FREE VIBRATIONS OF OPEN NONCIRCULAR CYLINDRICAL SHELL SEGMENTS

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CYLINDRICAL SHELL SEGMENTS

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NOMENCLATURE

$A_{m,n}$, $B_{m,n}$, $C_{m,n}$	Displacement constants
a	Radius of circular panel
D	Bending rigidity
D_j	jth constant for nondimensional curvature
E	Modulus of elasticity
G	Shear Modulus
h	Thickness of shell
j	Index of curvature constants
K	$Eh/(1-\mu^2)$
k	Number of terms necessary to express the curvature
L_x	Length of shell in x-direction
L_s	Arc length of panel
M_ξ	Moment stress resultant in ξ-direction
m,n	Indices on displacement summations
n_c	Number of circumferential modes for closed circular cylinder
$N_{\xi\eta}$	Shear stress resultant
N_ξ	Normal stress resultant
P	Maximum value of n for recurrence formulas
r	Radius of curvature
$\{S\}$	Column matrix for calculation of modal shapes

$[T]$	Square matrix for calculation of modal shapes
t	Time
V_ξ	Effective transverse shear resultant
v_ξ	$\partial v/\partial \xi$
s, x, z	Spatial coordinates
v, u, w	Orthogonal displacements
$[X]$	Frequency matrix
α	$(1+\mu)/2$
β_m	$m\pi L_s/L_x$
γ	$(1-\mu)/2$
δ_j^n	$= 0$ if $n < j$; $= 1$ if $n \geq j$
η	Nondimensional x-coordinate, x/L_x
θ_o	Subtended angle for circular panel
μ	Poisson's ratio
ξ	Nondimensional s-coordinate, s/L_s
ρ	Mass density
ω	natural frequency, radius/sec.
$\bar{\omega}$	Nondimensional natural frequency, $\left[\frac{12(1-\mu^2)\rho L_s{}^2\omega^2}{E}\right]^{\frac{1}{2}}$

CHAPTER I

INTRODUCTION

1.1 Discussion

Through the years, the study of the free vibrations of circular cylindrical shells has received much attention. These studies have resulted from the widespread industrial application of circular cylinders. Many times, the designer is faced with the problem of designing cylindrical shells which are not circular. The purpose of this study is to investigate the free vibrations of simply supported noncircular cylindrical shell panels.

The design of noncircular shells is of particular interest to designers of aircraft and submarine structures. In aircraft design, for example, the wing and leading-edge skin panels are usually noncircular. Helicopter blades are often two unstiffened, contoured skins joined together along their straight edges. There has also been a keen interest in submarine hulls with noncircular cross sections and noncircular skin panels between stiffeners.

Roof structures may be unstiffened, noncircular cylindrical shells. Cylinders which are designed to be circular, but become noncircular during fabrication, must sometimes be analyzed as noncircular shells.

The examples given are representative of "open" (as opposed to "closed") cylindrical shells. The present study was directed toward this class of cylindrical shells.

1.2 Background

For the discussions to follow, reference should be made to the geometry and nomenclature of Figure 1. The quantities appearing in Figure 1 are defined as follows: s, x, and z are the orthogonal coordinates; v, u, and w are corresponding displacement components; r is the variable radius of curvature; h is the shell thickness; L_x is the length of the shell in the x-direction; and L_s is the arc length of the cylindrical shell measured along the surface in the s-direction.

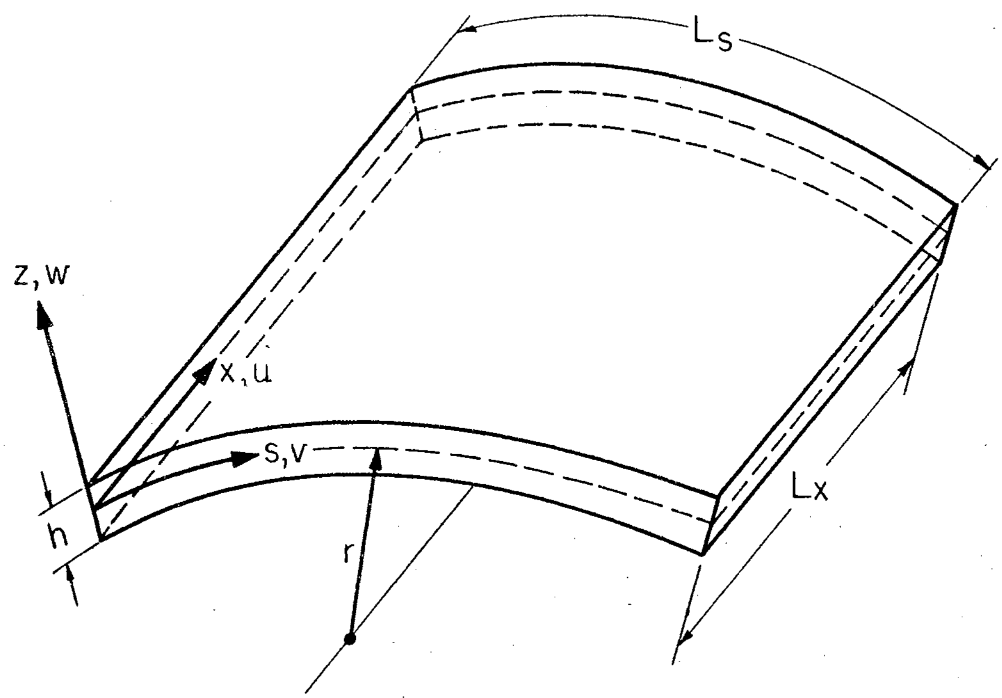

Figure 1. Elemental Geometry

In 1894, Rayleigh (1) derived an expression for the natural frequencies of closed circular cylinders with simply supported ends, considering only the circumferential modes. The results from his expression agreed with experimental data for cylinders vibrating in higher circumferential modes. For lower circumferential modes, his results were in error. Rayleigh's error occurred because he assumed the deformations to be inextensional. That is, the middle surface of the shell would only bend during small vibrations but not stretch. Using an explanation based on an energy approach, Arnold and Warburton (2) clarified the significance of Rayleigh's inextensional assumption.

Using this same energy approach, and including middle surface extensions, Arnold and Warburton (2) calculated natural frequencies for circular cylinders with simply supported ends. They verified the lower frequencies by experiments. Their study also explained why the lowest frequency of a cylinder could possibly occur at a high circumferential mode shape. Since their work, many papers with parametric studies and experimental data have been published for simply supported circular cylinders (3, 4, 5).

Stadler and Wang (6) solved for the natural frequencies of open circular panels. These panels were simply supported along their curved edges. They also satisfied the implied boundary conditions at the nodal lines of closed circular cylinders. A computer program was written to determine the natural frequencies of circular panels based on their method. This computer program helped verify that the method to be presented would give an exact solution.

The first attempts to solve noncircular cylindrical shell problems expressed the variable curvature as an infinite fourier series.

Marquerre (7), in 1951, assumed an infinite fourier cosine series for the curvature when he considered the stability of simply supported non-circular cylinders. This curvature expression is given by equation 1.1. That is,

$$\frac{1}{r} = \frac{1}{R_o}\left[1 + \sum_{j=2,4}^{\infty} \zeta_j \cos \frac{j\pi s}{S_o}\right] \tag{1.1}$$

where

R_o = Mean radius of curvature

ζ_j = Eccentricity constants dependent upon j

s = Coordinate along cross section

S_o = Arc length of closed cross section.

Malkina (8) assumed a nondimensional radius of curvature given by equation 1.2, when he studied the free vibrations of noncircular cylindrical shells. Equation 1.2 is

$$\frac{r}{R} = \frac{r_o}{R}\left[1 + \varepsilon \sum_{j=1}^{\infty} b_j \cos \frac{2j\pi s}{S_o}\right] \tag{1.2}$$

where

r_o = Mean radius of curvature

R = Arbitrary constant

ε = Eccentricity constant

b_j = Constants dependent upon j

S_o = Arc length of one half of closed cross section.

Kempner (9) verified the Donnell equations[1] for cylindrical shells having varying curvature, i.e. noncircular cylindrical shells. When solving the static and buckling problems, he assumed a curvature expression defined by

$$\frac{1}{r} = \frac{1}{r_o}\left[1 + \varepsilon \cos \frac{4\pi s}{L_o}\right] \tag{1.3}$$

where

r = Radius of curvature

r_o = Mean radius of curvature

ε = Eccentricity constant $|\varepsilon| \leq 1$

s = Coordinate along cross section

L_o = Circumference of cross section in s-direction.

It should be noted that this expression is a special case of equation 1.1. While this expression for the curvature includes a large class of problems, all arbitrary cross sections are not described accurately by equation 1.3. Although Marquerre's (7) curvature series is completely general, there may be cases where it would be difficult to generate. A power series could possibly give an easier and more accurate expression for the actual curvature.

Boyd (12) assumed the curvature expression as a finite power series. In order to nondimensionalize the curvature, the curvature was

---

[1]Donnell (10), in 1933, derived a simplified set of equilibrium equations for circular cylinders. In deriving this set of equations, Donnell made two simplifying assumptions. He first assumed that the transverse shear force makes a negligible contribution to the equilibrium of forces in the circumferential direction. As the ratio of the radius to the thickness of the shell increases, this assumption can be expected to improve in accuracy (11). In addition, he assumed that the circumferential displacements result in negligible contributions to the changes in curvature and twist.

multiplied by the arc length, L_s. The curvature was then assumed to be given by equation 1.4

$$\frac{L_s}{r} = \sum_{j=1}^{k} D_j \left(\frac{s}{L_s}\right)^{j-1} \tag{1.4}$$

where

D_j = Unitless constants dependent upon j

k = Number of terms necessary to express accurately the curvature.

Although not a part of this study, this curvature series can be obtained for any general cross section. From a known cross section, the curvature at a sufficient number of points can be calculated. With the curvature at these points known, equation 1.4 can be developed by any suitable technique, such as the least squares method (13).

For the special case of a circular cross section with a constant radius, r = a, k = 1 and equation 1.4 reduces to

$$\frac{L_s}{r} = D_1 = \frac{\theta_o a}{a} = \theta_o \tag{1.5}$$

where

θ_o = Angle subtended by the circular panel.

Another special case of equation 1.4 is the flat plate. For the flat plate, k in equation 1.4 is one, and $D_1 = 0$. Physically, this represents the case of an infinite radius of curvature.

In seeking solutions to Donnell's equations, Boyd (12) assumed the displacements for the static problem to take the form of a doubly infinite series. A trigonometric series in the x-direction and a power

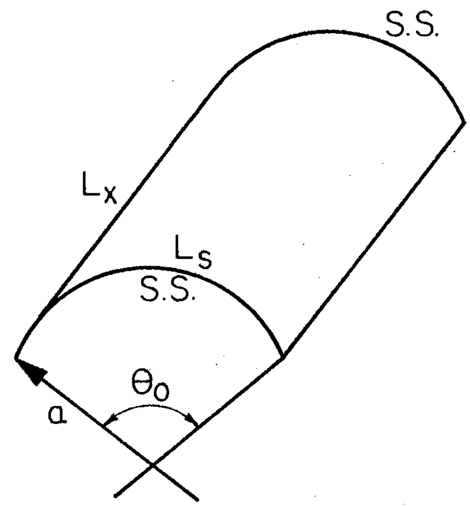

Figure 2. Circular Panel

series in the s-direction were assumed for the displacements.

1.3 Approach

The equations derived by Kempner for noncircular cylinders were modified to include the translational inertia terms. The displacements (u, v, w) were expanded in the doubly-infinite series, similar to the ones used by Boyd (12) in his study of statically loaded cylinders. A periodic function in the variable time was added to the displacement functions used by Boyd. The trigonmetric series satisfied the boundary conditions for the cylindrical shell panel simply supported along its curved edges. The straight edge boundary conditions were expressed in terms of the coefficients in the displacement series.

From the substitution of the assumed displacements and curvature functions into the equilibrium equations, the three recurrence formulas were obtained. From these recurrence formulas, and the eight equations stating the straight edge boundary conditions, the natural frequencies were calculated for circular and noncircular panels. Once the natural frequencies were known, the corresponding modal shapes were calculated.

CHAPTER II

FORMULATION OF THE SOLUTION

2.1 Introduction

Equations 2.1 are the familiar Donnell (10) equations of equilibrium for a freely-vibrating cylindrical shell. These equations were verified for a noncircular cylindrical shell by Kempner (9). The geometry of the shell was shown in Figure 1. The modified Donnell equations are

$$u_{xx} + \frac{1-\mu}{2} u_{ss} + \frac{1+\mu}{2} v_{xs} + \mu \left(\frac{w}{r}\right)_x - \frac{(1-\mu^2)\rho}{E} u_{tt} = 0$$

$$\frac{1-\mu}{2} v_{xx} + v_{ss} + \frac{1+\mu}{2} u_{xs} + \left(\frac{w}{r}\right)_s - \frac{(1-\mu^2)\rho}{E} v_{tt} = 0$$

$$\frac{1}{r}\left[\frac{w}{r} + v_s + \mu u_x\right] + \frac{h^2}{12} \nabla^4 w + \frac{(1-\mu^2)\rho}{E} w_{tt} = 0. \tag{2.1}$$

Subscripts indicate differentiation and

x, s, z, t = Longitudinal, circumferential, and transverse spatial coordinates, and time, respectively

u, v, w = Displacements in the x, s, z directions, respectively

r = Variable radius of curvature

μ = Poisson's ratio

ρ = Mass density.

For convenience, equations 2.1 are rewritten using the following nondimensional independent variables:

$$\eta = x/L_x$$
$$\xi = s/L_s$$
$$\alpha = \frac{1+\mu}{2} \qquad (2.2)$$
$$\gamma = \frac{1-\mu}{2}$$

where

L_x = Length of cylinder in x-direction

L_s = Arc length of cylinder in s-direction.

Equations 2.1 become

$$\left(\frac{L_s}{L_x}\right)^2 u_{\eta\eta} + \alpha\, u_{\xi\xi} + \left(\frac{L_s}{L_x}\right) v_{\eta\xi} + \mu\left(\frac{L_s}{L_x}\right)\left(\frac{L_s}{r}\, w\right)_{\eta} - \frac{(1-\mu^2)\rho L_s^{\,2}}{E}\, u_{tt} = 0 \qquad (2.3a)$$

$$\gamma\left(\frac{L_s}{L_x}\right)^2 v_{\eta\eta} + v_{\xi\xi} + \alpha\left(\frac{L_s}{L_x}\right) u_{\eta\xi} + \left(\frac{L_s}{r}\, w\right)_{\xi} - \frac{(1-\mu^2)\rho L_s^{\,2}}{E}\, v_{tt} = 0 \qquad (2.3b)$$

$$\frac{L_s}{r}\left[\frac{L_s}{r}\, w + v_{\xi} + \mu\left(\frac{L_s}{L_x}\right) u_{\eta}\right] + \frac{h^2}{12}\,\overline{\nabla}^4 w + \frac{(1-\mu^2)\rho L_s^{\,2}}{E}\, w_{tt} = 0 \qquad (2.3c)$$

where

$$\overline{\nabla}^4 w = \left(\frac{L_s}{L_x}\right)^2 \frac{1}{L_x^{\,2}}\, w_{\eta\eta\eta\eta} + \frac{2}{L_x^{\,2}}\, w_{\eta\eta\xi\xi} + \frac{1}{L_s^{\,2}}\, w_{\xi\xi\xi\xi}.$$

It should be noted that equations 2.3 are a set of three coupled partial differential equations.

As explained in the Introduction, it was assumed that the curvature of the cross section could be accurately expressed by equation 2.4.

$$\frac{L_s}{r} = \sum_{j=1}^{k} D_j \left(\frac{s}{L_s}\right)^{j-1} = \sum_{j=1}^{k} D_j \, \xi^{j-1} \tag{2.4}$$

where

D_j = Unitless constants describing the curvature

k = Number of terms necessary to express accurately the curvature.

Following the procedure of Boyd (12), one assumes the solutions for the displacements to be mixed, doubly infinite series which satisfy simply supported boundary conditions at $x = 0$ and $x = L_x$. These displacement series are shown in equation 2.5. They are

$$u(x,s,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{m,n} \left(\frac{s}{L_s}\right)^{n-1} \cos\frac{m\pi x}{L_x} \cos\omega_m t$$

$$v(x,s,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{m,n} \left(\frac{s}{L_s}\right)^{n-1} \sin\frac{m\pi x}{L_x} \cos\omega_m t \tag{2.5}$$

$$w(x,s,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{m,n} \left(\frac{s}{L_s}\right)^{n-1} \sin\frac{m\pi x}{L_x} \cos\omega_m t \, .$$

When the x and s coordinates are nondimensionalized by the expressions

$$\xi = s/L_s$$
$$\beta_m = m\,L_s/L_x \quad ,$$

equations 2.5 become

$$u(\eta,\xi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{m,n}\,\xi^{n-1}\cos\frac{\beta_m L_x}{L_s}\eta\,\cos\omega_m t$$

$$v(\eta,\xi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{m,n}\,\xi^{n-1}\sin\frac{\beta_m L_x}{L_s}\eta\,\cos\omega_m t \qquad (2.6)$$

$$w(\eta,\xi,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}\,\xi^{n-1}\sin\frac{\beta_m L_x}{L_s}\eta\,\cos\omega_m t$$

where

$A_{m,n}$, $B_{m,n}$, $C_{m,n}$ = Unknown constant coefficients with units of length.

The assumed displacement functions automatically satisfy the simply supported boundary conditions along the curved edges (η = 0,1). They do not, however, satisfy any specific boundary conditions along the straight edges (ξ = constant). The incorporation of arbitrary straight edge boundary conditions into the problem will be discussed in detail in Section 2.3.

2.2 Formulation of Recurrence Formulas

When the assumed displacement functions (2.6) are substituted into the equilibrium equations (2.3), three coupled algebraic recurrence formulas are obtained. There will be one recurrence formula for each equilibrium equation.

Substitution of equation 2.6 into equation 2.3a and simplification gives

$$\sum_{m=1}^{\infty}\Big[\sum_{n=1}^{\infty} - A_{m,n}\,\beta_m^{\,2}\,\xi^{n-1} + \gamma\sum_{n=1}^{\infty} A_{m,n}\,(n-1)(n-2)\,\xi^{n-3}$$

$$+ \alpha\sum_{n=1}^{\infty}(n-1)\,\beta_m\,B_{m,n}\,\xi^{n-2} + \mu\beta_m\sum_{j=1}^{K} D_j\,\xi^{j-1}\cdot\sum_{n=1}^{\infty} C_{m,n}\,\xi^{n-1}$$

$$+ \frac{(1-\mu^2)\rho L_s^{\,2}\omega_m^{\,2}}{E}\sum_{n=1}^{\infty} A_{m,n}\,\xi^{n-1}\Big]\cos\frac{\beta_m L_s}{L_x}\eta\;\cos\omega_m t = 0. \qquad (2.7)$$

Since this equation must be true for each m, it is therefore possible to write equation 2.7 as follows:

$$\sum_{n=1}^{\infty} - A_{m,n}\,\beta_m^{\,2}\,\xi^{n-1} + \gamma\sum_{n=1}^{\infty}(n-1)(n-2)\,A_{m,n}\,\xi^{n-3}$$

$$+ \alpha\sum_{n=1}^{\infty}(n-1)\,B_{m,n}\,\xi^{n-2} + \mu\beta_m\sum_{j=1}^{K} D_j\,\xi^{j-1}\cdot\sum_{n=1}^{\infty} C_{m,n}\,\xi^{n-1}$$

$$+ \frac{(1-\mu^2)\rho L_s^{\,2}\omega_m^{\,2}}{E}\sum_{n=1}^{\infty} A_{m,n}\,\xi^{n-1} = 0. \qquad (2.8)$$

For convenience, it is necessary to change the exponents of ξ to n-1 for each term in the previous equation. The following transformations are needed:

$$\sum_{n=1}^{\infty} (n-1)(n-2)\, A_{m,n}\, \xi^{n-3} = \sum_{p=1}^{\infty} p(p+1)\, A_{m,p+2}\, \xi^{p-1}$$

$$\sum_{n=1}^{\infty} (n-1)\, B_{m,n}\, \xi^{n-2} = \sum_{p=1}^{\infty} p\, A_{m,p+1}\, \xi^{p-1}$$

$$\sum_{j=1}^{\infty} D_j\, \xi^{j-1} \cdot \sum_{n=1}^{\infty} C_{m,n}\, \xi^{n-1} = \sum_{j=1}^{K} D_j \cdot \sum_{n=1}^{\infty} C_{m,n}\, \xi^{j+n-2}$$

$$= \sum_{j=1}^{K} D_j \cdot \sum_{p=1}^{\infty} \overset{p}{\delta}_j\, C_{m,p-j+1}\, \xi^{p-1} \qquad (2.9)$$

where

$$\overset{p}{\delta}_j = 0 \quad \text{if} \quad p < j$$

$$\overset{p}{\delta}_j = 1 \quad \text{if} \quad p \geq j \ .$$

Now, by letting the dummy index p become n, and substituting equations 2.9 into equation 2.8, one obtains

$$\sum_{n=1}^{\infty} \Bigg[-\beta_m^{\,2} A_{m,n} + \gamma\, n(n+1)\, A_{m,n+2} + \alpha\beta_m\, n\, B_{m,n+1}$$

$$(2.10)$$

$$+ \mu\beta_m \sum_{j=1}^{K} D_j\, \overset{n}{\delta}_j\, C_{m,n-j+1} + \frac{(1-\mu^2)\rho L_s^{\,2} \omega_m^{\,2}}{E} A_{m,n} \Bigg] \xi^{n-1} = 0 \ .$$

By equating the constant coefficients of all terms in this series to zero, one obtained the recurrence formula.

For each m and n,

$$-\beta_m^{\,2}\, A_{m,n} \;+\; \gamma\, n(n+1)\, A_{m,n+2} \;+\; \alpha\beta_m\, n\, B_{m,n+1}$$

$$+\; \mu\beta_m \sum_{j=1}^{\kappa} D_j\, \delta_j^{\,n}\, C_{m,n-j+1} \;+\; \frac{(1-\mu^2)\rho L_s^{\,2}\omega_m^{\,2}}{E}\, A_{m,n} \;=\; 0. \qquad (2.11)$$

Following the same procedure and using the second and third equilibrium equations, the second and third recurrence formulas are

$$-\gamma\beta_m^{\,2}\, B_{m,n} \;+\; n(n+1)\, B_{m,n+2} \;-\; \alpha\beta_m\, n\, A_{m,n+1}$$

$$+\; \sum_{j=1}^{\kappa} D_j\, \delta_{j-1}^{\,n} \cdot n\, C_{m,n+2-j} \;+\; \frac{(1-\mu^2)\rho\omega_m^{\,2} L_s^{\,2}}{E}\, B_{m,n} \;=\; 0 \qquad (2.12)$$

and

$$\left(\frac{h}{L_s}\right)^2 \beta_m^{\,4}\, C_{m,n} \;-\; 2\beta_m^{\,2}\left(\frac{h}{L_s}\right)^2 n(n+1)\, C_{m,n+2}$$

$$+\left(\frac{h}{L_s}\right)^2 n(n+1)(n+2)(n+3)\, C_{m,n+4} \;+\; 12\sum_{i=1}^{\kappa}\sum_{j=1}^{\kappa} D_i\, D_j\, \delta_i^{\,n}\, \delta_j^{\,n}\, C_{m,n+2-i-j}$$

$$-\; 12\mu\beta_m \sum_{j=1}^{\kappa} D_j\, \delta_j^{\,n}\, A_{m,n+1-j} \;+\; 12\sum_{j=1}^{\infty} D_j\, \delta_{j-1}^{\,n}\, (n-j+1)\, B_{m,n-j+2} \qquad (2.13)$$

$$-\; \frac{12(1-\mu^2)\rho L_s^{\,2}\omega_m^{\,2}}{E}\, C_{m,n} \;=\; 0\,.$$

When letting

$$\bar{\omega}^2 = \frac{12(1-\mu^2)\rho\omega_m{}^2 L_s{}^2}{E} , \tag{2.14}$$

the three recurrence formulas, (equations 2.11, 2.12, 2.13), become

$$\left(-\beta_m + \frac{\bar{\omega}^2}{12\beta_m} \right) A_{m,n} + \frac{\gamma(n)(n+1)}{\beta_m} A_{m,n+2} + \left(\frac{\alpha(n)}{\beta_m} \right) B_{m,n+1}$$

(2.15a)

$$+ \mu \sum_{j=1}^{\kappa} D_j \, \delta_j^n \, C_{m,n-j+1} = 0$$

$$-\alpha A_{m,n+1} + \left(\frac{-\gamma\beta_m + \frac{\bar{\omega}^2}{12\beta_m}}{n} \right) B_{m,n} + \frac{n+1}{\beta_m} B_{m,n+2}$$

(2.15b)

$$+ \sum_{j=1}^{\kappa} D_j \, \delta_{j-1}^n \, C_{m,n+2-j} = 0$$

$$- \frac{12\mu\beta_m}{(n+2)(n+3)} \sum_{j=1}^{\kappa} D_j \, \delta_j^n \, A_{m,n+1-j} + \frac{12}{(n+2)(n+3)} \sum_{j=1}^{\kappa} D_j \, \delta_j^n \, (n-j+1) B_{m,n-j+2}$$

$$+ \frac{\left[\left(\frac{h}{L_s} \right)^2 \beta_m{}^4 - \bar{\omega}^2 \right]}{(n+2)(n+3)} C_{m,n} - \frac{2n(n+1)}{(n+2)(n+3)} \beta_m{}^2 \left(\frac{h}{L_s} \right)^2 C_{m,n+2}$$

(2.15c)

$$+ \left(\frac{h}{L_s} \right)^2 n(n+1) \, C_{m,n+4} + 12 \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} D_i \, D_j \, \delta_i^n \, \delta_j^n \, C_{m,n+2-i-j} = 0.$$

If it is desired to neglect the in-surface inertia terms in the formulation of the vibration problem, the terms containing $\overline{\omega}^2$ are omitted from the first two recurrence formulas.

The recurrence formulas are general enough so that if eight unknown constants are found, then the remainder of the unknown constants can be found through the use of the three recurrence formulas. These eight unknown constants are obtained from the boundary conditions which will be discussed in the next section.

2.3 Boundary Conditions

Examination of equations 2.3 reveals that the highest order of the differential equation for ω is four; for u and v it is two. Since there are two independent variables, (η and ξ), sixteen independent boundary conditions are needed. There will be four along each of the four edges, giving a total of sixteen. When the displacement functions (2.6) were assumed, eight boundary conditions were established by the cosine and sine functions evaluated for $\eta = 0$ and $\eta = 1$. These conditions are:

$$u_\eta(0,\xi) = v(0,\xi) = w(0,\xi) = w_{\xi\xi}(0,\xi) = 0$$

$$u_\eta(1,\xi) = v(1,\xi) = w(1,\xi) = w_{\xi\xi}(1,\xi) = 0 .$$

Therefore, in order to solve for the eight constants, eight boundary conditions must be specified along the two boundaries of constant ξ.

The boundaries were chosen to correspond to $\xi = 0$ and 1, and the following boundary conditions were considered:

$$\left.\begin{array}{llll} u = 0 & \text{or} & N_{\xi\eta} = 0 \\ v = 0 & \text{or} & N_{\xi} = 0 \\ w = 0 & \text{or} & V_{\xi} = 0 \\ \beta_{\xi} = 0 & \text{or} & M_{\xi} = 0 \end{array}\right. \quad \text{at} \quad \xi = 0,1. \tag{2.16}$$

Next, it is necessary to substitute the assumed displacements into the above boundary conditions. These boundary conditions, expressed in terms of the displacements and nondimensional coordinates, are given by equation 2.17

$$\begin{aligned}
N_{\xi\eta} &= \frac{Gh}{L_s}\left[\frac{L_s}{L_x}\frac{\partial v}{\partial \eta} + \frac{\partial u}{\partial \xi}\right] \\
N_{\xi} &= \frac{K}{L_s}\left[\frac{\partial v}{\partial \xi} + \frac{L_s}{r} w + \mu\left(\frac{L_s}{L_x}\right)\frac{\partial u}{\partial \eta}\right] \\
V_{\xi} &= -\frac{D}{L_s^{\,3}}\left[\frac{\partial^3 w}{\partial \xi^3} + (2-\mu)\left(\frac{L_s}{L_x}\right)^2 \frac{\partial^3 w}{\partial \eta^2 \partial \xi}\right] \\
\beta_{\xi} &= \frac{1}{L_s}\left[\frac{L_s}{r} v - \frac{\partial w}{\partial \xi}\right] \\
M_{\xi} &= -\frac{D}{L_s^{\,2}}\left[\frac{\partial^2 w}{\partial \xi^2} + \mu\left(\frac{L_s}{L_x}\right)^2 \frac{\partial^2 w}{\partial \eta^2}\right]
\end{aligned} \tag{2.17}$$

where

$$\{K,D\} = \left\{h, \frac{h^3}{12}\right\} \frac{E}{1-\mu^2} .$$

For the two straight edges, ($\xi = 0,1$), all boundary conditions of equations 2.17 are listed (in a series expansion form) in Table I. At the edge $\xi = 0$, only the first term of the nondimensional curvature

TABLE I

SERIES EXPANSION OF GENERAL BOUNDARY CONDITIONS

Number	$\xi = 0$ Boundary Condition	$\xi = 0$ Series Condition	$\xi = 1$ Boundary Condition	$\xi = 1$ Series Condition
(1)	$u(0) = 0$ or $N_{\xi\xi}(0) = 0$	$A_{m,1} = 0$ $\beta_m B_{m,1} + A_{m,2} = 0$	$u(1) = 0$ or $N_{\xi\xi}(1) = 0$	$\sum_{n=1}^{\infty} A_{m,n} = 0$ $\sum_{n=1}^{\infty} (\beta_m B_{m,n} + n A_{m,n+1}) = 0$
(2)	$v(0) = 0$ or $N_{\xi}(0) = 0$	$B_{m,1} = 0$ $B_{m,2} + D_1 C_{m,1} + \mu\beta_m A_{m,1} = 0$	$v(1) = 0$ or $N_{\xi}(1) = 0$	$\sum_{n=1}^{\infty} B_{m,n} = 0$ $\sum_{n=1}^{\infty} (n B_{m,n+1} + \sum_{j=1}^{\kappa} D_j \delta_j^n C_{m,n-j+1} + \mu\beta_m A_{m,n}) = 0$
(3)	$w(0) = 0$ or $V_{\xi}(0) = 0$	$C_{m,1} = 0$ $6C_{m,4} - (2-\mu) \beta_m^2 C_{m,2} = 0$	$w(1) = 0$ or $V_{\xi}(1) = 0$	$\sum_{n=1}^{\infty} C_{m,n} = 0$ $\sum_{n=1}^{\infty} (n(n+1)(n+1) C_{m,n+3} - n(2-\mu) \beta_m^2 C_{m,n+1}) = 0$
(4)	$\beta_{\xi}(0) = 0$ or $M_{\xi}(0) = 0$	$D_1 B_{m,1} - C_{m,2} = 0$ $2C_{m,3} - \mu\beta_m^2 C_{m,1} = 0$	$\beta_{\xi}(1) = 0$ or $M_{\xi}(1) = 0$	$\sum_{n=1}^{\infty} \sum_{j=1}^{\kappa} D_j \cdot \delta_j^n B_{m,n-j+1} - n C_{m,n+1}) = 0$ $\sum_{n=1}^{\infty} (n(n+1) C_{m,n+2} - \mu\beta_m^2 C_{m,n}) = 0$

series is incorporated into the boundary conditions.

The boundary conditions actually incorporated into the computer program are given by equation 2.18.

$$\left.\begin{array}{lllll} u & = 0 & \text{or} & \dfrac{\partial u}{\partial \xi} & = 0 \\ v & = 0 & \text{or} & \dfrac{\partial v}{\partial \xi} & = 0 \\ w & = 0 & \text{or} & V_{\xi} & = 0 \\ \dfrac{\partial w}{\partial \xi} & = 0 & \text{or} & M_{\xi} & = 0 \end{array}\right. \quad \text{at} \quad \xi = 0,1 \tag{2.18}$$

Although these conditions are not as general as those given in equation 2.17, they are sufficiently general to enable one to consider a large class of useful problems. The series expansion form of these boundary conditions is given in Table II.

To summarize, the three recurrence formulas relating the unknown algebraic constants were presented; the eight necessary boundary conditions at constant ξ were found in order to get sufficient initial values to use the recurrence formulas; these boundary conditions were related to the unknown coefficients by the relationships found in Table II.

The next chapter will explain the combination of the recurrence formulas and boundary conditions, and the calculation of the natural frequencies and modal shapes.

TABLE II

SERIES EXPANSION OF ACTUAL BOUNDARY CONDITIONS

Number	$\xi = 0$		$\xi = 1$	
	Boundary Condition	Series Condition	Boundary Condition	Series Condition
(1)	$u(0) = 0$ or $\frac{\partial u(0)}{\partial \xi} = 0$	$A_{m,1} = 0$ $A_{m,2} = 0$	$u(1) = 0$ or $\frac{\partial u(1)}{\partial \xi} = 0$	$\sum_{n=1}^{\infty} A_{m,n} = 0$ $\sum_{n=1}^{\infty} n\, A_{m,n+1} = 0$
(2)	$v(0) = 0$ or $\frac{\partial v(0)}{\partial \xi} = 0$	$B_{m,1} = 0$ $B_{m,2} = 0$	$v(1) = 0$ or $\frac{\partial v(1)}{\partial \xi} = 0$	$\sum_{n=1}^{\infty} B_{m,n} = 0$ $\sum_{n=1}^{\infty} n\, B_{m,n+1} = 0$
(3)	$w(0) = 0$ or $V_{\xi}(0) = 0$	$C_{m,1} = 0$ $6C_{m,4} - (2-\mu)\, \beta_m^2\, C_{m,2} = 0$	$w(1) = 0$ or $V_{\xi}(1) = 0$	$\sum_{n=1}^{\infty} C_{m,n} = 0$ $\sum_{n=1}^{\infty} (n(n+1)(n+2)\, C_{m,n+3} - n\,(2-\mu)\, \beta_m^2\, C_{m,n+1}) = 0$
(4)	$\frac{\partial w(0)}{\partial \xi} = 0$ or $M_{\xi}(0) = 0$	$C_{m,2} = 0$ $2C_{m,3} - \mu\beta_m^2\, C_{m,1} = 0$	$\frac{\partial w(1)}{\partial \xi} = 0$ or $M_{\xi}(1) = 0$	$\sum_{n=1}^{\infty} n\, C_{m,n+1} = 0$ $\sum_{n=1}^{\infty} (n(n+1)\, C_{m,n+2} - \mu\beta_m^2\, C_{m,n}) = 0$

CHAPTER III

COMPUTER SOLUTION

3.1 Frequency Matrix

In the doubly infinite series expansion for the displacements (2.6), the m-summation denotes the vibrational mode shape in the η-direction. That is, the first mode in the η- direction is the case for m = 1. The n-summation includes an infinite number of constant coefficients for each m. If an infinite number of terms in the n-summation is used, the assumed displacement functions will converge to the exact solution. Examination of the three recurrence formulas (2.15) reveals that the magnitude of the coefficients will approach zero as n becomes large. Thus, this method will give sufficiently accurate displacements and natural frequencies if the n-summation is truncated where the omitted terms are negligible.

In Chapter II, the boundary conditions and the three algebraic recurrence formulas were discussed. The recurrence formulas resulted in n-wise coupling, for each m, of the constant coefficients of the displacement series. Because of this complexity, the constant coefficients of the recurrence formulas were not eliminated from the set of simultaneous equations.

The frequency matrix, denoted as X, was arranged as shown in equation 3.1. The frequency matrix is

$$\left[\begin{array}{c|c|c} \mathrm{BC}(1) & & \\ \mathrm{BC}(2) & & \\ \multicolumn{3}{c}{\text{Recurrence Formula 2.3a}} \\ \hline & \mathrm{BC}(3) & \\ & \mathrm{BC}(4) & \\ \multicolumn{3}{c}{\text{Recurrence Formula 2.3b}} \\ \hline & & \mathrm{BC}(5) \\ & & \mathrm{BC}(6) \\ & & \mathrm{BC}(7) \\ & & \mathrm{BC}(8) \\ \multicolumn{3}{c}{\text{Recurrence Formula 2.3c}} \end{array}\right] \bullet \left\{\begin{array}{c} A_{m,1} \\ \cdot \\ \cdot \\ A_{m,p+2} \\ \hline B_{m,1} \\ \cdot \\ \cdot \\ B_{m,p+2} \\ \hline C_{m,1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ C_{m,p+4} \end{array}\right\} = \left\{ 0 \right\} \tag{3.1}$$

$$\left[\; X \;\right] \left\{\begin{array}{c} A \\ B \\ C \end{array}\right\} = \left\{ 0 \right\}, \tag{3.2}$$

where

BC(1) = Boundary condition for u at $\xi = 0$

BC(2) = Boundary condition for u at $\xi = 1$

BC(3) = Boundary condition for v at $\xi = 0$

BC(4) = Boundary condition for v at $\xi = 1$

BC(5) = First boundary condition for w at $\xi = 0$

BC(6) = Second boundary condition for w at $\xi = 0$

BC(7) = First boundary condition for w at $\xi = 1$

BC(8) = Second boundary condition for w at $\xi = 1$.

For each mode shape in the η-direction, a corresponding value of m was obtained. For example, for the first mode shape in the η-direction, m was equal to one. For each value of m, each recurrence formula was repeated p number of times. That is, in each recurrence formula, n varies from one to p. Because there were two boundary conditions for u and v, there were (p + 2) coefficients for $A_{m,n}$ and $B_{m,n}$, respectively. Using the same argument for w, there were (p + 4) coefficients of $C_{m,n}$ in the frequency matrix.

The frequency matrix can be divided into 9 submatrices (see equation 3.3). For cases including in-surface inertia, the nondimensional $\bar{\omega}^2$ is found in submatrices X_{11}, X_{22}, and X_{33}. When in-surface inertia effects are neglected, only X_{33} includes the nondimensional frequency $\bar{\omega}^2$. The effects of curvature are found in the submatrices X_{13}, X_{23}, X_{31}, X_{32}, and X_{33}.

$$\left[\begin{array}{c|c|c} X_{11} & X_{12} & X_{13} \\ \hline X_{21} & X_{22} & X_{23} \\ \hline X_{31} & X_{32} & X_{33} \end{array}\right] \cdot \left\{\begin{array}{c} A \\ \hline B \\ \hline C \end{array}\right\} = \left\{ 0 \right\} . \qquad (3.3)$$

Upon further investigation, it should be noted that the overall frequency matrix is not of a convenient form. Because of the incorporation of the boundary conditions into the matrix, it is not of the usual eigenvalue form

$$\left| [X] - \omega^2 [I] \right| = 0 .$$

The rows containing the boundary conditions relate the unknown coefficients and are independent of the frequency $\bar{\omega}^2$. Even with rearranging the frequency matrix, this matrix cannot be made symmetric. Because of this unconventional manner of the frequency matrix, the solutions which solve the standard eigenvalue problems could not be used.

Because equation 3.3 is homogeneous, the natural frequencies are those values of $\bar{\omega}^2$ which satisfy the condition that the

$$\det \left| x \right| = 0 \; . \qquad (3.4)$$

In general, the frequency matrix is a 3p + 8 square matrix. Because a standard subroutine program was used to determine the determinant of the matrix, the frequency matrix was stored column-wise in the computer. This procedure of storing the matrix column-wise added to the flexibility of the program.

3.2 Determining the Natural Frequencies

In the previous section, it was noted that the frequency matrix is not symmetric, and standard eigenvalue solutions could not be used. The method used to satisfy the frequency criteria

$$\det \left| x \right| = 0$$

was to assume a value of $\bar{\omega}^2$ smaller than the actual value, and to calculate the value of the determinant of X (called DETS). Normally, the first value of $\bar{\omega}^2$ was zero, but other values can be used if desired. Then $\bar{\omega}^2$ was increased to $\bar{\omega}^2$ + BIT, where BIT was an incremental value read into the computer. A new value of the determinant (called DET) was calculated using the new nondimensional frequency, $\bar{\omega}^2$ + BIT.

If DET/DETS was positive, $\bar{\omega}^2$ was increased still another increment of BIT. DETS became the value of the determinant for $\bar{\omega}^2$ + BIT, formerly called DET. A new value of the determinant, DET, was again calculated, and the ratio of DET/DETS was checked. This procedure was continued until the ratio DET/DETS became negative.

When DET/DETS became negative, a root was known to be between the last two frequencies. Several iterations then took place, until the change in the old and new values of $\bar{\omega}^2$ became smaller than some known value of ERROR.

This method could not be used indiscriminately. If the initial value of BIT was too large, it was possible to skip two roots, and the value of DET/DETS would stay positive. Fortunately, it was found that the values of the determinant did not behave normally when this situation occurred. It was also necessary to insure that the initial value of BIT was not too small. If this was the case, it would take an excessive amount of computer time to reach the first root. With a little experience, it was possible to get excellent results with a minimum amount of computer time. The method for determining the roots of equation 3.4 could be improved. However, it was not the objective of this research to develop techniques for this operation.

Theoretically, there are 3p natural frequencies which satisfy the frequency criteria when in-surface inertia terms are included. Unfortunately, only the lower values of the roots could be obtained with any degree of accuracy, unless p was taken to be large. The higher values of the roots become quite inaccurate, and many may be imaginary. When these higher values are needed, a larger number of terms must be taken.

3.3 Calculating Modal Shapes

Once the natural frequency was calculated, the corresponding modal shape was found. As is the case for all free vibration problems, the true value of the deflections can not be calculated, but only the normalized modal shapes can be obtained. Therefore, only the ratios of the u, v, and w displacements were found.

Because a power series in the ξ-direction was used, a more restricted interpretation of the ratios of the displacements was needed.

The unknown coefficients of $A_{m,n}$, $B_{m,n}$, and $C_{m,n}$ were found as a function of the last $C_{m,n}$ coefficient. For convenience, this last coefficient was set equal to one.

Once the final $\bar{\omega}^2$ was calculated, the program used the actual frequency $\bar{\omega}^2$ and recalculated the elements in the X matrix. Then, a new submatrix of X, called T, was composed of all the elements of the final X matrix, except the last row and column. Another submatrix of X (a column matrix called S) comprised the last column of X, except for the last element. Therefore, the modal shape matrix could be written as

$$\left[\ T\ \right]\begin{Bmatrix} A_{m,1} \\ \vdots \\ C_{m,p+3} \end{Bmatrix} + \begin{Bmatrix} S \end{Bmatrix} \cdot C_{m,p+4} = 0$$

or

$$\begin{Bmatrix} A_{m,1} \\ \vdots \\ C_{m,p+3} \end{Bmatrix} = -\left[\ T\ \right]^{-1} \cdot \begin{Bmatrix} S \end{Bmatrix} \cdot C_{m,p+4} \ . \tag{3.5}$$

The previous equation, which solves for all of the unknown constants as a function of the last, becomes when $C_{m,p+4} = 1$,

$$\begin{Bmatrix} A_{m,1} \\ \vdots \\ C_{m,p+3} \end{Bmatrix} = -\left[\ T\ \right]^{-1} \begin{Bmatrix} S \end{Bmatrix} . \tag{3.6}$$

With the coefficients found, they were placed into the assumed displacement functions and the normalized displacements calculated at specific values of ξ and η.

CHAPTER IV

NUMERICAL RESULTS

4.1 Comparisons with Known Results for Circular Cylinders

Because the displacement series converged slowly, a large number of terms were required to obtain the natural frequencies. Therefore, it was necessary to enlist the aid of a computer to solve for the natural frequencies. The procedure outlined in this study was programed on an IBM 360/50 digital computer. A listing of this program is given in Appendix A.

One circular shell panel was studied in detail to verify the use of the power series method for calculating the natural frequencies of circular cylindrical shells. A second computer program was written, based on an exact solution for circular cylinders developed by Stadler and Wang (6), to compare with the frequencies obtained by the present method. In-surface inertia terms were included in this comparison. It was possible to use the same nondimensional shell parameters in both programs. The shell chosen for comparing the two methods had the following nondimensional properties:

$$\frac{L_s}{L_x} = 0.0785398 \qquad \frac{h}{L_s} = 0.025465 \qquad \frac{L_s}{r} = 0.392698$$

$$\mu = 0.29 \qquad u = v_\xi = w = w_{\xi\xi} = 0 \text{ at } \xi = 0,1 .$$

Each of the three recurrence formulas was written for each n ($n = 1, 2, \ldots, p$). Table III lists the nondimensional natural frequencies obtained for different values of p.

TABLE III

NATURAL FREQUENCIES VS. P

P	$\bar{\omega}^2$
5	.25933
6	.25933
7	.02995
8	.03081
9	.07676
10	.07464
11	.06246
12	.06250
15	.06301
20	.06301
25	.06301
30	.06301

$\bar{\omega}^2$ from Stadler and Wang = .06303

The two programs calculate natural frequencies which differed by only 0.01% for p = 15. A double-precision form of the power series method was examined, but the improved accuracy did not warrant the additional computer time. Thus, it was concluded that the power series method yielded the exact solution when sufficient number of terms were used.

Once the natural frequencies were calculated, the modal shapes could be found. The shell used to determine the accuracy of calculated modal shapes was the same as that studied by Arnold and Warburton (2). It had the following properties

$$\frac{L_s}{L_x} = 0.0967 \qquad \mu = 0.29 \qquad \frac{h}{L_s} = 0.0669 \qquad \frac{L_s}{r} = .785398$$

Comparisons of the ratios of the maximum displacements for the lowest frequency (m = 1) are given in Table IV.

TABLE IV

COMPARISON OF AMPLITUDE RATIOS

Amplitude Ratio	Arnold and Warburton	Power Series
$\frac{A}{C}$	0.024	0.024
$\frac{B}{C}$	0.25	0.25

Therefore, this method also provided accurate results when calculating the modal shapes.

4.2 Circular Cylinders

Section 4.1 established that the present method yields good results if a large number of terms in the power series is taken. It was then decided to hold the value of p at some practical limit and vary the nondimensional shell parameters. The results of these computer runs could then be compared with results from the solution of Stadler and Wang to establish practical accuracy bounds on the shell parameters. From the recurrence formulas (2.6), it is seen that the value of the constant coefficients depend upon the three geometric quantities: β_m, L_s/r, and h/L_s. For this investigation, the ratio L_s/r was fixed. The accuracy of the natural frequencies with in-surface inertia terms included was examined for different values of the ratio h/L_s.

For efficient utilization of the computer, the maximum practical value of p was 25. This value of p resulted in an 83 x 83 frequency matrix. The maximum value of p tried was 30. It was used only for special problems because of the large amount of computer time required to solve for the natural frequencies. For this study, the value of L_s/r for a circular panel was taken as

$$\frac{L_s}{r} = \frac{L_s}{a} = \frac{\pi}{8}$$

where a is the radius of panel.

Two different values of h/L_s were chosen as indicated on Figure 3. The boundary conditions used were the same as those implied by the

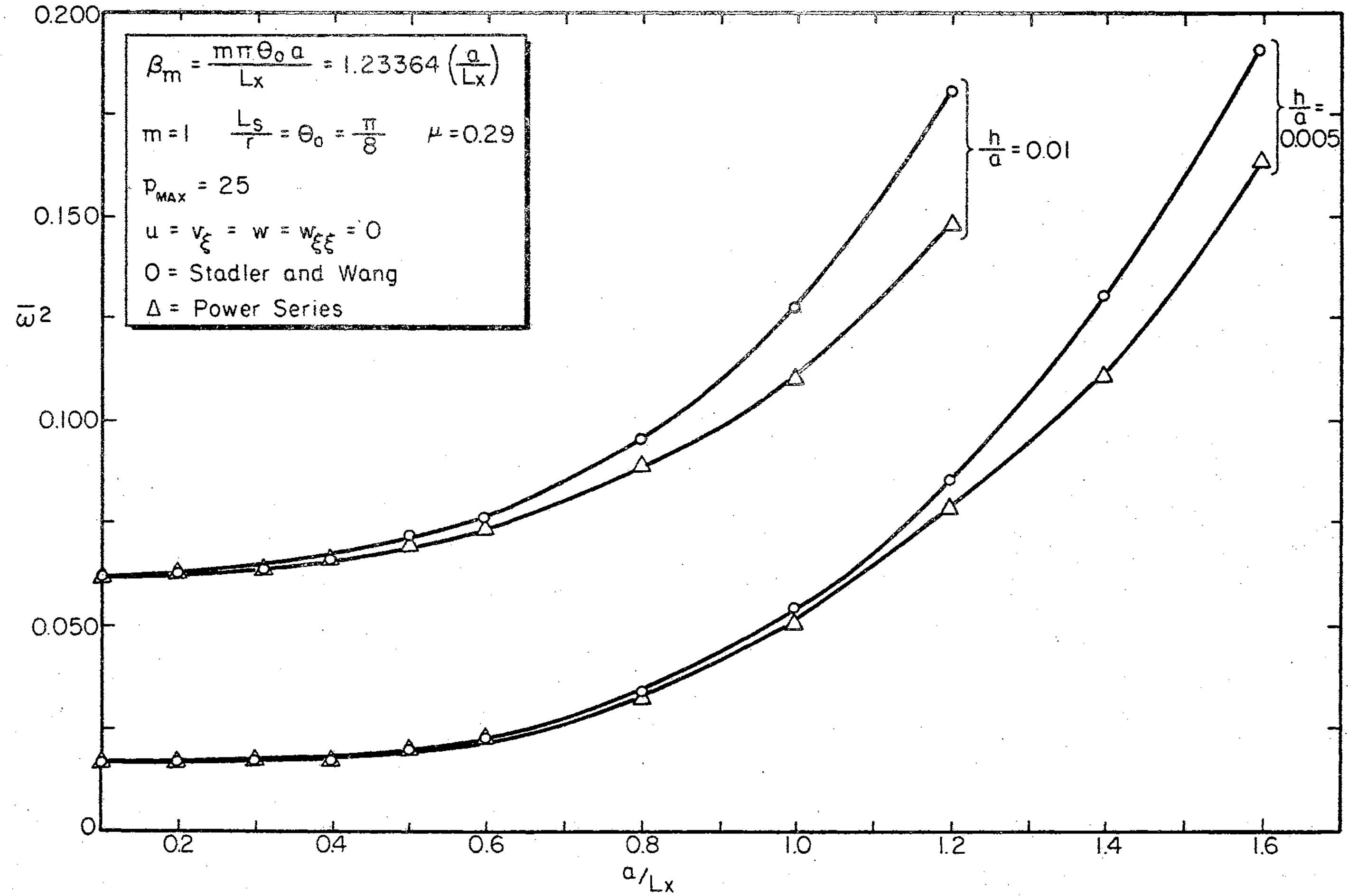

Figure 3. Nondimensional Frequencies Vs. a/L_x for ratios of h/a - In-Surface Inertia Included

solution obtained by Stadler and Wang. They are

$$u = v_\xi = w = w_{\xi\xi} = 0 \text{ at } \xi = 0,1.$$

The results of this study for m = 1 are shown in Figure 3[1]. As shown by Figure 3, greater accuracy was obtained for smaller values of a/L_x. For smaller values of h/a (thinner shells), the accuracy also improved. When frequencies for higher values of a/L_x are desired, p must be made larger. Theoretically, the frequency will converge to the exact solution; in practice, computer round-off error may make the exact solution impossible to obtain.

A similar case was studied with in-surface inertia terms neglected. Similar results were obtained. That is, for lower values of a/L_x, the natural frequencies were obtained with a higher degree of accuracy. Because the results were similar, a plot is not given when in-surface inertia terms were neglected.

With the wide use of closed circular cylinders in industry, the extension of the results of this method to closed circular cylinders

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[1]The following relationships will be helpful when using Figure 3:

$$\frac{h}{L_s} = \frac{h}{a} \cdot \frac{a}{L_s} = \frac{8}{\pi} \cdot \frac{h}{a}$$

$$\beta_m = \frac{m\pi L_s}{L_x} = \frac{m\pi L_s}{a} \cdot \frac{a}{L_x} = \frac{m\pi^2}{8} \cdot \frac{a}{L_x}$$

$$\bar{\omega}_m^{\,2} = \frac{12(1-\mu^2)\rho\omega_m^{\,2} L_s^{\,2}}{E} \quad .$$

would be most beneficial. When a closed circular cylinder vibrates, there are a number of nodal lines around the circumference. An equivalent panel for the closed circular cylinder was chosen between two adjacent nodal lines. The boundary conditions used for the equivalent panel were identical to the conditions existing along the nodal lines of the closed circular cylinder.

Figures 4, 5, 6, and 7 illustrate the values of the nondimensional natural frequencies for a closed circular shell vibrating in the first and third longitudinal modes, m = 1 and 3, respectively. The nondimensional natural frequencies in Figures 4 and 6 include in-surface inertia terms. For Figures 5 and 7, the in-surface inertia terms were neglected.

For this set of curves, the class of shells had the following properties:

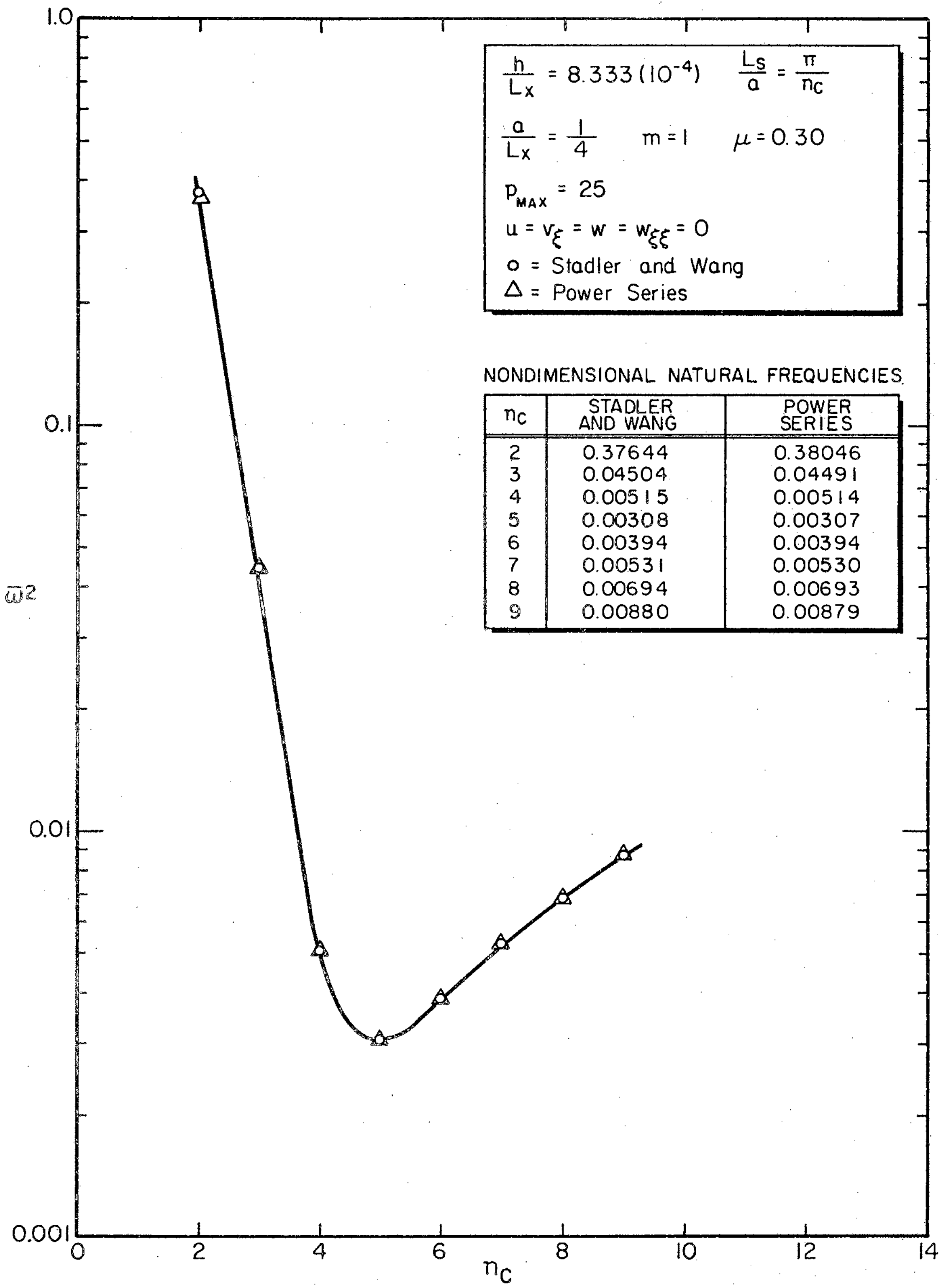
$$\frac{a}{L_x} = \frac{1}{4} \quad , \quad \mu = 0.3 \quad , \quad \frac{h}{L_x} = 0.008333.$$

The dimensionless shell parameters are found as follows:

$$\frac{L_s}{a} = \frac{\pi}{n_c}$$

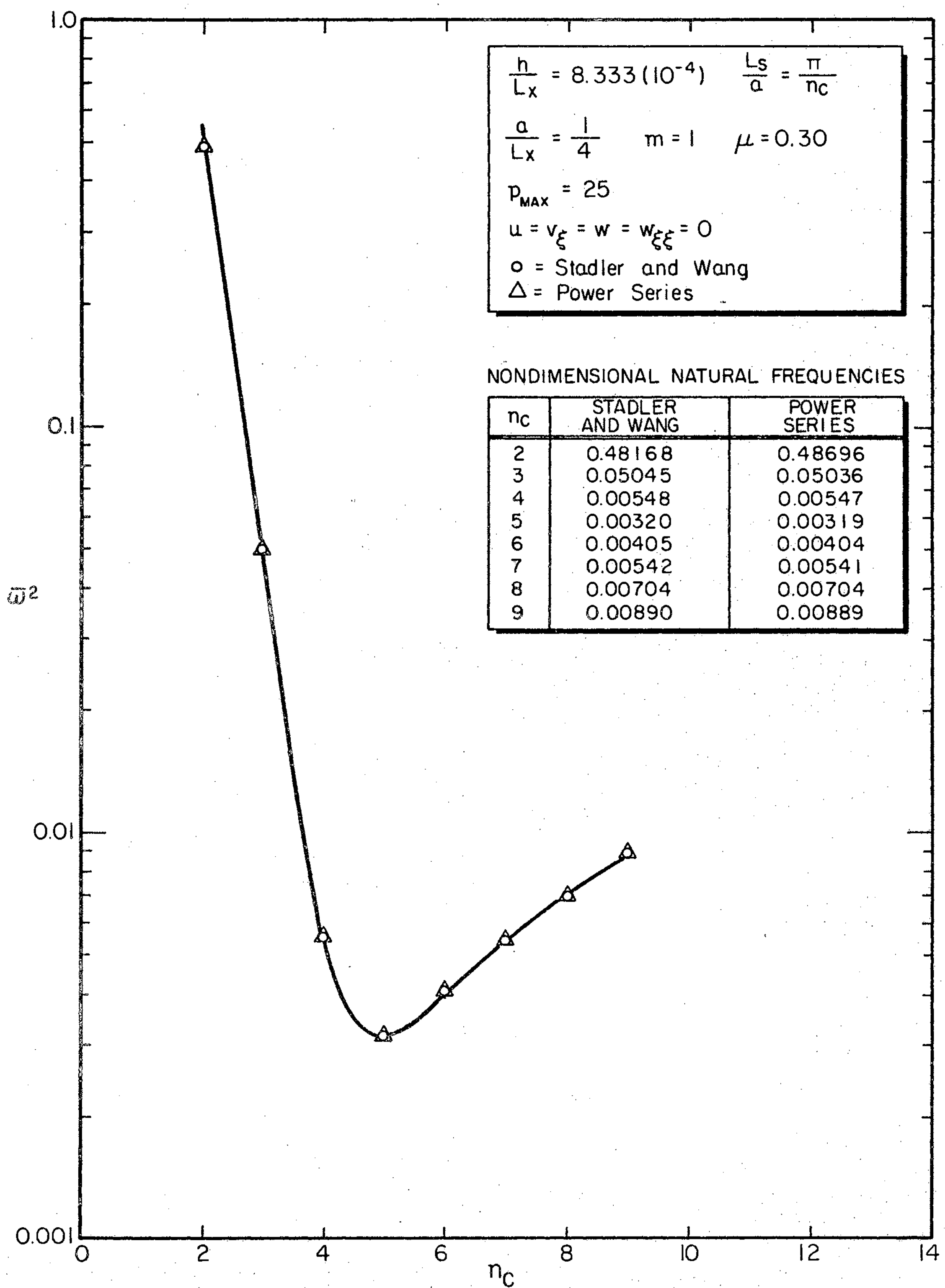
$$\beta_m = \frac{m\pi L_s}{L_x} = \frac{m\pi^2}{4n_c}$$

$$\frac{h}{L_s} = \frac{h}{L_x} \cdot \frac{L_x}{a} \cdot \frac{a}{L_s} = 0.03333 \frac{\pi}{n_c}$$



n_c	STADLER AND WANG	POWER SERIES
2	0.37644	0.38046
3	0.04504	0.04491
4	0.00515	0.00514
5	0.00308	0.00307
6	0.00394	0.00394
7	0.00531	0.00530
8	0.00694	0.00693
9	0.00880	0.00879

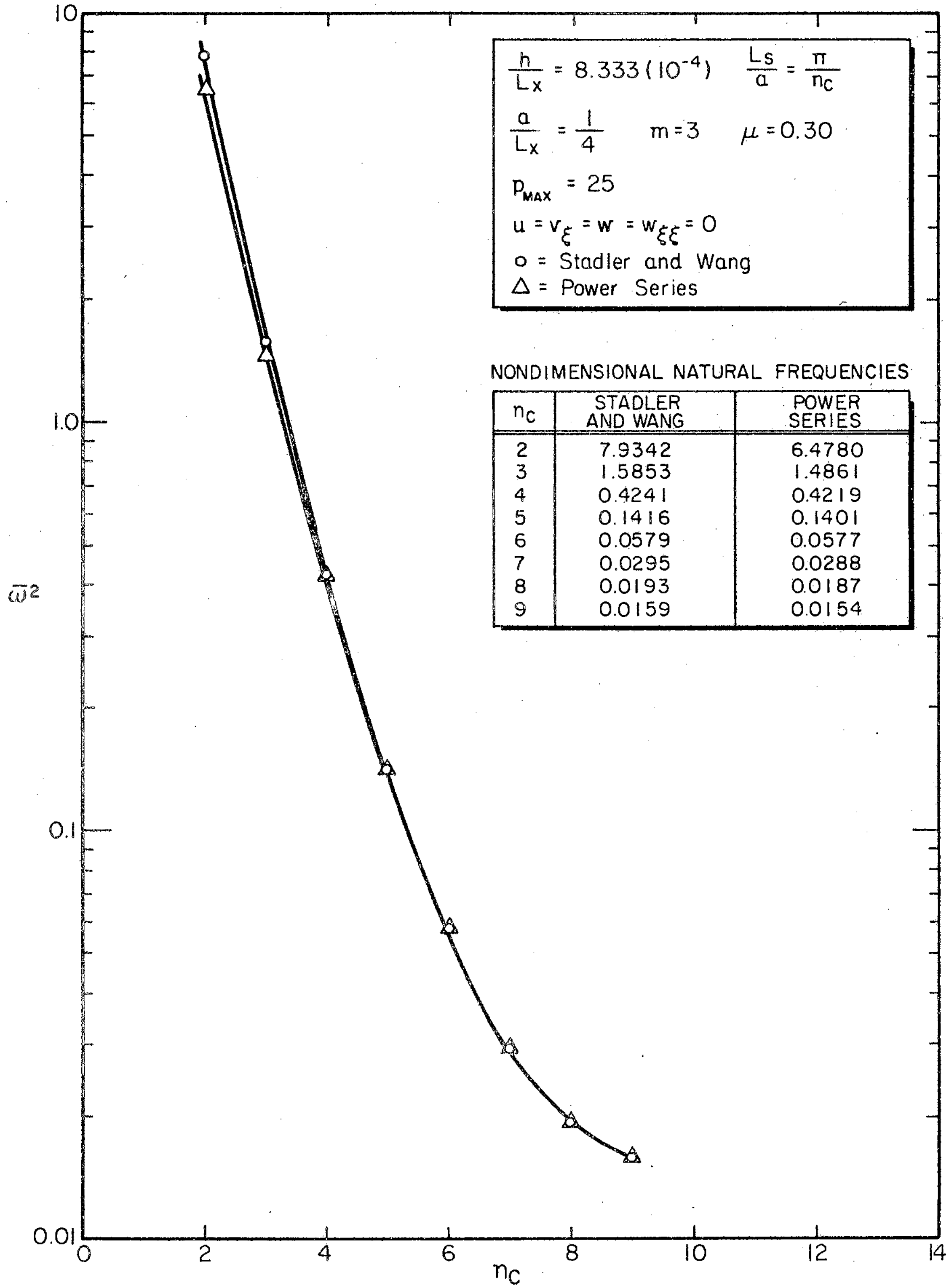
Figure 4. Nondimensional Frequencies for Closed, Simply Supported Circular Cylinder - m = 1, In-Surface Inertia Included



NONDIMENSIONAL NATURAL FREQUENCIES

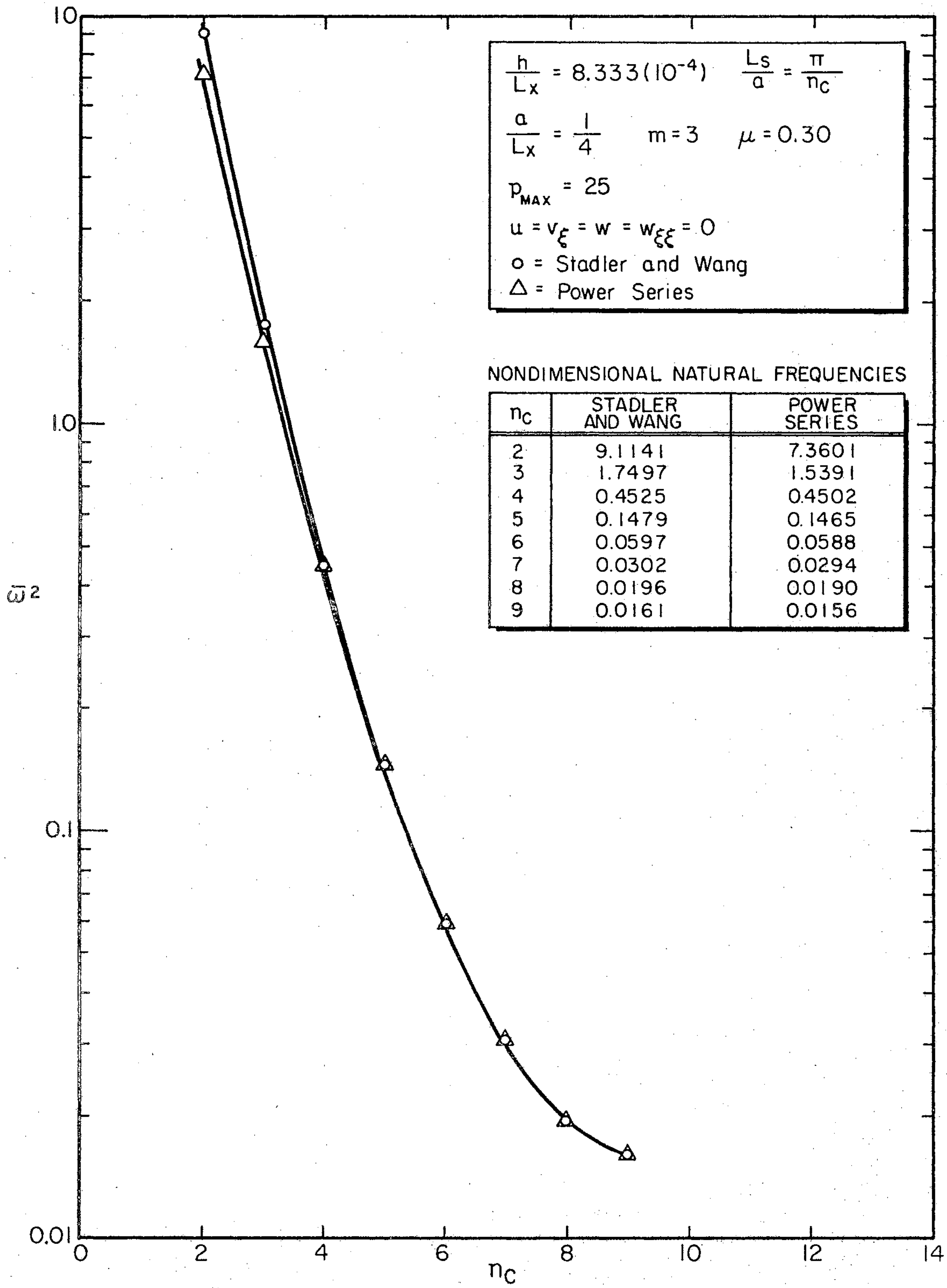
n_c	STADLER AND WANG	POWER SERIES
2	0.48168	0.48696
3	0.05045	0.05036
4	0.00548	0.00547
5	0.00320	0.00319
6	0.00405	0.00404
7	0.00542	0.00541
8	0.00704	0.00704
9	0.00890	0.00889

Figure 5. Nondimensional Frequencies for Closed, Simply Supported Circular Cylinder - m = 1, In-Surface Inertia Neglected



n_c	STADLER AND WANG	POWER SERIES
2	7.9342	6.4780
3	1.5853	1.4861
4	0.4241	0.4219
5	0.1416	0.1401
6	0.0579	0.0577
7	0.0295	0.0288
8	0.0193	0.0187
9	0.0159	0.0154

Figure 6. Nondimensional Frequencies for Closed, Simply Supported Circular Cylinder - m = 3 , In-Surface Inertia Included



NONDIMENSIONAL NATURAL FREQUENCIES

n_c	STADLER AND WANG	POWER SERIES
2	9.1141	7.3601
3	1.7497	1.5391
4	0.4525	0.4502
5	0.1479	0.1465
6	0.0597	0.0588
7	0.0302	0.0294
8	0.0196	0.0190
9	0.0161	0.0156

Figure 7. Nondimensional Frequencies for Closed, Simply Supported Circular Cylinder - m = 3, In-Surface Inertia Neglected

$$\bar{\omega}^2 = \frac{12(1-\mu^2)\rho\bar{\omega}^2 \left(\frac{\pi}{n_c} a\right)^2}{E}$$

where n_c is the number of the circumferential mode.

The maximum value of p was taken as 25. However, for higher n_c, the value of p could be reduced and still obtain accurate values for the frequencies. For the first mode shape in the η-direction ($m = 1$), all natural frequencies were accurately calculated for the second through the ninth circumferential modes (n_c = 2 to 9). For $m = 3$, the natural frequencies for circumferential modes 4 to 9 were obtained accurately. However, the natural frequencies for the second and third circumferential modes were not obtained accurately. For the second circumferential mode and neglecting in-surface inertias, the calculated frequency was 19% below that given by the Stadler and Wang method. For the third circumferential mode, the error was only 15%. If more terms were taken, this error would decrease for these lower circumferential modes. These results verify those given earlier for cases in which p was fixed. This indicates that the present method is limited practically to open panels or to the calculation of higher natural frequencies for closed oval shells.

The effect of neglecting in-surface inertia becomes apparent when comparing Figures 4-7. For lower circumferential modes, solutions which neglect in-surface inertia terms give higher natural frequencies than corresponding solutions which do not include in-surface inertia. For the higher circumferential modes, in-surface inertias do not appreciably affect the natural frequencies. These results have been observed by Armenakas (14), Ivanyuta and Finkel'shteyn (15) and many others (11).

The final study made for circular cylindrical shell panels used the following boundary conditions at constant ξ:

$$u = v = w = w_{\xi\xi} = 0 \text{ at } \xi = 0,1.$$

For a circular panel with this set of boundary conditions, the first mode shape, a predominately stretching mode, occurred at a much higher natural frequency. This behavior is different for circular panels extracted from vibrating circular shells. As previously noted, the straight edges of panels extracted from vibrating circular shells are allowed to move in during vibration and the lowest frequency coincides with the fundamental mode shape. It was found that the lowest frequency for the panels with the straight edges not allowed to move in vibrated in the second circumferential mode. See Figure 8 for the mode shapes at the lowest frequency.

In-surface inertia terms were neglected, and the panel was allowed to vibrate in the first longitudinal mode. The lowest vibrational frequencies are shown in Figure 9. When the modal shapes were calculated, the lowest frequency for cases with

$$v(\eta,0) = v(\eta,1) = 0$$

coincided with the second circumferential mode shape as shown in Figure 8b.

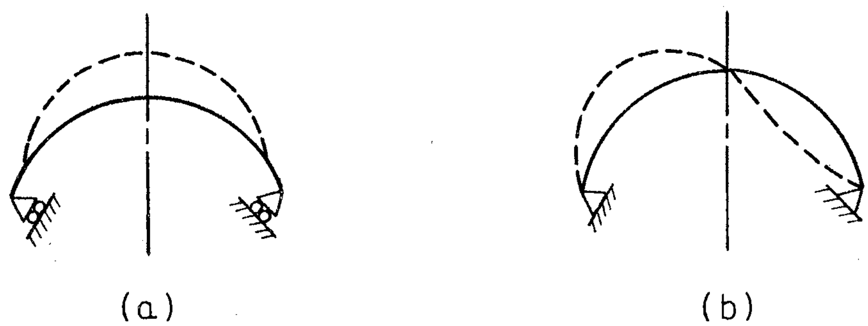


Figure 8. Mode Shapes at Lowest Frequencies

4.3 Noncircular Cylindrical Panels

For all practical purposes, there was very little experimental or analytical data with which to compare natural frequencies of noncircular cylinders. Therefore, it was decided to compare the natural frequencies of two noncircular cylinders which were mirror images of each other. From a physical point of view, one would expect these two cylinders to vibrate with the same natural frequencies. Also, using two mirror-image noncircular cylinders would imply that the D_j's for the nondimensional curvature would be numerically different but in reality describe the same shell. This objective was accomplished by using the following expressions for the nondimensional curvature:

$$\text{a)} \quad \frac{L_s}{r} = \frac{\pi}{4} + \frac{\pi}{16}\xi$$

and (4.1)

$$\text{b)} \quad \frac{L_s}{r} = \frac{5\pi}{16} - \frac{\pi}{16}\xi .$$

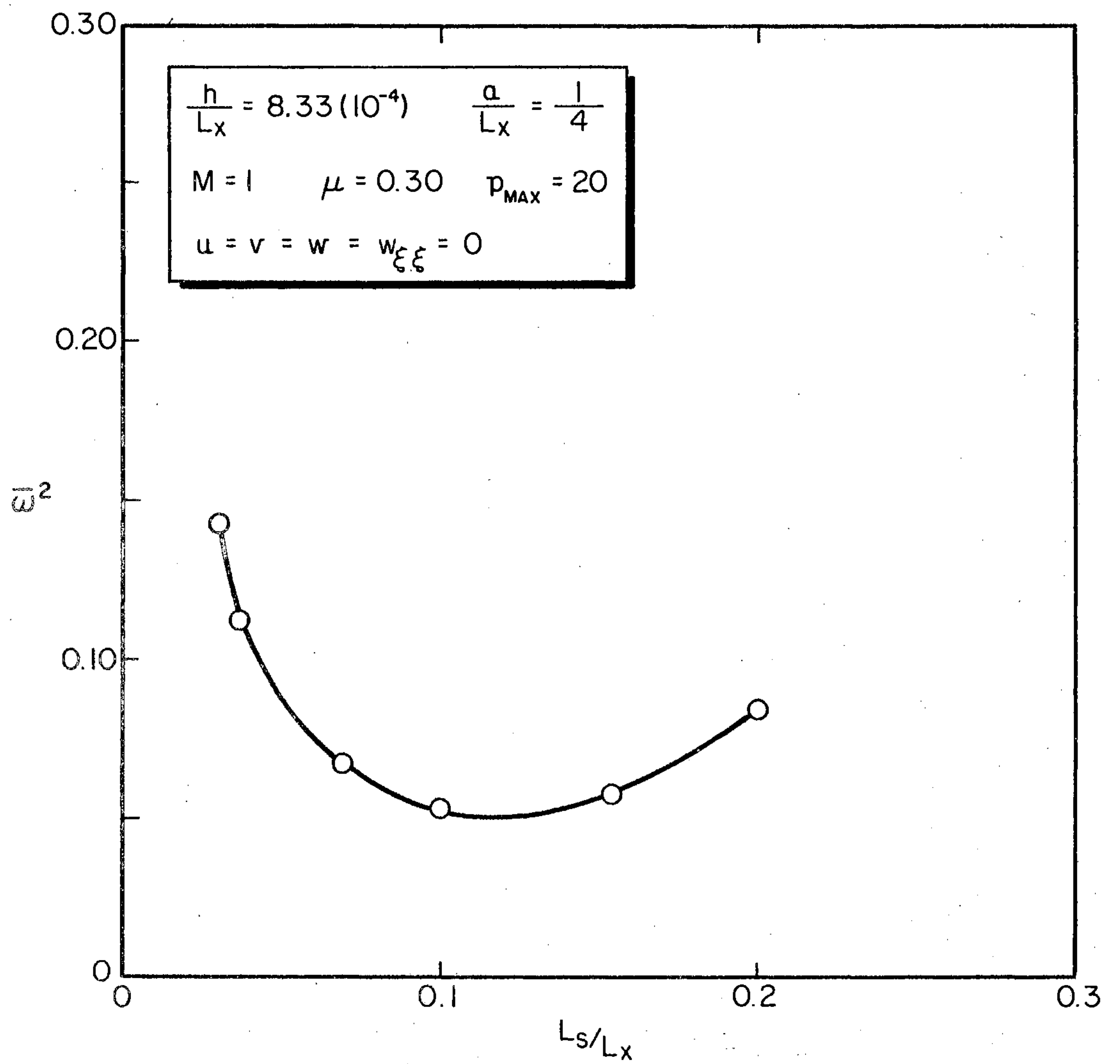


Figure 9. Lowest Nondimensional Natural Frequency (Second Mode Shape) - m = 1, In-Surface Inertia Neglected

It was found that the calculated natural frequencies were exactly the same for both cylinders. The fundamental nondimensional frequency for the parameters used was $\bar{\omega}_1^{\,2} = 0.03207$. The second nondimensional frequency had a value of $\bar{\omega}_2^{\,2} = 0.04243$.

Because only characteristic shapes are calculated for free vibrational problems, the exact value of the displacements were not found. Instead, for these cases the displacements are redefined as follows:

$$\bar{u} = \frac{u}{w_{max}} \,, \quad \bar{v} = \frac{v}{w_{max}} \,, \quad \bar{w} = \frac{w}{w_{max}} \,.$$

The maximum values of the modal shapes for the first two natural frequencies are shown in Figures 10 and 11. From equation 2.6 and setting $m = 1$, the maximum $\bar{v}$ and $\bar{w}$ displacements occur at $\eta = \frac{1}{2}$, and the maximum value of $\bar{u}$ occurs at $\eta = 0$.

From Figure 10, the maximum w displacement occurred at approximately $\xi = 0.43$ for the lowest natural frequency. The $\bar{v}$ displacement had approximately the same magnitude but different signs at $\xi = 0$ and 1. The $\bar{u}$ displacement had a similar shape to the $\bar{w}$ displacement but a much smaller magnitude. Because there was no available information on the modal shapes of noncircular cylinders in the literature, only a subjective analysis of these displacements could be made. From the analysis of circular cylinderical panels with the same boundary conditions, the displacements appear to be quite reasonable.

For the second natural frequency, the normalized displacements are shown in Figure 11. The normalized w displacements, $\bar{w}$, appear very reasonable. Boyd (12) studied a different but similar noncircular cylinder statically loaded by uniform pressure, and obtained a w deflection curve similar in shape to that shown for the second frequency

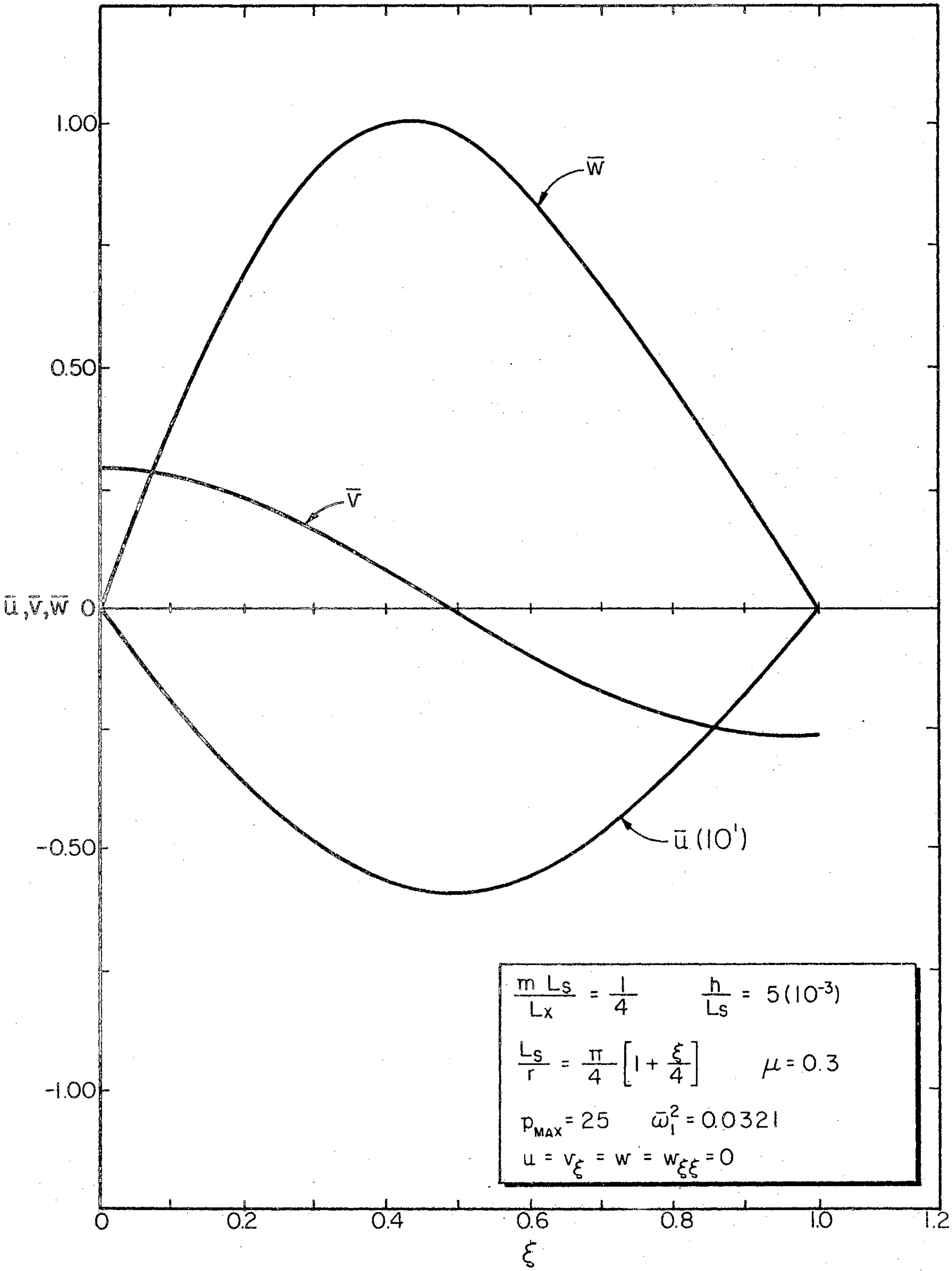


Figure 10. Normalized Displacements for First Circumferential Mode - m = 1, In-Surface Inertia Neglected

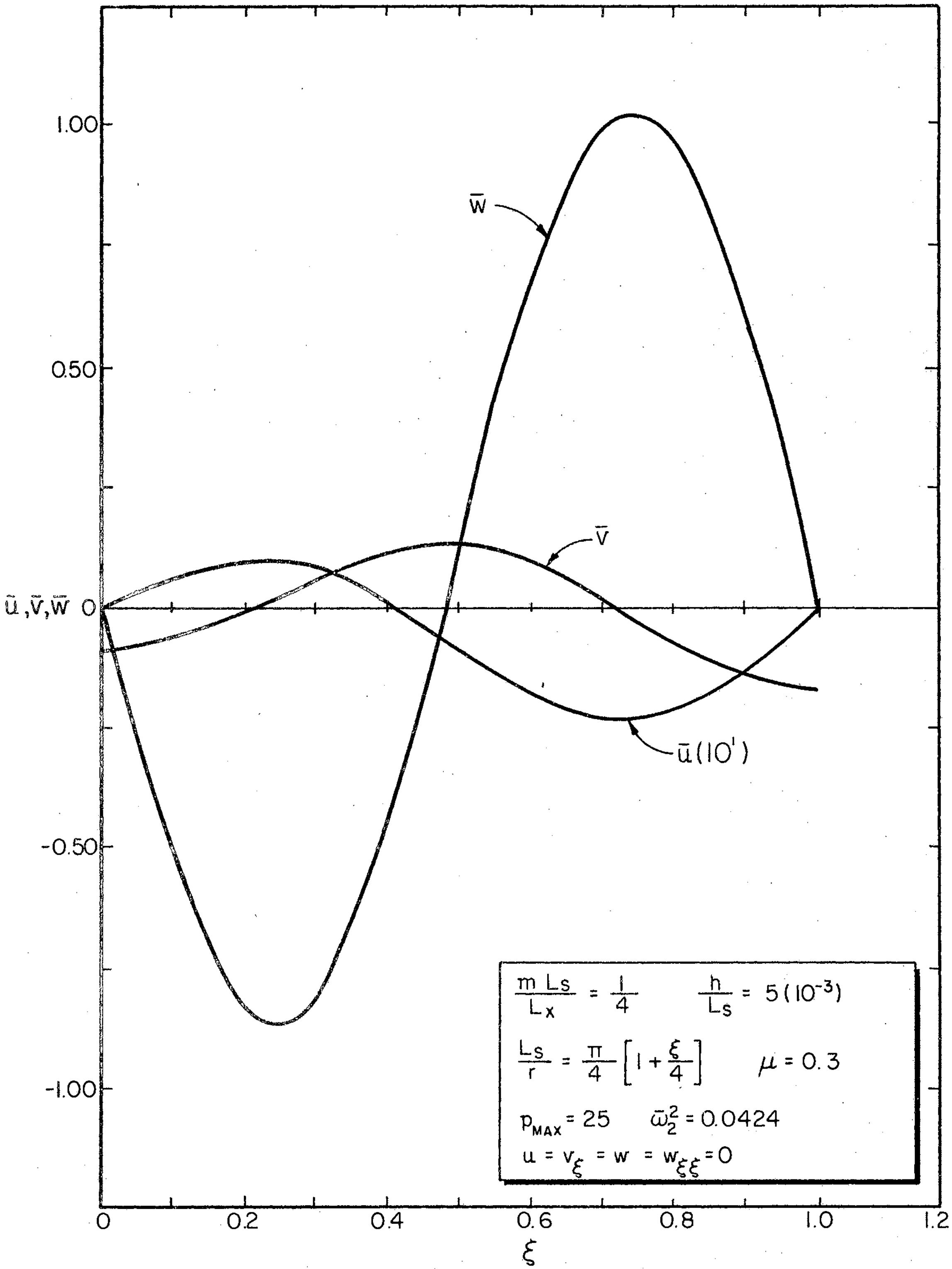


Figure 11. Normalized Displacements for Second Circumferential Mode - m = 1, In-Surface Inertia Neglected

mode. In both cases, the maximum displacements occurred when ξ was between 0.5 and 1.0. For circular panels vibrating in the second mode, the $\bar{w}$ displacements would be equal at $\xi = 0.25$ and 0.75.

Figure 12 shows the results obtained for a noncircular cylindrical panel with a curvature given by the expression

$$\frac{L_s}{r} = \frac{\pi}{4} + b\,\xi$$

where b is a variable constant.

The value of b was allowed to vary from 0.0 to 0.3. The boundary conditions along the straight edges were

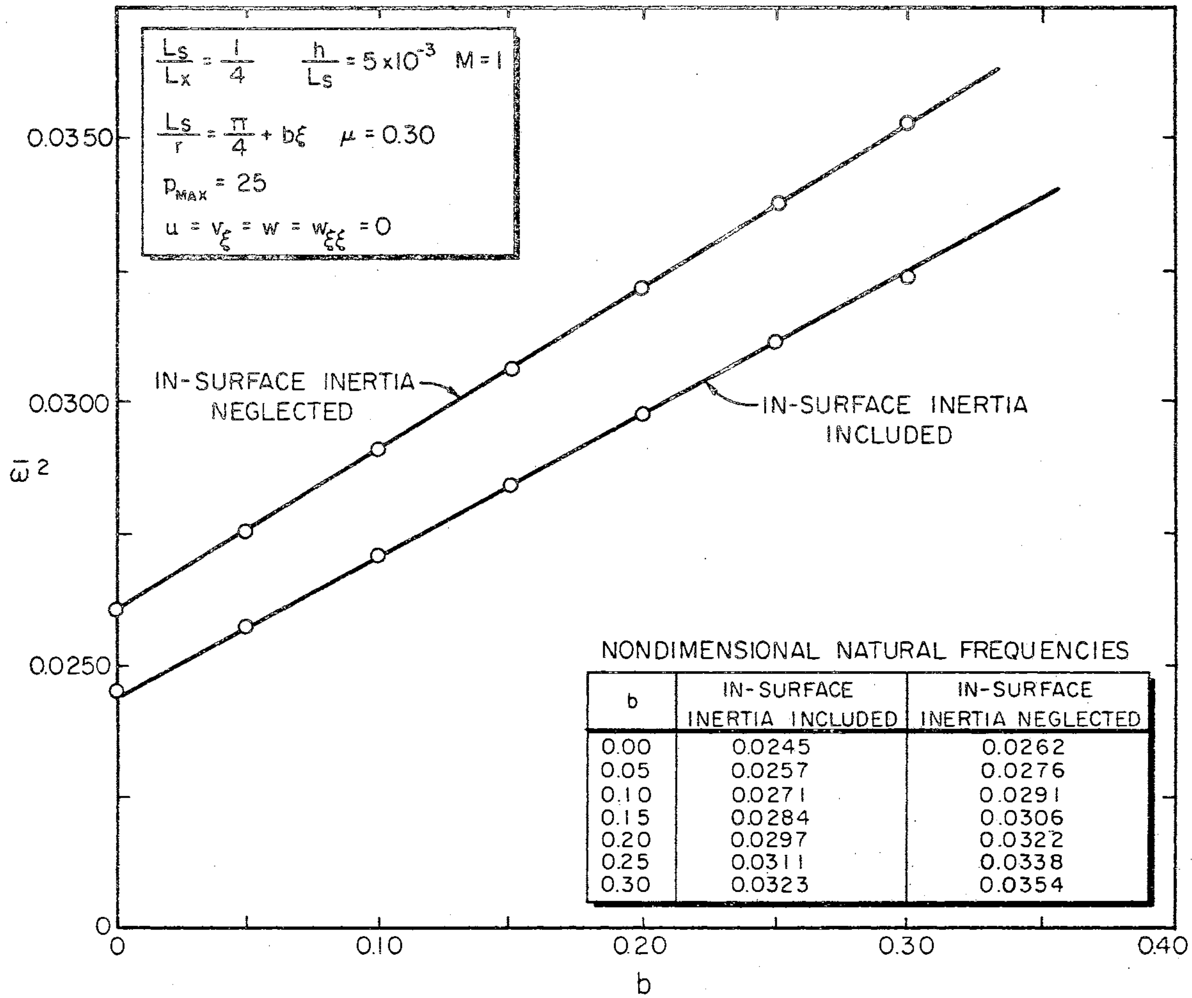
$$u = v_{\xi} = w = w_{\xi\xi} = 0 \quad \text{at} \quad \xi = 0,1.$$

The natural frequency vs. b was nearly linear with or without in-surface inertias included. Numerically, the two lines diverged for higher values of b, but the ratios of corresponding frequencies converged. When comparing the effects of in-surface inertias, higher natural frequencies were obtained when these effects were neglected. A similar result was obtained for circular cylinders.

Equivalent circular panels, which vibrate at the same natural frequencies as the corresponding noncircular panel, were found. The curvature of this equivalent circular panel was obtained for each value of b by the expression

$$\frac{L_s}{r} = \frac{\pi}{4} + b\,\xi_{equiv.}$$

The values of ξ_{equiv} are given in Table V. Thus, natural frequencies



b	IN-SURFACE INERTIA INCLUDED	IN-SURFACE INERTIA NEGLECTED
0.00	0.0245	0.0262
0.05	0.0257	0.0276
0.10	0.0271	0.0291
0.15	0.0284	0.0306
0.20	0.0297	0.0322
0.25	0.0311	0.0338
0.30	0.0323	0.0354

Figure 12. Lowest Nondimensional Frequency Vs. b - m = 1, In-Surface Inertia Included and Neglected

could be obtained for noncircular panels using an equivalent curvature with the methods available for circular panel.

TABLE V

VALUES OF ξ_{equiv} FOR SUBSTITUTION INTO CURVATURE EXPANSION $\frac{L_s}{r} = \frac{\pi}{4} + b\xi_{equiv}$ FOR EQUIVALENT CIRCULAR CYLINDRICAL PANELS

b	ξ_{equiv} Neglecting In-Surface Inertia	ξ_{equiv} Including In-Surface Inertia
0.05	0.46	0.46
0.10	0.47	0.48
0.15	0.48	0.48
0.20	0.47	0.47
0.25	0.48	0.48
0.30	0.48	0.48

The frequency and mode shape for the second circumferential mode were calculated for only one value of b. The value of this frequency and the corresponding mode shape appeared to be quite reasonable. Since its shape was similar to that shown by Figure 11, it was not shown.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary

A method has been presented in this thesis to determine the natural frequencies and mode shapes of noncircular cylindrical panels simply supported along their curved edges. A special case of a general noncircular panel is the circular panel. From this study, the following observations were made.

A summary of observations when the present method was applied to circular panels;

1. Through comparison of natural frequencies and modal shapes obtained by another method for identical circular panels, the power series method was shown to be valid.

2. For a given panel configuration and the value of p fixed, the frequencies of the lower circumferential modes were obtained more accurately than frequencies of higher circumferential modes.

3. For given panel properties and the value of p fixed, more accurate frequencies were obtained for lower longitudinal mode shapes (lower values of m).

4. When in-surface inertia terms were neglected, higher natural frequencies were obtained for identical panels. (This effect was more pronounced for higher values of the ratio L_s/L_x).

5. For a fixed value of p, the power series method calculated the natural frequencies more accurately for panels with smaller ratios of h/a and a/L_x.

6. With appropriate boundary conditions, natural frequencies for simply supported, closed, circular cylinders were found from the frequencies of circular panels. The straight edges of the panel coincided with the nodal lines of the closed cylinder. The panel was simply supported along its straight edge and allowed to move circumferentially in the plane of the surface.

For a fixed value of p and given shell properties, natural frequencies were obtained more accurately for panels coinciding with higher circumferential mode shapes and for lower values of m.

7. Natural frequencies were also obtained for circular panels with straight edges simply supported but not allowed to translate. For these boundary conditions, the lowest natural frequency corresponded to the second circumferential mode shape. The first circumferential mode shape was predominately a stretching mode and occurred at a much higher natural frequency.

A summary of observations when the present method was applied to <u>noncircular</u> panels;

8. Two noncircular, cylindrical panels with the same properties, but mirror images of each other, were found to vibrate at the same natural frequency.

9. The normalized modal displacements were calculated for the first two natural frequencies of the above noncircular panel. The normalized modal displacements for these frequencies were shown in Figures 10 and 11.

10. Natural frequencies for noncircular cylindrical panels were higher when in-surface inertia was neglected. This conclusion was also found for circular panels.

11. For panels which are initially circular and increase their curvature linearily with respect to ξ (increased b), the natural frequencies increased. This increase of frequencies is nearly linear with b.

5.2 Conclusions

The calculation of the natural frequencies and modal shapes for noncircular and circular panels was possible using this method. The method provided better accuracy for thinner shell panels and shell panels with high length to radius ratios. For circular panels, the boundary conditions along the straight edges affected the natural frequencies and modal shapes of the panel. For circular panels having straight edges which were simply supported and not allowed to translate, the lowest frequency corresponded to the second circumferential mode shape. Circular panels having straight edges which were simply supported but allowed in-surface translations, the lowest frequency was an order of magnitude lower than the previous case and corresponded to the first circumferential mode shape.

When comparing the natural frequencies resulting from the inclusions of in-surface inertia terms for noncircular panels, higher natural frequencies were calculated when in-surface inertia terms were neglected. An equivalent circular panel, which vibrated at the same natural frequency as the corresponding noncircular panel, was found.

5.3 Suggestions For Further Work

During this study, many interesting topics were noted which could be studied. As noted in Section 5.1, a large number of terms were taken in order for the natural frequencies to be calculated accurately. Because the frequency matrix was large, a large amount of computer time was necessary to calculate each natural frequency. To make this method more usable, a study should be made to determine a more efficient utilization of the computer. Examples for further study in this area are: 1) reducing the size of the frequency matrix to 3px3p by incorporating the boundary conditions along the straight edges into the frequency matrix; 2) more efficient methods of calculating the lower natural frequencies; and, 3) developing techniques which allow a larger class of shell panels to be worked.

Different dynamic equilibrium equations could be used, and the natural frequencies could be compared with those obtained from the Donnell equations. Examples of these equilibrium equations are the Flugge-Lur'e-Byrne (16) equations and the Morley (17) equations.

The study of forced vibrations could be an easy and valuable extension of this study. Such a study would be most beneficial for cases when a frequency generator was attached to a panel.

Variational and approximate methods could be used in solving the free vibration problem. Examples of these methods are: Rayleigh-Ritz, Galerkin, and Kantorovich. These methods are all approximate methods but could be used to solve for the natural frequencies of noncircular cylindrical panels.

In order for the investigation of noncircular cylindrical panels to be complete, one would desire the experimental verification of the natural frequencies calculated in this study.

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APPENDIX A

COMPUTER PROGRAM

```
C      REFERENCE :
C
C      ' VIBRATIONAL ANALYSIS OF NONCIRCULAR CYLINDRICAL SHELLS '
C
C
C      PURPOSE :
C
C           TO COMPUTE THE NATURAL FREQUENCIES OF NONCIRCULAR CYLINDRICAL SHELLS
C      AND THE U,V,AND W DISPLACEMENTS AT ANY POINT IN THE SHELL.
C
C
C      PROGRAMMER :
C
C
C      CARL  EDWARD  KURT
C      GRADUATE ASSISTANT
C      CIVIL ENGINEERING DEPARTMENT
C      OKLAHOMA STATE UNIVERSITY.
C
C
C      DESCRIPTION OF PARAMETERS :
C
C
C      BETAM   =  M PI LS/LX
C      BIT     =  FIRST INCREMENT OF THE TRIAL FREQUENCY.
C      D       =  AN ARRAY OF THE COEFFICIENTS OF THE CURVATURE POLYNOMIAL.
C      ERROR   =  PARAMETER DENOTING ACCURACY OF CALCULATED FREQUENCY.
C      HOLOS   =  H OVER LS SQUARED.
C      IUO     =  BOUNDARY CONDITION FOR U AT PSI = 0
C      IUL     =  BOUNDARY CONDITION FOR U AT PSI = 1
C      IVL     =  BOUNDARY CONDITION FOR V AT PSI = 0
C      IVL     =  BOUNDARY CONDITION FOR V AT PSI = 1
C      IWOO    =  FIRST BOUNDARY CONDITION FOR W AT PSI = 0
C      IWOT    =  SECOND BOUNDARY CONDITION FOR W AT PSI = 0
C      IWLO    =  FIRST BOUNDARY CONDITION FOR W AT PSI = 1
C      IWLT    =  SECOND BOUNDARY CONDITION FOR W AT PSI = 1
C
C
C           IN THE ABOVE BOUNDARY CONDITIONS THE VALUE REPRESENTS THE DEGREE OF
C      THE DERIVATIVE WRT PSI AT PSI = ZERO IF THE THIRD LETTER OF THE PARAMETER
C      IS 'O' OTHERWISE AT PSI = 1
C
C
C      NFINAL  =  IF NFINAL = 0 ; THE  PROGRAM WILL COMPUTE NATURAL FREQUENCY AND
C                 THE U,V AND W DISPLACEMENTS; OR IF NFINAL = 1 THE PROGRAM WILL
C                 COMPUTE U,V AND W DISPLACEMENTS ONLY.
C      NOA     =  NUMBER OF TIMES EACH RECURRENCE FORMULA IS USED.
C      NOC     =  NUMBER OF CASES TO BE RUN.
```

```
C      NOK      =  NUMBER OF TERMS PRESENT IN THE CURVATURE POLYNOMIAL.
C      NOMSD    =  NUMBER OF MODES IN THE 'S' DIRECTION.
C      NOSTEP   =  NUMBER OF POSITIONS FOR DISPLACEMENT CALCULATIONS IN 'S'
C                  DIRECTION.
C      OMEGAS   =  INITIALLY ASSUMED VALUE OF THE FREQUENCY; IF NOT READ IT WILL
C                  BE ASSUMED AS ZERO.
C      XMU      =  POISSON'S RATIO.
C      NN       =  SIZE OF FREQUENCY MATRIX ( 3 X NOA + 8 )
C      NCCS     =  COUNTER USED IN PROGRAMMING
C      NCDET    =  COUNTER USED IN PROGRAMMING
C      NCHM     =  COUNTER USED IN PROGRAMMING
C      NNJ      =  ELEMENT NUMBER IN FREQUENCY MATRIX
C      COEF     =  MATRIX OF KNOWN COEFFICIENTS FOR MODAL SHAPES.
C
C
C      SUBROUTINES REQUIRED :
C
C
C
C      1)   MATRIX INVERSION ( MINV )
C      2)   GENERAL MATRIX PRODUCT  ( GMPRD )
C
C
C      INPUT  FORMAT  SPECIFICATIONS :
C
C
C      1 ST  CARD ----- 'NOC' VALUE :  'NOC' VALUE (INTEGER)
C                                      TO BE PUNCHED IN FIRST 4 COLS.  RIGHT
C                                      JUSTIFIED. (I4)
C      2 ND  CARD ----- 'BETAM'; 'XMU'; 'HOLOS'; OMEGAS' :  ALL (REAL)
C                                      TO BE PUNCHED WITH A FORMAT OF (4F15.8)
C      3 RD  CARD -----'NOMSD' :  'NOMSD' VALUE (INTEGER)
C                                      TO BE PUNCHED IN FIRST 4 COLS.  RIGHT
C                                      JUSTIFIED. (I4)
C      4 TH  CARD ----- 'ERROR'; 'BIR' : BOTH VALUES (REAL)
C                                      TO BE PUNCHED WITH A FORMAT OF (2F15.8)
C      5 TH  CARD ----- 'IUO'; 'IUL'; 'IVO'; 'IVL'; 'IWOO'; 'IWOT'; 'IWLO';
C                                      'IWLT'  :  ALL VALUES  (INTEGERS)
C                                      TO BE PUNCHED RIGHT JUSTIFIED WITH A
C                                      FORMAT OF (8I4).
C      6 TH  CARD ----- 'NOA'; 'NOK' :  BOTH VALUES (INTEGERS)
C                                      TO BE PUNCHED RIGHT JUSTIFIED WITH A
C                                      FORMAT OF (2I4).
C      7 TH  CARD ----- 'D(I)' :  VALUE OF 'D(I)' (REAL)
C                                      TO BE PUNCHED RIGHT JUSTIFIED WITH A
C                                      FORMAT OF (F15.8).  THERE WILL BE 'NOK' OF
C                                      THESE CARDS.
C      8 TH  CARD ----- 'NFINAL'; 'NOSTEP':  BOTH VALUES (INTEGERS)
C                                      TO BE PUNCHED RIGHT JUSTIFIED WITH A
C                                      FORMAT OF (2I4).
C      9 TH  CARD ----- 'NOA'  :  REDEFINED VALUE OF 'NOA' (INTEGER) USED FOR
C                                      COMPUTATION OF DISPLACEMENTS. FORMAT(I4)
C      10TH  CARD ----- 'ENCREM' : USED ONLY FOR EACH 'S' MODE GREATER THAN ONE.
C                                      BIT FOR HIGHER FREQUENCIES EQUAL TO ENCREM
C                                      TIMES ORIGINAL BIT. FORMAT (F15.8).
```

```
C
C
C     IN PLANE INERTIA TERMS INCLUDED
C
      DIMENSION X(10000),LC(100),MC(100),D(20)
      DIMENSION T(10000), S(100), COEF(100)
      READ (5,905) NOC
      DO 500 NNOC = 1,NOC
      READ (5,900) BETAM, XMU , HOLOS ,OMEGAS
      READ(5,905) NOMSD
      READ(5,900) ERROR, BIT
      READ(5,903) IUO, IUL,IVO,IVL,IWOO,IWOT,IWLO,IWLT
      READ (5,905) NOA, NOK
      WRITE (6,930) BETAM, XMU, HOLOS,OMEGAS
      WRITE(6,955)
      WRITE(6,906) NOA, NOK
      WRITE(6,940) ERROR, NOMSD, BIT
      DO 15 I = 1,NOK
      READ(5,900) D(I)
15    WRITE(6,935)I,D(I)
      READ(5,904) NFINAL, NOSTEP
      WRITE(6,960) IUO, IUL
      WRITE (6,961) IVO,IVL
      WRITE (6,962) IWOO,IWOT,IWLO,IWLT
      GAMMA = (1.0 - XMU)/2.0
      ALPHA = (1.0 + XMU)/2.0
      BETAMS = BETAM*BETAM
      NN = 3*NOA + 8
      NNSQ = NN * NN
      XMBETA = XMU * BETAM
      NOAT = NOA + 2
      NOATT = 2* NOAT
      NOAF = NOA + 4
      NCDET = 1
      NCCS = 1
      NCHM = 1
      NOATS = NOAT
      NOAFS = NOAF
      NOATTS = NOATT
      NOAS = NOA
      NNS = NN
25    DO 30 I = 1,NNSQ
30    X(I) = 0.0
      IF(NFINAL.NE.1) GO TO 31
      READ(5,905) NOA
      WRITE(6,942) NOA
      NN = 3*NOA + 8
      NOAT = NOA + 2
      NOATT = 2*NOAT
      NOAF = NOA + 4
```

```
C      BOUNDARY CONDITIONS FOR U
C
C      U(0) = 0
31     IF(IU0.NE.0) GO TO 32
       NNJ = 1
       X(NNJ) = 1.0
       GO TO 34
C      U'(0) = 0
C
32     NNJ = NN + 1
       X(NNJ) = 1.0
34     IF(IUL.NE.0) GO TO 36
C      U(L) = 0
C
       DO 35 I=1,NOAT
       NNJ = (I-1)*NN + 2
35     X(NNJ) = 1.0
       GO TO 40
C      U'(L) = 0
C
36     DO 37 I=1,NOAT
       NNJ= (I-1)*NN + 2
37     X(NNJ) = I - 1
C      RECURRENCE FORMULA FOR X(1,1) AND X(1,2)
C
40     DO 50 I= 1,NOA
       XNZO = I*(I+1)
       XNO = I + 1
       NNJ = (I - 1)*NN + I + 2
       X(NNJ) = (-BETAM + (OMEGAS/(12.0*BETAM)))/XNZO
       NNJ = NNJ + 2*NN
       X(NNJ) = GAMMA/BETAM
       NNJ = NNJ + (NOA + 1)*NN
50     X(NNJ) = ALPHA/XNO
C      RECURRENCE FORMULA FOR X(1,3)
C
       DO 60 J = 1,NOK
       DO 60 I = 1,NOA
       IF(I.LT.J) GO TO 60
       XNZO = I*(I+1)
       NNJ = (2*NOA + 4 + I- J)*NN + I + 2
       X(NNJ) = XMU * D(J)/XNZO
60     CONTINUE
C      BOUNDARY CONDITIONS FOR V
C
C      V(0) = 0
       IF(IV0.NE.0) GO TO 64
       NNJ= NOAT*NN + NOAT + 1
       X(NNJ) = 1.0
       GO TO 66
C      V'(0) = 0
C
64     NNJ = (NOAT + 1)*NN + NOAT + 1
       X(NNJ) = 1.0
66     IF( IVL.NE.0) GO TO 68
```

```
C       V(L) = 0
C
        DO 67 I= 1,NOAT
        NNJ = (NOAT + I - 1)*NN + NOAT + 2
67      X(NNJ) = 1.0
        GO TO 90
C       V'(L) = 0
C
68      DO 69 I= 1,NOAT
        NNJ = (NOAT + I - 1)*NN + NOAT + 2
69      X(NNJ) = I - 1
C       RECURRENCE FORMULAS FOR X(2,1) AND X(2,2)
C
90      DO 95 I= 1,NOA
        NNJ = I*(NN+1) + NOA + 4
        XNZ = I
        XNO = I + 1
        X(NNJ) = -ALPHA
        NNJ = NNJ + (NOA+1)*NN
        X(NNJ) = (-GAMMA *BETAM + (OMEGAS/(12.0*BETAM)))/XNZ
        NNJ = NNJ + 2*NN
95      X(NNJ) = XNO/BETAM
C       RECURRENCE FORMULAS FOR X(2,3)
C
        DO 105 J = 1,NOK
        DO 105 I = 1,NOA
        IF((I+1).LT.J) GO TO 105
        NNJ = (2*NOA + 5 + I-J)*NN + NOA + 4+I
        X(NNJ) = D(J)/BETAM
105     CONTINUE
C
C       BOUNDARY CONDITIONS FOR W
C       IWOO
C
        NNJ = (IWOO + NOATT) * NN + NOATT + 1
        X(NNJ) = 1.0
C       IWOT
C
        IF(IWOT.NE.2) GO TO 110
C       W''(0) = 0
C
        NNJ = (NOATT + 2)*NN + NOATT + 2
        X(NNJ) = 1.0
        GO TO 112
C       VS(0) = 0
C
110     NNJ= (NOATT + 1)*NN + NOATT + 2
        X(NNJ) = (2.0 - XMU)*BETAMS
        NNJ = NNJ + 2*NN
        X(NNJ) = -6.0
C       W(L) = 0
C
112     IF(IWLO.NE.0) GO TO 114
```

```
      DO 113 I=1,NOAF
      NNJ = (NOATT + I -1) *NN + NOATT + 3
113   X(NNJ) = 1.0
      GO TO 116
C     W'(L) = 0
C
114   DO 115 I=2,NOAF
      NNJ = (NOATT + I - 1)*NN + NOATT + 3
115   X(NNJ) = I - 1
116   IF(IWLT.NE.2) GO TO 120
C     W''(L) = 0
C
      DO 117 I = 3,NOAF
      NNJ = (NOATT + I - 1)*NN + NOATT + 4
117   X(NNJ) = (I-1)*(I-2)
      GO TO 122
C     VS(L) = 0
C
120   NNJ = (NOATT + 1)*NN + NOATT + 4
      X(NNJ) = (2.0 - XMU)*BETAMS
      NNJ = NNJ + NN
      X(NNJ) = 2.0*(2.0 -XMU) * BETAMS
      NOAO = NOA + 1
      DO 121 I = 1,NOAO
      NNJ = (NOATT + 2 + I)*NN + NOATT + 4
      XN = I
121   X(NNJ) = (2.0 - XMU)*(XN + 2.0)*BETAMS -XN*(XN+1.)*(XN+2.0)
C     RECURRENCE FORMULAS FOR X(3,1) AND X(3,2)
C
122   DO 145 J= 1,NOK
      DO 145 I= 1,NOA
      XNTT = (I + 2)*(I + 3)
      NNJ = (I-J)*NN + 2*NOA + 8 +I
      IF(I.LT.J) GO TO 145
      X(NNJ) = -12.0* XMBETA*D(J)/XNTT
      NNJ = NNJ + (NOA+3)*NN
      XN = I+1
      XY = J
      X(NNJ) = 12.0*(XN-XY)*D(J)/XNTT
145   CONTINUE
C     RECURRENCE FORMULAS FOR X(3,3) WO D I TERMS
C
      DO 160 I =1,NOA
      NNJ = (2*NOA +3+I)*NN + 2*NOA + 8 + I
      XNTT = (I+3)*(I+2)
      XNTTOZ = I*(I+1)*(I+2)*(I+3)
      XNZO = I*(I+1)
      X(NNJ) = (HOLOS*BETAMS*BETAMS - OMEGAS)/XNTT
      NNJ = NNJ + 2*NN
      X(NNJ) =(-2.0* BETAMS*HOLOS/XNTT)*XNZO
      NNJ = NNJ + 2*NN
160   X(NNJ) = HOLOS *XNZO
```

```
C      RECURRENCE FORMULAS FOR X(3,3)
       DO 175 J = 1,NDK
       DO 175 K = 1,NDK
       DO 175 I = 1,NOA
       IF(I.LT.(J+K-1)) GO TO 175
       NNJ = (2*NOA + 5 + I-J-K)*NN + 2*NOA +8 +I
       XNTT = (I+2)*(I+3)
       X(NNJ) = X(NNJ) +(12.0*D(J)*D(K)/XNTT)
175    CONTINUE
       IF(NFINAL. NE. 1) GO TO 190
       NSEV = 3*NOA + 7
       DO 180 I= 1,NSEV
       DO 180 J= 1,NSEV
       II = (I-1)*NSEV + J
       JJ = (I-1)*NN + J
180    T(II) = X(JJ)
       DO 185 I=1,NSEV
       JJ = NSEV*NN + I
185    S(I) =-X(JJ)
       GO TO 325
190    CALL MINV(X,NN,DET,LC,MC)
       WRITE (6,920) DET , OMEGAS
       IF(NCDET.EQ.1) DETS = DET
       NCDET = 2
       IF(DET/DETS - 0.0) 285,300,250
250    DETS = DET
       AMEGAS = OMEGAS
260    OMEGAS = OMEGAS + BIT
       IF(ABS(OMEGAS-AMEGAS).LT. ERROR) GO TO 300
       GO TO 25
285    IF(NCCS.NE. 1) GO TO 290
       BMEGAS = OMEGAS
       ODET = DET
       OBIT = BIT
290    NCCS = 2
       CMEGAS = (DET*AMEGAS-DETS*OMEGAS)/(DET-DETS)
       IF(ABS(CMEGAS - OMEGAS).LT.ERROR) GO TO 300
       OMEGAS = CMEGAS
       BIT = BIT/10.0
       WRITE(6,945) AMEGAS , DET, OMEGAS
       IF(ABS(OMEGAS-AMEGAS).LT. ERROR) GO TO 300
       GO TO 25
300    WRITE(6,950) NCHM,CMEGAS,OMEGAS,DET
       OMEGAS = CMEGAS
       NFINAL = 1
       GO TO 25
325    CALL MINV(T,NSEV,DET,LC,MC)
       CALL GMPRD(T,S,COEF,NSEV,NSEV,1)
C      CALCULATION OF MODAL DISPLACEMENTS
       NSEVO = NSEV + 1
       COEF(NSEVO) = 1.0
       XNOSTP = NOSTEP
       NOAPT = NOA + 2
       NOAPF = NOA + 4
       WRITE(6,971) NOSTEP, COEF(1), COEF(NOA+3),COEF(NOATT+JJ
```

```
      DO 410 I = 1,NOSTEP
      U = 0.0
      V = 0.0
      W = 0.0
      XI = I
      SEE = XI/XNOSTP
      DO 350 J=1,NOAPT
      XJ = J
      SEEP = SEE**(XJ-1.0)
      U = U + SEEP*COEF(J)
      NOATJ = NOAPT + J
350   V = V + SEEP*COEF(NOATJ)
      DO 400 J=1,NOAPF
      XJ=J
      SEEP = SEE**(XJ-1.0)
      NOATJ = 2*NOAPT + J
400   W = W+SEEP*COEF(NOATJ)
      WRITE(6,970) I,NOSTEP,U,V,W
410     CONTINUE
      IF(NCHM.GE.NOMSD) GO TO 500
      NCHM = NCHM + 1
      READ (5,901) ENCREM
      NFINAL =0
      BIT = ENCREM*OBIT
      OMEGAS = AMEGAS + BIT
      DETS = ODET
      NOA = NOAS
      NN = NNS
      NOAF = NOAFS
      NOAT = NOATS
      NOATT = NOATTS
      NCCS = 1
      GO TO 25
900   FORMAT(  5F15.6)
903   FORMAT (8I4)
904   FORMAT(2I4,F15.0)
905   FORMAT (    3I4)
906   FORMAT(1X,' NUMBER OF TERMS USED IN CALCULATING THE FREQUENCY = ',
     1I4,'   NUMBER OF CURVATURE TERMS =',I4/)
920   FORMAT(1X,10E13.7)
930   FORMAT(1H3,6HBETAM=,E13.7, 6X, 5HXMU =,E13.7,6X,7HHOLOS =,E13.7, '
     1 INITIAL OMEGAS = 'E14.7/)
935   FORMAT ( 1X,2HD(,I2,3H) =,E14.7)
940   FORMAT(1HK,7HERROR =,E14.8,7HNOMSD =,I4,5HBIT =,E14.8/)
942   FORMAT(1H0,'NUMBER OF TERMS IN SERIES EXPANSION IN CALCULATING DEF
     1LECTIONS =',I4/)
945   FORMAT(1HK, 12HOLD AMEGAS =,E14.8,5HDET =,E14.8,12HNEW OMEGAS =,E1
     14.8)
950   FORMAT(1H0,8HS-MODE =,I4,18HNEW FINAL OMEGAS =,E14.8,18HOLD FINAL
     1OMEGAS =,E14.8,5HDET =,E14.8)
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955   FORMAT (1HL,'THIS PROGRAM INCLUDES INPLANE INERTIA EFFECTS'/)
960   FORMAT(1X,' BOUNDARY CONDITIONS FOR U(O) = ',I4,8X,'BOUNDARY CONDI
     1TIONS FOR U(L) =',I4/)
961   FORMAT(1X,' BOUNDARY CONDITIONS FOR V(O) = ',I4,8X,'BOUNDARY CONDI
     1TIONS FOR V(L) =',I4/)
962   FORMAT(1X,' BOUNDARY CONDITIONS FOR W(O) =',I4,'  AND  ',I4,'  BOU
     1NDARY CONDITIONS FOR W(L) = ',I4,'  AND  ',I4/)
970   FORMAT(1HO,' PSI COORDINATE = ',I4,'/',I4/,' U DISPLACEMENT =',9X,
     1E14.7/,' V DISPLACEMENT =',9X,E14.7/,' MAXIMUM W DISPLACEMENT = ',
     2E14.7/)
971   FORMAT(1HO,' PSI COORDINATE =     0/',I4/,' U DISPLACEMENT =',9X,E1
     14.7/,' V DISPLACEMENT =',9X,E14.7/,' MAXIMUM W DISPLACEMENT =',E14
     2.7/)
500   CONTINUE
      STOP
      END
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VITA

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Candidate for the Degree of

Doctor of Philosophy

Thesis: FREE VIBRATIONS OF OPEN NONCIRCULAR CYLINDRICAL SHELL SEGMENTS

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