RESOURCE ALLOCATION IN PROJECT STAGES TREATED AS A MARKOV PROCESS

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RESOURCE ALLOCATION IN PROJECT STAGES

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PREFACE

This dissertation is concerned with minimizing the total cost of allocating resources in the activities or stages of a project. A serial activity project is considered and treated as a finite state Markov Process. Three time estimates for each activity are used to define a triangular probability density function of completion time for the activity; the most likely time estimate is resource level modified. There is an allowable range in which the level of resource must lie for each activity. In addition, the over-all mean project duration is specified.

The resulting set of equations are linear in the case where the periodic review time interval is less than the most likely activity duration; this case is solved by linear programming techniques. The case where the review time interval is greater than the most likely activity duration yields nonlinear equations and is solved by Lagrange multiplier techniques.

I would like to take this opportunity to express my appreciation for the assistance and encouragement given me by the following members of my committee: Dr. John L. Folks, Dr. James E. Shamblin, Dr. G. T. Stevens, Dr. M. Palmer Terrell, and Dr. Jack M. Walden. I am particularly indebted to Dr. Shamblin for having directed me around the pitfalls that were inherent in this research study.

I would like to express appreciation to my wife, Marcia, for typing the rough draft and her encouragement in the preparation of this dissertation. I would also like to thank Miss Velda Davis for her valuable suggestions and typing the final draft.

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INTRODUCTION

A problem old to management is one of allocating resources among the various stages of a given project. Project in this context is a singular effort rather than a repetitive one completed on a production basis. Management must continually plan and schedule largely on the basis of experience with similar projects. This experience is varied and not always applicable; management needs more quantitative approaches that reduce value judgments and decision errors. It is this problem that shall be considered in this paper.

Historically, the first approach considered is that of Capital Budgeting where selection of projects to which time and money are allocated is well treated in the literature. For example, William Karush (1) has developed an algorithm for maximizing the payoff from activities using piecewise linear return functions subject to the allowable resource range of each activity. H. Martin Weingartner (2) considers the problem of project interrelationships such as mutual exclusion and interdependencies; the model he develops also includes non-linear utility functions. To anyone primarily concerned with Capital Budgeting, these papers provide a good approach to the problem and can yield further developments. However, this thesis is concerned with resource allocation in the stages of a selected project and the following references were found to be more relevant.

Critical Path Methods (CPM) were developed in the late 1950's and involve a graphical portrayal of the elements or stages and their interrelationships as a network; they include an arithmetic procedure which identifies the relative importance of each element in the over-all project. The budgeting of time and money within the stages or activities of a given project is treated in the literature by means of a cost versus activity time relationship for each project stage. For example, D. R. Fulkerson (3) considers a linear (with time) cost function defined by a normal activity time cost and a crash or expedited activity time cost as shown by the following Figure 1.

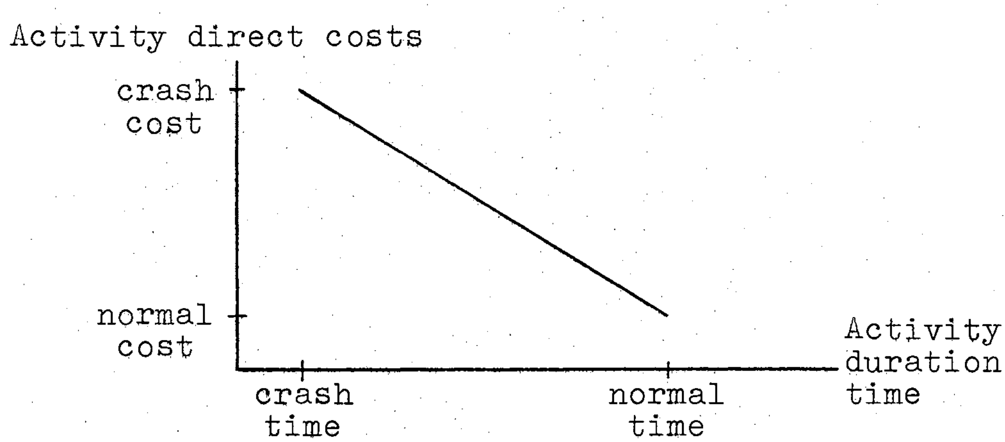


Figure 1. Linear Activity Cost Time Function

J. E. Kelley (4) develops a parametric linear programming model, using linear activity cost functions, by means of a computer search. All feasible solutions (including the minimum cost) are found for a given overall project duration. E. B. Berman (5) treats the case of a non-linear concave upward time-cost relationship of the following form:

$$C(t) = a + bt + \frac{c}{t-d}$$

where a, b, c, and d are positive constants and t is the activity time duration. It is seen that cost will increase for both short and long times in this type of expression. Berman has developed an iterative algorithm in which the resources are allocated so that the time-cost functions have equal slopes along a serial path. Berman considers uncertainty in one activity at a time along a serial path with the activity time duration distribution as portrayed in Figure 2.

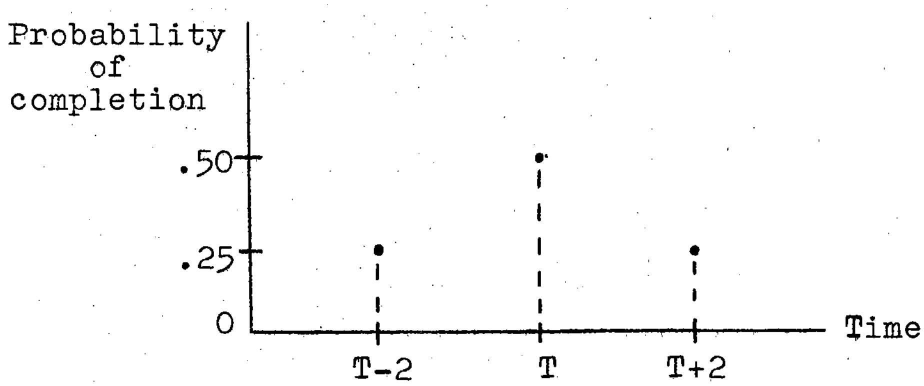


Figure 2. Activity Time Duration Probability Distribution

Berman finds the effect on the system is to shift the uncertain event and its predecessors to shorter duration times, thus maintaining the same over-all project time.

This thesis, in contrast, considers the activity or project stage to have a probabilistic time duration whose most likely completion time is a function of the resource level applied to the activity as illustrated in Figure 3. The shortest and longest time durations are assumed unaffected by the stage resource level.

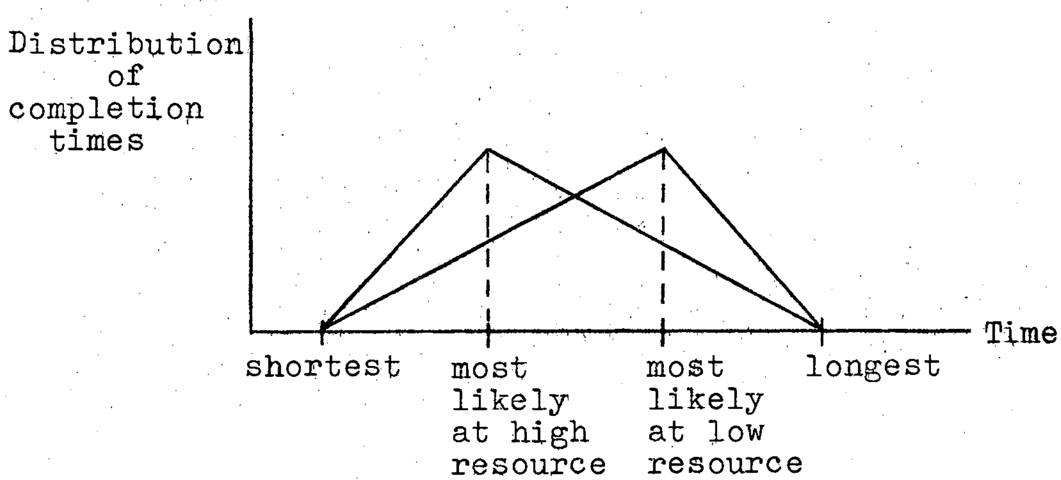


Figure 3. Distribution of Activity Completion Times

The type of system studied in this work is a project made up of a series of stages that are carried out consecutively with each stage being completed before the next one is permitted to begin. The nature of each stage dictates the allowable range of resource applied to it.

For example, there is a minimum as well as a maximum amount of money which must be spent in order to complete a stage. Spending less than the minimum money, the stage will not be completed and spending over the maximum will result in excessive waste. Furthermore, there is a defined length of time in which a project can be undertaken and completed. The optimization of the sequential stage project in this paper is based on constraining the expected duration of the over-all project to some specified time and adjusting the resource level at each stage to meet this constraint at minimum resource cost. This over-all time constraint is in addition to the individual stage resource constraints.

The approach used in analyzing this problem is to consider each stage of the project as having a certain probability of being completed during a given review time interval. Assuming the stages are independent and the probabilities of completion do not change with time or the project's progress, the state of the system can be described as a first order finite state Markov Process. Because the system is reviewed at constant time intervals, it can be treated as a discrete time Markov Chain. This treatment is possible since a finite Markov Chain is a stochastic or time varying process which moves through a finite number of states; the probability of entering a certain state only depending on the last state occupied. (For examples of Markov Processes, see Reference (6)).

CHAPTER II

MARKOV CHAIN FORMULATION

The project is considered as a series of sequential steps, or stages, each of which must be completed before the subsequent stage is started. The probability of completion of each stage during a given time interval can be represented as a Markov process probability transition matrix.

Considering the probabilities to be independent of time and history of the system the transition matrix takes the following form for a five stage project.

$$
\begin{array}{ll}
 & \text{Future Stage} \\
 & \begin{array}{ccccc} S1 & S2 & S3 & S4 & S5 \end{array} \\
\text{Present Stage} \quad \begin{array}{c} S1 \\ S2 \\ S3 \\ S4 \\ S5 \end{array} &
\begin{bmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\
P_{21} & P_{22} & P_{23} & P_{24} & P_{25} \\
P_{31} & P_{32} & P_{33} & P_{34} & P_{35} \\
P_{41} & P_{42} & P_{43} & P_{44} & P_{45} \\
P_{51} & P_{52} & P_{53} & P_{54} & P_{55}
\end{bmatrix} = \underline{P}
\end{array}
$$

Here, P_{ij} is the conditional probability of the project going from stage i to stage j during one review time interval.

Assuming that completed stages of the project cannot become undone, the P_{ij} terms equal zero where $i > j$. If it is assumed that the probabilities are essentially zero for completion of more than one stage during a review period, then P_{ij} terms equal zero where $j > i + 1$. With these assumptions, $\underline{P}$ becomes:

Future Stage

Present Stage	S1	S2	S3	S4	S5
S1	P_{11}	P_{12}	0	0	0
S2	0	P_{22}	P_{23}	0	0
S3	0	0	P_{33}	P_{34}	0
S4	0	0	0	P_{44}	P_{45}
S5	0	0	0	0	P_{55}

$= \underline{P}$

The only steps allowed are to stay or advance one stage; no more than one activity can be completed during a review time period. This can be insured by making the review time interval sufficiently small. The probability of two steps, which is desired to be negligible, is the following:

$$P_{i,\ i+2} = \int_0^T Pdf_i(t_i)\left[\int_0^{T-t_i} Pdf_{i+1}(t_{i+1})dt_{i+1}\right]dt_i$$

where T = time duration of review period and the Pdf(t)'s are the completion probability density functions of stages i and $i + 1$.

Since the row probabilities sum to unity in a transition matrix (the future states are completely described), $\underline{P}$ can be rewritten as

$$
\text{Present Stage}\quad
\begin{array}{c}
 \\ S1 \\ S2 \\ S3 \\ S4 \\ S5
\end{array}
\overset{\text{Future Stage}}{
\begin{array}{ccccc}
S1 & S2 & S3 & S4 & S5 \\
\hline
1-P_{12} & P_{12} & 0 & 0 & 0 \\
0 & 1-P_{23} & P_{23} & 0 & 0 \\
0 & 0 & 1-P_{34} & P_{34} & 0 \\
0 & 0 & 0 & 1-P_{45} & P_{45} \\
0 & 0 & 0 & 0 & 1
\end{array}}
= \underline{P}.
$$

Stage 5 is an absorbing stage and may be considered as completion of the project.

The matrix $\underline{P}$ is rearranged and partitioned as follows using the notation of Reference (6), Chapter III, for absorbing finite Markov Chains where the submatrix $\underline{Q}$ represents the process in transient states, submatrix $\underline{R}$ concerns the transition from transient to absorbing states, and submatrix $\underline{I}$ represents the absorbing states. Submatrix $\underline{0}$ consists of zeros.

$$
\underline{P} = \left[\begin{array}{c|c} \underline{I} & \underline{0} \\ \hline \underline{R} & \underline{Q} \end{array}\right]
=
\begin{array}{c}
 \\ S5 \\ S1 \\ S2 \\ S3 \\ S4
\end{array}
\left[\begin{array}{c|cccc}
S5 & S1 & S2 & S3 & S4 \\
1 & 0 & 0 & 0 & 0 \\
\hline
0 & 1-P_{12} & P_{12} & 0 & 0 \\
0 & 0 & 1-P_{23} & P_{23} & 0 \\
0 & 0 & 0 & 1-P_{34} & P_{34} \\
P_{45} & 0 & 0 & 0 & 1-P_{45}
\end{array}\right]
$$

where:

$\underline{I}$ is a unit matrix representing the absorbing stage or project completion.

$\underline{O}$ is a zero matrix representing the absorbing to non-absorbing stage. Once the project is completed it cannot become undone, hence $P_{ij} = 0$ here.

$\underline{R}$ is the non-absorbing to absorbing stage which is the probability of reaching completion.

$\underline{Q}$ is the non-absorbing to non-absorbing stage which represents the stage transition probabilities before completion.

The following is obtained from the $\underline{Q}$ matrix:

$$\underline{I} - \underline{Q} = \begin{array}{c} \\ S1 \\ S2 \\ S3 \\ S4 \end{array} \begin{array}{c} \begin{array}{cccc} S1 & S2 & S3 & S4 \end{array} \\ \begin{bmatrix} P_{12} & -P_{12} & 0 & 0 \\ 0 & P_{23} & -P_{23} & 0 \\ 0 & 0 & P_{34} & -P_{34} \\ 0 & 0 & 0 & P_{45} \end{bmatrix} \end{array}$$

By computing the fundamental matrix, $\underline{N} = [\underline{I} - \underline{Q}]^{-1}$ from the transient stages as below:

$$\underline{N} = [\underline{I} - \underline{Q}]^{-1} = \begin{array}{c} \\ S1 \\ S2 \\ S3 \\ S4 \end{array} \begin{array}{c} \begin{array}{cccc} S1 & S2 & S3 & S4 \end{array} \\ \begin{bmatrix} 1/P_{12} & 1/P_{23} & 1/P_{34} & 1/P_{45} \\ 0 & 1/P_{23} & 1/P_{34} & 1/P_{45} \\ 0 & 0 & 1/P_{34} & 1/P_{45} \\ 0 & 0 & 0 & 1/P_{45} \end{bmatrix} \end{array} .$$

The fundamental matrix $\underline{N}$ yields the mean number of times the system is in the transient states (see page 46, Reference (6)) from which the time per stage is found as:

$$\text{Time in stage } j = n_{ij} = 1/P_{j,j+1} \qquad j = 1, 2, 3, \text{ or } 4 \text{ and } i \leq j.$$

The total time then for the project becomes:

$$N = n_{11} + n_{12} + n_{13} + n_{14} = 1/P_{12} + 1/P_{23} + 1/P_{34} + 1/P_{45} = n_1 + n_2 + n_3 + n_4$$

when the i subscript is dropped. This is used as one of the constraint equations in the model being developed.

Note that the initial probability vector describing the system initially would always start the system at the first stage. The next chapter presents the method of computing individual probability terms.

CHAPTER III

TIME ESTIMATES AND PROBABILITY DENSITY FUNCTIONS

In place of having actual distributions of completion times for each of the project stages, it was decided to use an assumed distribution fitted to three time estimates as in the PERT System. The activity estimates, a, m, b, are defined as: a = the shortest time, m = the most likely time, and b = the longest time the stage is possible to take.

The reference "An Analytical Study of PERT Assumptions" by K. R. MacCrimmon and C. A. Rayavec (7) proposes the use of a triangular distribution as one alternative for the assumed distribution of completion times. They have found that the PERT model would have yielded approximately the same results using a triangular distribution instead of a beta distribution. The triangular distribution is completely defined by the three time estimates.

The resource application level is assumed to affect the most likely completion time only. The assumption that the shortest and longest times remain unchanged is justified since they are the results of unusual circumstances having taken place in the activity which are often beyond control.

The probability of transition to the next stage in one review time period denoted by $P_{i,i+1}$ is given by

$$P_{i,i+1} = \int_0^T f(t)dt \qquad (3\text{-}1)$$

where T is the length of the review time interval. For the triangular distribution (Figure 4),

$$P_{i,i+1} = \int_a^b f(t)dt = 1 \qquad (3\text{-}2)$$

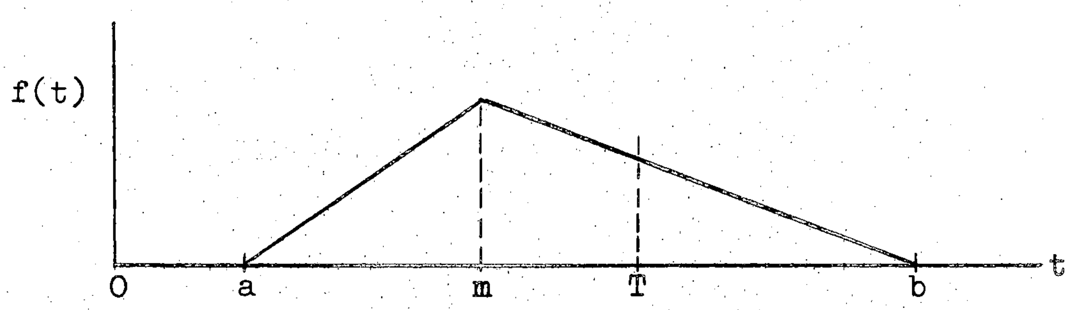


Figure 4. Triangular Distribution for Probable Activity Completion Times

$$\int_0^T f(t)dt = 0 \qquad 0 \leq T \leq a \qquad (3\text{-}3)$$

$$= \int_a^T K_1(t-a)dt \qquad a \leq T \leq m \qquad (3\text{-}4)$$

$$= \int_a^m K_1(t-a)dt + \int_m^T [K_2(t-m) \qquad (3\text{-}5)$$

$$+ K_1(m-a)]dt \qquad m < T < b$$

$$= 1 \qquad b \leq T \qquad (3\text{-}6)$$

where K_1 and K_2 are the slopes of the distribution between a and m and between m and b, respectively.

K_1 and K_2 are evaluated from the pdf boundary conditions, $f(t) = 0$ at $t = a$, and $t = b$, and by setting the area under the curve equal to one. The result is:

$$K_1 = \frac{2}{(m-a)(b-a)}$$

$$K_2 = \frac{-2}{(b-m)(b-a)}.$$

Hence, P[transition to next stage in one review time period]

$$= 0 \qquad 0 \leq T \leq a \qquad (3\text{-}7)$$

$$= \frac{(T-a)^2}{(m-a)(b-a)} \qquad a < T \leq m \qquad (3\text{-}8)$$

$$= 1 - \frac{(T-b)^2}{(b-m)(b-a)} \qquad m < T < b \qquad (3\text{-}9)$$

$$= 1. \qquad b \leq T \qquad (3\text{-}10)$$

Applying the resource modified modal time in the expressions (3-8) and (3-9) where $m = f(R)$ and R is the stage resource level yields:

$$P_{i,i+1} = \frac{(T-a)^2}{[f(R)-a](b-a)} \qquad a \leq T \leq f(R) \qquad (3\text{-}11)$$

$$= 1 - \frac{(T-b)^2}{[b-f(R)](b-a)} \qquad f(R) \leq T \leq b. \qquad (3\text{-}12)$$

Solving the Equations (3-11) and (3-12) for f(R) in terms of n_i, the mean time spent in each stage, yields the following:

$$f(R) = a + \frac{(T-a)^2}{P_{i,i+1}(b-a)} = a + \frac{n_i(T-a)^2}{b-a} \qquad a \leq T \leq f(R) \qquad (3\text{-}13)$$

$$= b - \frac{(T-b)^2}{(1-P_{i,i+1})(b-a)} = b - \frac{(T-b)^2}{(1-1/n_i)(b-a)}$$

$$f(R) \leq T \leq b. \qquad (3\text{-}14)$$

Consider the resource allocation for each stage to have both an upper and a lower bound; the resource applied must be between these limits. If more than the upper limit were allocated, the waste would be excessive; if less than the lower limit were applied, the activity could not be completed. These limits are designated R_H and R_L for the high and low levels. Corresponding to these resource level bounds are the most likely completion time estimates designated m_H and m_L, respectively.

The assumption of a linear relation between the most likely completion time m and the resource level R applied to the stage yields Equation (3-15).

$$f(R) = m = \frac{R-R_L}{R_H-R_L}(m_H - m_L) + m_L. \qquad (3\text{-}15)$$

Rearranging terms to solve for R in terms of f(R) yields Equation (3-16).

$$R = [f(R) - m_L] \cdot \frac{R_H - R_L}{m_H - m_L} + R_L. \qquad (3\text{-}16)$$

Introducing the i stage subscripts for all of the terms and combining Equations (3-13), (3-14), and (3-16) yields:

$$R_i = \left[a_i + \frac{n_i(T-a_i)^2}{b_i - a_i} - m_{Li}\right] \cdot \frac{R_{Hi} - R_{Li}}{m_{Hi} - m_{Li}} + R_{Li}$$

$$a_i \leq T \leq m_i \qquad (3\text{-}17)$$

and

$$R_i = \left[b_i - \frac{(T - b_i)^2}{(1 - 1/n_i)(b_i - a_i)} - m_{Li}\right] \cdot \frac{R_{Hi} - R_{Li}}{m_{Hi} - m_{Li}} + R_{Li}$$

$$m_i \leq T \leq b_i. \qquad (3\text{-}18)$$

Thus, the resource at stage i, R_i, is expressed in terms of the ranges of resource, of most likely completion and of activity time; the expected stage time n_i is the independent variable. These two equations take care of the complete stage activity time range.

Equation (3-17) is linear with respect to n_i and can be rewritten as follows where constants for the stage equation are combined into A_i and B_i.

$$R_i = A_i + B_i n_i \qquad a_i \leq T \leq m_i \qquad (3\text{-}19)$$

where $A_i = (a_i - m_{Li}) \cdot \frac{R_{Hi} - R_{Li}}{m_{Hi} - m_{Li}} + R_{Li}$ and

$$B_i = \frac{(T - a_i)^2}{b_i - a_i} \cdot \frac{R_{Hi} - R_{Li}}{m_{Hi} - m_{Li}}.$$

Equation (3-18) is not linear with respect to n_i and can be rewritten as follows where the stage equation constants are combined into D_i and E_i.

$$R_i = D_i - E_i \frac{n_i}{n_i - 1} \qquad m_i \leq T \leq b_i \qquad (3\text{-}20)$$

where $D_i = (b_i - m_{Li}) \cdot \frac{R_{Hi} - R_{Li}}{m_{Hi} - m_{Li}} + R_{Li}$

and

$$E_i = \frac{(T - b_i)^2}{b_i - a_i} \cdot \frac{R_{Hi} - R_{Li}}{m_{Hi} - m_{Li}}.$$

Equation (3-19) is linear with respect to n_i and represents the case where the review period T falls between the shortest (a) and most likely (m) completion times for stage i. The model formulated from this relationship is solved by linear programming techniques and will be referred to as Case I.

Equation (3-20) is nonlinear in n_i and represents the case where the review period T falls between the most likely (m) and the longest (b) completion times for stage i. The model formulated from this relationship is solved by Lagrange multiplier techniques and will be referred to as Case II.

The selection of the review time interval length must take into account the probability of completion of each stage during the interval. Every stage must have a finite probability of being completed. In addition, the probability of completion of two or more sequential stages during the interval must be essentially zero.

Both of these models are optimized by minimizing the total cost of resource used subject to the allowable range of resource application and a maximum expected over-all project duration.

CHAPTER IV

THE SOLUTION OF CASE I

Case I is the condition where the stage resource level is a linear function of the mean time spent in each stage. This occurs when the review period T lies between the shortest and most likely completion time ($a \leq T \leq m$).

Formulating this case as a linear programming problem for an M stage project (stage M = completion) results in the following:

Objective function

$$\text{Minimize:} \quad f = \sum_{i=1}^{M-1} C_i R_i \tag{4-1}$$

where C_i is the cost of the resource at the i^{th} stage and R_i is replaced by Equation (3-19), yielding

$$f = \sum_{i=1}^{M-1} C_i (A_i + B_i n_i)$$

Constraints

$$\text{Subject to:} \quad \sum_{i=1}^{M-1} n_i \leq N \tag{4-2}$$

$$\text{and} \qquad n_i \geq n_{i\ min} \qquad i = 1, 2, \ldots, M-1 \tag{4-3}$$

$$n_i \le n_{i\ max} \qquad i = 1, 2, \ldots, M-1 \tag{4-4}$$

where $n_{i\ min}$ and $n_{i\ max}$ are the allowable probable completion time ranges for the maximum and minimum stage resource allocation respectively.

The terms $n_{i\ min}$ and $n_{i\ max}$ are computed from Equations (4-5) and (4-6):

$$n_{i\ min} = \frac{(m_{Hi} - a_i)(b_i - a_i)}{(T - a_i)^2} \tag{4-5}$$

$$n_{i\ max} = \frac{(m_{Li} - a_i)(b_i - a_i)}{(T - a_i)^2} \tag{4-6}$$

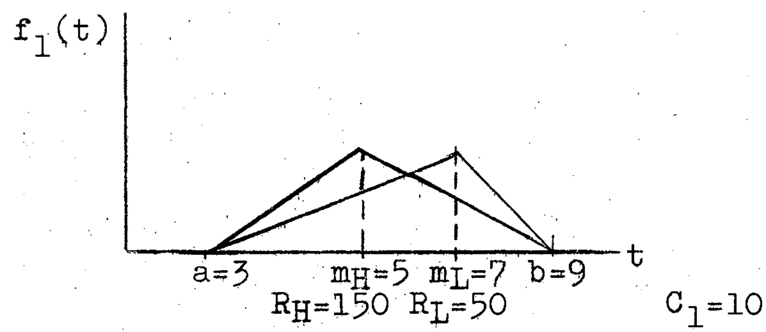
where m_{Li} and m_{Hi} are the most likely completion times for the lowest and highest levels of stage i resource application, respectively.

The following example serves to illustrate Case I. Consider a four stage research proposal (state 4 equals completion) whose over-all duration (N) is specified to be a mean of fifteen days. The activities are serial; each stage of the work must be completed before the succeeding stage can be started. The status of the proposal is reviewed at five day intervals (T). These times would correspond to a five day work week and a three week period in which the work is scheduled to be completed.

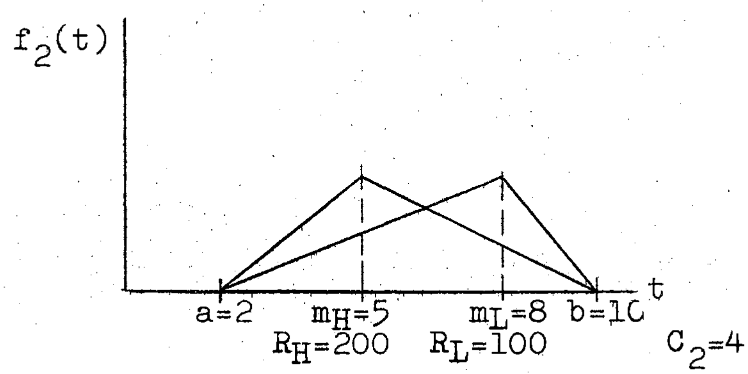
For Stage Number			
1	2	3	
3	2	3	a-shortest possible duration (days)
9	10	8	b-longest possible duration (days)
50	100	40	R_L-lowest stage resource allocation (man-hours)
7	8	7	m_L-most likely duration at lowest resource level (days)
150	200	80	R_H-highest stage resource allocation (man-hours)
5	5	5	m_H-most likely duration at highest resource level (days)
10	4	6	C-cost of resource (\$/hour)

The following graphically illustrates this data:

Stage 1



Stage 2



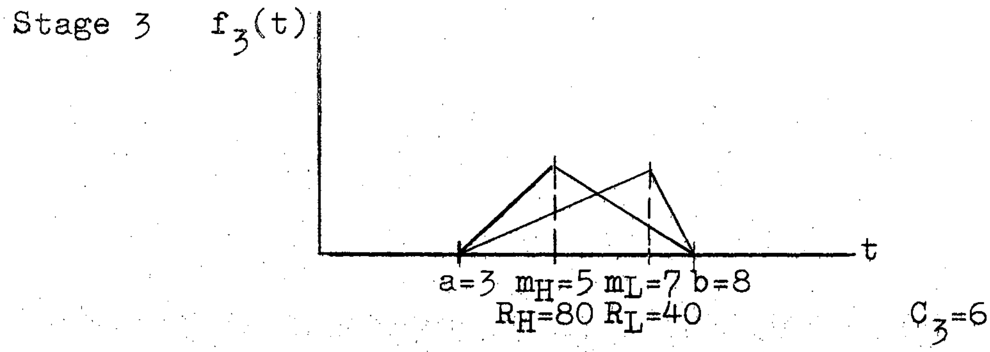


Figure 5. Distribution of Completion Times for Three Stage Example, Case I

Computing the coefficients of the objective function below:

$$A = (a - m_L)\left(\frac{R_H - R_L}{m_H - m_L}\right) + R_L \qquad \text{from Equation (3-19)}$$

$$B = \frac{(T - a)^2}{b - a}\,\frac{R_H - R_L}{m_H - m_L} \qquad \text{from Equation (3-19)}$$

yields the following stage values:

Stage 1	$A_1 = 250$,	$B_1 = -33.3$,	$C_1 = 10$
Stage 2	$A_2 = 300$,	$B_2 = -37.5$,	$C_2 = 4$
Stage 3	$A_3 = 120$,	$B_3 = -16$,	$C_3 = 6$.

So, the objective function (3-1) becomes Equation (4-7):

$$f = 10(250 - 33.3n_1) + 4(300 - 37.5n_2) + 6(120 - 16n_3)$$

$$= 4420 - 333n_1 - 150n_2 - 96n_3\,. \qquad (4\text{-}7)$$

Computing the constraint values $n_{i\ max}$ and $n_{i\ min}$ from the Equations (4-5) and (4-6) yields the following:

Stage 1	$n_{1\ max} = 6$	$n_{1\ min} = 3$
Stage 2	$n_{2\ max} = 5^{1/3}$	$n_{2\ min} = 2^{2/3}$
Stage 3	$n_{3\ max} = 5$	$n_{3\ min} = 2^{1/2}$

So the constraints (4-2), (4-3), and (4-4) become:

$$n_1 + n_2 + n_3 \leq 15 \quad (4\text{-}8)$$

$$n_1 \geq 3 \quad (4\text{-}9)$$

$$n_1 \leq 6 \quad (4\text{-}10)$$

$$n_2 \geq 2^{2/3} \quad (4\text{-}11)$$

$$n_2 \leq 5^{1/3} \quad (4\text{-}12)$$

$$n_3 \geq 2^{1/2} \quad (4\text{-}13)$$

$$n_3 \leq 5. \quad (4\text{-}14)$$

Thus, minimizing the objective function Equation (4-7) is equivalent to maximizing its variable portion and can be written as Equation (4-7a):

$$\text{maximize } Z = -\sum_{i=1}^{M-1} C_i B_i n_i \quad (4\text{-}7a)$$

$$= 333n_1 + 150n_2 + 96n_3$$

subject to the constraint Equations

$$n_1 + n_2 + n_3 + S_1 = 15 \quad (4\text{-}8a)$$

$$n_1 - S_2 + A_1 = 3 \quad (4\text{-}9a)$$

$$n_1 + S_3 = 6 \quad (4\text{-}10a)$$

$$3n_2 - S_4 + A_2 = 8 \quad (4\text{-}11a)$$

$$3n_2 + S_5 = 16 \quad (4\text{-}12a)$$

$$2n_3 - S_6 + A_3 = 5 \quad (4\text{-}13a)$$

$$2n_3 + S_7 = 10 \qquad (4\text{-}14a)$$

where terms S_1 through S_7 are slack variables and terms A_1, A_2, and A_3 are artificial variables. The solution of this system of equations is carried out in Appendix I by the Simplex Method and yields the following solution:

$n_1 = 6$ days	$S_2 = 3$	$S_1 = 0$	$A_1 = 0$
$n_2 = 5^{1/3}$ days	$S_4 = 8$	$S_3 = 0$	$A_2 = 0$
$n_3 = 3^{2/3}$	$S_6 = 2^{1/3}$	$S_5 = 0$	$A_3 = 0$
$Z = 3{,}152$	$S_7 = 2^{2/3}$		

with a total cost of

$$f = 4{,}420 - Z = \$1268.$$

The allowable range of resource cost for each stage for this solution is found from the linear programming final tableau as follows: for the non-basic variables, compute the ratio of cost increment to variable coefficient which lies in the row of the desired basic variable; add this ratio algebraically to the initial tableau cost coefficient of the basic variable. When this is done for all the non-basic variables, the minimum interval generated by these terms yields a limiting range for the respective basic variable.

Stage 1 $\quad B_1C_1 = 33 \cdot 10 = 333 \quad$ (-333 in tableau)

$$\frac{B_1C_1S_3}{a_{26}} = \frac{237}{1} = 237 \qquad -333 + 237 = -96$$

no lower limit

$$\therefore -B_1 C_1 < -96 \text{ or } C_1 > 2.88$$

Stage 2 $\quad B_2 C_2 = 37^{1/2} \cdot 4 = 150 \quad$ (-150 in tableau)

$$\frac{B_2 C_2 S_5}{a_{48}} = \frac{18}{1/3} = 54 \qquad -150 + 54 = -96$$

$$\therefore -B_2 C_2 < -96 \text{ or } C_2 > 2.56$$

Stage 3 $\quad B_3 C_3 = 16 \cdot 6 = 96 \quad$ (-96 in tableau)

$$\frac{B_3 C_3 S_1}{a_{64}} = \frac{96}{1} = 96 \qquad -96 + 96 = 0$$

$$\frac{B_3 C_3 S_3}{a_{66}} = \frac{237}{-1} = -237 \qquad -96 - 237 = -333$$

$$\frac{B_3 C_3 S_5}{a_{68}} = \frac{18}{-1/3} = -54 \qquad -96 - 54 = -150$$

$$\therefore -150 < -B_3 C_3 < 0 \text{ or } 9.38 > C_3 > 0$$

Therefore, for this solution the resource cost for Stage 1 must be greater than $2.88 per man-hour; the resource cost for Stage 2 must be greater than $2.56 per man-hour; the resource cost for Stage 3 must lie between zero and $9.38 per man-hour.

The linear programming final tableau also gives the incremental cost associated with each binding constraint, where $S_i = 0$. The over-all time restriction ($S_1 = 0$) has a cost of $96 per day associated with it. The maximum time constraint on Stage 1 ($S_3 = 0$) has a corresponding cost of

\$237 per day, while the maximum time constraint on Stage 3 ($S_6 = 0$) has a marginal cost of \$18 per day. This shows that restrictions, such as required man-hours, are most costly on Stage 1 of this example project.

In summary the results for each stage are:

Stage 1	2	3	
6	$5^{1/3}$	$3^{2/3}$	n-mean time in stage (days)
50	100	61.3	R-resource applied (man-hours)
500	400	368	CR-cost of resource applied (\$)

These results indicate the expected time each activity of the project will require. The resource level applied at each stage yields a minimum total resource cost subject to the specified time and resource constraints. This information, combined with the incremental costs associated with the binding time constraints, can enable a manager to more effectively plan the levels of effort in a cost-time trade off.

Case I, with its linear objective function, can be applied to any size sequential stage project. Each stage adds one variable and two constraint equations so the size of problem for feasible hand computation may be considered as five or six stages; machine computation is advised for larger systems.

CHAPTER V

THE SOLUTION OF CASE II

Case II is the condition where the stage resource level is not a linear function of the mean time spent in each stage, but varies as $n_i/(n_i - 1)$ from Equation (3-20). This occurs when the review period T lies between the most likely and longest completion time $(m \leq T \leq b)$.

Formulating this case as a nonlinear programming problem where the constraints are linear and the objective function is nonlinear results in the following for an M stage project where the M^{th} stage is completion.

Objective Function

Minimize
$$f = \sum_{i=1}^{M-1} C_i R_i \tag{5-1}$$

$$= \sum_{i=1}^{M-1} C_i \left(D_i - E_i \frac{n_i}{n_i - 1}\right)$$

where C_i is the resource cost at the i^{th} stage.

Constraints

Subject to:
$$\sum_{i=1}^{M-1} n_i \leq N \tag{5-2}$$

$$n_i \geq n_{i\,min} \qquad i = 1, 2, \ldots, M-1 \tag{5-3}$$

and
$$n_i \leq n_{i\,max} \qquad i = 1, 2, \ldots, M-1 \tag{5-4}$$

where n_i min and n_i max are, as in Case I, the range of stage completion times.

Equations (5-5) and (5-6) are obtained from Equation (3-12) and yield n_i min and n_i max:

$$n_{i\ min} = \frac{1}{1 - \frac{(T - b_i)^2}{(b_i - m_{Hi})(b_i - a_i)}} \tag{5-5}$$

$$n_{i\ max} = \frac{1}{1 - \frac{(T - b_i)^2}{(b_i - m_{Li})(b_i - a_i)}} \tag{5-6}$$

where m_{Hi} and m_{Li} are the stage most likely completion times for the highest and lowest resource levels, respectively.

Testing the objective function for convexity yields the following equations:

$$\frac{\partial f}{\partial n_i} = \sum_{i=1}^{M-1} \frac{C_i E_i}{(n_i - 1)^2} < 0, \text{ since } E_i < 0 \tag{5-7}$$

$$\frac{\partial^2 f}{\partial n_i^2} = -\sum_{i=1}^{M-1} \frac{2C_i E_i}{(n_i - 1)^3} > 0, \ n_i > 1 \text{ and } E_i < 0 \tag{5-8}$$

Since $\frac{\partial f}{\partial n_i} < 0$ and $\frac{\partial^2 f}{\partial n_i^2} > 0$ for $1 < n_i < \infty$

f is a monotonically decreasing convex function as n_i increases, as illustrated in Figure 6. The function does not have a local minimum because $\partial f / \partial n_i$ is always negative, hence the lowest cost occurs at large activity durations. Because the activity durations n_i are constrained, the solution of the system will have at least one constraint binding.

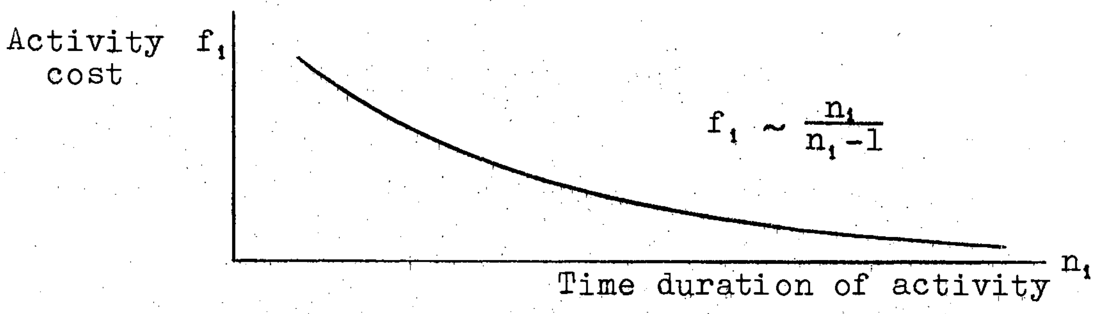


Figure 6. Cost Versus Time Duration of Activity for Case II

Writing Equations (5-1) through (5-4) in Lagrange Multiplier notation (8) and formulating the composite function:

$$F(n,\lambda) = f - \sum_{j=1}^{2M-1} \lambda_j g_j(n) \qquad (5\text{-}9)$$

$$= \sum_{i=1}^{M-1} C_i \left(D_i - E_i \frac{n_i}{n_i - 1}\right) - \lambda_1 \sum_{i=1}^{M-1} n_i - N \qquad (5\text{-}9)$$

$$- \sum_{i=1}^{M-1} \lambda_{i+1} (n_i - n_{i\ min}) - \sum_{i=1}^{M-1} \lambda_{i+M} (n_i - n_{i\ max})$$

where $g_j(n) = 0$ when the constraint is binding.

Equation (5-9) is the new objective function subject to 2M-1 constraints. Because the optimization is over M-1 activities (the M^{th} stage is completion) and each activity is subject to two time duration constraints, 2M-2 equations arise. The over-all project time duration constraint produces an additional equation, hence there is a total number of 2M-1. The λ_1 multiplier is associated with the over-all project constraint; λ_2 through λ_M are associated with the stage 1 through M-1 minimum time duration constraints, respectively, while λ_{M+1} through λ_{2M-1} are associated with the stage 1 through M-1 maximum time duration constraints, respectively.

Computing the partial derivatives of Equation (5-9) with respect to each variable and equating it to zero results in the following set of equations:

$$\frac{\partial F}{\partial n_i} = + \frac{C_i E_i}{(n_i - 1)^2} - \lambda_1 - \lambda_{i+1} - \lambda_{i+M} = 0 \qquad i = 1, 2, \ldots, M-1 \quad (5\text{-}10)$$

$$\frac{\partial F}{\partial \lambda_1} = - \sum_{i=1}^{M-1} (n_i - N) = 0 \quad (5\text{-}11)$$

$$\frac{\partial F}{\partial \lambda_{i+1}} = - (n_i - n_{i\ min}) = 0 \qquad i = 1, 2, \ldots, M-1 \quad (5\text{-}12)$$

$$\frac{\partial F}{\partial \lambda_{i+M}} = - (n_i - n_{i\ max}) = 0 \qquad i = 1, 2, \ldots, M-1. \quad (5\text{-}13)$$

For a given i in the Equation (5-10), either λ_{i+1} or λ_{i+M} will equal zero as, at most, only one of these constraints can be binding on the term n_i.

Consider the following numerical example to illustrate Case II.

Assume a project consisting of 4 sequential stages, where Stage 4 is completion, is desired to be completed in 15 days and that it is reviewed at five-day intervals (this example is similar to the one used in Case I).

For Stage Number

1	2	3	
1	4	2	a-shortest possible duration (days)
11	8	16	b-longest possible duration (days)
50	100	40	R_L-lowest stage resource allocation (man-hours)
5	7	5	m_L-most likely duration at lowest resource level (days)
150	200	80	R_H-highest stage resource allocation (man-hours)
3	6	4	m_H-most likely duration at highest resource level (days)
10	4	6	C-cost of resource ($/man-hour)

and M = 4, T = 5, N = 15.

Stage 1 and 3 have the review time T lying between m and b while Stage 2 was selected with T lying between a and m. Stage 2 was purposely chosen to have a relatively small probability of completion during one review time period because, in the model, it is necessary for the probability of completion of two or more activities during one review time period to be negligible. Computing the coefficients of the objective function from Equation (3-20) for Stages 1 and 3 where:

$$D = (b - m_L)\left(\frac{R_H - R_L}{m_H - m_L}\right) + R_L \quad \text{and} \quad E = \frac{(T-b)^2}{b-a}\left(\frac{R_H - R_L}{m_H - m_L}\right)$$

and from Equation (3-19) for Stage 2 where:

$$A = (a-m_L)\left(\frac{R_H-R_L}{m_H-m_L}\right) + R_L \quad \text{and} \quad B = \frac{(T-a)^2}{b-a}\left(\frac{R_H-R_L}{m_H-m_L}\right)$$

yields the following stage values:

Stage 1 $\quad D_1 = -250,\ E_1 = -180,\ C_1 = 10$

Stage 2 $\quad A_2 = +400,\ B_2 = -25,\ C_2 = 4$

Stage 3 $\quad D_3 = -400,\ E_3 = -2420/7,\ C_3 = 6.$

So, the objective function becomes:

$$f = 10\left(-250 + 180\frac{n_1}{n_1-1}\right) + 4(400 - 25n_2) +$$

$$6\left(-400 + \frac{2420}{7}\frac{n_3}{n_3-1}\right) = -3300 + \frac{1800n_1}{n_1-1} -$$

$$100n_2 + \frac{2074n_3}{n_3-1}.$$

Computing the constraint values $n_{i\,min}$ and $n_{i\,max}$ from the Equations (5-5) and (5-6) for Stages 1 and 3 and from Equations (4-5) and (4-6) for Stage 2 yields the following:

Stage 1 $\quad n_{1\,min} = 1^{9/11} \quad n_{1\,max} = 2^{1/2}$

Stage 2 $\quad n_{2\,min} = 8 \quad n_{2\,max} = 12$

Stage 3 $\quad n_{3\,min} = 3^{27/47} \quad n_{3\,max} = 4^{2/3}$

So the constraints become:

$$n_1 + n_2 + n_3 \leq 15 \tag{5-15}$$

$$n_1 \geq 1^{9/11} \tag{5-16}$$

$$n_2 \geq 8 \tag{5-17}$$

$$n_3 \geq 3^{27/47} \tag{5-18}$$

$$n_1 \leq 2^{1/2} \tag{5-19}$$

$$n_2 \leq 12 \tag{5-20}$$

$$n_3 \leq 4^{2/3} \tag{5-21}$$

Equations (5-9) through (5-13) can be written as the following for this example:

$$\begin{aligned} F(n,\lambda) = & -3300 + \frac{1800\ n_1}{n_1 - 1} + \frac{2074 n_3}{n_3 - 1} \\ & - \lambda_1 (n_1 + n_2 + n_3 - 8) - \lambda_2 (n_1 - 1.818) \\ & - \lambda_3 (n_2 - 8.000) - \lambda_4 (n_3 - 3.574) - \lambda_5 (n_1 - 2.500) \\ & - \lambda_6 (n_2 - 12.000) - \lambda_7 (n_3 - 4.666) \end{aligned} \tag{5-22}$$

$$\frac{\partial F}{\partial n_1} = - \frac{1800}{(n_1 - 1)^2} - \lambda_1 - \lambda_2 - \lambda_5 = 0 \tag{5-23}$$

$$\frac{\partial F}{\partial n_2} = - 100 - \lambda_1 - \lambda_3 - \lambda_6 = 0 \tag{5-24}$$

$$\frac{\partial F}{\partial n_3} = - \frac{2074}{(n_3 - 1)^2} - \lambda_1 - \lambda_4 - \lambda_7 = 0 \tag{5-25}$$

$$\frac{\partial F}{\partial \lambda_1} = - (n_1 + n_2 + n_3 - 15) = 0 \tag{5-26}$$

$$\frac{\partial F}{\partial \lambda_2} = - (n_1 - 1.818) = 0 \tag{5-27}$$

$$\frac{\partial F}{\partial \lambda_3} = - (n_2 - 8.000) = 0 \tag{5-28}$$

$$\frac{\partial F}{\partial \lambda_4} = - (n_3 - 3.574) = 0 \tag{5-29}$$

$$\frac{\partial F}{\partial \lambda_5} = - (n_1 - 2.500) = 0 \tag{5-30}$$

$$\frac{\partial F}{\partial \lambda_6} = -(n_2 - 12.000) = 0 \quad (5\text{-}31)$$

$$\frac{\partial F}{\partial \lambda_7} = -(n_3 - 4.666) = 0. \quad (5\text{-}32)$$

To solve this system of equations, assume initially that one constraint, Equation (5-26), is binding; therefore, λ_2 through λ_7 are equal to zero.

Solving for λ_1 from Equation (5-24) and substituting it into Equations (5-23) and (5-25) yields:

$$\lambda_1 = -100$$

$$n_1 = 1 + (-1800/\lambda_1)^{1/2} = 5.243$$

$$n_3 = 1 + (-2074/\lambda_1)^{1/2} = 5.554.$$

Substituting n_1 and n_3 into Equation (5-26) yields:

$$n_2 = 15 - 5.234 - 5.554 = 4.203.$$

Checking the values of n_i with the constraints

$$1.818 \leq n_1 \leq 2.500 \qquad n_1 = 5.243$$

$$8.000 \leq n_2 \leq 12.000 \qquad n_2 = 4.203$$

$$3.574 \leq n_3 \leq 4.666 \qquad n_3 = 5.554$$

n_1 and n_3 exceed their maximum limit while n_2 is less than its minimum limit. Therefore, constraint Equation (5-28) will be used for n_2 as it is the furthest outside of bounds:

$$n_2 = 8.000, \quad \lambda_3 \neq 0.$$

From Equation (5-24), $\lambda_3 = -\lambda_1 - 100.$

Substituting into Equation (5-26) yields:

$$1 + (-1800/\lambda_1)^{1/2} + 8.000 + 1 + (-2074/\lambda_1)^{1/2} - 15 = 0$$

which gives $\lambda_1 = -309$,

therefore,

$$n_1 = 1 + (-1800/\lambda_1)^{1/2} = 3.418$$

and

$$n_3 = 1 + (-2074/\lambda_1)^{1/2} = 3.595.$$

Checking the values of n_1 and n_3 with their constraints

$$1.818 \leq n_1 \leq 2.500 \qquad n_1 = 3.418$$

$$3.574 \leq n_3 \leq 4.666 \qquad n_3 = 3.595$$

n_1 is found to exceed its maximum limit. Therefore, constraint Equation (5-30) must be used yielding:

$$n_1 = 2.500, \lambda_5 \neq 0.$$

From Equation (5-23), $\lambda_5 = -\lambda_1 - \frac{1800}{(2.500-1)^2} = -\lambda_1 - 800.$

Solving for n_3 from Equation (5-26):

$$n_3 = 15 - 2.500 - 8.000 = 4.500$$

therefore,

$$n_1 = 2.500 \qquad n_2 = 8.000 \qquad n_3 = 4.500$$

which satisfies all of the constraints.

Obtaining λ_1 from Equation (5-25) yields:

$$\lambda_1 = -\frac{2074}{(4.500-1)^2} = -169.$$

Also,

$$\lambda_3 = -\lambda_1 - 100 = 169 - 100 = 69$$

and

$$\lambda_5 = -\lambda_1 - 800 = 169 - 800 = -631.$$

The Kuhn-Tucker condition which describes optimal solutions to nonlinear programming problems will now be used to test the optimality of this solution. The minimization of a convex function over convex constraint equations satisfying these conditions is an optimal solution. The conditions are:

$$\text{If } n_i^* > 0, \text{ then } \frac{\partial f}{\partial n_i} - \sum_{j=1}^{2M-1} \lambda_j \frac{\partial g_j}{\partial n_i} = 0 \qquad (5\text{-}33)$$

at $n_i = n_i^*$ for $i = 1, 2, \ldots, M-1$.

$$\text{If } n_i^* = 0, \text{ then } \frac{\partial f}{\partial n_i} - \sum_{j=1}^{2M-1} \lambda_j \frac{\partial g_j}{\partial n_i} \leq 0 \qquad (5\text{-}34)$$

at $n_i = n_i^*$ for $i = 1, 2, \ldots, M-1$.

Checking the solution in Equation (5-33) yields:

$$n_1 : \quad -\frac{1800}{(n_1-1)^2} - \lambda_1 - \lambda_2 - \lambda_5 =$$

$$= -\frac{1800}{(2.500-1)^2} + 169 - 0 + 631 = -800 + 169 + 631 = 0$$

$$n_2 : \quad -100 - \lambda_1 - \lambda_3 - \lambda_6 = -100 + 169 - 69 - 0 = 0$$

and

$$n_3 : \quad -\frac{2074}{(n_3-1)^2} - \lambda_1 - \lambda_4 - \lambda_7 =$$

$$-\frac{2074}{(4.500-1)^2} + 169 - 0 - 0 = -169 + 169 = 0$$

which satisfies the first condition and since $n_i^* > 0$, the second condition does not apply; therefore, the solution obtained is optimal.

The total resource cost then can be found from Equation (5-22) as:

$$F = -3300 + \frac{1800\ (2.500)}{2.500 - 1} - 100(8.000) + \frac{2074(4.500)}{4.500 - 1}$$

$$- (-169)(2.500 + 8.000 + 4.500 - 15)$$

$$- 0(2.500 - 1.818) - 69(8.000 - 8.000)$$

$$- 0(4.500 - 3.574) - (-631)(2.500 - 2.500)$$

$$- 0(8.000 - 12.000) - 0(4.500 - 4.666)$$

$$= -3300 + 3000 - 800 + 2667 = \$1567.$$

The λ_1 multiplier may be interpreted as the cost of an incremental change in the over-all project duration time. For example, a 1/10 day increase in the over-all project duration would reduce the project cost by $16.90 (approximately since this is a nonlinear system). Similarly, the λ_3 multiplier may be interpreted as the cost of an incremental change in the minimum time constraint of activity 2. Reducing the minimum time by 1/10 of a day would result in approximately a $6.90 reduction in project cost.

Finally, the λ_5 multiplier would represent the cost of an incremental change in the maximum time constraint of activity 1. A 1/10 of a day increase in the activity time duration would save approximately $63.10; of the three binding constraints, this one is the most costly. The sign of λ_3 differs from that of λ_1 and λ_5 because it is a minimum constraint whereas λ_1 and λ_5 are associated with maximum constraints. Summarizing the results for each stage:

Stage

1	2	3	
2.5	8.0	4.5	n-mean time in stage (days)
50.0	200.0	44.5	R-resource applied (man-hours)
500.0	800	267	CR-cost of resource applied ($)

where λ_1 = -169 over-all project duration constraint multiplier, λ_3 = 69, Stage 2 minimum duration constraint multiplier and λ_5 = -631, Stage 1 maximum duration constraint multiplier. The interpretation of these results is the same as for Case I discussed in the preceding chapter.

As seen from this example, determining the optimum for the nonlinear objective function is not particularly straightforward. Increasing the number of stages will further complicate the method computationally, but it should still be possible to obtain a constrained optimal solution. Bringing in constraints only as required will minimize the work for hand computation.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

This study has shown that it is possible to relate a sequential stage project to a Markov Process. The transient state probabilities represent the project stages while the absorbing state represents the project completion stage. By relating the probability of completion during a given time interval to the level of resource applied to a stage, a minimum resource cost is obtained for the overall project duration and stage resource constraints. The approach developed in this work can be used as a supplement to a relatively large scale project management tool such as PERT (9).

The use of this system as a practical management tool requires that the equations be programmed for digital computation. Computer logic can be used to select the appropriate equations when the range of the mode of the completion distribution includes the review time period.

Future work on this technique would include taking into consideration the possibility of more than one stage being completed during a review time period. This inclusion would be necessary if a distribution with a finite probability of very short activity duration were used.

Another area for further investigation would be the assignment of a probability of completion to the over-all project duration. A dynamic programming approach might be used for the model in this case.

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APPENDIX

The numerical example given in Chapter IV involves the maximimization of the following objective function subject to constraints.

Maximize $Z=1000n_1+150n_2+96n_3$ subject to:

$$n_1+n_2+n_3+S_1=10$$

$$n_1-S_2+A_1=2$$

$$n_1+S_3=3$$

$$3n_2-S_4+A_2=8$$

$$3n_2+S_5=16$$

$$2n_3-S_6+A_3=5$$

$$2n_3+S_7=10\,.$$

The following pages give the initial tableau and the five iterations obtaining the optimal solution by the Simplex Method.

APPENDIX

The numerical example given in Chapter IV involves the maximization of the following objective function subject to constraints:

Maximize $Z = 333n_1 + 150n_2 + 96n_3$

Subject to:

$$n_1 + n_2 + n_3 + S_1 = 15$$

$$n_1 - S_2 + A_1 = 3$$

$$n_1 + S_3 = 6$$

$$3n_2 - S_4 + A_2 = 8$$

$$3n_2 + S_5 = 16$$

$$2n_3 - S_6 + A_3 = 5$$

$$2n_3 + S_7 = 10.$$

The following pages give the initial tableau and the six iterations obtaining the optimal solution by the Simplex Method.

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
1	1	1	1	0	0	0	0	0	0	0	0	0	15	S_1	15
(1)	0	0	0	-1	0	0	0	0	0	1	0	0	3	A_1	3 ←
1	0	0	0	0	1	0	0	0	0	0	0	0	6	S_3	6
0	3	0	0	0	0	-1	0	0	0	0	1	0	8	A_2	
0	3	0	0	0	0	0	1	0	0	0	0	0	16	S_5	
0	0	2	0	0	0	0	0	-1	0	0	0	1	5	A_3	
0	0	2	0	0	0	0	0	0	1	0	0	0	10	S_7	
-333 ↑	-150	-96	0	0	0	0	0	0	0	+M	+M	+M	0	Z	

Initial Tableau

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
0	1	1	1	1	0	0	0	0	0	-1	0	0	12	S_1	12
1	0	0	0	-1	0	0	0	0	0	1	0	0	3	n_1	-3
0	0	0	0	(1)	1	0	0	0	0	-1	0	0	3	S_3	3 ←
0	3	0	0	0	0	-1	0	0	0	0	1	0	8	A_2	
0	3	0	0	0	0	0	1	0	0	0	0	0	16	S_5	
0	0	2	0	0	0	0	0	-1	0	0	0	1	5	A_3	
0	0	2	0	0	0	0	0	0	1	0	0	0	10	S_7	
0	-150	-96	0	-333	0	0	0	0	0	+M	+M	+M	1000	Z	
				↑											

First Iteration

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
0	1	1	1	0	-1	0	0	0	0	-2	0	0	9	S_1	9
1	0	0	0	0	1	0	0	0	0	0	0	0	6	n_1	
0	0	0	0	1	1	0	0	0	0	-1	0	0	3	S_2	
0	(3)	0	0	0	0	-1	0	0	0	0	1	0	8	A_2	8/3 ←
0	3	0	0	0	0	0	1	0	0	0	0	0	16	S_5	16/3
0	0	2	0	0	0	0	0	-1	0	0	0	1	5	A_3	
0	0	2	0	0	0	0	0	0	1	0	0	0	10	S_7	
0	-150 ↑	-96	0	0	333	0	0	0	0	M	M	M	2000	Z	

Second Iteration

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
0	0	1	1	0	-1	1/3	0	0	0	-2	-1/3	0	19/3	S_1	19/3
1	0	0	0	0	1	0	0	0	0	0	0	0	6	n_1	
0	0	0	0	1	1	0	0	0	0	-1	0	0	3	S_2	
0	1	0	0	0	0	-1/3	0	0	0	0	1/3	0	8/3	n_2	
0	0	0	0	0	0	1	1	0	0	0	-1	0	8	S_5	
0	0	(2)	0	0	0	0	0	-1	0	0	0	1	5	A_3	5/2 ←
0	0	2	0	0	0	0	0	0	1	0	0	0	10	S_7	10/2
0	0	-96 ↑	0	0	333	-50	0	0	0	M	M	M	2400	Z	

Third Iteration

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
0	0	0	1	0	-1	1/3	0	1/2	0	-2	-1/3	-1/2	23/6	S_1	23/2
1	0	0	0	0	1	0	0	0	0	0	0	0	6	n_1	
0	0	0	0	1	1	0	0	0	0	-1	0	0	3	S_2	
0	1	0	0	0	0	-1/3	0	0	0	0	1/3	0	8/3	n_2	-8
0	0	0	0	0	0	(1)	1	0	0	0	-1	0	8	S_5	8 ←
0	0	1	0	0	0	0	0	-1/2	0	0	0	1/2	5/2	n_3	
0	0	0	0	0	0	0	0	1	1	0	0	-1	5	S_7	
0	0	0	0	0	333	-50 ↑	0	-48	0	M	M	M	2640	Z	

Fourth Iteration

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
0	0	0	1	0	-1	0	-1/3	(1/2)	0	-2	0	-1/2	7/6	S_1	7/3 ←
1	0	0	0	0	1	0	0	0	0	0	0	0	6	n_1	
0	0	0	0	1	1	0	0	0	0	-1	0	0	3	S_2	
0	1	0	0	0	0	0	1/3	0	0	0	0	0	16/3	n_2	
0	0	0	0	0	0	1	1	0	0	0	-1	0	8	S_4	
0	0	1	0	0	0	0	0	-1/2	0	0	0	1/2	5/2	n_3	-5
0	0	0	0	0	0	0	0	1	1	0	0	-1	5	S_7	5
0	0	0	0	0	333	0	50	-48	0	M	M	M	3040	Z	
								↑							

Fifth Iteration

n_1	n_2	n_3	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_1	A_2	A_3	b	Var	θ
0	0	0	2	0	-2	0	-2/3	1	0	-4	0	-1	7/3	S_6	
1	0	0	0	0	1	0	0	0	0	0	0	0	6	n_1	
0	0	0	0	1	1	0	0	0	0	-1	0	0	3	S_2	
0	1	0	0	0	0	0	1/3	0	0	0	0	0	16/3	n_2	
0	0	0	0	0	0	1	1	0	0	0	-1	0	8	S_4	
0	0	1	1	0	-1	0	-1/3	0	0	-2	0	0	11/3	n_3	
0	0	0	-2	0	2	0	2/3	0	1	4	0	0	8/3	S_7	
0	0	0	96	0	237	0	18	0	0	M	M	M	3152	Z	

Sixth Iteration-Optimal

VITA

John Charles Even, Jr.

Candidate for the Degree of

Doctor of Philosophy

Thesis: RESOURCE ALLOCATION IN PROJECT STAGES TREATED AS A MARKOV PROCESS

Major Field: Engineering

Biographical:

Personal Data: Born in Evanston, Illinois, December 16, 1934, the son of Mr. and Mrs. John C. Even, Sr.

Education: Graduated from Roger Sullivan High School, Chicago, Illinois, in January, 1953; attended Northwestern University, Evanston, Illinois, from March, 1953-December, 1957; received the Bachelor of Science degree in December, 1957, with a major in Electrical Engineering; attended Northwestern University from January, 1958-June, 1959; received the Master of Science degree from Northwestern University in June, 1959, with a major in Electrical Engineering; attended Northwestern University from January, 1960-March, 1962 for Post Master's work; attended Oklahoma State University from June, 1965 to August, 1968, with a major in Industrial Engineering; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1969 as a Ford Foundation recipient.

Professional Experience: Aerial Measurements Laboratory, Northwestern University, Cooperative Student, June, 1954-September, 1957; Vapor Corporation, Chicago, Illinois, Part-time Engineer, December, 1957-June 1959; Senior Test Engineer, June 1959-September, 1962; Engineering Analyst, September 1962-September, 1965; Oklahoma State University, Stillwater, Oklahoma, School of Industrial Engineering and Management, one-fourth time Research, Boeing Research Project, September, 1965-May, 1967; Department of Engineering and Industrial Extension, full-time Editorial Assistant, June 1967-August, 1967, one-fourth time, September, 1967-January, 1968.

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