FACIAL CONES, LOCAL SIMILARITY AND INDECOMPOSABILITY OF POLYTOPES

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INDECOMPOSABILITY OF POLYTOPES

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PREFACE

This thesis is a study of polytopes in finite dimensional Euclidean space, E_n. Some of the results obtained, especially those in the first part of Chapter I, are also valid in a more general setting. However, a polytope is always a finite dimensional set and hence a finite dimensional setting is appropriate.

It should be pointed out that in some mathematical writing the term "polytope" may have a different or more general meaning than the one used here. Possibly the terminology "convex polytope" should be used for precision. However, to avoid a large number of repetitions of the word "convex", the shorter term "polytope" is used. This practice is also followed by Grünbaum, [2].

The terminology and symbolism used is either defined or is the same as that used in Grünbaum, [2], and Valentine, [7]. The end of a proof is marked by the symbol ▌.

Chapter I concerns itself with the support functional and the facial cones of a polytope. In Chapter II, more information is obtained about the facial cones which in turn are used to characterize local similarity of polytopes. The third chapter is a study of indecomposable and decomposable polytopes.

I would like to express my deep appreciation to Professor E. K. McLachlan for the inspiration he has provided over the past three years and his assistance in the preparation of this thesis. My thanks go to

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CHAPTER I

FACIAL CONES OF POLYTOPES

Let C be the collection of all compact convex sets in Euclidean n-space E_n. Then C can be given an algebraic structure by making the following definitions:

$$A + B = \{ a + b : a \in A, b \in B \},$$

$$\alpha A = \{ \alpha a : a \in A \} \text{ for any real } \alpha.$$

The collection C with these two operations has all of the defining properties of a real linear space except for the existence of additive inverses and the property $(\alpha + \beta)A = \alpha A + \beta A$ for α and β arbitrary real numbers. It is easy to see that an element of C has an additive inverse if and only if it is a singleton. The property $(\alpha + \beta)A = \alpha A + \beta A$ holds provided that $\alpha\beta \geq 0$. Proofs of these facts about the collection C can be found in Grünbaum, [2], p. 317. Although in general elements of C do not possess additive inverses, it can be shown that the cancellation law for addition holds, that is $A + C = B + C$ implies that $A = B$. For the proof of this fact, see Rådström, [4], p. 167.

Associated with each convex set K is a certain subset of K called the set of extreme points of K. The extreme points of a convex set will be very useful in the sequel and they are defined as follows:

Definition 1-1. Let K be a convex subset of E_n and let $x_0 \in K$. Then x_0 is an extreme point of K, written $x_0 \in ext(K)$, if and only if there do not exist two distinct points x_1 and x_2 of K and a real number t, $0 < t < 1$, such that $x_0 = tx_1 + (1 - t)x_2$.

A fundamental result about the extreme points of a set in $\mathcal{C}$ is the Krein - Milman theorem which is as follows:

Theorem 1-1. Let $K \in \mathcal{C}$. Then $K = conv[ext(K)]$.

The proof of this theorem appears in many places, for example by Grünbaum, [2], p. 18.

The set of extreme points of any scalar multiple of a convex set K is determined by the set of extreme points of K and the scalar as follows:

Theorem 1-2. Let K be a convex set in E_n and let α be any real number. Then $ext(\alpha K) = \alpha\ ext(K)$.

Proof: First note that in the case $\alpha = 0$, $ext(\alpha K) = \{0\} = \alpha\ ext(K)$. Suppose then that $\alpha \neq 0$ and let $\alpha x \in ext(\alpha K)$, $x \in K$ and assume that $x \notin ext(K)$. Then by Definition 1-1, there exists two distinct elements x_1 and x_2 of K and a real t, $0 < t < 1$, such that $x = tx_1 + (1 - t)x_2$. Then $\alpha x = t(\alpha x_1) + (1 - t)(\alpha x_2)$, which contradicts the supposition that $\alpha x \in ext(\alpha K)$. Therefore $x \in ext(K)$ and hence $\alpha x \in \alpha\ \text{ext}(K)$ and so $\text{ext}(\alpha K) \subset \alpha\ \text{ext}(K)$. This inclusion implies the reverse inclusion, for

$$\text{ext}(K) = \text{ext}[(1/\alpha)\cdot\alpha K] \subset (1/\alpha)\text{ext}(\alpha K)$$

which is equivalent to $\alpha\ \text{ext}(K) \subset \text{ext}(\alpha K)$. ∎

In view of Theorem 1-2, one might expect that

$$\text{ext}(A + B) = \text{ext}(A) + \text{ext}(B).$$

However in general this is not the case and it can only be concluded that $\text{ext}(A + B) \subset \text{ext}(A) + \text{ext}(B)$. Later on, in Theorem 1-18, it is determined exactly which elements of $\text{ext}(A) + \text{ext}(B)$ are in $\text{ext}(A + B)$ in the case where A and B are polytopes.

Theorem 1-3. For convex sets A and B in E_n,

$$\text{ext}(A + B) \subset \text{ext}(A) + \text{ext}(B).$$

Moreover, if $z \in \text{ext}(A + B)$, $z = x + y$ where $x \in A$ and $y \in B$, then $x \in \text{ext}(A)$ and $y \in \text{ext}(B)$.

Proof: Suppose that $x \notin \text{ext}(A)$. Then by Definition 1-1, there exist elements x_1 and x_2 in A, $x_1 \neq x_2$, and a real number t, $0 < t < 1$, such that $x = tx_1 + (1 - t)x_2$. Then

$$\begin{aligned} z &= x + y \\ &= tx_1 + (1 - t)x_2 + y \\ &= t(x_1 + y) + (1 - t)(x_2 + y), \end{aligned}$$

which contradicts the fact that $z \in \text{ext}(A + B)$. Therefore, $x \in \text{ext}(A)$ and similarly, $y \in \text{ext}(B)$. ∎

The following result is an easy consequence of Theorems 1-1 and 1-3, and will be used later to show that the sum of two polytopes is again a polytope. It also provides a practical method of obtaining $A + B$ from $\text{ext}(A)$ and $\text{ext}(B)$ (cf. Figure 1-1).

Theorem 1-4. If A and $B \in \mathcal{C}$, then $A + B = \text{conv}[\text{ext}(A) + \text{ext}(B)]$.

Proof: By Theorem 1-3, $\text{ext}(A + B) \subset \text{ext}(A) + \text{ext}(B)$, and therefore, using Theorem 1-1, $A + B \subset \text{conv}[\text{ext}(A) + \text{ext}(B)]$. Also, since $\text{ext}(A) + \text{ext}(B) \subset A + B$ and $A + B$ is convex,

$$\text{conv}[\text{ext}(A) + \text{ext}(B)] \subset A + B. \blacksquare$$

A very important concept to be used in the sequel is that of a face of a polytope. However, this concept can also be defined for any convex set as follows:

Definition 1-2. Let K be a convex subset of E_n. A set F is a face of K if and only if $F = K$, $F = \emptyset$, or $F = H \cap K$ where H is some supporting hyperplane of K. The faces K and $\emptyset$ are called improper faces of K. All other faces of K are called proper faces. If $\dim(F) = j$, then F is called a j-face of K. The 0-faces of K are also called exposed points of K, and the totality of such points is denoted by $\exp(K)$.

Later in the development many results will depend upon the concept of the support functional of a polytope. As in the case of Definition 1-2, this functional can be defined for any compact convex set as follows:

Definition 1-3. Let $K \in \mathcal{C}$. For any $x \in E_n$, define $f_K(x) = \sup_{y \in K} x \cdot y$.

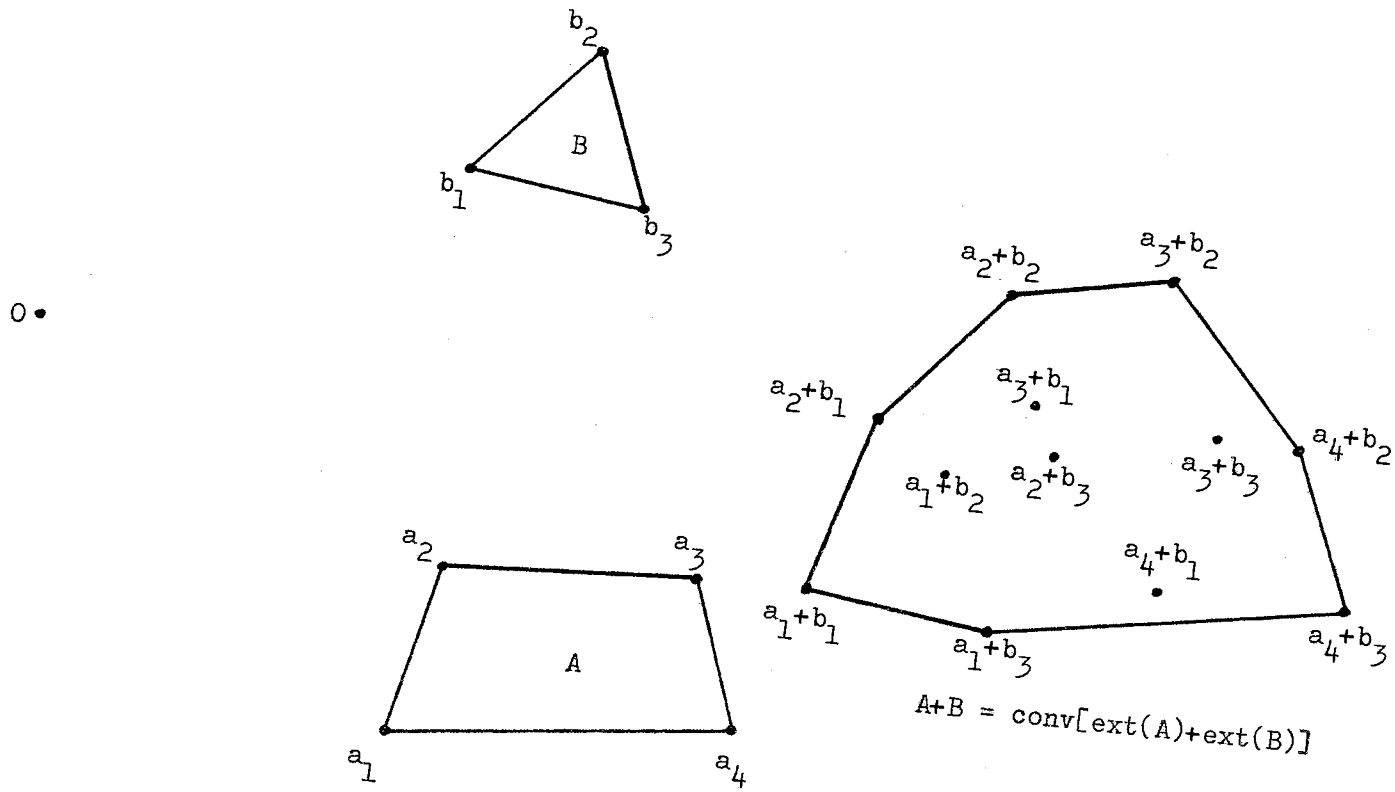
0
b_1
b_2
b_3
B
a_1
a_2
a_3
a_4
A
a_2+b_2
a_3+b_2
a_2+b_1
a_3+b_1
a_4+b_2
a_1+b_2
a_2+b_3
a_3+b_3
a_4+b_1
a_1+b_1
a_1+b_3
a_4+b_3
A+B = conv[ext(A)+ext(B)]

Figure 1-1.

The functional f_K is called the support functional of K.

Theorem 1-5. Support functionals have the following properties:

(a) real valued,

(b) sublinear,

(c) $f_A + f_B = f_{A+B}$,

(d) $f_{\alpha A} = \alpha f_A$ for $\alpha \geq 0$,

(e) $f_A = f_B$ implies that $A = B$,

(f) $A = \{x\colon x\cdot y \leq f_A(y)$ for all $y\}$.

For the proof see Valentine, [7], pp. 58-59 and p. 153.

If A and B are two compact convex sets in E_n, then so are the sets $A \cap B$ and $\mathrm{conv}(A \cup B)$, (cf. Valentine, [7], p. 30, Th. 3.10). By defining $A \wedge B = A \cap B$ and $A \vee B = \mathrm{conv}(A \cup B)$, the collection C with the operations $\wedge$ and $\vee$ form a lattice with respect to the order relation set inclusion. The next theorem considers the support functionals of $A \vee B$ and $A \wedge B$ in terms of f_A and f_B.

Theorem 1-6. Let A and $B \in C$. Then $f_{A\vee B} = \max(f_A, f_B)$. If also $A \cup B \in C$, then $f_{A\wedge B} = \min(f_A, f_B)$.

Proof: Let $x \in E_n$. Since $A \subset A \vee B$,

$$f_A(x) = \sup\{x\cdot y\colon y \in A\} \leq \sup\{x\cdot y\colon y \in A\vee B\} = f_{A\vee B}(x).$$

Similarly, $f_B(x) \leq f_{A\vee B}(x)$ and hence $\max[f_A(x), f_B(x)] \leq f_{A\vee B}(x)$. Now let $y \in \mathrm{conv}(A\cup B)$. Then there exists $y_1 \in A$, $y_2 \in B$ and a real t, $0 \leq t \leq 1$, such that $y = ty_1 + (1 - t)y_2$ (cf. Valentine, [7],

p. 16, Th. 1.25). Thus,

$$\begin{aligned} x\cdot y &= x\cdot[ty_1 + (1 - t)y_2] \\ &= t(x\cdot y_1) + (1 - t)(x\cdot y_2) \\ &\leq tf_A(x) + (1 - t)f_B(x) \\ &\leq t\max[f_A(x), f_B(x)] + (1 - t)\max[f_A(x), f_B(x)] \\ &= \max[f_A(x), f_B(x)], \end{aligned}$$

and therefore $f_{A\vee B}(x) \leq \max[f_A(x), f_B(x)]$.

Now assume that $A \cup B$ is convex and again let $x \in E_n$. Then $f_{A\wedge B}(x) = \sup_{y\in A\wedge B} x\cdot y \leq \sup_{y\in A} x\cdot y = f_A(x)$ and similarly, $f_{A\wedge B}(x) \leq f_B(x)$ and hence $f_{A\wedge B}(x) \leq \min[f_A(x), f_B(x)]$.

Now suppose that $f_{A\wedge B}(x) < \min[f_A(x), f_B(x)]$. Then $f_{A\wedge B}(x) < f_A(x)$ and $f_{A\wedge B}(x) < f_B(x)$. There exists $y_1 \in A$ and $y_2 \in B$ such that $f_A(x) = x\cdot y_1$ and $f_B(x) = x\cdot y_2$ (cf. Valentine, [7], p. 58, Th. 5.2). Let $t = \inf\{\alpha\colon 0 \leq \alpha \leq 1$ and $\alpha y_1 + (1-\alpha)y_2 \in A\}$ and define $y_t = ty_1 + (1 - t)y_2$. Then $y_t \in A$ since A is closed. If $t = 0$, then $y_t = y_2 \in B$. If $t > 0$, then $\alpha y_1 + (1-\alpha)y_2 \in B$ for $0 \leq \alpha < t$ since $A \cup B$ is convex and hence $y_t \in B$ since B is closed. Thus in any case, $y_t \in A \cap B$. Therefore, $x\cdot y_t \leq f_{A\wedge B}(x)$. However,

$$\begin{aligned} x\cdot y_t &= x\cdot[ty_1 + (1 - t)y_2] \\ &= t(x\cdot y_1) + (1 - t)(x\cdot y_2) \\ &> tf_{A\wedge B}(x) + (1 - t)f_{A\wedge B}(x) \\ &= f_{A\wedge B}(x), \end{aligned}$$

a contradiction. ∎

Now suppose that $K \in \mathcal{C}$ and $x_0 \in E_n$, $x_0 \neq 0$. It will be convenient to use the notation $H(K,x_0)$ to represent $\{x\colon x\cdot x_0 = f_K(x_0)\}$.

Thus $H(K,x_0)$ is a hyperplane in E_n. The following theorem gives more information about $H(K,x_0)$ and also shows the reason for calling f_K a support functional.

<u>Theorem 1-7.</u> The set $H(K,x_0)$ is a hyperplane of support for K. Conversely, if H is any hyperplane of support for K, then there exists an $x_0 \neq 0$ for which $H = H(K,x_0)$.

Proof: By Definition 1-3, $y \cdot x_0 \leq f_K(x_0)$ for every $y \in K$, so that $H(K,x_0)$ bounds K. That $H(K,x_0)$ supports K follows from Valentine, [7], (cf. p. 58, Th. 5.2).

Now suppose that $H = \{ x: f(x) = \alpha \}$ is any hyperplane of support for K. Suppose, without loss of generality, that $f(y) \leq \alpha$ for all $y \in K$. There exists some $x_0 \neq 0$ for which $f(x) = x \cdot x_0$ for all $x \in E_n$ (cf. Taylor, [6], pp. 44-45). Thus $y \cdot x_0 \leq \alpha$ for all $y \in K$, which implies that $f_K(x_0) \leq \alpha$. Also, if y_0 is any element of $H \cap K$, then $f_K(x_0) \geq y_0 \cdot x_0 = \alpha$. Therefore, $f_K(x_0) = \alpha$ and so $H = H(K,x_0)$. ∎

The remainder of this study will be concerned primarily with the concept of a polytope. Polytopes are defined as follows:

<u>Definition 1-4.</u> Let P be a subset of E_n. Then P is a polytope if and only if there exists a finite set A such that $P = \text{conv}(A)$.

There are several different characterizations of a polytope (for example, see Grünbaum, [2], pp. 31-32). The following characterization will be sufficient for the results to be obtained here.

Theorem 1-8. Let P be a subset of E_n. Then P is a polytope if and only if $P \in \mathcal{C}$ and ext(P) is finite.

Proof: Suppose first that P = conv(A) where A is finite. Clearly, P is convex and also P is compact (cf. Valentine, [7], p. 40, Th. 3.10). To show that ext(P) is finite, it is sufficient to show that ext(P) is contained in A. Suppose there exists some $x_0 \in \text{ext}(P) \setminus A$. Then $A \subset P \setminus \{x_0\}$ and it is easy to see that $P \setminus \{x_0\}$ is convex since $x_0 \in \text{ext}(P)$. This contradicts the fact that P, being the convex hull of A, is the smallest convex set containing A.

Now suppose that $P \in \mathcal{C}$ and that ext(P) is finite. Then Theorem 1-1 implies that P = conv[ext(P)] and hence P is a polytope. ∎

Let $\mathcal{P}$ denote the collection of all polytopes in E_n. Theorem 1-8 implies that $\mathcal{P} \subset \mathcal{C}$. Moreover, Theorem 1-2 implies that $\mathcal{P}$ is closed under scalar multiples and Theorem 1-4 shows that $\mathcal{P}$ is closed under sums. Therefore, $\mathcal{P}$ is an algebraic sub-structure of $\mathcal{C}$.

The next result shows that any face F of a polytope P is again a polytope and characterizes the extreme points of F in terms of the extreme points of P and the hyperplane of support for P which determines F.

Theorem 1-9. Let F be a face of a polytope P. Then F is a polytope. In fact, if $F = H \cap P$ where H is a hyperplane of support for P, then $\text{ext}(F) = H \cap \text{ext}(P)$.

The proof is given by Grünbaum, [2], (cf. p. 18, Th. 2). From Theorem 1-9, it is clear that a polytope has only a finite number of faces.

It can be shown that for any convex set K in E_n, $\exp(K) \subset \mathrm{ext}(K)$ (cf. Grünbaum, [2], p. 18). In general, this containment is proper, even for compact convex sets. An example of this in E_2 is given by Figure 1-2 in which K is the set obtained by taking the convex hull of a disk and a point not in the disk. The point x_0 in the figure is in $\mathrm{ext}(K)$ but not in $\exp(K)$.

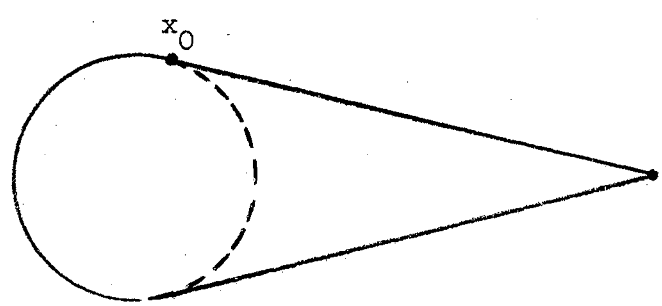

Figure 1-2.

The following theorem shows that for polytopes, extreme points and exposed points coincide.

Theorem 1-10. Let $P \in \mathcal{P}$. Then $\mathrm{ext}(P) = \exp(P)$.

The proof of this theorem follows easily from two theorems of Grümbaum, [2], (cf. Th. 3, p. 18 and Th. 9, p. 19).

The next two theorems characterize the support functional of a polytope.

Theorem 1-11. Let $P \in \mathcal{P}$ and $x \in E_n$. Then $f_P(x) = \max\{x \cdot v: v \in \text{ext}(P)\}$.

Proof: Clearly, $f_P(x) \geq \max\{x \cdot v: v \in \text{ext}(P)\}$. Let $y \in P$. By Theorem 1-1, $P = \text{conv}[\text{ext}(P)]$ and therefore $y = \sum_1^m \alpha_i v_i$ where for each i, $v_i \in \text{ext}(P)$, $\alpha_i \geq 0$, and $\sum_1^m \alpha_i = 1$. Then

$$x \cdot y = \sum_1^m \alpha_i (x \cdot v_i) \leq \sum_1^m \alpha_i [\max\{x \cdot v: v \in \text{ext}(P)\}] = \max\{x \cdot v: v \in \text{ext}(P)\}.$$

Therefore, $f_P(x) \leq \max\{x \cdot v: v \in \text{ext}(P)\}$. ∎

Theorem 1-11 shows that the support functional of a polytope is the maximum of a finite number of linear functionals. The next result establishes the converse of this statement.

Theorem 1-12. Suppose that f is a functional such that for all $x \in E_n$, $f(x) = \max\{x \cdot v: v \in A\}$ where A is a finite set. Then $f = f_P$, where $P = \text{conv}(A)$. Also, $A = \text{ext}(P)$ if and only if there does not exist a proper subset A_0 of A such that $f(x) = \max\{x \cdot v: v \in A_0\}$ for all x.

Proof: Recall that in the proof of Theorem 1-8, it was shown that $\text{ext}(P) \subset A$. Therefore, using Theorem 1-11,

$$f_P(x) = \max\{x \cdot v: v \in \text{ext}(P)\} \leq \max\{x \cdot v: v \in A\} = f(x).$$

But also, $f(x) = \max\{x \cdot v: v \in A\} \leq \sup\{x \cdot y: y \in P\} = f_P(x)$, and therefore $f(x) = f_P(x)$.

To prove the second part of the theorem, first suppose that $A = \text{ext}(P)$ and assume there exists a proper subset A_0 of A such that $f(x) = \max_{v \in A_0} x \circ v$. Then, as noted above, $ext(P) \subset A_0$, a contradiction.

Now suppose that $A \neq ext(P)$. Then $ext(P)$ is properly contained in A and therefore, in view of Theorem 1-11,

$$f(x) = \max\{x \circ v: v \in ext(P)\}, \text{ a contradiction.}\blacksquare$$

Theorem 1-11 prompts the following definition:

<u>Definition 1-5.</u> For $P \in \mathcal{P}$ and $v \in ext(P)$, let

$$C(P,v) = \{x: f_P(x) = x \circ v\}.$$

Also, if F is a face of P, let $C(P,F) = \bigcap\{C(P,v): v \in ext(F)\}$.

When no confusion arises, the notations C_v and C_F will be used instead of $C(P,v)$ and $C(P,F)$.

Theorems 1-13 and 1-14 give some useful information about the sets of Definition 1-5.

<u>Theorem 1-13.</u> For $P \in \mathcal{P}$ and any face F of P, C_F is a closed convex cone with vertex at the origin.

Proof: Let $v \in ext(P)$. From Theorem 1-11, it is clear that the functional f_P is continuous and hence the functional defined by $h(x) = f_P(x) - x \circ v$ is also continuous. Thus $h^{-1}(0) = C_v$ is a closed set.

Now if $x \in C_v$ and α is any nonnegative real number, then $f_P(\alpha x) = \alpha f_P(x) = \alpha(x \circ v) = (\alpha x) \circ v$, which implies that $\alpha x \in C_v$.

Now suppose that x and $y \in C_v$. Then

$$f_P(x + y) \leq f_P(x) + f_P(y) = x \circ v + y \circ v = (x + y) \circ v.$$

Also, $f_P(x + y) \geq (x + y) \circ v$ by Definition 1-3. Therefore, $x + y \in C_v$.

Thus C_v is a closed convex cone with vertex at 0 and therefore so is $C_F = \bigcap\{C_v : v \in \text{ext}(F)\}$. ∎

The cone C_F will be called the <u>facial cone</u> of P corresponding to F.

<u>Theorem 1-14.</u> Let $P \in \mathcal{P}$. Then $\bigcup\{C_v : v \in \text{ext}(P)\} = E_n$.

Proof: By Theorem 1-11, if $x \in E_n$, there exists some $v \in \text{ext}(P)$ such that $f_P(x) = x \cdot v$. Therefore, $x \in C_v \subset \bigcup\{C_v : v \in \text{ext}(P)\}$. ∎

The next two theorems give a simple geometric description of the facial cone C_F (see Figure 1-3 for an example in E_2).

<u>Theorem 1-15.</u> Let $P \in \mathcal{P}$ and suppose that F is a face of P. Let $x_0 \in E_n$, $x_0 \neq 0$. Then $x_0 \in C_F$ if and only if $F \subset H(P,x_0)$.

Proof: First suppose that $x_0 \in C_F$ and let $v \in \text{ext}(F)$. Then $x_0 \in C_v$ which means that $f_P(x_0) = x_0 \cdot v$. This says then that $v \in H(P,x_0)$. Therefore $\text{ext}(F) \subset H(P,x_0)$ and so $F \subset H(P,x_0)$.

Now suppose that $F \subset H(P,x_0)$ and let $v \in \text{ext}(F)$. Then $v \in H(P,x_0)$, which implies that $x_0 \cdot v = f_P(x_0)$. This means that $x_0 \in C_v$ and hence $x_0 \in C_F$. ∎

Let K be a convex set in E_n. In the sequel, the notation $\text{fh}(K)$ will be used to denote the minimal flat which contains K. Also, $\text{relint}(K)$ and $\text{relbd}(K)$ will denote respectively the interior and boundary of K, using the topology of K relative to $\text{fh}(K)$.

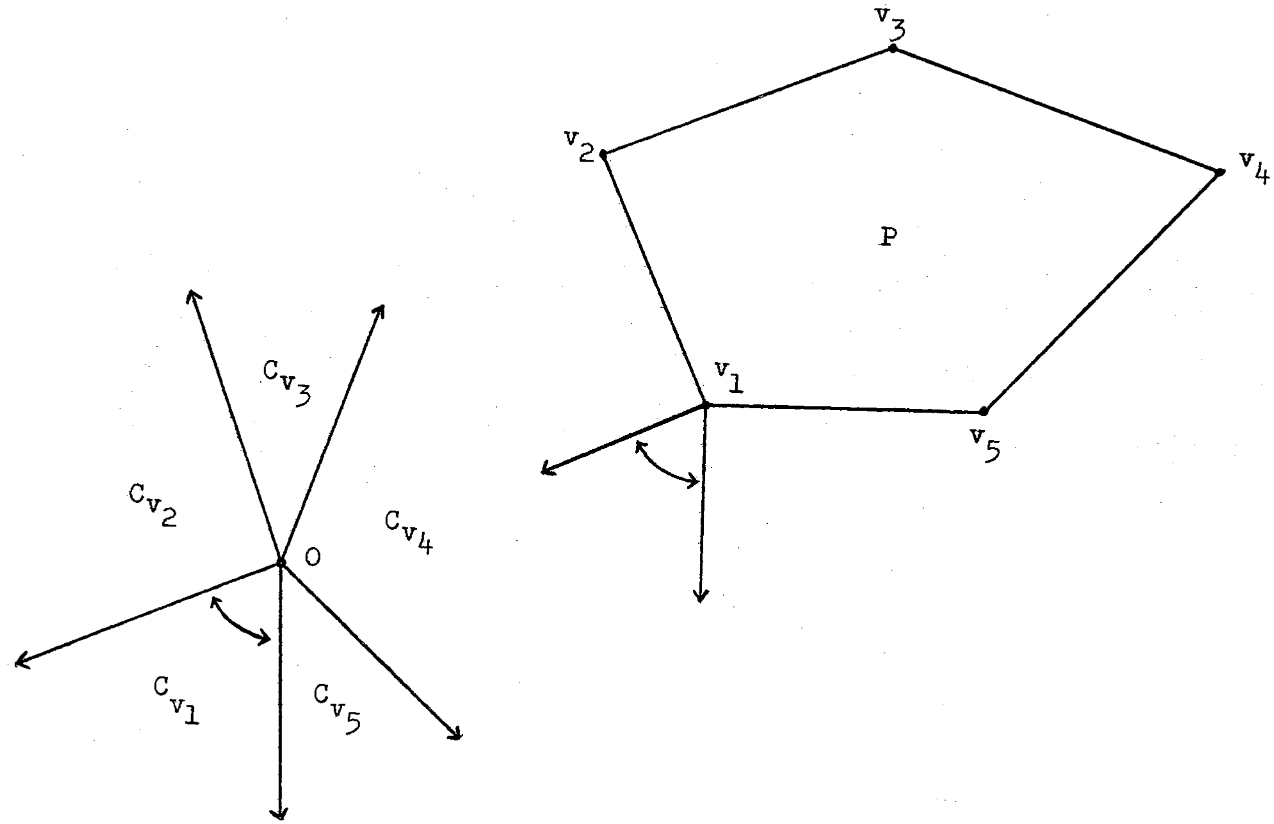

Figure 1-3.

Theorem 1-16. Let $P \in \mathcal{P}$ and suppose that F is a face of P. Let $x_0 \in En$, $x_0 \neq 0$. Then $x_0 \in \mathrm{relint}(C_F)$ if and only if $F = P \cap H(P,x_0)$.

Proof: Suppose first that $x_0 \in \mathrm{relint}(C_F)$. Then there exists some $\varepsilon > 0$ such that if $\| x - x_0 \| < \varepsilon$ and $x \in \langle C_F \rangle$, the linear span of C_F, then $x \in C_F$.

By Theorem 1-15, $F \subset P \cap H(P,x_0)$. Thus the proof will be complete if it can be shown that $P \cap H(P,x_0) \subset F$. Suppose that this is not the case. Then there exists some $v_0 \in \mathrm{ext}(P) \setminus \mathrm{ext}(F)$ such that $v_0 \in H(P,x_0)$. Choose $x_1 \neq 0$ such that $F = H(P,x_1) \cap P$. Then $x_1 \cdot v_0 < f_P(x_1) = x_1 \cdot v$ for each $v \in \mathrm{ext}(F)$. Also, $x_1 \in C_F$ by Theorem 1-15. Let $x_2 = \varepsilon(2 \|x_0 - x_1\|)^{-1}(x_1 - x_0)$ and let $x_3 = x_0 - x_2$. Then $x_3 \in \langle C_F \rangle$ and $\| x_3 - x_0 \| = \varepsilon/2 < \varepsilon$. The desired contradiction will be reached by showing that $x_3 \notin C_F$. To do this, let $v \in \mathrm{ext}(F)$. It is clearly sufficient to show that $x_3 \notin C_v$. This fact will be established by the following sequence of inequalities in which the first inequality implies that $x_3 \notin C_v$, each inequality is true if and only if the succeeding one is, and the last inequality is known to be true:

$$v \cdot x_3 < v_0 \cdot x_3,$$

$$x_3 \cdot (v_0 - v) > 0,$$

$$[x_0 + \varepsilon(2 \|x_0 - x_1\|)^{-1}(x_0 - x_1)] \cdot (v_0 - v) > 0,$$

$$x_0 \cdot (v_0 - v) + \varepsilon(2 \|x_0 - x_1\|)^{-1}(x_0 - x_1) \cdot (v_0 - v) > 0,$$

$$x_0 \cdot (v_0 - v) + x_1 \cdot (v - v_0) > 0,$$

$$x_1 \cdot (v - v_0) > 0,$$

$$x_1 \cdot v_0 < x_1 \cdot v.$$

Now for the proof of the converse implication, suppose that

$F = H(P,x_0) \cap P$. Given any $v \in ext(F)$ and $\bar{v} \in ext(P) \setminus ext(F)$, $x_0 \cdot (v - \bar{v}) > 0$. Therefore, there exists an $\varepsilon(v,\bar{v}) > 0$ such that if $\| x - x_0 \| < \varepsilon(v,\bar{v})$, then $x \cdot (v - \bar{v}) > 0$. Choose $\varepsilon = \min \varepsilon(v,\bar{v})$ where the minimum is taken over all choices of $v \in ext(F)$ and $\bar{v} \in ext(P) \setminus ext(F)$. Then for $\| x - x_0 \| < \varepsilon$, $x \cdot v > x \cdot \bar{v}$ holds whenever $v \in ext(F)$ and $\bar{v} \in ext(P) \setminus ext(F)$. Now suppose that $x \in \langle C_F \rangle$ and $\| x - x_0 \| < \varepsilon$. The proof will be complete if it can be shown that $x \in C_F$. Since $x \in \langle C_F \rangle = C_F - C_F$, $x = x_1 - x_2$ where x_1, $x_2 \in C_F$. Also, since $\| x - x_0 \| < \varepsilon$ and by use of Theorem 1-11, there exists some $v_0 \in ext(F)$ for which $f_P(x) = x \cdot v_0$. Now let v be any element of $ext(F)$. Then

$$
\begin{aligned}
x \cdot v &= x_1 \cdot v - x_2 \cdot v \\
&= f_P(x_1) - f_P(x_2) \\
&= x_1 \cdot v_0 - x_2 \cdot v_0 \\
&= x \cdot v_0 \\
&= f_P(x).
\end{aligned}
$$

Therefore, $x \in C_F$. ∎

<u>Corollary 1-16a.</u> With the hypothesis of Theorem 1-16, $x_0 \in relbd(C_F)$ if and only if F is a proper subset of $P \cap H(P,x_0)$.

Proof: This is true because $relbd(C_F) = C_F \setminus relint(C_F)$. ∎

<u>Corollary 1-16b.</u> If F and G are two faces of P and if F is not a subset of G, then $C_F \cap relint(C_G) = \emptyset$.

Proof: Suppose there exists some $x_0 \in C_F \cap relint(C_G)$. Then using

Theorems 1-15 and 1-16, $F \subset H(P,x_0) \cap P = G$, a contradiction. ∎

It will be shown later that for a polytope P in E_n, $\dim(C_v) = n$ for each $v \in \text{ext}(P)$, so that in view of Theorems 1-10 and 1-16, $\text{int}(C_v) \neq \emptyset$. Hence there exists a finite number of the sets C_v with non-empty interiors for which $E_n = \cup C_v$ and $f_P|_{C_v}$ is linear. The next theorem shows that all sublinear functionals of this type are support functionals for some polytope.

<u>Theorem 1-17.</u> Let f be a sublinear functional and suppose that there exists sets $C_1, C_2, \cdots, C_m$ and vectors $v_1, v_2, \cdots, v_m$ such that $E_n = \bigcup_1^m C_i$, $\text{int}(C_i) \neq \emptyset$ for each i, and $C_i = \{ x: f(x) = x \cdot v_i \}$. Then $f = f_P$ where $P = \text{conv}\{ v_1, v_2, \cdots, v_m \}$.

Proof: Let $x \in E_n$. There exists some i such that $x \in C_i$, that is, $f(x) = x \cdot v_i$. The proof will be complete if it can be shown that $x \cdot v_i \geq x \cdot v_j$, $j = 1, 2, \cdots, m$ by Theorem 1-11. Suppose for some j that $x \cdot v_i < x \cdot v_j$. Choose $y \in \text{int}(C_j)$. For $0 < t < 1$,

$$\begin{aligned} f[ty + (1 - t)x] &\leq tf(y) + (1 - t)f(x) \\ &= t(y \cdot v_j) + (1 - t)(x \cdot v_i) \\ &< t(y \cdot v_j) + (1 - t)(x \cdot v_j) \\ &= [ty + (1 - t)x] \cdot v_j . \end{aligned}$$

This contradicts the fact that $y \in \text{int}(C_j)$. ∎

The next theorem shows how the facial cones of Definition 1-5 can be used to characterize the faces of the sum of two polytopes. The following lemma simplifies the proof:

Lemma 1-18. Let A and $B \in \mathcal{C}$ and $x_0 \in E_n$, $x_0 \neq 0$. Then

$$H(A+B,x_0) \cap (A + B) = H(A,x_0) \cap A + H(B,x_0) \cap B.$$

Proof: First let $a+b \in H(A+B,x_0) \cap (A + B)$ where $a \in A$ and $b \in B$ and suppose that $a \notin H(A,x_0)$. Then $a \cdot x_0 < f_A(x_0)$, which implies that $f_{A+B}(x_0) = f_A(x_0) + f_B(x_0) > a \cdot x_0 + b \cdot x_0 = (a+b) \cdot x_0$, a contradiction. Thus $a \in H(A,x_0)$ and similarly, $b \in H(B,x_0)$.

Now let $x + y \in H(A,x_0) \cap A + H(B,x_0) \cap B$ where $x \in H(A,x_0) \cap A$ and $y \in H(B,x_0) \cap B$. Then $x + y \in A + B$ and

$$(x + y) \cdot x_0 = x \cdot x_0 + y \cdot x_0 = f_A(x_0) + f_B(x_0) = f_{A+B}(x_0)$$

shows that $x + y \in H(A+B,x_0)$. ∎

Theorem 1-18. Let P and $Q \in \mathcal{P}$ and let F and G be faces of P and Q, respectively. Then $F + G$ is a face of $P + Q$ if and only if $\text{relint}[C(P,F)] \cap \text{relint}[C(Q,G)] \neq \emptyset$.

Proof: First suppose that $F + G$ is a face of $P + Q$. Then by Theorem 1-7, there exists some $x_0 \neq 0$ such that $F + G = H(P+Q,x_0) \cap (P+Q)$. By Lemma 1-18, $F + G = H(P,x_0) \cap P + H(Q,x_0) \cap Q$. It will now be established that $F = H(P,x_0) \cap P$ and $G = H(Q,x_0) \cap Q$. Let $y \in F$ and suppose $y \notin H(P,x_0)$. Then $f_P(x_0) > y \cdot x_0$. Choose any $\bar{y} \in G$. Then $y + \bar{y} \in F + G$ and thus $(y + \bar{y}) \cdot x_0 = f_{P+Q}(x_0)$ since $F+G \subset H(P+Q,x_0)$. But since $f_P(x_0) > y \cdot x_0$ and $f_Q(x_0) \geq \bar{y} \cdot x_0$,

$$(y + \bar{y}) \cdot x_0 = y \cdot x_0 + \bar{y} \cdot x_0 < f_P(x_0) + f_Q(x_0) = f_{P+Q}(x_0),$$

a contradiction. Therefore, $F \subset H(P,x_0) \cap P$ and similarly, $G \subset H(Q,x_0) \cap Q$. Now since

$$F + H(Q,x_0) \cap Q \subset H(P,x_0) \cap P + H(Q,x_0) \cap Q = F + G,$$

it follows that $H(Q,x_0) \cap Q \subset G$ and similarly, $H(P,x_0) \cap P \subset F$. Therefore $F = H(P,x_0) \cap P$ and $G = H(Q,x_0) \cap Q$ and so by Theorem 1-16, $x_0 \in \text{relint}[C(P,F)] \cap \text{relint}[C(Q,G)]$.

Conversely, suppose that $x_0 \in \text{relint}[C(P,F)] \cap \text{relint}[C(Q,G)]$. By Theorem 1-16, $F = H(P,x_0) \cap P$ and $G = H(Q,x_0) \cap Q$. Therefore, using Lemma 1-18, $F + G = H(P,x_0) \cap P + H(Q,x_0) \cap Q = H(P+Q,x_0) \cap (P+Q)$. ∎

As was mentioned earlier, it will be shown later that $\text{int}(C_v) \neq \emptyset$ for each $v \in \text{ext}(P)$. In view of this, the preceding theorem states as a special case that if $v \in \text{ext}(P)$ and $w \in \text{ext}(Q)$, then $v+w \in \text{ext}(P+Q)$ if and only if $\text{int}[C(P,v)] \cap \text{int}[C(Q,w)] \neq \emptyset$ (cf. Theorem 1-3). An example in E_2 which illustrates this result is given in Figure 1-4. In the figure, $\text{int}(C_{v_4}) \cap \text{int}(C_{w_3}) \neq \emptyset$ and hence $v_4 + w_3 \in \text{ext}(P+Q)$, whereas C_{v_3} and C_{w_3} intersect only at the origin and hence $v_3 + w_3 \notin \text{ext}(P + Q)$.

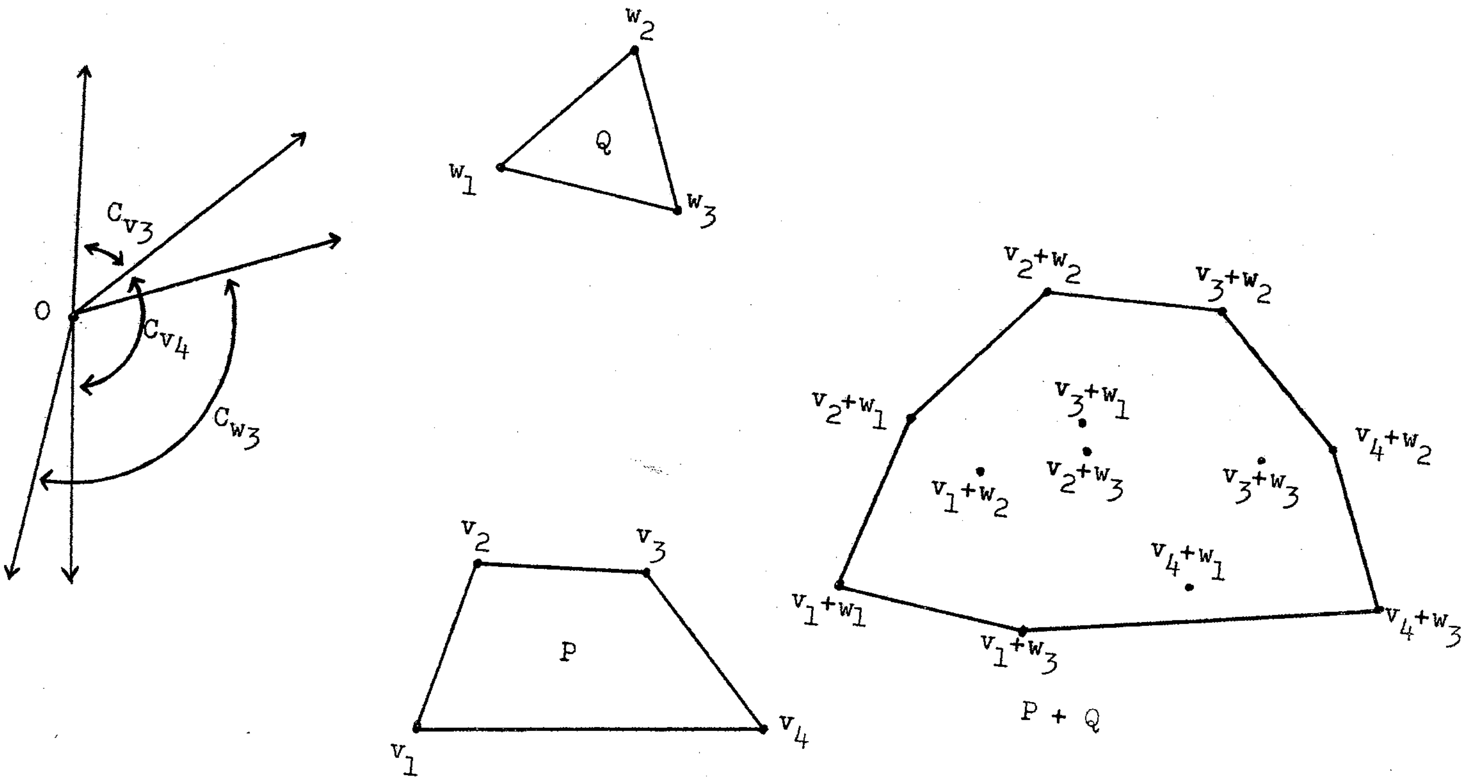
O
Cv3
Cv4
Cw3
w1
w2
w3
Q
v1
v2
v3
v4
P
v2+w2
v3+w2
v2+w1
v3+w1
v4+w2
v1+w2
v2+w3
v3+w3
v4+w1
v1+w1
v1+w3
v4+w3
P + Q

Figure 1-4.

CHAPTER II

RELATION OF LOCAL SIMILARITY TO FACIAL CONES

In this chapter, two equivalence relations, positive homothety and local similarity, will be defined on the collection $\mathcal{P}$.

<u>Definition 2-1.</u> Let P and $Q \in \mathcal{P}$. The polytope P is said to be positively homothetic to Q, written $P \sim Q$, if and only if there exists some $\alpha > 0$ and $x_0 \in E_n$ such that $P = \alpha Q + x_0$.

<u>Theorem 2-1.</u> Positive homothety is an equivalence relation.

Proof: (i) Since $P = 1 \cdot P + 0$, $P \sim P$.

(ii) Suppose that $P \sim Q$, say $P = \alpha Q + x_0$ where $\alpha > 0$. Then $Q = \frac{1}{\alpha}P - \frac{1}{\alpha}x_0$, which implies $Q \sim P$.

(iii) Suppose that $P \sim Q$ and $Q \sim R$, say $P = \alpha Q + x_0$ and $Q = \beta R + y_0$ where $\alpha > 0$ and $\beta > 0$. Then

$$\begin{aligned} P &= \alpha Q + x_0 \\ &= \alpha(\beta R + y_0) + x_0 \\ &= (\alpha\beta)R + (\alpha y_0 + x_0) \end{aligned}$$

shows that $P \sim R$. ∎

Suppose now that P and Q are two positively homothetic polytopes, say $P = \alpha Q + x_0$. By Theorem 1-3, $\text{ext}(P) \subset \text{ext}(\alpha Q) + x_0$. Also, since $C(\{x_0\},x_0) = E_n$, $C(\{x_0\},x_0) \cap C(\alpha Q,z) = C(\alpha Q,z)$ for any

$z \in ext(\alpha Q)$ and thus $ext(\alpha Q) + x_0 \subseteq ext(P)$ by use of Theorem 1-18. Therefore, $ext(P) = ext(\alpha Q) + x_0$ and hence by Theorem 1-2, $ext(P) = \alpha\ ext(Q) + x_0$. The next theorem shows that the facial cones of a polytope P remain invariant under positive homothety.

<u>Theorem 2-2.</u> If P and Q are positively homothetic, $P = \alpha Q + x_0$, then for any $w \in ext(Q)$, $C(Q,w) = C(P,\alpha w+x_0)$.

Proof: Let $x \in C(Q,w)$. This implies that $f_Q(x) = x \cdot w$. Then

$$\begin{aligned} f_P(x) &= f_{\alpha Q+x_0}(x) \\ &= \alpha f_Q(x) + f_{\{x_0\}}(x) \\ &= \alpha(x \cdot w) + x \cdot x_0 \\ &= (\alpha w + x_0) \cdot x \end{aligned}$$

and therefore $C(Q,w) \subset C(P,\alpha w+x_0)$. By symmetry, $C(P,\alpha w+x_0) \subset C(Q,w)$. ∎

<u>Definition 2-2.</u> The polytope P is said to be locally similar to the polytope Q, written $P \approx Q$, if and only if

$$\dim[H(P,x_0) \cap P] = \dim[H(Q,x_0) \cap Q]$$

for every $x_0 \neq 0$.

<u>Theorem 2-3.</u> Local similarity is an equivalence relation.

The proof of this theorem is immediate.

Suppose that P and Q are locally similar polytopes and that $v \in ext(P)$. Then by Theorem 1-10, there exists an $x_0 \neq 0$ such that $\{v\} = H(P,x_0) \cap P$. Corresponding to v is the face $H(Q,x_0) \cap Q$ of Q, and by the local similarity, $H(Q,x_0) \cap Q$ is an extreme point, say w,

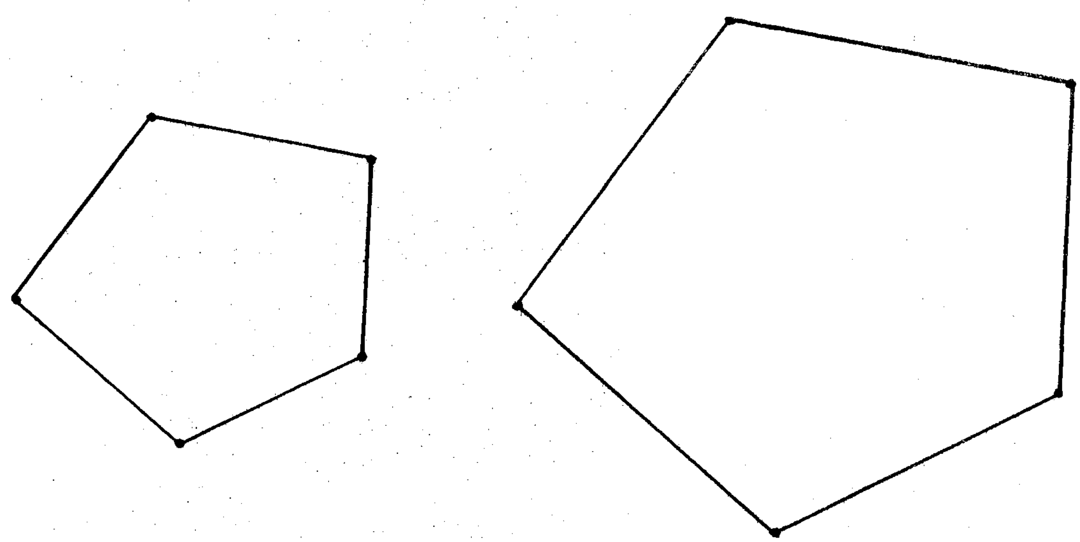

Figure 2-1. Positively Homothetic Polytopes in E_2.

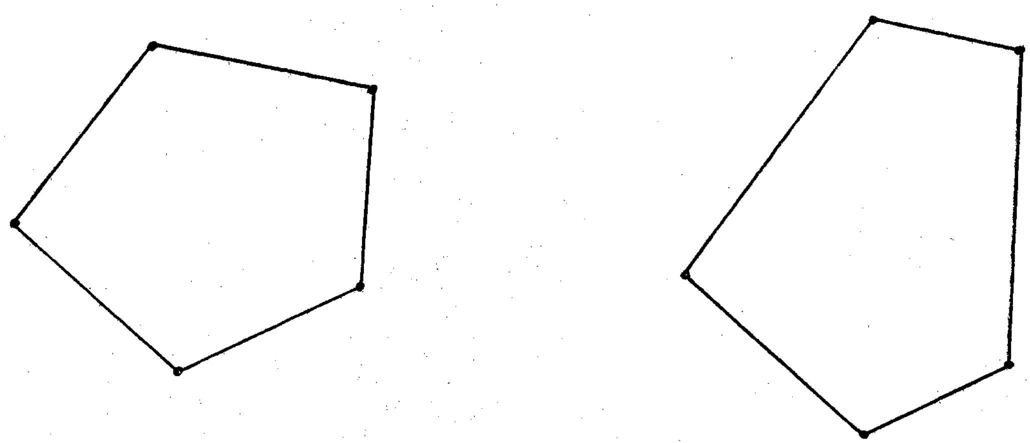

Figure 2-2. Locally Similar Polytopes in E_2.

of Q. In general, the vextor x_0 used to determine v is not unique, that is it may also be true that $\{v\} = H(P,x_1) \cap P$ where $x_1 \neq x_0$, and there is no immediate guarantee that $\{w\} = H(Q,x_1) \cap Q$, only that $H(Q,x_1) \cap Q$ is some extreme point of Q. The next theorem shows that the relationship described above is a function defined from ext(P) to ext(Q).

<u>Theorem 2-4.</u> Let P and Q be two locally similar polytopes and suppose that $v \in \text{ext}(P)$, $w \in \text{ext}(Q)$ where $\{v\} = H(P,x_0) \cap P$ and $\{w\} = H(Q,x_0) \cap Q$. If also $\{v\} = H(P,x_1) \cap P$, then $\{w\} = H(Q,x_1) \cap Q$.

Proof: Suppose that $H(Q,x_1) \cap Q = \{w'\}$ and $w' \neq w$. Then by Theorem 1-16, $x_0 \in \text{int}[C(P,v)] \cap \text{int}[C(Q,w')]$. By Corollary 1-16b, $x_1 \notin \text{int}[C(Q,w)]$. Choose x_2 on the line segment x_0x_1 such that $x_2 \in \text{bd}[C(Q,w)]$. Then by Corollary 1-16a, $\dim[H(Q,x_2) \cap Q] > 0$. Therefore, by the local similarity, $\dim[H(P,x_2) \cap P] > 0$. This contradicts the fact that since $\text{int}[C(P,v)]$ is convex, $x_2 \in \text{int}[C(P,v)]$, and hence $H(P,x_2) \cap P = \{v\}$ by Theorem 1-16. ∎

Although Theorem 2-4 only establishes that the relation that has been defined from ext(P) to ext(Q) is a function, it is actually a one to one correspondence. This is true because local similarity is symmetric and hence Theorem 2-4 shows that the inverse relation from ext(Q) to ext(P) is also a function.

The next two lemmas are required for the proof of Theorem 2-5 which characterizes local similarity in terms of the facial cones of Definition 1-5.

Lemma 2-5a. Let P be a polytope in E_n and suppose that $E \subset \text{ext}(P)$. If C_E denotes $\cap\{C_v : v \in E\}$, then $C_E \neq \{0\}$ if and only if there exists a face F of P such that $\dim(F) \leq n - 1$ and $\text{conv}(E) \subset F$.

Proof: First suppose that $x_0 \in C_E$, $x_0 \neq 0$. Define $F = H(P,x_0) \cap P$. Then F is a face of P whose dimension is not greater than $n - 1$ and since $x_0 \in C_E$, it follows that $E \subset F$ and hence $\text{conv}(E) \subset F$.

Now suppose that $\text{conv}(E) \subset F$ where F is a face of P, $\dim(F) \leq n - 1$. By Theorem 1-7, there exists some $x_0 \neq 0$ such that $F = H(P,x_0) \cap P$. Thus for each $v \in E$, $v \in \text{conv}(E) \subset F \subset H(P,x_0)$ and therefore $x_0 \cdot v = f_P(x_0)$ which implies that $x_0 \in C_v$. Hence $x_0 \in C_E$. ∎

Lemma 2-5b. Let P be a polytope in E_n and let F be a face of P. Let $L_F = \text{fh}(F) - y_0$ where $y_0 \in F$. Then L_F and $\langle C_F \rangle$ are orthogonal complements.

Proof: The result will first be established for the special case when $F = P$.

Case (1): Suppose $\dim(P) = n$. Then $L_P = E_n$. In this case, $C_P = \{0\}$ by Lemma 2-5a.

Case (2): Suppose $\dim(P) < n$. By Theorem 2-2, there is no loss of generality if it is assumed that $0 \in P$ so that $L_P = \langle P \rangle$. It will now be shown that $C_P = \langle P \rangle^\perp$. Let $x_0 \in C_P$, $x_0 \neq 0$. By Theorem 1-15, $P \subset H(P,x_0)$. Since $0 \in P \subset H(P,x_0)$, it follows that $f_P(x_0) = 0$. Now let $y \in P$. Then $y \in H(P,x_0)$ and therefore $y \cdot x_0 = f_P(x_0) = 0$. Thus $x_0 \in P^\perp = \langle P \rangle^\perp$.

Now let $x_1 \in \langle P \rangle^\perp = P^\perp$. This means that $x_1 \cdot y = 0$ for every

$y \in P$ and therefore $f_P(x_1) = 0$. Then for any $v \in \text{ext}(P)$, $x_1 \cdot v = 0 = f_P(x_1)$ which means that $x_1 \in C_v$. Therefore $x_1 \in C_P$. This completes the proof for the case $F = P$.

The remainder of the proof will be established by inducting downward on the dimension of F. Thus suppose that for some $k \leq \dim(P)$ it is true that if G is a face of P such that $\dim(G) = k$ then L_G and $\langle C_G \rangle$ are orthogonal complements and suppose that $\dim(F) = k - 1$. Again assume that $0 \in F$ so that $L_F = \langle F \rangle$. There exists a face G of P such that $F \subset G$ and $\dim(G) = k$. By the induction hypothesis, $\langle C_G \rangle = \langle G \rangle^{\perp}$. It must now be shown that $\langle C_F \rangle = \langle F \rangle^{\perp}$. Let $x_0 \in C_F$ and $v \in \text{ext}(F)$. Then $x_0 \cdot v = f_P(x_0)$ and since $F \subset H(P, x_0)$ by Theorem 1-15 and since $0 \in F$, $f_P(x_0) = 0$. Therefore $\text{ext}(F) \subset C_F{}^{\perp} = \langle C_F \rangle^{\perp}$ and thus $\langle F \rangle \subset \langle C_F \rangle^{\perp}$ which implies that $\langle C_F \rangle \subset \langle F \rangle^{\perp}$. This inclusion also gives $\dim(\langle C_F \rangle) \leq n - k + 1$. To complete the proof, it is sufficient to show that $\dim(\langle C_F \rangle) \geq n - k + 1$. Let $x_1 \in C_G$. Then using Theorem 1-15, $G \subset H(P, x_1) \cap P$ and thus F is properly contained in $H(P, x_1) \cap P$. Therefore by Corollary 1-16a, $x_1 \in \text{relbd}(C_F)$ and hence $C_G \subset \text{relbd}(C_F)$. Now this implies that $\dim(C_G) < \dim(C_F)$, for if not then $\dim(C_G) = \dim(C_F)$, which implies that $\text{fh}(C_G)$ and $\text{fh}(C_F)$ are the same, say $K = \text{fh}(C_G) = \text{fh}(C_F)$. Choose $\bar{x} \in \text{relint}(C_G)$, (cf. Grünbaum, [2], p. 9, Th. 7). Then there exists an $\varepsilon > 0$ such that if $\| x - \bar{x} \| < \varepsilon$ and $x \in K$, then $x \in C_G$. But since $\bar{x} \in \text{relbd}(C_F)$, there exists some $x \in K \setminus C_F \subset K \setminus C_G$ such that $\| x - \bar{x} \| < \varepsilon$, a contradiction. Thus $\dim(C_G) < \dim(C_F)$ and hence $\dim(\langle C_F \rangle) = \dim(C_F) > \dim(C_G) = \dim(\langle C_G \rangle) = \dim(\langle G \rangle^{\perp}) = n - k$. Therefore $\dim(C_F) \geq n - k + 1$. ∎

Theorem 2-5. Let P and $Q \in \mathcal{P}$. Then $P \approx Q$ if and only if there exists a one to one correspondence between $\text{ext}(P)$ and $\text{ext}(Q)$, say $v \longleftrightarrow w$, such that $C(P,v) = C(Q,w)$.

Proof: Suppose first that $P \approx Q$ and let $v \longleftrightarrow w$ be the one to one correspondence established by Theorem 2-4. To show that $C(P,v) = C(Q,w)$, it is sufficient to show that $\text{int}[C(P,v)] = \text{int}[C(Q,w)]$ (cf. Valentine, [7], p. 13, Th. 1.17). Let $x_0 \in \text{int}[C(P,v)]$. Then by Theorem 1-16, $H(P,x_0) \cap P = \{v\}$. Then by local similarity, $H(Q,x_0) \cap Q = \{w\}$, and thus $x_0 \in \text{int}[C(Q,w)]$, again by Theorem 1-16. Therefore, $\text{int}[C(P,v)] \subset \text{int}[C(Q,w)]$ and similarly, $\text{int}[C(Q,w)] \subset \text{int}[C(P,v)]$.

Now suppose that $v \longleftrightarrow w$ is any one to one correspondence between $\text{ext}(P)$ and $\text{ext}(Q)$ for which $C(P,v) = C(Q,w)$. Let $x_0 \in E_n$, $x_0 \neq 0$, and let $F = H(P,x_0) \cap P$, $G = H(Q,x_0) \cap Q$. Suppose that $\text{ext}(F) = \{v_1, v_2, \cdots, v_s\}$. It will now be shown that $\text{ext}(G) = \{w_1, w_2, \cdots, w_s\}$. Let $w \in \text{ext}(G)$. By Theorem 1-15, $x_0 \in C(Q,G) \subset C(Q,w) = C(P,v)$. Thus by Theorems 1-9 and 1-15, $v \in H(P,x_0) \cap \text{ext}(P) = \text{ext}(F) = \{v_1, v_2, \cdots, v_s\}$ and hence $w \in \{w_1, w_2, \cdots, w_s\}$.

Now let $w_j \in \{w_1, w_2, \cdots, w_s\}$. Then $v_j \in \text{ext}(F)$ and again using Theorem 1-15, $x_0 \in C(P,F) \subset C(P,v_j) = C(Q,w_j)$ which implies that $w_j \in H(Q,x_0) \cap \text{ext}(Q) = \text{ext}(G)$. Therefore, $\text{ext}(G) = \{w_1, w_2, \cdots, w_s\}$ and hence $C(P,F) = C(Q,G)$. Using Lemma 2-5b,

$$\dim(F) + \dim[C(P,F)] = n = \dim(G) + \dim[C(Q,G)],$$

and therefore $\dim(F) = \dim(G)$ which means that $P \approx Q$ by Definition 2-2. ∎

It was seen in Theorem 2-5 that the facial cones $C(P,v)$ of a polytope P play a fundamental role in the concept of local similarity. In Theorem 1-16 and its corollaries, these cones and their relative interiors and boundaries were described. Now a further investigation of these cones will be made by determining their extremal elements. First, two definitions concerning convex cones are in order.

Definition 2-3. Let C be a convex cone with vertex at the origin. A point $x_0 \neq 0$ of C is said to be an extremal element of C, written $x_0 \in \text{extr}(C)$, if and only if $x_0 = x_1 + x_2$ where $x_1, x_2 \in C$ implies that there exists positive real numbers α_1 and α_2 such that $x_1 = \alpha_1 x_0$ and $x_2 = \alpha_2 x_0$.

Definition 2-4. Let C be a convex cone with vertex at the origin. Then C is said to be salient if and only if there does not exist an $x_0 \in C$, $x_0 \neq 0$, for which also $-x_0 \in C$.

A simple consequence of these definitions is the following:

Theorem 2-6. A non-salient cone C has no extremal elements.

Proof: If C is non-salient, then there exists some $x_0 \in C$, $x_0 \neq 0$, for which also $-x_0 \in C$. Since $x_0 = 2x_0 + (-x_0)$ and there does not exist an $\alpha > 0$ such that $-x_0 = \alpha x_0$, it is clear that $x_0 \notin \text{extr}(C)$ and similarly, $\beta x_0 \notin \text{extr}(C)$ for all real β. Now consider any $x_1 \in C$ which is not a multiple of x_0. Then

$$x_1 = \tfrac{1}{2}(x_1 + x_0) + \tfrac{1}{2}(x_1 - x_0)$$

shows that $x_1 \notin \text{extr}(C)$ because $\frac{1}{2}(x_1+x_0)$ and $\frac{1}{2}(x_1-x_0)$ are in C

and $\frac{1}{2}(x_1 + x_0)$ is not a positive multiple of x_0 or else x_1 would be a multiple of x_0. ∎

Since non-salient cones have no extremal elements, it is appropriate to determine which of the cones C_F for a polytope P are salient.

<u>Theorem 2-7.</u> Let P be a polytope in E_n and suppose that F is any face of P. If $\dim(P) = n$, then each C_F is salient. If $\dim(P) < n$, then each C_F is non-salient.

Proof: Suppose that $\dim(P) = n$ and that C_F is non-salient for some face F of P. Then there exists an $x_0 \neq 0$ such that both x_0 and $-x_0 \in C_F$. Now since f_P is linear on C_F, $f_P(-x_0) = -f_P(x_0)$. By the definition of f_P, $y \cdot x_0 \leq f_P(x_0)$ for every $y \in P$ and also

$$y \cdot (-x_0) \leq f_P(-x_0) = -f_P(x_0)$$

which implies that $y \cdot x_0 \geq f_P(x_0)$ for all $y \in P$. Thus $P \subset H(P,x_0)$ and so $\dim(P) < n$, a contradiction.

Now suppose that $\dim(P) < n$. Then P is contained in some hyperplane, say $P \subset H(P,x_0)$, $x_0 \neq 0$. By Theorem 1-15, $x_0 \in C_F$. The proof will be complete if it can be shown that $-x_0 \in C_F$. Let $v \in \text{ext}(F)$ and let $y \in P$. Then since $P \subset H(P,x_0)$, $y \cdot x_0 = f_P(x_0) = v \cdot x_0$ and thus $y \cdot (-x_0) = v \cdot (-x_0)$. Therefore $f_P(-x_0) = v \cdot (-x_0)$ and hence $-x_0 \in C_v$. This implies then that $-x_0 \in C_F$. ∎

Now suppose that P is a polytope in E_n of dimension less than n. Then by Theorem 2-7, C_F is non-salient for each face F of P. The next result determines which of these facial cones are subspaces.

<u>Theorem 2-8.</u> If P is a polytope in E_n, $\dim(P) < n$, and F is a

face of P, then C_F is a subspace if and only if $F = P$.

Proof: If $F = P$, then it was shown in the proof of Lemma 2-5b that $C_F = [fh(P) - y_0]^{\perp}$ where $y_0 \in P$.

Suppose that C_F is a subspace. Since F is a face of P, there exists some $x_0 \neq 0$ such that $F = H(P,x_0) \cap P$. By Theorem 1-15, $x_0 \in C_F$ and therefore, since C_F is a subspace, $-x_0 \in C_F$. Now since f_P is linear on C_F, $f_P(-x_0) = -f_P(x_0)$. Let $y \in P$. Then $y \cdot x_0 \leq f_P(x_0)$ and $y \cdot (-x_0) \leq f_P(-x_0) = -f_P(x_0)$. Therefore $y \cdot x_0 = f_P(x_0)$ which implies that $P \subset H(P,x_0)$ and so $P = F$. ∎

The next result characterizes the extremal elements of the facial cone $C(P,F)$. Theorems 2-6 and 2-7 justify the requirement that $\dim(P) = n$ in the hypothesis.

<u>Theorem 2-9.</u> Let P be an n-dimensional polytope in E_n and let F be a face of P. A point $x_0 \neq 0$ of C_F is in $\mathrm{extr}(C_F)$ if and only if $\dim[H(P,x_0) \cap P] = n - 1$.

Proof: Suppose first that $x_0 \in \mathrm{extr}(C_F)$ and let $G = H(P,x_0) \cap P$. By Theorems 1-15 and 1-16, $F \subset G$ and $x_0 \in \mathrm{relint}(C_G)$. Since $C_G \subset C_F$, it is clear from Definition 2-3 that $x_0 \in \mathrm{extr}(C_G)$. Thus

$$x_0 \in \mathrm{relint}(C_G) \cap \mathrm{extr}(C_G),$$

and it will now be shown that this implies that $\dim(C_G) = 1$. Let $x_1 \in C_G$, $x_1 \neq 0$. Since $x_0 \in \mathrm{relint}(C_G)$, there exists an $\varepsilon > 0$ such that if $\| x - x_0 \| < \varepsilon$ and $x \in \langle C_G \rangle$, then $x \in C_G$. Define $x_2 = x_0 - (\varepsilon/2) \|x_1\|^{-1} x_1$. Then $\|x_2 - x_0\| < \varepsilon$ and $x_2 \in \langle C_G \rangle$ so that $x_2 \in C_G$. Thus $x_0 = x_2 + (\varepsilon/2) \|x_1\|^{-1} x_1$ and therefore

$(\varepsilon/2)\,\|x_1\|^{-1}x_1$ and hence x_1 is a positive multiple of x_0 since $x_0 \in \mathrm{extr}(C_G)$. This shows that $\dim(C_G) = 1$ and so $\dim(G) = n - 1$ by Lemma 2-5b.

Now suppose that $\dim(G) = n - 1$. Then using Theorem 1-16, Lemma 2-5b, and Theorem 2-7, it follows that $x_0 \in \mathrm{relint}(C_G)$ and that C_G is a ray. Suppose that $x_0 = x_1 + x_2$, $x_1, x_2 \in C_F$, and suppose that $x_1 \notin C_G$. Then there exists some $v \in \mathrm{ext}(G)$ such that $x_1 \cdot v < f_P(x_1)$. Now since f_P is linear on C_F,

$$x_0 \cdot v = x_1 \cdot v + x_2 \cdot v < f_P(x_1) + f_P(x_2) = f_P(x_0),$$

a contradiction. Therefore $x_1 \in C_G$ and similarly $x_2 \in C_G$ which implies that $x_0 \in \mathrm{extr}(C_F)$. ∎

CHAPTER III

INDECOMPOSABILITY OF POLYTOPES

In this chapter, the problem of expressing a given polytope P as a sum of other polytopes will be considered. This problem is motivated by the well-known fact that in E_2, every polytope can be written as a finite sum of simplices, i.e. points, line segments and triangles (cf. Yaglom and Boltyanski, [8], p.177). With this in mind, it is reasonable to make the following conjecture: In E_n, every polytope can be written as a finite sum of simplices. However, it has recently been shown that this conjecture is false for $n > 2$ (cf. Shephard, [5]). The next theorem gives a necessary condition for a polytope to be expressable as a finite sum of simplices.

<u>Theorem 3-1.</u> Let P be a polytope in E_n. If P can be expressed as a finite sum of simplices then all of its faces can be also.

Proof: Suppose that $P = \sum_1^t S_i$ where each S_i is a simplex. Let F be a face of P, say $F = H(P,x_0) \cap P$, $x_0 \neq 0$. Then by an easy generalization of Lemma 1-18,

$$F = H(P,x_0) \cap P = H(\sum_1^t S_i, x_0) \cap \sum_1^t S_i = \sum_1^t [H(S_i,x_0) \cap S_i],$$

and each $H(S_i,x_0) \cap S_i$ is a simplex. ▌

If all of the proper faces of a polytope P are expressable as a finite sum of simplices, it does not follow that P has this property.

This is true because as mentioned above, there exist polytopes in E_3 which cannot be decomposed into a finite sum of simplices, but every polytope in E_3 has the property that its proper faces all have such a decomposition.

Theorem 3-4 will provide a useful characterization of when one polytope is a summand of another. First, some preliminary definitions and results will be needed.

Definition 3-1. Let P and Q be polytopes in E_n. Then $P \geq Q$ if and only if $\dim[H(P,x_0) \cap P] \geq \dim[H(Q,x_0) \cap Q]$ for all $x_0 \neq 0$.

Comparing this definition with Definition 2-2, the following result is immediate.

Theorem 3-2. Let P and Q be polytopes in E_n. Then $P \approx Q$ if and only if $P \geq Q$ and $Q \geq P$.

The following theorem is the analogue of Theorem 2-5 and characterizes the relation $P \geq Q$ in terms of the facial cones.

Theorem 3-3. Let P and Q be polytopes in E_n. Then $P \geq Q$ if and only if there exists a function defined from ext(P) to ext(Q), say $v \to w$, such that $C(P,v) \subset C(Q,w)$.

Proof: Suppose first that $P \geq Q$. If $v \in \text{ext}(P)$, then for some $x_0 \neq 0$, $\{v\} = H(P,x_0) \cap P$. Since $P \geq Q$, $H(Q,x_0) \cap Q$ is some extreme point, say w, of Q. The same proof as that of Theorem 2-4 shows that the correspondence $v \to w$ is a function from $\text{ext}(P)$ to $\text{ext}(Q)$. That is, if also $\{v\} = H(P,x_1) \cap P$, then $\{w\} = H(Q,x_1) \cap Q$. Now to show that $C(P,v) \subset C(Q,w)$, it is sufficient to show that $\text{int}[C(P,v)] \subset \text{int}[C(Q,w)]$. Let $x_1 \in \text{int}[C(P,v)]$. Then by Theorem 1-16, $H(P,x_1) \cap P = \{v\}$. Therefore $H(Q,x_1) \cap Q = \{w\}$ and so again by Theorem 1-16, $x_1 \in \text{int}[C(Q,w)]$.

Now suppose that $v \to w$ is any function from $\text{ext}(P)$ to $\text{ext}(Q)$ for which $C(P,v) \subset C(Q,w)$. For any $x_0 \neq 0$, let $F = H(P,x_0) \cap P$ and $G = H(Q,x_0) \cap Q$. Suppose that $\text{ext}(F) = \{v_1, v_2, \cdots, v_m\}$. It will now be shown that $\text{ext}(G) = \{w_1, w_2, \cdots, w_m\}$. Let $w \in \text{ext}(G)$. Using Theorem 1-16, $x_0 \in \text{relint}[C(Q,G)] \subset \text{int}[C(Q,w)]$. Now by Theorem 1-14, there exists some $v \in \text{ext}(P)$ such that $x_0 \in C(P,v)$. Then since $C(P,v) \cap \text{int}[C(Q,w)] \neq \emptyset$, it follows that $C(P,v) \subset C(Q,w)$ and hence $v \to w$. Now since $x_0 \in C(P,v)$, $v \in \text{ext}(P) \cap H(P,x_0) = \text{ext}(F) = \{v_1, \cdots, v_m\}$ and therefore $w \in \{w_1, \cdots, w_m\}$.

Now let $w_j \in \{w_1, \cdots, w_m\}$. Then $v_j \in \text{ext}(F)$ and using Theorem 1-15, $x_0 \in C(P, F) \subset C(P,v_j) \subset C(Q,w_j)$, which implies that $w_j \in \text{ext}(Q) \cap H(Q,x_0) = \text{ext}(G)$. This completes the proof that $\text{ext}(G) = \{w_1, w_2, \cdots, w_m\}$. From this fact it follows that

$$C(P,F) = \bigcap_1^m C(P,v_j) \subset \bigcap_1^m C(Q,w_j) = C(Q,G).$$

Therefore, using Lemma 2-5b,

$$\dim(F) + \dim[C(P,F)] = n = \dim(G) + \dim[C(Q,G)],$$

and hence, $\dim(F) - \dim(G) = \dim[C(Q,G)] - \dim[C(P,F)] \geq 0$, which implies that $\dim(F) \geq \dim(G)$. ∎

Now suppose that $P \geq Q$ and suppose that vv' is an edge of P. Let $v \to w$ and $v' \to w'$. If $w \neq w'$, then ww' is an edge of Q parallel to vv'. Thus $w - w' = \alpha(v - v')$ for some $\alpha \geq 0$. If for every edge of P, the α so described satisfies the property $\alpha \leq 1$, then the notation $P \geqq Q$ (rather than $P \geq Q$) will be used.

Theorem 3-4. Let P and Q be polytopes in E_n. Then Q is a summand of P if and only if $P \geqq Q$.

The proof of this result is given by Shephard, [5]. See Figures 1-1 and 1-4 for examples in E_2.

It is easy to see that any polytope P always possesses summands, for if $0 \leq t \leq 1$ and x_0 is any fixed vector, then

$$P = [tP + x_0] + [(1-t)P - x_0].$$

However, in this type of decomposition, the summands are positively homothetic to P. This prompts the following definition:

Definition 3-2. Let P be a polytope in E_n. Then P is said to be decomposable if and only if P has a non-degenerate summand which is not positively homothetic to P. If P is not decomposable, then P is called indecomposable.

Examples of decomposable polytopes are abundant. Some examples of indecomposable polytopes will be given later. The following theorem characterizes indecomposability in terms of local similarity.

Theorem 3-5. Let P be a polytope in E_n. Then P is indecomposable if and only if $P \approx Q$ implies that $P \sim Q$.

Proof: Suppose first that P is indecomposable and let $P \approx Q$. Choose a real $\alpha > 0$ small enough so that $\alpha Q \leqq P$. Then by Theorem 3-4, αQ is a summand of P, and hence by Definition 3-2, $\alpha Q \sim P$ which implies $Q \sim P$.

Now suppose that $P \approx Q$ implies $P \sim Q$ and suppose that A is a non-degenerate summand of P. Then by Theorem 3-4, $A \leqq P$. Let $v \to a$ denote the correspondence defined by Theorem 3-3. Since $A \leqq P$, for each edge $v_i v_j$ of P, $a_i - a_j = \alpha_{ij}(v_i - v_j)$, $0 \leq \alpha_{ij} \leq 1$. Choose an $\alpha > 0$ such that $\alpha\alpha_{ij} < 1$ for each edge of P and let $A_0 = \alpha A$. Then $A_0 \leqq P$ and so by Theorem 3-4, there exists a polytope B_0 for which $P = A_0 + B_0$. Now also by Theorem 3-4, $B_0 \leqq P$. Let $x_i = \alpha a_i$ for each $a_i \in \text{ext}(A)$ and let the variable y (with subscripts) denote extreme points of B_0 where $v \to y$ is the correspondence from $\text{ext}(P)$ to $\text{ext}(B_0)$ defined by Theorem 3-3. Now for any edge $v_i v_j$ of P,

$$x_i - x_j = \delta_{ij}(v_i - v_j), \quad 0 \leq \delta_{ij} < 1 \quad (\delta_{ij} = \alpha\alpha_{ij}),$$

and

$$y_i - y_j = \beta_{ij}(v_i - v_j), \quad 0 \leq \beta_{ij} \leq 1.$$

Now using Lemma 1-18, $x_i + y_i = v_i$ and $x_j + y_j = v_j$ and thus $\delta_{ij} + \beta_{ij} = 1$ which implies that each $\beta_{ij} > 0$. This means that $B_0 \approx P$. Thus by assumption, $B_0 \sim P$, say $B_0 = \rho P + y_0$ where $\rho > 0$. Note that $\rho \leq 1$ since $B_0 \leqq P$. Thus,

$$\rho P + (1-\rho)P = P = A_0 + B_0 = A_0 + \rho P + y_0,$$

which implies that $A_0 = (1-\rho)P - y_0$, i.e. $A_0 \sim P$ and hence $A \sim P$. ∎

Theorem 3-5 will now be used to show that all simplices are indecomposable.

<u>Theorem 3-6.</u> Let P be a simplex in E_n. Then P is indecomposable.

Proof: Suppose that $P \approx Q$ where $\text{ext}(P) = \{v_1, v_2, \cdots, v_m\}$,

$\text{ext}(Q) = \{w_1, w_2, \dots, w_m\}$ and $v_i \leftrightarrow w_i$ is the correspondence between $\text{ext}(P)$ and $\text{ext}(Q)$ of Theorem 2-4. Now since P is a simplex, v_1v_2, v_2v_3, $\dots$, $v_{m-1}v_m$ and v_mv_1 are all edges of P. Since $P \approx Q$,

$$\begin{aligned} w_2 - w_1 &= \alpha_1(v_2 - v_1), \quad \alpha_1 > 0, \\ w_3 - w_2 &= \alpha_2(v_3 - v_2), \quad \alpha_2 > 0, \\ &\vdots \\ w_m - w_{m-1} &= \alpha_{m-1}(v_m - v_{m-1}), \quad \alpha_{m-1} > 0, \\ w_1 - w_m &= \alpha_m(v_1 - v_m), \quad \alpha_m > 0. \end{aligned}$$

Adding corresponding sides of these equalities yields

$$0 = (\alpha_m-\alpha_1)v_1+(\alpha_1-\alpha_2)v_2+(\alpha_3-\alpha_2)v_3+ \cdots +(\alpha_{m-2}-\alpha_{m-1})v_{m-1}+(\alpha_{m-1}-\alpha_m)v_m$$

$$\text{and } (\alpha_m-\alpha_1)+(\alpha_1-\alpha_2)+(\alpha_3-\alpha_2)+ \cdots +(\alpha_{m-2}-\alpha_{m-1})+(\alpha_{m-1}-\alpha_m) = 0.$$

Therefore since $\text{ext}(P)$ is affinely independent,

$$\alpha_m = \alpha_1 = \alpha_2 = \cdots = \alpha_{m-2} = \alpha_{m-1}.$$

Letting α denote this common value, it follows that

$$\begin{aligned} w_2 &= \alpha v_2 + (w_1 - \alpha v_1), \\ w_3 &= \alpha v_3 + (w_2 - \alpha v_2) = \alpha v_3 + (w_1 - \alpha v_1), \\ &\vdots \qquad\qquad\qquad\qquad \vdots \\ w_m &= \alpha v_m + (w_{m-1} - \alpha v_{m-1}) = \alpha v_m + (w_1 - \alpha v_1), \\ w_1 &= \alpha v_1 + (w_m - \alpha v_m) = \alpha v_1 + (w_1 - \alpha v_1). \end{aligned}$$

Therefore, $Q \sim P$. ∎

The following definition and theorem together with Theorem 3-6 will provide numerous examples of indecomposable polytopes which are not simplices and hence cannot be written as a sum of finitely many simplices.

<u>Definition 3-3.</u> Let P be a polytope in E_n and let $K = \{F_1, F_2, \dots, F_m\}$ be a collection of r-faces of P, $2 \le r \le n-1$, such that

$\dim(F_i \cap F_{i+1}) > 0$ for $i = 1, 2, \ldots, m-1$. Then K is called a chain of faces of P. The chain K is called an indecomposable chain if each F_i is indecomposable. Any extreme point or edge of F_1 is said to be connected by K to any extreme point or edge of F_m.

<u>Theorem 3-7.</u> Let P be a polytope in E_n. If there exists an edge of P to which each extreme point of P can be joined by an indecomposable chain of faces of P, then P is indecomposable.

The proof of this theorem is given by Shephard, [5].

In view of Theorems 3-6 and 3-7, it is clear that any pyramid in E_3 formed by taking the convex hull of a 2-polytope F and a point $x_0 \notin fh(F)$ is indecomposable (see Figure 3-1). For such a pyramid P, let x_1 be a point above one facet of P and below all the other facets of P. Then the polytope $P_1 = conv(P \cup \{x_1\})$ is indecomposable. This process can be repeated on one of the newly created facets of P_1, or one of the other facets of P, each time resulting in an indecomposable polytope (see Figure 3-1).

Theorems 3-6 and 3-7 also show that any simplicial polytope, i.e. one whose facets are all simplices, is indecomposable. The next result gives a necessary condition for indecomposability.

<u>Theorem 3-8.</u> Let P be an indecomposable polytope in E_n, $\dim(P) \geq 2$. Then for each edge E of P, there exists some $x_0 \in \langle C(P,E)\rangle$, $x_0 \neq 0$, such that $H(P,x_0) \cap P \in ext(P)$.

Proof: Suppose that there exists an edge E of P such that for all

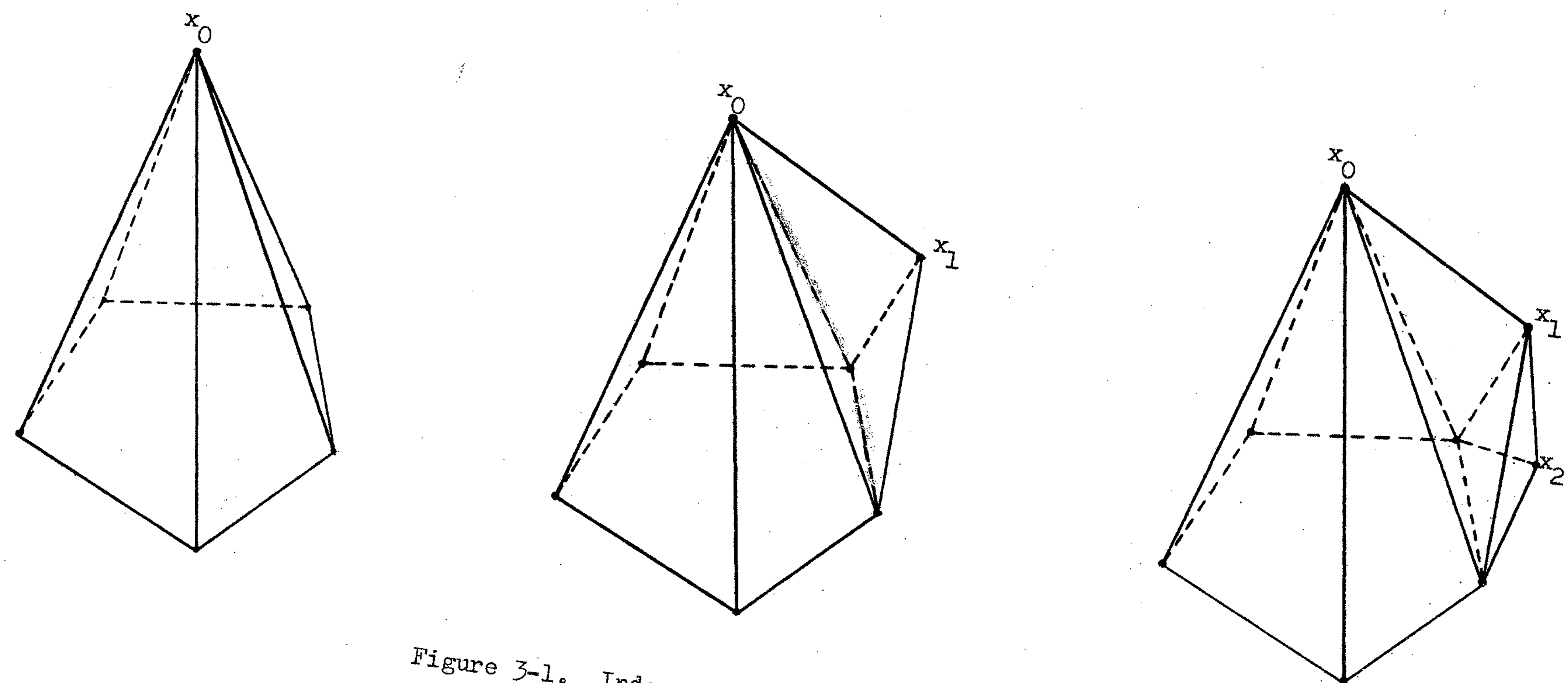

Figure 3-1. Indecomposable Polytopes in E_3.

$x_0 \in \langle C(P,E)\rangle$, $x_0 \neq 0$, $\dim[H(P,x_0) \cap P] \geq 1$. Choose $y_1, y_2 \in E$ such that $\|y_1 - y_2\| \leq \|y - y'\|$ for every edge yy' of P. Let $Q = y_1y_2$ and it will now be shown that Q is a summand of P. Let $x_0 \in E_n$, $x_0 \neq 0$. It will be shown first that $\dim[H(P,x_0) \cap P] \geq \dim[H(Q,x_0) \cap Q]$. If $\dim[H(Q,x_0) \cap Q] = 0$, this is true. Otherwise, $H(Q,x_0) \cap Q = Q$. Without loss of generality, assume that $0 \in Q$. Then using Theorem 1-15 and Lemma 2-5b, $x_0 \in C(Q,Q) = Q^{\perp} = E^{\perp} = \langle C(P,E)\rangle$. Therefore, by assumption, $\dim[HQ,x_0) \cap Q] = \dim(Q) = 1 \leq \dim[H(P,x_0) \cap P]$.

So far, it has been established that $Q \leq P$. By choosing y_1 and y_2 such that $\|y_1 - y_2\| \leq \|y - y'\|$ for all edges yy' of P, it is clear that $Q \leqq P$ and so by Theorem 3-4, Q is a summand of P which contradicts the assumption that P is indecomposable.∎

It will now be shown that certain types of transformations from E_n to E_n preserve indecomposability and decomposability.

<u>Theorem 3-9.</u> If f is a non-singular linear transformation from E_n to E_n and P is a polytope in E_n, then P is indecomposable if and only if $f(P)$ is indecomposable.

Proof: First of all, $f(P)$ is a polytope since

$$f(P) = f(\text{conv}[\text{ext}(P)]) = \text{conv}(f[\text{ext}(P)]),$$

(cf. Grünbaum, [2], p. 21, Th. 10). Suppose now that P is indecomposable and that $f(P) = A + B$. Then $P = f^{-1}(A + B) = f^{-1}(A) + f^{-1}(B)$. Thus $f^{-1}(A) = \alpha P + x_0$ and $f^{-1}(B) = \beta P + y_0$ where $\alpha > 0$ and $\beta > 0$. Therefore, $A = f(\alpha P + x_0) = \alpha f(P) + f(x_0)$, and similarly $B = \beta f(P) + f(y_0)$, which means that $f(P)$ is indecomposable.

Now if $f(P)$ is indecomposable, then P is indecomposable by applying the preceding argument to f^{-1}. ∎

Another type of transformation which preserves indecomposability and decomposability is the affine transformation defined as follows:

Definition 3-4. Let g be a transformation from E_n to E_n. Then g is called affine if and only if $g[\alpha x + (1-\alpha)y] = \alpha g(x) + (1-\alpha)g(y)$ for all real α and x and $y \in E_n$.

The next result characterizes the affine transformations in terms of linear transformations.

Theorem 3-10. Let g be a transformation from E_n to E_n. Then g is affine if and only if g has the form $g(x) = f(x) + x_0$ where f is linear and x_0 is fixed.

Proof: Suppose first that g can be expressed as $g(x) = f(x) + x_0$ where f is linear. Then,

$$\begin{aligned} g[\alpha x + (1-\alpha)y] &= f[\alpha x + (1-\alpha)y] + x_0 \\ &= \alpha f(x) + (1-\alpha)f(y) + x_0 \\ &= \alpha[f(x) + x_0] + (1-\alpha[f(y) + x_0] \\ &= \alpha g(x) + (1-\alpha)g(y). \end{aligned}$$

Now suppose that g is affine and define f as follows: $f(x) = g(x) - g(0)$. The proof will be completed by showing that f is linear.

(i) Homogeneity:

$$\begin{aligned} f(\alpha x) &= f[\alpha x + (1-\alpha)0] \\ &= g[\alpha x + (1-\alpha)0] - g(0) \\ &= \alpha g(x) + (1-\alpha)g(0) - g(0) \\ &= \alpha[g(x) - g(0)] \\ &= \alpha f(x). \end{aligned}$$

(ii) Additivity: $$\begin{aligned} f(x + y) &= f[2(\tfrac{1}{2}x + \tfrac{1}{2}y)] \\ &= 2f(\tfrac{1}{2}x + \tfrac{1}{2}y) \\ &= 2[g(\tfrac{1}{2}x + \tfrac{1}{2}y) - g(0)] \\ &= 2[\tfrac{1}{2}g(x) + \tfrac{1}{2}g(y) - g(0)] \\ &= [g(x) - g(0)] + [g(y) - g(0)] \\ &= f(x) + f(y). \blacksquare \end{aligned}$$

Now if g is an affine transformation, say $g(x) = f(x) + x_0$ where f is linear, then g is called non-singular if and only if f is non-singular. As in the case of linear transformations, an affine transformation is non-singular if and only if it is one to one and onto.

Theorem 3-11. Let g be a non-singular affine transformation from E_n to E_n and let P be a polytope in E_n. Then P is indecomposable if and only if $g(P)$ is indecomposable.

Proof: By Theorem 3-10, g has the form $g(x) = f(x) + x_0$ where f is a non-singular linear transformation. By Theorem 3-9, P is indecomposable if and only if $f(P)$ is indecomposable. Therefore, the theorem follows since $g(P) = f(P) + x_0$. $\blacksquare$

The remainder of this chapter will be concerned with characterizing the indecomposable polytopes in terms of their support functionals. To do this, it is necessary to consider the Steiner point of a polytope.

Definition 3-5. Let P be a polytope in E_n and let $S_{n-1} = \{x: \|x\|=1\}$. The Steiner point of P, $S(P)$, is defined as follows:

$$S(P) = \sum_{i=1}^{m} \frac{\mu[C(P,v) \cap S_{n-1}]}{\mu(S_{n-1})} v_i ,$$

where $\text{ext}(P) = \{v_1, v_2, \cdots, v_m\}$.

From this definition, and in view of Theorem 1-14 and Corollary 1-16b, it follows that $S(P)$ is a strictly positive convex combination of $\text{ext}(P)$ and hence $S(P) \in \text{relint}(P)$. Some other properties of the Steiner point are given in the following theorem.

<u>Theorem 3-12.</u> The Steiner point has the following properties:

(a) $S(P + Q) = S(P) + S(Q)$,

(b) $S(\alpha P) = \alpha S(P)$,

(c) $S(\{x\}) = x$.

For the proof of these facts, see Grünbaum, [2], p. 308.

From this point on, only polytopes P for which $S(P) = 0$ will be considered. In this setting, indecomposability has the following form:

<u>Theorem 3-13.</u> Let P be a polytope in E_n, $S(P) = 0$. Then P is indecomposable if and only if $P = Q + R$ where $S(Q) = S(R) = 0$ implies that $Q = \alpha P$ and $R = \beta P$ where $\alpha > 0$ and $\beta > 0$.

Proof: First suppose that P is indecomposable and that $P = Q + R$ where $S(Q) = S(R) = 0$. Then by Definition 3-2, $Q = \alpha P + x_0$ and $R = \beta P + x_1$ where $\alpha > 0$ and $\beta > 0$. Then

$$0 = S(Q) = S(\alpha P + x_0) = \alpha S(P) + S(\{x_0\}) = x_0,$$

and similarly, $x_1 = 0$.

Now suppose that $P = Q + R$ where $S(Q) = S(R) = 0$ implies that

Q and R are positive multiples of P. Let $P = A + B$. Then

$$0 = S(P) = S(A + B) = S(A) + S(B),$$

and hence $P = [A - S(A)] + [B - S(B)]$ and $S[A - S(A)] = S[B - S(B)] = 0$. Therefore, $A - S(A) = \alpha P$ and $B - S(B) = \beta P$ where $\alpha > 0$ and $\beta > 0$ which implies that $A \sim P$ and $B \sim P$ and hence P is indecomposable by Definition 3-2.∎

Now let $\mathcal{S} = \{f_P: P \in \mathcal{P} \text{ and } S(P) = 0\}$. The next result shows that $\mathcal{S}$ is a convex cone in the space of functionals on E_n.

<u>Theorem 3-14.</u> The set $\mathcal{S}$ is a convex cone.

Proof: (1) If f_P and $f_Q \in \mathcal{S}$, then $f_P + f_Q = f_{P+Q}$ and $S(P + Q) = S(P) + S(Q) = 0$ implies that $f_P + f_Q \in \mathcal{S}$.

(2) If $f_P \in \mathcal{S}$ and $\alpha \geq 0$, then $\alpha f_P = f_{\alpha P}$ and $S(\alpha P) = \alpha S(P) = 0$ implies that $\alpha f_P \in \mathcal{S}$.∎

The next theorem characterizes the indecomposable polytopes as those whose support functionals are extremal elements of $\mathcal{S}$.

<u>Theorem 3-15.</u> Let P be a polytope in E_n, $S(P) = 0$. Then P is indecomposable if and only if $f_P \in \text{extr}(\mathcal{S})$.

Proof: Suppose first that P is indecomposable and that $f_P = f_Q + f_R$ where $S(Q) = S(R) = 0$. Then $f_P = f_{Q+R}$ and thus $P = Q + R$ by Theorem 1-5, part (e). Therefore, by Theorem 3-13, $Q = \alpha P$ and $R = \beta P$ where $\alpha > 0$ and $\beta > 0$. Therefore $f_Q = f_{\alpha P} = \alpha f_P$ and $f_R = f_{\beta P} = \beta f_P$ which implies that $f_P \in \text{extr}(\mathcal{S})$ by Definition 2-3.

Now suppose that $f_P \in \text{extr}(\mathcal{S})$ and let $P = Q + R$ where

$S(Q) = S(R) = 0$. Then $f_P = f_{Q+R} = f_Q + f_R$, which implies that $f_Q = \alpha f_P$ and $f_R = \beta f_P$ where $\alpha > 0$ and $\beta > 0$. Thus $f_Q = f_{\alpha P}$ and $f_R = f_{\beta P}$ which implies that $Q = \alpha P$ and $R = \beta P$ and hence P is indecomposable by Theorem 3-13.∎

CHAPTER IV

SUMMARY AND CONCLUSIONS

Chapter I began with some elementary facts about convex sets which apply in particular to polytopes. Two characterizations of the support functional of a polytope were obtained. One of these characterizations established that $f_P(x) = \max\{x \cdot v: v \in ext(P)\}$. This prompted the definition of facial cones, the sets $C(P,F) = \{x: x \cdot v = f_P(x) \; \forall v \in ext(F)\}$ where F is any face of P.

It was shown that these facial cones were all convex cones with vertices at the origin. Also, useful characterizations of $C(P,F)$, $relint[C(P,F)]$, and $relbd[C(P,F)]$ were given using support hyperplanes for P. These facial cones were also used to characterize the faces of the sum of two polytopes.

In Chapter II, the facial cones were used to characterize local similarity of polytopes. The most important result needed was the fact that for any face F of a polytope P, such that $0 \in F$, $\langle F \rangle$ and $\langle C(P,F) \rangle$ are orthogonal complements.

It was then established that for n-dimensional polytopes in E_n, each facial cone $C(P,F)$ is salient whereas for lower dimensional polytopes in E_n, all of the facial cones are non-salient. This information was used to characterize the extremal elements of the facial cones.

In Chapter III, the concepts of indecomposability and decomposability of polytopes were defined. Shephard, [5], characterized when one polytope is a summand of another. This result was used to prove that a polytope P is indecomposable if and only if every polytope locally similar to P must be positively homothetic to P.

It was then shown that every simplex is indecomposable. In E_n, $n > 2$, this result together with a sufficient condition for indecomposability due to Shephard yields many examples of indecomposable polytopes that are not simplices. Another result provided a necessary condition for a polytope to be indecomposable in terms of its edges and extreme points.

In E_2, every polytope can be expressed as a finite sum of simplices, however this result does not generalize to higher dimensions. It was shown that a necessary condition for a polytope to be expressable as a finite sum of simplices is that each of its faces be expressable as a finite sum of simplices. An interesting problem which remains unsolved is that of characterizing the polytopes which can be expressed as a finite sum of simplices in E_n for $n > 2$.

It was established that non-singular linear and affine transformations preserve indecomposability and decomposability.

Finally, it was shown that the extremal elements of the convex cone of support functionals of polytopes with Steiner point at the origin are precisely those of the indecomposable polytopes. An unsolved problem concerning this result would be to characterize this cone of functionals and its extremal elements in such a way as to shed new light on indecomposability of polytopes.

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