# HOCHSCHILD, SHUKLA AND COTRIPLE COHOMOLOGIES OF LIE ALGEBRAS

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## PREFACE

The study of cohomologies of algebras (associative or non-associative) has been a central problem in homological algebra due to its important direct application to algebraic topology. On the other hand, it has been recognized during the last five years that the approach of categorical algebra has a penetrating influence on homological algebra (for example, see Eilenberg and Moore [5]).

It is known that the notion of a (co)triple, or equivalently, a pair of adjoint functors, provides a most convenient and practical tool for defining cohomologies (see Eilenberg and Moore [6], Barr [1], Barr and Beck [2], Beck [3], Shimada, Uehara and Brenneman [10]). Many known chomologies have been found to appear as cotriple cohomologies. In fact, it has been shown in [10] that cotriple cohomology can be interpreted as a relative cohomology theory, thus unifying the various cohomology theories for a specified category.

Glassman [7] has discussed, among other things, the Hochschild and Shukla cohomologies for associative algebras. Barr and Beck [1-2] have shown that these can be interpreted as cotriple cohomologies. The main purpose of this paper is to present two cotriple cohomologies for Lie algebras (non-associative); one is compared with the Hochschild cohomology for Lie algebras and the other is compared with Shukla's cohomology for Lie algebras. In order to facilitate the latter comparison, Shukla's work is presented in a categorical setting.

Chapter I contains a description of the Hochschild cohomology of Lie algebras in terms of their associative enveloping algebras, as defined by Cartan and Eilenberg [4]. In Chapter II Shukla's cohomology of Lie algebras [11] is generalized from the viewpoint of categorical algebra and some new results are obtained for the low-dimensional modules.

The two cotriples are constructed in Chapter III, one giving a cohomology for modules over an algebra and the other, presented by H. Uehara [12], giving a cohomology for Lie algebras. In Chapter IV it is shown that the former cotriple may be used to obtain the Hochschild cohomology of Lie algebras (with a dimension shift) and the cotriple cohomology for Lie algebras is shown to agree with Shukla's cohomology in the low-dimensional module.

A summary of results and a presentation of some problems for further research are given in Chapter V.

The notation and the terminology of MacLane [9] are used extensively in this paper. Numbers in brackets refer to an entry in the Bibliography.

I am indebted to my friend and adviser Professor Hiroshi Uehara. He has conveyed to me a spirit of mathematics and he has patiently encouraged and guided me in the writing of this paper. I am grateful to my wife Shirley; her love and her interest in mathematics have been a constant source of help to me. The National Science Foundation has partially supported me during my graduate studies and I wish to thank those who helped me obtain these funds.

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# CHAPTER I

## HOCHSCHILD COHOMOLOGY OF LIE ALGEBRAS

In this chapter, R is a commutative ring with unity 1. Tensor products and Hom functors will be over R unless indicated otherwise. H. Cartan and S. Eilenberg [4] have defined the cohomology groups of a Lie algebra L as the Hochschild cohomology groups of its enveloping algebra $L^e$. Since an explicit formulation will be needed in Chapter IV, this theory is included here; see Hochschild [8] and MacLane [9].

### 1. Lie Algebras, Enveloping Algebras, Modules over Lie Algebras

Definition 1.1: A graded R-module L is a Lie algebra if and only if there exists an R-homomorphism $[\,,\,]: L \otimes L \longrightarrow L$ of degree zero satisfying (1) $[x,x] = 0$ if the degree of x (denoted by $|x|$) is even, (2) $[x,y] = (-1)^{|x||y|+1}[y,x]$, and (3) $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$ (Jacobi identity). A non-graded Lie algebra is a Lie algebra L such that $L_n = 0$ for all $n>0$.

Remark: For a non-graded Lie algebra the three conditions above are (1) $[x,x] = 0$, (2) $[x,y] = -[y,x]$, (3) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in L$. Unless specified, all modules, algebras, and Lie algebras are graded.

Definition 1.2: If L and L' are Lie algebras, then a morphism of Lie algebras $f: L \rightarrow L'$ is an R-homomorphism of degree zero satisfying $f([x,y]) = [f(x),f(y)]$ for all $x,y \in L$.

The tensor algebra $T(L)$ of an R-module L is the R-module given by

$T(L)_o = R + \sum_{n=1}^{\infty} L_o^n$, ($L_o^n = L_o \otimes \cdots \otimes L_o$, n factors), and for each

$n>o$, $T(L)_n = \sum_{i_1+\cdots+i_n=n} L_{i_1} \otimes \cdots \otimes L_{i_n}$, where the multiplication $x.y$

in $T(L)$ is defined by $x \otimes y$.

Definition 1.3: The enveloping algebra $L^e$ of a Lie algebra L is the quotient algebra $T(L)/I$, where I is the two-sided ideal in $T(L)$ generated by elements of the form $x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y]$ for $x,y \in L$.

Definition 1.4: If L is a Lie algebra, then an R-module M is called a left L-module if and only if there exists an R-homomorphism $L \otimes M \xrightarrow{\cdot} M$ of degree zero satisfying $[x,y]\cdot m = x\cdot(y\cdot m) - (-1)^{|x||y|} y\cdot(x\cdot m)$ for all $x,y \in L$ and for all $m \in M$. Similarly, right L-modules may be defined.

Lemma 1.1: There exists an R-homomorphism $i:L \to L^e$ such that $i([x,y]) = i(x)\cdot i(y) - (-1)^{|x||y|} i(y)\cdot i(x)$ for all $x,y \in L$.

Proof: Consider $T(L)$ as an R-module, that is, $R + L + L \otimes L + \cdots$ and define $j:L \to T(L)$ by $j(x) = x$ for any $x \in L$. Define $i = pj$, where p is the natural projection $T(L) \to L^e$. Then $i([x,y])$ $= (pj)([x,y]) = p([x,y]) = p\ (x \otimes y - (-1)^{|x||y|} y \otimes x) = p(x) \otimes p(y)$ $- (-1)^{|x||y|} p(y) \otimes p(x) = (pj)(x) \otimes (pj)(y)$ $- (-1)^{|x||y|} (pj)(y) \otimes (pj)(x) = i(x)\cdot i(y) - (-1)^{|x||y|} i(y)\cdot i(x)$.

Proposition 1.1: M is a left L-module if and only if it is a left $L^e$ - module.

Proof: Assume that M is a left L-module. Then M has a left

$T(L)$-module structure given by $1\cdot m = m$, $(x_1 \otimes \cdots \otimes x_n)\cdot m = x_1\cdot(\cdots(x_{n-1}\cdot(x_n\cdot m))\cdots)$. Any element $\xi$ in the ideal $I$ of $T(L)$ is of the form $(\alpha(x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y])\beta)$, where $\alpha$, $\beta \in T(L)$ and $x, y \in L$. Then $\xi\cdot m = \alpha((x \otimes y)\cdot(\beta\cdot m) - (-1)^{|x||y|}(y \otimes x)\cdot(\beta\cdot m) - [x,y]\cdot(\beta\cdot m)) = \alpha(x\cdot(y\cdot m') - (-1)^{|x||y|} y\cdot(x\cdot m') - x\cdot(y\cdot m') + (-1)^{|x||y|} y\cdot(x\cdot m')) = 0$, where $m' = \beta\cdot m$. Therefore there is the induced left $L^e$-module structure for $M$. Conversely, assume that $M$ is a left $L^e$-module. Then there exists $\varphi: L^e \otimes M \longrightarrow M$. Define $\lambda = \varphi\circ(i \otimes 1_M)$, where $i$ is as in lemma 1.1. Therefore $\lambda([x,y] \otimes m) = i([x,y])\cdot m = [x,y]\cdot m$ by identifying $[x,y]$ and $i([x,y])$. Since $\varphi(i([x,y]) \otimes m) = (i(x)\cdot i(y))\cdot m - (-1)^{|x||y|}(i(y)\cdot i(x))\cdot m$ by lemma 1.1 and since $\lambda(x \otimes \lambda(y \otimes m) - (-1)^{|x||y|} y \otimes \lambda(x \otimes m)) = \varphi(i(x) \otimes \lambda(y \otimes m) - (-1)^{|x||y|} i(y) \otimes \lambda(x \otimes m)) = i(x)\cdot\lambda(y \otimes m) - (-1)^{|x||y|} i(y)\cdot\lambda(x \otimes m) = i(x)\cdot(i(y)\cdot m) - (-1)^{|x||y|} i(y)\cdot(i(x)\cdot m) = (i(x)\cdot i(y))\cdot m - (-1)^{|x||y|}(i(y)\cdot i(x))\cdot m$, we have $[x,y]\cdot m = x\cdot(y\cdot m) - (-1)^{|x||y|} y\cdot(x\cdot m)$. Hence $M$ is a left $L$ - module.

## 2. The Hochschild Cohomology of an Associative Algebra

In this section, let $\Lambda$ be an augmented $R$-algebra with augmentation $\epsilon: \Lambda \longrightarrow R$, unit $\eta: R \longrightarrow \Lambda$, and multiplication $\mu: \Lambda \otimes \Lambda \longrightarrow \Lambda$. Denote the augmentation ideal of $\Lambda$ by $Q$, that is, $Q = \ker \epsilon$.

<u>Definition 1.5</u>: The <u>bimodule bar resolution</u> $B(\Lambda,\Lambda)$ of $\Lambda$ is the chain complex

$$0 \longleftarrow \Lambda \underset{s_{-1}}{\overset{\zeta}{\rightleftarrows}} B_0(\Lambda,\Lambda) \underset{s_0}{\overset{\partial_1}{\rightleftarrows}} B_1(\Lambda,\Lambda) \rightleftarrows \cdots \rightleftarrows B_{n-1}(\Lambda,\Lambda) \underset{s_{n-1}}{\overset{\partial_n}{\rightleftarrows}} \cdots,$$

where $B_0(\Lambda,\Lambda) = \Lambda \otimes \Lambda$, $B_n(\Lambda,\Lambda) = \Lambda \otimes Q^n \otimes \Lambda$ ($Q^n = Q \otimes \cdots \otimes Q$, $n$ factors), $\zeta(\lambda[\ ]\lambda') = \lambda\lambda'$ and $\partial_n(\lambda[\lambda_1|\cdots|\lambda_n]\lambda')$

$$= \lambda\lambda_1[\lambda_2|\cdots|\lambda_n]\lambda' + \sum_{i=1}^{n-1} (-1)^i \lambda[\lambda_1|\cdots|\lambda_i\lambda_{i+1}|\cdots|\lambda_n]\lambda'$$

$+ (-1)^n \lambda[\lambda_1|\cdots|\lambda_{n-1}]\lambda_n\lambda'$ for $\lambda,\lambda' \varepsilon \Lambda$ and $\lambda_1,\cdots,\lambda_n \varepsilon Q$ for $n>o$. Following MacLane [9], the notations $\lambda[\lambda_1|\cdots|\lambda_n]\lambda'$ and $\lambda[\ ]\lambda'$ are used to denote $\lambda \otimes \lambda_1 \otimes \cdots \otimes \lambda_n \otimes \lambda'$ and $\lambda \otimes \lambda'$ respectively.

Remarks: Each $B_n(\Lambda,\Lambda)$ is a $\Lambda$- bimodule with scalar multiplication given by the multiplication in $\Lambda$;

namely ${}_\Lambda\varphi : \Lambda \otimes B_n(\Lambda,\Lambda) \longrightarrow B_n(\Lambda,\Lambda)$ is given by $\mu \otimes 1_{Q^n \otimes \Lambda}$ and

$\varphi_\Lambda : B_n(\Lambda,\Lambda) \otimes \Lambda \longrightarrow B_n(\Lambda,\Lambda)$ is given by $1_{\Lambda \otimes Q^n} \otimes \mu$. Since $B(\Lambda,\Lambda)$ has a contracting homotopy s defined by $s_{-1}(\lambda) = 1[\ ]\lambda$, and $s_n(\lambda[\lambda_1|\cdots|\lambda_n]\lambda') = 1[\bar{\lambda}|\lambda_1|\cdots|\lambda_n]\lambda'$ for $n \geq o$, where $\bar{\lambda} = \lambda - \eta\varepsilon(\lambda) \varepsilon Q$, it is an R-split exact resolution of $\Lambda$.

Definition 1.6: Let M be a $\Lambda$-bimodule. The Hochschild cohomology of $\Lambda$ with coefficients in M is defined by $H^n(\Lambda,M) = H^n(\mathrm{Hom}_{\Lambda-\Lambda}(B(\Lambda,\Lambda),M))$.

If $\Lambda$ is augmented, R can be considered as a left $\Lambda$-module by using the augmentation. Then we have

Definition 1.7: The left bar resolution $B(\Lambda)$ of the left $\Lambda$-module R is the chain complex

$$0 \longleftarrow R \underset{t_{-1}}{\overset{\varepsilon}{\rightleftarrows}} B_o(\Lambda) \underset{t_o}{\overset{d_1}{\rightleftarrows}} B_1(\Lambda) \rightleftarrows \cdots \rightleftarrows B_{n-1}(\Lambda) \underset{t_{n-1}}{\overset{d_n}{\rightleftarrows}} B_n(\Lambda) \rightleftarrows \cdots$$

where $B_o(\Lambda) = \Lambda$, $B_n(\Lambda) = \Lambda \otimes Q^n$, and $d_n(\lambda[\lambda_1|\cdots|\lambda_n]) = \lambda\lambda_1[\lambda_2|\cdots|\lambda_n]$

$$+ \sum_{i=1}^{n-1} (-1)^i \lambda[\lambda_1|\cdots|\lambda_i\lambda_{i+1}|\cdots|\lambda_n] \quad \text{for all } n>o.$$

The contracting homotopy t is given by $t_{-1}(1) = 1[\ ]$ and $t_n(\lambda[\lambda_1|\cdots|\lambda_n]) = 1[\bar{\lambda}|\lambda_1|\cdots|\lambda_n]$ for $n \geq 0$, where $\bar{\lambda} = \lambda - \eta\epsilon(\lambda)$. Therefore $B(\Lambda)$ is an R-split exact resolution of R.

It is well known that $B(\Lambda)$ is chain isomorphic to $B(\Lambda,\Lambda) \otimes_\Lambda R$ with differentials $\partial_n \otimes_\Lambda 1_R$.

If M is a left $\Lambda$-module and if $\Lambda$ is augmented then we may form the $\Lambda$-bimodule $M_\epsilon$ by pull-back along the augmentation. Specifically, define $M \otimes \Lambda \longrightarrow M$ by $m\cdot\lambda = \epsilon(\lambda)m$ for any $m \in M$ and for any $\lambda \in \Lambda$. Since $(\lambda\cdot m)\cdot\lambda' = \epsilon(\lambda')(\lambda\cdot m) = \lambda\cdot(\epsilon(\lambda')m) = \lambda\cdot(m\cdot\lambda')$, $M_\epsilon$ is a $\Lambda$-bimodule.

<u>Proposition 1.2</u>: If M is a left $\Lambda$-module and if $\Lambda$ is augmented, then the Hochschild cohomology of $\Lambda$ with coefficients in $M_\epsilon$ is isomorphic to $H^n(\mathrm{Hom}_\Lambda(B(\Lambda),M))$, where $B(\Lambda)$ is the left bar resolution of R.

<u>Proof</u>: See MacLane [9] on page 287.

## 3. The Hochschild Cohomology of a Lie Algebra

If L is a Lie algebra over R then $L^e$ is augmented. Since $T(L) = R + L + \cdots$, there is an augmentation $\epsilon'$: $T(L) \longrightarrow R$ defined by the projection on the direct summand R of T(L). Then $T(L) = \mathrm{Im}\ \eta' + \ker\ \epsilon'$, where $\eta'$: $R \longrightarrow T(L)$ is the natural injection. If $p: T(L) \longrightarrow L^e$ is the natural projection, then $\ker p \subset \ker \epsilon'$ so that there is the induced augmentation $\epsilon$: $L^e \longrightarrow R$ with the property $\epsilon p = \epsilon'$. Then $L^e = \mathrm{Im}\ \eta + \ker\ \epsilon$, where $\eta = p\eta'$.

<u>Definition 1.8</u>: Let L be a Lie algebra over R and let M be a left L-module. The <u>Hochschild cohomology of L with coefficients in M</u> is given by $H^n(L,M) = H^n(L^e, M_\epsilon)$.

It follows from propositions 1.1 and 1.2 that $H^n(L,M) \cong H^n(\mathrm{Hom}_{L^e}(B(L^e),M))$.

<u>Proposition 1.3</u>: Let L be a non-graded Lie algebra over R and let M be a left L-module. Then $H^o(L,M)$ is isomorphic to the R-module of invariant elements of M, denoted by $M^L$, and $H^1(L,M)$ is isomorphic to the R-module of all crossed homomorphisms $f: Q \longrightarrow M$ reduced modulo the principal crossed homomorphisms.

<u>Proof</u>: From the left bar resolution $B(L^e)$ of R we obtain the cochain complex:

$$0 \longrightarrow \mathrm{Hom}_{L^e}(R,M) \xrightarrow{\varepsilon^*} \mathrm{Hom}_{L^e}(L^e,M) \xrightarrow{d_1^*} \mathrm{Hom}_{L^e}(L^e \otimes Q,M) \xrightarrow{d_2^*} \quad \cdots,$$

where Q is the augmentation ideal of $L^e$. $H^o(L,M) = \ker d_1^* = \mathrm{im}\ \varepsilon^*$

$\cong \mathrm{Hom}_{L^e}(R,M)$, and $M^L = \{m \in M \ / \ \lambda m = 0 \text{ for all } \lambda \in Q\}$. Define

$\rho(f) = f(1)$ for all $f \in \mathrm{Hom}_{L^e}(R,M)$. Then $\lambda \cdot f(1) = f(\varepsilon(\lambda)\cdot 1) = 0$ for

all $\lambda \in Q$ so that $\rho(f) \in M^L$. Defining $\sigma(m)(r) = rm$ for all $m \in M$

and $r \in R$, we have $\sigma(m)(\lambda \cdot r) = \sigma(m)(\varepsilon(\lambda)r) = \varepsilon(\lambda)rm = \lambda \cdot rm$

$= \lambda \cdot \sigma(m)(r)$ so that $\sigma(m) \in \mathrm{Hom}_{L^e}(R,M)$. Since $\rho(\sigma(m)) = \sigma(m)(1)$

$= m$ and $\sigma(\rho(f))(r) = r\ \rho(f) = rf(1) = f(r)$, $\rho\sigma$ and $\sigma\rho$ are identities

and therefore $H^o(L,M) \cong M^L$. A 1-cocycle $g: L^e \otimes Q \longrightarrow M$ satisfies

$d_2^*(g) = gd_2 = 0$ so that $gd_2(1 \otimes \lambda_1 \otimes \lambda_2) = g(\lambda_1 \otimes \lambda_2 - 1 \otimes \lambda_1\lambda_2)$

$= \lambda_1 g(1 \otimes \lambda_2) - g(1 \otimes \lambda_1\lambda_2) = 0$. Define adjoint isomorphisms

$\mathrm{Hom}\ (Q,M) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}_{L^e}(L^e \otimes Q,M)$ by $\varphi(f)(a \otimes q) = af(q)$ and $\psi(g)(q) = g(1 \otimes q)$, where $f:Q \longrightarrow M$, $g:L^e \otimes Q \longrightarrow M$, $a \in L^e$, and $q \in Q$. Therefore for a 1-cocycle g we have $\lambda_1\ \psi(g)(\lambda_2) - \psi(g)(\lambda_1\lambda_2) = 0$, that is, $\lambda_1 f(\lambda_2) = f(\lambda_1\lambda_2)$, where $f:Q \longrightarrow M$ and $\lambda_1$, $\lambda_2 \in Q$. For a 1-coboundary $g:L^e \otimes Q \longrightarrow M$, there exists $c:L^e \longrightarrow M$ such that $d_1^*(c) = cd_1 = g$. Hence $cd_1(1 \otimes \lambda) = c(\lambda) = \lambda \cdot c(1) = g(1 \otimes \lambda) = \psi(g)(\lambda)$, that is, $f(\lambda) = \lambda m_o$ for any $\lambda \in Q$, where $f:Q \rightarrow M$ and $m_o = c(1)$.

CHAPTER II

# SHUKLA COHOMOLOGY OF LIE ALGEBRAS

In this chapter R is a commutative ring with unity 1 in which the element 2 is invertible, that is, there exists $r \in R$ such that $2r = 1$. Unless specified, all tensor products are over R and all modules, algebras and Lie algebras are graded.

U. Shukla in [11] has introduced a cohomology theory for Lie algebras over R. This theory is generalized here in a categorical setting and some new results are discussed at the end of the chapter.

## 1. The Category $(\mathfrak{L},\mathfrak{G})^{o}$

*Definition 2.1*: A Lie algebra L is a *graded differential Lie algebra* if and only if L is a graded differential R-module and $d([x,y]) = [d(x),y] + (-1)^{|x|}[x,d(y)]$. A *morphism of graded differential Lie algebras* $f:L \to L'$ is a morphism of Lie algebras satisfying $f_{n-1}d_n = d'_n f_n$ for all $n \geq 1$, where d and d' are the differentials in L and L' respectively.

*Definition 2.2*: For a fixed graded differential Lie algebra L, the *category of graded differential Lie algebras over L*, denoted by $(\mathfrak{L},L)$, has as its objects, morphisms $\gamma:L' \to L$ of graded differential Lie algebras, and as its morphisms, morphisms $f:L_1 \to L_2$ of graded differential Lie algebras such that the triangle
$$\begin{tikzcd} L_1 \arrow[rr, "f"] \arrow[dr, "\gamma_1"'] & & L_2 \arrow[dl, "\gamma_2"] \\ & L & \end{tikzcd}$$
commutes.

In particular, denote by $(\mathfrak{L},\mathfrak{g})^{o}$ the category of non-graded Lie algebras over $\mathfrak{g}$, that is, L in $\mathfrak{L}$ and $\mathfrak{g}$ are non-graded Lie algebras.

## 2. The Functor $\Delta$

Let $(\mathfrak{A},\mathfrak{g})$ denote the category of graded differential nonassociative algebras over a fixed non-graded Lie algebra $\mathfrak{g}$, defined analogous to definitions 2.1 and 2.2. This section presents the construction of a covariant functor $\Delta:(\mathfrak{L},\mathfrak{g})^{o} \longrightarrow (\mathfrak{A},\mathfrak{g})$.

Suppose $\gamma: L \to \mathfrak{g}$ is in $(\mathfrak{L},\mathfrak{g})^{o}$ and form the free R-module $R(\langle L \rangle)$ generated by the underlying set $\langle L \rangle$ of L. Define $\sigma_o: R(\langle L \rangle) \to L$ by $\sigma_o(\langle x \rangle) = x$ and extend by linearity. Let $\varepsilon = \gamma\sigma_o$; it is an R-homomorphism. Form the free R-module $R(\langle N_1 \rangle)$, where $N_1 = \ker \sigma_o$, and the R-homomorphism $\sigma_1: R(\langle N_1 \rangle) \longrightarrow N_1$ given by $\sigma_1(\langle x \rangle) = x$. Let $d_1 = k_1\sigma_1$, where $k_1: N_1 \longrightarrow R(\langle L \rangle)$ is the inclusion map. Form $R(\langle N_2 \rangle)$, where $N_2 = \ker d_1$, and define $\sigma_2: R(\langle N_2 \rangle) \longrightarrow N_2$ by $\sigma_2(\langle x \rangle) = x$ and $d_2 = k_2\sigma_2$, where $k_2: N_2 \longrightarrow R(\langle N_1 \rangle)$. Repeating this process we obtain the complex

$$\begin{tikzcd}
\cdots \arrow[r] & R(\langle N_2 \rangle) \arrow[rr, "d_2"] \arrow[dr, "\sigma_2"'] & & R(\langle N_1 \rangle) \arrow[rr, "d_1"] \arrow[dr, "\sigma_1"'] & & R(\langle L \rangle) \arrow[rr, "\sigma_o"] \arrow[dr, "\varepsilon"'] & & L \arrow[dl, "\gamma"] \\
& & N_2 \arrow[ur, "k_2"'] & & N_1 \arrow[ur, "k_1"'] & & \mathfrak{g} &
\end{tikzcd}$$

Then $\Delta(L): \cdots \longrightarrow R(\langle N_p \rangle) \xrightarrow{d_p} R(\langle N_{p-1} \rangle) \to \cdots \to R(\langle N_1 \rangle) \xrightarrow{d_1} R(\langle L \rangle)$ is a graded differential R-module. Let us define inductively a product in $\Delta(L)$. First $[\ \ ]:\Delta(L)_o \otimes \Delta(L)_o \to \Delta(L)_o$ is given by $[\langle x\rangle,\langle y\rangle] = \langle[x,y]\rangle$ using the Lie product in L and extending linearly. Then $\varepsilon$ is a morphism of nonassociative algebras because $\varepsilon([\langle x\rangle,\langle y\rangle])$ $= \gamma\sigma_o(\langle[x,y]\rangle) = \gamma([x,y]) = [\gamma(x),\gamma(y)] = [\gamma\sigma_o(\langle x\rangle), \gamma\sigma_o(\langle y\rangle)]$ $= [\varepsilon(\langle x\rangle),\varepsilon(\langle y\rangle)]$. Secondly, assume that for all p, q such that $p+q<n$, $R(\langle N_p\rangle) \otimes R(\langle N_q\rangle) \xrightarrow{[\ ]} R(\langle N_{p+q}\rangle)$ has been defined satisfying $d_{p+q}([\langle x\rangle,\langle y\rangle]) = [d_p(\langle x\rangle),\langle y\rangle] + (-1)^p [\langle x\rangle,d_q(\langle y\rangle)]$. Let $r+s = n$ and define $R(\langle N_r\rangle) \otimes R(\langle N_s\rangle) \xrightarrow{[\ ]} R(\langle N_n\rangle)$. It suffices to define $[\langle x\rangle,\langle y\rangle]$ for $x\in N_r$, $y\in N_s$. Consider $[d_r(\langle x\rangle),\langle y\rangle] + (-1)^r [\langle x\rangle,d_s(\langle y\rangle)]$ in $R(\langle N_{n-1}\rangle)$. Since $d_{n-1}([d_r(\langle x\rangle),\langle y\rangle]) + (-1)^r\, d_{n-1}([\langle x\rangle,d_s(\langle y\rangle)])$ $= [d_{r-1}d_r(\langle x\rangle),\langle y\rangle] + (-1)^{r-1} [d_r(\langle x\rangle),d_s(\langle y\rangle)]$ $+ (-1)^r [d_r(\langle x\rangle),d_s(\langle y\rangle)] + (-1)^{r+s-1} [\langle x\rangle,d_{s-1}d_s(\langle y\rangle)] = 0$, it is in $N_n$. Thus we may well define $[\langle x\rangle,\langle y\rangle] = \big\langle[d_r(\langle x\rangle),\langle y\rangle]$ $+ (-1)^r [\langle x\rangle,d_s(\langle y\rangle)]\big\rangle$ and extend linearly. Then $d_{r+s}([\langle x\rangle,\langle y\rangle])$ $= [d_r(\langle x\rangle),\langle y\rangle] + (-1)^r [\langle x\rangle,d_s(\langle y\rangle)]$, and so $\Delta(L)$ is a graded differential nonassociative algebra.

Define $\Delta(\gamma):\Delta(L)\to\mathfrak{g}$ by $\Delta(\gamma)_o = \varepsilon$. Then $\Delta(\gamma)$ is a chain map because $\varepsilon d_1 = \gamma\sigma_o d_1 = 0$. Since $\varepsilon$ is product-preserving, $\Delta(\gamma)$ is a morphism of graded differential nonassociative algebras.

Let f be a morphism in $(\mathfrak{L},\mathfrak{g})^o$, that is, the triangle

$$\begin{tikzcd} L \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & L' \arrow[dl, "\gamma'"] \\ & \mathfrak{g} & \end{tikzcd}$$

commutes. In order to define a morphism $\Delta(f)$ in $(\mathfrak{A},\mathfrak{g})$ such that

$$\begin{tikzcd} \Delta(L) \arrow[rr, "\Delta(f)"] \arrow[dr, "\Delta(\gamma)"'] & & \Delta(L') \arrow[dl, "\Delta(\gamma')"] \\ & \mathfrak{g} & \end{tikzcd}$$

commutes,

it suffices to define a chain map $g:\Delta(L)\to\Delta(L')$ inductively. First define $g_o:R(\langle L\rangle)\to R(\langle L'\rangle)$ by $g_o(\langle x\rangle) = \langle f(x)\rangle$. Then $g_o$ preserves the product in $\Delta(L)$ because $g_o([\langle x\rangle,\langle y\rangle]) = g_o(\langle[x,y]\rangle) = \langle f([x,y])\rangle$ $= \langle[f(x),f(y)]\rangle = [\langle f(x)\rangle,\langle f(y)\rangle] = [g_o(\langle x\rangle),g_o(\langle y\rangle)]$. Since $f\sigma_o(\langle x\rangle)$ $= f(x) = \sigma_o'(\langle f(x)\rangle) = \sigma_o' g_o(\langle x\rangle)$, $\sigma_o' g_o(n) = f\sigma_o(n) = 0$ for $n \in N_1$. Therefore $g_o(N_1) \subset N_1'$. Secondly, assume that for all p,q such that p+q<n, $g_{p+q}: R(\langle N_{p+q}\rangle) \longrightarrow R(\langle N'_{p+q}\rangle)$ has been defined such that $d'_{p+q}g_{p+q}$ $= g_{p+q-1}d_{p+q}$ and $g_{p+q}$ preserves the product in $\Delta(L)$. It follows immediately that $g_{p+q}(N_{p+q+1}) \subset N'_{p+q+1}$. Define $g_{p+q+1}(\langle x\rangle) = \langle g_{p+q}(x)\rangle$. Then $g_{p+q}d_{p+q+1}(\langle x\rangle) = g_{p+q}(x)$ and $d'_{p+q+1}g_{p+q+1}(\langle x\rangle) =$ $d'_{p+q+1}(\langle g_{p+q}(x)\rangle) = g_{p+q}(x)$ so that $gd = d'g$. Hence g is a chain map. Suppose r+s = n. Then $g_{r+s}([\langle x\rangle,\langle y\rangle]) = g_{r+s}(\langle [d_r(\langle x\rangle),\langle y\rangle]$

$$+ (-1)^r[\langle x\rangle,d_s(\langle y\rangle)]\rangle) = \langle g_{r+s-1}([d_r(\langle x\rangle),\langle y\rangle] + (-1)^r[\langle x\rangle,d_s(\langle y\rangle)])\rangle$$

$$= \langle[g_{r-1}d_r(\langle x\rangle),g_s(\langle y\rangle)] + (-1)^r[g_r(\langle x\rangle),g_{s-1}d_s(\langle y\rangle)]\rangle$$

$$= \langle[d_r'g_r(\langle x\rangle),g_s(\langle y\rangle)] + (-1)^r[g_r(\langle x\rangle),d_s'g_s(\langle y\rangle)]\rangle$$

$$= \langle[d_r'(\langle g_{r-1}(x)\rangle),\langle g_{s-1}(y)\rangle] + (-1)^r\,[\langle g_{r-1}(x)\rangle,d_s'(\langle g_{s-1}(y)\rangle)]\rangle$$

$= [\langle g_{r-1}(x)\rangle,\langle g_{s-1}(y)\rangle] = [g_r(\langle x\rangle),g_s(\langle y\rangle)]$. It follows that $\{g_n\}$ preserves the product in $\Delta(L)$. Define $\Delta(f)_n = g_n$. It is clear that $\Delta(f)$ is a morphism in $(\mathfrak{n},\mathfrak{g})$, that is, $\Delta(\gamma')\Delta(f) = \Delta(\gamma)$, because if $n>0$, $\Delta(\gamma')_n = \Delta(\gamma)_n = 0$ and if $n = 0$, $\Delta(\gamma')_o g_o(\langle x\rangle) = \varepsilon'(\langle f(x)\rangle)$ $= \gamma'\sigma_o'(\langle f(x)\rangle) = \gamma'(f(x)) = \gamma(x) = \gamma\sigma_o(\langle x\rangle) = \varepsilon(\langle x\rangle) = \Delta(\gamma)_o(\langle x\rangle)$. This completes the definition of the functor $\Delta$.

## 3. The Functor P

Definition 2.3: Let A be a graded nonassociative algebra with product denoted by [ ]. An R-submodule I of A is a <u>two-sided ideal</u> in A if and only if $[I,A] \subset I$ and $[A,I] \subset I$, that is, for all $x \in I_p$ and for all $a \in A_q$, $[x,a]$, $[a,x] \in I_{p+q}$. If $S \subset A$ is a graded set, then the <u>two-sided ideal generated by S</u>, denoted by I(S), is the

R-submodule of A generated by elements of the forms $x_o = s$ for all $s\epsilon S$, $x_1 = [a,x_o]$ and $x_1 = [x_o,a]$ for all $a\epsilon A$, and $x_n = [a,x_{n-1}]$ and $x_n = [x_{n-1},a]$ for all $a\epsilon A$ and for all $n>1$. Note that if A is a Lie algebra, $[a,x_n] = (-1)^{|a||x_n|+1}[x_n,a]$ for all $n\geq 1$ and so there will be half as many generators of the type $x_n$ for $n\geq 1$.

Lemma 2.1: If A is a graded differential nonassociative algebra, and if I(S) is a two-sided ideal generated by S in A which is closed under d, that is, $d(I(S)) \subset I(S)$, then A/I(S) is a graded differential nonassociative algebra.

Proof: The R-module structure of A/I(S) is given by $(A/(I(S))_n = A_n/I(S)_n$ and the product for A/I(S) is defined by $[\bar{x},\bar{y}] = \overline{[x,y]}$. Define $\bar{d}:A/I(S) \longrightarrow A/I(S)$ by $\bar{d}(\bar{x}) = \overline{d(x)}$. Since I(S) is closed under d, $\bar{d}$ is well-defined. If $|\bar{x}| = p$ and $|\bar{y}| = q$, then $\bar{d}_{p+q}([\bar{x},\bar{y}])$
$= \bar{d}_{p+q}(\overline{[x,y]}) = \overline{d_{p+q}([x,y])} = \overline{[d_p(x),y] + (-1)^p [x,d_q(y)]}$
$= \overline{[d_p(x),y]} + (-1)^p \overline{[x,d_q(y)]} = [\overline{d_p(x)},\bar{y}] + (-1)^p [\bar{x},\overline{d_q(y)}]$
$= [\bar{d}_p(\bar{x}),\bar{y}] + (-1)^p [\bar{x},\bar{d}_q(\bar{y})]$ and hence A/I(S) is a graded differential nonassociative algebra.

The construction of a functor $P:(\mathfrak{A},\mathfrak{g}) \longrightarrow (\mathfrak{L},\mathfrak{g})$ can now be described. Note that L in $\mathfrak{L}$ is graded but $\mathfrak{g}$ is non-graded. Let $\gamma:A \longrightarrow \mathfrak{g}$ be in $(\mathfrak{A},\mathfrak{g})$. Consider the graded set $S \subset A$ consisting of elements of the types (1) $[x,x]$ for $|x|$ even, (2) $[x,y] + (-1)^{|x||y|}[y,x]$, and (3) $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]]$ and form I(S) as in definition 2.3. Let $P(A) = A/I(S)$. If $d(I(S)) \subset I(S)$ then by lemma 2.1 P(A) is a graded differential nonassociative algebra. In order to show $d(I(S)) \subset I(S)$ it suffices to show it for the generators of I(S). Consider elements of the form $x_o = s$. For type (1), $d([x,x]) = [d(x),x] + (-1)^{|x|}[x,d(x)]$

$= [d(x),x] + (-1)^{|x||d(x)|}[x,d(x)]$ because $|x|$ is even. Therefore, $d([x,x]) \in S$. For type (2), $d([x,y] + (-1)^{|x||y|}[y,x])$
$= d([x,y]) + (-1)^{|x||y|}d([y,x]) = [d(x),y] + (-1)^{|x|}[x,d(y)]$
$+ (-1)^{|x||y|}[d(y),x] + (-1)^{|x||y|+|y|}[y,d(x)]$
$= ([d(x),y] + (-1)^{|d(x)||y|}[y,d(x)]) + (-1)^{|x|}([x,d(y)]$
$+ (-1)^{|d(y)||x|}[d(y),x])$ because $|y|(|x|+1) \equiv |y||d(x)| \bmod 2$ and $|x||y| \equiv |x|(1+|d(y)|) \bmod 2$. Therefore $d([x,y] + (-1)^{|x||y|}[y,x]) \in I(S)$.
For type (3), $d((-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]]$
$+ (-1)^{|z||y|}[z,[x,y]]) = (-1)^{|x||z|}d([x,[y,z]]) + (-1)^{|y||x|}d([y,[z,x]])$
$+ (-1)^{|z||y|}d([z,[x,y]]) = (-1)^{|x||z|}[d(x),[y,z]]$
$+ (-1)^{|x||z|+|x|}[x,d([y,z])] + (-1)^{|y||x|}[d(y),[z,x]]$
$+ (-1)^{|y||x|+|y|}[y,d([z,x])] + (-1)^{|z||y|}[d(z),[x,y]]$
$+ (-1)^{|z||y|+|z|}[z,d([x,y])] = (-1)^{|x||z|}[d(x),[y,z]]$
$+ (-1)^{|x||z|+|x|}[x,[d(y),z] + (-1)^{|y|}[y,d(z)]] + (-1)^{|y||x|}[d(y),[z,x]]$
$+ (-1)^{|y||x|+|y|}[y,[d(z),x] + (-1)^{|z|}[z,d(x)]] + (-1)^{|z||y|}[d(z),[x,y]]$
$+ (-1)^{|z||y|+|z|}[z,[d(x),y] + (-1)^{|x|}[x,d(y)]]$
$= (-1)^{|x||z|}[d(x),[y,z]] + (-1)^{|x||z|+|x|}[x,[d(y),z]]$
$+ (-1)^{|x||z|+|x|+|y|}[x,[y,d(z)]] + (-1)^{|y||x|}[d(y),[z,x]]$
$+ (-1)^{|y||x|+|y|}[y,[d(z),x]] + (-1)^{|y||x|+|y|+|z|}[y,[z,d(x)]]$
$+ (-1)^{|z||y|}[d(z),[x,y]] + (-1)^{|z||y|+|z|}[z,[d(x),y]]$
$+ (-1)^{|z||y|+|z|+|x|}[z,[x,d(y)]] = (-1)^{|x|}((-1)^{|x||z|}[x,[d(y),z]]$
$+ (-1)^{|d(y)||x|}[d(y),[z,x]] + (-1)^{|z||d(y)|}[z,[x,d(y)]])$
$+ (-1)^{|y|}((-1)^{|x||d(z)|}[x,[y,d(z)]] + (-1)^{|x||y|}[y,[d(z),x]]$
$+ (-1)^{|d(z)||y|}[d(z),[x,y]]) + (-1)^{|z|}((-1)^{|d(x)||z|}[d(x),[y,z]]$
$+ (-1)^{|y||d(x)|}[y,[z,d(x)]] + (-1)^{|z||y|}[z,[d(x),y]]$, which is in $I(S)$.
Consider elements of the form $x_1$. Since $d([a,x_o]) = [d(a),x_o]$ $+ (-1)^{|a|}[a,d(x_o)]$ and since, from the previous arguments, $d(x_o) = \Sigma s_i$,

where $s_i \varepsilon S$, we have $d([a,x_o]) = [d(a),x_o] + (-1)^{|a|}\Sigma\,[a,s_i]$, which is in $I(S)$. Similarly, $d([x_o,a]) \varepsilon I(S)$. Assume that $d(x_k) \varepsilon I(S)$ for all $k<n$. Then $d(x_n) = d([a,x_{n-1}]) = [d(a),x_{n-1}] + (-1)^{|a|}[a,d(x_{n-1})]$ $= [d(a),x_{n-1}] + (-1)^{|a|}\sum [a,y_i]$, where $y_i \varepsilon I(S)$. Therefore $d(x_n)$ $\varepsilon I(S)$. Similarly for $x_n = [x_{n-1},a]$. Hence by lemma 2.1, $P(A)$ is a graded differential nonassociative algebra. It is obviously a Lie algebra because of the construction of $I(S)$.

In order to define $P(\gamma):P(A)\rightarrow\mathfrak{g}$ it suffices to define $P(\gamma)_o:A_o/I(S)_o\rightarrow\mathfrak{g}$ because $\mathfrak{g}_n = 0$ for all $n>0$ and so $P(\gamma)_n$ is trivial for all $n>0$. Let $P(\gamma)_o(\bar{x}) = \gamma(x)$. Since $\gamma([x,x]) = [\gamma(x),\gamma(x)] = 0$ because $|\gamma(x)| = |x|$ is even, $\gamma([x,y] + [y,x]) = [\gamma(x),\gamma(y)]$ $+ [\gamma(y),\gamma(x)] = 0$, and $\gamma([x,[y,z]] + [y,[z,x]] + [z,[x,y]])$ $= [\gamma(x),[\gamma(y),\gamma(z)]] + [\gamma(y),[\gamma(z),\gamma(x)]] + [\gamma(z),[\gamma(x),\gamma(y)]] = 0$, $P(\gamma)_o$ is well-defined and the object transformation for $P$ has been given.

Let $f$ be a morphism in $(\mathfrak{A},\mathfrak{g})$, that is, the triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & A' \arrow[dl, "\gamma'"] \\ & \mathfrak{g} & \end{tikzcd}$$

commutes. In order to define a morphism $P(f)$ in $(\mathfrak{L},\mathfrak{g})$ it suffices to take $P(f)(\bar{x}) = \overline{f(x)}$ for all $x \varepsilon A$. $P(f)$ is well defined if $P(f)(I(S)) \subset I(S')$. We will omit the bars for the sake of typographical simplicity. $P(f)([x,x]) = f([x,x])$ $= [f(x),f(x)]$, $P(f)([x,y] + (-1)^{|x||y|}[y,x]) = f([x,y] + (-1)^{|x||y|}[y,x])$ $= [f(x),f(y)] + (-1)^{|f(x)||f(y)|}[f(y),f(x)]$, and $P(f)((-1)^{|x||z|}[x,[y,z]]$ $+ (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]])$ $= (-1)^{|f(x)||f(y)|}[f(x),[f(y),f(z)]] + (-1)^{|f(y)||f(x)|}[f(y),[f(z),f(x)]]$ $+ (-1)^{|f(z)||f(y)|}[f(z),[f(x),f(y)]]$ so $P(f)$ is well-defined.

Considering the differential $\tilde{d}$ induced by the differential $d$ in

A, we have $P(f)_{n-1}\bar{d}_n(\bar{x}) = P(f)_{n-1}(\overline{d_n(x)}) = \overline{f_{n-1}d_n(x)} = \overline{d'_n f_n(x)}$ $= \bar{d}'_n(\overline{f_n(x)}) = \bar{d}'_n P(f)_n(\bar{x})$ and so $P(f)$ is a chain map.

Also $P(f)$ is product-preserving since $P(f)([\bar{x},\bar{y}]) = P(f)(\overline{[x,y]})$ $= \overline{f([x,y])} = \overline{[f(x),f(y)]} = [\overline{f(x)},\overline{f(y)}] = [P(f)(\bar{x}),\ P(f)(\bar{y})]$.

Since $P(\gamma')P(f)(\bar{x}) = P(\gamma')(\overline{f(x)}) = \overline{\gamma' f(x)} = \overline{\gamma(x)} = P(\gamma)(\bar{x})$, $P(f)$ is the morphism in $(\mathfrak{L},\mathfrak{T})$ associated with f in $(\mathfrak{A},\mathfrak{T})$ and the morphism transformation for P is given. This completes the definition of the functor P.

## 4. The Functor D

<u>Definition 2.4</u>: Let A be a non-graded algebra. An <u>A-bicomplex</u> X is a bigraded A-module $X_{p,q}$, $p,q \geq 0$, together with A-homomorphisms $\partial'$, $\partial''$ of bidegrees $(0,-1)$ and $(-1,0)$ respectively, satisfying $\partial'\partial' = 0$, $\partial''\partial'' = 0$, and $\partial''\partial' + \partial'\partial'' = 0$.

<u>Definition 2.5</u>: The <u>category $\mathfrak{X}$ of A-bicomplexes</u> has as its objects A-bicomplexes and as its morphisms A-module homomorphisms $f: X \to Y$ of bidegree $(0,0)$ satisfying $\delta' f = f\partial'$ and $\delta'' f = f\partial''$, where $\delta'$ and $\delta''$ are morphisms of Y.

In summary, these two definitions may be illustrated by the diagram in which the following holds: $\partial'_{p,q}\partial'_{p,q+1} = 0$, $\partial''_{p,q}\partial''_{p+1,q} = 0$, $\partial''_{p,q-1}\partial'_{p,q} + \partial'_{p-1,q}\partial''_{p,q} = 0$ (similarly for $\delta'$ and $\delta''$), $\delta'_{p,q}f_{p,q}$ $= f_{p,q-1}\partial'_{p,q}$, $\delta''_{p,q}f_{p,q} = f_{p-1,q}\partial''_{p,q}$ (similarly for the other two squares in the diagram).

$$\begin{tikzcd}
 & & & X_{p+1,q} \arrow[d, "\partial''_{p+1,q}"] & \\
X_{p,q-1} \arrow[dr, "f_{p,q-1}"] \arrow[dddd, "\partial''_{p,q-1}"'] & & & X_{p,q} \arrow[lll, "\partial'_{p,q}"'] \arrow[dl, "f_{p,q}"] \arrow[dddd, "\partial''_{p,q}"] & X_{p,q+1} \arrow[l, "\partial'_{p,q+1}"'] \\
 & Y_{p,q-1} \arrow[d, "\delta''_{p,q-1}"] & Y_{p,q} \arrow[l, "\delta'_{p,q}"'] \arrow[d, "\delta''_{p,q}"] & & \\
 & Y_{p-1,q-1} & Y_{p-1,q} \arrow[l, "\delta'_{p-1,q}"'] & & \\
 & & & & \\
X_{p-1,q-1} \arrow[uur, "f_{p-1,q-1}"] & & & X_{p-1,q} \arrow[lll, "\partial'_{p-1,q}"'] \arrow[uul, "f_{p-1,q}"'] &
\end{tikzcd}$$

We are going to define a functor $D:(\mathfrak{L},\mathfrak{g}) \longrightarrow \mathfrak{X}$, where $\mathfrak{X}$ is the category of $\mathfrak{g}^e$-bicomplexes. Note that $\mathfrak{g}$ and its enveloping algebra $\mathfrak{g}^e$ are non-graded algebras.

For $\gamma:\Gamma \longrightarrow \mathfrak{g}$ in $(\mathfrak{L},\mathfrak{g})$ define $D(\gamma)_o$ as the trivial complex, that is, for all $m \geq o$ $D(\gamma)_{o,m} = 0$ and $\partial'_{o,m}:D(\gamma)_{o,m} \longrightarrow D(\gamma)_{o,m-1}$ is the trivial homomorphism. For $n > o$ consider the $\mathfrak{g}^e$-module

$\mathfrak{g}^e \otimes \underbrace{\Gamma \otimes \cdots \otimes \Gamma}_{n \text{ factors}}$ and define

$$D(\gamma)_{n,m} = \sum_{\substack{\alpha_1 + \cdots + \alpha_n = m \\ \alpha_i \geq o}} \mathfrak{g}^e \otimes \Gamma_{\alpha_1} \otimes \cdots \otimes \Gamma_{\alpha_n} \Big/ R_{n,m}$$ , where $R_{n,m}$

is the $\mathfrak{g}^e$-submodule generated by elements of the forms

(1) $1 \otimes x_1 \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \cdots \otimes x_n$

$+ (-1)^{|x_i||x_{i+1}|} 1 \otimes x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes \cdots \otimes x_n$ with $\sum_{i=1}^{n} x_i = m$

and $x_i \in \Gamma_{\alpha_i}$, and (2) $1 \otimes x_1 \otimes \ldots \otimes x \otimes x \otimes \ldots \otimes x_n$ with $|x|$ even.

Remark: Since $\frac{1}{2} \in R$ and, since for $|x|$ even,

$1 \otimes x_1 \otimes \ldots \otimes x \otimes x \otimes \ldots \otimes x_n + 1 \otimes x_1 \otimes \ldots \otimes x \otimes x \otimes \ldots \otimes x_n$

$= 2(1 \otimes x_1 \otimes \ldots \otimes x \otimes x \otimes \ldots \otimes x_n)$, the generators of the second type are redundant.

By way of example and in preparation for the computations carried out in Section 6, we have the following low-dimensional modules. Since $R_{1,m} = 0$, $D(\gamma)_{1,m} = \gamma^e \otimes \Gamma_m$ for all $m \geq 0$. The generators of $R_{2,o}$ are of the form $1 \otimes x \otimes x$, where $x \in \Gamma_o$ so that $D(\gamma)_{2,o}$ $= \gamma^e \otimes (\Gamma_o \wedge \Gamma_o)$. Since $R_{2,1}$ is generated by elements of the form $1 \otimes x \otimes y + 1 \otimes y \otimes x$, where $x \in \Gamma_o$ and $y \in \Gamma_1$, $D(\gamma)_{2,1} = \gamma^e \otimes \Gamma_o \otimes \Gamma_1$.

For $n > o$, define

$$\bar{\partial}'_{n,m} : \sum_{\alpha_1 + \ldots + \alpha_n = m} \gamma^e \otimes \Gamma_{\alpha_1} \otimes \ldots \otimes \Gamma_{\alpha_n} \longrightarrow$$

$$\sum_{\beta_1 + \ldots + \beta_n = m-1} \gamma^e \otimes \Gamma_{\beta_1} \otimes \ldots \otimes \Gamma_{\beta_n}$$

by $\bar{\partial}'(1 \otimes x_1 \otimes \ldots \otimes x_n)$

$$= (-1)^{n+1} \sum_{i=1}^{n} (-1)^{|x_1| + \ldots + |x_{i-1}|} 1 \otimes x_1 \otimes \ldots \otimes dx_i \otimes \ldots \otimes x_n,$$

where $d: \Gamma_{\alpha_i} \longrightarrow \Gamma_{\alpha_{i-1}}$. Then $\bar{\partial}'(1 \otimes x_1 \otimes \ldots \otimes x_i \otimes x_{i+1} \otimes \ldots \otimes x_n$

$+ (-1)^{|x_i||x_{i+1}|} 1 \otimes x_1 \otimes \ldots \otimes x_{i+1} \otimes x_i \otimes \ldots \otimes x_n)$

$$= (-1)^{n+1} \sum_{\substack{j=1\\ j\neq i,i+1}}^{n} (-1)^{|x_1|+\cdots+|x_{j-1}|}(1 \otimes x_1 \otimes \ldots \otimes dx_j \otimes \cdots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} 1 \otimes x_1 \otimes \cdots \otimes dx_j \otimes \cdots \otimes x_n)$$

$$+ (-1)^{n+1}(-1)^{|x_1|+\cdots+|x_i|}(1 \otimes x_1 \otimes \cdots \otimes x_i \otimes dx_{i+1} \otimes \cdots \otimes x_n$$

$$+(-1)^{|x_i||dx_{i+1}|} 1 \otimes x_1 \otimes \cdots \otimes dx_{i+1} \otimes x_i \otimes \cdots \otimes x_n)$$

$$+ (-1)^{n+1}(-1)^{|x_1|+\cdots+|x_{i-1}|}(1 \otimes x_1 \otimes \cdots \otimes dx_i \otimes x_{i+1} \otimes \cdots \otimes x_n$$

$$+ (-1)^{|x_{i+1}||dx_i|} 1 \otimes x_1 \otimes \cdots \otimes x_{i+1} \otimes dx_i \otimes \cdots \otimes x_n)$$ so that

$\bar{\partial}'(R_{n,m}) \subset R_{n,m-1}$ and $\bar{\partial}'$ induces $\partial'_{n,m} : D(\gamma)_{n,m} \quad D(\gamma)_{n,m-1}$.

Since $dd = 0$ and $|dx_i| = |x_i|-1$, $\partial'\partial'(1 \otimes x_1 \otimes \cdots \otimes x_n)$

$$= (-1)^{n+1} \sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_{i-1}|} \partial'(1 \otimes x_1 \otimes \cdots \otimes dx_i \otimes \cdots \otimes x_n)$$

$$= (-1)^{2n+1} \sum_{i=1}^{n} (-1)^{|x_1|+\cdots+|x_{i-1}|} \Big(\sum_{j=1}^{i-1} (-1)^{|x_1|+\cdots+|x_{j-1}|} 1 \otimes x_1 \otimes \cdots \otimes dx_j \otimes \cdots \otimes dx_i \otimes \cdots \otimes x_n$$

$$+ \sum_{j=i+1}^{n} (-1)^{|x_1|+\cdots+|x_{j-1}|-1} 1 \otimes x_1 \otimes \cdots \otimes dx_i \otimes \cdots \otimes dx_j \otimes \cdots \otimes x_n\Big)$$

$$= \sum_{1\le i<j\le n} (-1)^{|x_i|+\cdots+|x_{j-1}|} 1 \otimes x_1 \otimes \cdots \otimes dx_i \otimes \cdots \otimes dx_j \otimes \cdots \otimes x_n$$

$$- \sum_{1\le j<i\le n} (-1)^{|x_j|+\cdots+|x_{i-1}|} 1 \otimes x_1 \otimes \cdots \otimes dx_j \otimes \cdots \otimes dx_i \otimes \cdots \otimes x_n$$

$= 0$ because terms corresponding to $1 \le j_o < i_o \le n$ in the second summation and $i = j_o$, $j = i_o$ in the first summation are equal. Therefore $\partial'\partial' = 0$.

For $n>o$, define

$$\bar{\partial}''_{n,m}: \sum_{\alpha_1+\dots+\alpha_n=m} \mathfrak{g}^e \otimes \Gamma_{\alpha_1} \otimes \dots \otimes \Gamma_{\alpha_n} \longrightarrow$$

$$\sum_{\beta_1+\dots+\beta_{n-1}=m} \mathfrak{g}^e \otimes \Gamma_{\beta_1} \otimes \dots \otimes \Gamma_{\beta_{n-1}} \text{ by } \bar{\partial}''(1 \otimes x_1 \otimes \dots \otimes x_n)$$

$$= (-1)^{n+1} \Big( \sum_{i=1}^{n} (-1)^{i-1} \gamma(x_i) \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes x_n$$

$$- \sum_{1\le i<j\le n} (-1)^{p_{ij}} \otimes [x_i,x_j] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes x_n \Big),$$

where $\hat{x}_i$ denotes deletion of $x_i$, $p_{ij} = \sum_{p<i} |x_p||x_i|$

$+ \sum_{\substack{p<j \\ p\neq i}} |x_p||x_j| + i + j - 3$, $\gamma(x_i) = 0$ if $|x_i| > 0$ and $\gamma(x_i) \in \mathfrak{g}^e$ if $|x_i| = 0$. Then $\bar{\partial}''(1 \otimes x_1 \otimes \dots \otimes x_i \otimes x_{i+1} \otimes \dots \otimes x_n$

$$+ (-1)^{|x_i||x_{i+1}|} 1 \otimes x_1 \otimes \dots \otimes x_{i+1} \otimes x_i \otimes \dots \otimes x_n)$$

$$= (-1)^{n+1} \Big( \sum_{\substack{j=1 \\ j\neq i,i+1}}^{n} (-1)^{j-1} (\gamma(x_j) \otimes x_1 \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes x_i \otimes x_{i+1} \otimes \dots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} \gamma(x_j) \otimes x_1 \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes x_{i+1} \otimes x_i \otimes \dots \otimes x_n)$$

$$+ (-1)^{i-1} \gamma(x_i) \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes x_{i+1} \otimes \dots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} (-1)^i \gamma(x_i) \otimes x_1 \otimes \dots \otimes x_{i+1} \otimes \hat{x}_i \otimes \dots \otimes x_n$$

$$+ (-1)^i \gamma(x_{i+1}) \otimes x_1 \otimes \dots \otimes x_i \otimes \hat{x}_{i+1} \otimes \dots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} (-1)^{i-1} \gamma(x_{i+1}) \otimes x_1 \otimes \dots \otimes \hat{x}_{i+1} \otimes x_i \otimes \dots \otimes x_n\Big)$$

$$- (-1)^{n+1} \Big( \sum_{\substack{1 \le j < k \le n \\ j,k \ne i,i+1}} (-1)^{p_{jk}} (1 \otimes [x_j, x_k] \otimes x_1 \otimes \dots \otimes \hat{x}_j \otimes$$

$$\dots \otimes x_i \otimes x_{i+1} \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} 1 \otimes [x_j, x_k] \otimes x_1 \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes x_{i+1} \otimes x_i \otimes$$

$$\dots \otimes \hat{x}_k \otimes \dots \otimes x_n)$$

$$+ \sum_{j=1}^{i-1} (-1)^{p_{ji}} \otimes [x_j, x_i] \otimes x_1 \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_i \otimes x_{i+1} \otimes \dots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} \sum_{j=1}^{i-1} (-1)^{p_{j,i+1}} \otimes [x_j, x_i] \otimes x_1 \otimes \dots \otimes \hat{x}_j \otimes$$

$$\dots \otimes x_{i+1} \otimes \hat{x}_i \otimes \dots \otimes x_n)$$

$$- (-1)^{n+1} \Big( (-1)^{p_{i,i+1}} \otimes [x_i, x_{i+1}] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \hat{x}_{i+1} \otimes \dots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} (-1)^{p_{i+1,i}} \otimes [x_{i+1}, x_i] \otimes x_1 \otimes \dots \otimes \hat{x}_{i+1} \otimes \hat{x}_i \otimes$$

$$\dots \otimes x_n\Big)$$

$$= (-1)^{n+1} \left( \sum_{\substack{j=1 \\ j \neq i, i+1}}^{n} (-1)^{j-1} (\gamma(x_j) \otimes x_1 \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes \right.$$

$$\cdots \otimes x_n + (-1)^{|x_i||x_{i+1}|} \gamma(x_j) \otimes x_1 \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes$$

$$\left. \cdots \otimes x_n \right)$$

$$- \sum_{\substack{1 \leq j < k \leq n \\ j,k \neq i, i+1}} (-1)^{p_{jk}} (1 \otimes [x_j, x_k] \otimes x_1 \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_i \otimes x_{i+1} \otimes$$

$$\cdots \otimes \hat{x}_k \otimes \cdots \otimes x_n$$

$$+ (-1)^{|x_i||x_{i+1}|} 1 \otimes [x_j, x_k] \otimes x_1 \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_{i+1} \otimes x_i \otimes$$

$$\cdots \otimes \hat{x}_k \otimes \cdots \otimes x_n )$$

$$- \sum_{j=1}^{i-1} \left( (-1)^{\alpha} \otimes [x_j, x_i] \otimes x_1 \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes \hat{x}_i \otimes x_{i+1} \otimes \cdots \otimes x_n \right.$$

$$+ (-1)^{2|x_i||x_{i+1}|} (-1)^{\alpha+1} \otimes [x_j, x_i] \otimes x_1 \otimes \cdots \otimes \hat{x}_j \otimes$$

$$\left. \cdots \otimes x_{i+1} \otimes \hat{x}_i \otimes \cdots \otimes x_n \right)$$

$$- \left( (-1)^{\beta} \otimes [x_i, x_{i+1}] \otimes x_1 \otimes \cdots \otimes \hat{x}_i \otimes \hat{x}_{i+1} \otimes \cdots \otimes x_n \right.$$

$$\left. + (-1)^{2|x_i||x_{i+1}|} (-1)^{\beta+1} \otimes [x_i, x_{i+1}] \otimes x_1 \otimes \cdots \otimes \hat{x}_{i+1} \otimes \hat{x}_i \otimes \cdots \otimes x_n ) \right)$$

The last equality comes from the facts that $\gamma(x_i) = 0$ if $|x_i| > o$, $\gamma(x_{i+1}) = 0$ if $|x_{i+1}| > 0$, $\alpha = p_{ji}$, and $\beta = p_{i,i+1}$. Since the sums of all except the first two summations is zero, and these two summations are in $R_{n-1,m}$, we have $\bar{\partial}''(R_{n,m}) \subset R_{n-1,m}$ so that $\bar{\partial}''$ induces $\partial''_{n,m}: D(\gamma)_{n,m} \longrightarrow D(\gamma)_{n-1,m}$.

We wish to show that $\partial''\partial'' = 0$. By applying $\partial''\partial''$ to $1 \otimes x_1 \otimes \dots \otimes x_n$ we will obtain four types of summations, each of which is zero. First, we have

$$\sum_{1\le i<j\le n} \Big((-1)^{i+j-1}\, \gamma(x_i)\gamma(x_j) + (-1)^{i+j}\gamma(x_j)\gamma(x_i)$$

$$- (-1)^{p_{ij}}\gamma([x_i,x_j])\Big) \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes x_n.$$ If $|x_i| > 0$ or $|x_j| > 0$, then this sum is zero. If $|x_i| = |x_j| = 0$, then $p_{ij} = i + j - 3$. Hence the first factors of the tensor products are $(-1)^{i+j-1}\,(\gamma(x_i)\gamma(x_j) - \gamma(x_j)\gamma(x_i) - \gamma([x_i,x_j]))$, so that the sum is zero. Secondly, we obtain terms of the type

$$(-1)^{i-1}\,(-1)^{p}\,\gamma(x_i) \otimes [x_j,x_k] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes x_n$$

$$+ (-1)^{i}\,(-1)^{p_{jk}}\,\gamma(x_i) \otimes [x_j,x_k] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes x_n,$$

where $p = (|x_1| + \dots + |x_{j-1}|)|x_j| - |x_i||x_j| + (|x_1| + \dots + |x_{k-1}|)|x_k| - |x_i||x_k| + j + k - 5$ and $1\le i<j<k\le n$. If $|x_i| > 0$ then this sum is zero. If $|x_i| = 0$, then $i - 1 + p \equiv i + p_{jk} + 1 \pmod 2$ and the sum is zero. Similarly we may consider the cases $1\le i<k<j\le n$, $1\le j<i<k\le n$, $1\le j<k<i\le n$, $1\le k<i<j\le n$, and $1\le k<j<i\le n$ and obtain zero. Thirdly, we obtain terms of the type

$$(-1)^{p_{ij}}(-1)^{p} \otimes [x_k,x_e] \otimes [x_i,x_j] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes \hat{x}_e \otimes \dots \otimes x_n$$

$$+ (-1)^{p_{ke}}(-1)^{p'} \otimes [x_i,x_j] \otimes [x_k,x_e] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes \hat{x}_e \otimes \dots \otimes x_n,$$ where $p = (|x_1| + \dots + |x_{k-1}|)|x_k|$

$+ (|x_1| + \dots + |x_{e-1}|)|x_e| - |x_i||x_e| - |x_i||x_k| - |x_j||x_e|$

$- |x_j||x_k| + (|x_i| + |x_j|)(|x_k| + |x_e|) + k + e - 7$ and

$p' = (|x_1| + \dots + |x_{i-1}|)|x_i| + (|x_1| + \dots + |x_{j-1}|)|x_j| - |x_i||x_j|$

$+ (|x_k| + |x_e|)(|x_i| + |x_j|) + i + j - 1$. This sum is

$(-1)^{p+p_{ij} - (|x_i| + |x_j|)(|x_k| + |x_e|) - 6}$.

$(1 \otimes [x_k,x_e] \otimes [x_i,x_j] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes$

$\dots \otimes \hat{x}_e \otimes \dots \otimes x_n$

$+ (-1)^{|[x_i,x_j]||[x_k,x_e]|} 1 \otimes [x_i,x_j] \otimes [x_k,x_e] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes$

$\dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes \hat{x}_e \otimes \dots \otimes x_n)$, which is zero because

it is in $R_{n,m}$. Similarly for the other cases with $1 \le i,j,k,e \le n$.

Fourthly, we have terms of the type

$(-1)^{p_{ij}}(-1)^p \otimes [[x_i,x_j], x_k] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes$

$\dots \otimes x_n + (-1)^{p_{jk}}(-1)^{p'} \otimes [[x_j,x_k],x_i] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes \hat{x}_j \otimes$

$\dots \otimes \hat{x}_k \otimes \dots \otimes x_n + (-1)^{p_{ki}}(-1)^{p''} \otimes [[x_k,x_i]\ x_j] \otimes x_1 \otimes \dots \otimes \hat{x}_i \otimes$

$\dots \otimes \hat{x}_j \otimes \dots \otimes \hat{x}_k \otimes \dots \otimes x_n$, where $p = (|x_1| + \dots + |x_{k-1}|)|x_k|$

$- |x_i||x_k| - |x_j||x_k| + k - 2$, $p' = (|x_1| + \dots + |x_{i-1}|)|x_i| + i$,

$p'' = (|x_1| + \dots + |x_{j-1}|)|x_j| - |x_i||x_j| + j - 2$, and $1 \le i < j < k \le n$.

Combining terms, we obtain as second factors of the tensor products

$(-1)^{|x_i||x_k|} [[x_i,x_j], x_k] + (-1)^{|x_j||x_i|} [[x_j,x_k], x_i]$

$+ (-1)^{|x_k||x_j|} [[x_k,x_i], x_j]$, which are zero from the Jacobi identity.

Similarly for the other cases with $1 \le i,j,k, \le n$. Therefore

$\partial''\partial''(1 \otimes x_1 \otimes \ldots \otimes x_n) = 0.$

We have $(\partial'\partial'' + \partial''\partial')(1 \otimes x_1 \otimes \ldots \otimes x_n)$

$$= \partial' \left( (-1)^{n+1} \sum_{i=1}^{n} (-1)^{i-1} \gamma(x_i) \otimes x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes x_n \right.$$

$$\left. - (-1)^{n+1} \sum_{1 \leq i<j \leq n} (-1)^{p_{ij}} \otimes [x_i, x_j] \otimes x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_n \right)$$

$$+ \partial'' \left( (-1)^{n+1} \sum_{j=1}^{n} (-1)^{|x_1| + \ldots + |x_{j-1}|} 1 \otimes x_1 \otimes \ldots \otimes dx_j \otimes \ldots \otimes x_n \right).$$

By applying $\partial'$ and $\partial''$ we obtain three types of sums, each of which is zero. First, we have

$$\sum_{1 \leq i<j \leq n} (-1)^{i-1}(-1)^{|x_1| + \ldots + |x_{j-1}|} (1 - (-1)^{|x_i|}) \gamma(x_i) \otimes x_1 \otimes$$

$$\ldots \otimes \hat{x}_i \otimes \ldots \otimes dx_j \otimes \ldots \otimes x_n$$

$$+ \sum_{j=1}^{n} (-1)^{|x_1| + \ldots + |x_{j-1}|} (-1)^{j-1} \gamma(dx_j) \otimes x_1 \otimes \ldots \otimes \widehat{dx}_j \otimes \ldots \otimes x_n.$$

If $|x_i| > 0$ then $\gamma(x_i) = 0$ and if $|x_i| = 0$ then $1 - (-1)^{|x_i|} = 0$; in either case the first sum is zero. If $|dx_j| > 0$ then $\gamma(dx_j) = 0$ and if $|dx_j| > 0$ then $|x_j| > 1$ so that $\gamma_{j-1}(d_j(x_j)) = \gamma_j(x_j) = 0$; in either case the second sum is zero. Secondly, we obtain sums of the type

$$- (-1)^{n+1}(-1)^{n} \sum_{1 \leq i \leqslant j<k \leq n} (-1)^{p_{ij}}(-1)^{p} \otimes [x_i, x_j] \otimes x_1 \otimes \ldots \otimes \hat{x}_i \otimes$$

$$\cdots \otimes \hat{x}_j \otimes \cdots \otimes dx_k \otimes \cdots \otimes x_n$$

$$- (-1)^{n+1}(-1)^{n+1} \sum_{1\le i<j<k\le n} (-1)^{|x_1| + \cdots + |x_{k-1}|}(-1)^{p_{ij}} \otimes [x_i,x_j] \otimes x_1 \otimes$$

$\cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes dx_k \otimes \cdots \otimes x_n$, where $p = |x_i| + |x_j| + |x_1|$ $+\cdots+|x_{i-1}|+|x_{i+1}|+\cdots+|x_{j-1}|+|x_{j+1}|+\cdots+|x_{k-1}| = |x_1| +\cdots+ |x_{k-1}|$.

The sum is zero and similarly for the other cases with $1\le i,j,k\le n$.

Thirdly, we obtain sums of the type

$$- (-1)^{n+1}(-1)^n \sum_{1\le i<j\le n} (-1)^{p_{ij}} \otimes d([x_i,x_j]) \otimes x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes$$

$$\cdots \otimes x_n$$

$$- (-1)^{n+1}(-1)^{n+1} \sum_{1\le i<j\le n} (-1)^{|x_1| + \cdots + |x_{j-1}|} (-1)^p \otimes [x_i, dx_j] \otimes x_1 \otimes \cdots \otimes \hat{x}_i \otimes$$

$$\cdots \otimes \widehat{dx_j} \otimes \cdots \otimes x_n$$

$$- (-1)^{n+1}(-1)^{n+1} \sum_{1\le i<j\le n} (-1)^{|x_1| + \cdots + |x_{i-1}|} (-1)^{p'} \otimes [dx_i, x_j] \otimes x_1 \otimes$$

$\cdots \otimes \widehat{dx_i} \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n$, where

$p = (|x_1| +\cdots+ |x_{i-1}|)|x_i| + (|x_1| +\cdots+ |x_{j-1}|)|dx_j| - |x_i||dx_j|$ $+ i + j - 3$ and $p' = (|x_1| +\cdots+ |x_{i-1}|)|dx_i| + (|x_1|+\cdots+ |x_{j-1}|)|x_j|$ $- |x_i||x_j| + i + j - 3$. Hence this sum becomes

$$\sum_{1\le i<j\le n} (-1)^{p_{ij}} \otimes (d([x_i,x_j]) - [dx_i,x_j] - (-1)^{|x_i|} [x_i,dx_j]) \otimes x_1 \otimes$$

$\ldots \otimes \hat{x}_i \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_n$, which is zero from definition 2.1.

## 5. The Functor C

Let $\mathfrak{X}$ be the category of $\mathfrak{g}^e$-bicomplexes and let ${}_{\mathfrak{g}^e}\overline{\mathfrak{M}}$ be the category of $\mathfrak{g}^e$-complexes. By ${}_{\mathfrak{g}^e}\overline{\mathfrak{M}}$ we mean that the objects are graded $\mathfrak{g}^e$-modules $\{M_n\}$ with $\mathfrak{g}^e$-homomorphisms $\partial:M_n \to M_{n-1}$ such that $\partial\partial = 0$, and the morphisms are chain maps $f:M \to N$ of $\mathfrak{g}^e$-modules. Then $C:\mathfrak{X} \to {}_{\mathfrak{g}^e}\overline{\mathfrak{M}}$ is the standard condensation functor, described here for completeness.

For each object $X = \{X_{p,q}, \partial', \partial''\}$ in $\mathfrak{X}$ define

$C(X)_n = \sum_{p+q=n} X_{p,q}$. Hence, $C(X)$ is a graded $\mathfrak{g}^e$-module. Define $\partial_n:C(X)_n \to C(X)_{n-1}$ as follows: for $x \in X_{p,q}$, let $\partial_n(x) = \partial'_{p,q}(x) + \partial''_{p,q}(x)$ and extend over the direct sum. Then $\partial_{n-1}\partial_n(x)$
$= \partial_{n-1}(\partial'_{p,q}(x) + \partial''_{p,q}(x)) = \partial'_{p,q-1}\partial'_{p,q}(x) + \partial''_{p,q-1}\partial'_{p,q}(x)$
$+ \partial'_{p-1,q}\partial''_{p,q}(x) + \partial''_{p-1,q}\partial''_{p,q}(x) = 0$ from definition 2.4. Thus $\{C(X), \partial\}$ is in ${}_{\mathfrak{g}^e}\overline{\mathfrak{M}}$.

Let $f:X \to Y$ be in $\mathfrak{X}$, that is, $f_{p,q}:X_{p,q} \to Y_{p,q}$ commutes with the differentials in X and Y. Define $C(f)_n:C(X)_n \to C(Y)_n$ by extending $f_{p,q}$ over the direct sum, that is, $C(f)_n = \Sigma f_{p,q}$.

## 6. Shukla's Cohomology Modules

Consider the functor $E:(\mathfrak{L},\mathfrak{g})^o \longrightarrow {}_{\mathfrak{g}^e}\overline{\mathfrak{M}}$ defined by $E = CDP\Delta$,

$\bar{C}$, $\bar{D}$, $\bar{P}$, $\Delta$, as in the previous four sections. Let $\gamma: L \to \mathfrak{g}$ be in $(\mathfrak{L},\mathfrak{g})^o$ and let M be a fixed $\mathfrak{g}$-module. Note that by proposition 1.1, M is a $\mathfrak{g}^e$-module. Then $E(\gamma)$ is a $\mathfrak{g}^e$-complex:

$$E(\gamma)_o \xleftarrow{\partial_1} E(\gamma)_1 \longleftarrow \cdots \longleftarrow E(\gamma)_{n-1} \xleftarrow{\partial_n} E(\gamma)_n \longleftarrow \cdots .$$

Construct the complex $\mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma),M)$:

$$\mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma)_o,M) \xrightarrow{\partial_1^*} \mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma)_1,M) \to \cdots \to \mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma)_{n-1},M)$$

$$\xrightarrow{\partial_n^*} \mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma)_n,M) \to \cdots .$$

<u>Definition 2.6</u>: The <u>Shukla cohomology of $\gamma$ with coefficients in M</u> is defined by $H^n(\mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma),M)) = \ker \partial_{n+1}^*/\mathrm{im}\ \partial_n^*$ and will be denoted by $SH^n(\gamma,M)$.

In particular, if $\gamma = 1_{\mathfrak{g}}$, the notation $SH^n(\mathfrak{g},M)$ will be used. This is the case which is considered by Shukla in [11].

In view of our construction of E, $SH^n(\mathfrak{g},M)$ differs from Shukla's cohomology in dimensions zero and one. In fact, $SH^o(\mathfrak{g}, M) = 0$ although in [11] Shukla obtains $M^{\mathfrak{g}}$ for his zero-dimensional cohomology. This can be seen from the complex:

$$\mathrm{Hom}_{\mathfrak{g}^e}(E(\gamma),M): \quad 0 \xrightarrow{\partial_1^*} \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes V_o,M) \xrightarrow{\partial_2^*} \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes V_1,M)$$

$$+ \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes V_o \wedge V_o,M) \to \cdots,$$ where $V_o = \Delta(L)_o / I(S)_o$ and $V_1 = \Delta(L)_1 / I(S)_1$.

<u>Definition 2.7</u>: For a $\mathfrak{g}$- module M and for $\gamma: L \to \mathfrak{g}$ in $(\mathfrak{L},\mathfrak{g})^o$, $\mathrm{Der}(\gamma,M) = \{f \in \mathrm{Hom}(L,M) / f([x,y]) = \gamma(x)f(y) - \gamma(y)f(x)\}$ forms an R-module and is called the <u>module of derivations from L to M</u>.

In order to show that $SH^1(\gamma,M)$ is isomorphic to $\mathrm{Der}(\gamma,M)$ several lemmas will be established.

Lemma 2.2: $SH^1(\gamma, M) \cong \{f' \in \text{Hom}\,(V_o, M)/f'(\overline{<[x,y]>}) = \gamma(x)\, f'\,(\overline{<y>}) - \gamma(y)\, f'\,(\overline{<x>})$ for all $x, y \in L$ and $f'(\overline{n}) = 0$ for all $n \in N_1\}$.

Proof: Suppose $f \in \ker \partial_2^*$, that is, $f \in \text{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes V_o, M)$ such that $f\partial_2 = 0$. Since $\overline{<n>} \in V_1$ for any $n \in N_1$ and $\overline{<x>} \wedge \overline{<y>} \in V_o \wedge V_o$ for any $x, y \in L$, $f\partial_2(1 \otimes \overline{<n>}) = 0$ for all $n \in N_1$ and $f\partial_2(1 \otimes \overline{<x>} \wedge \overline{<y>}) = 0$ for all $x, y \in L$. Then $f\partial_2(1 \otimes \overline{<n>}) = f(\partial'_{1,1} + \partial''_{1,1})(1 \otimes \overline{<n>}) = f\partial'_{1,1}(1 \otimes \overline{<n>}) = f(1 \otimes \overline{d}_1(\overline{<n>})) = f(1 \otimes \overline{d_1(<n>)}) = f(1 \otimes \overline{n}) = 0$. Also $f\partial_2(1 \otimes \overline{<x>} \wedge \overline{<y>}) = f(\partial'_{2,o} + \partial''_{2,o})(1 \otimes \overline{<x>} \otimes \overline{<y>}) = f\partial''_{2,o}(1 \otimes \overline{<x>} \otimes \overline{<y>}) = f(\gamma(x) \otimes \overline{<y>} - \gamma(y) \otimes \overline{<x>} - 1 \otimes [\overline{<x>}, \overline{<y>}]) = f(\gamma(x) \otimes \overline{<y>} - \gamma(y) \otimes \overline{<x>} - 1 \otimes \overline{<[x,y]>}) = \gamma(x)f(1 \otimes \overline{<y>}) - \gamma(y)\, f(1 \otimes \overline{<x>}) - f(1 \otimes \overline{<[x,y]>}) = 0$. Hence $f(1 \otimes \overline{<[x,y]>}) = \gamma(x)f(1 \otimes \overline{<y>}) - \gamma(y)f(1 \otimes \overline{<x>})$. It is well known that there is an adjoint isomorphism $\rho: \text{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes V_o,\ M) \longrightarrow \text{Hom}\ (V_o, M)$ such that $\rho(f)(\zeta) = f'(\zeta) = f(1 \otimes \zeta)$ for $\zeta \in V_o$. It follows that $SH^1(\gamma, M) = \{f \in \text{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes V_o, M)/f(1 \otimes \overline{n}) = 0$ for $n \in N_1$, $f(1 \otimes \overline{<[x,y]>}) = \gamma(x)f(1 \otimes \overline{<y>}) - \gamma(y)f(1 \otimes \overline{<x>})$ for $x, y \in L\}$ is isomorphic to $D$, where $D = \{f' \in \text{Hom}(V_o, M)/f'(\overline{n}) = 0$ for $n \in N_1$, $f'(\overline{<[x,y]>}) = \gamma(x)f'(\overline{<y>}) - \gamma(y)f'(\overline{<x>})$ for $x, y \in L\}$.

Lemma 2.3: $\text{Der}(\gamma, M) \cong D$, where $D$ is the set described above.

Proof: We construct maps $\lambda$ and $\mu$, $\text{Der}(\gamma, M) \underset{\lambda}{\overset{\mu}{\rightleftarrows}} D$ such that $\mu\lambda$ and $\lambda\mu$ are identities. Define $\lambda$ by $\lambda(f')(x) = f'(\overline{<x>})$ for all $x \in L$. In order to see that $\lambda(f')$ is R-linear, consider $\sigma_o(<x+y> - <x> - <y>) = \sigma_o(<x + y>) - \sigma_o(<x>) - \sigma_o(<y>) = x + y - x - y = 0$. Hence $<x + y> - <x> - <y> = n \in N_1 = \ker \sigma_o$.

Therefore $\overline{n} = \overline{\langle x + y\rangle} - \overline{\langle x\rangle} - \overline{\langle y\rangle}$ so that $f'(\overline{\langle x + y\rangle})$ $= f'(\overline{\langle x\rangle} + \overline{\langle y\rangle} + \overline{n}) = f'(\overline{\langle x\rangle}) + f'(\overline{\langle y\rangle})$ because $f'$ is R-linear and $f'(\overline{n}) = 0$. Similarly $f'(r\,\overline{\langle x\rangle}) = rf'(\overline{\langle x\rangle})$ and $\lambda(f')$ is R-linear. Since $\lambda(f')([x,y]) = f'(\overline{\langle [x,y]\rangle}) = \gamma(x)f'(\overline{\langle y\rangle}) - \gamma(y)f'(\overline{\langle x\rangle})$ $= \gamma(x)\lambda(f')(y) - \gamma(y)\lambda(f')(x)$, $\lambda(f') \in \mathrm{Der}\ (\gamma,M)$. Define $\mu$ by $\mu(g)(\overline{\langle x\rangle}) = g(x)$, for $g \in \mathrm{Der}\ (\gamma,M)$. In order to see that $\mu(g)$ is well-defined, it suffices to show that $g$ kills the generators of $I(S)_o$. First, $g([x,x]) = \gamma(x)g(x) - \gamma(x)g(x) = 0$. Secondly, $g([x,y] + [y,x]) = g([x,y]) + g([y,x]) = \gamma(x)g(y) - \gamma(y)g(x)$ $+ \gamma(y)g(x) - \gamma(x)g(y) = 0$. Thirdly, $g([x, [y,z]] + [y, [z,x]]$ $+ [z, [x,y]]) = \gamma(x)g([y,z]) - \gamma([y,z])g(x) + \gamma(y)g([z,x])$ $- \gamma([z,x])g(y) + \gamma(z)g([x,y]) - \gamma([x,y])g(z) = \gamma(x)\gamma(y)g(z)$ $- \gamma(x)\gamma(z)g(y) - [\gamma(y),\gamma(z)]g(x) + \gamma(y)\gamma(z)g(x) - \gamma(y)\gamma(x)g(z)$ $- [\gamma(z),\gamma(x)]g(y) + \gamma(z)\gamma(x)g(y) - \gamma(z)\gamma(y)g(x) - [\gamma(x),\gamma(y)]g(z)$ $= (\gamma(y)\gamma(z) - \gamma(z)\gamma(y) - [\gamma(y),\gamma(z)])g(x) + (\gamma(z)\gamma(x) - \gamma(x)\gamma(z)$ $- [\gamma(z),\gamma(x)])g(y) + (\gamma(x)\gamma(y) - \gamma(y)\gamma(x) - [\gamma(x),\gamma(y)])g(z)$, which is zero from the definition of $\mathfrak{g}^e$. Since $g$ is R-linear, $\mu(g)(\overline{\langle x\rangle} + \overline{\langle y\rangle}) = \mu(g)(\overline{\langle x + y\rangle}) = g(x + y) = g(x) + g(y)$ $= \mu(g)(\overline{\langle x\rangle}) + \mu(g)(\overline{\langle y\rangle})$. Similarly, $\mu(g)(r\overline{\langle x\rangle}) = r\mu(g)(\overline{\langle x\rangle})$ so that $\mu(g)$ is R-linear. For $n \in N_I$, $\mu(g)(\overline{n}) = g(\sigma_o(n)) = 0$, and for $x, y \in L$, $\mu(g)(\overline{\langle [x,y]\rangle}) = g([x,y]) = \gamma(x)g(y) - \gamma(y)g(x) = \gamma(x)\mu(g)(\overline{\langle y\rangle})$ $- \gamma(y)\mu(g)(\overline{\langle x\rangle})$, so that $\mu(g) \in D$. Then $\mu(\lambda(f'))(\overline{\langle x\rangle}) = \lambda(f')(x)$ $= f'(\overline{\langle x\rangle})$ and $\lambda(\mu(g))(x) = \mu(g)(\overline{\langle x\rangle}) = g(x)$ and the desired identities are established.

<u>Theorem 2.1</u>: $SH^1(\gamma,M) \cong \mathrm{Der}\ (\gamma,M)$.

<u>Proof</u>: Immediate from lemmas 2.2 and 2.3.

CHAPTER III

# COTRIPLE COHOMOLOGY

S. Eilenberg and J. C. Moore [6] have shown that the concept of a cotriple on a category is a convenient tool for defining cohomology. This technique has been employed by Barr [1], Barr-Beck [2], and Shimada-Uehara-Brenneman [10].

## 1. Cotriples and Adjoint Functors

<u>Definition 3.1</u>: A <u>cotriple</u> $(G, \epsilon, \Delta)$ on a category $\mathfrak{A}$ consists of a functor $G: \mathfrak{A} \to \mathfrak{A}$ and natural transformations $\epsilon: G \to 1_{\mathfrak{A}}$, $\Delta: G \to G^2$ satisfying: (1) the compositions $G \xrightarrow{\Delta} G^2 \xrightarrow{G\epsilon} G$ and $G \xrightarrow{\Delta} G^2 \xrightarrow{\epsilon G} G$ are the identity and (2) the diagram

$$\begin{tikzcd} G \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & G^2 \arrow[d, "G\Delta"] \\ G^2 \arrow[r, "\Delta G"] & G^3 \end{tikzcd}$$

commutes.

The way to obtain a cotriple is from a pair of adjoint functors. Let $\mathfrak{A}$ and $\mathfrak{B}$ be pointed categories. If functors $T\colon \mathfrak{A} \to \mathfrak{B}$, $S\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ and natural transformations $\alpha\colon ST \to 1_{\mathfrak{A}}$, $\beta\colon 1_{\mathfrak{B}} \to TS$ satisfy $(\alpha S)(S\beta) = 1_S$ and $(T\alpha)(\beta T) = 1_T$, then T and S are called <u>adjoint functors</u> and we symbolize this by $(\alpha,\beta)\colon S \dashv T : (\mathfrak{A},\mathfrak{B})$ or simply by $S \dashv T$.

<u>Proposition 3.1</u>: If $(\alpha,\beta)\colon S \dashv T : (\mathfrak{A},\mathfrak{B})$, then $(ST,\alpha,S\beta T)$ is a cotriple on $\mathfrak{A}$.

<u>Proof</u>: Since S,T are functors and $\alpha,\beta$ are natural transformations, $ST\colon \mathfrak{A} \to \mathfrak{A}$ is a functor, $\alpha\colon ST \to 1_{\mathfrak{A}}$ and $S\beta T\colon ST \to STST = (ST)^2$ are natural transformations. From the definition of adjoint functors, $ST\alpha \circ S\beta T = S\circ(T\alpha \circ \beta T) = S\circ 1_T = 1_{ST}$ and $\alpha ST \circ S\beta T = (\alpha S \circ S\beta)\circ T = 1_S \circ T = 1_{ST}$. Since $\beta$ is a natural transformation, $TS\beta \circ \beta = \beta TS \circ \beta$ and so $(ST \circ S\beta T)\circ(S\beta T) = S\circ(TS\beta\circ\beta)\circ T = S\circ(\beta TS\circ\beta)\circ T = (S\beta T\circ ST)\circ(S\beta T)$.

2. The Standard Semi-Simplicial Complex

Let $\mathfrak{A}$ be a pointed category and let $(G,\epsilon,\Delta)$ be a cotriple on $\mathfrak{A}$. Define the <u>standard semi-simplicial complex</u> by

$$\mathbb{G}:\quad \cdots \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; G^3 \; \substack{\xrightarrow{\epsilon^0}\\ \xrightarrow{\epsilon^1}\\ \xrightarrow[\epsilon^2]{}} \; G^2 \; \substack{\xrightarrow{\epsilon^0}\\ \xrightarrow[\epsilon^1]{}} \; G \xrightarrow{\epsilon} 1,$$

where the face morphisms $\epsilon^i\colon G^{n+1} \longrightarrow G^n$ are defined by $\epsilon^i = G^i \epsilon\, G^{n-i}$ ($G^n = G\circ G\circ\cdots\circ G$, n factors) and the degeneracy morphisms $\Delta^i\colon G^{n+1} \longrightarrow G^{n+2}$ are defined by $\Delta^i = G^i\,\Delta\, G^{n-i}$ for $0 \le i \le n$. The verification that $\{G^n/n>0\}$ is a simplicial functor follows from the following:

Lemma 3.1: For $\epsilon^i, \Delta^i$ as defined above,

(i) $\epsilon^i\epsilon^j = \epsilon^{j-1}\epsilon^i$ for $i<j$,

(ii) $\Delta^i\Delta^j = \Delta^{j+1}\Delta^i$ for $i\le j$,

(iii) $\epsilon^i\Delta^j = \Delta^{j-1}\epsilon^i$ for $i<j$,

(iv) $\epsilon^i\Delta^i = 1$,

(v) $\epsilon^{i+1}\Delta^i = 1$,

(vi) $\epsilon^i\Delta^j = \Delta^j\epsilon^{i-1}$ for $i>j+1$.

Proof: (i) For i,j with $i<j$, $\epsilon^i\epsilon^j = G^i(\epsilon G^{j-i-1}\circ G^{j-i}\epsilon)\, G^{n-j}$ and $\epsilon^{j-1}\epsilon^i = G^i(G^{j-i-1}\epsilon\circ\epsilon G^{j-i})G^{n-j}$. Since $\epsilon$ is a natural transformation, so is $\epsilon G^{j-i-1}$. Hence $G^{j-i-1}\epsilon\circ\epsilon G^{j-i} = \epsilon G^{j-i-1}\circ G^{j-i}\epsilon$ and $\epsilon^i\epsilon^j = \epsilon^{j-1}\epsilon^i$. (ii) If $i\le j$, $\Delta^i\Delta^j = G^i(\Delta G^{j-i+1}\circ G^{j-i}\Delta)\, G^{n-j-2}$ and $\Delta^{j+1}\Delta^i = G^i(G^{j-i+1}\Delta\circ\Delta G^{j-i})G^{n-j-2}$. Since $\Delta$ is a natural transformation, we have $\Delta G^{j-i+1}\circ G^{j-i}\Delta = G^{j-i+1}\Delta\circ\Delta G^{j-i}$ so that $\Delta^i\Delta^j = \Delta^{j+1}\Delta^i$. (iii) If $i<j$, $\epsilon^i\Delta^j = G^i(\epsilon G^{j-i+1}\circ G^{j-i}\Delta)G^{n-j} = G^i(G^{j-i-1}\Delta\circ\epsilon G^{j-i})G^{n-j} = \Delta^{j-i}\epsilon^i$ because $\epsilon G^{j-i-1}$ is a natural transformation. (iv) $\epsilon^i\Delta^i = G^i(\epsilon G\circ\Delta)G^{n-i-1} = 1_{G^n}$. (vi) For $i>j+1$, $\epsilon^i\Delta^j = G^j(G^{i-j}\epsilon\circ\Delta G^{i-j-1})G^{n-i+1} = G^j(\Delta G^{i-j-2}\circ G^{i-j-1}\epsilon)G^{n-i+1} = \Delta^j\epsilon^{i-1}$ because $\Delta G^{i-j-2}$ is a natural transformation. (v) Follows from $G\epsilon\circ\Delta = 1_G$.

## 3. Cotriple Cohomology

Let $(G,\epsilon,\Delta)$ be a cotriple on a pointed category $\mathfrak{A}$ and let $T:\mathfrak{A}\longrightarrow Ab$ be a contravariant functor from $\mathfrak{A}$ to an abelian category Ab. Define the cochain complex

$$\mathbb{TG}:\quad \cdots \xleftarrow{\delta_{n+1}} TG^{n+1} \xleftarrow{\delta_n} \cdots \xleftarrow{\delta_2} TG^2 \xleftarrow{\delta_1} TG \longleftarrow 0$$

where $\delta_n = \sum_{i=0}^{n} (-1)^i T\epsilon^i$. This can be seen to be a complex from the following:

Lemma 3.2: For $n>0$, $\delta_{n+1} \cdot \delta_n = 0$.

Proof: From the definition of $\delta$ and lemma 3.1(i), $\delta_{n+1} \cdot \delta_n$

$$= \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} T(\varepsilon^i \varepsilon^j) = \sum_{i=0}^{n} T(\varepsilon^i \varepsilon^i) - \sum_{i=0}^{n} T(\varepsilon^i \varepsilon^{i+1})$$

$$+ \sum_{0 \le i < j \le n} (-1)^{i+j+1} T(\varepsilon^i \varepsilon^{j+1}) + \sum_{0 \le j < i \le n} (-1)^{i+j} T(\varepsilon^i \varepsilon^j)$$

$$= \sum_{i=o}^{n} (T(\varepsilon^i \varepsilon^i) - T(\varepsilon^i \varepsilon^i)) + \sum_{0 \le i < j \le n} (-1)^{i+j+1} T(\varepsilon^j \varepsilon^i)$$

$$+ \sum_{0 \le i < j \le n} (-1)^{i+j} T(\varepsilon^j \varepsilon^i) = 0.$$

Definition 3.2: For a cotriple $(G, \varepsilon, \Delta)$ on a pointed category $\mathfrak{A}$, a contravariant functor $T: \mathfrak{A} \longrightarrow Ab$, and any $A$ in $\mathfrak{A}$, the nth cotriple cohomology of $A$ is defined by $H^n(T\,\mathfrak{G}(A))$.

## 4. A Cotriple Cohomology for $\mathfrak{g}^e$-Modules

In this section, $\mathfrak{A} = {}_{\mathfrak{g}^e}\mathfrak{M}$, the category of left $\mathfrak{g}^e$-modules, $T = \mathrm{Hom}_{\mathfrak{g}^e}(\ ,M)$ for $M$ a fixed $\mathfrak{g}^e$-module, and $Ab = \mathfrak{M}$, the category of left R-modules.

Define $F: {}_{\mathfrak{g}^e}\mathfrak{M} \longrightarrow \mathfrak{M}$ to be the forgetful functor, that is, $F(M)$ is the underlying R-module for any $\mathfrak{g}^e$-module $M$ and $F(f)$ is the underlying R-homomorphism for any $\mathfrak{g}^e$-module homomorphism $f: M \longrightarrow M'$. Define $S: \mathfrak{M} \longrightarrow {}_{\mathfrak{g}^e}\mathfrak{M}$ by $S(N) = \mathfrak{g}^e \otimes N$ for any R-module $N$ and $S(g) = 1_{\mathfrak{g}^e} \otimes g$ for any R-homomorphism $g: N \longrightarrow N'$. The $\mathfrak{g}^e$-module structure of $\mathfrak{g}^e \otimes N$ is given by $m \otimes 1_N$, where $m$ is the multiplication in $\mathfrak{g}^e$.

Proposition 3.2: $S \dashv F$.

Proof: We make use of proposition 1.1, page 13 in [5]. Define the map $\lambda$: $\mathrm{Hom}_{\mathfrak{g}^e}(S(N),M) \longrightarrow \mathrm{Hom}\ (N,F(M))$, for any N in $\mathfrak{M}$ and for any M in ${}_{\mathfrak{g}^e}\mathfrak{M}$, by $\lambda(f)(n) = f(1 \otimes n)$, where $f: S(N) \longrightarrow M$, $n \in N$, and 1 is the unit in $\mathfrak{g}^e$. Since f is a $\mathfrak{g}^e$-homomorphism, it is R-linear and hence $\lambda(f)$ is an R-homomorphism.

Define $\mu$: $\mathrm{Hom}\ (N,F(M)) \longrightarrow \mathrm{Hom}_{\mathfrak{g}^e}(S(N),M)$ by $\mu(g)(a \otimes n) = a\cdot g(n)$, where $g: N \longrightarrow F(M)$, $n \in N$, and $a \in \mathfrak{g}^e$. Since g is an R-homomorphism and M is a $\mathfrak{g}^e$-module, $\mu(g)(a \otimes (n+n')) = a\cdot g(n+n') = a\cdot (g(n) + g(n'))$ $= a\cdot g(n) + a\cdot g(n') = \mu(g)(a \otimes n) + \mu(g)(a \otimes n')$, $\mu(g)(a \otimes rn)$ $= a\cdot g(rn) = a\cdot (rg(n)) = r(a\cdot g(n)) = r\mu(g)(a \otimes n)$, and $\mu(g)(m \otimes 1_N)(a \otimes a' \otimes n) = \mu(g)(aa' \otimes n) = aa'\cdot g(n) = a\cdot (a'\cdot g(n))$ $= \varphi(1_{\mathfrak{g}^e} \otimes \mu(g))(a \otimes a' \otimes n)$, where $\varphi(a \otimes x) = a\cdot x$. Hence $\mu(g)$ is a $\mathfrak{g}^e$-homomorphism. Since $\lambda\mu(g)(n) = \mu(g)(1 \otimes n) = 1\cdot g(n) = g(n)$, and $\mu\lambda(f)(a \otimes n) = a\cdot \lambda(f)(n) = a\cdot f(1 \otimes n) = f(a\cdot (1 \otimes n)) = f(a \otimes n)$, $\lambda$ is an isomorphism and $S \dashv F$.

From propositions 3.1 and 3.2 we have a cotriple on ${}_{\mathfrak{g}^e}\mathfrak{M}$. Specifically, the natural transformation $\alpha$: $SF \longrightarrow 1_{{}_{\mathfrak{g}^e}\mathfrak{M}}$ is given by $\alpha(M)(a \otimes m) = a\cdot m$ for any $a \in \mathfrak{g}^e$ and $m \in M$.

For any A in ${}_{\mathfrak{g}^e}\mathfrak{M}$, $H^n(\mathrm{Hom}_{\mathfrak{g}^e}(\mathbb{G}(A),M))$ is the nth cotriple cohomology of A, where $\mathbb{G}(A)_n = G^{n+1}(A) = (SF)^{n+1}(A)$ for $n \geq 0$.

## 5. A Cotriple Cohomology for Lie Algebras

In this section $\mathfrak{A} = (\mathfrak{L}, \mathfrak{g})^{\circ}$, $Ab = \mathfrak{M}$ and T will be defined as the composition of a covariant functor J: $(\mathfrak{L}, \mathfrak{g})^{\circ} \longrightarrow {}_{\mathfrak{g}^e}\mathfrak{M}$ and the contravariant functor $\mathrm{Hom}_{\mathfrak{g}^e}(\ \ ,M)$: ${}_{\mathfrak{g}^e}\mathfrak{M} \longrightarrow Ab$. The cotriple on $(\mathfrak{L}, \mathfrak{g})^{\circ}$ that will be presented, was defined by Uehara [12].

Let $(S^{\circ}, \mathfrak{g})$ denote the category of groupoids over a fixed non-graded Lie algebra $\mathfrak{g}$, that is, M in $S^{\circ}$ is a set with a binary operation, the objects in the category are functions $\rho: M \to \mathfrak{g}$ such that $\rho(x \cdot y) = [\rho(x), \rho(y)]$ for all $x, y \in M$, and the morphisms in the category are functions $f: M_1 \to M_2$ preserving the binary operation and satisfying the commutative diagram

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr, "\rho_1"'] & & M_2 \arrow[dl, "\rho_2"] \\ & \mathfrak{g} & \end{tikzcd}$$

Let $\gamma: L \to \mathfrak{g}$ be an object in $(\mathfrak{L}, \mathfrak{g})^{\circ}$. Define $U(L)$ to be the underlying set L with only its multiplicative structure and define $U(\gamma): U(L) \to \mathfrak{g}$ to be the product-preserving function from $U(L)$ to $\mathfrak{g}$. For a morphism $f: L_1 \to L_2$ in $(\mathfrak{L}, \mathfrak{g})^{\circ}$, define $U(f)$ to be the product-preserving function from $U(L_1)$ to $U(L_2)$.

Let $\rho: M \to \mathfrak{g}$ be an object in $(S^{\circ}, \mathfrak{g})$. In $R(\langle M\rangle)$, the free R-module generated by the underlying set of M, define $[\langle x\rangle, \langle y\rangle] = \langle x \cdot y\rangle$ for $x, y \in M$. Let I be the two-sided ideal in $R(\langle M\rangle)$ generated by elements of the forms $\langle x \cdot x\rangle$, $\langle x \cdot y\rangle + \langle y \cdot x\rangle$, $\langle x \cdot (y \cdot z)\rangle + \langle y \cdot (z \cdot x)\rangle + \langle z \cdot (x \cdot y)\rangle$ for all $x, y, z \in M$. Define $P(M) = R(\langle M\rangle)/I$ and define $P(\rho)(\overline{\langle x\rangle}) = \rho(x)$ for all $x \in M$ and extend by linearity. $P(M)$ is a non-graded Lie algebra. Since $\mathfrak{g}$ ~~is a Lie algebra~~ and $P(\rho)$ is R-linear, $P(\rho)(\overline{\langle x \cdot x\rangle}) = \rho(x \cdot x) = [\rho(x), \rho(x)] = 0$, $P(\rho)(\overline{\langle x \cdot y\rangle} + \overline{\langle y \cdot x\rangle}) = P(\rho)(\overline{\langle x \cdot y\rangle}) + P(\rho)(\overline{\langle y \cdot x\rangle}) = \rho(x \cdot y) + \rho(y \cdot x) = [\rho(x), \rho(y)] + [\rho(y), \rho(x)] = 0$, and $P(\rho)(\overline{\langle x \cdot (y \cdot z)\rangle} + \overline{\langle y \cdot (z \cdot x)\rangle} + \overline{\langle z \cdot (x \cdot y)\rangle})$

$= P(\rho)(\overline{\langle x\cdot(y\cdot z)\rangle}) + P(\rho)(\overline{\langle y\cdot(z\cdot x)\rangle}) + P(\rho)(\overline{\langle z\cdot(x\cdot y)\rangle}) = \rho(x\cdot(y\cdot z)) + \rho(y\cdot(z\cdot x)) + \rho(z\cdot(x\cdot y)) = [\rho(x),[\rho(y),\rho(z)]] + [\rho(y),[\rho(z),\rho(x)]] + [\rho(z),[\rho(x),\rho(y)]] = 0$. Therefore $P(\rho)$ is well defined. $P(\rho)$ is product-preserving because $P(\rho)([\overline{\langle x\rangle},\overline{\langle y\rangle}]) = P(\rho)(\overline{[\langle x\rangle,\langle y\rangle]})$ $= P(\rho)(\overline{\langle x\cdot y\rangle}) = \rho(x\cdot y) = [\rho(x),\rho(y)] = [P(\rho)(\overline{\langle x\rangle}),\ P(\rho)(\overline{\langle y\rangle})]$. For $f\colon M \longrightarrow M'$ in $(S^{\cdot},\mathfrak{g})$ define $P(f)(\overline{\langle x\rangle}) = \overline{\langle f(x)\rangle}$ for all $x \in M$. Then $P(\rho')\,P(f)(\overline{\langle x\rangle}) = P(\rho')(\overline{\langle f(x)\rangle}) = \rho' f(x) = \rho(x) = P(\rho)(\overline{\langle x\rangle})$.

<u>Proposition 3.3</u>: $P \dashv U$.

<u>Proof</u>: Define $\lambda\colon \mathrm{Hom}_{(\mathfrak{L},\mathfrak{g})^{o}}(P(\rho),\gamma) \longrightarrow \mathrm{Hom}_{(S^{\cdot},\mathfrak{g})}(\rho,U(\gamma))$ by $\lambda(\varphi)(m) = \varphi(\overline{\langle m\rangle})$ for any $\varphi\colon P(M) \longrightarrow L$ and for any $m \in M$. Since $\varphi$ is product-preserving, $\lambda(\varphi)(m\cdot m') = \varphi(\overline{\langle m\cdot m'\rangle}) = \varphi(\overline{[\langle m\rangle,\langle m'\rangle]})$ $= \varphi([\overline{\langle m\rangle},\overline{\langle m'\rangle}]) = [\varphi(\overline{\langle m\rangle}),\ \varphi(\overline{\langle m'\rangle})] = [\lambda(\varphi)(m),\lambda(\varphi)(m')]$ so that $\lambda(\varphi)$ is product-preserving. Since $U(\gamma\circ\varphi) = U(\gamma)\circ U(\varphi) = U(\gamma)\circ\varphi$ and since $\gamma\circ\varphi = P(\rho)$, we have $U(\gamma)\circ\lambda(\varphi)(m) = U(\gamma)\varphi(\overline{\langle m\rangle}) = U(\gamma\circ\varphi)(\overline{\langle m\rangle})$ $= U(P(\rho))(\overline{\langle m\rangle}) = U(\rho(m)) = \rho(m)$ and hence $U(\gamma)\circ\lambda(\varphi) = \rho$, that is, $\lambda(\varphi)$ is a morphism in $(S^{\cdot},\mathfrak{g})$.

Define $\mu\colon \mathrm{Hom}_{(S^{\cdot},\mathfrak{g})}(\rho,U(\gamma)) \longrightarrow \mathrm{Hom}_{(\mathfrak{L},\mathfrak{g})^{o}}(P(\rho),\gamma)$ by $\mu(\psi)(\overline{\langle m\rangle}) = \psi(m)$ and extend by linearity, where $\psi\colon M \longrightarrow U(L)$ and $m \in M$. In a manner similar to the demonstration that $P(\rho)$ is well-defined and product-preserving, it can be shown that $\mu(\psi)$ is well-defined and preserves products. Since $U(\gamma)\circ\psi = \rho$, $\gamma\circ\mu(\psi)(\overline{\langle m\rangle}) = \gamma\psi(m) = U(\gamma)\circ\psi(m) = \rho(m)$ $= P(\rho)(\overline{\langle m\rangle})$ so that $\mu(\psi)$ is a morphism in $(\mathfrak{L},\mathfrak{g})^{o}$. We have $\mu\lambda(\varphi)(\overline{\langle m\rangle}) = \lambda(\varphi)(m) = \varphi(\overline{\langle m\rangle})$ and $\lambda\mu(\psi)(m) = \mu(\psi)(\overline{\langle m\rangle}) = \psi(m)$. Therefore $\lambda$ is an isomorphism and $P \dashv U$.

From propositions 3.1 and 3.3 we have a cotriple on $(\mathfrak{L},\mathfrak{g})^{o}$; however we wish to formulate this cotriple explicitly for use in the next chapter. Let $\gamma\colon L \longrightarrow \mathfrak{g}$ be in $(\mathfrak{L},\mathfrak{g})^{o}$ and consider the diagram

$$\begin{tikzcd}
L \arrow[r, "\gamma"] & \mathfrak{g} \\
G(L) \arrow[u, "\varepsilon(\gamma)"] \arrow[ur, "G(\gamma)"']
\end{tikzcd}$$

where $G = P\,U$, that is, $G(L) = R(\langle L\rangle)/I$. From the theorem on adjoint functors in [5], page 13, we have $\mu(\psi) = \varepsilon(\gamma)\circ P(\psi)$. Therefore $\mu(\psi)(\overline{\langle m\rangle}) = \varepsilon(\gamma)\circ P(\psi)(\overline{\langle m\rangle})$, that is, $\psi(m) = \varepsilon(\gamma)(\overline{\langle\psi(m)\rangle})$ so that $\varepsilon(\gamma)(\overline{\langle x\rangle}) = x$ for any $x \in L$. Define $G(\gamma)(\overline{\langle x\rangle}) = \gamma(x)$ and extend linearly. $G(\gamma)$ is well-defined and preserves products. The above diagram commutes because $\gamma(\varepsilon(\gamma)(\overline{\langle x\rangle})) = \gamma(x) = G(\gamma)(\overline{\langle x\rangle})$. We then have $\varepsilon\colon G \longrightarrow 1_{(\mathfrak{L},\mathfrak{g})^{o}}$.

Let $\gamma\colon L \rightarrow \mathfrak{g}$ be in $(\mathfrak{L},\mathfrak{g})^{o}$ and define a $\mathfrak{g}^{e}$-homomorphism $d\colon \mathfrak{g}^{e}\otimes (L\wedge L) \longrightarrow \mathfrak{g}^{e}\otimes L$ by $d(1 \otimes x\wedge y) = \gamma(x) \otimes y - \gamma(y) \otimes x - 1 \otimes [x,y]$ for all $x,y \in L$, where $L\wedge L$ denotes the exterior product. Then $d$ is well-defined because $d(1 \otimes x\wedge x) = \gamma(x) \otimes x - \gamma(x) \otimes x - 1 \otimes [x,x] = 0$. We now define a covariant functor $J\colon (\mathfrak{L},\mathfrak{g})^{o} \longrightarrow {}_{\mathfrak{g}^{e}}\mathfrak{M}$ by $J(\gamma) = \mathfrak{g}^{e}\otimes L/\operatorname{im} d$. For a morphism $f\colon L \longrightarrow L'$ in $(\mathfrak{L},\mathfrak{g})^{o}$, define $J(f)\colon \mathfrak{g}^{e}\otimes L/\operatorname{im} d \longrightarrow \mathfrak{g}^{e}\otimes L'/\operatorname{im} d'$ to be the $\mathfrak{g}^{e}$-homomorphism induced by $1_{\mathfrak{g}^{e}}\otimes f$, where $d'\colon \mathfrak{g}^{e}\otimes(L'\wedge L') \longrightarrow \mathfrak{g}^{e}\otimes L'$. Since $(1_{\mathfrak{g}^{e}}\otimes f)(\gamma(x) \otimes y - \gamma(y) \otimes x - 1 \otimes [x,y]) = \gamma(x) \otimes f(y) - \gamma(y) \otimes f(x) - 1 \otimes f([x,y]) = \gamma'(f(x)) \otimes f(y) - \gamma'(f(y)) \otimes f(x) - 1 \otimes [f(x),f(y)]$, $J(f)(\operatorname{im} d) \subset \operatorname{im} d'$ so that $J(f)$ is well-defined.

For any $\gamma$ in $(\mathfrak{L},\mathfrak{g})^{o}$, $H^{n}(\operatorname{Hom}_{\mathfrak{g}^{e}}(J\,\mathfrak{G}(\gamma),M))$ is the nth cotriple cohomology of $\gamma$, where $\mathfrak{G}(\gamma)_n = G^{n+1}(\gamma) = (PU)^{n+1}(\gamma)$ for $n\geq 0$.

<u>Lemma 3.3</u>: For any $\gamma: L \to \mathfrak{g}$ in $(\mathfrak{L}, \mathfrak{g})^{\circ}$, $\mathrm{Hom}_{\mathfrak{g}^e}(J(\gamma), M)$ is isomorphic to Der $(\gamma, M)$.

<u>Proof</u>: Define $\rho$: $\mathrm{Hom}_{\mathfrak{g}^e}(J(\gamma), M) \longrightarrow \mathrm{Der}\ (\gamma, M)$ by $\rho(f)(x) = f(1 \otimes x)$, where $f: J(\gamma) \longrightarrow M$ and $x \in L$. Since $\rho(f)([x,y]) = f(1 \otimes [x,y]) = f(\gamma(x) \otimes y - \gamma(y) \otimes x) = \gamma(x)\ f(1 \otimes y) - \gamma(y) f(1 \otimes x) = \gamma(x)\rho(f)(y) - \gamma(y)\rho(f)(x)$, $\rho(f)$ is well-defined. Define $\sigma$: Der $(\gamma, M) \longrightarrow \mathrm{Hom}_{\mathfrak{g}^e}(J(\gamma), M)$ by $\sigma(g)(a \otimes x) = a \circ g(x)$ for $g: L \longrightarrow M$, $a \in \mathfrak{g}^e$, and $x \in L$. Since $\sigma(g)(\gamma(x) \otimes y - \gamma(y) \otimes x - 1 \otimes [x,y]) = \gamma(x)g(y) - \gamma(y)g(x) - g([x,y]) = 0$ and since $\sigma(g)(a' \circ (a \otimes x)) = \sigma(g)(a'a \otimes x) = a'a \circ g(x) = a' \circ (a \circ g(x)) = a' \circ \sigma(g)(a \otimes x)$ for all $x, y \in L$ and for all $a, a' \in \mathfrak{g}^e$, $\sigma(g)$ is well-defined. Therefore $\rho(\sigma(g))(x) = \sigma(g)(1 \otimes x) = g(x)$ and $\sigma(\rho(f))(a \otimes x) = a \circ \rho(f)(x) = a \circ f(1 \otimes x) = f(a \otimes x)$ so that $\rho$ is an isomorphism.

Hence for any $\gamma$ in $(\mathfrak{L}, \mathfrak{g})^{\circ}$, the nth cotriple cohomology of $\gamma$ with coefficients in a $\mathfrak{g}$-module M is given by $\tilde{H}^n(\gamma, M) = H^n(\mathrm{Der}(\mathfrak{G}(\gamma), M))$, where $\mathfrak{G}(\gamma)_n = (PU)^{n+1}\ (\gamma)$ for all $n\geq 0$.

In particular, if $\gamma = 1_{\mathfrak{g}}$, the notation $\tilde{H}^n(\mathfrak{g}, M)$ will be used.

CHAPTER IV

# COMPARISON OF COHOMOLOGY THEORIES

Let $(\mathfrak{L},\mathfrak{g})^{0}$ be the category of non-graded Lie algebras over $\mathfrak{g}$. Q is the augmentation ideal of $\mathfrak{g}^{e}$ and J is the functor defined in section five of Chapter III (denoted by III, 5). $H^{n}$, $SH^{n}$, and $\tilde{H}^{n}$ denote the Hochschild, Shukla, and cotriple cohomologies, respectively.

## 1. The Hochschild Cohomology of Lie Algebras as a Cotriple Cohomology

Let G = SF as in III, 4 and let M be a $\mathfrak{g}^{e}$-module.

__Definition 4.1:__ The __$n^{th}$ cotriple cohomology of $\mathfrak{g}$ with coefficients in M__ is given by $\tilde{H}^{n}(\mathfrak{g},M) = H^{n}(\mathrm{Hom}_{\mathfrak{g}^{e}}(\mathbb{G}(Q),M))$.

Note that this definition agrees with the cotriple cohomology in definition 3.2 because Q is a $\mathfrak{g}^{e}$-module.

__Lemma 4.1:__ For $1_{\mathfrak{g}}$ in $(\mathfrak{L},\mathfrak{g})^{0}$, $J(1_{\mathfrak{g}}) \cong Q$.

__Proof:__ Consider the diagram

$$\begin{tikzcd}
 & T(\mathfrak{g}) \otimes \mathfrak{g} \arrow[r, "m"] \arrow[d, "p \otimes 1"] & T(\mathfrak{g}) \arrow[d, "p"] \arrow[dr, "\epsilon'"] & \\
\mathfrak{g}^{e} \otimes \mathfrak{g} \wedge \mathfrak{g} \arrow[r, "d"] & \mathfrak{g}^{e} \otimes \mathfrak{g} \arrow[r, "\delta"] \arrow[d, "\Pi"] & \mathfrak{g}^{e} \arrow[r, "\epsilon"] & R \\
 & J(1_{\mathfrak{g}}) & &
\end{tikzcd}$$

where p, $\epsilon$, $\epsilon'$ are defined as in I, 3, $\Pi$ is the natural projection, and $\delta(a \otimes z) = az$ for $a \in \mathfrak{g}^e$ and $z \in \mathfrak{g}$. Since $\delta d(1 \otimes x \wedge y)$ $= \delta(x \otimes y - y \otimes x - 1 \otimes [x,y]) = xy - yx - [x,y] = 0$, $\operatorname{im} d \subset \ker \delta$. For any $w = \sum a_i \otimes x_i \in T(\mathfrak{g}) \otimes \mathfrak{g}$, define $m(w) = \sum a_i x_i \in T(\mathfrak{g})$. Then $m$ is a monomorphism because the multiplication in $T(\mathfrak{g})$ is defined by the tensor product. Since $p\,m = \delta(p \otimes 1)$, $\delta^{-1}(0)$ $= (p \otimes 1)m^{-1}p^{-1}(0)$. But $p^{-1}(0)$ is the two-sided ideal $I$ generated by elements of the form $xy - yx - [x,y]$, where $x,y \in \mathfrak{g}$, so that for any $\overline{w} \in \ker \delta$, $\overline{w} = (p \otimes 1)(w)$, where $w \in m^{-1}(I)$. Write $m(w) = \Sigma w_i$ with $w_i$ $= \alpha_i(x_i y_i - y_i x_i - [x_i,y_i])\beta_i \in I$. Since $T(\mathfrak{g})$ is augmented, $T(\mathfrak{g}) = R + Q'$ so that we may write $\beta_i = r_i + \beta_i'$, where $r_i \in R$ and $\beta_i' \in Q'$. Since $(p \otimes 1)m^{-1}(\beta_i') = 0$, $d(r_i p(\alpha_i) \otimes x_i \wedge y_i)$ $= (p \otimes 1)m^{-1}(w_i)$ so that $\overline{w} \in \operatorname{im} d$ and $\ker \delta \subset \operatorname{im} d$. For $\overline{a} \in \mathfrak{g}^e$ and $z \in \mathfrak{g}$, $\epsilon\delta(\overline{a} \otimes z) = \epsilon\delta(p(a) \otimes z) = \epsilon\delta(p \otimes 1)(a \otimes z) = \epsilon p m(a \otimes z)$ $= \epsilon' m(a \otimes z) = \epsilon'(az) = 0$ because $az \in T(\mathfrak{g})_n$, $n>0$, so that $\operatorname{im} \delta = Q$. Hence $J(1_{\mathfrak{g}}) = \mathfrak{g}^e \otimes \mathfrak{g}/\operatorname{im} d = \mathfrak{g}^e \otimes \mathfrak{g}/\ker \delta \cong \operatorname{im} \delta = Q$.

Lemma 4.2: $\widetilde{H}^o(\mathfrak{g},M) \cong \operatorname{Der}(1_{\mathfrak{g}},M)$.

Proof: From the chain complex

$$\mathfrak{G}(Q): \cdots \longrightarrow G^3(Q) \xrightarrow{\partial_2} G^2(Q) \xrightarrow{\partial_1} G(Q) \xrightarrow{\alpha(Q)} Q, \text{ where } \partial_n$$

$= \sum_{i=o}^{n} (-1)^i \alpha^i(Q)$ and $\alpha^i = G^i \alpha G^{n-i}$, we obtain the cochain complex

$$0 \longrightarrow \operatorname{Hom}_{\mathfrak{g}^e}(Q,M) \xrightarrow{\alpha(Q)^*} \operatorname{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes Q,M) \xrightarrow{\partial_1^*} \operatorname{Hom}_{\mathfrak{g}^e}(G^2(Q),M) \longrightarrow \cdots.$$

$\widetilde{H}^o(\mathfrak{g},M) = \ker \partial_1^* \cong \operatorname{Hom}_{\mathfrak{g}^e}(Q,M)$ so that by lemmas 3.3 and 4.1, $\widetilde{H}^o(\mathfrak{g},M) \cong \operatorname{Der}(1_{\mathfrak{g}},M)$.

Lemma 4.3: $\mathfrak{G}(Q)$ is an R-split exact resolution of Q.

Proof: $\mathfrak{G}(Q)$ is just the un-normalized bar resolution $\beta(\mathfrak{g}^e,Q)$ of Q, that is, $\beta_n(\mathfrak{g}^e,Q) = \mathfrak{g}^e \otimes (\mathfrak{g}^e)^n \otimes Q$ and $\partial_n: \beta_n(\mathfrak{g}^e,Q) \longrightarrow \beta_{n-1}(\mathfrak{g}^e,Q)$ is defined by $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$ with $d_i(\lambda_0 \otimes \lambda_1 \otimes \cdots \otimes \lambda_n \otimes q)$ $= \lambda_0 \otimes \cdots \otimes \lambda_i\lambda_{i+1} \otimes \cdots \otimes q$. By corollary 2.2 and theorem 2.1 on pages 281 and 282 in [9], $\mathfrak{G}(Q)$ is an R-split exact resolution of Q. An alternate proof of this lemma may be found in [10], corollary 3.2.

Theorem 4.1: $\tilde{H}^n(\mathfrak{q},M)$ is isomorphic to $\mathrm{Der}(1_{\mathfrak{q}},M)$ for $n = 0$ and to $H^{n+1}(\mathfrak{q},M)$ for $n>0$.

Proof: The first part is shown in lemma 4.2. For the second part consider the two chain complexes

$$\begin{array}{ccccccccccccc}
\cdots \to G^{n+2}(Q) & \xrightarrow{\partial_{n+1}} & G^{n+1}(Q) & \xrightarrow{\partial_n} & G^n(Q) & \to \cdots & \xrightarrow{\partial_1} & G(Q) & \xrightarrow{\partial_0} & \mathfrak{q}^e & \xrightarrow{\varepsilon} & R & \to 0 \\
 & & & & & & & & & & & \| & \\
\cdots \to \mathfrak{q}^e \otimes Q^{n+2} & \xrightarrow{d_{n+2}} & \mathfrak{q}^e \otimes Q^{n+1} & \xrightarrow{d_{n+1}} & \mathfrak{q}^e \otimes Q^n & \to \cdots & \xrightarrow{d_2} & \mathfrak{q}^e \otimes Q & \xrightarrow{d_1} & \mathfrak{q}^e & \xrightarrow{\varepsilon} & R & \to 0
\end{array}$$

where the lower complex is $B(\mathfrak{q}^e)$ and the upper complex is obtained from $\mathfrak{G}(Q)$ by cutting off $\alpha(Q)$ and splicing on the exact sequence $G(Q) = \mathfrak{q}^e \otimes Q \xrightarrow{\partial_0} \mathfrak{q}^e \xrightarrow{\varepsilon} R \to 0$ with $\partial_0(1 \otimes q) = q$. Both complexes are R-split exact resolutions of R so that by the comparison theorem they are chain equivalent. Therefore for $n>0$,

$\ker \partial_{n+1}^*/\mathrm{im}\, \partial_n^* \cong \ker d_{n+2}^*/\mathrm{im}\, d_{n+1}^*$ so that $\tilde{H}^n(\mathfrak{q},M) \cong H^{n+1}(\mathfrak{q},M)$.

## 2. Barr-Beck's Acyclic Model Theorem

Let $(G, \varepsilon, \Delta)$ be a cotriple on a category $\mathfrak{A}$ and let $\mathfrak{B}$ be a pre-additive category with kernels. Assume that $\mathbb{K} = \{K^n, \partial^n\}_{n\geq -1}$ is a cochain complex of functors, that is, $\mathbb{K}$ is the cochain complex:

$$0 \to K^{-1} \xrightarrow{\partial^{-1}} K^0 \xrightarrow{\partial^0} K^1 \xrightarrow{\partial^1} \cdots \to K^n \xrightarrow{\partial^n} K^{n+1} \to \cdots$$

where for each $n\geq -1$, $K^n: \mathfrak{A} \to \mathfrak{B}$ is a contravariant functor and $\partial^n: K^n \to K^{n+1}$ is a natural transformation satisfying $\partial^{n+1}\partial^n = 0$.

Definition 4.2: $\mathbb{K}$ is G-representable if and only if for each $n\geq 0$ $K^n\epsilon$: $K^n \longrightarrow K^nG$ is a coretraction, that is, there exists a natural transformation (retraction) $\theta^n$: $K^nG \longrightarrow K^n$ such that $\theta^n \circ K^n\epsilon = 1_{K^n}$. $\mathbb{K}$ is G-acyclic if and only if there exists a contracting homotopy $\{s^n\}_{n\geq -1}$ in the complex $\{K^nG, \partial^nG\}$, that is, for each $n\geq -1$ there exists a natural transformation $s^n$: $K^{n+1}G \longrightarrow K^nG$ satisfying $s^n \circ \partial^nG + \partial^{n-1}G \circ s^{n-1} = 1_{K^nG}$ for all $n\geq 0$.

Remark: For a chain complex of functors, the dual statements constitute the definitions of G-representability and G-acyclicity.

Theorem 4.2: If a complex $\mathbb{K}$ is G-representable and a complex $\mathbb{L}$ is G-acyclic and if f: $K^{-1} \longrightarrow L^{-1}$ is a natural transformation, then f can be extended to a natural chain transformation $F: \mathbb{K} \longrightarrow \mathbb{L}$ and any two extensions are chain homotopic.

Proof: See Barr-Beck [2] or Shimada-Uehara-Brenneman [10]. The latter proof shows that theorem 4.2 is the usual comparison theorem in relative homological algebra.

In particular, if $\mathbb{K}$ and $\mathbb{L}$ are both G-representable and G-acyclic and if $K^{-1} = L^{-1}$ then the extension is a chain equivalence.

Lemma 4.4: Let $(G,\epsilon,\Delta)$ be a cotriple on a pointed category $\mathfrak{A}$ and let $T: \mathfrak{A} \longrightarrow \mathfrak{L}$ be a contravariant functor, where $\mathfrak{L}$ is a preadditive category with kernels. Then the cochain complex

$$T\mathbb{G}: 0 \longrightarrow T \xrightarrow{\delta_0} TG \xrightarrow{\delta_1} TG^2 \longrightarrow \cdots \longrightarrow TG^n \xrightarrow{\delta_n} TG^{n+1} \longrightarrow \cdots, \text{ where}$$

$$\delta_n = \sum_{i=0}^{n} (-1)^i \, T\epsilon^i,$$ is G-representable and G-acyclic.

Proof: For $n\geq 0$, define $\theta^n$: $TG^{n+2} \longrightarrow TG^{n+1}$ by $\theta^n = TG^n\Delta$. Then $\theta^n \circ (TG^{n+1}\epsilon) = (TG^n\Delta)\circ(TG^{n+1}\epsilon) = T(G^{n+1}\epsilon\circ G^n\Delta) = TG^n(G\epsilon\circ\Delta) = TG^n(1_G)$ $= 1_{TG^{n+1}}$ so that $T\mathbb{G}$ is G-representable. For $n\geq -1$, define

$s^n: TG^{n+3} \longrightarrow TG^{n+2}$ by $s^n = (-1)^{n+1}TG^{n+1}\Delta$. Then $s^n\circ\delta_{n+1}G + \delta_n G\circ s^{n-1}$

$$= (-1)^{n+1}TG^{n+1}\Delta\circ\sum_{i=0}^{n+1}(-1)^i T\epsilon^i G + \sum_{i=0}^{n}(-1)^i T\epsilon^i G\circ(-1)^n TG^n\Delta$$

$$= T(\epsilon^{n+1}G\circ G^{n+1}\Delta) + \sum_{i=0}^{n}(-1)^{n+i+1}(T(G^i\epsilon G^{n-i+2}\circ G^{n+1}\Delta)$$

$- T(G^n\Delta\circ G^i\epsilon G^{n-i+1}))$. As in lemma 3.1, $\epsilon^{n+1}G\circ G^{n+1}\Delta = 1_{G^{n+2}}$ and $G^i\epsilon G^{n-i+2}\circ G^{n+1}\Delta = G^n\Delta\circ G^i\epsilon G^{n-i+1}$. Therefore $s^n\circ\delta_{n+1}G + \delta_n G\circ s^{n-1}$ $= 1_{TG^{n+2}}$ so that $T\mathfrak{G}$ is G-acyclic.

## 3. Shukla and Cotriple Cohomologies of Lie Algebras

Let G = PU as in III, 5 and let M be a $\mathfrak{g}$-module.

<u>Definition 4.3</u>: For any $\gamma$ in $(\mathfrak{L},\mathfrak{g})^o$, the <u>nth cotriple cohomology of $\gamma$ with coefficients in M</u> is given by $\tilde{H}^n(\gamma,M)$ $= H^n(\mathrm{Hom}_{\mathfrak{g}^e}(J\mathfrak{G}(\gamma),M))$.

Note that this definition agrees with the cotriple cohomology in definition 3.2 because $\mathrm{Hom}_{\mathfrak{g}^e}(J,M)\cong \mathrm{Der}(\ ,M)$ is a contravariant functor from $(\mathfrak{L},\mathfrak{g})^o$ to $\mathfrak{M}$.

<u>Theorem 4.3</u>: $\tilde{H}^o(\gamma,M)\cong \mathrm{Der}(\gamma,M)$ for $\gamma: L\longrightarrow\mathfrak{g}$ in $(\mathfrak{L},\mathfrak{g})^o$.

<u>Proof</u>: Consider the diagram

$$\begin{tikzcd}
\mathfrak{g}^e\otimes G^2(L)\wedge G^2(L) \arrow[r, "\delta"] & \mathfrak{g}^e\otimes G^2(L) \arrow[r, "p"] \arrow[d, dashed, "\beta"] & JG^2(\gamma) \arrow[r] \arrow[d, "d_1"] & 0 \\
\mathfrak{g}^e\otimes G(L)\wedge G(L) \arrow[r, "d"] & \mathfrak{g}^e\otimes G(L) \arrow[r, "q"] & JG(\gamma) \arrow[r] & 0
\end{tikzcd}$$

where $d_1 = J\epsilon G(\gamma) - JG\epsilon(\gamma)$ is the boundary operator in $J\mathfrak{G}$, p,q are natural projections, d is defined as in III, 5 and $\delta$ is defined analogous to d. By definition of the cokernel, the rows in the diagram are exact. We are going to define $\beta$ such that $q\ \beta = d_1 p$. Recall that we have a morphism $\epsilon G(\gamma)\colon G^2(L) \longrightarrow G(L)$ given by $\epsilon G(\gamma)(\overline{<\xi>}) = \xi$ for $\xi \in G(L)$, where $G^2(L) = R(<G(L)>)/I(S)$. In the diagram

$$\begin{tikzcd}
L \arrow[r, "\gamma"] & \mathfrak{g} \\
G(L) \arrow[u, "\epsilon(\gamma)"] \arrow[ur, "G(\gamma)"'] & G^2(L) \arrow[u, "G^2(\gamma)"'] \arrow[l, "\epsilon G(\gamma)"]
\end{tikzcd}$$

both triangles commute. Applying G to the upper triangle we obtain

$$\begin{tikzcd}
G(L) \arrow[r, "G(\gamma)"] & G(\mathfrak{g}) \\
G^2(L) \arrow[u, "G\epsilon(\gamma)"] \arrow[ur, "G^2(\gamma)"'] &
\end{tikzcd}$$

For any $\xi \in G(L)$, $\xi = \sum r_i \overline{<x_i>}$, where $r_i \in R$ and $x_i \in L$. Define $G\epsilon(\gamma)(\overline{<\xi>}) = \overline{<\epsilon(\gamma)(\xi)>}$. Therefore we have $\epsilon G(\gamma)(\overline{<\Sigma\ r_i\ \overline{<x_i>}>})$ $= \Sigma\ r_i\ \overline{<x_i>}$ and $G\epsilon(\gamma)(\overline{<\Sigma\ r_i \overline{<x_i>}>}) = \overline{<\Sigma\ r_i x_i>}$. Define $\beta(1 \otimes \overline{<\xi>})$ $= 1 \otimes \xi - 1 \otimes \overline{<\epsilon(\gamma)(\xi)>}$. Since $d_1\ p(1 \otimes \overline{<\xi>}) = d_1(\overline{1 \otimes \overline{<\xi>}}) = (J\epsilon G(\gamma)$ $- JG\epsilon(\gamma))(\overline{1 \otimes \overline{<\xi>}}) = \overline{1 \otimes \epsilon G(\gamma)(\overline{<\xi>})} - \overline{1 \otimes G\epsilon(\gamma)(\overline{<\xi>})} = \overline{1 \otimes \xi}$ $- \overline{1 \otimes \overline{<\epsilon(\gamma)(\xi)>}} = q(1 \otimes \xi - 1 \otimes \overline{<\epsilon(\gamma)(\xi)>} = q\beta(1 \otimes \overline{<\xi>})$, then $d_1 p = q\beta$. Let M be a $\mathfrak{g}$-module and form the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
\cdots & \mathrm{Hom}_{\mathfrak{g}^e}(JG^2(\gamma),M) \arrow[l] \arrow[d, "p^*"] & \mathrm{Hom}_{\mathfrak{g}^e}(JG(\gamma),M) \arrow[l, "d_1^*"'] \arrow[d, "q^*"] \\
 & \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes G^2(L),M) \arrow[d, "\delta^*"] & \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes G(L),M) \arrow[l, "\beta^*"'] \arrow[d, "d^*"] \\
 & \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes G^2(L) \wedge G^2(L),M) & \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes G(L) \wedge G(L),M)
\end{tikzcd}
$$

Since $\tilde{H}^o(\gamma,M) = \ker d_1^*$, the proof will be completed after we establish the following three lemmas.

<u>Lemma 4.5</u>: Ker $d_1^* \cong \ker d^* \cap \ker \beta^*$.

<u>Proof</u>: Define $\lambda$: $\ker d_1^* \longrightarrow \ker d^* \cap \ker \beta^*$ by $\lambda(f) = q^*(f)$. Since the columns in the diagram are exact, $q^*(f) \in \ker d^*$, and since $\beta^* q^*(f) = \beta^* d_1^*(f) = 0$, $q^*(f) \in \ker \beta^*$ so that $\lambda$ is well-defined. Define $\mu$: $\ker d^* \cap \ker \beta^* \longrightarrow \ker d_1^*$ by $\mu(g) = f$, where $q^*(f) = g$. Note that such an $f$ exists because $g \in \ker d^* = \mathrm{im}\, q^*$. Since $g \in \ker \beta^*$, $\beta^*(g) = \beta^* q^*(f) = p^* d_1^*(f) = 0$, and since $p^*$ is injective $d_1^*(f) = 0$ so that $\mu$ is well-defined. Therefore $\mu\lambda(f) = \mu(q^*(f)) = f$, $\lambda\mu(g) = q^*(\mu(g)) = q^*(f) = g$ and $\lambda$ is an isomorphism.

<u>Lemma 4.6</u>: Ker $d^* \cap \ker \beta^* \cong U$, where $U$ is the class of all R-homomorphisms $f'$: $G(L) \longrightarrow M$ satisfying $f'([\overline{\langle x\rangle},\overline{\langle y\rangle}]) = \gamma(x)f'(\overline{\langle y\rangle}) - \gamma(y)f'(\overline{\langle x\rangle})$ and $f'(\Sigma\, r_i\, \overline{\langle x_i\rangle}) = f'(\overline{\langle \Sigma\, r_i x_i\rangle})$ for all $x,y,\ x_i \in L$ and for all $r_i \in R$.

Proof: For any $f \in \ker d^* \wedge \ker \beta^*$, $\beta^*(f) = f\beta = 0$, that is, $f\beta(1 \otimes \overline{\langle\xi\rangle}) = f(1 \otimes \xi - 1 \otimes \overline{\langle\epsilon(\gamma)(\xi)\rangle}) = f(1 \otimes \Sigma r_i \overline{\langle x_i\rangle}) - f(1 \otimes \overline{\langle\Sigma r_i x_i\rangle}) = 0$, where $r_i \in R$ and $x_i \in L$. Using the adjoint isomorphisms $\mathrm{Hom}(G(L),M) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}_{\mathfrak{g}^e}(\mathfrak{g}^e \otimes G(L),M)$ and writing $\psi(f) = f'$ we have $f'(\Sigma r_i \overline{\langle x_i\rangle}) = f'(\overline{\langle\Sigma r_i x_i\rangle})$. Since $d^*(f) = fd = 0$, $fd(1 \otimes \overline{\langle x\rangle} \wedge \overline{\langle y\rangle}) = f(G(\gamma)(\overline{\langle x\rangle}) \otimes \overline{\langle y\rangle} - G(\gamma)(\overline{\langle y\rangle}) \otimes \overline{\langle x\rangle} - 1 \otimes [\overline{\langle x\rangle}, \overline{\langle y\rangle}]) = f(\gamma(x) \otimes \overline{\langle y\rangle}) - f(\gamma(y) \otimes \overline{\langle x\rangle}) - f(1 \otimes [\overline{\langle x\rangle},\overline{\langle y\rangle}]) = \gamma(x)\, f(1 \otimes \overline{\langle y\rangle}) - \gamma(y)\, f(1 \otimes \overline{\langle x\rangle}) - f(1 \otimes [\overline{\langle x\rangle},\overline{\langle y\rangle}]) = 0$. From the adjoint isomorphism $\psi$ we have $f'([\overline{\langle x\rangle},\overline{\langle y\rangle}]) = \gamma(x)f'(\overline{\langle y\rangle}) - \gamma(y)f'(\overline{\langle x\rangle})$, where $x,y \in L$.

Lemma 4.7: $U \cong V$, where $V$ is the class of all R-homomorphisms $f''\colon G(L) \longrightarrow M$ satisfying $f''([\overline{\langle x\rangle},\overline{\langle y\rangle}]) = \gamma(x)f''(\overline{\langle y\rangle}) - \gamma(y)f''(\overline{\langle x\rangle})$ for all $x,y \in L$ and $f''(\overline{n}) = 0$ for all $n \in \ker \sigma_0$, where $\sigma_0\colon R(\langle L\rangle) \longrightarrow L$ is given by $\sigma_0(\langle x\rangle) = x$.

Proof: The first condition of each class is the same. Let $f' \in U$ and take $n = \Sigma r_i \langle x_i\rangle$ such that $n \in \ker \sigma_0$, that is, $\Sigma r_i x_i = 0$. Since $f'(\overline{n}) = f'(\overline{\Sigma r_i \langle x_i\rangle}) = f'(\Sigma r_i \overline{\langle x_i\rangle}) = f'(\overline{\langle\Sigma r_i x_i\rangle}) = f'(\overline{\langle 0\rangle}) = 0$, $f' \in V$. Let $f'' \in V$ and take $n = \Sigma r_i \langle x_i\rangle - \langle\Sigma r_i x_i\rangle$. Since $\sigma_0(n) = \Sigma r_i x_i - \Sigma r_i x_i = 0$, $n \in \ker \sigma_0$ and since $f''(\overline{n}) = f''(\overline{\Sigma r_i \langle x_i\rangle} - \overline{\langle\Sigma r_i x_i\rangle}) = f''(\Sigma r_i \overline{\langle x_i\rangle}) - f''(\overline{\langle\Sigma r_i x_i\rangle}) = 0$, $f'' \in U$.

The proof of theorem 4.3 is now complete because by lemma 2.3, $V \cong \mathrm{Der}\ (\gamma,M)$.

Let $\mathbb{E}$ be the complex defined in II, 6 and let $G = PU$ as in III,5. Let $I_n\ C\ D\ P\ \Delta = E_n$, that is, $I_n\ C\ D\ P\ \Delta\ (\gamma) = E(\gamma)_n$.

Proposition 4.1: The cochain complex $\mathrm{Hom}_{\mathfrak{g}^e}(\mathbb{E},M)$ is G-representable.

<u>Proof</u>: Since $P \dashv U$ we have $\beta\colon 1 \longrightarrow UP$ so that $\beta U\colon U \longrightarrow UG$. Denoting $U(L)$ by $\langle L\rangle$, define $\beta\langle L\rangle : \langle L\rangle \longrightarrow \langle G(L)\rangle$ by $\beta\langle L\rangle(\langle x\rangle) = \langle\overline{\langle x\rangle}\rangle$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & R(\langle N_p\rangle) \arrow[r] & \cdots \arrow[r] & R(\langle N_1\rangle) \arrow[rr, "d_1"] \arrow[dr] & & R(\langle L\rangle) \arrow[rr, "\sigma_0"] \arrow[ddr, "\alpha"] \arrow[dddd, dashed, "c_0(L)"] & & L \arrow[ddl, "\gamma"'] \arrow[dddd, "f"] \\
& & & & N_1 \arrow[ur, "k_1"'] & & & \\
& & & & & & \mathfrak{g} & \\
& & & & N_1' \arrow[dr, "k_1'"] & & & \\
\cdots \arrow[r] & R(\langle N_p'\rangle) \arrow[r] & \cdots \arrow[r] & R(\langle N_1'\rangle) \arrow[rr, "d_1'"] \arrow[ur] & & R(\langle G(L)\rangle) \arrow[rr, "\sigma_0'"] \arrow[uur, "\alpha'"] & & G(L) \arrow[uul, "G(\gamma)"']
\end{tikzcd}$$

where $f(x) = \overline{\langle x\rangle}$, $N_1 = \ker \sigma_0$, $N_1' = \ker \sigma_0'$, $\alpha(\langle x\rangle) = \gamma(x) = \alpha'(\langle\overline{\langle x\rangle}\rangle)$. Then $\gamma\sigma_0 = \alpha$, $G(\gamma)\sigma_0' = \alpha'$, and $G(\gamma)f = \gamma$. Define $c_0(L)(\Sigma\, r_i\langle x_i\rangle) = \Sigma\, r_i \langle\overline{\langle x_i\rangle}\rangle$ so that $\alpha' c_0(L) = \alpha$ and therefore $f\,\sigma_0 = \sigma_0'\, c_0(L)$. If $n \in N_1$, $\sigma_0' c_0(L) = f\sigma_0(n) = 0$ so that $c_0(L)(N_1) \subset N_1'$. As in II, 2 we define inductively a product-preserving chain map $c(L)\colon \Delta(L) \longrightarrow \Delta G(L)$. To see that $c\colon \Delta \longrightarrow \Delta G$ is a natural transformation let $g\colon L \longrightarrow L'$ be a morphism of Lie algebras and consider the diagram

$$\begin{tikzcd}
\Delta(L) \arrow[r, "\Delta(g)"] \arrow[d, "c(L)"] & \Delta(L') \arrow[d, "c(L')"] \\
\Delta G(L) \arrow[r, "\Delta G(g)"] & \Delta G(L')
\end{tikzcd}$$

For $n = 0$, $c_0(L')\ \Delta(g)_0\ (\langle x\rangle) = c_0(L')(\langle g(x)\rangle) = \langle\overline{\langle g(x)\rangle}\rangle = \langle G(g)(\overline{\langle x\rangle})\rangle = \Delta G(g)_0\ (\langle\overline{\langle x\rangle}\rangle) = \Delta G(g)_0\ c_0(L)(\langle x\rangle)$. Assume that the diagram commutes for all $i<n$. Then $c_n(L')\ \Delta(g)_n(\langle x\rangle) = c_n(L')(\langle g_{n-1}(x)\rangle) = \langle c_{n-1}(L')(g_{n-1}(x))\rangle$ and $\Delta G(g)_n\ c_n(L)(\langle x\rangle) = \Delta G(g)_n\ (\langle c_{n-1}(L)(x)\rangle) = \langle\Delta G(g)_{n-1}\ (c_{n-1}(L)(x))\rangle$ so that by the induction hypothesis the diagram commutes for $n$ and hence $c$ is a

natural transformation. Define $\theta_n: E_n \longrightarrow E_n G$ by $\theta_n = I_n CDPc$. As in II, 2 we define inductively a product-preserving chain map. $r(L): \Delta G(L) \longrightarrow \Delta(L)$ induced by $\varepsilon(\gamma): G(L) \longrightarrow L$, that is, $r_0(L)(\langle\langle \overline{x} \rangle\rangle) = \langle \varepsilon(\gamma)(\langle \overline{x} \rangle) \rangle = \langle x \rangle$ and for $n>0$, $r_n(L)(\langle y \rangle) = \langle r_{n-1}(L)(y) \rangle$. As above, $r: \Delta G \longrightarrow \Delta$ is a natural transformation so that $E_n\varepsilon = I_n CDPr$ is a natural transformation. Then $r_0(L)\, c_0(L)(\langle x \rangle) = r_0(L)(\langle\langle \overline{x} \rangle\rangle) = \langle x \rangle$. Assuming that $r_i(L)c_i(L) = 1_{\Delta(L)_i}$ for all $i<n$, $r_n(L)c_n(L)(\langle x \rangle) = r_n(L)(\langle c_{n-1}(L)(x) \rangle) = \langle r_{n-1}(L)c_{n-1}(L)(x) \rangle = \langle x \rangle$ so that $rc = 1_\Delta$. Since $I_n CDP(rc) = I_n CDPr \circ I_n CDPc = I_n CDP1_\Delta$, $E_n\varepsilon \circ \theta_n = 1_{E_n}$ and the dual definition of G-representability is satisfied.

<u>Proposition 4.2</u>: The cochain complex $\mathrm{Hom}_{\mathfrak{g}^e}(J\mathfrak{G}, M)$ is G-representable and G-acyclic.

<u>Proof</u>: Immediate from lemmas 3.3 and 4.4.

<u>Proposition 4.3</u>: $SH^1(\gamma,M) \cong \tilde{H}^0(\gamma,M)$.

<u>Proof</u>: Immediate from theorems 2.1 and 4.3.

Let Der denote Der ( ,M) and let T denote $\mathrm{Hom}_{\mathfrak{g}^e}($ $,M)$, where M is a $\mathfrak{g}^e$-module. For $\mathbb{E}$, the Shukla complex defined in II,6, and for $J\mathfrak{G}$, the cotriple complex defined in III,5, we have the following diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathrm{Der} & \longrightarrow & TE_1 & \longrightarrow & TE_2 & \longrightarrow & \cdots & \longrightarrow & TE_n & \longrightarrow & \cdots \\
 & & \Vert & & & & & & & & & & \\
0 & \longrightarrow & \mathrm{Der} & \longrightarrow & TJG & \longrightarrow & TJG^2 & \longrightarrow & \cdots & \longrightarrow & TJG^n & \longrightarrow & \cdots
\end{array}$$

Both rows are cochain complexes of contravariant functors. The upper complex is G-representable from proposition 4.1, the lower complex is G-representable and G-acyclic from proposition 4.2, and the equality holds by proposition 4.3. In order to apply theorem 4.2, and hence to obtain isomorphisms between the Shukla and cotriple cohomology modules, it remains to show that the upper complex is G-acyclic.

## CHAPTER V

## SUMMARY AND CONCLUSIONS

This paper is concerned with a discussion of two cotriple cohomologies for Lie algebras and their relationship with the cohomology theories of Hochschild and Shukla. Categorical algebra is the principal tool used throughout this research.

An exposition of the Hochschild cohomology of Lie algebras is given in Chapter I and the low-dimensional modules are calculated. In Chapter II, a modification of Shukla's cohomology of Lie algebra is constructed in a categorical setting and the modules of dimension zero and one are obtained explicitly.

By the construction of adjoint functors, two cotriples are defined in Chapter III. From these cotriples, one on the category of modules over an algebra and the other on the category of non-graded Lie algebras over a fixed Lie algebra, cohomology theories are defined.

Finally, in Chapter IV, the complete comparison of the Hochschild cohomology of Lie algebras and the first cotriple cohomology is obtained by means of the comparison theorem. The one-dimensional Shukla cohomology module is shown to agree with the zero-dimensional cohomology module defined by the second cotriple. Using the terminology by Barr and Beck, the cochain complex employed for the calculation of the cotriple cohomology is shown to be G-representable and G-acyclic and that used for the calculation of the Shukla cohomologies is shown to be G-representable.

The complete comparison of the Shukla and cotriple cohomologies, using the Acyclic Model Theorem, requires the Shukla complex to be G-acyclic. This G-acyclicity has not been demonstrated and is proposed to further research.

Although Glassman [7] has discussed Dixmier's cohomology of Lie rings (Lie algebras over the integers), this theory has not been interpreted as a cotriple cohomology. It is proposed that such an interpretation can be accomplished by means of the cotriple obtained from the adjoint pair $(\mathfrak{L},\mathfrak{g}) \underset{L}{\overset{F}{\rightleftarrows}} ({}_R\mathfrak{M},\mathfrak{g})$, where $({}_R\mathfrak{M},\mathfrak{g})$ is the category of R-modules over $\mathfrak{g}$, F is the forgetful functor, L(M) is the free Lie algebra over M (see [4] , page 285), and $L(f):L(M) \longrightarrow \mathfrak{g}$ is the Lie algebra morphism satisfying $f = L(f)\bullet i$ for the inclusion monomorphism $i:M \longrightarrow L(M)$. This interpretation would provide a partial answer to MacLane's hope (see [9], page 317) that Dixmier's formulation might be simplified, and would provide further evidence that cotriple cohomology gives a unification to the various known cohomologies of Lie algebras, as has been conjectured in [10].

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