A MODERN GEOMETRIC DEVELOPMENT
FOR ELEMENTARY SCHOOL
MATHEMATICS TEACHERS

By

BILL R. GRIMES

Bachelor of Arts
Southeastern State College
Durant, Oklahoma
1950

Master of Teaching
Southeastern State College
Durant, Oklahoma
1956

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1959

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
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Thesis Approved:

Gerald K. Goff
Thesis Adviser

W. Ware Marsden

Jeanne Agnew

D. D. Durham
Dean of the Graduate College

PREFACE

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TABLE OF CONTENTS

Chapter		Page

CHAPTER I

INTRODUCTION

The traditional elementary school mathematics curriculum placed little or no emphasis on geometry. Consequently geometry has not been a part of the undergraduate preparation of prospective elementary teachers. As a result of the recent revolution in mathematics education, geometry has permeated the elementary school mathematics curriculum. A few years ago the study of geometry begin in the tenth grade. Today it may begin in kindergarten. Thus today's elementary school mathematics teacher is seriously handicapped without some formal preparation in the area of geometry.

Need for the Study

Evidence of the significant role of geometry in today's elementary school mathematics curriculum is found in current periodicals regarding mathematics education. Dr. Nicholas J. Vigilante reports as follows:

> As you survey THE ARITHMETIC TEACHER, for example, you may discover the following statistics: Between the years 1954 and 1960 it contains one article on the topic "elementary school geometry." In contrast, approximately twenty-five such articles appear between the years 1961 and 1966 ([12], p. 453).

As minimum preparation for elementary school mathematics teachers the Committee on the Undergraduate Program in Mathematics (CUPM) recommends four semester courses, one of which is geometry. The geometry course is described as follows:

> INTUITIVE FOUNDATIONS OF GEOMETRY. A study of space, plane, and line as sets of points, considering separation properties and simple closed curves; the triangle, rectangle, circle, sphere, and the other figures in the plane and space considered as sets of points with their properties developed intuitively; the concept of deduction and the beginning of deductive theory based on the properties that have been identified in the intuitive development; concepts of measurement of the circle, volumes of familiar solids (/9/, p. 990).

These recommendations assume a full year of algebra and a full year of geometry in secondary school.

As a consequence of the new role of geometry in the elementary school curriculum, many colleges and universities now include an undergraduate course in geometry for elementary teachers. Traditionally undergraduate texts in geometry were designed primarily for mathematics majors. Thus there is a need for resource material designed for elementary teachers in the area of geometry.

Statement of the Problem

Specifically stated the problem in this study is to present a geometric development that is:

1. Consistent with the spirit of modern mathematics education.
2. Appropriate as resource material for pre-service and in-service training for elementary school mathematics teachers.

The second criteria presents the subproblem of identifying the geometric concepts that are present in the elementary school mathematics curriculum.

Scope

Since elementary teachers are expected to be proficient in a great many areas, the time allotted to the study of geometry will be limited. Thus the concepts to be studied should be well defined. A survey of selected series of elementary school mathematics texts reveals that more than one hundred geometric concepts are now included in the elementary school mathematics curriculum. Certainly these concepts should be included in a geometric development designed for elementary teachers. Thus the concepts that are present in the elementary school mathematics curriculum provides a basis for the development.

Modern mathematics education involves more than the presentation of basic concepts such as vocabulary, facts and principles. The basic concepts must be related so as to expose the structure of the discipline. In summarizing a discussion regarding the importance of structure, Dr. Jerome S. Bruner comments as follows:

> . . . . . the curriculum of a subject should be determined by the most fundamental understanding that can be achieved of the underlying principles that give structure to that subject. Teaching specific topics or skills without making clear their context in the broader fundamental structure of a field of knowledge is uneconomical . . . . (/2/, p. 31)

Regarding knowledge transfer, Dr. Robert M. Gagné comments:

> The student needs to be encouraged to "think about" the relationships among various categories of knowledge he has acquired and to make his own applications to new situations and problems. . . . there is a good deal of agreement that knowledge should be used for thinking and that thinking fosters transfer of knowledge (/5/, p. 256).

The development of structure and continuity of thought is facilitated by the preciseness of language inherent in modern set terminology and deductive inference. Consequently an introduction to set terminology and deductive reasoning will be included in the study. Additional concepts will be introduced only when they are needed to preserve the continuity of the development. Many undergraduates enter college without having had geometry at the secondary level. Consequently no previous knowledge of geometry is assumed.

Procedure

The concepts that provide the nucleus of the study were determined by surveying seven selected elementary school mathematics series. The results of the survey are summarized in Appendix A.

To provide a basis for a logical development the nature of deductive reasoning is introduced in Chapter II. The remainder of the study is a discourse relating the concepts determined by the survey. New concepts will be introduced on an intuitive basis and subsequently will be precisely defined or classified as undefined. The postulates and definitions that are a part of this discourse are listed in Appendix B.

In order that the development be in harmony with the spirit of modern mathematics, precision of language will be emphasized throughout.

CHAPTER II

BASIC CONCEPTS

Precision, in the expression of abstract concepts and in the application of the logical processes, is an essential characteristic of a geometrical discourse. This precision is attained through the use of special terminology and symbols which eliminate the normal ambiguity in everyday language. This chapter is concerned with basic linguistic devices that will be used throughout the discourse.

Undefined Terms

Words and symbols are invariably defined in terms of other words and symbols. An understanding of the definition of a particular word is contingent upon a prior understanding of the other words in the definition. Is it possible to give a series of explicit definitions covering every technical word in a particular discourse? Certainly in any such series there would have to be a beginning, that is, a first definition. Consequently any technical word appearing in this first definition must be considered as undefined. To illustrate, an attempt will be made to determine the meaning of the word "point" from an ordinary dictionary. Point--a <u>place</u> considered as to its <u>position</u> only, a <u>spot</u>. The key words in this "definition" are place, position and spot. For these words the dictionary gives the following definitions:

Place--a particular or specifiable spot.

Position--the manner in which anything is placed. Spot; site; place.

Spot--a small extent of space, any particular place.

Space--the aggregate of points.

An understanding of the word "position" depends on a prior understanding of the words "place" and spot." "Place" is a synonym of "spot" and the definition of "spot" involves the word "space." Thus an understanding of "point" is ultimately subject to an understanding of "space," but the definition of "space" depends on the word "point." The effort to define the word "point" results in an endless circular process. If this circular process is to be avoided, either the word "point" or the word "space" should be considered as undefined.

Words that are undefined are not to be considered as meaningless. Before one person can communicate with another there must be some ideas that are understood by both and for which no definitions are necessary. Words which symbolize these common ideas are referred to as primitive or undefined terms.

The primitives used in a particular discourse are optional, as may be seen in the illustration above. If the word "space" is designated as an undefined term then the word "point" is defined. If "point" is considered as an undefined term then "space" is defined. Thus the reader may find the undefined terms in this discourse different from those found in some other discussion of geometry. Those undefined terms that symbolize geometric ideas will be pointed out as they occur throughout the discussion. In many instances certain conditions will be imposed on a primitive term to help create a mutual understanding regarding its use.

Sets

It will be convenient to have a word that will be used to indicate that some objects or things are to be considered together. The word most commonly used in mathematics for this purpose is the word "set." The word set will be taken as an undefined term. It will be used to indicate that a collection of objects have certain properties in common. Once the desired properties are stipulated the objects in the collection will be determined. Those objects having the stipulated properties, and no others, will be called elements of the set. For example, the collection of whole numbers less than 5 is a set and its elements are 0, 1, 2, 3 and 4.

Since frequent references are made to various sets, it is convenient to have names for sets just as names are used to distinguish people. Capital letters will be used as names for sets. For example, if one wished to refer to the set of symbols in the decimal system, that is, the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, it would be convenient to have a name for this collection. Suppose this set is named "D." Then D represents the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and could be used to denote this set rather than to list the elements. The brackets "$\{\}$" are used to enclose the elements of a set when they are listed, and the elements are separated by commas. Frequently it is necessary to indicate the elements of a set when it is impossible or inconvenient to list all of the elements. The set of counting numbers is a set of this type. If the letter "C" is used to name this set the bracket notation can be utilized to symbolize C as $\{1, 2, 3, 4, \ldots\}$.

The initial element is listed as well as enough subsequent elements to indicate a pattern. The series of three dots following the last element is called an ellipsis and indicates that the last element is not listed and indeed there is no last element. If "H" denotes the set whose elements are the first one-hundred counting numbers, it is convenient to symbolize H as $\{1, 2, 3, \ldots, 99, 100\}$. In this case the ellipsis indicates that some of the elements are not listed. The symbol "ε" is often used in lieu of the phrase "is an element of." Thus the statement "3 is an element of the set C" could be symbolized "$3 \varepsilon C$." The symbol "$\notin$" is read "is not an element of." The statement "0 is not an element of H" is symbolized "$0 \notin H$."

The "Equals" Relation

The word "equal" will be used to describe the relation between different names for the same thing. Intuitively "equal" symbolized "=" means "the same as." Thus the statement "A = B" means that A names the same thing that B names. It will be assumed that the equals relation has the following properties:

[For all sets]

1. A = A, the reflexive property.
2. If A = B then B = A, the symmetric property.
3. If A = B and B = C then A = C, the transitive property.

Any relation that has all of these three properties is called an equivalence relation. Thus the equals relation is an equivalence relation.

If A and B are names for sets then "A = B" means that A and B are different names for the same set. That is, the elements of set A are exactly the same as the elements of set B. A statement such as

"A and B are two sets and $A = B$" is self-contradictory. If A and B are two sets then they must be different and hence $A \neq B$. (The symbol $\neq$ means not equal.) If $A = B$ then there is just one set and two names for that set. The word equals as used here is not appropriate to describe a relation between two physical objects. Two sets of dishes may be alike in many ways, but if there are two sets then they are not equal. It will be assumed that if A and B name the same thing, that is $A = B$, then A may be substituted for B or B may be substituted for A in any expression in which either A or B occurs.

Subsets

Let $A = \{2,4,6,7,8,9\}$ and $B = \{2,6,9\}$. Notice that every element of set B is also an element of set A; that is, set B is a part of set A. The word subset is used to describe this relation. The set B is a subset of the set A if every element of B is an element of A. This relation is symbolized $B \subset A$. According to this definition every set is a subset of itself. If B is a subset of A and there is at least one element in A that is not in B then B is called a proper subset of A.

Union and Intersection

In a study of the real numbers the word "operation" is used to refer to a way of thinking of two numbers so as to obtain a number. Addition and multiplication are operations on real numbers. The word "operation" will also be used to refer to a way of thinking about two sets to obtain a set. The operations on sets called union and intersection will be useful in this discussion. It will be assumed that it is possible to perform these operations on any two sets and that the result will always be a set.

Let $S = \{1,2,4,5,7\}$ and $T = \{2,3,5,6,8\}$. The distinct elements of these two sets are the numbers 1, 2, 3, 4, 5, 6, 7, 8. This collection of numbers is a new set formed by considering the distinct elements of S and T. This new set is called the union of the sets S and T and is symbolized $S \cup T$. The union of two sets A and B is defined to be the set consisting of all of the elements that belong to A or B or both. If $A = \{a, b, c\}$ and $B = \{1, 2\}$ then $A \cup B = \{a, b, c, 1, 2\}$.

Referring to the sets S and T above, note that the numbers 2 and 5 are elements of both sets. Thus a set may be formed from S and T by taking those elements that are common to both sets. This set is called the intersection of S and T and is symbolized $S \cap T$. The intersection of M and N is defined to be the set consisting of all elements that belong to both M and N. If $M = \{2, 3, a, b\}$ and $N = \{1, 3, a, c\}$ then $M \cap N = \{3, a\}$.

The Empty Set

In many instances two sets have no elements in common; for example: Consider $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. However, the operation intersection has been defined as an operation on sets. Consequently $A \cap B$ must be some set. But there are no elements that are in both A and B. A set which contains no elements will be called the "empty set." The symbol $\{\}$ will be used to denote the empty set. Thus if $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ then $A \cap B = \{\}$.

Universal Set

In situations in which set terminology is used it is often convenient to have in mind a set which contains all possible elements from which particular sets may be selected. This set is called the universal set and is symbolized by the capital letter U. The universal set may vary in different situations. If the letters in various words in the English language are considered as sets, than a universal set is the English alphabet. In the study of arithmetic a universal set is the set of real numbers.

Statements

The intellectual process of deriving conclusions from previously accepted premises is called deduction. Most of the conclusions of mathematics are a result of deduction. A primary ingredient in the deductive process is a type of sentence called a statement.

A sentence is defined as "a unit of speech consisting of a meaningful arrangement of words, or merely a word, that expresses an assertion, a question, a command, a wish or an exclamation." Sentences which are assertions are of particular concern here. Such sentences will be called statements. It is assumed that statements are either true or false but not both. Sentences which express a single assertion are called simple statements. Sentences which contain two or more assertions are called compound statements.

The sentence "The world is round." is a simple statement that is true. The sentence "The world is flat." is a simple statement that is false. Sentences such as "Stop.", "Is the world round?", and "Write a short paragraph describing statements." are not statements since they do not contain assertions.

All statements have the property of being either true or not true.

Statements that are always true are classified as true statements. Statements that are sometimes true and sometimes not true and statements that are always not true are classified as false statements. This true or false classification of a statement is called its truth value.

It is frequently desirable to combine two or more simple statements so that a new sentence is formed. (Recall that a simple statement is a sentence.) Since each of the simple statements involved contains an assertion, the new sentence will contain two or more assertions and thus is a compound statement. When simple statements are combined to form compound statements, the simple statements will then be called the components of the compound statement. The truth value of a compound statement is determined by the truth values of its components and the way these components are connected.

Connectives--AND, OR

There are two common connectives that will be used extensively in this discourse. One of these is the word "and." When the word "and" is used to connect simple statements, the truth value of the resulting compound statement will be "true" if the truth value of each component has truth value "true." Otherwise, the truth value of the compound statement is false. If two statements are connected with the connective "and" the resulting statement is called a conjunction.

Example 1. The compound sentence "The world is round and the sun rises in the east." is true (has truth value "true") since both components are true.

Example 2. The compound statement "The world is flat and the sun rises in the east." is false (has truth value "false") since one of the components is false.

Example 3. The compound statement "The world is flat and the sun rises in the west," has truth value "false" since both components are false.

In some compound statements, the role of the connective "and" is not as apparent as in the three examples above. Two such cases are of particular interest in this discourse.

Example 4. Consider the compound statement "A baseball is round and hard." Here the word "and" indicates that two assertions are being made about the subject "baseball." Hence it is a compound statement. The simple statements implied are: "A baseball is round." and "A baseball is hard."

Example 5. Consider the true statement "A crow is a black bird." Is this a compound statement? Notice that if the word "red" is substituted for the word "black" the statement is no longer true. If the word "dog" is substituted for the word "bird" the statement again becomes false. It is apparent that the words "black" and "bird" are both pertinent in the description of a crow. Thus two assertions are made about the subject crow; that is, "A crow is a bird." and "A crow is black." Thus the statement "A crow is a black bird." is a compound statement. Here the connective is the word "and" even though it is not present in the statement.

The other connective that is of interest here is the word "or." The word "or" is used to indicate the presence of alternatives. When this word is used to connect simple statements, the resulting compound statement will be true if at least one of the components is true. If all components are false, the compound statement is false. A compound statement formed by using "or" to combine two statements is called a disjunction.

Example 6. The compound statement "The world is round or the sun rises in the east." has truth value "true" since both components are true.

Example 7. The compound statement "The world is flat or the sun rises in the west." is false since both components are false.

Example 8. The compound statement "The world is flat or the sun rises in the east." is true since the second component is true.

A device called a truth table provides a graphic representation of the relation between the truth value of a compound statement and the truth values of its components. For brevity, capital letters are used to symbolize simple statements. Let A represent some simple statement and let B represent a second simple statement. Then the conjunction of the two statements is symbolized "A and B" and the disjunction is symbolized "A or B." Figure 1 below is the truth table for the conjunction and Figure 2 is the truth table for the disjunction. There are four rows in the table since for each of the two possible truth values of one component there are two truth values of the second component for a total of four possible combinations. The first and second columns give the truth values of the components. The third column gives the truth value of the compound statement for each possible combination of truth values of the components.

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

Figure 2

Negation

The negation (denial) of a statement is indicated by the presence of the word "not" or the phrase "it is false" immediately preceding the statement. Thus the negation of the statement "The world is round." is the statement "It is false that the world is round." or more conveniently "The world is not round." If the letter A represents a statement, then not-A represents the negation of the statement. Two of the basic assumptions of deduction occur in connection with a statement and its negation.

1. A statement is true or the negation of the statement is true but not both.
2. A compound statement which is the conjunction of a simple statement and the negation of the simple statement is always false.

Figures 3 and 4 illustrate the assumptions 1 and 2, respectively.

Example 9. As an illustration of the use of truth tables in determining the truth value of compound statements, consider the statement: "Joe is not a Texan or Joe is a citizen of the United States." Is this statement always true, always false, or sometimes true and sometimes false? The letter A will be used for the statement "Joe is a Texan." and the letter B for the statement "Joe is a citizen of the United States." The statement "Joe is not a Texan." would be represented by not-A. The compound statement, "Joe is not a Texan or Joe is a citizen of the United States.", then becomes "not-A or B." There are two possible truth values for each of the components. Either Joe is a Texan, in which case "not-A" is false; or Joe is not a Texan, in which case "not-A" is true. Similarly, Joe is a citizen and B is true or Joe is not a citizen and B is false. The resulting truth table is shown in Figure 5.

A	not-A	A or not-A
T	F	T
F	T	T

Figure 3

A	not-A	A and not-A
T	F	F
F	T	F

Figure 4

	A	not-A	B	not-A or B
1.	T	F	T	T
2.	T	F	F	F
3.	F	T	T	T
4.	F	T	F	T

Figure 5

Figure 5 indicates that the compound statement is true in each case except the second. A close look at the second row of the truth table is instructive. In this case, "not-A" is false, meaning that it is false that Joe is not a Texan and thus Joe is a Texan. Also, "B" is false, meaning that Joe is not a citizen. This gives, "Joe is a Texan and Joe is not a citizen." Since all Texans are citizens, this situation is impossible. Therefore, it is not surprising that the resulting compound statement is false.

Conditional Statements

In deduction it is frequently desirable to make assertions subject to certain conditions. As an example consider the sentence, "Tomorrow is Saturday, if today is Friday." This sentence contains the two simple statements: "Tomorrow is Saturday." and "Today is Friday." Since the sentence contains two simple statements it is a compound statement and thus has a truth value.

Suppose the owner of a professional football team issues the following statement: "The coach will receive a raise if the team wins ten games." This amounts to an assertion on the part of the owner that he will perform a specific act subject to a stated condition. In the event that the team does win ten games, the coach has every right to expect a raise. Suppose the team wins only nine games. Does the coach get a raise? The statement issued by the owner makes no assertion about what he will do in the event that the team fails to win ten games. Presumably, he could grant the coach a raise but he is under no obligation to do so. In any event, assuming that the owner is reliable, one of the following must be true (and possibly both): (1) the team does not win ten games, (2) the coach receives a raise.

Intuitively, it seems that the statement: I. "The coach will receive a raise if the team wins ten games." amount to the same thing as the statement: II. "Either the coach receives a raise or the team does not win ten games." This suggests that the truth values for statement II be used to assign truth values to statement I. Since statement II is a disjunction, its truth values are known. It will be convenient to symbolize the statements. Let A represent "the coach will receive a raise" and B represent "the team wins ten games," then not-B represents "the team does not win ten games." Statements I and II may then be written as follows:

I. A if B

II. A or not-B

In the truth table below the fourth column gives the truth values for statement II. The fifth column assigns truth values to statement I.

	A	B	not-B	A or not-B	A if B
1.	T	T	F	T	T
2.	F	T	F	F	F
3.	T	F	T	T	T
4.	F	F	T	T	T

Figure 6

Compound statements of the form, A if B, are called conditional statements. A conditional statement results when two simple statements are connected by the conjunction "if." The simple statement preceding the connective "if" will be called the "assertion." The simple statement following the connective "if" will be called the condition.

The first row of Figure 6 shows that the truth value of the conditional statement is true provided the assertion is true when the contition holds (has truth value true). Also if it is known that the conditional statement is true and the condition holds, it follows that the assertion is true. Rows three and four show that the truth value of the conditional statement is also true when the condition fails to hold regardless of the truth value of the assertion. Consequently no conclusions may be derived concerning the assertion when the condition fails to hold. The second row of the table shows that the truth value of the conditional statement is false in any case in which the condition holds and the assertion is false.

Example 10. The conditional statement "Joe is a citizen of the United States if Joe is a Texas." is true since the assertion is true provided the condition is satisfied. In this example the assertion is about Joe and is subject to a condition that is imposed on Joe. Notice that in the event that the condition is not satisfied the assertion may or may not be true. Indeed Joe may be an "Okie" and yet still be a citizen of the United States. On the other hand Joe could be a Martian and would not be a citizen of the United States.

Example 11. The conditional statement "Joe is a Texan if Joe is a citizen of the United States." is false since the condition could be satisfied and yet the assertion be false. That is, Joe could be a citizen of the United States and also be an "Okie" and thus would not be a "Texan."

Often the connective "if" is implied but not present in a statement. The statement "All dogs are four-legged animals" could be written as the conditional statement "An animal has four legs if it is a dog."

The conditional statement was introduced initially in the form "A if B" to emphasize that the assertion A is subject to the condition B. In mathematic textbooks and in common language the conditional is often written in the form "If B then A." The two forms will be considered equivalent in this discourse and both will be used. Conditional statements are often referred to as implications and the condition is said to imply the assertion.

Converse

When a conditional statement "A if B" is altered by interchanging the assertion and the condition, the resulting statement, "B if A," is called the converse of the original statement. The converse then becomes a conditional statement in its own right since it contains the assertion "B" subject to the condition "A." The converse of the conditional statement "Tomorrow is Saturday if today is Friday." is the conditional statement "Today is Friday if tomorrow is Saturday."

The truth value of the converse should be determined in the same manner as in any conditional statement. That is, the truth value is "true" if the truth value of the assertion is true when the truth value of the condition is "true." The truth value is "false" if the truth value of the assertion is "false" and the truth value of the condition is "true."

Example 12. The conditional statement "Joe is a Texan if Joe is a citizen of the United States" has truth value "false" as noted in example 11. The converse "Joe is a citizen of the United States if Joe is a Texan" has truth value "true" as noted in example 10. Thus a conditional statement and its converse do not always have the same truth value.

Contrapositive

The contrapositive of the conditional statement "A if B" is the conditional statement "not-B if not-A." That is, the contrapositive of a conditional statement is obtained by writing a new conditional statement whose assertion is the negation of the condition of the original statement, and whose condition is the negation of the assertion of the original statement. The truth value of the contrapositive may be determined from a truth table.

1 A	2 B	3 not-A	4 not-B	5 A if B	6 not-B if not-A
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

Figure 7

In this table the truth values in column 5 are obtained by considering columns 1 and 2. The truth values in column 6 are obtained by considering columns 3 and 4. Note that for any possible combination of truth values of the components A and B, the truth value of the statement is the same as the truth value of its contrapositive. Thus the truth value of a conditional statement is always the same as the truth value of the contrapositive statement. Statements having this characteristic are said to be equivalent statements and may be used interchangeably.

Example 13. The conditional statement, "Joe is a citizen of the United States if Joe is a Texan," has truth value true. The contrapositive is the conditional statement: "Joe is not a Texan, if Joe is not a citizen of the United States." Notice that if the condition is satisfied (has truth value "true") then the assertion is true. Thus the contrapositive is a true statement as was the original conditional statement.

Example 14. Suppose the conditional statement is "Joe is a Texan if Joe is a citizen of the United States," which is false from example 11. The contrapositive is "Joe is not a citizen of the United States if Joe is not a Texan." The condition is satisfied if Joe is an "Okie" but the assertion is false. Thus the truth value of the contrapositive is "false" as was the original statement.

Observe that in examples 13 and 14 the truth value of the contrapositive is the same as the truth value of the original conditional statement. This proves nothing of course, but perhaps it will make the definition of the truth value of the contrapositive seem plausible.

Definitions

Definitions play a vital role in the study of mathematics. Therefore, an analysis of the term "definition" is in order. The definition of the term "definition" given by Whitehead and Russell provides a good starting point for such an analysis.

> A definition is a declaration that a newly-introduced symbol or combination of symbols is to mean the same as a certain other combination of symbols, the meaning of which is already known (/13/, p. 11).

That which is to be defined is called the "definiendum." The combination of symbols that ~~constitutes~~ the definiendum is called the "definiens." Once the definition is made the definiendum and the definiens become synonymous in the sense that either may replace the other in a discourse.

Notice that the definiens must be present before a definition can be formulated. Consequently the meaning which is to be attached to the definiendum has already been symbolized. Thus, definitions are theoretically unnecessary. They are, however, very convenient. Without definitions language would become extremely cumbersome. Often a dozen or more words would be necessary where one word would suffice.

In addition to being convenient, definitions add precision to language. Often a term which is to be defined is familiar in a vague sort of way but the precise meaning may be unknown. For example consider the statement, "An even number is a number like two, four, six, etc." This indicates some knowledge of "even number" but leaves some important questions unanswered. Is twenty even? If so, in what way is twenty like two, four and six? Is nine even? If not, then apparently nine is unlike two, four and six. But in what way is nine unlike two or four? A precise definition of "even number" provides the answers to these questions.

As previously stated, the definiendum may be substituted for, or replaced by, the definiens in any discourse. In most instances the definiendum consists of only one or two words, while the definiens consist of a phrase or the conjunction of two or more phrases stating the characteristics of the definiendum. Since the definiendum and the definiens are interchangeable, it follows that the definiendum must inherit all of the characteristics set forth in the definiens, and nothing more. On the other hand, the definiens must include all of the characteristics of the definiendum. To illustrate, consider the following definition.

Example 15. "An even number is a number that is divisible by two." (It should be understood that in referring to numbers as being even or not even, only integers are considered.) The definiendum is "even number" and the definiens is the phrase "a number that is divisible by two." The interchangeability of the definiendum and the definiens permits the derivation of two conditional statements from the definition, both having truth value true.

1. If a number is an even number then the number is divisible by two.
2. If a number is divisible by two then it is an even number.

Since these two conditionals are true, the contrapositive of each is also true. Hence we obtain two additional conditional statements.

3. If a number is not divisible by two then the number is not even.
4. If a number is not an even number then it is not divisible by two.

Statements (2) and (3) are the most useful in this particular definition and indeed the conjunction of these two conditional statements is often taken as the definition of even number. That is,

5. "If a number is divisible by two then the number is even and if a number is not divisible by two then the number is not even."

Statement (5) is often written in the following abbreviated form.

6. A number is even if and only if the number is divisible by two.

The form used in statement (6) is the one most often used in mathematical definitions. It is referred to as the "if and only if" form of a definition.

If a definition is not given in the if and only if form then the sentence giving the definition should be preceded by the word "definition." To illustrate this point, consider the two following statements:

(a) An even number is a number that is divisible by two.

(b) A dog is an animal that has four legs.

Statements (a) and (b) have exactly the same form. Statement (a) was given as the definition of "even number" in example 15 but certainly statement (b) does not define "dog." Note that example 15 was preceded by a statement that a definition of "even number" was forthcoming. Thus statement (a) is acceptable as a definition of "even number" and statement (b) would be acceptable as a definition of "dog" only when preceded by notification that it is to be considered a definition.

The word "definition" preceding statement (a) would inform the reader that two assertions were intended. Namely,

1. "If a number is even then it is divisible by two.", and

2. "If a number is divisible by two then the number is even."

If the word "definition" did not precede the statement, then only the first assertion above could be considered.

The two statements that may be derived directly from a definition depends on the manner in which the definition is stated. This is illustrated by considering two forms of the definition of Tuesday.

1. Definition: Tuesday is the day following Monday.
2. Today is Tuesday if and only if yesterday was Monday.

From the first form the following statements may be derived:

(a) If yesterday was Monday then today is Tuesday.

(b) If today is Tuesday then yesterday was Monday.

The second (if and only if) form implies the statements:

(c) If yesterday was Monday then today is Tuesday.

(d) If yesterday was not Monday then today is not Tuesday.

Note that (a) and (c) are exactly the same while (b) is the contrapositive of (d) and (d) is the contrapositive of (b). Since the contrapositive always has the same truth value as the statement from which it is derived, the two forms of the definition convey precisely the same information.

From the foregoing discussion it is apparent that a knowledge of conditional statements is a valuable aid in extracting information from a definition.

Proofs

A primary function of mathematics is to establish certain results by means of a deductive argument usually referred to as a "proof." Any idea which is capable of being believed, doubted or denied is a possible subject of a proof. In this discussion only ideas which can be written as conditional statements will be used as subjects for proofs. Ordinarily a number of conditional statements are present in a complete proof. Therefore in order to avoid confusion between the statements that are a part of the proof and the statement which is the subject of the proof, the latter will be referred to as a theorem or a proposition.

The object of a proof of any theorem is to establish that the associated conditional statement has truth value true. Thus an understanding of "proof" requires a prior knowledge of the conditions under which a conditional statement is true. Recall that a conditional statement has two components, a condition and an assertion. Both the condition and the assertion are statements and thus are either true or false. The conditional statement is true provided the assertion is always true when the condition holds (has truth value true). Thus for the purpose of proving a theorem the truth value of the condition is fixed. It is always assumed to be true. Since the condition is always assumed true it is often referred to as the hypothesis of the theorem. The ultimate problem in any proof is to establish that the assertion will invariably be true under the hypothesis that the condition holds. Once a theorem has been proved one may always conclude that the assertion is true provided the condition holds. The assertion is called the conclusion of the theorem. If, in a particular situation, the condition of a theorem does not hold then the truth value of the assertion is undetermined and no conclusion may be derived from the theorem. The theorem, "The sum of two even numbers is an even number.", enables one to conclude that A + B names an even number in the event that both A and B name even numbers. This conclusion does not depend on knowing precisely what even number is named by A or by B. On the other hand if either A or B or both are odd numbers then the theorem provides no conclusion about A + B.

It is not really difficult to state what must be done in order to prove a theorem. One simply assumes the condition and then must establish that under that assumption the assertion must be true. The difficulty arises in establishing the truth of the assertion. Having assumed the condition, how does one establish the truth of the assertion? The conditional statement is the basic element in this process. Conditional statements are the vehicles on which one moves from the assumption of the condition to the truth of the assertion.

Just as one needs words to define words, conditionals are needed to prove theorems. It was previously noted that every mathematical definition gives rise to two conditional statements. Thus the definition is an important source of conditionals. Also, every theorem in itself gives rise to at least one conditional. Consequently, once a theorem is proved, the conditional of that theorem is then available as an aid in proving other theorems. This source becomes significant as soon as a number of theorems have been proved.

The third and last source of conditional statements originates in much the same way as the set of undefined terms. In a particular area of study certain properties are intuitively apparent. These properties are analogous to the rules of a game. They are to be agreed upon initially by all concerned and thereafter accepted without question. In most instances it is possible to state these properties as conditional statements. These properties are often referred to as laws, axioms, or postulates. In the following discussions the term "postulate" will be used to refer to a statement that is to be accepted without proof. Some familiar examples of this type of statement are the following properties of the real number system.

1. If a and b are real numbers then $a + b = b + a$; i.e., the commutative law for addition of real numbers.

2. If a, b and c are real numbers, then $a + (b + c) = (a + b) + c$; i.e., the associative law for addition.

A proof is a reasoning process in which certain statements, known or assumed to be true, are used to establish the truth of some other statement. The initial statement in a proof is usually the hypothesis of the theorem to be proved. This is followed by an additional statement that is closely related to the initial statement. The next step is to combine these statements in a way that will produce an additional true statement. This process continues until the conclusion of the theorem is established as a true statement.

The process of combining statements to obtain additional statements is justified by a set of rules, called rules of inference. Two rules of inference and two basic laws of logic will be needed for this discourse. They are as follows.

1. The Rule of Detachment: If the conditional statement "If A then B" is accepted as true and the condition "A" is accepted as true then the assertion "B" must also be accepted as true.

2. The Rule of Indirect Proof: If a contradiction can be derived as a result of assuming the denial of the conclusion of a proposition, then the conclusion of the proposition is true.

3. The Law of Contradiction: If A is any statement then A and the denial of A cannot both be true. That is, the conjunction "A and not-A" is always false.

4. The Law of the Excluded Middle: If A is any statement then either

A or the denial of A must be true. That is, the disjunction "A or not-A" is always true.

The rule of detachment suggests a starting point for a proof. First one writes the hypothesis of the theorem to be proved as a simple statement. As previously noted this statement is to be accepted as true. The next step, if possible, is to write a conditional statement which is known to be true and which has as its condition the hypothesis of the theorem. The rule of detachment then permits one to write as a third statement the assertion of the conditional stated in the second step. To illustrate, consider the following theorem.

Theorem. The square of an even number is an even number. (The square of the number a is denoted a^2 and is obtained by multiplying the number by itself.)

First, note that the theorem is not stated as a conditional and so should be restated.

Restatement. If a names an even number then a^2 names an even number.

The first step in the proof is a statement of the condition of the theorem.

1. a is an even number.

The definition of even number provides a true conditional which has statement 1 as its condition.

2. If a is an even number then a is divisible by 2.

The rule of detachment may be applied to statements 1 and 2 to obtain:

3. a is divisible by 2.

The next step is to produce a conditional statement with statement 3 as a condition. Again a definition provides the statement.

Definition. A number is divisible by two if and only if the number can be expressed in the form $2 \cdot n$ where n is some integer. From this definition one may extract the statement:

4. If a is divisible by two then $a = 2 \cdot n$ where n is an integer.

Then by the rule of detachment we have,

5. $a = 2 \cdot n$.

From statement 5 and the properties of multiplication on the set of real numbers we may write

6. $a \cdot a = 2n \cdot 2n$ or

$a^2 = 2 \cdot (2n \cdot n) = 2 \cdot k$ where k is the product $2n \cdot n$.

The number k is an integer since the product of integers is invariably an integer. This expresses the square of a in the form 2k where k is an integer. The definition of "divisible" yields a second condition which is pertinent to the above argument.

7. If $a^2 = 2n$ where n is some integer than a^2 is divisible by 2.

From statement 6, a^2 is the product of 2 and some integer so by the rule of detachment,

8. a^2 is divisible by 2.

According to the definition of "even number,"

9. If a^2 is divisible by 2 then a^2 is even.

By applying the rule of detachment to statements 8 and 9 one obtains,

10. a^2 is even.

Thus the argument began with the hypothesis of the theorem and by using conditional statements and the rule of detachment the conclusion of the theorem was established.

The above proof is an example of what is referred to as a "direct proof." In a direct proof one begins with the hypothesis and proceeds directly to the conclusion. It is sometimes advantageous to use a slightly different technique called an "indirect proof." The indirect proof is based on the two basic laws of logic and the rule of indirect proof.

The procedure for an indirect proof is as follows:

1. Assume that the hypothesis of the proposition is true.
2. Assume that the denial of the conclusion of the proposition is true.
3. Use the same procedure as in a direct proof to arrive at a contradiction of the form "A and not-A."
4. Apply the rule of indirect proof to conclude that the conclusion of the proof is true.

The law of the excluded middle is not actually a part of the above outline. Its function in an indirect proof is to motivate the rule of indirect proof. The thinking is as follows. The second step assumes that the denial of a statement is true. This assumption leads to a contradiction and this suggests that the assumption was false. But according to the law of the excluded middle either the denial of the statement in step 2 or the statement itself must hold. Since the denial of the statement turns out false the statement (which is the conclusion of the proposition) must be true. This type of proof is difficult but nevertheless very useful. An example will be given in the proof of the first theorem in the next chapter.

An understanding of the conditional statement and the rule of detachment is a prerequisite to the construction of a proof. However, this understanding is useful only to those with a background of information related to the subject of the proof. Consequently, before attempting a geometric proof it will be necessary to acquire some basic concepts of geometry. In what is to follow it will be assumed that the reader is familiar with the rational number system of elementary school mathematics. No previous knowledge of geometry is assumed.

CHAPTER III

POINTS, LINES, PLANES AND SPACE

Points

Geometry is a branch of mathematics which investigates the relations, properties and measurement of sets of points. Thus the fundamental entities of geometry are sets of points. The sets of points to be considered can be divided into three major categories: the line, the plane and space. All other sets will be subsets of one or more of these three. This chapter is concerned with some of the properties of these three universal sets. Since points are the elements of each set to be considered, the discussion will begin with the word point.

No doubt the reader has used the word "point" as a noun on many occasions and thus it is a familiar term. In Chapter II, it was noted that the dictionary definition of the word "point" depends on certain other technical words which in turn were defined in terms of "points." It seems that it is impossible to define "point" without introducing other words that are less meaningful. Therefore, the word "point" will be considered as an undefined term. Using "point" as an undefined term, it is possible to define space.

Definition 3-1: Space is the set of all points.

Thus space is a set. Every element of space is a point and every point is an element of space.

In a particular model of space, points could be described as positions. This description suggests that points are fixed relative to a particular model of space. For example, if the model of space is the earth and its atmosphere, then the geographical center of the United States is a model of a point in this space. The position of the geographical center of the United States is fixed relative to all other positions in the model. Thus this point is fixed relative to every other point in the model.

In this discourse points will be assumed to be fixed relative to a space. Points will be represented by a dot "•". Since sets are denoted by capital letters and points are elements of sets, points are denoted by lower case letters. If a particular point is denoted by the letter "a" it will be referred to as point a. Since space is the set of all points, every set of points is a subset of space. Indeed, every set of points other than space is a proper subset of space. One of the most important proper subsets of space is the line.

Lines

If one places a ruler or straightedge on a sheet of paper and moves a pencil along the edge, a mark is left in each position that the pencil occupies. Each of these positions is occupied by a point which will be represented by the mark left by the pencil. The resulting configuration (Figure 8) must then be a representation of a set of points. The arrows on either end of the drawing in Figure 8 are to indicate that the representation should extend indefinitely. Any set of points that can be represented as in Figure 8 will be called a line. Any such physical representation of a line will be limited by the edges of the paper but the reader should be aware that the line itself is not limited. Thus every line contains an unlimited number of points. It is possible to draw many different figures like Figure 8, and thus there are many different lines. Capital letters will be used to symbolize sets of points such as lines, etc. If L symbolizes a certain line, that line will be referred to as line L.

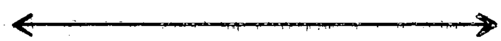


Figure 8

The preceding discussion is not a definition of a line. In fact the term "line," like the term "point," will be undefined. However, it is now possible to assert one definite characteristic of the concept "line."

<u>Postulate 3-1</u>. If "L" is a line, then "L" is a set of points.

This postulate is a conditional statement about the term "line" guaranteeing that every line is a set of points. Is this property a definition? Recall that definitions must be reversible. The converse of Postulate 3-1 is, "If 'L' is a set of points, then 'L' is a line." But certainly there are sets of points that are not lines. Figure 9 is a model of a set of points that cannot be represented as in Figure 8. Thus the converse of Postulate 3-1 is not a true statement. Postulate 3-1 is not reversible and therefore could not be a definition. Figure 10 is a combination of Figures 8 and 9. From this model it may be seen that for any line there are points that are not in the line. This observation suggests a second property of lines.

Figure 9

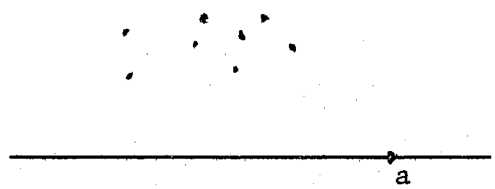


Figure 10

Postulate 3-2. A line is a proper subset of space.

One frequently hears or reads references to a line passing through points. The phrase "passing through" suggests that the line is in motion. But lines are sets of points and points do not move. Hence lines cannot move. Thus in this setting it would be improper to speak of a line passing through a point. However, it will be convenient to refer to certain points that are parts of certain lines. Postulate 3-1 provides a device for such reference. By Postulate 3-1, lines are sets of points. If "L" is a particular line and "a" is a point of the line, then "a" is just an element of the set "L" and may be symbolized "$a \in L$."

That is, "a" is an element of "L" or "L" contains "a."

For any line L, Postulate 3-2 implies that there are points that are not in L. It is often convenient to refer to certain sets of points that belong to the same line. The term "collinear" is used to describe sets of points that belong to the same line.

Definition 3-2. A set "S" of points is said to be collinear if and only if every point of the set belongs to the same line.

Consider a set of exactly two points, as the points a and b in Figure 10. With some convenient straightedge draw as many lines as possible so that each line will contain both of the points a and b. If more than one is found, a straightedge was not used. The drawing should look like Figure 11. This experiment suggests a third property for lines.

Figure 11

Postulate 3-3. If a and b are two different points, then there is exactly one line that contains both a and b.

The phrase "exactly one" is used to emphasize that there is one but no more than one. Thus if a is some point in Texas and b is some point in Alaska, then there is a line (only one) that contains both a and b. Since a line is completely determined by two points it is advantageous to be able to refer to a line L in terms of two of the points on L. If a and b are point in L, then the symbol $\overleftrightarrow{ab}$ may be used to represent L whenever it is convenient. In view of Postulate 3-3 and the definition of collinear, two points are always collinear. Thus the symbol $\overleftrightarrow{mn}$ will represent the line containing points m and n, the symbol $\overleftrightarrow{hk}$ represents the line containing points h and k, etc.

Postulates 3-1 and 3-2 function primarily as language aids. They are a consequence of a need for convenient ways to express certain ideas. Postulate 3-3 imposes a condition on the concept "line." Its effect is to force lines to be what is commonly referred to as "straight."

It is now possible to obtain a fourth property of lines by applying some deductive reasoning to the concepts that are available.

Suppose that L is a line and a is a point of L. Let M be a second line, $M \neq L$, that also contains the point a. Thus there are two different lines, L and M, and a point a that is an element of both. In the language of sets a is an element of the intersection of the sets L and M. That is $a \in L \cap M$.

Is there another point that also belongs to both L and M? Deductive reasoning will provide the answer. In order to facilitate understanding, the steps in the thought process will be listed numerically. The first step will be a statement of the situation leading to the question.

1. L and M are two different lines and a is a point that is in both L and M.
2. Assume that b is a point different from a and that b is in both L and M.
3. Thus $a \in L \cap M$ and $b \in L \cap M$.

4. Since $a \in L \cap M$, then $a \in L$ and $a \in M$.

5. Since $b \in L \cap M$, then $b \in L$ and $b \in M$,

6. Thus $a \in L$ and $b \in L$, or L contains both a and b.

7. Also, $a \in M$ and $b \in M$, or M contains both a and b.

8. By Postulate 3-2, there is only one line that contains both a and b.

9. Therefore since both M and L contain the points a and b it must be concluded that L and M are the same line.

10. Thus an impossible situation arises. In step 1, L and M are different lines but step 9 contends that L and M are really the same line. One of these statements must be false.

11. The statement in step 1 cannot be false since it is just a statement of the conditions that lead to the question, "Is there a point other than a that is in both L and M?"

12. Thus the conclusion in step 9 that L and M are the same line must be false.

13. Notice that step 2 makes the assumption that the answer to the question was "yes." That is, that b was a point different from a and that b was in both L and M.

14. This assumption leads to a false conclusion.

15. It is only reasonable then to conclude that the assumption was false.

16. But if this assumption is false, then there are no points other than a that are in both L and M.

This conclusion is stated formally in Postulate 4.

<u>Postulate 3-4</u>. If two distinct lines L and M intersect, then the intersection is exactly one point.

Postulates 3-1, 3-2 and 3-3 are agreements based on observation and intuition. Postulate 3-4 is a logical conclusion of applying deductive reasoning to the concepts previously developed. Such a process is called a formal proof. To distinguish the properties that are established by a formal proof from properties that are agreements, the former will be referred to as theorems. Consequently Postulate 3-4 should be renamed as Theorem 3-1.

Theorem 3-1. If two distinct lines intersect, then the intersection is exactly one point.

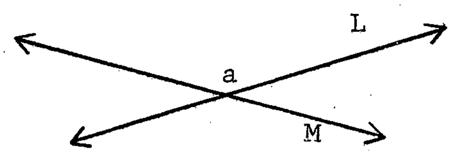


Figure 12

Figure 12 is a model illustrating Theorem 3-1. In observing Figure 12 one would likely conclude that indeed the lines L and M do not intersect in more than one point. Thus Theorem 3-1 might have been obtained as a result of intuition as was the case with Postulates 3-1 and 3-2. What is the purpose of a lengthy argument to obtain an obvious conclusion?

The deductive process is an essential part of geometry. Thus it is important that a student of geometry become familiar with this process. His concern is as much with the method of obtaining conclusions as it is with the conclusions themselves. It seems appropriate that one's first experience with deductive arguments should be in situations in which the conclusions are compatible with his intuition. One is not likely to develop much faith in deductive reasoning if his initial conclusions are unbelievable.

Planes

Consider a set of physical objects that contains the top of a table, the floor of a room, and the surface of a lake. It is common to use the phrase "flat surface" in describing each of these objects. Each of these objects contains positions and thus each is a physical representation of a set of points. Each of these sets is in some way different from just random collections of points and each one is in some way like each of the others. The term "plane" will be used to describe sets of points whose physical representations are commonly referred to as flat surfaces. It is frequently desirable to draw some form of representation of a plane. A drawing like Figure 13 will be used for this purpose, and capital letters such as P, Q, etc., will be used to designate a particular plane.

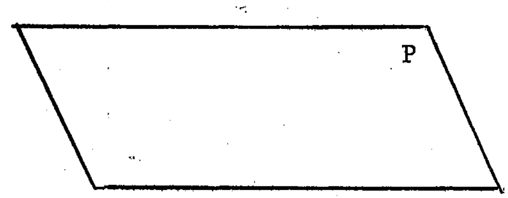


Figure 13

The representation in Figure 13 is misleading in the sense that it suggests that a plane has limitations when in reality a plane extends indefinitely. Thus any physical representation of a plane would be incomplete. Floors and lakes have boundaries and a table top has a "dropping-off" place or edge. Planes have no boundaries and it would be impossible to "drop off" the edge of a plane since there is no edge.

If the edge of a straight ruler is placed on the top of a desk the ruler will contact the desk at every point on the edge of the ruler. If the ruler is turned or moved in any way so that two points of its edge remain on the desk top then every point on the edge of the ruler will be in contact with the desk. Note that the edge of the ruler is a physical representation of a line. This experiment suggests the first postulate for planes.

Postulate 3-5. If a plane contains two points of a line, then the plane contains every point of the line; that is, the plane contains the entire line.

The reader is reminded that a plane is actually a set of points and as such is an abstraction. Postulate 3-5 is suggested by physical representations of a line and a plane. Once stated, the postulate imposes a condition on the abstract concept "plane." Its effect is to force the plane to conform to one's intuitive notion of a "flat surface."

According to Postulate 3-4, two points determine a line. How many points are necessary to determine a plane? Suppose a door swings on its hinged edge. The hinged edge is a subset of a line that contains the three collinear points a, b, and c, and more (see Figure 14). As the door is swung to its various possible positions it represents a different plane in each position. Note that points a, b and c are all elements of each of the planes represented by the door. Therefore when three (or more) points are collinear, they do not determine a plane.

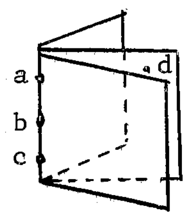


Figure 14

Now suppose the door is to contain a specific point such as d in Figure 14. The door now becomes stationary and thus represents only one plane. Only two of the points, say a and c, are necessary to determine the line that contains the hinged edge. The three points a, c and d fix the position of the door and therefore determine a plane.
<u>Postulate 3-6</u>. If a, b and c are noncollinear points then there is exactly one plane that contains a, b and c. Stated another way, any three noncollinear points determine a plane.

It has been established that a set of collinear points does not determine a plane. Every line is a set of collinear points. Therefore a line does not determine a plane. Would a line and a point not on the line determine a plane? Consider a line L and a point c not in L. Every line contains many points, so choose any two points a and b in L.

Now a and b are in L and c is not in L. Since there is only one line that contains a and b, namely L, and c is not in that line, it follows that a, b and c are noncollinear. Therefore by Postulate 3-6 the points a, b and c determine a plane. Remember than the points a, b and c were obtained by first having a line and a point not in the line. This proves the following theorem.

Theorem 3-2. A line and a point not in the line determine a plane.

According to Theorem 3-1, if two lines intersect, the intersection is one point. Every line contains more than one point and hence contains at least two points. Thus the union of two intersecting lines will be a set containing at least three points. These observations together with Theorem 3-2 are useful in the proof of Theorem 3-3.

Theorem 3-3. If L and M are two lines that intersect, then L and M determine a plane (see Figure 15).

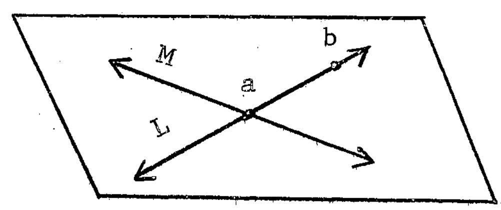


Figure 15

Proof:

1. Let a be the point of intersection of lines L and M. That is, $L \cap M = \{a\}$.

2. Since every line contains at least two points, there is some point b in L other than a.
3. The intersection of L and M contains only one point, namely a.
4. Hence the only point in both L and M is the point a.
5. Since b is in L and b is not a, then b is not in M.
6. Thus b is a point not in M.
7. Therefore by Theorem 3-2, M and b determine a plane.
8. The line M and the point b were obtained as a result of having two intersecting lines.
9. Thus two intersecting lines determine a plane.

If a plane P is determined by the three points a, b and c, then the three points lie in the plane. By Postulate 3-3 each pair of points determine a line and thus points a, b and c determine lines $\overleftrightarrow{ab}$, $\overleftrightarrow{ac}$ and $\overleftrightarrow{bc}$. By Postulate 3-5, if two points of a line are in a plane then the line is in the plane. Thus since $a \in P$ and $b \in P$, then $\overleftrightarrow{ab} \subset P$. Similarly $\overleftrightarrow{ac} \subset P$ and $\overleftrightarrow{bc} \subset P$. Hence a plane determined by three points contains at least three lines. If a plane is determined by two intersecting lines, then it contains at least the two lines that determine it. If a plane is determined by a line and a point not in the line, it is easy to argue that the plane contains at least three lines.

Postulate 3-7. Every plane contains more than one line.

In view of Postulate 3-7, every plane contains at least two lines. By Postulate 3-1, every line contains an unlimited number of points. Consider a plane P and two lines L and M so that $L \subset P$ and $M \subset P$ (see Figure 16). Take a point $a \in M$ and a point $b \in L$. By Postulate 3-3, $\overleftrightarrow{ab}$ is a line and by Postulate 3-5, $\overleftrightarrow{ab} \subset P$. If c is any other point in L then $\overleftrightarrow{ac} \subset P$. Thus there are at least as many lines in P as there are points in L. Since there are an unlimited number of points in L, every plane contains an unlimited number of lines.

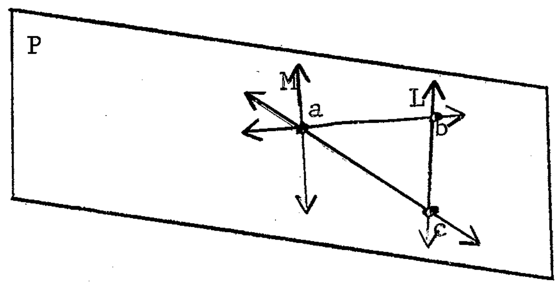


Figure 16

Consider a plane P and a point a so that a is not an element of P. As a convenient physical representation, think of the top of a desk as the plane P and a point on the ceiling of the room as the point a. Take any point b in P. By Postulate 3-3, points a and b determine the line $\overleftrightarrow{ab}$. Since b is in P, the line $\overleftrightarrow{ab}$ intersects the plane P in at least one point. Is it possible for $\overleftrightarrow{ab} \cap P$ to contain more than one point? The answer is no, as is stated formally in Theorem 3-4.

<u>Theorem 3-4</u>. Let P be a plane and b be any point in P. If a is any point not in P, then $\overleftrightarrow{ab} \cap P$ contains exactly one point (see Figure 17).

The intersection of $\overleftrightarrow{ab}$ and P certainly contains at least one point since b is in P and b is on $\overleftrightarrow{ab}$. Thus the theorem will be proved if it can be shown that the intersection does not contain more than one point. The following simple observation will facilitate the argument. In order for $\overleftrightarrow{ab} \cap P$ to contain more than one point, it must contain at least two points. Hence if $\overleftrightarrow{ab} \cap P$ does not contain at least two points then it could not contain more than one point.

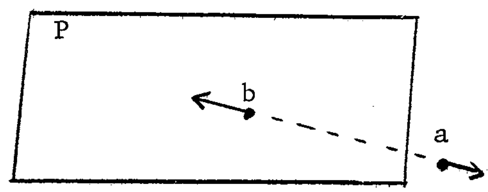


Figure 17

Proof of Theorem 3-4:

1. By hypothesis,

   (i) $a \in \overleftrightarrow{ab}$ and $a \notin P$

   (ii) $b \in \overleftrightarrow{ab}$ and $b \in P$ and therefore $b \in (\overleftrightarrow{ab} \cap P)$

2. Suppose c is a third point such that $c \in (\overleftrightarrow{ab} \cap P)$. That is, $c \in \overleftrightarrow{ab}$ and $c \in P$.

3. By Postulate 3-3, there is only one line that contains a and b and $\overleftrightarrow{ab}$ is this line.

4. Since c is on this one line, then $\overleftrightarrow{bc} = \overleftrightarrow{ab}$.

5. By Postulate 3-5, if two points of a line are in a plane, then the entire line is in the plane.

6. But b is in P by hypothesis and c is in P by the assumption in step (2).

7. Therefore, $\overleftrightarrow{bc} \subset P$.

8. But $\overleftrightarrow{ab} = \overleftrightarrow{bc}$, so $\overleftrightarrow{ab} \subset P$.

9. This implies that $a \in P$.

10. This implies the impossible situation that $a \in P$ and $a \notin P$.

11. This situation is a result of the supposition in step (2), so this supposition must be false.

12. But if step (2) is false then there is no point other than b that is on $\overleftrightarrow{ab}$ and also in plane P.

13. Therefore the intersection of $\overleftrightarrow{ab}$ and P contains exactly one point.

By Postulate 3-5, if two points of a line are in a plane then the entire line is in the plane. By Theorem 3-4, if a line L contains a point a that is not in a plane P and also contains a point b that is in plane P, then $L \cap P = \{b\}$. Combining Postulate 3-5 and Theorem 3-4 yields the following theorem.

<u>Theorem 3-5</u>. If a line L and a plane P intersect, then the intersection is either the line L or a set containing exactly one point. (See Figure 18.)

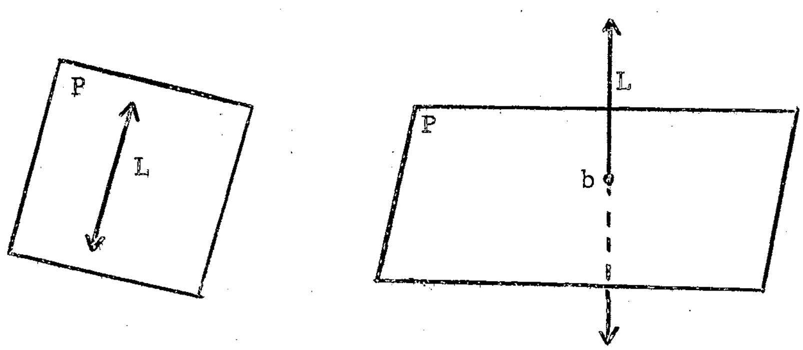


Figure 18

This theorem is similar to Theorem 3-1 which states: "If two lines intersect, the intersection is exactly one point." Theorems 3-1 and 3-5 are both conditional statements. The conclusion in Theorem 3-1 is subject to a condition that two lines intersect. The conclusion in Theorem 3-5 is subject to a condition that a line and a plane intersect. The presence of these conditions suggests that (1) lines do not always intersect, and (2) a given line does not always intersect a given plane.

Physical representations of both possibilities are present in any classroom. Consider the line determined by the edge of the floor at the bottom of the north wall of a room and the line determined by the edge of the floor at the bottom of the south wall. These lines do not intersect (assuming that the room is square). Notice that both lines are in the floor and thus are in the same plane. Such lines are called parallel lines.

Definition 3-3. Two lines are parallel if and only if they are in the same plane and their intersection is empty.

The symbol "$||$" is often used to symbolize the word parallel. That is, $L||M$ means that L and M are parallel.

To illustrate the second possibility above, consider the floor of the room and the line on the ceiling at the top of the north wall. This line does not intersect the plane of the floor.

Definition 3-4. A line L and a plane P are parallel if and only if their intersection is empty.

The definition of parallel lines insists that the lines must lie in the same plane. The lines determined by the north edge of the floor and the line determined by the west edge of the ceiling do not intersect. These lines are not parallel because they do not lie in the same plane. Such lines are called skew lines.

Definition 3-5. Two lines are skew if and only if their intersection is empty and they do not lie in the same plane.

The definition of parallel lines, like any definition, implies two conditional statements.

1. If two lines lie in the same plane and do not intersect, then they are parallel.

2. If two lines are parallel, then they lie in the same plane and do not intersect.

The second statement provides another means of determining a plane. Let L and M be two parallel lines; that is, $L \| M$. Let a and b be two points in L and c be a point in M. Then a, b and c are not all in the same line since no point in M is in L. Thus a, b and c are non-collinear. By Postulate 3-6, there is exactly one plane that contains a, b and c but any plane that contains L and M must contain a, b and c. Hence there is exactly one plane that contains L and M. This proves the following theorem.

Theorem 3-6. Two parallel lines determine exactly one plane.

Consider a plane P and a point a such that a is not in P. Let L be any line in P. By Theorem 3-2, line L and point a determine exactly one plane, call this plane Q. The point a is in plane Q but not in the plane P so that Q and P are different planes. The line L was in P by choice. It is in the plane Q since line L and point a determine Q. Thus line L is in both P and Q and hence is in the intersection of P and Q. Could there be any points in the intersection of P and Q other than those points in line L? Suppose b is a point in plane P and b is not in L. Then by Theorem 3-2, line L and point b lie in just one plane and since they are both in plane P, then P must be the one plane that contains them. Hence no other plane contains both L and b. Since line L is in plane Q, it follows that point b is not in plane Q. Thus there are no points in the intersection of P and Q other than the points of the line L.

Theorem 3-7. If two planes intersect, then the intersection is a line.

Definition 3-6. Two planes are parallel if and only if their intersection is empty.

The ceiling and the floor of a classroom provide an illustration of parallel planes. The floor and a wall of a room determine two planes that intersect and the line of intersection contains an edge of the floor and also the bottom of the wall.

From the definition of parallel planes and Theorem 3-6, it follows that any two planes are either parallel or they intersect in a line. Thus it is impossible for two planes to intersect in a single point. Suppose three planes are considered. Could the intersection of three planes be a point? Notice the line determined by the intersection of two walls of a room. This line does not lie in the plane determined by the ceiling and therefore by Theorem 3-5 must intersect this plane in exactly one point. Thus it is possible for three planes to intersect in a single point.

Three of more planes could intersect in a line as may be illustrated by three or more pages of a book. The different pages can be positioned so as to determine different planes and each of the planes contains the line determined by the binding of the book. Thus the intersection of three planes could be either a point or a line.

In considering two planes it was noted that the planes are either parallel or they intersect. It is possible for three or more planes in a set of planes to be mutually parallel in the sense that each plane in the set is parallel to every other plane in the set. This situation is illustrated by considering the planes determined by the floors of a building having three or more stories. On the other hand suppose that it is known that three particular planes are not mutually parallel. Is there some set of points that belongs to each of the three planes?

Consider a tent of the type illustrated in Figure 19. The walls determine different planes and the floor determines a third plane.

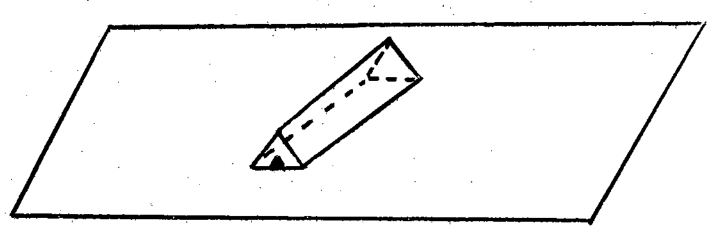


Figure 19

Certainly the three planes are not parallel, yet there is no point that is in all three planes. Thus the intersection of the three planes is the empty set. Note that each pair of planes in this set of planes intersect in a line.

In the beginning of this chapter it was noted that geometry is concerned with the study of point sets. Thus the point is the fundamental building block of geometry. Space was defined as the set of all points. Consequently space is the universal set in the study of geometry in the sense that every geometric entity is a point set and thus is a subset of space.

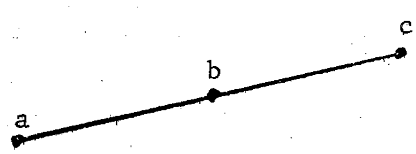
CHAPTER IV

SUBSETS OF THE PLANE

This chapter is concerned with certain sets of points which will be subsets of a plane. In some cases the sets under consideration will also be subsets of a line.

Betweenness

Consider the points a, b and c in Figure 20 (a) and the points x, y and z in Figure 20 (b). Note that the points a, b and c are collinear while the points x, y and z are noncollinear.



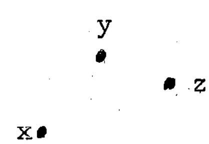


Figure 20

If three points are situated as in Figure 20 (a), it is common to refer to one of the three, b in this case, as being between the other two. Is one of the points in Figure 20 (b) between the other two? If so, which one? Notice that in Figure 20 (a) there is no doubt about which point is between the other two. The point b is between a and c, but a is not between b and c nor is c between a and b. If the term "between" is applied to one of the points in Figure 20 (b), then it might just as well be applied to either of the other two. Thus the term "between" does not seem to be applicable to the sets of points in Figure 20 (b). Under what circumstances is it appropriate to apply the term "between" to one element of a set of points? The question could be answered with a definition but such a definition would necessitate the introduction of other concepts, the definitions of which would involve more concepts, etc. Consequently the term "between" will be considered an undefined term. Certain properties will facilitate the use of the concept.

Postulate 4-1. If a is between b and c, then a, b and c are collinear.

Notice that this statement is not reversible. Consider Figure 21. Points a, b and c are collinear but a is not between b and c.

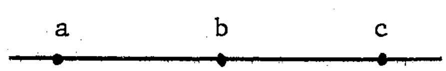


Figure 21

The notation a-b-c will be used to indicate that b is between a and c. Similarly b-a-c means that a is between b and c, and a-c-b means that c is between a and b. It is intuitively apparent that if a is between b and c, then a is between c and b. Thus a-c-b and b-c-a mean the same thing.

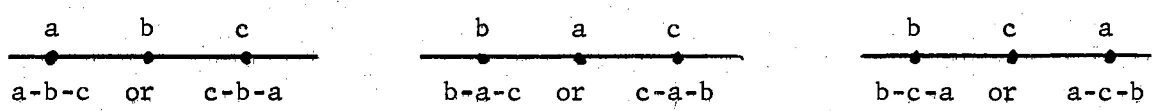


Figure 22

A study of Figure 22 suggests the following postulate.

<u>Postulate 4-2</u>. If a, b and c are three points in the same line, then exactly one of the points is between the other two.

Consider points a and b in line L as in Figure 23 (a). Since the line L extends indefinitely, intuition suggests the following:

1. There is some point d so that a-b-d. (See Figure 23 (b).)
2. There is a point c so that c-a-b. (See Figure 23 (c).)
3. There is a point e so that a-e-b. (See Figure 23 (d).)

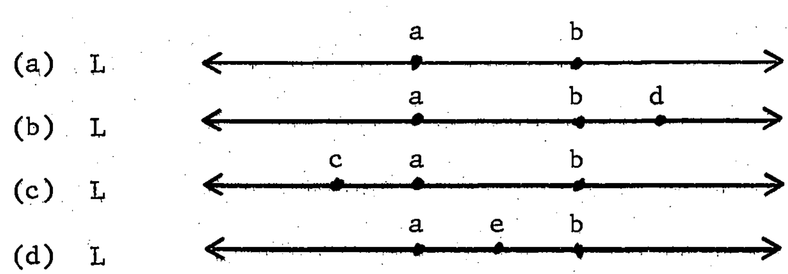


Figure 23

Notice Figure 23 (d). There is no difficulty in finding a place for point e so that e is between a and b. Is it possible for a and b to be so "close" together that there is no place for point e so that e is between a and b? Intuition may suggest that if a is taken "next to" b then there would be no place for point e. However, intuition is not always dependable. It is possible (but not appropriate here) to prove that the point a cannot be "next to" the point b and thus there is always a place for a point between a and b. This is formally stated in the third part of Postulate 4-3.

Postulate 4-3. If a and b are two points, then:

1. There is a point d so that a-b-d,
2. There is a point c so that c-a-b, and
3. There is a point e so that a-e-b.

Segments

Let a and b be any two points. By Postulate 3-3 there is exactly one line $\overleftrightarrow{ab}$ that contains both points. If c is any other point in $\overleftrightarrow{ab}$ then (1) a-c-b, (2) c-a-b or (3) a-b-c. Thus some of the points of $\overleftrightarrow{ab}$ are between a and b and some are not.

Definition 4-1. The set consisting of the points a and b and all of the points between a and b is called a segment.

The points a and b are called the endpoints of the segment and the points between a and b are called interior points of the segment. The segment whose endpoints are a and b is denoted "$\overline{ab}$", and the interior of segment $\overline{ab}$ is denoted I $\overline{ab}$. Since the set of points between a and b is the same set as the set of points between b and a, it follows that I $\overline{ab}$ = I $\overline{ba}$ and $\overline{ab} = \overline{ba}$. Since definitions are reversible the following four conditional statements are implied by the definition of segment.

1. If p is an interior point of the segment $\overline{ab}$, then p is between a and b.

   Symbolically: If $p \in I\ \overline{ab}$, then a-p-b.

2. If p is not between a and b, then p is not an interior point of segment $\overline{ab}$.

   Symbolically: If not a-p-b, then $p \notin I\ \overline{ab}$.

3. If p is between a and b, then p is an interior point of segment $\overline{ab}$.

   Symbolically: If a-p-b, then $p \in I\ \overline{ab}$.

4. If p is not an interior point of segment $\overline{ab}$, then p is not between a and b.

   Symbolically: If $p \notin I$ ab, then not a-p-b.

The definition of "segment" and Postulate 4-1 provides a basis for a proof of the following theorem.

<u>Theorem 4-1</u>. The segment $\overline{ab}$ is a subset of the line $\overleftrightarrow{ab}$.

According to the definition of "subset," it is necessary to prove that every point of the segment $\overline{ab}$ is a point of the line $\overleftrightarrow{ab}$.

Proof:

1. Let p be any point of segment $\overline{ab}$.
2. Then $p = a$, $p = b$ or $p \in I\ \overline{ab}$.
3. If $p = a$ or $p = b$, then $p \in \overleftrightarrow{ab}$.
4. If $p \in I\ \overline{ab}$, then p is between a and b.
5. Therefore by Postulate 4-1: p, a and b are collinear and $p \in \overleftrightarrow{ab}$.
6. Thus every point of segment $\overline{ab}$ is a point of line $\overleftrightarrow{ab}$.

Is $\overline{ab}$ a proper subset of $\overleftrightarrow{ab}$? That is, are there points in line $\overleftrightarrow{ab}$

that are not in segment $\overline{ab}$? Recall from Postulate 4-3 that if a and b are any two points then there is a point d such that a-b-d. Then $d \in \overleftrightarrow{ab}$, but $d \notin \overline{ab}$; thus there are points in line $\overleftrightarrow{ab}$ that are not in segment $\overline{ab}$. Therefore the segment $\overline{ab}$ is a proper subset of the line $\overleftrightarrow{ab}$. In fact every pair of points in a line L determines a segment in L. In Figure 24 the points a, b, c and d in L determine segments $\overline{ab}$, $\overline{ac}$, $\overline{ad}$, $\overline{bc}$, $\overline{bd}$, and $\overline{cd}$.

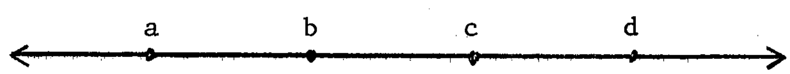


Figure 24

Rays

Consider a line L and a point a and L. (See Figure 25.) Recall that the line L extends indefinitely. Thus there is an unlimited number of points in the line on either side of a. The set of points of L on one side of a together with the point a is called a ray and the point a is called the endpoint of the ray. Since there is a set of points on either side of a it is apparent that the point a in the line L determines two rays on L. Each of these rays has the point a as its endpoint. Some means of identifying a particular one of these rays is needed. Let b and c be points in L so that a is between b and c. (See Figure 26.) Consider the two rays in L determined by the point a. The point b is in one of the rays and the point c is in the other. This suggests the possibility of determining a particular ray in terms of its endpoint and one other point of the ray. That is, there is just one ray in line L with endpoint a that contains the point b and just one ray in L with endpoint a that contains point c. If, as in Figure 26, a is between b and c these two rays will be different. The symbol $\overrightarrow{ab}$, which is read ray ab, denotes the ray with endpoint a that contains b, and the symbol $\overrightarrow{ac}$ denotes the ray with endpoint a that contains c. The rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are called opposite rays.

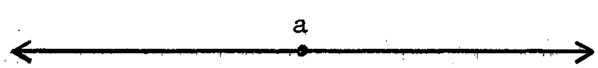


Figure 25

Figure 26

Apparently each point in the line L in Figure 26 belongs to one of the rays $\overrightarrow{ab}$ or $\overrightarrow{ac}$. If a particular point in L is considered, how does one determine which ray it is in? A functional definition of the concept "ray" is needed. Since a ray is a set of points, it will be defined in terms of the points that it contains. Consider the ray $\overrightarrow{ab}$ in line L in Figure 27. The point q is in $\overrightarrow{ab}$ and also in segment $\overline{ab}$. The points labeled p_1, p_2, and p_3 are also in $\overrightarrow{ab}$. Notice that b is between a and p_1, b is between a and p_2 and b is between a and p_3. The point r is not in $\overrightarrow{ab}$. Note that r is not in segment $\overline{ab}$ and that b is not between a and r.

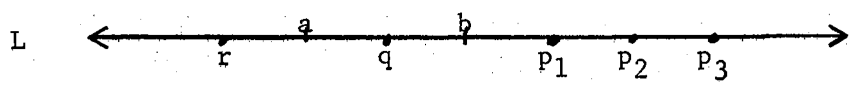


Figure 27

<u>Definition 4-2</u>. The ray $\overrightarrow{ab}$ is the union of the segment $\overline{ab}$ and the set of all point p such that b is between a and p.

Another way of describing a ray is as follows: Consider a line L and points a, b and c in L so that a is between b and c. Think of the point a as separating the line L into two parts. One part would be the set of all points in L that are on the same side of a that b is on. The other part is the set of all points in L that are on the same side of a that c is on. Notice that the point a is not in either set. Each of the sets described is called a half-line. In Figure 28 the point a is circled to indicate that it is not in either half-line. The half-line on the b side of a is called the half-line determined by a that contains b. The other is called the half-line determined by a that contains c. The ray $\overrightarrow{ab}$ is the union of the point a and the half-line determined by a that contains b.

The ray $\overrightarrow{ab}$ contains many points other than a and b but only one endpoint, namely a. Thus there are many ways of symbolizing a particular ray with endpoint a but each symbol must contain the letter a. In Figure 29, b, c and d are all on the same side of a and hence the symbols $\overrightarrow{ab}$, $\overrightarrow{ac}$ and $\overrightarrow{ad}$ all symbolize the same ray.

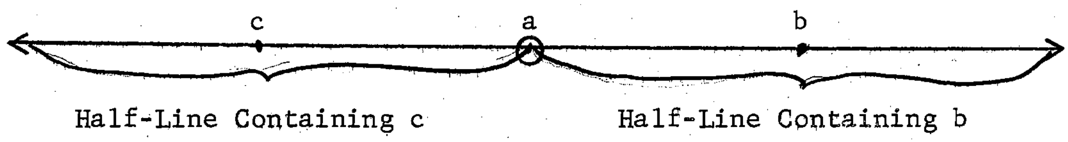


Figure 28

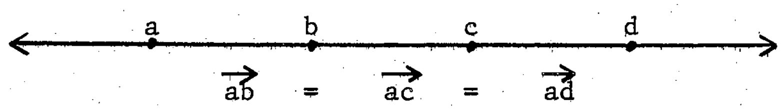


Figure 29

Angles

If two rays lie in the same line their union will be the line or a ray or two disjoint rays. Their intersection will be a point, segment, ray or the empty set. Thus no new types of point sets occur as a result of taking the union or intersection of two rays in the same line.

The most important situation arising from the union of two rays occurs when the rays have a common endpoint but do not lie in the same line. Let a, b and c be three non-collinear points as in Figure 30. Consider the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ having the common endpoint a. Since a, b and c are non-collinear, the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are not in the same line. The set of points consisting of the union of these two rays is called an angle.

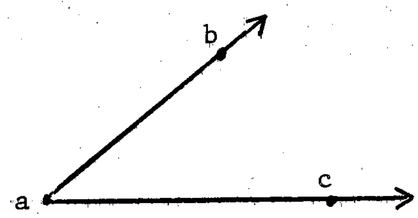


Figure 30

Definition 4-3. An angle is the union of two non-collinear rays having the same endpoint.

The symbol "$\measuredangle$" is used to denote angle. If $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are non-collinear rays then the angle formed by $\overrightarrow{ab} \cup \overrightarrow{ac}$ is symbolized $\measuredangle$ bac or $\measuredangle$ cab. The three letters used to symbolize the angle are the three letters used to denote the two rays involved. The middle letter in the symbol will always be the common endpoint of the two rays. The first letter will be a point in one of the rays (either one) other than the endpoint, while the third letter will be a point in the other ray. The two rays whose union makes the angle are called the sides of the angle and their common endpoint is called the vertex of the angle. In Figure 31, the sides of $\measuredangle$ yxz are $\overrightarrow{xy}$ and $\overrightarrow{xz}$ and the vertex is the point x. As previously noted there are many ways of symbolizing a particular ray.

Thus in Figure 31, $\overrightarrow{xy} = \overrightarrow{xa}$ and $\overrightarrow{xz} = \overrightarrow{xb}$. But $\overrightarrow{xy} \cup \overrightarrow{xz} = \measuredangle\, yxz$ so $\overrightarrow{xa} \cup \overrightarrow{xb} = \measuredangle\, yxz$. By definition, $\overrightarrow{xa} \cup \overrightarrow{xb} = \measuredangle\, axb$, and therefore $\measuredangle\, yxz = \measuredangle\, axb$. Thus there are many ways to symbolize a particular angle but the vertex must always appear as the middle letter in each symbol. In Figure 31, $\measuredangle\, axb = \measuredangle\, yxb = \measuredangle\, axz = \measuredangle\, yxz$.

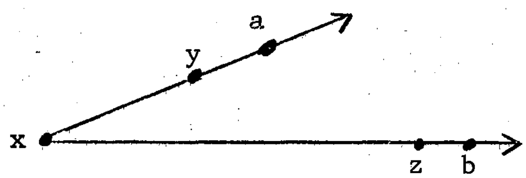


Figure 31

If $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are two rays whose union is an angle, then according to the definition of angle, ab and ac lie in different lines. Thus the lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{ac}$ are different lines that intersect in the point a. According to Theorem 3-2 two intersecting lines determine a plane. Since the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are subsets of the lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{ac}$ it follows that $\overrightarrow{ab} \cup \overrightarrow{ac}$ is a subset of the unique plane that is determined by the lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{ac}$. Thus an angle is a subset of exactly one plane.

Considerable confusion arises concerning the points of a plane that are actually a part of a particular angle and the points of the plane that are not part of the angle. In Figure 32 one may be inclined to say that the point p is "in" angle abc. However, $\measuredangle\, abc$ is a set of points and hence if p is in $\measuredangle\, abc$, it must be one of the points of the set. Yet p is not in $\overrightarrow{ba}$ or $\overrightarrow{bc}$ and therefore p is not an element of $\overrightarrow{ba} \cup \overrightarrow{bc}$. Thus it is incorrect to state that p is in $\measuredangle$ bac.

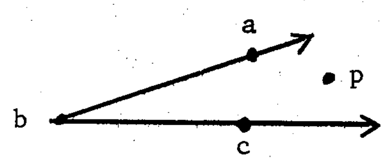


Figure 32

In viewing a model of an angle in a plane as in Figure 33, three sets of points are in evidence. First the angle itself, second the points in the interior of the angle and third the remaining points in the plane, called the exterior of the angle. It is a simple matter to indicate a particular point in the plane and state which of the three sets that the point is in. But how does one describe these sets in mathematical language? The angle has been defined and thus an adequate description is available. Some notion of separation of the points in a plane will provide a means of describing the interior and exterior of an angle.

Separation in the Plane

Consider a line L in a plane M and let a be a point in L. Let b be a point in M such that b is not in L. By Postulate 4-3 there is a point c such that a is between b and c. Since a is between b and c, by Postulate 4-1, a, b and c are collinear. The point b was not in L so the line $\overleftrightarrow{ab}$ is different from L and therefore intersects L in only one point, namely point a. Since c is in line $\overleftrightarrow{ab}$ it follows that c is not in line L. The situation is pictured in Figure 34.

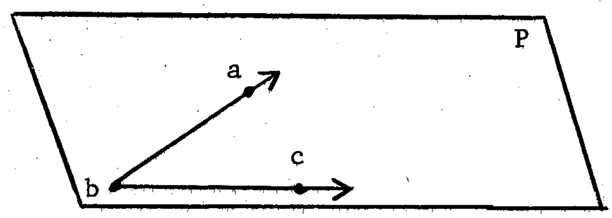


Figure 33

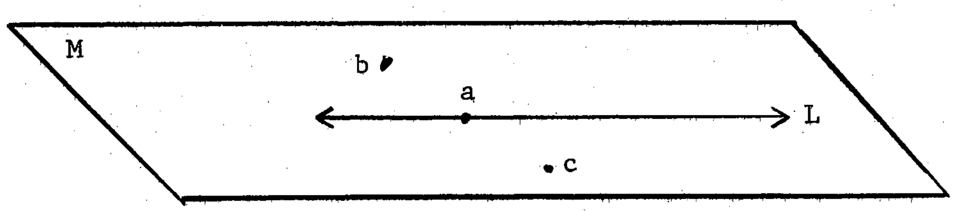


Figure 34

Since the line $\overleftrightarrow{bc}$ intersects the line L in a point a which is between b and c, b and c are said to be on opposite sides of L. Thus the plane M is separated into three subsets. One subset is the set of all points that are on the same side of L that b is on. Another is the set of points that are on the same side of L that c is on. The third subset is the line L. The set of points in the c-side of L is called the half-plane determined by L that contains c. The set of points in the b-side of L is called the half-plane determined by L that contains b. The line L is said to separate the plane M into two half-planes. If c and b are points on opposite sides of L, the half-plane that contains c is called the c-side of L. The half-plane that contains b is called the b-side of L.

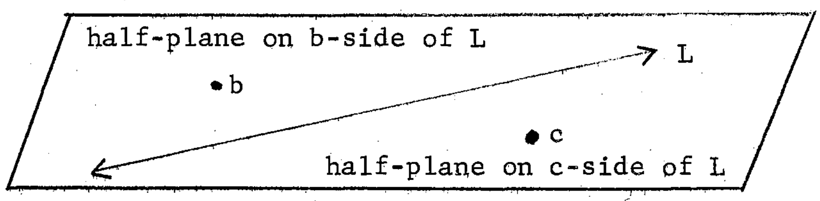


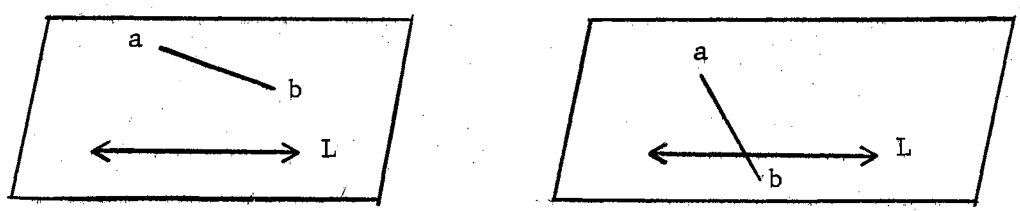
Figure 35

Postulate 4-3. Let L be a line in a plane M. If a and b are two points of M such that a and b are not in L, then a and b are on the same side of L if and only if $\overline{ab} \cap L = \{\}$.

The following four conditional statements may be extracted from the if and only if statement of Postulate 4-3.

1. If $\overline{ab} \cap L = \{\}$, then a and b are on the same side of L.
2. If a and b are on the same side of L, then $\overline{ab} \cap L = \{\}$.
3. If a and b are on opposite sides of L, then $\overline{ab} \cap L \neq \{\}$.
4. If $\overline{ab} \cap L \neq \{\}$, then a and b are on opposite sides of L.

The concept of a line separating a plane provides a means of indicating the positions of points relative to certain lines. This in turn provides a basis for stating precisely the conditions that will guarantee that a particular point is in the interior of a particular angle.



a and b on same side of L   a and b on opposite sides of L

Figure 36

Interior and Exterior of an Angle

Consider an $\measuredangle abc$ in a plane. (See Figure 37.) From the definition of "angle" points a, b and c are non-collinear and $\measuredangle abc = \overrightarrow{ba} \cup \overrightarrow{bc}$. The ray $\overrightarrow{ba}$ is a subset of the line $\overleftrightarrow{ba}$ and the ray $\overrightarrow{bc}$ is a subset of the line $\overleftrightarrow{bc}$. Since a, b and c are noncollinear, $\overleftrightarrow{ba}$ and $\overleftrightarrow{bc}$ are different lines. Thus c is not in $\overleftrightarrow{ba}$ and a is not in $\overleftrightarrow{bc}$. Consequently $\overleftrightarrow{ba}$ determines two half-planes one of which contains the point c. This half-plane, the c-side of $\overleftrightarrow{ab}$, is indicated in Figure 38 by the horizontal shadings. In a similar manner the line $\overleftrightarrow{bc}$ determines two half-planes, one of which contains the point a. The a-side of $\overleftrightarrow{bc}$ is illustrated in Figure 39 by the vertical shadings.

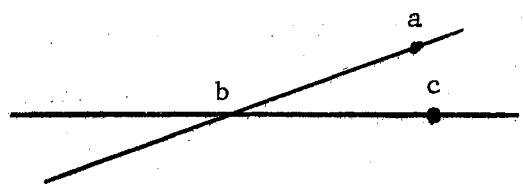


Figure 37

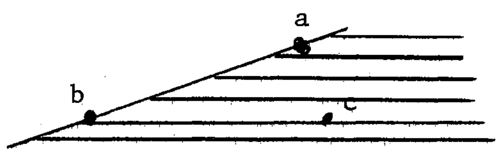


Figure 38

Figure 39

The union of the half-planes in Figures 38 and 39 is shown in Figure 40. The section of Figure 40 that contains both vertical and horizontal shadings is that subset of the plane which is common to both the a-side of $\overleftrightarrow{bc}$ and the c-side of $\overleftrightarrow{ab}$. Notice that any point that is in the a-side of $\overleftrightarrow{bc}$ and also in the c-side of $\overleftrightarrow{ab}$ is a point that, intuitively speaking, is in the interior of $\measuredangle$ abc. This suggests the following definition of the interior of an angle.

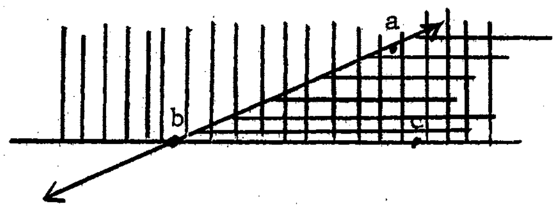


Figure 40

Definition 4-4. A point p is an element of the interior of $\measuredangle$ abc if and only if p is in the a-side of $\overleftrightarrow{bc}$ and p is in the c-side of $\overleftrightarrow{ab}$. (See Figure 41.)

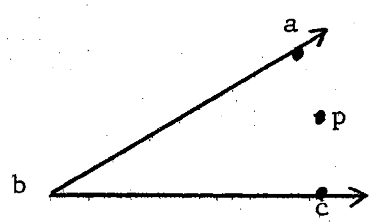


Figure 41

Recall that the angle $\measuredangle$ abc is the union of the rays $\overrightarrow{ba}$ and $\overrightarrow{bc}$. Thus any point that is in $\measuredangle$ abc must be a point in $\overrightarrow{ba}$ or $\overrightarrow{bc}$. According to the definition of interior of an angle, any point of the interior of $\measuredangle$ abc must be in a half-plane determined by $\overleftrightarrow{ba}$ and also in a half-plane determined by $\overleftrightarrow{bc}$. But a half-plane and the line determining it have no points in common. Consequently if a point q is in the interior of $\measuredangle$ abc, the q is not in $\overleftrightarrow{ba}$ and q is not in $\overleftrightarrow{bc}$. Therefore q is not in $\measuredangle$ abc. If a point p is in $\measuredangle$ abc then p is in $\overrightarrow{ba}$ or p is in $\overrightarrow{bc}$. If p is in $\overrightarrow{ba}$, then p is not in any half-plane determined by $\overleftrightarrow{ba}$ and therefore it is not in the interior of $\measuredangle$ abc. If p is not in $\overrightarrow{ba}$ but p is in $\measuredangle$ abc, then p must be in $\overrightarrow{bc}$. In this case p is not in any half-plane determined by $\overleftrightarrow{bc}$ and so p is not in the interior of $\measuredangle$ abc. Thus an angle and its interior are disjoint subsets of the plane.

Definition 4-5. If $\measuredangle$ abc is a subset of a plane M, then the set consisting of all points of M that are not in $\measuredangle$ abc or the interior of $\measuredangle$ abc is called the exterior of $\measuredangle$ abc.

Thus every angle of a plane separates the plane into three disjoint subsets, the interior of the angle, the exterior of the angle and the angle.

Convex Sets

According to Postulate 4-3, if a and b lie on the same side of a line L, then the segment $\overline{ab}$ does not intersect L. To say that a and b lie on the same side of L is equivalent to saying that a and b lie in the same half-plane. Thus a half-plane is a set of points such that if a and b are any two points in the set, then the segment $\overline{ab}$ is also in the set.

Definition 4-6. A set of points S is said to be a convex set if and only if for every two points a and b of S the segment $\overline{ab}$ is also in S.

If L is a line and a and b are points in L, then $\overline{ab}$ is a subset of L. Therefore a line is a convex set. Other examples of convex sets are rays, planes and half-planes. An angle is not a convex set for if p and q are points such that p is in one of the sides of the angle and q is in the other side, then the interior of segment $\overline{pq}$ is not a subset of the angle. (See Figure 42.)

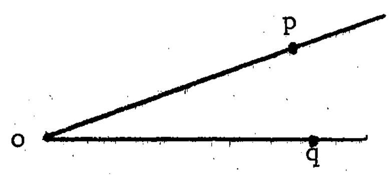


Figure 42

<u>Theorem 4-2</u>. If A is a convex set and B is a convex set, then $A \cap B$ is a convex set.

Proof: According to the definition of "convex set," it must be established that if p and q are any two points in the set $A \cap B$, then the segment $\overline{pq}$ is also in $A \cap B$.

1. Let p and q be any two points in $A \cap B$.
2. By the definition of intersection, $p \in A$, $q \in A$, $p \in B$ and $q \in B$.
3. Since A is convex, $p \in A$ and $q \in A$, then $\overline{pq} \subset A$.
4. Since B is convex, $p \in B$ and $q \in B$, then $\overline{pq} \subset B$.
5. Again by the definition of intersection, since $\overline{pq} \subset A$ and $\overline{pq} \subset B$, then $\overline{pq} \subset A \cap B$.
6. Therefore if p and q are any two points of $A \cap B$, then $\overline{pq}$ is a

subset of $A \cap B$.

7. Hence $A \cap B$ is a convex set.

Notice that steps 3 and 4 in the proof are a result of the hypothesis that A and B are both convex sets. The following examples show that if A and B are not convex, then $A \cap B$ may or may not be convex.

Example 1. The intersection of $\measuredangle$ abc and $\measuredangle$ adc is a set consisting of the two points a and c. The set $\{a,c\}$ is not convex. (See Figure 43.)

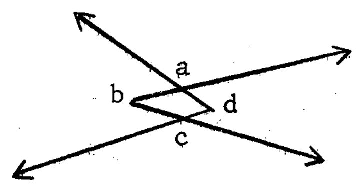


Figure 43

Example 2. The intersection of $\measuredangle$ abc and $\measuredangle$ dbc as shown in Figure 44 is the ray $\overrightarrow{bc}$ and the ray $\overrightarrow{bc}$ is a convex set.

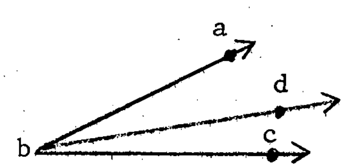


Figure 44

Theorem 4-3. The interior of an angle is a convex set.

Proof: This theorem is a direct consequence of Theorem 4-2 and the definition of the interior of an angle. Half-planes are convex sets and the interior of an angle is the intersection of two half-planes. Therefore by Theorem 4-2 the interior of an angle is a convex set.

Simple Closed Curves

The word "curve" like the words point and line is a common term. From an intuitive point of view, a curve is a continuous path. Of course this would not suffice for a definition since the words continuous and path have not been defined. Intuition plays an important role in the study of mathematics but, as was previously suggested, intuition is not always reliable. To illustrate, consider Figures 45 and 46. Intuition might lead one to refer to Figure 45 as a curved line and Figure 46 as a straight line. But this terminology is not consistent with the concepts that have been previously developed. If Figure 45 represents a line (curved or otherwise), then Figure 47 represents two lines both of which contain points a and b. According to Postulate 3-3, Chapter III, there is only one line that contains points a and b. Therefore Figure 45 does not represent a line. This discussion suggests that intuition can be misleading, but it should not be abandoned. Indeed the idea of a curve will be considered on an intuitive basis only since a precise definition would depend on concepts that are not appropriate here.

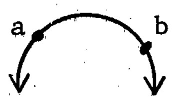


Figure 45

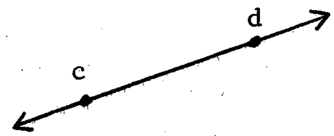


Figure 46

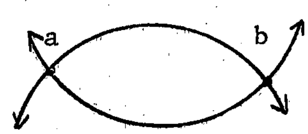
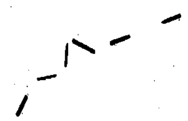


Figure 47

The sets of points represented by Figures 46 and 47 are both curves. As noted in Chapter III, a set of points like Figure 46 is a line. This set is now referred to as a curve. This implies that all lines are curves. Segments, rays and angles are also referred to as curves. The set of points in Figure 45 is a curve which is neither a line, a segment, a ray or an angle. Planes and half-planes are examples of sets of points that are not curves. The point sets represented in Figure 48 and Figure 49 are not curves; whereas the point set in Figure 50 is a curve. In what ways does the set of points in Figure 50 differ from those in Figures 48 and 49? Notice that it is possible to move a pencil from any point in Figure 50 to any other point in the set without removing the pencil from the paper. This is not possible with the point set in Figure 48. One might say intuitively speaking that the point set in Figure 48 has gaps in it. The set of points in

Figure 50 has no gaps in it. The set of points in Figure 49 has no gaps but it has a characteristic which is commonly called "thickness."



(Not a curve)

Figure 48

(Not a curve)

Figure 49

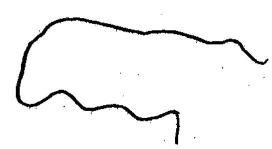


Figure 50

These observations suggest the following description of a curve. This is a description only, the term "curve" is to be considered as an undefined term.

A curve is described as any set of points which has the following properties:

1. In any physical representation of a curve it is possible to trace a pencil from any point in the set to any other point in the set without removing the pencil from the set.
2. A curve has no thickness.
3. A curve contains more than one point.

If it is possible to trace a curve in a way such that the pencil ultimately returns to its original position without retracing its route then the curve is a closed curve. If it is possible to trace a curve in a way such that the pencil ultimately returns to its original point and not trace the same point twice, then the curve is a simple closed curve. Figure 51 represents a simple closed curve. The curve in Figure 52 is closed but not simple. Figure 53 represents a simple closed curve.

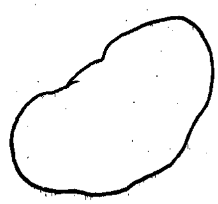


Figure 51

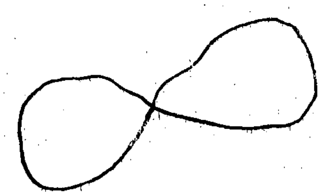


Figure 52

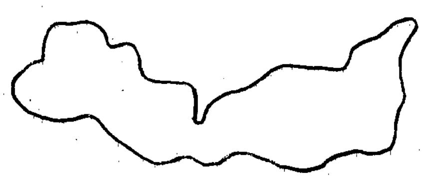


Figure 53

Polygons

While there are many forms of simple closed curves, only two types will be considered in this discourse. One of these is the circle which will be considered in a later chapter. Of immediate interest is a form of the simple closed curve which consists of the union of segments.

Definition 4-7. A polygon is a simple closed curve which is the union of segments.

Note that the definition of "polygon" requires that it be a simple closed curve and that it is the union of segments. Thus the simple closed curve illustrated in Figure 53 is not a polygon since it is not the union of segments. Figure 54 illustrates a simple curve which is the union of segments, but it is not closed and therefore, is not a polygon.

Figure 54

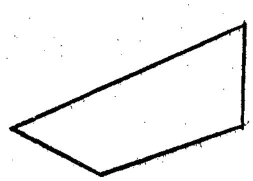


Figure 55

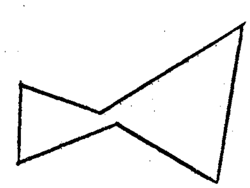


Figure 56

The simple closed curves illustrated in Figures 55 and 56 are both polygons.

The segments whose union constitutes a polygon will be called the sides of the polygon. If two sides of a polygon have a point in common, then they will be called adjacent sides and the common point will be called a vertex. The vertices (plural for vertex) of a polygon are endpoints of segments. Each vertex will be named by the common endpoint of the two segments that determine it. Two vertices that are in the same segment will be referred to as consecutive vertices. The polygon illustrated in Figure 57 is the set of points consisting of $\overline{ab} \cup \overline{bc} \cup \overline{cd} \cup \overline{da}$. The verticex are a, b, c and d. All of the vertices of a polygon are used in naming the polygon. For convenience the letters naming consecutive vertices will be listed consecutively in naming the polygon. Thus the polygon in Figure 57 could be named dcba or dabc if d is chosen as the first vertex to be named. Any vertex could be named first and in each case there are two distinct orders for listing the other vertices. Thus there are 2n ways of naming a polygon having n sides. With this notation it would be incorrect to refer to Figure 57 as the polygon acbd since this suggests that the segments $\overline{ac}$ and $\overline{bd}$ are sides of the polygon.

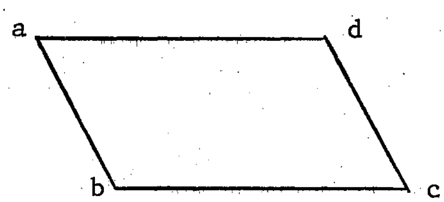


Figure 57

A particular polygon is classified according to the number of sides that it has. The prefix "poly" is a Greek form meaning many. If a polygon has more than four sides the Greek form for the particular number of sides involved may be substituted for the form "poly" to indicate the number of sides that the polygon has. Thus a pentagon is a polygon having five sides, a hexagon is a polygon having six sides, et cetera. The term "quadrilateral" is used to denote a polygon having four sides and the term "triangle" denotes a polygon having three sides. The polygons most frequently encountered in elementary geometry are the quadrilaterals and the triangles.

The study of the point sets introduced in this chapter is facilitated by the congruence relation. This concept will be considered in the next chapter.

CHAPTER V

CONGRUENCE

Much of the success of modern industry may be attributed to the interchangeability of components. A defective part of a Swiss made watch is readily replaceable in a local jewelry store. A similar situation exists regarding most of the mechanical devices used by modern society. This situation is a result of manufacturers producing large quantities of items that are alike in size and shape.

Experiences with physical objects as suggested above provide an intuitive basis for considering sets of points that are alike in size and shape. In the study of geometry the word congruent is used to describe the relation between sets of points that are alike in size and shape. The reader will recall that the equals relation discussed in Chapter III applied only to different names for the same set. If A and B are point sets such that A = B, then A and B are the same set and therefore are alike in every possible way. Hence they are congruent. The congruence relation is broader in that it describes a relation that may exist between sets that are not equal.

Congruent Segments

Initial consideration of congruent sets of points will be focused upon segments. Intuitively speaking "congruent segments" are segments that are alike in size and shape. This is to be considered only as a description. "Congruence" as applied to segments will be considered an undefined term. All segments have the same shape, thus two segments are congruent if and only if they are the same size. Hence the problem is determining whether or not they are the same size.

Consider the segments $\overline{ab}$ and $\overline{cd}$ in Figure 58. How does one determine if $\overline{ab}$ is the same size as $\overline{cd}$? Remember that $\overline{ab}$ and $\overline{cd}$ are point sets and points do not move. Thus it is not possible to place $\overline{ab}$ over $\overline{cd}$ or $\overline{cd}$ over $\overline{ab}$ to see if they are the same size. However, it is possible to take some convenient model of a segment (commonly called a straightedge) and make a copy of $\overline{ab}$ on the model. This may be accomplished by placing the straightedge alongside $\overline{ab}$ and making points m and n on the straightedge to correspond to the points a and b. The model may then be moved alongside $\overline{cd}$ so that the point m is on point c. If the point n of the model falls on the point d then the segments $\overline{ab}$ and $\overline{cd}$ are the same size and thus are congruent. The symbol "$\cong$" is used to symbolize "is congruent to." Thus the statement, "$\overline{ab}$ is congruent to $\overline{cd}$" is symbolized "$\overline{ab} \cong \overline{cd}$."

Figure 58

Since the procedure described above involves a physical operation, only approximate accuracy may be expected. Therefore, it could not serve as a basis for asserting with certainty that a given pair of segments are congruent. At best, the process justifies the assertion that two segments appear to be congruent. Indeed, nothing in the foregoing discussion guarantees that congruent segments even exist. The existence of congruent segments is assured by the following postulate.

<u>Postulate 5-1</u>. If $\overline{ab}$ is any segment and $\overrightarrow{cd}$ is any ray, then there exists exactly one point p in $\overrightarrow{cd}$ such that the segment $\overline{cp}$ is congruent to the segment $\overline{ab}$.

The phrase "is congruent to" when applied to two segments indicates that the segments are related in some way and thus is a relation on segments. Since the term "congruent" is undefined, this is an undefined relation. The next postulate assigns three useful properties to the congruence relation.

<u>Postulate 5-2</u>. For all segments,

(a) $\overline{ab} \cong \overline{ab}$.

(b) If $\overline{ab} \cong \overline{cd}$ then $\overline{cd} \cong \overline{ab}$.

(c) If $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{pq}$ then $\overline{ab} \cong \overline{pq}$.

Statements (a), (b) and (c) of Postulate 5-2 are respectively the reflexive, symmetric and transitive properties of an equivalence relation. Thus the relation "congruence" on segments is an equivalence relation. Since for any segment $\overline{ab}$, $\overline{ab} = \overline{ba}$, it follows from (a) that $\overline{ab} \cong \overline{ba}$. Also if $\overline{ab} \cong \overline{cd}$ then $\overline{ab} \cong \overline{dc}$ since $\overline{cd} \cong \overline{dc}$.

The reader should note that the congruence relation is not the same as the equals relation. Consider the segment $\overline{ab}$ and the ray $\overrightarrow{cd}$ in Figure 59. According to Postulate 5-1 there is a point p in $\overrightarrow{cd}$ such that $\overline{ab} \cong \overline{cp}$. $\overline{ab}$ and $\overrightarrow{cd}$ are disjoint sets and hence $\overline{ab}$ and $\overline{cp}$ are disjoint. But two sets are equal only if they contain the same elements. Thus $\overline{ab} \neq \overline{cp}$.

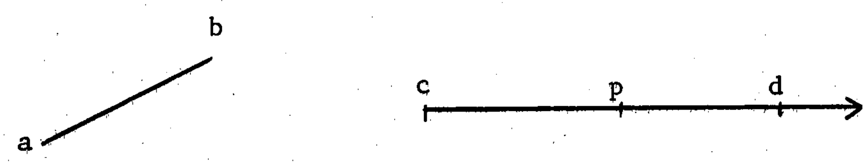


Figure 59

Congruent Angles

The undefined relation "congruence" on segments provides a basis for formulating a definition of congruent angles.

<u>Definition 5-1</u>. Let $\measuredangle$ abc and $\measuredangle$ mno be two given angles. Let p be the point in ray $\overrightarrow{nm}$ such that $\overline{np} \cong \overline{ba}$, and q be the point in ray $\overrightarrow{no}$ such that $\overline{nq} \cong \overline{bc}$. Then $\measuredangle$ mno $\cong$ $\measuredangle$ abc if and only if $\overline{pq} \cong \overline{ac}$. (See Figure 60.)

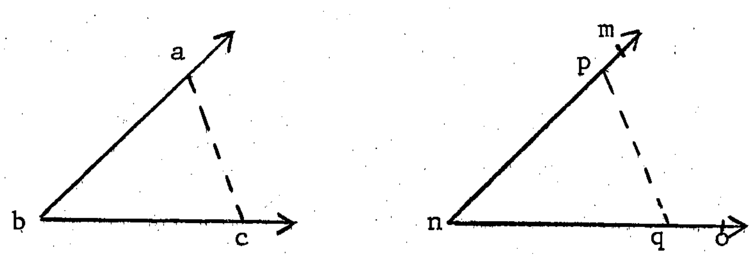


Figure 60

Note that this definition involves three pairs of congruent segments, namely $\overline{nq} \cong \overline{bc}$, $\overline{np} \cong \overline{ba}$ and $\overline{pq} \cong \overline{ac}$. Postulate 5-1 guarantees the existence of a point p in $\overrightarrow{nm}$ such that $\overline{np} \cong \overline{ba}$ and a point q in $\overrightarrow{no}$ such that $\overline{nq} \cong \overline{bc}$. There is, however, no assurance that the segment $\overline{pq}$ determined by these points is congruent to $\overline{ac}$. Consequently previous developments do not provide for the existence of congruent angles.

<u>Postulate 5-3</u>. Let $\measuredangle$ abc be any angle and L be any line in a plane M. Let H be one of the half-planes in M determined by L. If $\overrightarrow{pq}$ is any ray in L, then there exists exactly one ray $\overrightarrow{pr}$ with r in H such that $\measuredangle$ rpq $\cong$ $\measuredangle$ abc.

Postulate 5-2 states that congruence is an equivalence relation on segments. This postulate along with the definition of congruent angles makes it possible to prove that the congruence relation is an equivalence relation on angles.

<u>Theorem 5-1</u>. For all angles,

(a) $\measuredangle$ abc $\cong$ $\measuredangle$ abc.

(b) If $\measuredangle$ abc $\cong$ $\measuredangle$ mno, then $\measuredangle$ mno $\cong$ $\measuredangle$ abc.

(c) If $\measuredangle$ abc $\cong$ $\measuredangle$ mno and $\measuredangle$ mno $\cong$ $\measuredangle$ pqr, then $\measuredangle$ abc $\cong$ $\measuredangle$ pqr.

Proof:

(a) 1. By Postulate 5-2(a), $\overline{ba} \cong \overline{ba}$, $\overline{bc} \cong \overline{bc}$ and $\overline{ac} \cong \overline{ac}$.

2. Therefore by the definition of congruent angles, $\measuredangle$ abc $\cong$ abc.

(b) 1. $\measuredangle$ abc $\cong$ $\measuredangle$ mno by hypothesis. (See Figure 61.)

2. By Postulate 5-1 there exists $p \epsilon \overrightarrow{ba}$ such that $\overline{bp} \cong \overline{nm}$ and $q \epsilon \overrightarrow{bc}$ such that $\overline{bq} \cong \overline{no}$.

3. Since $p \epsilon \overrightarrow{ba}$ and $q \epsilon \overrightarrow{bc}$, $\measuredangle$ pbq = $\measuredangle$ abc.

4. Therefore by substitution in step 1, $\measuredangle$ pqb $\cong$ $\measuredangle$ mno.

5. Thus, $\overline{bp} \cong \overline{nm}$, $\overline{bq} \cong \overline{no}$ and $\measuredangle$ pbq = $\measuredangle$ mno; therefore by the definition of congruent angles $\overline{pq} \cong \overline{mo}$.

6. By Postulate 5-2(b), $\overline{nm} \cong \overline{bp}$, $\overline{no} \cong \overline{bq}$ and $\overline{mo} \cong \overline{pq}$.

7. Therefore, by definition of congruent angles $\sphericalangle$ mno $\cong$ $\sphericalangle$ pbq.

8. Since $\sphericalangle$ abc = $\sphericalangle$ pbq, one may substitute $\sphericalangle$ abc for $\sphericalangle$ pbq in step 7 and obtain $\sphericalangle$ mno $\cong$ $\sphericalangle$ abc as was to be proved.

The proof of part (c) is similar and will be omitted.

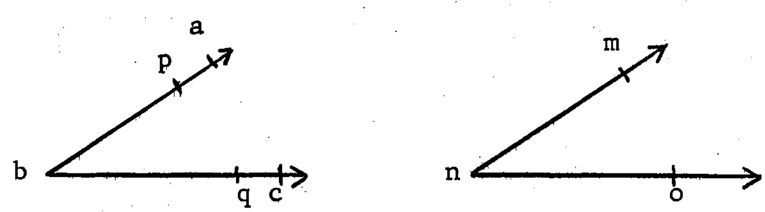


Figure 61

Triangles

Definition 5-2. If a, b and c are three noncollinear points then $ab \cup bc \cup ca$ is a triangle. (symbolized Δ abc)

Thus the triangle bac is a polygon having three sides, namely $\overline{ab}$, $\overline{bc}$ anc $\overline{ca}$. The segments $\overline{ac}$ and $\overline{ab}$ are subsets of rays $\overline{ar}$ and $\overline{aq}$ respectively. (See Figure 62.) These rays have a common endpoint a. Let b be a point on $\overrightarrow{aq}$ and c be a point on $\overrightarrow{ar}$, then $\overrightarrow{ab} \cup \overrightarrow{ac}$ = $\sphericalangle$ bac. Since the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are determined by the sides $\overline{ab}$ and $\overline{ac}$, $\sphericalangle$ bac is determined by the triangle and hence is said to be an angle of Δ abc. This designation is somewhat misleading in that it suggests that $\sphericalangle$ bac is a subset of Δ abc. Let p be a point on $\overrightarrow{ac}$ such that c is between a and p. Then p ϵ $\overline{ac}$, p ϵ $\overline{ab}$ and p ϵ $\overline{bc}$. Thus p ϵ $\sphericalangle$ bac but p $\not\epsilon$ Δ abc.

Therefore ∡ bac is not a subset of △ abc. Thus an angle of a triangle is not a subset of the triangle. The two angles, ∡ abc and ∡ bca, are also angles of △ abc. Since each angle determines a vertex, a triangle has three vertices. Thus a triangle has three sides, three angles and three vertices.

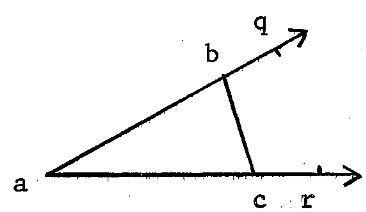


Figure 62

Congruent Triangles

The congruence relation between two triangles will be defined such that it will involve a correspondence between certain parts of the two triangles. In this correspondence comparable parts will be paired, that is, a particular side of one triangle with a specified side of the other and a particular angle in one with a specific angle of the other. The parts that are paired will be referred to as corresponding parts. The order in which the vertices are listed in naming the triangles will determine the particular correspondence to be considered. The symbol "↔" will mean "corresponds to." For the correspondence, △ abc ↔ △mno, the vertices a and b of △ abc determine the segment $\overline{ab}$ while the vertices m and n of $\triangle$ mno determine the segment $\overline{mn}$ (see Figure 63). The segments $\overline{ab}$ and $\overline{mn}$ will be designated as corresponding parts. The vertices a and c of $\triangle$ abc determine segment $\overline{ac}$ and vertices m and o of $\triangle$ mno determine segment $\overline{mo}$. Segments $\overline{ac}$ and $\overline{mo}$ are corresponding parts. Similarly $\overline{bc}$ and $\overline{no}$ are corresponding parts. The correspondence between the angles will be determined by the position of the vertices in naming the triangles. Again in the correspondence $\triangle$ abc $\leftrightarrow$ $\triangle$ mno the angle with vertex a corresponds to the angle with vertex m, the angle with vertex b corresponds to the angle with vertex n and the angle with vertex c corresponds to the angle with vertex o.

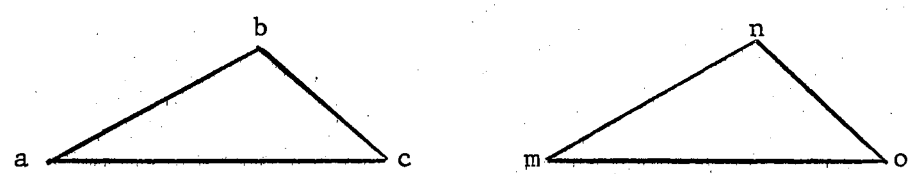


Figure 63

The correspondence $\triangle$ abc $\leftrightarrow$ $\triangle$ mno implies the set of correspondences listed below.

$\overline{ab} \leftrightarrow \overline{mn}$	$\measuredangle abc \leftrightarrow \measuredangle mno$
$\overline{ac} \leftrightarrow \overline{mo}$	$\measuredangle bca \leftrightarrow \measuredangle nom$
$\overline{bc} \leftrightarrow \overline{no}$	$\measuredangle cab \leftrightarrow \measuredangle omn$

Any correspondence between $\triangle$ abc and $\triangle$ mno which implies this set of corresponding parts will be said to be equivalent to the correspondence $\triangle$ abc $\leftrightarrow$ $\triangle$ mno. Thus the correspondence $\triangle$ bca $\leftrightarrow$ $\triangle$ nom is equivalent to $\triangle$ abc $\leftrightarrow$ $\triangle$ mno. The two correspondences $\triangle$ abc $\leftrightarrow$ $\triangle$ mno and $\triangle$ abc $\leftrightarrow$ $\triangle$ nom are not equivalent since in the former $\overline{ab} \leftrightarrow \overline{mn}$ and in the latter $\overline{ab} \leftrightarrow \overline{no}$. Invariably when symbolizing a congruence between two triangles a particular correspondence will be implied. Thus the congruence $\triangle$ abc $\cong$ $\triangle$ mno implies the correspondence $\triangle$ abc $\leftrightarrow$ $\triangle$ mno.

<u>Definition 5-3</u>. $\triangle$ abc $\cong$ $\triangle$ mno if and only if $\overline{ab} \cong \overline{mn}$, $\overline{bc} \cong \overline{no}$ and $\overline{ac} \cong \overline{mo}$.

<u>Theorem 5-2</u>. If $\triangle$ abc $\cong$ $\triangle$ mno then $\measuredangle$ abc $\cong$ $\measuredangle$ mno, $\measuredangle$ bca $\cong$ $\measuredangle$ nom and $\measuredangle$ cab $\cong$ $\measuredangle$ omn.

Proof. In Figure 64 the marks indicate the parts that are known to be congruent.

Figure 64

1. By Definition 5-2, $\overline{ab} \cong \overline{mn}$, $\overline{bc} \cong \overline{no}$ and $\overline{ac} \cong \overline{mo}$.
2. Therefore c is the point on ray $\overrightarrow{ac}$ such that $\overline{ac} \cong \overline{mo}$, and b is the point on ray $\overrightarrow{ab}$ such that $\overline{ab} \cong \overline{mn}$.
3. But $\overline{bc} \cong \overline{no}$, therefore $\measuredangle$ cab $\cong$ $\measuredangle$ omn.

The proof for the other two angles is identical except for notation and will be omitted.

<u>Theorem 5-3</u>. The congruence relation on triangles is an equivalence relation, that is, for all triangles,

(a) $\triangle abc \cong \triangle abc$

(b) If $\triangle abc \cong \triangle mno$, then $\triangle mno \cong \triangle abc$

(c) If $\triangle abc \cong \triangle mno$ and $\triangle mno \cong \triangle pqr$, then $\triangle abc \cong \triangle pqr$.

Proof: The proof for part (c) will be given; parts (a) and (b) are similar and will be omitted. Refer to Figure 65.

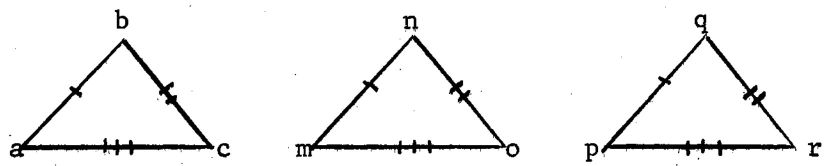


Figure 65

1. Since $\triangle abc \cong \triangle mno$, Definition 5-2 implies that $\overline{ab} \cong \overline{mn}$, $\overline{bc} \cong \overline{no}$ and $\overline{ac} \cong \overline{mo}$; and
2. Since $\triangle mno \cong \triangle pqr$, Definition 5-2 implies that $\overline{mn} \cong \overline{pq}$, $\overline{no} \cong \overline{qr}$ and $\overline{mo} \cong \overline{pr}$.
3. By Postulate 5-2 (c), $\overline{ab} \cong \overline{mn}$ and $\overline{mn} \cong \overline{pq}$ implies that $\underline{\overline{ab} \cong \overline{pq}}$; $\overline{bc} \cong \overline{no}$ and $\overline{no} \cong \overline{qr}$ implies that $\underline{\overline{bc} \cong \overline{qr}}$; $\overline{ac} \cong \overline{mo}$ and $\overline{mo} \cong \overline{pr}$ implies that $\underline{\overline{ac} \cong \overline{pr}}$.
4. Thus $\overline{ab} \cong \overline{pq}$, $\overline{bc} \cong \overline{qr}$ and $\overline{ac} \cong \overline{pr}$; hence by Definition 5-2, $\triangle abc \cong \triangle pqr$.

Definition 5-3 gives the only criteria thus far available for establishing a congruence between two triangles. That is, two triangles are congruent under a particular correspondence if the three pairs of corresponding sides are congruent. Are there other sets of conditions that are sufficient to establish a congruence between two triangles? The answer is, yes, and each of the next two theorems provides such a set of conditions.

<u>Theorem 5-4</u>. If, for the correspondence $\triangle abc \leftrightarrow \triangle mno$, $\overline{ab} \cong \overline{mn}$, $\overline{ac} \cong \overline{mo}$ and $\measuredangle bac \cong \measuredangle nmo$, then $\triangle abc \cong \triangle mno$.

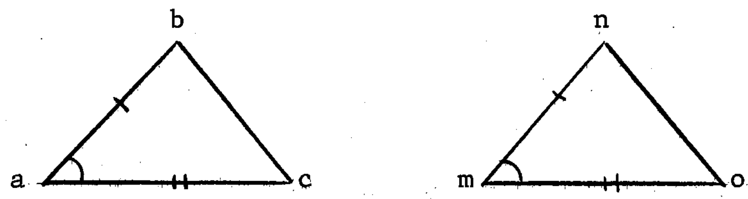


Figure 66

Proof: Refer to Figure 66.

1. By hypothesis, $\overline{ac} \cong \overline{mo}$ and $\overline{ab} \cong \overline{mn}$.
2. Thus c is the point on ray $\overrightarrow{ac}$ such that $\overline{ac} \cong \overline{mo}$, and b is the point on ray $\overrightarrow{ab}$ such that $\overline{ab} \cong \overline{mn}$.
3. But by hypothesis $\measuredangle bac \cong \measuredangle nmo$.
4. Therefore from Definition 5-1, $\overline{bc} \cong \overline{no}$.
5. Thus, for the correspondence $\triangle abc \leftrightarrow \triangle mno$, $\overline{ab} \cong \overline{mn}$, $\overline{ac} \cong \overline{mo}$ and $\overline{bc} \cong \overline{no}$; therefore $\triangle abc \cong \triangle mno$ by Definition 5-2.

For a $\triangle abc$, the side $\overline{ab}$ is a subset of one ray of $\measuredangle bac$ while the side of $\overline{ac}$ is a subset of the other ray of $\measuredangle bac$. Thus the sides $\overline{ab}$ and $\overline{ac}$ determine $\measuredangle bac$ which is often referred to as the included angle relative to these two sides. With this terminology the above theorem may be stated, "If for a particular correspondence two sides and the included angle of one triangle are congruent respectively to the corresponding two sides and included angle of another triangle, then the triangles are congruent under the indicated correspondence." This is often referred to as the side-angle-side theorem or more briefly S.A.S.

Theorem 5-5. If for the correspondence $\triangle abc \leftrightarrow \triangle mno$, $\angle bac \cong \angle nmo$, $\overline{ac} \cong \overline{mo}$ and $\angle bca \cong \angle nom$, then $\triangle abc \cong \triangle mno$.

Proof: Refer to Figure 67. This proof is quite different from any previously encountered and perhaps more difficult. Observe that side $\overline{mo}$ of $\triangle mno$ lies in a line L which in turn determines two half-planes. The ray $\overrightarrow{mn}$ lies in one of these half-planes, which will be called H. Since $\overline{mo} \cong \overline{ac}$ and $\angle nmo \cong \angle bac$, if $\overline{mn}$ was congruent to $\overline{ab}$ then the S.A.S. theorem would apply and $\triangle abc$ would be congruent to $\triangle mno$. Thus if it could be established that $\overline{mn} \cong \overline{ab}$, the theorem could be proved. Of course it isn't known that $\overline{mn} \cong \overline{ab}$, but by Postulate 5-1 there is exactly one point $p \in \overline{mn}$ such that $\overline{mp} \cong \overline{ab}$. The problem is to show that p and n are the same point.

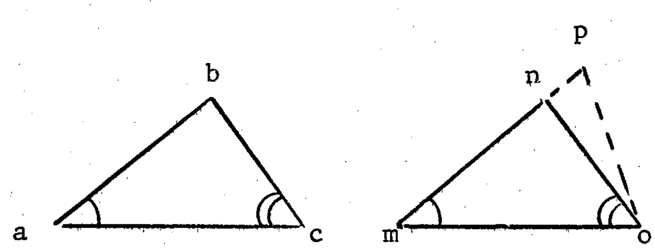


Figure 67

In Figure 67 let L be the line that contains $\overline{mo}$ and let H be the half-plane determined by L than contains n.

1. $\measuredangle$ nmo $\cong$ $\measuredangle$ bac, $\overline{mo} \cong \overline{ac}$ and $\measuredangle$ mon $\cong$ $\measuredangle$ acb by hypothesis.
2. Let p be the point on $\overrightarrow{mn}$ such that $\overline{mp} \cong \overline{ab}$.
3. Since p ϵ $\overrightarrow{mn}$, $\overrightarrow{mp} = \overrightarrow{mn}$ and $\measuredangle$ pmo = $\measuredangle$ nmo.
4. But $\measuredangle$ nmo $\cong$ $\measuredangle$ bac, so by substitution $\measuredangle$ pmo $\cong$ $\measuredangle$ nmo.
5. Thus for Δ mpo and Δ abc, $\overline{mp} \cong \overline{ab}$, $\measuredangle$ pmo $\cong$ $\measuredangle$ bac and $\overline{mo} \cong \overline{ac}$.
6. Therefore, by Theorem 5-4, Δ mpo $\cong$ Δ abc.
7. This implies that $\measuredangle$ mop $\cong$ $\measuredangle$ acb and from (1) $\measuredangle$ mon $\cong$ $\measuredangle$ acb.
8. Thus $\overrightarrow{on}$ is a ray in H such that $\measuredangle$ nom $\cong$ $\measuredangle$ acb, and $\overrightarrow{op}$ is a ray in H such that $\measuredangle$ pom $\cong$ $\measuredangle$ acb.
9. According to Postulate 5-3 there is only one such ray, therefore $\overrightarrow{op} = \overrightarrow{on}$.
10. Therefore p ϵ $\overrightarrow{on}$ and by step (3) p ϵ $\overrightarrow{mn}$.
11. Hence p ϵ $\overrightarrow{on} \cap \overrightarrow{mn}$, but two rays intersect in at most one point and $\overrightarrow{on}$ and $\overrightarrow{mn}$ intersect at n.
12. Therefore p = n.
13. Hence Δ mno $\cong$ Δ mpo; but Δ mpo $\cong$ Δ abc from step (6) so Δ mno $\cong$ Δ abc.

This theorem states that if two angles and the included side of one triangle are congruent to the corresponding two angles and included side of a second triangle, then the triangles are congruent. It is referred to as the angle-side-angle theorem or A.S.A.

Congruent Polygons

Consider the polygons abcde and mnopq in Figure 68 and the correspondence abcde $\leftrightarrow$ mnopq. The segment determined by two nonconsecutive vertices of a polygon is called a diagonal. The segments $\overline{ac}$ and $\overline{ad}$ are diagonals of polygon abcde. The corresponding diagonals of polygon mnopq are $\overline{mo}$ and $\overline{mp}$. The diagonals $\overline{ac}$ and $\overline{ad}$ partition polygon abcde and its interior into three triangles and $\overline{mo}$ and $\overline{mp}$ partition polygon mnopq and its interior into three triangles. If the corresponding triangles determined by this partition are congruent then the polygons are congruent.

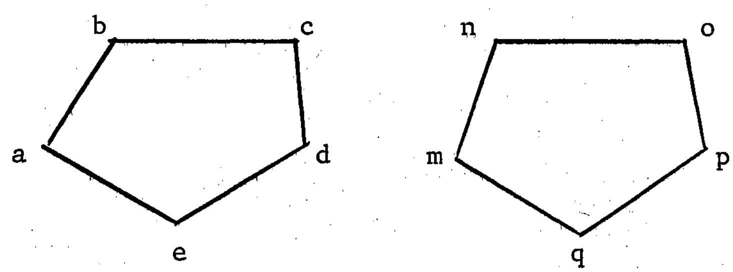


Figure 68

<u>Definition 5-4</u>. Two polygons are congruent if and only if there is a partition of the polygons into triangles such that the corresponding triangles are congruent.

The congruence relation provides a basis for developing many additional concepts in geometry. In the next chapter this relation will be used in the classification of triangles and angles.

CHAPTER VI

CLASSIFICATION OF ANGLES AND TRIANGLES

One frequently encounters references to right angles, perpendicular lines and to classes of triangles such as isosceles triangles, equilateral triangles and right triangles. These classifications are a consequence of the congruence relation and will be explored in this chapter.

Isosceles Triangles

Suppose $\measuredangle$ paq is a given angle and $\overline{mn}$ is a given segment. There is a point b on ray $\overrightarrow{ap}$ such that $\overline{ab} \cong \overline{mn}$ and there is a point c on ray $\overrightarrow{aq}$ such that $\overline{ac} \cong \overline{mn}$. Since the congruence relation is an equivalence relation it follows that $\overline{ab} \cong \overline{ac}$. The points b and c determine segment $\overline{bc}$ and $\overline{ab} \cup \overline{bc} \cup \overline{ca}$ is a triangle. (See Figure 69.) For the triangle thus determined, two of the sides are congruent segments. Any triangle that has this characteristic is called an isosceles triangle.

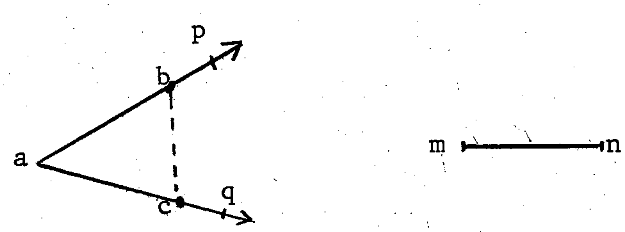


Figure 69

<u>Definition 6-1</u>: A triangle is an isosceles triangle if and only if two of its sides are congruent segments.

When referring to a triangle such as $\triangle$ abc, it is convenient for one to describe a particular angle in terms of a certain side. The $\measuredangle$ bac with vertex a is referred to as the angle opposite the side $\overline{bc}$, $\measuredangle$ abc is opposite $\overline{ac}$ and $\measuredangle$ acb is opposite $\overline{ab}$. Conversely, side $\overline{bc}$ is opposite $\measuredangle$ bac, $\overline{ac}$ is opposite $\measuredangle$ abc and $\overline{ab}$ is opposite $\measuredangle$ acb.

<u>Theorem 6-1</u>. If two sides of a triangle are congruent, then the angles opposite these sides are congruent; i.e., in $\triangle$ abc if $\overline{ab} \cong \overline{cb}$, then $\measuredangle$ cab $\cong$ $\measuredangle$ acb.

Proof: Since the congruence relation is an equivalence relation, every triangle is congruent to itself. That is, $\triangle$ abc $\cong$ $\triangle$ abc under the correspondence $\triangle$ abc $\leftrightarrow$ $\triangle$ abc. This correspondence is called the identity correspondence, every part corresponds to itself. Under certain conditions a triangle will be congruent to itself under some correspondence other than the identity correspondence. This provides a basis for a proof of Theorem 6-1. Refer to Figure 70.

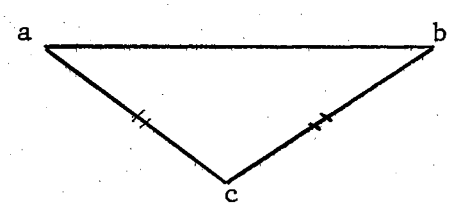


Figure 70

1. Consider the correspondence $\triangle abc \leftrightarrow \triangle cba$.

2. For this correspondence $\overline{ab} \leftrightarrow \overline{cb}$, $\measuredangle abc \leftrightarrow \measuredangle cba$ and $\overline{cb} \leftrightarrow \overline{ab}$.

3. i. $\overline{ab} \cong \overline{cb}$ by hypothesis.

   ii. $\measuredangle abc = \measuredangle cba$, hence $\measuredangle abc \cong \measuredangle cba$.

   iii. $\overline{cb} \cong \overline{ab}$ by the symmetric property of the congruence relation.

4. Therefore $\triangle abc \cong \triangle cba$ by the S.A.S. theorem.

5. For the correspondence $\triangle abc \leftrightarrow \triangle cba$, $\measuredangle cba \leftrightarrow \measuredangle acb$, and therefore $\measuredangle cab \cong \measuredangle acb$ by Theorem 5-4.

In an isosceles triangle the side opposite the angle determined by the two congruent sides is called the base. Thus in Figure 71 $\overline{ab}$ is the base. The angle opposite the base is called the vertex angle. Each of the two angles that has as its vertex a point in the base is called a base angle. In Figure 72 the base angles are $\measuredangle cab$ and $\measuredangle acb$. With this terminology, Theorem 6-1 may be stated: "The base angles of an isosceles triangle are congruent."

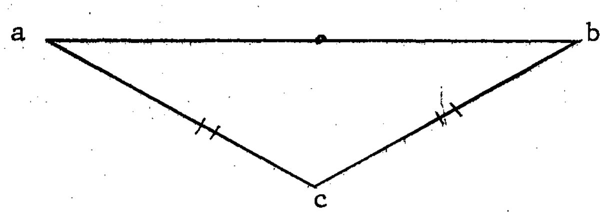


Figure 71

Theorem 6-2. If two angles of a triangle are congruent, the sides opposite these angles are congruent; i.e., in $\triangle abc$ if $\measuredangle cab \cong \measuredangle acb$, then $\overline{cb} \cong \overline{ab}$.

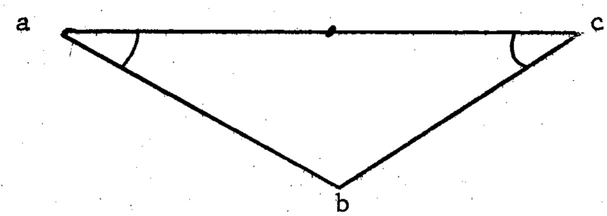


Figure 72

Proof:

1. Consider the correspondence $\triangle abc \leftrightarrow \triangle cba$.

2. Then $\measuredangle cab \leftrightarrow \measuredangle acb$, $\overline{ac} \leftrightarrow \overline{ca}$ and $\measuredangle acb \leftrightarrow \measuredangle cab$ (see Figure 73).

3. i. $\measuredangle cab \cong \measuredangle acb$ by hypothesis.

   ii. $\overline{ac} = \overline{ca}$, hence $\overline{ac} \cong \overline{ca}$.

   iii. $\measuredangle acb \cong \measuredangle cab$ by the symmetric property.

4. Therefore $\triangle abc \cong \triangle cba$ by the A.S.A. theorem.

5. Then $\overline{ab} \cong \overline{cb}$ by Definition 5-2.

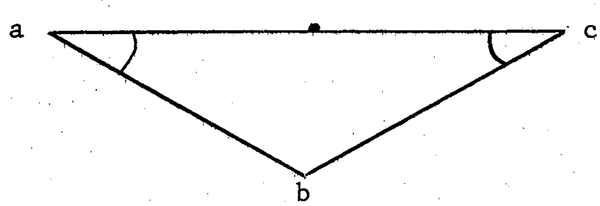


Figure 73

Equilateral Triangles

All isosceles triangles have two congruent sides. A special class of isosceles triangles is one which consists of triangles that have all three sides congruent. These are called equilateral triangles.

Definition 6-2: $\triangle$ abc is an equilateral triangle if and only if $\overline{ab} \cong \overline{bc}$ and $\overline{bc} \cong \overline{ac}$, that is if all of its sides are congruent.

Thus if $\triangle$ abc is an equilateral triangle, then any two of its sides are congruent. Consequently every equilateral triangle is also an isosceles triangle.

Theorem 6-3: If $\triangle$ abc is equilateral, then $\measuredangle$ abc $\cong$ $\measuredangle$ bac and $\measuredangle$ bac $\cong$ $\measuredangle$ bca, or $\measuredangle$ abc $\cong$ $\measuredangle$ bac $\cong$ $\measuredangle$ bca.

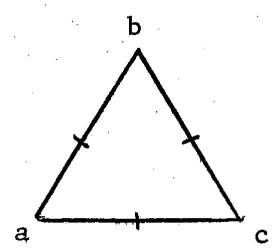


Figure 74

Proof:

1. $\overline{ac} \cong \overline{bc}$ by hypothesis, hence $\measuredangle$ abc $\cong$ $\measuredangle$ bac by Theorem 5-6.
2. $\overline{bc} \cong \overline{ba}$ by hypothesis, hence $\measuredangle$ bac $\cong$ $\measuredangle$ bca.
3. Therefore $\measuredangle$ abc $\cong$ $\measuredangle$ bac $\cong$ $\measuredangle$ bca from step (1), step (2) and the transitive property of the congruence relation.

4. Thus each of the three angles of an equilateral triangle is congruent to each of the other two.

<u>Theorem 6-4</u>: If in $\triangle$ abc, $\sphericalangle$ abc $\cong$ $\sphericalangle$ bac $\cong$ $\sphericalangle$ bca, then $\triangle$ abc is equilateral.

Proof: (Refer to Figure 75.)

1. $\sphericalangle$ abc $\cong$ $\sphericalangle$ bac by hypothesis; therefore by Theorem 5-7, $\overline{ac} \cong \overline{bc}$.
2. $\sphericalangle$ bac $\cong$ $\sphericalangle$ bca by hypothesis; hence by Theorem 5-7, $\overline{bc} \cong \overline{ba}$.
3. Therefore $\triangle$ abc is equilateral by Definition 6-2.

Not all triangles are isosceles and if a triangle is not isosceles then it is not equilateral. If a triangle is not isosceles it is called a scalene triangle. Thus triangles are classified as:

i. Isosceles if two sides are congruent segments.

ii. Equilateral if all three sides are congruent segments.

iii. Scalene if no two sides are congruent segments.

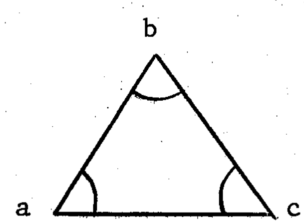


Figure 75

Midpoints

Suppose $\overline{ab}$ is a segment and L is a line as in Figure 76. Let m, p and n be points of L such that p is between m and n. Then $\overrightarrow{pn}$ and $\overrightarrow{pm}$

are opposite rays. According to Postulate 5-1, there exists a point d in $\overrightarrow{pn}$ such that $\overline{pd} \cong \overline{ab}$. Also by Postulate 5-1 there exists a point c in $\overrightarrow{pm}$ such that $\overline{pc} \cong \overline{ab}$. According to Postulate 5-2 (b), the congruence relation is symmetric; hence $\overline{pc} \cong \overline{ab}$ implies that $\overline{ab} \cong \overline{pc}$. Thus $\overline{pc} \cong \overline{ab}$ and $\overline{ab} \cong \overline{pc}$. Applying Postulate 5-2 (c) one obtains $\overline{pd} \cong \overline{pc}$. The points c and d determine the segment $\overline{cd}$. Since p is an interior point of $\overline{cd}$, $\overline{cp} \cup \overline{pd} = \overline{cd}$, and $\overline{cp} \cong \overline{pd}$; the point p is called the midpoint of cd.

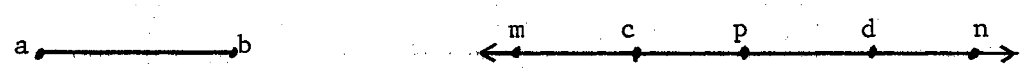


Figure 76

<u>Definition 6-3</u>: A point p is the midpoint of a segment $\overline{ab}$ if and only if p is an interior point of $\overline{ab}$ and $\overline{ap} \cong \overline{pb}$.

<u>Postulate 6-1</u>: If $\overline{ab}$ is any segment then $\overline{ab}$ has exactly one midpoint.

Interior Points of Angles and Triangles

Let $\triangle$ abc be any triangle and let p be a point in the interior of $\measuredangle$ bac. Then p could be in the interior of each of the other two angles of the triangle as in Figure 77, or p could be in the interior of $\measuredangle$ bac but not in the interior of either of the other two angles of the triangle as in Figure 78 (a) and (b). In Figure 77, p appears to be in the interior of the triangle whereas in Figure 78, p does not appear to be in the interior of the triangle.

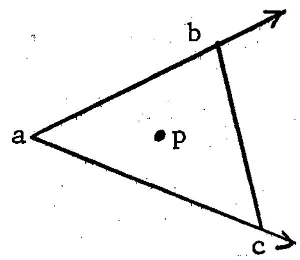


Figure 77

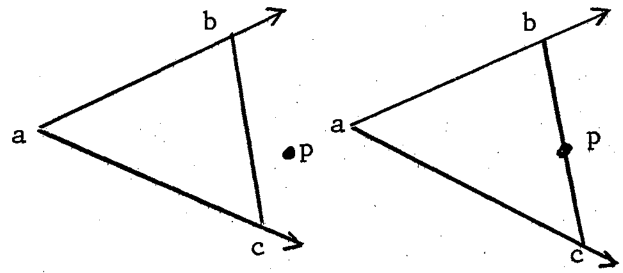


Figure 78

Definition 6-4: A point p is in the interior of $\triangle$ abc if and only if p is in the interior of each of the three angles of the triangle.

Thus a point of a triangle is not in the interior of the triangle since any point of a triangle belongs to at least one of the sides. If q is a point in the segment $\overline{ac}$ of $\triangle$ abc then q is not in the interior of either $\measuredangle$ bac or $\measuredangle$ bca. (See Figure 79.)

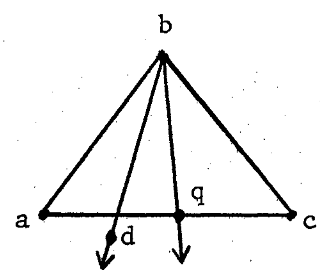


Figure 79

The point q in Figure 79 does appear to be in the interior of $\measuredangle$ abc. Further consideration of Figure 79 suggests that if d is a point in the interior of $\angle$ abc, then the ray $\overrightarrow{bd}$ is in the interior $\angle$ abc, and that any ray in the interior of $\angle$ abc will contain an interior point of side $\overline{ac}$. These properties are not logical consequences of the previous development and thus are based on intuition. They are important to further development and therefore will be postulated.

<u>Postulate 6-2</u>: Let $\triangle$ abc be any triangle and for definiteness consider $\angle$ bac, then:

1. If q is an interior point of $\overline{bc}$, then q is in the interior of $\angle$ bac.
2. If d is any point in the interior of $\angle$ bac, then every point of the ray $\overrightarrow{ad}$ except a is in the interior of $\angle$ bac. This ray is said to be in the interior of $\angle$ bac.
3. If $\overrightarrow{ao}$ is any ray in the interior of $\angle$ bac, then $\overrightarrow{ao}$ intersects the side $\overline{bc}$ in an interior point of $\overline{bc}$.

Bisectors of Angles

Let $\angle$ paq be any angle and $\overline{mn}$ be any segment. There is a point b on ray $\overrightarrow{ap}$ such that $\overline{ab} \cong \overline{mn}$ and there is a point c on ray $\overrightarrow{aq}$ such that $\overline{ac} \cong \overline{mn}$. Thus $\overline{ab} \cong \overline{ac}$ and $\overline{ab} \cup \overline{ac} \cup \overline{bc}$ is an isosceles triangle. Therefore for any given $\angle$ paq there exists an isosceles triangle having $\angle$ paq as its vertex angle.

<u>Definition 6-5</u>: The bisector of $\angle$ paq is a ray $\overrightarrow{ao}$ in the interior of $\angle$ paq such that $\angle$ pao $\cong$ $\angle$ oaq. Stated another way, the ray $\overrightarrow{ao}$ is the bisector of $\angle$ paq if and only if ray $\overrightarrow{ao}$ is in the interior of $\angle$ paq and $\angle$ pao $\cong$ $\angle$ oaq.

<u>Theorem 6-5</u>: For any $\angle$ paq, then $\angle$ paq has a unique bisector.

Proof: Theorem 6-5 is a conditional statement with a compound statement for a conclusion. That is, $\measuredangle$ paq has a bisector and the bisector is unique. Thus the proof must establish both the existence of an angle bisector and the uniqueness. The proof for the existence is given first.

1. For the given angle, $\measuredangle$ paq, let $\triangle$ abc be an isosceles triangle with $\measuredangle$ paq as its vertex angle. (See Figure 80.)

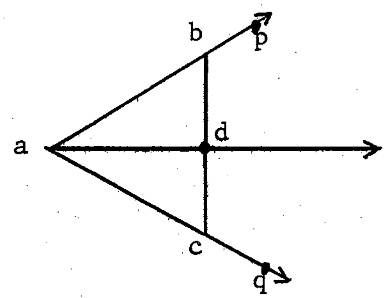


Figure 80

2. According to Postulate 6-1 there exists a midpoint d of segment $\overline{bc}$.
3. Since $d \in \overline{bc}$, by Postulate 5-5 ray $\overrightarrow{ad}$ is in the interior of $\measuredangle$ bac.
4. Points b, d and c are in $\overline{bc}$ and $a \notin \overline{bc}$; hence a, d and b are noncollinear and points a, b and c are noncollinear.
5. Therefore $\overline{ad} \cup \overline{db} \cup \overline{ab} = \triangle$ adb and $\overline{ad} \cup \overline{dc} \cup \overline{ac} = \triangle$ adc.
6. $\overline{ab} \cong \overline{ac}$ since $\triangle$ abc is isosceles.
7. $\overline{ad} \cong \overline{ad}$ by the reflexive property, and $\overline{db} \cong \overline{dc}$ since d is the midpoint of $\overline{bc}$.

8. Thus for the correspondence $\triangle adb \leftrightarrow \triangle adc$, the corresponding sides are congruent segments.
9. Therefore $\triangle adb \cong \triangle adc$ by the definition of congruent triangles.
10. Therefore $\measuredangle bad \cong \measuredangle cad$ since they are corresponding angles of congruent triangles (Theorem 5-2).

This proves the existence of an angle bisector. It remains to prove that there is only one angle bisector. The approach will be to consider any angle bisector and prove that it is the one whose existence was just established.

1. Let $\triangle abc$ be an isosceles triangle with vertex at the given angle, $\measuredangle paq$. (See Figure 81.)

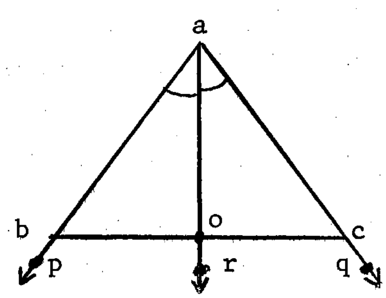


Figure 81

2. Let ray $\overrightarrow{an}$ be any angle bisector of $\measuredangle bac$, then ray $\overrightarrow{an}$ is in the interior of $\measuredangle bac$ from the definition of angle bisector.
3. By Postulate 5-5, the ray $\overrightarrow{an}$ intersects $\overline{bc}$ in an interior point, call it o.
4. Then b, o and c are distinct points on $\overline{bc}$ and $a \notin \overline{bc}$.
5. Therefore points a, o and b are noncollinear, and points a, o and

c are noncollinear.

6. Thus $\overline{ao} \cup \overline{ob} \cup \overline{ab} = \triangle$ aob and $\overline{ao} \cup \overline{oc} \cup \overline{ac} = \triangle$ aoc.

7. Since $\overrightarrow{ao}$ is an angle bisector, ∡ bao ≅ ∡ cao.

8. $\overline{ao} \cong \overline{ao}$ by the reflexive property and since $\triangle$ bac is isosceles, $\overline{ab} \cong \overline{ac}$.

9. Then for the correspondence $\triangle$ aob ↔ $\triangle$ aoc, two sides and included angle of $\triangle$ aob are congruent to the corresponding two sides and included angles of $\triangle$ aoc.

10. Therefore $\triangle$ aob ≅ $\triangle$ aoc by the S.A.S. theorem.

11. Therefore $\overline{bo} \cong \overline{oc}$ since they are corresponding sides of congruent triangles.

12. The point o is the midpoint of $\overline{bc}$ by the definition of midpoint.

13. Thus any ray that bisects ∡ paq intersects $\overline{bc}$ at its midpoint.

14. Since there is only one midpoint, it follows that there is only one angle bisector.

Supplementary Angles

Suppose $\overrightarrow{ab}$, $\overrightarrow{ac}$ and $\overrightarrow{ad}$ are distinct rays having the common endpoint a, such that the three rays determine three angles, ∡ bac, ∡ bad and ∡ cad. (See Figure 82.) All of these angles have the same vertex, namely a. The angles in {∡ bac, ∡ bad} have a common side $\overrightarrow{ba}$, the angles in {∡ bad, ∡ cad} have a common side $\overrightarrow{ad}$, and the angles in {∡ bac, ∡ cad} have a common side $\overrightarrow{ac}$. Thus there are three distinct pairs of angles, each pair having a common vertex and a common side. However, the pair of angles, ∡ bac and ∡ cad, are different from the other two pairs. Their interiors are disjoint sets. These two angles are called adjacent angles and each is said to be adjacent to the other.

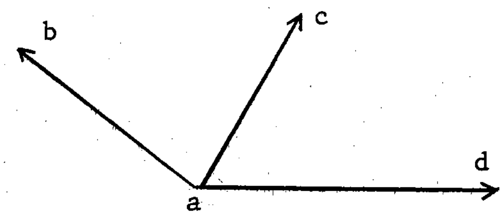


Figure 82

Definition 6-6: Two angles are adjacent angles if and only if they have a common vertex, a common side and their interiors are disjoint sets.

If a pair of angles, $\measuredangle$ abc and $\measuredangle$ cbd, are adjacent angles, it is conceivable that the sides $\overrightarrow{ba}$ and $\overrightarrow{bd}$ lie in the same line. Then since $\overrightarrow{ba}$ and $\overrightarrow{bd}$ are different collinear rays having the same endpoint, they must be opposite rays. Adjacent angles having the property that their noncommon sides lie on opposite rays are called supplementary adjacent angles and each one is said to be a supplement of the other. (See Figure 83.)

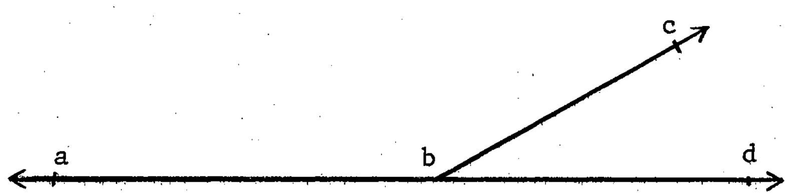


Figure 83

<u>Definition 6-7</u>: Two angles are supplementary adjacent angles if and only if (1) they are adjacent angles, and (2) their noncommon sides are opposite rays.

It is important to note that supplementary angles always occur in pairs. Each angle in the pair is a supplement of the other. Thus "supplementary" is a symmetric relation.

Definition 6-7 provides for a pair of supplementary angles only in the event that the angles are adjacent. In Figure 84 $\sphericalangle$ mno is a supplement of $\sphericalangle$ onp. If $\sphericalangle$ abc $\cong$ $\sphericalangle$ mno and $\sphericalangle$ def $\cong$ $\sphericalangle$ onp, $\sphericalangle$ abc and $\sphericalangle$ def seem to be related in a manner similar to the relation between $\sphericalangle$ mno and $\sphericalangle$ onp. However, they are not supplementary according to Definition 6-7 since they are not adjacent. It will be convenient to have a definition of supplementary angles that will include pairs of angles that are not adjacent. A basis for such a definition is provided by the definition of Supplementary Adjacent Angles.

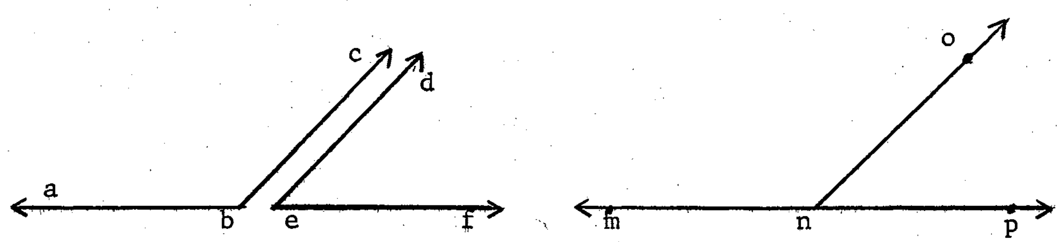


Figure 84

<u>Definition 6-8</u>: Let $\sphericalangle$ mno be any angle and let $\overrightarrow{ns}$ be the ray opposite $\overrightarrow{nm}$. Then $\sphericalangle$ abc is a supplement of $\sphericalangle$ mno if and only if $\sphericalangle$ abc $\cong$ $\sphericalangle$ ons.

That is, $\measuredangle$ abc is a supplement of $\measuredangle$ mno if and only if $\measuredangle$ abc is congruent to an angle that is adjacent to and a supplement of $\measuredangle$ mno. (See Figure 85.)

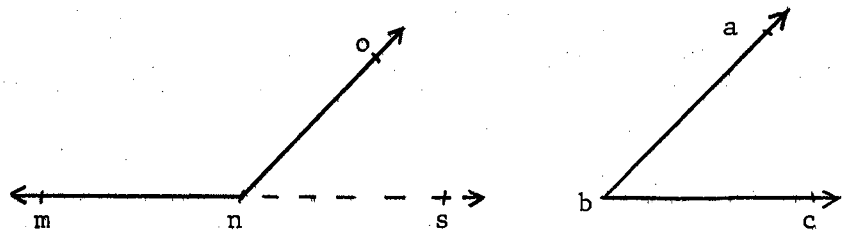


Figure 85

In Figure 86 let $\measuredangle$ mno be a given angle and $\overrightarrow{ns}$ be the ray opposite $\overrightarrow{nm}$. Hence $\measuredangle$ ons is adjacent and supplementary to $\measuredangle$ mno. Then from Definition 6-8:

(1) If $\measuredangle$ abc is any angle such that $\measuredangle$ abc $\cong$ $\measuredangle$ ons, then $\measuredangle$ abc is a supplement of $\measuredangle$ mno, that is, every angle that is congruent to $\measuredangle$ ons is a supplement of $\measuredangle$ mno.

(2) If $\measuredangle$ abc is any angle such that $\measuredangle$ abc is a supplement of $\measuredangle$ mno, then $\measuredangle$ abc $\cong$ $\measuredangle$ ons, that is, any angle that is a supplement of $\measuredangle$ mno is congruent to $\measuredangle$ ons.

The statement numbered (2) is the key to the proof of the following theorem.

<u>Theorem 6-6</u>: Let $\measuredangle$ mno be a given angle. If $\measuredangle$ abc is a supplement of $\measuredangle$ mno and $\measuredangle$ pqr is a supplement of $\measuredangle$ mno, then $\measuredangle$ abc $\cong$ $\measuredangle$ pqr. That is, supplements of the same angle are congruent.

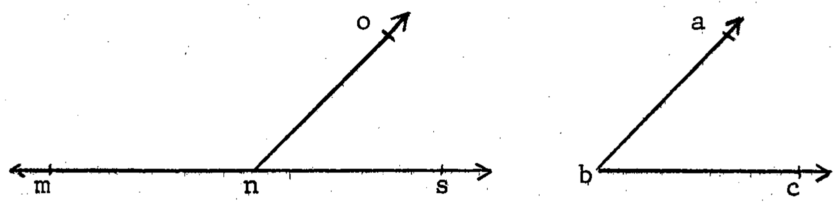


Figure 86

Proof: Refer to Figure 87.

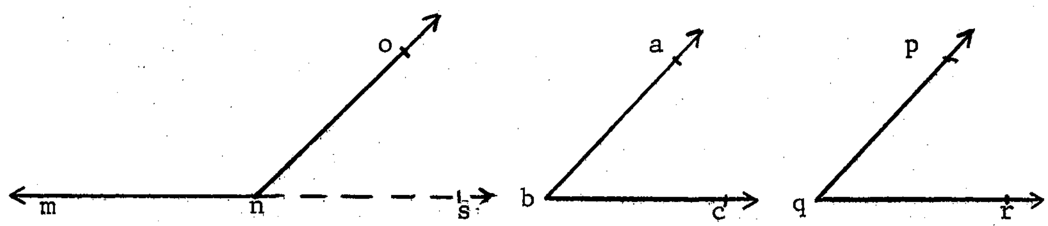


Figure 87

1. Let $\overrightarrow{ns}$ be the ray opposite $\overrightarrow{nm}$.
2. $\measuredangle$ abc is a supplement of $\measuredangle$ mno by hypothesis; hence $\measuredangle$ abc $\cong$ $\measuredangle$ ons by Definition 6-8.
3. $\measuredangle$ pqr is a supplement of $\measuredangle$ mno by hypothesis; hence $\measuredangle$ pqr $\cong$ $\measuredangle$ ons by Definition 6-8.
4. $\measuredangle$ ons $\cong$ $\measuredangle$ pqr by the symmetric property of the congruence relation.
5. Then $\measuredangle$ abc $\cong$ $\measuredangle$ ons from step (2) and $\measuredangle$ ons $\cong$ $\measuredangle$ pqr from step (5); hence $\measuredangle$ abc $\cong$ $\measuredangle$ pqr by the transitive property of the congruence

relation in angles.

This theorem implies that two angles that are supplements of the same angle are congruent. It is also possible to prove that two angles that are supplements of congruent angles are congruent.

<u>Theorem 6-7</u>: If ∡ abc ≅ ∡ pqr, and

(1) ∡ efg is a supplement of ∡ abc,

(2) ∡ mno is a supplement of ∡ pqr,

then ∡ efg ≅ ∡ mno.

Proof: Refer to Figure 88.

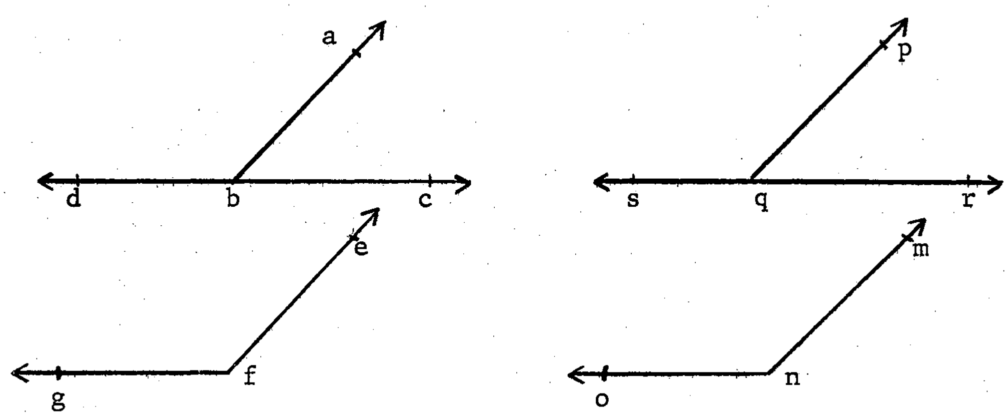


Figure 88

1. ∡ abc ≅ ∡ pqr by hypothesis.
2. ∡ efg is a supplement of ∡ abc and ∡ mno is a supplement of ∡ pqr by hypothesis.
3. Let $\overrightarrow{bd}$ be the ray opposite $\overrightarrow{bc}$.
4. Then ∡ abd is a supplement of ∡ abc.

5. Therefore $\measuredangle$ abd $\cong$ $\measuredangle$ efg by Theorem 6-6.

6. If $\overrightarrow{qs}$ is opposite $\overrightarrow{qr}$, then $\measuredangle$ pqs is a supplement of $\measuredangle$ pqr and $\measuredangle$ pqs $\cong$ $\measuredangle$ mno by Theorem 6-6.

7. Since $\measuredangle$ pqr $\cong$ $\measuredangle$ abc and $\measuredangle$ abc is a supplement of $\measuredangle$ abd, then by Definition 6-8, $\measuredangle$ pqr is a supplement of $\measuredangle$ abd.

8. Hence $\measuredangle$ abd is a supplement of $\measuredangle$ pqr.

9. But $\measuredangle$ pqs is a supplement of $\measuredangle$ pqr, hence $\measuredangle$ pqs $\cong$ $\measuredangle$ abd by Theorem 6-6.

10. From step (5) $\measuredangle$ efg $\cong$ $\measuredangle$ abd and from step (6) $\measuredangle$ mno $\cong$ $\measuredangle$ pqs.

11. Then by the transitive property $\measuredangle$ efg $\cong$ $\measuredangle$ mno.

Vertical Angles

Let L and M be two lines that intersect at point p. Let a and b be points in L such that a-p-b (p is between a and b) and let r and s be points in M such that r-p-s. Thus four angles are determined. (See Figure 89.)

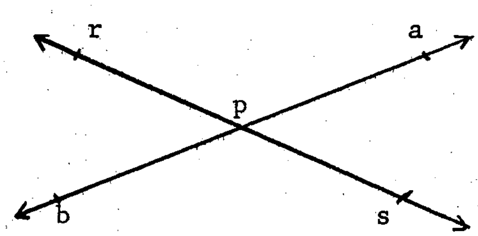


Figure 89

The angles in {$\measuredangle$ aps, $\measuredangle$ apr} have a common vertex p and a common side $\overrightarrow{pa}$. Hence, they are adjacent angles. The rays $\overrightarrow{ps}$ and $\overrightarrow{pr}$ are opposite rays, therefore $\sphericalangle$ aps and $\sphericalangle$ apr are also supplementary. Similarly, $\sphericalangle$ apr and $\sphericalangle$ rpb are supplementary adjacent angles, $\sphericalangle$ rpb and $\sphericalangle$ bps are supplementary adjacent angles and $\sphericalangle$ bps and $\sphericalangle$ spa are supplementary adjacent angles. The angles in $\{\sphericalangle \text{ aps}, \sphericalangle \text{ rpb}\}$ are not adjacent since they do not have a common side. Similarly $\sphericalangle$ apr and $\sphericalangle$ spb are not adjacent. These two pairs of angles are the pairs of nonadjacent angles determined by two intersecting lines. Each pair is called a pair of vertical angles. Note that pairs of vertical angles are always determined by two intersecting lines. Further, vertical angles have a common vertex and their sides determine pairs of opposite rays.

<u>Definition 6-9</u>: Two angles determined by two intersecting lines are vertical angles if and only if their sides determine pairs of opposite rays (see Figure 90).

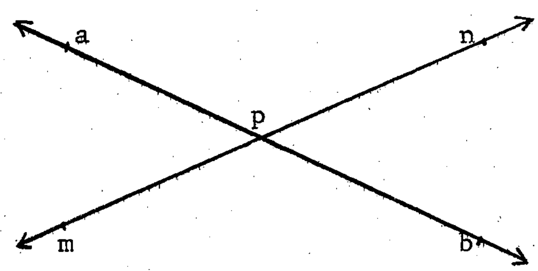


Figure 90

<u>Theorem 6-8</u>. If two angles are vertical angles, then they are congruent.

Proof: Refer to Figure 90. For definiteness suppose $\overrightarrow{pa}$ and $\overrightarrow{pb}$ are opposite rays, and $\overrightarrow{pn}$ and $\overrightarrow{pm}$ are opposite rays.

1. $\overrightarrow{pa}$, $\overrightarrow{pn}$ and $\overrightarrow{pb}$ are distinct rays since $\overleftrightarrow{ab}$ and $\overleftrightarrow{mn}$ are distinct lines.
2. Thus $\measuredangle$ npa and $\measuredangle$ npb are distinct angles having a common vertex p and a common side $\overline{pn}$.
3. The interior of $\measuredangle$ npb is a subset of the b-side of $\overleftrightarrow{mn}$, and the interior of $\measuredangle$ npa is a subset of the a-side of $\overleftrightarrow{mn}$; hence the interiors of $\measuredangle$ npb and $\measuredangle$ npa are disjoint sets.
4. $\overrightarrow{pa}$ and $\overrightarrow{pb}$ are opposite rays; hence $\measuredangle$ npb and $\measuredangle$ npa are supplementary adjacent angles.
5. Similarly, $\measuredangle$ npa and $\measuredangle$ apm are supplementary adjacent angles.
6. Thus $\measuredangle$ apm and $\measuredangle$ npb are both supplements of $\measuredangle$ npa.
7. Therefore $\measuredangle$ apm $\cong$ $\measuredangle$ npb by Theorem 6-6.

Perpendicular Lines

From the previous section, if two lines intersect in a point, then two pair of vertical angles are determined and four pair of supplementary adjacent angles are determined. In every case each pair of vertical angles is a pair of congruent angles. It is conceivable that a pair of supplementary adjacent angles are congruent angles. Suppose in Figure 91 that $\measuredangle$ apm $\cong$ $\measuredangle$ apn. Since $\measuredangle$ apn and $\measuredangle$ mpb are vertical angles, $\measuredangle$ mpb $\cong$ $\measuredangle$ apn; hence $\measuredangle$ mpb $\cong$ $\measuredangle$ apm. Similarly $\measuredangle$ mpb $\cong$ $\measuredangle$ npb and $\measuredangle$ npb $\cong$ $\measuredangle$ apn. Thus if two lines intersect such that two adjacent supplementary angles are congruent, then each of the four angles determined is congruent to each of the other three and the two lines are said to be perpendicular lines.

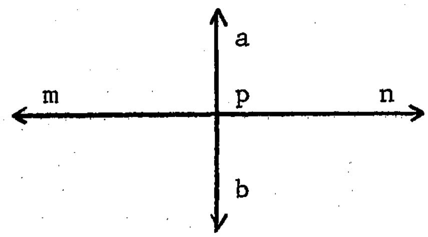


Figure 91

<u>Definition 6-10</u>. Two intersecting lines L and M are perpendicular lines if and only if the adjacent angles determined are congruent.

The symbol "$\perp$" denotes the relation "is perpendicular to." If L and M are lines and L$\perp$M, then M$\perp$L; that is, "perpendicular" is a symmetric relation. This relation is neither transitive nor reflexive and therefore is not an equivalence relation.

Two rays or two segments are perpendicular if and only if the lines containing them are perpendicular lines. Thus two rays or two segments could be perpendicular even if they are disjoint sets. In Figure 92 $\overline{ab} \perp \overline{cd}$ and $\overline{ab} \perp \overline{cd}$ if and only if $\overline{ab} \perp \overline{cd}$.

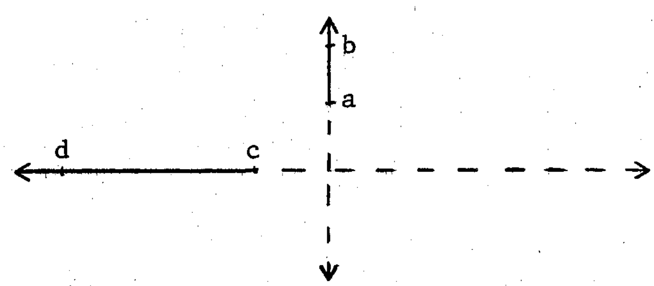


Figure 92

Right Angles

Suppose $\overleftrightarrow{ab} \perp \overleftrightarrow{cd}$ and $\overleftrightarrow{ab} \cap \overleftrightarrow{cd} = p$ (see Figure 93). Then $\overrightarrow{pc}$ is opposite $\overrightarrow{pd}$, hence $\measuredangle$ apc and $\measuredangle$ pad are supplementary adjacent angles. According to Definition 6-8 every angle that is a supplement of $\measuredangle$ apd is congruent to $\measuredangle$ apc. But $\measuredangle$ apc $\cong$ $\measuredangle$ apd by Definition 6-10, hence every angle that is congruent to $\measuredangle$ apc is congruent to $\measuredangle$ apd. Therefore, every angle that is a supplement of $\measuredangle$ apd is congruent to $\measuredangle$ apd. Stated another way, $\measuredangle$ apd is congruent to every angle that is supplementary to it. An angle that is congruent to every angle that is supplementary to it, is called a right angle. Thus $\measuredangle$ apd is a right angle. From Theorem 6-6, angles that are supplements of the same angle are congruent. Since the congruence relation is transitive, if an angle is congruent to one of its supplements it is congruent to every angle that is supplementary to it, and consequently is a right angle.

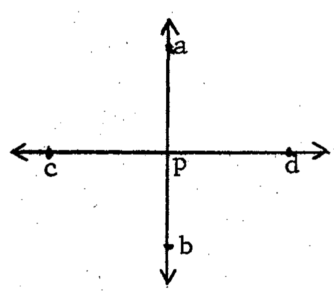


Figure 93

Definition 6-11: $\measuredangle$ abc is a right angle if and only if $\measuredangle$ abc is congruent to one of its supplements.

The above definition defines a right angle but does not guarantee that right angles exist.

<u>Theorem 6-9</u>: Right angles exist.

Proof: Refer to Figure 94.

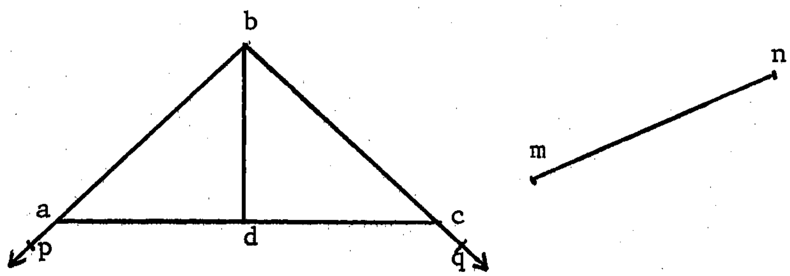


Figure 94

1. Let $\sphericalangle pbq$ be any angle and $\overline{mn}$ be any segment.
2. Let a be the point on $\overrightarrow{bp}$ such that $\overline{ba} \cong \overline{mn}$ and c be the point on $\overrightarrow{bq}$ such that $\overline{bc} \cong \overline{mn}$.
3. Then $\overline{ba} \cup \overline{ac} \cup \overline{cb} = \triangle abc$. $\overline{ba} \cong \overline{bc}$ so $\triangle abc$ is isosceles.
4. Let d be the midpoint of $\overline{ac}$, then $\overline{ad} \cong \overline{cd}$.
5. Points d and b determine $\overline{bd}$.
6. $\overline{bd} \cong \overline{bd}$, $\overline{ad} \cong \overline{cd}$ and $\overline{ba} \cong \overline{bc}$; hence $\triangle bda \cong \triangle bdc$ by definition.
7. Therefore $\sphericalangle bda \cong \sphericalangle bdc$ by Theorem 5-2.
8. $\overrightarrow{da}$ and $\overrightarrow{dc}$ are opposite rays; hence $\sphericalangle bdc$ and $\sphericalangle bda$ are adjacent supplementary angles.
9. Therefore $\sphericalangle bda$ is congruent to its supplement $\sphericalangle bdc$, and hence $\sphericalangle bda$ is a right angle.

Note that $\measuredangle$ bdc is also congruent to its supplement $\measuredangle$ bda, and thus is also a right angle. These two right angles are congruent from step (7).

Let $\measuredangle$ abc be a right angle and $\overrightarrow{bd}$ be the ray opposite $\overrightarrow{bc}$. (See Figure 95.) Then $\measuredangle$ abd is adjacent and supplementary to $\measuredangle$ abc. Since $\measuredangle$ abc is a right angle, it is congruent to every angle that is supplementary to it. Hence $\measuredangle$ abc $\cong$ $\measuredangle$ abd, and by the symmetric property $\measuredangle$ abd $\cong$ $\measuredangle$ abc. Thus $\measuredangle$ abd is congruent to one of its supplements and therefore is a right angle. This is formally stated in Theorem 6-10.

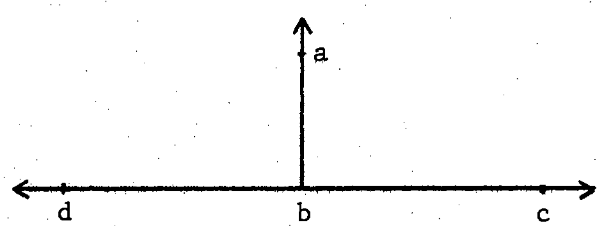


Figure 95

<u>Theorem 6-10</u>: If two adjacent angles are supplementary and one is a right angle, then the other is also a right angle and these two right angles are congruent.

Let $\measuredangle$ abc be a right angle and $\overrightarrow{bd}$ be opposite $\overrightarrow{bc}$. Let $\measuredangle$ mno be any angle that is a supplement of $\measuredangle$ abc. (See Figure 96.) Then $\measuredangle$ abc is a supplement of $\measuredangle$ mno. But $\measuredangle$ mno $\cong$ $\measuredangle$ abd since $\measuredangle$ abd is a supplement of $\measuredangle$ abc, and $\measuredangle$ abd $\cong$ $\measuredangle$ abc. Therefore $\measuredangle$ mno $\cong$ $\measuredangle$ abc. This $\measuredangle$ mno is congruent to one of its supplements, and hence is congruent to every angle that is supplementary to it. Therefore $\measuredangle$ mno is a right angle. But $\measuredangle$ mno was any supplement of the right angle, $\measuredangle$ abc. This argument proves the following theorem.

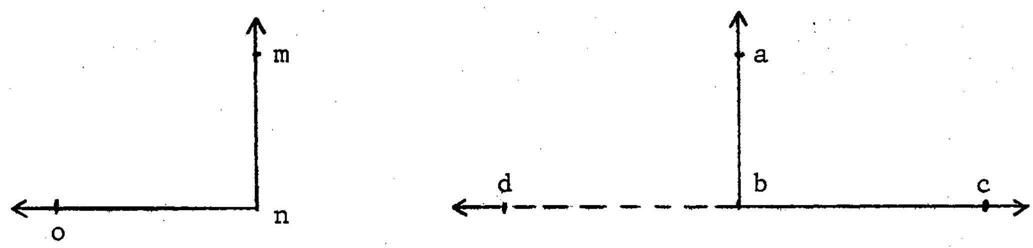


Figure 96

<u>Theorem 6-11</u>: Every angle that is a supplement of a right angle is a right angle.

Suppose $\measuredangle$ abc is any right angle and $\measuredangle$ pqr is any angle congruent to $\measuredangle$ abc. Let $\overrightarrow{bd}$ be the ray opposite $\overrightarrow{ba}$. (See Figure 97.) Then $\measuredangle$ cbd is a supplement of $\measuredangle$ abc and hence $\measuredangle$ abc $\cong$ $\measuredangle$ cbd. Since $\measuredangle$ pqr = $\measuredangle$ abc and $\measuredangle$ abc $\cong$ $\measuredangle$ cbd, it follows that $\measuredangle$ pqr $\cong$ $\measuredangle$ cbd. Therefore $\measuredangle$ pqr is a supplement of $\measuredangle$ abc, and hence $\measuredangle$ abc is a supplement of $\measuredangle$ pqr. Thus $\measuredangle$ pqr is congruent to one of its supplements. Therefore $\measuredangle$ pqr is a right angle. This proves the following theorem.

<u>Theorem 6-12</u>: Every angle that is congruent to a right angle is a right angle.

From Theorem 6-10, if two adjacent angles are supplements and one is a right angle, then the other is a right angle, and the two angles are congruent. This provides a pair of right angles that are also congruent angles. From Theorem 6-11, every supplement of a right angle is a right angle and from Theorem 6-6 all supplements of any given angle are congruent angles. This suggests that there are many right angles that are also congruent angles. In fact, intuitively it seems that all right angles are congruent. It is possible to prove that all right angles are congruent, but the proof involves concepts that have not been developed in this discourse. Consequently this property of right angles will be postulated.

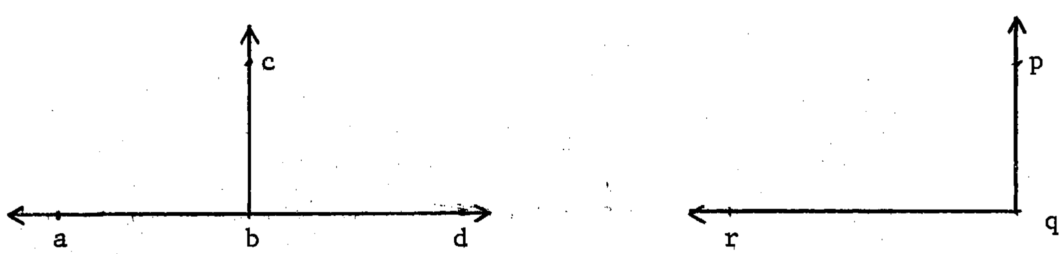


Figure 97

Postulate 6-3: All right angles are congruent.

It was previously noted that if two lines intersect, they determine adjacent supplementary angles. Also if two of these adjacent angles are congruent, then each of the four angles determined is congruent to each of the other three and the two lines are called perpendicular lines. Thus if the lines are perpendicular, then each of the four angles determined is congruent to one of its supplements and therefore is a right angle. Since all of the angles are right angles, perpendicular lines are said to intersect at right angles.

If $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are two rays such that $\overrightarrow{ab} \cup \overrightarrow{ac} = \measuredangle$ bac is a right angle and $\overrightarrow{ad}$ is opposite $\overrightarrow{ac}$, then $\measuredangle$ bad is a right angle by Theorem 6-10. If $\overrightarrow{ae}$ is opposite $\overrightarrow{ab}$, then $\measuredangle$ cae and $\measuredangle$ dae are also right angles (see Figure 98). Furthermore $\overrightarrow{ae} \cup \overrightarrow{ab} = \overleftrightarrow{be}$ and $\overrightarrow{ac} \cup \overrightarrow{ad} = \overleftrightarrow{cd}$. Thus if $\overrightarrow{ab} \cup \overrightarrow{ac}$ is a right angle then the lines containing these rays intersect so as to form congruent adjacent angles and therefore are perpendicular. Hence perpendicular lines intersect at right angles and the sides of a right angle determine perpendicular lines. This is stated formally in Theorem 6-13.

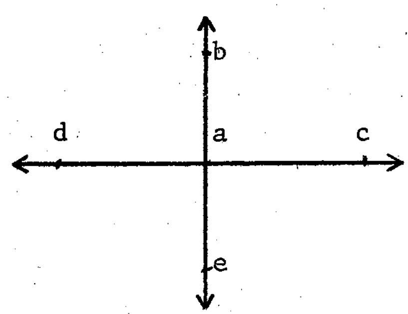


Figure 98

<u>Theorem 6-13</u>. Two lines are perpendicular if and only if the unions of noncollinear rays with endpoints at their point of intersection are right angles.

By Theorem 6-9 right angles exist. Consequently in view of Theorem 6-13, perpendicular lines exist. Furthermore if L is a line and m and p are points of L, then in one of the half-planes determined by L there is exactly one ray $\overrightarrow{pq}$ such that $\measuredangle$ mpq is a right angle.

Thus there is only one line through p that is perpendicular to L.

Theorem 6-14: If L is a given line and p is a point in L, then there exists one and only one line through p that is perpendicular to L.

Right Triangles

Every triangle has three sides and determines three angles. If one of the angles determined is a right angle, then the triangle is called a right triangle. In Figure 99 suppose $\measuredangle$ acb is a right angle. Then $\triangle$ abc is a right triangle. The side opposite the right angle ($\overline{ab}$ in this model) is called the hypotenuse. The other two sides are called legs. If the legs are congruent segments, then the triangle is isosceles, and thus is an isosceles right triangle. Like any isosceles triangle the angles opposite the congruent sides are congruent angles.

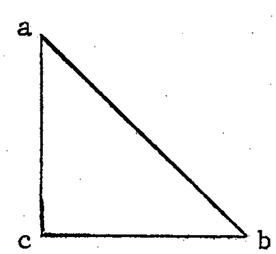


Figure 99

It was previously established that if a triangle is equilateral then each of its three determined angles is congruent to the other two. Thus if a right triangle was also equilateral, then each of its determined angles would be a right angle. Concepts developed in the next chapter will show that a triangle cannot determine more than one right angle. Therefore a right triangle could not be equilateral.

Acute Angles and Obtuse Angles

Suppose $\measuredangle$ abc is a right angle and $\overrightarrow{bd}$ is a ray such that d is in the interior of $\measuredangle$ abc. Then $\measuredangle$ abd is called an acute angle. (See Figure 100.) If $\overrightarrow{be}$ is a ray such that $\overrightarrow{bc}$ is in the interior of $\measuredangle$ abe then $\measuredangle$ abe is called an obtuse angle.

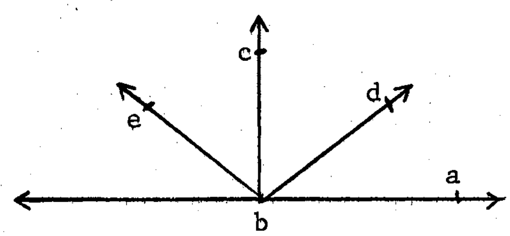


Figure 100

<u>Definition 6-12</u>: Let $\measuredangle$ abc be a right angle:

1. $\measuredangle$ abd is an acute angle if and only if $\overrightarrow{bd}$ is in the interior of $\measuredangle$ abc.
2. $\measuredangle$ abe is an obtuse angle if and only if $\overrightarrow{bc}$ is in the interior of $\measuredangle$ abe.

This terminology gives rise to a further classification of triangles. If one of the angles determined by a triangle is an obtuse angle then the triangle is called an obtuse triangle. If all of the angles determined by a triangle are acute angles then the triangle is called an acute triangle.

CHAPTER VII

PARALLELS AND QUADRILATERALS

Since lines extend indefinitely, physical models of lines do not exist. A model like Figure 101 is often useful for reference in a discussion pertaining to one or more lines. The fact that the lines L and M do not intersect in the model should not be interpreted to mean that they do not intersect at some point not shown in the model. It does suggest the possibility that they do not intersect, but this could be determined only if certain conditions are established guaranteeing that they are disjoint sets.

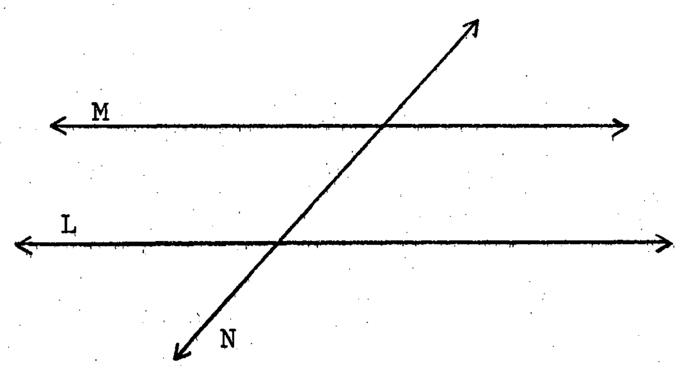


Figure 101

The word "coplanar" is used to describe sets of points that are in the same plane. In particular the sets L, M and N in Figure 101 are coplanar.

Transversals and Associated Angles

The study of parallel lines is facilitated by the introduction of certain terminology regarding pairs of lines intersected by a third line and the associated angles formed by these lines. In Figure 102 the line N intersecting the lines L and M is called a transversal of lines L and M. If L and M are any two coplanar lines and N is a third line intersecting L and M in distinct points then N is called a transversal of the lines L and M.

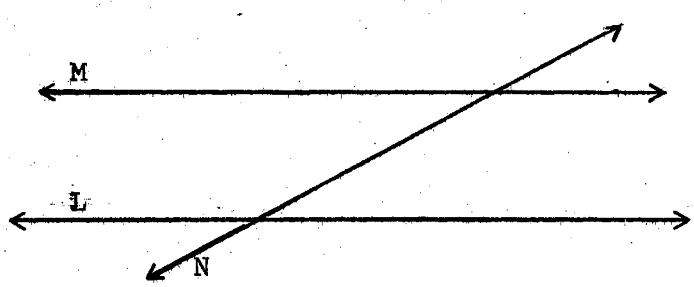


Figure 102

If L and M are distinct lines and N is a transversal intersecting L in point a and intersecting M in point b, four angles are determined with vertex b (see Figure 103). Let m and n be points in L such that m - a - n. Let c and d be points in M such that c - b - d. Consider the set $\{\measuredangle \text{ mab}, \measuredangle \text{ dba}\}$. The ray $\overrightarrow{ba}$ is a side of $\measuredangle$ dba and the ray $\overrightarrow{ab}$

is a side of $\measuredangle$ mab. The sides $\overrightarrow{am}$ of $\measuredangle$ mab and $\overrightarrow{bd}$ of $\measuredangle$ dba, are on opposite sides of the transversal N. The angles in $\{\measuredangle \text{ mab}, \measuredangle \text{ dba}\}$ are called alternate interior angles.

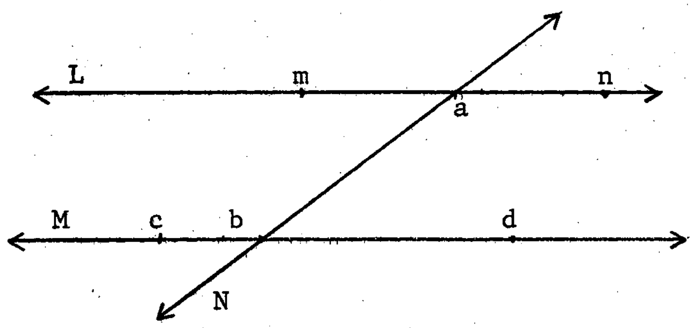


Figure 103

<u>Definition 7-1</u>: Let $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ be two coplanar lines and let $\overleftrightarrow{ab}$ be a transversal intersecting $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ in the distinct points a and b respectively (see Figure 103). Then $\{\measuredangle \text{ bam}, \measuredangle \text{ abd}\}$ is a set of alternate interior angles if and only if m and d are on opposite sides of the transversal $\overleftrightarrow{ab}$. If c - b - d and m - a - n then $\{\measuredangle \text{ nab}, \measuredangle \text{ cba}\}$ is also a set of alternate interior angles.

If m and d are on the same side of the transversal as in Figure 104, then the angles in $\{\measuredangle \text{ mab}, \measuredangle \text{ dba}\}$ are called interior angles on the same side of the transversal.

<u>Definition 7-2</u>: Let $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ be cut by a transversal $\overleftrightarrow{ab}$ such that $\measuredangle$ mab and $\measuredangle$ dba are alternate interior angles. The angles in $\{\measuredangle \text{ mab}, \measuredangle \text{ cbq}\}$ are corresponding angles if and only if c and q are

points such that the angles in {∡ cbq, ∡ dba} are vertical angles (see Figure 105).

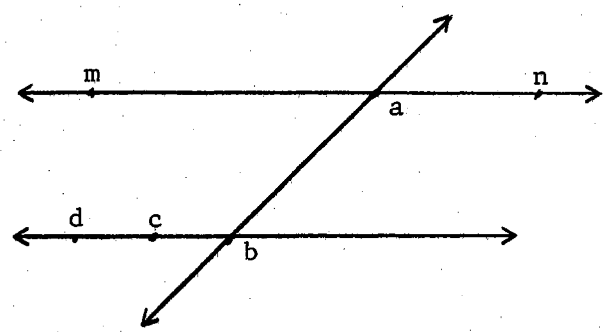


Figure 104

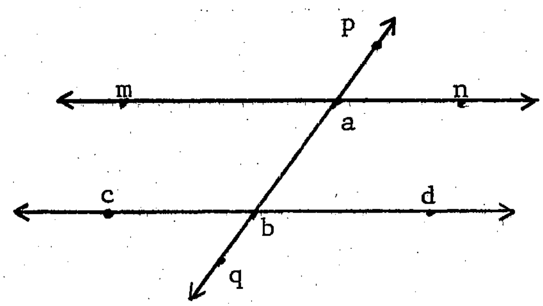


Figure 105

In Figure 105, {∡ nab, ∡ dbq} is a set of corresponding angles.

With each angle of a pair of alternate interior angles there is an associated vertical angle. These two angles considered as a pair are called alternate exterior angles. In Figure 105, ∡ pan and ∡ cbq are alternate exterior angles.

Parallel Lines

Parallel lines were defined in Chapter III. For convenience the definition will be repeated.

Definition 7-3: Two lines are parallel if and only if they are coplanar and their intersection is empty.

Let $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ be two lines cut by a transversal intersecting the two lines at points a and b such that: (1) a is between m and n, (2) b is between c and d, and (3) $\{\measuredangle mab, \measuredangle abd\}$ is a set of congruent alternate interior angles. Suppose $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ intersect at p (see Figure 106). For definiteness assume p is on the d-side of $\overleftrightarrow{ab}$. On the ray $\overleftrightarrow{bc}$ opposite $\overleftrightarrow{bd}$ let q be the point such that $\overline{bq} \cong \overline{ap}$. For the correspondence $\triangle pab \leftrightarrow \triangle qba$, $\overline{ab} \cong \overline{ba}$ and $\overline{bq} \cong \overline{ap}$. The angles in $\{\measuredangle qba, \measuredangle pab\}$ are supplements of the congruent angles in $\{\measuredangle abd, \measuredangle mab\}$; hence $\measuredangle qba \cong \measuredangle pab$ by Theorem 6-6. Therefore $\triangle pab \cong \triangle qba$ by the S.A.S. theorem. Then by Theorem 5-2 $\measuredangle qab \cong \measuredangle abp$. Since $p \in \overline{bd}$, $\measuredangle abp \cong \measuredangle abd$ and therefore $\measuredangle qab \cong \measuredangle abd$ by the transitive property. But $\measuredangle abd \cong \measuredangle mab$ by hypothesis; hence $\measuredangle qab \cong \measuredangle mab$. Thus the rays $\overrightarrow{aq}$ and $\overrightarrow{am}$ are on the same side of $\overleftrightarrow{ab}$ and form congruent angles with ray $\overrightarrow{ab}$. According to Postulate 5-3 there is only one such ray. Therefore $\overrightarrow{aq} = \overrightarrow{am}$. This means that $q \in \overleftrightarrow{mn}$. Thus if $\overleftrightarrow{cd}$ intersects $\overleftrightarrow{mn}$ in point p then $\overleftrightarrow{cd}$ intersects $\overleftrightarrow{mn}$ in point q. This implies that two distinct lines intersect in two points which is impossible. Therefore $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ do not intersect and thus are parallel. This argument proves Theorem 7-1.

Theorem 7-1. If two coplanar lines are cut by a transversal such that a pair of congruent alternate interior angles are determined, then the lines are parallel.

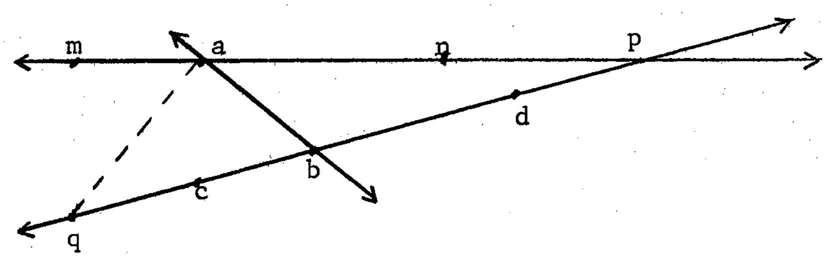


Figure 106

This theorem and Postulate 5-3 make it possible to establish the existence of parallel lines. Let $\overleftrightarrow{ab}$ be a line and let q be any point not in $\overleftrightarrow{ab}$. Let $\overleftrightarrow{mn}$ be a line containing q and intersecting $\overleftrightarrow{ab}$ at p such that a - p - b. (See Figure 107.) By Postulate 5-3 there is exactly one ray $\overrightarrow{qc}$ on the b-side of $\overleftrightarrow{mn}$ such that $\measuredangle$ cpq $\cong$ $\measuredangle$ npa. Let $\overleftrightarrow{qc}$ be the line containing $\overleftrightarrow{qc}$. Then $\overleftrightarrow{qc}$ and $\overleftrightarrow{ab}$ are two lines cut by the transversal $\overleftrightarrow{pq}$ such that a pair of alternate interior angles are congruent. Thus $\overleftrightarrow{qc}$ is parallel to $\overleftrightarrow{ab}$ by Theorem 7-1.

Theorem 7-1 asserts that the lines are parallel if the alternate interior angles are congruent. It makes no assertion in the event that the angles are not congruent. Thus the fact that there is only one ray $\overrightarrow{qc}$ such that $\measuredangle$ cqp $\cong$ $\measuredangle$ npa does not guarantee that there is only one line containing q that is parallel to $\overleftrightarrow{ab}$. Intuitively it seems that there is only one such line and therefore the uniqueness of this line will be postulated.

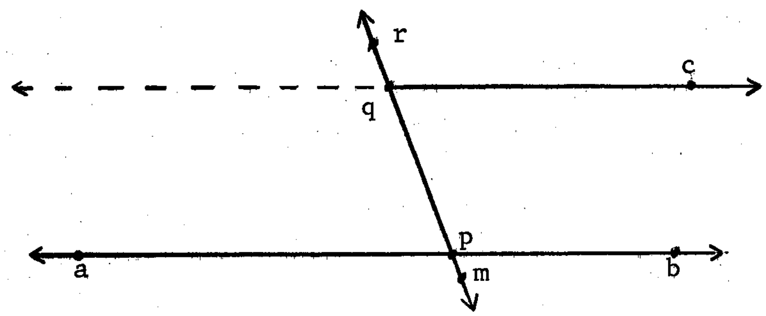


Figure 107

Postulate 7-1: If L is a line and p is a point not in L, then there exists one and only one line M containing p and coplanar with L such that M is parallel to L (symbolized L $\parallel$ M).

Postulate 7-1 is a simplified form of Euclid's famous fifth postulate. It is the distinguishing characteristic of Euclidean Geometry.

Theorem 7-2: If L and M are parallel lines and N $\neq$ L is a line such that N$\cap$L = p then N intersection M is not empty.

Proof: Refer to Figure 108.

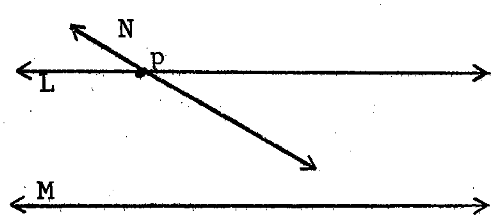


Figure 108

1. Assume that the assertion of the theorem is false, that is $N \cap M = \{\ \}$.
2. Then $N \parallel M$ by definition.
3. Therefore L and N are two distinct lines through p and parallel to M.
4. This is contradiction of Postulate 7-1.
5. Therefore $N \cap M$ is not empty by the rule of indirect proof.

If $L \parallel M$ then $L \cap M = \{\ \}$, but then $M \cap L = \{\ \}$ hence $M \parallel L$. Thus the parallel relation is symmetric. The next theorem will establish that the relation is also transitive.

<u>Theorem 7-3</u>: If (1) L, M and N are coplanar, (2) $L \parallel M$, and (3) $M \parallel N$, then $L \parallel N$.

Proof: See Figure 109.

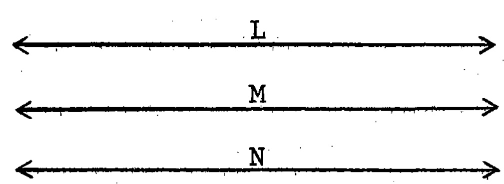


Figure 109

1. If $L \cap N$ is not empty then since $N \parallel M$, L must intersect M by Theorem 7-2.
2. But $L \parallel M$ by hypothesis, hence $L \cap M = \{\ \}$.
3. Therefore $L \cap N = \{\ \}$ hence $L \parallel N$.

If the symmetric property is applied to the third condition of

Theorem 7-3, conditions two and three are (2) $L \parallel M$ and (3) $N \parallel M$. The assertion remains $L \parallel N$. In this form the theorem states that if two lines are parallel to a third line then they are parallel.

<u>Theorem 7-4</u>: If two parallel lines are cut by a transversal both pairs of alternate interior angles determined are congruent.

Proof: In Figure 110 let $\overleftrightarrow{ab}$ be parallel to $\overleftrightarrow{mn}$ and $\overleftrightarrow{co}$ be a transversal intersecting $\overleftrightarrow{ab}$ and $\overleftrightarrow{mn}$ at points c and o respectively. With this notation the theorem may be stated, "If $\overleftrightarrow{ab} \parallel \overleftrightarrow{mn}$ then $\measuredangle$ aco $\cong$ $\measuredangle$ con and $\measuredangle$ bco $\cong$ $\measuredangle$ com."

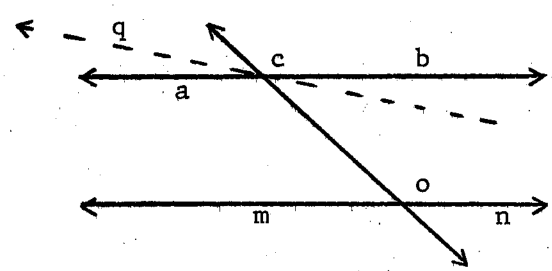


Figure 110

1. $\overleftrightarrow{ab} \parallel \overleftrightarrow{mn}$ by hypothesis.
2. Assume $\measuredangle$ aco is not congruent to $\measuredangle$ con.
3. By Postulate 5-3 there exists exactly one ray $\overrightarrow{cq}$ such that $\measuredangle$ qco $\cong$ $\measuredangle$ con.
4. Since $\measuredangle$ aco is not congruent to $\measuredangle$ con then $\overrightarrow{cq} \neq \overrightarrow{ca}$.
5. But by Theorem 7-1 the line $\overleftrightarrow{cq} \parallel \overleftrightarrow{mn}$.
6. Therefore there are two lines through c parallel to $\overleftrightarrow{mn}$ and this is

a contradiction of Postulate 7-1.

7. Therefore $\sphericalangle$ aco $\cong$ $\sphericalangle$ con.

8. $\sphericalangle$ bco is a supplement of $\sphericalangle$ aco and $\sphericalangle$ com is a supplement of $\sphericalangle$ con, thus $\sphericalangle$ bco $\cong$ $\sphericalangle$ com since they are supplements of congruent angles.

Theorem 7-4 is the converse of Theorem 7-1. The next theorem is an if and only if statement regarding corresponding angles and is a consequent of Theorems 7-4 and 7-1.

<u>Theorem 7-5</u>: If two coplanar lines are cut by transversal then the lines are parallel if and only if the angles of a pair of corresponding angles are congruent.

Proof: Let $\overleftrightarrow{ab}$ and $\overleftrightarrow{mn}$ be two lines and $\overleftrightarrow{cd}$ be a transversal intersecting $\overleftrightarrow{ab}$ and $\overleftrightarrow{mn}$ in points p and q such that a-p-b, m-q-n, c-p-d and p-q-d. Using the symbols in Figure 111 the theorem may be stated:

(1) If $\overleftrightarrow{ab} \parallel \overleftrightarrow{mn}$ then $\sphericalangle$ apc $\cong$ $\sphericalangle$ pqm and,

(2) If $\sphericalangle$ apc $\cong$ $\sphericalangle$ pqm then $\overleftrightarrow{ab} \parallel \overleftrightarrow{mn}$.

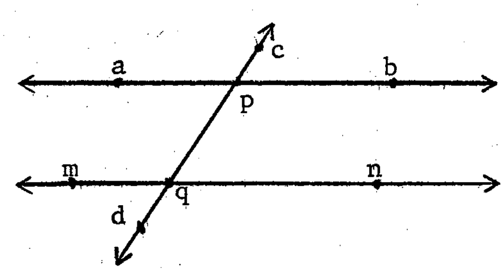


Figure 111

Proof of (1):

1. $\overleftrightarrow{ab} \parallel \overleftrightarrow{mn}$ by hypothesis.

2. $\measuredangle$ bpq $\cong$ $\measuredangle$ pqm by Theorem 7-4.

3. $\measuredangle$ apc $\cong$ $\measuredangle$ bpq since they are vertical angles.

4. Therefore $\measuredangle$ apc $\cong$ $\measuredangle$ pqm by the transitive property.

Proof of (2):

1. $\measuredangle$ apc $\cong$ $\measuredangle$ pqm by hypothesis.

2. $\measuredangle$ bpq $\cong$ $\measuredangle$ apc since vertical angles are congruent.

3. Therefore $\measuredangle$ bpq $\cong$ $\measuredangle$ pqm by the transitive property.

4. Hence $\overline{ab} \parallel \overline{mn}$ by Theorem 7-1.

Suppose the lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ are perpendicular to the line $\overleftrightarrow{mn}$ at the points a and c respectively. (See Figure 112.) Let ray $\overrightarrow{cp}$ be opposite $\overrightarrow{cd}$. The $\measuredangle$ cab is a right angle and $\measuredangle$ acp is a right angle. Therefore $\measuredangle$ bac $\cong$ $\measuredangle$ acp by Postulate 6-3. Thus the lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ are cut by a transversal such that a pair of alternate interior angles are congruent. Therefore $\overleftrightarrow{ab} \parallel \overleftrightarrow{cd}$. This proves the following theorem.

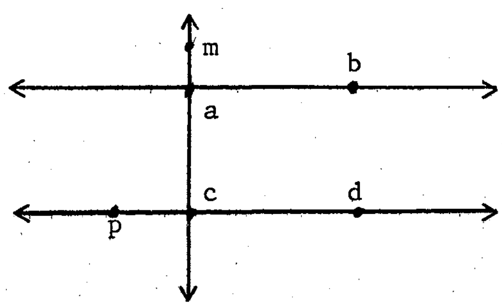


Figure 112

<u>Theorem 7-6</u>: If two coplanar lines are perpendicular to the same line then they are parallel.

Every triangle is the union of three segments which are determined by three noncollinear points. These three points determine a set of three lines such that any two lines in the set intersect the third line in distinct points and also intersect each other. Thus with any triangle there is associated a set of three lines such that each pair of lines in the set are intersecting lines. Therefore no two lines in the set are parallel and thus according to the contrapositive of Theorem 7-6 no two lines in the set are perpendicular to the same line. In Figure 113 lines L and M are not parallel. Therefore they intersect and a triangle is formed. If in Figure 114 the lines N and K are perpendicular to line H then they do not intersect. Therefore these three lines could not contain the sides of a triangle.

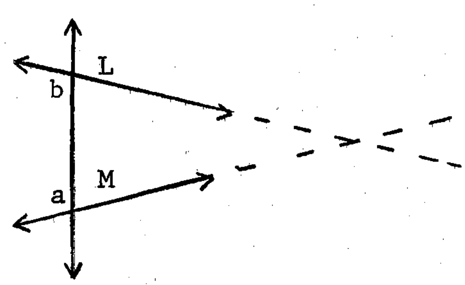


Figure 113

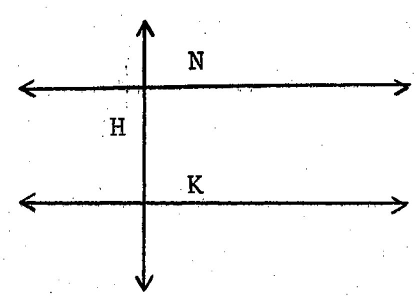


Figure 114

If two of the angles of a triangle were right angles then the two lines determined by two of its sides would be perpendicular to the line determined by the third side. According to the above discussion this is impossible. Therefore not more than one of the angles of a triangle is a right angle.

Theorem 7-7: A triangle has at most one right angle.

Theorems 7-1, 7-5, and 7-6 provide three sets of conditions that are sufficient to guarantee that two lines are parallel. Theorems 7-4 and 7-5 both contain assertions that are subject to the condition that the two lines are parallel. The next theorem is an if and only if type theorem. It will give a fourth condition that is sufficient to guarantee that two lines are parallel and will give an additional consequent of the parallel relation existing between two lines.

Theorem 7-8: Let L and M be two lines and N a transversal. (1) If the interior angles on the same side of the transversal are supplementary, then L $\parallel$ M. (2) If L $\parallel$ M then the interior angles on the same side of the transversal are supplementary.

Proof of (1): In Figure 115, let N intersect L and M in a and b respectively. Let m and n be points of L such that m - a - n. Let c and d be points of M such that c - b - d and c is on the m-side of N. Then $\measuredangle$ mab and $\measuredangle$ abc are interior angles on the same side of the transversal and are supplementary by hypothesis.

1. $\measuredangle$ nab is a supplement of $\measuredangle$ mab since $\overrightarrow{an}$ is opposite $\overrightarrow{am}$.
2. Therefore $\measuredangle$ nab $\cong$ $\measuredangle$ abc since they are both supplements of $\measuredangle$ mab.
3. But $\measuredangle$ nab and $\measuredangle$ abc are alternate interior angles, hence L $\parallel$ M by Theorem 7-1.

Proof of (2): Refer to Figure 115.

1. L $\parallel$ M by hypothesis.
2. Therefore $\measuredangle$ mab $\cong$ $\measuredangle$ abd by Theorem 7-4.
3. $\overrightarrow{bc}$ is opposite $\overrightarrow{bd}$, therefore $\measuredangle$ cba is a supplement of $\measuredangle$ abd.

4. Therefore $\measuredangle$ cba is a supplement of $\measuredangle$ mab.

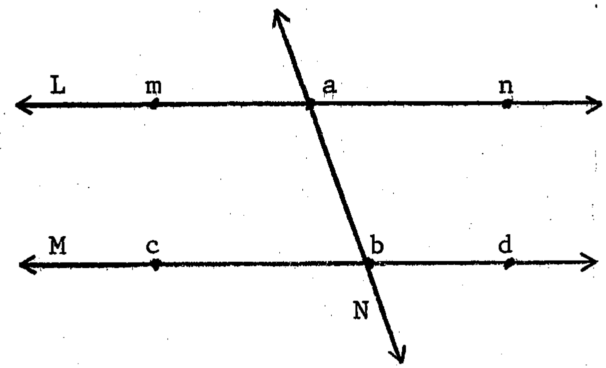


Figure 115

Quadrilaterals

In Chapter IV a quadrilateral is defined as a polygon having four sides. A polygon is a simple closed curve that is the union of segments. Thus the sides of a quadrilateral are segments. The points of intersection of the sides are called vertices, thus every quadrilateral has four vertices. Two sides will be called adjacent sides if their intersection is a vertex. If their intersection is the empty set then they are called opposite sides.

In Figure 116 $\overline{ad}$ and $\overline{bc}$ are opposite sides since they do not intersect. $\overline{ab}$ and $\overline{bc}$ are adjacent sides since $\overline{ab} \cap \overline{bc} = b$. Similarly, $\overline{bc}$ and $\overline{cd}$ are adjacent, $\overline{cd}$ and $\overline{da}$ are adjacent, with $\overline{da}$ and $\overline{ab}$ also adjacent. If two vertices are in the same segment they are called consecutive vertices. In accordance with Chapter IV regarding the naming of polygons, a quadrilateral may be named by listing the vertices in any order such that consecutive letters name consecutive vertices. Thus the quadrilateral in Figure 116 could be named abcd, adcb, or any one of six other names. In a quadrilateral, the nonconsecutive vertices are called opposite vertices. Thus in naming a quadrilateral the nonconsecutive letters listed always name opposite vertices.

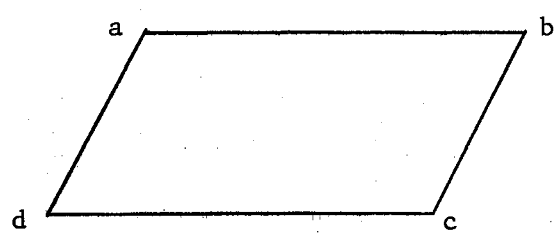


Figure 116

The segment determined by opposite vertices is called a diagonal. In Figure 117, a and c are opposite vertices and $\overline{ac}$ is a diagonal. Also b and d are opposite vertices and $\overline{bd}$ is a diagonal.

Each pair of adjacent sides of a quadrilateral determine a pair of rays having one vertex of the quadrilateral as a common end point. The angle formed by the union of these two rays is called an angle of the quadrilateral. Thus each quadrilateral has four angles. If two angles of a quadrilateral are such that their vertices are opposite vertices of the quadrilateral then they are opposite angles. If their vertices are consecutive vertices of the quadrilateral they are called consecutive angles of the quadrilateral. In Figure 117, angles in $\{\measuredangle \text{ bad}, \measuredangle \text{ bcd}\}$ are opposite angles. The angles in $\{\measuredangle \text{ bad}, \measuredangle \text{ adc}\}$ are consecutive angles.

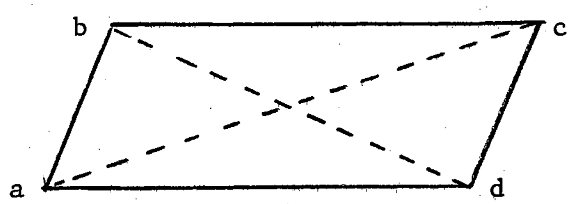


Figure 117

A quadrilateral is the union of noncollinear segments and thus could not be a convex set. Nevertheless, it is convenient to refer to certain types of quadrilaterals as convex quadrilaterals.

<u>Definition 7-4</u>: A quadrilateral is a convex quadrilateral if and only if its sides are such that no side of the quadrilateral intersects the line determined by the opposite side.

The quadrilateral abcd in Figure 118 is a convex quadrilateral. The quadrilateral mnop in Figure 119 is not convex since $\overleftrightarrow{mn} \cap \overleftrightarrow{op}$ is not the empty set but $\overline{mn}$ and $\overline{op}$ are opposite sides of the quadrilateral. The quadrilaterals considered in this discussion will be convex quadrilaterals. Thus any quadrilateral will be assumed to be a convex quadrilateral unless otherwise specified.

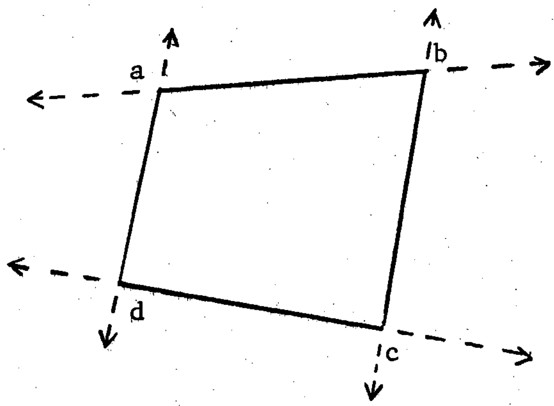


Figure 118

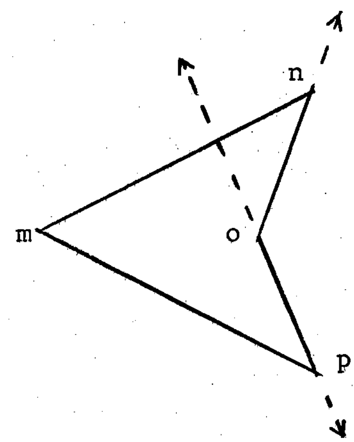


Figure 119

<u>Definition 7-5</u>: If abcd is a quadrilateral, then a point is in the interior of abcd if and only if it is in the interior of each of the angles of the quadrilateral.

This implies that the interior of a quadrilateral is the intersection of the interior of the four angles of the quadrilateral. Since the interior of an angle is a convex set by Theorem 4-3 and the intersection of convex sets is convex from Theorem 4-2, it follows that the interior of a quadrilateral is a convex set.

If abcd is a quadrilateral as in Figure 120, the points b and a are on the same side of $\overrightarrow{dc}$ and the points b and c are on the same side of $\overrightarrow{da}$. Then b is in the interior of $\measuredangle$ cda and hence $\overrightarrow{db}$ is in the interior of $\measuredangle$ cda. Therefore every interior point of the diagonal $\overline{db}$ is in the interior of $\measuredangle$ cda. Similarly every interior point of the diagonal $\overline{bd}$ (=$\overline{db}$) is in the interior of $\measuredangle$ abc. By Postulate 6-2 every interior point of $\overline{bd}$ is in the interior of $\measuredangle$ bad and also in the interior of $\measuredangle$ bcd. Thus every interior point of the diagonal of a quadrilateral is in the interior of the quadrilateral.

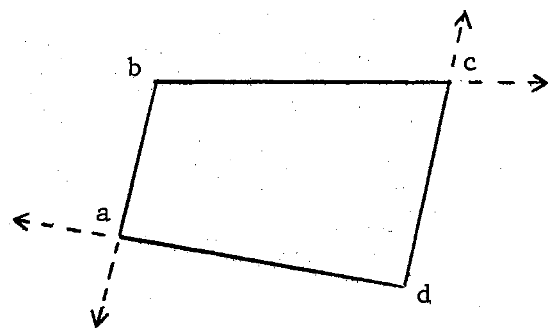


Figure 120

Trapezoids

A special class of quadrilaterals is that in which a pair of opposite sides determines subsets of parallel lines.

Definition 7-6: Two segments $\overline{ab}$ and $\overline{cd}$ are parallel segments if and only if $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ are parallel lines.

Definition 7-7: A quadrilateral is a trapezoid if and only if at least one pair of opposite sides of the quadrilateral consists of parallel segments.

In Figure 121 $\overline{ab} \parallel \overline{dc}$ and hence abcd is a trapezoid.

Every trapezoid has four sides and two of the sides are parallel segments. If the other two sides are non-parallel and congruent then the trapezoid is called an isosceles trapezoid. (See Figure 122.)

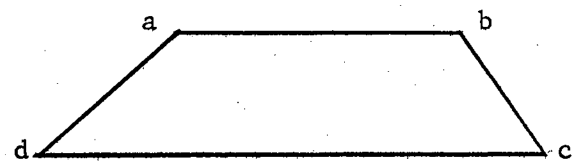


Figure 121

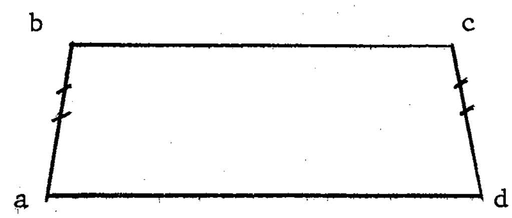


Figure 122

Parallelograms

A further classification of quadrilaterals occurs if both pairs of opposite sides of a quadrilateral are parallel segments.

<u>Definition 7-8</u>: A quadrilateral is a parallelogram if and only if the opposite sides of the quadrilateral are parallel segments.

Thus every parallelogram is also a convex quadrilateral and hence the terminology just developed regarding vertices and sides of quadrilaterals applies to parallelograms.

A segment joining the points a and c of parallelogram abcd is a diagonal and since every parallelogram is a convex quadrilateral, every interior point of the diagonal is in the interior of the parallelogram. Then $\overline{ac} \cup \overline{ab} \cup \overline{cb} = \Delta$ abc and $\overline{ac} \cup \overline{ad} \cup \overline{cd} = \Delta$ adc. (See Figure 123.)

It will be established that these two triangles are congruent and this conclusion will be used to prove two interesting properties of parallelograms.

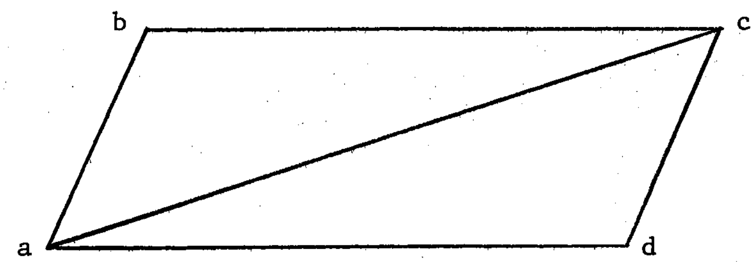


Figure 123

Theorem 7-9: The opposite sides of a parallelogram are congruent segments and the opposite angles of a parallelogram are congruent angles.

Proof: As suggested above a diagonal will be used in the proof. With a particular diagonal it will be possible to prove two pair of opposite sides and one pair of opposite angles congruent. The proof for the other pair of angles would differ only in that the other diagonal would be used and thus will be omitted. Refer to Figure 124.

1. abcd is a parallelogram by hypothesis.
2. Then $\overline{ad} \parallel \overline{bc}$ and $\overline{ab} \parallel \overline{dc}$ by Definition 7-8.
3. $\overleftrightarrow{ad} \parallel \overleftrightarrow{bc}$ and $\overleftrightarrow{ab} \parallel \overleftrightarrow{dc}$ by Definition 7-6.
4. The diagonal $\overline{ac}$ intersects lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{dc}$ in points a and c respectively and intersects lines $\overleftrightarrow{ad}$ and $\overleftrightarrow{bc}$ in points a and c

respectively.

5. Therefore $\overleftrightarrow{ac}$ is a transversal of $\overleftrightarrow{ab}$ and $\overleftrightarrow{dc}$.

6. The angles in $\{\sphericalangle\ acd, \sphericalangle\ cab\}$ are alternate interior angles and since $\overleftrightarrow{ab} \parallel \overleftrightarrow{dc}$, $\sphericalangle\ acd \cong \sphericalangle\ cab$ by Theorem 7-4.

7. $\overleftrightarrow{ac}$ is also a transversal of $\overleftrightarrow{ad}$ and $\overleftrightarrow{cb}$.

8. The angles in $\{\sphericalangle\ cad, \sphericalangle\ acb\}$ are alternate interior angles and since $\overleftrightarrow{ad} \parallel \overleftrightarrow{cb}$, $\sphericalangle\ cad \cong \sphericalangle\ acb$ by Theorem 7-4.

9. $\overline{ac} \cong \overline{ac}$ by the reflexive property of the congruence relation.

10. Thus for the correspondence $\triangle\ cad \leftrightarrow \triangle\ acb$, $\sphericalangle\ cad \cong \sphericalangle\ acb$, $\overline{ac} \cong \overline{ac}$ and $\sphericalangle\ acd \cong \sphericalangle\ cab$.

11. Therefore $\triangle\ cad \cong \triangle\ acb$ by the A.S.A. theorem.

12. For this correspondence $\overline{cd} \leftrightarrow \overline{ab}$ and $\overline{ad} \leftrightarrow \overline{cb}$ thus $\overline{cd} \cong \overline{ab}$ and $\overline{ad} \cong \overline{cb}$ by Definition 5-3.

13. $\sphericalangle\ adc \leftrightarrow \sphericalangle\ cba$ hence $\sphericalangle\ adc \cong \sphericalangle\ cba$ by Theorem 5-2.

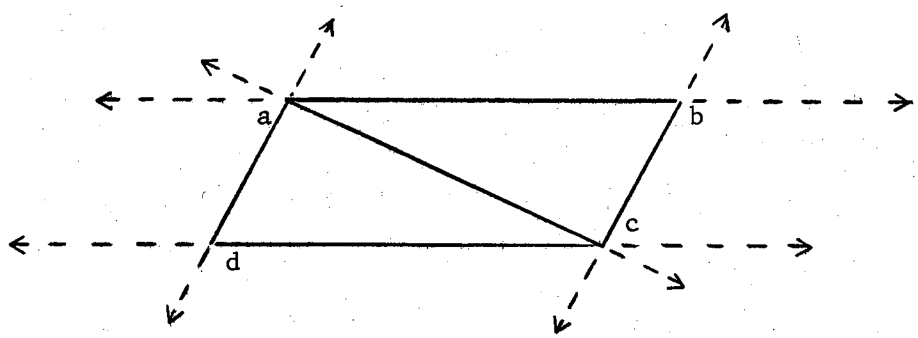


Figure 124

In the parallelogram abcd, the line determined by any side, side

$\overline{ab}$ for definiteness, is a transversal of the lines determined by its adjacent sides, $\overline{bc}$ and $\overline{ad}$ (see Figure 125). Since $\overleftrightarrow{bc} \parallel \overleftrightarrow{ad}$, the angles in $\{\measuredangle\ cba, \measuredangle\ dab\}$ are supplementary angles by Theorem 7-8.

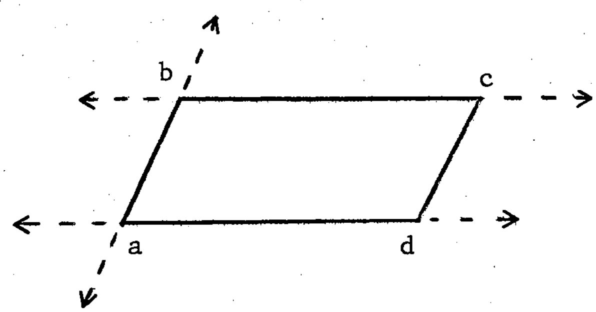


Figure 125

Theorem 7-10. Any two consecutive angles determined by a parallelogram are supplementary.

If two consecutive angles of a parallelogram are congruent, then each of the angles is congruent to one of its supplements and is therefore a right angle. Also if one angle of a parallelogram is a right angle then each angle consecutive to it is a right angle since consecutive angles are supplementary. By Theorem 7-9, the opposite angles of a parallelogram are congruent. Thus if one angle of a parallelogram is a right angle then all angles of the parallelogram are right angles.

Definition 7-9. A rectangle is a parallelogram which determines right angles.

Thus the set of rectangles is a subset of the set of parallelograms. A further classification occurs if two adjacent sides of a parallelogram are congruent. If two adjacent sides of a parallelogram are congruent then all sides are congruent since the opposite sides are congruent by Theorem 7-8. If all sides of a parallelogram are congruent, then it is called a rhombus. If all sides of a parallelogram are congruent and all determined angles are right angles then it is called a square (see Figure 126).

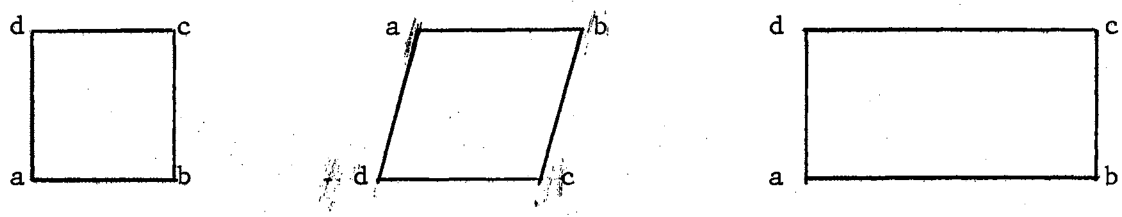


Figure 126

Definition 7-8 provides a set of conditions sufficient to guarantee that a quadrilateral is a parallelogram, namely that each pair of opposite sides is a pair of parallel segments. Two other sets of conditions sufficient to establish that a quadrilateral is a parallelogram are provided by the next two theorems.

Theorem 7-11: If two opposite sides of a quadrilateral are parallel and also congruent, then the quadrilateral is a parallelogram.

Proof: In Figure 127 let $\overline{ab}$ and $\overline{dc}$ be the two sides that are parallel and congruent as given in the hypothesis of the theorem. Let $\overline{ac}$ be the diagonal determined by the opposite vertices a and c.

1. $\overline{ab} \parallel \overline{dc}$ by hypothesis, therefore $\overleftrightarrow{ab} \parallel \overleftrightarrow{cd}$ by Definition 7-6.

2. The angles in $\{\measuredangle\, bac, \measuredangle\, dca\}$ are alternate interior angles and thus $\measuredangle\, bac \cong \measuredangle\, dca$ by Theorem 7-4.

3. $\overline{ab} \cong \overline{dc}$ by hypothesis.

4. $\overline{ac} \cong \overline{ac}$ by the reflexive property of the congruence relation.

5. For the correspondence $\triangle\, bac \leftrightarrow \triangle\, dca$, $\overline{ba} \leftrightarrow \overline{dc}$, $\measuredangle\, bac \leftrightarrow \measuredangle\, dca$ and $\overline{ac} \leftrightarrow \overline{ca}$.

6. Therefore $\triangle\, bac \cong \triangle\, dca$ by the S.A.S. theorem.

7. Therefore $\measuredangle\, dac \cong \measuredangle\, bca$.

8. Hence $\overline{ad} \parallel \overline{cb}$ by Theorem 7-1.

9. Therefore abcd is a parallelogram by Definition 7-8.

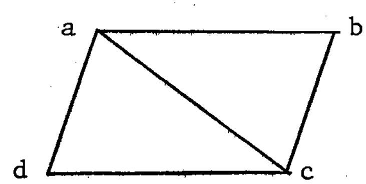


Figure 127

<u>Theorem 7-12</u>. If the opposite sides of a quadrilateral are congruent segments, then the quadrilateral is a parallelogram.

Proof: In Figure 128 $\overline{ad} \cong \overline{bc}$ and $\overline{ad} \cong \overline{dc}$.

1. $\overline{ad} \cong \overline{cb}$ and $\overline{ba} \cong \overline{dc}$ by hypothesis.

2. $\overline{bd} \cong \overline{db}$ by the reflexive property.

3. Therefore $\triangle\, adb \cong \triangle\, cbd$ by Definition 5-3.

4. Then $\measuredangle\, cbd \cong \measuredangle\, adb$ by Theorem 5-2.

5. Then $\overline{bc} \parallel \overline{ad}$ by Theorem 7-1.

6. Thus segments $\overline{bc}$ and $\overline{da}$ are parallel and congruent, hence abcd is a parallelogram by Theorem 7-11.

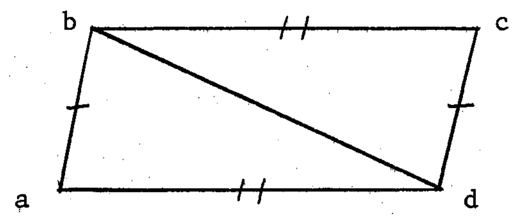


Figure 128

CHAPTER VIII

CIRCLES AND GEOMETRIC CONSTRUCTIONS

Circles

The triangles and quadrilaterals considered in previous chapters are elements of a set of simple closed curves that are polygons. There are many simple closed curves that are not polygons. The most commonly used simple closed curves, other than the polygons, are the circles. There are many physical models of circles such as the rim of a wheel, a wedding band, the top of a coffee cup, etc. These are models only since circles are point sets and therefore are abstractions. The congruence relation provides a basis for a definition of "circle."

Definition 8-1. Let o be a point in a plane M and $\overline{ab}$ be a segment. The set of all points p in the plane M such that $\overline{op} \cong \overline{ab}$ is a circle.

The point o is called the center and any segment $\overline{oq}$ such that q is a point in the circle is called a radius. Thus a radius of a circle is a segment. Every circle has many radii (plural for radius). Each radius is congruent to a given segment and all segments congruent to a given segment are congruent. Therefore all radii of the same circle are congruent. In Figure 129 the points p, q, r and s are points in the circle, that is, they are elements of the point set that is the circle. The point o is the center of the circle. Segments $\overline{op}$, $\overline{oq}$, $\overline{or}$ and $\overline{os}$ are radii. Note that o is not an element of the circle; thus it is incorrect to say that o is in the circle. Similarly, the interior points of the radius $\overline{op}$ or any other radius are not in the circle. The center of a circle and all interior points of any segment that is a radius of the circle are in the interior of the circle. A circle with center o will be called circle O. If a particular capital letter is used to name a circle, it is understood that the corresponding lower case letter refers to the center of that particular circle.

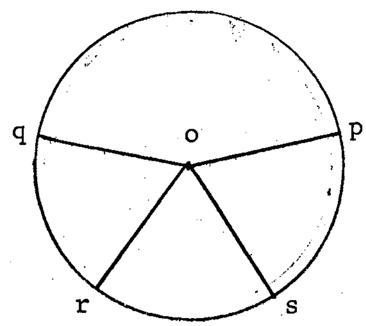


Figure 129

<u>Definition 8-2</u>. If O is a circle in a plane M, then a point $p \in M$ is in the interior of circle O if and only if the segment $\overline{op}$ does not intersect the circle.

Thus a circle and its interior are disjoint sets. If a point q in the plane of a circle is neither in the circle or in the interior of the circle then q is in the exterior of the circle. If q is in the exterior of a circle with center o, then $\overline{oq}$ intersects the circle.

If two circles are such that they have the same center, but their radii are not congruent, they are called concentric circles (see

Figure 130). If two circles have different centers and congruent radii they are congruent circles.

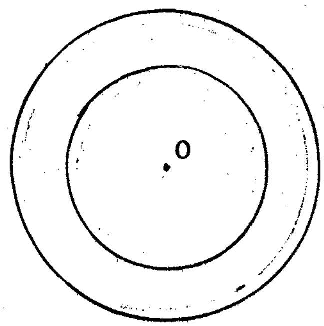


Figure 130

<u>Definition 8-3</u>. If O is a circle with center o and radius $\overline{op}$ and K is a circle with center k and radius $\overline{kq}$, then circle O $\cong$ circle K if and only if $\overline{op} \cong \overline{kq}$.

In Figure 131 circle O $\cong$ circle K if and only if $\overline{op} \cong \overline{kq}$.

Figure 131

A segment that is determined by two points in a circle is called a chord. If a chord contains the center of the circle, then it is called a diameter. If p is any point in a circle with radius $\overline{ab}$ and center o, then $\overleftrightarrow{op}$ is a line. Let m be a point in $\overleftrightarrow{op}$ such that m-o-p. In the ray $\overrightarrow{om}$ there is exactly one point q such that $\overline{oq} \cong \overline{ab}$. Since $\overline{oq} \cong \overline{ab}$, q is in the circle by definition. Therefore $\overline{qp}$ is a diameter. Thus if p is any point in a circle, then there exists exactly one point q in the circle such that $\overline{pq}$ is a diameter. In Figure 132, $\overline{mn}$ is a chord and $\overline{pq}$ is a chord that is also a diameter.

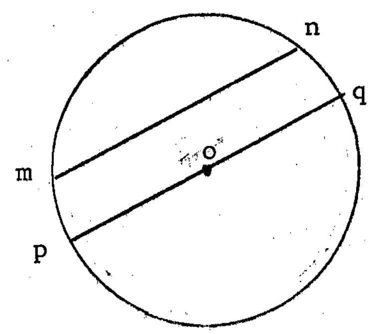


Figure 132

The needle of a compass is a physical model of the diameter of a circle. In any position the needle represents a particular diameter $\overline{pq}$. If the needle rotates on an axis at the center to a different position, it then represents a different diameter, say $\overline{mn}$. Evidently the needle does not change size or shape as it rotates. This suggests that the diameters $\overline{pq}$ and $\overline{mn}$ are congruent segments (see Figure 133).

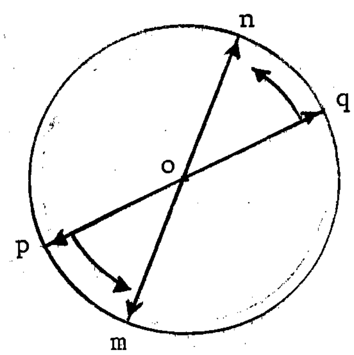


Figure 133

Postulate 8-1. If $\overline{pq}$ and $\overline{mn}$ are diameters of the same circle, then $\overline{pq} \cong \overline{mn}$.

If a and b are distinct points in a circle with center o such that o, a and b are noncollinear, then $\overrightarrow{oa} \cup \overrightarrow{ob} = \measuredangle aob$. The sides of this angle are determined by the radii $\overline{oa}$ and $\overline{ob}$ and its vertex is the center of the circle. Any angle which has its vertex at the center of a circle is called a central angle. In Figure 134, $\measuredangle$ aob is a central angle. The set of points consisting of a and b together with all points in the circle that are in the interior of $\measuredangle$ aob is called a minor arc of the circle. Points a and b will be called endpoints of the arc. A minor arc with endpoints a and b will be referred to as arc ab. The set of points consisting of a and b together with all points of the circle that are in the exterior of $\measuredangle$ aob is called a major arc. A major arc with endpoints a and b will be referred to as major arc ab. If a and b are endpoints of a diameter, than a, o and b are collinear; hence no central angle is determined. The points a and b determine the line $\overleftrightarrow{ab}$. The set of all points of the circle that are in the same side of $\overleftrightarrow{ab}$ together with a and b is called a semicircle. Thus a semicircle is a special arc with its endpoints in a diameter. Since two points in a circle always determine two arcs, a third point is used in symbolizing a particular arc. If a and b are endpoints of an arc and c is a point in the arc, then the arc is symbolized $\overparen{acb}$. In Figure 134 $\overparen{apb}$ is a minor arc and $\overparen{aqb}$ is a major arc.

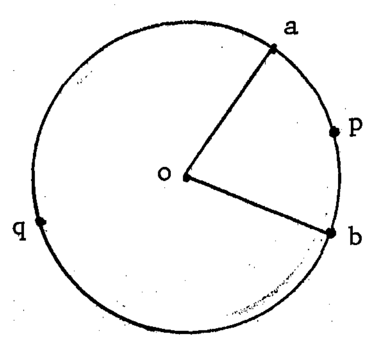


Figure 134

<u>Theorem 8-1</u>. If a, b and c are three collinear points and o is any point not in the line containing these points, then at most two of the segments $\overline{oa}$, $\overline{ob}$ and $\overline{oc}$ are congruent.

Proof: An indirect argument will be given. $\overline{oa}$, $\overline{ob}$ and $\overline{oc}$ will be assumed to be congruent, and it will be shown that this assumption leads to a contradiction. Refer to Figure 135. For definiteness, assume a-b-c.

1. a, b and c are collinear and $o \notin \overleftrightarrow{ab}$ by hypothesis.
2. $\overline{oa} \cong \overline{ob} \cong \overline{oc}$ by assumption.
3. Then $\triangle$ aoc is isosceles by definition.

4. Therefore $\measuredangle$ oac $\cong$ $\measuredangle$ oca by Theorem 6-1.

5. Similarly $\triangle$ aob is isosceles and $\measuredangle$ oba $\cong$ $\measuredangle$ oab.

6. $\measuredangle$ oab = $\measuredangle$ oac hence $\measuredangle$ oab $\cong$ $\measuredangle$ oac.

7. Then $\measuredangle$ oba $\cong$ $\measuredangle$ oca by the transitive property.

8. But $\triangle$ obc is isosceles; hence $\measuredangle$ obc $\cong$ $\measuredangle$ ocb.

9. $\measuredangle$ ocb = $\measuredangle$ oca, hence $\measuredangle$ ocb $\cong$ $\measuredangle$ oca.

10. Then $\measuredangle$ obc $\cong$ $\measuredangle$ oca by steps (8) and (9) and the transitive property.

11. Therefore $\measuredangle$ oba $\cong$ $\measuredangle$ obc by steps (7) and (9) and the transitive property.

12. Hence $\measuredangle$ oba is congruent to one of its supplements and therefore is a right angle by definition.

13. From step (7), $\measuredangle$ oca $\cong$ $\measuredangle$ oba and from steps (5) and (6), $\measuredangle$ oac $\cong$ $\measuredangle$ oba; hence $\measuredangle$ oac and $\measuredangle$ oca are right angles.

14. Thus $\triangle$ oac has two right angles and this contradicts Theorem 7-7. (A triangle has at most one right angle.)

15. Therefore the assumption in step (2) is false, hence at most two of the segments $\overline{oa}$, $\overline{ob}$ and $\overline{oc}$ are congruent.

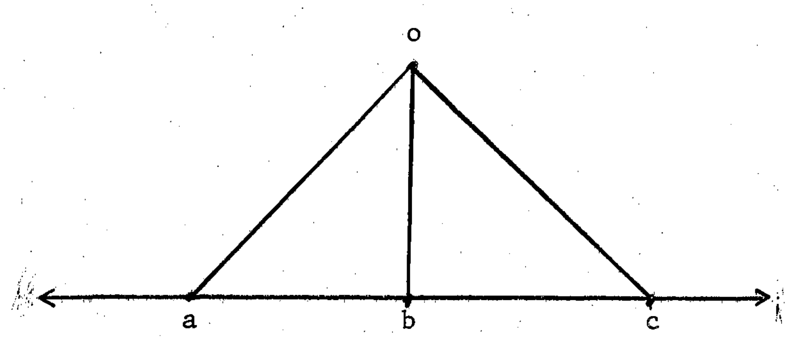


Figure 135

The contrapositive of Theorem 8-1 is "if a, b, c and o are distinct points such that $\overline{oa} \cong \overline{ob} \cong \overline{oc}$, then a, b and c are noncollinear points." If a, b and c are distinct points in a circle O, then $\overline{oa}$, $\overline{ob}$ and $\overline{oc}$ are congruent segments and therefore a, b and c are noncollinear.

Theorem 8-2. If a, b and c are any three distinct points in a circle, then a, b and c are noncollinear.

As a consequent of Theorem 8-2, any three points in a circle determine a triangle. Each vertex of this triangle is a point in the circle and its sides are chords. A triangle which has all of its vertices in a circle is said to be inscribed in the circle. More generally, any polygon which has all of its vertices in a circle is called an inscribed polygon. If the vertex of an angle is in a circle and its sides intersect the circle in points other than the vertex, then the angle is called an inscribed angle. An angle is said to be inscribed in an arc if its vertex is a point in the arc and its sides intersect the circle at the endpoints of the arc. In Figure 136 ∡ abc is inscribed in the arc $\widehat{abc}$.

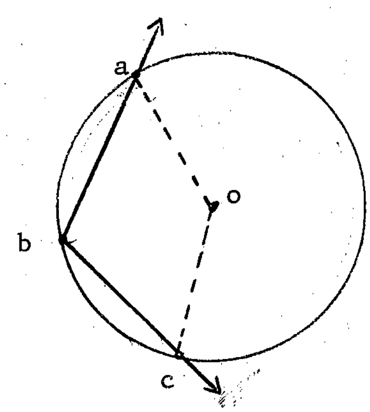


Figure 136

<u>Theorem 8-3</u>. Any angle inscribed in a semicircle is a right angle.

Proof: Refer to Figure 137. The problem is to show that $\measuredangle$ acb is a right angle.

1. The arc $\overset{\frown}{acb}$ is a semicircle by hypothesis.
2. Hence $\overline{ab}$ is a diameter.
3. Let d be the point in the circle such that $\overline{cd}$ is a diameter.
4. Consider $\triangle$ aoc and $\triangle$ bod under the correspondence $\triangle$ aoc $\leftrightarrow$ $\triangle$ bod.
5. $\overline{ao} \cong \overline{bo}$ and $\overline{oc} \cong \overline{od}$ since they are all radii of the same circle.
6. $\measuredangle$ aoc $\cong$ bod by Theorem 6-8. (Vertical angles are congruent.)
7. Therefore $\triangle$ aoc $\cong$ $\triangle$ bod by the S.A.S. theorem.
8. Hence $\overline{ac} \cong \overline{bd}$ by Definition 5-3 and $\measuredangle$ aco $\cong$ $\measuredangle$ bdo by Theorem 5-2.
9. Relative to the lines $\overleftrightarrow{ac}$ and $\overleftrightarrow{bd}$ the line $\overleftrightarrow{cd}$ is a transversal and, $\measuredangle$ aco and $\measuredangle$ bdo are alternate interior angles.
10. Therefore $\overleftrightarrow{ac} \parallel \overleftrightarrow{bd}$ by Theorem 7-1.
11. Then acbd is a quadrilateral with a pair of opposite sides parallel and congruent.
12. Therefore acbd is a parallelogram by Theorem 7-11.
13. Then by Theorem 7-10, $\measuredangle$ acb and $\measuredangle$ dbc are supplementary angles.
14. Consider the correspondence $\triangle$ acb $\leftrightarrow$ $\triangle$ dbc.
15. $\overline{ac} \cong \overline{db}$ from step (8), $\overline{ab} \cong \overline{dc}$ since they are both diameters of the same circle and $\overline{cb} \cong \overline{cb}$ by the reflexive property.
16. Hence $\triangle$ acb $\cong$ $\triangle$ dbc.
17. Therefore $\measuredangle$ acb $\cong$ $\measuredangle$ dbc by Theorem 5-2.
18. Thus $\measuredangle$ acb is congruent to one of its supplements and therefore $\measuredangle$ acb is a right angle by Definition 6-11.

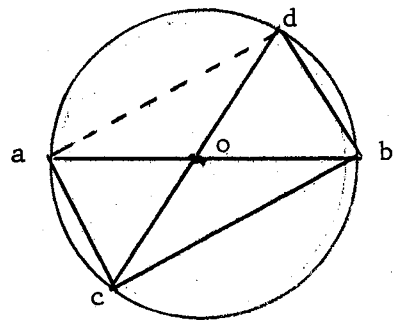


Figure 137

The two circles in Figure 138 intersect in two points. In Figure 139 the two circles have one common point, while the two circles in Figure 140 are disjoint sets. In Figure 138 $\overline{oq} \cap \overline{kp} = \overline{qp}$. In Figure 139 $\overline{oq} \cap \overline{kp} = \overline{pq}$. In Figure 140 $\overline{oq} \cap \overline{kp}$ is empty. These observations suggest the following postulate.

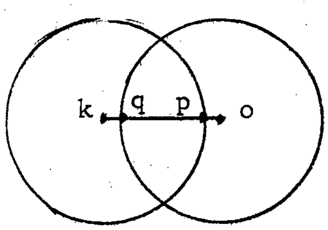


Figure 138

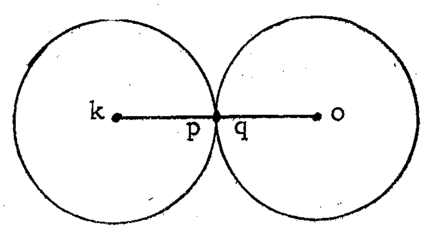


Figure 139

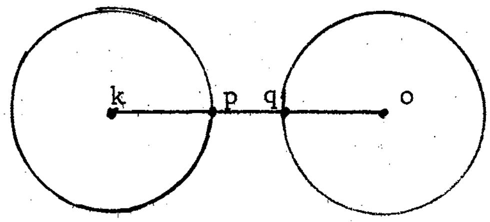


Figure 140

Postulate 8-2. The Two Circle Postulate. Let $\overline{ok}$, $\overline{ab}$ and $\overline{mn}$ be distinct segments. Let q be the point in $\overrightarrow{ok}$ such that $\overline{oq} \cong \overline{ab}$ and p be the point in $\overrightarrow{ko}$ such that $\overline{kp} \cong \overline{mn}$, then:

(1) If $\overline{oq} \cap \overline{kp}$ is a segment, then the circle with center o and radius $\overline{ab}$ intersects the circle with center k and radius $\overline{mn}$ in two points on opposite sides of $\overleftrightarrow{ok}$. If the two circles intersect on one side of $\overleftrightarrow{ok}$, then they will intersect in the opposite side of $\overleftrightarrow{ok}$.

(2) If $\overline{oq} \cap \overline{kp}$ is a point, then the two circles in statement (1) intersect in one point and are called tangent circles.

(3) If $\overline{oq} \cap \overline{kp} = \{ \}$, then the circles do not intersect.

Geometric Construction

In this section techniques will be developed for constructing models of some of the point sets encountered in the previous chapters. The procedures used will be justified by the definitions, postulates and theorems that have been established. The only tools necessary are pencil, compass and unmarked straight edge.

Construction 8-1. On a given ray construct a segment congruent to a given segment.

Procedure: According to Postulate 5-1, there exists exactly one segment $\overline{ab}$ in ray $\overrightarrow{ap}$ such that $\overline{ab}$ is congruent to a given segment. From Definition 8-1 all radii of the same circle are congruent. Let $\overrightarrow{ap}$ be the given ray and $\overline{mn}$ be the given segment (see Figure 141). Place the point of the compass at m and adjust the compass so that the pencil is on point n. Without changing the compass adjustment, move the point of the compass to point a and swing an arc intersecting ray $\overrightarrow{ap}$ at b. This arc is a part of a circle with radius $\overline{mn}$. Therefore for every point p in the arc, $\overline{ap} \cong \overline{mn}$. Thus $\overline{ab} \cong \overline{mn}$.

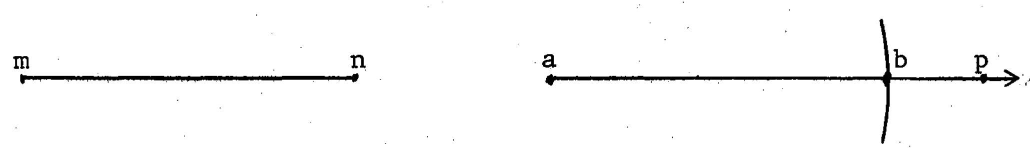


Figure 141

According to Postulate 6-1, every segment has a midpoint. A line intersecting a segment at its midpoint is said to bisect the segment. If the line bisects the segment and is also perpendicular to the segment, then the line is called the perpendicular bisector of the segment.

<u>Construction 8-2</u>. Construct the perpendicular bisector of a given segment.

Procedure: Let $\overline{mn}$ be the given segment. With m as center and $\overline{mn}$ as radius, construct a circle and with n as center and $\overline{mn}$ as radius construct a circle (see Figure 142). Then $\overline{nm} \cap \overline{mn} = \overline{mn}$ and the two circle postulate applies. Therefore circles M and N intersect in two points on opposite sides of $\overleftrightarrow{mn}$. Let a and b be the points of intersection of the two circles. The segments $\overline{am}$, $\overline{mb}$, $\overline{bn}$ and $\overline{na}$ are all radii of congruent circles and hence are congruent. Therefore manb is a parallelogram by Theorem 7-12. Since every parallelogram is a convex quadrilateral, every interior point of segment $\overline{ab}$ is in the interior of manb. For the correspondence $\triangle$ amb $\leftrightarrow$ $\triangle$ anb, $\overline{am} \cong \overline{an}$, $\overline{mb} \cong \overline{nb}$ and $\overline{ab} \cong \overline{ab}$. Hence $\triangle$ amb $\cong$ $\triangle$ anb. Therefore $\angle$ mab $\cong$ $\angle$ nab. Points a and b are on opposite sides of $\overleftrightarrow{mn}$, so $\overline{ab}$ intersects $\overleftrightarrow{mn}$. Every interior point of $\overline{ab}$ is in the interior of parallelogram ambn, therefore it follows that $\overline{ab}$ intersects $\overline{mn}$. Let $\overline{ab} \cap \overline{mn} = p$, then $p \in \overleftrightarrow{ab}$; hence $\angle$ map $\cong$ $\angle$ nap and $\overline{ap} \cong \overline{ap}$. Thus $\triangle$ apn $\cong$ $\triangle$ apm by the S.A.S. theorem. Therefore $\overline{mp} \cong \overline{pn}$; hence p is the midpoint of $\overline{mn}$. Also $\angle$ apm $\cong$ $\angle$ apn and they are adjacent supplementary angles. Thus $\angle$ apm is a right angle and $\angle$ apn is a right angle. Also $\overleftrightarrow{ab} \perp \overleftrightarrow{mn}$. Therefore $\overleftrightarrow{ab}$ is the perpendicular bisector of $\overline{mn}$.

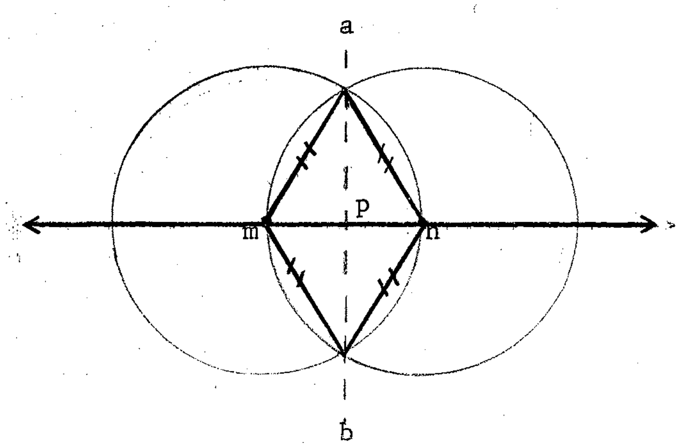


Figure 142

In Chapter II it was noted that two points are required to determine a line. In Construction 8-2 it was necessary to determine a particular line and thus it was necessary to determine two points. The next construction also requires the construction of a line but in this case one point of the line is known.

<u>Construction 8-3</u>. Construct a line containing a given point such that it is perpendicular to a given line.

Procedure: There are two possibilities: (1) the given point is in the given line, or (2) the given point is not in the given line. Case (1) will be considered first.

1. Let L be the given line and p be the given point in L.
2. With p as center and any radius swing an arc intersecting L in points m and n as in Figure 143.
3. Then $\overline{mp} \cong \overline{pn}$ since they are radii of the same circle.
4. Place the point at m and adjust the radius of the compass so that an arc centered at m will intersect the line at n.
5. Using this radius, circles centered at n and m will intersect in two points according to the two circle postulate.
6. Let q be the point of intersection on one side of L.
7. $\overleftrightarrow{qp}$ is perpendicular to L.

To justify the contention in step (7), consider $\triangle$ qmp and $\triangle$ qnp. $\overline{mq} \cong \overline{nq}$ since they are radii of congruent circles. $\overline{mp} \cong \overline{np}$ from step (3) and $\overline{qp} \cong \overline{qp}$. Therefore $\triangle$ qmp $\cong$ qnp; hence $\measuredangle$ qpm $\cong$ qpn by Theorem 5-2. $\overrightarrow{pm}$ and $\overrightarrow{pn}$ are opposite rays, so $\measuredangle$ qpm and $\measuredangle$ qpn are supplementary congruent angles. Therefore each is a right angle and hence $\overleftrightarrow{qp} \perp$ L.

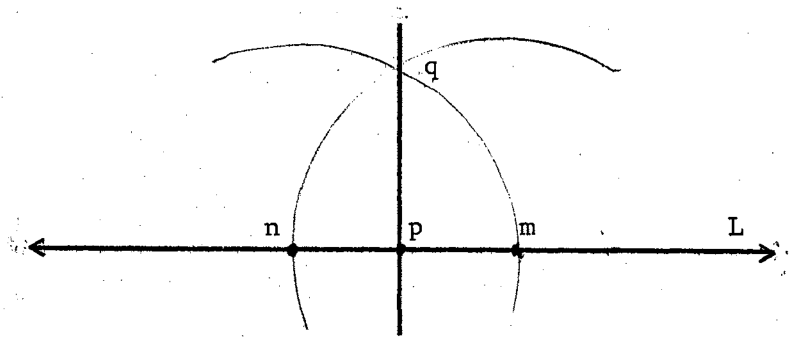


Figure 143

Case (2).

1. Let L be the given line and let a be the given point, $a \notin L$.
2. Adjust the compass so that an arc with center a will intersect L in two points m and n (see Figure 144).
3. Using $\overline{am} \cong \overline{an}$ as radii, construct circles centered at m and n. Since these circles intersect at a they also intersect at a point on the side of L opposite a. Call this second point of intersection b.
4. $\overleftrightarrow{ab} \perp L$.

Statement (4) may be justified by the argument used in Construction 8-2.

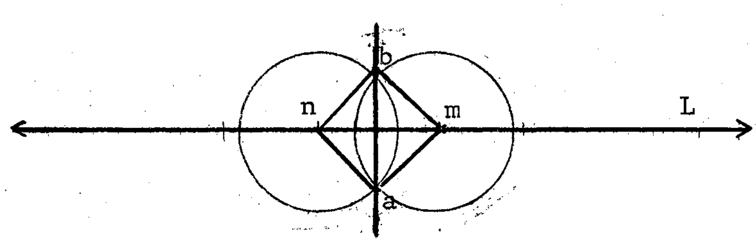


Figure 144

<u>Construction 8-4</u>. Construct the bisector of a given angle.

1. Let $\measuredangle$ paq be the given angle.
2. With center a and any radius swing an arc intersecting $\overrightarrow{ap}$ at b and $\overrightarrow{aq}$ at c, thus $\overline{ab} \cong \overline{ac}$.
3. With radius $\overline{bc}$ and center b swing an arc in the interior of $\measuredangle$ paq, but not in the interior of Δ abc.
4. With radius $\overline{bc}$ and center c swing an arc intersecting the arc in step (3). (These arcs intersect by the two-circle postulate.) (See Figure 145.)
5. Call this point of intersection d.
6. $\overrightarrow{ad}$ is the bisector of $\measuredangle$ paq.

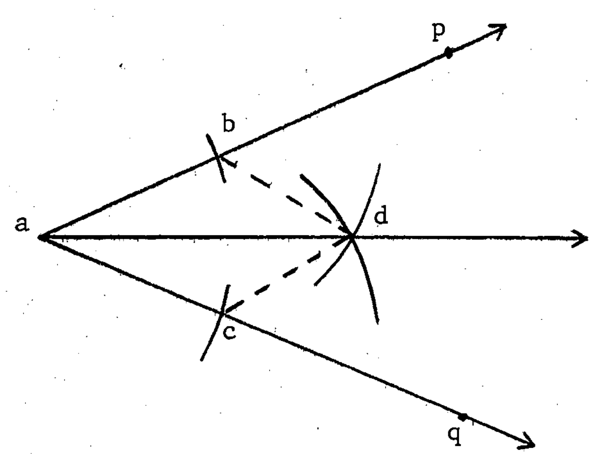


Figure 145

To justify the statement in step (7) consider Δ abd and Δ acd.

$\overline{ab} \cong \overline{ac}$ from step (2). $\overline{bd} \cong \overline{cd}$ since they are radii of congruent circles and $\overline{ad} \cong \overline{ad}$. Therefore $\triangle$ abd $\cong$ $\triangle$ acd by Definition 5-3. Then $\angle$ bad $\cong$ $\angle$ cad by Theorem 5-3 and therefore $\overrightarrow{ad}$ is the bisector of $\angle$ paq by Definition 6-5.

<u>Construction 8-5</u>. Construct an angle congruent to a given angle.

Procedure:

1. Let $\angle$ man be the given angle and $\overrightarrow{op}$ be any ray.
2. With center a and any radius swing an arc intersecting $\overrightarrow{am}$ in b and $\overrightarrow{an}$ in c. Then $\angle$ bac = $\angle$ man (see Figure 146).
3. With center at o and radius $\overline{ab} \cong \overline{ac}$, construct a circle intersecting $\overline{op}$ in d.
4. With d as center and $\overline{bc}$ as radius swing an arc intersecting the circle in q.
5. $\angle$ qod $\cong$ $\angle$ bac and therefore $\angle$ qod $\cong$ $\angle$ man.

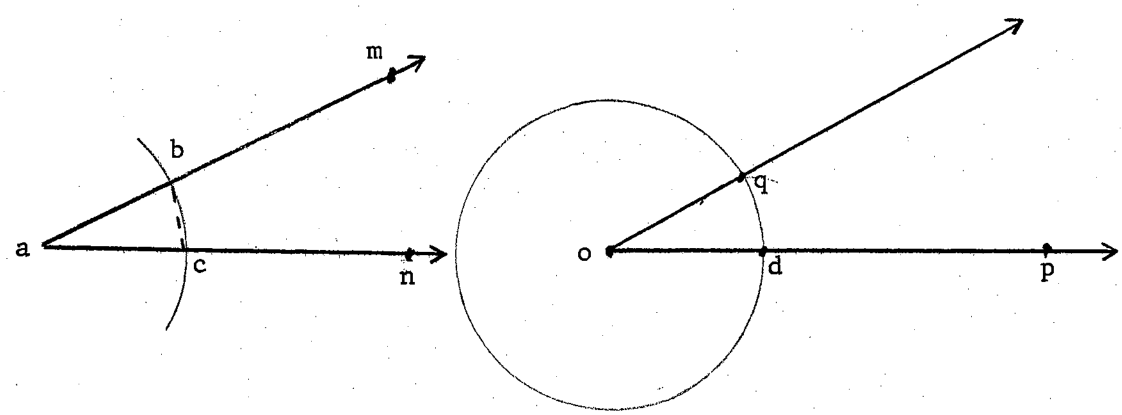


Figure 146

The statement in step (5) is justified by the definition of congruent angles. The point d is the point in $\overrightarrow{op}$ such that $\overline{od} \cong \overline{ac}$ and q is the point in $\overrightarrow{oq}$ such that $\overline{oq} \cong \overline{ab}$, since the segments are radii of congruent circles. $\overline{dq} \cong \overline{cb}$ for the same reason. Therefore $\measuredangle$ qod $\cong$ $\measuredangle$ bac by Definition 5-1.

<u>Construction 8-6</u>. Construct a triangle congruent to a given triangle.

Procedure:

1. Let $\triangle$ abc be the given triangle and $\overrightarrow{pd}$ be any ray (see Figure 147).
2. On $\overrightarrow{pd}$ construct $\overline{pq} \cong \overline{ab}$.
3. With vertex p construct $\measuredangle$ qpm $\cong$ $\measuredangle$ bac.
4. On $\overrightarrow{pm}$ construct $\overline{pn} \cong \overline{ac}$.
5. Then $\overline{pq} \cong \overline{ab}$, $\measuredangle$ qpn $\cong$ $\measuredangle$ bac and $\overline{pn} \cong \overline{ac}$.
6. Therefore $\triangle$ qpn $\cong$ $\triangle$ bac by the S.A.S. theorem.

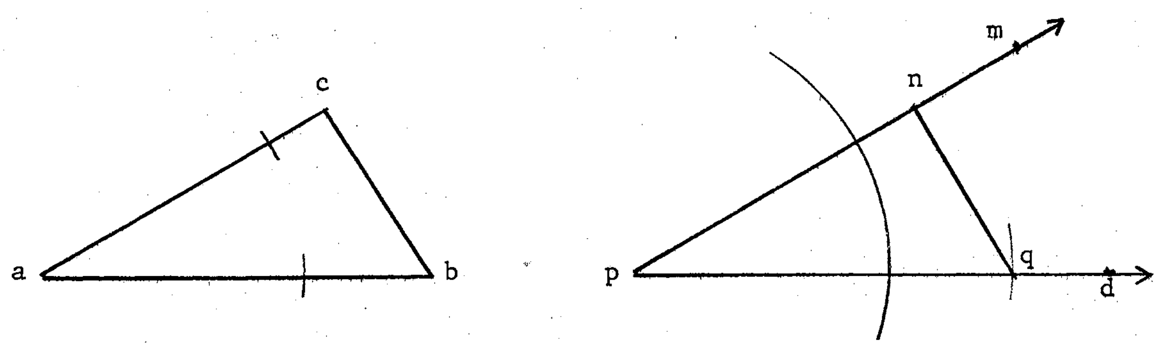


Figure 147

<u>Construction 8-7</u>. Construct a right triangle.

Procedure:

1. Let $\overrightarrow{ap}$ be any ray.
2. At point a in line $\overleftrightarrow{ap}$ construct a line $\overleftrightarrow{aq}$ perpendicular to $\overleftrightarrow{ap}$.
3. $\measuredangle$ qap is a right angle; therefore, $\triangle$ qap is a right triangle since one of its angles is a right angle (see Figure 148).

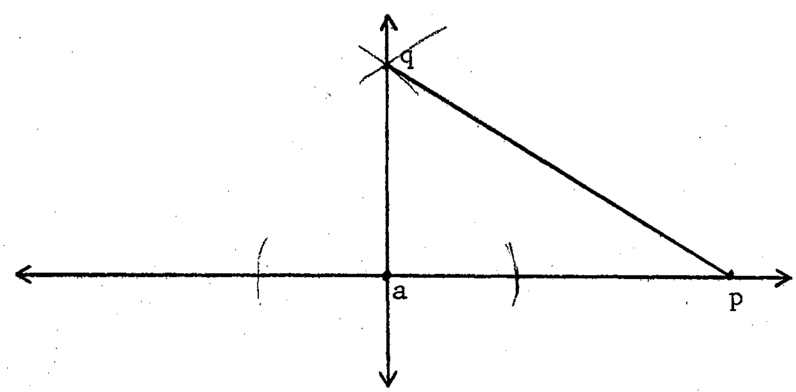


Figure 148

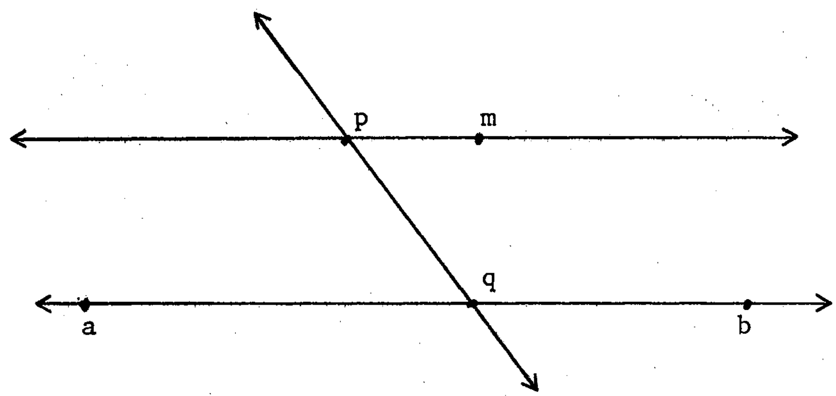
<u>Construction 8-8</u>. Construct a line parallel to a given line that contains a given point not in the given line.

Procedure:

1. Let $\overleftrightarrow{ab}$ be a given line and p be a point such that $p \notin \overleftrightarrow{ab}$ (see Figure 149).
2. Let $\overleftrightarrow{pq}$ be any line intersecting $\overleftrightarrow{ab}$ and for definiteness, assume a-q-b.
3. With vertex p construct $\measuredangle$ qpm $\cong$ $\measuredangle$ pqa such that m and a are on opposite sides of $\overleftrightarrow{pq}$.
4. Then $\overleftrightarrow{pq}$ is a transversal of $\overleftrightarrow{ab}$ and $\overleftrightarrow{pm}$ and $\measuredangle$ qpm and $\measuredangle$ pqa are

alternate interior angles.

5. Since $\measuredangle$ qpm $\cong$ $\measuredangle$ pqa, line $\overleftrightarrow{mp}$ is parallel to line $\overleftrightarrow{ab}$ by Theorem 7-1.



CHAPTER IX

MEASURE

The previous chapters have been concerned with non-metric geometry, that is, geometry without measure. In this chapter the concept of measure of point sets will be considered. Measure involves the correlation of point sets with positive real numbers. The point sets are the entities to be measured and for a particular point set the number assigned to it in any correlation process is called the measure of the point set.

Units of Measure

The above paragraph suggests that a particular point set may be assigned more than one number as a measure. This is a result of the existence of more than one "standard unit of measure." The first step in measuring any point set is to select a particular point set, preferably of the same shape as the set to be measured, as a standard unit of measure. This standard unit of measure is assigned the number 1. The set to be measured is then compared in some way to the standard unit and a number is assigned based on this comparison. Thus the use of different standards results in the assignment of different numbers to the same point set. For example, consider the question, "What is the measure of a yard stick?" The answer could be 1 or 3 or 36 depending on the standard unit of measure. If the standard unit is the yard, then the answer is 1. If the standard unit is the foot, then the answer is 3, and if the standard unit is the inch, then the answer is 36. Note that for a particular standard unit only one number is assigned as the measure. In the discussion to follow a particular unit will be assumed in every instance. Subject to this assumption, if a point set has a measure, then its measure is unique. The number of standard units that constitute a point set S is the measurement of S, denoted M(S).

Measuring Segments

The measure that is assigned to a particular point set depends on the size of the set to be measured relative to the standard unit of measure. The standard unit of measure for segments is some arbitrarily chosen segment which is assigned the number 1 as a measure. Any segment to be measured is compared to this unit segment and assigned a measure based on this comparison. The measure of a segment $\overline{ab}$ will be denoted "ab". Since congruent segments have the same size it is reasonable to assign the same measure to each segment in any set of congruent segments.

<u>Postulate 9-1</u>. Two segments have the same measure if and only if the two segments are congruent. In symbols, ab = cd if and only if $\overline{ab} \cong \overline{cd}$.

Suppose $\overline{ab}$ is a unit segment and $\overrightarrow{mn}$ is any ray (see Figure 150). Let p be the point in $\overrightarrow{mn}$ such that $\overline{mp} \cong \overline{ab}$. For definiteness, assume m-p-n. Let q be the point in $\overrightarrow{pq}$ such that $\overline{pq} \cong \overline{ab}$. It follows from Postulate 9-1 that mp = pq = ab and since $\overline{ab}$ is a unit segment ab = 1. Thus mp = 1 and pq = 1. Note that the model was constructed so that $\overline{mp} \cap \overline{pq} = p$, that is, the intersection of $\overline{mp}$ and $\overline{pq}$ is a single point.

Furthermore $\overline{mp}$ and $\overline{pq}$ are subsets of the same line.

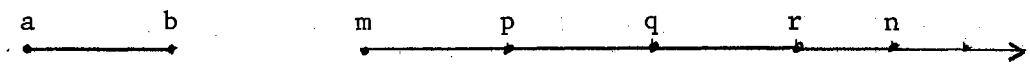


Figure 150

Postulate 9-2. If two segments are subsets of the same line and their intersection is a point, then the measure of their union is the sum of their individual measures.

In Figure 150 $\overline{mq} = \overline{mp} \cup \overline{pq}$. It follows from Postulate 9-2 that $mq = mp + pq = 2$.

To extend this process, suppose that point q in Figure 150 is between p and n. Let r be the point in $\overrightarrow{qn}$ such that $\overline{qr} \cong \overline{ab}$. Then $\overline{qr} \cap \overline{mq} = q$ and $\overline{mr} = (\overline{mp} \cup \overline{pq}) \cup \overline{qr}$; therefore by Postulate 9-2 $mr = (mp + pq) + qr$. But $mp = 1$, $pq = 1$ and $qr = 1$, therefore $mr = (1+1)+1 = 2+1 = 3$.

In general suppose $\overline{ab}$ is a unit segment and $\overrightarrow{pq}$ is a ray. Let the set $\{p_0, p_1, p_2, \ldots, p_{12}\}$ be a set of points in $\overrightarrow{pq}$ such that $p_0 = p$ and such that the segment determined by any two consecutive points in the set is congruent to $\overline{ab}$. (Consecutive points are points having consecutive subscripts.) Since the congruence relation is transitive it follows that $\overline{p_0p_1} \cong \overline{p_1p_2} \cong \overline{p_2p_3}$ etc., or that $\overline{p_ip_{i+1}} \cong \overline{p_jp_{j+1}}$ for all i and j in $\{0,1,2,\ldots,11\}$. Since each of these segments is congruent to the unit segment $\overline{ab}$ it follows from Postulate 9-1 that each segment has measure 1 (see Figure 151).

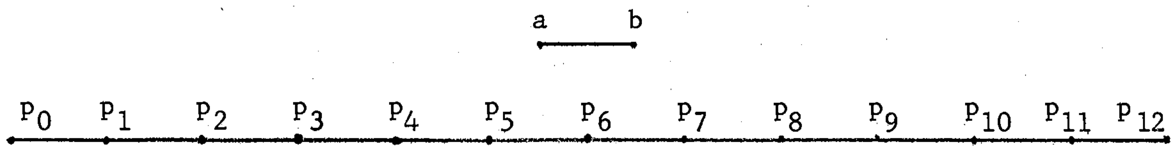


Figure 151

Thus $p_0p_1 = 1$ and from Postulate 9-1 $p_0p_2 = p_0p_1 + p_1p_2 = 2$. Similarly $p_0p_3 = p_0p_2 + p_2p_3 = 2+1 = 3$ and $p_0p_4 = 4$ etc. Thus the measure of $p_op_j = j$ for all $j \epsilon \{1,2,...,12\}$.

The segment $\overline{p_0p_{12}}$ together with the points $\{p_0,p_1,\cdots,p_{12}\}$ is called a ruler. The subscript of the point p_j is the measure of the segment $\overline{p_0p_j}$. For example $p_0p_7 = 7$ (see Figure 152).

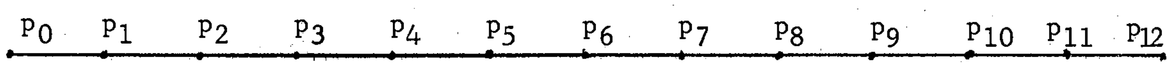


Figure 152

This particular ruler is a twelve unit ruler. An n-unit ruler may be constructed by constructing n congruent copies of some chosen unit segment on a ray such that the consecutive segments determined have exactly one point in common. The measurement of an n-unit ruler is n units. If the point set is a segment and the unit is specified, then the measurement of the segment is referred to as the length of the segment.

The ruler described above is useful for determining the approximate measure of a given segment. Let $\overline{mn}$ be a given segment and $\overline{p_0p_6}$ be a six-inch ruler. In Figure 153 the segment $\overline{mn}$ is approximately congruent to $\overline{p_0p_4}$; hence $\overline{mn}$ is approximately 4. Using the symbol " $\approx$ " for "approximately equal to" this is written $mn \approx 4$. In Figure 154, the segment $\overline{cd}$ is compared to the six-unit ruler $\overline{p_0p_6}$. It appears that the measure of $\overline{p_0p_3}$ is less than the measure of cd and that the measure of $\overline{cd}$ is less than the measure of $\overline{p_0p_4}$. Using the symbol " $<$ " for the phrase "is less than," $p_0p_3 < cd < p_0p_4$. But $p_0p_3 = 3$ and $p_0p_4 = 4$; hence $3 < cd < 4$. Thus the numbers 3 and 4 are both approximations of the measure of $\overline{cd}$. A better approximation of $\overline{cd}$ may be obtained by partitioning $\overline{p_3p_4}$ as follows.

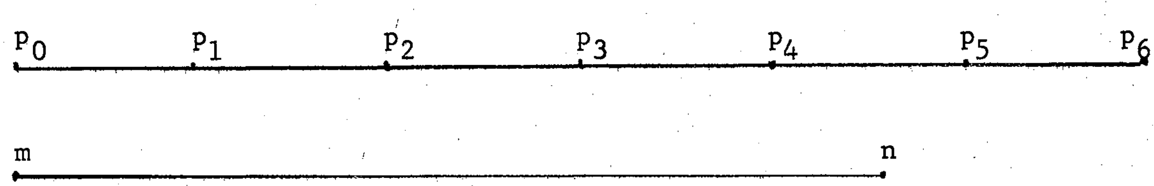


Figure 153

Let p be the midpoint of $\overline{p_3p_4}$. Then $\overline{p_3p} \cup \overline{pp_4} = \overline{p_3p_4}$ and $p_3p \cap pp_4 = p$. So from Postulate 9-2, $p_3p_4 = p_3p + pp_4$. Hence $p_3p + pp_4 = 1$.

But $\overline{p_3p} = \overline{pp_4}$ by the definition of midpoint, so $p_3p = pp_4$ by Postulate 9-1. It follows that $p_3p = \frac{1}{2}$. Then by Postulate 9-2, $p_0p = p_0p_3 + p_3p = 3+\frac{1}{2} = 3\frac{1}{2}$. Figure 154 suggests that $\overline{cd}$ is approximately congruent to $\overline{p_0p}$; therefore $cd \approx 3\frac{1}{2}$.

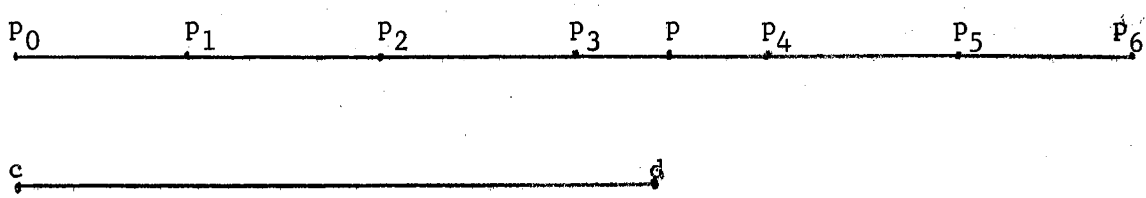


Figure 154

This example suggests that it is advantageous to partition a ruler into subunits. For definiteness consider a six-unit ruler $\overline{p_0p_6}$ such that $\overline{p_ip_{i+1}}$ is congruent to some unit segment $\overline{ab}$ for each $i \in \{0,1,\ldots,5\}$. Let m_i be the midpoint of $\overline{p_{i-1}p_i}$ for each $i \in \{1,2,\ldots,6\}$ (see Figure 155). Then $\overline{p_{i-1}m_i} = m_ip_i$, $i \in \{1,2,\ldots,6\}$ and in each case $\overline{p_{i-1}m_i} \cap \overline{m_ip_i} = m_i$. Thus by Postulate 9-1 $p_{i-1}m_i = m_ip_i$ and by Postulate 9-2 $p_{i-1}p_i = p_{i-1}m_i + m_ip_i$. Since $p_{i-1}p = 1$ it follows that $p_{i-1}m_i = \frac{1}{2}$. Then $p_0m_i = p_0p_{i-1} + \frac{1}{2} = i-1+\frac{1}{2} = i-\frac{1}{2}$, $i \in \{1,2,\ldots,6\}$. Thus $p_0m_3 = 3-\frac{1}{2} = 2\frac{1}{2}$, $p_0m_6 = 6-\frac{1}{2} = 5\frac{1}{2}$, etc. In this process each unit segment in the ruler is subdivided into two subunits each having measure $\frac{1}{2}$. If each of these subunits is partitioned at its midpoint the resulting segments will each have measure $\frac{1}{4}$. In general if this process is repeated n times the measure of each subunit thus determined is $\frac{1}{2}n$. An ordinary ruler uses the "inch" as a standard unit and each unit is subdivided into sixteen subunits. The measurement of length of each subunit is 1/16 inch. Thus an ordinary ruler may be used to determine the measurement of a model of a segment to the nearest sixteenth of an inch. For example if one measures a given segment and contends that the measurement is 3 3/16 inches, this means that the measurement is greater than 3 2/16 inches but less than 3 4/16 inches.

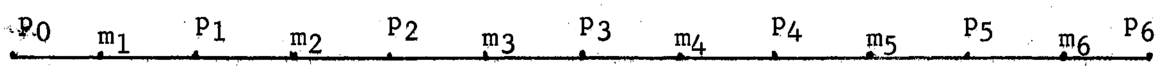


Figure 155

The calibrations on measuring instruments are always rational numbers. In measuring a segment one places the initial point of the measuring instrument at one endpoint of the segment and attempts to match the other endpoint of the segment with one of the calibrated points on the measuring instrument. It may happen that no calibrated point on the instrument matches the second endpoint of the segment. In this case one chooses the calibrated mark that seems most appropriate in the situation and assigns the corresponding number of units as the measurement of the segment. Since this number is a rational number, measurement is referred to as a rational approximation. The accuracy of the approximation depends on the precision of the instrument and the care with which it is used.

Measuring Angles

The process of measuring angles is analogous to the process of measuring segments. An angle is arbitrarily chosen as a standard unit of measure and a given angle is measured by comparing it to the standard unit. As with segments, congruent angles have the same measure.

<u>Postulate 9-3</u>. Two angles have the same measure if and only if they are congruent.

If $M(\measuredangle abc) = M(\measuredangle mno)$ then $\measuredangle abc$ and $\measuredangle mno$ have the same measure and if $\measuredangle abc$ and $\measuredangle mno$ have the same measure, then $M(\measuredangle abc) = M(\measuredangle mno)$.

<u>Postulate 9-4</u>. Let $\overrightarrow{ab}$ be a ray and $\overrightarrow{ap}$ and $\overrightarrow{aq}$ be two rays with p and q in one of the half-planes determined by $\overleftrightarrow{ab}$. If $\measuredangle bap$ and $\measuredangle paq$ are adjacent angles, then $M(\measuredangle baq) = M(\measuredangle bap) + M(\measuredangle paq)$ (see Figure 156).

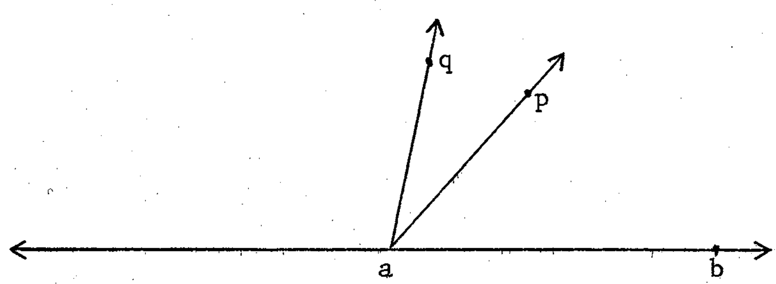


Figure 156

A common device used to measure angles is the protractor. The protractor is calibrated in standard units called a "degree." Consider a semicircle with center o and radius $\overline{oa}$. Let $\overline{ab}$ be a diameter and $\overleftrightarrow{od}$ be a line perpendicular to $\overline{ab}$ at point o. Let c be the point of intersection of $\overleftrightarrow{od}$ and the semicircle (see Figure 157). Then $\measuredangle$ aoc is a right angle. Suppose the arc $\widehat{ac}$ is subdivided into ninety arcs such that if p and q are consecutive points in the subdivision and m and n are consecutive points, then $\overline{pq} \cong \overline{mn}$. If x and y are any two consecutive points in this subdivision, then the measurement of $\measuredangle$ xoy is 1 degree. Thus M($\measuredangle$ poq) = 1 degree and M($\measuredangle$ mon) = 1 degree. Each pair of consecutive points together with point o determines an angle with measurement equal 1 degree. Since there are ninety such angles determined by this subdivision, M($\measuredangle$ aoc) = 90 degrees. Thus the measurement of a right angle is 90^{o} (the symbol "^{o}" means "degree").

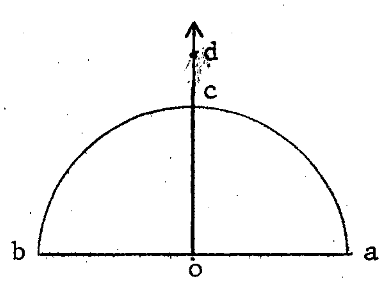


Figure 157

If the arc $\widehat{cb}$ is subdivided as was arc $\widehat{ac}$, one obtains 180 points on the semicircle. These points are numbered consecutively beginning with point a using the set of integers $\{0,1,2,\ldots,180\}$. The semicircle together with the points described is a protractor.

The approximate measure of any angle is obtained by placing the protractor so that 0 is at the vertex of the angle and $\overline{oa}$ coincides with one side of the angle. One of the points of the protractor will correspond approximately with a point on the other side of the angle. The number associated with that point is the measure of the angle. In Figure 158 M($\measuredangle$ mon) = 20° and M($\measuredangle$ mop) = 105°. By definition the sides of an angle are not subsets of the same line. Consequently the measurement of any angle is less than 180°. If $\measuredangle$ mox is any acute angle, then $\overrightarrow{ox}$ is in the interior of $\measuredangle$ aoc and thus M($\measuredangle$ mox) $< 90^{\circ}$. If $\measuredangle$ moy is any obtuse angle, then $\overrightarrow{oc}$ is in the interior of $\measuredangle$ moy; hence M($\measuredangle$ moy) $> 90^{\circ}$.

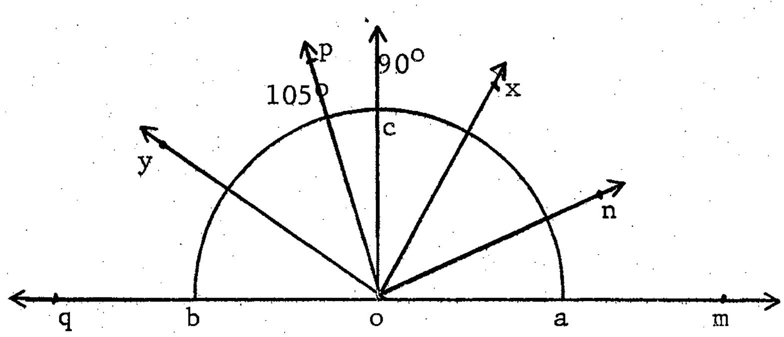


Figure 158

In Figure 158 $\measuredangle$ mop and $\measuredangle$ poq are supplementary angles. The sum of the measurements of these two angles is 180°. The measurement of $\measuredangle$ mop = 105° and the measurement of $\measuredangle$ poq = 85°. If $\measuredangle$ rst is any supplement of $\measuredangle$ mop, then $\measuredangle$ rst $\cong$ $\measuredangle$ poq and thus M($\measuredangle$ rst) = 85°. This suggests the following postulate.

<u>Postulate 9-5</u>. Two angles are supplementary if and only if the sum of their measurements is 180^{o}.

<u>Theorem 9-1</u>. The sum of the measurements of the angles determined by a triangle is 180^{o}.

Proof: Refer to Figure 159.

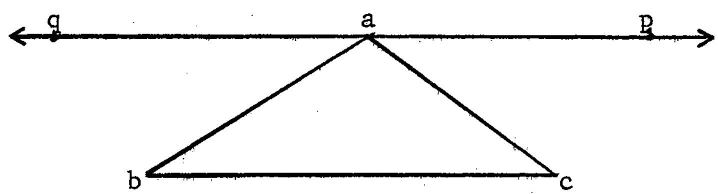


Figure 159

1. Let Δ abc be any triangle and $\overleftrightarrow{ap}$ be the line containing a that is parallel to $\overleftrightarrow{bc}$.
2. Let q be a point in $\overleftrightarrow{ap}$ such that q-a-p.
3. Then $\measuredangle$ qab $\cong$ $\measuredangle$ abc and $\measuredangle$ pac $\cong$ $\measuredangle$ acb by Theorem 7-1.
4. Therefore M($\measuredangle$ qab) = M($\measuredangle$ abc) and M($\measuredangle$ pac) = M($\measuredangle$ acb) by Postulate 9-3.
5. Since $\overleftrightarrow{qa} \parallel \overleftrightarrow{cb}$, $\measuredangle$ qac is a supplement of $\measuredangle$ acb by Theorem 7-8.
6. Therefore M($\measuredangle$ qac) + M($\measuredangle$ acb) = 180^{o} by Postulate 9-5.
7. By Postulate 9-4 M($\measuredangle$ qac) = M($\measuredangle$ qab) + M($\measuredangle$ bac).
8. By substitution from step 7 to step 6, M($\measuredangle$ qab) + M($\measuredangle$ bac) + M($\measuredangle$ acb) = 180^{o}.

9. Substituting from step 4 to step 8, M($\measuredangle$ abc) + M($\measuredangle$ bac) + M($\measuredangle$ acb) = 180°.

Suppose Δ abc is a right triangle and $\measuredangle$ abc is a right angle. Then M($\measuredangle$ acb) + M($\measuredangle$ cab) + M($\measuredangle$ abc) = 180° by Theorem 9-1. But M($\measuredangle$ abc) = 90°. Therefore by substitution, ($\measuredangle$ acb) + M($\measuredangle$ cba) + 90° = 180° and therefore M($\measuredangle$ acb) + M($\measuredangle$ cab) = 180° - 90° = 90°. Thus M($\measuredangle$ acb) < 90° and M($\measuredangle$ cab) < 90°; hence each of these angles is an acute angle.

<u>Definition 9-1</u>. Two angles are complementary angles if and only if the sum of their measurements is 90°.

Since the sum of the measurements of the acute angles of a right triangle is 90° it follows from Definition 9-1 that the acute angles of a right triangle are complementary.

<u>Theorem 9-2</u>. If two angles are complements of congruent angles, then they are congruent.

Proof: Refer to Figure 160.

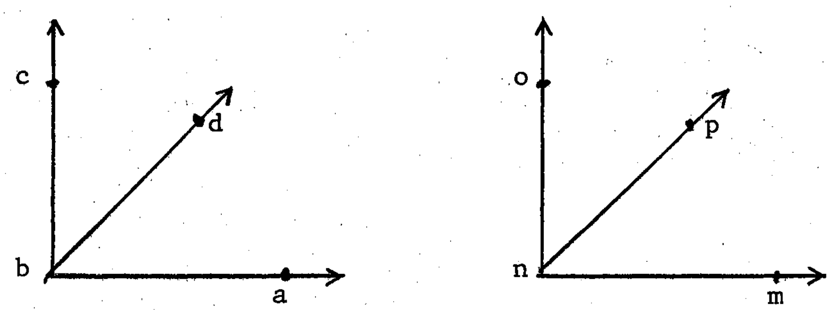


Figure 160

1. By hypothesis:

   (i) $M(\measuredangle abd) + M(\measuredangle dbc) = 90^{\circ}$.

   (ii) $M(\measuredangle mnp) + M(\measuredangle pno) = 90^{\circ}$.

   (iii) $\measuredangle abd \cong \measuredangle mnp$.

2. From Postulate 9-3, $M(\measuredangle abd) = M(\measuredangle mnp)$.
3. From step 1(i), $M(\measuredangle dbc) = 90^{\circ} - M(\measuredangle abd)$.
4. From step 1(ii), $M(\measuredangle pno) = 90^{\circ} - M(\measuredangle mnp)$.
5. Substituting from step 2 into step 4, $M(\measuredangle pno) = 90^{\circ} - M(\measuredangle abd)$.
6. Therefore from steps 3 and 5, $M(\measuredangle dbc) = M(\measuredangle pno)$.
7. From Postulate 9-3, $\measuredangle dbc \cong \measuredangle pno$.

Measuring Polygons

Every polygon partitions the plane into three disjoint subsets, the polygon, its interior and its exterior. The union of the polygon and its interior is called a region and the polygon is the boundary of the region.

The measure of the boundary of a polygon is called the perimeter. Since a polygon is the union of segments the perimeter of a polygon is just the sum of the measures of the segments which constitute the polygon. If abcde is a polygon, then the perimeter of abcde = ab + bc + cd + de + ea.

The measure of a region requires the introduction of a standard unit of measure whose shape is similar to the shape of a region. The standard unit commonly used for measuring regions is the unit square. The unit square is a square such that each of its sides is congruent to a unit segment. The measure assigned the unit square is 1 and its measurement is 1 square unit. The measure of a region is commonly referred to as the area of the polygon that determines the region. Thus the area of a unit square is 1.

Postulate 9-6. If two polygons are congruent then their measures are equal.

Rectangles

Suppose that a rectangle abcd is such that ab = 2 and ad = 3 (see Figure 161). Let m be the midpoint of $\overline{ab}$ and n be the midpoint of $\overline{cd}$. Then mb = ma = 1 and nc = nd = 1. Similarly let r and s be points of $\overline{bc}$ such that br = rs = sc = 1 and let h and j be points of $\overline{ad}$ such that ah = hj = jd = 1. Then $\overline{br} \parallel \overline{ah}$ and $\overline{br} \cong \overline{ah}$ hence abrh is a parallelogram. Similarly rsjh and scdj are parallelograms. Since $\overline{ab} \perp \overline{ad}$ and $\overline{rh}$ and $\overline{sj}$ are both parallel to $\overline{ab}$ it follows that $\overline{rh}$ and $\overline{sj}$ are both perpendicular to $\overline{ad}$. Therefore each of the three parallelograms is a rectangle. Let $\overline{mn} \cap \overline{rh}$ = p and $\overline{mn} \cap \overline{sj}$ = q. Then $\overline{mb} \parallel \overline{pr} \parallel \overline{qs} \parallel \overline{nc}$ and $\overline{ma} \parallel \overline{ph} \parallel \overline{qj} \parallel \overline{nd}$. Further $\overline{ma} \cong \overline{nd}$ and $\overline{ma} \parallel \overline{nd}$; hence mnda is a parallelogram. Therefore $\overline{mn} \parallel \overline{ad}$ but $\overline{ad} \parallel \overline{bc}$ hence $\overline{mn} \parallel \overline{ad} \parallel \overline{bc}$. Since $\overline{rh}$ and $\overline{sj}$ are both perpendicular to $\overline{ad}$, then $\overline{rh}$ and $\overline{sj}$ are both perpendicular to $\overline{ad}$, then $\overline{rh}$ and $\overline{sj}$ are both perpendicular to $\overline{mn}$. It follows that each of the six quadrilaterals in Figure 161 is a rectangle. Further their sides are congruent segments with measure 1; hence each is a unit square. Therefore it is reasonable to assign the number 6 as the measure of the rectangle abcd. Note that this number is the product of the measure of $\overline{ab}$ and the measure of $\overline{ad}$.

Postulate 9-7. If a region R is partitioned into n subregions such that the interiors of the subregions are disjoint, then the measure of the region R is the sum of the measures of the subregions.

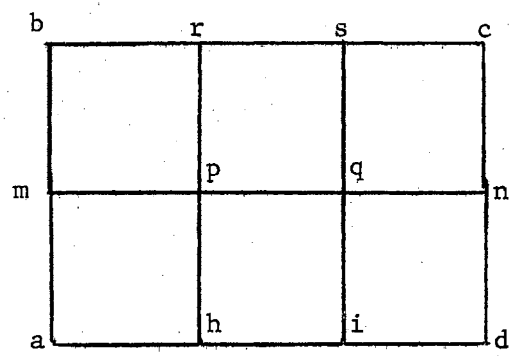


Figure 161

Let the unit square abcd be partitioned as in Figure 162, such that j is the midpoint of $\overline{ad}$ and h is the midpoint of $\overline{bc}$. Thus $aj = bh = jd = hc = \frac{1}{2}$ and $ab = jh = dc = 1$. For the correspondence $\triangle abh \leftrightarrow \triangle jhc$, $\overline{ab} \cong \overline{jh}$, $\angle abc \cong \angle jhc$ (both are right angles) and $\overline{bh} \cong \overline{hc}$. Therefore $\triangle abh \cong \triangle jhc$. Similarly $\triangle ahj \cong \triangle jcd$. Hence abhj $\cong$ jhcd and thus by Postulate 9-6 M(abhj) = M(jhcd). Then by Postulate 9-7 M(abhj) + M(abcd) = 1 square unit. Therefore M(abhj) = $\frac{1}{2}$ square unit, and M(jhcd) = $\frac{1}{2}$ square unit. Again the measure of the rectangular region is the product of the measures of its adjacent sides. This discussion provides an intuitive basis for the following definition.

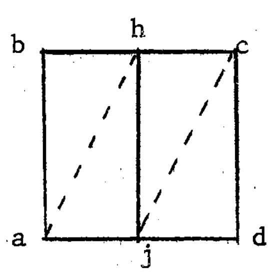


Figure 162

<u>Definition 9-2</u>. The measurement of a rectangular region is the product of the measurements of two adjacent sides. A(bcde) denotes the area of the polygon having vertices b, c, d and e.

The measurements of a pair of adjacent sides of a rectangle are commonly referred to as the length (ℓ) and the width (w) of the rectangle and the measurement of a region is called the area (A). Using this symbolism, Definition 9-2 provides a formula for determining the measurement of a rectangular region. Thus if abcd is a rectangle such that $M(\overline{ab}) = \ell$ units and $M(\overline{ad})$ = w units, then A(abcd) $= \ell \cdot w$ square units. Thus a formula is used to determine the measure of a region rather than a tool such as a ruler or a protractor.

Since in actual practice the measures of the sides of a rectangle are approximations, only the approximate measure of a rectangular region can be determined.

The formula for determining the area of a rectangle provides a basis for formulas for the area of regions determined by parallelograms and triangles.

Parallelograms

Suppose abcd is a parallelogram as in Figure 163. Construct $\overline{bp} \perp \overleftrightarrow{ad}$, and $\overline{nc} \perp \overleftrightarrow{ad}$. Then $\overline{bp} \parallel \overline{cn}$ since they are perpendicular to the same line. Further $\measuredangle$ bpd and $\measuredangle$ cnd are right angles. Therefore bpnc is a rectangle. The opposite sides of a rectangle are congruent segments, therefore bp = nc. If a segment has its endpoints in the lines containing the opposite side of a parallelogram and is perpendicular to one side of the parallelogram, it is perpendicular to the other side. Such a segment is called an altitude of the parallelogram relative to either of the sides that contains an endpoint of the segment. Thus $\overline{bp}$ is an altitude of the parallelogram abcd relative to sides $\overline{bc}$ and $\overline{ad}$. Similarly $\overline{cn}$ is an altitude of the parallelogram abcd relative to the sides $\overline{bc}$ and $\overline{ad}$.

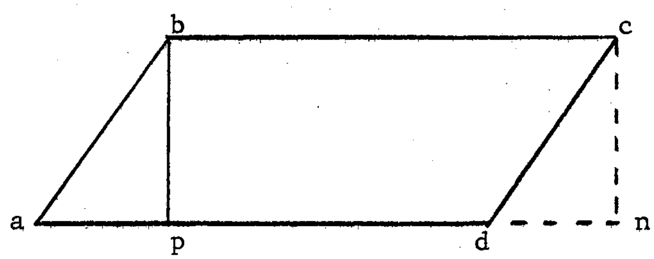


Figure 163

By Definition 9-2 A(bpnc) = $M(\overline{bc}) \cdot M(\overline{bp})$ and by Postulate 9-7 A(abcd) = A(abp) + A(bpdc). Consider the correspondence $\triangle$ apb $\leftrightarrow$ $\triangle$ dnc. Segments $\overline{ab}$ and $\overline{dc}$ are opposite sides of a parallelogram; hence $\overline{ab} \cong \overline{dc}$. Relative to the parallel lines $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ and the transversal $\overleftrightarrow{an}$, $\measuredangle$ bap and $\measuredangle$ cdn are corresponding angles and therefore $\measuredangle$ bap $\cong$ $\measuredangle$ cdn. Since $\measuredangle$ bpa and $\measuredangle$ cnd are right angles, $\triangle$ bpa and $\triangle$ cnd are right triangles. Therefore $\measuredangle$ abp is a complement of $\measuredangle$ bap and $\measuredangle$ dcn is a complement of $\measuredangle$ cdn. Since $\measuredangle$ bap $\cong$ $\measuredangle$ cdn it follows from Theorem 9-2 that $\measuredangle$ abp $\cong$ $\measuredangle$ dcn. Thus $\triangle$ apb $\cong$ $\triangle$ dnc by the A.S.A. theorem. Therefore by Postulate 9-6 A(dnc) = A(apb). Since A(abcd) = A(abp) + A(bpdc) by substitution, A(abcd) = A(dnc) + A(bpdc). But the union of the regions dnc and bpdc is the region bpnc. Then by Postulate 9-7 A(dnc) + A(bpdc) = A(bpnc). Therefore A(abcd) = A(bpnc). But

A(bpnc) = $M(\overline{bc}) \cdot M(\overline{bp})$, hence A(abcd) = $M(\overline{bc}) \cdot M(\overline{bp})$. Thus the area of the region determined by the parallelogram abcd is the product of the measurement of one side and the measurement of an altitude relative to that side. The measure of the altitude is called the height (h) and the relative side is called the base. The measurement of the base is symbolized b.

Theorem 9-3. The measurement of a region determined by a parallelogram is the product of the measurement of one of the sides and the measurement of an altitude relative to that side.

In symbols, if the measurement of the base of a parallelogram abcd is b units and the measurement of the altitude is h units, then A(abcd) = b · h square units.

Triangles

Let $\triangle$ abc be any triangle and $\overline{am}$, $\overline{cn}$ and $\overline{bo}$ be segments such that $\overline{am} \perp \overleftrightarrow{bc}$, $\overline{cn} \perp \overleftrightarrow{ab}$ and $\overline{bo} \perp \overleftrightarrow{ac}$ as in Figure 164. Each of the segments $\overline{am}$, $\overline{cn}$ and $\overline{bo}$ are called altitudes of the triangle. An altitude is a segment having one endpoint at a vertex and the other endpoint in the line determined by the side opposite that vertex such that the segment is perpendicular to the line. The measurement of the altitude is called the height (h).

Consider $\triangle$ abc and the line $\overleftrightarrow{ap}$ such that $\overleftrightarrow{ap} \parallel \overline{bc}$. Let d be the point in $\overline{ap}$ such that $\overline{ad} \cong \overline{bc}$ (see Figure 165). Then abcd is a quadrilateral such that a pair of opposite sides are parallel and congruent. By Theorem 7-11, abcd is a parallelogram. Let $\overline{am}$ be an altitude of the parallelogram abcd. Then $\overline{am} \perp \overleftrightarrow{bc}$ and therefore $\overline{am}$ is an altitude of $\triangle$ abc. Consider the correspondence $\triangle$ abc $\leftrightarrow$ $\triangle$ cda. The segments $\overline{ab}$

and $\overline{cd}$ are opposite sides of a parallelogram, hence $\overline{ab} \cong \overline{cd}$. Similarly $\overline{bc} \cong \overline{da}$. Since the congruence relation is reflexive, $\overline{ac} \cong \overline{ac}$. Therefore the corresponding sides of $\triangle$ abc and $\triangle$ cda are congruent segments and hence $\triangle$ abc $\cong$ $\triangle$ cda. Hence, from Postulate 9-6, A($\triangle$ abc) = A($\triangle$ cda). By Postulate 9-7, A($\triangle$ abc) + A($\triangle$ cda) = A(abcd) and by Theorem 9-3 M(abcd) = M($\overline{bc}$) $\cdot$ M($\overline{am}$). Thus A($\triangle$ abc) + A($\triangle$ cda) = M($\overline{bc}$) $\cdot$ M($\overline{am}$). Then by substitution A($\triangle$ abc) + A($\triangle$ abc) = M($\overline{bc}$) $\cdot$ M($\overline{am}$) or 2A($\triangle$ abc) = M($\overline{bc}$) $\cdot$ M($\overline{am}$). Therefore A($\triangle$ abc) = $\frac{1}{2}$M($\overline{bc}$) $\cdot$ M($\overline{am}$). This proves the following theorem.

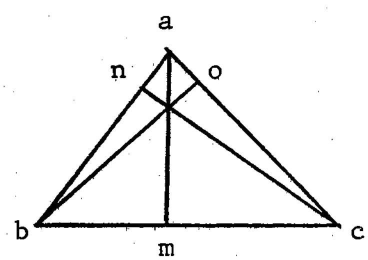


Figure 164

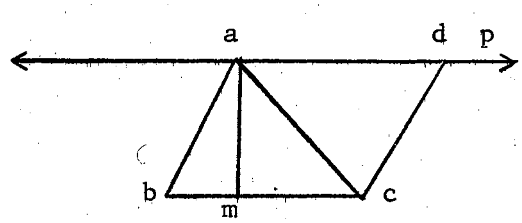


Figure 165

<u>Theorem 9-4</u>. The measurement of a triangular region is one-half the product of the measurement of one side and the measurement of the altitude to that side.

In symbols let $\overline{bc}$ be the base of Δ abc and $\overline{am}$ be the altitude to the side $\overline{bc}$. If $M(\overline{bc}) = b$ and $M(\overline{am}) = h$, then $A(\Delta\ abc) = \frac{1}{2}b \cdot h$.

The Pythagorean Theorem

The formula for the measurement of a rectangular region provides a basis for developing a relation that exists between the sides of a right triangle.

<u>Theorem 9-5</u>. The Pythogorean Theorem. The square of the measure of the hypotenuse of a right triangle is equal to the sum of the squares of the measures of the other two sides. In symbols, if Δ acb is a right triangle such that c is the measure of the hypotenuse, b is the measure of one side and a is the measure of the other side, then $a^2 + b^2 = c^2$.

Proof: Let $\overline{mn}$ be a segment and q be a point in $\overline{mn}$ such that mq = a and qn = b. Let mnop be a square having $\overline{mn}$ as one side. Since the sides of a square are congruent segments, $\overline{mn} \cong \overline{no} \cong \overline{op} \cong \overline{pm}$ (see Figure 166). Let r, s and t be points on $\overline{no}$, $\overline{op}$, and $\overline{pm}$ respectively such that mq = nr = os = pt = a. Then qn = ro = sp = tm = b.

1. Consider Δ tmq, Δ qnr, Δ ros, and Δ spt.
2. Each is a right triangle, $\overline{tm} \cong \overline{qn} \cong \overline{ro} \cong \overline{sp}$ and $\overline{mq} \cong \overline{nr} \cong \overline{os} \cong \overline{pt}$; therefore the four triangles are mutually congruent by the S.A.S. theorem.
3. Thus $\overline{tq} \cong \overline{qr} \cong \overline{rs} \cong \overline{st}$. Let the measure of each of these be denoted by c.

4. Since $\measuredangle$ mqt and $\measuredangle$ tqn are supplementary angles, M($\measuredangle$ mqt) + M($\measuredangle$ tqn) = 180°.

5. But M($\measuredangle$ tqn) = M($\measuredangle$ tqr) + M($\measuredangle$ rqn); hence M($\measuredangle$ mqt) + M($\measuredangle$ tqr) + M($\measuredangle$ rqn) = 180°.

6. Since $\measuredangle$ mqt and $\measuredangle$ mtq are complementary, M($\measuredangle$ mqt) + M($\measuredangle$ mtq) = 90°.

7. But $\measuredangle$ nqr $\cong$ $\measuredangle$ mtq, so M($\measuredangle$ nqr) = M($\measuredangle$ mtq).

8. Therefore M($\measuredangle$ mqt) + M($\measuredangle$ nqr) = 90° by substitution.

9. From steps 8 and 5 it follows that M($\measuredangle$ tqr) = 90°.

10. Therefore $\overline{qt} \perp \overline{qr}$.

11. Similarly $\overline{rs} \perp \overline{qr}$.

12. Therefore $\overline{qt} \parallel \overline{rs}$ and since $\overline{qt} \cong \overline{rs}$ it follows from Theorem 7-11 that the quadrilateral tqrs is a parallelogram.

13. Since $\measuredangle$ tqr is a right angle and $\overline{tq} \cong \overline{qr} \cong \overline{rs} \cong \overline{st}$, the parallelogram tqrs is a square.

14. From step 3 and Definition 9-2, A(tqrs) = $c \cdot c = c^2$ square units.

15. Since mn = a + b, A(mnop) = $(a + b)(a + b) = a^2 + 2ab + b^2$ square units.

16. From steps 1 and 2 and Theorem 9-4, A($\triangle$ tmq) = A($\triangle$ qnr) = A($\triangle$ ros) = A($\triangle$ spt) = $\frac{1}{2}ab$ square units.

17. From Postulate 9-7 A(mnop) = A(tqrs) + A($\triangle$ tmq) + A($\triangle$ qnr) + A($\triangle$ ros) + A($\triangle$ spt).

18. By substitution, $a^2 + 2ab + b^2 = c^2 + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}ab$ or $a^2 + 2ab + b^2 = c^2 + 2ab$.

19. Therefore $a^2 + b^2 = c^2$.

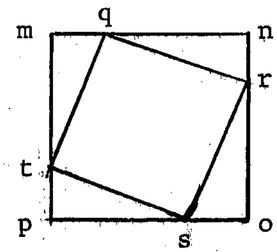


Figure 166

Measuring Circles

The boundary of a circular region is called the circumference and its measure is symbolized by the letter "c". Experimentation with various circular objects having different diameters will suggest that the ratio of the measure of the circumference to the measure of the diameter is the same number for all circles. One may conduct such experiments by measuring the circumference and diameter of vegetable cans or similar circular objects with a tape measure and dividing the measure of the circumference by the measure of the diameter. In advanced mathematics it is established that this ratio is the same for all circles and is approximated by the number 3 1/7. This number is denoted by the Greek letter π. Using decimals, $\pi \approx 3.1416$. If the measure of the diameter of a circle is symbolized by the letter "d", then $c/d = \pi$. The measure of the radius of a circle is denoted r and since a diameter is the union of 2 radii, $d = 2r$. Therefore $c/d = c/2r = \pi$ and $c = 2\pi r$. The expression $c = 2\pi r$ is the formula for the measure of the circumference of a circle in terms of the radius.

Consider a circle with center o and an inscribed polygon

$p_1p_2 \cdots p_{20}$ such that $p_ip_{i+1} \cong p_jp_{j+1}$ for each i and $j \varepsilon \{1,2,\ldots,20\}$. That is, the sides of the polygon are mutually congruent segments (see Figure 167).

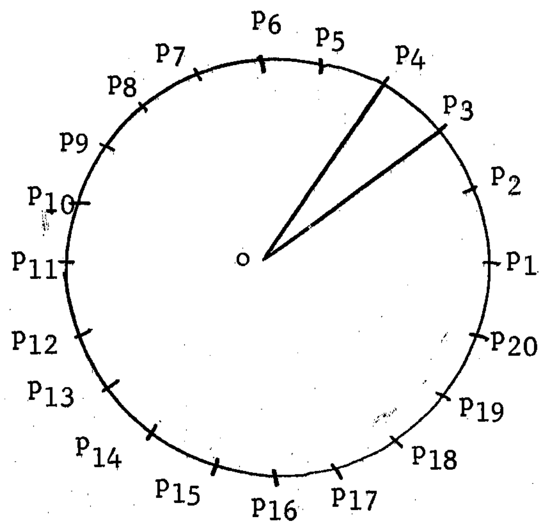


Figure 167

Such a polygon is called a regular polygon, that is a polygon is a regular polygon if all sides and angles are congruent. A regular polygon having n sides is called an n-gon. If p_i and p_{i+1} are consecutive vertices of an inscribed n-gon and o is the center of the circle, then p_iop_{i+1} is a triangle. Any other triangle determined by consecutive vertices of the n-gon and the center of the circle is congruent to $\Delta\, p_iop_{i+1}$ since the corresponding sides are congruent segments. Thus $\Delta\, p_3op_4$ is one such triangle. Thus an inscribed n-gon determines a set of n mutually congruent triangles in the interior of the circle. The perimeter of the n-gon is an approximation of the measure of the circumference of the circle and the sum of the areas of the triangular regions is an approximation of the area of the circular region. Larger values of n give rise to better approximations. For large values of n, the measure of the altitude of the associated triangle is approximately r.

Suppose the measure of the side of an inscribed n-gon is ℓ. Then, the perimeter is approximately $n \cdot \ell$. Thus $c \approx n \cdot \ell$. The area of each of the associated triangular regions is approximately $\frac{1}{2} \cdot r \cdot \ell$; thus the area of the circular region is approximately $n \cdot \frac{1}{2} \cdot r \cdot \ell = \frac{1}{2} r \cdot n \cdot \ell$. But $n \cdot \ell \approx c$ and $c = 2\pi r$; hence the area of the circular region is approximately $\frac{1}{2} \cdot r \cdot c = \frac{1}{2} \cdot r \cdot 2\pi r = \pi r^2$. Since the n-gon is inscribed in the circular region, the perimeter of the n-gon is less than the circumference of the circle. That is, $n \cdot \ell < c$. Also the altitude of each associated triangle is less than the radius of the circle. Consequently the area of any inscribed n-gonal region is less than the number πr^2.

Now consider an n-gon such that the sides are tangent to the circle, called a circumscribed n-gon (see Figure 168). For large values of n the perimeter is an approximation of the circumference of the circle, but is greater than the circumference. If each side has measure s then $n \cdot s > c$, but $n \cdot s \approx c$. The area of the n-gonal region is $n \cdot \frac{1}{2} \cdot r \cdot s = \frac{1}{2} \cdot r \cdot n \cdot s$ but $n \cdot s \approx c = 2\pi r$; hence the area of the circular region is approximately $\frac{1}{2} r \cdot 2\pi r = \pi r^2$. In this case the area of the n-gonal region is greater than πr^2 since $n \cdot s > c$. Hence the area of any circumscribed n-gonal region is greater than the number πr^2.

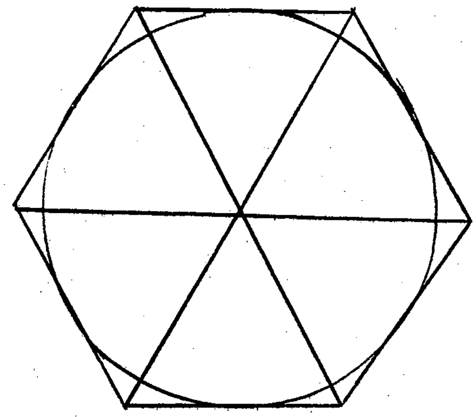


Figure 168

The examples in the last two paragraphs suggest the following definition.

<u>Definition 9-3</u>. If r is the measure of the radius of a circle, then the area of the circular region is πr^2 , that is $A(O) = \pi r^2$.

CHAPTER X

POINT SETS IN 3-SPACE

The point sets previously considered have been subsets of the plane. This chapter will consider certain point sets that are not subsets of a plane.

A line may be thought of as a set in 1-space. Subsets of the plane are sets in 2-space. In Chapter II, it was noted that two intersecting lines determine a plane. In particular two perpendicular lines determine a plane and thus 3 mutually perpendicular lines determine 3-space. In the remainder of this discussion 3-space will be referred to as space.

A plane partitions space into three disjoint subsets. One is the plane and the other two are half-spaces. Two points are in the same half-space if and only if the segment determined by the points does not intersect the plane that determines the half-space.

In Chapter II it was noted that if two planes intersect, then the intersection is a line. Suppose planes M and N intersect in line $\overleftrightarrow{ab}$. The line $\overleftrightarrow{ab}$ determines two half-planes in M. Let c be a point in one of these half-planes. Similarly let d be a point in one of the half-planes in N determined by $\overleftrightarrow{ab}$. The union of the line $\overleftrightarrow{ab}$, the c-side of $\overleftrightarrow{ab}$ and the d-side of $\overleftrightarrow{ab}$ is called a dihedral angle (see Figure 169). The line $\overleftrightarrow{ab}$ is called the edge of the dihedral angle and the two half-planes are called faces of the dihedral angle.

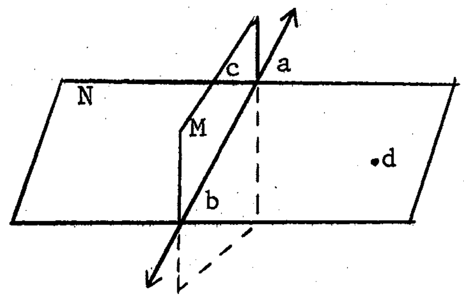


Figure 169

Let m be a point in the edge $\overleftrightarrow{ab}$ of a dihedral angle. Let p be a point in one face of the angle and q be a point in the other face such that $\overleftrightarrow{mp} \perp \overleftrightarrow{ab}$ and $\overleftrightarrow{mq} \perp \overleftrightarrow{ab}$. Then $\measuredangle$ pmq is called a plane angle of the dihedral angle. A dihedral angle is a right dihedral angle if and only if the associated plane angle is a right angle.

Definition 10-1. Two planes are perpendicular if and only if they determine a right dihedral angle.

Let M be a plane and $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ be lines in M such that $\overleftrightarrow{ab} \cap \overleftrightarrow{cd} = p$. The line $\overleftrightarrow{pq}$ is perpendicular to M if and only if $\overleftrightarrow{pq} \perp \overleftrightarrow{ab}$ and $\overleftrightarrow{pq} \perp \overleftrightarrow{cd}$.

Definition 10-2. A line $\overleftrightarrow{nq}$ is perpendicular to plane M at a point p if and only if $\overleftrightarrow{nq}$ is perpendicular to at least two distinct lines in M that contain p (see Figure 170).

Simple Closed Surfaces

A simple closed surface is a set of points that partitions space into three disjoint subsets, called the interior of the surface, the exterior of the surface, and the surface. A simple closed surface which is the union of a finite number of polygonal regions is called a polyhedron. The polygonal regions are called faces of the polyhedron and any segment determined by the intersection of two faces is called an edge of the polyhedron.

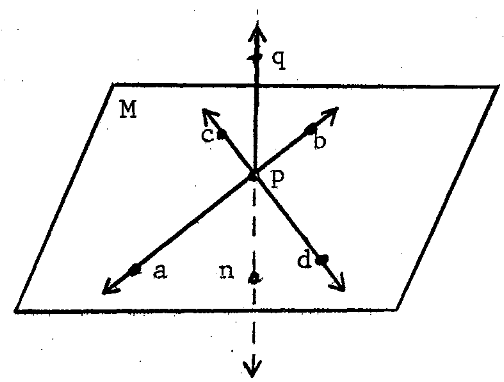


Figure 170

Prisms

A proper subset of the set of polyhedrons is the set of prisms. A prism is a polyhedron such that two of its faces are congruent polygons in parallel planes and its other faces are regions determined by parallelograms. The two parallel congruent faces are called bases of the prism and all other faces are called lateral faces. If the bases of a prism are parallelograms, then the prism is a parallelepiped. If the bases are rectangles, then the prism is a rectangular prism, and if the bases are triangular, then the prism is a triangular prism (see Figure 171). An altitude of a prism is a segment having its endpoints in the two planes determined by the bases such that the segment is perpendicular to the two planes. Each lateral face of a prism determines a plane that intersects the planes determined by the bases. Thus each lateral face together with a base determines a dihedral angle. If each dihedral angle so determined is a right dihedral angle then the prism is a <u>right prism</u>. That is, a prism is a <u>right prism</u> if and only if its lateral faces are perpendicular to its bases. If each face of a right prism is a square, then the prism is a <u>cube</u>.

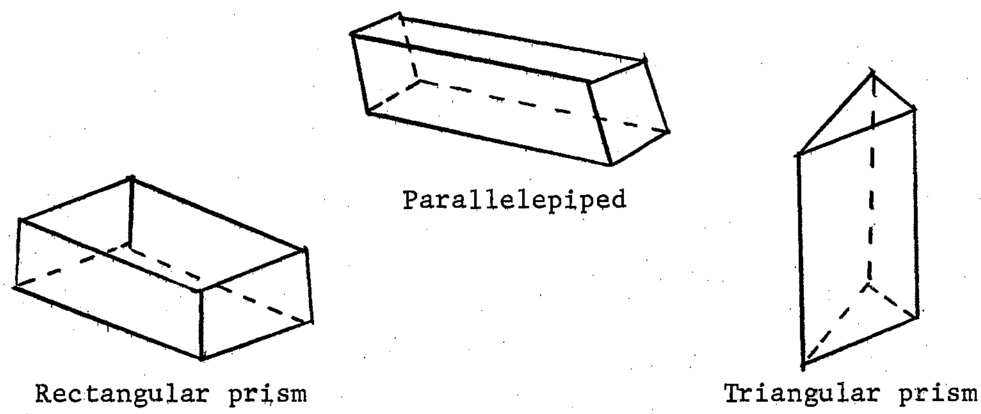


Figure 171

Cylinders

Consider two parallel planes M and N. Let C be a simple closed curve in M and H be a simple closed curve in N such that $C \cong H$. Let p be a point in C and q be a point in H. Let S be the union of all segments $\overline{xy}$ such that $x \in C$, $y \in H$ and $\overline{xy} \parallel \overline{pq}$. Then the union of region C, region H and region S is a <u>cylinder</u>. The regions determined by the curves C and H are called the bases of the cylinder. If it is possible for each segment in S to be perpendicular to the planes determined by the bases, then the cylinder is a right cylinder. If the curves C and H are circles, then the cylinder is a circular cylinder. An ordinary vegetable can is a model of a right circular cylinder (see Figure 172). An altitude of a cylinder is a segment having its endpoints in the plane determined by the bases such that the segment is perpendicular to these planes. The set S is called the lateral surface of the cylinder.

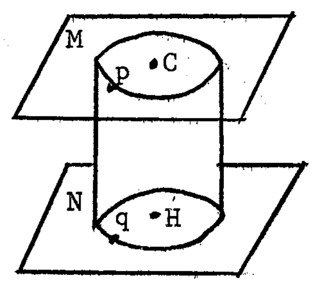


Figure 172

Cones

Consider a simple closed curve C in a plane M and a point p such that $p \notin M$. The union of the region determined by the curve C with the set of all segments determined by points in curve C and the point p is a <u>cone</u>. The <u>altitude</u> of the cone is a segment with p as one endpoint and the other endpoint in the plane of curve C such that the segment is perpendicular to the plane. The point p is called the <u>vertex</u> and the region determined by C is called the <u>base</u> of the cone. If the curve C is a circle then the cone is a circular cone. If the intersection of the altitude of a circular cone with the plane of the base is the center of the base, then the cone is a right circular cone (see Figure 173).

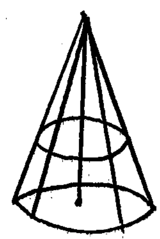


Figure 173

Pyramids

Consider a cone in which the curve C is a polygon abcd in a plane M and a point p not in M. Any two consecutive vertices of the polygon together with p determines a triangle. The union of the triangular regions thus determined with the polygonal region is called a <u>pyramid</u> (see Figure 174). The polygonal region is called the base of the pyramid. The triangular regions are called lateral faces and p is called the vertex of the pyramid. The altitude of the pyramid is the segment $\overline{pq}$ where p is the vertex, q is in the plane of the base and $\overline{pq}$ is perpendicular to the plane of the base.

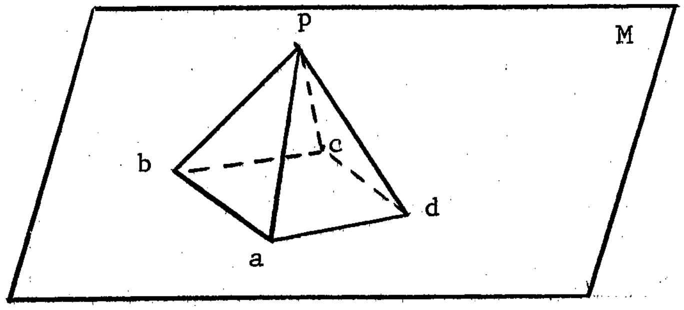


Figure 174

Sphere

The last simple closed surface to be considered is the sphere. Let o be a point in space and $\overline{ab}$ be a segment. The set of all points p in space such that $\overline{op} \cong \overline{ab}$ is a <u>sphere</u>. If q is any point in the sphere, then $\overline{oq}$ is a radius of the sphere. The point o is called the center of the sphere (see Figure 175). A diameter is a segment that has its endpoints on the sphere and contains the center of the sphere.

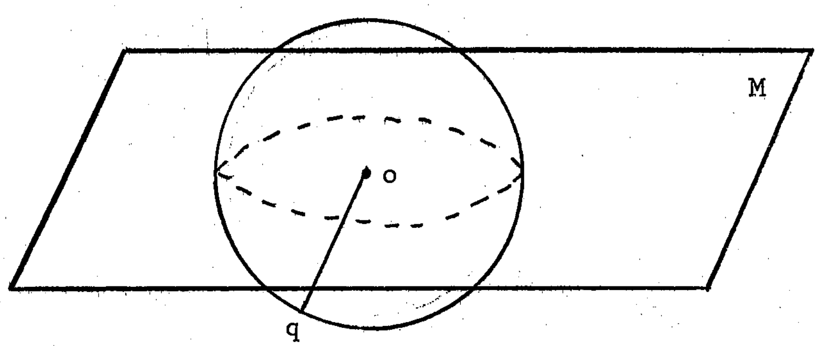


Figure 175

If a plane intersects a sphere, then the intersection is a point or a circle. If the intersection is a point, then the plane is said to be tangent to the sphere. If a plane that intersects a sphere contains the center of the sphere, then the intersection of the plane and the sphere is called a great circle of the sphere.

Volume

The measurement of a region in space determined by a simple closed surface is called its volume. The standard unit of measure for volume is a unit cube, that is a cube such that each edge has length 1 unit. The volume of a unit cube is 1 unit · 1 unit · 1 unit = 3 cubic units. The volume of a region in space is the number of unit cubes that constitute the region.

Prisms

Consider a prism having a parallelogram for a base and an altitude having measurement 1 unit. If the measurement of one side of the parallelogram is b units and the measurement of the altitude is h units, then the area of the parallelogram is $b \cdot h$ square units. That is the number of square units that constitute the base is $b \cdot h$. Since the measurement of the altitude of the prism is 1 unit, then each square unit in the base gives rise to a cubic unit in the region. Thus the volume is $1 \cdot (b \cdot h)$ cubic units (see Figure 176).

In general if the area of the base of a prism is $b \cdot h$ square units and the measurement of the altitude is a units, then the volume is given by the formula $V = a \cdot b \cdot h$ (see Figure 177).

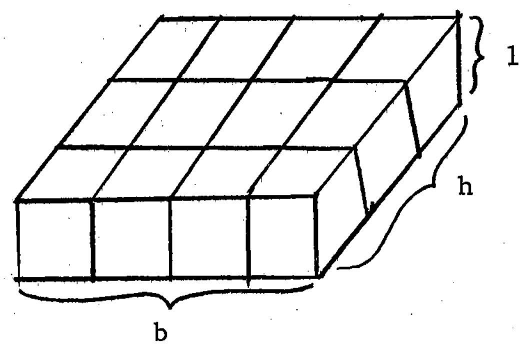


Figure 176

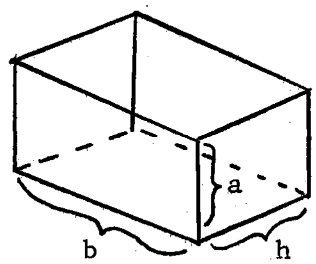


Figure 177

If a prism is a triangular prism, then its base is a triangle. If the measurement of one side of the triangle is b units and the measurement of the altitude to that side is h units, then the area of the base of the prism is $\frac{1}{2}b \cdot h$ square units. If the measurement of the altitude of the prism is a, then the volume is $\frac{1}{2}b \cdot h \cdot a$ cubic units.

Cylinders, Cones and Pyramids

The formula for the volume of a circular cylinder is determined in the same manner as the formulas for the volumes of prisms. If the measurement of the radius of the base of a cylinder is r units, then the area of the base is πr^2 square units. If the measurement of the altitude is h, then the volume is given by $V = \pi r^2 \cdot h$ cubic units.

The formula for the volume of a cone results from a relation that exists between the cone and a cylinder such that the altitude of the cone is congruent to the altitude of the cylinder and the base of the cone is congruent to the base of the cylinder. In Figure 178 the measurement of the radius of the base of the cone is r units and the measurement of the radius of the base of the cylinder is r units. The measurement of the altitudes of both regions is h units. If a cone and a cylinder are related in this manner, then the volume of the cone is 1/3 of the volume of the cylinder. Thus the volume of a cone is given by the formula $V = 1/3 \pi r^2 \cdot h$ cubic units.

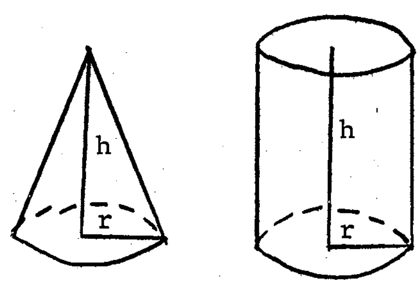


Figure 178

The volume of a pyramid is related to the volume of a prism in the same way that the volume of a cone is related to the volume of a cylinder. Thus for a given pyramid and a given prism, if the base of the pyramid is congruent to the base of the prism and their altitudes are congruent, then the volume of the pyramid is 1/3 the volume of the prism. Therefore the volume of a pyramid is given by the formula $V = 1/3\ A_b \cdot a$ cubic units where A_b represents the number of square units in the base. This number may be determined from one of the previous formulas if the base is a parallelogram or a triangle.

Spheres

The volume of a sphere is related to the volume of a cylinder and the volume of a cone as follows. Suppose the measurement of the radius of a given sphere is r units. Consider a cylinder and a cone such that the measurement of the radius of the base of each is r units and such that the measure of the altitude is 2r units. Then the volume of the sphere is the volume of the cylinder minus the volume of the cone. Hence the formula for the volume of the sphere is:

$$V = \pi r^2 \cdot (2r) - 1/3\ \pi r^2 \cdot (2r)$$
$$= 2\pi r^3 - 2/3\ \pi r^3$$
$$= 4/3\ \pi r^3 \text{ cubic units}$$

(see Figure 179).

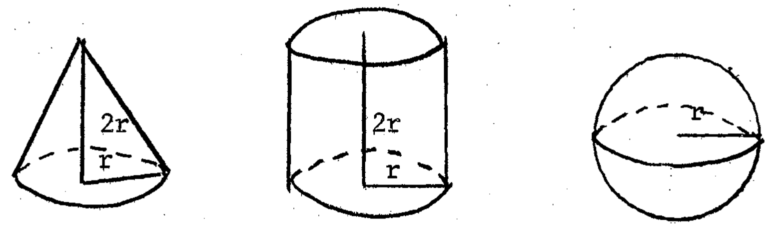


Figure 179

CHAPTER XI

CONCLUSION

The development in this discourse was structured specifically to be of assistance in the training of elementary school mathematics teachers. It is hoped that the study of this material will provide the reader with sufficient depth of understanding to treat, with confidence, the geometric concepts that occur in the elementary school mathematics curriculum. No topics are included that are not relevant to the development of such an understanding. This does not mean that the development consists only of material that is included in the elementary school mathematics curriculum. Modern mathematics education requires on the part of the teacher a much greater comprehension than would be expected from elementary school students.

The topics that were included in the study were considered appropriate in the sense that each belongs to one or more of the following categories.

1. Topics that occur in the elementary school curriculum.
2. Topics that contribute to an understanding of the relations that exist between the geometric concepts that occur in the elementary school mathematics curriculum.
3. Topics that help provide a basis for independent study in mathematics.

Relative to today's textbooks, a reasonably accurate determination of topics in the first category is possible. However, it is only reasonable to conclude that curricula will continue to change. Consequently the concepts in tomorrow's texts may be expected to differ from the concepts found in present editions. This suggests that the topics in the second and third categories are the most significant for elementary school mathematics teachers.

Modern pedagogical procedures stress the importance of the structure of a discipline. Regarding the nature of structure, Bruner comments as follows:

> Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure, in short, is to learn how things are related (/2/, p. 7).

Mathematics education has often been criticized for offering students a collection of seemingly unrelated topics to be memorized. An objective of modern mathematics education is to present the subject in a way such that new ideas are developed through past experiences and understandings and also in a manner that may contribute to the development of related concepts to be learned. Teaching mathematics in this manner requires an understanding of structure. Therefore relations such as congruence, parallelism and perpendicularity are emphasized far more in this study than in the elementary school curriculum.

In view of the ever changing curriculum it would be inappropriate to expect that preservice training could be extensive enough to last a lifetime. As a consequence a teacher must expect to continue his own education indefinitely. Certainly most teachers will be involved in some formal training beyond the Bachelor's degree, either in in-service study or in summer school. Nevertheless, it is important that every teacher be prepared to do independent study in any field in which he teaches. It follows that formal courses should be designed so as to prepare the student to do independent study. Thus topics in the third category are appropriate in this study.

Mathematics is by its nature a deductive science. Consequently an understanding of the nature of deductive inference is prerequisite to independent study in mathematics. This development places a great deal of emphasis on deductive techniques such as conditional statements, the nature of definitions, direct proofs and indirect proofs. While these topics are not a part of the elementary school curriculum they are emphasized here for two reasons:

1. An understanding of these topics contributes to an understanding of the structure of geometry.
2. Most mathematics texts assume that the reader is familiar with these techniques and therefore an understanding of these topics is essential to independent study in mathematics.

An effort has been made to make this material readable. The proofs and explanations are detailed. The sequence of the development is such that new ideas are introduced in terms of previous understandings or are based on intuition and experience. However, the mathematical background of those for whom it is intended is limited. Consequently this material is not intended for independent study but rather for use in an organized class under the supervision of a competent instructor.

In the interest of providing material for independent study in the area of elementary school geometry this writer suggests as a further study a programmed development of geometry designed specifically for the elementary school mathematics teacher.

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APPENDIX A

SUMMARY OF GEOMETRIC CONCEPTS OCCURRING IN SELECTED ELEMENTARY SCHOOL MATHEMATICS TEXTS

Concept	Number of Series in which Concept Occurs	Number of Texts in which Concept Occurs	Grade Level(s)
Acute Angles	2	3	5-6
Acute Triangles	1	1	5
Altitude	3	4	5-6
Angles	7	24	2-6
Arcs	3	5	5-6
Area	7	16	3-6
Area of Circle	4	6	3, 5-6
Area of Triangle	5	7	3, 5-6
Area of Rectangle	7	14	3-6
Area of Square	6	11	3, 5-6
Base of Parallelogram	3	5	5-6
Base of Pyramid	1	1	6
Base of Triangle	4	6	5-6
Center of Circle	4	9	2-6
Central Angle	3	3	4,6
Chord	4	4	3,6
Circle	7	31	1-6
Circumference	2	2	5-6
Concentric Circles	1	1	6
Cone	4	5	4-6
Congruence	4	11	2-6
Congruent Angles	4	9	2,3,5,6
Congruent Segments	3	9	2-6
Congruent Triangles	2	6	2,3,5,6

Concept	Number of Series in which Concept Occurs	Number of Texts in which Concept Occurs	Grade Level(s)
Congruent Polygons	1	1	3
Construction	3	5	5-6
Bisect Angle	2	3	5-6
Circle	3	3	5-6
Copy Angle	2	4	5-6
Copy Segment	3	4	5-6
Copy Triangle	3	4	5-6
Parallel Lines	1	1	6
Perpendicular Bisector	2	3	5-6
Perpendicular Lines	2	2	6
Right Angle	1	1	6
Triangles	3	3	5-6
Cube	7	12	3-6
Cubic Measure	5	5	6
Cylinder	4	6	4-6
Diagonal	4	6	4-6
Diameter	7	12	3-6
Dihedral Angles	1	1	5
Edge of a Cube	4	7	3-6
Endpoint	4	14	1-6
Equilateral Triangle	5	11	3-6
Face of a Cube	4	7	3-6
Equiangular Triangle	1	2	5-6
Height	5	8	1,5,6

Concept	Number of Series in which Concept Occurs	Number of Texts in which Concept Occurs	Grade Level(s)
Half-plane	1	1	5
Hexagon	3	5	4-6
Hypotenuse	2	4	4-6
Inscribed Angle	2	2	4,6
Intersecting Lines	7	10	2,4-6
Intersecting Planes	2	2	4-5
Isosceles Triangle	5	9	3-6
Isosceles Right Triangle	1	1	4
Legs of a Right Triangle	1	2	4,6
Length	7	41	1-6
Lines	7	27	1-6
Measurement	7	32	1-6
Measurement of Angles	5	8	4-6
Measurement of Circles	2	3	5-6
Measurement of Cubes	2	2	4,6
Measurement of Cylinders	1	1	5
Measurement of Squares	5	5	4-6
Measurement of Triangles	3	3	5-6
Midpoint	2	3	4-5
Obtuse Angle	2	3	5-6
Parallel	6	10	4-6
Parallel Lines	6	14	2,4-6
Parallelograms	5	8	4-6
Pentagon	3	4	4-6

Concept	Number of Series in which Concept Occurs	Number of Texts in which Concept Occurs	Grade Level(s)
Perimeter	7	17	3-6
Perpendicular Bisector	2	2	6
Perpendicular Lines	6	10	4-6
Pi	1	1	6
Planes	7	14	4-6
Plane Figure	4	6	4-6
Points	7	29	1-6
Polygon	7	18	2-6
Prism	6	9	4-6
Protractor	4	6	4-6
Pyramid	6	8	4-6
Pythagorean Theorem	2	3	4,6
Quadrilateral	6	20	1-6
Radius	6	11	3-6
Ray	6	17	2-6
Rectangle	7	32	1-6
Rectangular Prism	7	11	4-6
Rectangular Pyramid	6	7	4-6
Region	5	13	1-6
Rhombus	2	4	4-6
Right Angle	7	24	2-6
Right Triangle	6	15	3,4-6
Segment	7	26	1-6
Sides	4	16	1-6

Concept	Number of Series in which Concept Occurs	Number of Texts in which Concept Occurs	Grade Level(s)
Simple Closed Figure	4	11	3-6
Skew Lines	1	1	4
Space	2	4	2-4,6
Space Figures	2	3	4-6
Space Geometry	1	2	5-6
Sphere	5	10	3-6
Square	7	32	1-6
Sums of Angle Measurement	2	2	6
Supplementary Angle	1	1	6
Tangent Circles	1	1	6
Trapezoid	2	4	4-6
Triangle	7	35	1-6
Triangular Prism	6	7	4-6

APPENDIX B

LIST OF DEFINITIONS

DEFINITIONS

Definition

3-1. Space is the set of all points.

3-2. A set "S" of points is said to be collinear if and only if every point of the set belongs to the same line.

3-3. Two lines are parallel if and only if they are in the same plane and their intersection is empty.

3-4. A line L and a plane P are parallel if and only if their intersection is empty.

3-5. Two lines are skew if and only if their intersection is empty and they do not lie in the same plane.

3-6. Two planes are parallel if and only if their intersection is empty.

4-1. The set consisting of the points a and b and all of the points between a and b is called a segment.

4-2. The ray $\overrightarrow{ab}$ is the union of the segment $\overline{ab}$ and the set of all points p such that b is between a and p.

4-3. An angle is the union of two noncollinear rays having the same endpoint.

4-4. A point p is an element of the interior of $\measuredangle$ abc if and only if p is in the a-side of $\overleftrightarrow{bc}$ and p is in the c-side of $\overleftrightarrow{ab}$.

4-5. If $\measuredangle$ abc is a subset of a plane M, then the set consisting of all points of M that are not in $\measuredangle$ abc or the interior of $\measuredangle$ abc is called the exterior of $\measuredangle$ abc.

4-6. A set of points S is said to be a convex set if and only if for every two points a and b of S the segment $\overline{ab}$ is also in S.

4-7. A polygon is a simple closed curve which is the union of segments.

5-1. Let $\measuredangle$ abc and $\measuredangle$ mno be two given angles. Let p be the point in ray $\overrightarrow{nm}$ such that $\overline{np} \cong \overline{ba}$, and q be the point in ray $\overrightarrow{no}$ such that $\overline{nq} \cong \overline{bc}$. Then $\measuredangle$ mno $\cong$ $\measuredangle$ abc if and only if $\overline{pq} \cong \overline{ac}$.

5-2. If a, b and c are three noncollinear points, then $\overline{ab} \cup \overline{bc} \cup \overline{ca}$ is a triangle.

Definition

5-3. $\triangle$ abc $\cong$ $\triangle$ mno if and only if $\overline{ab} \cong \overline{mn}$, $\overline{bc} \cong \overline{no}$ and $\overline{ac} \cong \overline{mo}$.

5-4. Two polygons are congruent if and only if there is a partition of the polygon into triangles such that the corresponding triangles are congruent.

6-1. A triangle is an isosceles triangle if and only if two of its sides are congruent segments.

6-2. $\triangle$ abc is an equilateral triangle if and only if $\overline{ab} \cong \overline{bc}$ and $\overline{bc} \cong \overline{ac}$, that is if all of its sides are congruent.

6-3. A point p is the midpoint of a segment $\overline{ab}$ if and only if p is an interior point of $\overline{ab}$ and $\overline{ap} \cong \overline{pb}$.

6-4. A point p is in the interior of $\triangle$ abc if and only if p is in the interior of each of the three angles of the triangle.

6-5. The bisector of $\measuredangle$ paq is a ray $\overrightarrow{ao}$ in the interior of $\measuredangle$ paq such that $\measuredangle$ pao $\cong$ $\measuredangle$ oaq.

6-6. Two angles are adjacent angles if and only if they have a common vertex, a common side and their interiors are disjoint sets.

6-7. Two angles are supplementary adjacent angles if and only if (1) they are adjacent angles, and (2) their non-common sides are opposite rays.

6-8. Let $\measuredangle$ mno be any angle and let $\overrightarrow{ns}$ be the ray opposite $\overrightarrow{nm}$. Then $\measuredangle$ abc is a supplement of $\measuredangle$ mno if and only if $\measuredangle$ abc $\cong$ $\measuredangle$ ons. That is, $\measuredangle$ abc is a supplement of $\measuredangle$ mno if and only if $\measuredangle$ abc is congruent to an angle that is adjacent to and a supplement of $\measuredangle$ mno.

6-9. Two angles determined by two intersecting lines are vertical angles if and only if their sides determine pairs of opposite rays.

6-10. Two intersecting lines L and M are perpendicular lines if and only if the adjacent angles determined are congruent.

6-11. $\measuredangle$ abc is a <u>right angle</u> if and only if $\measuredangle$ abc is congruent to one of its supplements.

6-12. Let $\measuredangle$ abc be a right angle: (1) $\measuredangle$ abd is an acute angle if and only if $\overrightarrow{bd}$ is in the interior of $\measuredangle$ abc. (2) $\measuredangle$ abe is an obtuse angle if and only if $\overrightarrow{bc}$ is in the interior of $\measuredangle$ abe.

Definition

7-1. Let $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ be two coplanar lines and let $\overleftrightarrow{ab}$ be a transversal intersecting $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ in the distinct points a and b respectively. Then {$\measuredangle$ bam, $\measuredangle$ abd} is a set of alternate interior angles if and only if m and d are on opposite sides of the transversal $\overleftrightarrow{ab}$.

7-2. Let $\overleftrightarrow{mn}$ and $\overleftrightarrow{cd}$ be cut by a transversal $\overleftrightarrow{ab}$ such that $\measuredangle$ mab and $\measuredangle$ dba are alternate interior angles. The angles in {$\measuredangle$ mab, $\measuredangle$ cbq} are corresponding angles if and only if c and q are points such that the angles in {$\measuredangle$ cbq, $\measuredangle$ dba} are vertical angles.

7-3. Two lines are parallel if and only if they are coplanar and their intersection is empty.

7-4. A quadrilateral is a convex quadrilateral if and only if its sides are such that no side of the quadrilateral intersects the line determined by the opposite side.

7-5. If abcd is a quadrilateral then a point is in the interior of abcd if and only if it is in the interior of each of the angles of the quadrilateral.

7-6. Two segments $\overline{ab}$ and $\overline{cd}$ are parallel segments if and only if $\overleftrightarrow{ab}$ and $\overleftrightarrow{cd}$ are parallel lines.

7-7. A quadrilateral is a trapezoid if and only if at least one pair of opposite sides of the quadrilateral are parallel segments.

7-8. A quadrilateral is a parallelogram if and only if the opposite sides of the quadrilateral are parallel segments.

7-9. A rectangle is a parallelogram which determines right angles.

8-1. Let o be a point in a plane M and $\overline{ab}$ be a segment. The set of all points p in the plane M such that $\overline{op} \cong \overline{ab}$ is a circle.

8-2. If O is a circle in a plane M, then a point $p \in M$ is in the interior of circle O if and only if the segment $\overline{op}$ does not intersect the circle.

8-3. If O is a circle with center o and radius $\overline{op}$ and K is a circle with center k and radius $\overline{kq}$, then circle O $\cong$ circle K if and only if $\overline{op} \cong \overline{kq}$.

9-1. Two angles are complementary angles if and only if the sum of their measurements is 90^{o}.

Definition

9-2. The measurement of a rectangular region is the product of the measurements of two adjacent sides.

9-3. If r is the measurement of the radius of a circle, then the area of the circular region is πr^2, that is $A(\odot) = \pi r^2$.

10-1. Two planes are perpendicular if and only if they determine a right dihedral angle.

10-2. A line $\overleftrightarrow{nq}$ is perpendicular to plane M at point p if and only if $\overleftrightarrow{nq}$ is perpendicular to at least two distinct lines in M that contain p.

APPENDIX C

LIST OF POSTULATES

Postulate

3-1. If "L" is a line, then "L" is a set of points.

3-2. A line is a proper subset of space.

3-3. If a and b are two different points, then there is exactly one line that contains both a and b.

3-4. If two distinct lines L and M intersect, then the intersection is exactly one point.

3-5. If a plane contains two points of a line, then the plane contains every point of the line; that is, the plane contains the entire line.

3-6. If a, b and c are noncollinear points then there is exactly one plane that contains a, b and c.

3-7. Every plane contains more than one line.

4-1. If a is between b and c, then a, b and c are collinear.

4-2. If a, b and c are three points in the same line, then exactly one of the points is between the other two.

4-3. Let L be a line in a plane M. If a and b are two points of M such that a and b are not in L, then a and b are on the same side of L if and only if $\overline{ab} \cap L = \{\ \}$.

5-1. If $\overline{ab}$ is any segment and $\overrightarrow{cd}$ is any ray, then there exists exactly one point p in $\overrightarrow{cd}$ such that the segment $\overline{cp}$ is congruent to the segment $\overline{ab}$.

5-2. For all segments: (a) $\overline{ab} \cong \overline{ab}$, (b) if $\overline{ab} \cong \overline{cd}$ then $\overline{cd} \cong \overline{ab}$, and (c) if $\overline{ab} \cong \overline{cd}$ and $\overline{cd} \cong \overline{pq}$ then $\overline{ab} \cong \overline{pq}$.

5-3. Let $\measuredangle$ abc be any angle and L be any line in a plane M. Let H be one of the half-planes in M determined by L. If $\overrightarrow{pq}$ is any ray in L, then there exists exactly one ray $\overrightarrow{pr}$ with r in H such that $\measuredangle$ rpq $\cong$ $\measuredangle$ abc.

6-1. If $\overline{ab}$ is any segment, then $\overline{ab}$ has exactly one midpoint.

6-2. Let $\triangle$ abc be any triangle and for definiteness consider $\measuredangle$ bac, then: (1) If q is an interior point of $\overline{bc}$, then q is in the interior of $\measuredangle$ bac. (2) If d is any point in the interior of $\measuredangle$ bac, then every point of the ray $\overrightarrow{ad}$ except a is in the interior of $\measuredangle$ bac. (3) If $\overrightarrow{ao}$ is any ray in the interior of $\measuredangle$ bac, then $\overrightarrow{ao}$ intersects the side $\overline{bc}$ is an interior point of $\overline{bc}$.

6-3. All right angles are congruent.

Postulate

7-1. If L is a line and p is a point not in L, then there exists one and only one line M containing p, and coplanar with L such that M is parallel to L.

8-1. If $\overline{pq}$ and $\overline{mn}$ are diameters of the same circle, then $\overline{pq} \cong \overline{mn}$.

8-2. The Two Circle Postulate. Let $\overline{ok}$, $\overline{ab}$ and $\overline{mn}$ be distinct segments. Let q be the point in $\overrightarrow{ok}$ such that $\overline{oq} \cong \overline{ab}$ and p be the point in $\overrightarrow{ko}$ such that $\overline{kp} \cong \overline{mn}$, then: (1) If $\overline{oq} \cap \overline{kp}$ is a segment, then the circle with center o and radius $\overline{ab}$ intersects the circle with center k and radius $\overline{mn}$ in two points on opposite sides of $\overleftrightarrow{ok}$. If the two circles intersect on one side of $\overleftrightarrow{ok}$, then they will intersect in the opposite side of $\overleftrightarrow{ok}$. (2) If $\overline{oq} \cap \overline{kp}$ is a point, then the two circles in statement (1) intersect in one point and are called tangent circles. (3) If $\overline{op} \cap \overline{kp} = \{\ \}$, then the circles do not intersect.

9-1. Two segments have the same measure if and only if the two segments are congruent.

9-2. If two segments are subsets of the same line and their intersection is a point, then the measure of their union is the sum of their individual measures.

9-3. Two angles have the same measure if and only if they are congruent.

9-4. Let $\overrightarrow{ab}$ be a ray and $\overrightarrow{ap}$ and $\overrightarrow{aq}$ be two rays with p and q in one of the half-planes determined by $\overleftrightarrow{ab}$. If $\measuredangle$ bap and $\measuredangle$ paq are adjacent angles, then M($\measuredangle$ bap) = M($\measuredangle$ bap) + M($\measuredangle$ paq).

9-5. Two angles are supplementary if and only if the sum of their measurements is 180°.

9-6. If two polygons are congruent then their measures are equal.

9-7. If a region R is partitioned into n subregions such that the interiors of the subregions are disjoint, then the measure of the region R is the sum of the measures of the subregions.

VITA

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Bill R. Grimes

Candidate for the Degree of

Doctor of Education

Thesis: A MODERN GEOMETRIC DEVELOPMENT FOR ELEMENTARY SCHOOL MATHEMATICS TEACHERS

Major Field: Higher Education - Mathematics

Biographical:

Personal Data: Born in Buffalo, Oklahoma, March 12, 1927, son of Raymond Walter and Ethel A. Grimes.

Education: Attended grade school in Yuba and Achille, Oklahoma; graduated from Achille High School in 1945; received Bachelor of Arts degree from Southeastern State College, Durant, Oklahoma with a major in Mathematics in May, 1950; received Master of Teaching degree from Southeastern State College in July, 1956; received Master of Science in Mathematics degree at Oklahoma State University in May, 1959; completed requirements for Doctor of Education degree in May, 1969.

Professional Experience: United States Army, 1945-46; Buffalo, Oklahoma, High School Mathematics instructor, 1950-51; Fox, Oklahoma, High School Mathematics instructor, 1951-58; Fellow in National Science Foundation Academic Year Institute, Oklahoma State University, 1958-59; Southeastern State College, Durant, Oklahoma, instructor department of mathematics, 1959-63; Science Faculty Fellow, Oklahoma State University, 1963-65; Assistant Professor of Mathematics, Southeastern State College, 1965-68.