ij} = \alpha + (1-\alpha) = 1.$$

Similarly the column sums are 1. Therefore, $\alpha A + (1-\alpha)B \,\epsilon\, K_n$ for every α, $0 \leq \alpha \leq 1$ and K_n is convex.

Definition 3.8. The convex hull of a finite set of points in E^n, $x_1, \ldots, x_p$, is denoted as $H(x_1, \ldots, x_p)$ and

$$H(x_1, \ldots, x_p) = \{ \sum_{i=1}^{p} \alpha_i x_i : \sum_{i=1}^{p} \alpha_i = 1, \ \alpha_i \geq 0, \text{ for every } i \}.$$

In addition there exist points y_i, $i=1, \ldots, r$ such that $y_i \,\epsilon\, H(x_1, \ldots, x_p)$, $y_i \notin H(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_r)$ and $H(x_1, \ldots, x_p) = H(y_1, \ldots, y_r)$. These points y_i, $i=1, \ldots, r$ are called vertices of the convex hull.

Thus in E^2, $H((1,2), (3,4), (2,0))$ is the triangle with vertices at (1,2) (3,4) and (2,0). Also from Definition 3.8 we see that for any point x_o in this triangle,

$$x_o = \alpha_1(1,2) + \alpha_2(3,4) + \alpha_3(2,0)$$

where $\alpha_i \geq 0$, $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

We now state an important theorem given by G. Birkhoff [2]. An English version of the proof can be found in [22], page 97-98.

Theorem 3.9. The set K_n is the convex hull of the set of permutation matrices. Moreover, the permutation matrices are the vertices of K_n.

This theorem means that if A is a doubly stochastic matrix then A can be expressed in the form

$$A = \sum_{i=1}^{m} \alpha_i P_i \qquad (28)$$

where

$$\sum_{i=1}^{m} \alpha_i = 1,$$

$\alpha_i \geq 0$ and P_i are permutation matrices $i=1, \ldots, m$. The right hand side of (28) is called a convex combination of the P_i, $i=1, \ldots, m$. Notice that there are at most n! permutation matrices which implies $m \leq n!$. Actually m is smaller than n!, $n \neq 2$, as can be seen in the following results.

Definition 3.10. The dimension of a convex set is the dimension of the smallest euclidean n-space containing it. That is, it is the maximum number of linearly independent vectors contained in the set.

Thus, the triangle in the example above has dimension 2 since the smallest euclidean n-space containing it is E^2.

Theorem 3.11. The dimension of the convex set K_n is $(n-1)^2$. For the proof of this theorem see Marcus and Minc, [22], page 99-101.

Theorem 3.12. Every point x_o in the convex hull of dimension m of a finite set of points in E^n, $m \leq n$ with vertices $y_1, \ldots, y_r$ can be expressed as a convex combination of at most m + 1 vertices.

The proof of this theorem is omitted.

We can now state a corollary of Theorem 3.12 which gives the maximum number of permutation matrices needed to express any doubly stochastic matrix as a convex combination of permutation matrices. It is the following.

Corollary 3.13. If $A \in K_n$ then A can be expressed as a convex combination of at most $[(n-1)^2 + 1]$ permutation matrices.

Proof: This follows from Theorems 3.11 and 3.12.

For an example of Corollary 3.13 suppose $A \in K_3$ such that

$$A = \begin{bmatrix} 1/6 & 2/6 & 3/6 \\ 3/6 & 2/6 & 1/6 \\ 2/6 & 2/6 & 2/6 \end{bmatrix}.$$

Then if P_{ijk} denotes the permutation matrix with 1's in the positions i, j, k of rows 1, 2, and 3 respectively, then we have

$$A = 2/6P_{312} + 1/6P_{321} + 1/6P_{231} + 1/6P_{213} + 1/6P_{123}.$$

Definition 3.14. Let $a = (a_1, \ldots, a_n)$ be a fixed non-zero element of E^n and $x = (x_1, \ldots, x_n) \in E^n$. Then the set

$$\{x \in E^n : a_1x_1 + \ldots + a_nx_n = \beta\}$$

where β is a real number is called a hyperplane in E^n. The sets $\{x \in E^n : a_1x_1 + \ldots + a_nx_n \leq \beta\}$ and $\{x \in E^n : a_1x_1 + \ldots + a_nx_n \geq \beta\}$ are called closed half-spaces. If the equality is left out in the above sets, then they are called open half-spaces.

Thus, a hyperplane in E^n divides E^n into three disjoint complementary sets: the hyperplane itself and two open half-spaces.

Hyperplanes are closed convex sets in E^n. For example in E^2, if $a = (3, 2)$ and $\beta = 5$ then the set $\{(x, y) : 3x + 2y = 5\}$ is a hyperplane. That is, in E^2 hyperplanes are lines. In E^3, hyperplanes are two-dimensional planes.

We can now prove the following important theorem.

Theorem 3.15. Let $X = (x_{ij}) \in E^{n^2}$ and A be the intersection of the closed half-spaces defined by the sets $\{X \in E^{n^2} : x_{ij} \geq 0\}$, $i, j = 1, \ldots, n$. Also let H be the intersection of the hyperplanes defined by the sets

$$\{X \in E^{n^2} : \sum_{j=1}^{n} x_{ij} = 1\}, \; i=1, \ldots, n,$$

and G be the intersection of the hyperplanes defined by the sets

$$\{X \in E^{n^2} : \sum_{i=1}^{n} x_{ij} = 1\}, \; j=1, \ldots, n.$$

Then $K_n = A \cap H \cap G$.

Proof: First let $X \in K_n$. Then $x_{ij} \geq 0$ which implies $X \in A$. Also the row and column sums are 1 which means $X \in H$ and $X \in G$. Thus $X \in A \cap H \cap G$ and $K_n \subset A \cap H \cap G$. Next suppose $X \in A \cap H \cap G$ and X is written in matrix notation. Then $X \in A$ implies X is a non-negative matrix and $X \in H \cap G$ implies X is doubly stochastic. Thus $A \cap H \cap G \subset K_n$ and $K_n = A \cap H \cap G$.

We now collect the last few results and are able to make some important observations on the nature of the set of doubly stochastic matrices and the permanent function defined on this set. The results for the set K_n are:

1) It is convex;

2) It is the convex hull of permutation matrices;

3) The permutation matrices are the vertices;

4) It has dimension $(n-1)^2$; (29)

5) It is the intersection of a finite number of closed sets and hence closed;

6) It is bounded (consider the euclidean n^2-sphere of radius n).

Properties (29.5) and (29.6) tell us that K_n is compact. From the definition of the permanent function it is continuous. Thus we have a continuous function defined on a compact set when we consider the permanent function over K_n. Hence it must assume its maximum and minimum value on K_n. This fact makes the Van der Waerden inequality more interesting since the minimum must occur at a doubly stochastic matrix. The maximum is not quite so difficult to find as can be seen from the following theorem.

Theorem 3.16. If $A \in K_n$ then $p(A) \leq 1$ with equality if and only if A is a permutation matrix.

Proof: By definition

$$p(A) = \sum^{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Consider

$$\prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} .$$

Then every addend of p(A) occurs in this product. Since all the addends are non-negative and the row and column sums are 1, we have

$$p(A) \leq \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \prod_{i=1}^{n} 1 = 1.$$

If A is a permutation matrix then by Theorem 2.8, $p(A) = 1$ and equality holds. Next assume A is doubly stochastic and $p(A) = 1$. Then

$$\sum^{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} = \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$$

implies that A is a permutation matrix. The details are omitted. Thus, the proof is complete.

Let us now examine in more detail the problem of finding the minimum value of the permanent function over K_n. Several lower bounds have been computed and are continuing to be improved. For example, if $A \epsilon K_n$ then using Theorem 3.4, $p(A) \geq 0$, and zero is a lower bound. A more interesting lower bound is contained in the following theorem.

<u>Theorem 3.17</u>. If $A \epsilon K_n$, then $p(A) \geq [(n-1)^2 + 1]^{1-n}$.

Proof: By Corollary 3.13, A can be expressed as a convex combination of at most $(n-1)^2 + 1$ permutation matrices. Thus

$$A = \sum_{i=1}^{k} \alpha_i P_i, \quad k \leq (n-1)^2 + 1$$

where

$$\sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i > 0$$

for every i and P_i are permutation matrices. Then

$$p(A) = p\left(\sum_{i=1}^{k} \alpha_i P_i \right) \geq \sum_{i=1}^{k} \alpha_i^n p(P_i) = \sum_{i=1}^{k} \alpha_i^n,$$

by Theorems 3.3, 2.5, and 2.8.

Define

$$f(\alpha_1, \ldots, \alpha_k) = \sum_{i=1}^{k} \alpha_i^n .$$

Then the minimum of f subject to the constraints given above occurs whenever $\alpha_1 = \ldots = \alpha_k = 1/k$. Thus

$$\sum_{i=1}^{k} \alpha_i^n \geq \sum_{i=1}^{k} (1/k)^n = k^{1-n}.$$

But $1/k \geq 1/[(n-1)^2 + 1]$ since $k \leq (n-1)^2 + 1$. Therefore

$$p(A) \geq [(n-1)^2 + 1]^{1-n},$$

and the proof is complete.

Some additional background material concerning doubly stochastic matrices will now be stated. For a good reference as to what is known about this type of matrix see Marcus and Minc [22]. We proceed with the results which will be useful to us.

Theorem 3.18. If $A \in K_n$ then A has 1 as a characteristic root.

Theorem 3.19. If $A \in K_n$ then there exists n-square permutation matrices P and Q such that

$$PAQ = \sum_{j=1}^{h} \cdot A_j$$

where $\Sigma \cdot$ denotes the direct sum (defined on page 41), A_j is a doubly stochastic matrix for every j, and h is the number of characteristic roots of absolute value 1.

Theorem 3.20. If $A \in K_n$ then

$$\{\max \prod_{i=1}^{n} a_{i\sigma(i)} : \sigma \text{ runs over all permutations of } 1, \ldots, n\} \geq n^{-n}$$

with equality if and only if $A = J_n$.

Note that the theorem is misstated for the case of equality in Marcus and Minc [22] page 131.

Theorem 3.20 gives us another lower bound for the permanent function over the set K_n. Namely, $p(A) \geq n^{-n}$ since

$$p(A) \geq \prod_{i=1}^{n} a_{i\sigma(i)}$$

for every permutation σ of $1, \ldots, n$ by Theorem 3.4. This result will not be stated formally as it is a special case of the next theorem which gives what is believed to be the best lower bound for the permanent function over K_n.

Theorem 3.21. If $A \in K_n$ and has at least h characteristic roots of absolute value 1, then

$$p(A) \geq (n - h + 1)^{-(n-h+1)} \tag{30}$$

with equality if and only if A is a permutation matrix.

In order to prove Theorem 3.21 the following lemma is needed.

Lemma 3.22. If

$$\sum_{i=1}^{h} n_i = n, \quad n_i \geq 1, \quad i = 1, \ldots, h$$

then

$$\prod_{i=1}^{h} n_i^{n_i} \leq (n - h + 1)^{n-h+1}$$

with equality if and only if $n_i = 1$ for $h - 1$ values of i.

Proof: The proof is by induction on h. It is easily seen to be true for $h = 1$ and the proof is quite instructive for the case $h = 2$. Hence consider $h = 2$. Then $n_1 + n_2 = n$, $n_i \geq 1$, $i = 1, 2$. Define the function

$$f(x) = x^x(n - x)^{n-x}, \; 1 \leq x \leq n - 1.$$

Then

$$f'(x) = f(x) \ln \frac{x}{n-x} \text{ and } f'(x) = 0$$

when and only when $x = n/2$. But

$$f''(x) = f(x) \left[\left(\ln \frac{x}{n-x} \right)^2 + \frac{n}{x(n-x)} \right]$$

and $f''(x) > 0$ for all x, $1 \leq x \leq n - 1$. Thus f(x) is concave upward with a minimum occuring at $x = n/2$, and the maximum value must occur at the endpoints of the interval, $1 \leq x \leq n - 1$. That is, max $f(x) = f(1)$ or $f(n-1)$. But $f(1) = f(n-1) = (n-1)^{n-1}$ and hence

$$x^x(n - x)^{n-x} \leq (n - 1)^{n-1}.$$

Thus, when $x = n_1$ we have

$$n_1^{n_1} n_2^{n_2} \leq (n - 1)^{n-1}.$$

Equality occurs only at the endpoints since the curve is concave upward. That is, only when $n_1 = 1$ or $n_2 = 1$.

Next suppose the lemma is true for $h - 1$. We consider the case for h. Now

$$\begin{aligned} n_1^{n_1} \cdots n_h^{n_h} &= n_1^{n_1} (n_2^{n_2} \cdots n_h^{n_h}) \\ &\leq n_1^{n_1} [(n-n_1) - (h-1) + 1]^{[(n-n_1)-(h-1)+1]} \\ &= n_1^{n_1} [(n-h+2) - n_1]^{[(n-h+2)-n_1]} \end{aligned}$$

$$\leq [(n-h+2) - 1]^{[(n-h+2)-1]}$$

$$= (n-h+1)^{n-h+1}. \qquad (31)$$

In (31) the first inequality is true because of the induction hypothesis, and the second inequality is true since it is the case $h = 2$ proved above. Now equality holds in the first case of (31) if and only if $h - 2$ of the $n_2, \ldots, n_h$ are 1. Equality holds in the second case of (31) if and only if either $n_1 = 1$ or $n_1 = n-h+1$. In either event of the second case there is exactly $h-1$ of the n_i, $i=1, \ldots, h$ equal to 1. Hence, the lemma is true for h, and therefore true by induction for all positive integers.

Proof of Theorem 3.21: By Theorem 3.19 there exist permutation matrices P_1 and Q_1 such that

$$P_1AQ_1 = \sum_{j=1}^{h} \cdot A_j$$

where A_j is n_j-square doubly stochastic. Applying Theorem 3.20 to each A_j $j=1, \ldots, h$ we have

$$\max_{\sigma} \prod_{i=1}^{n_j} (A_j)_{i\sigma(i)} \geq n_j^{-n_j} \qquad (32)$$

where σ runs over all permutations of $1, \ldots, n_j$. Multiplying the products (32) together gives

$$\max_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} \geq \prod_{j=1}^{h} n_j^{-n_j} \qquad (33)$$

where σ runs over all permutations of $1, \ldots, n$. But by Lemma 3.22,

$$\prod_{j=1}^{h} n_j^{-n_j} \geq (n - h + 1)^{-(n-h+1)},$$

and Theorem 3.4 implies

$$p(A) \geq \max_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Thus $p(A) \geq (n-h+1)^{n-h+1}$.

For the case of equality we have

$$p(A) = \max_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}$$

if and only if A is a permutation matrix. Thus the only possibility for equality in (31) is for permutation matrices. Since for this type of matrix $n = h$ we see that $(n-h+1)^{-(n-h+1)} = 1$ and equality does result. This concludes the proof.

The above theorem can be improved somewhat if the matrix A is indecomposable.

Definition 3.23. An n-square matrix A is said to be decomposable if there exists a permutation matrix P such that

$$PAP' = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

where B and D are square matrices. If A is not decomposable, then it is said to be indecomposable.

If A is indecomposable then Theorem 3.19 can be improved so that PAQ can be written as the direct sum of h, n/h-square doubly stochastic matrices. That is, each of the A_j's has the same order, and h divides n [22], page 130. With this remark we are able to prove the following theorem.

Theorem 3.24. If A is an indecomposable doubly stochastic matrix that has h characteristic roots of absolute value 1, then

$$p(A) \geq (h/n)^n$$

with equality if and only if A is a permutation matrix.

Proof: By Theorem 3.19 and the above remark there exist permutation matrices P and Q such that

$$PAQ = \sum_{j=1}^{h} \cdot A_j$$

where each A_j is n/h-square doubly stochastic. Now applying Theorem 3.20 to each A_j, j=1, . . . , h we have

$$\max_{\sigma} \prod_{i=1}^{n/h} (A_j)_{i\sigma(i)} \geq (n/h)^{-n/h},$$

where σ denotes a permutation of 1, . . . , n/h. Thus

$$\begin{aligned} p(A) \geq \max_{\sigma_1} \prod_{i=1}^{n} a_{i\sigma_1(i)} &= \prod_{j=1}^{h} \max_{\sigma} \prod_{i=1}^{n/h} (A_j)_{i\sigma(i)} \\ &\geq \prod_{j=1}^{h} (n/h)^{-n/h} \\ &= (h/n)^n \end{aligned}$$

where σ_1 denotes a permutation of 1, . . . , n.

Equality again results since

$$p(A) = \max_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}$$

if and only if A is a permutation matrix and h = n in this case.

At this point several lower bounds have been given for the permanent function over the set K_n. In order to compare these bounds let us consider the 4-square matrices A, B, and J_4 where

$$A = I_2 \dotplus \begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

and

$$B = [1] \dotplus \begin{bmatrix} 1/6 & 2/6 & 3/6 \\ 3/6 & 1/6 & 2/6 \\ 2/6 & 3/6 & 1/6 \end{bmatrix} .$$

Then h = 3, 2, and 1 in A, B, and J_4 respectively, and the lower bounds given by the above theorems and Conjecture 3.2 for the permanents of matrices A, B and J_4 are exhibited in the following table.

	A	B	J_4
Theorem 3.17	.001	.001	.001
Theorem 3.21	.250	.037	.004
Theorem 3.24	-	-	.004
Conjecture 3.2	.094	.094	.094
Value of the Permanent	.556	.250	.094

Theorem 3.24 does not give bounds for A and B since they are decomposable matrices. In general we see that as n becomes larger all of our lower bounds approach 0. Also the bounds given by Theorems 3.21 and 3.24 depend heavily on h. That is, these bounds become better as h approaches n.

In the last few years considerable effort has been made to find a solution to the Van der Waerden conjecture. As a result of this research many interesting theorems have appeared, and some special cases have been proved. For example, it is known to be true whenever n = 2 or 3, [28], and it is also known for the special class of positive semi-definite symmetric matrices, [30]. We shall now demonstrate the proofs for the special cases n = 2 and 3. For n = 2, the proof reduces to the problem of finding the minimum of a polynomial.

Theorem 3.25. If $A \in K_2$ then $p(A) \geq 1/2$ with equality if and only if $A = J_2$.

Proof: Let $A \in K_2$ and suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $a + b = 1$, $c + d = 1$, $a + c = 1$, and $b + d = 1$. Thus $a = d$ and $b = c$. Therefore $p(A) = a^2 + b^2$ subject to the constraint $b = 1 - a$, or $p(A) = a^2 + (1 - a)^2$. Consider the function

$$f(x) = x^2 + (1 - x)^2, \quad 0 \leq x \leq 1.$$

Then the minimum occurs at $x = 1/2$, and the curve is concave upward which implies that this minimum is uniquely obtained. Thus whenever $a = 1/2$ the minimum value of the permanent function occurs, and this minimum is unique if and only if $A = J_2$. Therefore

$$p(A) \geq p(J_2) = 1/2.$$

The proof for the case n = 3 is a little more complicated, and we will state some of the main results which are given by Marcus and

Newman [28] in the general situation before we proceed with case. The proofs of these theorems are lengthy and will be omitted.

Theorem 3.26. If $A \in K_n$ such that $p(A) = \min \{p(S) : S \in K_n\}$ then A is indecomposable.

Theorem 3.27. If $A \in K_n$ such that $p(A) = \min \{p(S) : S \in K_n\}$ then

(a) $p(A(i/j)) = p(A)$, if $a_{ij} \neq 0$,

(b) $p(A(i/j)) = p(A) + \beta$, if $a_{ij} = 0$ where $\beta \geq 0$ and is independent of i, j.

Notice that the matrix J_n satisfies conditions (a) and (b).

Theorem 3.28. If $A \in K_n$ has positive entries and $p(A) = \min \{p(S) : S \in K_n\}$ then $A = J_n$.

This is an important theorem and tells us that the minimum is obtained uniquely at J_n for matrices with positive entries. This does not, however, help us in case some entries are zero. In this case the following result is known.

Theorem 3.29. If $A \in K_n$ such that $p(A) = \min \{p(S) : S \in K_n\}$ and A has at least 1 zero entry then all its zeros cannot occur in a fixed row.

Thus if $A \in K_n$ with only one zero entry then Theorem 3.29 tells us p(A) is not the minimum of the permanents over the set K_n.

We now state and prove Conjecture 3.2 for n = 3.

Theorem 3.30. If $A \in K_3$, then $p(A) \geq 2/9$ with equality if and only if $A = J_3$.

Proof: Let $B \in K_3$ such that $p(B) = \min \{p(S) : S \in K_3\}$. Then the proof consists of showing B cannot have any zero entries. First suppose B has at least 4 zero entries. By considering all possible cases then B must be a permutation matrix or else $p(B) = 1/2$. In either case it cannot be a minimizing matrix since $p(J_3) = 2/9$.

Secondly, assume B has 3 zero entries. Then no 2 of these can occur in the same row or column since this would imply B has more than 3 zero entries. Therefore without loss of generality assume

$$b_{ii} = 0, \quad i=1,2,3$$

$$b_{ij} > 0, \quad i \neq j.$$

Now by Theorem 3.27

$$p(A) = b_{23}b_{31} = b_{21}b_{32} = b_{12}b_{31} = b_{13}b_{32} = b_{12}b_{23} = b_{13}b_{21}.$$

Thus

$$b_{12} = b_{23} = b_{31} = x, \quad b_{21} = b_{13} = b_{32} = y$$

and

$$B = \begin{bmatrix} 0 & x & y \\ y & 0 & x \\ x & y & 0 \end{bmatrix}$$

where $x + y = 1$. Then $p(B) = x^3 + (1-x)^3$, $0 < x < 1$, and the minimum occurs at $x = 1/2$. This gives $p(B) = 1/4$ as the minimum. But $p(J_3) = 2/9 < p(B)$, a contradiction.

Next assume B has 2 zero entries. Again they cannot occur in the same row or column so without loss of generality assume they are $b_{11} = b_{22} = 0$. Then by Theorem 3.27

$$b_{21}b_{32} = b_{12}b_{31} = b_{12}b_{21} = b_{21}b_{13} = b_{12}b_{23}$$

and

$$b_{32} = b_{12} = b_{13} = x, \quad b_{23} = b_{31} = b_{21} = y.$$

Then

$$B = \begin{bmatrix} 0 & x & x \\ y & 0 & y \\ y & x & b_{33} \end{bmatrix},$$

and $x = y = 1/2$ implies $b_{33} = 0$, a contradiction.

Finally B cannot have just one zero since this contradicts Theorem 3.29. Thus the only possibility for B is a matrix with positive entries. Therefore using Theorem 3.28, $B = J_3$ and $p(B) = 2/9$. Hence if $A \in K_3$ then

$$p(A) \geq p(J_3) = 2/9,$$

and the proof is complete.

The Permanent as an Inner Product

We shall now investigate a new approach which has been used to prove many matrix inequalities involving the permanent function. This approach is a difficult one, but it has the saving grace of yielding some rather significant results in which other methods of proof have not succeeded. The main idea is to exhibit the permanent as an inner product on a special kind of vector space. This then enables us to use the Cauchy-Schwarz inequality as one of our main tools.

We begin by introducing the preliminary material necessary to prove these results.

Let V be an n-dimensional vector space with elements in the complex field. Usually a and b will denote complex scalars with x and y vectors.

Definition 3.31. An inner product in a vector space V is a complex valued function of ordered pairs of vectors x and y whose value at x and y is denoted as (x, y), such that

(a) $(x, y) = \overline{(y, x)}$,

(b) $(ax_1 + bx_2, y) = a(x_1, y) + b(x_2, y)$,

(c) $(x, ay_1 + by_2) = \bar{a}(x, y_1) + \bar{b}(x, y_2)$,

(d) $(x, x) \geq 0$ and $(x, x) = 0$ if and only if $x = 0$.

The inner product defined on V which we will want to use is given as

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i \tag{34}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. It is easy to see that (34) satisfies the conditions of Definition 3.31. An inner product space is a vector space with an inner product. Thus the vector space V with the inner product (34) is an inner product space and is called a unitary space. The norm of $x \in V$ is defined to be $(x, x)^{1/2}$ and is denoted as $\|x\|$.

The Cauchy-Schwarz inequality can now be stated.

Theorem 3.32. If x and y are vectors of the unitary space V then

$$|(x, y)| \leq \|x\| \ \|y\|$$

with equality if and only if x and y are linearly dependent.

Definition 3.33. Let $M_n(V)$ denote the space of n-multilinear functionals defined on V. That is, $g \in M_n(V)$ if and only if g is a complex-valued function defined on the cartesian product of V taken n times with itself such that g satisfies the following property:

$$g(x_1, \ldots, x_{i-1}, ax_i + bx_i', x_{i+1}, \ldots, x_n) \\ = ag(x_1, \ldots, x_i, \ldots, x_n) + bg(x_1, \ldots, x_i', \ldots, x_n) \tag{35}$$

for each x_i, $x_i' \in V$, $i=1, \ldots, n$.

Thus if $x_i = (x_{i1}, \ldots, x_{in})$, $i=1, \ldots, n$, then the n-multilinear functional β defined as $\beta(x_1, \ldots, x_n) = p(X)$ is an element of $M_n(V)$ where p is the permanent of the matrix X with rows x_i, $i=1, \ldots, n$. The fact that β satisfies condition (35) is a consequence of Theorems 2.4 and 2.14. The fact that $M_n(V)$ forms a vector space using the definition of vector addition and scalar multiplication defined as follows:

$$(f + g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n)$$

$$(bf)(x_1, \ldots, x_n) = bf(x_1, \ldots, x_n)$$

with $f, g \in M_n(V)$ and b a complex scalar is easily seen. We now form the dual space of $M_n(V)$, that is, the space of all complex-valued linear functionals on $M_n(V)$ and denote it as $V^{(n)}$. Certainly $\beta \in V^{(n)}$ if $\beta(f) = 0$ for every $f \in M_n(V)$. For non-trivial examples of elements of $V^{(n)}$ consider the functionals defined by using elements of V. That is, if $x_1, \ldots, x_n \in V$ then define $\alpha = x_1 \otimes \cdots \otimes x_n$ such that if $f \in M_n(V)$ then

$$\alpha(f) = f(x_1, \ldots, x_n).$$

Certainly we see that α is defined for each element of $M_n(V)$ and for scalars a and b with $g \in M_n(V)$ we have

$$\alpha(af + bg) = (af + bg)(x_1, \ldots, x_n)$$
$$= af(x_1, \ldots, x_n) + bg(x_1, \ldots, x_n)$$
$$= a\alpha(f) + b\alpha(g).$$

Thus α is linear and belongs to $V^{(n)}$. We will use the notation $x_1 \otimes \cdots \otimes x_n$ since it has the advantage of specifying the set of vectors from V.

For a more specific example, let $n = 3$ and consider the element $x_1 \otimes x_2 \otimes x_3$ of $V^{(3)}$ defined by $x_1 = (2, 1, 0,)$, $x_2 = (-1, 4, 5)$, and $x_3 = (1, 1, 2)$. Then the value of $x_1 \otimes x_2 \otimes x_3$ at $\beta \in M_3(V)$ where β is defined above is

$$x_1 \otimes x_2 \otimes x_3(\beta) = \beta(x_1, x_2, x_3) = 29.$$

The following theorem gives the full nature of $V^{(n)}$.

Theorem 3.34. Every element of $V^{(n)}$ is of the form $x_1 \otimes \cdots \otimes x_n$ for some set of vectors $x_1, \ldots, x_n$ from V.

For the proof of this theorem and a more detailed discussion of $V^{(n)}$ see [33].

We now define an inner product on $V^{(n)}$ in terms of the inner product (34).

Let $x_1, \ldots x_n, y_1, \ldots, y_n \in V$ then $x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \in V^{(n)}$ and define

$$(x_1, \otimes \cdots \otimes x_n, \; y_1 \otimes \cdots \otimes y_n) = \prod_{i=1}^{n} (x_i, y_i). \tag{36}$$

We now define two operators on the space $V^{(n)}$. By an operator we mean a linear mapping of $V^{(n)}$ into itself.

First we define the permutation operator $P(\sigma)$ on $V^{(n)}$ such that

$$P(\sigma)(x_1 \otimes \cdots \otimes x_n) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \tag{37}$$

where σ denotes a permutation of the numbers $1, \ldots, n$.

Next we define another operator T_n on $V^{(n)}$ such that

$$T_n(x_1 \otimes \cdots \otimes x_n) = 1/n! \sum_{\sigma} P(\sigma)(x_1 \otimes \cdots \otimes x_n) \tag{38}$$

where σ ranges over all permutations of $1, \ldots, n$.

<u>Lemma 3.35.</u> If σ' denotes any permutation of $1, \ldots, n$ then

$$P(\sigma')[T_n(x_1 \otimes \cdots \otimes x_n)] = T_n(x_1 \otimes \cdots \otimes x_n).$$

Proof: By (38) and linearity we have

$$P(\sigma')[T_n(x_1 \otimes \cdots \otimes x_n)] = P(\sigma')[1/n! \sum_{\sigma} P(\sigma)(x_1 \otimes \cdots \otimes x_n)]$$

$$\begin{aligned} P(\sigma')[T_n(x_1 \otimes \cdots \otimes x_n)] &= 1/n! \sum_{\sigma} P(\sigma')[P(\sigma)(x_1 \otimes \cdots \otimes x_n)] \\ &= 1/n! \sum_{\sigma} P(\sigma)(x_1 \otimes \cdots \otimes x_n) \\ &= T_n(x_1 \otimes \cdots \otimes x_n) \end{aligned}$$

since $P(\sigma)$ operating on the sum rearranges the elements in that sum but does not change it.

<u>Lemma 3.36.</u> The operator T_n is such that $T_n^2 = T_n$.

Proof: By (38) and linearity we have

$$\begin{aligned} T_n^2(x_1 \otimes \cdots \otimes x_n) &= T_n[1/n! \sum_{\sigma} P(\sigma)(x_1 \otimes \cdots \otimes x_n)] \\ &= 1/n! \sum_{\sigma} T_n[P(\sigma)(x_1 \otimes \cdots \otimes x_n)] \\ &= 1/n! \sum_{\sigma} 1/n! \sum_{\sigma'} P(\sigma')[P(\sigma)(x_1 \otimes \cdots \otimes x_n)] \end{aligned} \tag{39}$$

But

$$\sum_{\sigma'} P(\sigma')[P(\sigma)(x_1 \otimes \cdots \otimes x_n)] = \sum_{\sigma'} P(\sigma')(x_1 \otimes \cdots \otimes x_n)$$

and there are exactly n! such sums since there are n! permutations σ of the numbers $1, \ldots, n$. Thus (39) reduces to

$$1/n! \sum_{\sigma'} P(\sigma')(x_1 \otimes \cdots \otimes x_n)$$

and $T_n^2 = T_n$ as was to be shown.

Let V be a unitary space. Then for each $y \in V$ there exists a linear functional y' defined as $y'(x) = (x, y)$ for every $x \in V$. Now if A is an operator on V then for each fixed y there exists a linear functional y^* with the defining property that for each $x \in V$,

$$y^*(x) = y'(Ax) = (Ax, y). \tag{40}$$

If we now allow y' to vary over all linear functionals of V then this makes correspond to each y' a y^*, depending on y'. We will use the notation

$$y^* = A^*y'.$$

The defining property of A^* is

$$A^*y'(x) = (x, A^*y). \tag{41}$$

Now from (40) and (41) we have

$$(x, A^*y) = A^*y'(x) = y^*(x) = (Ax, y). \tag{42}$$

We now state two useful properties about linear operators.

Lemma 3.37. The operator A is such that $A = A^{**}$.

Lemma 3.38. The operator A is hermitian ($A = A^*$) if and only if $(Ax, y) = (x, Ay)$ for all $x, y \in V$ where V is a unitary space.

Another property of the operator T_n is given in the following lemma.

Lemma 3.39. T_n is hermitian.

Proof: By Lemma 3.38 we must show that $(T_n\alpha, \beta) = (\alpha, T_n\beta)$ for all $\alpha, \beta \in V^{(n)}$. Using linearity and definitions (38) and (36) we have

$$(T_n(x_1 \otimes \cdots x_n),\ y_1 \otimes \cdots \otimes y_n) = 1/n! \sum_{\sigma} (P(\sigma)(x_1 \otimes \cdots \otimes x_n), y_1 \otimes \cdots \otimes y_n)$$

$$= 1/n! \sum_{\sigma} \prod_{i=1}^{n} (x_{\sigma(i)}, y_i)$$

$$= 1/n! \sum_{\sigma} \prod_{i=1}^{n} (x_i, y_{\sigma(i)})$$

$$= 1/n! \sum_{\sigma} (x_1 \otimes \cdots \otimes x_n, P(\sigma)(y_1 \otimes \cdots \otimes y_n))$$

$$= (x_1 \otimes \cdots \otimes x_n,\ 1/n! \sum_{\sigma} P(\sigma)(y_1 \otimes \cdots \otimes y_n))$$

$$= (x_1 \otimes \cdots \otimes x_n,\ T_n(y_1 \otimes \cdots \otimes y_n)),$$

and the proof is complete.

At this point we are ready to see how the last few results are connected to the permanent. Thus let us prove an important theorem.

Theorem 3.40. Let x_i and y_i, $i=1, \ldots, n$, be arbitrary vectors in V and define $a_{ij} = (x_i, y_j)$. If $A = (a_{ij})$ then

$$(T_n(x_1 \otimes \cdots \otimes x_n),\ T_n(y_1 \otimes \cdots \otimes y_n)) = 1/n!\, p(A). \tag{43}$$

Proof: Using Lemma 3.37, result (42), Lemmas 3.39, and

3.36, and Definitions (38), (36), and 1.1 in that order we have

$$(T_n(x_1 \otimes \cdots \otimes x_n),\ T_n(y_1 \otimes \cdots \otimes y_n)) = (T_n^* T_n(x_1 \otimes \cdots \otimes x_n), y_1 \otimes \cdots \otimes y_n)$$

$$= (T_n^2(x_1 \otimes \cdots \otimes x_n),\ y_1 \otimes \cdots \otimes y_n)$$

$$= (T_n(x_1 \otimes \cdots \otimes x_n),\ y_1 \otimes \cdots \otimes y_n)$$

$$= 1/n! \sum_{\sigma} (P(\sigma)(x_1 \otimes \cdots \otimes x_n), y_1 \otimes \cdots \otimes y_n)$$

$$= 1/n! \sum_{\sigma} \prod_{i=1}^{n} (x_{\sigma(i)}, y_i)$$

$$= 1/n! \sum_{\sigma} \prod_{i=1}^{n} a_{\sigma(i)i}$$

$$= 1/n!\, p(A)$$

where σ ranges over all permutations of $1, \ldots, n$. Thus the proof is complete.

Notice that Theorem 3.40 states that given any set of vectors x_i and y_i, $i=1, \ldots, n$, from V a matrix A can be found such that the permanent of this matrix represents an inner product of two vectors in the space $V^{(n)}$. Now the question naturally arises as to whether or not the permanent of any n-square matrix A can be expressed as the inner product of two vectors from $V^{(n)}$. By a reversal of the steps in the proof of Theorem 3.40 we see that this can be done if there exists vectors x_i and y_i, $i=1, \ldots, n$, from V such that $A = (a_{ij}) = ((x_i, y_j))$. To find such vectors of V it is the same as finding n-square matrices X and Y such that $A = XY^*$, since the ij-th element of A is just

$$\sum_{k=1}^{n} x_{ik} \overline{y}_{jk}.$$

This can always be done by letting $A = X$ and $Y = I$. Then there exist vectors $x_i = A_{(i)}$ and $y_i = I_{(i)}$, $i=1,\ldots,n$, such that $A = ((x_i, y_i))$. We state this result formally in the next theorem.

Theorem 3.41. Given an n-square matrix A then there exist vectors x_i and y_i, $i=1,\ldots,n$, from V such that

$$p(A) = n!(T_n(x_1 \otimes \cdots \otimes x_n), \ T_n(y_1 \otimes \cdots \otimes y_n)).$$

In order for Theorems 3.40 and 3.41 to be of maximum use something about the vectors x_i and y_i usually needs to be known. Thus we have the following theorem.

Theorem 3.42. If A is an n-square positive semi-definite hermitian matrix then there exist vectors $x_1, \ldots, x_n$ from V such that $a_{ij} = (x_i, x_j)$ for every i and j.

The proof is omitted, but can be found in [26], page 184.

We now state and prove an interesting theorem given by Marcus and Minc [23] and [25], which is a generalization of the Van der Waerden inequality for the class of positive semi-definite symmetric matrices.

Theorem 3.43. If A is an n-square positive semi-definite hermitian matrix with row sums r_i, $i=1,\ldots,n$, and if

$$r = \sum_{i=1}^{n} r_i \neq 0$$

then

$$p(A) \geq n! \ \frac{\prod_{i=1}^{n} |r_i|^2}{r^n} \ . \tag{44}$$

Equality holds in (44) if and only if

(a) $\rho(A) = 1$, or

(b) A has a zero row.

Proof: By Theorem 3.42 there exists vectors x_i, $i=1,\ldots,n$ such that $A = (a_{ij}) = ((x_i, x_j))$. Now using (43) and Theorem 3.32 we have

$$1/n!p(A) = (T_n(x_1 \otimes \cdots \otimes x_n), T_n(x_1 \otimes \cdots \otimes x_n)) \tag{45}$$

$$= (T_n(x_1 \otimes \cdots \otimes x_n), T_n(x_1 \otimes \cdots \otimes x_n))(u/\|u\|, u/\|u\|)$$

$$\geq |(T_n(x_1 \otimes \cdots \otimes x_n), u/\|u\|)|^2$$

where u is any non-zero vector from $V^{(n)}$. Next let

$$v = \sum_{i=1}^{n} x_i.$$

Then

$$(v,v) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i, x_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} r_i = r.$$

Now $r \neq 0$ implies $\|v\| \neq 0$ since $\|v\|^2 = r$. Also $r > 0$. Let $u = T_n(v \otimes \cdots \otimes v)$. Then by (43)

$$\|u\|^2 = (T_n(v \otimes \cdots \otimes v), T_n(v \otimes \cdots \otimes v)) = 1/n!p(B)$$

where $B = (b_{ij}) = (v, v) = r$ for all i and j, $i, j = 1, \ldots, n$. But by Corollary 2.7, $p(B) = n!r^n$. Thus $\|u\|^2 = r^n$. Now using (45) we have

$$1/n!p(A) \geq \left| (T_n(x_1 \otimes \cdots \otimes x_n), \frac{T_n(v \otimes \cdots \otimes v)}{r^{n/2}} \right|^2$$

$$= |(T_n(x_1 \otimes \cdots \otimes x_n), T_n(v \otimes \cdots \otimes v))|^2 / r^n$$

$$= |1/n!p((x_i, v))|^2/r^n.$$

But

$$(x_i, v) = (x_i, \sum_{j=1}^{n} x_j) = \sum_{j=1}^{n} (x_i, x_j) = \sum_{j=1}^{n} a_{ij} = r_i.$$

Thus we are interested in finding the permanent of a matrix whose i-th row is $(r_i, \ldots, r_i)$. By Theorem 2.6

$$p((x_i, v)) = n! \prod_{i=1}^{n} r_i ,$$

and therefore

$$1/n!p(A) \geq |1/n! \cdot n! \prod_{i=1}^{n} r_i|^2/r^n = [\prod_{i=1}^{n} |r_i|^2]/r^n,$$

and

$$p(A) \geq n! \frac{\prod_{i=1}^{n} |r_i|^2}{r^n} .$$

The proofs for the cases involving equality are quite involved and will be omitted.

As a direct consequence of this theorem the Van der Waerden inequality can be proved for the case of positive semi-definite symmetric matrices.

<u>Corollary 3.44</u>. If $A \in K_n$ is positive semi-definite symmetric, then $p(A) \geq n!/n^n$, with equality if and only if $A = J_n$.

Proof: Since $A \in K_n$ then the row sums are 1 and r = n in Theorem 3.43. Thus using (44) we have $p(A) \geq n!/n^n$. Certainly if $A = J_n$ equality occurs. Next assume B is a matrix satisfying the hypothesis such that $p(B) = n!/n^n$. Then Theorem 3.43 implies $\rho(B) = 1$. Therefore, every row of B is a multiple of the first row. Let

$B_{(1)} = (b_{11}, \ldots, b_{1n})$. Then

$$\sum_{j=1}^{n} b_{1j} = 1$$

and consider row i of B, i=2, ..., n. Thus $B_{(i)} = (db_{11}, \ldots, db_{1n})$ where d is a non-zero scalar. But

$$\sum_{j=1}^{n} db_{1j} = 1$$

implies d = 1. Hence

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{11} & \cdots & b_{1n} \end{bmatrix} .$$

But the columns sum to 1 which implies $nb_{1j} = 1$ or $b_{1j} = 1/n$ and $B = J_n$. Thus the case of equality is proved.

The usefulness of expressing the permanent of a matrix as an inner product of two vectors from $V^{(n)}$ has been demonstrated in the proof of Theorem 3.43. This particular technique of proof is not restricted to the permanent function only. For example, if T_n' is defined to be

$$T_n'(x_1 \otimes \cdots \otimes x_n) = 1/n! \sum_{\sigma} \epsilon(\sigma) P(\sigma)(x_1 \otimes \cdots \otimes x_n)$$

where $\epsilon(\sigma) = \pm 1$ according as σ is even or odd, then

$$(T_n'(x_1 \otimes \cdots \otimes x_n),\ T_n'(y_1 \otimes \cdots \otimes y_n)) = 1/n! d((x_i, y_j)). \qquad (46)$$

Using the result (46) the classical inequality known as the Hadamard determinant theorem has been proved by Marcus and Minc [21]. This theorem was done by J. Hadamard in 1893, [9] and will prove to be useful to us. We state it without proof.

Theorem 3.45. If A is an n-square positive semi-definite hermitian matrix then

$$d(A) \leq \prod_{i=1}^{n} a_{ii} ,$$

with equality if and only if A has a zero row (or column) or A is a diagonal matrix.

We now state a theorem given by Marcus [14] in 1964 which will allow us to prove the so called Hadamard theorem for permanents.

Theorem 3.46. If A is an n-square positive semi-definite hermitian matrix then

$$na_{ii}p(A(i/i)) \geq p(A) \geq a_{ii}p(A(i/i)), \quad 1 \leq i \leq n. \tag{47}$$

Equality holds in (47) if A has a zero row (or column). If A has no zero row (or column) then the lower inequality holds if and only if a_{ii} is the only non-zero entry in row and column i of A; the upper equality holds if and only if the rank of A is 1.

The proof of this theorem uses techniques similar to the proof of Theorem 3.43, but is much longer. Hence, it is omitted.

We are now ready to state and prove the permanent analog to the Hadamard determinant theorem.

Theorem 3.47. If A is an n-square positive semi-definite hermitian matrix then

$$p(A) \geq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if A has a zero row (or column) or A is a diagonal matrix.

Proof: By Theorem 3.46 the lower inequality gives

$$p(A) \geq a_{nn} p(A(n/n)).$$

Now consider the matrix A(n/n). This matrix is an (n-1)-square positive semi-definite hermitian matrix. Therefore applying (47) again we have

$$p(A(n/n)) \geq a_{(n-1)(n-1)} p(A(n-1, n/n-1, n)).$$

Thus,

$$p(A) \geq a_{(n-1)(n-1)} a_{nn} p(A(n-1, n/n-1, n)).$$

Now applying (47) (n-1)-times yields

$$p(A) \geq \prod_{i=1}^{n} a_{ii} .$$

Using the results of Theorems 2.3 and 2.8 equality occurs if A has a zero row (or column) or A is a diagonal matrix. Now suppose

$$p(A) = \prod_{i=1}^{n} a_{ii}$$

and A does not have a zero row (or column). Then

$$\prod_{i=1}^{n} a_{ii} = p(A) \geq a_{11} p(A(1/1)) \geq a_{11} a_{22} p(A(1, 2/1, 2)) \geq \ldots \geq \prod_{i=1}^{n} a_{ii} ,$$

which implies

$$\prod_{i=1}^{n} a_{ii} = a_{11} p(A(1/1)) = p(A).$$

Now by Theorem 3.46 this is true if and only if a_{11} is the only non-zero entry in row and column 1 of A. Next we have

$$\prod_{i=2}^{n} a_{ii} = p(A(1/1)) \geq a_{22}p(A(1,2/1,2)) \geq \ldots \geq \prod_{i=2}^{n} a_{ii},$$

which implies a_{22} is the only non-zero entry in row and column 2 of A. Using this technique (n-1)-times we have a_{ii} as the only non-zero entry in row and column i of A, $i=1,\ldots,n$. Thus A is a diagonal matrix, and the proof is complete.

Corollary 3.48. If A is an n-square positive semi-definite hermitian matrix then $p(A) \geq d(A)$ with equality if and only if A has a zero row (or column) or A is a diagonal matrix.

Proof: This is a direct result of Theorems 3.45 and 3.47.

Most of the inequalities we have investigated thus far deal with special types of matrices. The reason for this is probably clear since most of the results place restrictions which can only be obtained by certain kinds of matrices. There are very few results which can be proved for permanental inequalities in the general case. One such inequality is the following.

Theorem 3.49. Let x_i and y_i, $i=1,\ldots,n$ be vectors from the n-dimensional unitary space V. Then

$$|p((x_i, y_j))|^2 \leq p((x_i, x_j))p((y_i, y_j)), \tag{48}$$

with equality if and only if

(a) the zero vector occurs in one of the sets x_i or y_i, or

(b) the zero vector does not occur in either set x_i or y_i, and there exist non-zero scalars d_i, $i=1,\ldots,n$, and a permutation σ of $1,\ldots,n$ such that $y_i = d_i x_{\sigma(i)}$, $i=1,\ldots,n$.

Proof: By using the result (43) and the Cauchy-Schwarz inequality we see that

$$|1/n!p((x_i, y_j))| \le [1/n!p((x_i, x_j))]^{1/2}[1/n!p((y_i, y_j))]^{1/2}.$$

Thus by squaring both sides (48) is obtained. The cases for equality involve arguments about the vectors of $V^{(n)}$ being linearly dependent and are omitted.

Theorem 3.50. Let A and B be n-square complex matrices. Then

$$|p(AB)|^2 \le p(AA^*)p(B^*B), \tag{49}$$

with equality in (49) if and only if

(a) a row of A or a column of B is zero, or

(b) A and B are nonsingular and there exists a diagonal matrix D and a permutation matrix P such that $A^* = BDP$.

Proof: Let e_i denote the basis vector of V with 1 in position i and zero elsewhere. Then by using (42) the ij-th element of the matrix AB is

$$(AB)_{ij} = (ABe_j, e_i) = (Be_j, A^*e_i).$$

Now by applying (48) and (42) we have

$$\begin{aligned} |p(AB)|^2 &= |p((Be_j, A^*e_i))|^2 \le p((Be_i, Be_j))p((A^*e_i, A^*e_j)) \\ &= p((B^*Be_i, e_j))p((AA^*e_i, e_j)) = p((B^*B)_{ji})p((AA^*)_{ji}) \\ &= p(B^*B)p(AA^*). \end{aligned}$$

Using the results of Theorem 3.49 we see that if (a) or (b) holds then equality holds. Thus suppose equality holds in (49). Then by Theorem

3.49 the zero vector must occur in the sets Be_i, A^*e_i, i=1, . . . , n. That is, among the rows of A or columns of B, or there exist scalars d_i, i=1, . . . , n, and a permutation σ of 1, . . . , n such that $A^*e_i = d_i Be_{\sigma(i)}$. But this is the same as saying $A^* = BDP$ which was to be shown.

Definition 3.51. A nonsingular n-square matrix A is said to be unitary if $AA^* = A^*A = I$.

Theorem 3.52. If A is unitary then $|p(A)| \leq 1$, with equality if and only if A has exactly one entry of modulus 1 in each row and column.

Proof: From (49) it follows that

$$|p(A)| = |p(AI)| \leq [p(AA^*)p(I^*I)]^{1/2}.$$

But $p(AA^*) = p(I) = 1$ which implies $|p(A)| \leq 1$.

Now suppose $|p(A)| = 1$. Then Theorem 3.50 implies $I = DPA$ where D and P are diagonal and permutation matrices respectively. Also $A^*A = I$ implies $A^* = DP$. Thus $A^*A = DPP^*D^* = DD^* = I$ and $d_{ii}\bar{d}_{ii} = 1$. This means $|d_{ii}| = 1$, and therefore $A = P^*D^*$ and has exactly one entry of modulus 1 in each row and column. If A has only one entry of modulus 1 in each row and column, it is easy to see $|p(A)| = 1$.

Some General Inequalities

Several techniques of proof have been demonstrated in proving some of the inequalities encountered thus far. Of course these techniques do not cover all the various methods of proof which have been used or can be used. They do serve to illustrate the fact that many times the proofs are of a non-trivial nature and involve background material which can be found in a variety of different mathematical branches. There are many inequalities involving the permanent function which have not been touched upon at this point. In order to give an awareness of the more important results we will resort to stating these inequalities and giving some illustrative examples. Most of the results which follow is current research in the sense that it has been done in the last few years.

One of the more easily applied results which has been obtained recently is the following theorem.

Theorem 3.53. Let A be a n-square matrix and for $i=1, \ldots, n$, let

$$r_i = \sum_{j=1}^{n} |a_{ij}|,$$

and $q_i = \max |a_{ij}|$, $1 \leq j \leq n$. Then

$$|p(A)| \leq \prod_{i=1}^{n} (r_i + q_i)/2 = 1/2^n \prod_{i=1}^{n} (r_i + q_i). \tag{50}$$

This theorem was proved by Jurkat and Ryser [12] in a lengthly article in 1966. It is a generalization of a theorem of Minc [32] given in 1963, which was proved for (0, 1)-matrices. That is, matrices all of whose elements are either 0 or 1. Minc's theorem is the following.

Theorem 3.54. Let A be an n-square (0, 1)-matrix and let

$$r_i = \sum_{j=1}^{n} a_{ij}, \quad i=1, \ldots, n.$$

Then

$$p(A) \leq \prod_{i=1}^{n} (r_i + 1)/2 = 1/2^n \prod_{i=1}^{n} (r_i + 1)$$

with equality if and only if A is a permutation matrix.

As an example of Theorem 3.53 consider the matrix A where

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 4 & 0 & 2 & -4 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 5 & -1 \end{bmatrix}.$$

Then $p(A) = -8$ and using (50) we have $r_1 = 4$, $r_2 = 10$, $r_3 = 2$, and $r_4 = 8$, while $q_1 = 2$, $q_2 = 4$, $q_3 = 1$, and $q_4 = 5$. Thus

$$\prod_{i=1}^{n} (r_i + q_i)/2 = (3)(7)(3/2)(13/2) = 819/4,$$

and

$$|p(A)| = 8 \leq 819/4.$$

Thus it can be seen from this example that the inequality (50) is somewhat weak in general, but equality does occur for permutation matrices.

Another upper bound for the permanent of a general n-square matrix which has been obtained by Beasley and Brenner [1] in 1968, but which is not quite so easy to use is given in the next theorem.

<u>Theorem 3.55.</u> Let A be an n-square matrix such that $a_{nn} \neq 0$. Then

$$|p(A)| \leq |a_{nn}| p(B)$$

where B is the (n-1)-square matrix such that

$$b_{ij} = |a_{ij}| + \frac{|a_{nj}|\,|a_{in}|}{|a_{nn}|}$$

for $i=1,\ldots,n-1$, $j=1,\ldots,n-1$.

For the 4x4 example A given above we have

$$|p(A)| = 8 \leq |a_{nn}|p(B) = |-1|(160) = 160,$$

where

$$B = \begin{bmatrix} 4 & 1 & 5 \\ 12 & 0 & 22 \\ 0 & 1 & 1 \end{bmatrix}.$$

Beasley and Brenner, in the same paper as above, have also generalized an inequality of Jurkat and Ryser which again uses the elements of the matrix to obtain an upper bound for the permanent function. It is easier to apply than that of Theorem 3.55.

Theorem 3.56. If A is an n-square matrix then

$$|p(A)| \leq \prod_{i=1}^{n} S_i^{(i)}$$

where $S_i^{(i)}$ is defined to be the sum of the i largest absolute values of the elements from row i of A.

For the matrix A defined above this upper bound gives

$$|p(A)| = 8 \leq \prod_{i=1}^{4} S_i^{(i)} = (2)(8)(2)(8) = 256.$$

Thus Theorem 3.55 gives the lowest upper bound for the matrix A, but none of them is very sharp. For the matrix J_3, the upper bounds given by Theorems 3.53, 3.55, and 3.56 are .296, .296, and .222, respectively. Thus, in this case Theorem 3.56 gives the lowest upper bound. In general it appears difficult to tell which will give the best upper bound for a given matrix with the possible exception that Theorem 3.55 seems to give as good as or better results than Theorem 3.53. The ease of finding the bound given by Theorem 3.53 still makes it quite useful.

Some other upper bounds which were obtained in 1964 by Marcus and Minc [23] relate the permanent to some of the more usual matrix invariants. This is done though at the expense of requiring the matrix A to be normal.

Definition 3.57. An n-square matrix A is said to be normal if $AA^* = A^*A$.

The inequality is now stated.

Theorem 3.58. If A is an n-square normal matrix with characteristic roots $\alpha_1, \ldots, \alpha_n$ then

$$|p(A)| \leq 1/n \sum_{i=1}^{n} |\alpha_i|^n.$$

For an example of Theorem 3.58 consider the matrix C such that

$$C = \begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}.$$

Then C is symmetric which implies it is normal and has characteristic roots 4, -2, -2. Thus

$$|p(C)| = 4 \leq 1/3(|4|^3 + |-2|^3 + |-2|^3) = 80/3.$$

Using Theorems 3.58 and 3.50 the following two upper bounds have been obtained for doubly stochastic matrices using the rank of the matrix, [23].

Theorem 3.59. If $A \in K_n$ is normal then

$$p(A) \leq \rho(A)/n. \tag{51}$$

Equality holds in (51) if and only if A is a permutation matrix or n = 2 and $A = J_2$.

If the condition that A is normal is dropped the following inequality can be proved.

Theorem 3.60. If $A \in K_n$ then $p(A) \leq [\rho(A)/n]^{1/2}$, with equality if and only if A is a permutation matrix.

The inequalities proved and stated in this chapter represent some of the more important results which a comprehensive study of this subject would reveal. They do not, however, cover all the results such as some theorems and corollaries which led to the development of the stated inequalities and usually involve very special cases. Also no attempt has been made to cover all the inequalities which are not directly connected with the bounding of the permanent of a matrix. For example, inequalities involving subpermanents, [5, 6, 16]; inequalities involving the direct product of matrices, [4, 15]; or inequalities involving the square root of a matrix, [30]. With the exception of Theorem 3.54 no mention has been made of some important inequalities for (0, 1)-matrices. This type of matrix will be examined in more detail in the next chapter.

CHAPTER IV

APPLICATIONS AND PROBLEMS OF THE PERMANENT FUNCTION

Introduction

In the presentation thus far we have not given any real need for the study of the permanent function. To some, just the fact that it is there and has some interesting properties is enough. To others some type of motivation is needed. This motivation usually centers around the idea of its usefulness in either some future application, or else some physical situation exists in which this study can play a part in explaining this situation. Fortunately a study of the permanent function has something to offer to both needs. The usefulness of the permanent can be found mainly in a branch of mathematics known as combinatorial theory. Combinatorial mathematics consists essentially of the study of the arrangement of elements into sets. Usually two general types of problems present themselves in this area. The first of these is concerned with the existence of a prescribed arrangement of elements of the set; while in the second problem a prescribed arrangement is known, but the exact number of distinct arrangements is not known. Thus a study is attempted to count the number of these arrangements. A simple example will help explain these concepts. Suppose we are given five rectangles of dimension 1" by 2", and six 1" squares. The problem is to arrange the rectangles and squares so that they form a 4" square. Whether or not this can be done is called an existence problem. In this example, the existence question can be solved by putting the six 1" squares together to form three 1" by 2" rectangles to yield eight 1" by 2" rectangles which can be arranged into a 4" square, (Figure 1). Now the second question is how many different arrangements exist. That is, how many distinct arrangements can we have knowing that there exists at least one arrangement of the rectangles and squares into a 4" square. This second problem is the one in which the permanent function has made a contribution.

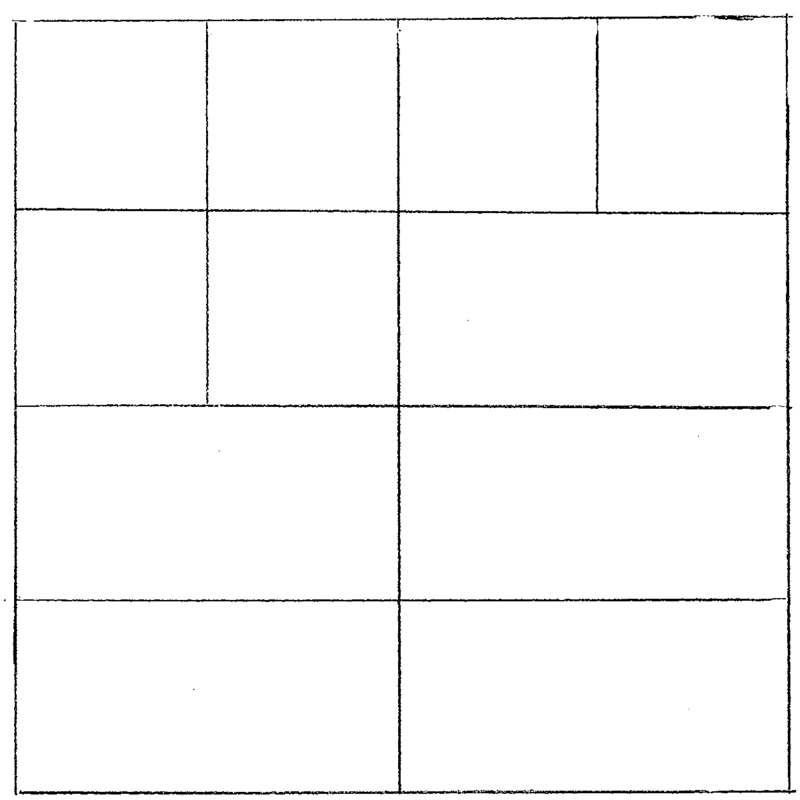


Figure 1.

Applications of the Permanent

We now explain the type of situation where the permanent function can be used. We begin with the definition of a system of distinct representatives.

Definition 4.1. Let S be an n-set with distinct elements $a_1, \ldots, a_n$. (That is, $S = \{a_1, a_2, \ldots, a_n\}$). Let $S_1, \ldots, S_n$ be n subsets of S and suppose for some permutation σ of $1, \ldots, n$, $a_{\sigma(i)} \in S_i$ for $i=1, \ldots, n$. Then the element $a_{\sigma(i)}$ represents the set S_i, and the subsets are said to have a system of distinct representatives.

Notice that if $i \neq j$, then $a_i \neq a_j$, but the subsets S_i, $i=1, \ldots, n$, are not required to be distinct. Consider the following example of this concept. Let $S = \{1, 3, 5, 7, 9\}$ with subsets $S_1 = \{1, 5\}$, $S_2 = \{3, 7\}$, $S_3 = \{1, 5\}$, $S_4 = \{1, 3, 5, 7\}$, and $S_5 = \{1, 3, 5, 9\}$. Then a system of distinct representatives is given by $B = (5, 7, 1, 3, 9)$ for $(S_1, S_2, S_3, S_4, S_5)$. The system B is not the only one since $(5, 3, 1, 7, 9)$, $(1, 3, 5, 7, 9)$, and $(1, 7, 5, 3, 9)$ will also do the job. If we replace S_5 by $S_5' = \{1, 3, 5\}$, then no system of distinct representatives exists since there are only four elements and five subsets.

The subsets $(S_1, S_2, S_3, S_4, S_5)$ can be completely described by means of a (0, 1)-matrix called an incidence matrix.

Definition 4.2. Let S be an n-set with distinct elements $a_1, \ldots, a_n$. Let $S_1, \ldots, S_n$ be n subsets of S and define the n-square matrix A such that $a_{ij} = 1$ if $a_j \in S_i$ and $a_{ij} = 0$ if $a_j \notin S_i$. Then the matrix A is called the incidence matrix of the subsets $S_1, \ldots, S_n$ of S.

Notice that the 1's of row i of the incidence matrix A specifies the elements of set S_i, while the 1's in column j of A specifies to what sets the element a_j belongs. The incidence matrix of the above example is

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

By trial we found for this example that there were four systems of distinct representatives for the subsets $(S_1, S_2, S_3, S_4, S_5)$. The permanent of the incidence matrix A is also 4, which suggests that there may be some connection between the permanent of the incidence matrix and the number of systems of distinct representatives. The following theorem by H. J. Ryser [40] shows that this is indeed the case.

Theorem 4.3. Let $S_1, \ldots, S_n$ be subsets of an n-set S. Let A be the incidence matrix for $S_1, \ldots, S_n$. Then p(A) is the number of systems of distinct representatives for $S_1, \ldots, S_n$.

Proof: The proof follows from the definition of the permanent function since the only non-zero addends are +1 and represent a system of distinct representatives. Also such system of distinct representatives is represented by a +1 addend since it is just some permutation σ in the sum of p(A). Therefore, all systems of distinct representatives are counted, and p(A) is the sum of them.

Thus, we see that the permanent function appears in any combinatorial setting where a count of the number of systems of distinct representatives is required.

Theorem 4.3 is illustrated by the following examples.

The representatives of five groups of people, French, English, Negro, Spanish, and German are sponsoring a meeting with some famous baseball players in New York City. Milltown, U. S. A., can send five boys, one from each of its five baseball teams, as representatives to this meeting. Milltown wants to be fair and represent not only the five teams but also the five groups as well (which we will abbreviate as F, E, N, S, G). It is found that each team does not represent all of the above groups. The various groups represented by the teams T_i, $i=1,\ldots,5$, are: $T_1 = \{F, E, N, G\}$, $T_2 = \{E, N\}$, $T_3 = \{E, G, S\}$, $T_4 = \{F, E, N\}$, and $T_5 = \{F, S\}$. Rather than force one particular team to send a boy representing one group it is suggested that all the possibilities be written down and put in a hat from which one possibility will be drawn and then the teams will abide by this selection. Now, how many ways can a representative be chosen, one from each team and one from each group? The number of ways this can be done is just the number of systems of distinct representatives for the subsets, $T_1, \ldots, T_5$, of the set $\{F, E, N, S, G\}$. The incidence matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and $p(A) = 7$. Thus the number of ways this can be done is 7. These possibilities are then computed and one is drawn.

For another example consider a projective plane of order 2. We define these concepts.

Definition 4.4. A projective plane π is a mathematical system composed of two entities called points and lines that satisfies the following postulates:

(a) Two distinct points of π are on one and only one line of π;

(b) Two distinct lines of π pass through one and only one point of π;

(c) There exist four distinct points of π, no three of which are on the same line;

(d) There exist four distinct lines of π, no three of which pass through the same point.

A projective plane is said to be finite if it contains only a finite number of points. The order of a finite projective plane π is said to be n if given any line L, the total number of points on L is n + 1. The totality of lines and points in a projective plane of order n is $n^2 + n + 1$. The smallest projective plane is of order 2. Thus it has seven points which can be represented by the numbers $\{1, 2, 3, 4, 5, 6, 7\}$. The lines of this plane can be represented by the following subsets: $L_1 = \{1, 2, 4\}$, $L_2 = \{2, 3, 5\}$, $L_3 = \{3, 4, 6\}$, $L_4 = \{4, 5, 7\}$, $L_5 = \{5, 6, 1)$, $L_6 = \{6, 7, 2\}$, and $L_7 = \{7, 1, 3\}$. This system of sets and subsets can be described by an incidence matrix of order 7. It is as follows.

$$P = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} . \qquad (52)$$

Each line of the projective plane of order 2 has three points on it. Thus each point could be a representative of one of these lines. In fact, the number of distinct ways of representing the lines $L_1, \ldots, L_7$, by the points $1, \ldots, 7$, is given by $p(P)$. The number of ways this can be done is 24. That is, $p(P) = 24$. The matrix P will be useful to us later.

Theorem 4.3 is a corollary of a theorem of H. J. Ryser [40]. In order to write it in a more general form a new definition of the permanent function is required which includes matrices which are not square. The definition includes the case of square matrices. It is as follows.

Definition 4.5. Let A be an mxn matrix such that $m \leq n$. Then the permanent of A, denoted as p(A), is defined as

$$p(A) = \sum_{\sigma} \prod_{i=1}^{m} a_{i\sigma(i)}$$

where the summation extends over all $\binom{n}{m}m!$ permutations σ which can be found from choosing m distinct integers from the numbers $1, \ldots, n$.

Using this definition Theorem 4.3 can be written as follows.

Theorem 4.6. Let $S_1, \ldots, S_m$ be subsets of an n-set S with $m \leq n$. Let A be the incidence matrix for $S_1, \ldots, S_m$. Then p(A) is the number of systems of distinct representatives for $S_1, \ldots, S_m$.

Proof: The proof is a direct result of Definition 4.5.

Theorem 4.6 has a little broader application than Theorem 4.3 in that the number of subsets does not have to be the same as the number of elements in the n-set. The following example illustrates this point.

Five stocks $(S_1, S_2, S_3, S_4, S_5)$ are to be sold in blocks. There are four corporations interested in buying these stocks but due to government regulations they can buy only one stock each. Each of the corporations (C_1, C_2, C_3, C_4) is not interested in all five stocks, but their interests are listed as: $C_1 = \{S_1, S_2, S_5\}$, $C_2 = \{S_1, S_2, S_3\}$, $C_3 = \{S_2, S_4\}$, and $C_4 = \{S_1, S_2, S_3, S_4, S_5\}$. The corporations have agreed among themselves to each buy one stock of their interest. How many distinct ways can this be done? The incidence matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Using Definition 4.5 we have $p(A) = 20$. Thus, there are 20 distinct ways this can be done.

The permanent function has also been used in solving several interesting problems in combinatorial theory where the problem is essentially unaltered by a relabeling of the items under consideration. The following material is basic in stating this type of problem.

Let S be a set of n elements. If $a \in S$ then assign to each element of S a unique weight $w(a)$ which is a complex number. Let

$P = \{P_1, \ldots, P_q\}$ be q properties concerned with the elements of S. If $P_{i_1}, \ldots, P_{i_r}$ denotes an r-subset of P then let $W(P_{i_1}, \ldots, P_{i_r})$ be the sum of the weights of the elements of S which satisfy each of the properties P_{i_s}, $s=1, \ldots, r$. If no elements of S satisfy all the properties P_{i_s}, $s=1, \ldots, r$, then let $W(P_{i_1}, \ldots, P_{i_r}) = 0$. Next let $W(r) = \Sigma\, W(P_{i_1}, \ldots, P_{i_r})$ where the summation extends over all the r-subsets of P. Let W(0) be the sum of the weights of the elements of S. We are now ready to prove the following fundamental theorem.

Theorem 4.7. If E(m), $m=0, 1, \ldots, q$, denotes the sum of the weights of the elements of an n-set S which satisfy exactly m of the properties $P = \{P_1, \ldots, P_q\}$ then

$$E(m) = W(m) - \binom{m+1}{m}W(m+1) + \binom{m+2}{m}W(m+2) - \ldots + (-1)^{q-m}\binom{q}{m}W(q). \tag{53}$$

Proof: Let $a \in S$ and suppose a satisfies exactly t of the properties of P. First consider the case where $t < m$. Then w(a) is not an addend of W(j) for any $j > t$. Thus w(a) does not contribute any weight to the right hand side of (53).

Next consider the case where $t = m$. That is, a satisfies exactly m of the properties of P. Then w(a) is an addend of W(m) and not an addend of W(m+j), $j > 0$. Hence a contributes w(a) to the right hand side of (53).

Finally consider the case $t > m$. Then a contributes $\binom{t}{m}w(a)$ to W(m), $\binom{t}{m+1}w(a)$ to W(m+1), ..., and finally w(a) to W(t). Thus the amount a contributes to the right hand side is just

$$\left[\binom{t}{m} - \binom{m+1}{m}\binom{t}{m+1} + \binom{m+2}{t}\binom{t}{m+2} - \ldots + (-1)^{t-m}\binom{t}{m}\binom{t}{t}\right]w(a). \tag{54}$$

But

$$\binom{k}{m}\binom{t}{k} = \binom{t}{m}\binom{t-m}{t-k},$$

for $m \leq k \leq t$. Hence rewriting (54) we have

$$\binom{t}{m}\left[\binom{t-m}{t-m} - \binom{t-m}{t-m-1} + \binom{t-m}{t-m-2} - \dots + (-1)^{t-m}\binom{t-m}{0}\right]w(a). \qquad (55)$$

But in the binomial expansion of $(x+y)^n$ we have

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \dots + \binom{n}{n}y^n.$$

Thus if $x = 1$, $y = -1$, then

$$\binom{n}{0} - \binom{n}{1} + \dots + (-1)^n\binom{n}{n} = 0.$$

Hence the expression in the brackets of (55) is 0. Thus a contributes 0 to the right hand side of (53). Therefore, for each $a \in S$, a contributes to the right hand side only if a satisfies exactly m of the properties of P. Hence the right hand side of (53) is the sum of the weights of the elements of S that satisfy exactly m of the properties of P, and the proof is complete.

An example will help clarify the concepts connected with Theorem 4.7.

Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $P = \{P_1, P_2, P_3, P_4\}$ where P_1 is the property that the numeral is 0; P_2 is the property that the numeral is odd; P_3 is the property that the numeral is even; and P_4 is the property that the numeral is 0, 2, 4, or 8. For each element a of S assign the weight $w(a) = 1$. Then $E(2)$ denotes the sum of the weights of the elements of S which satisfy exactly 2 of the properties P_1, P_2, P_3, P_4. Now $W(2) = 6$, $W(3) = 1$, and $W(4) = 0$ so that by (53)

$$E(2) = W(2) - \binom{3}{2}W(3) + \binom{4}{2}W(4) = 3.$$

This example is simple enough so that the answer can be verified by trial since the only elements that satisfy exactly 2 of the properties are 2, 4, and 8. Hence the sum of their weights would be 3.

Corollary 4.8. Let E(0) denote the sum of the weights of the elements of S that satisfy none of the properties of P. Then

$$E(0) = W(0) - W(1) + W(2) - \ldots + (-1)^q W(q).$$

Proof: This is the special case of Theorem 4.7 whenever $m = 0$.

Corollary 4.8 can be applied to solve a classical problem by Montmort, known as, "le probleme des recontres" which asks for the number of derangements of ordered items from their natural order.

Definition 4.9. Let $(a_1, \ldots, a_n)$ be a permutation of n elements labeled $1, \ldots, n$. Then the permutation is said to be a derangement if $a_i \neq i$, $i=1, \ldots, n$.

Let d_n denote the number of these derangements. To evaluate d_n consider the following. Let S be the set of permutations of the numbers $1, \ldots, n$. If $(a_1, \ldots, a_n)$ denotes one of these permutations then for each $i=1, \ldots, n$, let P_i be the property that the permutation has $a_i = i$. Let the weight of each permutation be 1. It is noted by the definition of E(0) that $d_n = E(0)$. Then $W(P_{i_1}, \ldots, P_{i_r}) = (n-r)!$ and

$$W(r) = \binom{n}{r}(n-r)! = n!/r! .$$

Now by using Corollary 4.8 we have

$$d_n = E(0) = W(0) - W(1) + W(2) - \dots + (-1)^n W(n) \qquad (56)$$

$$= n! - n! + n!/2! - \dots + (-1)^n n!/n!$$

$$= n! \, [1 - 1/1! + 1/2! - \dots + (-1)^n 1/n!].$$

Theorem 4.10. Let J denote the n-square matrix with every entry equal to 1. Then $d_n = p(J - I_n)$.

Proof: Each of the addends in the expansion of $p(J - I_n)$ corresponds to one of the permutations σ of $1, \dots, n$. If the addend is such that $\sigma(i) = i$ then the product is 0 since it contains an element from the main diagonal which is zero. If $\sigma(i) \neq i$ for every $i = 1, \dots, n$ then the product is 1, and thus the sum counts all of the derangements which was to be shown.

Theorem 4.10 can be used to find the number of derangements in some interesting problems. For example, suppose five letters are written and five envelopes are prepared for the letters. A two year old child puts the letters in the envelopes and seals them. What is the probability that no one receives the correct letter? The number of ways the letters can be put in the envelopes is 5!. The number of ways the letters can be put in the envelopes with no one receiving the correct letter is $p(J - I_5)$. Thus the probability is given by

$$1/5! \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}^{+} = 11/30 = .367.$$

This problem is interesting in the sense that it does not really matter whether there are 5, 10, or 100 letters and envelopes for the probability that no one receives the correct letter is essentially the same. If there were 10 letters, for example, the probability is

$$\frac{p(J - I_{10})}{10!} = .368.$$

The reason for this follows from (56) which shows that

$$d_n = n!\,[1 - 1/1! + 1/2! - \dots + (-1)^n/n!] \approx n!e^{-1}.$$

Thus the probability that given n letters no one of which is put in the right envelope is approximately $1/e$, $n > 2$.

Another example of the use of the permanent function in probability can be seen in the following. Suppose the distinct points a_i, $i=1,\dots,n$, have distinct particles q_i, $i=1,\dots,n$, one located at each of the points. These points are connected to one another in some fashion so that the particles can move from one point to another. The following information is known about the motion of the particles. The probability that particle q_i moves from point a_i to a_j is given by p_{ij}. Thus we can form an n-square matrix $P = (p_{ij})$. Now, what is the probability that the ultimate arrangement of the particles is such that there is one and only one particle at each of the points a_i, $i=1,\dots,n$? If σ denotes a permutation of $1,\dots,n$, then the probability that $q_{\sigma(1)},\dots,q_{\sigma(n)}$ move to the points $a_1,\dots,a_n$ is given by

$$\prod_{i=1}^{n} p_{\sigma(i)i} .$$

Thus the probability that the particles distribute themselves one at each point in some order is given by $\sum_{\sigma}\prod_{i=1}^{n} p_{\sigma(i)i}$ which is just $p(P)$.

Evaluation of the Permanent

In the applications involving the permanent function that have been presented in the previous section it has been necessary to evaluate the permanent of several matrices. The evaluation of some of these such as the incidence matrix of the projective plane of order 2 has not been easy. Essentially the method which has been used is the definition coupled with the expansion theorem for permanents using rows or columns (Theorem 2.13). This method is good if n is small, say $n < 6$, but becomes long and tedious if n is larger. Certainly the applications we have given are not restricted to small values of n, so that one of the main problems connected with the theory of permanents is a simple procedure for the evaluation of the permanent of a matrix. So far such a procedure does not exist, but recently several attempts have been made to solve this problem. A method proved by H. J. Ryser [40] seems to be effective when used with the aid of a computer but not of much help when the computation is done by hand. It is given in the next theorem.

<u>Theorem 4.11</u>. Let A be an n-square matrix. Let A_r denote the matrix A with r columns replaced by zeros. Let $S(A_r)$ denote the product of the row sums of A_r and $\Sigma S(A_r)$ denote the sums of $S(A_r)$ over all the $\binom{n}{r}$ choices of the r columns of A. Then

$$p(A) = S(A) - \Sigma S(A_1) + \Sigma S(A_2) - \dots + (-1)^{n-1} \Sigma S(A_{n-1}). \qquad (57)$$

Proof: Let $T = \{1, \dots, n\}$ and $S = T x T x \dots x T$ where T appears as a factor n times. Let the weight of each element $(j_1, \dots, j_n)$ of S be

$$w(j_1, \dots, j_n) = a_{1j_1} \dots a_{nj_n}$$

where $A = (a_{ij})$. Let P_i, $i=1,\ldots,n$, be the property that the element $(j_1,\ldots,j_n)$ does not contain the integer i. Now $W(P_{i_1},\ldots,P_{i_r})$ is equal to the sum of the weights of the elements of S which satisfy each of the properties $P_{i_1},\ldots,P_{i_r}$. Letting A_r denote the matrix obtained from A by replacing the columns numbered $i_1,\ldots,i_r$, by 0's, then

$$W(P_{i_1},\ldots,P_{i_r}) = S(A_r)$$

and hence as used in Theorem 4.7,

$$W(r) = \Sigma\, S(A_r).$$

Now p(A) is the function which is equal to the sum of the weights of the elements of S which consist of all permutations σ of the numbers $1,\ldots,n$. But if $(j_1,\ldots,j_n)$ is a permutation of $1,\ldots,n$ then $(j_1,\ldots,j_n)$ does not satisfy any of the properties P_i, $i=1,\ldots,n$, since the integer i, $i=1,\ldots,n$ appears in each permutation of $1,\ldots,n$. Thus, p(A) is the function which is equal to the sum of the weights of the elements which do not satisfy any of the properties P_i, $i=1,\ldots,n$. Hence, by Corollary 4.8 we have $p(A) = E(0)$ and

$$p(A) = S(A) - \Sigma\, S(A_1) + \Sigma\, S(A_2) - \ldots + (-1)^{n-1}\Sigma\, S(A_{n-1}) + (-1)^n \Sigma\, S(A_n).$$

But $S(A_n) = 0$. Hence (57) is proved.

One of the advantages of using (57) is that it reduces the number of addends in the expansion of the permanent from n! to 2^n-1.

As an example of Theorem 4.11 let us find p(B) where

$$B = \begin{bmatrix} 2 & 0 & 1 & 5 \\ -2 & 1 & 1 & 2 \\ 0 & 1 & 5 & -3 \\ -1 & 2 & 0 & 1 \end{bmatrix}.$$

Then by (57),

$$p(B) = S(B) - \Sigma S(B_1) + \Sigma S(B_2) - \Sigma S(B_3)$$

$$= (8)(2)(3)(2) - [(6)(4)(3)^2 + (8)(1)(2)(0) + (7)(1)(-2)(2) + (3)(0)(6)(1)]$$

$$+[(6)(3)(2)(1) + (5)(3)^2(-2) + (1)(2)^2(6) + (7)(0)^2(-3) + (3)(-1)^2(5)$$

$$+(2)(-1)(1)^2] - [(5)(2)(-3)(1) + (1)^2(5)(0) + (0)(1)^2(2) + (2)(-2)(0)(-1)]$$

$$= 96 - 188 - 17 + 30$$

$$= -79$$

Paul Nikolai [37] has computed the permanents of some rather large incidence matrices which represent a (v, k, λ)-configuration using (57).

Definition 4.12. Given a set S with v elements and v subsets $S_1, \ldots, S_v$, of S each containing k elements such that every distinct pair of subsets has λ elements in common is called a (v, k, λ)-configuration. In statistics this is called a balanced incomplete block design.

One of the amazing facts about (v, k, λ)-configurations is that the determinant of the incidence matrix A representing the configuration is such that

$$|d(A)| = k(k-\lambda)^{(v-1)/2}. \tag{58}$$

That is, the determinant is a function of the parameters v, k, and λ. Using (58) enables one to compute the value of the determinant rather easily.

Equation (58) also suggests the possibility that the permanent of an incidence matrix representing a (v, k, λ)-configuration might be a function of the parameters v, k, and λ. If so this might be the first break in trying to find a simple formula for the permanent of incidence matrices. Nikolai answered this question in the negative whenever he computed the permanents of two nonisomorphic (15, 7, 3)-configurations and found them to be different. By nonisomorphic it is meant that one (15, 7, 3)-configuration cannot be transformed to another by permutations of rows and columns. It is known to be true that for $v < 15$, all (v, k, λ)-configurations are isomorphic, hence his reason for using $v = 15$. To give an example of the magnitude of the problem of computing the permanent of an incidence matrix consider a (21, 5, 1)-configuration associated with a projective plane of order 4. It took better than four hours using a UNIVAC scientific computer and what Nikolai describes as a very efficient program which used formula (57) to compute the permanent of this matrix in 1960.

In 1966, Jurkat and Ryser, [12], found another method of evaluating the permanent function. It is called the economy equation for permanents but could just as well be called the factorization method since it expresses the permanent of an n-square matrix as a product of n matrices. It is based on a recurrence formula which means it is a special type of relationship involving a quantity with integer parameters. This relationship is such that it may be used to evaluate the quantity from given initial values and from previously computed values. For example, Pascal's triangle method for finding binomial coefficients is a recurrence formula. Before stating the theorem, we will give examples of how the factorization is used for the cases $n = 2, 3$, and 4. If A is a 2-square matrix then

$$p(A) = [a_{11}, a_{12}] \begin{bmatrix} a_{22} \\ a_{21} \end{bmatrix} .$$

If A is a 3-square matrix then

$$p(A) = [a_{11}, a_{12}, a_{13}] \begin{bmatrix} a_{22} & a_{23} & 0 \\ a_{21} & 0 & a_{23} \\ 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{33} \\ a_{32} \\ a_{31} \end{bmatrix} .$$

If A is a 4-square matrix then

$$p(A) = [a_{11}, a_{12}, a_{13}, a_{14}] \begin{bmatrix} a_{22} & a_{23} & a_{24} & 0 & 0 & 0 \\ a_{21} & 0 & 0 & a_{23} & a_{24} & 0 \\ 0 & a_{21} & 0 & a_{22} & 0 & a_{24} \\ 0 & 0 & a_{21} & 0 & a_{22} & a_{23} \end{bmatrix}$$

$$\times \begin{bmatrix} a_{33} & a_{34} & 0 & 0 \\ a_{32} & 0 & a_{34} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{31} & 0 & 0 & a_{34} \\ 0 & a_{31} & 0 & a_{33} \\ 0 & 0 & a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} a_{44} \\ a_{43} \\ a_{42} \\ a_{41} \end{bmatrix} .$$

The result is now stated for the general case.

Theorem 4.13. If A is an n-square matrix then

$$p(A) = P_1(A_{(1)})P_2(A_{(2)}) \ldots P_n(A_{(n)}) \qquad (59)$$

where $P_i(A_{(i)})$ is a matrix with elements from row i of A and zeros. The dimension of $P_i(A_{(i)})$ is $\binom{n}{i-1} \times \binom{n}{i}$ and is defined as

$$P_i(A_{(i)}) = \begin{bmatrix} P_{i-1}(a_{i2}, \ldots, a_{in}) & 0 \\ & \\ a_{i1} I_{\binom{n-1}{i-1}} & P_i(a_{i2}, \ldots, a_{in}) \end{bmatrix}$$

for i=2, . . . , n-1, and where

$$P_j(a_{ik}, \ldots, a_{in}) = \begin{bmatrix} P_{j-1}(a_{i(k+1)}, \ldots, a_{in}) & 0 \\ & \\ a_{ik} I_{\binom{n-k}{j-1}} & P_j(a_{i(k+1)}, \ldots, a_{in}) \end{bmatrix}$$

j =i , i-1, . . . , 2, k=2, . . . , n-2. The initial values are given by

$$P_1(a_{im}, \ldots, a_{in}) = [a_{im}, \ldots, a_{in}], \quad m=2, \ldots, n-1$$

and

$$P_j(a_{i(n-j+1)}, \ldots, a_{in}) = \begin{bmatrix} a_{in} \\ \cdot \\ \cdot \\ \cdot \\ a_{i(n-j+1)} \end{bmatrix}.$$

For the cases where i = 1 or n we have

$$P_1(A_{(1)}) = [a_{11}, \ldots, a_{1n}] \quad \text{and} \quad P_n(A_{(n)}) = \begin{bmatrix} a_{nn} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \end{bmatrix}.$$

The proof is omitted.

As an example of (59) we shall find the permanent of

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 4 & 0 \\ 2 & 5 & 6 \end{bmatrix}.$$

Now

$$p(A) = P_1(A_{(1)}) \begin{bmatrix} P_1(a_{22}, a_{23}) & 0 \\ & \\ a_{21}I_2 & P_2(a_{22}, a_{23}) \end{bmatrix} P_3(A_{(3)}),$$

where $P_1(a_{22}, a_{23})$ and $P_2(a_{22}, a_{23})$ are computed from the initial values. Thus

$$p(A) = [2, -1, 3] \begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix} = 81.$$

The dimensions of the matrices which are factors of p(A) can be computed by using Pascal's triangle. In the above example the dimension of A is 3. Thus using the row of Pascal's triangle for $n = 3$ we obtain the coefficients 1, 3, 3, 1. The dimensions of the matrices of p(A) are 1x3, 3x3, and 3x1.

Formula (59) gives us another way to evaluate the permanent of a matrix. Research into the effectiveness of this procedure needs to be done to determine the advantages or disadvantages of this method over that of Theorem 4.11. Jurkat and Ryser have suggested that formula (59) is more efficient than formula (57) in evaluation of permanents. Seemingly, the argument could just as well go the other way, since if n is very large then the dimensions of the matrices of the factors of (59) become large. For example, if A is a matrix of order 8 then the dimensions of the factors of p(A) are 1x8, 8x28, 28x56, 56x70, 70x56, 56x28, 28x8, and 8x1. This would rule out computation by hand except in the smallest cases. A computer programer possibly could devise an efficient program using the recurrence relationships in (59) to bring out its effectiveness.

The two methods we have discussed are not the only recent attempts which have been made to evaluate the permanent of a matrix. In 1960, M. F. Tinsley [42] determined for a certain class of (0, 1)-matrices those matrices for which the permanent and determinant are equal in absolute value. For such matrices the permanent could be evaluated by the determinant. Before stating this result we first give an extension of one of Tinsley's theorems.

Theorem 4.14. Let A be an n-square matrix such that the only non-zero addends found in the expansion of p(A) are those representing an even (odd) permutation σ of $1, \ldots, n$, then $p(A) = d(A)$, $(p(A) = -d(A))$.

Proof: If all the non-zero addends are those belonging to even permutations then

$$d(A) = \sum_{\sigma} \prod_{i=1}^{n} \epsilon(\sigma) a_{i\sigma(i)} = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} = p(A),$$

since $\epsilon(\sigma) = 1$ for all non-zero entries. The other case follows since $\epsilon(\sigma) = -1$ for all odd permutations.

The importance of Theorem 4.14 as a tool to evaluate permanents is minimal since knowing that all the non-zero addends are even or odd is about the same as expanding by the definition in the first place.

If the hypothesis of Theorem 4.14 is satisfied for (0, 1)-matrices then $p(A) = |d(A)|$ since $p(A) \geq 0$. Tinsley [42] extended this result a bit further and found the following theorem for (0, 1)-circulants. We first define a circulant.

Definition 4.15. An n-square matrix A of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

is called a circulant.

The incidence matrix (52) of the projective plane of order 2 is a circulant.

<u>Theorem 4.16</u>. Let A be an n-square (0, 1)-circulant with k ones, $k \geq 3$, in each row and column. If $k > 3$, then $p(A) > |d(A)|$. If $k = 3$, then $p(A) = |d(A)|$ if and only if after suitable permutations of the rows and columns A can be reduced to the direct sum of the matrix P in (52) taken e times where e is a positive integer. In this case $n = 7e$ and $p(A) = (24)^e$.

The proof is omitted.

This theorem does help evaluate certain (0, 1)-circulants such as P of (52). In this case we have by (58)

$$p(A) = |d(A)| = k(k - \lambda)^{(v-1)/2} = 3(3 - 1)^{(7-1)/2} = 24.$$

Since the permanent of (0, 1)-matrices plays a role in combinatorial applications and is usually not easy to compute, it causes some of the permanential inequalities of Chapter III to assume a more useful status. In particular, Theorems 3.54, 3.55, and 3.56 can be used to give upper bounds for permanents of incidence matrices. The only lower bound that we have at this point is the obvious one given by

Theorem 3.4 which is zero. This situation can be improved somewhat with the following inequality which gives both an upper and lower bound for the permanent of (0, 1)-matrices. It is proved by Jurkat and Ryser [12].

Theorem 4.17. Let A be an n-square (0, 1)-matrix. Let r_i be the sum of the elements in row i of A. Then

$$\prod_{i=1}^{n} \max(r_i + 1 - i,\ 0) \leq p(A) \leq \prod_{i=1}^{n} \min(n + 1 - i,\ r_i).$$

To illustrate Theorem 4.17, let us compute upper and lower bounds for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The matrix A is triangular and thus p(A) = 1. The lower bound is

$$\prod_{i=1}^{4} \max(r_i + 1 - i,\ 0) = (1)(1)(1)(1) = 1.$$

The upper bound is

$$\prod_{i=1}^{4} \min(4 + 1 - i,\ r_i) = (1)(2)(2)(1) = 4.$$

Thus $1 \leq p(A) \leq 4$.

Another upper bound for the special class of (0, 1)-matrices which are (v, k, λ)-configurations was proved by Marcus and Newman [30] in 1962. We state it without proof.

Theorem 4.18. If A is the incidence matrix of a (v, k, λ)-

configuration, then

$$p(A) < v!\,[(k-t)/v]^v \sum_{r=0}^{v} [(kt+t^2)/\lambda]^r/r!\,, \tag{60}$$

where $t = \sqrt{k-\lambda}$.

As an example of this theorem consider the matrix P in (52) which is a (7, 3, 1)-configuration. Then p(P) = 24 while the upper bound given by (60) is 56.346 when rounded off to three decimals. Thus

$$p(P) = 24 < 56.346.$$

The upper bound given in (60) is difficult to compute but does have the advantage of giving some upper bounds smaller than pervious theorems for some cases. For example, the upper bound given for P by Theorem 4.17 is 486 which is more than eight times that of (60).

Problems

Connected with the comprehensive study of any area in mathematics are a number of interesting problems. The permanent function is no exception to this rule. Thus no study would be complete unless some attempt is made to examine the problems in this area and to indicate the direction of current research. There are two types of problems which will be of concern in the concluding part of this chapter. The first type will be called solved problems, while the second type will be termed advanced problems. Solved problems will denote problems which have a known solution and either follow from theorems already established or appear in print somewhere. Advanced problems will denote statements whose validity or exposition has not been established. Thus, advanced problems should indicate what is now current research, and the solution of any of these problems could be considered a contribution to mathematical knowledge.

Solved Problems

Problem 1. (Thomas Muir, [34]) Show that if $k = 2\pi/2n+1$ then for odd positive values of n,

$$\begin{vmatrix} \cos k & \cos 2k & \dots & \cos nk \\ \cos 2k & \cos 4k & \dots & \cos 2nk \\ \vdots & \vdots & & \vdots \\ \cos nk & \cos 2nk & \dots & \cos n^2k \end{vmatrix}^{+} = -1/2^n.$$

Problem 2. (Cayley, [34]) Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}.$$

Suppose the elements of A are non-zero and $d(A) = 0$. Using Theorem 2.21 prove that if

$$B = \begin{bmatrix} 1/a & 1/b & 1/c \\ 1/d & 1/e & 1/f \\ 1/g & 1/h & 1/k \end{bmatrix},$$

then

$$p(B)d(B) = \begin{bmatrix} 1/a^2 & 1/b^2 & 1/c^2 \\ 1/d^2 & 1/e^2 & 1/f^2 \\ 1/g^2 & 1/h^2 & 1/k^2 \end{bmatrix}.$$

Problem 3. (Thomas Muir, [35]) Prove by induction that if A is an n-square matrix then

$$\sum_{j=0}^{n} \sum_{\alpha \in Q_{j,n}} (-1)^j p(A[\alpha/\alpha])d(A(\alpha/\alpha)) = 0,$$

where $p(A[\alpha/\alpha]) = 1$ and $d(A(\alpha/\alpha)) = d(A)$ when $\alpha \in Q_{0,n}$, while $p(A[\alpha/\alpha]) = p(A)$ and $d(A(\alpha/\alpha)) = 1$ when $\alpha \in Q_{n,n}$.

Problem 4. (Perfect, [38]) A real valued function f(x) defined on a convex set U is said to be convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in U$ and $0 \leq \alpha \leq 1$. Prove that the permanent is not a convex function on the convex set K_n.

Problem 5. (W. B. Jurkat, [25]). H. J. Ryser has conjectured that if $A, B \in K_n$ then $p(AB) \leq \min \{p(A), p(B)\}$. Show that this statement is not true.

Problem 6. (Brualdi and Newman, [5]). If $0 \leq \alpha \leq 1$ and $A \in K_n$ then

$$p(\alpha I_n + (1 - \alpha)A) \leq \alpha + (1 - \alpha)p(A).$$

Problem 7. (Marcus and Minc, [20]) Prove that if $A \in K_n$ and $A \neq J_n$, then at most $(n-1)(n-1)!$ terms in the expansion of p(A) have a common non-zero value.

Problem 8. (Marcus and Minc, [20]) Prove that if $A \in K_n$ then there exists a permutation σ of $1, \ldots, n$, such that

$$\prod_{i=1}^{n} s_{i\sigma(i)} \geq 1/n^n.$$

This is a direct consequence of Van der Waerden's conjecture.

Problem 9. (Marcus and Minc, [25]) A positive semi-definite hermitian n-square matrix A can be factored such that $A = TT^*$, where T is an n-square upper triangular matrix. Using this fact together with Theorem 3.50 show that $p(A) \geq d(A)$.

This is a different technique of proof than the one used in Corollary 3.48.

Problem 10. (Minc, [31]) Let H_n denote the set of positive semi-definite hermitian n-square matrices which have $e = (1, 1, \ldots, 1)$ as a characteristic vector. Prove that if $A \in H_n$ and the row sums of A are all equal to γ then

$$p(A) \geq n!(\gamma/n)^n,$$

with equality if and only if either a row of A is zero or A is a non-negative multiple of J_n.

Note that this is an extension of Corollary 3.44.

Problem 11. (Peter M. Gibson, [8]) A matrix A is said to be substochastic if it has non-negative entries with each row sum no greater than 1. Prove that if A is an n-square substochastic matrix then

$$p(I_n - A) \geq 0,$$

using Theorem 4.11.

This problem was first proposed by Marcus and Minc [30] for the case of doubly stochastic matrices. It was proved by Brualdi and Newman [7] for the case of row stochastic matrices, Advanced Problems

Problem 12. (Marcus, [25]) Prove or disprove that if A is an n-square matrix with all positive entries and the n! terms in the expansion of p(A) take on at most r different values then $\rho(A) \leq r$.

This has been proven for $n < 5$ by H. Minc and R. Westwick, but these results are unpublished.

Problem 13. (Van der Waerden, [44]) Prove or disprove that if A is an n-square doubly stochastic matrix then

$$p(A) \geq n!/n^n,$$

with equality if and only if $A = J_n$.

Problem 14. (Marcus and Newman, [28]) Prove or disprove that if A is a doubly stochastic matrix then there does not exist a positive number β, independent of i and j such that

$$p(A(i/j)) = p(A), \quad a_{ij} \neq 0,$$

and

$$p(A(i/j)) = p(A) + \beta, \quad a_{ij} = 0.$$

This result is false with additional hypothesis, (Theorem 3.27).

Problem 15. (Marcus and Newman, [25]) Prove or disprove that if A is a positive semi-definite n-square hermitian matrix and $1 \leq k \leq n$ then

$$p(A) \geq p(A[1, \ldots, k/1, \ldots, k])p(A[k+1, \ldots, n/k+1, \ldots, n]).$$

This is true whenever $k = 1$, (Theorem 3.46).

Problem 16. (Marcus, [25]) Prove or disprove that if A is an mk-square positive semi-definite hermitian matrix partitioned as follows:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}$$

in which each A_{ij}, $i,j=1,\ldots,m$ is k-square and B is the m-square matrix such that $B = (b_{ij}) = (p(A_{ij}))$ then $p(A) \geq p(B)$.

Problem 17. (Minc, [32]) Prove or disprove that if A is an n-square (0, 1)-matrix and

$$r_i = \sum_{j=1}^{n} a_{ij} \neq 0, \quad i=1,\ldots,n,$$

then

$$p(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i},$$

with equality if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of matrices all of whose entries are 1.

This inequality is known to be true for all (0, 1)-matrices with $r_i < 7$, $i=1,\ldots,n$.

Problem 18. (H. J. Ryser, [25]) Prove or disprove that if U is the set of v-square (0, 1)-matrices with k ones in each row and column, then the minimal value of the permanent over the set U occurs for one of the incidence matrices of a (v, k, λ)-configuration.

Problem 19. (Herbert Wilf, [46]) Prove or disprove that if A is an incidence matrix of a (v, k, λ)-configuration with $k > 3$, then

$$p(A) > |d(A)|.$$

If k = 3, then $p(A) = |d(A)|$ if and only if after suitable permutations of the rows and columns A can be reduced to the direct sum of the matrix P of (52) taken e times where e is a positive integer.

This is a generalization of Theorem 4.16.

In addition to the above mentioned unsolved problems there are two studies which arise from the results of this paper that would be of some benefit if they were carried out. The author states these as problems 20 and 21.

Problem 20. A comparison of the two computational formulas, Theorems 4.11 and 4.13 needs to be made to determine which formula is the shortest and most effective for computer use.

Problem 21. In Chapter III several inequalities have been given which specify upper and lower bounds for the permanents of various classes of matrices. Some attempt has been made to compare these bounds with examples, but no rigorous effort has been made to determine "best" upper and lower bounds in the various cases. Possibly by a thorough comparison of these bounds to each other, some could be eliminated, and others could be classified to enable the user of these inequalities to select the best one for his particular problem.

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