A UNIFIED STATE-SPACE APPROACH TO RLCT TWO-PORT TRANSFER FUNCTION SYNTHESIS

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A UNIFIED STATE-SPACE APPROACH TO RLCT TWO-PORT

TRANSFER FUNCTION SYNTHESIS

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CHAPTER I

INTRODUCTION

<u>1.1 Statement of the Problem.</u> As automation becomes more common in all aspects of industrial, scientific, and domestic processes, the digital computer is relied upon to accomplish more and more tasks. It is obvious then that the spectrum of processes, from the simplest to the most sophisticated, must be "systematized" to allow programming for computer control. This "systematizing" implies development of unified approaches in all procedures.

Synthesis of transfer functions is a primary consideration in any process to be controlled. Adaptation of the synthesis procedure to digital programming is probably the next consideration. At the present time transfer functions cannot be synthesized with two-port linear networks utilizing a unified procedure in the complex frequency or s-domain and the synthesis procedure varies depending on the function. Neither are the present synthesis procedures easily programmable on the digital computer. It is recognized that, in general, a topological approach to the synthesis problem offers greater insight than the complex frequency approach; that the state-model provides network topological information; and that the state-model lends itself to digital computer application.

The state-model of a general linear RLC network has the form:

$$\frac{d}{dt} X_1 = A_1X_1 + B_1Y_1^* + C_1 \frac{d}{dt} Y_1^*$$
$$\underline{Y}_1^* = P_1X_1 + Q_1Y_1^* + R_1 \frac{d}{dt} Y_1^* \qquad (1.1.1)$$

where Y_1^* and $\underline{Y}_1^*$ corresponds to the terminal variables of the network and the state-vector, X_1, consists of the branch capacitor voltages and chord inductor currents.

Equation 1.1.1 corresponds to the general RLC network, however we shall consider the following reduced state-model which will simplify the synthesis procedure.

$$\frac{d}{dt} X = AX + BY^*$$
$$\underline{Y}^* = PX + QY^* \qquad (1.1.2)$$

This state-model corresponds to a RLC network which has certain topological restrictions. These restrictions allow the above mentioned procedural simplifications and they are presented later.

The Laplace transform of Equation 1.1.2 yields

$$\underline{Y}^*(s) = [P(sU-A)^{-1}B + Q]Y^*(s) \qquad (1.1.3)$$

Then the characteristic polynomial of the restricted network is

$$|sU-A| = 0 \qquad (1.1.4)$$

The problem considered in this dissertation is to obtain a state-model of the form given in Equation 1.1.2 from the s-domain transfer functions: (1) short circuit transfer admittance, $y_{12}(s)$, (2) open circuit transfer impedance, $z_{12}(s)$, or (3) voltage transfer function, T(s). At the same time the state-model should correspond to a RLC network with or without transformers. Further the procedure should be the same regardless of the type of transfer function. This is achieved here and since this is a state-space approach to transfer function synthesis, it is programmable on the digital computer.

1.2 Previous Work in this Area. Lucal (25) developed a procedure to realize special classes of driving-point and transfer functions as three-terminal RC networks. Guillemin (17) and Dasher (11) presented other RC synthesis procedures resulting in two-port networks. Fialkow and Gerst (14) have presented the conditions necessary for a transfer function to be realizable. Ho (18,19) developed a transfer function synthesis procedure resulting in two-port RLC networks and it was based on a matrix factorization technique. Yengst (31), Karni (20), and Weinberg (33) have written texts which present some of these synthesis procedures. Included in these texts is Guillemin's two-element-kind parallel ladder realization of transfer functions.

It must be emphasized that the synthesis procedures mentioned above are complex frequency domain procedures.

Since Bashkow (4) first defined the A-matrix in 1957 for use in network and system theory, there has been a growing interest in the state-model concept of system analysis and synthesis. Bryant (6) in 1962 considered the explicit form of Bashkow's A-matrix and determined a general matrix expression of the state-model for a RLC network. Brown (5) in 1963 considered the derivative-explicit differential equations for a RLC network which has drivers. Rauch (29) in 1963 dealt with the realization of time-domain models of real linear bielement systems. Dervisoglu (12,13) in 1964 considered the problem of realizing the A-matrix for a RLC network under the condition that the number of state-variables is equal to the number of reactive elements in the network and the resistive subnetwork is connected. Levy and Brown (24) in 1965 considered the time-domain description of a class of RLC graphs in terms of a first-derivative-explicit system of differential equations. Kuh and Rohrer (22) in 1965 reviewed the state-variable approach to network analysis. Layton (23) in 1966 considered state equation descriptions of passive networks. Bacon (2) in 1966 established the constraints on the topology and element values of the n-port RLC network which are necessary and sufficient for the network to have a given time-response. Daniel and Grigsby (10) in 1966 presented a procedure for testing a given state-model for realizability as a passive RLC one-port network. Anderson and Newcomb (1) in 1968 have given state-space procedures for positive real matrices using RLC networks, transformers, and gyrators. However all of the state-space synthesis procedures are given for positive real matrices of functions.

Marshall (26) in 1966 presented the synthesis of a doubly terminated ladder network which is lossless except for resistances in the terminal branches. This synthesis is accomplished by transforming a singly terminated network into an equivalent doubly terminated network by successive perturbations of an associated tridiagonal matrix. Navot (27) in 1967 developed a procedure for finding certain subclasses of tridiagonal matrices with prescribed eigenvalues. Yarlagadda (34) presented a procedure for obtaining a tridiagonal matrix with prescribed eigenvalues and a transformation matrix that transforms the obtained tridiagonal to a desired form. This transformed matrix is then equated to the general A-matrix that Bryant (6) determined. This yields a portless ladder network with a characteristic polynomial that has roots equal to the eigenvalues of the obtained tridiagonal matrix. If there are n eigenvalues then this procedure yields a ladder network that has one resistor in a terminal branch and n reactive elements. The ideas presented in the above three papers are fundamental to this thesis.

1.3 Research Necessary to Develope the Synthesis Procedure. In the research to develope a procedure for the synthesis of two-port transfer functions it is found that not only is the characteristic polynomial realization a problem, but that obtaining the transmission zeros or numerator polynomial is also a problem.

Using the one resistor network developed by Yarlagadda (34), it is found that not all degrees of numerator polynomial can be obtained when all possible port locations are tried. Therefore a realization of the characteristic polynomial is developed that will yield a portless ladder network with two resistors, one in each of the terminal branches, and n reactive elements. This realization procedure uses a tridiagonal matrix presented by Navot (27) and a transformation presented by Yarlagadda (34). The resulting portless ladder network yields all possible numerator degrees by properly selecting the port terminals. This location is not unique for each numerator degree. Therefore a systematic procedure is developed to locate an acceptable position for every possible numerator polynomial.

The A-matrix of Equation 1.1.2 must have a special form to yield a ladder network in the characteristic polynomial realization procedure. This form is obtained with the tridiagonal realization and transformation mentioned above. Now it is seen that the B and P matrices of Equations 1.1.2 and 1.1.3 have certain relationships with this A-matrix. Therefore when the transfer function is obtained from the state-model, these relationships with respect to the numerator of the transfer function have to be determined. However, it is first found advantageous to develope an algorithm that yields the [adj(sU-A)] by simple recursive calculations. With this done the general expressions for the state-model transfer function numerators are determined and this gives the desired relationships between the A, B, and P matrices.

The component values of the ladder network resulting from the synthesis procedure are now obtained by the following method. By using the A, B, and P matrices above with Bryant's (6) general state-model and the relationships determined above, a set of non-linear algebraic equations are determined whose unknowns include the ladder network component values. A solution which will always work for this set of equations is developed.

In the research that developes the above set of non-linear algebraic equations, it is found to be necessary to interconnect ladder networks in such a manner so that all of the transfer function numerator coefficients are satisfied. A procedure is established as to how many ladder networks must be interconnected and what these interconnections must be. This procedure is based on the coefficients of the transfer function numerator and the type of transfer function.

These interconnections probably require the addition of transformers to satisfy the validity test. Therefore a technique is developed to utilize the transformers in the synthesis procedure to yield interconnected ladder networks with corresponding components in the ladders which have unified values.

These research results are enumerated below.

1. The restrictions on the s-domain transfer function are determined.
2. The topology restrictions on the resulting synthesized networks are determined.
3. A realizable state-model for the restricted RLC network is derived.
4. From 3. certain properties are determined.
5. An "A" matrix is derived which has the properties of 4 (27,35).
6. Using the "A" matrix of 5., a RLC network corresponding to the given characteristic polynomial is obtained.
7. Drivers are properly inserted into the networks of 6. to yield the transmission zeros of the transfer functions.
8. The network component values are determined and transformers are added to satisfy the validity test and/or the numerator coefficient magnitude and polarity.

Chapter II presents the necessary procedure for the synthesis of the transfer functions.

Chapter III presents the short circuit transfer admittance synthesis procedure in considerable detail.

Chapter IV presents the open circuit transfer impedance synthesis procedure rather briefly since this is the dual of the material in Chapter III.

Chapter V presents the voltage transfer function synthesis procedure rather briefly. This is because this transfer function uses many of the procedures of the other two transfer functions.

Chapter VI lists the conclusions.

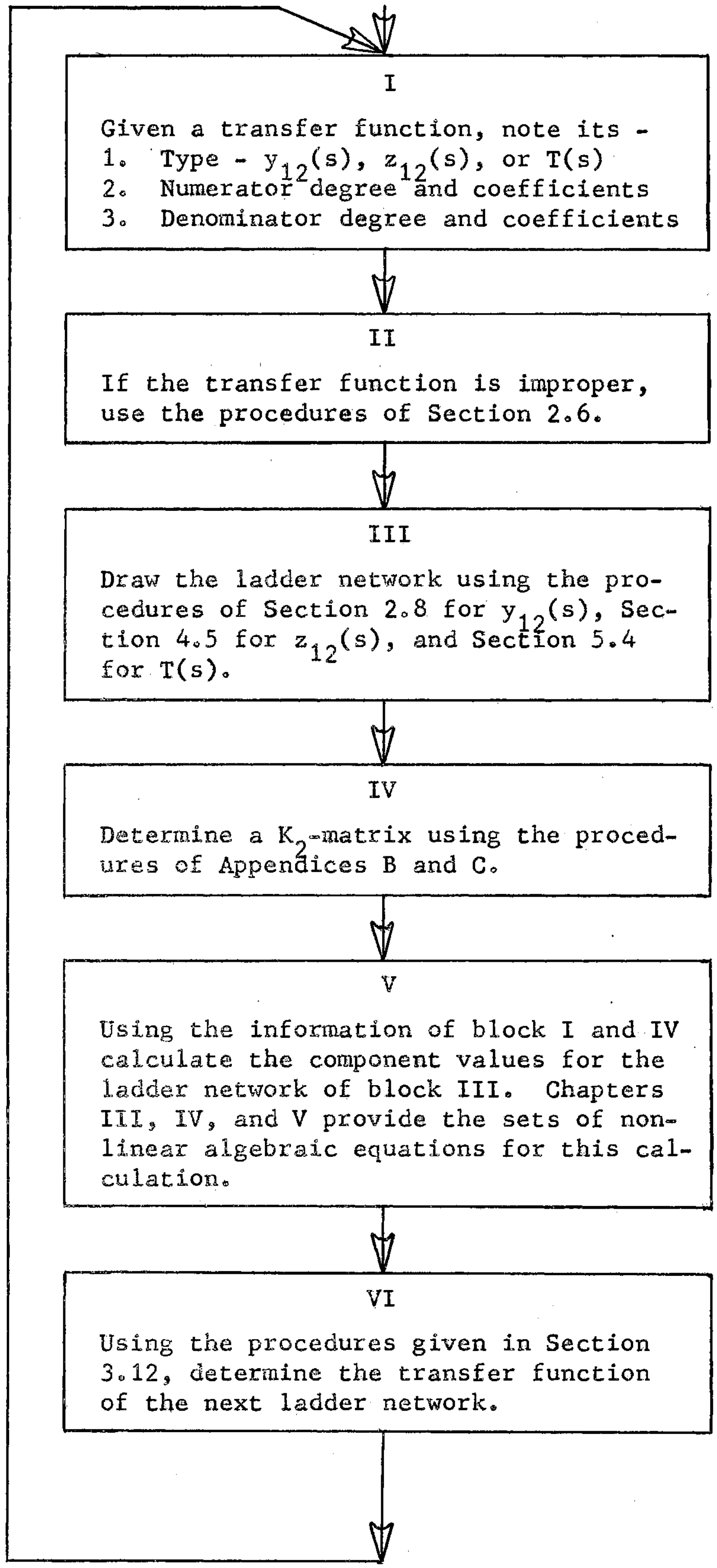

Figure 1.4.1 Synthesis Procedure Flow Graph

1.4 Synthesis Procedure Flow Graph. The synthesis procedure is presented briefly in block diagram form in Figure 1.4.1. Each block is programmed with a finite number of analytical equations and/or decision statements which are presented in the sections listed.

1.5 Definitions. Given below are definitions that are felt necessary for better comprehension in reading this paper.

1. Improper function - A rational function that is a ratio of polynomials with the degree of the numerator polynomial greater than or equal to the degree of the denominator polynomial.
2. Proper function - A rational function that is a ratio of polynomials with the degree of the numerator polynomial less than the degree of the denominator polynomial.
3. Unimodular or E-matrix - A matrix, B_{ij}, of real elements where the determinant of every square submatrix of B_{ij} is 1, -1, or 0. Also every entry of B_{ij} will be 1, -1, or 0 (30).
4. Minimal network - A synthesized network with n reactive elements that results from a characteristic polynomial of degree n.
5. Positive semidefinite matrix - A matrix is positive semidefinite if and only if each principal minor is non-negative (33).
6. Dual networks - If N and N^* are dual two-port networks, then the short circuit admittance matrix of either network is equal to the open circuit impedance matrix of the other (30).
7. Hurwitz polynomial - A polynomial with no zeros in the right half plane (33).
8. Strictly Hurwitz polynomial - A polynomial with no zeros on the imaginary axis or the right half plane (33).

CHAPTER II

STATE-MODELS, TRANSFER FUNCTIONS, CHARACTERISTIC POLYNOMIAL REALIZATION, AND FUNDAMENTAL CIRCUIT EQUATIONS

2.1 Introduction. Before presenting the synthesis procedures, it is necessary to discuss some topics that are basic ideas to the synthesis but which would confuse the presentation if left until later. These topics are the network topology restrictions, the desired state-models, and the transfer function derivations. The first topic discussed will be the topology restrictions, since these will affect the state-model derivations.

2.2 Synthesized Network Topology Restrictions. Simplification of the synthesis procedure is allowed by placing certain restrictions upon the network topology. This simplification is accomplished when the topology restrictions allow the state-model, representing the network, to be written in a form that displays desirable interrelationships and matrix structure. These characteristics will be discussed further in Chapter III.

The topological restrictions are:

1. Both branch resistors and chord resistors will not be permitted in the same fundamental circuits.
2. Circuits of capacitors with or without voltage drivers will not be permitted.

3. Cut-sets of inductors with or without current drivers will not be permitted.

These restrictions will now be used in the derivation of the various desired state-models.

2.3 "Modified" General State-Model. The network topology restrictions of Section 2.2 and the state-model given in Appendix A which represents a general network will now be combined to obtain the desired "modified" general state-model. However before this is done, the matrix elements are defined as follows:

V_a - Branch voltage source vector consisting of the two across variables, v_{a_1} and v_{a_2}.

V_{bc} - Branch capacitor vector consisting of the across variables, v_{bc_1}, v_{bc_2}, ..., and v_{bc_k}.

V_{br} - Branch resistor vector consisting of the across variable, v_{br_1}.

V_{cr} - Chord resistor vector consisting of the two across variables, v_{cr_1} and v_{cr_2}.

$V_{c\ell}$ - Chord inductor vector consisting of the across variables, $v_{c\ell_1}$, $v_{c\ell_2}$, ..., and $v_{c\ell_j}$.

V_t - Chord current source vector consisting of the two across variables, v_{t_1} and v_{t_2}.

B_{ij} - Submatrices of a unimodular matrix with element values of 0 or ± 1.

I_a - Branch voltage source vector consisting of the two through variables, i_{a_1} and i_{a_2}.

I_{bc} - Branch capacitor vector consisting of the through variables, i_{bc_1}, i_{bc_2}, ..., and i_{bc_k}.

I_{br} - Branch resistor vector consisting of the through variable, i_{br_1}.

I_{cr} - Chord resistor vector consisting of the two through variables, i_{cr_1}, and i_{cr_2}.

$I_{c\ell}$ - Chord inductor vector consisting of the through variables, $i_{c\ell_1}$, $i_{c\ell_2}$, ..., and $i_{c\ell_j}$.

I_t - Chord current source vector consisting of the two through variables, i_{t_1} and i_{t_2}.

Applying the network topology restrictions of Section 2.2 to the fundamental circuit equation of Appendix A results in the following reduced circuit equations, cut-set equations, and component equations.

$$\left[\begin{array}{ccc|ccc} B_{21} & B_{22} & 0 & U & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & U & 0 \\ B_{41} & B_{42} & B_{43} & 0 & 0 & U \end{array}\right] \begin{bmatrix} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \\ V_t \end{bmatrix} = 0 \qquad (2.3.1)$$

$$\left[\begin{array}{ccc|ccc} U & 0 & 0 & -B_{21}^T & -B_{31}^T & -B_{41}^T \\ 0 & U & 0 & -B_{22}^T & -B_{32}^T & -B_{42}^T \\ 0 & 0 & U & 0 & -B_{33}^T & -B_{43}^T \end{array}\right] \begin{bmatrix} I_a \\ I_{bc} \\ I_{br} \\ \hline I_{cr} \\ I_{c\ell} \\ I_t \end{bmatrix} = 0 \qquad (2.3.2)$$

$$\begin{bmatrix} C_b & 0 \\ 0 & L_c \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} I_{bc} \\ V_{c\ell} \end{bmatrix} \qquad (2.3.3)$$

$$\begin{bmatrix} V_{br} \\ I_{cr} \end{bmatrix} = \begin{bmatrix} R_b & 0 \\ 0 & G_c \end{bmatrix} \begin{bmatrix} I_{br} \\ V_{cr} \end{bmatrix} \qquad (2.3.4)$$

Using these sets of equations and following a procedure similar to Appendix A, the "modified" general state-model can be formulated and is shown below.

$$\frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-1} B_{22}^T G_c B_{22} & C_b^{-1} B_{32}^T \\ \hline -L_c^{-1} B_{32} & -L_c^{-1} B_{33} R_b B_{33}^T \end{array}\right] \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$$

(2.3.5a)

$$+ \left[\begin{array}{c|c} -C_b^{-1} B_{22}^T G_c B_{21} & C_b^{-1} B_{42}^T \\ \hline -L_c^{-1} B_{31} & -L_c B_{33} R_b B_{43}^T \end{array}\right] \begin{bmatrix} V_a \\ I_t \end{bmatrix}$$

$$\begin{bmatrix} I_a \\ V_t \end{bmatrix} = \left[\begin{array}{c|c} -B_{21}^T G_c B_{22} & B_{31}^T \\ \hline -B_{42} & -B_{43} R_b B_{33}^T \end{array}\right] \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} + \left[\begin{array}{c|c} -B_{21}^T G_c B_{21} & B_{41}^T \\ \hline -B_{41} & -B_{43} R_b B_{43}^T \end{array}\right] \begin{bmatrix} V_a \\ I_t \end{bmatrix}$$

(2.3.5b)

Note that C_b and L_c are diagonal matrices with real positive elements and therefore their inverses will exist. Equation 2.3.5a is the set of differential equations in the state-model and Equation 2.3.5b is the set of algebraic equations. A change of variables in the state vector will be necessary so that the desired "modified" general state-model is obtained that displays the interrelationships mentioned previously and is as follows

$$\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \begin{bmatrix} C_b^{-\frac{1}{2}} & 0 \\ 0 & L_c^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} \qquad (2.3.6)$$

Note than since C_b and L_c are diagonal matrices with positive entries, $C_b^{-\frac{1}{2}}$ and $L_c^{-\frac{1}{2}}$ will be diagonal matrices with their entries chosen to be positive. The terminal variables V_a^*, I_a^*, I_t^*, and V_t^* are related to V_a, I_a, I_t, and V_t by

$$\begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} = \begin{bmatrix} V_a \\ I_t \end{bmatrix}$$

$$\begin{bmatrix} I_a^* \\ V_t^* \end{bmatrix} = \begin{bmatrix} -I_a \\ -V_t \end{bmatrix}$$

Using the change of variables indicated in Equation 2.3.6 in the state-model of Equation 2.3.5 and utilizing the terminal variables, the desired "modified" general state-model results as shown.

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}}B_{32}C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{33}R_b B_{33}^T L_c^{-\frac{1}{2}} \end{array}\right] \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{22}^T G_c B_{21} & C_b^{-\frac{1}{2}}B_{42}^T \\ \hline -L_c^{-\frac{1}{2}}B_{31} & -L_c^{-\frac{1}{2}}B_{33}R_b B_{43}^T \end{array}\right] \begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \qquad (2.3.7a)$$

$$\begin{bmatrix} I_a^* \\ V_t^* \end{bmatrix} = \left[\begin{array}{c|c} B_{21}^T G_c B_{22} C_b^{-\frac{1}{2}} & -B_{31}^T L_c^{-\frac{1}{2}} \\ \hline B_{42} C_b^{-\frac{1}{2}} & B_{43} R_b B_{33}^T L_c^{-\frac{1}{2}} \end{array}\right] \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \left[\begin{array}{c|c} B_{21}^T G_c B_{21} & -B_{41}^T \\ \hline B_{41} & B_{43} R_b B_{43}^T \end{array}\right] \begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \tag{2.3.7b}$$

This state-model is the basic expression in the synthesis procedure. As comparisons will be made later with this state-model, it is written in a simplified form.

$$\frac{d}{dt} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} B_{111} & B_{121} \\ B_{211} & B_{221} \end{bmatrix} \begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \tag{2.3.8a}$$

$$\begin{bmatrix} I_a^* \\ V_t^* \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \tag{2.3.8b}$$

This will ease the task of identifying a particular part of the state-model. Note that

$$B_{111} = -P_{11}^T$$

$$B_{221} = -P_{22}^T$$

$$B_{121} = P_{21}^T$$

$$B_{211} = P_{21}^T$$

A reduced form of this state-model will be used in each of the derived transfer functions to follow.

2.4 State-Models and Transfer Functions. An idea of the unified approach to transfer function synthesis considered in this thesis is obtained when the "modified" general state-model of Equation 2.3.7 is compared with the state-models derived for each of the transfer functions: short circuit transfer admittance, $Y_{12}(s)$, open circuit transfer impedance, $Z_{12}(s)$, and voltage transfer function, $T(s)$. The assumed network driver configuration for each type of transfer function is of interest as they are unique. Each transfer function will be considered separately as it is felt that the "derived" state-models are significant enough to warrant this approach. The short circuit transfer admittance is considered first.

2.4.1 Short Circuit Transfer Admittance, $Y_{12}(s)$. In determining the short circuit transfer admittance by the state-space approach, it is first necessary to derive the "desired" state-model that represents the network with the correct driver configuration. Therefore when determining $Y_{12}(s)$ by the state-space approach, the driver configuration for the 2-port network is assumed to be that of Figure 2.4.1. The reason for this assumption will be apparent later.

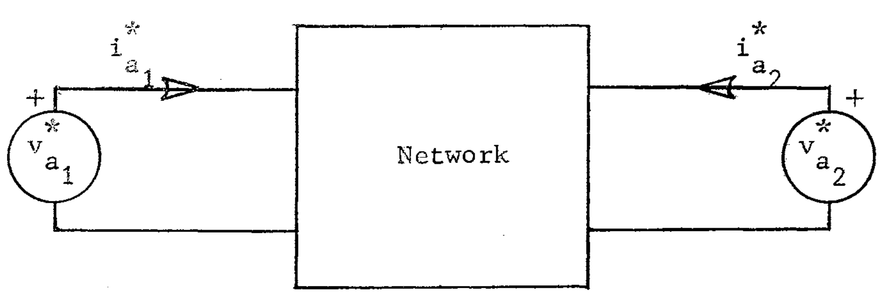

Figure 2.4.1 Network Driver Configuration for $Y_{12}(s)$

In deriving this "desired" state-model, the fundamental circuit equations can be written in reduced form from those in Equation 2.3.1, which were for the "modified" general state-model and did not consider the restriction on the driver configuration. This restriction being that the network only has voltage sources and no current sources. With this further restriction on the driver configuration, the fundamental circuit equations will be

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \qquad (2.4.1)$$

Now determining the equivalent cut-set equations and component equations, similar to Equations 2.3.2, 2.3.3, and 2.3.4, and following a procedure similar to Appendix A, we have the "desired" state-model

$$\frac{d}{dt}\left[\begin{array}{c} V'_{bc} \\ I'_{c\ell} \end{array}\right] = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}} B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}} B_{22}^T L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}} B_{22} C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}} B_{23} R_b B_{23}^T L_c^{-\frac{1}{2}} \end{array}\right] \left[\begin{array}{c} V'_{bc} \\ I'_{c\ell} \end{array}\right] \qquad (2.4.2a)$$

$$+ \left[\begin{array}{c} -C_b^{-\frac{1}{2}} B_{12}^T G_c B_{11} \\ \hline -L_c^{-\frac{1}{2}} B_{21} \end{array}\right] \left[V_a^* \right]$$

$$\left[I_a^*\right] = \left[B_{11}^T G_c B_{12} C_b^{-\frac{1}{2}} \;\middle|\; -B_{21}^T L_c^{-\frac{1}{2}}\right]\begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} + \left[B_{11}^T G_c B_{11}\right]\left[V_a^*\right] \quad (2.4.2b)$$

The similarities between Equations 2.3.7 and 2.4.2 are readily obvious. Using Equations 2.3.8 for identification purposes, it is noted that the part of the state-models of Equations 2.3.7a and 2.4.2a that correspond to the K_{11}, K_{12}, and K_{22} elements would be identical if the same fundamental circuit equation subscript notation had been used. Also the B_{111} and B_{211} elements of Equation 2.3.8 would be the same as the coefficient matrix multiplied times V_a^* of Equation 2.4.2a except that the subscript notation has been changed. This comparison is also true for the P_{11}, P_{12}, and R_{11} submatrices. This reduced form of Equation 2.4.2 could then be obtained by setting I_t^* and V_t^* equal to zero in Equation 2.3.7, which agrees with the network driver configuration of Figure 2.4.1.

For purposes of identification in showing the interrelationships of the state-model of Equation 2.4.2, it is written in the simplified form as shown

$$\frac{d}{dt}\begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix}\begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} + \begin{bmatrix} B_{111} \\ B_{211} \end{bmatrix}\left[V_a^*\right] \quad (2.4.3a)$$

$$\left[I_a^*\right] = \begin{bmatrix} P_{11} & P_{12} \end{bmatrix}\begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} + \left[R_{11}\right]\left[V_a^*\right] \quad (2.4.3b)$$

It should be noted that elements B_{111}, B_{211}, P_{11}, P_{12}, and R_{11} of Equation 2.4.3 are not identical with elements B_{111}, B_{211}, P_{11}, P_{12}, and R_{11} of Equation 2.3.8. By inspection the relationship between the elements, K_{12}^T and $-K_{12}$, of Equation 2.4.2a is obvious. Also note that

$$B_{111} = -P_{11}^T$$

$$B_{211} = P_{12}^T$$

To derive the short circuit transfer admittance, $Y_{12}(s)$, from the state-model of Equation 2.4.3, it is more desirable to write the state-model in a form that lists the voltage sources individually. The $\underline{B}$, P, and R matrices have been expanded to allow the proper matrix multiplication, which is as follows

$$\frac{d}{dt}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix}\begin{bmatrix} v^*_{a_1}(t) \\ v^*_{a_2}(t) \end{bmatrix} \tag{2.4.4a}$$

$$\begin{bmatrix} i^*_{a_1}(t) \\ i^*_{a_2}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_{11} & 0 \\ 0 & \mathcal{R}_{22} \end{bmatrix}\begin{bmatrix} v^*_{a_1}(t) \\ v^*_{a_2}(t) \end{bmatrix} \tag{2.4.4b}$$

To facilitate the presentation, Equation 2.4.4 is written in the simple form

$$\frac{d}{dt} X(t) = K_2 X(t) + \mathcal{B}\, V_a^*(t) \tag{2.4.5a}$$

$$I_a^*(t) = \mathcal{P}\, X(t) + \mathcal{R}\, V_a^*(t) \tag{2.4.5b}$$

Taking the Laplace transform of Equation 2.4.5 and solving for the transformed state-vector, X(s), we have

$$X(s) = (sU-K_2)^{-1}\mathcal{B}\, V_a^*(s)$$

$$I_a^*(s) = \mathcal{P}\, X(s) + \mathcal{R}\, V_a^*(s)$$

Now,

$$I_a^*(s) = \mathcal{D}(s) V_a^*(s) + \mathcal{R}\, V_a^*(s)$$

where

$$\mathcal{D}(s) = \mathcal{P}\,(sU-K_2)^{-1}\mathcal{B} \tag{2.4.6}$$

and after expanding I^*, V^*, $\mathcal{D}$, and $\mathcal{R}$

$$\begin{bmatrix} i_{a_1}^*(s) \\ i_{a_2}^*(s) \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{11}(s) & \mathcal{D}_{12}(s) \\ \mathcal{D}_{21}(s) & \mathcal{D}_{22}(s) \end{bmatrix} \begin{bmatrix} v_{a_1}^*(s) \\ v_{a_2}^*(s) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_{11} & 0 \\ 0 & \mathcal{R}_{22} \end{bmatrix} \begin{bmatrix} v_{a_1}^*(s) \\ v_{a_2}^*(s) \end{bmatrix} \tag{2.4.7}$$

The short circuit transfer admittance function is defined as

$$Y_{12}(s) = \left. \frac{i_{a_1}^*(s)}{v_{a_2}^*(s)} \right|_{v_{a_1}^*(s) = 0} \tag{2.4.8}$$

and $Y_{12}(s)$ can be obtained from Equation 2.4.7, which is

$$Y_{12}(s) = \mathcal{D}_{12}(s) \tag{2.4.9}$$

if $\mathcal{R}$ has the form shown in Equation 2.4.7.

Simplification of the state-space synthesis procedure results when $\mathcal{R}$ has the form

$$\mathfrak{R} = \begin{bmatrix} \mathfrak{R}_{11} & 0 \\ 0 & \mathfrak{R}_{22} \end{bmatrix}$$

because $Y_{12}(s)$ is then only equal to the proper function, $\mathfrak{D}_{12}(s)$, of Equation 2.4.7. Therefore $\mathfrak{R}$ will only enter the synthesis procedure as a restriction of the matrix form and its non-zero element values will not be important. This implies that $\mathfrak{D}(s)$ in Equation 2.4.6 will be the matrix that is to be presented in detail as is done in Section 2.5.

Now we shall justify the network configuration as shown in Figure 2.4.1. Considering the state-model of Equation 2.4.4 and the s-domain solution for the complementary variables of Equation 2.4.7, it is obvious that the driver configuration of Figure 2.4.1 is the only configuration that will allow the calculation of $Y_{12}(s)$ by Equation 2.4.8 when applied to Equation 2.4.7.

The discussion of the open circuit transfer impedance is presented next which will be similar to the discussion on the short circuit transfer admittance.

2.4.2 Open Circuit Transfer Impedance, $Z_{12}(s)$. In determining the open circuit transfer impedance by the state-space approach, it is first necessary to derive the "desired" state-model that represents the network with the correct driver configuration. Therefore when determining $Z_{12}(s)$ by the state-space approach, the driver configuration for the 2-port network is assumed to be that of Figure 2.4.2. The reason for this assumption will be similar to the reasoning in Section 2.4.1. Note that for $Z_{12}(s)$ current sources will be used while for

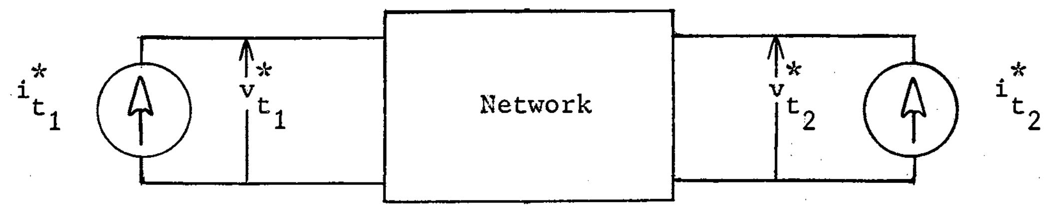

Figure 2.4.2 Network Driver Configuration For $Z_{12}(s)$

$Y_{12}(s)$ in Figure 2.4.1 voltage sources were used.

In deriving this "desired" state-model, the fundamental circuit equations can be written in reduced form from those in Equation 2.3.1, which were for the "modified" general state-model and did not consider the restriction on the driver configuration. This restriction being that the network has only current sources and no voltage sources. With this further restriction on the driver configuration, the fundamental circuit equations will be

$$\begin{bmatrix} B_{11} & 0 & U & 0 & 0 \\ B_{21} & B_{22} & 0 & U & 0 \\ B_{31} & B_{32} & 0 & 0 & U \end{bmatrix} \begin{bmatrix} V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \\ V_{t} \end{bmatrix} = 0 \qquad (2.4.10)$$

The cut-set equations and the component equations are given in Equation 2.3.4. Using these equations and following a procedure similar to Appendix A, we have the "desired" state-model

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{11}^T G_c B_{11} C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}}B_{21}^T L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}}B_{21}C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{22}R_b B_{22}^T L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} \tag{2.4.11a}$$

$$+\begin{bmatrix} C_b^{-\frac{1}{2}}B_{31}^T \\ -L_c^{-\frac{1}{2}}B_{22}R_b B_{32}^T \end{bmatrix}\begin{bmatrix} I_t^* \end{bmatrix}$$

$$\begin{bmatrix} V_t^* \end{bmatrix} = \left[\begin{array}{c|c} B_{31}C_b^{-\frac{1}{2}} & B_{32}R_b B_{22}^T L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} B_{32}R_b B_{32}^T \end{bmatrix}\begin{bmatrix} I_t^* \end{bmatrix} \tag{2.4.11b}$$

The similarities between Equations 2.3.7 and 2.4.11 are readily obvious. Using Equation 2.3.8 for identification purposes, it is noted that the part of the state-models of Equations 2.3.7a and 2.4.11a that correspond to the K_{11}, K_{12}, and K_{22} elements would be identical if the same fundamental circuit equation subscript notation had been used. Also that B_{121} and B_{221} elements of Equation 2.3.8 would be the same as the coefficient matrix multiplied times I_t^* of Equation 2.4.11a except that the subscript notation has been changed. This comparison is also true for the P_{21}, P_{22}, and R_{22} submatrices. This reduced form of Equation 2.4.11 could then be obtained by setting V_a^* and I_a^* equal to zero in Equation 2.3.7, which agrees with the network driver configuration of Figure 2.4.2.

For purposes of identification in showing the interrelationships of the state-model of Equation 2.4.11, it is written in the following simplified form

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} B_{121} \\ B_{221} \end{bmatrix}\begin{bmatrix} I_t^* \end{bmatrix} \tag{2.4.12a}$$

$$\begin{bmatrix} V_t^* \end{bmatrix} = \begin{bmatrix} P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} R_{22} \end{bmatrix}\begin{bmatrix} I_t^* \end{bmatrix} \tag{2.4.12b}$$

It should be noted that elements B_{121}, B_{221}, P_{21}, P_{22}, and R_{22} of Equation 2.4.12 are not identical with elements B_{121}, B_{221}, P_{21}, P_{22}, and R_{22} of Equation 2.3.8. By inspection the relationship between the elements, K_{12}^T and $-K_{12}$, of Equation 2.4.11a is obvious. Also note that

$$B_{121} = P_{21}^T$$

$$B_{221} = -P_{22}^T$$

To derive the open circuit transfer impedance, $Z_{12}(s)$, from the state-model of Equation 2.4.11, it is more desirable to write the state-model in a form that lists the current sources individually. The $\underline{B}$, P, and R matrices have been expanded to allow the proper matrix multiplication, which is as follows

$$\frac{d}{dt}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix}\begin{bmatrix} i^*_{t_1}(t) \\ i^*_{t_2}(t) \end{bmatrix}$$

(2.4.13a)

$$\begin{bmatrix} v^*_{t_1}(t) \\ v^*_{t_2}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_{11} & 0 \\ 0 & \mathcal{R}_{22} \end{bmatrix} \begin{bmatrix} i^*_{t_1}(t) \\ i^*_{t_2}(t) \end{bmatrix} \qquad (2.4.13b)$$

Note that the $\mathcal{B}_{ij}$'s, $\mathcal{P}_{ij}$'s, and $\mathcal{R}_{ii}$'s of Equation 2.4.13 are different from the elements of like notation in Equation 2.4.4. To facilitate the presentation, Equation 2.4.13 is written in the simple form

$$\frac{d}{dt} X(t) = K_2 X(t) + \mathcal{B}\, I^*_t(t) \qquad (2.4.14a)$$

$$V^*_t(t) = \mathcal{P}\, X(t) + \mathcal{R}\, I^*_t(t) \qquad (2.4.14b)$$

Taking the Laplace transform of Equation 2.4.14 and solving for the complementary variable, $V^*(s)$, by a procedure similar to that presented in Section 2.4.1 will yield

$$V^*_t(s) = \mathcal{D}\,(s) I^*_t(s) + \mathcal{R}\, I^*_t(s)$$

where

$$\mathcal{D}\,(s) = \mathcal{P}\,(sU-K_2)^{-1}\mathcal{B} \qquad (2.4.15)$$

and after expanding V^*, I^*, $\mathcal{D}$, and $\mathcal{R}$

$$\begin{bmatrix} v^*_{t_1}(s) \\ v^*_{t_2}(s) \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{11}(s) & \mathcal{D}_{12}(s) \\ \mathcal{D}_{21}(s) & \mathcal{D}_{22}(s) \end{bmatrix} \begin{bmatrix} i^*_{t_1}(s) \\ i^*_{t_2}(s) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_{11} & 0 \\ 0 & \mathcal{R}_{22} \end{bmatrix} \begin{bmatrix} i^*_{t_1}(s) \\ i^*_{t_2}(s) \end{bmatrix} \qquad (2.4.16)$$

It should be noted that Equations 2.4.15 and 2.4.6 will be identical if the entries of $\mathcal{P}$ and $\mathcal{B}$ are left as general unknowns and the K_2's need to be determined from the same polynomial as shown in Appendices B and C. This observation will be used later in the synthesis procedure.

The open circuit transfer impedance function is defined as

$$Z_{12}(s) = \left. \frac{v^*_{t_1}(s)}{i^*_{t_2}(s)} \right|_{i^*_{t_1}(s) = 0} \tag{2.4.17}$$

and $Z_{12}(s)$ can be obtained from Equation 2.4.16, which is

$$Z_{12}(s) = \mathfrak{D}_{12}(s) \tag{2.4.18}$$

if $\mathfrak{R}$ has the form shown in Equation 2.4.16. The presentation concerning the importance of the form of $\mathfrak{R}$ is very similar to the one given in Section 2.4.1.

Simplification of the state-space synthesis procedure results when $\mathfrak{R}$ has the form shown in Equation 2.4.16, because $Z_{12}(s)$ is then only equal to the proper function, $\mathfrak{D}_{12}(s)$, of Equation 2.4.16. Therefore $\mathfrak{R}$ will only enter the synthesis procedure as a restriction of the matrix form and its non-zero element values will not be important. This implies that $\mathfrak{D}(s)$ in Equation 2.4.15 will be the matrix that is to be presented in detail as is done in Section 2.5.

Justification for the choice of the network driver configuration is apparent from the state-model of Equation 2.4.13 and the s-domain solution for the complementary variables of Equation 2.4.16, since the driver configuration of Figure 2.4.2 is the only configuration that will allow the calculation of $Z_{12}(s)$ by Equation 2.4.17 when applied to Equation 2.4.16. The voltage transfer function, which is presented next, will have some differences from $Y_{12}(s)$ and $Z_{12}(s)$.

2.4.3 Voltage Transfer Function, T(s). In determining the voltage transfer function by the state-space approach, it is first necessary to derive the "desired" state-model that represents the network with the correct driver configuration, which is given in Figure 2.4.3. The

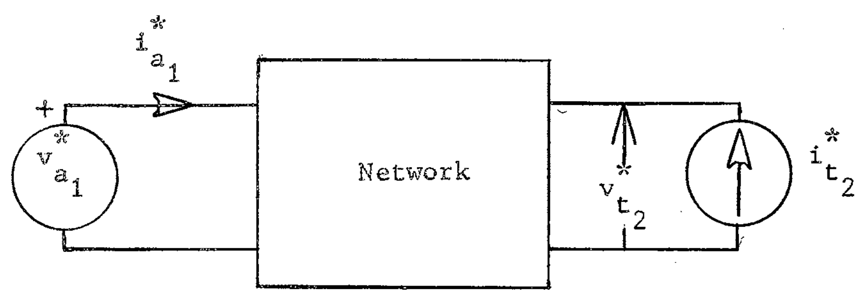

Figure 2.4.3 Network Driver Configuration For T(s)

reason for this assumption will be similar to the reasoning used in the choice of drivers in Sections 2.4.1 and 2.4.2. Note that while $Y_{12}(s)$ used voltage sources and $Z_{12}(s)$ used current sources, T(s) calculations will be made with a voltage source on the input-port and a current source on the output-port as shown in Figure 2.4.3. As may be thought, this will complicate the state-model.

Since there is both a voltage source and current source, the fundamental circuit equations used in deriving the "desired" state-model will be the same as those in Equation 2.3.1. Also the "desired" state-model will be the same as the "modified" general state-model of

Equation 2.3.7. The same interrelationships that were identified with the aid of Equation 2.3.8 will still be true for the T(s) state-model.

To derive the voltage transfer function, T(s), from the state-model of Equation 2.3.8, it is more desirable to write this state-model in a form that lists the individual voltage source and current source as shown

$$\frac{d}{dt}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix}\begin{bmatrix} v^*_{a_1}(t) \\ i^*_{t_2}(t) \end{bmatrix} \quad (2.4.19a)$$

$$\begin{bmatrix} i^*_{a_1}(t) \\ v^*_{t_2}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_{11} & 0 \\ 0 & \mathcal{R}_{22} \end{bmatrix}\begin{bmatrix} v^*_{a_1}(t) \\ i_{t_2}(t) \end{bmatrix} \quad (2.4.19b)$$

Note that the $\mathcal{B}_{ij}$'s, $\mathcal{P}_{ij}$'s, and $\mathcal{R}_{ii}$'s of Equation 2.4.19 are different from the elements of like notation in Equations 2.4.4 and 2.4.13. To facilitate the presentation, Equation 2.4.19 is written in the simple form

$$\frac{d}{dt} X(t) = K_2 X(t) + \mathcal{B}\, Y^*(t) \quad (2.4.20a)$$

$$\underline{Y}^*(t) = \mathcal{P}\, X(t) + \mathcal{R}\, Y^*(t) \quad (2.4.20b)$$

Taking the Laplace transform of Equation 2.4.20 and solving for the complementary variable, $\underline{Y}^*(s)$, by a procedure similar to that presented in Sections 2.4.1 and 2.4.2 will yield

$$\underline{Y}^*(s) = \mathcal{D}\,(s) Y^*(s) + \mathcal{R}\, Y^*(s)$$

where

$$\mathcal{Q}(s) = \mathcal{P}\,(sU-K_2)^{-1}\mathcal{B} \qquad (2.4.21)$$

and after expanding $\underline{Y}^*$, Y^*, $\mathcal{Q}$, and $\mathcal{R}$

$$\begin{bmatrix} i^*_{a_1}(s) \\ v^*_{t_2}(s) \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_{11}(s) & \mathcal{Q}_{12}(s) \\ \mathcal{Q}_{21}(s) & \mathcal{Q}_{22}(s) \end{bmatrix} \begin{bmatrix} v^*_{a_1}(s) \\ i^*_{t_2}(s) \end{bmatrix} + \begin{bmatrix} \mathcal{R}_{11} & 0 \\ 0 & \mathcal{R}_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1}(s) \\ i^*_{t_2}(s) \end{bmatrix} \qquad (2.4.22)$$

It should be noted that Equation 2.4.6, 2.4.15, and 2.4.21 will be identical if the entries of $\mathcal{P}$ and $\mathcal{B}$ are left as general unknowns and the K_2's need to be determined from the same polynomial as shown in Appendices B and C. This observation will be used in the synthesis procedure.

The voltage transfer function is defined as

$$T(s) = \left.\frac{v^*_{t_2}(s)}{v^*_{a_1}(s)}\right|_{i^*_{t_2}(s) = 0} \qquad (2.4.23)$$

and T(s) can be obtained from Equation 2.4.22, which is

$$T(s) = \mathcal{Q}_{21}(s) \qquad (2.4.24)$$

If $\mathcal{R}$ has the form shown in Equation 2.4.22. Although we have only considered a T(s) corresponding to a ratio of two voltages, we could also consider a current transfer function, in which case

$$T'(s) = \left. \frac{i^*_{a_1}(s)}{i^*_{t_2}(s)} \right|_{v^*_{a_1}(s) = 0}$$

This $T'(s)$ can also be obtained from Equation 2.4.22. The analysis of the current transfer function follows along the same lines as that for the voltage transfer function, therefore it will not be considered further.

The form of $\Re$ is important in that it simplifies the state-space synthesis procedure. The presentation to justify this simplification is the same as given in Sections 2.4.1 and 2.4.2. Because of the form of $\Re$ another topology restriction must be imposed, which is not to permit the voltage and current sources in the same fundamental circuit, that is in Equation 2.3.1

$$B_{41} = 0$$

This forces $\Re$ to the desired form of

$$\Re = \begin{bmatrix} \Re_{11} & 0 \\ 0 & \Re_{22} \end{bmatrix}$$

This characteristic will be discussed further in the restrictions given in Section 6.1.

The matrix $\mathfrak{D}(s)$ consists of elements of proper functions. $\mathfrak{D}(s)$ as given in Equations 2.4.6, 2.4.15, and 2.4.21 is the same algebraic matrix expression and it will be presented at length in the next section.

2.5 $\mathcal{D}$ (s) and [Adj(sU-K_2)] Algorithm. In sections 2.4.1, 2.4.2, and 2.4.3 the expression

$$\mathcal{D}(s) = \mathcal{P}(sU-K_2)^{-1}\mathcal{B} \qquad (2.5.1a)$$

$$\mathcal{D}(s) = \begin{bmatrix} \mathcal{D}_{11}(s) & \mathcal{D}_{12}(s) \\ \mathcal{D}_{21}(s) & \mathcal{D}_{22}(s) \end{bmatrix} \qquad (2.5.1b)$$

was shown to be important in the transfer function analysis. It must also be pointed out that the matrix, $\mathcal{D}$ (s), is a significant matrix in the state-space synthesis procedure because $\mathcal{D}$ (s) determines the numerator functions of the transfer functions. This can be shown by expanding the matrices $\mathcal{B}$, $\mathcal{P}$, and $(sU-K_2)^{-1}$ which are multiplied together to yield $\mathcal{D}$ (s). Now

$$\mathcal{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{bmatrix} \qquad (2.5.2)$$

$$\mathcal{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \end{bmatrix} \qquad (2.5.3)$$

$$(sU-K_2)^{-1} = \frac{adj(sU-K_2)}{\triangle} \qquad (2.5.4a)$$

$$(sU-K_2)^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (2.5.4b)$$

where

$$\Delta = |sU-K_2|$$

By observing Equations 2.4.4, 2.4.13, and 2.4.19 it can be shown that $\mathfrak{B}$ and $\mathfrak{P}$ should have the dimensions as shown in Equations 2.5.2 and 2.5.3. The entries in $\mathfrak{B}$ and $\mathfrak{P}$ are unknown constants and $(sU-K_2)^{-1}$ can be written as shown in Equation 2.5.4. Then Equation 2.5.1 can be written as

$$\mathfrak{D}(s) = \frac{1}{\Delta}\,\mathfrak{P}\,[\mathrm{adj}(sU-K_2)]\,\mathfrak{B} \qquad (2.5.5a)$$

or

$$\mathfrak{D}(s) = \begin{bmatrix} \frac{\mathfrak{D}'_{11}(s)}{\Delta} & \frac{\mathfrak{D}'_{12}(s)}{\Delta} \\ \frac{\mathfrak{D}'_{21}(s)}{\Delta} & \frac{\mathfrak{D}'_{23}(s)}{\Delta} \end{bmatrix} \qquad (2.5.5b)$$

and the transfer functions can then be shown as

$$Y_{12}(s) = \frac{\mathfrak{D}'_{12}(s)}{\Delta} \qquad (2.5.5c)$$

when the drivers are voltage sources,

$$Z_{12}(s) = \frac{\mathfrak{D}'_{12}(s)}{\Delta} \qquad (2.5.5d)$$

when the drivers are current sources, and

$$T(s) = \frac{\mathfrak{D}'_{21}(s)}{\Delta} \tag{2.5.5e}$$

when the input is a voltage source and the output is a current source. This illustrates the significance of $\mathfrak{D}(s)$ in determining the numerator functions of the transfer functions.

From Equation 2.5.5a, it is apparent that the entries of the matrix, $\mathcal{P}\,[\mathrm{adj}(sU-K_2)]\,\mathcal{B}$, must be determined for future use. Combining Equations 2.5.2, 2.5.3, and 2.5.4 (b) into Equation 2.5.5a yields

$$\mathfrak{D}(s) = \frac{1}{\Delta} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{n1} & b_{n2} \end{bmatrix} \tag{2.5.5f}$$

which is written in expanded form in Equation 2.5.5g.

From Equation 2.5.5a it is apparent that the entries of the matrix, $[\mathrm{adj}(sU-K_2)]$, must be determined analytically. Therefore an algorithm has been developed that will give the elements which are functions of s in the $[\mathrm{adj}(sU-K_2)]$. However it is first necessary to present some relationships of K_2 and K_1, where K_1 is the matrix from which K_2 is transformed.

In Appendix B it has been shown that by using Navot's (27) method it is possible to determine from a given polynomial, D(s), a matrix,

$$\mathcal{Q}(s) = \frac{1}{\Delta}\left[\begin{array}{c|c}
(p_{11}a_{11} + p_{12}a_{21} + p_{13}a_{31} + \cdots + p_{1n}a_{n1})b_{11} & (p_{11}a_{11} + p_{12}a_{21} + p_{13}a_{31} + \cdots + p_{1n}a_{n1})b_{12} \\
+ (p_{11}a_{12} + p_{12}a_{22} + p_{13}a_{32} + \cdots + p_{1n}a_{n2})b_{21} & +(p_{11}a_{12} + p_{12}a_{22} + p_{13}a_{32} + \cdots + p_{1n}a_{n2})b_{22} \\
+ (p_{11}a_{13} + p_{12}a_{23} + p_{13}a_{33} + \cdots + p_{1n}a_{n3})b_{31} & +(p_{11}a_{13} + p_{12}a_{23} + p_{13}a_{33} + \cdots + p_{1n}a_{n3})b_{32} \\
\vdots & \vdots \\
+ (p_{11}a_{1n} + p_{12}a_{2n} + p_{13}a_{3n} + \cdots + p_{1n}a_{nn})b_{n1} & +(p_{11}a_{1n} + p_{12}a_{2n} + p_{13}a_{3n} + \cdots + p_{1n}a_{nn})b_{n2} \\
\hline
(p_{21}a_{11} + p_{22}a_{21} + p_{23}a_{31} + \cdots + p_{2n}a_{n1})b_{11} & (p_{21}a_{11} + p_{22}a_{21} + p_{23}a_{31} + \cdots + p_{2n}a_{n1})b_{12} \\
+ (p_{21}a_{12} + p_{22}a_{22} + p_{23}a_{32} + \cdots + p_{2n}b_{n2})b_{21} & +(p_{21}a_{12} + p_{22}a_{22} + p_{23}a_{32} + \cdots + p_{2n}a_{n2})b_{22} \\
+ (p_{21}a_{13} + p_{22}a_{23} + p_{23}a_{33} + \cdots + p_{2n}b_{n3})b_{31} & +(p_{21}a_{13} + p_{22}a_{23} + p_{23}a_{33} + \cdots + p_{2n}a_{n3})b_{32} \\
\vdots & \vdots \\
+ (p_{21}a_{1n} + p_{22}a_{2n} + p_{23}a_{3n} + \cdots + p_{2n}b_{nn})b_{n1} & +(p_{21}a_{1n} + p_{22}a_{2n} + p_{23}a_{3n} + \cdots + p_{2n}a_{nn})b_{n2}
\end{array}\right] \tag{2.5.5g}$$

$(sU-K_1)$, whose determinant will be equal to $D(s)$ and that K_1 can be represented in general as

$$K_1 = \begin{bmatrix} -f_o & k_1 & & & & \\ -k_1 & 0 & k_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & -k_{n-2} & 0 & k_{n-1} \\ & & & & -k_{n-1} & -f_n \end{bmatrix} \tag{2.5.6}$$

(Note that when $n = 1$, $K_1 = -f_o$.) Also in Appendix C it has been shown that by using Yarlagadda's (34) transformation, it is possible to determine the matrix, K_2, so that

$$K_2 = L^T K_1 L \tag{2.5.7}$$

and K_2 is shown when n is odd

$$K_2 = \begin{array}{c} \\ \left[\begin{array}{cccccc|cccc} (1) & (2) & (3) & \cdots & (\frac{n-1}{2}) & (\frac{n+1}{2}) & (\frac{n+3}{2}) & (\frac{n+5}{2}) & \cdots & (n) \\ -f_o & & & & & & k_1 & & & \\ & 0 & & & & & -k_2 & k_3 & & \\ & & 0 & & & & & -k_4 & \cdot & \\ & & & \ddots & & & & & \ddots & \\ & & & & 0 & & & & & k_{n-2} \\ & & & & & -f_n & & & & -k_{n-1} \\ \hline -k_1 & k_2 & & & & & & & & \\ & -k_3 & k_4 & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & & -k_{n-2} & k_{n-1} & & & & \end{array}\right] \end{array} \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ \vdots \\ (\frac{n-1}{2}) \\ (\frac{n+1}{2}) \\ (\frac{n+3}{2}) \\ (\frac{n+5}{2}) \\ \vdots \\ (n) \end{array}$$

(2.5.8a)

with

$$L = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccccc} (1) & (2) & (3) & \cdots & (\frac{n+1}{2}) & (\frac{n+3}{2}) & (\frac{n+5}{2}) & (\frac{n+7}{2}) & \cdots & (n) \end{array}$$

$$L = \left[\begin{array}{cccccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & . & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & . & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & . & 0 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ \vdots \\ (n-1) \\ (n) \end{array}$$

(2.5.8b)

and when n is even

$$\begin{array}{cccccccccc} (1) & (2) & \cdots & (\frac{n}{2}) & (\frac{n}{2}+1) & (\frac{n}{2}+2) & \cdots & (n-1) & (n) \end{array}$$

$$K_2 = \left[\begin{array}{cccc|cccc} -f_o & & & & k_1 & & & \\ & 0 & & & -k_2 & k_3 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 0 & & & -k_{n-2} & k_{n-1} \\ \hline -k_1 & k_2 & & & 0 & & & \\ & -k_3 & \ddots & & & \ddots & & \\ & & \ddots & k_{n-2} & & & \ddots & \\ & & & -k_{n-1} & & & & -f_n \end{array}\right] \begin{array}{c} (1) \\ (2) \\ \vdots \\ (\frac{n}{2}) \\ (\frac{n}{2}+1) \\ (\frac{n}{2}+2) \\ \vdots \\ (n-1) \\ (n) \end{array}$$

(2.5.9a)

with

$$\begin{array}{cccccccccc} (1) & (2) & (3) & \cdots & (\frac{n}{2}) & (\frac{n}{2}+1) & (\frac{n}{2}+2) & (\frac{n}{2}+3) & \cdots & (n) \end{array}$$

$$L = \left[\begin{array}{cccccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ \vdots \\ (n-1) \\ (n) \end{array}$$

(2.5.9b)

and

$$(sU-K_2)^{-1} = L^T(sU-K_1)^{-1}L \qquad (2.5.10)$$

Equation (2.5.10) is of interest since the algorithm will be for determining of the adj$(sU-K_1)$ and the transformation to the adj$(sU-K_2)$ will be made after this adjoint matrix is determined. The reasons for this approach are that cofactors of K_1 are easier to evaluate than those of K_2 and the matrix, $(sU-K_1)$, has the same form whether n is even or odd.

Given an (nxn)-matrix, K_1, that represents a general n^{th} order polynomial, D(s), the matrix $(sU-K_1)$ can be written as

$$(sU-K_1) = \begin{bmatrix} (s+f_o) & -k_1 & & & & & & \\ k_1 & s & -k_2 & & & & & \\ & k_2 & s & -k_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & k_{n-3} & s & -k_{n-2} & & \\ & & & & k_{n-2} & s & -k_{n-1} \\ & & & & & k_{n-1} & (s+f_n) \end{bmatrix} \qquad (2.5.11)$$

and the algorithm for the matrix, adj$(sU-K_1)$, is given in Equation 2.5.12 on the following page. Note that $[a_{ij}]$ identifies the element locations. In using this algorithm the (1,n)-element is the first element to be evaluated. Then the first row should be next, from right to left. Next the n^{th} column should be determined, from top to bottom. Next the main diagonal and all elements above it are obtained. And last the elements below the main diagonal are determined. It should be noted that

$$
\operatorname{adj}(sU-K_1) = \left[\begin{array}{cccccc}
[a_{11}] & [a_{1,j}] & & [a_{1,n-2}] & [a_{1,n-1}] & [a_{1,n}] \\
\left[\frac{1}{k_1}(sa_{12}+k_2a_{13})\right]\cdots & \left[\frac{1}{k_j}(sa_{1,j+1}+k_{j+1}a_{1,j+2})\right]\cdots & & \left[\frac{1}{k_{n-2}}(sa_{1,n-1}+k_{n-1}a_{1,n})\right] & \left[\frac{a_{1,n}}{k_{n-1}}(s+f_n)\right] & \left[\prod_{i=1}^{n-1} k_i\right] \\
 & \vdots & & & & \\
 & & & [a_{2,n-2}] & [a_{2,n-1}] & [a_{2,n}] \\
\cdot & [a_{k,j}] & & & & \\
 & \left[\frac{a_{1,j}a_{k,n}}{a_{1,n}}\right] & & \left[\frac{a_{1,n-2}a_{2,n}}{a_{1,n}}\right] & \left[\frac{a_{1,n-1}a_{2,n}}{a_{1,n}}\right] & \left[\frac{a_{1,n}}{k_1}(s+f_o)\right] \\
 & \cdot & & & [a_{3,n-1}] & [a_{3,n}] \\
 & & & & \left[\frac{a_{1,n-1}a_{3,n}}{a_{1,n}}\right] & \left[\frac{1}{k_2}(sa_{2,n}+k_1a_{1,n})\right] \\
 & & \cdot & & & \vdots \\
 & & & [a_{j,j}] & & [a_{j,n}] \\
(-1)^{n+\ell+1}a_{1,n-\ell} & & & \left[\frac{a_{1,j}a_{j,n}}{a_{1,n}}\right] & & \left[\frac{1}{k_{j-1}}(sa_{j-1,n}+k_{j-2}a_{j-2,n})\right] \\
\vdots & (-1)^{n+m}a_{2,n-m} & & & \cdot & \vdots \\
 & \vdots & (-1)^{n+r+q}a_{r,n-q} & & & \\
(-1)^{n}a_{1,n-3} & & \vdots & & \cdot & \\
(-1)^{n+1}a_{1,n-2} & (-1)^{n}a_{2,n-2} & & & \cdot & [a_{n,n}] \\
(-1)^{n}a_{1,n-1} & (-1)^{n+1}a_{2,n-1} & & & & \\
(-1)^{n+1}a_{1,n} & (-1)^{n}a_{2,n} \quad \cdots & (-1)^{n+r}a_{r,n} & \cdot & \cdot \quad \cdot & \left[\frac{1}{k_{n-1}}(sa_{n-1,n}+k_{n-2}a_{n-2,n})\right]
\end{array}\right]
$$

(2.5.12)

$$a_{(n-q),r} = (-1)^{n+r+q} a_{r,(n-q)} \; ; \quad \begin{array}{l} r = 1, 2, \ldots, n \\ q = 0, 1, 2, \ldots, n-1 \end{array}$$

The proof of this algorithm is simple to see, but it requires excessive space and so is not given here. This algorithm is illustrated by an example which is presented below. Consider a 5th order polynomial, D(s) which is represented by $(sU-K_1)$ as shown

$$(sU-K_1) = \begin{bmatrix} (s+f_o) & -k_1 & & & \\ k_1 & s & -k_2 & & \\ & k_2 & s & -k_3 & \\ & & k_3 & s & -k_4 \\ & & & k_4 & (s+f_5) \end{bmatrix} \qquad (2.5.13)$$

then the algorithm of Equation 2.5.12 yields

$$\mathrm{adj}(sU-K_1) = \begin{bmatrix} \overset{[a_{11}]}{\left[\frac{1}{k_1}(sa_{12}+k_2a_{13})\right]} & \overset{[a_{12}]}{\left[\frac{1}{k_2}(sa_{13}+k_3a_{14})\right]} & \overset{[a_{13}]}{\left[\frac{1}{k_3}(sa_{14}+k_4a_{15})\right]} & \overset{[a_{14}]}{\left[\frac{a_{15}}{k_4}(s+f_5)\right]} & \overset{[a_{15}]}{\left[\prod_{i=1}^{4} k_i\right]} \\ -a_{12} & \overset{[a_{22}]}{\left[\frac{a_{12}a_{25}}{\prod_{i=1}^{4} k_i}\right]} & \overset{[a_{23}]}{\left[\frac{a_{13}a_{25}}{\prod_{i=1}^{4} k_i}\right]} & \overset{[a_{24}]}{\left[\frac{a_{14}a_{25}}{\prod_{i=1}^{4} k_i}\right]} & \overset{[a_{25}]}{\left[\frac{a_{15}}{k_1}(s+f_o)\right]} \\ a_{13} & -a_{23} & \overset{[a_{33}]}{\left[\frac{a_{13}a_{35}}{\prod_{i=1}^{4} k_i}\right]} & \overset{[a_{34}]}{\left[\frac{a_{14}a_{35}}{\prod_{i=1}^{4} k_i}\right]} & \overset{[a_{35}]}{\left[\frac{1}{k_2}(sa_{25}+k_1a_{15})\right]} \\ -a_{14} & a_{24} & -a_{34} & \overset{[a_{44}]}{\left[\frac{a_{14}a_{45}}{\prod_{i=1}^{4} k_i}\right]} & \overset{[a_{45}]}{\left[\frac{1}{k_3}(sa_{35}+k_2a_{25})\right]} \\ a_{15} & -a_{25} & a_{35} & -a_{45} & \overset{[a_{55}]}{\left[\frac{1}{k_4}(sa_{45}+k_3a_{35})\right]} \end{bmatrix}$$

(2.5.14)

$$
\mathrm{adj}(sU-K_1) = \begin{bmatrix}
\begin{array}{c}[a_{11}]\\ [s^4+f_5s^3\\ +(k_2^2+k_3^2+k_1^2)s^2\\ +f_5(k_2^2+k_3^2)s\\ +k_2^2k_4^2]\end{array} &
\begin{array}{c}[a_{12}]\\ k_1[s^3+f_5s^2\\ +(k_3^2+k_4^2)s\\ +k_3^2f_5]\end{array} &
\begin{array}{c}[a_{13}]\\ k_1k_2(s^2+f_5s+k_4^2)\end{array} &
\begin{array}{c}[a_{14}]\\ k_1k_2k_3(s+f_5)\end{array} &
\begin{array}{c}[a_{15}]\\ (k_1k_2k_3k_4)\end{array} \\
\begin{array}{c}[a_{21}]\\ -a_{12}\end{array} &
\begin{array}{c}[a_{22}]\\ s^4+f_5s^3\\ +(k_3^2+k_4^2+f_of_5)s^2\\ +[k_3^2f_5+f_o(k_3^2+k_4^2)]s\\ +f_of_5k_3^2\end{array} &
\begin{array}{c}[a_{23}]\\ k_2[s^3+(f_o+f_5)s^2\\ +(f_of_5+k_4^2)s\\ +f_ok_4^2]\end{array} &
\begin{array}{c}[a_{24}]\\ k_2k_3[s^2+(f_o+f_5)s\\ +f_of_5]\end{array} &
\begin{array}{c}[a_{25}]\\ k_2k_3k_4(s+f_o)\end{array} \\
\begin{array}{c}[a_{31}]\\ a_{13}\end{array} &
\begin{array}{c}[a_{32}]\\ -a_{23}\end{array} &
\begin{array}{c}[a_{33}]\\ [s^4+(f_o+f_5)s^3+(k_1^2+k_4^2\\ +f_of_5)s^2+(f_ok_4^2+f_5k_1^2)s\\ +(k_1^2+k_4^2)]\end{array} &
\begin{array}{c}[a_{34}]\\ k_3[s^3+(f_o+f_5)s^2\\ +(f_of_5+k_1^2)\ s+f_5k_1^2]\end{array} &
\begin{array}{c}[a_{35}]\\ k_3k_4(s^2+f_os+k_1^2)\end{array} \\
\begin{array}{c}[a_{41}]\\ -a_{14}\end{array} &
\begin{array}{c}[a_{42}]\\ a_{24}\end{array} &
\begin{array}{c}[a_{43}]\\ -a_{34}\end{array} &
\begin{array}{c}[a_{44}]\\ s^4+(f_o+f_5)s^3\\ +(k_1^2+k_2^2+f_of_5)s^2+[f_ok_2^2\\ +f_5(k_1^2+k_2^2)]s\\ +f_of_5k_2^2\end{array} &
\begin{array}{c}[a_{45}]\\ k_4[s^3+f_os^2\\ +(k_1^2+k_2^2)s\\ +k_2^2f_o]\end{array} \\
\begin{array}{c}[a_{51}]\\ a_{15}\end{array} &
\begin{array}{c}[a_{52}]\\ -a_{25}\end{array} &
\begin{array}{c}[a_{53}]\\ a_{35}\end{array} &
\begin{array}{c}[a_{54}]\\ -a_{45}\end{array} &
\begin{array}{c}[a_{55}]\\ [s^4+f_os^3+(k_1^2+k_2^2+k_3^2)s^2\\ +f_o(k_2^2+k_3^2)s\\ +k_1^2k_3^2]\end{array}
\end{bmatrix}
\tag{2.5.15}
$$

which results in Equation 2.5.15. It can be seen that the $[adj(sU-K_1)]$ agrees with Equation 2.5.15.

It should be noted that (n-1) is the highest degree of any element of the (nxn)-matrix, $[adj(sU-K_1)]$. This implies that the numerator function degree of any transfer function obtained while using the $[adj(sU-K_1)]$ will be at most one less than the denominator function degree. How this affects the state-space synthesis procedure will be presented in Section 2.6.

2.6 Obtaining Proper Transfer Functions. As has been observed previously, the state-space synthesis is simplified to the point of feasibility if the transfer function is a proper function. As Balabanian (3) has shown, physically realizable transfer functions to be synthesized will be encountered that are improper functions, with a numerator function degree a maximum of one greater than the denominator function degree. Therefore an s-domain synthesis procedure is presented that will reduce the numerator function degree until the transfer function left to be synthesized is a proper function. The first transfer function to be presented that will be modified by this procedure is the short circuit transfer admittance, $Y_{12}(s)$.

2.6.1 $Y_{12}(s)$ Modification. If a short circuit transfer admittance to be synthesized is encountered which is an improper function as shown

$$Y_{12}(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_o}{b_n s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_o} \quad (2.6.1)$$

the first step is to divide the denominator into the numerator until the remainder is a proper function as shown

$$Y_{12}(s) = c_1 s + c_o + \frac{d_{n-1}s^{n-1} + \ldots + d_1 s + d_o}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \quad (2.6.2)$$

where

$$c_1 = \frac{a_{n+1}}{b_n}$$

$$c_o = \frac{a_n b_n - a_{n+1} b_{n-1}}{b_n^2}$$

and let

$$Y_{12}^1(s) = c_1 s \quad (2.6.5)$$

$$Y_{12}^2(s) = c_o \quad (2.6.6)$$

$$Y_{12}^3(s) = \frac{d_{n-1}s^{n-1} + \ldots + d_1 s + d_o}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \quad (2.6.7)$$

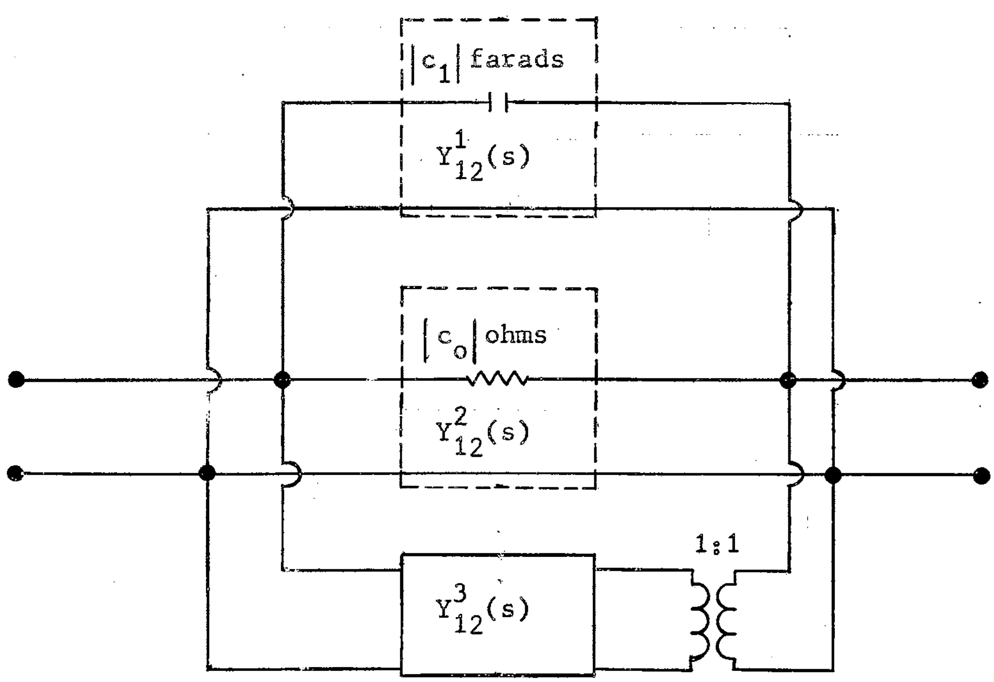

Figure 2.6.1 Modified Network, $Y_{12}(s)$

Now as shown in Weinberg (33), if $Y_{12}^{1}(s)$, $Y_{12}^{2}(s)$, and $Y_{12}^{3}(s)$ are placed in parallel while observing the validity test, the short circuit transfer admittance of the total network will be $Y_{12}(s)$ of Equation 2.6.1 and the resultant network is shown in Figure 2.6.1. The transformer may or may not be necessary. Now $Y_{12}^{3}(s)$ is a proper function and can be realized by the state-space synthesis procedure of Chapter III.

2.6.2 $Z_{12}(s)$ Modification. If an open circuit transfer impedance to be synthesized is encountered which is an improper function as shown

$$Z_{12}(s) = \frac{e_{n+1}s^{n+1} + e_n s^n + e_{n-1}s^{n-1} + \dots + e_1 s + e_o}{f_n s^n + f_{n-1}s^{n-1} + \dots + f_1 s + f_o} \quad (2.6.8)$$

the first step is to divide the denominator into the numerator until the remainder is a proper function as shown

$$Z_{12}(s) = g_1 s + g_o + \frac{h_{n-1}s^{n-1} + \dots + h_1 s + h_o}{f_n s^n + f_{n-1}s^{n-1} + \dots + f_1 s + f_o} \quad (2.6.9)$$

where

$$g_1 = \frac{e_{n+1}}{f_n}$$

$$g_o = \frac{e_n f_n - e_{n+1} f_{n-1}}{f_n^2}$$

and let

$$Z_{12}^{1}(s) = g_1 s \quad (2.6.10)$$

$$Z_{12}^{2}(s) = g_o \quad (2.6.11)$$

$$Z_{12}^{3}(s) = \frac{h_{n-1}s^{n-1} + \ldots + h_1 s + h_o}{f_n s^n + f_{n-1}s^{n-1} + \ldots + f_1 s + f_o} \qquad (2.6.12)$$

Now as shown in Weinberg (33), if $Z_{12}^{1}(s)$, $Z_{12}^{2}(s)$, and $Z_{12}^{3}(s)$ are placed in series while observing the validity test, the open circuit transfer impedance of the total network will be $Z_{12}(s)$ of Equation 2.6.8 and the resultant network is shown in Figure 2.6.2. The transformer may or may not be necessary. Now $Z_{12}^{3}(s)$ is a proper function and can be realized by the state-space synthesis procedure of Chapter IV.

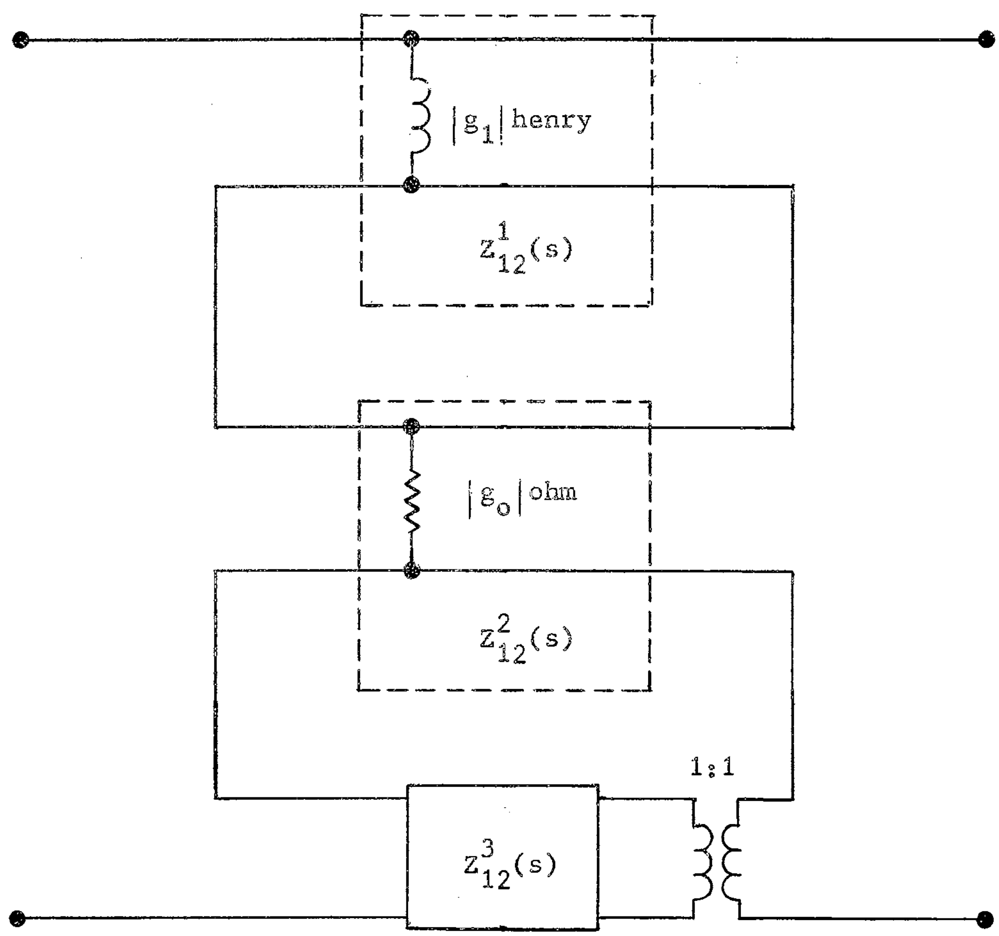

Figure 2.6.2 Modified Network, $Z_{12}(s)$

2.6.3 T(s) Modification. Balabanian (3) has shown that a physically realizable voltage transfer function that is an improper function will have a numerator function degree that at most equals the denominator function degree. Therefore if a voltage transfer function is encountered which is an improper function, it can be written as

$$T(s) = \frac{k_n s^n + \ldots + k_1 s + k_o}{\ell_n s^n + \ldots + \ell_1 s + \ell_o} \qquad (2.6.13)$$

The first step in obtaining a proper function is to divide the denominator into the numerator until the remainder is a proper function as shown

$$T(s) = r_o + \frac{m_{n-1} s^{n-1} + \ldots + m_1 s + m_o}{\ell_n s^n + \ell_{n-1} s^{n-1} + \ldots + \ell_1 s + \ell_o} \qquad (2.6.14)$$

where

$$r_o = \frac{k_n}{\ell_n}$$

and let

$$T^1(s) = r_o \qquad (2.6.15)$$

$$T^2(s) = \frac{m_{n-1} s^{n-1} + \ldots + m_1 s + m_o}{\ell_n s^n + \ell_{n-1} s^{n-1} + \ldots + \ell_1 s + \ell_o} \qquad (2.6.16)$$

Now as shown in Weinberg (33), if $T^1(s)$ and $T^2(s)$ are placed with their inputs in parallel and their outputs in series while observing the validity test, the voltage transfer function of the total network will be T(s) of Equation 2.6.13 and the resultant network is shown in Figure 2.6.3. Now $T^2(s)$ is a proper function and can be realized by the state-space synthesis procedure of Chapter V.

At this point the topics that are basic ideas to the synthesis procedure have been discussed and we are ready to proceed to the state-space synthesis procedures.

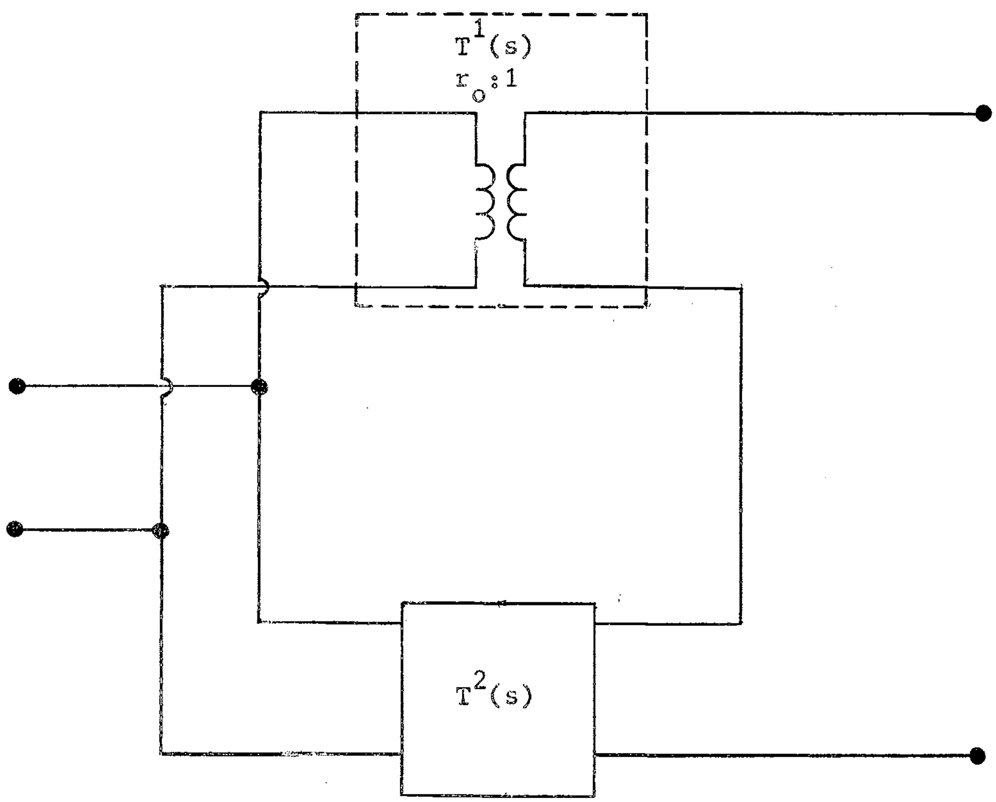

Figure 2.6.3 Modified Network, T(s)

2.7 Realization of Characteristic Polynomial. Yarlagadda (34) has presented the realization of a characteristic polynomial (see Equation 1.1.4) that resulted in a ladder network with one resistive element.

This presentation required the network topology restrictions of Section 2.2, assumed a K_2-matrix like that of Equation 2.5.8a only with f_n equal to zero, and assumed a state-model like that of Equation 2.4.5 with the sources equal to zero. This reference showed that since the K_2-matrix can always be obtained from a given characteristic polynomial, we can relate the characteristic polynomial and the network. In this reference, given a characteristic polynomial of degree, n, only one resistor in addition to the n reactive elements will result from this synthesis procedure. This one resistor and n reactive elements network, as can be seen in Appendix D, is inadequate for the general synthesis procedure of transfer functions as presented here. Therefore the following presentation will be a synthesis procedure of the characteristic polynomial resulting in two resistive elements in addition to the n reactive elements. Now the theorem.

Theorem 1: Let D(s) be a polynomial with constant coefficients. If D(s) has roots with non-positive real parts, then a state-model with no sources and a D(s) as its characteristic polynomial can be obtained. This state-model can be realized by a port-less network with a minimum number of n reactive elements consisting of inductors, capacitors, and two resistors.

The proof of this theorem is vital to the state-space synthesis and it now follows.

<u>2.7.1 Theorem Proof.</u> In Section 2.5 and Appendices B and C, it has been shown that a polynomial with constant coefficients and roots with non-positive real parts can yield a K_2-matrix as shown in Equation 2.5.8a or 2.5.9a. With this K_2-matrix and the presentation

of Section 2.4, a state-model which has the characteristic polynomial, D(s), can be given by

$$\frac{d}{dt} X(t) = K_2 X(t) \tag{2.7.1a}$$

$$\frac{d}{dt}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix} \begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} \tag{2.7.1b}$$

$$\frac{d}{dt}\begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}} B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}} B_{32}^T L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}} B_{32} C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}} B_{33} R_b B_{33}^T L_c^{-\frac{1}{2}} \end{array}\right] \begin{bmatrix} V'_{bc}(t) \\ I'_{c\ell}(t) \end{bmatrix}$$

(2.7.1c)

where the fundamental circuit equations for this state-model are

$$\left[\begin{array}{cc|cc} B_{22} & 0 & U & 0 \\ B_{32} & B_{33} & 0 & U \end{array}\right] \left[\begin{array}{c} V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \tag{2.7.2}$$

Consider the degree of D(s) to be n and n is odd. Then from Equation 2.5.8a

$$K_2 = \left[\begin{array}{ccccc|ccccc} -f_o & & & & & k_1 & & & & \\ & 0 & & & & -k_2 & k_3 & & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & 0 & & & & -k_{n-3} & k_{n-2} & \\ & & & & -f_n & & & & -k_{n-1} & \\ \hline -k_1 & k_2 & & & & & & & & \\ & -k_3 & \ddots & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & k_{n-3} & & & & & & \\ & & & -k_{n-2} & k_{n-1} & & & & & \end{array}\right] \quad (2.7.3)$$

Equate like parts of the partitioned matrix K_2 from Equations 2.7.1c and 2.7.3. This will yield

$$-C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}} = \begin{bmatrix} -f_o & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -f_n \end{bmatrix} \quad (2.7.4a)$$

$$C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \ddots & \ddots & \\ & & -k_{n-3} & k_{n-2} \\ & & & -k_{n-1} \end{bmatrix} \quad (2.7.4b)$$

$$-L_c^{-\frac{1}{2}}B_{33}R_b B_{33}^T L_c^{-\frac{1}{2}} = 0 \quad (2.7.4c)$$

It is noted that C_b, L_c, G_c, and R_b are diagonal matrices with positive entries and that the B_{ij}'s are unimodular matrices with element values of ± 1 or 0. Since C_b and L_c are positive diagonal matrices, the matrices $C_b^{-\frac{1}{2}}$ and $L_c^{-\frac{1}{2}}$ are diagonal with entries chosen to be positive.

To obtain the network which yields the state-model of Equation 2.7.1, the fundamental circuit equations of Equation 2.7.2 must be obtained. These can be determined by decomposing Equation 2.7.4 to yield the B_{ij}'s, G_c, and R_b matrices.

To decompose $(C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}})$ of Equation 2.7.4a, first remove the $C_b^{-\frac{1}{2}}$'s by premultiplication and postmultiplication of both sides of the Equation by $C_b^{\frac{1}{2}}$. This can be done since they are positive diagonal matrices. Assume $C_b^{-\frac{1}{2}}$ is a (mxm)-matrix. This yields

$$B_{22}^T G_c B_{22} = \begin{bmatrix} e_{11} & & & \\ & 0 & & \\ & & \ddots & \\ & & 0 & \\ & & & e_{m,m} \end{bmatrix} \tag{2.7.5}$$

where e_{11} and $e_{m,m}$ are functions of f_o, f_n, and $C_b^{-\frac{1}{2}}$. Since B_{22} is a unimodular matrix, G_c is a positive diagonal matrix, and $(B_{22}^T G_c B_{22})$ is the matrix being considered, it is observed that these are the conditions necessary of applying Cederbaum's (8) algorithm. Using this algorithm for the decomposition of $(B_{22}^T G_c B_{22})$ will yield

$$B_{22}^T G_c B_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{11} & 0 \\ 0 & e_{m,m} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.7.6}$$

where B_{22} is a (2xm)-matrix and G_c is a (2x2)-matrix as shown

$$B_{22} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{2.7.7}$$

$$G_c = \begin{bmatrix} e_{11} & 0 \\ 0 & e_{m,m} \end{bmatrix} \tag{2.7.8a}$$

or we shall identify G_c for simplicity as

$$G_c = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \tag{2.7.8b}$$

Cederbaum (8) showed that the decomposition is <u>essentially</u> unique, which can be seen from Equation 2.7.6 where it is possible to select other signs in B_{22} or permute the rows and columns and still obtain a correct decomposition. Further this decomposition gives a B_{22} which is a non-redundant unimodular or e-matrix.

Substituting the B_{22} of Equation 2.7.7 and the G_c of Equation 2.7.8b into $-(C_b^{-\frac{1}{2}} B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}})$ results in

$$-C_b^{-\frac{1}{2}} B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}} = -\begin{bmatrix} C_1' & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \\ & & & C_m' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1' & & & \\ & \bullet & & \\ & & \bullet & \\ & & & \bullet \\ & & & C_m' \end{bmatrix} \tag{2.7.9}$$

Now Equation 2.7.4a with Equation 2.7.9 yields

$$\begin{bmatrix} -f_o & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -f_n \end{bmatrix} = \begin{bmatrix} -g_1(C_1')^2 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -g_2(C_m')^2 \end{bmatrix} \qquad (2.7.10)$$

and this equation implies that

$$f_o = g_1(C_1')^2 \qquad (2.7.11a)$$

$$f_n = g_2(C_m')^2 \qquad (2.7.11b)$$

Now considering Equation 2.7.4b, where $C_b^{-\frac{1}{2}}$ and $L_c^{-\frac{1}{2}}$ are positive diagonal matrices and B_{32} is a unimodular matrix, it can be seen that the decomposition of $(C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}})$ will be

$$C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} C_1' & & & \\ & \ddots & & \\ & & & C_m' \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} L_1' & & & \\ & \ddots & & \\ & & & L_r' \end{bmatrix}$$

(2.7.12)

where $C_b^{-\frac{1}{2}}$ is a (mxm)-matrix, $L_c^{-\frac{1}{2}}$ is a (rxr)-matrix and B_{32} is a unique (rxm)-matrix as shown

$$B_{32} = \begin{bmatrix} 1 & -1 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & -1 & 1 \end{bmatrix} \qquad (2.7.13)$$

Now Equation 2.7.4b with Equation 2.7.12 yields

$$\begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \ddots & \ddots & \\ & & -k_{n-3} & k_{n-2} \\ & & & -k_{n-1} \end{bmatrix} = \begin{bmatrix} C_1'L_1' & & & \\ -C_2'L_1' & C_2'L_2' & & \\ & \ddots & \ddots & \\ & & -C_{m-1}'L_{r-1}' & C_{m-1}'L_r' \\ & & & C_m'L_r' \end{bmatrix} \qquad (2.7.14)$$

Equation 2.7.14 implies that

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ k_2 &= C_2'L_1' \\ k_3 &= C_2'L_2' \\ &\vdots \\ k_{n-3} &= C_{m-1}'L_{r-1}' \\ k_{n-2} &= C_{m-1}'L_r' \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \qquad (2.7.15)$$

Since $L_c^{-\frac{1}{2}}$ is a non-singular matrix, Equation 2.7.4c implies that

$$B_{33}R_bB_{33}^T = 0 \qquad (2.7.16)$$

From the fundamental circuit equations, B_{33} is a submatrix which corresponds to the resistors in the positive diagonal matrix, R_b. This implies that if either B_{33} or R_b is zero, then the other is zero also. Therefore the equality in Equation 2.7.16 is satisfied when

$$B_{33} = 0 \tag{2.7.17a}$$

and

$$R_b = 0 \tag{2.7.17b}$$

The fundamental circuit equations for the case when n is odd can now be written utilizing the submatrices shown in Equations 2.7.7, 2.7.13, and 2.7.17a. In symbolic form these are

$$\left[\begin{array}{cc|cc} B_{22} & 0 & U & 0 \\ B_{32} & B_{33} & 0 & U \end{array}\right] \left[\begin{array}{c} V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \tag{2.7.18a}$$

which can be written as

$$\left[\begin{array}{ccccccc|c|cccccccc} 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 & 0 & \cdot & & & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 1 & \cdot & & & \cdot & \cdot & 0 \\ \hline 1 & -1 & & & & & & 0 & & & 1 & & & & & \\ & 1 & \cdot & & & & & \cdot & & & & 1 & & & & \\ & & \cdot & \cdot & & & & \cdot & & & & & \cdot & & & \\ & & & \cdot & \cdot & & & \cdot & & & & & & \cdot & & \\ & & & & \cdot & -1 & & 0 & & & & & & & 1 & \\ & & & & & 1 & -1 & 0 & & & & & & & & 1 \end{array}\right] \left[\begin{array}{c} v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline v_{br} \\ \hline v_{cr_1} \\ v_{cr_2} \\ v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r} \end{array}\right] = 0 \tag{2.7.18b}$$

From these fundamental circuit equations, it is easy to construct the network graph which is shown in Figure 2.7.1 and the resulting network of Figure 2.7.2 (21). This then is the network which has a characteristic polynomial, D(s), of degree n with n odd. For the coefficients

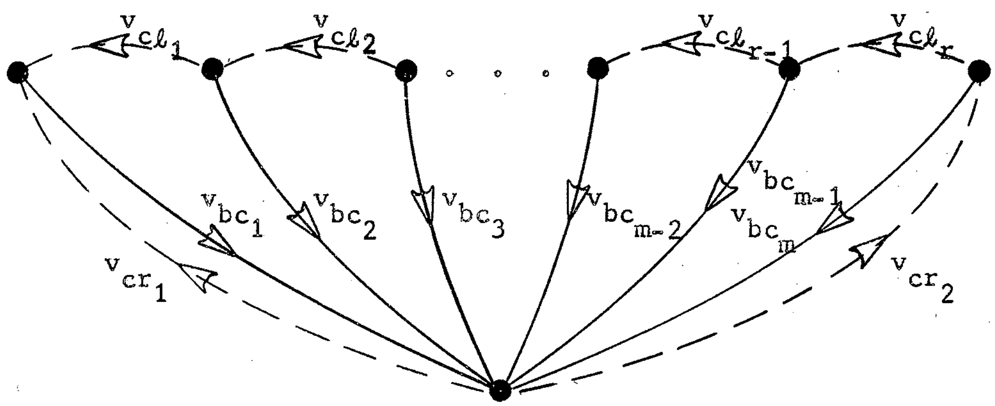

Figure 2.7.1 Network Graph When n Is Odd

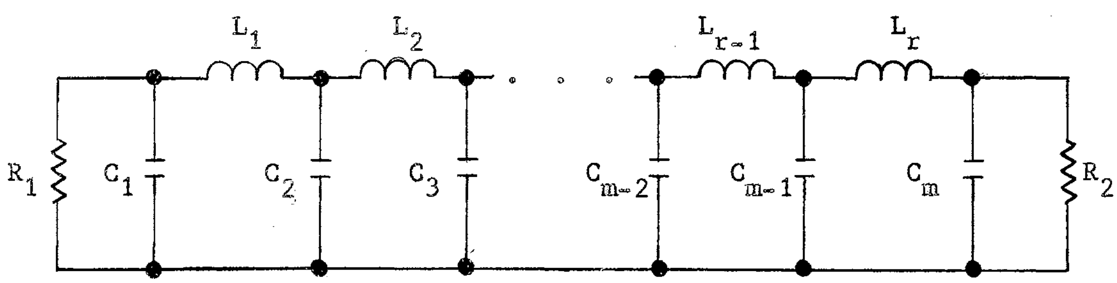

Figure 2.7.2 Synthesized Network

of D(s) to be the same as those in the characteristic polynomial obtained from the network of Figure 2.7.2, a solution of Equations 2.7.11 and 2.7.15 must be determined.

It can be seen from Equations 2.7.1, 2.7.2, and 2.7.3 that if there are m capacitors, r inductors, and the degree of D(s) is n, then $m + r = n$. Equations 2.7.11 and 2.7.15 involve $n + 2$ unknowns and $n + 1$ equations. Therefore a solution for this set of equations is obtained when one of the unknowns is assigned an arbitrary value, such as:

$$C_1' = 1 \qquad (2.7.19a)$$

This value with Equation 2.7.15 implies that

$$L_1' = k_1 \qquad (2.7.19b)$$

$$C_i' = \frac{k_2 k_4 \cdots k_{2(i-1)}}{k_1 k_3 \cdots k_{2i-3}} \; ; \; 1 < i \leq m \qquad (2.7.19c)$$

$$L_i' = \frac{k_1 k_3 \cdots k_{2i-1}}{k_2 k_4 \cdots k_{2(i-1)}} \; ; \; 1 < i \leq r \qquad (2.7.19d)$$

where

$$m = \frac{n + 1}{2}$$

$$r = \frac{n - 1}{2}$$

Since a change of variables was made in Equation 2.3.6, the C_i's and L_i's of the synthesized network of Figure 2.7.2 will still have to be calculated by

$$C_i = (C_i')^{-2} \qquad (2.7.20a)$$

$$L_i = (L_i')^{-2} \qquad (2.7.20b)$$

From Equations 2.7.11 and 2.7.19, g_1 and g_2 are evaluated by

$$g_1 = f_o \tag{2.7.21a}$$

$$g_2 = \left[\frac{k_2 k_4 \cdots k_{n-1}}{k_1 k_3 \cdots k_{n-2}}\right]^{-2} f_n \tag{2.7.21b}$$

where

$$R_1 = \frac{1}{g_1}$$

$$R_2 = \frac{1}{g_2}$$

Note that this solution is not unique since C_1' was arbitrarily chosen. The network has n reactive elements and two resistors as stated in the theorem. This completes the proof when n is odd.

Now consider the degree of D(s) to be n and n is even. Then from Equation 2.5.9a

$$K_2 = \left[\begin{array}{cccc|cccc} -f_o & & & & k_1 & & & \\ & 0 & & & -k_2 & k_3 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 0 & & & -k_{n-2} & k_{n-1} \\ \hline -k_1 & k_2 & & & 0 & & & \\ & -k_3 & \ddots & & & \ddots & & \\ & & \ddots & k_{n-2} & & & 0 & \\ & & & -k_{n-1} & & & & -f_n \end{array}\right] \tag{2.7.22}$$

Again equate like parts of the partitioned matrix K_2 from Equations 2.7.1c and 2.7.22. This will yield

$$-C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}} = \begin{bmatrix} -f_o & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 0 \end{bmatrix} \qquad (2.7.23a)$$

$$C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} k_1 & & & & \\ -k_2 & k_3 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & -k_{n-2} & k_{n-1} \end{bmatrix} \qquad (2.7.23b)$$

$$-L_c^{-\frac{1}{2}}B_{33}R_b B_{33}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} 0 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 0 \\ & & & & & -f_n \end{bmatrix} \qquad (2.7.23c)$$

Again the fundamental circuit equations of Equation 2.7.2 are obtained by decomposing Equation 2.7.23. Applying a similar procedure as was used in the decomposition of $C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}}$ of Equation 2.7.4a, $(C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}})$ of Equation 2.7.23a can be decomposed as

$$-C_b^{-\frac{1}{2}}B_{22}^T G_c B_{22} C_b^{-\frac{1}{2}} = -\begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & C_m' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{bmatrix} g_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & C_m' \end{bmatrix}$$

(2.7.24)

where B_{22} is a (1xm)-matrix and G_c is a (1x1)-matrix as shown

$$B_{22} = [1 \ 0 \ . \ . \ . \ 0] \tag{2.7.25a}$$

$$G_c = [g_1] \tag{2.7.25b}$$

Now Equation 2.7.23 with Equation 2.7.24 yields

$$\begin{bmatrix} -f_o & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 0 \end{bmatrix} = \begin{bmatrix} -g_1(C_1')^2 & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 0 \end{bmatrix} \tag{2.7.26a}$$

and this equation implies that

$$f_o = g_1(C_1')^2 \tag{2.7.26b}$$

Now considering Equation 2.7.23b, it can be seen that the decomposition of $(C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}})$ will be

$$C_b^{-\frac{1}{2}}B_{32}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & C_m' \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} L_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & L_r' \end{bmatrix} \tag{2.7.27a}$$

where B_{32} is a unique (rxm)-matrix as shown

$$B_{32} = \begin{bmatrix} 1 & -1 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & -1 \\ & & & & & 1 \end{bmatrix} \tag{2.7.27b}$$

Now Equation 2.7.23b with Equation 2.7.27a yields

$$\begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \ddots & \ddots & \\ & & -k_{n-2} & k_{n-1} \end{bmatrix} = \begin{bmatrix} C_1'L_1' & & & \\ -C_2'L_1' & C_2'L_2' & & \\ & \ddots & \ddots & \\ & & -C_m'L_{r-1}' & C_m'L_r' \end{bmatrix} \tag{2.7.28a}$$

and this equation implies

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ k_2 &= C_2'L_1' \\ k_3 &= C_2'L_2' \\ &\vdots \\ k_{n-2} &= C_m'L_{r-1}' \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{2.7.28b}$$

Considering Equation 2.7.23c and applying a similar procedure as was used in the decomposition of $C_b^{-\frac{1}{2}}B_{22}^TG_cB_{22}C_b^{-\frac{1}{2}}$ of Equation 2.7.4a, $(L_c^{-\frac{1}{2}}B_{33}R_bB_{33}^TL_c^{-\frac{1}{2}})$ of Equation 2.7.23c can be decomposed as

$$-L_c^{-\frac{1}{2}}B_{33}R_bB_{33}^TL_c^{-\frac{1}{2}} = -\begin{bmatrix} L_1' & & & \\ & \ddots & & \\ & & \ddots & \\ & & & L_r' \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} r_2 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} L_1' & & & \\ & \ddots & & \\ & & \ddots & \\ & & & L_r' \end{bmatrix}$$

(2.7.29a)

where B_{33} is a (1xr)-matrix and R_b is a (1x1)-matrix as shown

$$B_{33} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{2.7.29b}$$

$$R_b = [r_2] \tag{2.7.29c}$$

Now Equation 2.7.23c with Equation 2.7.29 yields

$$\begin{bmatrix} 0 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & 0 & \\ & & & & -f_n \end{bmatrix} = \begin{bmatrix} 0 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & 0 & \\ & & & & -r_2(L_r')^2 \end{bmatrix} \tag{2.7.30a}$$

and this equation implies that

$$f_n = r_2(L_r')^2 \tag{2.7.30b}$$

The fundamental circuit equations for the case when n is even can now be written utilizing the submatrices shown in Equations 2.7.25a, 2.7.27b, and 2.7.29b. In symbolic form these are

$$\left[\begin{array}{cc|cc} B_{22} & 0 & U & 0 \\ B_{32} & B_{33} & 0 & U \end{array}\right] \left[\begin{array}{c} V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \tag{2.7.31a}$$

which can be written as

$$
\left[\begin{array}{ccccc|c|cccccc}
1 & 0 & \cdot & \cdot & 0 & 0 & 1 & & & & & \\ \hline
1 & -1 & & & & 0 & & 1 & & & & \\
 & 1 & \cdot & & & \cdot & & & \cdot & & & \\
 & & \cdot & \cdot & & \cdot & & & & \cdot & & \\
 & & & \cdot & \cdot & \cdot & & & & & \cdot & \\
 & & & \cdot & -1 & 0 & & & & & 1 & \\
 & & & & 1 & 1 & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r}
\end{array}\right] = 0 \qquad (2.7.31b)
$$

From these fundamental circuit equations, it is easy to construct the network graph which is shown in Figure 2.7.3 and the resulting network of Figure 2.7.4 (21). This then is the network which has a characteristic polynomial, D(s), of degree n with n even. For the coefficients

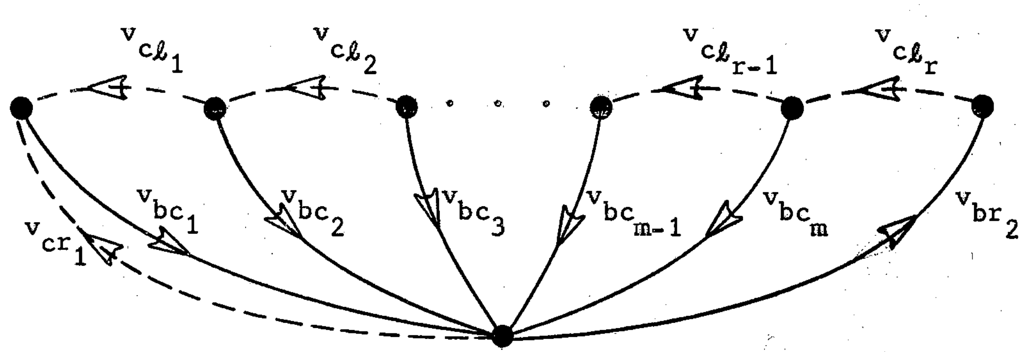

Figure 2.7.3 Network Graph When n Is Even

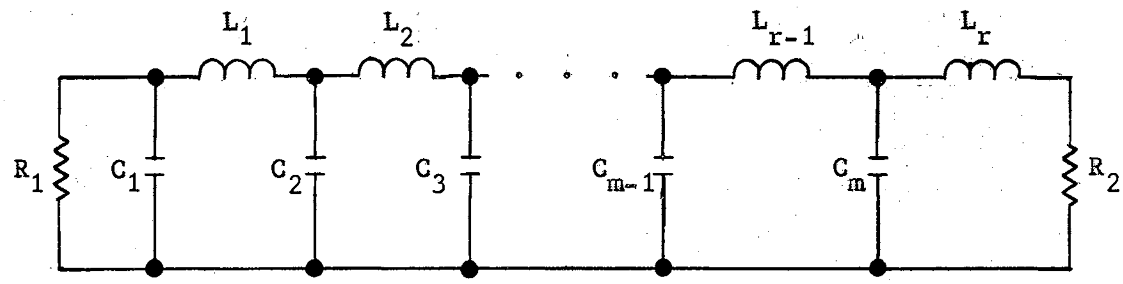

Figure 2.7.4 Synthesized Network

of D(s) to be the same as those in the characteristic polynomial obtained from the network of Figure 2.7.4, a solution of Equations 2.7.26b, 2.7.28b, and 2.7.30b must be determined.

It can be seen from Equations 2.7.1, 2.7.2, and 2.7.22 that if there are m capacitors, r inductors, and the degree of D(s) is n, then $m + r = n$. Equations 2.7.26b, 2.7.28b, and 2.7.30b involve $n + 2$ unknowns and $n + 1$ equations. Therefore a solution for this set of equations is obtained when one of the unknowns is assigned an arbitrary value, such as:

$$C_1' = 1 \tag{2.7.32a}$$

This value with Equation 2.7.28b implies that

$$L_1' = k_1 \tag{2.7.32b}$$

$$C_i' = \frac{k_2 k_4 \cdots k_{2(i-1)}}{k_1 k_3 \cdots k_{2i-3}} \; ; \; 1 < i \leq m \tag{2.7.32c}$$

$$L_i' = \frac{k_1 k_3 \cdots k_{2i-1}}{k_2 k_4 \cdots k_{2(i-1)}} \; ; \; 1 < i \leq r \tag{2.7.32d}$$

where

$$m = \frac{n}{2}$$

$$r = \frac{n}{2}$$

Again because of the change of variables made in Equation 2.3.6, the C_i's and L_i's will be calculated by

$$C_i = (C_i')^{-2} \tag{2.7.33a}$$

$$L_i = (L_i')^{-2} \tag{2.7.33b}$$

From Equations 2.7.26b and 2.7.30b, g_1 and r_2 are evaluated by

$$g_1 = f_o \tag{2.7.34a}$$

$$r_2 = \left[\frac{k_1 k_2 \cdots k_{n-1}}{k_2 k_4 \cdots k_{n-2}}\right]^{-2} f_n \tag{2.7.34b}$$

where

$$R_1 = \frac{1}{g_1}$$

$$R_2 = r_2$$

Note that this solution is not unique since C_1' was arbitrarily chosen. The network has n reactive elements and two resistors as stated in the theorem. This completes the proof when n is even.

It has been shown that a polynomial with constant coefficients and roots with non-positive real parts can be represented by the state-model of Equation 2.7.1, and that this state-model can be used to obtain the port-less network of either Figure 2.7.2 or Figure 2.7.4 with a minimum number of elements consisting of inductors, capacitors, and two resistors. Thus we have related the characteristic polynomial to a network and the proof is complete.

<u>2.8 Transfer Functions And Their Resulting Fundamental Circuit Equations.</u> In deriving the port-less ladder network that represents a characteristic polynomial, the fundamental circuit equations of Equation 2.7.18b or 2.7.31b were obtained. The synthesis procedure presented uses these ladder networks as the basic networks from which to build the synthesized networks. Appendix D shows the transfer admittance resulting from various ladder networks. There are six classifications which includes all of the possible transfer admittances that will be synthesized. These classifications are:

1. Case I - the transfer admittance numerator degree is odd and the denominator degree is odd.
2. Case II - the transfer admittance numerator degree is even, but not zero, and the denominator degree is odd.
3. Case II - Special - the transfer admittance numerator is a constant and the denominator degree is odd.
4. Case III - the transfer admittance numerator degree is odd and the denominator degree is even.
5. Case IV - the transfer admittance numerator degree is even, but not zero, and the denominator degree is even.
6. Case IV - Special - the transfer admittance numerator is a constant and the denominator degree is even.

These classifications will now be discussed individually. First we will consider the simplest cases which are Case II - Special and Case IV-Special.

Note 2.8.1: As is well known for all cases, all of the numerator coefficients will be negative for the orientation of the drivers as shown in Section 2.4.1. It can be shown that to obtain all positive numerator coefficients, just reverse the orientation of one driver.

2.8.1 Case II - Special. The transfer admittance of the Case II - Special classification, as can be shown by using the material in Appendix D is written as

$$y_{12}(s) = \frac{a_o}{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \qquad (2.8.1)$$

where

a_o = negative constant

n = odd integer

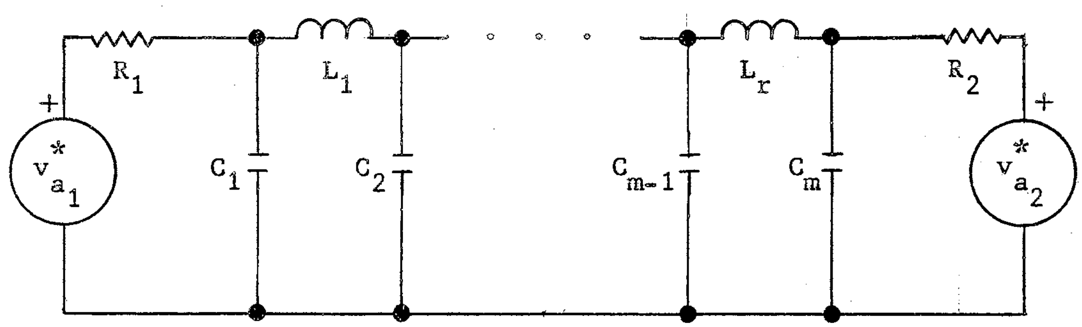

Figure 2.8.1 Network for Case II - Special

The ladder network for this case is shown in Figure 2.8.1 with the resulting network graph shown in Figure 2.8.2. This network graph yields the following fundamental circuit equations in symbolic form as

$$
\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right]
\left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \tag{2.8.2a}
$$

which can be written as

$$
\left[\begin{array}{cc|cccccc|c|cccccccc}
-1 & 0 & 1 & 0 & \cdots & & 0 & 0 & 0 & 1 & & & & & & & \\
0 & -1 & 0 & 0 & \cdots & & 0 & 1 & 0 & & 1 & & & & & & \\
\hline
0 & 0 & 1 & -1 & & & & & 0 & & & 1 & & & & & \\
0 & 0 & & 1 & \cdot & & & & 0 & & & & 1 & & & & \\
\vdots & \vdots & & & \ddots & \ddots & & & \vdots & & & & & \ddots & & & \\
0 & 0 & & & & \cdot & -1 & & 0 & & & & & & & 1 & \\
0 & 0 & & & & & 1 & -1 & 0 & & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c} v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline v_{br} \\ \hline v_{cr_1} \\ v_{cr_2} \\ v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r} \end{array}\right] = 0 \tag{2.8.2b}
$$

where $n = m + r$

For a_o positive, see Note 2.8.1. Now we will consider Case IV - Special.

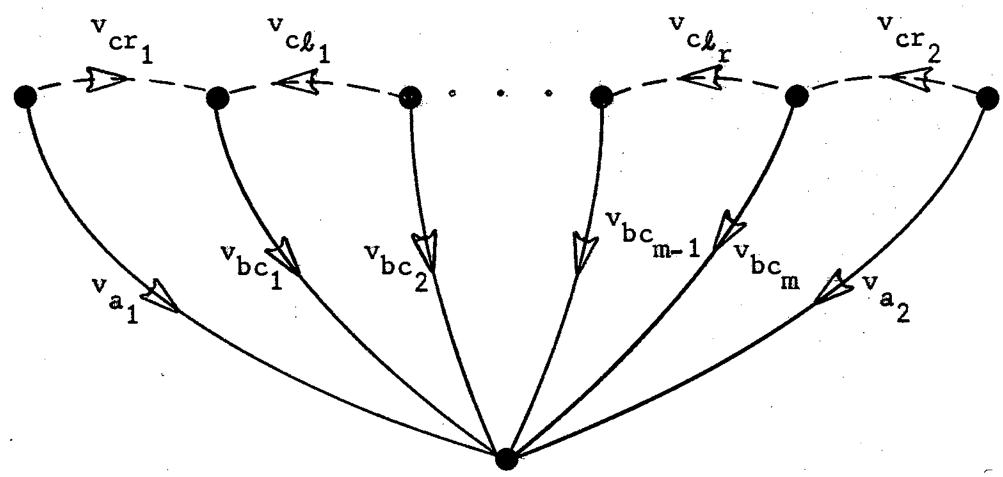

Figure 2.8.2 Network Graph for Case II - Special

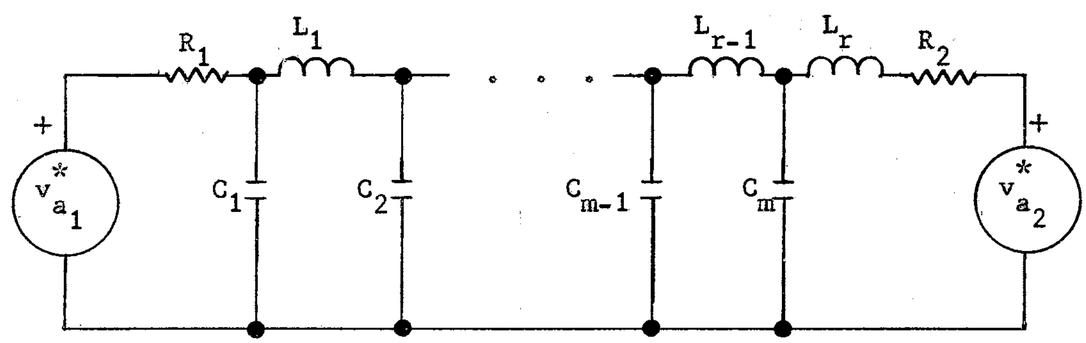

Figure 2.8.3 Network for Case IV - Special

2.8.2 Case IV - Special. The transfer admittance of the Case IV - Special classification, as can be shown by using the material in Appendix D, is written

$$y_{12}(s) = \frac{a_o}{s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_o} \qquad (2.8.3)$$

where

$$a_o = \text{negative constant}$$

$$n = \text{even integer}$$

The ladder network for this case is shown in Figure 2.8.3 with the resulting network graph shown in Figure 2.8.4. This network yields

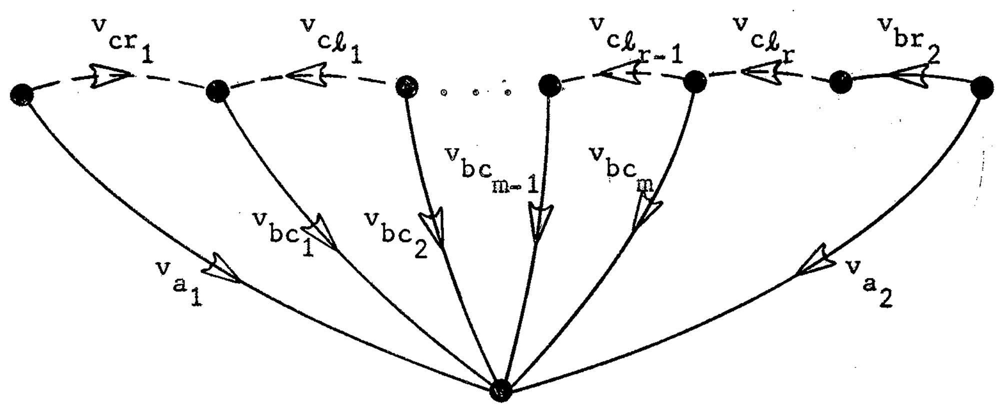

Figure 2.8.4 Network Graph for Case IV - Special

the following fundamental circuit equations in symbolic form as

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \tag{2.8.4a}$$

which can be written as

$$
\left[\begin{array}{cc|cccccc|c|cccccc}
-1 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 & & & & & \\
\hline
0 & 0 & 1 & -1 & & & & & 0 & & 1 & & & & \\
0 & 0 & & 1 & \cdot & & & & 0 & & & 1 & & & \\
\cdot & \cdot & & & \cdot & \cdot & & & \cdot & & & & \cdot & & \\
\cdot & \cdot & & & & \cdot & \cdot & & \cdot & & & & & \cdot & \\
\cdot & \cdot & & & & & & & \cdot & & & & & & \cdot \\
0 & 0 & & & & & \cdot & -1 & 0 & & & & & 1 & \\
0 & -1 & & & & & & 1 & 1 & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_m} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ v_{c\ell_1} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r}
\end{array}\right] = 0
\qquad (2.8.4b)
$$

where

$$n = m + r$$

For a_o positive, see Note 2.8.1. Now we will consider Case I.

2.8.3 Case I. The transfer admittance of the Case I classification, as can be shown by using the material in Appendix D, is written as

$$y_{12}(s) = \frac{a_{x_1} s^{x_1} + a_{x_1-1} s^{x_1-1} + \dots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_o} \qquad (2.8.5)$$

where

a_i = negative constant; $i = o, 1, \dots, x_1$

x_1 = odd integer

n = odd integer

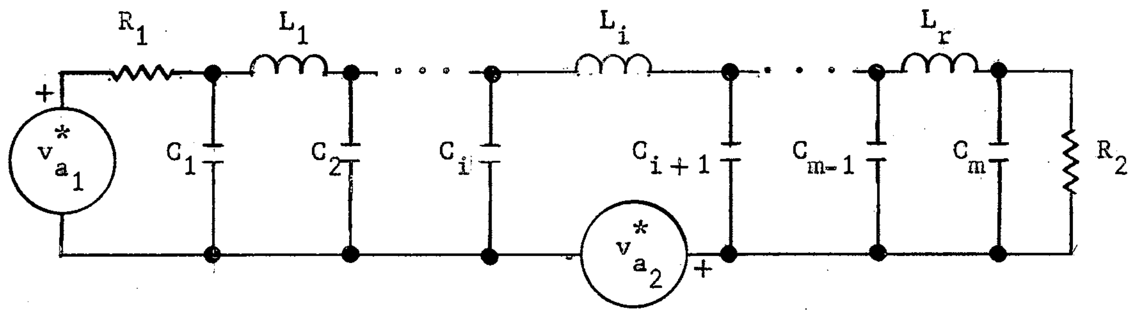

Figure 2.8.5 Network for Case I

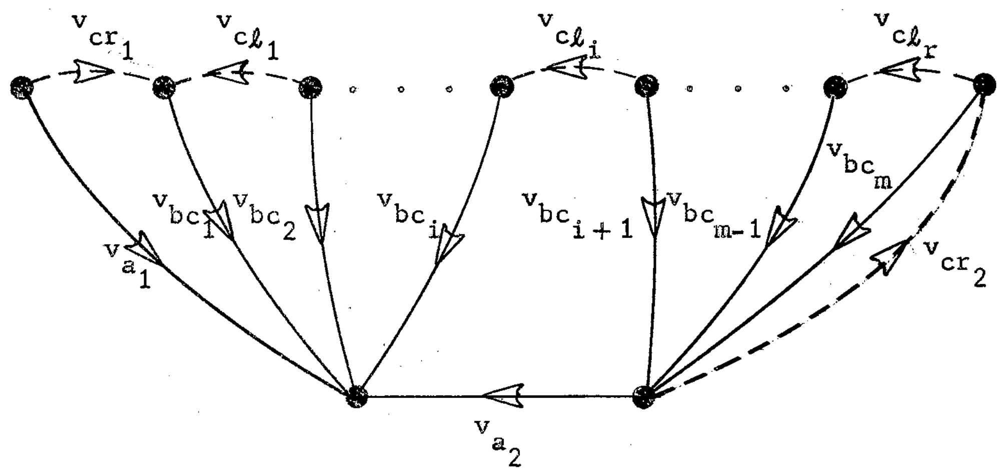

Figure 2.8.6 Network Graph for Case I

The ladder network for this case is shown in Figure 2.8.5 with (i) representing the mesh consisting of the elements C_i, L_i, C_{i+1}, and $v_{a_2}^*$. Note that x_1 determines the mesh in which the driver $v_{a_2}^*$ must be located to yield the desired numerator degree of $y_{12}(s)$ in Equation 2.8.5. (i) is related to x_1 by

$$i = \frac{n-x_1}{2} \tag{2.8.6a}$$

$$n = m + r \tag{2.8.6b}$$

$$m = r + 1 \tag{2.8.6c}$$

The graph corresponding to this network is given in Figure 2.8.6. The fundamental circuit equations corresponding to this graph are symbolically represented by

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \tag{2.8.7a}$$

Since there are no branch resistors in the graph, R_b and B_{23} will be zero, and the fundamental circuit equations can be written as

$$\left[\begin{array}{cc|cccccccc|cccccccc} -1 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & & & 0 & 0 & 1 & & 1 & & & & & \\ \hline 0 & 0 & 1 & -1 & & & & & & & & & 1 & & & & & \\ 0 & 0 & & 1 & -1 & & & & & & & & & 1 & & & & \\ \cdot & \cdot & & & \ddots & \ddots & & & & & & & & & \ddots & & & \\ 0 & -1 & & & & 1 & -1 & & & & & & & & & 1 & & \\ 0 & 0 & & & & & 1 & -1 & & & & & & & & & 1 & \\ \cdot & \cdot & & & & & & \ddots & \ddots & & & & & & & & \ddots & \\ 0 & 0 & & & & & & & 1 & -1 & & & & & & & 1 & \\ 0 & 0 & & & & & & & & 1 & -1 & & & & & & & 1 \end{array}\right] \left[\begin{array}{c} v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_i} \\ v_{bc_{i+1}} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline v_{cr_1} \\ v_{cr_2} \\ v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_i} \\ v_{c\ell_{i+1}} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r} \end{array}\right] = 0$$

(2.8.7b)

where

$$n = m + r$$

$$i \leq r$$

and the i^{th} row of B_{21} has the -1 element in column 2. For the a_i's of Equation 2.8.5 to be positive, see Note 2.8.1. Now we will consider Case II.

<u>2.8.4 Case II.</u> The transfer admittance of the Case II classification, as can be shown by using the material in Appendix D, is written as

$$y_{12}(s) = \frac{a_{x_2} s^{x_2} + a_{x_2-1} s^{x_2-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \qquad (2.8.8)$$

where

a_i = negative constant; $i = 0, 1, \ldots, x_2$

x_2 = even integer

$x_2 \neq 0$

n = odd integer

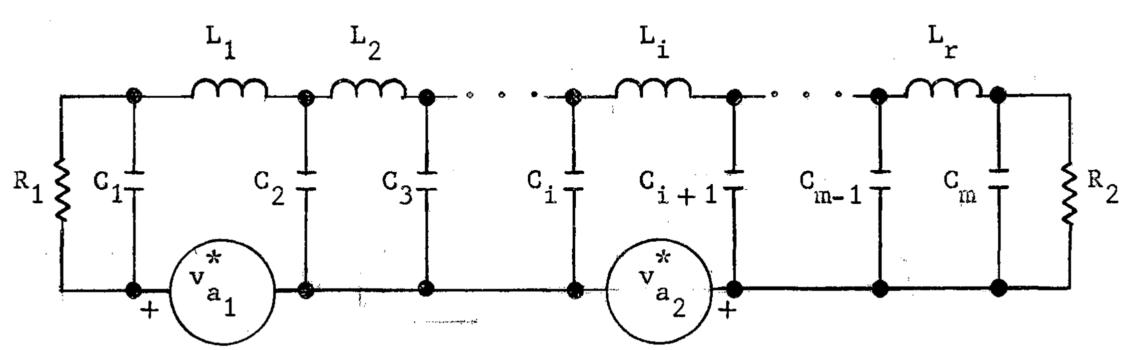

Figure 2.8.7 Network for Case II

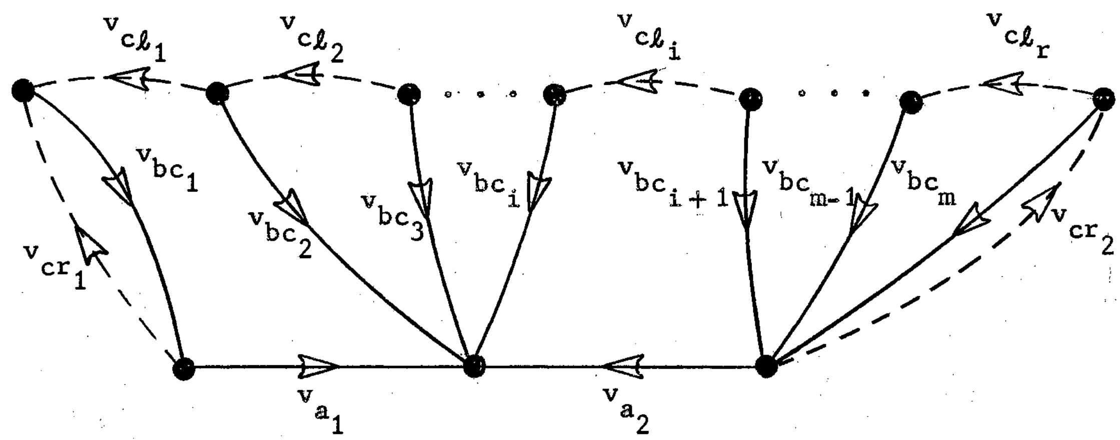

Figure 2.8.8 Network Graph for Case II

The ladder network for this case is shown in Figure 2.8.7 with (i) representing the mesh consisting of the elements C_i, L_i, C_{i+1}, and $v^*_{a_2}$. Note that x_2 determines the mesh in which the driver $v^*_{a_2}$ must be located to yield the desired numerator degree of $y_{12}(s)$ in Equation 2.8.8. (i) is related to x_2 by

$$i = \frac{n - x_2 + 1}{2} \qquad (2.8.9a)$$

Further

$$n = m + r \qquad (2.8.9b)$$

$$m = r + 1 \qquad (2.8.9c)$$

The graph corresponding to this network is given in Figure 2.8.8.

The fundamental circuit equations corresponding to this graph are symbolically represented as

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \begin{bmatrix} V_a \\ V_{bc} \\ \hline V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{bmatrix} = 0 \qquad (2.8.10a)$$

Since there are no branch resistors in the graph, R_b and B_{23} will be zero, and the fundamental circuit equations can be written as

$$\left[\begin{array}{cc|cccccccc|cccccccc} 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \; 0 & 1 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \; 1 & & 1 & & & & & & \\ \hline 1 & 0 & 1 & -1 & & & & & & & & & 1 & & & & & \\ 0 & 0 & & 1 & -1 & & & & & & & & & 1 & & & & \\ \cdot & \cdot & & & \ddots & \ddots & & & & & & & & & \ddots & & & \\ 0 & -1 & & & & 1 & -1 & & & & & & & & & 1 & & \\ 0 & 0 & & & & & 1 & -1 & & & & & & & & & 1 & \\ \cdot & \cdot & & & & & & \ddots & \ddots & & & & & & & & & \ddots \\ 0 & 0 & & & & & & & 1 & -1 & & & & & & & & 1 \\ 0 & 0 & & & & & & & & 1 \; -1 & & & & & & & & \quad 1 \end{array}\right] \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_i} \\ v_{bc_{i+1}} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline v_{cr_1} \\ v_{cr_2} \\ v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_i} \\ v_{c\ell_{i+1}} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r} \end{bmatrix} = 0$$

(2.8.10b)

$$n = m + r$$

$$i \leq r$$

and the i^{th} row of B_{21} has the -1 element in column 2. For the a_i's of Equation 2.8.8 to be positive, see Note 2.8.1. Now we will consider Case III.

2.8.5 Case III. The transfer admittance of the Case III classification, as can be shown by using the material in Appendix D, is written

$$y_{12}(s) = \frac{a_{x_3} s^{x_3} + a_{x_3 - 1} s^{x_3 - 1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \qquad (2.8.11)$$

where

a_i = negative constant; $i = 0, 1, \ldots, x_3$

x_3 = odd integer

n = even integer

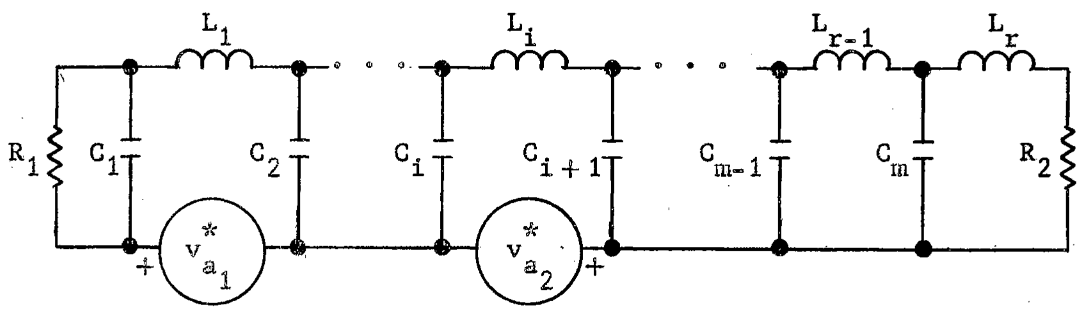

Figure 2.8.9 Network for Case III

The ladder network for this case is shown in Figure 2.8.9 with (i) representing the mesh consisting of the elements C_i, L_i, C_{i+1}, and $v^*_{a_2}$. Note that x_3 determines the mesh in which the driver $v^*_{a_2}$ must be located to yield the desired numerator degree of $y_{12}(s)$ in Equation 2.8.11. (i) is related to x_3 by

$$i = \frac{n - x_3 + 1}{2} \tag{2.8.12a}$$

Further

$$n = m + r \tag{2.8.12b}$$

$$m = r \tag{2.8.12c}$$

The graph corresponding to this network is given in Figure 2.8.10.

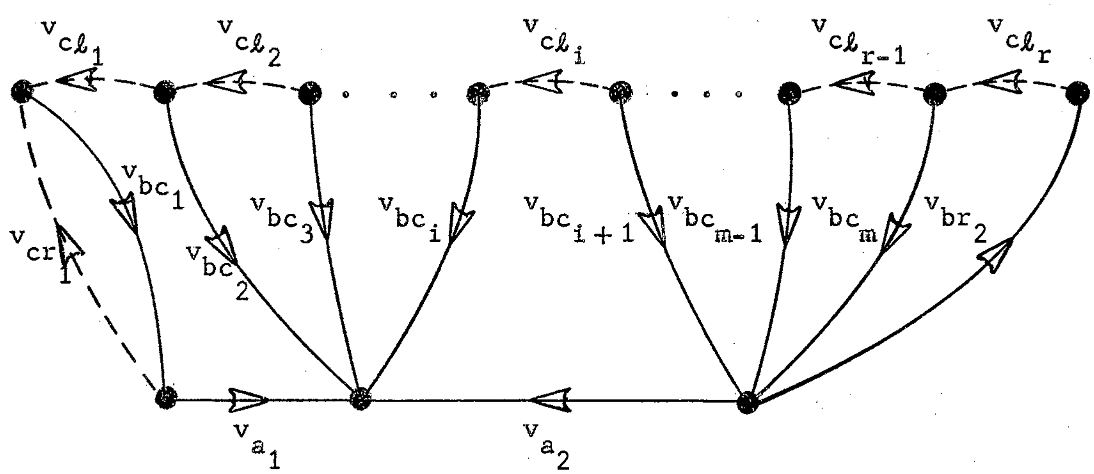

Figure 2.8.10 Network Graph for Case III

The fundamental circuit equations corresponding to this graph are symbolically represented as

$$\begin{bmatrix} B_{11} & B_{12} & 0 & | & U & 0 \\ B_{21} & B_{22} & B_{23} & | & 0 & U \end{bmatrix} \begin{bmatrix} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{bmatrix} = 0 \tag{2.8.13a}$$

which can be written as

$$
\left[\begin{array}{cc|ccccccccc|c|ccccccc}
0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & & 0 & 0 & 0 & 1 & & & & & & \\
\hline
1 & 0 & 1 & -1 & & & & & & & & 0 & & 1 & & & & & \\
0 & 0 & & 1 & -1 & & & & & & & 0 & & & 1 & & & & \\
\cdot & \cdot & & & \cdot & \cdot & & & & & & \cdot & & & & \cdot & & & \\
\cdot & \cdot & & & & \cdot & \cdot & & & & & \cdot & & & & & \cdot & & \\
\cdot & \cdot & & & & & \cdot & \cdot & & & & \cdot & & & & & & \cdot & \\
0 & -1 & & & & & & 1 & -1 & & & 0 & & & & & & 1 & \\
0 & 0 & & & & & & & 1 & -1 & & 0 & & & & & & & 1 \\
\cdot & \cdot & & & & & & & & \cdot & \cdot & \cdot & & & & & & & \cdot \\
\cdot & \cdot & & & & & & & & & \cdot & \cdot & & & & & & & \cdot \\
\cdot & \cdot & & & & & & & & & \cdot & \cdot & & & & & & & \cdot \\
0 & 0 & & & & & & & & 1 & -1 & 0 & & & & & & & 1 \\
0 & 0 & & & & & & & & & 1 & 1 & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ v_{a_2} \\ \hline
v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_i} \\ v_{bc_{i+1}} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline
v_{br_2} \\ \hline
v_{cr_1} \\ \hline
v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_i} \\ v_{c\ell_{i+1}} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r}
\end{array}\right] = 0 \qquad (2.8.13b)
$$

$$n = m + r$$

$$i \leq r$$

and the i^{th} row of B_{21} has the -1 element in column 2. For the a_i's of Equation 2.8.11 to be positive, see Note 2.8.1. Now we will consider Case IV.

2.8.6 Case IV. The transfer admittance of the Case IV classification, can be shown by using the material in Appendix D, is written

$$y_{12}(s) = \frac{a_{x_4} s^{x_4} + a_{x_4-1} s^{x_4-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \qquad (2.8.14)$$

where

a_i = negative constant; $i = 0, 1, \ldots, x_4$

x_4 = even integer

$x_4 \neq 0$

n = even integer

The ladder network for this case is shown in Figure 2.8.11 with (i) representing the mesh consisting of the elements C_i, L_i, C_{i+1}, and $v^*_{a_2}$.

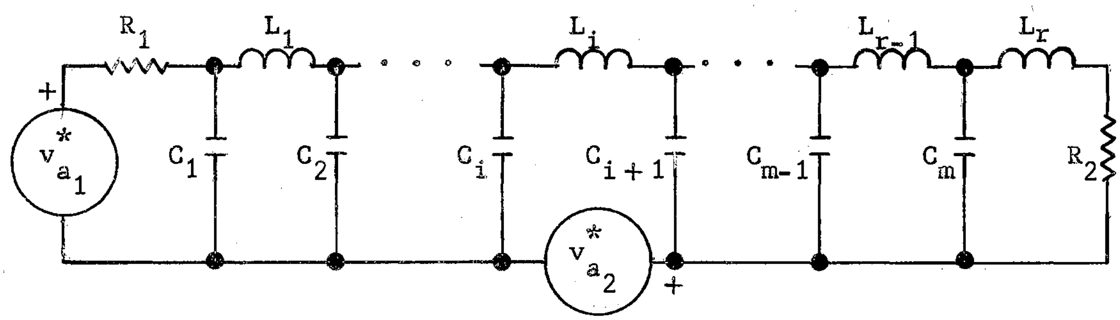

Figure 2.8.11 Network for Case IV

Note that x_4 determines the mesh in which the driver $v^*_{a_2}$ must be located to yield the desired numerator degree of $y_{12}(s)$ in Equation 2.8.14. (i) is related to x_4 by

$$i = \frac{n - x_4}{2} \tag{2.8.15a}$$

Further

$$n = m + r \tag{2.8.15b}$$

$$m = r \tag{2.8.15c}$$

The graph corresponding to this network is given in Figure 2.8.12.

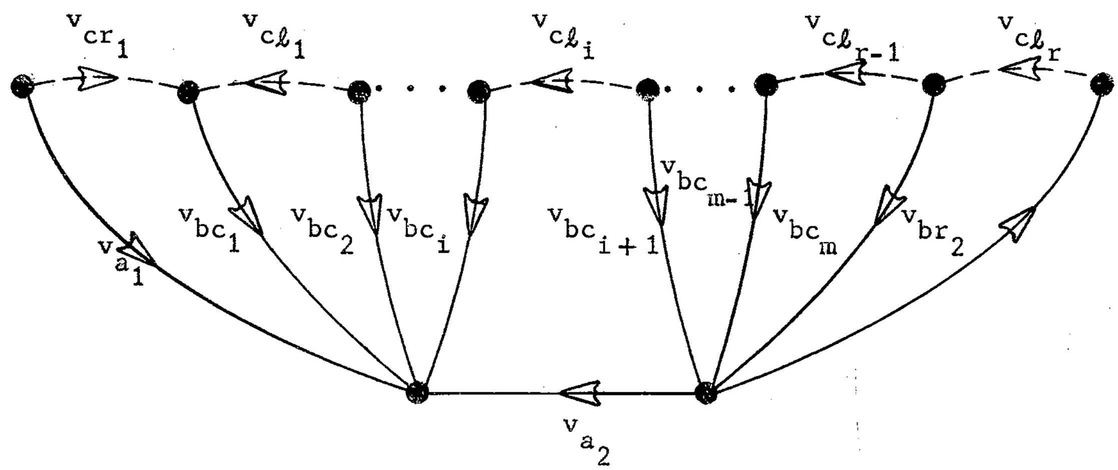

Figure 2.8.12 Network Graph for Case IV

The fundamental circuit equations corresponding to this graph are symbolically represented by

$$\begin{bmatrix} B_{11} & B_{12} & 0 & | & U & 0 \\ B_{21} & B_{22} & B_{23} & | & 0 & U \end{bmatrix} \begin{bmatrix} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{bmatrix} = 0 \tag{2.8.16a}$$

which can be written as

$$
\left[\begin{array}{cc|cccccccc|c|cccccccc}
-1 & 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & & & & & & & \\
\hline
0 & 0 & 1 & -1 & & & & & & & 0 & & 1 & & & & & & \\
0 & 0 & & 1 & -1 & & & & & & 0 & & & 1 & & & & & \\
\vdots & \vdots & & & \ddots & \ddots & & & & & \vdots & & & & \ddots & & & & \\
0 & -1 & & & & 1 & -1 & & & & 0 & & & & & 1 & & & \\
0 & 0 & & & & & 1 & -1 & & & 0 & & & & & & 1 & & \\
\vdots & \vdots & & & & & & \ddots & \ddots & & \vdots & & & & & & & \ddots & \\
0 & 0 & & & & & & & 1 & -1 & 0 & & & & & & & & 1 \\
0 & 0 & & & & & & & & 1 & 1 & & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \vdots \\ v_{bc_i} \\ v_{bc_{i+1}} \\ \vdots \\ v_{bc_{m-1}} \\ v_{bc_m} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ \hline v_{c\ell_1} \\ v_{c\ell_2} \\ \vdots \\ v_{c\ell_i} \\ v_{c\ell_{i+1}} \\ \vdots \\ v_{c\ell_{r-1}} \\ v_{c\ell_r}
\end{array}\right] = 0
\qquad (2.8.16b)
$$

$$n = m + r$$

$$i \leq r$$

and the i^{th} row B_{21} has the -1 element in column 2. For the a_i's of Equation 2.8.14 to be positive, see Note 2.8.1.

In the next chapter these results will be used in the synthesis procedure.

CHAPTER III

SYNTHESIS OF THE SHORT CIRCUIT TRANSFER ADMITTANCE, $Y_{12}(s)$

3.1 Introduction. This chapter will present the state-space approach to the synthesis of the short circuit transfer admittance, $Y_{12}(s)$, using the concepts presented in Chapter II. Only the general RLC case and LC case will be considered. The restrictions on the network topology and on the s-domain transfer function will be presented as they apply to the short circuit transfer admittance only. The desired state-model for $Y_{12}(s)$ synthesis will be given and using this state-model certain short circuit transfer admittances will be synthesized to illustrate the developed procedures.

3.2 Restrictions. The restrictions are of two types; those on the s-domain short circuit transfer admittance function to be synthesized and those of the resulting network that exhibits the given $Y_{12}(s)$. Discussions of these restrictions in general are presented throughout Chapter II. They are presented below as they apply only to the short circuit transfer admittance, $Y_{12}(s)$.

The short circuit transfer admittance function restrictions are:

1. The degree of the numerator polynomial can be no more than one greater than the degree of the denominator polynomial.
2. The coefficients of the numerator polynomial must be real and finite.

3. The denominator polynomial must be a strictly Hurwitz polynomial (32).

If $Y_{12}(s)$ is an improper function, Section 2.6.1 presents the necessary modification to complete before the state-space synthesis procedure can be applied to the proper function portion of $Y_{12}(s)$ which is represented as $Y_{12}^{3}(s)$ in Equation 2.6.7.

The topological restrictions on the network to be synthesized from the proper function $Y_{12}^{3}(s)$ are:

1. Both branch resistors and chord resistors will not be permitted in the same fundamental circuits.
2. Circuits of capacitors with or without voltage drivers will not be permitted.
3. Cut-sets of inductors will not be permitted.
4. The network driver configuration must be that of Figure 2.4.1.

<u>3.3 State-Models and $[\mathrm{adj}(sU-K_2)]$ with n Odd.</u> It is desirable to recall the state-model developed in Section 2.4.1 as it will be used in the synthesis procedure. This state-model is given here for ready reference.

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{12}^{T}G_cB_{12}C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}}B_{22}^{T}L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}}B_{22}C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{23}R_bB_{23}^{T}L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \left[\begin{array}{c} -C_b^{-\frac{1}{2}}B_{12}^{T}G_cB_{11} \\ \hline -L_c^{-\frac{1}{2}}B_{21} \end{array}\right]\begin{bmatrix} V_a^{*} \end{bmatrix} \tag{3.3.1a}$$

$$\left[I_a^*\right] = \left[B_{11}^T G_c B_{12} C_b^{-\frac{1}{2}} \;\vdots\; -B_{21}^T L_c^{-\frac{1}{2}}\right] \begin{bmatrix} V_{bc}' \\ I_{c\ell}' \end{bmatrix} + \left[B_{11}^T G_c B_{11}\right]\left[V_a^*\right] \quad (3.3.1b)$$

Our objective at this time will be to derive a state-model of the form in Equation 3.3.1 from the proper function $Y_{12}^3(s)$. From here onwards for simplicity, $Y_{12}^3(s)$ will be referenced as $y_{12}(s)$.

Let us consider a $y_{12}(s)$ which has a denominator polynomial of odd degree, n. Now using the developments presented in Sections 2.4 and 2.5 (Equations 2.4.5-2.4.9 and 2.5.1-2.5.10), it can be seen that this can be represented in the state-model as

$$\frac{d}{dt}\begin{bmatrix} v_{bc_1}' \\ \vdots \\ v_{bc_m}' \\ \hline i_{c\ell_{m+1}}' \\ \vdots \\ i_{c\ell_{m+r}}' \end{bmatrix} = \left[\begin{array}{cccc|cccc} -f_o & & & & k_1 & & & \\ & 0 & & & -k_2 & \ddots & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 0 & & & \ddots & k_{n-2} \\ & & & -f_n & & & & -k_{n-1} \\ \hline -k_1 & k_2 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & & & & \\ & & -k_{n-2} & k_{n-1} & & & & \end{array}\right] \begin{bmatrix} v_{bc_1}' \\ \vdots \\ v_{bc_m}' \\ \hline i_{c\ell_{m+1}}' \\ \vdots \\ i_{c\ell_{m+r}}' \end{bmatrix}$$

$$+ \begin{bmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \\ \hline b_{m+1,1} & b_{m+1,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} \begin{bmatrix} v_{a_1}^* \\ v_{a_2}^* \end{bmatrix}$$

(3.3.2a)

$$\begin{bmatrix} i^*_{a_1} \\ i^*_{a_2} \end{bmatrix} = \left[\begin{array}{ccc|ccc} p_{11} & \cdots & p_{1,m} & p_{1,m+1} & \cdots & p_{1,m+r} \\ p_{21} & \cdots & p_{2,m} & p_{2,m+1} & \cdots & p_{2,m+r} \end{array}\right] \left[\begin{array}{c} v'_{bc_1} \\ \vdots \\ v'_{bc_m} \\ \hline i'_{c\ell_{m+1}} \\ \vdots \\ i'_{c\ell_{m+r}} \end{array}\right] + \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix}$$

(3.3.2b)

where n is odd, $n = m + r$ and $m = r + 1$. Observe that there are m capacitor voltages and r inductor currents in the state-vector. As was done in Equation 2.4.5, Equation 3.3.2 can be written in a symbolic form as

$$\frac{d}{dt} X = K_2 X + \mathcal{B} V^*_a \tag{3.3.3a}$$

$$I^*_a = \mathcal{P} X + R V^*_a \tag{3.3.3b}$$

Now from Equations 2.4.6 and 2.5.5

$$\mathcal{D}(s) = \mathcal{P}(sU-K_2)^{-1}\mathcal{B} \tag{3.3.4a}$$

$$\mathcal{D}(s) = \frac{1}{\Delta} \mathcal{P}\left[\mathrm{adj}(sU-K_2)\right]\mathcal{B} \tag{3.3.4b}$$

$$\mathcal{D}(s) = \begin{bmatrix} \mathcal{D}_{11}(s) & \mathcal{D}_{12}(s) \\ \mathcal{D}_{21}(s) & \mathcal{D}_{22}(s) \end{bmatrix} \tag{3.3.4c}$$

where

$$\Delta = \left| sU-K_2 \right|$$

and

$$y_{12}(s) = \frac{\mathfrak{D}'_{12}(s)}{\Delta} \qquad (3.3.5)$$

Equations 3.3.4b and 3.3.5 show that $\mathfrak{D}'_{12}(s)$ is the product of certain elements of $\mathcal{P}$, $\mathcal{B}$, and $[adj(sU-K_2)]$. Therefore one of the major ideas in this synthesis procedure will be to select the correct network from those of section 2.8 which will result in a network graph that specifies the elements of $\mathcal{P}$ and $\mathcal{B}$ so that $\mathcal{P}\,[adj(sU-K_2)]\,\mathcal{B}$ will yield the desired $\mathfrak{D}'_{12}(s)$. Now we shall obtain the matrix $[adj(sU-K_2)]$. From Appendix C it was shown that

$$[adj(sU-K_2)] = L^T[adj(sU-K_1)]L \qquad (3.3.6)$$

where the matrix $[adj(sU-K_1)]$ is given in Equation 2.5.12 and can be written in symbolic form as

$$[adj(sU-K_1)] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n} \\ a_{21} & a_{22} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \qquad (3.3.7)$$

Using the transformation matrix, L, of Equation 3.5.8b and Equation 3.3.7 in Equation 3.3.6 yields

$$[\mathrm{adj}(sU-K_2)] = \left[\begin{array}{cccc|cccc} a_{11} & a_{13} & \cdots & a_{1,n} & a_{12} & a_{14} & \cdots & a_{1,n-1} \\ a_{31} & a_{33} & \cdots & a_{3,n} & a_{32} & a_{34} & \cdots & a_{3,n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,3} & \cdots & a_{n,n} & a_{n,2} & a_{n,4} & \cdots & a_{n,n-1} \\ \hline a_{21} & a_{23} & \cdots & a_{2,n} & a_{22} & a_{24} & \cdots & a_{2,n-1} \\ a_{41} & a_{43} & \cdots & a_{4,n} & a_{42} & a_{44} & \cdots & a_{4,n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,3} & \cdots & a_{n-1,n} & a_{n-1,2} & a_{n-1,4} & \cdots & a_{n-1,n-1} \end{array}\right] \tag{3.3.8}$$

where the a_{ij} elements of Equation 3.3.8 are the same a_{ij} elements of Equation 2.5.12.

Now substitute the $\mathfrak{B}$ and $\mathfrak{P}$ of Equation 3.3.2 and the $[\mathrm{adj}(sU-K_2)]$ of Equation 3.3.8 into Equation 3.3.4b and it can be shown that the $\mathfrak{D}'_{12}(s)$ of Equation 3.3.5 will be as shown in Equation 3.3.9 on the following page. Equations 3.3.1, 3.3.2, 3.3.7, and 3.3.9 will be used frequently in the synthesis procedure that follows.

<u>3.4 Synthesis of $y_{12}(s)$, Case II - Special.</u> Several features of the synthesis procedure will become apparent with the synthesis of $y_{12}(s)$ that has a denominator function degree of n, which is odd, and a numerator function that is a real and finite constant. The synthesis procedure will produce an unbalanced ladder network as in Section 2.8.1 that has two resistors, n reactive elements, and no transformers. Note that in the special case of $n = 1$, we will have one resistor and one reactive element. This will be a subclass of the general case which will become evident later. A $y_{12}(s)$ with these characteristics is presented first as it will have the simplest synthesis procedure and it will give a good overall idea of the synthesis approach used in the more complex short circuit transfer admittance to be synthesized.

$$
\begin{aligned}
\mathcal{D}'_{12}(s) = \; & (p_{11}a_{11} + p_{12}a_{31} + p_{13}a_{51} + \cdots + p_{1,m}a_{n,1} + p_{1,m+1}a_{21} + p_{1,m+2}a_{41} + \cdots + p_{1,n}a_{n-1,1})b_{12} \\
& + (p_{11}a_{12} + p_{12}a_{32} + p_{13}a_{52} + \cdots + p_{1,m}a_{n,2} + p_{1,m+1}a_{22} + p_{1,m+2}a_{42} + \cdots + p_{1,n}a_{n-1,2})b_{m+1,2} \\
& + (p_{11}a_{13} + p_{12}a_{33} + p_{13}a_{53} + \cdots + p_{1,m}a_{n,3} + p_{1,m+1}a_{23} + p_{1,m+2}a_{43} + \cdots + p_{1,n}a_{n-1,3})b_{22} \\
& + (p_{11}a_{14} + p_{12}a_{34} + p_{13}a_{54} + \cdots + p_{1,m}a_{n,4} + p_{1,m+1}a_{24} + p_{1,m+2}a_{44} + \cdots + p_{1,n}a_{n-1,4})b_{m+2,2} \\
& + (p_{11}a_{15} + p_{12}a_{35} + p_{13}a_{55} + \cdots + p_{1,m}a_{n,5} + p_{1,m+1}a_{25} + p_{1,m+2}a_{45} + \cdots + p_{1,n}a_{n-1,5})b_{32} \\
& \quad \vdots \\
& + (p_{11}a_{1,n-1} + p_{12}a_{3,n-1} + p_{13}a_{5,n-1} + \cdots + p_{1,m}a_{n,n-1} + p_{1,m+1}a_{2,n-1} + p_{1,m+2}a_{4,n-1} + \cdots + p_{1,n}a_{n-1,n-1})b_{m+r,2} \\
& + (p_{11}a_{1,n} + p_{12}a_{3,n} + p_{13}a_{5,n} + \cdots + p_{1,m}a_{n,n} + p_{1,m+1}a_{2,n} + p_{1,m+2}a_{4,n} + \cdots + p_{1,n}a_{n-1,n})b_{m,2}
\end{aligned}
$$

n = odd integer

$n = m + r$

$m = r + 1$

(3.3.9)

Assume that the Case II - Special short circuit transfer admittance to be synthesized is

$$y_{12}(s) = \frac{a_o}{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \tag{3.4.1}$$

where a_o is an arbitrary real coefficient and the denominator polynomial is a strictly Hurwitz polynomial. In Section 2.8.1 a ladder network has been given that yields a transfer admittance like that of Equation 3.4.1. Also the fundamental circuit equations for this network are given in Equation 2.8.2. Utilizing this information allows the determination of the element values of this circuit and completion of the synthesis procedure.

Proceeding as in Section 2.7.1 and considering the partitioned matrix parts of Equations 3.3.1 and 3.3.2, it is observed that the corresponding parts of the partitioned matrices can be equated such as

$$-C_b^{-\frac{1}{2}} B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}} = \begin{bmatrix} -f_o & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & -f_n \end{bmatrix} \tag{3.4.2a}$$

$$C_b^{-\frac{1}{2}} B_{22}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \ddots & \ddots & \\ & & -k_{n-3} & k_{n-2} \\ & & & -k_{n-1} \end{bmatrix} \tag{3.4.2b}$$

$$-L_c^{-\frac{1}{2}}B_{23}R_bB_{23}^TL_c^{-\frac{1}{2}} = 0 \tag{3.4.2c}$$

$$-C_b^{-\frac{1}{2}}B_{12}^TG_cB_{11} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \end{bmatrix} \tag{3.4.2d}$$

$$-L_c^{-\frac{1}{2}}B_{21} = \begin{bmatrix} b_{m+1,1} & b_{m+1,2} \\ b_{m+2,1} & b_{m+2,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} \tag{3.4.2e}$$

$$-C_b^{-\frac{1}{2}}B_{12}^TG_cB_{11} = -\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,m} \\ p_{21} & p_{22} & \cdots & p_{2,m} \end{bmatrix}^T \tag{3.4.2f}$$

$$-L_c^{-\frac{1}{2}}B_{21} = \begin{bmatrix} p_{1,m+1} & p_{1,m+2} & \cdots & p_{1,m+r} \\ p_{2,m+1} & p_{2,m+2} & \cdots & p_{2,m+r} \end{bmatrix}^T \tag{3.4.2g}$$

$$B_{11}^TG_cB_{11} = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \tag{3.4.2h}$$

There are four points that must be considered. First, the unimodular matrices, B_{ij}, are known from Equation 2.8.2. Second, a set of independent algebraic equations can be obtained from these sets of matrix equations that will yield the network component values. Third, Equations 3.4.2d, 3.4.2e, and 2.5.5g will designate which element of Equation 3.3.8 will be a factor of the numerator constant. And fourth, in Equation 3.4.2h only the form of the matrix product need to be considered and the values of r_{11} and r_{22} ignored as shown in Equations 2.4.7 through 2.4.9.

Before obtaining equations in terms of component values, consider Equation 3.3.1 and its development in Chapter II. It is noted that C_b, L_c, R_b, and G_c are diagonal matrices with positive entries, that the B_{ij}'s are unimodular matrices with elements ± 1 or 0, and that the state-vector in Equation 3.3.2 implies that C_b will be a diagonal matrix of order m and L_c will be a diagonal matrix of order r. Since C_b and L_c are positive diagonal matrices, the matrix $C_b^{-\frac{1}{2}}$ is a diagonal matrix of order m and the matrix $L_c^{-\frac{1}{2}}$ is a diagonal matrix of order r whose entries are chosen to be positive.

Utilizing the discussion above while considering Equation 3.4.2a and the unimodular matrix, B_{12}, of Equation 2.8.2, it is possible to decompose $(C_b^{-\frac{1}{2}} B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}})$ in the same manner as the decomposition of Equation 2.7.4a and this decomposition results in

$$-C_b^{-\frac{1}{2}} B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}} = - \begin{bmatrix} C_1' & & \\ & \ddots & \\ & & C_m' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1' & & \\ & \ddots & \\ & & C_m' \end{bmatrix} \tag{3.4.3}$$

Equation 3.4.2a with Equation 3.4.3 yields

$$\begin{bmatrix} -f_o & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ & & & & -f_n \end{bmatrix} = \begin{bmatrix} -g_1(C_1')^2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ & & & & -g_2(C_m')^2 \end{bmatrix} \tag{3.4.4a}$$

and this equation implies that

$$f_o = g_1(C_1')^2 \tag{3.4.4b}$$

$$f_n = g_2(C_m')^2 \tag{3.4.4c}$$

Considering Equation 3.4.2b and using the B_{22}' submatrix of Equation 2.8.2, the decomposition of $(C_b^{-\frac{1}{2}}B_{22}^T L_c^{-\frac{1}{2}})$ is the same as in Equation 2.7.12 and is

$$C_b^{-\frac{1}{2}}B_{22}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} C_1' & & \\ & \ddots & \\ & & C_m' \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix} \begin{bmatrix} L_1' & & \\ & \ddots & \\ & & L_r' \end{bmatrix} \tag{3.4.5}$$

Equation 3.4.2b with Equation 3.4.5 yields

$$\begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \ddots & \ddots & \\ & & -k_{n-3} & k_{n-2} \\ & & & -k_{n-1} \end{bmatrix} = \begin{bmatrix} C_1'L_1' & & & \\ -C_2'L_1' & C_2'L_2' & & \\ & \ddots & \ddots & \\ & & -C_{m-1}'L_{r-1}' & C_{m-1}'L_r' \\ & & & -C_m'L_r' \end{bmatrix} \tag{3.4.6a}$$

Equation 3.4.6a implies that

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ k_2 &= C_2'L_1' \\ k_3 &= C_2'L_2' \\ &\vdots \\ k_{n-3} &= C_{m-1}'L_{r-1}' \\ k_{n-2} &= C_{m-1}'L_r' \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{3.4.6b}$$

Considering Equation 3.4.2c, it follows from a similar argument to that given in Section 2.7 that $R_b = 0$ and $B_{23} = 0$.

Considering Equation 3.4.2d and using the B_{11}, G_c and B_{12} submatrices from Equation 2.8.2, the decomposition of $(C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11})$ will be

$$-C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11} = - \begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & C_m' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad (3.4.7)$$

Equation 3.4.2d with Equation 3.4.7 yields

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \\ b_{m-1,1} & b_{m-1,2} \\ b_{m,1} & b_{m,2} \end{bmatrix} = \begin{bmatrix} g_1 C_1' & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & g_2 C_m' \end{bmatrix} \qquad (3.4.8a)$$

and this implies that b_{11} and $b_{m,2}$ are the only two non-zero entries of this matrix and are

$$b_{11} = g_1 C_1' \qquad (3.4.8b)$$

$$b_{m,2} = g_2 C_m' \qquad (3.4.8c)$$

The B_{11} of Equation 3.4.7 also satisfies the matrix product of Equation 3.4.2b, since

$$B_{11}^T G_c B_{11} = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \qquad (3.4.9)$$

Consider Equation 3.4.2e with the B_{21} submatrix of Equation 2.8.2 where $B_{21} = 0$ and this results in

$$\begin{bmatrix} b_{m+1,1} & b_{m+1,2} \\ b_{m+2,1} & b_{m+2,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} = 0 \qquad (3.4.10)$$

Equation 3.4.2f with Equation 3.4.8a yields

$$\begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & b_{m,2} \end{bmatrix} = - \begin{bmatrix} p_{11} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & p_{2,m} \end{bmatrix}^T \qquad (3.4.11a)$$

and this equation implies

$$b_{11} = -p_{11} \qquad (3.4.11b)$$

$$b_{m,2} = -p_{2,m} \qquad (3.4.11c)$$

Equation 3.4.2g with Equation 3.4.10 yields

$$\begin{bmatrix} p_{1,m+1} & p_{1,m+2} & \cdots & p_{1,m+r} \\ p_{2,m+1} & p_{2,m+2} & \cdots & p_{2,m+r} \end{bmatrix} = 0 \qquad (3.4.12)$$

Using the $\mathcal{P}$ and $\mathcal{B}$ of Equation 3.3.3 with Equations 3.4.8a, 3.4.10, 3.4.11a, and 3.4.12 yields

$$\mathcal{B} = \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & b_{m,2} \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{3.4.13}$$

$$\mathcal{P} = \left[\begin{array}{cccccc|ccc} p_{11} & 0 & \cdots & 0 & 0 & & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & p_{2,m} & & 0 & \cdots & 0 \end{array}\right] \tag{3.4.14}$$

Substituting the entries of Equations 3.4.13 and 3.4.14 into Equation 3.3.9 yields

$$\mathcal{D}'_{12}(s) = p_{11}a_{1,n}b_{m,2} \tag{3.4.15a}$$

where $a_{1,n}$ is an entry of the matrix of Equation 3.3.8 and has been determined in Equation 2.5.12 as

$$a_{1,n} = \prod_{i=1}^{n-1} k_i \tag{3.4.15b}$$

Therefore $\mathcal{D}'_{12}(s)$ can be written as

$$\mathcal{D}'_{12}(s) = b_{m,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right] \tag{3.4.16a}$$

and from Equations 3.3.5 and 3.4.1, we have

$$\mathcal{D}'_{12}(s) = a_o \tag{3.4.16b}$$

which yields the result

$$a_o = b_{m,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right] \qquad (3.4.16c)$$

Recall that this development assumes the network of Figure 2.8.1.

Now the network component values must be determined and Equations 3.4.4b & c, 3.4.6b, 3.4.8b & c, 3.4.11b & c, and 3.4.16c provide the following set of (n + 6) non-linear algebraic equations with the (n+6) unknowns, g_1, g_2, C_1', ..., C_m', L_1', ..., L_r', b_{11}, $b_{m,2}$, p_{11}, and $p_{2,m}$.

$$a_o = b_{m,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right] \qquad (3.4.17a)$$

$$f_o = g_1(C_1')^2 \qquad (3.4.17b)$$

$$f_n = g_2(C_m')^2 \qquad (3.4.17c)$$

$$b_{11} = g_1C_1' \qquad (3.4.17d)$$

$$b_{m,2} = g_2C_m' \qquad (3.4.17e)$$

$$b_{11} = -p_{11} \qquad (3.4.17f)$$

$$b_{m,2} = -p_{2,m} \qquad (3.4.17g)$$

$$\left.\begin{array}{l} k_1 = C_1'L_1' \\ \vdots \\ k_{n-1} = C_m'L_r' \end{array}\right\} \qquad (3.4.17h)$$

A solution to this set of non-linear algebraic equations can be found in the following manner. Equations 3.4.17d and 3.4.17e can be written as

$$g_1 = \frac{b_{11}}{C_1'} \qquad (3.4.18a)$$

$$g_2 = \frac{b_{m,2}}{C_m'} \qquad (3.4.18b)$$

Then substitute g_1 and g_2 into Equations 3.4.17b and 3.4.17c to yield

$$f_o = b_{11}C_1' \qquad (3.4.19a)$$

$$f_n = b_{m,2}C_m' \qquad (3.4.19b)$$

Solving for b_{11} and $b_{m,2}$ in Equation 3.4.19 gives

$$b_{11} = \frac{f_o}{C_1'} \qquad (3.4.20a)$$

$$b_{m,2} = \frac{f_n}{C_m'} \qquad (3.4.20b)$$

From Equations 3.4.17f and 3.4.20a

$$p_{11} = -\frac{f_o}{C_1'} \qquad (3.4.21)$$

Using Equation 3.4.17h, it can be shown that

$$C_m' = \frac{k_{n-1}k_{n-3}k_{n-5}\cdots k_1C_1'}{k_{n-2}k_{n-4}k_{n-6}\cdots k_2} \qquad (3.4.22)$$

Note that this is the same solution as that given in Equation 2.7.19c. Substituting $b_{m,2}$ of Equation 3.4.20b and p_{11} of Equation 3.4.21 into

Equation 3.4.17a yields

$$a_o = \left[\frac{f_n}{C_m'}\right]\left[\frac{-f_o}{C_1'}\right](k_{n-1}k_{n-2}k_{n-3} \dots k_1) \tag{3.4.23}$$

Substituting C_m' of Equation 3.4.22 into Equation 3.4.23 and solving for C_1' results in

$$C_1' = \left[\frac{-f_o f_n}{a_o}\right]^{\frac{1}{2}}(k_{n-2}k_{n-4}k_{n-6} \dots k_1) \tag{3.4.24}$$

where a_o is a negative real constant. Substituting C_1' into Equation 3.4.17 will yield the other unknowns. Observe that C_1' is positive and real, since f_o, f_n, k_1, ..., k_{n-1} are positive and real. Note that in Equations 3.4.3, 3.4.5, and 3.4.7 a change of variables has been made of

$$C_i' = C_i^{-\frac{1}{2}} \tag{3.4.25a}$$

$$L_i' = L_i^{-\frac{1}{2}} \tag{3.4.25b}$$

From these equations the network component values of the synthesized network of Figure 2.8.1 will be given by

$$C_i = (C_i')^{-2} \tag{3.4.26a}$$

$$L_i = (L_i')^{-2} \tag{3.4.26b}$$

$$R_i = \frac{1}{g_i} \tag{3.4.26c}$$

It should be noted that for a ladder network of the type shown in Figure 2.8.1 the numerator polynomial of $y_{12}(s)$ will have negative coefficients (32). This implies that a_o of Equation 3.4.1 must be negative. It is shown below that a_o as calculated in Equation 3.4.16c will always be negative. Considering Equation 3.4.17 it is observed that b_{11} and $b_{m,2}$ must be positive constants if only physically realizable components are to be in the synthesized network. With b_{11} a positive constant, Equation 3.4.17f implies that p_{11} is a negative constant. With

$$p_{11} < 0$$

$$b_{m,2} > 0$$

$$k_i > 0; \; i = 1, \ldots, n-1$$

then

$$a_o = b_{m,2} p_{11} \left[\prod_{i=1}^{n-1} k_i \right] < 0$$

An important point to be considered is that for the denominator polynomial of a given $y_{12}(s)$, Navot's (27) method of Appendix B will yield an infinite number of real positive values for f_o, f_n, k_1, ..., k_{n-1}. This implies that there are infinitely many real positive values of g_1, g_2, C_1, ..., C_m, L_1, ..., L_r possible in the synthesized network that yield the same $y_{12}(s)$ and these are dependent upon the users manipulation of Navot's method.

If a_o in Equation 3.4.1 is positive, B_{11}, b_{11}, and/or $b_{m,2}$ will reflect the change in sign and should be handled accordingly. Or a

1:-1 transformer can be added to either port to yield a positive a_o. This completes the presentation of the synthesis of $y_{12}(s)$ with a constant numerator and a denominator polynomial of odd degree. It is felt that a summary of the synthesis procedure should follow.

3.5 Synthesis Procedure for Case II - Special Summarized. The synthesis procedure for synthesizing a Case II - Special short circuit transfer admittance, $y_{12}(s)$, with a constant numerator and a denominator polynomial of odd degree is given in the following enumerated steps.

1. Take the proper function, $y_{12}(s)$, and use Appendices B and C to obtain the matrix, K_2, of Equations 3.3.2 and 3.3.3.
2. Write the state-model of $y_{12}(s)$ in the form of Equation 3.3.2 using the element values determined in Step 1. Leave matrices V^*, I^*, $\mathcal{B}$, $\mathcal{P}$, and $\mathcal{R}$ in general terms as was done in Equation 3.3.2. This will allow you to determine their sizes and will yield the number of capacitors and inductors.
3. If desired, write the resulting fundamental circuit equations using Equation 2.8.2 as a guide.
4. Draw the resulting ladder network using Figure 2.8.1 as a guide.
5. Solve the set of (n + 6) non-linear algebraic equations of Equation 3.4.17 by first solving for C_1' of Equation 3.4.24 and then using this result to solve for the other unknowns of Equation 3.4.17.
6. Obtain the synthesized network component values from Equation 3.4.26, which completes the synthesis of the transfer admittance.

An example will now be presented to illustrate this synthesis procedure.

3.6 Synthesis Example of $y_{12}(s)$ with $n = 3$. It is instructive to observe a Case II - Special example, therefore a $y_{12}(s)$ with a constant numerator and an odd denominator function degree is given to be synthesized. Let

$$y_{12}(s) = \frac{-4}{s^3 + 5s^2 + 17s + 25} \tag{3.6.1}$$

Using Appendices B and C, a possible K_2 matrix is

$$K_2 = \left[\begin{array}{cc|c} -f_o & 0 & k_1 \\ 0 & -f_3 & -k_2 \\ \hline -k_1 & k_2 & 0 \end{array}\right] \tag{3.6.2a}$$

$$K_2 = \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & -4 & -3 \\ \hline -2 & 3 & 0 \end{array}\right] \tag{3.6.2b}$$

It is now possible to write the state-model for this transfer admittance using the results of Sections 3.3 and 3.4.

$$\frac{d}{dt}\left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] = \left[\begin{array}{cc|c} -f_o & 0 & k_1 \\ 0 & -f_3 & -k_2 \\ \hline -k_1 & k_2 & 0 \end{array}\right]\left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] + \left[\begin{array}{cc} b_{11} & 0 \\ 0 & b_{22} \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c} v^*_{a_1} \\ v^*_{a_2} \end{array}\right] \tag{3.6.3a}$$

$$\begin{bmatrix} i^*_{a_1} \\ i^*_{a_2} \end{bmatrix} = \left[\begin{array}{cc|c} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \end{array}\right] \left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] + \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.6.3b)$$

or

$$\frac{d}{dt}\left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] = \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & -4 & -3 \\ \hline -2 & 3 & 0 \end{array}\right] \left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] + \left[\begin{array}{cc} b_{11} & 0 \\ 0 & b_{22} \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.6.3c)$$

$$\begin{bmatrix} i^*_{a_1} \\ i^*_{a_2} \end{bmatrix} = \left[\begin{array}{cc|c} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \end{array}\right] \left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] + \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.6.3d)$$

From Section 2.8.1, the fundamental circuit equations in symbolic form are

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \qquad (3.6.4a)$$

which for this example can be written as

$$\left[\begin{array}{cc|cc|c|ccc} -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \hline v_{br} \\ \hline v_{cr_1} \\ v_{cr_2} \\ v_{c\ell_1} \end{array}\right] = 0 \qquad (3.6.4b)$$

and after reducing the network graph of Figure 2.8.2 to fit this example, it is shown in Figure 3.6.1 while the synthesized network is shown in Figure 3.6.2.

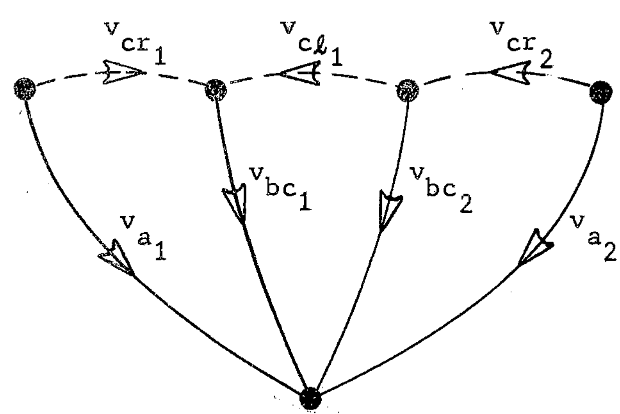

Figure 3.6.1 Example Network Graph

Now the following nine non-linear algebraic equations, which are similar to those of Equation 3.4.17, will be

$$-4 = 6b_{22}p_{11}$$

$$1 = g_1(C_1')^2$$

$$4 = g_2(C_2')^2$$

$$b_{11} = g_1C_1'$$

$$b_{22} = g_2C_2' \qquad (3.6.5)$$

$$b_{11} = -p_{11}$$

$$b_{22} = -p_{22}$$

$$2 = C_1'L_1'$$

$$3 = C_2'L_1'$$

These can be solved by starting with Equation 3.4.24 which yields

$$C_1' = \left[\frac{(-1)(4)}{-4}\right]^{\frac{1}{2}}(2)$$

and resulting in the synthesized network of Figure 3.6.2 with component values of

$$R_1 = 4 \text{ ohms}$$

$$R_2 = 9/4 \text{ ohms}$$

$$L_1 = 1 \text{ henry} \qquad (3.6.6)$$

$$C_1 = \tfrac{1}{4} \text{ farad}$$

$$C_2 = 1/9 \text{ farad}$$

and state-model element values of

$$b_{11} = \tfrac{1}{2}$$

$$b_{22} = 4/3$$

$$p_{11} = -\tfrac{1}{2} \qquad (3.6.7)$$

$$p_{22} = -4/3$$

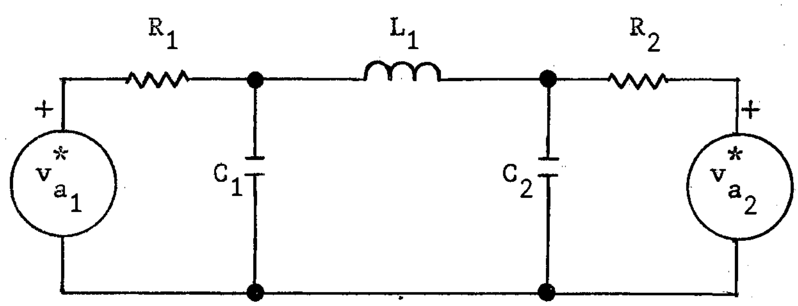

Figure 3.6.2 Example Synthesized Network

To check these results, $y_{12}(s)$ is determined in algebraic form from the synthesized network as

$y_{12}(s) =$

$$\frac{-1}{R_1R_2C_1C_2L_1s^3 + L_1(R_1C_1 + R_2C_2)s^2 + [L_1 + R_1R_2(C_1 + C_2)]s + (R_1 + R_2)} \qquad (3.6.8a)$$

and substituting in the calculated component values yields

$$y_{12}(s) = \frac{-1}{\frac{1}{4}s^3 + 5/4s^2 + 17/4s + 25/4} \qquad (3.6.8b)$$

or

$$y_{12}(s) = \frac{-4}{s^3 + 5s^2 + 17s + 25} \qquad (3.6.8c)$$

which is equal to the short circuit transfer admittance of Equation 3.6.1 that was to be synthesized.

When the degree of the denominator polynomial of $y_{12}(s)$ is even, there are some small differences in the synthesis procedure and in the synthesized network. These are presented in the next section.

3.7 State-Models, $[\mathrm{adj}(sU-K_2)]$ and Synthesis of Case IV - Special $y_{12}(s)$. This section will be similar to the presentations in Sections 3.3-3.5. Whereas these sections were concerned with the synthesis of a Case II - Special short circuit transfer admittance, $y_{12}(s)$, this section will present the synthesis procedure for a $y_{12}(s)$ that is a Case IV - Special.

The state-model of Equation 3.3.1 will be used again in the same role that it was in Sections 3.3 and 3.4. Also the developments of Sections 2.4 and 2.5 are used to obtain this state-model that represents $y_{12}(s)$ as

$$\frac{d}{dt}\begin{bmatrix} v'_{bc_1} \\ v'_{bc_2} \\ \vdots \\ v'_{bc_m} \\ \hline i'_{c\ell_{m+1}} \\ i'_{c\ell_{m+2}} \\ \vdots \\ i'_{c\ell_{m+r}} \end{bmatrix} = \left[\begin{array}{cccc|cccc} -f_o & & & & k_1 & & & \\ & 0 & & & -k_2 & k_3 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 0 & & & -k_{n-2} & k_{n-1} \\ \hline -k_1 & k_2 & & & 0 & & & \\ & -k_3 & \ddots & & & \ddots & & \\ & & \ddots & k_{n-2} & & & 0 & \\ & & & -k_{n-1} & & & & -f_n \end{array}\right] \begin{bmatrix} v'_{bc_1} \\ v'_{bc_2} \\ \vdots \\ v'_{bc_m} \\ \hline i'_{c\ell_{m+1}} \\ i'_{c\ell_{m+2}} \\ \vdots \\ i'_{c\ell_{m+r}} \end{bmatrix}$$

$$+ \left[\begin{array}{cc} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \\ \hline b_{m+1,1} & b_{m+1,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{array}\right] \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.7.1a)$$

$$\begin{bmatrix} i^*_{a_1} \\ i^*_{a_2} \end{bmatrix} = \left[\begin{array}{ccc|ccc} p_{11} & \cdots & p_{1,m} & p_{1,m+1} & \cdots & p_{1,m+r} \\ p_{21} & \cdots & p_{2,m} & p_{2,m+1} & \cdots & p_{2,m+r} \end{array}\right] \begin{bmatrix} v'_{bc_1} \\ \vdots \\ v'_{bc_m} \\ \hline i'_{c\ell_{m+1}} \\ \vdots \\ i'_{c\ell_{m+r}} \end{bmatrix} + \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.7.1b)$$

where n is even, $n = m + r$ and $m = r$. Again there are m capacitor voltages and r inductor currents. As presented in Section 3.3, the matrix $\mathcal{P}$ $[\mathrm{adj}(sU-K_2)]$ $\mathcal{B}$ will yield the numerator of $y_{12}(s)$ and so the $[\mathrm{adj}(sU-K_2)]$ and $\mathcal{D}'_{12}(s)$ must be obtained before proceeding with the synthesis procedure. Using the transformation matrix, L, of Equation 2.5.9b and the $[\mathrm{adj}(sU-K_1)]$ of Equation 3.3.7 in Equation 3.3.6 yields

$$[\mathrm{adj}(sU-K_2)] = \left[\begin{array}{cccc|cccc} a_{11} & a_{13} & \cdots & a_{1,n-1} & a_{12} & a_{14} & \cdots & a_{1,n} \\ a_{31} & a_{33} & \cdots & a_{3,n-1} & a_{32} & a_{34} & \cdots & a_{3,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,2} & a_{n-1,4} & \cdots & a_{n-1,n} \\ \hline a_{21} & a_{23} & \cdots & a_{2,n-1} & a_{22} & a_{24} & \cdots & a_{2,n} \\ a_{41} & a_{43} & \cdots & a_{4,n-1} & a_{42} & a_{44} & \cdots & a_{4,n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,2} & a_{n,4} & \cdots & a_{n,n} \end{array}\right] \qquad (3.7.2)$$

where the a_{ij} elements of Equation 3.7.2 are the same a_{ij} elements of Equation 2.5.12.

Now substitute the $\mathcal{B}$ and $\mathcal{P}$ of Equation 3.7.1 and the $[\mathrm{adj}(sU-K_2)]$ of Equation 3.7.2 into Equation 3.3.4b and it can be shown that the $\mathcal{D}'_{12}(s)$ of Equation 3.3.5 will be as shown in Equation 3.7.3 on the following page. Equations 3.3.1, 3.7.1, 3.3.7, and 3.7.3 will be used frequently in the synthesis procedure for the Case IV - Special $y_{12}(s)$.

Assume that the Case IV - Special short circuit transfer admittance to be synthesized is

$$y_{12}(s) = \frac{c_o}{s^n + d_{n-1}s^{n-1} + \ldots + d_1 s + d_o} \qquad (3.7.4)$$

$$
\begin{aligned}
D'_{12}(s) = \; & (p_{11}a_{11} + p_{12}a_{31} + p_{13}a_{51} + \cdots + p_{1,m}a_{n-1,1} + p_{1,m+1}a_{21} + p_{1,m+2}a_{41} + \cdots + p_{1,n}a_{n,1})b_{12} \\
& + (p_{11}a_{12} + p_{12}a_{32} + p_{13}a_{52} + \cdots + p_{1,m}a_{n-1,2} + p_{1,m+1}a_{22} + p_{1,m+2}a_{42} + \cdots + p_{1,n}a_{n,2})b_{m+1,2} \\
& + (p_{11}a_{13} + p_{12}a_{33} + p_{13}a_{53} + \cdots + p_{1,m}a_{n-1,3} + p_{1,m+1}a_{23} + p_{1,m+2}a_{43} + \cdots + p_{1,n}a_{n,3})b_{22} \\
& + (p_{11}a_{14} + p_{12}a_{34} + p_{13}a_{54} + \cdots + p_{1,m}a_{n-1,4} + p_{1,m+1}a_{24} + p_{1,m+2}a_{44} + \cdots + p_{1,n}a_{n,4})b_{m+2,2} \\
& + (p_{11}a_{15} + p_{12}a_{35} + p_{13}a_{55} + \cdots + p_{1,m}a_{n-1,5} + p_{1,m+1}a_{25} + p_{1,m+2}a_{45} + \cdots + p_{1,n}a_{n,5})b_{32} \\
& \qquad \vdots \\
& + (p_{11}a_{1,n-1} + p_{12}a_{3,n-1} + p_{13}a_{5,n-1} + \cdots + p_{1,m}a_{n-1,n-1} + p_{1,m+1}a_{2,n-1} + p_{1,m+2}a_{4,n-1} + \cdots + p_{1,n}a_{n,n-1})b_{m,2} \\
& + (p_{11}a_{1,n} + p_{12}a_{3,n} + p_{13}a_{5,n} + \cdots + p_{1,m}a_{n-1,n} + p_{1,m+1}a_{2,n} + p_{1,m+2}a_{4,n} + \cdots + p_{1,n}a_{n,n})b_{m+r,2}
\end{aligned}
$$

n = even integer

$n = m + r$

$m = r$

(3.7.3)

where c_o is an arbitrary real negative coefficient and the denominator polynomial is a strictly Hurwitz polynomial of even degree n. Since this transfer admittance is a Case IV - Special type, the synthesis procedure will produce an unbalanced ladder network with two resistors, n reactive elements, and no transformers as shown in Section 2.8.2. The fundamental circuit equations for this Case IV - Special are given in Equation 2.8.4. Again this information will be used in determining the network component values as was done in the previous section when n was odd.

Proceeding as in Section 3.4, the corresponding parts of the partitioned matrices of Equations 3.3.1 and 3.7.1 are equated to yield

$$-C_b^{-\frac{1}{2}} B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}} = \begin{bmatrix} -f_o & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 0 \end{bmatrix} \tag{3.7.5a}$$

$$C_b^{-\frac{1}{2}} B_{22}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \bullet & \bullet & \\ & & \bullet & \bullet \\ & & \bullet & \bullet \\ & & -k_{n-2} & k_{n-1} \end{bmatrix} \tag{3.7.5b}$$

$$-L_c^{-\frac{1}{2}} B_{23} R_b B_{23}^T L_c^{-\frac{1}{2}} = \begin{bmatrix} 0 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & 0 & \\ & & & & -f_n \end{bmatrix} \tag{3.7.5c}$$

$$-C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \end{bmatrix} \qquad (3.7.5d)$$

$$-C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11} = -\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,m} \\ p_{21} & p_{22} & \cdots & p_{2,m} \end{bmatrix}^T \qquad (3.7.5e)$$

$$-L_c^{-\frac{1}{2}}B_{21} = \begin{bmatrix} b_{m+1,1} & b_{m+1,2} \\ b_{m+2,1} & b_{m+2,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} \qquad (3.7.5f)$$

$$-L_c^{-\frac{1}{2}}B_{21} = \begin{bmatrix} p_{1,m+1} & p_{1,m+2} & \cdots & p_{1,m+r} \\ p_{2,m+1} & p_{2,m+2} & \cdots & p_{2,m+r} \end{bmatrix}^T \qquad (3.7.5g)$$

$$B_{11}^T G_c B_{11} = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \qquad (3.7.5h)$$

The solution of the network component values can be obtained from these equations by considering the four points which were presented in Section 3.4 immediately after Equation 3.4.2. As in Section 3.4, the matrices, $C_b^{-\frac{1}{2}}$, $L_c^{-\frac{1}{2}}$, G_c, and R_b, will be diagonal with positive entries and the B_{ij}'s will be unimodular or E-matrices with elements ± 1 or 0. $C_b^{-\frac{1}{2}}$ will be a matrix of order m and $L_c^{-\frac{1}{2}}$ will be a matrix of order r.

Considering Equation 3.7.5a and applying a similar procedure as in Section 3.4, $(C_b^{-\frac{1}{2}}B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}})$ can be decomposed as

$$-C_b^{-\frac{1}{2}}B_{12}^TG_cB_{12}C_b^{-\frac{1}{2}} = -\begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & C_m' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} g_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & C_m' \end{bmatrix} \tag{3.7.6}$$

Now Equation 3.7.5a with Equation 3.7.6 yields

$$\begin{bmatrix} -f_o & 0 & & \\ & \cdot & & \\ & & \cdot & \\ & & & 0 \end{bmatrix} = \begin{bmatrix} -g_1(C_1')^2 & 0 & & \\ & \cdot & & \\ & & \cdot & \\ & & & 0 \end{bmatrix} \tag{3.7.7a}$$

and this matrix equation implies that

$$f_o = g_1(C_1')^2 \tag{3.7.7b}$$

Now considering Equation 3.7.5b, it can be seen that the decomposition of $(C_b^{-\frac{1}{2}}B_{22}^TL_c^{-\frac{1}{2}})$ will be

$$C_b^{-\frac{1}{2}}B_{22}^TL_c^{-\frac{1}{2}} = \begin{bmatrix} C_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & C_m' \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} L_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & L_r' \end{bmatrix} \tag{3.7.8}$$

Now Equation 3.7.5b with Equation 3.7.8 yields

$$\begin{bmatrix} k_1 & & & \\ -k_2 & k_3 & & \\ & \ddots & \ddots & \\ & & -k_{n-2} & k_{n-1} \end{bmatrix} = \begin{bmatrix} C_1'L_1' & & & \\ -C_2'L_1' & C_2'L_2' & & \\ & \ddots & \ddots & \\ & & -C_m'L_{r-1}' & C_m'L_r' \end{bmatrix} \tag{3.7.9a}$$

and this matrix equation implies that

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ k_2 &= C_2'L_1' \\ k_3 &= C_2'L_2' \\ &\vdots \\ k_{n-2} &= C_m'L_{r-1}' \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{3.7.9b}$$

Considering Equation 3.7.5c and again using a similar procedure as in Section 3.4, $(L_c^{-\frac{1}{2}}B_{23}R_bB_{23}^TL_c^{-\frac{1}{2}})$ can be decomposed as

$$-L_c^{-\frac{1}{2}}B_{23}R_bB_{23}^TL_c^{-\frac{1}{2}} = -\begin{bmatrix} L_1' & & \\ & \ddots & \\ & & L_r' \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} r_2 \end{bmatrix}\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}\begin{bmatrix} L_1' & & \\ & \ddots & \\ & & L_r' \end{bmatrix} \tag{3.7.10}$$

Now Equation 3.7.5c with Equation 3.7.10 yields

$$\begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & -f_n \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & -r_2(L_r')^2 \end{bmatrix} \tag{3.7.11a}$$

and this matrix equation implies that

$$f_n = r_2(L_r')^2 \tag{3.7.11b}$$

Now considering Equation 3.7.5d with the B_{11}, B_{12} and G_c of Equation 2.8.4, it can be seen that the decomposition of $(C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11})$ will be

$$-C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11} = -\begin{bmatrix} C_1' & & \\ & \ddots & \\ & & C_m' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} g_1 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} \tag{3.7.12}$$

Equation 3.7.5d with Equation 3.7.12 yields

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \end{bmatrix} = \begin{bmatrix} g_1C_1' & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{3.7.13a}$$

and this matrix equation implies

$$b_{11} = g_1C_1' \tag{3.7.13b}$$

and the remaining b_{ij}'s in Equation 3.7.13a are identically zero. Equation 3.7.5d and c with Equation 2.7.13a implies that the only non-zero entry in Equation 3.7.5e is

$$p_{11} = -b_{11} \tag{3.7.14}$$

Considering Equation 3.7.5f, it can be seen that using the B_{21} of Equation 2.8.4 the decomposition of $(L_c^{-\frac{1}{2}}B_{21})$ will be

$$-L_c^{-\frac{1}{2}}B_{21} = -\begin{bmatrix} L_1' & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot\; L_r' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.7.15}$$

Equation 3.7.5f with Equation 3.7.15 yields

$$\begin{bmatrix} b_{m+1,1} & b_{m+1,2} \\ \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & L_r' \end{bmatrix} \tag{3.7.16a}$$

and this equation implies that the only non-zero entry is

$$b_{m+r,2} = L_r' \tag{3.7.16b}$$

Equation 3.7.5f with Equation 3.7.5g implies that the only non-zero entry is

$$b_{m+r,2} = p_{2,m+r} \tag{3.7.17}$$

When B_{11} of Equation 2.8.4 is substituted into Equation 3.7.5h, the correct form results as shown

$$B_{11}^T G_c B_{11} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} g_1 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} \tag{3.7.18a}$$

$$\begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} g_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.7.18b}$$

Substituting the zero entries of Equations 3.7.13a and 3.7.16a into the $\mathcal{P}$ and $\mathcal{B}$ matrices of Equation 3.3.3 yields

$$\mathcal{P} = \left[\begin{array}{cccc|cccc} p_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & p_{2,m+r} \end{array}\right] \tag{3.7.19}$$

and

$$\mathcal{B} = \left[\begin{array}{cc} b_{11} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & b_{m+r,2} \end{array}\right] \tag{3.7.20}$$

Substituting the b_{ij}'s and p_{ij}'s of Equations 3.7.19 and 3.7.20 into Equation 3.7.3 yields

$$\mathcal{D}_{12}'(s) = p_{11} a_{1,n} b_{m+r,2} \tag{3.7.21a}$$

where $a_{1,n}$ is an entry of the matrix of Equation 3.7.2 and has been determined in Equation 2.5.12 as

$$a_{1,n} = \prod_{i=1}^{n-1} k_i \tag{3.7.21b}$$

Therefore $\mathcal{D}_{12}'(s)$ can be written as

$$\mathcal{D}_{12}'(s) = b_{m+r,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right] \quad (3.7.22a)$$

and from Equations 3.3.5 and 3.7.4

$$\mathcal{D}_{12}'(s) = c_o \quad (3.7.22b)$$

which yields the desired result

$$c_o = b_{m+r,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right] \quad (3.7.22c)$$

Recall that this development assumes the network of Figure 2.8.3.

Now the network component values must be determined. Equations 3.7.7b, 3.7.9b, 3.7.11b, 3.7.13b, 3.7.14, 3.7.16b, 3.7.17, and 3.7.22a provide the following set of (n + 6) non-linear algebraic equations with the (n + 6) unknowns, g_1, r_2, C_1', ..., C_m', L_1', ..., L_r', b_{11}, $b_{m+r,2}$, p_{11}, and $p_{2,m+r}$,

$$c_o = b_{m+r,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right] \quad (3.7.23a)$$

$$f_o = g_1(C_1')^2 \quad (3.7.23b)$$

$$f_n = r_2(L_r')^2 \quad (3.7.23c)$$

$$b_{11} = g_1C_1' \quad (3.7.23d)$$

$$b_{m+r,2} = L_r' \quad (3.7.23e)$$

$$b_{11} = -p_{11} \tag{3.7.23f}$$

$$b_{m+r,2} = p_{2,m+r} \tag{3.7.23g}$$

$$\left.\begin{array}{l} k_1 = C_1'L_1' \\ \vdots \\ k_{n-1} = C_m'L_r' \end{array}\right\} \tag{3.7.23h}$$

A solution to this set of non-linear algebraic equations can be found in a manner similar to that presented in Section 3.4. This results in

$$C_m' = \frac{k_{n-2}k_{n-4}k_{n-6} \cdots k_2C_1'}{k_{n-3}k_{n-5}k_{n-7} \cdots k_1} \tag{3.7.24}$$

and

$$C_1' = \left[\frac{-f_o}{c_o}\right]^{\frac{1}{2}} (k_{n-1}k_{n-3}k_{n-5} \cdots k_1) \tag{3.7.25}$$

where c_o is a negative real constant. Substituting C_1' into Equation 3.7.23 will yield the other unknowns. Observe that C_1' is positive and real, since f_o, f_n, k_1, ..., k_{n-1} are positive and real. As in Section 3.4 after the C_i''s and L_i''s are determined, the C_i's and L_i's of the synthesized network of Figure 2.8.3 will be calculated by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = 1/g_1$, and $R_2 = r_2$.

Again note that for a ladder network of the type in Figure 2.8.3, the numerator polynomial of $y_{12}(s)$ will have negative coefficients. Therefore c_o of Equation 3.7.4 must be negative. It is shown below that c_o as calculated by Equation 3.7.23a will always be negative. Considering Equations 3.7.23d and 3.7.23e it is observed that b_{11} and $b_{m+r,2}$ must be positive constants if only physically realizable components are to be in the synthesized network. With b_{11} a positive constant, Equation 3.7.23f implies that p_{11} is a negative constant. With

$$p_{11} < 0$$

$$b_{m+r,2} > 0$$

$$k_i > 0; \; i = 1, \ldots, n-1$$

then

$$c_o = b_{m+r,2} p_{11} \left[\prod_{i=1}^{n-1} k_i \right] < 0$$

3.8 Synthesis Example of $y_{12}(s)$ with $n = 4$. As in Section 3.6, it is felt that a brief example of a Case IV - Special transfer admittance will be instructive. The $y_{12}(s)$ to be synthesized has a negative constant numerator and an even denominator function degree of 4. Let

$$y_{12}(s) = \frac{-5}{s^4 + 2s^3 + 10s^2 + 10s + 17} \tag{3.8.1}$$

Using Appendices B and C, a possible K_2 matrix is

$$K_2 = \left[\begin{array}{cc|cc} -f_o & 0 & k_1 & 0 \\ 0 & 0 & -k_2 & k_3 \\ \hline -k_1 & k_2 & 0 & 0 \\ 0 & k_3 & 0 & -f_4 \end{array}\right] \tag{3.8.2a}$$

$$K_2 = \left[\begin{array}{cc|cc} -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ \hline -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{3.8.2b}$$

Using the state-model of Equation 3.7.1, substituting in the K_2 matrix of Equation 3.8.2b, and using the results of Section 3.7 yields

$$\frac{d}{dt}\begin{bmatrix} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \\ i'_{c\ell_2} \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ \hline -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right] \begin{bmatrix} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \\ i'_{c\ell_2} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & b_{42} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.8.3a)$$

$$\begin{bmatrix} i^*_{a_1} \\ i^*_{a_2} \end{bmatrix} = \left[\begin{array}{cc|cc} p_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{24} \end{array}\right] \begin{bmatrix} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \\ i'_{c\ell_2} \end{bmatrix} + \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \qquad (3.8.3b)$$

From Section 2.8.2, the fundamental circuit equations in symbolic form are

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \begin{bmatrix} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{bmatrix} = 0 \qquad (3.8.4a)$$

which for this example can be written as

$$\left[\begin{array}{cc|cc|c|ccc} -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ v_{c\ell_1} \\ v_{c\ell_2} \end{array}\right] = 0 \qquad (3.8.4b)$$

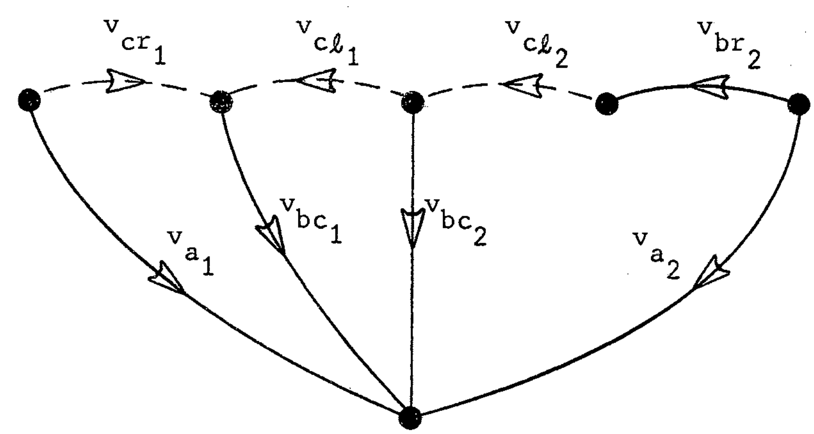

Figure 3.8.1 Example Network Graph

after reducing the network graph of Figure 2.8.3 to fit this example, the network graph is shown in Figure 3.8.1 while the synthesized network is shown in Figure 3.8.2.

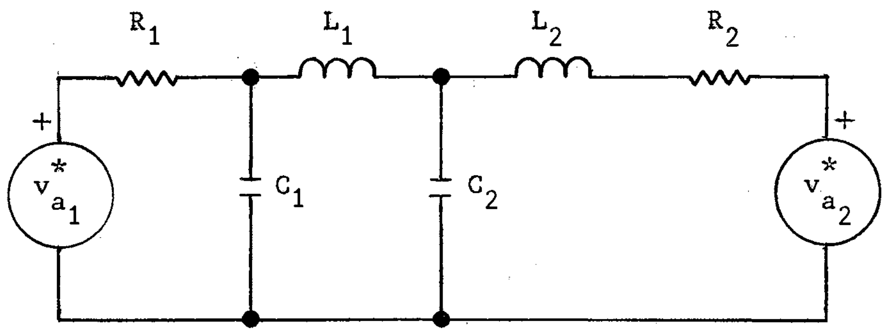

Figure 3.8.2 Example Synthesized Network

Using the solution of Equations 3.7.23 and 3.7.25 yields for the component values

$$R_1 = \frac{16}{5} \text{ ohm}$$

$$R_2 = 1/5 \text{ ohm}$$

$$L_1 = 4/5 \text{ henry}$$

$$L_2 = 1/5 \text{ henry}$$

$$C_1 = 5/16 \text{ farad}$$

$$C_2 = 5/4 \text{ farad}$$

and for the state-model element values

$$b_{11} = \frac{\sqrt{5}}{4}$$

$$b_{42} = \sqrt{5}$$

$$p_{11} = -\frac{\sqrt{5}}{4}$$

$$p_{24} = \sqrt{5}$$

Since short circuit transfer admittances with other than constant numerators must be considered, these will be presented next.

3.9 Synthesis of $y_{12}(s)$ with Numerator Degree Greater than Zero. In the previous sections of this chapter, transfer admittances with constant numerators were considered. However it is common to have a transfer admittance numerator polynomial with a degree greater than zero. The synthesis procedure for such transfer admittance functions will use much of the presentation for zero degree numerators. This will be seen in the following material.

First it is assumed that if the $y_{12}(s)$ to be synthesized is an improper function, the procedures of Section 2.6.1 have been executed until a proper function is left to be considered.

Next it must be pointed out that this synthesis procedure will satisfy only one coefficient in the numerator polynomial per ladder network. Therefore if the numerator degree is (i) then in general there will be (i) ladder networks paralleled in the resulting network that synthesizes the transfer admittance. This paralleling procedure is justified with validity test remarks in Weinberg (33).

It will be necessary to present the peculiarities of each of the Cases, I through IV, of Section 2.8. Then an outline of the synthesis procedure will be presented and last, examples will be presented to illustrate the synthesis procedure.

3.9.1 Case I. For this case, the numerator and denominator degrees are both odd, and the network and fundamental circuit equations to be used are given in Section 2.8.3 with the transfer admittance of Equation 2.8.5. Since n is odd, the state-model to be used is given in

Equation 3.3.2. Because the fundamental circuit equations for this case differ from those in Case II - Special only in submatrices B_{11} and B_{21}, the results in Equations 3.4.4 and 3.4.6 will apply to this case and are

$$f_o = g_1(C_1')^2 \tag{3.9.1a}$$

$$f_n = g_2(C_m')^2 \tag{3.9.1b}$$

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ &\vdots \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{3.9.1c}$$

Since B_{11} for Case I is

$$B_{11} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.9.2}$$

from Equation 3.4.2d it follows that

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \end{bmatrix} = \begin{bmatrix} g_1C_1' & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{3.9.3a}$$

This matrix equation implies that b_{11} is the only non-zero element and equals

$$b_{11} = g_1C_1' \tag{3.9.3b}$$

Considering Equation 3.4.2e with the B_{21} of Equation 2.8.7 yields

$$\begin{bmatrix} b_{m+1,1} & b_{m+1,2} \\ \vdots & \vdots \\ b_{m+i,1} & b_{m+i,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & L_i' \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \qquad (3.9.4a)$$

which implies that $b_{m+i,2}$ is the only non-zero element and equals

$$b_{m+i,2} = L_i' \qquad (3.9.4b)$$

Equation 3.4.2d with Equation 3.4.2f implies that

$$b_{11} = -p_{11} \qquad (3.9.5)$$

Equation 3.4.2e with Equation 3.4.2g implies that

$$b_{m+i,2} = p_{2,m+i} \qquad (3.9.6)$$

Now following a procedure similar to that in Section 3.4, $\mathfrak{B}$ and $\mathfrak{P}$ can be written as

$$\mathfrak{B} = \left[\begin{array}{cc} b_{11} & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & b_{m+i,2} \\ \vdots & \vdots \\ 0 & 0 \end{array}\right] \qquad (3.9.7)$$

$$\rho = \left[\begin{array}{ccc|cccc} p_{11} & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & p_{2,m+i} & \cdots & 0 \end{array}\right] \tag{3.9.8}$$

Substituting the entries of $\mathcal{B}$ and ρ from Equations 3.9.7 and 3.9.8 into Equation 3.3.9 yields

$$\mathcal{D}'_{121}(s) = p_{11}a_{1,2i}b_{m+i,2} \tag{3.9.9}$$

where

$$1 \leq i \leq r$$

and from Equation 2.8.6a

$$2i = n - x_1$$

$a_{1,2i}$ has been determined in Equation 2.5.12. It can be shown that the $a_{1,2i}$ element is a polynomial of (n-2i) degree, which for this case is equal to x_1. This implies that $\mathcal{D}'_{121}(s)$ of Equation 3.9.9 can be written as

$$\mathcal{D}'_{121}(s) = c_{x_1}s^{x_1} + c_{x_1-1}s^{x_1-1} + \cdots + c_1 s + c_o \tag{3.9.10a}$$

while from Equation 2.8.5

$$\mathcal{D}'_{12}(s) = a_{x_1}s^{x_1} + a_{x_1-1}s^{x_1-1} + \cdots + a_1 s + a_o \tag{3.9.10b}$$

Once the K_2 matrix is determined, the coefficients in $a_{1,2i}$ are fixed. Further the coefficients in $\mathcal{D}'_{121}(s)$ will be fixed once $b_{m+i,2}$ and p_{11} are determined. Under these two conditions it can be seen that only one coefficient of $\mathcal{D}'_{121}(s)$ can be equated to a "like" coefficient of $\mathcal{D}'_{12}(s)$ and, in general

$$\mathcal{D}'_{121}(s) \neq \mathcal{D}'_{12}(s)$$

Therefore we shall equate only the "like" coefficients, c_{x_1} and a_{x_1}, in Equation 3.9.10.

From Equations 2.5.12 and 3.9.9 it can be shown that

$$c_{x_1} = p_{11} b_{m+i,2} \left[\prod_{j=1}^{2i-1} k_j \right] \tag{3.9.11}$$

and so

$$a_{x_1} = p_{11} b_{m+i,2} \left[\prod_{j=1}^{2i-1} k_j \right] \tag{3.9.12}$$

The element values in the synthesized network which yields the transfer admittance

$$y_{121} = \frac{\mathfrak{D}'_{121}(s)}{\triangle}$$

can be obtained from the following (n + 6) non-linear algebraic equations:

$$a_{x_1} = p_{11} b_{m+i,2} \left[\prod_{j=1}^{2i-1} k_j \right] \tag{3.9.13a}$$

$$f_o = g_1 (C'_1)^2 \tag{3.9.13b}$$

$$f_n = g_2 (C'_m)^2 \tag{3.9.13c}$$

$$\left.\begin{aligned} k_1 &= C'_1 L'_1 \\ &\vdots \\ k_{n-1} &= C'_m L'_r \end{aligned}\right\} \tag{3.9.13d}$$

$$b_{11} = g_1 C_1' \tag{3.9.13e}$$

$$b_{m+i,2} = L_i' \tag{3.9.13f}$$

$$b_{11} = -p_{11} \tag{3.9.13g}$$

$$b_{m+i,2} = p_{2,m+i} \tag{3.9.13h}$$

Using the set of equations in Equation 3.9.13d, it can be shown that

$$L_i' = \frac{k_1 k_3 \cdots k_{2i-1}}{k_2 k_4 \cdots k_{2(i-1)} C_1'}, \tag{3.9.14}$$

And from Equations 3.9.13b, e, and g it can be shown that

$$p_{11} = -\frac{f_o}{C_1'} \tag{3.9.15}$$

Now using Equations 3.9.14 and 3.9.15 in Equations 3.9.13a and 3.9.13f yields

$$C_1' = \left[\frac{-f_o}{a_{x_1}}\right]^{\frac{1}{2}} k_1 k_3 \cdots k_{2i-1} \tag{3.9.16}$$

Substituting this value of C_1' into Equation 3.9.13 will yield all of the unknowns in Equation 3.9.13. Then the synthesized network component values can be obtained by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, and $R_i = 1/g_i$.

The above procedure yields

$$y_{121}(s) = \frac{\mathcal{D}_{121}'(s)}{\Delta}$$

where $\mathcal{D}'_{121}(s)$ only satisfies the a_{x_1} coefficient in $\mathcal{D}'_{12}(s)$. The other coefficients of $\mathcal{D}'_{121}(s)$ will be determined from Equation 3.9.9. Most likely these will not be equal to the desired coefficients of $\mathcal{D}'_{12}(s)$. Therefore the next coefficient will have to be satisfied by placing in parallel with the network just synthesized, another synthesized network that will yield a coefficient which is the difference between a_{x_1-1} and c_{x_1-1} resulting from the first synthesized network. This procedure will possibly have to be repeated until there are $(x_1 + 1)$ synthesized networks in parallel, as will be shown in Sections 3.11 and 3.12. It should be noted that each new synthesized network placed in parallel will yield a transfer admittance numerator degree that is one less than the previous network, however all of the networks will yield transfer admittance denominators that are identical.

The procedure above is also applicable to a $\mathcal{D}'_{12}(s)$ where one or more of the coefficients are zero. When writing Equation 3.9.10b, be sure and include the zero coefficients as such and complete the synthesis procedure as presented above.

Note that it is not necessary to write down the fundamental circuit equations or the state-model in order to execute the synthesis procedure. However, if desired, they can be obtained.

3.9.2 Case II. The transfer admittance for Case II is given in Equation 2.8.8 and the numerator degree is even with the denominator degree being odd. The network and the fundamental circuit equations to be used are given in Section 2.8.4. Since n is odd, the state-model to be used is given in Equation 3.3.2. Because the fundamental circuit equations for this case differ from those in Case II - Special only in submatrices B_{11} and B_{21}, the results in Equations 3.4.4 and 3.4.6 will apply to this case and are

$$f_o = g_1(C_1')^2 \tag{3.9.17a}$$

$$f_n = g_2(C_m')^2 \tag{3.9.17b}$$

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ &\vdots \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{3.9.17c}$$

From Equation 2.8.10, $B_{11} = 0$. Then from Equation 3.4.2d it follows that

$$\begin{bmatrix} b_{11} & b_{12} \\ \vdots & \vdots \\ b_{m,1} & b_{m,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{3.9.18}$$

Considering Equation 3.4.2e with the B_{21} of Equation 2.8.10 yields

$$\begin{bmatrix} b_{m+1,1} & b_{m+1,2} \\ \vdots & \vdots \\ b_{m+i,1} & b_{m+i,2} \\ \vdots & \vdots \\ b_{m+r,1} & b_{m+r,2} \end{bmatrix} = \begin{bmatrix} -L_1' & 0 \\ \vdots & \vdots \\ 0 & L_i' \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{3.9.19a}$$

and this matrix equation implies that $b_{m+1,1}$ and $b_{m+i,2}$ are the only non-zero elements and are equal to

$$b_{m+1,1} = -L_1' \tag{3.9.19b}$$

$$b_{m+i,2} = L_i^{'} \qquad (3.9.19c)$$

Equations 3.4.2d, 3.4.2f, and 3.9.18 imply that

$$\begin{bmatrix} p_{11} & \cdots & p_{1,m} \\ p_{21} & \cdots & p_{2,m} \end{bmatrix} = 0 \qquad (3.9.20)$$

Equation 3.4.2e with Equation 3.4.2g implies that

$$b_{m+1,1} = p_{1,m+1} \qquad (3.9.21a)$$

$$b_{m+i,2} = p_{2,m+i} \qquad (3.9.21b)$$

Now following a procedure similar to that in Section 3.4, $\mathbb{B}$ and $\mathbb{P}$ can be written as

$$\mathbb{B} = \left[\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline b_{m+1,1} & 0 \\ \vdots & \vdots \\ 0 & b_{m+i,2} \\ \vdots & \vdots \\ 0 & 0 \end{array}\right] \qquad (3.9.22)$$

$$\mathbb{P} = \left[\begin{array}{ccc|ccccc} 0 & \cdots & 0 & p_{1,m+1} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & p_{2,m+i} & \cdots & 0 \end{array}\right] \qquad (3.9.23)$$

Substituting the entries of $\mathbb{B}$ and $\mathbb{P}$ from Equations 3.9.22 and 3.9.23 into Equation 3.3.9 yields

$$\mathfrak{D}'_{121}(s) = p_{1,m+1}a_{2,2i}b_{m+i,2} \tag{3.9.24}$$

where

$$1 \leq i \leq r + 1$$

and from Equation 2.8.9a

$$2i = n - x_2 + 1$$

$a_{2,2i}$ has been determined in Equation 2.5.12. It can be shown that the $a_{2,2i}$ element is a polynomial of (n - 2i + 1) degree, which for this case, is equal to x_2. This implies that $\mathfrak{D}'_{121}(s)$ of Equation 3.9.24 can be written as

$$\mathfrak{D}'_{121}(s) = c_{x_2}s^{x_2} + c_{x_2-1}s^{x_2-1} + \ldots + c_1 s + c_o \tag{3.9.25a}$$

while from Equation 2.8.8

$$\mathfrak{D}'_{12}(s) = a_{x_2}s^{x_2} + a_{x_2-1}s^{x_2-1} + \ldots + a_1 s + a_o \tag{3.9.25b}$$

Again as in Section 3.9.1 only one set of "like" coefficients in Equations 3.9.25a and b can be equated. As before the coefficients of the s^{x_2} terms shall be equated.

From equations 2.5.12 and 3.9.24 it can be shown that

$$c_{x_2} = p_{1,m+1}b_{m+i,2}; \quad i = 1 \tag{3.9.26a}$$

and

$$c_{x_2} = p_{1,m+1}b_{m+i,2}\left[\prod_{j=2}^{2i-1} k_j\right]; \quad i \geq 2 \tag{3.9.26b}$$

Note that when i = 1, there is no k_j term in the c_{x_2} expression. As shown in Equation 2.8.9a, $x_2 = n - 1$ and therefore only the diagonal terms of Equation 2.5.12 are considered. It can be seen the coefficients of the s^{x_2} terms in the diagonal entries are equal to one. Thus no k_j terms.

Since a_{x_2} is being equated to c_{x_2}, then

$$a_{x_2} = p_{1,m+1} b_{m+i,2}; \quad i = 1 \tag{3.9.27a}$$

and

$$a_{x_2} = p_{1,m+1} b_{m+i,2} \left[\prod_{j=2}^{2i-1} k_j \right]; \quad i \geq 2 \tag{3.9.27b}$$

The element values in the synthesized network which yields the polynomial, $\mathfrak{D}'_{121}(s)$, can be obtained from the following (n + 6) non-linear algebraic equations:

$$\left.\begin{aligned} a_{x_2} &= p_{1,m+1} b_{m+i,2}; \quad i = 1 \\ a_{x_2} &= p_{1,m+1} b_{m+i,2} \left[\prod_{j=2}^{2i-1} k_j \right]; \quad i \geq 2 \end{aligned}\right\} \tag{3.9.28a}$$

$$f_o = g_1 (C'_1)^2 \tag{3.9.28b}$$

$$f_n = g_2 (C'_m)^2 \tag{3.9.28c}$$

$$\left.\begin{aligned} k_1 &= C'_1 L'_1 \\ &\vdots \\ k_{n-1} &= C'_m L'_r \end{aligned}\right\} \tag{3.9.28d}$$

$$b_{m+1,1} = -L_1'$$ (3.9.28e)

$$b_{m+i,2} = L_i'$$ (3.9.28f)

$$b_{m+1,1} = p_{1,m+1}$$ (3.9.28g)

$$b_{m+i,2} = p_{2,m+i}$$ (3.9.28h)

Using the set of equations in Equation 3.9.13d, it can be shown that

$$L_i' = \frac{k_1 k_3 \ldots k_{2i-1}}{k_2 k_4 \ldots k_{2(i-1)} C_1'}$$ (3.9.29)

and

$$L_1' = \frac{k_1}{C_1'}$$ (3.9.30)

From Equations 3.9.28e, 3.9.28g, and 3.9.30

$$p_{1,m+1} = \frac{k_1}{C_1'}$$ (3.9.31)

Now using Equations 3.9.28f, 3.9.29, and 3.9.31 with Equation 3.9.28a when $i \geq 2$, it can be shown that

$$C_1' = (-a_{x_2})^{-\frac{1}{2}} k_1 k_3 \ldots k_{2i-1}$$ (3.9.32)

and using Equations 3.9.28e, 3.9.28f, and 3.9.28g with Equation 3.9.28a when $i = 1$, it can be shown that

$$L_1' = (-a_{x_2})^{\frac{1}{2}}$$ (3.9.33)

Substituting the value of C_1' or L_1' into Equation 3.9.28 will yield all of the unknowns. Then the synthesized network component values can be obtained by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, and $R_i = 1/g_i$.

The above procedure yields

$$y_{121} = \frac{\mathfrak{D}_{121}'(s)}{\Delta}$$

where $\mathfrak{D}_{121}'(s)$ only satisfies the a_{x_2} coefficient in $\mathfrak{D}_{12}'(s)$. To satisfy the other coefficients of $\mathfrak{D}_{12}'(s)$, a procedure like that in Section 3.9.1 should be followed. This procedure is shown in the examples of Sections 3.11 and 3.12.

3.9.3 Case III. The transfer admittance for Case III is given in Equation 2.8.11 and the numerator degree is odd while the denominator degree is even. The network and the fundamental circuit equations to be used are given in Section 2.8.5. Since n is even, the state-model to be used is given in Equation 3.7.1. Because the fundamental circuit equations for this case differ from those in Case IV - Special only in submatrices B_{11} and B_{21}, the results in Equations 3.7.7, 3.7.9, and 3.7.11 will apply to this case and are

$$f_o = g_1(C_1')^2 \qquad (3.9.34a)$$

$$\left.\begin{array}{l} k_1 = C_1'L_1' \\ \vdots \\ k_{n-1} = C_m'L_r' \end{array}\right\} \qquad (3.9.34b)$$

$$f_n = r_2(L_r')^2 \qquad (3.9.34c)$$

From Equation 2.8.13 it is seen that the submatrices B_{11} and B_{21} have the same entries as in Case II. This implies that the following equations will result

$$b_{m+1,1} = -L_1' \tag{3.9.35}$$

$$b_{m+i,2} = L_i' \tag{3.9.36}$$

$$\mathcal{B} = \left[\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline b_{m+1,1} & 0 \\ \vdots & \vdots \\ 0 & b_{m+i,2} \\ \vdots & \vdots \\ 0 & 0 \end{array}\right] \tag{3.9.37}$$

$$\mathcal{P} = \left[\begin{array}{ccc|ccccc} 0 & \cdots & 0 & p_{1,m+1} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & p_{2,m+i} & \cdots & 0 \end{array}\right] \tag{3.9.38}$$

$$\mathcal{Q}_{121}'(s) = p_{1,m+1} a_{2,2i} b_{m+i,2} \tag{3.9.39}$$

$$\mathcal{Q}_{121}'(s) = c_{x_3} s^{x_3} + c_{x_3-1} s^{x_3-1} + \cdots + c_1 s + c_o \tag{3.9.40a}$$

$$\mathcal{Q}_{12}'(s) = a_{x_3} s^{x_3} + a_{x_3} s^{x_3} + \cdots + a_1 s + a_o \tag{3.9.40b}$$

and

$$a_{x_3} = p_{1,m+1} b_{m+i,2}; \quad i = 1 \tag{3.9.41a}$$

or

$$a_{x_3} = p_{1,m+1} b_{m+i,2} \left[\prod_{j=2}^{2i-1} k_j \right]; \quad i \geq 2 \tag{3.9.41b}$$

The element values in the synthesized network which yields the polynomial, $\mathfrak{D}'_{121}(s)$, can be obtained from the following (n + 6) non-linear algebraic equations:

$$\left.\begin{array}{l} a_{x_3} = p_{1,m+1} b_{m+i,2};\ i = 1 \\ \\ a_{x_3} = p_{1,m+1} b_{m+i,2} \left[\prod_{j=2}^{2i-1} k_j \right];\ i \geq 2 \end{array}\right\} \quad (3.9.42a)$$

$$f_o = g_1 (C_1')^2 \quad (3.9.42b)$$

$$\left.\begin{array}{l} k_1 = C_1' L_1' \\ \vdots \\ k_{n-1} = C_m' L_r' \end{array}\right\} \quad (3.9.42c)$$

$$f_n = r_2 (L_r')^2 \quad (3.9.42d)$$

$$b_{m+1,1} = -L_1' \quad (3.9.42e)$$

$$b_{m+i,2} = L_i' \quad (3.9.42f)$$

$$b_{m+1,1} = p_{1,m+1} \quad (3.9.42g)$$

$$b_{m+i,2} = p_{2,m+i} \quad (3.9.42h)$$

By comparing Equation 3.9.42 with Equation 3.9.28, it can be seen that

$$C_1' = (-a_{x_3})^{-\frac{1}{2}} k_1 k_3 \ \ldots \ k_{2i-1};\ i \geq 2 \quad (3.9.43)$$

or

$$L_1' = (-a_{x_3})^{\frac{1}{2}};\ i = 1 \quad (3.9.44)$$

Substituting the value of $C_1^{'}$ or $L_1^{'}$ into Equation 3.9.42 will yield all of the unknowns. Again $C_i = (C_i^{'})^{-2}$, $L_i = (L_i^{'})^{-2}$, $R_1 = 1/g_1$, and $R_2 = r_2$.

This synthesis procedure satisfies only the coefficient, a_{x_3}. For every other coefficient in the transfer admittance numerator that is to be satisfied, other ladder networks will have to be synthesized and placed in parallel as discussed in Section 3.9.1.

3.9.4 Case IV. For this case the transfer admittance is given in Equation 2.8.14 and the numerator and denominator degrees are both even. Also the network and fundamental circuit equations to be used are given in Section 2.8.6. Since n is even, the state-model to be used is given in Equation 3.7.1. Because the fundamental circuit equations for this case differ from those in Case IV - Special only in submatrix B_{21}, the results in Equations 3.7.7, 3.7.9, 3.7.11, and 3.7.13 will apply to this case and are

$$f_o = g_1(C_1^{'})^2 \qquad (3.9.45a)$$

$$\left.\begin{aligned} k_1 &= C_1^{'}L_1^{'} \\ &\vdots \\ k_{n-1} &= C_m^{'}L_r^{'} \end{aligned}\right\} \qquad (3.9.45b)$$

$$f_n = r_2(L_r^{'})^2 \qquad (3.9.45c)$$

$$b_{11} = g_1C_1^{'} \qquad (3.9.45d)$$

From Equation 2.8.16 it is seen that the submatrix B_{21} has the same entries as in Case I. This implies that the following equations will result

$$b_{m+i,2} = L_i' \tag{3.9.46}$$

$$\mathcal{B} = \left[\begin{array}{cc} b_{11} & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & b_{m+i,2} \\ \vdots & \vdots \\ 0 & 0 \end{array}\right] \tag{3.9.47}$$

$$\mathcal{P} = \left[\begin{array}{ccc|ccccc} p_{11} & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & p_{2,m+i} & \cdots & 0 \end{array}\right] \tag{3.9.48}$$

$$\mathcal{D}_{121}'(s) = p_{11}a_{1,2i}b_{m+i,2} \tag{3.9.48}$$

$$\mathcal{D}_{121}'(s) = c_{x_4}s^{x_4} + c_{x_4-1}s^{x_4-1} + \ldots + c_1 s + c_o \tag{3.9.49a}$$

$$\mathcal{D}_{12}'(s) = a_{x_4}s^{x_4} + a_{x_4-1}s^{x_4-1} + \ldots + a_1 s + a_o \tag{3.9.49b}$$

and

$$a_{x_4} = p_{11}b_{m+i,2}\left[\prod_{j=1}^{2i-1} k_j\right] \tag{3.9.50}$$

The (n + 6) non-linear algebraic equations used to determine the network component values are

$$a_{x_4} = p_{11}b_{m+i,2}\left[\prod_{j=1}^{2i-1} k_j\right] \tag{3.9.51a}$$

$$f_o = g_1(C_1')^2 \tag{3.9.51b}$$

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ &\vdots \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{3.9.51c}$$

$$f_n = r_2(L_r')^2 \tag{3.9.51d}$$

$$b_{11} = g_1C_1' \tag{3.9.51e}$$

$$b_{m+i,2} = L_i' \tag{3.9.51f}$$

$$b_{11} = -p_{11} \tag{3.9.51g}$$

$$b_{m+i,2} = p_{2,m+i} \tag{3.9.51h}$$

By comparing Equation 3.9.51 with Equation 3.9.13, it can be seen that

$$C_1' = \left[\frac{-f_o}{a_{x_4}}\right]^{\frac{1}{2}} k_1k_3 \cdots k_{2i-1} \tag{3.9.52}$$

Substituting this value of C_1' into Equation 3.9.51 will yield all of the unknowns. Again $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = 1/g_1$, and $R_2 = r_2$.

This synthesis procedure satisfies only the coefficient, a_{x_4}. For every other coefficient in the transfer admittance numerator that is to be satisfies, other ladder networks will have to be synthesized and placed in parallel as discussed in Section 3.9.1.

3.10 Synthesis Procedure Outline. The synthesis procedures for synthesizing Case I, Case II, Case III, and Case IV short circuit transfer admittances are given in outline form in the following enumerated steps.

1. Take the proper function, $y_{12}(s)$, and use the methods of Appendices B and C to obtain the matrix, K_2, of either Equation 3.3.2 or 3.7.1 depending on n being odd or even.
2. If desired, the state-model can be written using the K_2 of Step 1, and the $\mathcal{B}$ and $\mathcal{P}$ of the appropriate case in Section 3.9.
3. If desired, the fundamental circuit equations can be written using the results of the appropriate case in Section 2.8.
4. Draw the ladder network that will result using the appropriate case in Section 2.8.
5. Use the explicit solutions of the appropriate set of (n + 6) non-linear algebraic equations given for each case in Section 3.9.
6. Obtain the network component values from:
   (a.) $C_i = (C_i')^{-2}$; $1 \leq i \leq m$ and $L_i = (L_i')^{-2}$; $1 \leq i \leq r$ for Cases I, II, III, and IV.
   (b.) $R_i = 1/g_i$ for Case I and II.
   (c.) $R_1 = 1/g_1$ and $R_2 = r_2$ for Cases III and IV.
7. If there is more than one coefficient in the numerator to be satisfied, return to Step 2 and continue through the other steps as presented in Section 3.9.1.

<u>3.11 Example of Case I Synthesis.</u> Consider the Case I short circuit transfer admittance of

$$y_{12}(s) = \frac{a_1 s + a_o}{s^3 + b_2 s^2 + b_1 s + b_o} \tag{3.11.1}$$

where a_1 and a_o are arbitrary real coefficients, the numerator degree is one, and the denominator degree is three.

We shall synthesize $y_{12}(s)$ in two stages. In the first stage, a network will be obtained which has the transfer admittance of

$$y'_{12}(s) = \frac{a_1 s + a'_o}{s^3 + b_2 s^2 + b_1 s + b_o} \tag{3.11.2}$$

where a'_o will be accepted as generated from the synthesis procedure. In the second stage, a network will be obtained which has the transfer admittance of

$$y''_{12}(s) = \frac{a_o - a'_o}{s^3 + b_2 s^2 + b_1 s + b_o} \tag{3.11.3}$$

Whether a positive or negative constant results in the numerator of Equation 3.11.3 depends on whether a'_o is greater than or less than a_o. These two networks will be paralleled using an isolation transformer, if necessary, [See Weinberg (33)] to obtain the desired transfer admittance of Equation 3.11.1.

The particular transfer admittance to be synthesized is

$$y_{12}(s) = \frac{-2s - 12}{s^3 + 5s^2 + 17s + 25} \tag{3.11.4}$$

Using the material of Section 2.8.3 and Equation 3.11.4

$$n = 3$$

$$x_1 = 1$$

$$i = 1$$

$$m = 2$$

$$r = 1$$

and the synthesized network is shown in Figure 3.11.1 with the funda-

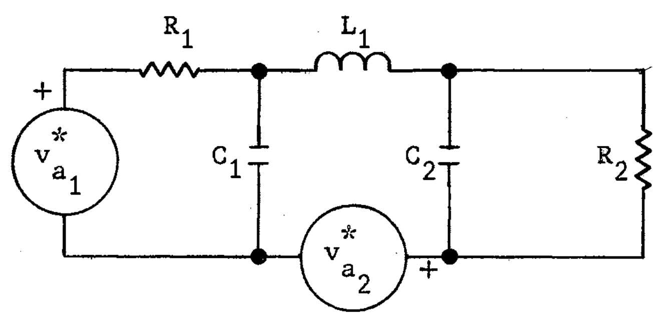

Figure 3.11.1 Network for $y'_{12}(s)$

mental circuit equations in symbolic form as

$$\begin{bmatrix} B_{11} & B_{12} & 0 & \vdots & U & 0 \\ B_{21} & B_{22} & B_{23} & \vdots & 0 & U \end{bmatrix} \begin{bmatrix} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{bmatrix} = 0 \tag{3.11.5a}$$

which can be written as

$$\left[\begin{array}{cc|cc|c|ccc} -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} v_{a_1} \\ v_{a_2} \\ \hline v_{bc_1} \\ v_{bc_2} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ v_{cr_2} \\ v_{c\ell_1} \end{array}\right] = 0 \qquad (3.11.5b)$$

Using the material of Section 3.9.1 and the results above yield

$$\mathcal{B} = \left[\begin{array}{cc} b_{11} & 0 \\ 0 & 0 \\ \hline 0 & b_{32} \end{array}\right] \qquad (3.11.6)$$

$$\mathcal{P} = \left[\begin{array}{cc|c} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \end{array}\right] \qquad (3.11.7)$$

Using the K_2 matrix of Section 3.6 and Equations 3.11.6 and 3.11.7, the state-model for $y'_{12}(s)$ will be

$$\frac{d}{dt}\left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] = \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & -4 & -3 \\ \hline -2 & 3 & 0 \end{array}\right] \left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] + \left[\begin{array}{cc} b_{11} & 0 \\ 0 & 0 \\ \hline 0 & b_{32} \end{array}\right] \left[\begin{array}{c} v^*_{a_1} \\ v^*_{a_2} \end{array}\right] \qquad (3.11.8a)$$

$$\begin{bmatrix} i^*_{a_1} \\ i^*_{a_2} \end{bmatrix} = \left[\begin{array}{cc|c} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \end{array}\right] \left[\begin{array}{c} v'_{bc_1} \\ v'_{bc_2} \\ \hline i'_{c\ell_1} \end{array}\right] + \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix} \begin{bmatrix} v^*_{a_1} \\ v^*_{a_2} \end{bmatrix} \tag{3.11.8b}$$

Using the above results, the 9 non-linear equations of Equation 3.9.13 can be written as

$$\begin{aligned} -2 &= p_{11}b_{32}(2) \\ 1 &= g_1(C'_1)^2 \\ 4 &= g_2(C'_2)^2 \\ 2 &= C'_1L'_1 \\ 3 &= C'_2L'_1 \\ b_{11} &= g_1C'_1 \\ b_{32} &= L'_1 \\ b_{11} &= -p_{11} \\ b_{32} &= p_{23} \end{aligned} \tag{3.11.9}$$

Now their solution can be obtained by using Equation 3.9.16

$$C'_1 = \left[\frac{-1}{-2}\right]^{\frac{1}{2}}(2)$$

$$C'_1 = \sqrt{2}$$

and the network component values are

$$\begin{aligned} C_1 &= \tfrac{1}{2} \text{ farad} \\ C_2 &= 2/9 \text{ farad} \\ L_1 &= \tfrac{1}{2} \text{ henry} \\ R_1 &= 2 \text{ ohms} \\ R_2 &= 9/8 \text{ ohms} \end{aligned} \tag{3.11.10}$$

with state-model element values of

$$\begin{aligned} b_{11} &= \frac{1}{\sqrt{2}} \\ b_{32} &= \sqrt{2} \\ p_{11} &= -\frac{1}{\sqrt{2}} \\ p_{23} &= \sqrt{2} \end{aligned} \tag{3.11.11}$$

From Equation 3.9.9, a_{12} is the element in Equation 3.3.8 that will be synthesized in $\mathfrak{D}'_{12}(s)$ by this network. From Equation 2.5.12, a_{12} is

$$a_{12} = \frac{a_{13}}{k_2}(s + f_3) \tag{3.11.12a}$$

$$a_{12} = 2(s + 4) \tag{3.11.12b}$$

From the above,

$$y'_{12}(s) = \frac{-2s - 8}{s^3 + 5s^2 + 17s + 25} \tag{3.11.13}$$

and then from Equations 3.11.2, 3.11.3, and 3.11.13

$$y''_{12}(s) = \frac{-4}{s^3 + 5s^2 + 17s + 25} \tag{3.11.14}$$

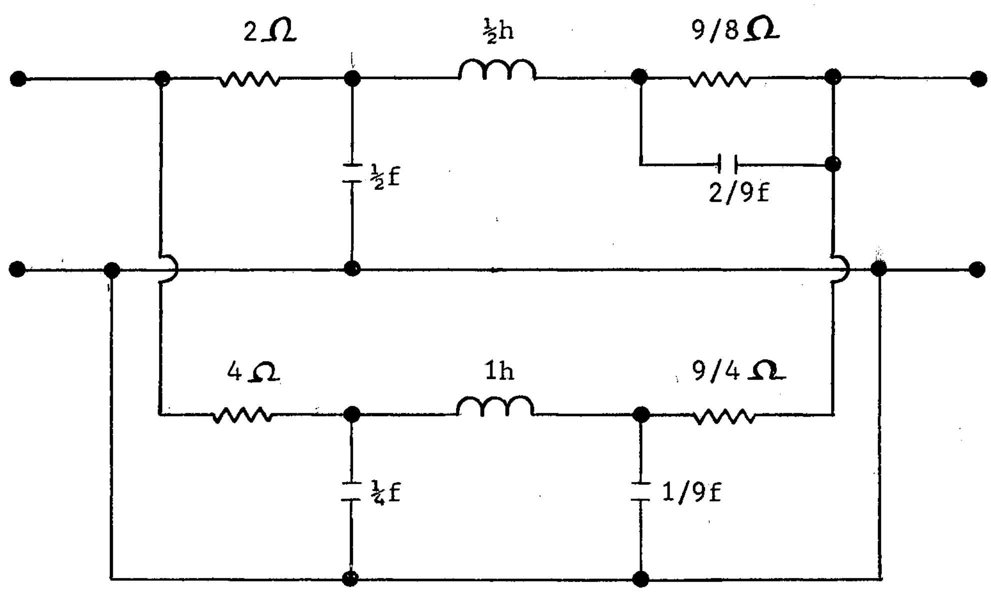

Figure 3.11.2 Synthesis of Case I $y_{12}(s)$

Since the synthesis of $y''_{12}(s)$ is the same as the example of Section 3.6, it will not be repeated here. The network that was synthesized from $y_{12}(s)$ is shown in Figure 3.11.2.

It should be pointed out that if the resulting constant in the numerator of $y'_{12}(s)$ would have been less than -12, $y''_{12}(s)$ would need a positive constant in its numerator. This would be accomplished by synthesizing $y''_{12}(s)$ with a negative numerator constant and then placing a 1:-1 transformer at one end of the synthesized network.

If the constant term in the numerator of Equation 3.11.4 would have been zero instead of -12, the synthesis procedure would have been the same as shown above. Equation 3.11.14 would have had -8 as the constant in the numerator and a 1:-1 transformer would be needed.

It is worthwhile to point out that the synthesis procedure can be completed without writing down the state-model and the fundamental circuit equations. This approach will be used in the next example.

3.12 Synthesis Example of $y_{12}(s)$ with Cases III and IV. Consider the short circuit transfer admittance of

$$y_{12}(s) = \frac{-2s^3 - 5s^2 - 3s - 9}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.1)$$

As in Section 3.11, the synthesis of this transfer admittance will be done in four stages corresponding to the four coefficients in the $y_{12}(s)$ numerator. This will be done by first synthesizing each of the transfer admittances

$$y'_{12}(s) = \frac{a'_3 s^3 + a'_2 s^2 + a'_1 s + a'_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.2a)$$

$$y''_{12}(s) = \frac{a''_2 s^2 + a''_1 s + a''_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.2b)$$

$$y^3_{12}(s) = \frac{a^3_1 s + a^3_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.2c)$$

$$y^4_{12}(s) = \frac{a^4_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.2d)$$

where

$$-2 = a'_3 \qquad (3.12.2e)$$

$$-5 = a'_2 + a''_2 \qquad (3.12.2f)$$

$$-3 = a'_1 + a''_1 + a^3_1 \qquad (3.12.2g)$$

$$-9 = a'_o + a''_o + a^3_o + a^4_o \qquad (3.12.2h)$$

These transfer admittances will then be put in parallel with transformers added if necessary.

First $y'_{12}(s)$ is to be synthesized. This is a Case III transfer admittance, therefore the material of Section 2.8.5 is applicable and yields

$$\begin{aligned} n &= 4 \\ x_3 &= 3 \\ i &= 1 \\ m &= 2 \\ r &= 2 \end{aligned} \tag{3.12.3}$$

and the synthesized network is shown in Figure 3.12.1.

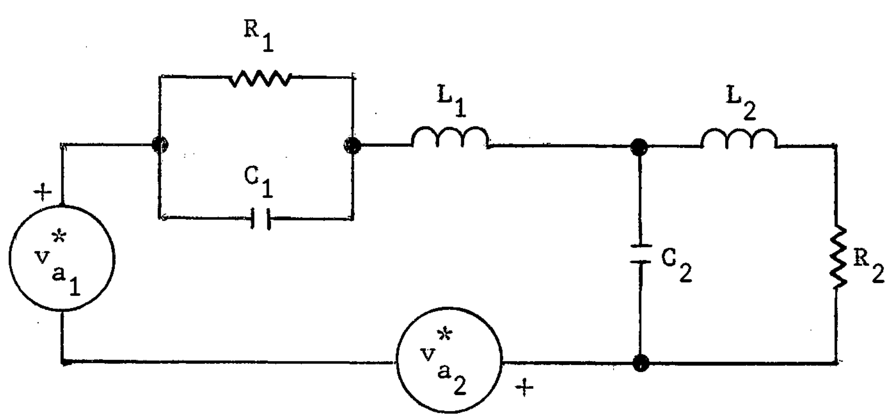

Figure 3.12.1 Network for $y'_{12}(s)$

Since the denominator of Equation 3.12.2 is identical to that of the example transfer admittance of Section 3.8, the K_2 matrix of Equation 3.8.2 will also be used here and is

$$K_2 = \left[\begin{array}{cc|cc} -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ \hline -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{3.12.4}$$

Using the above results, the 10 non-linear equations of Equation 3.9.42 can be written as

$$\begin{aligned} -2 &= p_{13}b_{32} \\ 1 &= g_1(C_1')^2 \\ 2 &= C_1'L_1' \\ 1 &= C_2'L_1' \\ 2 &= C_2'L_2' \\ 1 &= r_2(L_2')^2 \\ b_{31} &= -L_1' \\ b_{32} &= L_1' \\ b_{31} &= p_{13} \\ b_{32} &= p_{23} \end{aligned} \tag{3.12.5}$$

Now their solution can be obtained by using Equation 3.9.44

$$\begin{aligned} L_1' &= [-(-2)]^{\frac{1}{2}} \\ L_1' &= \sqrt{2} \end{aligned} \tag{3.12.6}$$

and the network component values are

$$C_1 = \tfrac{1}{2} \text{ farad}$$
$$C_2 = 2 \text{ farads}$$
$$L_1 = \tfrac{1}{2} \text{ henry}$$
$$L_2 = 1/8 \text{ henry} \qquad (3.12.7)$$
$$R_1 = 2 \text{ ohms}$$
$$R_2 = 1/8 \text{ ohm}$$

From Equation 3.9.39

$$\mathcal{D}'_{121}(s) = p_{13}a_{22}b_{32}$$

and Equation 2.5.12 yields

$$a_{22} = s^3 + 2s^2 + 5s + 4$$

and this results in

$$y'_{12}(s) = \frac{-2s^3 - 4s^2 - 10s - 8}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.8)$$

Upon comparing Equation 3.12.8 with Equation 3.12.2a, the a'_i coefficients can be obtained and are

$$a'_3 = -2$$
$$a'_2 = -4$$
$$a'_1 = -10$$
$$a'_o = -8$$

Now from Equations 3.12.2b and f and the above results, $y''_{12}(s)$ can be written as

$$y_{12}''(s) = \frac{-s^2 + a_1''s + a_o''}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.9)$$

This is a Case IV transfer admittance, therefore the material of Section 2.8.5 is applicable and yields

$$\begin{aligned} n &= 4 \\ x_4 &= 2 \\ i &= 1 \\ m &= 2 \\ r &= 2 \end{aligned} \qquad (3.12.10)$$

The synthesized network is shown in Figure 3.12.2.

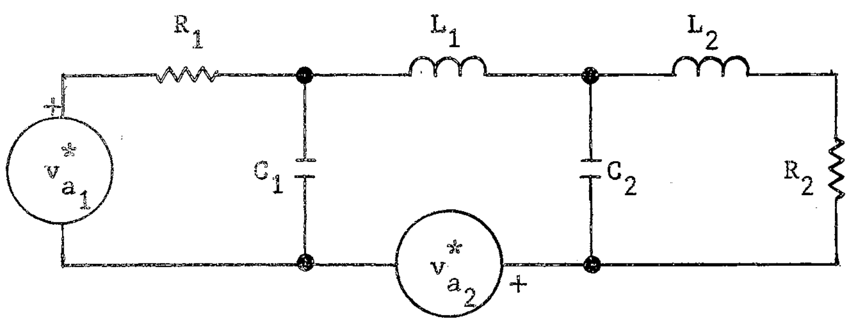

Figure 3.12.2 Network for $y_{12}''(s)$

It is not necessary, but for ease of computation, use the same K_2 matrix of Equation 3.12.4 with the above results and then the 10 non-linear equations of Equation 3.9.51 can be written as

$$-1 = p_{11}b_{32}(2)$$

$$1 = g_1(C_1')^2$$

$$2 = C_1'L_1'$$

$$1 = C_2'L_1'$$

$$2 = C_2'L_2' \tag{3.12.11}$$

$$1 = r_2(L_2')^2$$

$$b_{11} = g_1C_1'$$

$$b_{32} = L_1'$$

$$b_{11} = -p_{11}$$

$$b_{32} = p_{23}$$

Their solution can be obtained by using Equation 3.9.52

$$C_1' = \left[\frac{-1}{-1}\right]^{\frac{1}{2}}(2)$$

$$C_1' = 2 \tag{3.12.12}$$

and the network component values are

$$C_1 = \tfrac{1}{4} \text{ farad}$$

$$C_2 = 1 \text{ farad}$$

$$L_1 = 1 \text{ henry}$$

$$L_2 = \tfrac{1}{4} \text{ henry} \tag{3.12.13}$$

$$R_1 = 4 \text{ ohms}$$

$$R_2 = \tfrac{1}{4} \text{ ohm}$$

From Equation 3.9.48

$$\mathfrak{D}'_{121}(s) = p_{11}a_{12}b_{32}$$

and Equation 2.5.12 yields

$$a_{12} = 2(s^2 + s + 4)$$

and this results in

$$y''_{12}(s) = \frac{- s^2 - s - 4}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.14)$$

Upon comparing Equation 3.12.14 with Equation 3.12.2b, the a''_i coefficients are

$$a''_2 = -1$$
$$a''_1 = -1$$
$$a''_o = -4$$

Now from Equations 3.12.2c and g and the above results, $y^3_{12}(s)$ can be written as

$$y^3_{12}(s) = \frac{8s + a^3_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.15)$$

This is a Case III transfer admittance with a 1:-1 transformer added after the synthesis procedure is completed. Therefore the material of Section 2.8.5 is applicable and yields

$$
\begin{aligned}
n &= 4 \\
x_3 &= 1 \\
i &= 2 \\
m &= 2 \\
r &= 2
\end{aligned}
\qquad (3.12.16)
$$

The synthesized network that will yield negative coefficients is shown in Figure 3.12.3.

Again, for ease of computation, use the same K_2 matrix of Equation 3.12.4 with the above results and then the 10 non-linear equations of

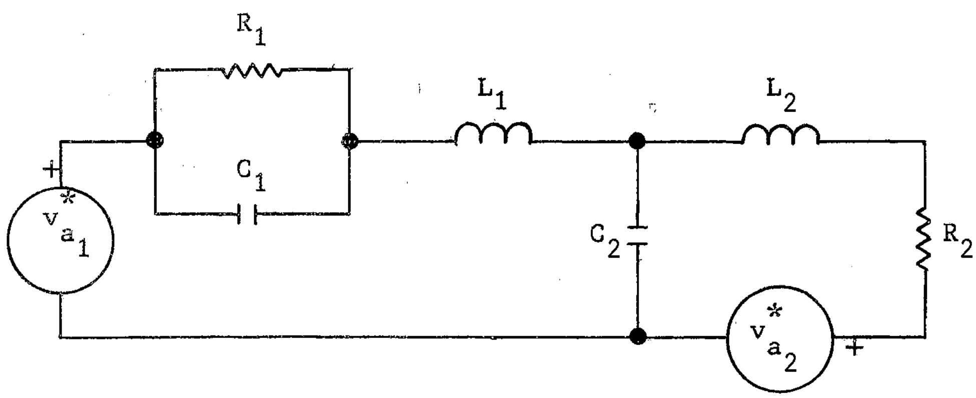

Figure 3.12.3 Network for $y^3_{121}(s)$

Equation 3.9.42 can be written as

$$-8 = p_{13}b_{42}(2)$$

$$1 = g_1(C_1')^2$$

$$2 = C_1' L_1'$$

$$1 = C_2' L_1'$$

$$2 = C_2' L_2'$$

$$1 = r_2 (L_2')^2$$

$$b_{31} = -L_1'$$

$$b_{42} = L_2'$$

$$b_{31} = p_{13}$$

$$b_{42} = p_{24} \qquad (3.12.17)$$

Now their solution can be obtained by using Equation 3.9.43

$$C_1' = [-(-8)]^{-\frac{1}{2}}(4)$$
$$C_1' = \sqrt{2} \qquad (3.12.18)$$

and the network component values are

$$C_1 = \tfrac{1}{2} \text{ farad}$$
$$C_2 = 2 \text{ farads}$$
$$L_1 = \tfrac{1}{2} \text{ henry} \qquad (3.12.19)$$
$$L_2 = 1/8 \text{ henry}$$
$$R_1 = 2 \text{ ohms}$$
$$R_2 = 1/8 \text{ ohm}$$

It should be noted that it was just a coincidence that these are the same network component values as those for $y_{12}'(s)$.

From Equation 3.9.39

$$\mathfrak{D}'_{121}(s) = p_{13}a_{24}b_{42}$$

and Equation 2.5.12 yields

$$a_{24} = 2(s+1)$$

and this results in

$$y^3_{121}(s) = \frac{-8s - 8}{s^4 + 2s^3 + 10s^2 + 10s + 17}$$

but by placing a 1:-1 transformer at either port, the following transfer admittance results

$$y^3_{12}(s) = \frac{8s + 8}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.20)$$

Upon comparing Equation 3.12.20 with Equation 3.12.2c, the a^3_i coefficients are

$$a^3_1 = 8$$
$$a^3_o = 8$$

Now from Equations 3.12.2d and h and the above results, $y^4_{12}(s)$ can be written as

$$y^4_{12}(s) = \frac{-5}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (3.12.21)$$

Since this is the same transfer admittance as that synthesized in Section 3.8, only the network and component values are given in Figure 3.12.4. If the paralleling of these four synthesized transfer admittances satisfies the validity test then

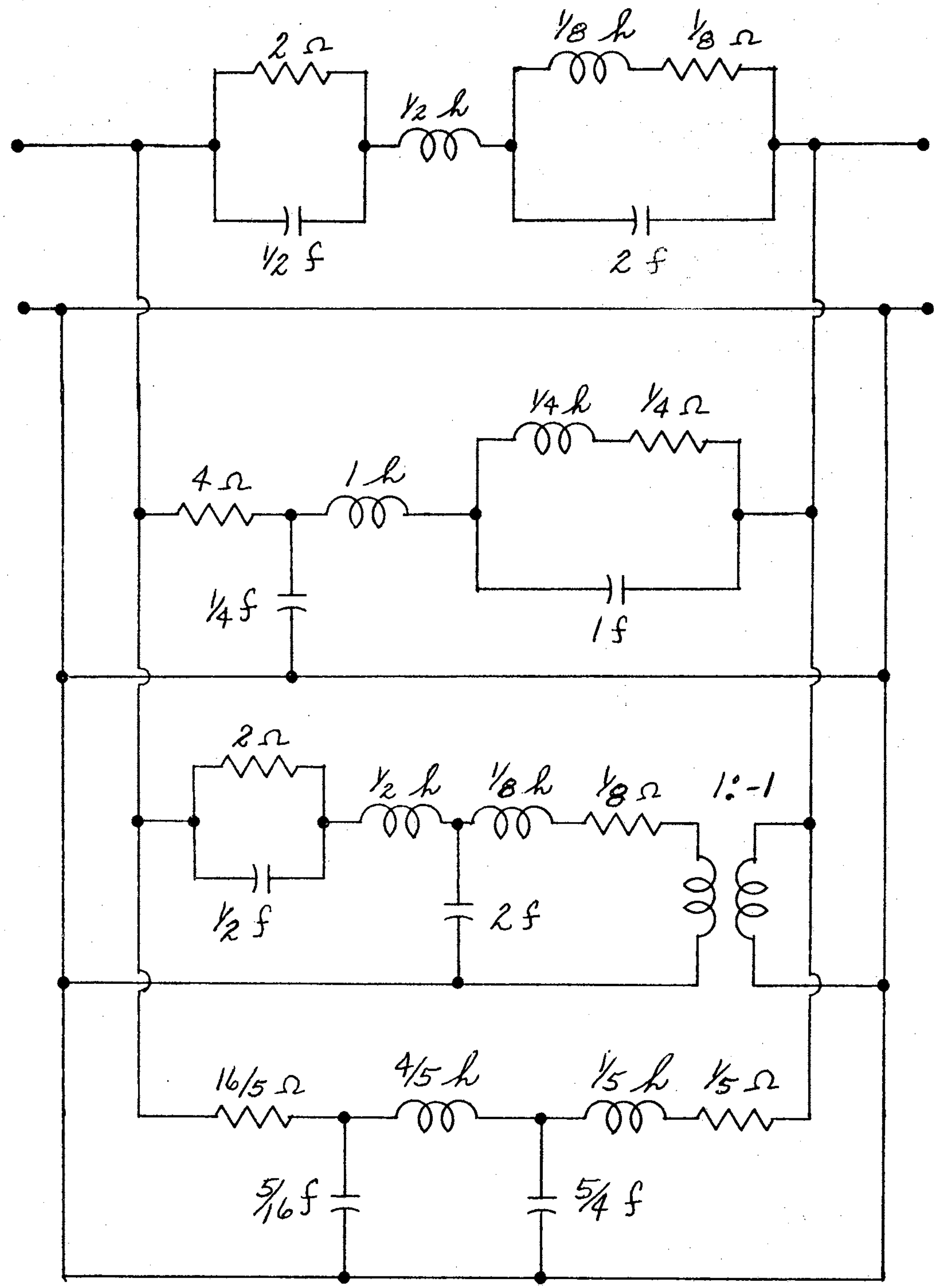

Figure 3.12.5 Network for $y_{12}(s)$

$$y_{12}(s) = y'_{12}(s) + y''_{12}(s) + y^{3}_{12}(s) + y^{4}_{12}(s)$$

The resultant network is shown in Figure 3.12.5 and the synthesis procedure is complete.

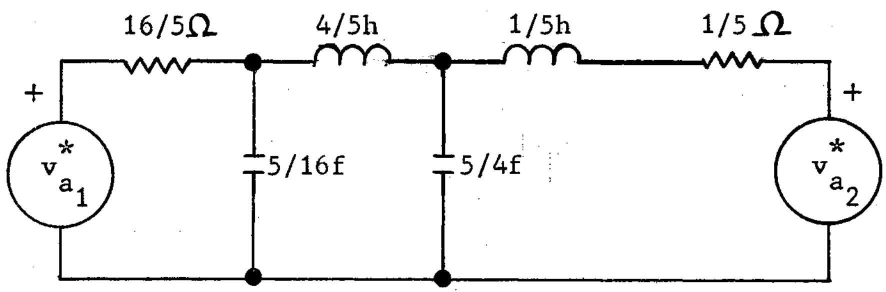

Figure 3.12.4 Network for $y^{4}_{12}(s)$

3.13 Unified Element Value Synthesis. In the previous sections, transfer admittance synthesis is obtained by paralleling several ladder networks. The first ladder network satisfies the first coefficient in the transfer admittance numerator. The first and second ladder networks satisfy the second coefficient and the rest are satisfied in a like manner. Further the characteristic equations of each of these ladder networks are identical. In Section 2.8 we observed that a transfer admittance with a numerator polynomial of given degree can be achieved by inserting two drivers at specified locations in the synthesized network of the characteristic polynomial. This implies that if we are not interested in the magnitude of the first coefficient in the numerator of the resulting transfer admittance, then by proper insertion of the drivers into any synthesized network of the characteristic polynomial we can always realize a given transfer admittance which has a desired numerator degree. The first coefficient of the numerator can be altered to a desired value by connecting a two-port transformer with the proper turns ratio.

This idea can be implemented into the general synthesis procedure by modifying one equation and adding one equation to the (n + 6) nonlinear algebraic equations that are used to obtain the network component values. This can be shown by considering the Case III development of Section 3.9.

If a $n_1:n_2$ transformer is placed on the output port of the Case III ladder network, Equation 3.9.39 becomes

$$\mathfrak{D}'_{121}(s) = \frac{n_1}{n_2} p_{1,m+1} a_{2,2i} b_{m+i,2} \qquad (3.13.1)$$

This will change Equation 3.9.41 to

$$a_{x_3} = \frac{n_1}{n_2} p_{1,m+1} b_{m+i,2} \qquad (3.13.2a)$$

$$a_{x_3} = \frac{n_1}{n_2} p_{1,m+1} b_{m+i,2} \left[\prod_{j=1}^{2i-1} k_j \right] \qquad (3.13.2b)$$

and this places the added unknown, n_1/n_2, into the set of equations. Next another equation must be added. A possibility which is chosen for ease of computation is $C'_1 = 1$ or $L'_1 = 1$. Now a solution to the modified set of equations of Equation 3.9.42 is available. If this modification is done to the synthesis procedure for each ladder and the same equation is added each time, then "like" components for each ladder will have identical values.

This will be illustrated by synthesizing the transfer admittance of Equation 3.12.1 using the procedure just presented. In each case, the added equation will be $C_1' = 1$ and the unknown, n_1/n_2, will be found from Equation 3.13.2. It is understood that the selection of n_1 (or n_2) is arbitrary. Using the procedure above results in the synthesized network of Figure 3.13.1. A similar approach can be used with the other three cases.

It would appear that this idea could have an application in fabricating transfer functions with integrated circuit chips.

<u>3.14 Synthesis with a Modified K_2-Matrix.</u> It would appear that a general synthesis procedure would result from a similarity transformation on the K_2-matrix that would interchange the role of the capacitor voltages and the inductor currents of the state-model. However it does not and this is shown by the following presentation.

Perlis (28) has shown that two similar matrices exhibit the same characteristic polynomial. Consider the two matrices, K_2 and K_2', which are related by the similarity transformation, P, such that

$$K_2' = P^{-1}K_2P \tag{3.14.1}$$

where K_2 is given in Equations 2.5.8a and 2.5.9a and is given here in symbolic form for ready reference.

$$K_2 = \begin{bmatrix} -K_{11} & K_{12}^T \\ -K_{12} & -K_{22} \end{bmatrix} \tag{3.14.2}$$

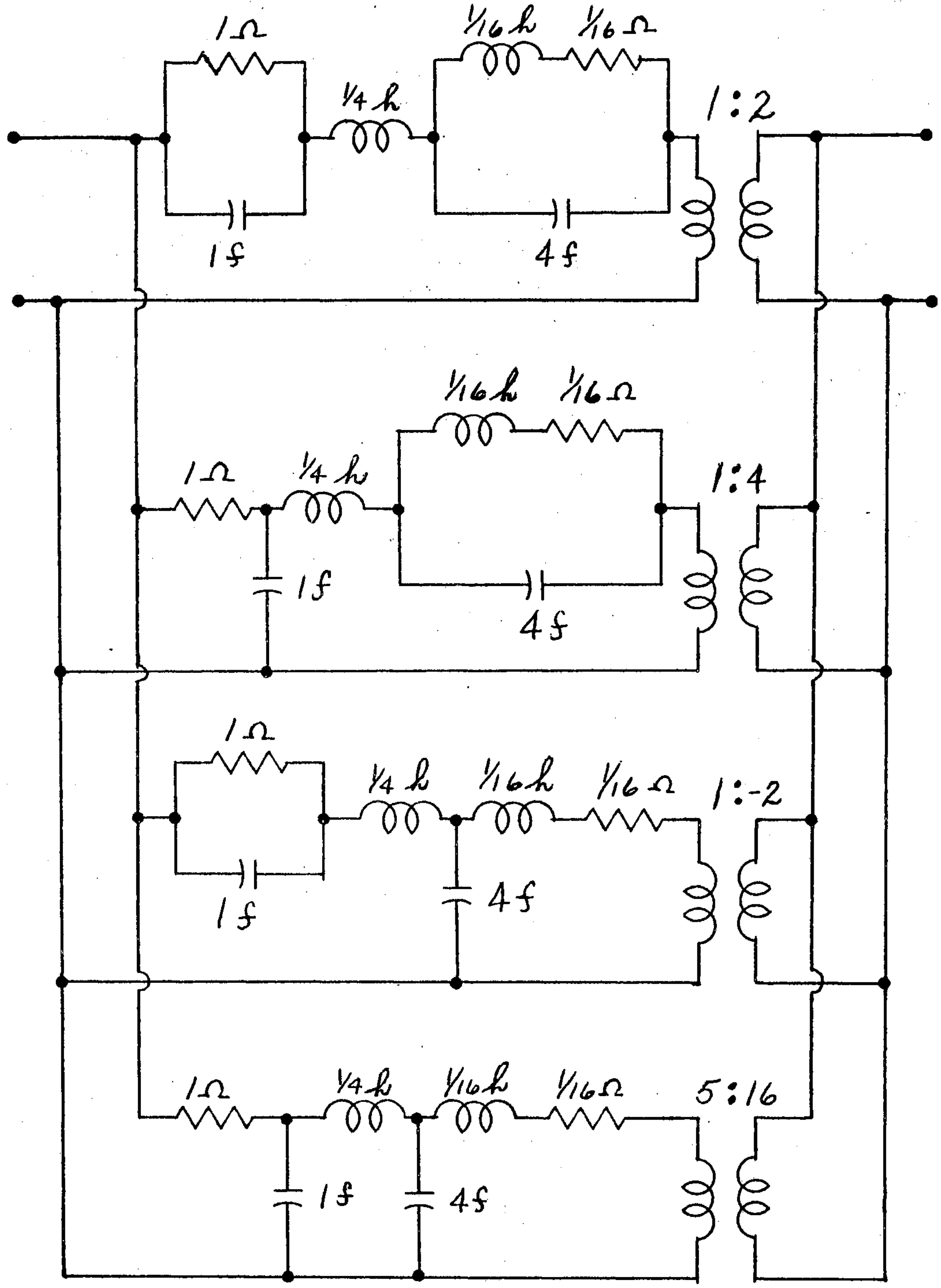

Figure 3.13.1 Unified Network for $y_{12}(s)$

The above mentioned interchange of roles can be achieved with a K_2' of

$$K_2' = \begin{bmatrix} -K_{22} & -K_{12} \\ K_{12}^T & -K_{11} \end{bmatrix} \qquad (3.14.3)$$

which can be obtained with a similarity transformation of

$$P = \begin{bmatrix} 0 & U_1 \\ U_2 & 0 \end{bmatrix} \qquad (3.14.4)$$

where the U_i's are unit matrices.

It can be seen that if the K_2' of Equation 3.14.3 were used in Equation 3.3.2a, there would be r capacitor voltages and m inductor currents. Another way to reflect this transformation and still be able to use the previous material is to write Equation 3.3.1 as

$$\frac{d}{dt}\begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} = \left[\begin{array}{c|c} -L_c^{-\frac{1}{2}}B_{23}R_bB_{23}^TL_c^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{22}C_b^{-\frac{1}{2}} \\ \hline C_b^{-\frac{1}{2}}B_{22}^TL_c^{-\frac{1}{2}} & -C_b^{-\frac{1}{2}}B_{12}^TG_cB_{12}C_b^{-\frac{1}{2}} \end{array}\right] \begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} + \left[\begin{array}{c} -L_c^{-\frac{1}{2}}B_{21} \\ \hline -C_b^{-\frac{1}{2}}B_{12}^TG_cB_{11} \end{array}\right] \begin{bmatrix} V_a^* \end{bmatrix} \qquad (3.14.5a)$$

$$\begin{bmatrix} I_a^* \end{bmatrix} = \left[\begin{array}{c|c} -B_{21}^TL_c^{-\frac{1}{2}} & B_{11}^TG_cB_{12}C_b^{-\frac{1}{2}} \end{array}\right] \begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} + \begin{bmatrix} B_{11}^TG_cB_{11} \end{bmatrix} \begin{bmatrix} V_a^* \end{bmatrix} \qquad (3.14.5b)$$

The fundamental circuit equations for this state-model are

$$\left[\begin{array}{ccc|cc} B_{11} & B_{12} & 0 & U & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U \end{array}\right] \left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \end{array}\right] = 0 \qquad (3.14.6)$$

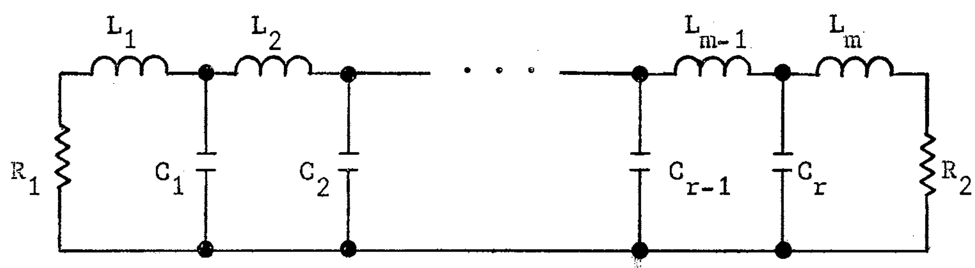

Figure 3.14.1 Ladder Network for n odd

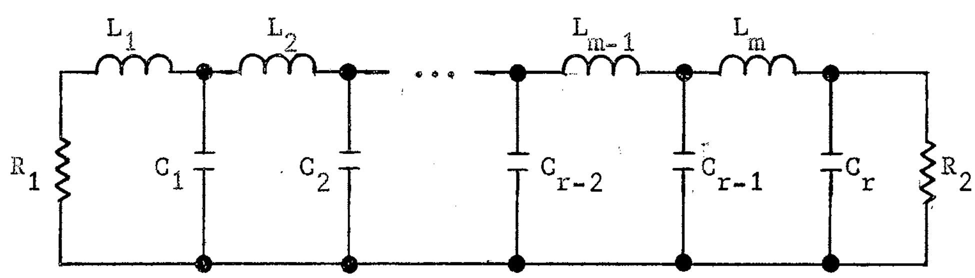

Figure 3.14.2 Ladder Network for n even

Following a procedure similar to Section 2.7 and realizing the ladder networks of the characteristic polynomial that is placed in the state-model form shown above, results in the ladder networks shown in Figure 3.14.1 and Figure 3.14.2. As in Section 2.8, the drivers can be inserted into these networks to obtain the transfer admittances. However it can be shown that it is impossible to obtain a transfer admittance with a numerator degree of one for the ladder network of Figure 3.14.1. This implies that we will not obtain a general synthesis procedure using the K_2' matrix. However special cases are possible and these need further investigation.

It is interesting to note that the ladder network corresponding to the even order K_2' matrix, will yield the same synthesis procedure as presented in the previous sections.

<u>3.15 One Resistor Ladder Networks.</u> Investigation of ladder networks with one resistor was made to see if these could be used in a general synthesis procedure. It was found that they will not yield a general procedure as is shown.

Following a procedure similar to Section 2.7 and realizing the ladder networks of the characteristic polynomial as done by Yarlagadda (34), results in the ladder networks shown in Figure 3.15.1 and Figure 3.15.2.

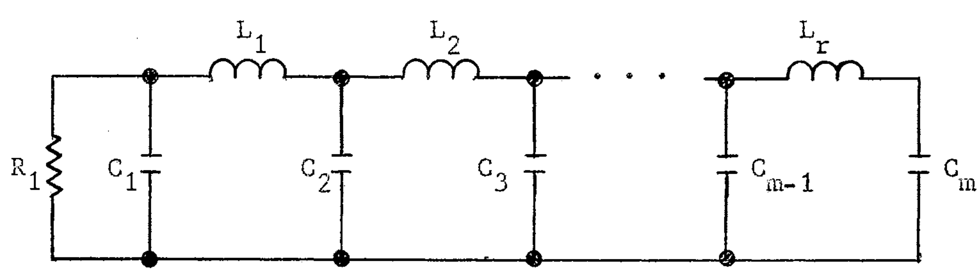

Figure 3.15.1 Ladder Network for n odd

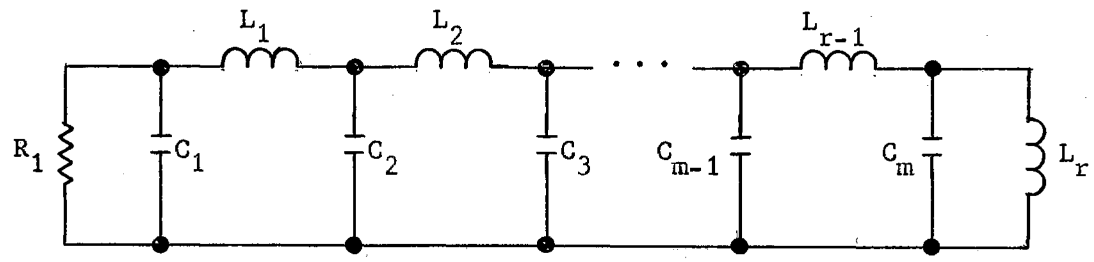

Figure 3.15.2 Ladder Network for n even

It can be shown that it is impossible to obtain a transfer admittance with a constant in the numerator for the ladder network of Figure 3.15.1. However it is possible while using these one resistor ladder networks to obtain transfer admittances with other coefficients in the numerator which may be used in conjunction with the two resistor ladder networks.

Since one resistor ladder networks do not give a general synthesis procedure for transfer admittances and they are a special case of the general case (Special Case of the two resistor ladder networks), there will be no further discussion concerning them.

3.16 Special Case - LC Transfer Function Synthesis. Previously we have restricted the synthesis procedure to transfer admittance functions that have strictly Hurwitz characteristic polynomials. Here we shall consider transfer admittance functions with characteristic polynomials which have roots on the imaginary axis.

Given a transfer admittance function with a Hurwitz polynomial such as

$$y_{12}(s) = \frac{N(s)}{D(s)} \tag{3.16.1}$$

It is possible to factor the denominator into two components. One that is strictly Hurwitz, $D_1(s)$, and one that has roots only on the imaginary axis, $D_2(s)$. Then Equation 3.16.1 can be written

$$y_{12}(s) = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)} \tag{3.16.2a}$$

$$y_{12}(s) = y_{121}(s) + y_{122}(s) \tag{3.16.2b}$$

Now $y_{121}(s)$ can be synthesized by the procedures presented previously. The $y_{122}(s)$ characteristic polynomial realization will be handled in the manner that Yarlagadda (34) has presented with the drivers being inserted to obtain the transmission zeros according to the ideas of Appendix D.

Now to complete the synthesis of $y_{12}(s)$, parallel the networks obtained for $y_{121}(s)$ and $y_{122}(s)$.

CHAPTER IV

SYNTHESIS OF THE OPEN CIRCUIT TRANSFER IMPEDANCE, $Z_{12}(s)$

4.1 Introduction. This chapter will briefly present the state-space approach to the synthesis of the open circuit transfer impedance, $z_{12}(s)$, using the concepts presented in Chapters II and III. The brevity of this chapter results from the duality property existing between the short circuit transfer admittance and the open circuit transfer impedance (30). Therefore just the results will be presented with only the duality property being given as justification.

4.2 Restrictions. The s-domain restrictions for the open circuit transfer impedance will be the same as those given in Section 3.2. The topological restrictions on the network to be synthesized from the transfer impedance will be similar to those given in Section 3.2 and are:

1. Both branch resistors and chord resistors will not be permitted in the same fundamental cut-sets.
2. Circuits of capacitors will not be permitted.
3. Cut-sets of inductors with or without current drivers will not be permitted.
4. The network driver configuration must be that of Figure 2.4.2.

4.3 State-Models. It is desirable to recall the state-model developed in Section 2.4.2 and is given here for ready reference.

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{11}^T G_c B_{11} C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}}B_{21}^T L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}}B_{21}C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{22}R_b B_{22}^T L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \left[\begin{array}{c} C_b^{-\frac{1}{2}}B_{31}^T \\ \hline -L_c^{-\frac{1}{2}}B_{22}R_b B_{32}^T \end{array}\right]\begin{bmatrix} I_t^* \end{bmatrix}$$

(4.3.1a)

$$\begin{bmatrix} V_t^* \end{bmatrix} = \left[\begin{array}{c|c} B_{31}C_b^{-\frac{1}{2}} & B_{32}R_b B_{22}^T L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \begin{bmatrix} B_{32}R_b B_{32}^T \end{bmatrix}\begin{bmatrix} I_t^* \end{bmatrix} \quad (4.3.1b)$$

The fundamental circuit equations for this state-model are

$$\left[\begin{array}{cc|ccc} B_{11} & 0 & U & 0 & 0 \\ B_{21} & B_{22} & 0 & U & 0 \\ B_{31} & B_{32} & 0 & 0 & U \end{array}\right]\left[\begin{array}{c} V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \\ V_t \end{array}\right] = 0 \quad (4.3.2)$$

To utilize the presentation in the previous chapters and by use of the duality principle, we shall write the state-model in the following form

$$\frac{d}{dt}\begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} = \left[\begin{array}{c|c} -L_c^{-\frac{1}{2}}B_{22}R_b B_{22}^T L_c^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{21}C_b^{-\frac{1}{2}} \\ \hline C_b^{-\frac{1}{2}}B_{21}^T L_c^{-\frac{1}{2}} & -C_b^{-\frac{1}{2}}B_{11}^T G_c B_{11} C_b^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix}$$

$$+ \left[\begin{array}{c} -L_c^{-\frac{1}{2}}B_{22}R_b B_{32}^T \\ \hline C_b^{-\frac{1}{2}}B_{31}^T \end{array}\right]\begin{bmatrix} I_t^* \end{bmatrix}$$

(4.3.3a)

$$\left[V_t^*\right] = \left[B_{32}R_bB_{22}^TL_c^{-\frac{1}{2}} \;\middle|\; B_{31}C_b^{-\frac{1}{2}}\right]\begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} + \left[B_{32}R_bB_{32}^T\right]\left[I_t^*\right] \qquad (4.3.3b)$$

and written in symbolic form as

$$\frac{d}{dt}X = K_2X + \mathcal{B}\, I_t^* \qquad (4.3.4a)$$

$$V_t^* = \mathcal{P}\, X + \mathcal{R}\, I_t^* \qquad (4.3.4b)$$

When the open circuit transfer impedance denominator is of odd degree, then

$$K_2 = \left[\begin{array}{cccccc|ccccc}
-f_o & & & & & & k_1 & & & & \\
& 0 & & & & & -k_2 & \bullet & & & \\
& & \bullet & & & & & \bullet & \bullet & & \\
& & & \bullet & & & & & \bullet & \bullet & \\
& & & & 0 & & & & & \bullet & k_{n-2} \\
& & & & & -f_n & & & & & -k_{n-1} \\
\hline
-k_1 & k_2 & & & & & & & & & \\
& \bullet & \bullet & & & & & & & & \\
& & \bullet & \bullet & & & & & & & \\
& & & \bullet & \bullet & & & & & & \\
& & & & -k_{n-2} & k_{n-1} & & & & &
\end{array}\right] \qquad (4.3.5)$$

and when the denominator degree is even, then

$$K_2 = \left[\begin{array}{cccccc|cccccc} -f_o & & & & & & k_1 & & & & & \\ & 0 & & & & & -k_2 & k_3 & & & & \\ & & \bullet & & & & & \bullet & \bullet & & & \\ & & & \bullet & & & & & \bullet & \bullet & & \\ & & & & \bullet & & & & & \bullet & \bullet & \\ & & & & & 0 & & & & & -k_{n-2} & k_{n-1} \\ \hline -k_1 & k_2 & & & & & 0 & & & & & \\ & -k_3 & \bullet & & & & & \bullet & & & & \\ & & \bullet & \bullet & & & & & \bullet & & & \\ & & & \bullet & \bullet & & & & & \bullet & & \\ & & & & \bullet & k_{n-2} & & & & & 0 & \\ & & & & & -k_{n-1} & & & & & & -f_n \end{array}\right] \quad (4.3.6)$$

4.4 <u>Realization of Characteristic Polynomial.</u> Using a procedure similar to that in Section 2.7 with the state-model of Equation 4.3.3a without the driver and written as

$$\frac{d}{dt}\begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} = \left[\begin{array}{c|c} -L_c^{-\frac{1}{2}}B_{22}R_bB_{22}^TL_c^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}}B_{21}C_b^{-\frac{1}{2}} \\ \hline C_b^{-\frac{1}{2}}B_{21}^TL_c^{-\frac{1}{2}} & -C_b^{-\frac{1}{2}}B_{11}^TG_cB_{11}C_b^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} I'_{c\ell} \\ V'_{bc} \end{bmatrix} \quad (4.4.1)$$

it is possible to synthesize a network that exhibits the characteristic polynomial that is obtained when the K_2 matrices of Equations 4.3.5 and 4.3.6 are substituted into Equation 4.3.7. These synthesized networks are shown in Figures 4.4.1 and 4.4.2. As should be expected, these networks are the duals of those shown in Figures 2.7.2 and 2.7.4.

As in Section 2.7 this procedure yields the following equations when n is odd:

$$
\begin{aligned}
f_o &= r_1(L_1^{'})^2 \\
f_n &= r_2(L_m^{'})^2 \\
k_1 &= L_1^{'}C_1^{'} \\
k_2 &= L_2^{'}C_1^{'} \\
&\vdots \\
k_{n-1} &= L_m^{'}C_r^{'}
\end{aligned}
\qquad (4.4.2)
$$

and as in Equation 2.7.19, a solution to this set of equations is obtained when one of the unknowns is assigned an arbitrary value, such as

$$L_1^{'} = 1 \qquad (4.4.3)$$

The remaining unknowns in the set of equations of Equation 4.4.2 can be solved in a manner similar to that of Section 2.7. The element values of the network components are related to these unknowns by $C_i = (C_i^{'})^{-2}$, $L_i = (L_i^{'})^{-2}$, $R_1 = r_1$ and $R_2 = r_2$.

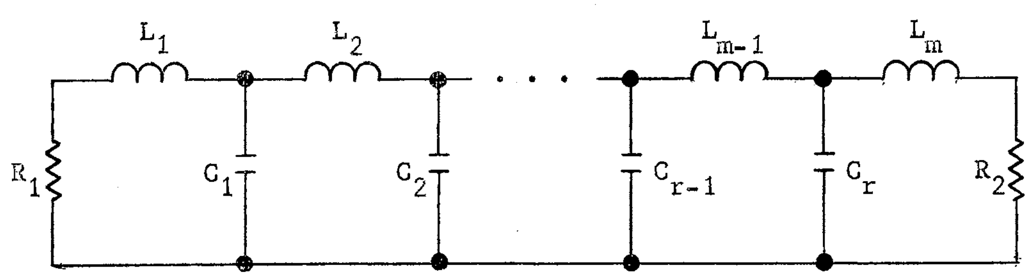

Figure 4.4.1 Polynomial Realization with n odd

When the polynomial degree is even, the network of Figure 4.4.2 and the following set of non-linear equations result.

$$
\begin{aligned}
f_o &= r_1(L_1')^2 \\
k_1 &= L_1'C_1' \\
&\vdots \\
k_{n-1} &= L_m'C_r' \\
f_n &= g_2(C_r')^2
\end{aligned}
\qquad (4.4.4)
$$

and as in Equation 2.7.32, a solution to this set of equations can be obtained in a similar manner as above. Let

$$L_1' = 1 \qquad (4.4.5)$$

and the remaining variables of Equation 4.4.4 can then be determined. The network component values are related to these variables by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = r_1$, and $R_2 = 1/g_2$.

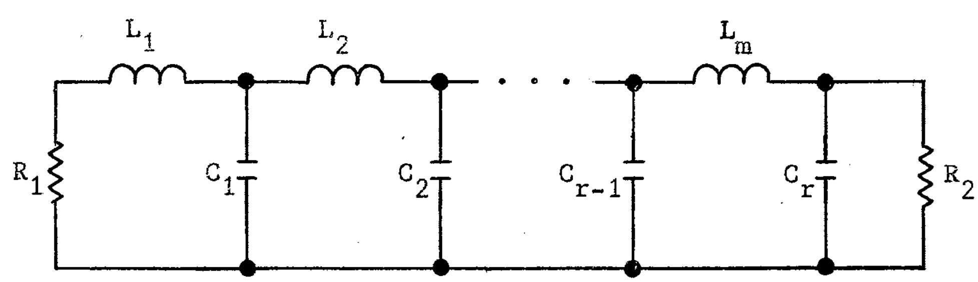

Figure 4.4.2 Polynomial Realization with n even

4.5 Synthesis of $z_{12}(s)$. Since the presentation of the transfer impedance is based on the principle of duality with respect to the transfer admittance, the transfer impedances are again classified into six types. These classifications are for transfer impedances which are proper functions. If the given $z_{12}(s)$ is an improper function, it is assumed that the procedures of Section 2.6.2 are followed until a resulting proper function is obtained. Further the denominator polynomial of $z_{12}(s)$ must be a strictly Hurwitz polynomial.

Now each of the classifications will be presented.

4.5.1 Case II - Special. For this transfer impedance case, the numerator is a constant and the denominator function is of odd degree and is written as

$$z_{12}(s) = \frac{a_o}{s^n + b_{n-1}s^{n-1} + \dots + b_1 s + b_o} \tag{4.5.1}$$

where

a_o = positive constant

n = odd integer

Using the information in Chapter III and the principle of duality, we have the relationships:

$$n = m + r$$

$$m = r + 1$$

where

n = degree of characteristic polynomial

m = number of inductors

r = number of capacitors

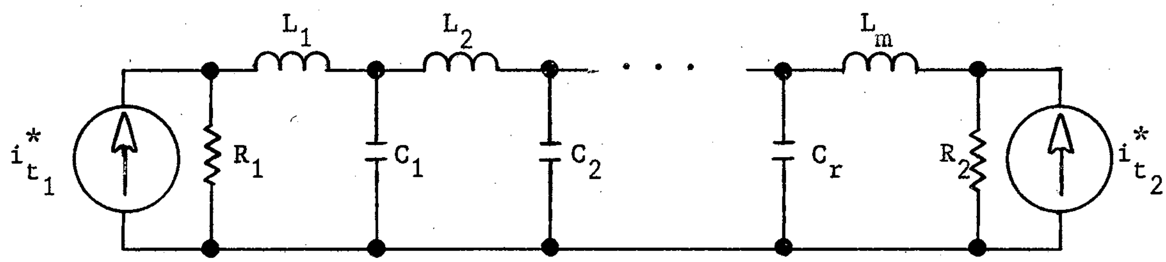

Figure 4.5.1 Case II - Special Realization of $z_{12}(s)$

Using the procedures of Chapter II and III with Equation 4.5.1 results in the network of Figure 4.5.1 and the following set of non-linear equations.

$$a_o = b_{m,2} p_{11} \left[\prod_{i=1}^{n-1} k_i \right]$$

$$f_o = r_1 (L_1')^2$$

$$f_n = r_2 (L_m')^2$$

$$k_1 = L_1' C_1'$$

$$k_2 = L_2' C_1'$$

$$\vdots$$

$$k_{n-2} = L_{m-1}' C_r'$$

$$k_{n-1} = L_m' C_r'$$

$$b_{11} = r_1 L_1'$$

$$b_{m,2} = -r_2 L_m'$$

$$b_{11} = -p_{11}$$

$$b_{m,2} = p_{2,m} \tag{4.5.2}$$

This set of equations has the solution of

$$L_1' = \left[\frac{f_o f_n}{a_o}\right]^{\frac{1}{2}} (k_1 k_3 \ldots k_{n-2}) \tag{4.5.3}$$

After the C_i''s and L_i''s are determined the component values still have to be determined by

$$C_i = (C_i')^{-2}$$

$$L_i = (L_i')^{-2} \tag{4.5.4}$$

$$R_i = r_i$$

<u>4.5.2 Case IV - Special.</u> For this transfer impedance case, the numerator is a constant and the denominator function is of even degree and is written as

$$z_{12}(s) = \frac{a_o}{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \tag{4.5.5}$$

where

a_o = positive constant

n = even integer

$n = m + r$

$m = r$

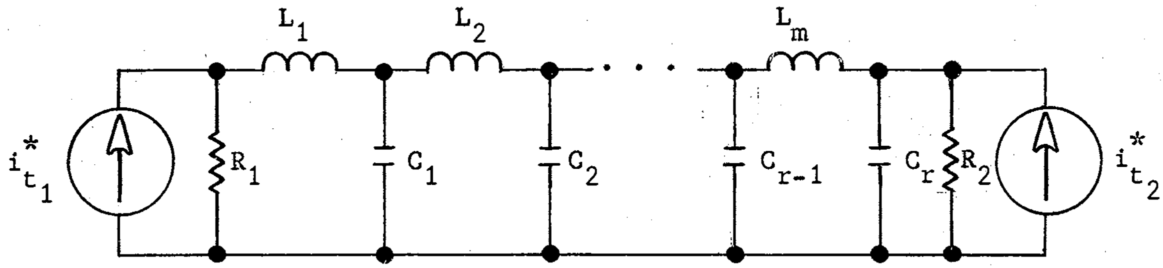

Figure 4.5.2 Case IV - Special Realization of $z_{12}(s)$

Using the procedures of Chapters II and III with Equation 4.5.5 results in the network of Figure 4.5.2 and the following set of non-linear equations.

$$a_o = b_{m+r,2}p_{11}\left[\prod_{i=1}^{n-1} k_i\right]$$

$$f_o = r_1(L_1')^2$$

$$k_1 = L_1'C_1'$$

$$k_2 = L_2'C_1'$$

$$\vdots$$

$$k_{n-2} = L_m'C_{r-1}'$$

$$k_{n-1} = L_m'C_r'$$

$$f_n = g_2(C_r')^2$$

$$b_{11} = r_1L_1'$$

$$b_{m+r,2} = -C_r'$$

$$b_{11} = -p_{11}$$

$$b_{m+r,2} = p_{2,m+r} \tag{4.5.6}$$

This set of equations has the solution of

$$L_1' = \left[\frac{f_o}{a_o}\right]^{\frac{1}{2}} (k_1 k_3 \ldots k_{n-1}) \tag{4.5.7}$$

where $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = r_1$, and $R_2 = 1/g_2$.

4.5.3 Case I. For this transfer impedance case, the numerator and denominator polynomials are both of odd degree. This transfer impedance is written as

$$z_{12}(s) = \frac{a_{x_1} s^{x_1} + a_{x_1-1} s^{x_1-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \tag{4.5.8}$$

where

a_i = positive constant; $i = 0, 1, \ldots, x_1$

x_i = odd integer

n = odd integer

$n = m + r$

$m = r + 1$

$i = \frac{n - x_1}{2}$

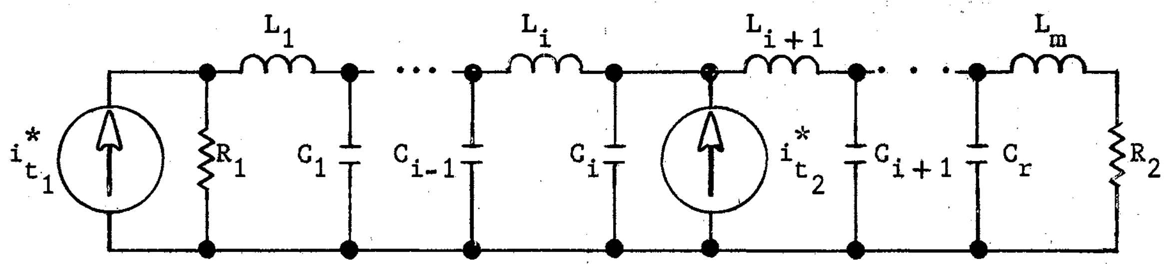

Figure 4.5.3 Case I Realization of $z_{12}(s)$

Using the procedures of Chapters II and III with Equation 4.5.8 results in the network of Figure 4.5.3 and the following set of non-linear equations.

$$\begin{aligned}
a_{x_1} &= p_{11} b_{m+i,2} \left[\prod_{j=1}^{2i-1} k_j \right] \\
f_o &= r_1 (L_1')^2 \\
f_n &= r_2 (L_m')^2 \\
k_1 &= L_1' C_1' \\
&\vdots \\
k_{n-1} &= L_m' C_r' \\
b_{11} &= r_1 L_1' \\
b_{m+i,2} &= -C_i' \\
b_{11} &= -p_{11} \\
b_{m+i,2} &= p_{2,m+i}
\end{aligned} \tag{4.5.9}$$

This set of equations has the solution of

$$L_1' = \left[\frac{f_o}{a_{x_1}}\right] k_1 k_3 \ldots k_{2i-1} \tag{4.5.10}$$

where $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, and $R_i = r_i$.

4.5.4 Case II. This transfer impedance case has a numerator polynomial of even degree and a denominator polynomial of odd degree and is written as

$$z_{12}(s) = \frac{a_{x_2}s^{x_2} + a_{x_2-1}s^{x_2-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \tag{4.5.11}$$

where

a_i = positive constant; $i = 0, 1, \ldots, x_2$

x_2 = even integer

$x_2 \neq 0$

n = odd integer

$n = m + r$

$m = r + 1$

$i = \frac{n-x_2+1}{2}$

Using the procedures of Chapters II and III with Equation 4.5.11 results in the network of Figure 4.5.4 and the following set of nonlinear equations.

$$a_{x_2} = p_{1,m+1} b_{m+i,2}; \quad i = 1$$

$$a_{x_2} = p_{1,m+1} b_{m+i,2} \left[\prod_{j=2}^{2i-1} k_j \right]; \quad i \geq 2$$

$$f_o = r_1 (L_1')^2$$

$$f_n = r_2 (L_m')^2$$

$$k_1 = L_1' C_1' \tag{4.5.12}$$

$$\vdots$$

$$k_{n-1} = L_m' C_r'$$

$$b_{m+1,1} = -C_1'$$

$$b_{m+i,2} = -C_i'$$

$$b_{m+1,1} = p_{1,m+1}$$

$$b_{m+i,2} = p_{2,m+i}$$

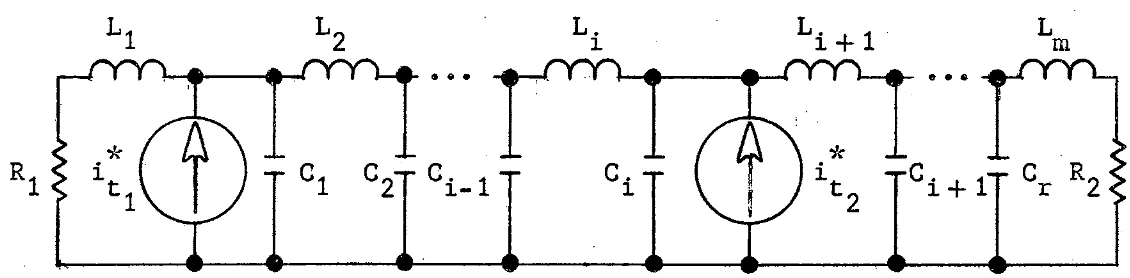

Figure 4.5.4 Case II Realization of $z_{12}(s)$

This set of equations has the solution of

$$C_1' = (a_{x_2})^{\frac{1}{2}};\ i = 1 \qquad (4.5.13a)$$

$$L_1' = (a_{x_2})^{-\frac{1}{2}} k_1 k_3 \ldots k_{2i-1};\ i \geq 2 \qquad (4.5.13b)$$

where $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, and $R_i = 1/g_i$.

<u>4.5.5 Case III.</u> This transfer impedance case has a numerator polynomial of odd degree and a denominator polynomial of even degree and is written as

$$z_{12}(s) = \frac{a_{x_3} s^{x_3} + a_{x_3-1} s^{x_3-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \qquad (4.5.14)$$

where

a_i = positive constant; $i = 0, 1, \ldots, x_3$

x_3 = odd integer

n = even integer

$n = m + r$

$m = r$

$i = \frac{n - x_3 + 1}{2}$

Using the procedure of Chapters II and III with Equation 4.5.14 results in the network of Figure 4.5.5 and the following set of non-linear equations.

$$a_{x_3} = p_{1,m+1} b_{m+i,2}; \quad i = 1$$

$$a_{x_3} = p_{1,m+1} b_{m+i,2} \left[\prod_{j=2}^{2i-1} k_j \right]; \quad i \geq 2$$

$$f_o = r_1 (L_1')^2$$

$$k_1 = L_1' C_1'$$

$$\vdots$$

$$k_{n-1} = L_m' C_r' \qquad (4.5.15)$$

$$f_n = g_2 (C_r')^2$$

$$b_{m+i,2} = -C_1'$$

$$b_{m+1,1} = -C_i'$$

$$b_{m+1,1} = p_{1,m+1}$$

$$b_{m+i,2} = p_{2,m+i}$$

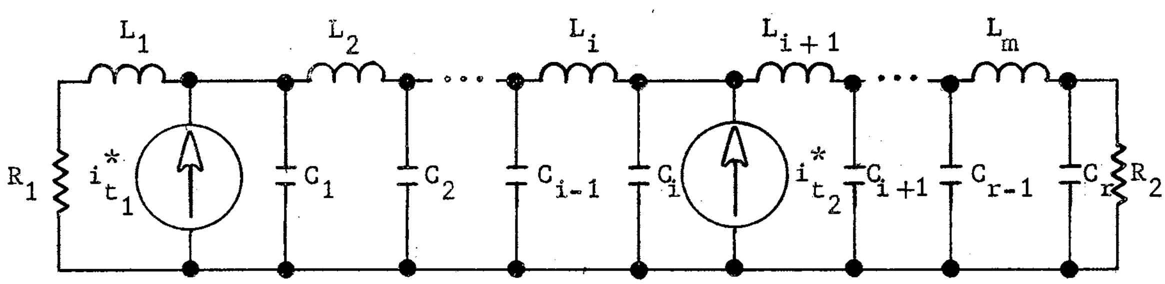

Figure 4.5.5 Case III Realization of $z_{12}(s)$

This set of equations has the solution of

$$C_1' = (a_{x_3})^{\frac{1}{2}};\ i = 1 \tag{4.5.16a}$$

$$L_1' = (a_{x_3})^{-\frac{1}{2}} k_1 k_3 \ldots k_{2i-1};\ i \geq 2 \tag{4.5.16b}$$

where $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = r_1$, and $R_2 = 1/g_2$.

4.5.6 Case IV. This transfer impedance case has numerator and denominator polynomials which are both of even degree. This transfer impedance is written as

$$z_{12}(s) = \frac{a_{x_4} s^{x_4} + a_{x_4-1} s^{x_4-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \tag{4.5.17}$$

where

a_i = positive constant; $i = 0, 1, \ldots, x_4$

x_4 = even integer

$x_4 \neq 0$

n = even integer

$n = m + r$

$m = r$

$i = \frac{n-x_4}{2}$

Using the procedures of Chapter II and III with Equation 4.5.17 results in the network of Figure 4.5.6 and the following set of non-linear equations.

$$a_{x_4} = p_{11} b_{m+i,2} \left[\prod_{j=1}^{2i-1} k_j \right]$$

$$f_o = r_1 (L_1')^2$$

$$k_1 = L_1' C_1'$$

$$\vdots$$

$$k_{n-1} = L_m' C_r'$$

$$f_n = g_2 (C_r')^2 \qquad (4.5.18)$$

$$b_{11} = r_1 L_1'$$

$$b_{m+i,2} = -C_i'$$

$$b_{11} = -p_{11}$$

$$b_{m+i,2} = p_{2,m+i}$$

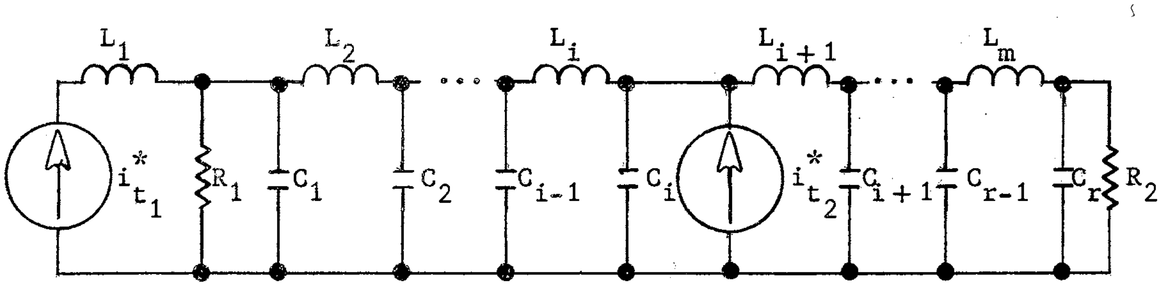

Figure 4.5.6 Case IV Realization of $z_{12}(s)$

This set of equations has the solution of

$$L_1' = \left[\frac{f_o}{a_{x_4}}\right]^{\frac{1}{2}} k_1 k_3 \cdots k_{2i-1} \tag{4.5.19}$$

where $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = r_1$, and $R_2 = 1/g_2$.

4.6 Synthesis of a General Open Circuit Transfer Impedance.

Recall that for a general short circuit transfer admittance, the synthesis procedure as presented in Section 3.9.1 satisfied, in general, only one numerator coefficient of the transfer admittance per ladder network. Also when paralleling these ladder networks, it was sometimes found necessary to add a 1:1 transformer so that the validity test would be satisfied. Or a 1:-1 transformer was sometimes added to obtain positive coefficients in the numerator polynomial.

The synthesis procedure for a general open circuit transfer impedance will be very similar to that for the transfer admittance. Each resulting ladder network will, in general, satisfy only one numerator coefficient of the transfer impedance. If the numerator degree is x_i, then there will be a maximum of $(x_i + 1)$ ladder networks placed in series to satisfy the numerator coefficients. When putting the ladder networks in series, it will sometimes be necessary to add a 1:1 transformer so that the validity test will be satisfied. Sometimes a 1:-1 transformer will have to be added to obtain negative numerator coefficients.

A unified element value synthesis procedure for transfer impedances is very similar to that for transfer admittances as presented in Section 3.13. This procedure for transfer impedances will not be presented here because it is so similar to that for transfer admittances. An open circuit transfer impedance synthesis example will now be presented to illustrate the above procedures.

4.7 Synthesis Example of $z_{12}(s)$. Consider the open circuit transfer impedance of

$$z_{12}(s) = \frac{2s^3 + 5s^2 + 3s + 9}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.1)$$

The four transfer impedances, whose ladder network will be placed in series, are

$$z'_{12}(s) = \frac{a'_3 s^3 + a'_2 s^2 + a'_1 s + a'_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.2a)$$

$$z''_{12}(s) = \frac{a''_2 s^2 + a''_1 s + a''_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.2b)$$

$$z^3_{12}(s) = \frac{a^3_1 s + a^3_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.2c)$$

$$z^4_{12}(s) = \frac{a^4_o}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.2d)$$

where

$$2 = a'_3 \qquad (4.7.2e)$$

$$5 = a'_2 + a''_2 \qquad (4.7.2f)$$

$$3 = a'_1 + a''_1 + a^3_1 \qquad (4.7.2g)$$

$$9 = a'_o + a''_o + a^3_o + a^4_o \qquad (4.7.2h)$$

So as to utilize the material of Section 3.12, a K_2 matrix for the denominator polynomial will be determined as

$$K_2 = \left[\begin{array}{cc|cc} -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ \hline -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{4.7.3}$$

This K_2 matrix will be used in the synthesis of each ladder network of this example.

Using the material of Section 4.5.5, realization of $z'_{12}(s)$ yields the network of Figure 4.7.1 and

$$z'_{12}(s) = \frac{2s^3 + 4s^2 + 10s + 8}{s^4 + 2s^3 + 10s^2 + 10s + 17} \tag{4.7.4}$$

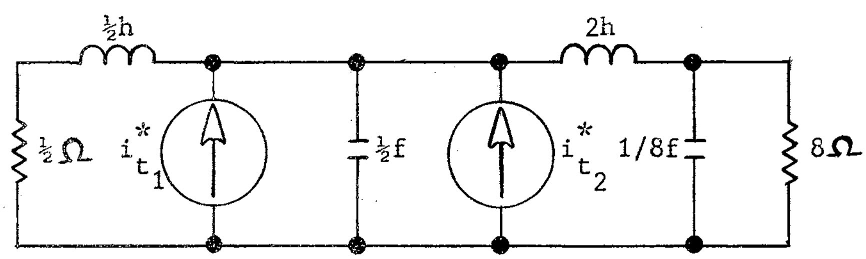

Figure 4.7.1 Network for $z'_{12}(s)$

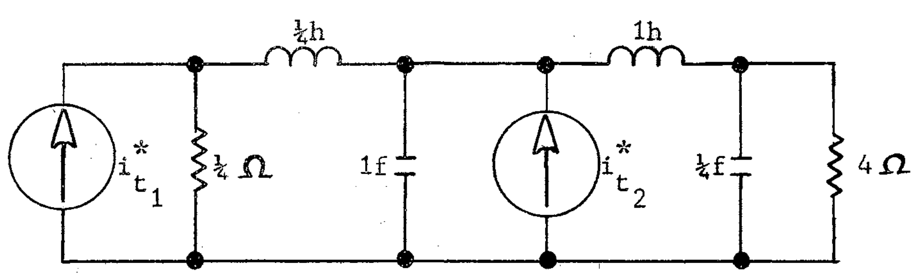

Figure 4.7.2 Network for $z''_{12}(s)$

Using the material of Section 4.5.6 realization of $z''_{12}(s)$ yields the network of Figure 4.7.2 and

$$z''_{12}(s) = \frac{s^2 + s + 4}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.5)$$

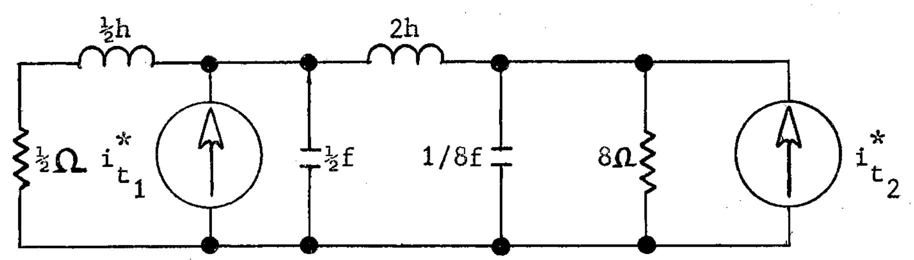

Figure 4.7.3 Network for $z^3_{121}(s)$

Since $z^3_{12}(s)$ must have a negative coefficient, the network of Figure 4.7.3 is first realized. This yields positive coefficients and is labeled $z^3_{121}(s)$. When this ladder network is placed in series with the others, a 1:-1 transformer will be added at one of its ports. When this transformer is added to the network of Figure 4.7.3, its transfer impedance will be

$$z^3_{12}(s) = \frac{-\,8s - 8}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.5)$$

Now $z^4_{12}(s)$ must be

$$z^4_{12}(s) = \frac{5}{s^4 + 2s^3 + 10s^2 + 10s + 17} \qquad (4.7.6)$$

and this is realized using the material of Section 4.5.2 with the network of Figure 4.7.4.

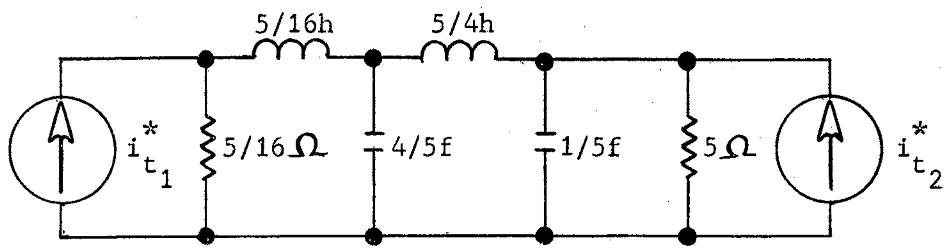

Figure 4.7.4 Network of $z^4_{12}(s)$

The network that synthesizes the transfer impedance of Equation 4.7.1 is given in Figure 4.7.5. The port connections that are shown in this figure are necessary to satisfy the validity test.

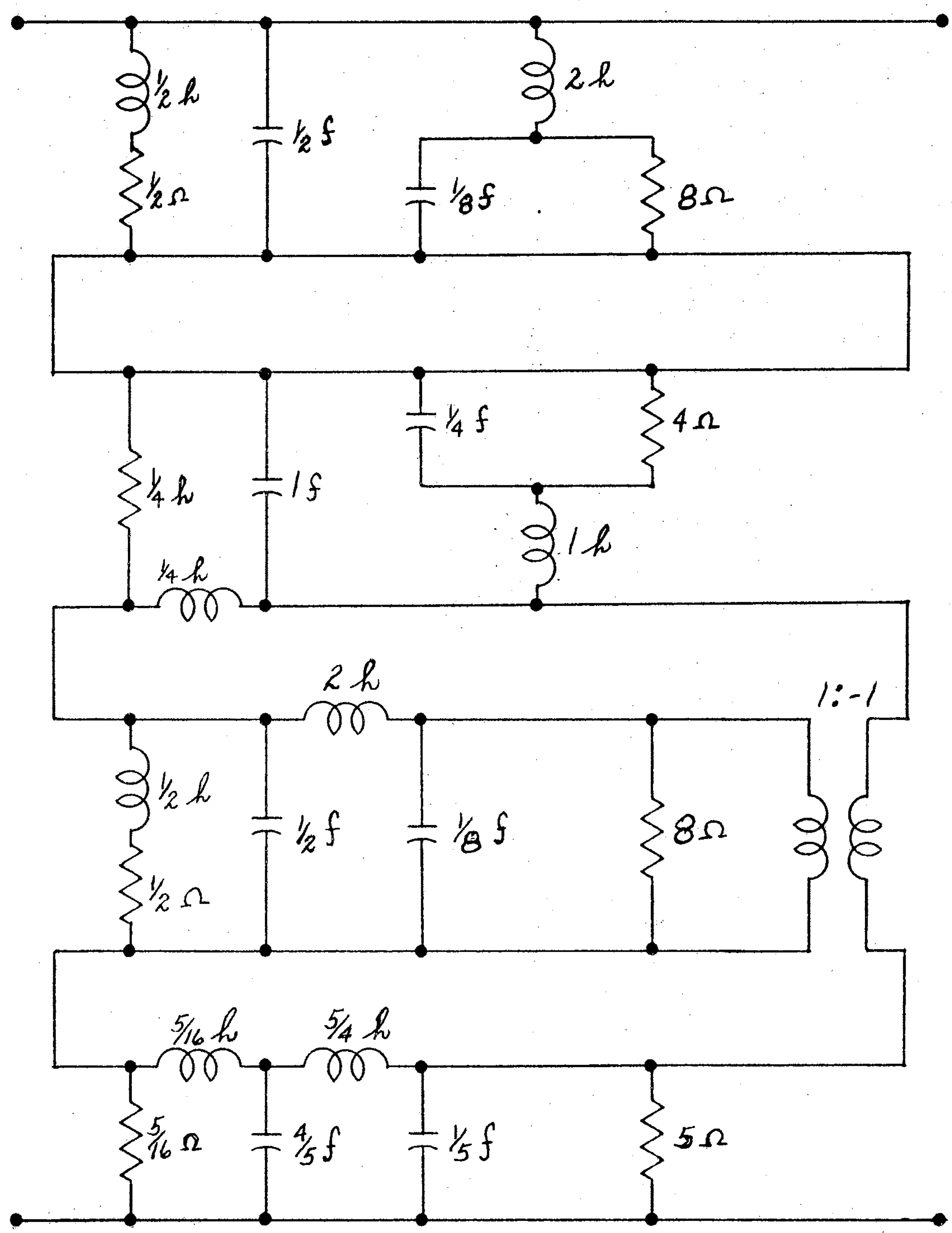

Figure 4.7.5 Network of $z_{12}(s)$

CHAPTER V

SYNTHESIS OF THE VOLTAGE TRANSFER FUNCTION, T(s)

5.1 Introduction. This chapter will briefly present the state-space approach of the voltage transfer function, T(s), using the concepts presented in Chapters II, III, and IV. The brevity of this chapter results from the similarities between the voltage transfer function and the transfer admittances and impedances. Therefore primarily just the results will be presented with the property of similarity being given as the justification.

5.2 Restrictions. The s-domain restrictions for the voltage transfer function, T(s), are:

1. The degree of the numerator polynomial can not be greater than the degree of the denominator (3). If the numerator degree equals the denominator degree, see Section 2.6.3.
2. The coefficients of the numerator polynomial must be real and finite.
3. The denominator polynomial must be a strictly Hurwitz polynomial (32).

The topological restrictions on the network to be synthesized from the proper function, T(s), are:

1. Both branch resistors and chord resistors will not be permitted in the same fundamental circuits.
2. Circuits of capacitors with or without voltage drivers will not be permitted.
3. Cut-sets of inductors with or without current drivers will not be permitted.
4. The network driver configuration must be that of Figure 2.4.3.

<u>5.3 State-Models, K_2-Matrices, and $\mathfrak{D}'_{21}(s)$.</u> It is desirable to recall the state-model developed in Section 2.4.3 to be used in the state-space synthesis of the voltage transfer function. This state-model is given here for ready reference.

$$\frac{d}{dt}\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{12}^T G_c B_{12} C_b^{-\frac{1}{2}} & C_b^{-\frac{1}{2}} B_{22}^T L_c^{-\frac{1}{2}} \\ \hline -L_c^{-\frac{1}{2}} B_{22} C_b^{-\frac{1}{2}} & -L_c^{-\frac{1}{2}} B_{23} R_b B_{23}^T L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix}$$

$$+\left[\begin{array}{c|c} -C_b^{-\frac{1}{2}}B_{12}^T G_c B_{11} & C_b^{-\frac{1}{2}} B_{32}^T \\ \hline -L_c^{-\frac{1}{2}} B_{21} & -L_c^{-\frac{1}{2}} B_{23} R_b B_{33}^T \end{array}\right]\begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \tag{5.3.1a}$$

$$\begin{bmatrix} I_a^* \\ V_t^* \end{bmatrix} = \left[\begin{array}{c|c} B_{11}^T G_c B_{12} C_b^{-\frac{1}{2}} & -B_{21}^T L_c^{-\frac{1}{2}} \\ \hline B_{32} C_b^{-\frac{1}{2}} & B_{33} R_b B_{23}^T L_c^{-\frac{1}{2}} \end{array}\right]\begin{bmatrix} V'_{bc} \\ I'_{c\ell} \end{bmatrix} + \left[\begin{array}{c|c} B_{11}^T G_c B_{11} & -B_{31}^T \\ \hline B_{31} & B_{33} R_b B_{33}^T \end{array}\right]\begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \tag{5.3.1b}$$

The fundamental circuit equations for this state-model are

$$\left[\begin{array}{ccc|ccc} B_{11} & B_{12} & 0 & U & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & U & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 & U \end{array}\right] \left[\begin{array}{c} V_a \\ V_{bc} \\ V_{br} \\ \hline V_{cr} \\ V_{c\ell} \\ V_t \end{array}\right] = 0 \qquad (5.3.2)$$

Writing Equation 5.3.1 in symbolic form as in Equation 2.4.20 yields

$$\frac{d}{dt} X = K_2 X + \mathcal{B}\, Y^* \qquad (5.3.3a)$$

$$\underline{Y}^* = \mathcal{P}\, X + \mathcal{R}\, Y^* \qquad (5.3.3b)$$

Following a procedure like that in Section 2.4.3 yields

$$\mathcal{D}(s) = \mathcal{P}\,(sU-K_2)^{-1}\mathcal{B} \qquad (5.3.4a)$$

$$\mathcal{D}(s) = \begin{bmatrix} \mathcal{D}_{11}(s) & \mathcal{D}_{12}(s) \\ \mathcal{D}_{21}(s) & \mathcal{D}_{22}(s) \end{bmatrix} \qquad (5.3.4b)$$

and as shown in Section 2.5

$$T(s) = \mathcal{D}_{21}(s) \qquad (5.3.5a)$$

$$T(s) = \frac{\mathcal{D}'_{21}(s)}{\Delta} \qquad (5.3.5b)$$

where

$$\Delta = |sU-K_2| \qquad (5.3.5c)$$

and

$$\mathcal{D}'(s) = \mathcal{P}\,[\mathrm{adj}(sU-K_2)]\,\mathcal{B} \qquad (5.3.5d)$$

$$\mathfrak{D}'(s) = \begin{bmatrix} \mathfrak{D}'_{11}(s) & \mathfrak{D}'_{12}(s) \\ \mathfrak{D}'_{21}(s) & \mathfrak{D}'_{22}(s) \end{bmatrix} \qquad (5.3.5e)$$

As in the previous chapter, if the voltage transfer function has an odd denominator degree, then the K_2-matrix will be

$$K_2 = \left[\begin{array}{cccc|cccc} -f_o & & & & k_1 & & & \\ & 0 & & & -k_2 & \ddots & & \\ & & \ddots\ 0 & & & \ddots & \ddots & k_{n-2} \\ & & & -f_n & & & & -k_{n-1} \\ \hline -k_1 & k_2 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -k_{n-2} & k_{n-1} & & & & \end{array}\right] \qquad (5.3.6a)$$

When the denominator degree is even, then the K_2-matrix will be

$$K_2 = \left[\begin{array}{cccc|cccc} -f_o & & & & k_1 & & & \\ & 0 & & & -k_2 & k_3 & & \\ & & \ddots & & & \ddots & \ddots & \\ & & & 0 & & & -k_{n-2} & k_{n-1} \\ \hline -k_1 & k_2 & & & 0 & & & \\ & -k_3 & \ddots & & & \ddots & & \\ & & \ddots & k_{n-2} & & & 0 & \\ & & & -k_{n-1} & & & & -f_n \end{array}\right] \qquad (5.3.6b)$$

Using these K_2-matrices in the same manner as shown in Equations 3.3.8, 3.7.2, and 5.3.5d yields the $\mathfrak{D}'_{21}(s)$ of Equation 5.3.7 when the T(s) denominator degree is odd and that of Equation 5.3.8 when the denominator degree is even.

$$
\begin{aligned}
D'_{21}(s) = \; & (p_{21}a_{11} + p_{22}a_{31} + p_{23}a_{51} + \cdots + p_{2,m}a_{n,1} + p_{2,m+1}a_{21} + p_{2,m+2}a_{41} + \cdots + p_{2,n}a_{n-1,1})b_{11} \\
& + (p_{21}a_{12} + p_{22}a_{32} + p_{23}a_{52} + \cdots + p_{2,m}a_{n,2} + p_{2,m+1}a_{22} + p_{2,m+2}a_{42} + \cdots + p_{2,n}a_{n-1,2})b_{m+1,1} \\
& + (p_{21}a_{13} + p_{22}a_{33} + p_{23}a_{53} + \cdots + p_{2,m}a_{n,3} + p_{2,m+1}a_{23} + p_{2,m+2}a_{43} + \cdots + p_{2,n}a_{n-1,3})b_{21} \\
& + (p_{21}a_{14} + p_{22}a_{34} + p_{23}a_{54} + \cdots + p_{2,m}a_{n,4} + p_{2,m+1}a_{24} + p_{2,m+2}a_{44} + \cdots + p_{2,n}a_{n-1,4})b_{m+2,1} \\
& + (p_{21}a_{15} + p_{22}a_{35} + p_{23}a_{55} + \cdots + p_{2,m}a_{n,5} + p_{2,m+1}a_{25} + p_{2,m+2}a_{45} + \cdots + p_{2,n}a_{n-1,5})b_{31} \\
& \qquad \vdots \\
& + (p_{21}a_{1,n-1} + p_{22}a_{3,n-1} + p_{23}a_{5,n-1} + \cdots + p_{2,m}a_{n,n-1} + p_{2,m+1}a_{2,n-1} + p_{2,m+2}a_{4,n-1} + \cdots + p_{2,n}a_{n-1,n-1})b_{m+r,1} \\
& + (p_{21}a_{1,n} + p_{22}a_{3,n} + p_{23}a_{5,n} + \cdots + p_{2,m}a_{n,n} + p_{2,m+1}a_{2,n} + p_{2,m+2}a_{4,n} + \cdots + p_{2,n}a_{n-1,n})b_{m,1}
\end{aligned}
$$

n = odd integer

$n = m + r$

$m = r + 1$

(5.3.7)

$$
\begin{aligned}
\mathcal{Q}'_{21}(s) = \; & (p_{21}a_{11} + p_{22}a_{31} + p_{23}a_{51} + \cdots + p_{2,m}a_{n-1,1} + p_{2,m+1}a_{21} + p_{2,m+2}a_{41} + \cdots + p_{2,n}a_{n,1})b_{11} \\
& + (p_{21}a_{12} + p_{22}a_{32} + p_{23}a_{52} + \cdots + p_{2,m}a_{n-1,2} + p_{2,m+1}a_{22} + p_{2,m+2}a_{42} + \cdots + p_{2,n}a_{n,2})b_{m+1,1} \\
& + (p_{21}a_{13} + p_{22}a_{33} + p_{23}a_{53} + \cdots + p_{2,m}a_{n-1,3} + p_{2,m+1}a_{23} + p_{2,m+2}a_{43} + \cdots + p_{2,n}a_{n,3})b_{21} \\
& + (p_{21}a_{14} + p_{22}a_{34} + p_{23}a_{54} + \cdots + p_{2,m}a_{n-1,4} + p_{2,m+1}a_{24} + p_{2,m+2}a_{44} + \cdots + p_{2,n}a_{n,4})b_{m+2,1} \\
& + (p_{21}a_{15} + p_{22}a_{35} + p_{23}a_{55} + \cdots + p_{2,m}a_{n-1,5} + p_{2,m+1}a_{25} + p_{2,m+2}a_{45} + \cdots + p_{2,n}a_{n,5})b_{31} \\
& \quad \vdots \\
& + (p_{21}a_{1,n-1} + p_{22}a_{3,n-1} + p_{23}a_{5,n-1} + \cdots + p_{2,m}a_{n-1,n-1} + p_{2,m+1}a_{2,n-1} + p_{2,m+2}a_{4,n-1} + \cdots + p_{2,n}a_{n,n-1})b_{m,1} \\
& + (p_{21}a_{1,n} + p_{22}a_{3,n} + p_{23}a_{5,n} + \cdots + p_{2,m}a_{n-1,n} + p_{2,m+1}a_{2,n} + p_{2,m+2}a_{4,n} + \cdots + p_{2,n}a_{n,n})b_{m+r,1}
\end{aligned}
$$

$n =$ even integer

$n = m + r$

$m = r$

(5.3.8)

Also using the above K_2-matrices to obtain the realizations of the characteristic polynomials of the voltage transfer functions will result, as before, in the networks of Figures 2.7.2 and 2.7.4. These then will be the basic networks in which the drivers will be inserted in the appropriate fashion as to yield the desired numerator and denominator degrees in the voltage transfer function.

5.4 Synthesis of T(s). Since the presentation of the voltage transfer function is so similar to that for the transfer admittance and impedance, the same classification of the proper functions to be synthesized will be used here. It will not be necessary to consider the Case II - Special T(s) since this case will be included in the Case II T(s).

Now each of the classifications will be presented.

5.4.1 Case IV - Special. For this transfer function case, the numerator is a constant and the denominator function is of even degree and is written as

$$T(s) = \frac{a_o}{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_o} \qquad (5.4.1)$$

where

a_o = positive constant

n = even integer

$n = m + r$

$m = r$

A transfer function of the form in Equation 5.4.1 will result if the network drivers are inserted into the network realization of the characteristic polynomial as shown in Figure 5.4.1 (See Appendix D).

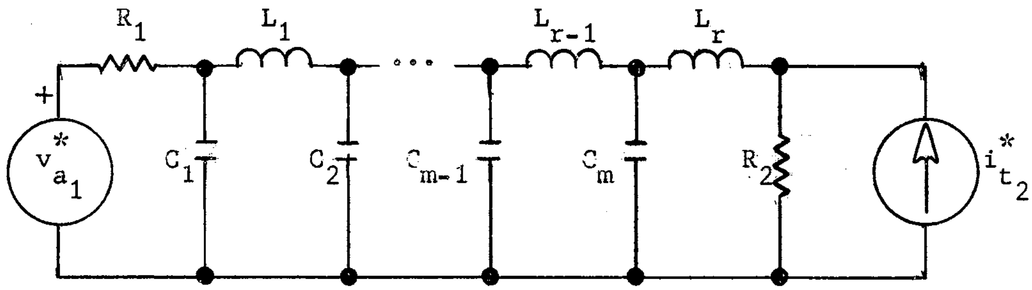

Figure 5.4.1 Case IV - Special Realization of T(s)

This network yields the fundamental circuit equations of

$$\left[\begin{array}{c|ccccc|c|c|cccc|c} -1 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & & & & & \\ \hline 0 & 1 & -1 & & & & 0 & & 1 & & & & \\ 0 & & 1 & \ddots & & & 0 & & & 1 & & & \\ \vdots & & & \ddots & \ddots & & \vdots & & & & \ddots & & \\ 0 & & & & \ddots & -1 & 0 & & & & & 1 & \\ 0 & & & & & 1 & 1 & & & & & & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & -1 & & & & & & 1 \end{array}\right] \left[\begin{array}{c} v_{a_1} \\ \hline v_{bc_1} \\ \vdots \\ v_{bc_m} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ \hline v_{c\ell_1} \\ \vdots \\ v_{c\ell_r} \\ \hline v_{t_2} \end{array}\right] = 0 \qquad (5.4.2)$$

which is written in symbolic form as shown in Equation 5.3.2.

Using these fundamental circuit equations, the state-model of Equation 5.3.1 and the K_2-matrix of Equation 5.3.6b with the procedures of Chapters II and III results in the set of equations given in Equation 3.7.5 which can be written as

$$f_o = g_1(C_1')^2 \qquad (5.4.5a)$$

$$\left.\begin{aligned} k_1 &= C_1' L_1' \\ &\vdots \\ k_{n-1} &= C_m' L_r' \end{aligned}\right\} \tag{5.4.5b}$$

$$f_n = r_2 (L_r')^2 \tag{5.4.5c}$$

$$b_{11} = g_1 C_1' \tag{5.4.5d}$$

$$b_{m+r,2} = r_2 L_r' \tag{5.4.5e}$$

$$b_{11} = -p_{11} \tag{5.4.5f}$$

$$b_{m+r,2} = -p_{2,m+r} \tag{5.4.5g}$$

Using the non-zero b_{ij} and $p_{k\ell}$ of Equation 5.4.5 in Equation 5.3.8 and noting that $n = m + r$, we have

$$\mathfrak{D}_{21}'(s) = p_{2,n} a_{n,1} b_{11} \tag{5.4.6}$$

From Equation 2.5.12

$$a_{n,1} = -a_{1,n} \tag{5.4.7a}$$

and

$$-a_{1,n} = -\left[\prod_{j=1}^{n-1} k_j\right] \tag{5.4.7b}$$

From Equations 5.3.5b, 5.4.1, and 5.4.6

$$a_o = -p_{2,n} b_{11} \left[\prod_{j=1}^{n-1} k_j\right] \tag{5.4.8}$$

Using the procedures of Chapter III, it would appear that Equation 5.4.8 could again be used in the set of (n + 6) non-linear equations that were solved to obtain the synthesized network component values. This is not true as can be seen when Equations 5.4.5a and d are combined to obtain

$$b_{11} = \frac{f_o}{C_1'} \tag{5.4.9}$$

and Equations 5.4.5 b, c, and e are manipulated to obtain

$$p_{2,m+r} = \frac{-f_n k_{n-2} k_{n-4} \cdots k_2 C_1'}{k_{n-1} k_{n-3} \cdots k_1} \tag{5.4.10}$$

Then Equations 5.4.8, 5.4.9, and 5.4.10 imply that

$$a_o = f_o f_n k_{n-2}^2 k_{n-4}^2 \cdots k_2^2 \tag{5.4.11}$$

and in general this will be an inequality if a_o is specified.

The solution to this problem is to use a procedure like that of Section 3.13. By placing a $n_1{:}n_2$ transformer at either port of the synthesized ladder modifies Equation 5.4.8 to

$$a_o = -N\, p_{2,n} b_{11} \left[\prod_{j=1}^{n-1} k_j \right] \tag{5.4.12}$$

where

$$N = \frac{n_2}{n_1}$$

This introduces another unknown to the set of (n + 6) equations; therefore, arbitrarily add another equation such as

$$g_1 = 1 \tag{5.4.13}$$

Now the remaining variables of the set of equations of Equation 5.4.5 can be determined and the network component values are related to these variables by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = 1/g_1$, and $R_2 = r_2$.

<u>5.4.2 Case I.</u> For this transfer function case, the numerator and denominator degrees are both odd and the transfer function is written

$$T(s) = \frac{a_{x_1} s^{x_1} + a_{x_1-1} s^{x_1-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \tag{5.4.14}$$

where

a_j = positive constant; $j = 0, 1, \ldots, x_1$

x_1 = odd integer

n = odd integer

$n = m + r$

$m = r + 1$

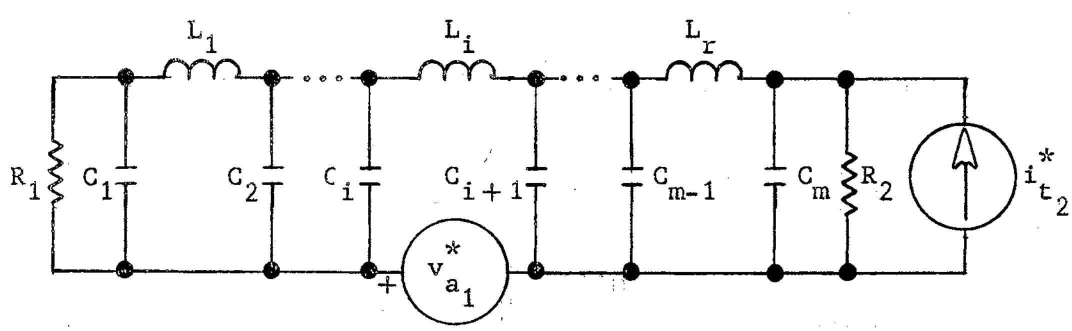

Figure 5.4.2 Case I Realization of T(s)

A transfer function of the form shown in Equation 5.4.14 will result if the network drivers are inserted into the network realization of the characteristic polynomial as shown in Figure 5.4.2 (See Appendix D) where

$$i = \frac{x_1 + 1}{2} \tag{5.4.15}$$

This network yields the fundamental circuit equations of

$$\left[\begin{array}{c|ccccc|c|cc|cccc|c}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & & & & & & \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & & 1 & & & & & \\
\hline
0 & 1 & -1 & & & & 0 & & & 1 & & & & \\
\vdots & & 1 & \ddots & & & 0 & & & & 1 & & & \\
1 & & & \ddots & \ddots & & \vdots & & & & & \ddots & & \\
\vdots & & & & \ddots & -1 & 0 & & & & & & 1 & \\
0 & & & & 1 & -1 & 0 & & & & & & & 1 \\
\hline
0 & 0 & 0 & \cdots & 0 & 1 & 0 & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ \hline
v_{bc_1} \\ \vdots \\ v_{bc_m} \\ \hline
v_{br} \\ \hline
v_{cr_1} \\ v_{cr_2} \\ \hline
v_{c\ell_1} \\ \vdots \\ v_{c\ell_r} \\ \hline
v_{t_2}
\end{array}\right] = 0 \tag{5.4.16}$$

which is written in symbolic form as shown in Equation 5.3.2.

Using these fundamental circuit equations, the state-model of Equation 5.3.1 and the K_2-matrix of Equation 5.3.6a with the procedures of Chapters II and III results in the set of equations given in Equation 3.4.2 which can be manipulated to yield

$$f_o = g_1(C_1')^2 \tag{5.4.17a}$$

$$f_n = g_2(C_m')^2 \tag{5.4.17b}$$

$$\left.\begin{aligned} k_1 &= C_1' L_1' \\ &\vdots \\ k_{n-1} &= C_m' L_r' \end{aligned}\right\} \tag{5.4.17c}$$

$$b_{m,2} = C_m' \tag{5.4.17d}$$

$$b_{m+i,1} = -L_i' \tag{5.4.17e}$$

$$b_{m,2} = p_{2,m} \tag{5.4.17f}$$

$$b_{m+i,1} = p_{1,m+i} \tag{5.4.17g}$$

Using the non-zero b_{ij} and $p_{k\ell}$ of Equation 5.4.17 in Equation 5.3.7 yields

$$\mathfrak{D}_{211}'(s) = p_{2,m} a_{n,2i} b_{m+i,1} \tag{5.4.18a}$$

$$\mathfrak{D}_{211}'(s) = c_{x_1} s^{x_1} + \ldots + c_1 s + c_o \tag{5.4.18b}$$

From Equation 2.5.12 it can be seen that

$$a_{n,2i} = (-1)^{n+2i} a_{2i,n} \tag{5.4.19a}$$

and this implies that

$$a_{n,2i} = (-1) a_{2i,n} \tag{5.4.19b}$$

Using a procedure similar to that of Section 3.9.1 with Equations 2.5.12, 5.4.18, and 5.4.19 yields

$$c_{x_1} = (-1) p_{2,m} b_{m+i,1} \left[\prod_{j=1}^{n-2i} k_{n-j} \right] \tag{5.4.20}$$

As for the Case IV - Special transfer function, if a_{x_1} is equated to the c_{x_1} of Equation 5.4.20, in general, there will result an inequality. As in Section 5.4.1, the solution is to place a $n_{11}:n_{21}$ transformer at either port of the synthesized ladder and this results in

$$a_{x_1} = (-1)N_1 p_{2,m} b_{m+i,1}\left[\prod_{j=1}^{n-2i} k_{n-j}\right] \tag{5.4.21}$$

where

$$N_1 = \frac{n_{21}}{n_{11}}$$

and with an arbitrary choice for an added equation of

$$g_1 = 1 \tag{5.4.22}$$

Now the remaining variables of the set of equations of Equation 5.4.17 can be determined and the network component values are related to these variables by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, and $R_i = 1/g_i$.

5.4.3 Case II. For this transfer function case, the numerator degree is even and the denominator degree is odd with the transfer function written as

$$T(s) = \frac{a_{x_2} s^{x_2} + a_{x_2-1} s^{x_2-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \tag{5.4.23}$$

where

a_j = positive constants; $j = 0, 1, \ldots, x_2$

x_2 = odd integer including zero

n = odd integer

$$n = m + r$$

$$m = r + 1$$

A transfer function of the form shown in Equation 5.4.23 will result if the network drivers are inserted into the network realization of the characteristic polynomial as shown in Figure 5.4.3 (See Appendix D) where

$$i = \frac{n - x_2 + 1}{2} \tag{5.4.24}$$

This network yields the fundamental circuit equations of

$$\left[\begin{array}{c|ccccc|c|cc|ccccc|c}
-1 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & & 1 & & & & & & \\
\hline
0 & 1 & -1 & & & & 0 & & & 1 & & & & & \\
0 & & 1 & \ddots & & & 0 & & & & 1 & & & & \\
\vdots & & & \ddots & \ddots & & \vdots & & & & & \ddots & & & \\
0 & & & & \ddots & -1 & 0 & & & & & & 1 & & \\
0 & & & & 1 & -1 & 0 & & & & & & & 1 & \\
\hline
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ \hline
v_{bc_1} \\ \vdots \\ v_{bc_m} \\ \hline
v_{br} \\ \hline
v_{cr_1} \\ v_{cr_2} \\ \hline
v_{c\ell_1} \\ \vdots \\ v_{c\ell_m} \\ \hline
v_{t_2}
\end{array}\right] = 0 \tag{5.4.25}$$

which is written in symbolic form as shown in Equation 5.3.2.

Using these fundamental circuit equations, the state-model of Equation 5.3.1 and the K_2-matrix of Equation 5.3.6a with the procedures of Chapters II and III results in the set of equations given in Equation 3.4.2 which can be manipulated to yield

$$f_o = g_1(C_1')^2 \tag{5.4.25a}$$

$$f_n = g_2(C_m')^2 \tag{5.4.25b}$$

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ &\vdots \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{5.4.25c}$$

$$b_{11} = g_1C_1' \tag{5.4.25d}$$

$$b_{i,2} = C_i' \tag{5.4.25e}$$

$$b_{11} = -p_{11} \tag{5.4.25f}$$

$$b_{i,2} = p_{2,i} \tag{5.4.25g}$$

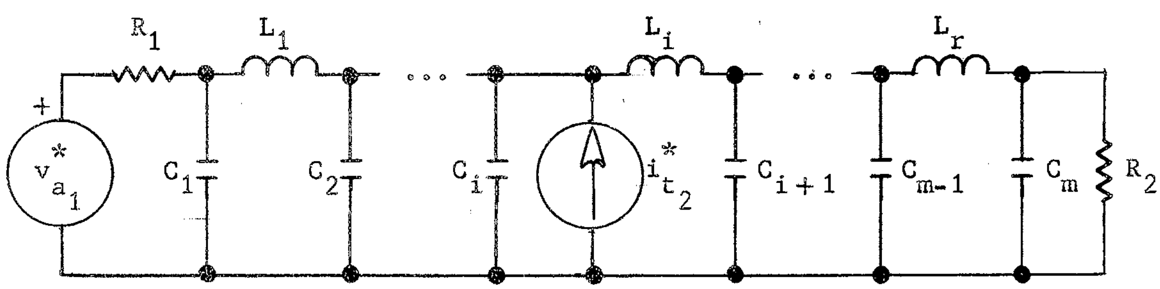

Figure 5.4.3 Case II Realization of T(s)

Using the non-zero b_{ij} and $p_{k\ell}$ of Equation 5.4.25 in Equation 5.3.7 yields

$$\mathfrak{D}_{211}'(s) = p_{2,i}a_{2i-1,1}b_{11} \tag{5.4.26a}$$

$$\mathfrak{D}'_{211}(s) = c_{x_2} s^{x_2} + \ldots + c_1 s + c_o \qquad (5.4.26b)$$

From Equation 2.5.12 it can be seen that

$$a_{2i-1,1} = (-1)^{2(n+1-i)} a_{1,2i-1} \qquad (5.4.27a)$$

and this implies that

$$a_{2i-1,1} = a_{1,2i-1} \qquad (5.4.27b)$$

Using a procedure similar to that of Section 3.9.2 with Equations 2.5.12, 5.4.26, and 5.4.27 yields

$$c_{x_2} = p_{2,i} b_{11}; \; i = 1 \qquad (5.4.28a)$$

$$c_{x_2} = p_{2,i} b_{11} \left[\prod_{j=1}^{2(i-1)} k_j \right]; \; i \geq 2 \qquad (5.4.28b)$$

As for the Case I transfer function, if a_{x_2} is equated to the c_{x_2} of Equation 5.4.28, in general, there will result an inequality. As in Section 5.4.2, the solution is to place a $n_{12}:n_{22}$ transformer at either port of the synthesized ladder and this results in

$$a_{x_2} = N_2 p_{2,i} b_{11}; \; i = 1 \qquad (5.4.29a)$$

$$a_{x_2} = N_2 p_{2,i} b_{11} \left[\prod_{j=1}^{2(i-1)} k_j \right]; \; i \geq 2 \qquad (5.4.29b)$$

where

$$N_2 = \frac{n_{22}}{n_{12}}$$

and with an arbitrary choice for an added equation of

$$g_1 = 1 \tag{5.4.30}$$

Now the remaining variables of the set of equations of Equation 5.4.25 can be determined and the network component values are related to these variables by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, and $R_i = 1/g_i$.

5.4.4 Case III. For this transfer function case, the numerator degree is odd and the denominator degree is even with the transfer function written as

$$T(s) = \frac{a_{x_3} s^{x_3} + a_{x_3 - 1} s^{x_3 - 1} + \dots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_o} \tag{5.4.31}$$

where

a_j = positive constant; $j = 0, 1, \dots, x_3$

x_3 = odd integer

n = even integer

$n = m + r$

$m = r$

A transfer function of the form shown in Equation 5.4.31 will result if the network drivers are inserted into the network realization of the characteristic polynomial as shown in Figure 5.4.4 (See Appendix D.) where

$$i = \frac{x_1 + 1}{2} \tag{5.4.32}$$

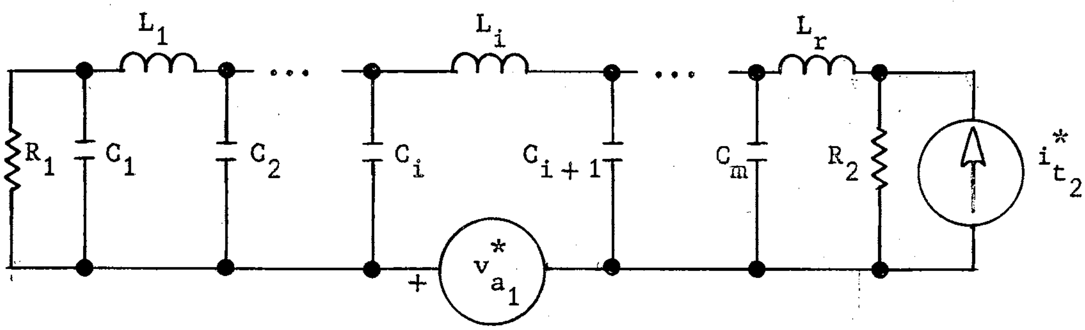

Figure 5.4.4 Case III Realization of T(s)

This network yields the fundamental circuit equations of

$$\left[\begin{array}{c|ccccc|c|c|ccccc|c}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & & & & & & \\ \hline
0 & 1 & -1 & & & & 0 & & 1 & & & & & \\
\vdots & & 1 & \ddots & & & 0 & & & 1 & & & & \\
1 & & & \ddots & \ddots & & \vdots & & & & \ddots & & & \\
\vdots & & & & \ddots & -1 & 0 & & & & & 1 & & \\
0 & & & & & 1 & 1 & & & & & & 1 & \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & -1 & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ \hline v_{bc_1} \\ \vdots \\ v_{bc_m} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ \hline v_{c\ell_1} \\ \vdots \\ v_{c\ell_r} \\ \hline v_{t_2}
\end{array}\right] = 0$$

(5.4.33)

which is written in symbolic form as shown in Equation 5.3.2.

Using these fundamental circuit equations, the state-model of Equation 5.3.1 and the K_2-matrix of Equation 5.3.6b with the procedures of Chapters II and III results in the set of equations given in Equation 3.7.5 which can be manipulated to yield

$$f_o = g_1(C_1')^2 \tag{5.4.34a}$$

$$\left.\begin{aligned} k_1 &= C_1'L_1' \\ &\vdots \\ k_{n-1} &= C_m'L_r' \end{aligned}\right\} \tag{5.4.34b}$$

$$f_n = r_2(L_r')^2 \tag{5.4.34c}$$

$$b_{m+i,1} = -L_i' \tag{5.4.34d}$$

$$b_{m+r,2} = r_2L_r' \tag{5.4.34e}$$

$$b_{m+i,1} = p_{1,m+i} \tag{5.4.34f}$$

$$b_{m+r,2} = -p_{2,m+r} \tag{5.4.34g}$$

Using the non-zero b_{ij} and $p_{k\ell}$ of Equation 5.4.34 in Equation 5.3.8 and noting that $n = m + r$, we have

$$\mathfrak{D}_{211}'(s) = p_{2,n}a_{n,2i}b_{m+i,1} \tag{5.4.35a}$$

$$\mathfrak{D}_{211}'(s) = c_{x_3}s^{x_3} + \ldots + c_1s + c_o \tag{5.4.35b}$$

From Equation 2.5.12 it can be seen that

$$a_{n,2i} = (-1)^{n+2i}a_{2i,n} \tag{5.4.36a}$$

and this implies that

$$a_{n,2i} = a_{2i,n} \tag{5.4.36b}$$

Using a procedure similar to that of Section 3.9.3 with Equations 2.5.12, 5.4.35, and 5.4.36 yields

$$c_{x_3} = p_{2,n} b_{m+i,1} \; ; \; i = r \tag{5.4.37a}$$

$$c_{x_3} = p_{2,n} b_{m+i,1} \left[\prod_{j=1}^{n-2i} k_{n-j} \right] \; ; \; i < r \tag{5.4.37b}$$

As for the Case II transfer function, if a_{x_3} is equated to the c_{x_3} of Equation 5.4.37, in general, there will result an inequality. As in Section 5.4.3, the solution is to place a n_{13}:n_{23} transformer at either port of the synthesized ladder and this results in

$$a_{x_3} = N_3 p_{2,n} b_{m+i,1} \; ; \; i = r \tag{5.4.38a}$$

$$a_{x_3} = N_3 p_{2,n} b_{m+i,1} \left[\prod_{j=1}^{n-2i} k_{n-j} \right] \; ; \; i < r \tag{5.4.38b}$$

where

$$N_3 = \frac{n_{23}}{n_{13}}$$

and with an arbitrary choice for an added equation of

$$g_1 = 1 \tag{5.4.39}$$

Now the remaining variables of the set of equations of Equation 5.4.34 can be determined and the network component values are related to these variables by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = 1/g_1$ and $R_2 = r_2$.

5.4.5 Case IV. For this transfer function case, the numerator and denominator degrees are both even and the transfer function is written as

$$T(s) = \frac{a_{x_4} s^{x_4} + a_{x_4-1} s^{x_4-1} + \ldots + a_1 s + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_o} \qquad (5.4.40)$$

where

a_j = positive constants; j = 0, 1, ..., x_4

x_4 = even integer

$x_4 \neq 0$

n = even integer

n = m + r

m = r

A transfer function of the form shown in Equation 5.4.40 will result if the network drivers are inserted into the network realization of the characteristic polynomial as shown in Figure 5.4.5 (See Appendix D.) where

$$i = \frac{n - x_4}{2} + 1 \qquad (5.4.41)$$

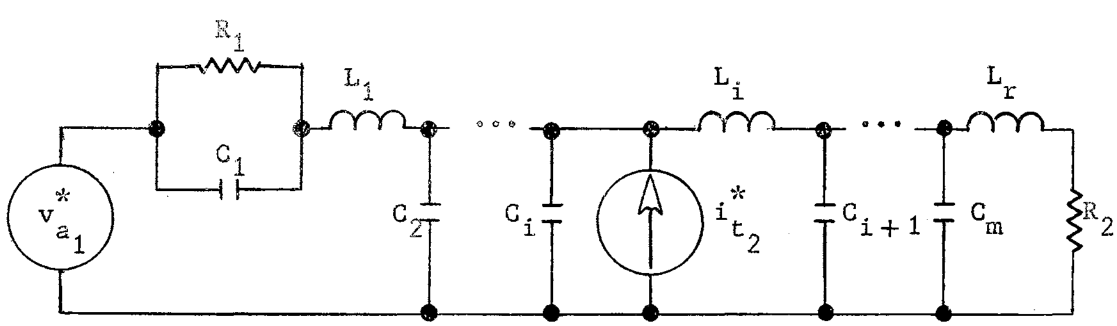

Figure 5.4.5 Case IV Realization of T(s)

This network yields the fundamental circuit equation of

$$\left[\begin{array}{c|ccccc|c|c|ccccc|c}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & & & & & & \\
\hline
1 & 1 & -1 & & & & 0 & & 1 & & & & & \\
0 & & 1 & \ddots & & & 0 & & & 1 & & & & \\
\vdots & & & \ddots & \ddots & & \vdots & & & & \ddots & & & \\
0 & & & & \ddots & -1 & 0 & & & & & 1 & & \\
0 & & & & & 1 & 1 & & & & & & 1 & \\
\hline
0 & 0 & \cdots & 1 & \cdots & 0 & 0 & & & & & & & 1
\end{array}\right]
\left[\begin{array}{c}
v_{a_1} \\ \hline v_{bc_1} \\ \vdots \\ v_{bc_m} \\ \hline v_{br_2} \\ \hline v_{cr_1} \\ \hline v_{c\ell_1} \\ \vdots \\ v_{c\ell_m} \\ \hline v_{t_2}
\end{array}\right] = 0 \qquad (5.4.42)$$

which is written in symbolic form as shown in Equation 5.3.2.

Using these fundamental circuit equations, the state-model of Equation 5.3.1 and the K_2-matrix of Equation 5.3.6b with the procedures of Chapters II and III results in the set of equations given in Equation 3.7.5 which can be manipulated to yield

$$f_o = g_1(C_1')^2 \qquad (5.4.43a)$$

$$\left.\begin{array}{l} k_1 = C_1'L_1' \\ \quad\vdots \\ k_{n-1} = C_m'L_r' \end{array}\right\} \qquad (5.4.43b)$$

$$f_n = r_2(L_r')^2 \qquad (5.4.43c)$$

$$b_{m+1,1} = -L_1' \qquad (5.4.43d)$$

$$b_{i,2} = C_i' \qquad (5.4.43e)$$

$$b_{m+1,1} = p_{1,m+1} \qquad (5.4.43f)$$

$$b_{i,2} = p_{2,i} \qquad (5.4.43g)$$

Using the non-zero b_{ij} and $p_{k\ell}$ of Equation 5.4.43 in Equation 5.3.8 yields

$$\mathfrak{D}'_{211}(s) = p_{2,i}a_{2i-1,2}b_{m+1,1} \qquad (5.4.44a)$$

$$\mathfrak{D}'_{211}(s) = c_{x_4}s^{x_4} + \ldots + c_1 s + c_o \qquad (5.4.44b)$$

From Equation 2.5.12 it can be seen that

$$a_{2i-1,2} = (-1)^{2(n-i)+1}a_{2,2i-1} \qquad (5.4.45a)$$

and this implies that

$$a_{2i-1,2} = (-1)a_{2,2i-1} \qquad (5.4.45b)$$

Using a procedure similar to that of Section 3.9.4 with Equations 2.5.12, 5.4.44, and 5.4.45 yields

$$c_{x_4} = (-1)p_{2,i}b_{m+1,1}\left[\prod_{j=2}^{2(i-1)} k_j\right] \qquad (5.4.46)$$

As for the Case III transfer function, if a_{x_4} is equated to the c_{x_4} of Equation 5.4.46, in general, there will result an inequality. As in Section 5.4.4, the solution is to place a $n_{14}:n_{24}$ transformer at either port of the synthesized ladder and this results in

$$a_{x_4} = (-1)N_4 p_{2,i} b_{m+1,1} \left[\prod_{j=2}^{2(i-1)} k_j \right] \qquad (5.4.47)$$

where

$$N_4 = \frac{n_{24}}{n_{14}}$$

and with an arbitrary choice for an added equation of

$$g_1 = 1 \qquad (5.4.48)$$

Now the remaining variables of the set of equations of Equation 5.4.43 can be determined and the network component values are related to these variables by $C_i = (C_i')^{-2}$, $L_i = (L_i')^{-2}$, $R_1 = 1/g_1$, and $R_2 = r_2$.

5.5 Synthesis of a General Voltage Transfer Function. Recall that for a general short circuit transfer admittance the synthesis procedure as presented in Section 3.9.1 satisfied, in general, only one numerator coefficient of the transfer admittance per ladder network. Also when paralleling these ladder networks, it was sometimes found necessary to add a $n_1{:}n_2$ transformer to either satisfy the validity test or to obtain positive coefficients in the numerator polynomial.

The synthesis procedure for a general voltage transfer function will be very similar to that given for transfer admittances in Section 3.9.1. Each resulting ladder network will, in general, satisfy only one numerator coefficient of the transfer function. If the numerator degree is x_i, then there will be a maximum of $(x_i + 1)$ ladder networks placed in parallel-series to satisfy the numerator coefficients (33). When connecting the ladder networks in parallel-series, it will not be necessary to consider the validity test since each ladder will contain a transformer. However it may be necessary sometimes to change the positive turns ratio on a transformer to a negative turns ratio so as to obtain negative numerator coefficients.

This synthesis procedure will result in a unified element value network if the same equation is added for each ladder synthesis as presented in Section 3.13.

A voltage transfer function synthesis example will now be presented to illustrate the above procedures.

5.6 Synthesis Example of T(s). Consider the voltage transfer function of

$$T(s) = \frac{4s^2 + 8s + 3}{s^3 + 5s^2 + 17s + 25} \tag{5.6.1}$$

So as to utilize previous material, the K_2-matrix of Equation 3.6.2 will be used with the procedures of Sections 5.4.2 and 5.4.3. These procedures yield the three transfer functions of

$$T_1(s) = \frac{4s^2 + 16s + 36}{s^3 + 5s^2 + 17s + 25} \tag{5.6.2a}$$

with

$$N_2' = 4$$

$$T_2(s) = \frac{-8s - 8}{s^3 + 5s^2 + 17s + 25} \tag{5.6.2b}$$

with

$$N_1 = -8/9$$

$$T_3(s) = \frac{-25}{s^3 + 5s^2 + 17s + 25} \tag{5.6.2c}$$

with

$$N_2'' = \frac{-25}{9}$$

and the synthesized network shown in Figure 5.6.1.

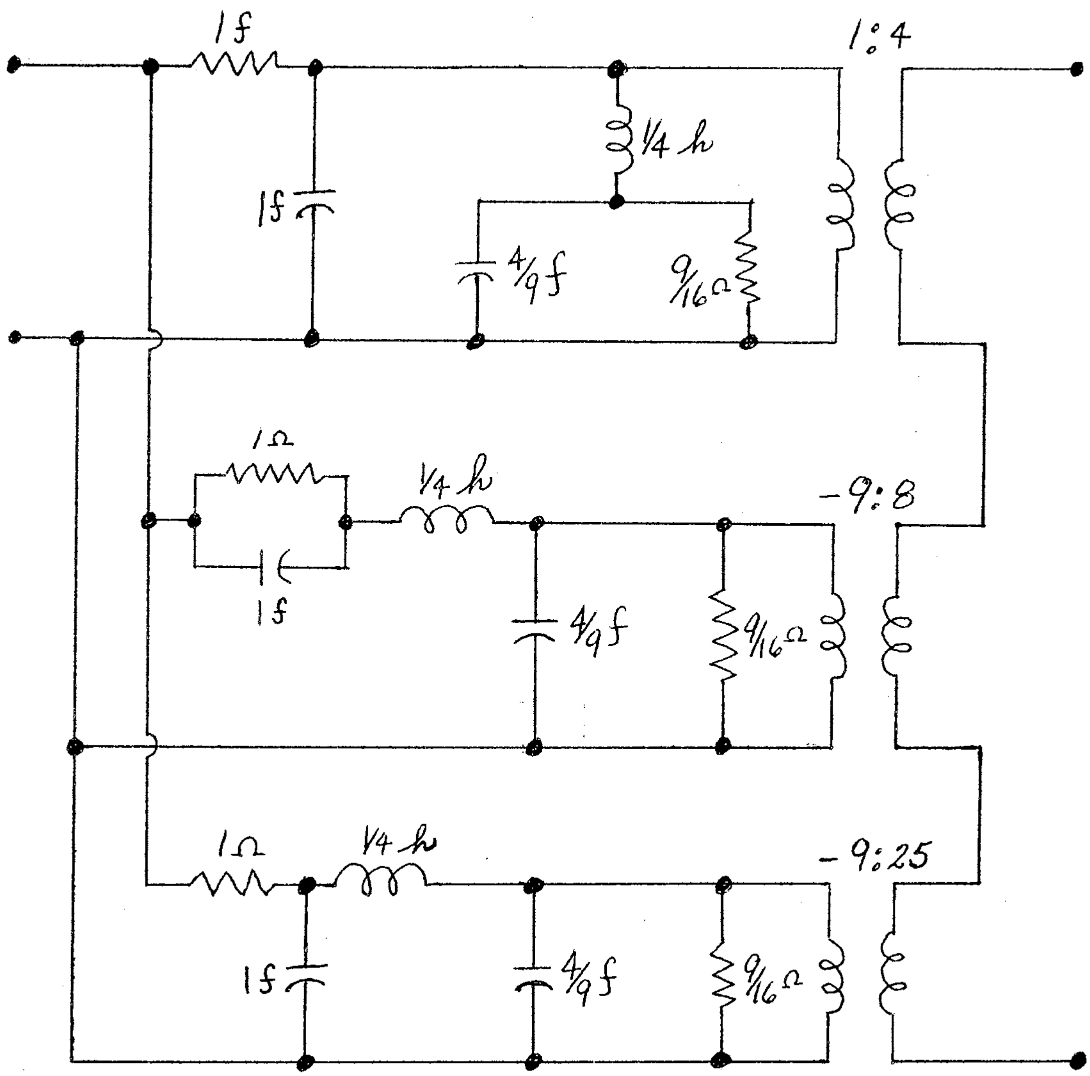

Figure 5.6.1 Realization of Example T(s)

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary. The objective of this study was to develope a state-space approach to the synthesis of the following transfer functions; short circuit transfer admittance, open circuit transfer impedance, and voltage transfer function.

To develope this synthesis procedure, in Chapter II it was necessary to derive an algorithm that would yield $(sU-K_1)^{-1}$, where K_1 is a tridiagonal matrix that represents a ladder network. Also in this chapter it was necessary to present the realization procedures for a characteristic polynomial that yields a ladder network with two resistors.

Chapters III, IV, and V present a unified state-space synthesis procedure for the realization of $y_{12}(s)$, $z_{12}(s)$, and $T(s)$ with the following properties:

1. This realization results in nonminimal networks made up of ladder networks connected in parallel, series or parallel-series and with transformers.
2. There are restrictions on the s-domain transfer functions to be synthesized.
3. There are restrictions on the topology of the resulting synthesized network.

4. There are no restrictions on the transmission zeros of the transfer functions.

5. In general the resulting networks are not minimal except for the Special Cases.

6. Each ladder network is planar, but in general, the total resulting network will not be planar.

7. It is possible to obtain all of the ladder networks in the total synthesized network with unified component values.

8. There are special cases that can be realized with one resistor per ladder network.

9. The network driver configuration for each transfer function is presented. Also since the location of the drivers in each ladder network is not unique, an acceptable location is presented.

10. A tridiagonal K_1-matrix is utilized with the derived state-model.

11. This synthesis procedure is programmable for the digital computer.

6.2 Conclusions. It is possible to derive a unified state-space procedure for synthesizing s-domain transfer functions. This procedure determines a state-model that can be represented by a RLC network with or without transformers and the network component values can be determined. Also one procedure is used for all three types of s-domain transfer functions.

The four advantageous results of this synthesis procedure are:

1. This is a unified synthesis procedure.

2. There are no restrictions on the transfer function transmission zeros.

3. This synthesis procedure can be programmed for the digital computer.

4. No previous knowledge of graph theory or state-models is necessary to use this synthesis procedure.

The two disadvantages in the results obtained are:

1. The synthesized networks are not minimal.

2. Transformers are required in most cases.

6.3 Recommendations for Further Study. An interesting extension of this study would be to determine a K_2-matrix that would represent lattice networks and then determine if this would eliminate the requirement of needing transformers.

Another idea to investigate is whether for a given state-model it is possible to be transformed into a recognizable form that yields paralleled ladder networks.

A straight forward investigation would be the determination of what types of transfer functions can be synthesized by one resistor ladder networks, by permuted state-vectors in the derived state-models, or by inserting the drivers into different locations of the ladder networks than were presented in this study.

Another area of investigation is to determine how Navot's method of Appendix B might be manipulated to regulate the values of the network components in the synthesized ladder networks.

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APPENDIX A

GENERAL STATE-MODEL REPRESENTATION FOR RLC NETWORKS

Recently the following has been given which presents a general state-model representation for RLC networks (35).

Consider an n-port network consisting of two-terminal RLC components only, and let the type of the drivers (voltage or current) at the ports be specified. The state-model for such a network has been derived by several authors, and the explicit expressions for the state-model in terms of a general circuit equation are given in several papers (6,5). In this appendix the results and details are given that can be obtained from one of the references or that can be derived.

By a proper selection of the tree, the fundamental circuit equations for an RLC network can be written as:

$$\left[\begin{array}{cccc|cccc} B_{11} & B_{12} & 0 & 0 & U & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 & U & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} & 0 & 0 & U & 0 \\ B_{41} & B_{42} & B_{43} & B_{44} & 0 & 0 & 0 & U \end{array}\right] \begin{bmatrix} V_a \\ V_{bc} \\ V_{br} \\ V_{b\ell} \\ \hline V_{cc} \\ V_{cr} \\ V_{c\ell} \\ V_t \end{bmatrix} = 0 \qquad (A.1)$$

where V_a-voltage source vector, V_{bc}-voltage vector of branch capacitors, V_{br}-voltage vector of branch resistors, $V_{b\ell}$-voltage vector of branch inductors, V_{cc}-voltage vector of chord capacitors, V_{cr}-voltage vector of chord resistors, $V_{c\ell}$-voltage vector of chord inductors, and V_t-voltage vector of current sources. In Equation A.1 U represents the identity matrix. The cut-set equations can be written in terms of their complementary variables. Let the terminal equations of the components be written in the form:

$$\begin{bmatrix} C_b & & & \\ & L_c & & \\ & & L_b & \\ & & & C_c \end{bmatrix} \frac{d}{dt} \begin{bmatrix} V_{bc} \\ I_{c\ell} \\ I_{b\ell} \\ V_{cc} \end{bmatrix} = \begin{bmatrix} I_{bc} \\ V_{c\ell} \\ V_{b\ell} \\ I_{cc} \end{bmatrix} \qquad (A.2)$$

and

$$\begin{bmatrix} V_{br} \\ I_{cr} \end{bmatrix} = \begin{bmatrix} R_b & 0 \\ 0 & G_c \end{bmatrix} \begin{bmatrix} I_{br} \\ V_{cr} \end{bmatrix} \qquad (A.3)$$

where C_b, C_c, L_b, L_c, R_b, and G_c are diagonal matrices, having positive entries. The system of Equations A.1-A.3 can always be reduced to the state-model form by eliminating the variables I_{bc}, $I_{b\ell}$, I_{br}, I_{cr}, $V_{c\ell}$, V_{cc}, V_{br}, and V_{cr} using the circuit and cut-set equations.

If we let $I_t^* = I_t$, $V_t^* = -V_t$, $V_a^* = V_a$, and $I_a^* = -I_a$, where the star variables represent the vectors of terminal variables for an n-port RLC network, the final form of the state-model is

$$\frac{d}{dt}\begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix} = \left[\begin{array}{c|c} -(C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{22} & (C_b + B_{12}^T C_c B_{12})^{-1}[B_{32}^T - B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b B_{33}^T] \\ \hline (L_c + B_{34} L_b B_{34}^T)^{-1}[B_{33} R_b B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{22} - B_{32}] & -(L_c + B_{34} L_b B_{34}^T)^{-1} B_{33} R_b [U - B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b] B_{33}^T \end{array}\right] \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} -(C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{21} & -(C_b + B_{12}^T C_c B_{12})^{-1}[B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b B_{43}^T - B_{42}^T] \\ \hline (L_c + B_{34} L_b B_{34}^T)^{-1}[B_{33} R_b B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{21} - B_{31}] & -(L_c + B_{34} L_b B_{34}^T)^{-1} B_{33} R_b [U - B_{23}^T (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b] B_{43}^T \end{array}\right] \begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} -(C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11} & 0 \\ \hline 0 & -(L_c + B_{34} L_b B_{34}^T)^{-1} B_{34} L_b B_{44}^T \end{array}\right] \frac{d}{dt}\begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \qquad \text{(A.3a)}$$

$$\begin{bmatrix} I_a^* \\ V_t^* \end{bmatrix} = \left[\begin{array}{c|c} \begin{gathered} B_{21}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{22} \\ - B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{22} \end{gathered} & \begin{gathered} -B_{31}^T + B_{21}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b B_{33}^T \\ + B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1}[B_{32}^T \\ - B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b B_{33}^T] \end{gathered} \\ \hline \begin{gathered} B_{44} L_b B_{34}^T (L_c + B_{34} L_b B_{34}^T)^{-1}[B_{33} R_b B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{22} - B_{32}] \\ - B_{43} R_b B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{22} + B_{42} \end{gathered} & \begin{gathered} B_{43} R_b [U - B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b] B_{33}^T \\ - B_{44} L_b B_{34}^T (L_c + B_{34} L_b B_{34}^T)^{-1} B_{33} R_b [U \\ - B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b] B_{33}^T \end{gathered} \end{array}\right] \begin{bmatrix} V_{bc} \\ I_{c\ell} \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} \begin{gathered} B_{21}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{21} \\ - B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{22}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{21} \end{gathered} & \begin{gathered} B_{21}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b B_{43}^T - B_{41}^T \\ - B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1}[B_{22}^T G_c (U \\ + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b B_{43}^T - B_{42}^T] \end{gathered} \\ \hline \begin{gathered} B_{41} - B_{43} R_b B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{21} \\ + B_{44} L_b B_{34}^T (L_c + B_{34} L_b B_{34}^T)^{-1}[B_{33} R_b B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{21} \\ - B_{31}] \end{gathered} & \begin{gathered} B_{43} R_b [U - B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b] B_{43}^T \\ - B_{44} L_b B_{34}^T (L_c + B_{34} L_b B_{34}^T)^{-1} B_{33} R_b [U \\ - B_{23}^T G_c (U + B_{23} R_b B_{23}^T G_c)^{-1} B_{23} R_b] B_{43}^T \end{gathered} \end{array}\right] \begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix}$$

$$+ \left[\begin{array}{c|c} B_{11}^T C_c B_{11} - B_{11}^T C_c B_{12} (C_b + B_{12}^T C_c B_{12})^{-1} B_{12}^T C_c B_{11} & 0 \\ \hline 0 & -B_{44} L_b B_{34}^T (L_c + B_{34} L_b B_{34}^T)^{-1} B_{34} L_b B_{44}^T + B_{44} L_b B_{44}^T \end{array}\right] \frac{d}{dt}\begin{bmatrix} V_a^* \\ I_t^* \end{bmatrix} \qquad \text{(A.3b)}$$

APPENDIX B

TRIDIAGONAL MATRICES

B.1 Introduction. Several authors have been interested in the close association between ladder networks and tridiagonal matrices (27, 34, 26). This appendix is concerned with presenting Navot's (27) method of obtaining a tridiagonal matrix from a strictly Hurwitz polynomial that has eigenvalues which are equal to the roots of the polynomial.

B.2 Navot's Method. Given a strictly Hurwitz polynomial, called the primary polynomial,

$$H_n(s) = s^n + h_{n-1}s^{n-1} + \dots + h_o \qquad (B.2.1)$$

from which a tridiagonal matrix is to be obtained, it is necessary to first generate a secondary polynomial from $H_n(s)$. This secondary polynomial, $G_n(s)$, is written

$$G_n(s) = s^n + g_{n-1}s^{n-1} + \dots + g_o \qquad (B.2.2)$$

and is obtained by the relation

$$G_n(s)G_n(-s) = H_n(s)H_n(-s)-c \qquad (B.2.3)$$

where

$$0 < c \leq m \tag{B.2.4}$$

and

$$m = \min\left\{\left|H_n(j\omega)\right|^2;\ -\infty < \omega < \infty\right\} \tag{B.2.5}$$

After determining c from Equations B.2.4 and B.2.5, then Equation B.2.3 is used to obtain the g_i of Equation B.2.2. From Equation B.2.4 it is obvious that c will not be unique, which in turn implies that the g_i will not either.

Next construct the rational function

$$W(s) = \frac{H_n(s) + G_n(s)}{H_n(s) - G_n(s)} \tag{B.2.6}$$

which is used to obtain a continued fraction expansion of the form

$$\frac{2}{(h_{n-1}-g_{n-1})[1+W(s)]} = \frac{H_n(s) - G_n(s)}{(h_{n-1}-g_{n-1})[H_n(s)]} \tag{B.2.7a}$$

$$= \cfrac{1}{f_1 + s + \cfrac{f_2}{s + \cfrac{f_3}{s + \ddots + \cfrac{f_n}{s + f_{n+1}}}}} \tag{B.2.7b}$$

where

$$K = \begin{bmatrix} -f_1 & -f_2 & & & & \\ 1 & 0 & -f_3 & & & \\ & 1 & 0 & -f_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & -f_n \\ & & & & 1 & -f_{n+1} \end{bmatrix} \tag{B.2.8}$$

and K has eigenvalues equal to the roots of $H_n(s)$.

B.3 Example. Given the strictly Hurwitz primary polynomial

$$H_3(s) = s^3 + 2s^2 + 3s + 1 \tag{B.3.1}$$

and defining $f(\omega)$ as

$$f(\omega) = \left|H_3(j\omega)\right|^2$$

then

$$f(\omega) = \omega^6 - 2\omega^4 + 5\omega^2 + 1$$

and

$$\frac{df(\omega)}{d\omega} = 0$$

yields $\omega = 0$ for the argument value where $f(\omega)$ is a minimum which implies that

$$m = 1$$

and

$$0 < c \leq 1 \tag{B.3.2}$$

Arbitrarily choose c = 0.5 and then

$$G_3(s)G_3(-s) = - s^6 - 2s^4 - 5s^2 + 0.5 \qquad \text{(B.3.3)}$$

Assume

$$G_3(s) = s^3 + as^2 + bs + d \qquad \text{(B.3.4)}$$

then

$$G_3(s)G_3(-s) = - s^6 - (2b-a^2)s^4 - (b^2-2ad)s^2 + d^2 \qquad \text{(B.3.5)}$$

Equating coefficients of Equations B.3.3 and B.3.5 yields

$$d = .707 \qquad \text{(B.3.6a)}$$

and

$$a^4 + 4a^2 - 5.66a - 16 = 0 \qquad \text{(B.3.6b)}$$

A solution is

$$a = 1.875 \qquad \text{(B.3.7)}$$

which implies that

$$b = 2.76 \qquad \text{(B.3.8)}$$

Now from Equations B.3.4, B.3.6a, B.3.7, and B.3.8

$$G_3(s) = s^3 + 1.875s^2 + 2.76s + .707 \qquad \text{(B.3.9)}$$

From Equations B.2.7a, B.3.1, and B.3.9

$$\frac{H_3(s)-G_3(s)}{(h_2-g_2)H_3(s)} = \frac{s^2 + 1.92s + 2.344}{s^3 + 2s^2 + 3s + 1} \qquad \text{(B.3.10a)}$$

and this result with Equation B.2.7b yields

$$\frac{s^2 + 1.92s + 2.344}{s^3 + 2s^2 + 3s + 1} = \cfrac{1}{f_1 + s + \cfrac{f_2}{s + \cfrac{f_3}{s + f_4}}} \qquad \text{(B.3.10b)}$$

Evaluating the f_i above and substituting into Equation B.2.8 yields

$$K = \begin{bmatrix} -.08 & -.503 & 0 \\ 1 & 0 & -2.34 \\ 0 & 1 & -1.92 \end{bmatrix} \qquad \text{(B.3.11)}$$

This tridiagonal matrix will yield the primary polynomial of Equation B.3.1. There will be an error in the h_o coefficient due to slide rule inaccuracies.

APPENDIX C

TRANSFORMATION FOR TRIDIAGONAL MATRICES

C.1 Introduction. The tridiagonal matrix obtained in Appendix B is not of the same form as the K-matrix of the "modified" general state-model of Section 2.3. The desired form being

$$K = \begin{bmatrix} -K_{11} & K_{12} \\ -K_{12}^T & -K_{22} \end{bmatrix} \tag{C.1.1}$$

C.2 Transformation Procedure. The procedure to transform the K-matrix of Appendix B into the form of Equation C.1.1 will be done in two steps. The first step is to use a similarity transformation on the initial tridiagonal matrix

$$K_1 = \begin{bmatrix} -f_1 & -f_2 & & & & \\ 1 & 0 & -f_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & -f_n \\ & & & & 1 & -f_{n+1} \end{bmatrix} \tag{C.2.1}$$

to obtain the tridiagonal matrix

$$K_2 = \begin{bmatrix} -f_1 & f_2 & & & & \\ -f_2 & 0 & f_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & -f_{n-1} & 0 & f_n \\ & & & & -f_n & -f_{n+1} \end{bmatrix} \qquad (C.2.2)$$

Then another similarity transformation is made in step two to obtain a K-matrix of the form in Equation C.1.1.

C.2.1 Similarity Transformation of Step One. Perlis (28) has shown that in a similarity transformation the K_2 of

$$K_2 = PK_1P^{-1} \qquad (C.2.3)$$

will have the same eigenvalues as K_1. There can be determined a transformation matrix, P, such that when given K_1 of Equation C.2.1 and

$$P = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

that

$$K_2 = \begin{bmatrix} -f_1 & \frac{-f_2 d_1}{d_2} & & & & \\ \frac{d_2}{d_1} & 0 & \frac{-f_3 d_2}{d_3} & & & \\ & \frac{d_3}{d_2} & 0 & \frac{-f_4 d_3}{d_4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{d_{n-1}}{d_{n-2}} & 0 & \frac{-f_n d_{n-1}}{d_n} \\ & & & & \frac{d_n}{d_{n-1}} & -f_{n+1} \end{bmatrix} \tag{C.2.4}$$

Equating like entries from Equations C.2.2 and C.2.4 and solving for the d_i entries yields

$$\left[\frac{d_2}{d_1}\right]^2 = f_2$$

$$\left[\frac{d_3}{d_2}\right]^2 = f_3$$

$$\vdots$$

$$\left[\frac{d_n}{d_{n-1}}\right]^2 = f_n$$

This set of (n-1) equations has n unknowns. Therefore a solution is

$$\begin{aligned} d_1 &= 1 \\ d_2 &= (f_2)^{\frac{1}{2}} \\ d_3 &= (f_2 f_3)^{\frac{1}{2}} \\ &\vdots \\ d_n &= (f_2 f_3 \ldots f_{n-1} f_n)^{\frac{1}{2}} \end{aligned} \qquad (C.2.5)$$

Now K_2 can be obtained from K_1 and is

$$K_2 = \begin{bmatrix} -f_1 & -\sqrt{f_2} & & & & & \\ \sqrt{f_2} & 0 & -\sqrt{f_3} & & & & \\ & \sqrt{f_3} & 0 & -\sqrt{f_4} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{f_{n-1}} & 0 & -\sqrt{f_n} \\ & & & & \sqrt{f_n} & -f_{n+1} \end{bmatrix} \qquad (C.2.6)$$

C.2.2 Similarity Transformation of Step Two. Using the P of Equation C.2.3 to obtain a similarity transformation from K_2 to K has been done by Yarlagadda (35) and requires when n is odd that

$$P = \begin{array}{c} \begin{matrix} (1) & (2) & (3) & \ldots & (\frac{n+1}{2}) & (\frac{n+3}{2}) & (\frac{n+5}{2}) & (\frac{n+7}{2}) & \ldots & (n) \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & . & 0 \\ . & . & . & \ldots & . & . & . & . & . & . \\ . & . & . & \ldots & . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & . & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & . & 0 \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ . \\ . \\ (n-1) \\ (n) \end{matrix} \end{array}$$

(C.2.7)

The similarity transformation

$$K = P^T K_2 P \qquad (C.2.8)$$

will yield

$$
K = \begin{array}{c}
\begin{array}{cccccc|cccc}
(1) & (2) & (3) & \cdots & (\frac{n-1}{2}) & (\frac{n+1}{2}) & (\frac{n+3}{2}) & (\frac{n+5}{2}) & \cdots & (n)
\end{array}
\end{array}
$$

$$
K = \left[\begin{array}{cccccc|cccc}
-f_1 & & & & & & f_2 & & & \\
 & 0 & & & & & -f_3 & f_4 & & \\
 & & 0 & & & & & -f_5 & \ddots & \\
 & & & \ddots & & & & & \ddots & \\
 & & & & 0 & & & & \ddots & f_{n-1} \\
 & & & & & -f_{n+1} & & & & -f_n \\
\hline
-f_2 & f_3 & & & & & & & & \\
 & -f_4 & f_5 & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & & \ddots & \ddots & & & & & \\
 & & & & -f_{n-1} & f_n & & & &
\end{array}\right]
\begin{array}{c}
(1) \\ (2) \\ (3) \\ \vdots \\ (\frac{n-1}{2}) \\ (\frac{n+1}{2}) \\ (\frac{n+3}{2}) \\ (\frac{n+5}{2}) \\ \vdots \\ \vdots \\ (n)
\end{array}
$$

(C.2.9)

When n is even this similarity transformation requires that

$$
\begin{array}{cccccccccc}
(1) & (2) & (3) & \cdots & (\frac{n}{2}) & (\frac{n}{2}+1) & (\frac{n}{2}+2) & (\frac{n}{2}+3) & \cdots & (n)
\end{array}
$$

$$
P = \left[\begin{array}{cccccccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right]
\begin{array}{c}
(1) \\ (2) \\ (3) \\ (4) \\ (5) \\ \cdot \\ \cdot \\ (n-1) \\ (n)
\end{array}
$$

(C.2.10)

which yields

$$K = \begin{array}{c} \\ \end{array}\left[\begin{array}{ccccc|ccccc} (1) & (2) & \cdots & & (\frac{n}{2}) & (\frac{n}{2}+1) & (\frac{n}{2}+2) & \cdots & (n-1) & (n) \\ -f_1 & & & & & f_2 & & & & \\ & 0 & & & & -f_3 & f_4 & & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & & 0 & & & & -f_{n-1} & f_n \\ \hline -f_2 & f_3 & & & & 0 & & & & \\ & -f_4 & \ddots & & & & 0 & & & \\ & & \ddots & \ddots & & & & \ddots & & \\ & & & & f_{n-1} & & & & 0 & \\ & & & & -f_n & & & & & -f_{n+1} \end{array}\right]\begin{array}{c} \\ (1) \\ (2) \\ \vdots \\ (\frac{n}{2}) \\ (\frac{n}{2}+1) \\ (\frac{n}{2}+2) \\ \vdots \\ (n-1) \\ (n) \end{array}$$

(C.2.11)

This completes the second step as the K-matrices of Equations C.2.9 and C.2.11 are of the form of that in Equation C.1.1.

C.2.3 Another Transformation. In Section 2.5 an algorithm is given that determines $(sU-K_2)^{-1}$ and Perlis (28) has shown that

$$(sU-K)^{-1} = P^T(sU-K_2)^{-1}P \qquad \text{(C.2.12)}$$

APPENDIX D

TRANSFER FUNCTION TRANSMISSION ZEROS

D.1 Introduction. In the synthesis procedure presented in the preceding chapters, the characteristic polynomial of the three different transfer functions was realized in a ladder network. The numerator polynomial was obtained by the drivers being inserted at the appropriate locations of the ladder network. The primary concern was that the driver insertions yield the desired degree in the numerator polynomial. Since the number of transmission zeros is equal to the degree of the numerator polynomial, this appendix will present a method for determining the transmission zeros of ladder networks.

D.2 Transmission Zeros. Seshu and Reed (31) present the following theorem on transmission zeros.

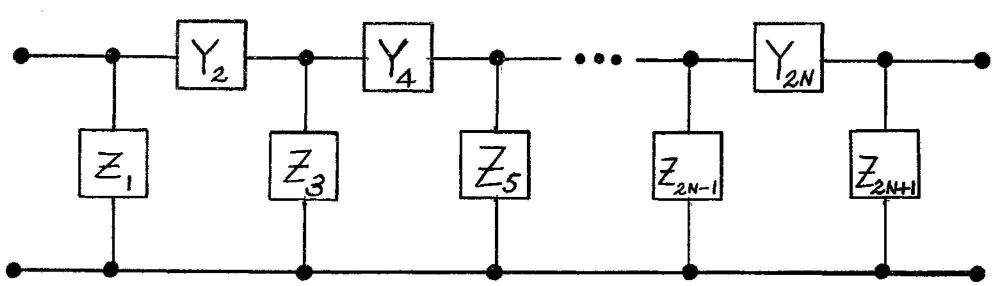

Figure D.2.1 Ladder Network

Theorem D.2.1 - The zeros of transmission of the ladder network in Figure D.2.1 are contained among the zeros of Y_2, Y_4, ..., Y_{2n} and Z_1, Z_3, ..., Z_{2n+1}.

Chen (9) has given the following two theorems concerning zeros of transmission.

Theorem D.2.2 - The poles of the impedance $Z_1(s)$, as shown in Figure D.2.2, of a series branch other than the series branch $Z_2(s)$ is a zero of the open circuit transfer impedance $z_{12}(s)$ of that ladder network.

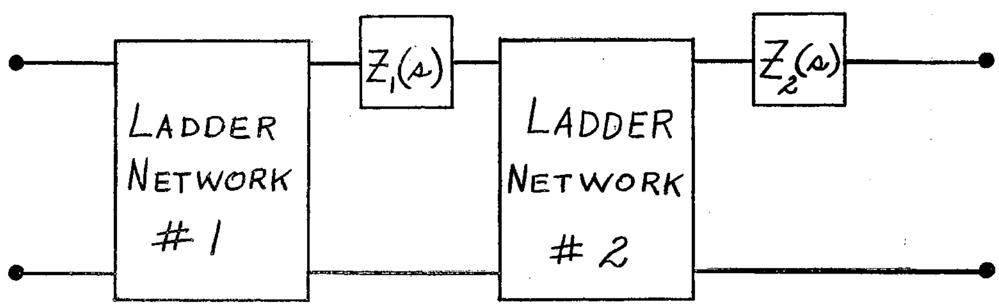

Figure D.2.2 Ladder Network for Theorem D.2.2

Theorem D.2.3 - The poles of the admittance $Y_1(s)$, as shown in Figure D.2.3, of a parallel branch other than the parallel branch $Y_2(s)$ is a zero of the short circuit transfer admittance $y_{12}(s)$ of that ladder network.

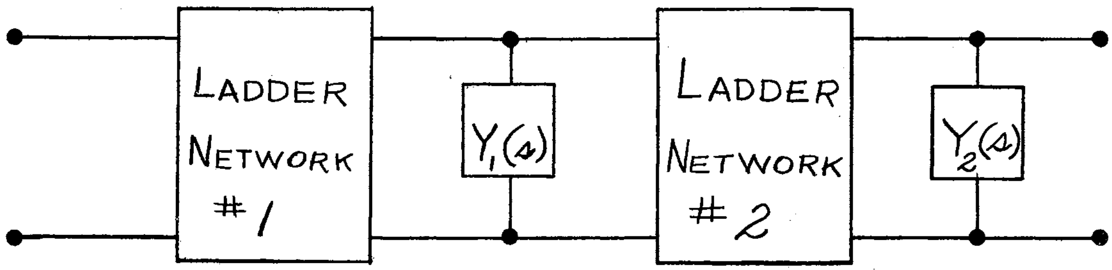

Figure D.2.3 Ladder Network for Theorem D.2.3

Consider the ladder network of Figure D.2.4 and the chain parameters or general circuit parameters can be written

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} \tag{D.2.1}$$

where

$$A = 1 + Z_1Y_2 + Z_1Y_4 + Z_3Y_4 + Z_1Y_2Z_3Y_4$$

$$B = Z_1 + Z_3 + Z_5 + Z_1Y_2Z_3 + Z_1Y_2Z_5 + Z_1Y_4Z_5 + Z_3Y_4Z_5 + Z_1Y_2Z_3Y_4Z_5$$

$$C = Y_2 + Y_4 + Y_2Z_3Y_4$$

$$D = 1 + Y_2Z_3 + Y_2Z_5 + Y_4Z_5 + Y_2Z_3Y_4Z_5$$

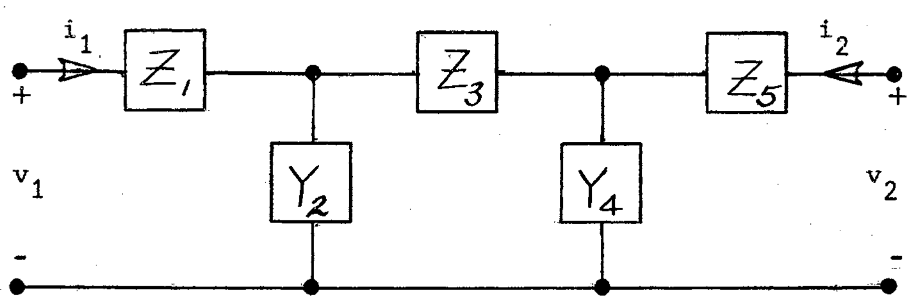

Figure D.2.4 Ladder Network for Chain Parameters

It can be seen that

$$T(s) = \frac{1}{A} \qquad (D.2.2)$$

$$y_{12}(s) = - \frac{1}{B} \qquad (D.2.3)$$

$$z_{12}(s) = \frac{1}{C} \qquad (D.2.4)$$

From Equations D.2.1 and D.2.2 it can be seen that poles of the Z_i and Y_j will be the zeros of transmission for the voltage transfer function. Note that the poles of Z_5 are not considered. Note that if Z_1 is a short, then the poles of Y_2 are not considered.

From Equations D.2.1 and D.2.3 it can be seen that the poles of the Z_i and Y_j will be the zeros of transmission for the short circuit transfer admittance. Note that if Z_5 is a short circuit then the poles of Y_4 are not considered. This comment is also true for Z_1 and Y_2.

From Equations D.2.1 and D.2.3 it can be seen that the poles of Z_3 and the Y_i will be the zeros of transmission for the open circuit transfer impedance. Note that the poles of Z_1 and Z_5 are not considered.

VITA

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