ECONOMIC CAPACITY PLANNING

FOR VARIABLE DEMAND

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NOMENCLATURE

X	A random variable representing the demand against a productive process, assumed to be represented by some probability density function, f(x), which in turn implies a time domain.
c	The decision variable representing the level of productive capacity, under control of the decision maker.
L	Cost per unit for production up to the level of c (applicable for $X \leq c$).
P	Cost per unit for production in excess of the level of c (applicable for $X > c$).
t	The degree of flexibility allowed, being a multiple of the basic time periods for which the probability distribution functions of X apply.
f(x)	The probability distribution function for the random variable X.
F(x)	The cumulative distribution function for the random variable X.
$f^t(x)$	The t^{th} convolution of f(x).
$F^t(x)$	The cumulative distribution of $f^t(x)$.
$f_x(s)$	The Laplace transform of f(x).
$f_x(z)$	The z transform of f(x).

CHAPTER I

INTRODUCTION

Operating a production process requires some form of decision making. In simple cases, the decisions may be obvious. However, in the more general (and usual) case, a decision is often one of many in a sequence of a decision making process, and once made, has a relationship to the feasibility of remaining decisions. This research is concerned with one such decision from a sequence.

Statement of Problem

This research is set in a production context, with the production manager as the primary decision maker. Sales, advertising, profit, transportation, and many other important aspects of the business are thus removed from his control and his concentration is upon minimizing production costs, given that he has some indication of the magnitude of his expected level of production for some time period into the future.

The following assumptions are made in framing the context of the problem:

1. The decision is a planning decision, and once made can be used as an input to such operational decisions as the scheduling of regular and overtime hours at work centers, levels of

inprocess inventories, timing of raw material requirements, etc.

2. Two basic alternatives are available; regularly scheduled production with its associated cost parameter and what will be referred to as premium capacity. Premium capacity can be the utilization of the same resources available for regular capacity, such as overtime or multiple shift work, or it may be considered as subcontracting a portion of the total work or the buying of production from an outside source to supplement that available on a regular basis. Thus the type of premium capacity applicable will also carry its specific cost parameter.

3. The input to the decision is a probabilistic distribution of demand, obtained from a separate planning decision dealing with forecasting of future demand for the firm's goods and services.

4. The lead time in which demand must be honored is specified and determines the length in time of the planning horizon.

Chapter II will discuss in more detail the meaning and interrelationships that exist between capacity, costs, demand, and time in the production process. For now the problem can be visualized in a broad sense as in Figure 1.

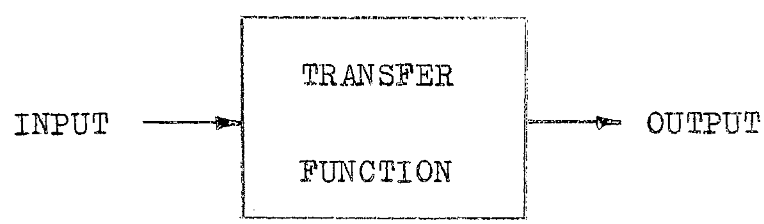


Figure 1. A Planning Decision Process

For the problem at hand, the input is the probabilistic demand into the future, the transfer function is a mathematical model (developed in Chapter III) that relates the variables involved and the output is the economic capacity level (the decision).

Determining the economic level of capacity direct from the expected load to be placed upon that capacity would seem to have usefulness, rather than considering the capacity level as fixed and then working with the often complex production scheduling problem of fitting whatever demand occurs against the production facility. It is this aspect which places the decision of this research at the planning level. No attempt will be made to integrate the decision into a complex working system for any given production system, but its usefulness, once developed, will be discussed later.

Literature Search

Most textbooks written in the area of production planning or production control do an adequate job of outlining the areas of interest to the production environment. They fail, in general, however, to effectively do more than expose the existance of relationships between main areas. Abramowitz [1] presents a thorough concept of capacity as it relates to a very general production model. Magee and Boodman [9] present a strong development of the role of inventory in the production system. In most cases, the textbook approach eventually comes down to decision making procedures dealing with either deterministic or probabilistic demand, imposed upon some assumed level of capacity and the development of operational procedures for then scheduling the demand onto the current capacity. It is the intent of this research to explore a way in which the capacity level itself can be economically evaluated, at a planning level, prior to the scheduling problem.

Attempts to more explicitly involve the interrelationships between demand, work force (a form of capacity) and inventory levels can be found in the technical journals. One of the earliest involves the work of Holt, Modigliani and Simon [7] in applying a linear decision rule (developed mathematically in [6]) to a paint factory. The cost function utilized is:

$$C_N = \sum_{t=1}^{\infty} [(C_1 - C_6)\ W_t + C_2\ (W_t - W_{t-1} - C_{11})^2$$

$$+ C_3(P_t - C_4W_t)^2 + C_5P_t + C_{12}P_tW_t$$

$$+ C_7(I_t - C_8 - C_9O_t)^2 + C_{13}]$$

where the subscript t indicates the time period; W_t, the work force level; P_t, the aggregate production; I_t, the net inventory; and O_t, the ordered shipments. All C values represent cost parameters except C_{13} which is a constant cost term not affected by scheduling decisions. Their [6] explanation of the problem is given as:

> The problem we then face is the following: To choose a decision rule (strategy) for making production and labor force decisions in successive

> time periods that will minimize the expected value of total costs over a large number of periods. Since costs are influenced by the interaction between current actions and future orders, forecasts of the future are indispensable even though such forecasts are subject to errors. The passage of time makes new information available which allows improvements in the accuracy of the forecasts. The design of an optimal decision rule should take these considerations into account.
>
> In general, however, future orders are uncertain; that is to say, information about orders in each future period may be cast in the form of a probability distribution.

Minimization of the cost function was based upon using each future period's expected number of orders (demand upon the capacity) and then by taking the total derivative of the cost function. This resulted in a system of equations which was solved by matrix inversion procedures. Solution was possible by this procedure due to the quadratic form of the cost function.

Fetter [4] applied linear programming procedures to the problem of long range capacity planning, given future demands. His model was based upon capacity having the following market characteristics:

1. Ownership implies a long term capital investment.
2. Leasing is an available alternative and implies a different capital commitment than ownership.
3. Capacity is available on a short term or spot basis.
4. Capacity prices are variable over time and may vary in terms of their relative relationship for the various alternatives.

His objective function minimized the present worth of all future costs, subject to stated constraints (among others) that required demand be met and capacity be retired at the appropriate time determined by replacement policy methods.

Other studies are available that attack the production smoothing problem (the relationship between production, inventory and demand) such that costs over some planning horizon are minimum [8]. Studies involving employment planning in the face of varying demand have also been made [11].

In [7], the model was applied to past years data and results were compared with what would have been, had the model been in effect, compared to what actually occurred. A cost savings of 8.5 percent per year in favor of the model resulted. An underlying assumption of this study involved the ability of the firm to manipulate the level of the work force and the magnitude of the overtime hours simultaneously, a condition not always available. Also, capacity was defined as work force alone.

Implicit in Fetter's model [4] is a planning horizon sufficiently long enough to account for the retirement of installed capacity (and that purchased during the planning horizon) or the inclusion of them outside the model for shorter planning horizons. Since machinery life is often expressed in years, this complicates the forecasting problem, as accuracy of forecasts over several years become unreliable. Also, giving the demand of future periods a probabilistic nature, increases the number of constraints to the problem significantly.

The smoothing studies attempt to distribute the demand across the planning horizon to some assumed level of available capacity, that capacity in itself not being a variable.

It is the author's opinion that the capacity level itself can be viewed as a variable, at least at the planning stage, and that it can be economically determined allowing interactions of demand in interim periods of the planning horizon to occur. Once this has been determined, then the approaches of smoothing could be applied.

CHAPTER II

THE PRODUCTION PROCESS

General

The production process for manufacturing (as opposed to services) can be depicted as in Figure 2 [1]. Decisions necessitated by this dynamic environment can, in general, be grouped into two main categories; planning and operational. In this research, attention is directed at the planning level for the specific purpose of determining capacity levels economically. This decision then becomes an input to other required decisions until finally, operational decisions that involve detailed production schedules, inventory policies and other related information such that a fundamental operational schedule can be set for some planning horizon.

Demand

Demand against a productive process is usually measured in such common terms as manhours, machine hours, standard hours, etc. Thus once forecasts have been made, regardless of the unit of measurement used (dollars, units, tons) these should be converted into units compatable with the controlling function of the production process that is

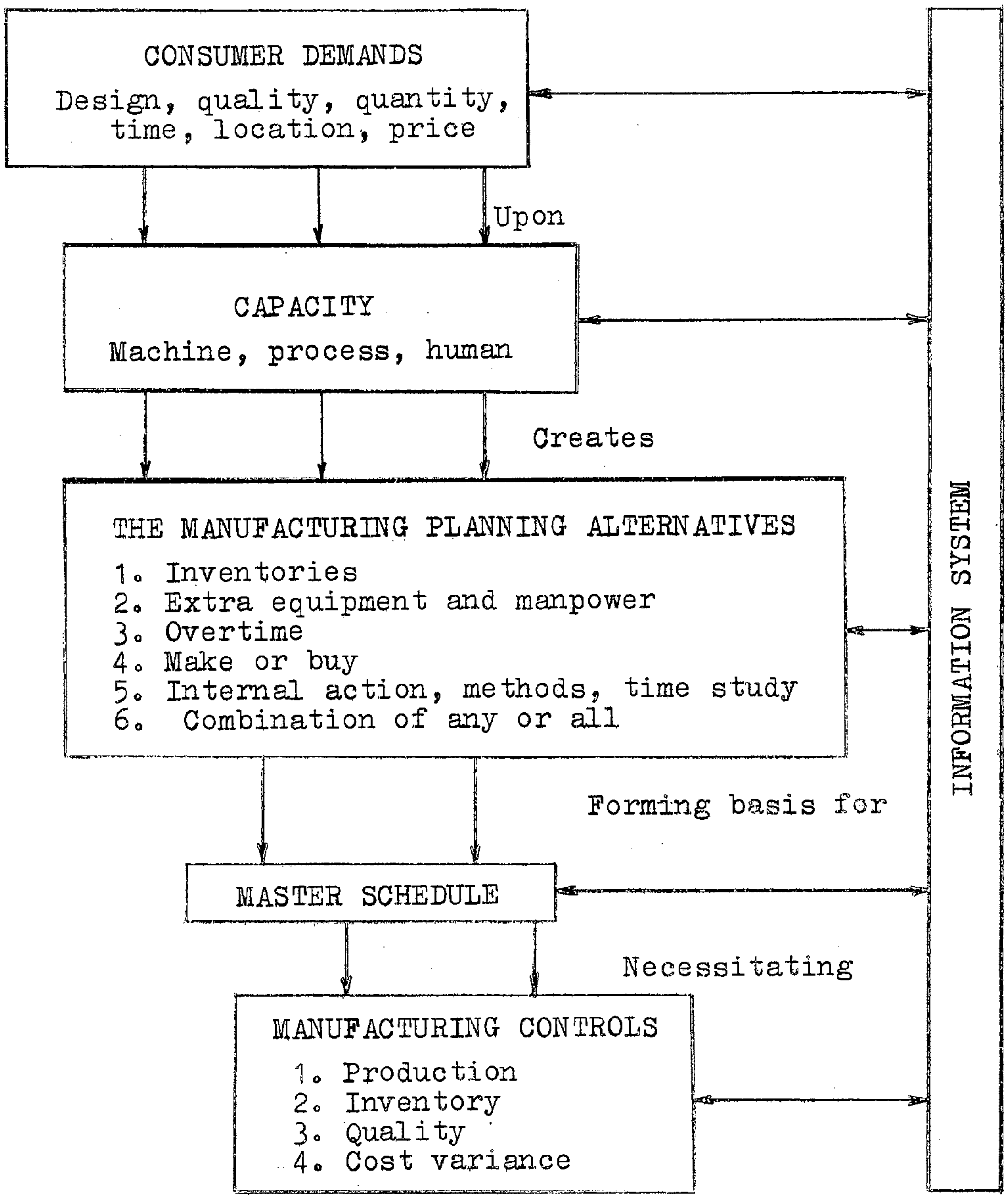


Figure 2. The Manufacturing Process

responsible for creating the utility possessed by the firm's goods and services.

In this research interest is in determining a capacity level, economically, in view of a variable demand placed upon that capacity. There exists considerable difficulty in obtaining the input required to start such analysis, but certain procedures have been developed, generally under the name of "forecasting methods." Abramowitz and Magee[1, 9] present most of these procedures and discuss their relationship to the production planning and control areas. Such forecasts are seldom taken as point estimates of the actual demand. They can be viewed as random variables, possessing a mean and finite variance. They may be assumed to follow some well known (with respect to behavior) probability distribution function such as the normal, Poisson, exponential, etc. Or they may be empirically described based mainly on historical records. The degree to which these random variables behave as related to major pricing changes, advertising efforts and significant changes in the economy could be considered in applied cases.

Assuming that the probability distribution functions are available, the decision of setting a capacity level for some time period into the future must be made. Naturally, the decision is related to the time period for which the distribution functions are considered valid and the time period for which the decision on capacity levels is to be optimal. This requires two considerations. Given that a demand has occurred, when should it be met? (i.e., How much lead time does the production manager have?) This variable will be referred to as "flexibility" and is discussed later separately. The other consideration that arises is the distribution of demand over multiples of the time periods for the demand distributions as given. This last consideration is handled by finding the convolutions of demand. The theory and development of convolution is presented in Appendix A along with specific examples of various probability distribution functions which have been convoluted. Here concern is directed towards understanding what the convolution procedure accomplishes.

Consider a simple demand distribution for a manufacturing process based on a monthly time basis:

	Demand (x)	Probability p(x)
$x_{(min)}$	1200 manhours	0.15
	1300 manhours	0.40
	1400 manhours	0.35
$x_{(max)}$	1500 manhours	0.10

The probability distribution of demand for a six-month period is desired under the assumption that in each subsequent month the demand will follow the same distribution. In a statistical context, a random sample of size six is taken from the basic monthly distribution and the distribution of the sum of the random variables is to be determined. Intuition indicates that the minimum demand for the six-month period will be at least six times $x_{(min)}$ and the maximum demand will be at most six times $x_{(max)}$. What about the probabilities of these events? The minimum will occur with probability $P[x_{(min)}]^6 = (0.15)^6$, and the maximum with probability $P[x_{(max)}]^6 = (0.10)^6$. For small problems one can enumerate all possible outcomes of the sum and find their probability of occurrence by using the conditional probabilities that comprise the specific outcome event. As this becomes quite cumbersome, the convolution procedure which finds the distribution of the sum for all values may be invoked to completely determine the distribution of the sum.

Note that the minimum and maximum values occur with much smaller probabilities than in the basic one-month time period. These facts would suggest that the variance of the six-month distribution is larger than for the basic one-month distribution. By the same logic, knowing that the distribution for the six-month period must sum to unity, one suspects that there is a larger probability in the six-month case than in the one-month case, associated with the event that the actual demand will be within $\pm$ Y percent of the mean.

What has been asserted is nothing more than the powerful conclusions of the Central Limit Theorem. This useful theorem is applicable when n (the sample size, and in the above discussion six) approaches infinity. Further, this theorem asserts that this sum is distributed normally. As long as it is not always true that a particular few of the member random variables dominate the sum, then the random variables need not come from identically the same distribution. Why then become concerned with the convolution procedure? The answer is to have an applicable procedure for finding demand distribution when only a small number of time periods are to be considered. The decision as when to use the convolution concept and as to when the Central Limit Theorem is applicable will depend upon the circumstances to which the models (yet to be developed) would apply (see Example 1 in Appendix A). Concern here is that there is a way to generate demand distributions for <u>any</u> multiple number of time periods for which the basic demand distributions are known.

A final observation closes this discussion of demand distributions. If for some reason, such as strong seasonal factors, a completely different demand distribution is to follow in a subsequent period, these two densities can be convoluted together and this resultant can be convoluted with yet another different density for yet another period. This procedure may be indefinitely followed. When the assumption that each period has essentially the same density holds, the mathematics are simply less cumbersome than otherwise.

Capacity

Capacity is a concept, involving an understanding of factors that affect it and time as a parameter. What is an effective capacity today may be obsolete tomorrow due to changes over time in the factors that affect it. The relationship of these factors to capacity can be represented as in Figure 3 [1].

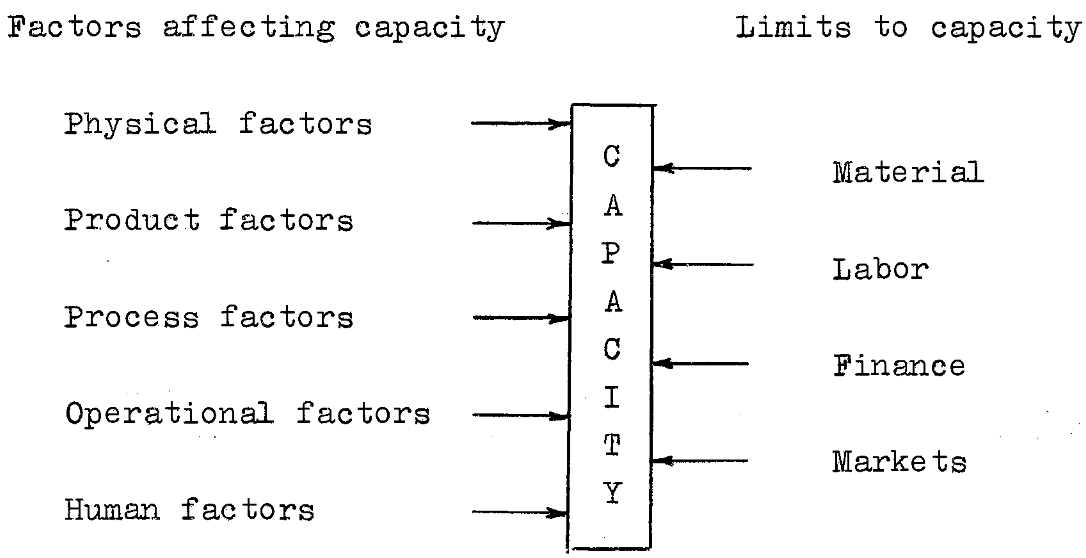


Figure 3. Major Factor Categories in Capacity Analysis and the Major Outside Limits to Capacity Change

Furthermore, the following definitions are helpful [1]:

Capacity--The maximum output of acceptable goods or services that a machine or process is capable of producing without the influence of external or internal factors.

Effective Capacity--The total goods and services that can be produced at a given time period with specific operating conditions, work intensity, product mix, product specifications, plant, and equipment.

Efficiency--The relationship between the output actually achieved and the effective capacity. This relationship is usually expressed as a percentage.

The restraints of the factors of Figure 3 give a measure of effective capacity. For the purpose at hand, capacity as required by a probabilistic demand is sought. That considerable effect is necessarily required to make effective capacity as near to capacity as possible is not at all to be de-emphasized. To the contrary, once an economic capacity level as a function of expected demand has been determined, it may well be that the best way to obtain that level is by re-evaluation and adjustment of the factors affecting capacity such that effective capacity can be increased to the desired economic capacity level. Modern industrial engineering techniques in the areas of plant design and layout, materials handling, environmental factor analysis, standardization and simplification, quality design and control, and effective incentives; all are the tools available for attacking the factors affecting capacity.

Capacity and Demand

Typically, the relationship of profit (or loss) as a result of a <u>given</u> demand can be shown by break-even charts. Figure 4 indicates three possible profitable events:

1. A demand (1) occurs that is less than 100 percent capacity. If the demand is above the break-even point a profit will result, otherwise a loss is incurred. For the case depicted a profit of (ab) results.

2. A demand (2) occurs equal to 100 percent capacity (probabilistically rare). Unit costs are minimum, and the process is operating at its highest level. A profit of (cd) results.

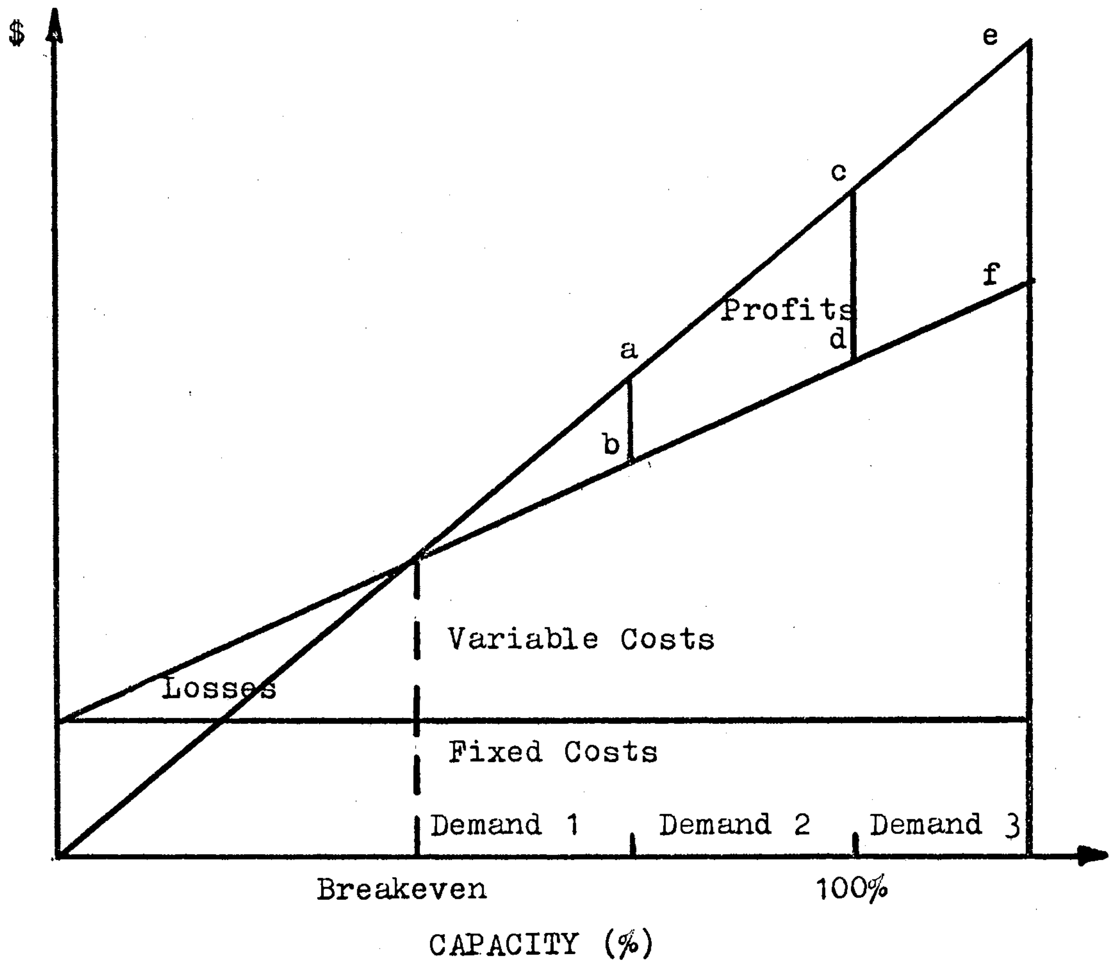


Figure 4. Breakeven Chart Relating Demand and Capacity Levels

3. A demand (3) occurs that is greater than 100 percent capacity. Supplementary means must be used if demand is to be met.

The last situation is more generally the case and affords production management the more complex decision environment.

Break-even analysis requires classifying costs, such that total costs can be separated into those that are fixed and those that are variable. For purposes of this study, fixed expenses are a function of time and variable expenses are a function of operational volume (all fixed expenses become variable if time is increased sufficiently). The planning horizons that are to be meaningful will be limited in length by the ability of the forecasting methods to project into the future accurately, and will be considered as "short term" relative to the term of fixed expenses. Thus, the emphasis of the production manager for a given planning horizon is upon control of the variable costs.

Variable costs in the short term are largely constituted of direct labor and direct material charges. Of these, his control of direct material is largely the control of scrap and rework, a quality control function. Given that demand will be met (or some specified level of it), material will be expensed in some proportion to that level and the better the scrap and rework job, the smaller the proportion. Direct labor is however a different matter, in break-even analysis the labor rate is assumed to be a constant per unit of demand. In actuality, all the labor going into any given demand may not occur at the same rate per unit of time, as when overtime hours are scheduled. Also, the fixed expenses for any given planning horizon will be expensed, whether recovered or not. Recognizing the random variation of demand over time, the decision of a desired capacity level can be represented by Figure 5.

Interpreting the figure, total expected costs (TEC) are a function of the capacity level (c) and demand for some planning horizon of interest is probabilistic. Once a c value has been selected, it will be expensed at a rate of L ($/unit), but demand greater than c must be met by utilizing some supplemental means with an associated rate of P ($/unit). Then if c_1 is selected as the capacity level and a demand equivalent to c_4 occurs, c_1 units of the total will be expensed at $L/unit, and $(c_4 - c_1)$ will be expensed at $P/unit, if total demand is met. At the other extreme, if capacity is selected at c_4 and a demand equivalent to c_1 occurs, an expense of $\$Lc_4$ will be expensed but only $\$Lc_1$ would have been required. Clearly the problem of selecting the "best" c_i depends upon the relative ratio of L and P and the shape of the probabilistic demand. Models to handle this problem are presented in Chapter III.

Flexibility

Flexibility as used here implies the amount of time available to the production manager to plan his capacity level such that all demand (or a managerially specified percent of it) occurring within the planning horizon is met.

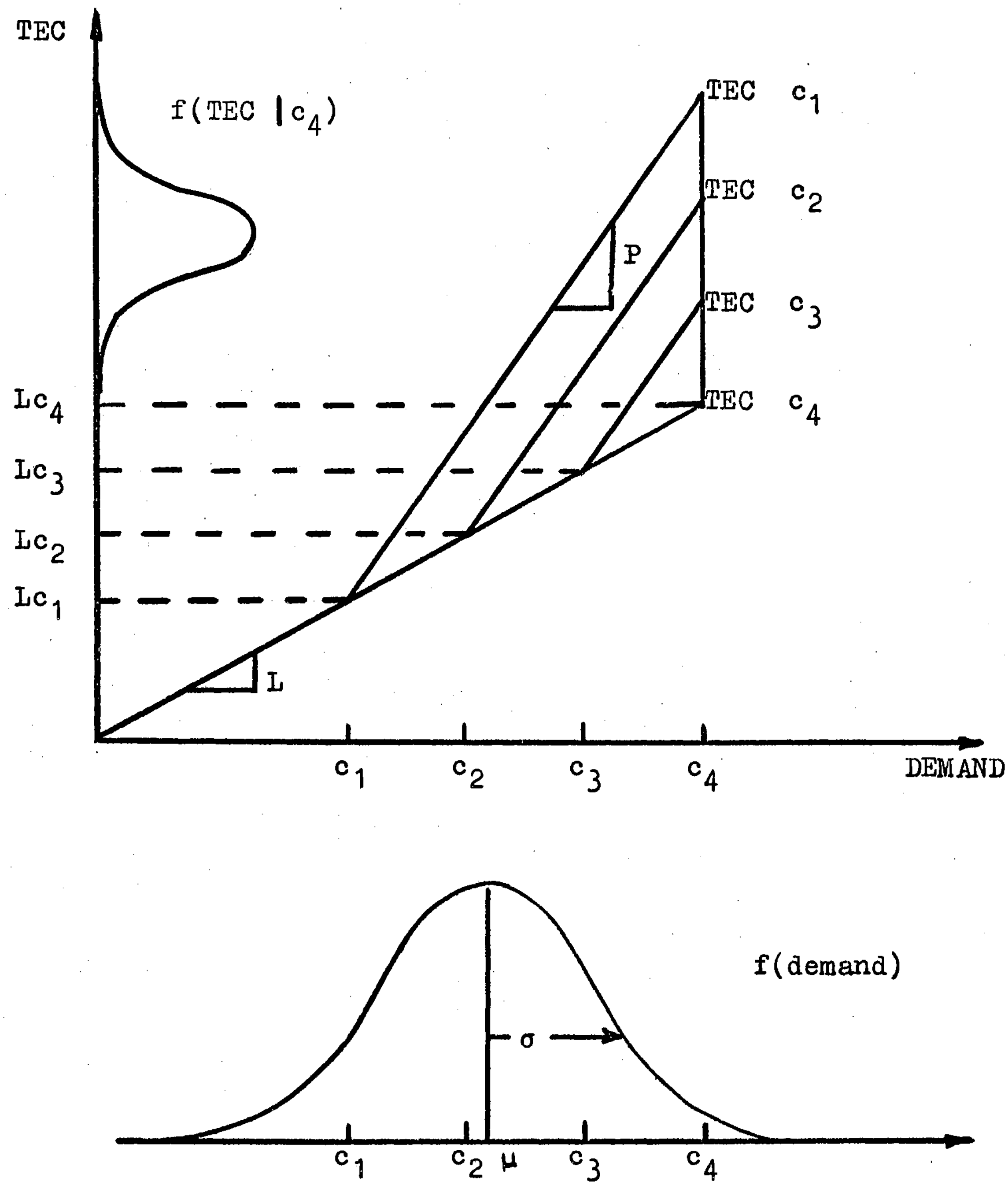


Figure 5. Total Expected Costs as a Function of c, L and P; Given That Demand is Probabilistic

The degree of flexibility could be determined by the market or by internal management policy or by some combination of both. Flexibility, as used in this context, is not to be confused with any ability to shift work or capacity levels. Such decisions fall into the category of operational decisions.

Three general degrees of flexibility are to be considered. No flexibility means that a unit demand must be met within the same time period of the minimum demand distribution. In this case, a unit of demand in any one day must be completed on that same day where the basic probability distribution function is in units of demand per day. The more general case is that of intermediate flexibility, in which case, a unit of demand in any one period must be met within some finite multiple of that time period, for example, any unit of demand on any given day must be met within say, five days. In this context the total distribution of demand within the flexibility period is assumed to all be due, which requires that in any given period of time of duration equal to the flexibility period, the work deferred from previous periods into the current planning horizon is not significantly different in magnitude from that which will be deferred from this planning period into the next.

The extreme and highly theoretical limit of flexibility is that of infinite or full flexibility. This implies that a unit of demand is only required to be processed, but due at any future time period. This non practical case has only academic usefulness.

Service Level

It may be a management decision in certain circumstances that not all work demanded will be processed. It is logical to turn away work which must be processed at the most expensive productive rate. Recalling that associated with a probability distribution function, $f(x)$, is its cumulative distribution function, $F(x)$, then if only $\alpha F(x)$ is considered, where α can be thought of as a service level factor, then this decision can easily be handled.

Conclusion

In closing this chapter, the need to approach the capacity problem from a planning stage will be defended. Three things can happen, either the economic capacity will be above, on, or below the effective capacity level considered available at that point in time. If the model yields an economic capacity level above the <u>current</u> considered attainable effective capacity level, either additional capacity is called for <u>or</u> a re-evaluation and adjustment of the factors causing effective capacity to be below full potential can be initiated. This problem most likely involves an economic evaluation of alternatives available and would be related to the investment limitations that are imposed upon the firm. For the economic level to fall exactly on the current considered attainable effective capacity level would be rare, but at least would give indication of the need for planning now for the next planning horizon depending upon an expected increase or decrease in demand. When the economic level is below the current attainable effective capacity level, this points to need for consideration of stimulation of demand or the creation of new products in order that unit costs may be decreased.

The magnitude of the difference between the economic level and the current effective level of course could determine what courses of action are reasonable. But by having measured this magnitude, allowing the probabilistic nature of future demand to interact, such that future courses of action may be implemented at an early point in time, may well be the best defense of the method. In other words, planning in the manner suggested by this research for some fixed planning horizon may yield as much useful information for such areas as long range expansion plans and the timing of improvement projects in the industrial engineering area as it does for just the immediate planning horizon under consideration.

CHAPTER III

MODEL DEVELOPMENT

The purpose of this chapter is to define the logic of the models, which will explicitly be utilized in Chapters IV and V.

A model, in most cases a mathematical expression relating various dependent and independent variables, is nothing more than a logical expression. As stated in Chapter I, the objective is to meet demand, but to do so in such a way that costs are minimized. It is not surprising then that the model concept shall be in terms of cost per time period, since the input to the model is a demand which implicitly carries with it a time domain, (i.e., units/day, units/year, etc.) and the production process requires time which is chargable as cost per unit time.

The measure of effectiveness is to be total expected cost. The decision needs to be made, for planning purposes, at what level should capacity be set, when the demand against that capacity itself is a random variable. An understanding of expected values imply that future demand may occur over a rather wide range of values and for any given planning horizon the decision may not have been the best, after the fact. But the problem is one under risk, not certainty. Indeed in subsequent planning horizons, the decision will need to be repeated, and if the capacity level is selected such that total expected costs are minimized within each planning horizon, then over the sequence of decisions, the total costs would be minimized across all planning horizons.

Statement equations will attempt to emphasize the logic involved, and these equations will be followed by symbolic ones expressing the relationships among the parameters.

No Flexibility

$$\begin{bmatrix}\text{Total Expected} \\ \text{Cost}\end{bmatrix} = \begin{bmatrix}\text{Cost of Work Performed} \\ \text{on Regularly Available} \\ \text{Capacity}\end{bmatrix} + \begin{bmatrix}\text{Cost of Work} \\ \text{Performed on} \\ \text{Premium Capacity}\end{bmatrix}$$

$$TEC = Lc + P \int_{c}^{\infty} (x-c)\ f(x)dx \qquad x \sim \text{continuous}$$

$$TEC = Lc + P \sum_{c}^{\infty} (x-c)\ f(x) \qquad x \sim \text{discrete}$$

As discussed in Chapter II, a fixed effective capacity level can be considered as a fixed investment within the production planning horizon. The cost represented by the first term in the above expression will be incurred, whether or not it is utilized. Since no flexibility exists, even if the random variable of input (x) is less than c, Lc dollars will be committed and charged against the production process.

Modern production management systems yield several examples. Work forces are not flexible with respect to variations in work load over short periods, sometimes clearly specified in union contracts and sometimes by design of the controlling management philosophy to maintain a skilled work force. Fully or highly automated processes, once set at <u>some</u> capacity level, represent costs in many cases, that depreciate more with respect to time and productive obsolescence than with utilization.

The second term appearing in the TEC equation implies its dimensions as cost/time period, the time period compatible with f(x).

As P is a "premium" cost per unit of work, the integral should yield a quantity in units. Note that the lower limit of the integral is at c, the threshold for premium work as all levels of $x \leq c$ can be handled by the available capacity. The term $(x - c)$ is the amount of demand by which x has exceeded c and therefore is a random variable in itself since x is a random variable. The behavior of x is determined by f(x), its distribution function. Thus the integral gives the expectation of the number of units applicable to the P cost parameter.

Intermediate Flexibility

[Total Expected Cost] = [Cost of Work Performed on Regularly Available Capacity] + [Cost of Work Performed on Premium Capacity]

$$TEC = Lc + P \int_{c}^{\infty} (X-c)\ f(X)\ dX \qquad X \sim \text{continuous}$$

$$TEC = Lc + \sum_{c+1}^{\infty} (X-c)\ f(X) \qquad X \sim \text{discrete}$$

In the intermediate case, the demand over some multiple of the time periods for which the basic probability distribution(s) of demand is(are) valid, requires use of the convolutions of demand as discussed in Chapter II. In the first term, c is expressed as units of capacity over the entire planning horizon and the second term follows the same logic as above, except that the distribution function of demand is now the convoluted distribution function.

Production to inventory within any one basic time period <u>could</u> <u>be</u> feasible, up to c/t units, but in effect the total demand is constituted as a random sample of size t from either a parent population or from up to t different populations, each with their specific parameters. Thus $f^t(x)$ is utilized to determine the behavior of total demand, f(X), for the entire planning horizon (t units of time in length). As mentioned in Chapter II, some carry forward from the previous planning horizon would be expected and the likelihood of unused capacity ($X < c$) will depend upon the level of capacity that is established. In any case, as the measure of effectiveness is cost, the same arguments concerning Lc as a fixed cost, regardless of X, still hold.

Full Flexibility

This highly impractical level of flexibility illustrates the limiting case. In Chapter II full flexibility was defined to imply an infinite amount of time at the discretion of the production manager as to when all work must be finished. If an attempt is made to convolute the basic distribution functions an infinite number of times, the result is a rather flat normal curve, one with an infinite mean and variance. So the approach must be altered. Consider the effect of full flexibility upon the production manager within any basic time period. He knows that he would like to keep his capacity at full utilization, but any demand above that level would be deferred until some subsequent period in which he had "slack time" due to a demand less than his capacity level. Intuitively, he may suspect that if his capacity level is at the mean of the total demand, then in any given time period, he could balance work carried forward with other periods in which demand levels are less than the mean, and in the long run never be required to process work on a premium time basis.

To "prove" this intuitive thought, consider the Law of Large Numbers [3, 10]. In words, the law states that as t gets large (approaches infinity), the probability that the average of t independent experimental values of the random variable (demand) differs from the expected value of demand by more than any nonzero ϵ goes to zero. Symbolically:

Let

$$\bar{\bar{X}} = \frac{1}{t} \sum_{i=1}^{t} x_i$$

then

$$E(\bar{\bar{X}}) = \mu$$

and from the Law of Large Numbers

$$\lim_{t \to \infty} \text{Prob} \left[\, | \bar{\bar{X}} - \mu | \geq \varepsilon \right] = 0$$

or alternatively,

$$\lim_{t \to \infty} \text{Prob} \left[\, | \bar{\bar{X}} - \mu | < \varepsilon \right] = 1.0 \quad .$$

Thus for the full flexibility case, c would be set at the mean of the convoluted distribution function. This fact is used only in the limit to converge the optimal capacity level as a function of an increasing t.

Service Level

If the decision is made to only process up to some percentage of the total demand, say α, then the upper limit of the integral (summation) sign need only be changed to α from the theoretical ∞. In this manner, all work turned down $(1 - \alpha)$ would be that applicable to the more expensive processing rate. This modification is independent of the flexibility rule imposed.

CHAPTER IV

CONTINUOUS CASE

The random variable of demand is considered to be represented by continuous probability distributions which result from, or are forecasts of, demand over the planning horizon. The approach will be to minimize the total expected costs per planning horizon under the three general cases of flexibility, in order.

No Flexibility

In this limited case, the forecast must be made to project the demand over a minimum period of time. All work received is due out within the same time period for which it was forecasted, and no work may be carried forward into the next period. The model from Chapter III for this case is:

$$TEC = Lc + P \int_{c}^{\infty} (x-c)\ f(x)\ dx \quad . \qquad (4.1)$$

To minimize TEC, with respect to the mathematical variable c, the first derivative is set to zero:

$$0 = \frac{d(TEC)}{dc} = L + P \left[\int_{c}^{\infty} \frac{\partial[(x-c)\ f(x)]}{\partial c}\ dx + [(\infty-c)\ f(x)] \frac{d(\infty)}{dc} - [(c-c)\ f(x)] \frac{d(c)}{dc} \right]$$

$$0 = L + P \int_{c}^{\infty} (-1)\ f(x)\ dx$$

since

$$F_x(c) = \int_{-\infty}^{c} f(x)\ dx\ , \text{ and } \int_{c}^{\infty} f(x)\ dx = 1.0 - F_x(c)$$

then

$$0 = L + P\ [F_x(c) - 1]$$

or

$$1 - \frac{L}{P} = F_x(c) \quad . \tag{4.2}$$

Thus the decision criteria for minimizing expected costs as a function of c is given by equation (4.2) and is seen to be a function of L, P and the cumulative of total demand.

Discussion

Before developing the models for the other flexibility cases, a more detailed explanation than given in Chapter III of the second term of equation (4.1) is in order.

$$P \int_{c}^{\infty} (x-c)\ f(x)\ dx \quad .$$

Repeating briefly the discussion of Chapter III, P is the applicable cost coefficient to the quantity of work expected to exceed the capacity level c. Dimensionally the expression yields:

$$\frac{\$}{\text{unit}} \times \frac{\text{expected \# of units}}{\text{planning horizon}} \quad .$$

Thus the integral should give a measure in quantity. A small example should aid in understanding.

Let x be distributed uniformly on the interval [4,10]. Arbitrarily set c = 6 (this <u>could</u> be optimal for some value of L and P but interest here is directed towards the expected quantity (x-c)). Define a new random variable y by the transformation:

$$y = 0 \qquad \text{for} \qquad x \leq 6$$

$$y = x - 6 \qquad \text{for} \qquad x > 6 \quad .$$

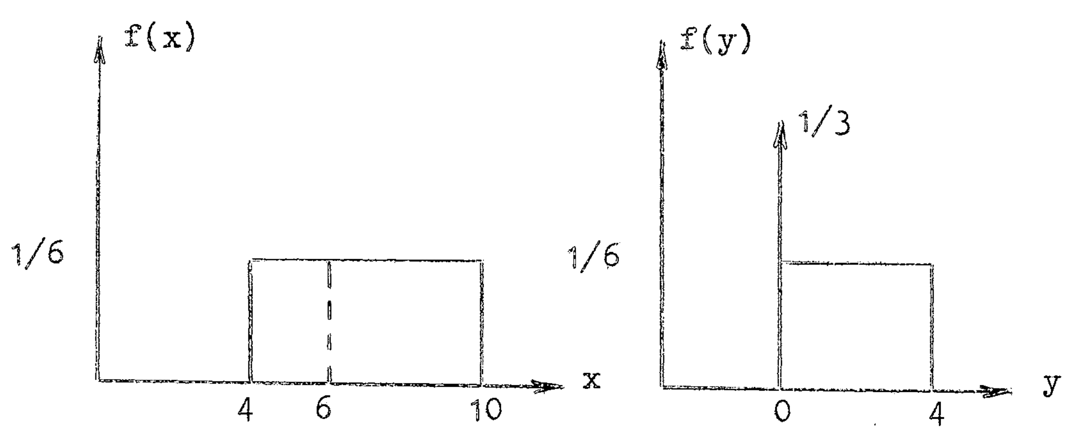
Thus y becomes the number of units within the planning horizon that must be "carried over" above the set capacity of six and processed at the P rate/unit, as any level of demand $\leq$ 6 could be processed within c at the lower rate L.

It will be shown that:

$$P \int_{c}^{\infty} (x-c) \; f(x) \; dx = P \int_{-\infty}^{\infty} y \; f(y) \; dy \quad .$$

The right hand term clearly appears as the expected value of y, E(Y), and P E(Y) represents the expected cost applicable to that portion of work processed on "premium" time.

For the example:



Notice that f(y) appears both discrete and continuous. Recognizing the essence of the transformation, this is understandable. For all values of $x \leq c$, y is defined to be zero. For the set value of $c = 6$, $F_x(6) = 1/3$, or in words one-third of the time, there would be no work done on "P time," if x is uniform on [4, 10] and $c = 6$. Or stated differently, $4 \leq x \leq 6$ maps into $y = 0$. Note also that

$$\int_{0^+}^{\infty} f(y)\, dy$$

is not unity inasmuch as the event $(y > 0)$ is not certain (one-third of the time y is zero). But expressing y as the sum of an impulse function[1] (for the discrete portion) and as a regular integral (for the continuous portion) the sum over the y-domain is unity; i.e.,

$$\int_{-\infty}^{\infty} f(y)\, dy = \lim_{\epsilon \to 0} \int_{0-\epsilon}^{0+\epsilon} f(y)\ \delta\ (y-0)\ dy + \int_{0+\epsilon}^{4} f(y)\ dy$$

$$= \underset{\epsilon \to 0}{f(0)} \int_{0-\epsilon}^{0+\epsilon} \delta(y-0)\ dy + \int_{0+\epsilon}^{4} (1/6)\ dy$$

$$= 1/3\ [1] + \int_{0}^{4} (1/6)\ dy$$

$$= 1/3 + \frac{y}{6}\Bigg|_{0}^{4} = 1/3 = 4/6 = 1.0 \quad .$$

[1]See Appendix B for a brief discussion of the impulse function.

Thus y as described is a distribution function, as it is everywhere positive on its domain and sums to unity over its domain.

Now to show:

$$P \int_{c}^{\infty} (x-c)\ f(x)\ dx = P \int_{-\infty}^{\infty} y\ f(y)\ dy \quad .$$

First the left hand term is evaluated;

$$\int_{c}^{\infty} (x-c)\ f(x)\ dx = \int_{6}^{10} (x-6)(1/6)\ dx = 1\ 1/3 \ .$$

Now for the right hand side; let

$$y\ f(y) = g(y)$$

then

$$\int y\ f(y)\ dy = \int g(y)\ dy$$

$$= \lim_{\epsilon \to 0} \int_{0-\epsilon}^{0+\epsilon} g(y)\ \delta(y-0)\ dy + \int_{0+\epsilon}^{\infty} g(y)\ dy$$

$$= g(y) \lim_{\epsilon \to 0} \int_{0-\epsilon}^{0+\epsilon} \delta(y-0)\ dy + \int_{0+\epsilon}^{4} y(1/6)\ dy$$

$$= y\ f(y)\ 1.0 + 1/6 \lim_{\epsilon \to 0} \int_{0+\epsilon}^{4} y\ dy$$

$$= (0)(1/3) + 8/6 = 1\ 1/3 \quad .$$

Thus, the expected quantity is given by the second term of expression (4.1). Since the event $y = 0$ has nonzero probability, this probability has a weighting effect in the E(Y), as expected.

Intermediate Flexibility

In the intermediate case, the convolution of the basic demand forecasts within the planning horizon must be dealt with. The model from Chapter III is:

$$TEC = Lc + P \int_{c}^{\infty} (X-c)\ f(X)\ dX \quad . \qquad (4.3)$$

In (4.3), f(X) is the resultant or convoluted distribution function of total demand within the planning horizon, which implies

$$X = \sum_{i=1}^{t} x_i \quad .$$

Letting * represent the mathematical operation of convolution;

$$f(X) = f(x_1) * f(x_2) \ . \ . \ . * f(x_t)$$

where x_i represents the distribution function of demand for the i^{th} basic period and t represents the number of such basic periods allowed by the flexibility level. In the special case where

$$f(x_1) = f(x_2) = \ . \ . \ . \ f(x_t)$$

then

$$f(X) = f^t(x_i) \qquad i = 1, 2, \ldots t \quad . \qquad (4.4)$$

Otherwise, the model is the same as in the no flexibility case, and the remarks concerning the expected quantity carried over into the higher processing rate, P, still apply.

Proceeding as before, the model is minimized with respect to the mathematical variable c:

$$0 = \frac{d(TEC)}{dc} = L + P \left[\int_{c}^{\infty} \frac{\partial[(X-c)\ f(X)]}{\partial c}\ dX \right.$$

$$\left. + [(\infty-c)\ f(X)]\ \frac{d(\infty)}{dc} - (c-c)\ f(X)\ \frac{d(c)}{dc} \right]$$

$$0 = L + P \int_{c}^{\infty} (-1)\ f(X)\ dX$$

$$0 = L - P\ [1 - F_X(c)]$$

or

$$1 - \frac{L}{P} = F_X(c) \tag{4.5}$$

which is general in nature and for the special case given by equation (4.4) becomes

$$1 - \frac{L}{P} = F^{t}_{x_i}\ (c) \qquad i = 1,\ 2,\ .\ .\ .\ t. \tag{4.6}$$

Again, the decision criteria for minimizing expected costs as a function of c is given by either equation (4.5) or (4.6) and is seen to be a function of L, P and the cumulative of total demand.

Full Flexibility

As shown in Chapter III, the Law of Large Numbers implies that the economic capacity level c would be set at the mean of the convoluted distribution function. Thus, for t time periods

$$\text{TEC}\ (t) = L\ \mu_X\ ,$$

and for one time period

$$\text{TEC}\ (1) = \frac{L}{t}\ \mu_X\ .$$

Obviously, if full flexibility were allowed, no work would be processed at the more expenseive rate P, it would only be deferred into the future. Thus as t is allowed to increase, the c value, in the limit, becomes the mean of the total distribution function, regardless of L and P. The convergence of c will be more evident in the examples of Chapter VI.

CHAPTER V

DISCRETE CASE

In this chapter, the random variable of demand is assumed to only take on discrete values. The basic approach is the same as in Chapter IV except for this difference.

No Flexibility

Under the no flexibility assumption, all work forecasted for some basic planning horizon is due within that same period. The model from Chapter III is repeated for this case:

$$TEC = Lc + P \sum_{c+1}^{\infty} (x-c)\, f(x) \quad . \qquad (5.1)$$

The mathematical variable c must take on the possible values of x, and one or at most two, of these possible values will be optimal in the sense that it will yield a lower TEC value than all others. (When two optimal values of c occur, they have equal TEC values and are equally optimal.) Identifying the optimal value of c as c_o, i.e.:

$$\text{Optimal } \left\{TEC\right\} = TEC_{c_o} = L_{c_o} + P \sum_{c_o+1}^{\infty} (x-c_o)\, f(x) \qquad (5.2a)$$

and

$$TEC_{c_o+1} > TEC_{c_o} \qquad (5.2b)$$

$$TEC_{c_o-1} > TEC_{c_o} \tag{5.2c}$$

by definition of optimal.

The approach will be to find TEC_{c_o+1} and TEC_{c_o-1} in terms of TEC_{c_o}.

$$TEC_{c_o+1} = L(c_o+1) + P \sum_{x=(c_o+1)+1}^{\infty} [x - (c_o+1)] f(x)$$

$$= Lc_o + L + P \sum_{x=c_o+2}^{\infty} [x - (c_o+1)] f(x)$$

$$= Lc_o + L + P \sum_{x=c_o+1}^{\infty} [x - (c_o+1)] f(x)$$

$$- P[(c_o+1) - (c_o+1)] f(c_o+1)$$

$$= Lc_o + L + P \sum_{x=c_o+1}^{\infty} [x-c_o] f(x) - P \sum_{x=c_o+1}^{\infty} f(x)$$

$$= Lc_o + P \sum_{x=c_o+1}^{\infty} (x-c_o) f(x) + L - P[1 - F(c_o)]$$

since

$$\sum_{x=c_o+1}^{\infty} f(x) = 1 - F(c_o)$$

$$TEC_{c_o+1} = TEC_{c_o} + L - P + P\,F(c_o)\,. \tag{5.3}$$

Proceeding similarly for TEC_{c_o-1}:

$$TEC_{c_o-1} = L(c_o-1) + P \sum_{x=(c_o-1)+1}^{\infty} [x-(c_o-1)]\, f(x)$$

$$= Lc_o - L + P \sum_{x=c_o}^{\infty} [x - (c_o-1)]\, f(x)$$

$$= Lc_o - L + P \sum_{x=c_o+1}^{\infty} [x - (c_o-1)]\, f(x)$$

$$+ P[c_o - (c_o-1)]\, f(c_o)$$

$$= Lc_o - L + P \sum_{x=c_o+1}^{\infty} (x-c_o)\, f(x) + P \sum_{x=c_o+1}^{\infty} f(x)$$

$$+ P\, f(c_o)$$

$$= TEC_{c_o} - L + P \sum_{x=c_o}^{\infty} f(x) - Pf(c_o) + Pf(c_o)$$

$$TEC_{c_o-1} = TEC_{c_o} - L + P\,[1 - F(c_o-1)] \qquad (5.4)$$

since

$$\sum_{x=c_o}^{\infty} f(x) = 1 - F(c_o-1) \quad .$$

From the relationships (5.2) it follows.

$$TEC_{c_o+1} - TEC_{c_o} > 0$$

$$TEC_{c_o-1} - TEC_{c_o} > 0$$

substituting (5.3) and (5.4), this becomes

$$L - P + P\,F(c_o) > 0$$

$$-L + P - P\,F(c_o - 1) > 0$$

which can be combined into:

$$F(c_o - 1) < 1 - \frac{L}{P} < F(c_o) \quad . \qquad (5.5)$$

In application, it may turn out that the term $\left(1 - \frac{L}{P}\right)$ is equal to either $F(c_o-1)$ or $F(c_o)$, but obviously not both. If c_o is such that

$$F(c_o - 1) = 1 - \frac{L}{P} < F(c_o)$$

then both c_o and c_o-1 are equally optimal, and if c_o is such that

$$F(c_o - 1) < 1 - \frac{L}{P} = F(c_o)$$

then c_o and c_o+1 are equally optimal.

The important observation is that the cumulative distribution of the random demand through equation (5.5) allows for the determination of c_o.

Intermediate Flexibility

Intermediate flexibility allows for either the same basic distribution of demand to follow in subsequent periods or a series of different distributions to occur within the time domain encompassing the planning horizon.

From Chapter III, the model for this case is

$$TEC = Lc + P \sum_{c+1}^{\infty} (x-c)\, f^t(x) \quad . \qquad (5.6)$$

The only difference in this case than in the previous section is that the convoluted distribution of demand will appear in the decision criteria. Either the total possible demand that is expected to occur over the time domain of the planning horizon is the t^{th} convolution of the basic demand distribution or a derived convolution of total demand due to different (up to t) basic demand distributions existing within the planning horizon.

The development follows the same logic of the previous section with respect to some value of demand being optimal, and leads to the decision criteria:

$$F^t(c_o-1) < 1 - \frac{L}{P} < F^t(c_o) \quad . \qquad (5.7)$$

The same possibilities exist for two optimal values of c to occur in specific cases when the inequality is not met but occurs as an equality. In any case, given intermediate flexibility, the convoluted cumulative distribution function, through (5.7), allows for the determination of c_o.

Attention should be directed to the fact that as certain discrete distributions are increasingly convoluted, the resultant takes on a normal form, although the function naturally remains discrete. In these cases, computational ease may be gained by using the normal distribution to approximate the resultant, the error being a function of the original discrete distributions and the number of time periods encompassing the planning horizon.

Full Flexibility

The same argument made in Chapter IV relative to the behavior of a convoluted continuous distribution function as the number of convolutions increase indefinitely holds in the discrete case. Only the method of performing the convolutions differ. Thus it can be stated that, in the limit, as t increases indefinitely, the economic capacity level c_o would be set at the mean of the convoluted distribution function.

CHAPTER VI

APPLICATIONS

In this chapter, the decision criteria of Chapters IV and V will be used with hypothetical examples. In this way, the interactions of basic demand distributions with each other in determining the total demand over the planning horizon and the effect of increased flexibility should be clarified. Although the examples deal with rather common distribution functions, the approach would be no different for more complex cases.

Example 1. Normally Distributed Demand

Consider: $X \sim N\ (100,\ 10)/\text{wk}$.

It is desired to find the optimal capacity level, c, for various sets of L and P values, for various levels of allowed flexibility. Whatever the values of L and P, P will be measured proportional to L, for computational ease. Table 1 gives the values for which calculations are presented.

Consider first, t = 1, or the no flexibility case. The decision criteria for the continuous case from Chapter IV is:

$$1 - \frac{L}{P} = F_X(c) \quad . \qquad (4.2)$$

TABLE I

L, P VALUES AND DEGREE OF FLEXIBILITY FOR EXAMPLE 1

Curve	L	P	L/P	Flexibility (in weeks)
1	1.0	1.5	2/3	1, 3, 6, 20, 30, 40
2	1.0	2.0	1/2	1, 3, 6, 20, 30, 40
3	1.0	3.0	1/3	1, 3, 6, 20, 30, 40

As $F_x(c)$ represents the cumulative distribution of x, the transformation to the standard unit normal will be used:

$$Z_N = \frac{c - \mu}{\sigma} \qquad \text{(c is some value of x)}$$

and in general

$$(Z_N)(\sigma) + \mu = c \tag{6.1}$$

for the first case, t = 1 (L = 1.0 and P = 1.5);

$$F_x(c) = 1 - 2/3 = 1/3 = 0.333 = \int_{-\infty}^{c} f(x)\, dx$$

which implies a Z_N of -0.43. Thus

$$(-0.43)(\sqrt{10}) + 100 = c = 98.64 \quad .$$

Increasing the flexibility, and holding L and P fixed, causes no difficulty, if equation (6.1) is modified to be compatible with the increased flexibility and the decision criteria is changed to the intermediate case, as follows: since

$$f(x_1) = f(x_2) = \ldots f(x_t) \ , \tag{4.4}$$

applies

$$f(X) = f^t(x_i) \qquad i = 1, 2, \ldots t \qquad (4.4)$$

which implies

$$f^t(x_i) \sim N(t\ 100,\ t\ 10)$$

and the decision criteria is equation (4.6):

$$1 - \frac{L}{P} = F^t_{x_i}(c) \qquad i = 1, 2, \ldots t \qquad (4.6)$$

but so long as L and P remain fixed, (4.6) yields

$$1 = \frac{L}{P} = \frac{1}{3} = 0.333$$

for all cases of flexibility, and in general, equation (6.1) can be expressed:

$$c = (Z_N)(\sigma_{t\ wks}) + \mu_{t\ wks}$$

or

$$c/wk = \frac{(Z_N)\ \sqrt{t\ \sigma^2_{wk}} + t(\mu_{wk})}{t} \quad . \qquad (6.2)$$

Using equation (6.2), as many points on the t axis (flexibility) as desired may be computed, since Z_N remains constant for fixed L and P. The optimal c is then measured as a percentage of the weekly mean (the basic demand mean).

The results for L = 1.0 and P = 1.5 can be summarized as in Table II.

Now consider the second curve, L = 1.0, P = 2.0, Equation (4.6) yields:

$$1 - \frac{L}{P} = F^t_{x_i}(c) = 0.50$$

which implies a Z_N of zero, and equation (6.2) becomes:

$$c/wk = \frac{0 + t(\mu_{wk})}{t} = \mu_{wk} \quad .$$

This interesting result shows the optimal capacity level to be invariant for this particular set of L and P values, regardless of the flexibility. This is intuitively appealing; as long as it costs twice as much to process work that is carried over above an established capacity level, why not set the capacity level at such a point that the probability of work being carried over into the higher rate P, is one-half.

TABLE II

OPTIMAL c/wk FOR FIXED L = 1.0 AND P = 1.5 (Curve # 1)

L = 1.0	P = 1.5
t	c/wk
1	98.64
3	99.21
6	99.45
20	99.70
30	99.75
40	99.79

Comparing this result with that obtained for the first curve (L = 1.0, P = 1.5), the pattern of behavior for c is starting to appear. The lower the P value, the larger the portion of work that will be carried forward into it, and as P increases, the smaller this portion, as will be seen by analyzing curve three.

For the third curve, L = 1.0, P = 3.0, equation (4.6) yields:

$$1 - \frac{L}{P} = F_{x_i}^{t}(c) = \frac{2}{3} = 0.666$$

which implies a Z_N of +0.43, and equation (6.2) becomes:

$$c/wk = \frac{(+0.43)\sqrt{t\ \sigma_{wk}^2} + t(\mu_{wk})}{t}$$

and as before, as many points as desired on the t axis can be generated, those calculated are given in Table III.

The three curves considered so far are depicted graphically in Figure 6.

TABLE III

OPTIMAL c/wk FOR L = 1.0 AND P = 3.0 (Curve # 3)

L = 1.0	P = 3.0
t	c/wk
1	103.16
3	100.79
6	100.55
20	100.30
30	100.25
40	100.22

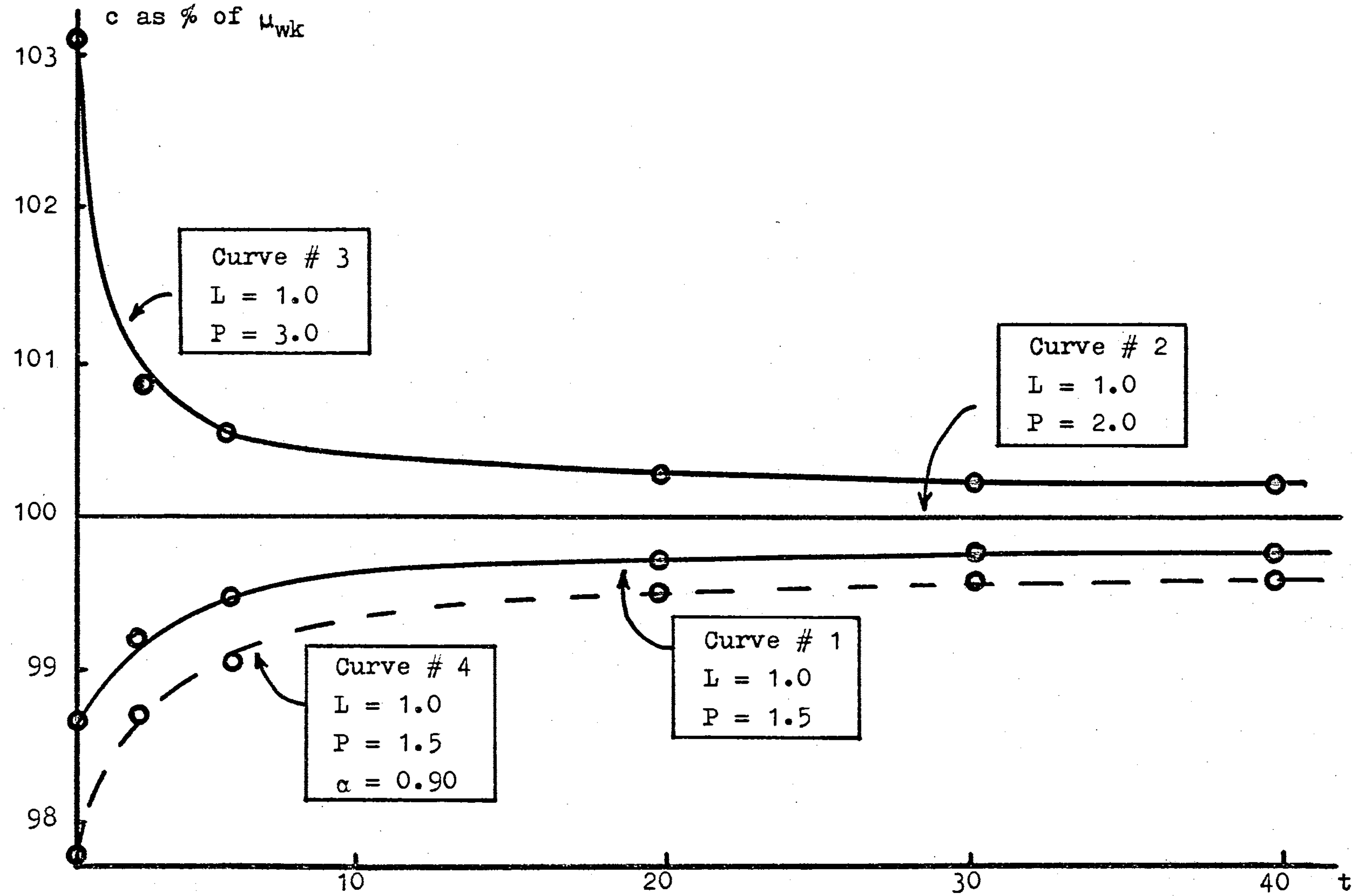


Figure 6. Optimal Capacity as a Function of L, P and Flexibility

Example 2. Normally Distributed Demand With A Service Level Imposed

The effect of imposing a service level will be developed in this example. The model from Chapter III is repeated:

$$TEC = Lc + P \int_{c}^{\infty} (x-c)\ f(x)\ dx \ .$$

The service level implies that only a portion of the expected demand is desired to be processed, and thus the upper limit of the integral becomes:

$$TEC = Lc + P \int_{c}^{\alpha} (x-c)\ f(x)\ dx$$

proceeding as before:

$$0 = \frac{d(TEC)}{dc} = L - P \int_{c}^{\alpha} (f(x)\ dx$$

since

$$\int_{c}^{\alpha} f(x)\ dx = F_x\ (\alpha) - F_x(c)$$

$$0 = L - P\ [F_x(\alpha) - F_x(c)]$$

or

$$F_x(\alpha) - \frac{L}{P} = F_x(c) \quad . \tag{6.3}$$

Thus, $F_x(\alpha)$ replaces the constant 1.0 in the decision criteria expression.

For convoluted cases, the decision criteria would be:

$$F_x^t(\alpha) - \frac{L}{P} = F_x^t(c) \quad . \tag{6.4}$$

Thus equations (6.3) or (6.4) provide the determination of the optimal c.

Applying this result to curve one of the previous example, with α set at 0.90, the following results are obtained:

$$F_x(0.90) - \frac{2}{3} = F_x(c) = 0.90 - 0.67 = 0.23$$

which implies a Z_N of −0.74. Equation (6.2) gives:

$$c/wk = \frac{(-0.74)\ \sqrt{t\ \sigma_{wk}^2}\ + t\ (\mu_{wk})}{t}$$

and the generated points are given in Table IV.

TABLE IV

OPTIMAL c/wk FOR L = 1.0, P = 1.5 AND α = 0.90
(Curve # 4)

L = 1.0	P = 1.5	α = 0.90
t		c/wk
1		97.66
3		98.65
6		99.04
20		99.48
30		99.57
40		99.64

Comparison of Tables II and IV indicate the effect of α as lowering the optimal c value, which is logical, since the percent of demand turned away, $(1 - \alpha)$, would be that applicable to the higher processing rate.

Example 3. Uniformly Distributed Demand

Consider Example 1 of Appendix A in which the uniform distribution on [0,1] was convoluted three times (results are graphically reproduced in Figure 7). As the resulting curve is quickly approaching a normal shape, if this distribution were to be studied as a function of increasing flexibility, clearly the normal could be used to approximate the actual distribution. Calculations for various degrees of flexibility will be presented, using both the actual distribution and the normal as an approximation for the cost values of L = 1.0 and P = 1.5 (Curve 1) and L = 1.0 and P = 3.0 (Curve 2). The inputs for the example are summarized in Figure 7. The actual method would yield the optimal value by integrating the cumulative function up to c such that the probability of a demand be $\leq (1 - L/P)$. Using the approximate method, the fixed Z_N value (for any set of L and P values) would imply the optimal value by use of equation (6.2):

$$c/wk = \frac{(Z_N)\ \sqrt{t\ \sigma_{wk}^2} + t\ (\mu_{wk})}{t} \quad .$$

The results obtained by both methods are summarized in Table V.

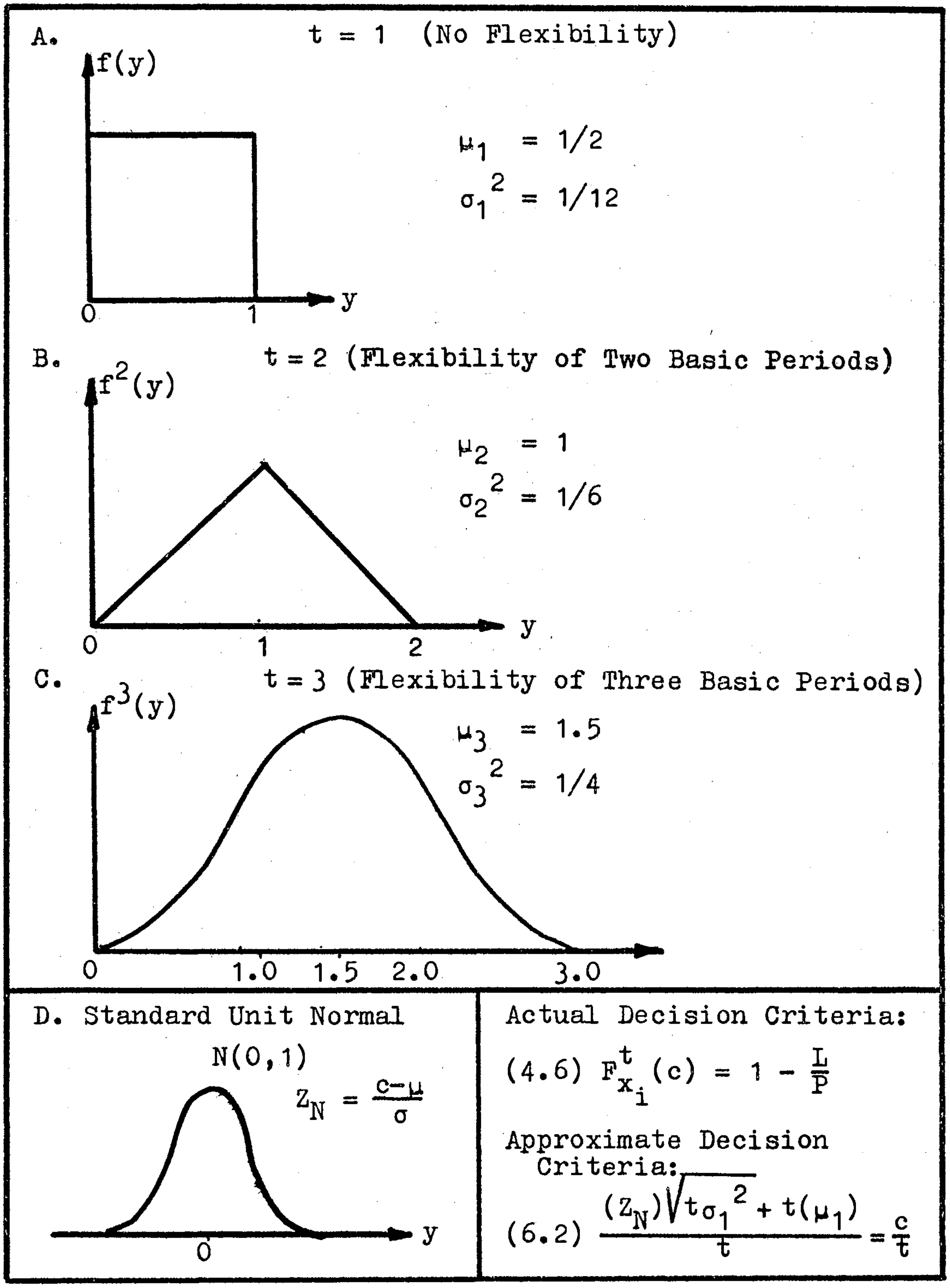


Figure 7. Inputs for Actual Versus Normal Approximation to Optimal Capacity Determination

TABLE V

ACTUAL VERSUS APPROXIMATE COMPARISONS FOR INCREASING FLEXIBILITY

Flexibility	Actual Method	Approximate Method	Per Cent Error
Curve # 1 L=1.0 P=1.5			
1	0.333	0.376	12.90
2	0.408	0.412	0.98
3	0.427	0.428	Nil
Curve # 2 L=1.0 P=3.0			
1	0.667	0.624	6.45
2	0.592	0.588	0.68
3	0.573	0.572	Nil

Example 4. Uniform Discrete Demands--

Fixed Flexibility

Suppose that a planning horizon is to encompass one year and that three four-month forecasts are available, each different and considered independent. The particular demand interest is measured in discrete units. The distributions are as follows:

Forecast period 1: $f(x_1) = \begin{cases} 1/3 & x_1 = 10, 11, 12 \\ 0 & \text{elsewhere} \end{cases}$

Forecast period 2: $f(x_2) = \begin{cases} 1/4 & x_2 = 12, 13, 14, 15 \\ 0 & \text{elsewhere} \end{cases}$

Forecast period 3: $f(x_3) = \begin{cases} 1/6 & x_3 = 8, 9, 10, 11, 12, 13 \\ 0 & \text{elsewhere} \end{cases}$.

Let $u = x_1 + x_2$. The z transform of u is given as:[1]

$$f_u(z) = f_{x_1}(z)\, f_{x_2}(z)$$

since

$$f_{x_1}(z) = 1/3\ z^{10} + 1/3\ z^{11} + 1/3\ z^{12}$$

and

$$f_{x_2}(z) = 1/4\ z^{12} + 1/4\ z^{13} + 1/4\ z^{14} + 1/4\ z^{15}$$

then

$$f_u(z) = [1/3\ z^{10} + 1/3\ z^{11} + 1/3\ z^{12}]$$

$$[1/4\ z^{12} + 1/4\ z^{13} + 1/4\ z^{14} + 1/4\ z^{15}]$$

$$f_u(z) = 1/12[z^{22} + 2z^{23} + 3z^{24} + 3z^{25} + 2z^{26} + z^{27}]$$

which implies that:

$$f(u) = \begin{cases} 1/12 & u = 22, 27 \\ 2/12 & u = 23, 26 \\ 3/12 & u = 24, 25 \\ 0 & \text{elsewhere} \end{cases} .$$

Now define

$$w = u + x_3$$

then

$$f_w(z) = f_u(z)\, f_{x_3}(z)$$

---

[1]For convenience, Appendix B contains a general discussion of Transform Theory.

since

$$f_{x_3}(z) = 1/6\ z^8 + 1/6\ z^9 + 1/6\ z^{10} + 1/6\ z^{11} + 1/6\ z^{12} + 1/6\ z^{13}$$

and $f_u(z)$ has been obtained above, then:

$$f_w(z) = [1/12(z^{22} + 2z^{23} + 3z^{24} + 3z^{25} + 2z^{26} + z^{27})] [1/6(z^8 + z^9 + z^{10} + z^{11} + z^{12} + z^{13})]$$

$$f_w(z) = 1/72[z^{30} + 3z^{31} + 6z^{32} + 9z^{33} + 11z^{34} + 12z^{35} + 11z^{36} + 9z^{37} + 6z^{38} + 3z^{39} + z^{40}]$$

which is represented by Figure 8.

The decision criteria of Chapter V is:

$$F^t(c_o - 1) < 1 - \frac{L}{P} < F^t(c_o) \qquad (5.7)$$

and the cumulative of W can be used to specify the solution of c_o as is shown in Table VI. The behavior of c_o to various L and P values is shown in Figure 9.

Discussion

Certain characteristics of the economic capacity determination problem are becoming clearer after the results of the examples are considered. Table VII briefly summarizes some of these characteristics.

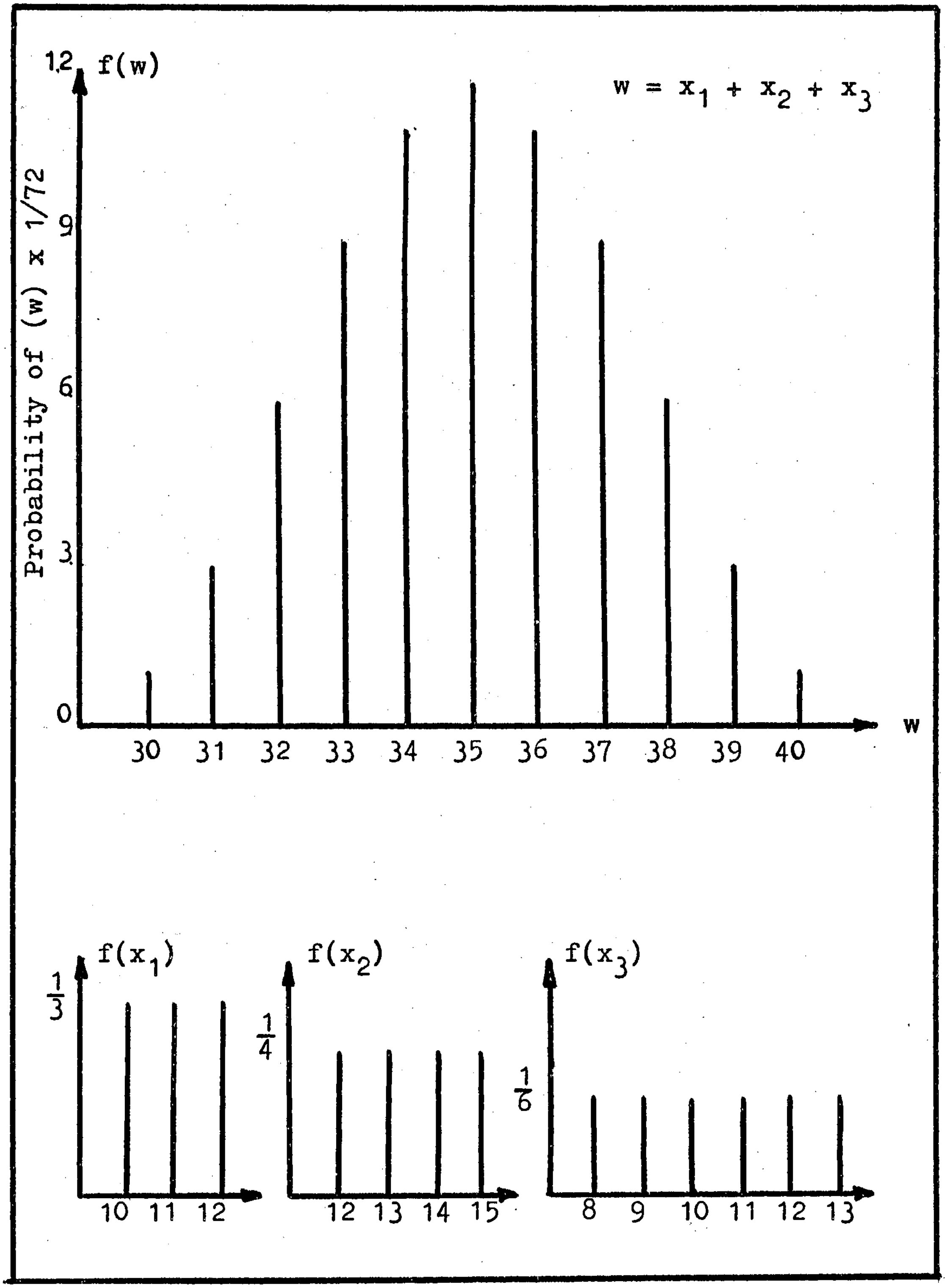


Figure 8. Probability Distribution of w, Where w is the Sum of Three Uniform Discrete Random Variables

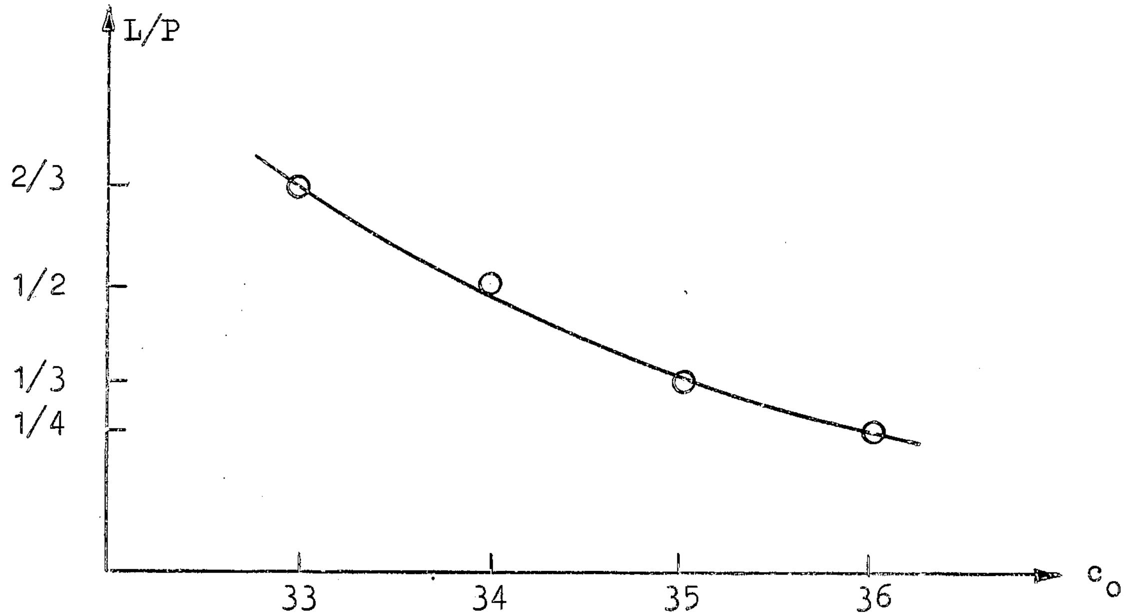


Figure 9. Relationship of c_o for Fixed Flexibility to Various L and P Values

TABLE VI

OPTIMAL c_o/yr FOR VARIOUS L AND P VALUES

W	f(w)	F(w)	c_o	L	P	$1 - \frac{L}{P}$
30	1/72	1/72				
31	3/72	4/72				
32	6/72	10/72				
33	9/72	19/72	33	1.0	1.5	0.33
34	11/72	30/72	34	1.0	2.0	0.50
35	12/72	42/72	35	1.0	3.0	0.66
36	11/72	53/72	36	1.0	4.0	0.75
37	9/72	62/72				
38	6/72	68/72				
39	3/72	71/72				
40	1/72	72/72				

TABLE VII

OBSERVATIONS ON EXAMPLES

Example	Conditions	Main Observations
1	X ~ Normal	1) effect of L and P with respect to level setting 2) rapid approach of optimal towards μ as flexibility increased
2	Same as 1 except service level imposed	1) effect of level on μ, below the unimposed case
3	X ~ Uniform	1) rapid convergence of convoluted demand towards normality 2) magnitude of error of approximate method is small as flexibility increases
4	X ~ Uniformly Discrete Fixed Flexibility	1) rapid approach towards normal shape, even within short flexibility 2) usefulness of transform theory in obtaining total demand within planning horizon

CHAPTER VII

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The capacity determination decision has been discussed within the context of production management, and described as a planning decision, made prior to operating decisions. The input for such a decision is the forecast of demand on the capacity system, projected from the future. Recognizing the variability involved, it was proposed that such forecasts be described probabilistically. It was further proposed that the cost of processing demand be considered in two general classes, regular and premium.

Given the probabilistic nature of future demand and the applicable costs associated with processing all or a portion of that demand, a decision criteria of minimizing expected costs was presented. In general, the decision criteria was seen to be a function of the relative costs and the flexibility allowed in processing work with respect to time. Examples were presented, under various conditions, which exemplified application of the models.

Using the decision criteria, it was demonstrated that:

1. The optimal capacity level, for given inputs, can be obtained.
2. The optimal level rapidly approaches the mean of

total demand within the planning horizon as flexibility is allowed to increase.

3. The relative location of the optimal level to the mean of total demand within the planning horizon is a function of the L and P cost ratio. When P is low relative to L, the optimal capacity level falls below the mean level and when P is high relative to L, it falls above the mean level.

These basic observations from the application of the models are intuitively logical, but go further than to just reinforce intuition. The models could be used to generate complete families of curves for various cost ratios and for various degrees of flexibility, and thus, numerically yield the range of optimality for parameter changes. Such a sensitivity analysis could also be made relative to forecast errors on future demands.

In addition to the obvious extension of the analysis to test the optimality range via sensitivity, certain other recommendations become apparent:

1. Relaxation of the requirement that future forecasts conform to probability distributions and the application of either non-parametric analysis or quantitative distribution free analysis based upon only partial information.

2. Relate the model results to the appropriate timing of major additions to capacity (merging the model with engineering economic analysis in

such a way that major outlays for expansion are economically timed). Such an analysis could impose both budget restrictions and upper limits on allowable premium time available within any planning horizon.

3. Relaxation of only two classes of costs relative to processing to include mixes of premium types.
4. Extension of the results of the models as inputs to the operational decision area, where the scheduling of total demand within the planning horizon is accomplished.

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APPENDIX A

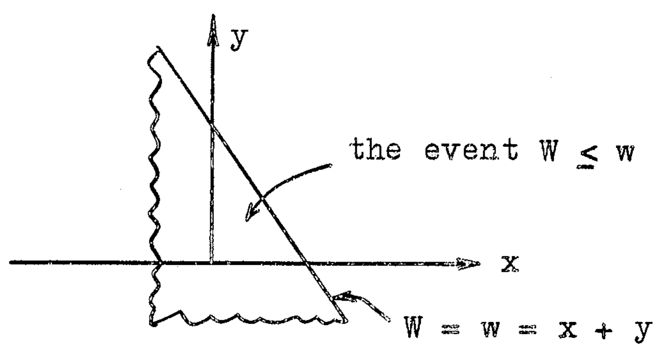
CONVOLUTION [3, 5, 10]

General Case

Given: X and Y are random variables

$f(x,y)$ is the joint pdf of X and Y.

The pdf of W is desired where $W = X + Y$. The event space is:



$$Pr(W \leq w) = F(w) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f(x,y)\, dx\, dy$$

$$f(w) = \frac{dF(w)}{dw} = \int_{x=-\infty}^{\infty} dx \frac{d}{dw}\left[\int_{y=-\infty}^{w-x} f(x,y)\, dy\right]$$

performing the derivative

$$f(w) = \int_{x=-\infty}^{\infty} f(x, w-x)\, dx \quad .$$

In general, the integral can be analyzed no further without specific knowledge of $f(x,y)$, the joint distribution function.

Case: X and Y Independent:

Independence $\Rightarrow f(x,y) = f(x)\, f(y)$

$$\therefore \quad f(w) = \int_{x=-\infty}^{\infty} f(x)\, f(w-x)\, dx \quad .$$

This integral is known as the convolution of $f(x)$ and $f(y)$. Integrating over x first rather than y, results in the equivalent expression:

$$f(w) = \int_{y=-\infty}^{\infty} f(y)\, f(w-y)\, dy \quad .$$

The above integrals may also be obtained by the following approach:

$$f(x,y) = f(x)\, f(y) \quad .$$

Let

$$z = x + y \qquad w = x$$

then

$$y = z - x \qquad x = w \quad .$$

The Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$$f(z,w) = f(w) \cdot f(z-w) \cdot |J| = f(w)\, f(z-w) \quad .$$

To obtain the density of z, integrate out w:

$$f(z) = \int_{-\infty}^{\infty} f(w)\ f(z-w)\ dw$$

which, except for the symbols is the same integral.

To illustrate some of these concepts and to draw attention to the approximation of convolution to the result obtained via the Central Limit Theorem, the remainder of this appendix will deal with examples.

Example 1. Convolution of Uniform
Distribution Function

Let

$$x_i \sim \text{Uniform on } [0,1]$$

$$f(x_i) = \begin{cases} 1 & 0 \leq x_i \leq 1 \\ 0 & \text{elsewhere} \end{cases} .$$

Define:

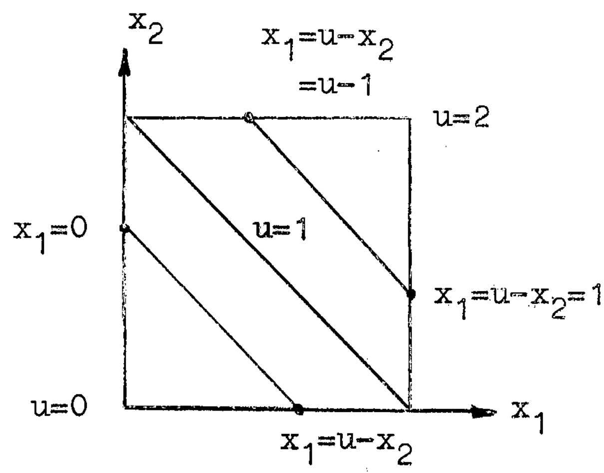
$$U = X_1 + X_2 .$$

The convolution integral yields the pdf of U:

$$f(u) = \int_{-\infty}^{\infty} f(u-x_1)\ f(x_1)\ dx_1 .$$

In evaluating the convolution integral, care must be taken to consider the domain for which the density functions are non-zero.

The range space of u, R_u is [0,2].



With the aid of the sketch, the limits of integration with respect to x_1 can be obtained.

Note also:

$$u = x_1 + x_2$$

$$u - x_1 = x_2$$

$$f(u - x_1) = f(x_2) = \begin{cases} 1 & 0 \leq x_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$0 \leq x_2 \leq 1 \quad \Rightarrow \quad 0 \leq u - x_1 \leq 1$$

which is equivalent to

$$u - 1 \leq x_1 \leq u$$

and

$$f(u) = \int_{-\infty}^{\infty} f(u - x_1)\, f(x_1)\, dx_1$$

becomes

$$f(u) = \int_{u-1}^{u} f(u - x_1)\, f(x_1)\, dx_1 \quad .$$

Consider the range space of u in two intervals:

1. $0 \leq u \leq 1$

2. $1 \leq u \leq 2$.

1. For $0 \leq u \leq 1$:

   a. The lower limit $(u-1)$ is bound $-1 \leq u - 1 \leq 0$, but $f(x_1) = 0$ for all values of $x_1 < 0$; this implies that $u - 1 = 0$ is the only valid lower limit for $f(x_1)$ to remain non-vanishing.

   b. The upper limit (u) is bound $0 \leq u \leq 1$, and $f(x_1)$ is non-vanishing over this entire domain; this implies the upper limit of u is valid.

Then for $0 \leq u \leq 1$:

$$f(u) = \int_0^u f(u - x_1)\, f(x_1)\, dx_1$$

$$f(u) = \int_0^u (1)(1)\, dx_1 = x_1 \Big|_0^u = u$$

or

$$f(u) = u \qquad 0 \leq u \leq 1 \quad .$$

2. For $1 \leq u \leq 2$:

$$f(u) = \int_{u-1}^u f(u - x_1)\, f(x_1)\, dx_1$$

   a. The lower limit $(u-1)$ is bound $0 \leq u - 1 \leq 1$, and $f(x_1)$ is non-vanishing for this domain.

   b. The upper limit (u) is bound $1 \leq u \leq 2$, but $f(x_1) = 0$ for all $x > 1$; this implies that only $u = 1$ need be considered.

Then for $1 \leq u \leq 2$:

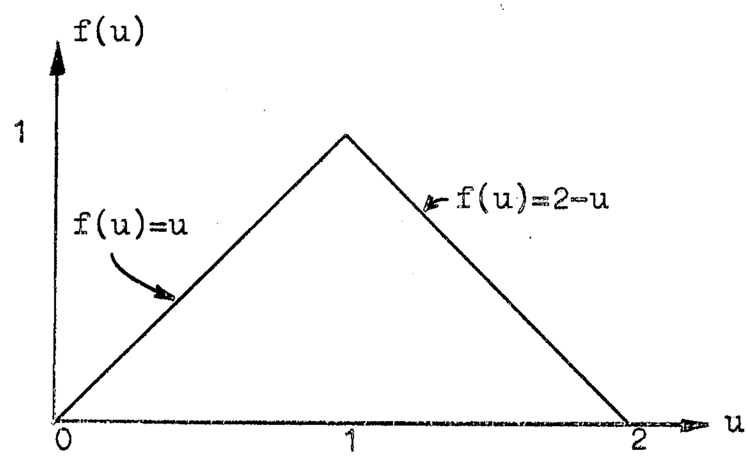
$$f(u) = \int_{u-1}^{1} f(u - x_1)\, f(x_1)\, dx_1$$

$$f(u) = \int_{u-1}^{1} (1)(1)\, dx_1 = x_1 \Big|_{u-1}^{1} = 1 - (u-1) = 2 - u$$

or

$$f(u) = 2 - u \quad \text{for} \quad 1 \leq u \leq 2 \ .$$

The result is



Notice, beginning with $f(x_i)$ uniform on [0,1] the sum of two x_i yields a triangular density function. Assume it is desired to convolute the result above u, with yet a third observation of x_i.

That is, define $Y = U + X_3 = (X_1 + X_2) + X_3$. The pdf of Y is desired, recognizing the range space of Y, $R_y = [0,3]$. By the convolution integral:

$$f(y) = \int_{-\infty}^{\infty} f(y - u)\, f(u)\, du$$

$$f(x_3) = f(y - u) = \begin{cases} 1 & 0 \leq y - u \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$0 \leq y - u \leq 1 \quad \Rightarrow \quad y - 1 \leq u \leq y$$

then

$$f(y) = \int_{y-1}^{y} f(y - u)\, f(u)\, du \quad .$$

Again consider the range space in intervals:

1. $0 \leq y \leq 1$
2. $1 \leq y \leq 2$
3. $2 \leq y \leq 3$.

1. For $0 \leq y \leq 1$:

   a. The lower limit (y-1) is bound $-1 \leq y - 1 \leq 0$, f(u) is non-vanishing only for 0 in this domain; this implies the only valid lower limit is 0.

   b. The upper limit (y) is bound $0 \leq y \leq 1$, and f(u) is non-vanishing over this domain; this implies y is a valid upper limit.

2. For $1 \leq y \leq 2$:

   a. The lower limit (y-1) is bound $0 \leq y - 1 \leq 1$, f(u) is non-vanishing over this domain; this implies y-1 is a valid lower limit.

   b. The upper limit (y) is bound $1 \leq y \leq 2$, f(u) is non-vanishing over this domain; this implies y is a valid upper limit.

3. For $2 \leq y \leq 3$:

   a. The lower limit (y-1) is bound $1 \leq y - 1 \leq 2$, f(u) is non-vanishing over this domain; this implies that y-1 is a valid lower limit.

b. The upper limit (y) is bound $2 \leq y \leq 3$, f(u) is non-vanishing only for y = 2 of this domain; this implies that y = 2 is the only valid upper limit.

Thus

$$f(y) = \int_{-\infty}^{\infty} f(y-u)\, f(u)\, du$$

$$= \int_{0}^{y} f(y-u)\, f(u)\, du \,, \qquad \text{for} \quad 0 \leq y \leq 1,$$

$$f(y) = \int_{y-1}^{y} f(y-u)\, f(u)\, du \,, \qquad \text{for} \quad 1 \leq y \leq 2,$$

$$f(y) = \int_{y-1}^{2} f(y-u)\, f(u)\, du \,, \qquad \text{for} \quad 2 \leq y \leq 3.$$

Previously, the following have already been found:

$$f(u) = u \qquad 0 \leq u \leq 1$$

$$f(u) = 2 - u \qquad 1 \leq u \leq 2.$$

Then

$$(1) \quad \int_{0}^{y} (1)\, f(u)\, du = \int_{0}^{y} u\, du = \frac{y^2}{2} \qquad 0 \leq y \leq 1.$$

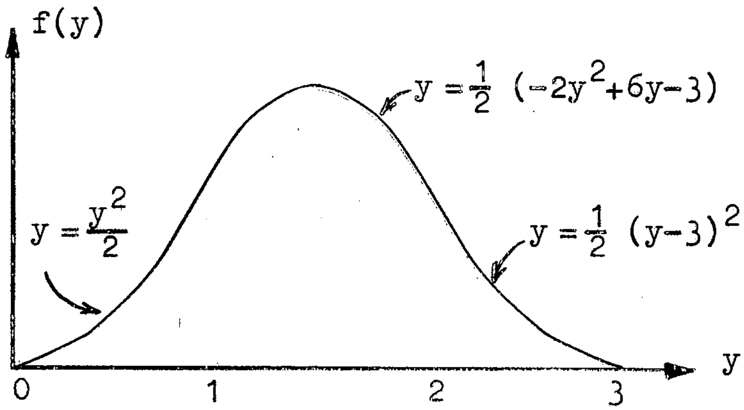
$$(2) \quad \int_{y-1}^{y} (1)\, f(u)\, du = \int_{y-1}^{1} u\, du + \int_{1}^{y} (2-u)\, du \qquad 1 \leq y \leq 2$$

$$= -y^2 + 3y - \frac{3}{2} = \frac{1}{2}\,(-2y^2 + 6u - 3) \quad .$$

$$(3) \int_{y-1}^{2} (1)\ f(u)\ du = \int_{y-1}^{2} (2 - u)\ du \qquad 2 \le y \le 3$$

$$= 2u - \frac{u^2}{2}\Big|_{y-1}^{2} = \frac{1}{2}(y-3)^2 \quad .$$

The result is:



Note the similarity of $y = x_1 + x_2 + x_3$ to a normal curve when in fact x_i are uniform [0,1]. This "evidence" of the applicability of the Central Limit Theorem is rewarding.

Example 2. Convolution of a Normal
Distribution Function

Let: $X \sim N\ (0,1) \qquad f(x) = \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}$

$Y \sim N\ (0,1) \qquad f(y) = \frac{1}{\sqrt{2\pi}}\ e^{-y^2/2}$,

Define:

$$Z = X + Y \quad .$$

The convolution integral yields the pdf of Z.

$$f(z) = \int_{-\infty}^{\infty} f(x)\ f(z - x)\ dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-(z-x)^2/2}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[x^2+z^2-2zx+x^2]}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \int_{-\infty}^{\infty} e^{-(x^2-zx)}\ dx \quad .$$

Complete the square in the exponent to obtain

$$x^2 - zx = \left[\left(x - \frac{z}{2}\right)^2 - \frac{z^2}{4}\right]$$

then

$$f(z) = \frac{1}{2\pi} e^{-z^2/2}\ e^{z^2/4} \int_{-\infty}^{\infty} e^{-1/2[\sqrt{2}\ (x - z/2)]^2}\ dx \quad .$$

Letting

$$\sqrt{2}\ (x - z/2) = u$$

$$dx = \frac{du}{\sqrt{2}}$$

$$f(z) = \frac{1}{\sqrt{2\pi}\ \sqrt{2}} e^{-z^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\ du$$

now

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2}\ du = 1.0 \quad .$$

Thus

$$f(z) = \frac{1}{\sqrt{2\pi}\ \sqrt{2}}\ e^{-1/2(z/\sqrt{2})^2} \quad ,$$

but this is the pdf of a random variable with distribution N(0,2).

Thus, via convolution we obtain a result normally obtained by the moment generating function technique.

APPENDIX B

TRANSFORM THEORY AND THE UNIT IMPULSE FUNCTION [2, 3]

Laplace Transform

The Laplace transform (sometimes referred to as the s transform or the exponential transform) is defined as:

$$\mathcal{L}[f(x)] = f_x(s) = E(e^{-sx}) = \int_{-\infty}^{\infty} e^{-sx} f(x)\, dx \,.$$

Its usefulness in probability theory will be illustrated by example 1, but basically is an alternative way to determine the distribution function of a sum of random variables.

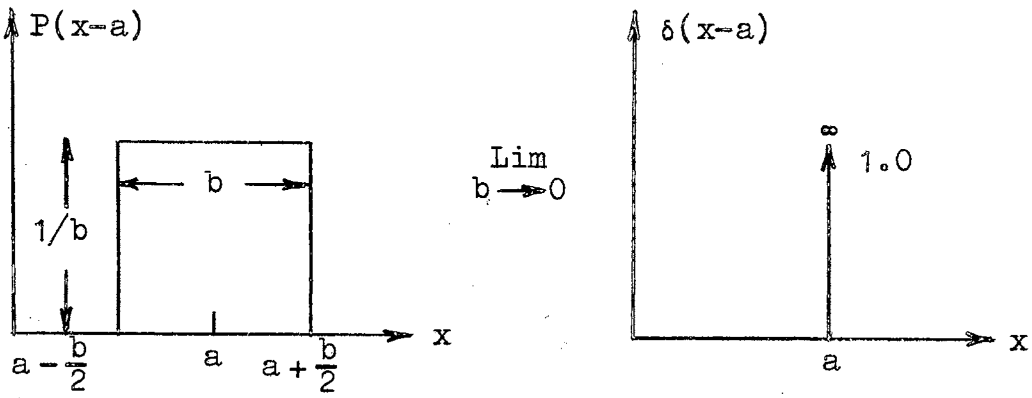
The Impulse Function and Z Transforms

The impulse function $\delta(x)$ is defined by the relations:

(1) $\delta(x - a) = 0 \qquad x \neq a$

(2) $\int_{a-\epsilon}^{a+\epsilon} \delta(x - a)\, dx = 1.0 \qquad \epsilon > 0 \,.$

It is generated by starting with a rectangular pulse of unit area and considering the limit as the width of the pulse goes to zero.



The usefulness of the impulse function in probability lies in the fact that a discrete distribution can be expressed in a continuous form. This is possible by utilizing condition (2).

Consider:

$$\int_a^c f(x)\ \delta(x - b)\ dx \qquad a < b < c\ .$$

The impulse occurs at b, and is within the limits of the integral. But the impulse function is zero except at x = b, which implies:

$$\int_a^c f(x)\ \delta(x - b)\ dx = \int_{b-\varepsilon}^{b+\varepsilon} f(x)\ \delta(x - b)\ dx \qquad \varepsilon > 0\ .$$

As $\varepsilon \to 0$, if f(x) is continuous at x = b, the change in f(x) becomes small and approaches f(b), then:

$$\int_{b-\varepsilon}^{b+\varepsilon} f(x)\ \delta(x - b)\ dx \underset{e \to 0}{\longrightarrow} f(b) \int_{b-\varepsilon}^{b+\varepsilon} \delta(x-b)\ dx \qquad \varepsilon > 0\ .$$

The last integral is equal to unity, thus

$$\int_a^c f(x)\ \delta(x - b)\ dx = f(b) \qquad a < b < c\ .$$

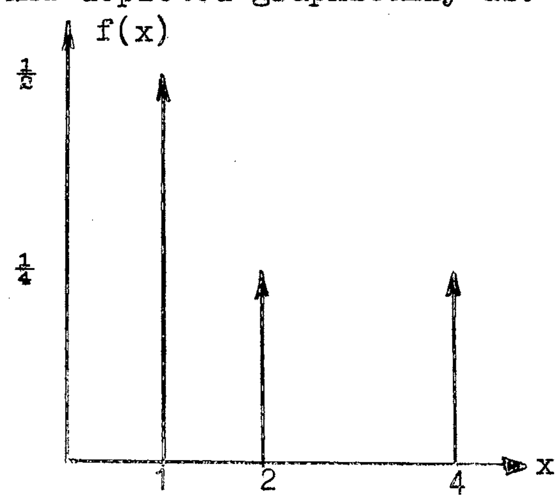
Using this relationship, consider:

$$f(x) = \begin{cases} 0.50 & x = 1 \\ 0.25 & x = 2,\ 4 \\ 0 & \text{elsewhere} \end{cases}$$

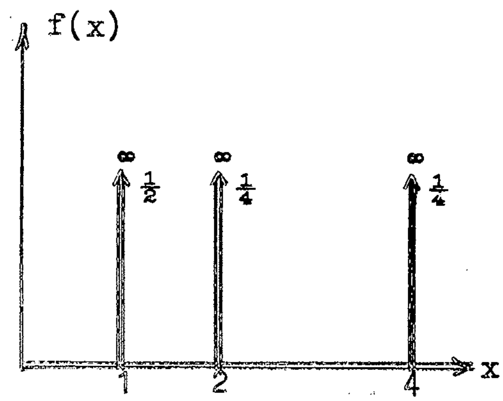
which can be expressed as:

$$f(x) = 0.50\ \delta(x - 1) + 0.25\ \delta(x - 2) + 0.25\ \delta(x - 4)$$

and depicted graphically as:



discrete



continuous

Having expressed f(x) in a continuous form, the Laplace transform can now be obtained:

$$\mathcal{L}[f(x)] = f_x(s) = \int_{-\infty}^{\infty} e^{-sx}\ f(x)\ dx$$

$$= 0.50\ e^{-s} + 0.25\ e^{-2s} + 0.25\ e^{-4s}\ .$$

The Laplace transform is defined for the distribution function of any random variable. However, for random variables which are discrete and which take on only non-negative integer value (as in the case of discrete demand units), a special transform has been defined and called either the discrete or z transform, given as:

$$f_x(z) = E(z^x) = \sum_{x=0}^{\infty} z^x f(x) \quad .$$

Applying this transform to the previous example,

$$f_x(z) = \sum_{x=0}^{\infty} z^x f(x) = 0.50 z + 0.25 z^2 + 0.25 z^4$$

and, as can be seen, z has simply replaced e^{-s}.

Transforms of the Probability Density Function for the Sum of Independent Random Variables

Continuous Case: Let

$W = X + Y$, x and y independent random variables

$X \sim f(x)$ X,Y Continuous

$Y \sim f(y)$.

The Laplace transform of W is

$$f_w(s) = E(e^{-sw}) = E(e^{-s(x+y)})$$

$$f_w(s) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} e^{-sx} e^{-sy} f(x,y) \, dxdy$$

$$f_w(s) = \int_{x=-\infty}^{\infty} e^{-sx} f(x)\, dx \int_{y=-\infty}^{\infty} e^{-sy} f(y)\, dy$$

$$f_w(s) = f_x(s)\, f_y(s) \quad . \qquad (B.1)$$

Discrete Case: Let

$$W = X + Y$$

$$X \sim f(x) \qquad \qquad X, Y \text{ Discrete}$$

$$Y \sim f(y) \quad .$$

The z transform of W is

$$f_w(z) = \sum_{w=0}^{\infty} z^w f(x,y) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} z^{(x+y)} f(x)\, f(y)$$

$$f_w(z) = \sum_{x=0}^{\infty} z^x f(x) \sum_{y=0}^{\infty} z^y f(y)$$

$$f_w(z) = f_x(z)\, f_y(z) \quad . \qquad (B.2)$$

Example 1: This example will find the distribution of the sum of two normal distribution functions using Laplace transforms, a result previously obtained in Example 2 of Appendix A, via convolution methods.

Let

$$X \sim N\ (\mu_x\ \sigma_x^2)$$

$$Y \sim N\ (\mu_y\ \sigma_y^2) \quad .$$

Define $Z = X + Y$, and the pdf of Z is desired. The Laplace transform of X is

$$f_x(s) = e^{s^2\sigma_x{}^2/2 - sE(x)}$$

and for Y

$$f_y(s) = e^{s^2\sigma_y{}^2/2 - sE(y)} \quad .$$

By the relationship (B.1),

$$f_z(s) = f_x(s)\; f_y(s)$$

$$f_z(s) = e^{s^2\sigma_x{}^2/2 - sE(x)} \; e^{s^2\sigma_y{}^2/2 - sE(y)}$$

$$f_z(s) = e^{s^2/2[(\sigma_x{}^2+\sigma_y{}^2)] - s(E(x)+E(y))} \quad .$$

Notice that $f_z(s)$ is in the form of both $f_x(s)$ and $f_y(s)$, except that the variance of z is the sum of $\sigma_x{}^2$ and $\sigma_y{}^2$ and the mean of z is the sum of μ_x and μ_y. If, as in Example 2 of Appendix A,

$$X = Y \sim N\ (0,1)$$

then

$$Z \sim N\ (0,2) \quad .$$

VITA

3

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